MATH 95: ELEMENTARY ALGEBRA II



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Elementary Algebra II

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Detailed Licensing

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1.1: Review- Solving Linear Equations

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1.1.1: SIMPLIFYING ALGEBRAIC EXPRESSIONS

LEARNING OBJECTIVES

- Apply the distributive property to simplify an algebraic expression.
- Identify and combine like terms.

DISTRIBUTIVE PROPERTY

The properties of real numbers are important in our study of algebra because a variable is simply a letter that represents a real number. In particular, the distributive property states that given any real numbers $a, b,$ and c,

$$a(b+c) = ab+ac \tag{1.1.1.1}$$

This property is applied when simplifying algebraic expressions. To demonstrate how it is used, we simplify $2(5-3)$ in two ways, and observe the same correct result.

Certainly, if the contents of the parentheses can be simplified, do that first. On the other hand, when the contents of parentheses cannot be simplified, multiply every term within the parentheses by the factor outside of the parentheses using the distributive property. Applying the distributive property allows you to multiply and remove the parentheses.

Simplify:

$$5(7y+2).$$

Solution:

Multiply 5 times each term inside the parentheses.

$$\begin{aligned} 5(7y+2) &= 5\cdot 7y+5\cdot 2 \\ &= 35y+10 \end{aligned}$$

Answer:

$$35y+10$$

EXAMPLE 1.1.1.2

Simplify:

$$-3(2x^2+5x+1).$$

Solution:

Multiply -3 times each of the coefficients of the terms inside the parentheses.

Answer:

$-6x^2-15x-3$

EXAMPLE 1.1.1.3

Simplify:

$$5(-2a+5b)-2c.$$

Solution:

Apply the distributive property by multiplying only the terms grouped within the parentheses by 5.

$$\begin{aligned} &5(-2a+5b)-2c \\ &= 5\cdot(-2a)+5\cdot 5b-2c \\ &= -10a+25b-2c \end{aligned}$$

***Figure* 1.1.1.1**

Answer:

$$-10a+25b-2c$$

Because multiplication is commutative, we can also write the distributive property in the following manner:

$$(b+c)a = ba+ca \tag{1.1.1.2}$$

EXAMPLE 1.1.1.4

Simplify:

$$(3x-4y+1)\cdot 3\,.$$

Solution:

Multiply each term within the parentheses by 3.

$$\begin{aligned}(3x-4y+1)\cdot 3 &= 3x\cdot 3-4y\cdot 3+1\cdot 3\\ &= 9x-12y+3\end{aligned}$$

Answer:

$$9x-12y+3$$

Division in algebra is often indicated using the fraction bar rather than with the symbol (÷). And sometimes it is useful to rewrite expressions involving division as products:

$$\frac{x}{5} = \frac{1x}{5} = \frac{1}{5}\cdot x$$
$$\frac{3ab}{7} = \frac{3}{7}\cdot\frac{ab}{1} = \frac{3}{7}\cdot ab$$
$$\frac{x+y}{3} = \frac{1}{3}\cdot\frac{(x+y)}{1} = \frac{1}{3}\cdot(x+y)$$

Rewriting algebraic expressions as products allows us to apply the distributive property.

EXAMPLE 1.1.1.5

Divide:

\(\frac{25x^{2}-5x+10}{5}.

Solution:

First, treat this as $\frac{1}{5}$ times the expression in the numerator and then distribute.

$$\begin{aligned}\frac{25x^2-5x+10}{5} &= \frac{1}{5}\cdot\frac{(25x^2-5x+10)}{1}\\ &= \frac{1}{5}\cdot(25x^2-5x+10) && \textit{Multiply each term by } \frac{1}{5}.\\ &= \frac{1}{5}\cdot 25x^2-\frac{1}{5}\cdot 5x+\frac{1}{5}\cdot 10 && \textit{Simplify.}\\ &= 5x^2-x+2\end{aligned}$$

Alternate Solution:

Think of 5 as a common denominator and divide each of the terms in the numerator by 5:

$$\begin{aligned}\frac{25x^2-5x+10}{5} &= \frac{25x^2}{5}-\frac{5x}{5}+\frac{10}{5}\\ &= 5x^2-x+2\end{aligned}$$

Answer:

$$5x^2-x+2$$

We will discuss the division of algebraic expressions in more detail as we progress through the course.

EXERCISE 1.1.1.1

Simplify:

$$\tfrac{1}{3}(-9x+27y-3)\,.$$

Answer

$-3x+9y-1$

COMBINING LIKE TERMS

Terms with the same variable parts are called like terms, or similar terms. Furthermore, constant terms are considered to be like terms. If an algebraic expression contains like terms, apply the distributive property as follows:

$$2a+3a=(2+3)a=5a$$
$$7xy-5xy=(7-5)xy=2xy$$
$$10x^2+4x^2-6x^2=(10+4-6)x^2=8x^2$$

In other words, if the variable parts of terms are ***exactly the same***, then we may add or subtract the coefficients to obtain the coefficient of a single term with the same variable part. This process is called combining like terms. For example,

$$3a^2b+2a^2b=5a^2b$$

Notice that the variable factors and their exponents do not change. Combining like terms in this manner, so that the expression contains no other similar terms, is called simplifying the expression. Use this idea to simplify algebraic expressions with multiple like terms.

EXAMPLE 1.1.1.6

Simplify:

$$3a+2b-4a+9b\ .$$

Solution:

Identify the like terms and combine them.

$$\begin{aligned} 3a+2b-4a+9b &= 3a-4a+2b+9b && \textit{Commutative property of addition} \\ &= -1a+11b && \textit{Combine like terms.} \\ &= -a+11b \end{aligned}$$

Answer:

$$-a+11b$$

In the previous example, rearranging the terms is typically performed mentally and is not shown in the presentation of the solution.

EXAMPLE 1.1.1.7

Simplify:

$$x^2+3x+2+4x^2-5x-7\ .$$

Solution:

Identify the like terms and add the corresponding coefficients.

$$\begin{aligned} &1x^2+3x+2+4x^2-5x-7 && \textit{Identify like terms.} \\ &=5x^2-2x-5 && \textit{Combine like terms.} \end{aligned}$$

Answer:

$$5x^2-2x-5$$

EXAMPLE 1.1.1.8

Simplify:

$$5x^2y-3xy^2+4x^2y-2xy^2\ .$$

Solution:

Remember to leave the variable factors and their exponents unchanged in the resulting combined term.

$$\begin{aligned} &5x^2y-3xy^2+4x^2y-2xy^2 \\ &=9x^2y-5xy^2 \end{aligned}$$

Answer:

$$9x^2y-5xy^2$$

EXAMPLE 1.1.1.9

Simplify:

$$\tfrac{1}{2}a-\tfrac{1}{3}b+\tfrac{3}{4}a+b\,.$$

To add the fractional coefficients, use equivalent coefficients with common denominators for each like term.

$$\begin{aligned}\frac{1}{2}a-\frac{1}{3}b+\frac{3}{4}a+1b &= \frac{1}{2}a+\frac{3}{4}a-\frac{1}{3}b+1b\\ &= \frac{2}{4}a+\frac{3}{4}a-\frac{1}{3}b+\frac{3}{3}b\\ &= \frac{5}{4}a+\frac{2}{3}b\end{aligned}$$

Answer:

$$\tfrac{5}{4}a+\tfrac{2}{3}b$$

EXAMPLE 1.1.1.10

Simplify:

$$-12x(x+y)^3+26x(x+y)^3\,.$$

Solution:

Consider the variable part to be $x(x+y)^3$. Then this expression has two like terms with coefficients -12 and 26.

$$\begin{aligned}&-12x(x+y)^3+26x(x+y)^3 \qquad \textit{Add the coefficients.}\\ &=14x(x+y)^3\end{aligned}$$

Answer:

$$14x(x+y)^3$$

EXERCISE 1.1.1.2

Simplify:

$$-7x+8y-2x-3y\,.$$

Answer

$-9x+5y$

DISTRIBUTIVE PROPERTY AND LIKE TERMS

When simplifying, we will often have to combine like terms after we apply the distributive property. This step is consistent with the order of operations: multiplication before addition.

EXAMPLE 1.1.1.11

Simplify:

$$2(3a-b)-7(-2a+3b).$$

Solution:

Distribute 2 and -7 and then combine like terms.

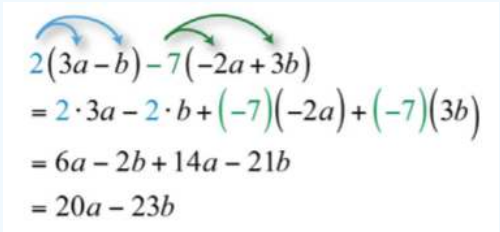


***Figure* 1.1.1.2**

Answer:

$$20a - 23b$$

In the example above, it is important to point out that you can remove the parentheses and collect like terms because you multiply the second quantity by -7, not just by 7. To correctly apply the distributive property, think of this as adding -7 times the given quantity, $2(3a - b) + (-7)(-2a + 3b)$.

? EXERCISE 1.1.1.3

Simplify:

$$-5(2x - 3) + 7x.$$

Answer

$-3x + 15$

Often we will encounter algebraic expressions like $+(a + b)$ or $-(a + b)$. As we have seen, the coefficients are actually implied to be $+1$ and -1, respectively, and therefore, the distributive property applies using $+1$ or -1 as the factor. Multiply each term within the parentheses by these factors:

$$+(a + b) = +1(a + b) = (+1)a + (+1)b = a + b \tag{1.1.1.3}$$

$$-(a + b) = -1(a + b) = (-1)a + (-1)b = -a - b \tag{1.1.1.4}$$

This leads to two useful properties,

$$+(a + b) = a + b \tag{1.1.1.5}$$

$$-(a + b) = -a - b \tag{1.1.1.6}$$

✔ EXAMPLE 1.1.1.12

Simplify:

$$5x - (-2x^2 + 3x - 1)\,.$$

Solution:

Multiply each term within the parentheses by -1 and then combine like terms.

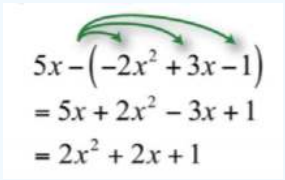


***Figure* 1.1.1.3**

Answer:

$$2x^2 + 2x + 1$$

When distributing a negative number, all of the signs within the parentheses will change. Note that $5x$ in the example above is a separate term; hence the distributive property does not apply to it.

✔ EXAMPLE 1.1.1.13

Simplify:

$$5 - 2(x^2 - 4x - 3)\,.$$

Solution:

The order of operations requires that we multiply before subtracting. Therefore, distribute -2 and then combine the constant terms. Subtracting $5 - 2$ first leads to an incorrect result, as illustrated below:

Incorrect!	*Correct!*
$5-2(x^2-4x-3)$	$5-2(x^2-4x-3)$
$=3(x^2-4x-3)$	$=5-2x^2+8x+6$
$=3x^2-12x-9$ x	$=-2x^2+8x+11$ ✓

Answer:

$$-2x^2+8x+11$$

NOTE

It is worth repeating that you must ***follow the order of operations***: multiply and divide before adding and subtracting!

EXERCISE 1.1.1.4

Simplify:

$$8-3(-x^2+2x-7)\,.$$

Answer

$3x^2-6x+29$

EXAMPLE 1.1.1.14

Subtract $3x-2$ from twice the quantity $-4x^2+2x-8$.

Solution:

First, group each expression and treat each as a quantity:

$$(3x-2) \quad \text{and} \quad (-4x^2+2x-8)$$

Next, identify the key words and translate them into a mathematical expression.

$$\overbrace{2\cdot(-4x^2+2x-8)}^{\textit{Twice the quantity...}} \overbrace{-(3x-2)}^{\textit{Subtract from...}}$$

***Figure* 1.1.1.4**

Finally, simplify the resulting expression.

Answer:

$-8x^2+x-14$

KEY TAKEAWAYS

- The properties of real numbers apply to algebraic expressions, because variables are simply representations of unknown real numbers.
- Combine like terms, or terms with the same variable part, to simplify expressions.
- Use the distributive property when multiplying grouped algebraic expressions, $a(b+c)=ab+ac$.
- It is a best practice to apply the distributive property only when the expression within the grouping is completely simplified.
- After applying the distributive property, eliminate the parentheses and then combine any like terms.
- Always use the order of operations when simplifying.

EXERCISE 1.1.1.5 DISTRIBUTIVE PROPERTY

Multiply.

1. $3(3x-2)$
2. $12(-5y+1)$
3. $-2(x+1)$
4. $5(a-b)$
5. $\frac{5}{8}(8x-16)$
6. $-\frac{3}{5}(10x-5)$

7. $(2x+3)\cdot 2$
8. $(5x-1)\cdot 5$
9. $(-x+7)(-3)$
10. $(-8x+1)(-2)$
11. $-(2a-3b)$
12. $-(x-1)$
13. $\frac{1}{3}(2x+5)$
14. $-\frac{3}{4}(y-2)$
15. $-3(2a+5b-c)$
16. $-(2y^2-5y+7)$
17. $5(y^2-6y-9)$
18. $-6(5x^2+2x-1)$
19. $7x^2-(3x-11)$
20. $-(2a-3b)+c$
21. $3(7x^2-2x)-3$
22. $\frac{1}{2}(4a^2-6a+4)$
23. $-\frac{1}{3}(9y^2-3y+27)$
24. $(5x^2-7x+9)(-5)$
25. $6(\frac{1}{3}x^2-\frac{1}{6}x+\frac{1}{2})$
26. $-2(3x^3-2x^2+x-3)$
27. $\frac{20x+30y-10z}{10}$
28. $\frac{-4a+20b-8c}{4}$
29. $\frac{3x^2-9x+81}{-3}$
30. $\frac{15y^2+20y-5}{5}$

Answer

1. $9x-6$
3. $-2x-2$
5. $5x-10$
7. $4x+6$
9. $3x-21$
11. $-2a+3b$
13. $\frac{2}{3}x+\frac{5}{3}$
15. $-6a-15b+3c$
17. $5y^2-30y-45$
19. $7x^2-3x+11$
21. $21x^2-6x-3$
23. $-3y^2+y-9$
25. $2x^2-x+3$
27. $2x+3y-z$
29. $-x^2+3x-27$

? EXERCISE 1.1.1.6 DISTRIBUTIVE PROPERTY

Translate the following sentences into algebraic expressions and then simplify.

1. Simplify two times the expression $25x^2-9$.
2. Simplify the opposite of the expression $6x^2+5x-1$.
3. Simplify the product of 5 and x^2-8.
4. Simplify the product of -3 and $-2x^2+x-8$.

Answer

1. $50x^2 - 18$
3. $5x^2 - 40$

? EXERCISE 1.1.1.7 COMBINING LIKE TERMS

Simplify.

1. $2x - 3x$
2. $-2a + 5a - 12a$
3. $10y - 30 - 15y$
4. $\frac{1}{3}x + \frac{5}{12}x$
5. $-\frac{1}{4}x + \frac{4}{5} + \frac{3}{8}x$
6. $2x - 4x + 7x - x$
7. $-3y - 2y + 10y - 4y$
8. $5x - 7x + 8y + 2y$
9. $-8\alpha + 2\beta - 5\alpha - 6\beta$
10. $-6\alpha + 7\beta - 2\alpha + \beta$
11. $3x + 5 - 2y + 7 - 5x + 3y$
12. $-y + 8x - 3 + 14x + 1 - y$
13. $4xy - 6 + 2xy + 8$
14. $-12ab - 3 + 4ab - 20$
15. $\frac{1}{3}x - \frac{2}{5}y + \frac{2}{3}x - \frac{3}{5}y$
16. $\frac{3}{8}a - \frac{2}{7}b - \frac{1}{4}a + \frac{3}{14}b$
17. $-4x^2 - 3xy + 7 + 4x^2 - 5xy - 3$
18. $x^2 + y^2 - 2xy - x^2 + 5xy - y^2$
19. $x^2 - y^2 + 2x^2 - 3y$
20. $\frac{1}{2}x^2 - \frac{2}{3}y^2 - \frac{1}{8}x^2 + \frac{1}{5}y^2$
21. $\frac{3}{16}a^2 - \frac{4}{5} + \frac{1}{4}a^2 - \frac{1}{4}$
22. $\frac{1}{5}y^2 - \frac{3}{4} + \frac{7}{10}y^2 - \frac{1}{2}$
23. $6x^2y - 3xy^2 + 2x^2y - 5xy^2$
24. $12x^2y^2 + 3xy - 13x^2y^2 + 10xy$
25. $-ab^2 + a^2b - 2ab^2 + 5a^2b$
26. $m^2n^2 - mn + mn - 3m^2n + 4m^2n^2$
27. $2(x + y)^2 + 3(x + y)^2$
28. $\frac{1}{5}(x + 2)^3 - \frac{2}{3}(x + 2)^3$
29. $-3x(x^2 - 1) + 5x(x^2 - 1)$
30. $5(x - 3) - 8(x - 3)$
31. $-14(2x + 7) + 6(2x + 7)$
32. $4xy(x + 2)^2 - 9xy(x + 2)^2 + xy(x + 2)^2$

Answer

1. $-x$
3. $-5y - 30$
5. $\frac{1}{8}x + \frac{4}{5}$
7. y
9. $-13\alpha - 4\beta$
11. $-2x + y + 12$
13. $6xy + 2$
15. $x - y$
17. $-8xy + 4$
19. $3x^2 - y^2 - 3y$
21. $\frac{7}{16}a^2 - \frac{21}{20}$

23. $8x^2y - 8xy^2$
25. $6a^2b - 3ab^2$
27. $5(x+y)^2$
29. $2x(x^2-1)$
31. $-8(2x+7)$

EXERCISE 1.1.1.8 MIXED PRACTICE

Simplify.

1. $5(2x-3)+7$
2. $-2(4y+2)-3y$
3. $5x-2(4x-5)$
4. $3-(2x+7)$
5. $2x-(3x-4y-1)$
6. $(10y-8)-(40x+20y-7)$
7. $\frac{1}{2}y-\frac{3}{4}x-(\frac{2}{3}y-\frac{1}{5}x)$
8. $\frac{1}{5}a-\frac{3}{4}b+\frac{3}{15}a-\frac{1}{2}b$
9. $\frac{2}{3}(x-y)+x-2y$
10. $-\frac{1}{3}(6x-1)+\frac{1}{2}(4y-1)-(-2x+2y-\frac{1}{6})$
11. $(2x^2-7x+1)+(x^2+7x-5)$
12. $6(-2x^2+3x-1)+10x^2-5x$
13. $-(x^2-3x+8)+x^2-12$
14. $2(3a-4b)+4(-2a+3b)$
15. $-7(10x-7y)-6(8x+4y)$
16. $10(6x-9)-(80x-35)$
17. $10-5(x^2-3x-1)$
18. $4+6(y^2-9)$
19. $\frac{3}{4}x-(\frac{1}{2}x^2+\frac{2}{3}x-\frac{7}{5})$
20. $-\frac{7}{3}x^2+(-\frac{1}{6}x^2+7x-1)$
21. $(2y^2-3y+1)-(5y^2+10y-7)$
22. $(-10a^2-b^2+c)+(12a^2+b^2-4c)$
23. $-4(2x^2+3x-2)+5(x^2-4x-1)$
24. $2(3x^2-7x+1)-3(x^2+5x-1)$
25. $x^2y+3xy^2-(2x^2y-xy^2)$
26. $3(x^2y^2-12xy)-(7x^2y^2-20xy+18)$
27. $3-5(ab-3)+2(ba-4)$
28. $-9-2(xy+7)-(yx-1)$
29. $-5(4\alpha-2\beta+1)+10(\alpha-3\beta+2)$
30. $\frac{1}{2}(100\alpha^2-50\alpha\beta+2\beta^2)-\frac{1}{5}(50\alpha^2+10\alpha\beta-5\beta^2)$

Answer

1. $10x-8$
3. $-3x+10$
5. $-x+4y+1$
7. $-\frac{11}{20}x-\frac{1}{6}y$
9. $\frac{5}{3}x-\frac{8}{3}y$
11. $3x^2-4$
13. $3x-20$
15. $-118x+25y$
17. $-5x^2+15x+15$
19. $-\frac{1}{2}x^2+\frac{1}{12}x+\frac{7}{5}$

21. $-3y^2-13y+8$
23. $-3x^2-32x+3$
25. $-x^2y+4xy^2$
27. $-3ab+10$
29. $-10\alpha-20\beta+15$

EXERCISE 1.1.1.9 MIXED PRACTICE

Translate the following sentences into algebraic expressions and then simplify.

1. What is the difference of $3x-4$ and $-2x+5$?
2. Subtract $2x-3$ from $5x+7$.
3. Subtract $4x+3$ from twice the quantity $x-2$.
4. Subtract three times the quantity $-x+8$ from $10x-9$.

Answer

1. $5x-9$
3. $-2x-7$

EXERCISE 1.1.1.10 DISCUSSION BOARD TOPICS

1. Do we need a distributive property for division, $(a+b)\div c$? Explain.
2. Do we need a separate distributive property for three terms, $a(b+c+d)$? Explain.
3. Explain how to subtract one expression from another. Give some examples and demonstrate the importance of the order in which subtraction is performed.
4. Given the algebraic expression $8-5(3x+4)$, explain why subtracting $8-5$ is not the first step.
5. Can you apply the distributive property to the expression $5(abc)$? Explain why or why not and give some examples.
6. How can you check to see if you have simplified an expression correctly? Give some examples.

Answer

1. Answers may vary
3. Answers may vary
5. Answers may vary

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1.1.2: SOLVING LINEAR EQUATIONS- PART 1

LEARNING OBJECTIVES

- Identify linear equations with one variable and verify their solutions.
- Use the properties of equality to solve basic linear equations.
- Use multiple steps to solve linear equations by isolating the variable.
- Solve linear equations where the coefficients are fractions or decimals.

LINEAR EQUATIONS WITH ONE VARIABLE AND THEIR SOLUTIONS

Learning how to solve various algebraic equations is one of the main goals in algebra. This section introduces the basic techniques used for solving linear equations with one variable.

An equation is a statement indicating that two algebraic expressions are equal. A linear equation with one variable, x, is an equation that can be written in the general form $ax+b=0$, where a and b are real numbers and $a \neq 0$. Here are some examples of linear equations, all of which are solved in this section:

$$x+3=-5 \qquad 2x-5=15 \qquad \tfrac{5}{3}x+2=-8$$

A solution to a linear equation is any value that can replace the variable to produce a true statement. The variable in the linear equation $2x+3=13$ is x, and the solution is $x=5$. To verify this, substitute the value 5 for x and check that you obtain a true statement.

$$\begin{aligned} 2x+13 &= 13 \\ 2(5)+3 &= 13 \\ 10+3 &= 13 \\ 13 &= 13 \quad \checkmark \end{aligned}$$

Alternatively, when an equation is equal to a constant, we can verify a solution by substituting the value for the variable and show that the result is equal to that constant. In this sense, we say that solutions satisfy the equation.

EXAMPLE 1.1.2.1

Is $x=3$ a solution to $-2x-3=-9$?

Solution:

$$-2x-3=-2(3)-3=-6-3=-9 \quad \checkmark$$

Answer:

Yes, it is a solution, because $x=3$ satisfies the equation.

EXAMPLE 1.1.2.2

Is $a=-\frac{1}{2}$ a solution to $-10a+5=25$?

Solution:

$$-10a+5=-10\left(-\tfrac{1}{2}\right)+5=5+5=10 \neq 25 \quad x$$

Answer:

No, it is not a solution, because $a=-\frac{1}{2}$ does not satisfy the equation.

Recall that when evaluating expressions, it is a good practice to first replace all variables with parentheses, then substitute the appropriate values. By making use of parentheses we avoid some common errors when working the order of operations.

EXAMPLE 1.1.2.3

Is $y=-3$ a solution to $2y-5=-y-14$?

Solution:

Answer:

Yes, it is a solution, because $y=-3$ produces a true statement.

? EXERCISE 1.1.2.1

Is $x = -3$ a solution to $-2x + 5 = -1$?

Answer

No

SOLVING BASIC LINEAR EQUATIONS

We begin by defining equivalent equations as equations with the same solution set. Consider the following two linear equations and check to see if the solution is $x = 7$.

$$\begin{array}{rl|rl} 3x-5 &= 16 & 3x &= 21 \\ 3(7)-5 &= 16 & 3(7) &= 21 \\ 21-5 &= 16 & 21 &= 21 \quad \checkmark \\ 16 &= 16 \quad \checkmark & & \end{array}$$

Here we can see that the two linear equations $3x - 5 = 16$ and $3x = 21$ are equivalent because they share the same solution set, namely, $\{7\}$. The goal is to develop a systematic process to find equivalent equations until the variable is isolated:

$$\left.\begin{array}{r} 3x-5 = 16 \\ 3x = 21 \\ x = 7 \end{array}\right\} \quad \textit{Equivalent equations}$$

To do this, use the properties of equality. Given algebraic expressions A and B, where c is a real number, we have the following:

$$\begin{array}{ll} \textbf{Addition Property of Equality}: & \textit{If } A = B, \textit{ then } A+c = B+c \\ \textbf{Subtraction Property of Equality:} & \textit{If } A = B, \textit{ then } A-c = B-c \\ \textbf{Multiplication Property of Equality:} & \textit{If } A = B, \textit{ then } cA = cB \\ \textbf{Division Property of Equality:} & \textit{If } A = B, \textit{ then } \frac{A}{c} = \frac{B}{c}\ c \neq 0 \end{array}$$

NOTE

Multiplying or dividing both sides of an equation by 0 is carefully avoided. Dividing by 0 is undefined and multiplying both sides by 0 results in the equation $0 = 0$.

To summarize, equality is retained and you obtain an equivalent equation if you add, subtract, multiply, or divide ***both sides*** of an equation by any nonzero real number. The technique for solving linear equations involves applying these properties in order to ***isolate the variable*** on one side of the equation. If the linear equation has a constant term, then we add to or subtract it from both sides of the equation to obtain an equivalent equation where the variable term is isolated.

✔ EXAMPLE 1.1.2.4

Solve:

$$x + 3 = -5.$$

Solution:

To isolate the variable x on the left side, subtract 3 from both sides.

Answer:

The solution is $x = -8$.

To check that this is true, substitute -8 into the original equation and simplify to see that it is satisfied: $x + 3 = -8 + 3 = -5 \quad \checkmark$.

In the previous example, after subtracting 3 from both sides, you get $x + 0 = -8$. By the additive identity property of real numbers, this is equivalent to $x = -8$. This step is often left out in the presentation of the solution.

If the ***variable term*** of the equation (including the coefficient) is isolated, then apply the multiplication or division property of equality to obtain an equivalent equation with the ***variable*** isolated. In other words, our goal is to obtain an equivalent equation with x or $1x$ isolated on one side of the equal sign.

EXAMPLE 1.1.2.5

Solve:

$$-5x = -35.$$

Solution:

The coefficient of x is -5, so divide both sides by -5.

Answer:

The solution is $x = 7$. Perform the check mentally by substituting 7 for x in the original equation.

In the previous example, after dividing both sides by $-5, x$ is left with a coefficient of 1, because $\frac{-5}{-5} = 1$. In fact, when we say "isolate the variable," we mean to change the coefficient of the variable to 1, because $1x = 7$ is equivalent to $x = 7$. This step is often left out of the instructional examples even though its omission is sometimes a source of confusion.

Another important property is the symmetric property: for any algebraic expressions A and B,

$$\text{If } A = B, \text{ then } B = A \tag{1.1.2.1}$$

The equation $2 = x$ is equivalent to $x = 2$. It does not matter on which side we choose to isolate the variable.

EXAMPLE 1.1.2.6

Solve:

$$2 = 5 + x\,.$$

Solution:

Isolate the variable x by subtracting 5 from both sides of the equation.

Answer:

The solution is -3, and checking the solution shows that $2 = 5 - 3\,.$

EXERCISE 1.1.2.2

Solve:

$$6 = x - 4$$

Answer

$x = 10$

ISOLATING THE VARIABLE IN TWO STEPS

A linear equation of the form $ax + b = c$ takes two steps to solve. First, use the appropriate equality property of addition or subtraction to isolate the variable term. Next, isolate the variable using the equality property of multiplication or division. Checking solutions in the following examples is left to the reader.

EXAMPLE 1.1.2.7

Solve:

$$2x - 5 = 15\,.$$

Solution:

$$\begin{aligned} 2x - 5 &= 15 \\ 2x - 5{\color{cyan}+5} &= 15{\color{cyan}+5} && \textit{Add 5 to both sides.} \\ 2x &= 20 \\ \frac{2x}{\color{cyan}2} &= \frac{20}{\color{cyan}2} && \textit{Divide both sides by 2.} \\ x &= 10 \end{aligned}$$

Answer:

The solution is 10.

EXAMPLE 1.1.2.8

Solve:

$$-3x-2=9.$$

Solution:

$$\begin{aligned} -3x-2 &= 9 \\ -3x-2+2 &= 9+2 && \textit{Add 2 to both sides.} \\ -3x &= 11 \\ \frac{-3x}{-3} &= \frac{11}{-3} && \textit{Divide both sides by } -3. \\ x &= -\frac{11}{3} \end{aligned}$$

Answer:

The solution is $-\frac{11}{3}$.

EXAMPLE 1.1.2.9

Solve:

$$6-5y=-14.$$

Solution:

When no sign precedes the term, it is understood to be positive. In other words, think of this as $+6-5y=-14$. Begin by subtracting 6 from both sides of the equal sign.

Answer:

The solution is 4.

EXAMPLE 1.1.2.10

Solve:

$$3x+\tfrac{1}{2}=\tfrac{2}{3}.$$

Solution:

$$\begin{aligned} 3x+\frac{1}{2} &= \frac{2}{3} \\ 3x+\frac{1}{2}-\frac{1}{2} &= \frac{2}{3}-\frac{1}{2} && \textit{Subtract } \frac{1}{2} \textit{ from both sides.} \\ 3x &= \frac{2\cdot 2}{3\cdot 2}-\frac{1\cdot 3}{2\cdot 3} && \textit{Obtain equivalent fractions with a common denominator.} \\ 3x &= \frac{4}{6}-\frac{3}{6} \\ 3x &= \frac{1}{6} \\ \frac{3x}{3} &= \frac{6}{3} && \textit{Divide both sides by 3.} \\ x &= \frac{1}{6}\div 3=\frac{1}{6}\cdot\frac{1}{3}=\frac{1}{18} \end{aligned}$$

Answer:

The solution is $\frac{1}{18}$.

EXAMPLE 1.1.2.11

Solve:

$$3-y=1.$$

Solution:

Recall that $-y$ is equivalent to $-1y$; divide both sides of the equation by -1.

$$\begin{aligned} -y &= -2 \\ \frac{-1y}{-1} &= \frac{-2}{-1} \\ y &= 2 \end{aligned}$$

Alternatively, multiply both sides of $-y = -2$ by -1 and achieve the same result:

Answer:

The solution is 2.

In summary, to retain equivalent equations, we must perform the same operation on both sides of the equation. First, apply the addition or subtraction property of equality to isolate the variable term and then apply the multiplication or division property of equality to isolate the variable on one side of the equation.

? EXERCISE 1.1.2.3

Solve:

$$-7x + 6 = 27.$$

Answer

$x = -3$

MULTIPLYING BY THE RECIPROCAL

To solve an equation like $\frac{3}{4}x = 1$, we can isolate the variable by dividing both sides by the coefficient. For example

$$\begin{aligned} \frac{3}{4}x &= 1 \\ \frac{\frac{3}{4}x}{\frac{3}{4}} &= \frac{1}{\frac{3}{4}} \\ x &= 1 \cdot \frac{4}{3} \\ x &= \frac{4}{3} \end{aligned}$$

On the left side of the equal sign, the fraction cancels. On the right side, we have a complex fraction and multiply by the reciprocal of the coefficient. You can save a step by recognizing this and start by multiplying both sides of the equation by the reciprocal of the coefficient.

$$\begin{aligned} \frac{3}{4}x &= 1 \\ \frac{4}{3} \cdot \frac{3}{4}x &= \frac{4}{3} \cdot 1 \\ x &= \frac{4}{3} \end{aligned}$$

Recall that the product of reciprocals is 1, in this case $\frac{4}{3} \cdot \frac{3}{4} = 1$, leaving the variable isolated.

✓ EXAMPLE 1.1.2.12

Solve:

$$\frac{5}{3}x + 2 = -8.$$

Solution:

Isolate the variable term using the addition property of equality and then multiply both sides of the equation by the reciprocal of the coefficient $\frac{5}{3}$.

Answer:

The solution is -6.

EXAMPLE 1.1.2.13

Solve:

$$-\tfrac{4}{5}x - 5 = 15.$$

Solution:

$$\begin{aligned} -\frac{4}{5}x - 5 &= 15 \\ -\frac{4}{5}x - 5{\color{cyan}+5} &= 15{\color{cyan}+5} \\ -\frac{4}{5}x &= 20 \end{aligned}$$

The reciprocal of $-\frac{4}{5}$ is $-\frac{5}{4}$ because $(-\frac{5}{4})(-\frac{4}{5}) = +\frac{20}{20} = 1$. Therefore, to isolate the variable x, multiply both sides by $-\frac{5}{4}$.

Answer:

The solution is -25.

EXERCISE 1.1.2.4

Solve:

$$\tfrac{2}{3}x - 9 = -4.$$

Answer

$x = \frac{15}{2}$

KEY TAKEAWAYS

- Linear equations with one variable can be written in the form $ax + b = 0$, where a and b are real numbers and $a \neq 0$.
- To "solve a linear equation" means to find a numerical value that can replace the variable and produce a true statement.
- The properties of equality provide tools for isolating the variable and solving equations.
- To solve a linear equation, first isolate the *variable term* by adding the opposite of the constant term to both sides of the equation. Then isolate the *variable* by dividing both sides of the equation by its coefficient.
- After isolating a variable term with a fraction coefficient, solve by multiplying both sides by the reciprocal of the coefficient.

EXERCISE 1.1.2.5 SOLUTIONS TO LINEAR EQUATIONS

Is the given value a solution to the linear equation?

1. $x - 6 = 20; x = 26$
2. $y + 7 = -6; y = -13$
3. $-x + 5 = 17; x = 12$
4. $-2y = 44; y = 11$
5. $4x = -24; x = -6$
6. $5x - 1 = 34; x = -7$
7. $-2a - 7 = -7; a = 0$
8. $-\frac{1}{3}x - 4 = -5; x = -3$
9. $-\frac{1}{2}x + \frac{2}{3} = -\frac{1}{4}; x = \frac{11}{6}$
10. $-8x - 33 = 3x; x = 3$
11. $3y - 5 = -2y - 15; y = -2$
12. $3(2x + 1) = -4x - 3; x = -\frac{1}{2}$
13. $\frac{1}{2}y - \frac{1}{3} = \frac{1}{3}y + \frac{1}{6}; y = 3$
14. $-\frac{4}{3}y + \frac{1}{9} = -\frac{2}{3}y - \frac{1}{9}; y = \frac{1}{3}$

Answer

1. Yes
3. No
5. Yes

7. Yes
9. Yes
11. Yes
13. Yes

EXERCISE 1.1.2.6 SOLVING BASIC LINEAR EQUATIONS

Solve.

1. $x+3=13$
2. $y-4=22$
3. $-6+x=12$
4. $9+y=-4$
5. $x-\frac{1}{2}=\frac{1}{3}$
6. $x+\frac{2}{3}=-\frac{1}{5}$
7. $x+2\frac{1}{2}=3\frac{1}{3}$
8. $-37+x=-37$
9. $4x=-44$
10. $-9x=63$
11. $-y=13$
12. $-x=-10$
13. $-9x=0$
14. $-3a=-33$
15. $27=18y$
16. $14=-7x$
17. $31.5.6a=-39.2$
18. $-1.2y=3.72$
19. $\frac{1}{3}x=-\frac{1}{2}$
20. $-\frac{t}{12}=\frac{1}{4}$
21. $-\frac{7}{3}x=\frac{1}{2}$
22. $\frac{x}{5}=-3$
23. $\frac{4}{9}y=-\frac{2}{3}$
24. $-\frac{5}{8}y=-\frac{5}{2}$

Answer

1. 10
3. 18
5. $\frac{5}{6}$
7. $\frac{5}{6}$
9. -11
11. -13
13. 0
15. $\frac{3}{2}$
17. -7
19. $-\frac{3}{2}$
21. $-\frac{3}{14}$
23. $-\frac{3}{2}$

EXERCISE 1.1.2.7 SOLVING LINEAR EQUATIONS

Solve.

1. $5x + 7 = 32$
2. $4x - 3 = 21$
3. $3a - 7 = 23$
4. $12y + 1 = 1$
5. $21x - 7 = 0$
6. $-3y + 2 = -13$
7. $-5x + 9 = 8$
8. $22x - 55 = -22$
9. $4.5x - 2.3 = 6.7$
10. $1.4 - 3.2x = 3$
11. $9.6 - 1.4y = -10.28$
12. $4.2y - 3.71 = 8.89$
13. $3 - 2y = -11$
14. $-4 - 7a = 24$
15. $-10 = 2x - 5$
16. $24 = 6 - 12y$
17. $\frac{5}{6}x - \frac{1}{2} = \frac{2}{3}$
18. $\frac{1}{2}x + \frac{1}{3} = \frac{2}{5}$
19. $4a - \frac{2}{3} = -\frac{1}{6}$
20. $\frac{3}{5}x - \frac{1}{2} = \frac{1}{10}$
21. $-\frac{4}{5}y + \frac{1}{3} = \frac{1}{15}$
22. $-\frac{9}{16}x + \frac{4}{3} = \frac{4}{3}$
23. $-x + 5 = 14$
24. $-y - 7 = -12$
25. $75 - a = 200$
26. $15 = 5 - x$
27. $-8 = 4 - 2x$
28. $33 - x = 33$
29. $18 = 6 - y$
30. $-12 = -2x + 3$
31. $-3 = 3.36 - 1.2a$
32. $0 = -3.1a + 32.55$
33. $\frac{1}{4} = -\frac{3}{8} + 10x$
34. $70 = 50 - \frac{1}{2}y$

Answer

1. 5
3. 10
5. $\frac{1}{3}$
7. $\frac{1}{5}$
9. 2
11. 14.2
13. 7
15. $-\frac{5}{2}$
17. $\frac{7}{5}$
19. $\frac{1}{8}$
21. $\frac{1}{3}$
23. -9

25. -125
27. 6
29. -12
31. 5.3
33. $\frac{1}{16}$

EXERCISE 1.1.2.8 SOLVING LINEAR EQUATIONS

Translate the following sentences into linear equations and then solve.

1. The sum of $2x$ and 5 is equal to 15.
2. The sum of $-3x$ and 7 is equal to 14.
3. The difference of $5x$ and 6 is equal to 4.
4. Twelve times x is equal to 36.
5. A number n divided by 8 is 5.
6. Six subtracted from two times a number x is 12.
7. Four added to three times a number n is 25.
8. Three-fourths of a number x is 9.
9. Negative two-thirds times a number x is equal to 20.
10. One-half of a number x plus 3 is equal to 10.

Answer

1. $2x+5=15; x=5$
3. $5x-6=4; x=2$
5. $\frac{n}{8}=5; n=40$
7. $3n+4=25; n=7$
9. $-\frac{2}{3}x=20; x=-30$

EXERCISE 1.1.2.9 SOLVING LINEAR EQUATIONS

Find a linear equation of the form $ax+b=0$ with the given solution, where a and b are integers. (Answers may vary.)

1. $x=2$
2. $x=-3$
3. $x=-\frac{1}{2}$
4. $x=\frac{2}{3}$

Answer

1. $x-2=0$
3. $2x+1=0$

EXERCISE 1.1.2.10 DISCUSSION BOARD TOPICS

1. How many steps are needed to solve any equation of the form $ax+b=c$? Explain.
2. Instead of dividing by 6 when $6x=12$, could you multiply by the reciprocal of 6? Does this always work?

Answer

1. Answers may vary

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1.1.3: SOLVING LINEAR EQUATIONS- PART II

LEARNING OBJECTIVES

- Solve general linear equations.
- Identify and solve conditional equations, identities, and contradictions.
- Clear decimals and fractions from equations.
- Solve literal equations or formulas for a given variable.

COMBINING LIKE TERMS AND SIMPLIFYING

Linear equations typically are not given in standard form, so solving them requires additional steps. These additional steps include simplifying expressions on each side of the equal sign using the order of operations.

SAME-SIDE LIKE TERMS

We will often encounter linear equations where the expressions on each side of the equal sign can be simplified. Typically, this involves combining same-side like terms. If this is the case, then it is best to simplify each side first before solving.

EXAMPLE 1.1.3.1

Solve:

$$-4a+2-a=3-2\ .$$

Solution:

First, combine the like terms on each side of the equal sign.

Answer:

The solution is $\frac{1}{5}$.

OPPOSITE-SIDE LIKE TERMS

Given a linear equation in the form $ax+b=cx+d$, we begin by combining like terms on opposite sides of the equal sign. To combine opposite-side like terms, use the addition or subtraction property of equality to effectively "move terms" from one side to the other so that they can be combined.

EXAMPLE 1.1.3.2

Solve:

$$-2y-3=5y+11\ .$$

Solution:

To "move" the term $5y$ to the left side, subtract it on both sides.

$$\begin{aligned} -2y-3 &= 5y+11 \\ -2y-3-5y &= 5y+11-5y \qquad \textit{Subtract } 5y \textit{ from both sides.} \\ -7y-3 &= 11 \end{aligned}$$

From here, solve using the techniques developed previously.

Always check to see that the solution is correct by substituting the solution back into the original equation and simplifying to see if you obtain a true statement.

$$\begin{aligned} -2y-3 &= 5y+11 \\ -2(-2)-3 &= 5(-2)+1 \\ 4-3 &= -10+11 \\ 1 &= 1 \quad \checkmark \end{aligned}$$

Answer:

The solution is -2.

GENERAL GUIDELINES FOR SOLVING LINEAR EQUATIONS

When solving linear equations, the goal is to determine what value, if any, will produce a true statement when substituted in the original equation. Do this by isolating the variable using the following steps:

Step 1: Simplify both sides of the equation using the order of operations and combine all same-side like terms.

Step 2: Use the appropriate properties of equality to combine opposite-side like terms with the variable term on one side of the equation and the constant term on the other.

Step 3: Divide or multiply as needed to isolate the variable.

Step 4: Check to see if the answer solves the original equation.

✔ EXAMPLE 1.1.3.3

Solve:

$$-\tfrac{1}{2}(10y-2)+3=14$$

Solution:

Simplify the linear expression on the left side before solving.

To check,

$$\begin{aligned} -\frac{1}{2}(10(-2)-2)+3 &= 14 \\ -\frac{1}{2}(-20-2)+3 &= 14 \\ -\frac{1}{2}(-22)+3 &= 14 \\ 11+3 &= 14 \\ 14 &= 14 \quad \checkmark \end{aligned}$$

Answer:

The solution is -2.

✔ EXAMPLE 1.1.3.4

Solve:

$$5(3x+2)-2=-2(1-7x)\ .$$

Solution:

First, simplify the expressions on both sides of the equal sign.

Answer:

The solution is -10. The check is left as an exercise.

? EXERCISE 1.1.3.1

Solve:

$$6-3(4x-1)=4x-7\ .$$

Answer

$x=1$

CONDITIONAL EQUATIONS, IDENTITIES, AND CONTRADICTIONS

There are three different types of equations. Up to this point, we have been solving conditional equations. These are equations that are true for particular values. An identity is an equation that is true for all possible values of the variable. For example,

$$x=x \quad \textit{Identity}$$

has a solution set consisting of all real numbers, R. A contradiction is an equation that is never true and thus has no solutions. For example,

$$x+1=x \quad \textit{Contradiction}$$

has no solution. We use the empty set, $\emptyset$, to indicate that there are no solutions.

If the end result of solving an equation is a true statement, like $0 = 0$, then the equation is an identity and any real number is a solution. If solving results in a false statement, like $0 = 1$, then the equation is a contradiction and there is no solution.

EXAMPLE 1.1.3.5

Solve:

$$4(x+5)+6=2(2x+3)\ .$$

Solution:

$$\begin{aligned} 4(x+5)+6 &= 2(2x+3) && \textit{Distribute.}\\ 4x+20+6 &= 4x+6 && \textit{Combine same - side like terms.}\\ 4x+26 &= 4x+6 && \textit{Combine opposite - side like terms.}\\ 4x+26-4x &= 4x+6-4x\\ 26 &= 6 \quad x && \textit{False} \end{aligned}$$

Answer:

$\emptyset$. Solving leads to a false statement; therefore, the equation is a contradiction and there is no solution.

EXAMPLE 1.1.3.6

Solve:

$$3(3y+5)+5=10(y+2)-y\ .$$

Solution:

$$\begin{aligned} 3(3y+5)+5 &= 10(y+2)-y && \textit{Distribute.}\\ 9y+15+5 &= 10y+20-y && \textit{Combine same - side like terms.}\\ 9y+20 &= 9y+20 && \textit{Combine opposite - side like terms.}\\ 9y+20-9y &= 9y+20-9y\\ 20 &= 20 \quad \checkmark && \textit{True} \end{aligned}$$

Answer:

R. Solving leads to a true statement; therefore, the equation is an identity and any real number is a solution.

If it is hard to believe that any real number is a solution to the equation in the previous example, then choose your favorite real number, and substitute it in the equation to see that it leads to a true statement. Choose $x = 7$ and check:

Solve:

$$-2(3x+1)-(x-3)=-7x+1\ .$$

Answer

R

CLEARING DECIMALS AND FRACTIONS

The coefficients of linear equations may be any real number, even decimals and fractions. When decimals and fractions are used, it is possible to use the multiplication property of equality to clear the coefficients in a single step. If given decimal coefficients, then multiply by an appropriate power of 10 to clear the decimals. If given fractional coefficients, then multiply both sides of the equation by the least common multiple of the denominators (LCD).

EXAMPLE 1.1.3.7

Solve:

$$2.3x+2.8=-1.2x+9.8\ .$$

Solution:

Notice that all decimal coefficients are expressed with digits in the tenths place; this suggests that we can clear the decimals by multiplying both sides by 10. Take care to distribute 10 to each term on both sides of the equation.

$$
\begin{aligned}
10\cdot(2.3x+2.8) &= 10\cdot -1.2x+9.8 && \textit{Multiply both sides by } 10.\\
10\cdot 2.3x+10\cdot 2.8 &= 10\cdot(-1.2x)+10\cdot 9.8 &&\\
23x+28 &= -12x+98 && \textit{Integer coefficients}\\
23x+28+12x &= -12x+98+12x && \textit{Solve.}\\
35x+28 &= 98 &&\\
35x+28-28 &= 98-28 &&\\
35x &= 70 &&\\
\frac{35x}{35} &= \frac{70}{35} &&\\
x &= 2 &&
\end{aligned}
$$

Answer:

The solution is 2.

EXAMPLE 1.1.3.8

Solve:

$$\tfrac{1}{3}x+\tfrac{1}{5}=\tfrac{1}{5}x-1\,.$$

Solution:

Clear the fractions by multiplying both sides by the least common multiple of the given denominators. In this case, the LCM $(3,5)=15$.

Answer:

The solution is -9.

It is important to know that these techniques only work for equations. ***Do not try to clear fractions when simplifying expressions***. As a reminder

Expression	*Equation*
$\frac{1}{2}x+\frac{5}{3}$	$\frac{1}{2}x+\frac{5}{3}=0$

Solve equations and simplify expressions. If you multiply an expression by 6, you will change the problem. However, if you multiply both sides of an equation by 6, you obtain an equivalent equation.

Incorrect	*Correct*
$\frac{1}{2}x+\frac{5}{3}$	$\frac{1}{2}x+\frac{5}{3}=0$
$\neq 6\cdot\left(\frac{1}{2}x+\frac{5}{3}\right)$	$6\cdot\left(\frac{1}{2}x+\frac{5}{3}\right)=6\cdot 0$
$=3x+10$ x	$3x+10=10$ ✓

LITERAL EQUATIONS (LINEAR FORMULAS)

Algebra lets us solve whole classes of applications using literal equations, or formulas. Formulas often have more than one variable and describe, or model, a particular real-world problem. For example, the familiar formula $D=rt$ describes the distance traveled in terms of the average rate and time; given any two of these quantities, we can determine the third. Using algebra, we can solve the equation for any one of the variables and derive two more formulas.

$$
\begin{aligned}
D &= rt\\
\frac{D}{r} &= \frac{rt}{r} && \textit{Divide both sides by } r.\\
\frac{D}{r} &= t &&
\end{aligned}
$$

If we divide both sides by r, we obtain the formula $t=Dr$. Use this formula to find the time, given the distance and the rate

$$
\begin{aligned}
D &= rt\\
\frac{D}{t} &= \frac{rt}{t} && \textit{Divide both sides by } t.\\
\frac{D}{t} &= r &&
\end{aligned}
$$

If we divide both sides by t, we obtain the formula $r=Dt$. Use this formula to find the rate, given the distance traveled and the time it takes to travel that distance. Using the techniques learned up to this point, we now have three equivalent formulas relating distance, average

rate, and time:

$$D = rt \qquad t = \frac{D}{r} \qquad r = \frac{D}{t}$$

When given a literal equation, it is often necessary to solve for one of the variables in terms of the others. Use the properties of equality to isolate the indicated variable.

EXAMPLE 1.1.3.9

Solve for a:

$$P = 2a + b\,.$$

Solution:

The goal is to isolate the variable a.

$$\begin{aligned} P &= 2a + b \\ P - b &= 2a + b - b \qquad \textit{Subtract } b \textit{ from both sides.} \\ P - b &= 2a \\ \frac{P-b}{2} &= \frac{2a}{2} \qquad \textit{Divide both sides by } 2. \\ \frac{P-b}{2} &= a \end{aligned}$$

Answer:

$$a = \tfrac{P-b}{2}$$

EXAMPLE 1.1.3.10

Solve for y:

$$z = \tfrac{x+y}{2}\,.$$

Solution:

The goal is to isolate the variable y.

$$\begin{aligned} z &= \frac{x+y}{2} \\ 2 \cdot z &= 2 \cdot \frac{x+y}{2} \qquad \textit{Multiply both sides by } 2. \\ 2z &= x + y \\ 2z - x &= x + y - x \qquad \textit{Subtract } x \textit{ from both sides.} \\ 2z - x &= y \end{aligned}$$

Answer:

$$y = 2z - x$$

EXERCISE 1.1.3.3

Solve for b:

$$2a - 3b = c\,.$$

Answer

$b = \tfrac{2a-c}{3}$

KEY TAKEAWAYS

- Solving general linear equations involves isolating the variable, with coefficient 1, on one side of the equal sign.
- The steps for solving linear equations are:
  - Simplify both sides of the equation and combine all same-side like terms.
  - Combine opposite-side like terms to obtain the variable term on one side of the equal sign and the constant term on the other.
  - Divide or multiply as needed to isolate the variable.
  - Check the answer.

- Most linear equations that you will encounter are conditional and have one solution.
- If solving a linear equation leads to a true statement like $0 = 0$, then the equation is an identity and the solution set consists of all real numbers, R.
- If solving a linear equation leads to a false statement like $0 = 5$, then the equation is a contradiction and there is no solution, $\emptyset$.
- Clear fractions by multiplying both sides of a linear equation by the least common multiple of all the denominators. Distribute and multiply all terms by the LCD to obtain an equivalent equation with integer coefficients.
- Given a formula, solve for any variable using the same techniques for solving linear equations. This works because variables are simply representations of real numbers.

? EXERCISE 1.1.3.4 CHECKING FOR SOLUTIONS

Is the given value a solution to the linear equation?

1. $2(3x+5)-6=3x-8; x=-4$
2. $-x+17-8x=9-x; x=-1$
3. $4(3x-7)-3(x+2)=-1; x=\frac{1}{3}$
4. $-5-2(x-5)=-(x+3); x=-8$
5. $7-2(\frac{1}{2}x-6)=x-1; x=10$
6. $3x-\frac{2}{3}(9x-2)=0; x=\frac{4}{9}$

Answer

1. Yes
3. No
5. Yes

? EXERCISE 1.1.3.5 SOLVING LINEAR EQUATIONS

Solve.

1. $4x-7=7x+5$
2. $-5x+3=-8x-9$
3. $3x-5=2x-17$
4. $-2y-52=3y+13$
5. $-4x+2=7x-20$
6. $4x-3=6x-15$
7. $9x-25=12x-25$
8. $12y+15=-6y+23$
9. $1.2x-0.7=3x+4.7$
10. $2.1x+6.1=-1.3x+4.4$
11. $2.02x+4.8=14.782-1.2x$
12. $-3.6x+5.5+8.2x=6.5+4.6x$
13. $\frac{1}{2}x-\frac{2}{3}=x+\frac{1}{5}$
14. $\frac{1}{3}x-\frac{1}{2}=-\frac{1}{4}x-\frac{1}{3}$
15. $-\frac{1}{10}y+\frac{2}{5}=\frac{1}{5}y+\frac{3}{10}$
16. $x-\frac{20}{3}=\frac{5}{2}x+\frac{5}{6}$
17. $\frac{2}{3}y+\frac{1}{2}=\frac{5}{8}y+\frac{37}{24}$
18. $\frac{1}{3}+\frac{4}{3}x=\frac{10}{7}x+\frac{1}{3}-\frac{2}{21}x$
19. $\frac{8}{9}-\frac{11}{18}x=\frac{7}{6}-12x$
20. $\frac{1}{3}-9x=\frac{4}{9}+\frac{1}{2}x$
21. $12x-5+9x=44$
22. $10-6x-13=12$
23. $-2+4x+9=7x+8-2x$
24. $20x-5+12x=6-x+7$
25. $3a+5-a=2a+7$
26. $-7b+3=2-5b+1-2b$
27. $7x-2+3x=4+2x-2$

28. $-3x+8-4x+2=10$
29. $6x+2-3x=-2x-13$
30. $3x-0.75+0.21x=1.24x+7.13$
31. $-x-2+4x=5+3x-7$
32. $-2y-5=8y-6-10y$
33. $\frac{1}{10}x-\frac{1}{3}=\frac{1}{30}-\frac{1}{15}x-\frac{7}{15}$
34. $\frac{5}{8}-\frac{4}{3}x+\frac{1}{3}=-\frac{3}{9}x-\frac{1}{4}+\frac{1}{3}x$

Answer

1. -4
3. -12
5. 2
7. 0
9. -3
11. 3.1
13. $-\frac{26}{15}$
15. $\frac{1}{3}$
17. 25
19. $-\frac{5}{2}$
21. $\frac{7}{3}$
23. -1
25. $\emptyset$
27. $\frac{1}{2}$
29. -3
31. R
33. $-\frac{3}{5}$

? EXERCISE 1.1.3.6 SOLVING LINEAR EQUATIONS INVOLVING PARENTHESES

Solve.

1. $-5(2y-3)+2=12$
2. $3(5x+4)+5x=-8$
3. $4-2(x-5)=-2$
4. $10-5(3x+1)=5(x-4)$
5. $9-(x+7)=2(x-1)$
6. $-5(2x-1)+3=-12$
7. $3x-2(x+1)=x+5$
8. $5x-3(2x-1)=2(x-3)$
9. $-6(x-1)-3x=3(x+8)$
10. $-\frac{3}{5}(5x+10)=\frac{1}{2}(4x-12)$
11. $3.1(2x-3)+0.5=22.2$
12. $4.22-3.13(x-1)=5.2(2x+1)-11.38$
13. $6(x-2)-(7x-12)=14$
14. $-9(x-3)-3x=-3(4x+9)$
15. $3-2(x+4)=-3(4x-5)$
16. $12-2(2x+1)=4(x-1)$
17. $3(x+5)-2(2x+3)=7x+9$
18. $3(2x-1)-4(3x-2)=-5x+10$
19. $-3(2a-3)+2=3(a+7)$
20. $-2(5x-3)-1=5(-2x+1)$

21. $\frac{1}{2}(2x+1)-\frac{1}{4}(8x+2)=3(x-4)$
22. $-\frac{2}{3}(6x-3)-\frac{1}{2}=\frac{3}{2}(4x+1)$
23. $\frac{1}{2}(3x-1)+\frac{1}{3}(2x-5)=0$
24. $\frac{1}{3}(x-2)+\frac{1}{5}=\frac{1}{9}(3x+3)$
25. $-2(2x-7)-(x+3)=6(x-1)$
26. $10(3x+5)-5(4x+2)=2(5x+20)$
27. $2(x-3)-6(2x+1)=-5(2x-4)$
28. $5(x-2)-(4x-1)=-2(3-x)$
29. $6(3x-2)-(12x-1)+4=0$
30. $-3(4x-2)-(9x+3)-6x=0$

Answer

1. $\frac{1}{2}$
3. 8
5. $\frac{4}{3}$
7. $\emptyset$
9. $-\frac{3}{2}$
11. 5
13. -14
15. 2
17. 0
19. $-\frac{10}{9}$
21. 3
23. 1
25. $\frac{17}{11}$
27. $\emptyset$
29. $\frac{7}{6}$

? EXERCISE 1.1.3.7 LITERAL EQUATIONS

Solve for the indicated variable.

1. Solve for w: $A=l\cdot w$.
2. Solve for a: $F=ma$.
3. Solve for w: $P=2l+2w$.
4. Solve for r: $C=2\pi r$.
5. Solve for b: $P=a+b+c$.
6. Solve for C: $F=\frac{9}{5}C+32$.
7. Solve for h: $A=\frac{1}{2}bh$.
8. Solve for t: $I=Prt$.
9. Solve for y: $ax+by=c$.
10. Solve for h: $S=2\pi r^2+2\pi rh$.
11. Solve for x: $z=\frac{2x+y}{5}$.
12. Solve for c: $a=3b-\frac{2c}{3}$.
13. Solve for b: $y=mx+b$.
14. Solve for m: $y=mx+b$.
15. Solve for y: $3x-2y=6$.
16. Solve for y: $-5x+2y=12$.
17. Solve for y: $\frac{x}{3}-\frac{y}{5}=1$.
18. Solve for y: $\frac{3}{4}x-\frac{1}{5}y=\frac{1}{2}$.

Answer

1. $w = \frac{A}{l}$
3. $w = \frac{P-2l}{2}$
5. $b = P - a - c$
7. $h = \frac{2A}{b}$
9. $y = \frac{-ax+c}{b}$
11. $x = \frac{5z-y}{2}$
13. $b = y - mx$
15. $y = \frac{3x-6}{2}$
17. $y = \frac{5x-15}{3}$

? EXERCISE 1.1.3.8 LITERAL EQUATIONS

Translate the following sentences into linear equations and then solve.

1. The sum of $3x$ and 5 is equal to the sum of $2x$ and 7.
2. The sum of $-5x$ and 6 is equal to the difference of $4x$ and 2.
3. The difference of $5x$ and 25 is equal to the difference of $3x$ and 51.
4. The sum of $\frac{1}{2}x$ and $\frac{3}{4}$ is equal to $\frac{2}{3}x$.
5. A number n divided by 5 is equal to the sum of twice the number and 3.
6. Negative ten times a number n is equal to the sum of three times the number and 13.

Answer

1. $3x + 5 = 2x + 7$; $x = 2$
3. $5x - 25 = 3x - 51$; $x = -13$
5. $\frac{n}{5} = 2n + 3$; $n = -\frac{5}{3}$

? EXERCISE 1.1.3.9 DISCUSSION BOARD TOPICS

1. What is the origin of the word algebra?
2. What is regarded as the main business of algebra?
3. Why is solving equations such an important algebra topic?
4. Post some real-world linear formulas not presented in this section.
5. Research and discuss the contributions of Diophantus of Alexandria.
6. Create an identity or contradiction of your own and share on the discussion board. Provide a solution and explain how you found it.

Answer

1. Answers may vary
3. Answers may vary
5. Answers may vary

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1.1.E: REVIEW EXERCISES AND SAMPLE EXAM

REVIEW EXERCISES

? EXERCISE 1.1.E.1 INTRODUCTION TO ALGEBRA

Evaluate.

1. $2x+7$, where $x=-4$
2. $-4x+1$, where $x=-2$
3. $\frac{2}{3}y-\frac{1}{2}$, where $y=\frac{3}{5}$
4. $-\frac{3}{4}y+\frac{5}{3}$, where $y=\frac{2}{3}$
5. b^2-4ac, where $a=5, b=-2$, and $c=\frac{1}{2}$
6. b^2-4ac, where $a=-\frac{1}{4}, b=-1$, and $c=-3$
7. $2x^2-x+3$, where $x=-3$
8. $5x^2-2x+4$, where $x=-1$
9. Calculate the simple interest earned on a 3-year investment of \$750 at an annual interest rate of 8%.
10. A bus traveled for $1\frac{2}{3}$ hours at an average speed of 48 miles per hour. What distance did the bus travel?
11. Calculate the area of a rectangle with dimensions $4\frac{1}{2}$ feet by 6 feet.
12. Calculate the volume of a rectangular box with dimensions $4\frac{1}{2}$ feet by 6 feet by 1 foot.

Answer

1. -1
3. $-\frac{1}{10}$
5. -6
7. 24
9. \$180
11. 27 square feet

? EXERCISE 1.1.E.2 SIMPLIFYING ALGEBRAIC EXPRESSIONS

Multiply.

1. $-5(3x-2)$
2. $(6x-9)\cdot 3$
3. $\frac{3}{4}(4x^2-8x+32)$
4. $-20(\frac{1}{10}x^2-\frac{2}{5}x-\frac{5}{4})$
5. $-(3a-2b+5c-1)$
6. $-6(y^3+3y^2-7y+5)$

Answer

1. $-15x+10$
3. $3x^2-6x+24$
5. $-3a+2b-5c+1$

? EXERCISE 1.1.E.3 SIMPLIFYING ALGEBRAIC EXPRESSIONS

Simplify.

1. $5a-7b-3a+5b$
2. $6x^2-4x+7x^2-3x$
3. $\frac{3}{5}xy+\frac{1}{2}-\frac{1}{10}xy-\frac{1}{4}$
4. $-\frac{3}{4}a-\frac{4}{21}b+\frac{1}{3}a-\frac{1}{7}b$
5. $a^2b+2ab^2-7a^2b+9ab^2$

6. $y^2-3y+5-y^2+9$
7. $-8(8x-3)-7$
8. $7-(6x-9)$
9. $2(3x^2-2x+1)-(5x-7)$
10. $(2y^2+6y-8)-(5y^2-12y+1)$
11. $6-3(a-2b)+7(5a-3b)$
12. $10-5(x^2-x+1)-(3x^2+5x-1)$
13. Subtract $5x-1$ from $2x-3$.
14. Subtract $x-3$ from twice the quantity $x-1$.

Answer

1. $2a-2b$
3. $\frac{1}{2}xy+\frac{1}{4}$
5. $-6a^2b+11ab^2$
7. $-64x+17$
9. $6x^2-9x+9$
11. $32a-15b+6$
13. $-3x-2$

? EXERCISE 1.1.E.4 SOLVING LINEAR EQUATIONS: PART 1

Is the given value a solution to the linear equation?

1. $-x+3=-18; x=-15$
2. $4x-3=-3x; x=-2$
3. $8x+2=5x+1; x=-\frac{1}{3}$
4. $2x+4=3x-2; x=-1$

Answer

1. No
3. Yes

? EXERCISE 1.1.E.5 SOLVING LINEAR EQUATIONS: PART 1

Solve.

1. $y+23=25$
2. $-3x=54$
3. $\frac{x}{4}=8$
4. $\frac{5}{2}x=\frac{2}{3}$
5. $7x-5=-54$
6. $-2x+7=43$
7. $7x+3=0$
8. $4x+5=5$
9. $1=10-3x$
10. $10-5y=15$
11. $7-y=28$
12. $33-x=16$
13. $\frac{5}{6}x+\frac{1}{3}=\frac{3}{2}$
14. $-\frac{2}{3}y+\frac{1}{5}=-\frac{1}{3}$
15. The sum of $9x$ and 6 is 51.
16. The difference of $3x$ and 8 is 25.

Answer

1. 2

3. 32
5. -7
7. $-\frac{3}{7}$
9. 3
11. -21
13. $\frac{7}{5}$
15. 5

? EXERCISE 1.1.E.6 SOLVING LINEAR EQUATIONS: PART II

Solve.

1. $5x-2=3x+6$
2. $7x+1=2x-29$
3. $14x+1=15x-11$
4. $6y-13=3+7y$
5. $8y+6-3y=22-3y$
6. $12-5y+6=y-6$
7. $5-2(7x-1)=2x+1$
8. $10-5(x-1)=5-x$
9. $2x-(3x-4)=7-x$
10. $9x-3(2x+1)=3x-3$
11. $2(5x-2)-3(2x+1)=5(x-3)$
12. $3(5x-1)-4(x-4)=-5(2x+10)$
13. $\frac{3}{2}(4x-3)+\frac{1}{4}=1$
14. $\frac{3}{4}-\frac{1}{6}(4x-9)=2$
15. $\frac{2}{3}(9x-3)+\frac{1}{2}=3(2x-\frac{1}{2})$
16. $1-\frac{5}{4}(4x-1)=5(\frac{1}{2}-x)$
17. The sum of $4x$ and 3 is equal to the difference of $7x$ and 8.
18. The difference of $5x$ and 1 is equal to the sum of $12x$ and 1.
19. Solve for x: $y=9x+1$
20. Solve for y: $5x+2y=3$
21. Solve for l: $P=2l+2w$
22. Solve for b: $A=\frac{1}{2}bh$

Answer

1. 4
3. 12
5. 2
7. $\frac{3}{8}$
9. Ø
11. 8
13. $\frac{7}{8}$
15. R
17. $\frac{11}{3}$
19. $x=\frac{y-1}{9}$
21. $l=\frac{P-2w}{2}$

EXERCISE 1.1.E.7 APPLICATIONS OF LINEAR EQUATIONS

1. A larger integer is 3 more than twice a smaller integer. If their sum is 39, then find the integers.
2. A larger integer is 5 more than 3 times a smaller integer. If their sum is 49, then find the integers.
3. The sum of three consecutive odd integers is 45. Find the integers.
4. The sum of three consecutive even integers is 72. Find the integers.
5. The sum of three consecutive integers is 60. Find the integers.
6. The length of a rectangle is 7 centimeters less than twice its width. If the perimeter measures 46 centimeters, then find the dimensions of the rectangle.
7. A triangle has sides whose measures are consecutive even integers. If the perimeter is 24 meters, then find the measure of each side.
8. The circumference of a circle measures 24π inches. Find the radius of the circle.
9. Mary invested $\$1,800$ in two different accounts. One account earned 3.5% simple interest and the other earned 4.8%. If the total interest after 1 year was $\$79.25$, then how much did she invest in each account?
10. James has \$6 in dimes and quarters. If he has 4 fewer quarters than he does dimes, then how many of each coin does he have?
11. Two brothers leave the house at the same time traveling in opposite directions. One averages 40 miles per hour and the other 36 miles per hour. How long does it take for the distance between them to reach 114 miles?
12. Driving to her grandmother's house, Jill made several stops and was only able to average 40 miles per hour. The return trip took 2 hours less time because she drove nonstop and was able to average 60 miles per hour. How long did it take Jill to drive home from her grandmother's house?

Answer

1. $12, 27$
3. $13, 15, 17$
5. $19, 20, 21$
7. 6 meters, 8 meters, 10 meters
9. Mary invested \$550 at 3.5% and $\$1,250$ at 4.8%.
11. They will be 114 miles apart in $1\frac{1}{2}$ hours.

EXERCISE 1.1.E.8 RATIO AND PROPORTION APPLICATIONS

Solve.

1. $\frac{3}{4} = \frac{n}{8}$
2. $\frac{7}{3} = \frac{2}{8n}$
3. $\frac{6}{n} = \frac{30}{11}$
4. $\frac{n}{5} = \frac{2}{3}$
5. $\frac{3n-1}{3} = \frac{1}{2}$
6. $\frac{4}{2n+5} = -\frac{1}{3}$
7. $-3 = \frac{1}{n-1}$
8. $\frac{2}{n-6} = \frac{1}{2n+1}$
9. Find two numbers in the proportion 4 to 5 whose sum is 27.
10. A larger number is 2 less than twice a smaller number. If the two numbers are in the proportion 5 to 9, then find the numbers.
11. A recipe calls for $1\frac{1}{2}$ teaspoons of vanilla extract for every 3 cups of batter. How many teaspoons of vanilla extract should be used with 7 cups of batter?
12. The ratio of female to male employees at a certain bank is 4 to 5. If there are 80 female employees at the bank, then determine the total number of employees.

Answer

1. 6
3. $\frac{11}{5}$
5. $\frac{5}{6}$
7. $\frac{2}{3}$
9. $12, 15$

11. $3\frac{1}{2}$ teaspoons

EXERCISE 1.1.E.9 RATIO AND PROPORTION APPLICATIONS

If triangle ABC is similar to triangle RST, then find the remaining two sides given the following.

1. $a = 4, b = 9, c = 12$, and $s = 3$
2. $b = 7, c = 10, t = 15$, and $r = 6$
3. At the same time of day, a pole casts a 27-foot shadow and 4-foot boy casts a 6-foot shadow. Calculate the height of the pole.
4. An equilateral triangle with sides measuring 10 units is similar to another equilateral triangle with scale factor of $2:3$. Find the perimeter of the unknown triangle.

Answer

1. $t = 4, r = \frac{4}{3}$
3. 18 feet

EXERCISE 1.1.E.10 INTRODUCTION TO INEQUALITIES AND INTERVAL NOTATION

Graph all solutions on a number line and provide the corresponding interval notation.

1. $x < -1$
2. $x \leq 10$
3. $x \geq 0$
4. $x > -2$
5. $-\frac{1}{2} \leq x < \frac{3}{2}$
6. $-20 < x < 30$
7. $x < 5$ or $x \geq 15$
8. $x < 2$ or $x > 0$

Answer

1. $(-\infty, -1)$

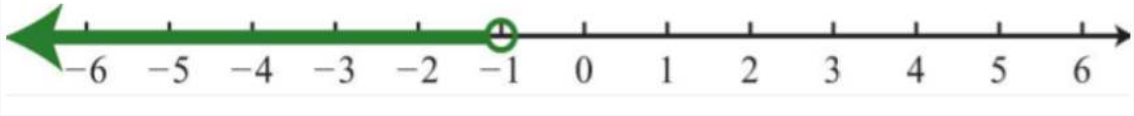


Figure 2.E.1

3. $[0, \infty)$

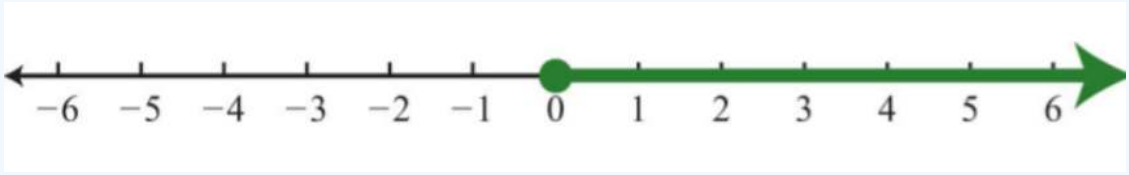


Figure 2.E.2

5. $[-\frac{1}{2}, \frac{3}{2})$

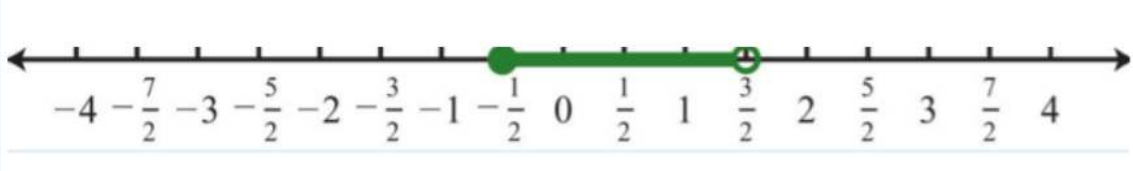


Figure 2.E.3

7. $(-\infty, 5) \cup [15, \infty)$



Figure 2.E.4

? EXERCISE 1.1.E.11 INTRODUCTION TO INEQUALITIES AND INTERVAL NOTATION

Determine the inequality given the answers expressed in interval notation.

1. $(-\infty, 3)$
2. $[-4, \infty)$
3. $(-2, 2)$
4. $(-3, 8]$
5. $(-\infty, 1) \cup [3, \infty)$
6. $(-\infty, -8] \cup [8, \infty)$

Answer

1. $x < 3$
3. $-2 < x < 2$
5. $x < 1$ or $x \geq 3$

? EXERCISE 1.1.E.12 LINEAR INEQUALITIES (ONE VARIABLE)

Solve and graph. In addition, present the solution set in interval notation.

1. $x + 2 > -1$
2. $-4x \geq 16$
3. $9x + 4 \leq -5$
4. $5x - 7 < 13$
5. $7x + 5 - 8x \geq 15$
6. $5x - 6 + 3x < 2 + 9x - 5$
7. $3x - (x - 4) > x + 4$
8. $3(2x - 1) - 3(x - 2) \leq 2(x + 4)$
9. $2 - 5(x - 4) > 12$
10. $3x - 5(x - 2) \geq 11 - 5x$
11. $-1 < 2x + 5 \leq 11$
12. $-2 \leq \frac{1}{4}x - \frac{7}{2} \leq 2$
13. $5x + 3 < -2$ or $6x - 5 \geq 7$
14. $20 - 3x \leq 5$ or $5 - 2x \geq 25$

Answer

1. $x > -3; (-3, \infty)$

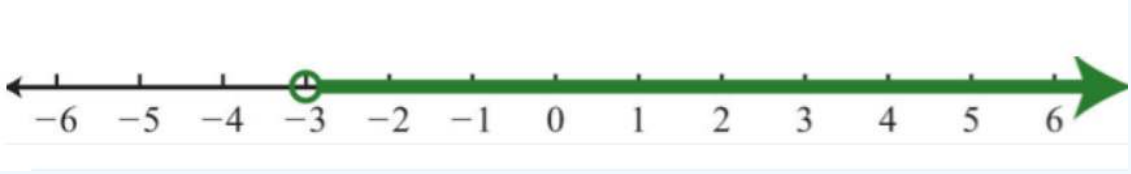


Figure 2.E.5

3. $x \leq -1; (-\infty, -1]$

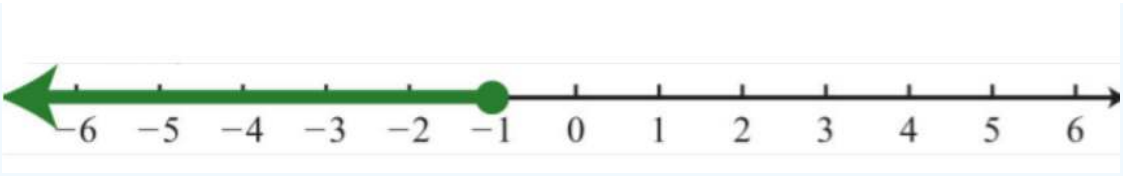


Figure 2.E.6

5. $x \leq -10; (-\infty, -10]$

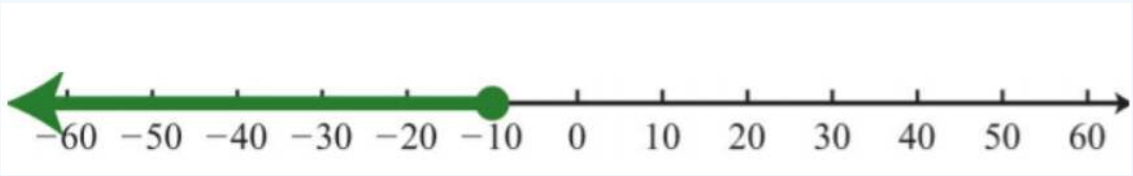


Figure 2.E.7

7. $x > 0; (0, \infty)$

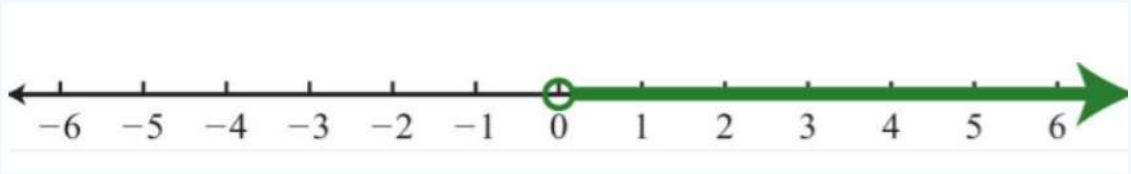


Figure 2.E.8

9. $x < 2; (-\infty, 2)$

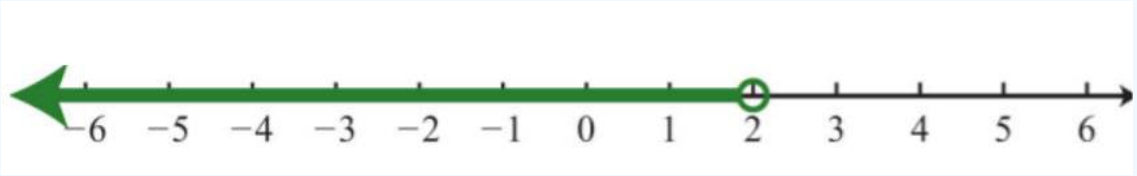


Figure 2.E.9

11. $-3 < x \leq 3; (-3, 3]$



Figure 2.E.10

13. $x < -1$ or $x \geq 2; (-\infty, -1) \cup [2, \infty)$

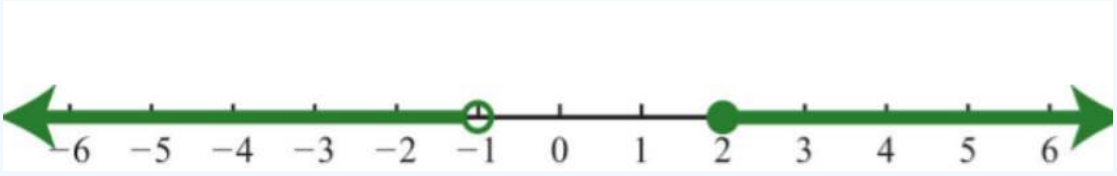


Figure 2.E.11

SAMPLE EXAM

EXERCISE 1.1.E.13

1. Evaluate b^2-4ac, where $a=-1, b=-2 and \(c=\frac{1}{2}$.
2. Determine the area of a triangle given that the base measures 10 centimeters and the height measures 5 centimeters. $(A=\frac{1}{2}bh)$

Answer

1. 6

EXERCISE 1.1.E.14

Simplify.

1. $5-2(4x-1)$
2. $\frac{1}{4}x-\frac{2}{3}y+\frac{1}{2}x-\frac{3}{5}y$
3. $(5a+4ab-2b)-(3a+2ab-3b)$
4. $3x-(x^2+5x-1)+(x^2-x+4)$

Answer

1. $-8x+7$
3. $2ab+2a+b$

EXERCISE 1.1.E.15

Solve.

1. $2-5x=27$
2. $\frac{1}{2}x-\frac{3}{4}=-\frac{1}{8}$
3. $5x-7=3x-5$
4. $3(y-3)-(4y+2)=1$
5. $5(x-2)-3(x+2)=2x-3$
6. $\frac{5}{8}=\frac{n}{32}$
7. $\frac{3}{n+1}=-\frac{6}{4}$
8. Solve for b: $A=a+2b$.

Answer

1. -5
3. 1
5. Ø
7. -3

EXERCISE 1.1.E.16

Solve and graph the solution set. In addition, present the solution set in interval notation.

1. $2x+3>23$
2. $5(-2x+1)\leq 35$
3. $4(3x-2)<3(2x+1)+1$
4. $-9\leq 3(x+4)\leq 21$
5. $6(x-\frac{1}{3})<-2$ or $\frac{1}{5}(x+10)\geq 3$

Answer

1. $x>10; (10,\infty)$

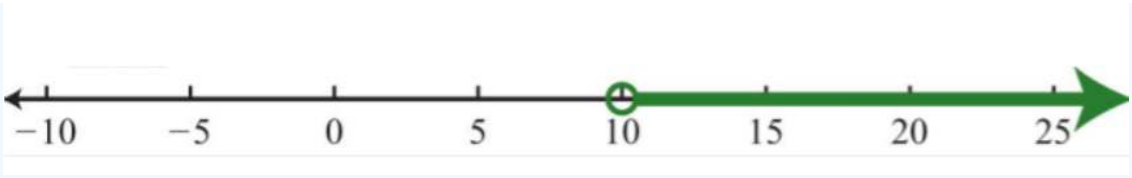


Figure 2.E.12

3. $x < 2; (-\infty, 2)$



Figure 2.E.13

5. $x < 0$ or $x \geq 5; (-\infty, 0) \cup [5, \infty)$ \)

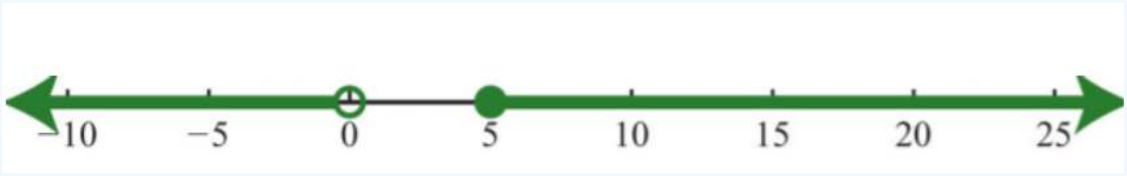


Figure 2.E.14

EXERCISE 1.1.E. 17

1. An algebra student earns $75, 79$, and 89 points on the first three quizzes. What must she score on the fourth quiz to earn an average of at least 80?
2. The sum of three consecutive odd integers is 117. Find the integers.
3. The length of a rectangle is 6 inches less than twice the width. If the perimeter measures 39 inches, then find the dimensions of the rectangle.
4. Millie invested her $\$5,350$ savings in two accounts. One account earns 5% annual interest and the other earns 6.2% in annual interest. If she earned $\$317.30$ simple interest in 1 year, then how much was in each account?
5. Because of traffic, Joe was only able to drive an average of 42 miles per hour on the trip to a conference. He was able to average 63 miles per hour on the return trip and it took 1 hour less time. How long did it take Joe to drive home from the conference?
6. A graphic designer wishes to crop an image in the width-to-height ratio of $3:2$. If the height is required to be 400 pixels, then to how many pixels should the width be set?

Answer

2. The three odd integers are $37, 39$, and 41.
4. Millie invested $\$1,200$ in the account earning 5% annual interest and $\$4,150$ in the account earning 6.2%.
6. The width should be set to 600 pixels.

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1.2: Review- Graphing

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1.2.1: RECTANGULAR COORDINATE SYSTEM

LEARNING OBJECTIVES

- Plot points using the rectangular coordinate system.
- Calculate the distance between any two points in the rectangular coordinate plane.
- Determine the midpoint between any two points.

RECTANGULAR COORDINATE SYSTEM

The rectangular coordinate system consists of two real number lines that intersect at a right angle. The horizontal number line is called the x-axis, and the vertical number line is called the y-axis. These two number lines define a flat surface called a plane, and each point on this plane is associated with an ordered pair of real numbers (x, y). The first number is called the ***x*-coordinate**, and the second number is called the ***y*-coordinate**. The intersection of the two axes is known as the origin, which corresponds to the point $(0, 0)$.

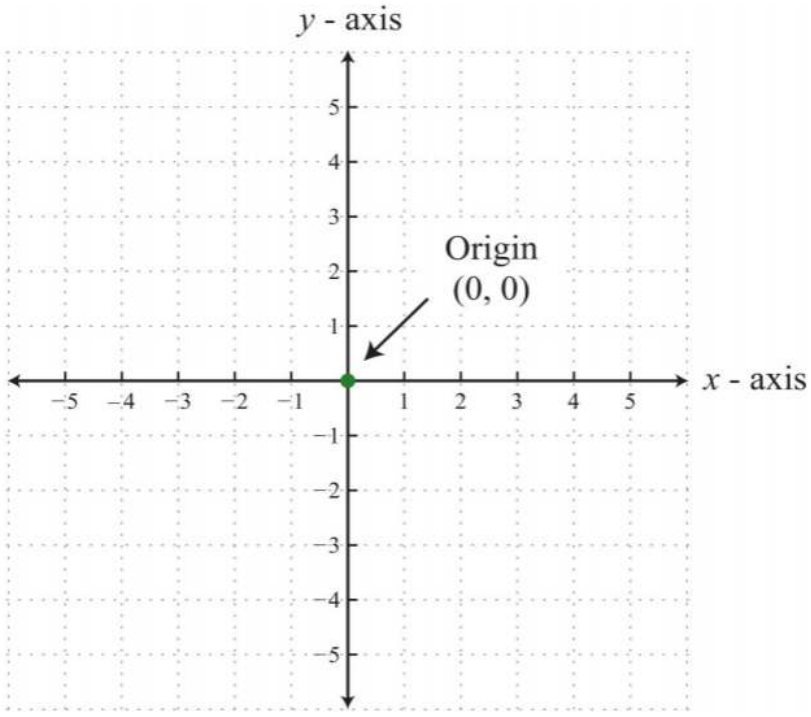


Figure 1.2.1.1

An ordered pair (x, y) represents the position of a point relative to the origin. The x-coordinate represents a position to the right of the origin if it is positive and to the left of the origin if it is negative. The y-coordinate represents a position above the origin if it is positive and below the origin if it is negative. Using this system, every position (point) in the plane is uniquely identified. For example, the pair $(2, 3)$ denotes the position relative to the origin as shown:

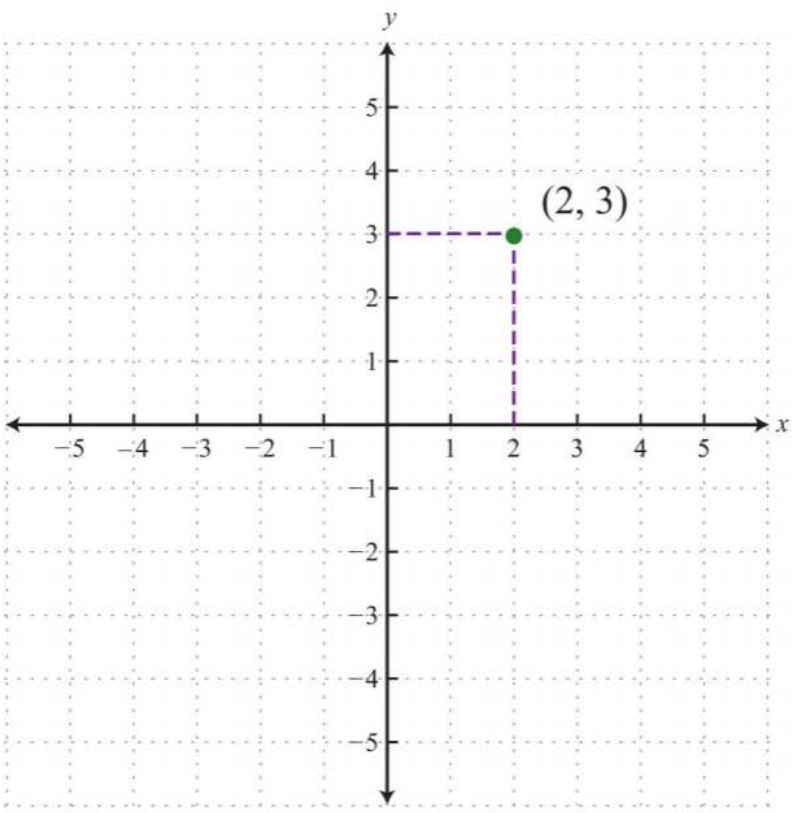


Figure 1.2.1.2

This system is often called the Cartesian coordinate system, named after the French mathematician René Descartes (1596– 1650).

The x- and y-axes break the plane into four regions called quadrants, named using roman numerals I, II, III, and IV, as pictured. In quadrant I, both coordinates are positive. In quadrant II, the x-coordinate is negative and the y-coordinate is positive. In quadrant III, both coordinates are negative. In quadrant IV, the x-coordinate is positive and the y-coordinate is negative.

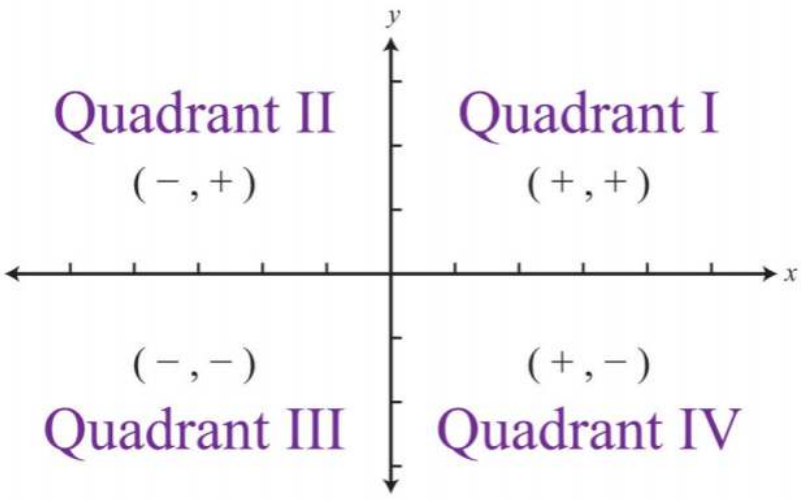


Figure 1.2.1.3

EXAMPLE 1.2.1.1

Plot the ordered pair $(-3, 5)$ and determine the quadrant in which it lies.

Solution:

The coordinates $x = -3$ and $y = 5$ indicate a point 3 units to the left of and 5 units above the origin.

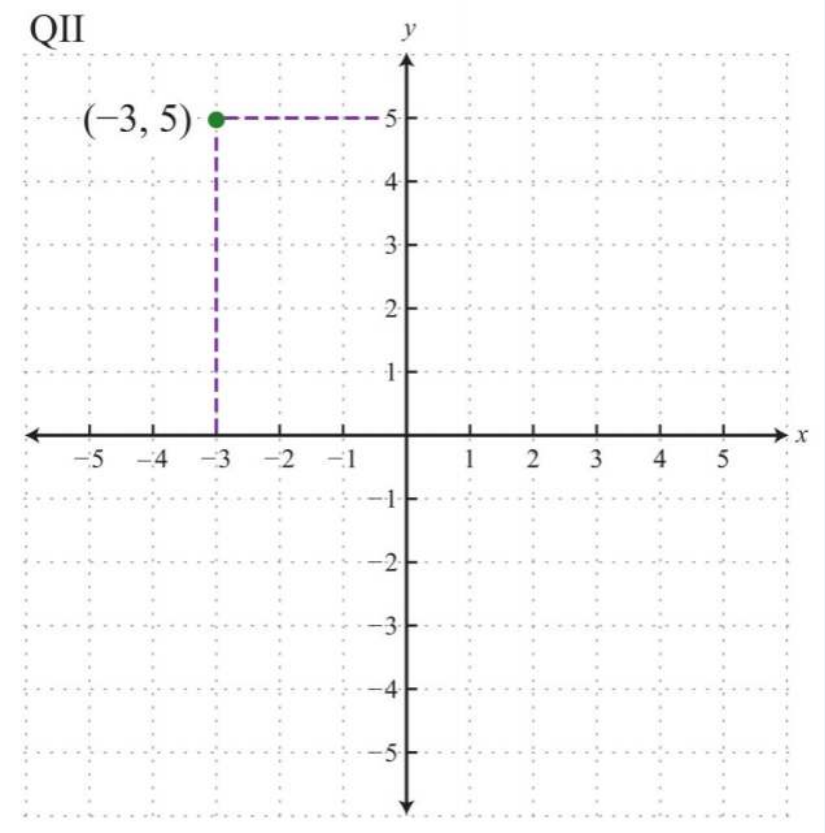


Figure 1.2.1.4

Answer:

The point is plotted in quadrant II (QII) because the x-coordinate is negative and the y-coordinate is positive.

Ordered pairs with 0 as one of the coordinates do not lie in a quadrant; these points are on one axis or the other (or the point is the origin if both coordinates are 0). Also, the scale indicated on the x-axis may be different from the scale indicated on the y-axis. Choose a scale that is convenient for the given situation.

EXAMPLE 1.2.1.2

Plot this set of ordered pairs:

$$\{(4,0),(-6,0),(0,3),(-2,6),(-4,-6)\}.$$

Solution:

Each tick mark on the x-axis represents 2 units and each tick mark on the y-axis represents 3 units.

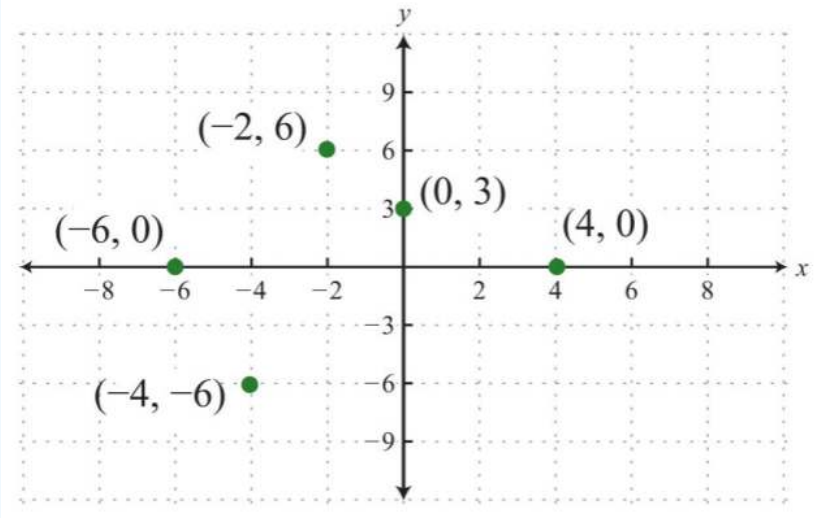


Figure 1.2.1.5

EXAMPLE 1.2.1.3

Plot this set of ordered pairs:

$$\{(-6,-5),(-3,-3),(0,-1),(3,1),(6,3)\}.$$

Solution:

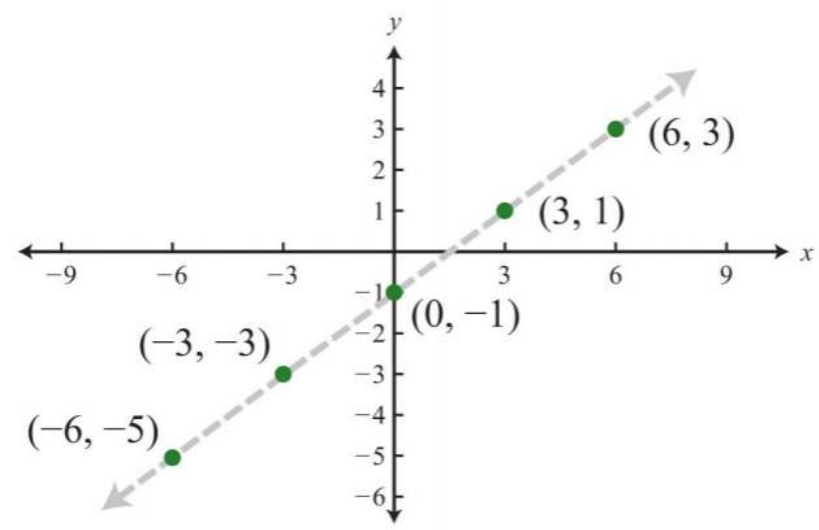


Figure 1.2.1.6

In this example, the points appear to be collinear, or to lie on the same line. The entire chapter focuses on finding and expressing points with this property.

EXERCISE 1.2.1.1

Plot the set of points $\{(5,3),(-3,2),(-2,-4),(4,-3)\}$ and indicate in which quadrant they lie.

Answer

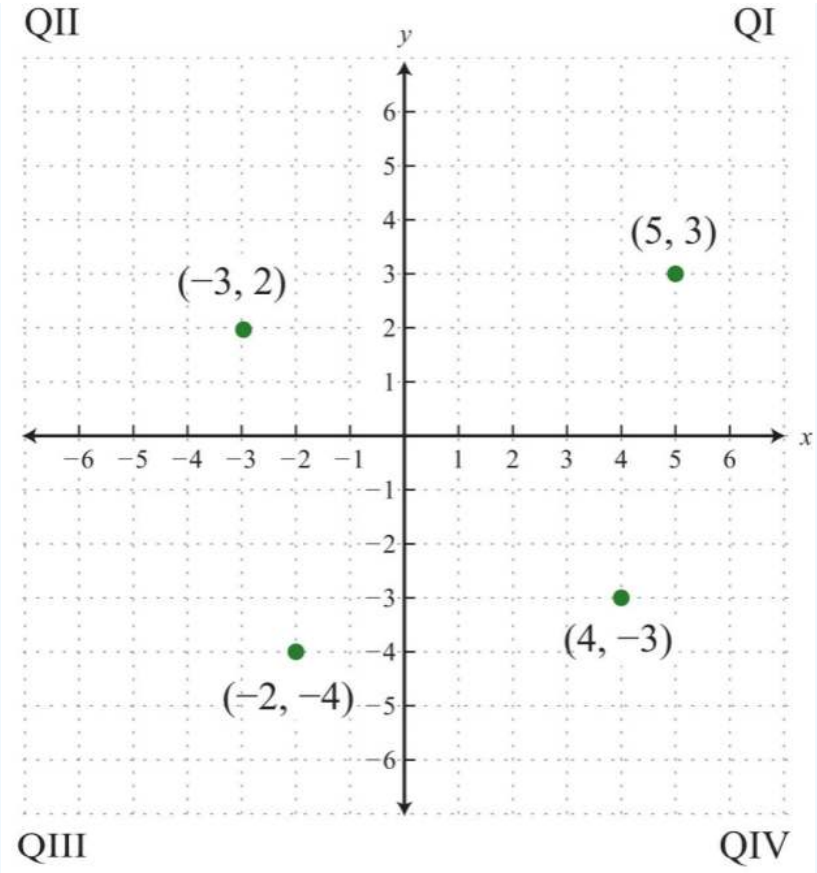


Figure 1.2.1.7

Graphs are used in everyday life to display data visually. A line graph consists of a set of related data values graphed on a coordinate plane and connected by line segments. Typically, the independent quantity, such as time, is displayed on the x-axis and the dependent quantity, such as distance traveled, on the y-axis.

EXAMPLE 1.2.1.4

The following line graph shows the number of mathematics and statistics bachelor's degrees awarded in the United States each year since 1970.

https://math.libretexts.org/@go/page/186895

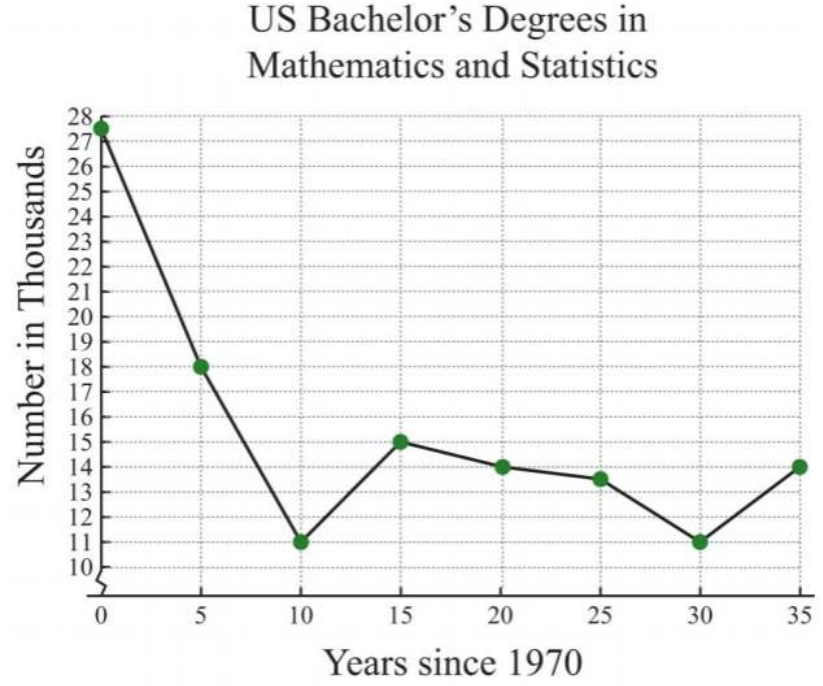


Figure 1.2.1.8*: Source: Digest of Education Statistics.*

a. How many mathematics and statistics bachelor's degrees were awarded in 1975?

b. In which years were the number of mathematics and statistics degrees awarded at the low of $11,000$?

Solution:

a. The scale on the x-axis represents time since 1970, so to determine the number of degrees awarded in 1975, read the y-value of the graph at $x = 5$.

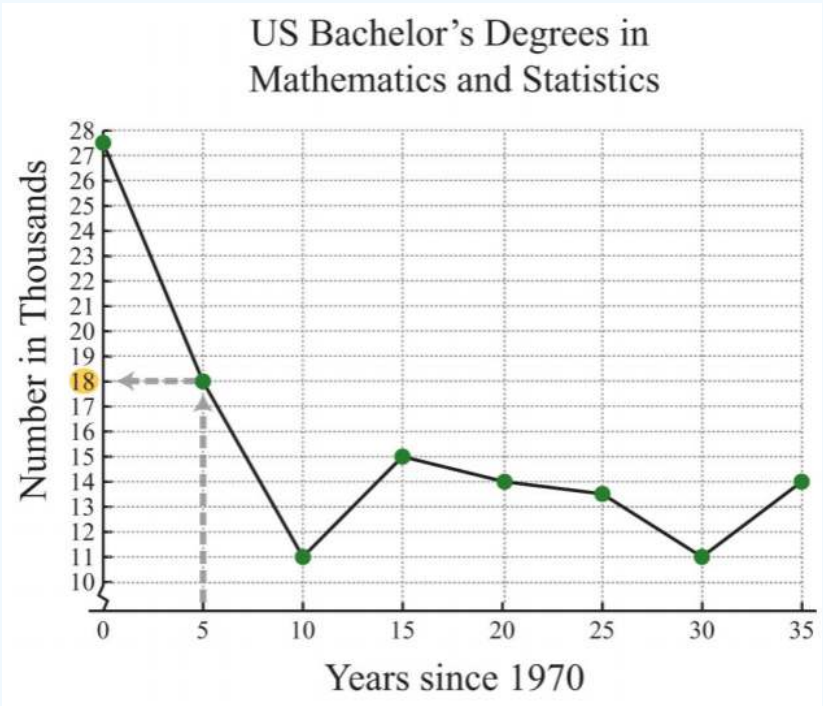


Figure 1.2.1.9*: Source: Digest of Education Statistics.*

The y-value corresponding to $x = 5$ is 18. The graph indicates that this is in thousands; there were $18,000$ mathematics and statistics degrees awarded in 1975.

b. To find the year a particular number of degrees was awarded, first look at the y-axis. In this case, $11,000$ degrees is represented by 11 on the y-axis; look to the right to see in which years this occurred.

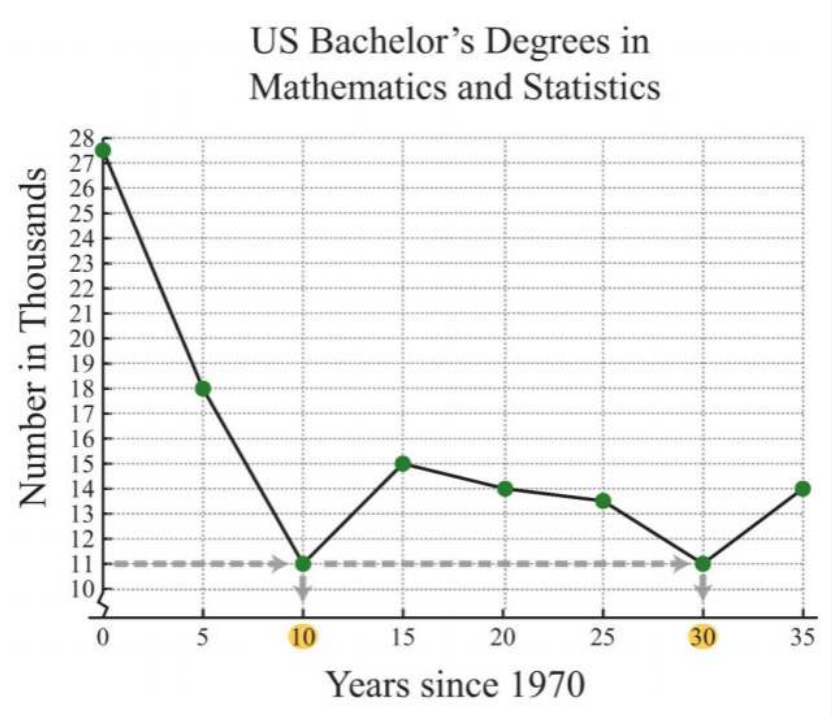


Figure 1.2.1.10: *Source: Digest of Education Statistics.*

The y-value of 11 occurs at two data points, one where $x = 10$ and the other where $x = 30$. These values correspond to the years 1980 and 2000, respectively.

Answers:

a. In the year 1975, $18,000$ mathematics and statistics degrees were awarded.
b. In the years 1980 and 2000, the lows of $11,000$ mathematics and statistics degrees were awarded.

DISTANCE FORMULA

Frequently you need to calculate the distance between two points in a plane. To do this, form a right triangle using the two points as vertices of the triangle and then apply the Pythagorean theorem. Recall that the **Pythagorean theorem** states that if given any right triangle with legs measuring a and b units, then the square of the measure of the hypotenuse c is equal to the sum of the squares of the legs: $a^2 + b^2 = c^2$. In other words, the hypotenuse of any right triangle is equal to the square root of the sum of the squares of its legs.

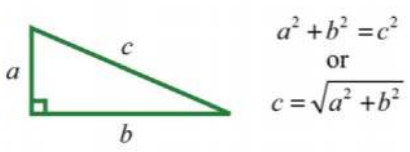


Figure 1.2.1.11

EXAMPLE 1.2.1.5

Find the distance between $(-1, 2)$ and $(3, 5)$.

Solution:

Form a right triangle by drawing horizontal and vertical lines through the two points. This creates a right triangle as shown below:

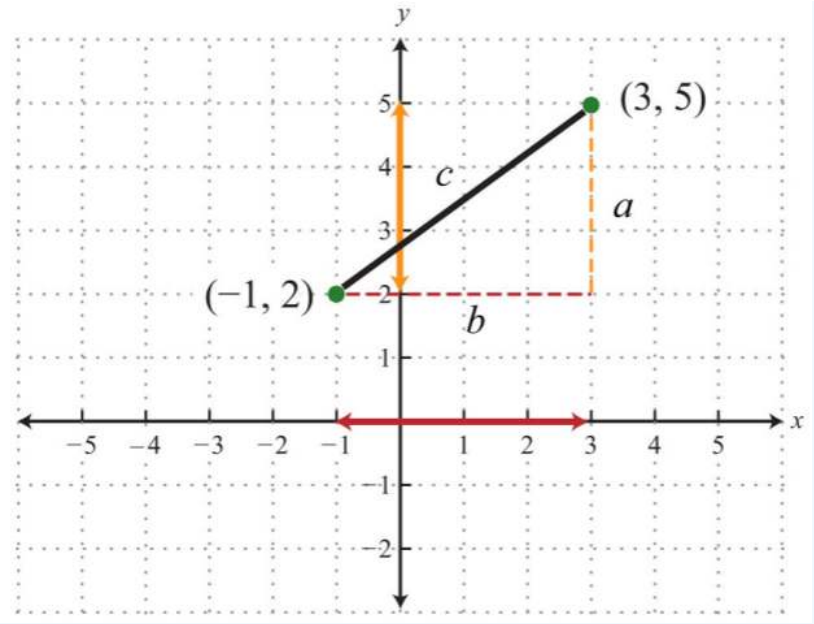


Figure 1.2.1.12

The length of leg b is calculated by finding the distance between the x-values of the given points, and the length of leg a is calculated by finding the distance between the given y-values.

$$a = 5 - 2 = 3\text{units}$$
$$b = 3 - (-1) = 3 + 1 = 4\text{units}$$

Next, use the Pythagorean theorem to find the length of the hypotenuse.

$$\begin{aligned} c &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \text{ units} \end{aligned}$$

Answer:

The distance between the two points is 5 units.

Generalize this process to produce a formula that can be used to algebraically calculate the distance between any two given points.

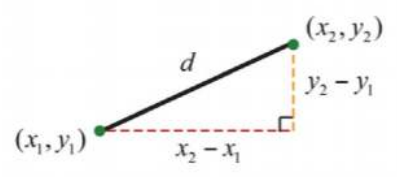


Figure 1.2.1.13

Given two points, (x_1, y_1) and (x_2, y_2), then the distance, d, between them is given by the distance formula:

$$\sqrt{(x_2 + x_1)^2 + (y_2 + y_1)^2} \tag{1.2.1.1}$$

EXAMPLE 1.2.1.6

Calculate the distance between $(-3, -1)$ and $(-2, 4)$.

Solution:

Use the distance formula.

x_1, y_1	x_2, y_2
$(-3, -1)$	$(-2, 4)$

It is a good practice to include the formula in its general form as a part of the written solution before substituting values for the variables. This improves readability and reduces the chance for errors.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-2-(-3))^2+(4-(-1))^2} \\ &= \sqrt{(-2+3)^2 + (4+1)^2} \\ &= \sqrt{(1)^2 + (5)^2} \\ &= \sqrt{1+25} \\ &= \sqrt{26} \end{aligned}$$

Answer:

$\sqrt{26}$ units

EXERCISE 1.2.1.2

Calculate the distance between $(-7, 5)$ and $(-1, 13)$.

Answer

10 units

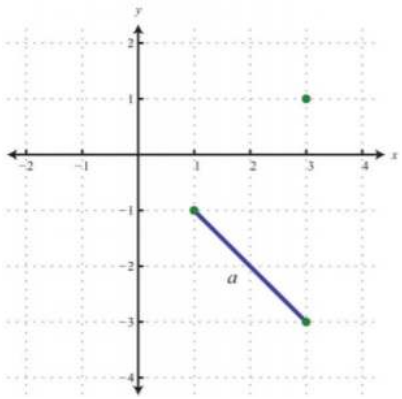
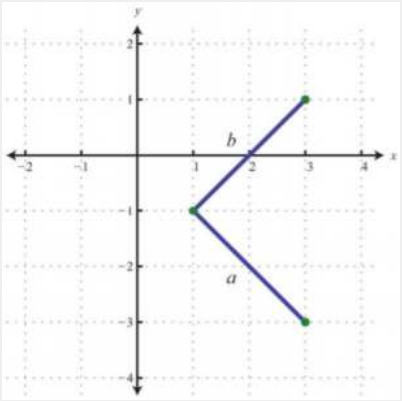
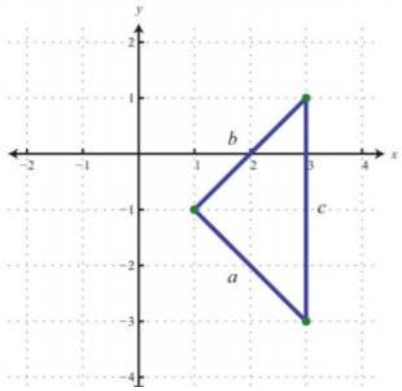
EXAMPLE 1.2.1.7

Do the three points $(1, -1), (3, -3)$, and $(3, 1)$ form a right triangle?

Solution:

The Pythagorean theorem states that having side lengths that satisfy the property $a^2 + b^2 = c^2$ is a necessary and sufficient condition of right triangles. In other words, if you can show that the sum of the squares of the leg lengths of the triangle is equal to the square of the length of the hypotenuse, then the figure must be a right triangle. First, calculate the length of each side using the distance formula.

Table 1.2.1.1

Geometry	Calculation
 Figure 1.2.1.14	Points: $(1,-1)$ and $(3,-3)$ $a=\sqrt{(3-1)^2+(-3-(-1))^2}$ $=\sqrt{(a)^2+(-3+1)^2}$ $=\sqrt{4+(-2)^2}$ $=\sqrt{4+4}$ $=\sqrt{8}$
 Figure 1.2.1.15	Points: $(1,-1)$ and $(3,1)$ $b=\sqrt{(3-1)^2+(1-(-1))^2}$ $=\sqrt{2^2+(1+1)^2}$ $=\sqrt{4+(2)^2}$ $=\sqrt{4+4}$ $=\sqrt{8}$
 Figure 1.2.1.16	Points: $(3,-3)$ and $(3,1)$ $c=\sqrt{(3-3)^2+(1-(-3))^2}$ $=\sqrt{(0)^2+(1+3)^2}$ $=\sqrt{0+(4)^2}$ $=\sqrt{16}$ $=4$

Now we check to see if $a^2+b^2=c^2$.

$$\begin{aligned} a^2+b^2&=c^2\\ (\sqrt{8})^2+(\sqrt{8})^2&=(4)^2\\ \sqrt{64}+\sqrt{64}&=16\\ 8+8&=16\\ 16&=16 \quad \checkmark \end{aligned}$$

Answer:

Yes, the three points form a right triangle. In fact, since two of the legs are equal in length, the points form an isosceles right triangle.

MIDPOINT FORMULA

The point that bisects the line segment formed by two points, (x_1, y_1) and (x_2, y_2), is called the midpoint and is given by the following formula:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \tag{1.2.1.2}$$

The midpoint is an ordered pair formed by finding the average of the x-values and the average of the y-values of the given points.

EXAMPLE 1.2.1.8

Calculate the midpoint between $(-1, -2)$ and $(7, 4)$.

Solution:

First, calculate the average of the x- and y-values of the given points.

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-1, -2) & (7, 4) \end{array}$$

$$\frac{x_1 + x_2}{2} = \frac{-1 + 7}{2} = \frac{6}{2} = 3$$

$$\frac{y_1 + y_2}{2} = \frac{-2 + 4}{2} = \frac{2}{2} = 1$$

Next, form the midpoint as an ordered pair using the averaged coordinates.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$(3, 1)$$

Answer:

$$(3, 1)$$

To verify that this is indeed the midpoint, calculate the distance between the two given points and verify that the result is equal to the sum of the two equal distances from the endpoints to this midpoint. This verification is left to the reader as an exercise.

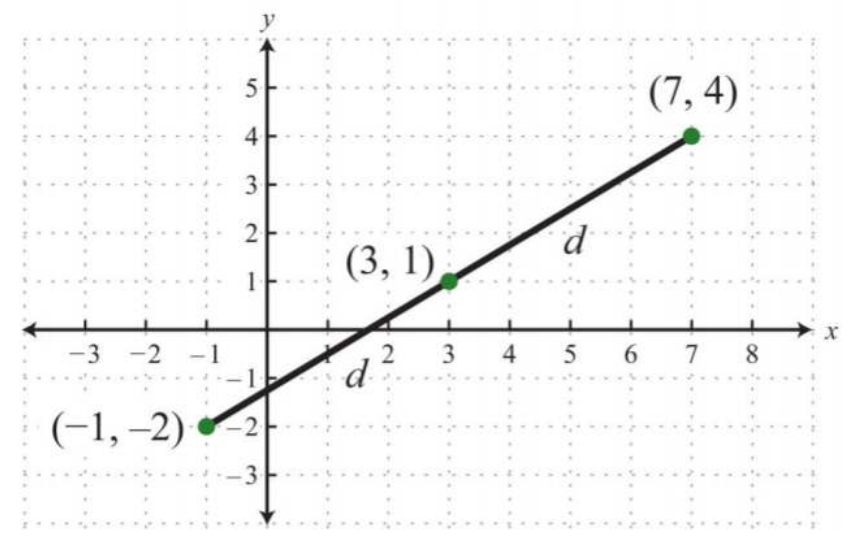


Figure 1.2.1.17

EXERCISE 1.2.1.3

Find the midpoint between $(-6, 5)$ and $(6, -11)$.

Answer

$(0, -3)$

KEY TAKEAWAYS

- Use the rectangular coordinate system to uniquely identify points in a plane using ordered pairs (x, y). Ordered pairs indicate position relative to the origin. The x-coordinate indicates position to the left and right of the origin. The y-coordinate indicates position above or below the origin.
- The scales on the x-axis and y-axis may be different. Choose a scale for each axis that is appropriate for the given problem.
- Graphs are used to visualize real-world data. Typically, independent data is associated with the x-axis and dependent data is associated with the y-axis.
- The Pythagorean theorem gives us a necessary and sufficient condition of right triangles. Given a right triangle, then the measures of the sides satisfy $a^2 + b^2 = c^2$. Conversely, if the sides satisfy $a^2 + b^2 = c^2$, then the triangle must be a right triangle.
- The distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, is derived from the Pythagorean theorem and gives us the distance between any two points, (x_1, y_1) and (x_2, y_2), in a rectangular coordinate plane.
- The midpoint formula, $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, is derived by taking the average of each coordinate and forming an ordered pair.

? EXERCISE 1.2.1.4 ORDERED PAIRS

Give the coordinates of points $A, B, C, D,$ and E.

1.

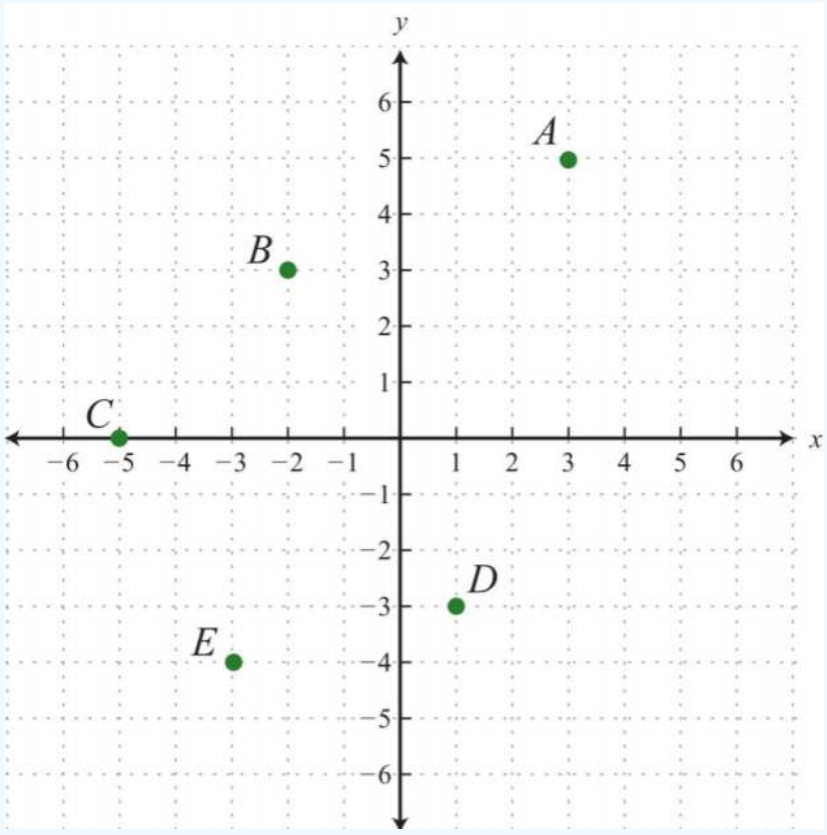


Figure 1.2.1.18

2.

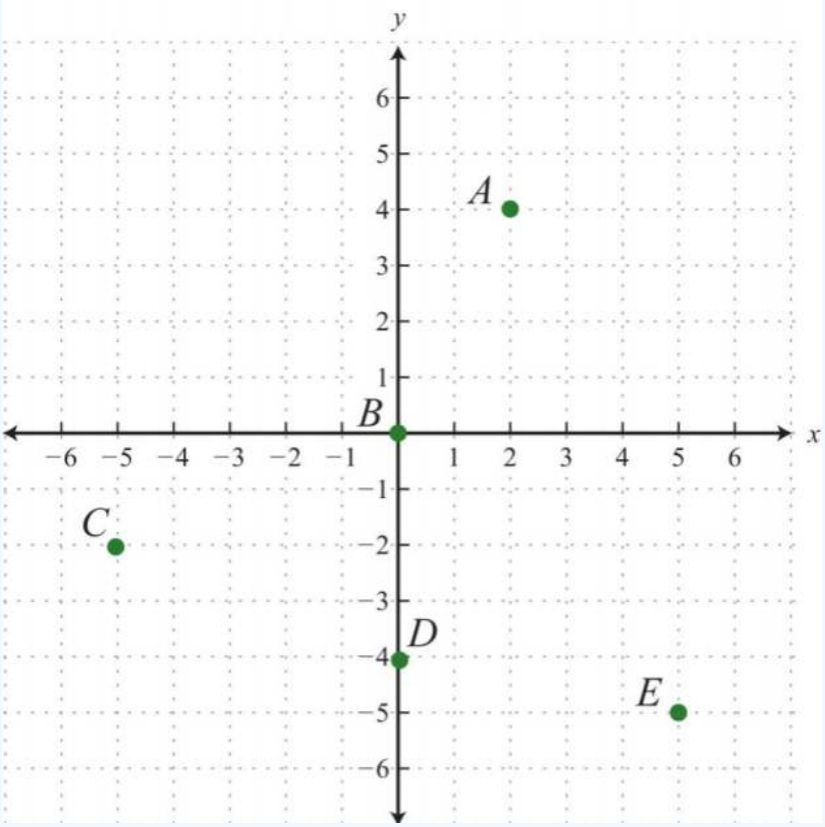


Figure 1.2.1.19

3.

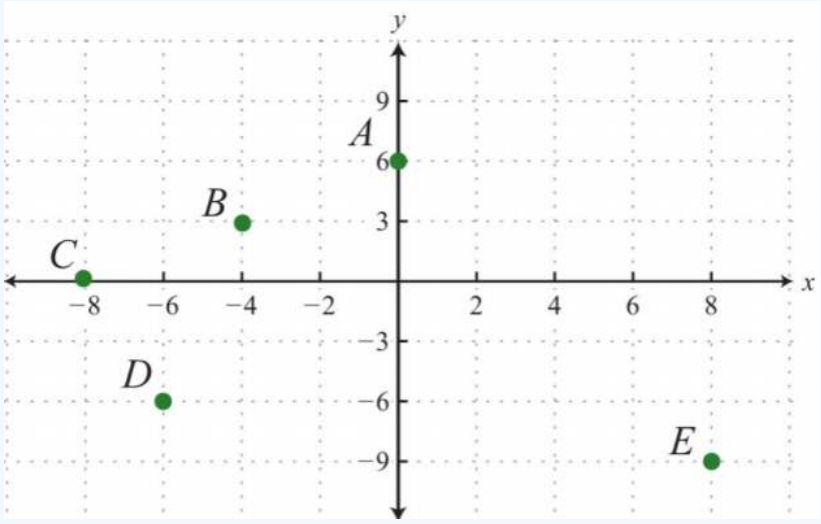


Figure 1.2.1.20

4.

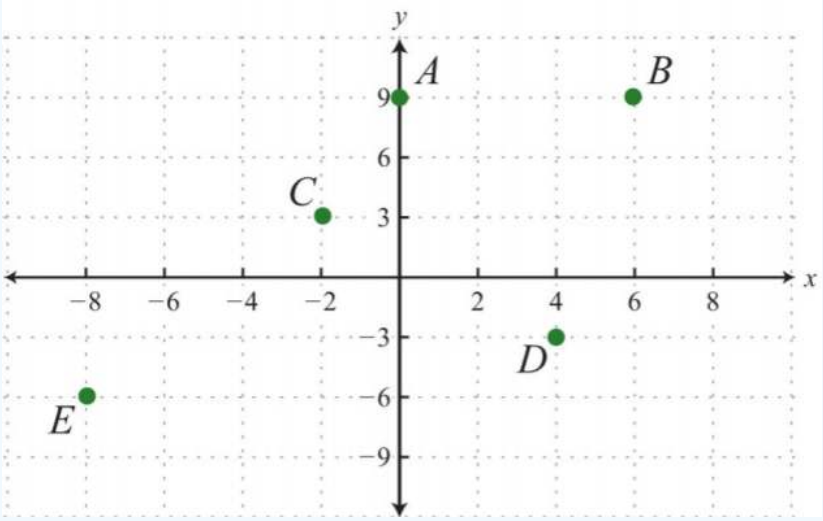


Figure 1.2.1.21

5.

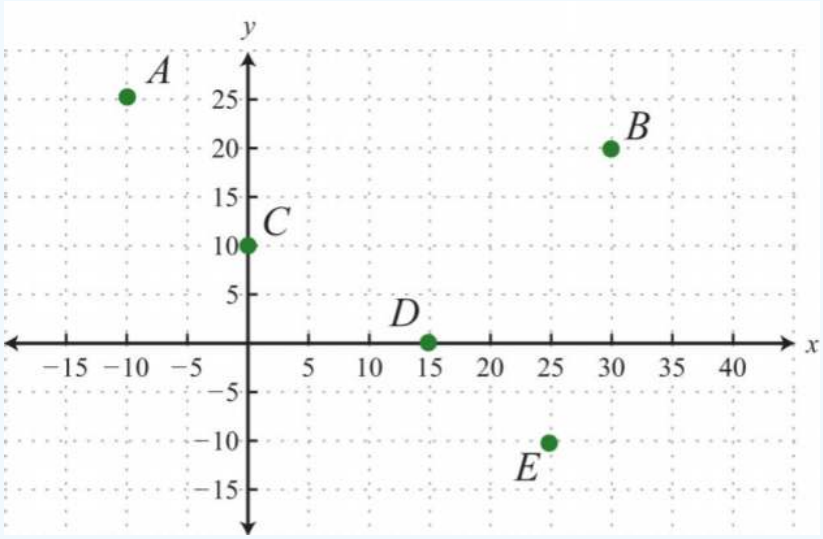


Figure 1.2.1.22

6.

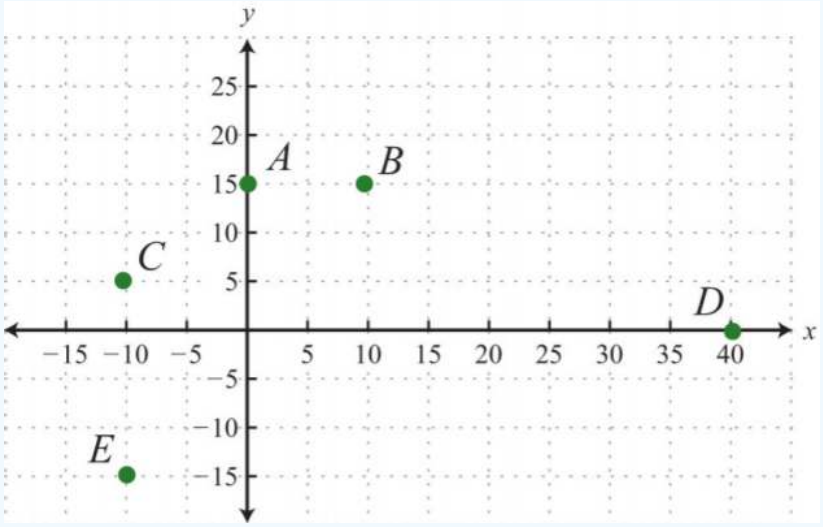


Figure 1.2.1.23

Answer

1. $A:(3,5); B:(-2,3); C:(-5,0); D:(1,-3); E:(-3,-4)$
3. $A:(0,6); B:(-4,3); C:(-8,0); D:(-6,-6); E:(8,-9)$
5. $A:(-10,25); B:(30,20); C:(0,10); D:(15,0); E:(25,-10)$

EXERCISE 1.2.1.5 ORDERED PAIRS

Graph the given set of ordered pairs.

1. $\{(-4,5),(-1,1),(-3,-2),(5,-1)\}$
2. $\{(-15,-10),(-5,10),(15,10),(5,-10)\}$
3. $\{(-2,5),(10,0),(2,-5),(6,-10)\}$
4. $\{(-8,3),(-4,6),(0,-6),(6,9)\}$
5. $\{(-10,5),(20,-10),(30,15),(50,0)\}$
6. $\{(-\frac{5}{3},-\frac{1}{2}),(-\frac{1}{3},\frac{1}{2}),(\frac{2}{3},-1),(\frac{5}{3},1)\}$
7. $\{(-\frac{3}{5},-\frac{4}{3}),(\frac{2}{5},\frac{4}{3}),(1,-\frac{2}{3}),(0,1)\}$
8. $\{(-3.5,0),(-1.5,2),(0,1.5),(2.5,-1.5)\}$
9. $\{(-0.8,0.2),(-0.2,-0.4),(0,-1),(0.6,-0.4)\}$
10. $\{(-1.2,-1.2),(-0.3,-0.3),(0,0),(0.6,0.6),(1.2,1.2)\}$

Answer

1.

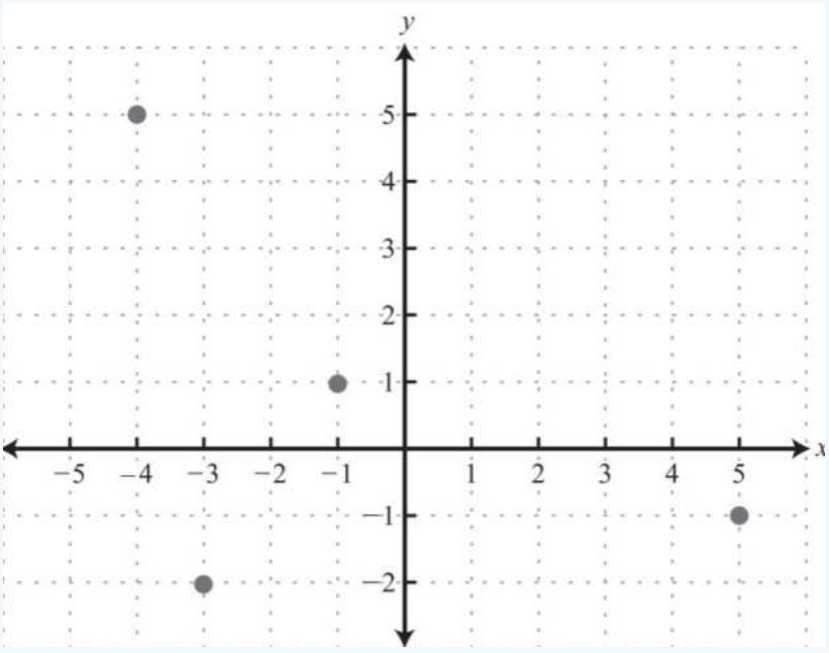


Figure 1.2.1.24

3.

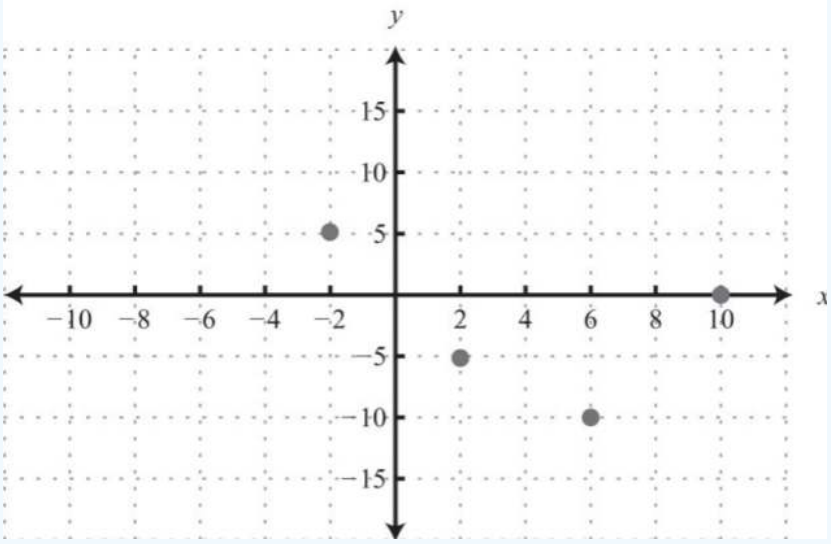


Figure 1.2.1.25

5.

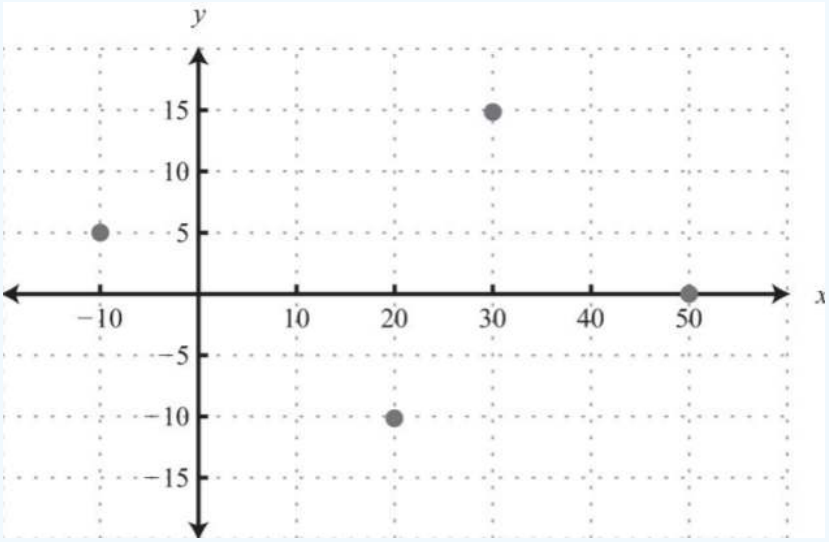


Figure 1.2.1.26

7.

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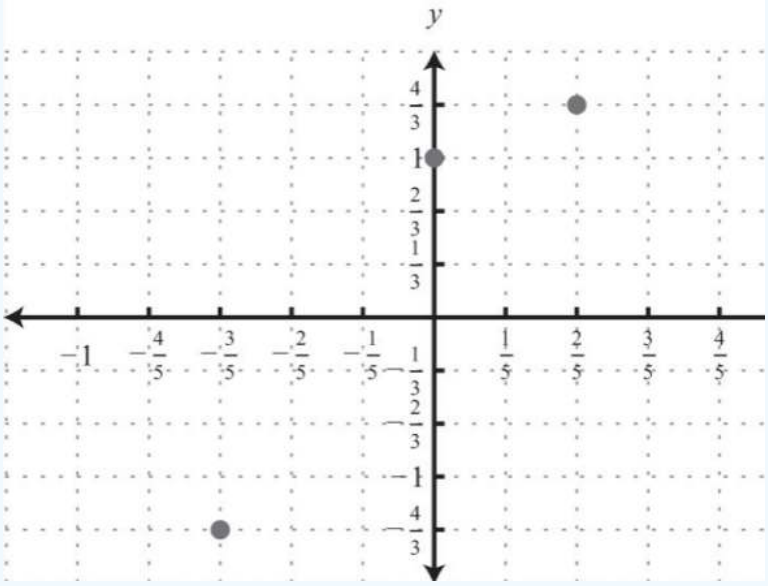


Figure 1.2.1.27

9.

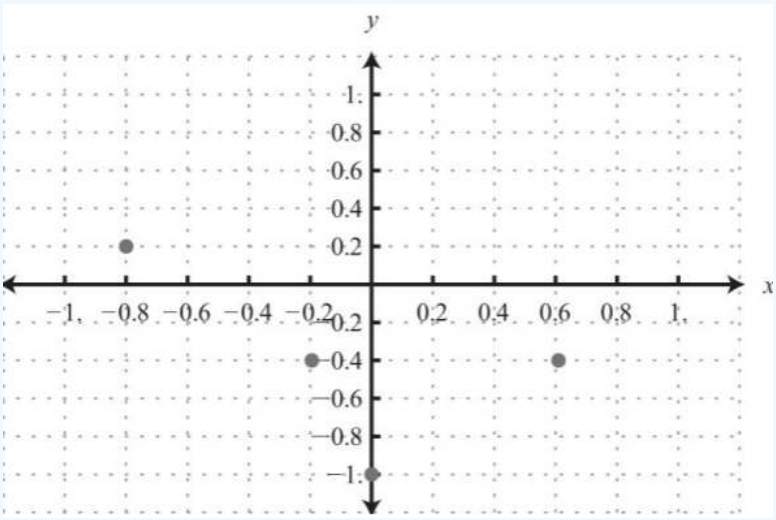


Figure 1.2.1.28

? EXERCISE 1.2.1.6 ORDERED PAIRS

State the quadrant in which the given point lies.

1. $(-3, 2)$
2. $(5, 7)$
3. $(-12, -15)$
4. $(7, -8)$
5. $(-3.8, 4.6)$
6. $(17.3, 1.9)$
7. $(-\frac{1}{8}, -\frac{5}{8})$
8. $(\frac{3}{4}, -\frac{1}{4})$
9. $x > 0$ and $y < 0$
10. $x < 0$ and $y < 0$
11. $x < 0$ and $y > 0$
12. $x > 0$ and $y > 0$

Answer

1. QII
3. QIII
5. QII
7. QIII
9. QIV
11. QII

EXERCISE 1.2.1.7 ORDERED PAIRS

The average price of a gallon of regular unleaded gasoline in US cities is given in the following line graph. Use the graph to answer the following questions.

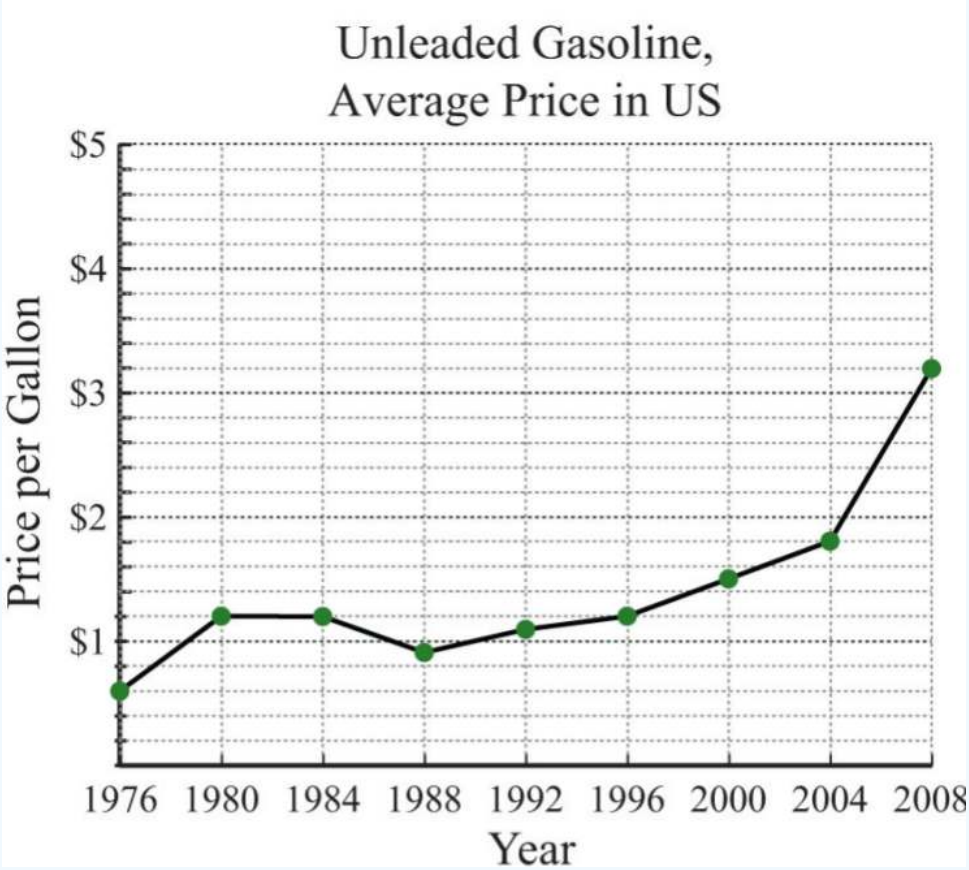


Figure 1.2.1.29: *Source: Bureau of Labor Statistics*

1. What was the average price of a gallon of unleaded gasoline in 2004?
2. What was the average price of a gallon of unleaded gasoline in 1976?
3. In which years were the average price of a gallon of unleaded gasoline $1.20?
4. What is the price increase of a gallon of gasoline from 1980 to 2008?
5. What was the percentage increase in the price of a gallon of unleaded gasoline from 1976 to 1980?
6. What was the percentage increase in the price of a gallon of unleaded gasoline from 2000 to 2008?

Answer

1. $1.80
3. 1980 to 1984, 1996
3. 100%

EXERCISE 1.2.1.8 ORDERED PAIRS

The average price of all-purpose white flour in US cities from 1980 to 2008 is given in the following line graph. Use the graph to answer the questions that follow.



Figure 1.2.1.30: *Source: Bureau of Labor Statistics*

1. What was the average price per pound of all-purpose white flour in 2000?
2. What was the average price per pound of all-purpose white flour in 2008?
3. In which year did the price of flour average $0.25 per pound?
4. In which years did the price of flour average $0.20 per pound?
5. What was the percentage increase in flour from the year 2000 to 2008?
6. What was the percentage increase in flour from the year 1992 to 2000?

Answer

1. $0.30
3. 1992
5. 67%

EXERCISE 1.2.1.9 ORDERED PAIRS

Given the following data, create a line graph.

1. The percentage of total high school graduates who enrolled in college.

Table 1.2.1.2: *Source: Digest of Education Statistics*

Year	Percentage
1969	36%
1979	40%
1989	47%
1999	42%

2. The average daily temperature given in degrees Fahrenheit in May.

Table 1.2.1.3

Exam	Temperature
8:00 am	60
12:00 pm	72
4:00 pm	75
8:00 pm	67
12:00 am	60
4:00 am	55

Answer

1.

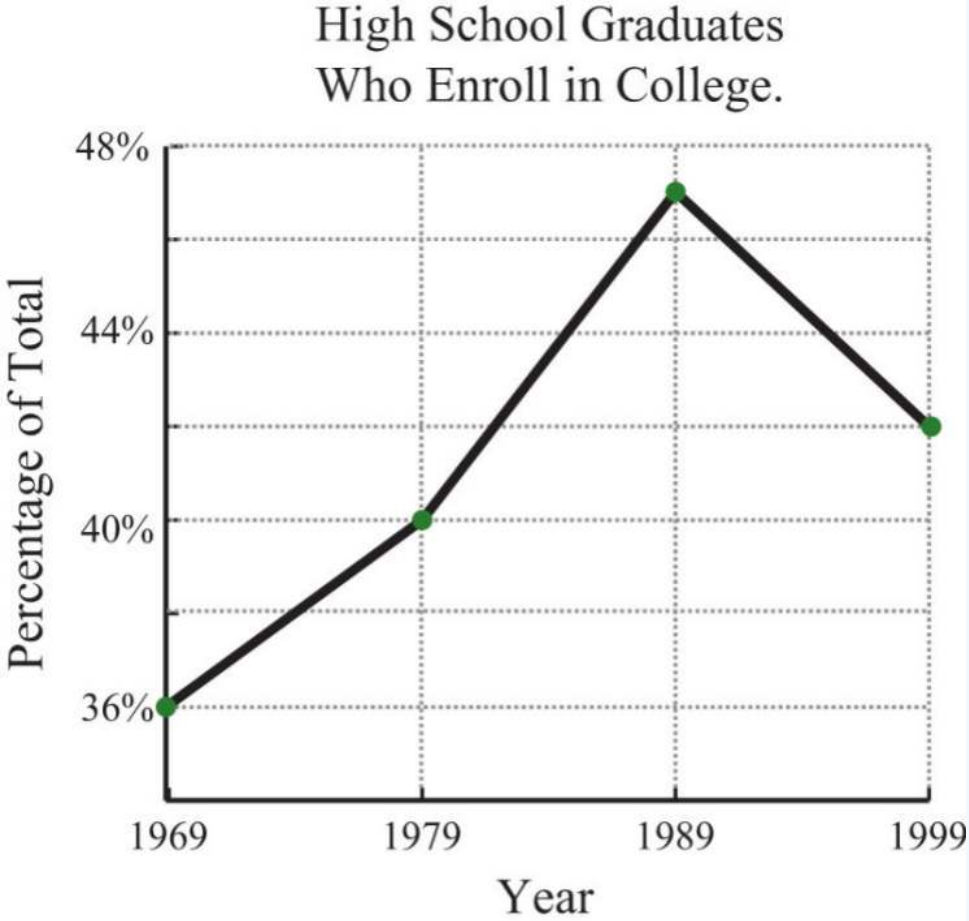


Figure 1.2.1.31

? EXERCISE 1.2.1.10 ORDERED PAIRS

Calculate the area of the shape formed by connecting the following set of vertices.

1. $\{(0,0),(0,3),(5,0),(5,3)\}$
2. $\{(-1,-1),(-1,1),(1,-1),(1,1)\}$
3. $\{(-2,-1),(-2,3),(5,3),(5,-1)\}$
4. $\{(-5,-4),(-5,5),(3,5),(3,-4)\}$
5. $\{(0,0),(4,0),(2,2)\}$
6. $\{(-2,-2),(2,-2),(0,2)\}$
7. $\{(0,0),(0,6),(3,4)\}$
8. $\{(-2,0),(5,0),(3,-3)\}$

Answer

1. 15 square units
3. 28 square units
5. 4 square units
7. 9 square units

EXERCISE 1.2.1.11 DISTANCE FORMULA

Calculate the distance between the given two points.

1. $(-5,3)$ and $(-1,6)$
2. $(6,-2)$ and $(-2,4)$
3. $(0,0)$ and $(5,12)$
4. $(-6,-8)$ and $(0,0)$
5. $(-7,8)$ and $(5,-1)$
6. $(-1,-2)$ and $(9,22)$
7. $(-1,2)$ and $(-\frac{7}{2},-4)$
8. $(-\frac{1}{2},\frac{1}{3})$ and $(\frac{5}{2},-\frac{11}{3})$
9. $(-\frac{1}{3},\frac{2}{3})$ and $(1,-\frac{1}{3})$
10. $(\frac{1}{2},-\frac{3}{4})$ and $(\frac{3}{2},\frac{1}{4})$
11. $(1,2)$ and $(4,3)$
12. $(2,-4)$ and $(-3,-2)$
13. $(-1,5)$ and $(1,-3)$
14. $(1,-7)$ and $(5,-1)$
15. $(-7,-3)$ and $(-1,6)$
16. $(0,1)$ and $(1,0)$
17. $(-0.2,-0.2)$ and $(1.8,1.8)$
18. $(1.2,-3.3)$ and $(2.2,-1.7)$

Answer

1. 5 units
3. 13 units
5. 15 units
7. $\frac{13}{2}$ units
9. $\frac{5}{3}$ units
11. $\sqrt{10}$ units
13. $2\sqrt{17}$ units
15. $3\sqrt{13}$ units
17. 2.8 units

EXERCISE 1.2.1.12 DISTANCE FORMULA

For each problem, show that the three points form a right triangle.

1. $(-3,-2),(0,-2)$, and $(0,4)$
2. $(7,12),(7,-13)$, and $(-5,-4)$
3. $(-1.4,0.2),(1,2)$, and $(1,-3)$
4. $(2,-1),(-1,2)$, and $(6,3)$
5. $(-5,2),(-1,-2)$, and $(-2,5)$
6. $(1,-2),(2,3)$, and $(-3,4)$

Answer

1. Proof
3. Proof

5. Proof

EXERCISE 1.2.1.13 DISTANCE FORMULA

Isosceles triangles have two legs of equal length. For each problem, show that the following points form an isosceles triangle.

1. $(1,6),(-1,1)$, and $(3,1)$
2. $(-6,-2),(-3,-5)$, and $(-9,-5)$
3. $(-3,0),(0,3)$, and $(3,0)$
4. $(0,-1),(0,1)$, and $(1,0)$

Answer

1. Proof
3. Proof

EXERCISE 1.2.1.14 DISTANCE FORMULA

Calculate the area and the perimeter of the triangles formed by the following set of vertices.

1. $\{(-4,-5),(-4,3),(2,3)\}$
2. $\{(-1,1),(3,1),(3,-2)\}$
3. $\{(-3,1),(-3,5),(1,5)\}$
4. $\{(-3,-1),(-3,7),(1,-1)\}$

Answer

1. Perimeter: 24 units; area: 24 square units
3. Perimeter: $8+4\sqrt{2}$ units; area: 8 square units

EXERCISE 1.2.1.15 MIDPOINT FORMULA

Find the midpoint between the given two points.

1. $(-1,6)$ and $(-7,-2)$
2. $(8,0)$ and $(4,-3)$
3. $(-10,0)$ and $(10,0)$
4. $(-3,-6)$ and $(-3,6)$
5. $(-10,5)$ and $(14,-5)$
6. $(0,1)$ and $(2,2)$
7. $(5,-3)$ and $(4,-5)$
8. $(0,0)$ and $(1,1)$
9. $(-1,-1)$ and $(4,4)$
10. $(3,-5)$ and $(3,5)$
11. $(-\frac{1}{2},-\frac{1}{3})$ and $(\frac{3}{2},\frac{7}{3})$
12. $(\frac{3}{4},-\frac{2}{3})$ and $(\frac{1}{8},-\frac{1}{2})$
13. $(\frac{5}{3},\frac{1}{4})$ and $(-\frac{1}{6},-\frac{3}{2})$
14. $(-\frac{1}{5},-\frac{5}{2})$ and $(\frac{7}{10},-\frac{1}{4})$
15. Given the right triangle formed by the vertices $(0,0),(6,0)$, and $(6,8)$, show that the midpoints of the sides form a right triangle.
16. Given the isosceles triangle formed by the vertices $(-10,-12),(0,12)$, and $(10,-12)$, show that the midpoints of the sides also form an isosceles triangle.
17. Calculate the area of the triangle formed by the vertices $(-4,-3),(-1,1)$, and $(2,-3)$. (Hint: The vertices form an isosceles triangle.)
18. Calculate the area of the triangle formed by the vertices $(-2,1),(4,1)$, and $(1,-5)$.

Answer

1. $(-4,2)$
3. $(0,0)$

5. $(2,0)$
7. $(\frac{9}{2},-4)$
9. $(\frac{3}{2},\frac{3}{2})$
11. $(\frac{1}{2},1)$
13. $(\frac{3}{4},-\frac{5}{8})$
15. 12 square units

EXERCISE 1.2.1.16 DISCUSSION BOARD TOPICS

1. Research and discuss the life and contributions to mathematics of René Descartes.
2. Research and discuss the history of the right triangle and the Pythagorean theorem.
3. What is a Pythagorean triple? Provide some examples.
4. Explain why you cannot use a ruler to calculate distance on a graph.
5. How do you bisect a line segment with only a compass and a straightedge?

Answer

1. Answers may vary
3. Answers may vary
5. Answers may vary

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1.2.2: GRAPH BY PLOTTING POINTS

LEARNING OBJECTIVES

- Verify solutions to linear equations with two variables.
- Graph lines by plotting points.
- Identify and graph horizontal and vertical lines.

SOLUTIONS TO EQUATIONS WITH TWO VARIABLES

A linear equation with two variables has standard form $ax+by=c$, where a, b, and c are real numbers and a and b are not both 0. Solutions to equations of this form are ordered pairs (x, y), where the coordinates, when substituted into the equation, produce a true statement.

EXAMPLE 1.2.2.1

Determine whether $(1,-2)$ and $(-4,1)$ are solutions to $6x-3y=12$.

Solution:

Substitute the x- and y-values into the equation to determine whether the ordered pair produces a true statement.

$Check\ (1,-2)$	$Check\ (-4,1)$
$6x-3y=12$	$6x-3y=12$
$6(1)-3(-2)=12$	$6(-4)-3(1)=12$
$6+6=12$	$-24-3=12$
$12=12$ ✓	$-27=12$ x

Answer:

$(1,-2)$ is a solution, and $(-4,1)$ is not.

It is often the case that a linear equation is given in a form where one of the variables, usually y, is isolated. If this is the case, then we can check that an ordered pair is a solution by substituting in a value for one of the coordinates and simplifying to see if we obtain the other.

EXAMPLE 1.2.2.2

Are $(\frac{1}{2},-3)$ and $(-5,14)$ solutions to $y=2x-4$?

Solution:

Substitute the x-values and simplify to see if the corresponding y-values are obtained.

$x=\frac{1}{2}$	$x=-5$
$y=2x-4$	$y=2x-4$
$=2(\frac{1}{2})-4$	$=2(-5)-4$
$=1-4$	$=-10-4$
$=-3$ ✓	$=-14$ x

Answer:

$(\frac{1}{2},-3)$ is a solution, and $(-5,14)$ is not.

EXERCISE 1.2.2.1

Is $(6,-1)$ a solution to $y=-\frac{2}{3}x+3$?

Answer

Yes

When given linear equations with two variables, we can solve for one of the variables, usually y, and obtain an equivalent equation as follows:

Written in this form, we can see that y depends on x. Here x is the independent variable and y is the dependent variable.

$$\left.\begin{aligned} 6x-3y &= 12 \\ y &= 2x-4 \end{aligned}\right\} \textit{Equivalent equations}$$

The linear equation $y=2x-4$ can be used to find ordered pair solutions. If we substitute any real number for x, then we can simplify to find the corresponding y-value. For example, if $x=3$, then $y=2(3)-4=6-4=2$, and we can form an ordered pair solution, $(3,2)$. Since there are infinitely many real numbers to choose for x, the linear equation has infinitely many ordered pair solutions (x,y).

EXAMPLE 1.2.2.3

Find ordered pair solutions to the equation $5x-y=14$ with the given x-values $\{-2,-1,0,4,6\}$.

Solution:

First, solve for y.

Next, substitute the x-values in the equation $y=5x-14$ to find the corresponding y-values.

$x-value\ y-value\ Solution x=-2\ (-2,-24)\ x=-1\ (-1,-19)\ x=0\ (0,-14)\ x=4\ (4,6)\ x=6\ (6,16)$ Table 1.2.2.1

Answer:

$$\{(-2,-24),(-1,-19),(0,-14),(4,6),(6,16)\}$$

In the previous example, certain x-values are given, but that is not always going to be the case. When treating x as the independent variable, we can choose any values for x and then substitute them into the equation to find the corresponding y-values. This method produces as many ordered pair solutions as we wish.

EXAMPLE 1.2.2.4

Find five ordered pair solutions to $6x+2y=10$.

Solution:

First, solve for y.

$$\begin{aligned} 6x+2y &= 10 \\ 6x+2y-6x &= 10-6x \\ 2y &= -6x+10 \\ \frac{2y}{2} &= \frac{-6x+10}{2} \\ y &= \frac{-6x}{2}+\frac{10}{2} \\ y &= -3x+5 \end{aligned}$$

Next, choose any set of x-values. Usually we choose some negative values and some positive values. In this case, we will find the corresponding y-values when x is $\{-2,-1,0,1,2\}$. Make the substitutions required to fill in the following table (often referred to as a t-chart):

x	y		Solutions
−2	11	$y=-3(-2)+5=6+5=11$	(−2, 11)
−1	8	$y=-3(-1)+5=3+5=8$	(−1, 8)
0	5	$y=-3(0)+5=0+5=5$	(0, 5)
1	2	$y=-3(1)+5=-3+5=2$	(1, 2)
2	−1	$y=-3(2)+5=-6+5=-1$	(2, −1)

Figure 1.2.2.1

Answer:

$\{(-2,11),(-1,8),(0,5),(1,2),(2,-1)\}$. Since there are infinitely many ordered pair solutions, answers may vary depending on the choice of values for the independent variable.

? EXERCISE 1.2.2.2

Find five ordered pair solutions to $10x - 2y = 2$.

Answer

$\{(-2,-11),(-1,-6),(0,-1),(1,4),(2,9)\}$ (answers may vary)

GRAPH BY PLOTTING POINTS

Since the solutions to linear equations are ordered pairs, they can be graphed using the rectangular coordinate system. The set of all solutions to a linear equation can be represented on a rectangular coordinate plane using a straight line connecting at least two points; this line is called its graph. To illustrate this, plot five ordered pair solutions, $\{(-2,11),(-1,8),(0,5),(1,2),(2,-1)\}$, to the linear equation $6x + 2y = 10$.

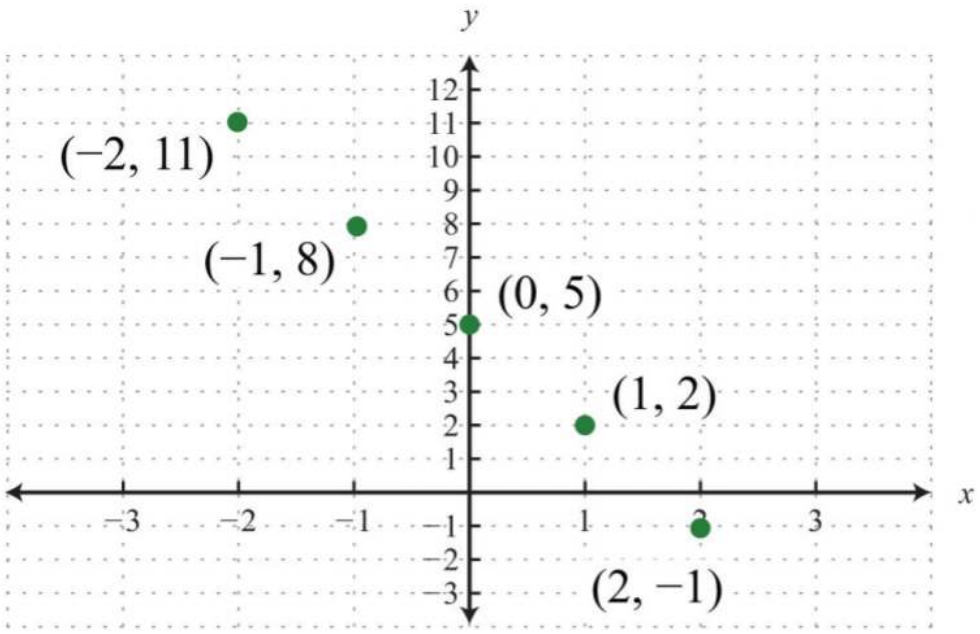


Figure 1.2.2.2

Notice that the points are collinear; this will be the case for any linear equation. Draw a line through the points with a straightedge, and add arrows on either end to indicate that the graph extends indefinitely.

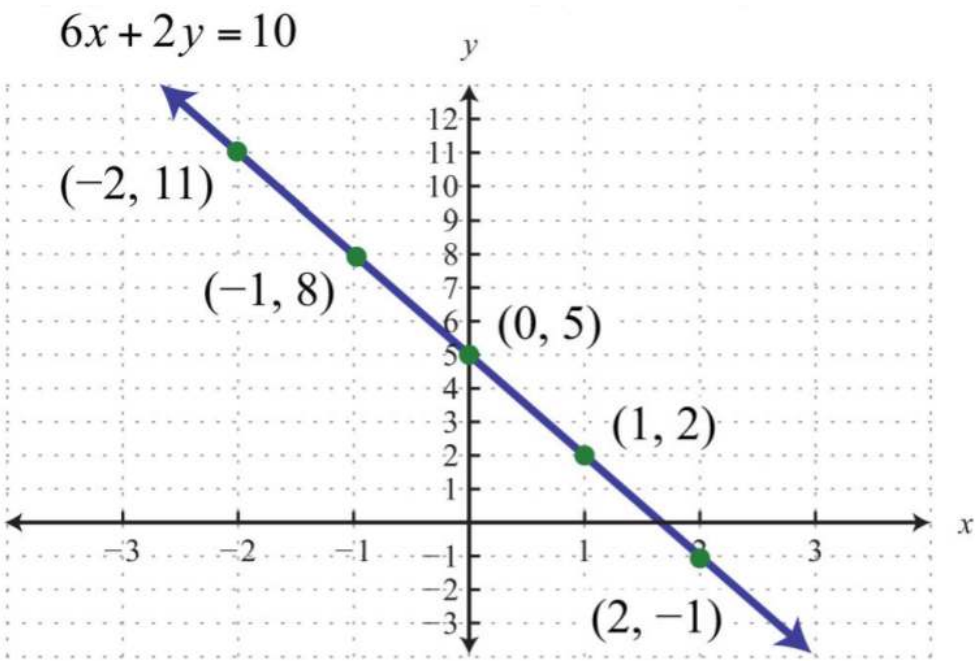


Figure 1.2.2.3

The resulting line represents all solutions to $6x + 2y = 10$, of which there are infinitely many. The steps for graphing lines by plotting points are outlined in the following example.

EXAMPLE 1.2.2.5

Find five ordered pair solutions and graph:

$$10x - 5y = 10\,.$$

Solution:

Step 1: Solve for y.

Step 2: Choose at least two x-values and find the corresponding y-values. In this section, we will choose five real numbers to use as x-values. It is a good practice to choose 0 and some negative numbers, as well as some positive numbers.

Figure 1.2.2.4

Five ordered pair solutions are $\{(-2,-6),(-1,-4),(0,-2),(1,0),(2,2)\}$

Step 3: Choose an appropriate scale, plot the points, and draw a line through them using a straightedge. In this case, choose a scale where each tick mark on the y-axis represents 2 units because all the y-values are multiples of 2.

Answer:

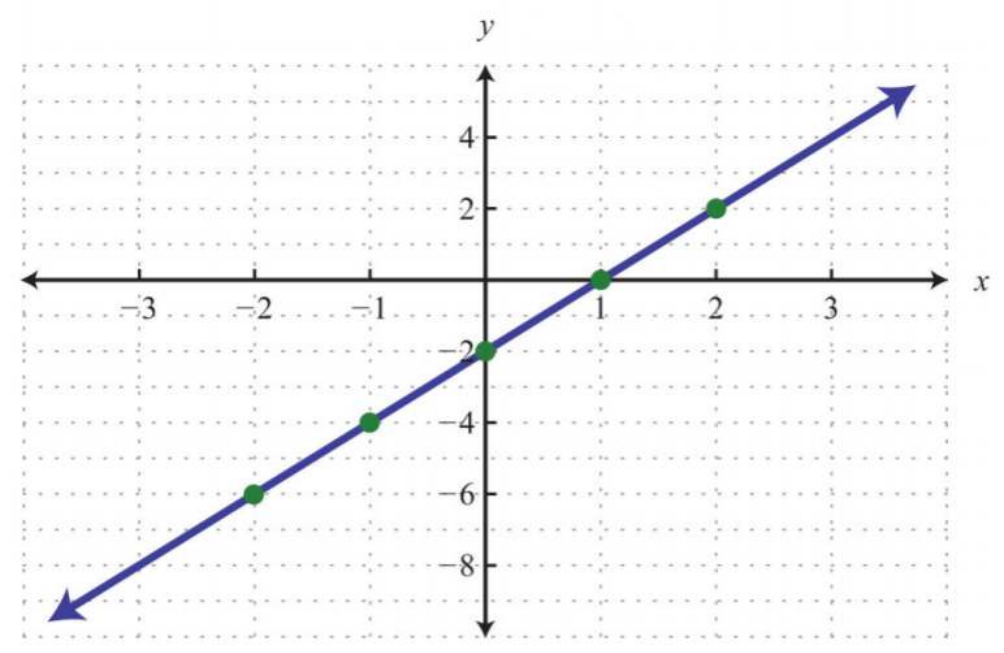


Figure 1.2.2.5

It will not always be the case that y can be solved in terms of x with integer coefficients. In fact, the coefficients often turn out to be fractions.

EXAMPLE 1.2.2.6

Find five ordered pair solutions and graph:

$$-5x+2y=10.$$

Solution:

$$\begin{aligned} -5x+2y &= 10 \\ -5x+2y+5x &= 10+5x \\ 2y &= 5x+10 \\ frac2y2 &= \frac{5x+10}{2} \\ y &= \frac{5x}{2}+\frac{10}{2} \\ y &= \frac{5}{2}x+5 \end{aligned}$$

Remember that you can choose any real number for the independent variable x, so choose wisely here. Since the denominator of the coefficient of the variable x is 2, you can avoid fractions by choosing multiples of 2 for the x-values. In this case, choose the set of x-values $\{-6,-4,-2,0,2\}$ and find the corresponding y-values.

x	y
-6	-10
-4	-5
-2	0
0	5
2	10

$$y=\frac{5}{2}(-6)+5=-15+5=-10$$

$$y=\frac{5}{2}(-4)+5=-10+5=-5$$

$$y=\frac{5}{2}(-2)+5=-5+5=0$$

$$y=\frac{5}{2}(0)+5=0+5=5$$

$$y=\frac{5}{2}(2)+5=5+5=10$$

Figure 1.2.2.6

Five solutions are $\{(-6,-10),(-4,-5),(-2,0),(0,5),(2,10)\}$. Here we choose to scale the x-axis with multiples of 2 and the y-axis with multiples of 5.

Answer:

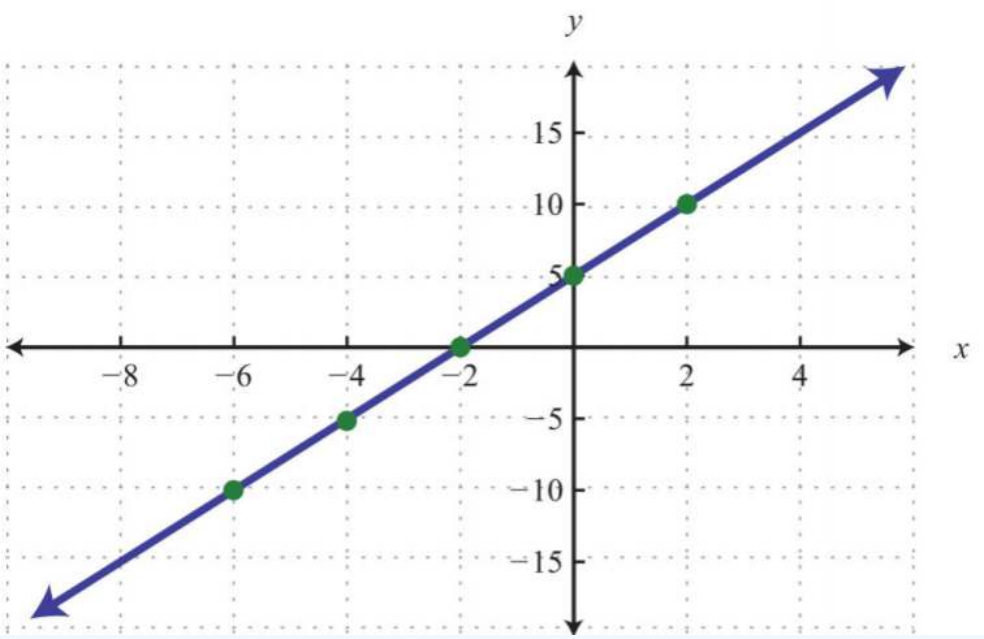


Figure 1.2.2.7

? EXERCISE 1.2.2.3

Find five ordered pair solutions and graph: $x+2y=6$.

Answer

$\{(-2,4),(0,3),(2,2),(4,1),(6,0)\}$

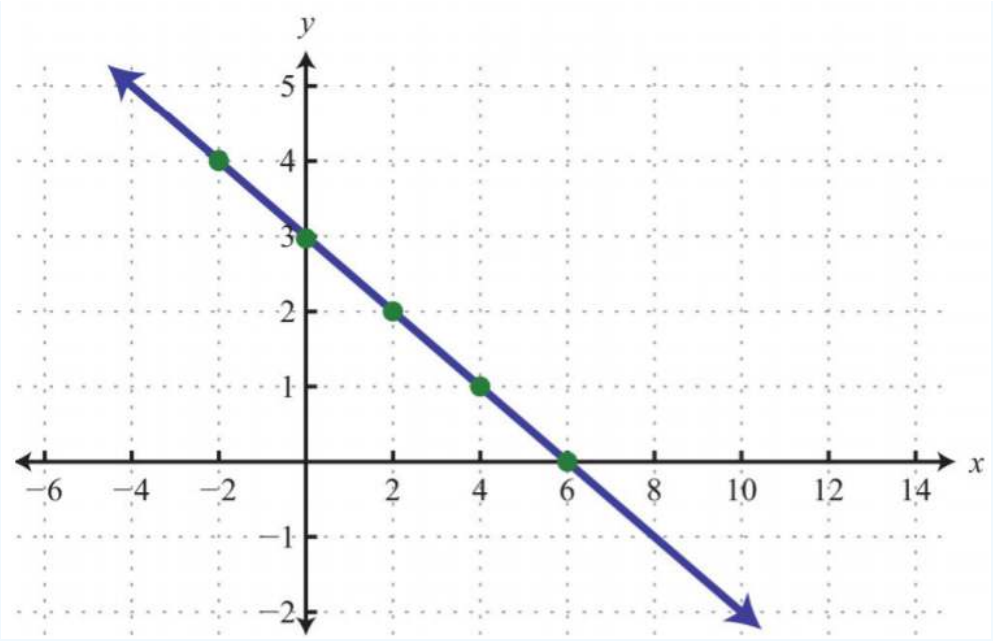


Figure 1.2.2.8

HORIZONTAL AND VERTICAL LINES

We need to recognize by inspection linear equations that represent a vertical or horizontal line.

EXAMPLE 1.2.2.7

Graph by plotting five points: $y = -2$.

Solution:

Since the given equation does not have a variable x, we can rewrite it with a 0 coefficient for x.

$$y = 0x - 2$$

Choose any five values for x and see that the corresponding y-value is always -2.

x	y	
−2	−2	$y = 0(-2) - 2 = 0 - 2 = -2$
−1	−2	$y = 0(-1) - 2 = 0 - 2 = -2$
0	−2	$y = 0(0) - 2 = 0 - 2 = -2$
1	−2	$y = 0(1) - 2 = 0 - 2 = -2$
2	−2	$y = 0(2) - 2 = 0 - 2 = -2$

Figure 1.2.2.9

We now have five ordered pair solutions to plot $\{(-2,-2),(-1,-2),(0,-2),(1,-2),(2,-2)\}$.

Answer:

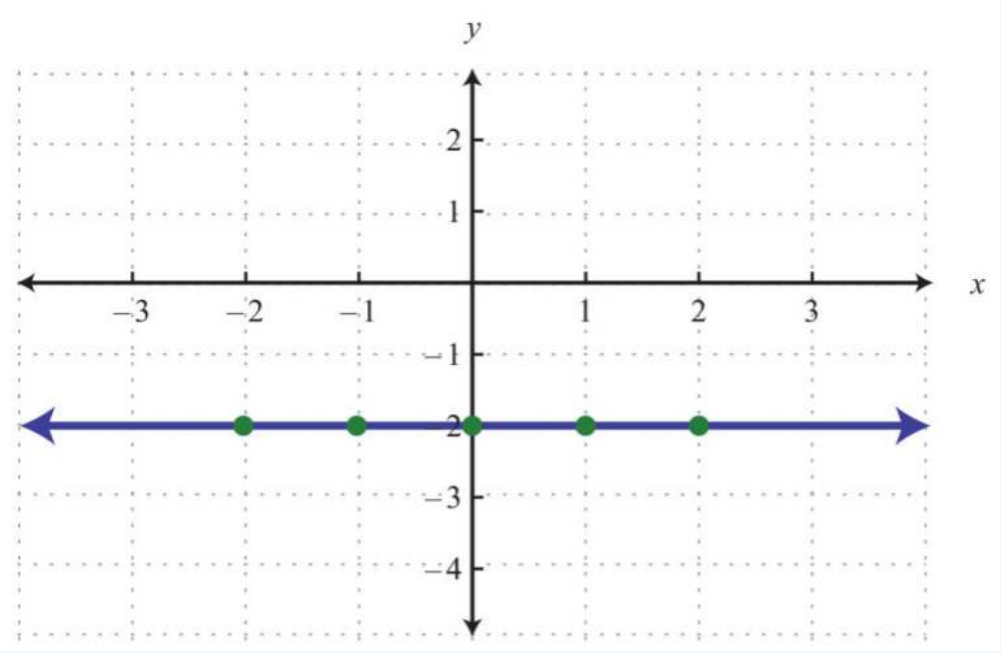


Figure 1.2.2.10

When the coefficient for the variable x is 0, the graph is a horizontal line. In general, the equation for a horizontal line can be written in the form $y = k$, where k represents any real number.

EXAMPLE 1.2.2.8

Graph by plotting five points: $x = 3$.

Solution:

Since the given equation does not have a variable y, rewrite it with a 0 coefficient for y.

$$x = 0y + 3$$

Choose any five values for y and see that the corresponding x-value is always 3.

x	y	
3	−2	$x = 0(-2) + 3 = 0 + 3 = 3$
3	−1	$x = 0(-1) + 3 = 0 + 3 = 3$
3	0	$x = 0(0) + 3 = 0 + 3 = 3$
3	1	$x = 0(1) + 3 = 0 + 3 = 3$
3	2	$x = 0(2) + 3 = 0 + 3 = 3$

Figure 1.2.2.11

We now have five ordered pair solutions to plot: $\{(3,-2),(3,-1),(3,0),(3,1),(3,2)\}$.

Answer:

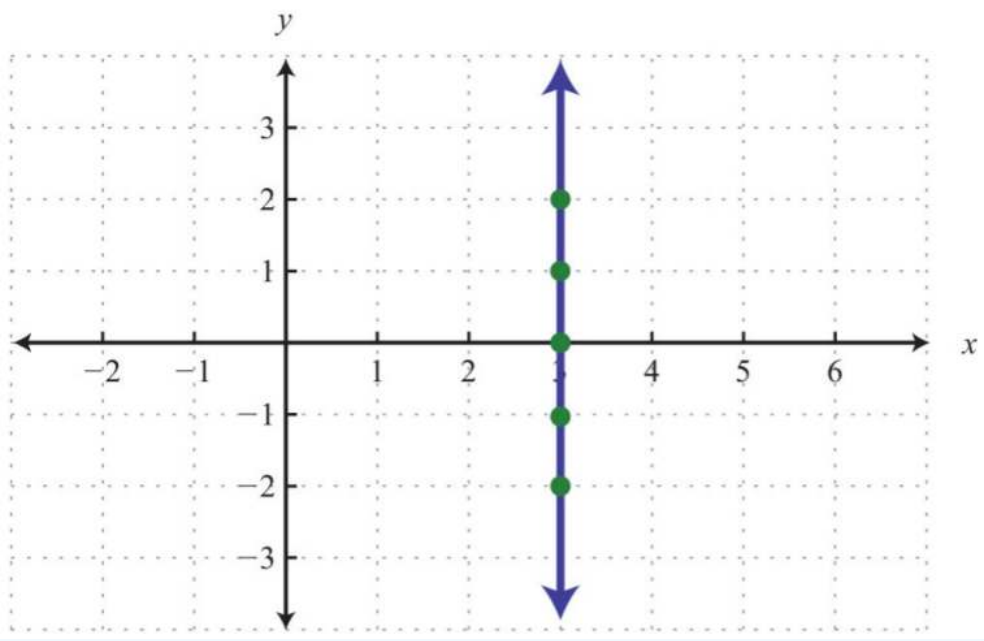


Figure 1.2.2.12

When the coefficient for the variable y is 0, the graph is a vertical line. In general, the equation for a vertical line can be written as $x = k$, where k represents any real number.

To summarize, if k is a real number.

$$y = k \quad \textit{Horizontal line} \tag{1.2.2.1}$$

$$x = k \quad \textit{Vertical line} \tag{1.2.2.2}$$

? EXERCISE 1.2.2.4

Graph $y = 5$ and $x = -2$ on the same set of axes and determine where they intersect.

Answer

$(-2, 5)$

KEY TAKEAWAYS

- Solutions to linear equations with two variables $ax + by = c$ are ordered pairs (x, y), where the coordinates, when substituted into the equation, result in a true statement.
- Linear equations with two variables have infinitely many ordered pair solutions. When the solutions are graphed, they are collinear.
- To find ordered pair solutions, choose values for the independent variable, usually x, and substitute them in the equation to find the corresponding y-values.
- To graph linear equations, determine at least two ordered pair solutions and draw a line through them with a straightedge.
- Horizontal lines are described by $y = k$, where k is any real number.
- Vertical lines are described by $x = k$, where k is any real number.

? EXERCISE 1.2.2.5 SOLUTIONS TO LINEAR SYSTEMS

Determine whether the given point is a solution.

1. $5x - 2y = 4; (-1, 1)$
2. $3x - 4y = 10; (2, -1)$
3. $-3x + y = -6; (4, 6)$
4. $-8x - y = 24; (-2, -3)$
5. $-x + y = -7; (5, -2)$
6. $9x - 3y = 6; (0, -2)$

7. $\frac{1}{2}x+\frac{1}{3}y=-\frac{1}{6};(1,-2)$
8. $\frac{3}{4}x-\frac{1}{2}y=-1;(2,1)$
9. $4x-3y=1;(\frac{1}{2},\frac{1}{3})$
10. $-10x+2y=-\frac{9}{5};(\frac{1}{5},\frac{1}{10})$
11. $y=\frac{1}{3}x+3;(6,3)$
12. $y=-4x+1;(-2,9)$
13. $y=\frac{2}{3}x-3;(0,-3)$
14. $y=-\frac{5}{8}x+1;(8,-5)$
15. $y=-\frac{1}{2}x+\frac{3}{4};(-\frac{1}{2},1)$
16. $y=-\frac{1}{3}x-\frac{1}{2};(\frac{1}{2},-\frac{2}{3})$
17. $y=2;(-3,2)$
18. $y=4;(4,-4)$
19. $x=3;(3,-3)$
20. $x=0;(1,0)$

Answer

1. No
3. Yes
5. Yes
7. Yes
9. Yes
11. No
13. Yes
15. Yes
17. Yes
19. Yes

? EXERCISE 1.2.2.6 SOLUTIONS TO LINEAR SYSTEMS

Find the ordered pair solutions given the set of x-values.

1. $y=-2x+4;\{-2,0,2\}$
2. $y=12x-3;\{-4,0,4\}$
3. $y=-\frac{3}{4}x+\frac{1}{2};\{-2,0,2\}$
4. $y=-3x+1;\{-\frac{1}{2},0,\frac{1}{2}\}$
5. $y=-4;\{-3,0,3\}$
6. $y=\frac{1}{2}x+\frac{3}{4};\{-\frac{1}{4},0,\frac{1}{4}\}$
7. $2x-3y=1;\{0,1,2\}$
8. $3x-5y=-15;\{-5,0,5\}$
9. $-x+y=3;\{-5,-1,0\}$
10. $\frac{1}{2}x-\frac{1}{3}y=-4;\{-4,-2,0\}$
11. $\frac{3}{5}x+\frac{1}{10}y=2;\{-15,-10,-5\}$
12. $x-y=0;\{10,20,30\}$

Answer

1. $\{(-2,8),(0,4),(2,0)\}$
3. $\{(-2,2),(0,1/2),(2,-1)\}$
5. $\{(-3,-4),(0,-4),(3,-4)\}$
7. $\{(0,-\frac{1}{3}),(1,\frac{1}{3}),(2,1)\}$
9. $\{(-5,-2),(-1,2),(0,3)\}$
11. $\{(-15,110),(-10,80),(-5,50)\}$

EXERCISE 1.2.2.7 SOLUTIONS TO LINEAR SYSTEMS

Find the ordered pair solutions, given the set of y-values.

1. $y = \frac{1}{2}x - 1; \{-5, 0, 5\}$
2. $y = -\frac{3}{4}x + 2; \{0, 2, 4\}$
3. $3x - 2y = 6; \{-3, -1, 0\}$
4. $-x + 3y = 4; \{-4, -2, 0\}$
5. $\frac{1}{3}x - \frac{1}{2}y = -4; \{-1, 0, 1\}$
6. $\frac{3}{5}x + \frac{1}{10}y = 2; \{-20, -10, -5\}$

Answer

1. $\{(-8, -5), (2, 0), (12, 5)\}$
3. $\{(0, -3), (\frac{4}{3}, -1), (2, 0)\}$
5. $\{(-\frac{27}{2}, -1), (-12, 0), (-\frac{21}{2}, 1)\}$

EXERCISE 1.2.2.8 GRAPHING LINES

Given the set of x-values $\{-2, -1, 0, 1, 2\}$, find the corresponding y-values and graph them.

1. $y = x + 1$
2. $y = -x + 1$
3. $y = 2x - 1$
4. $y = -3x + 2$
5. $y = 5x - 10$
6. $5x + y = 15$
7. $3x - y = 9$
8. $6x - 3y = 9$
9. $y = -5$
10. $y = 3$

Answer

1.

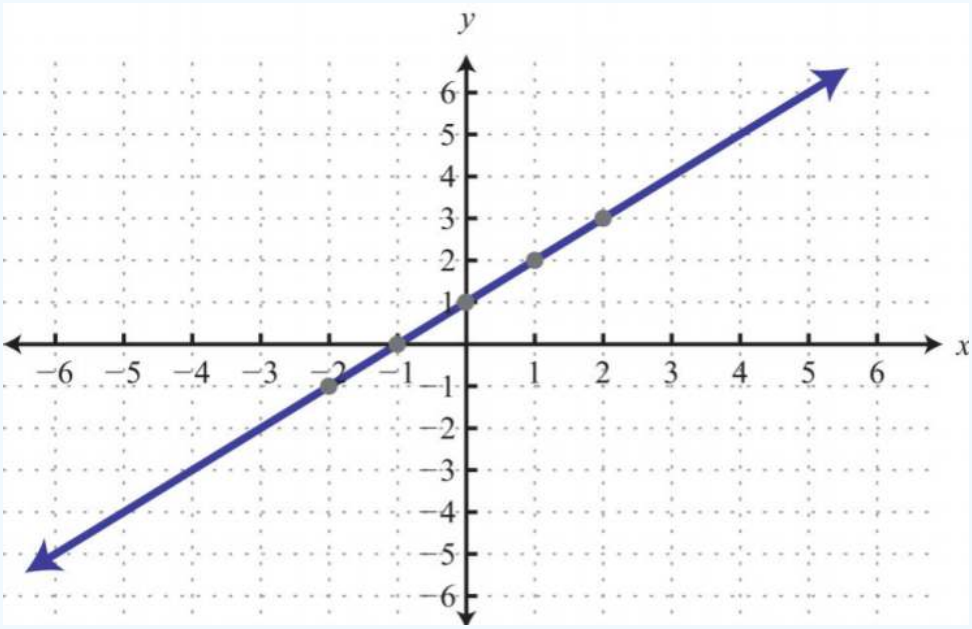


Figure 1.2.2.13

3.

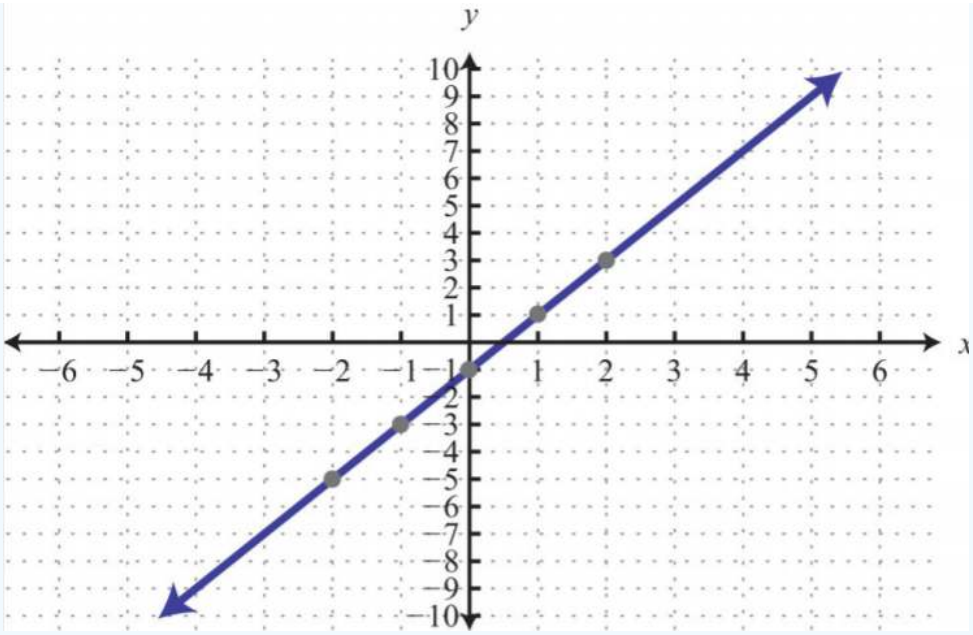


Figure 1.2.2.14

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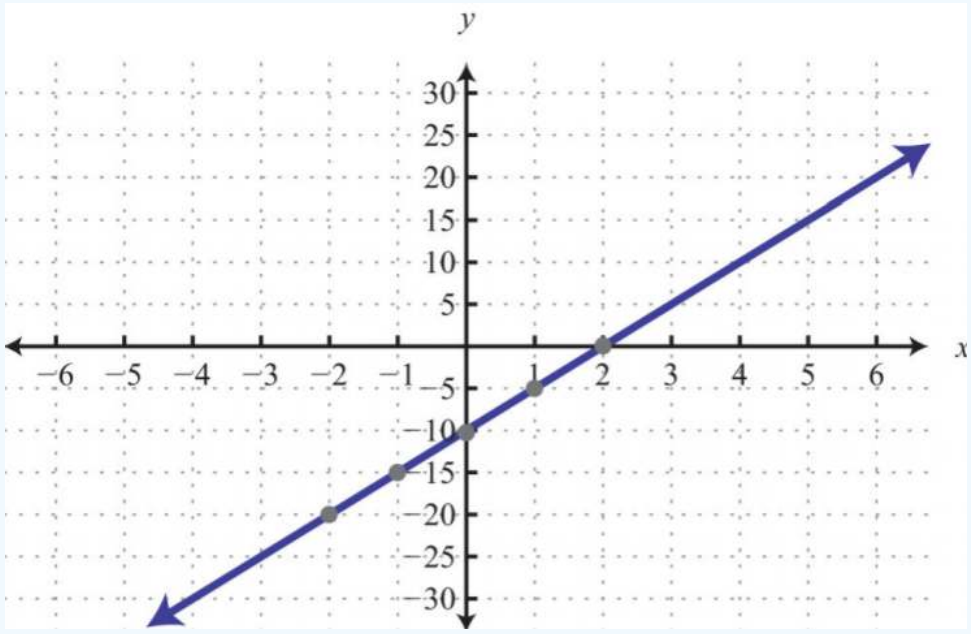


Figure 1.2.2.15

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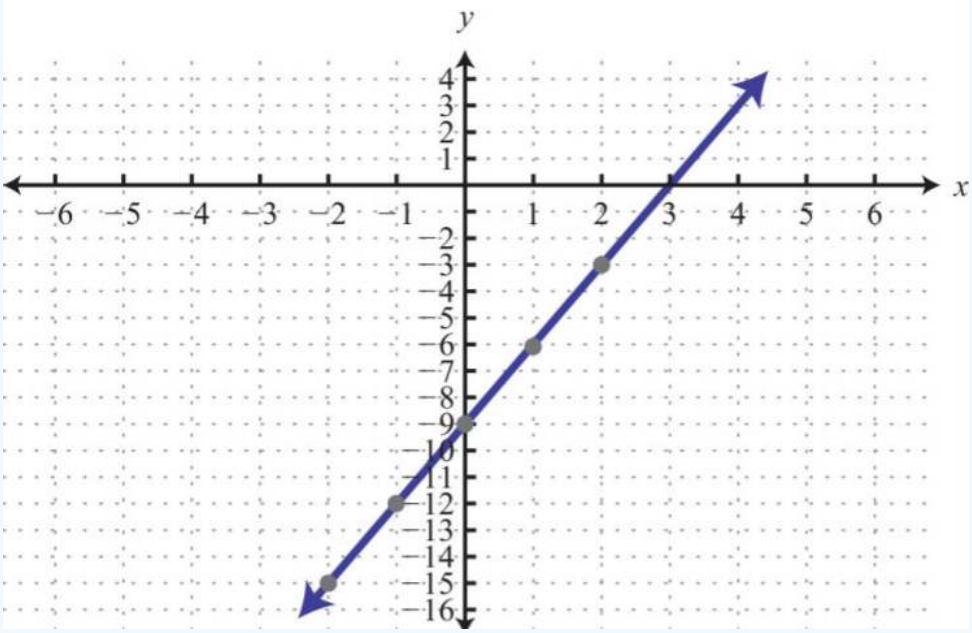


Figure 1.2.2.16

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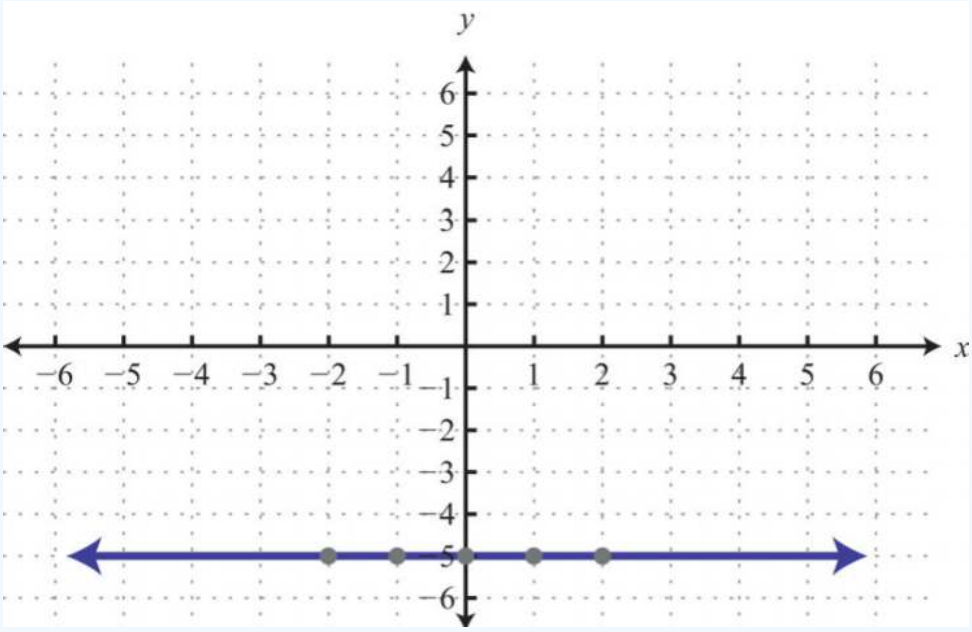


Figure 1.2.2.17

? EXERCISE 1.2.2.9 GRAPHING LINES

Find at least five ordered pair solutions and graph.

1. $y = 2x - 1$
2. $y = -5x + 3$
3. $y = -4x + 2$
4. $y = 10x - 20$
5. $y = -\frac{1}{2}x + 2$
6. $y = \frac{1}{3}x - 1$
7. $y = \frac{2}{3}x - 6$
8. $y = -\frac{2}{3}x + 2$

9. $y = x$
10. $y = -x$
11. $-2x + 5y = -15$
12. $x + 5y = 5$
13. $6x - y = 2$
14. $4x + y = 12$
15. $-x + 5y = 0$
16. $x + 2y = 0$
17. $\frac{1}{10}x - y = 3$
18. $\frac{3}{2}x + 5y = 30$

Answer

1.

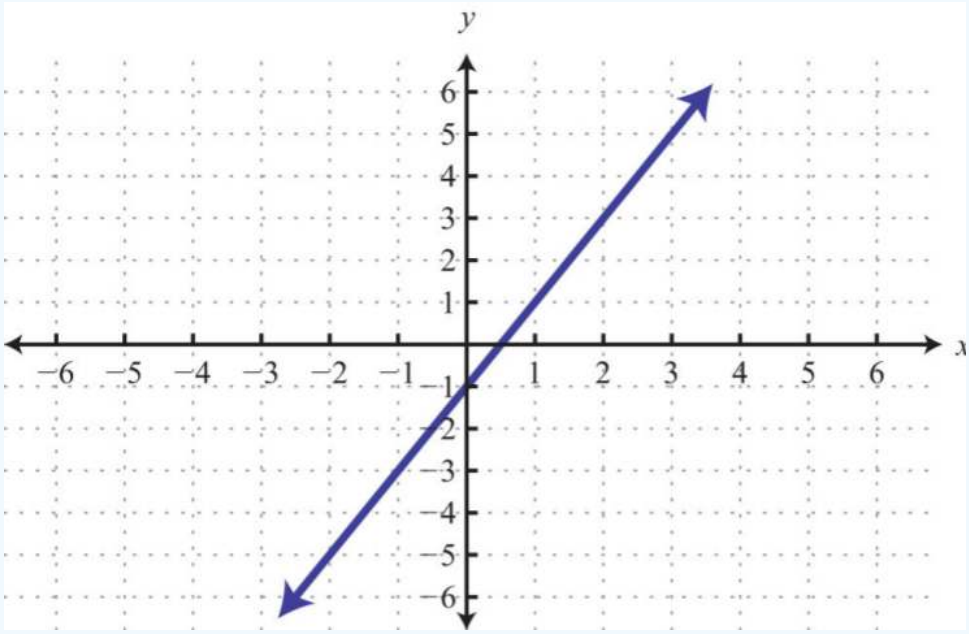


Figure 1.2.2.18

3.

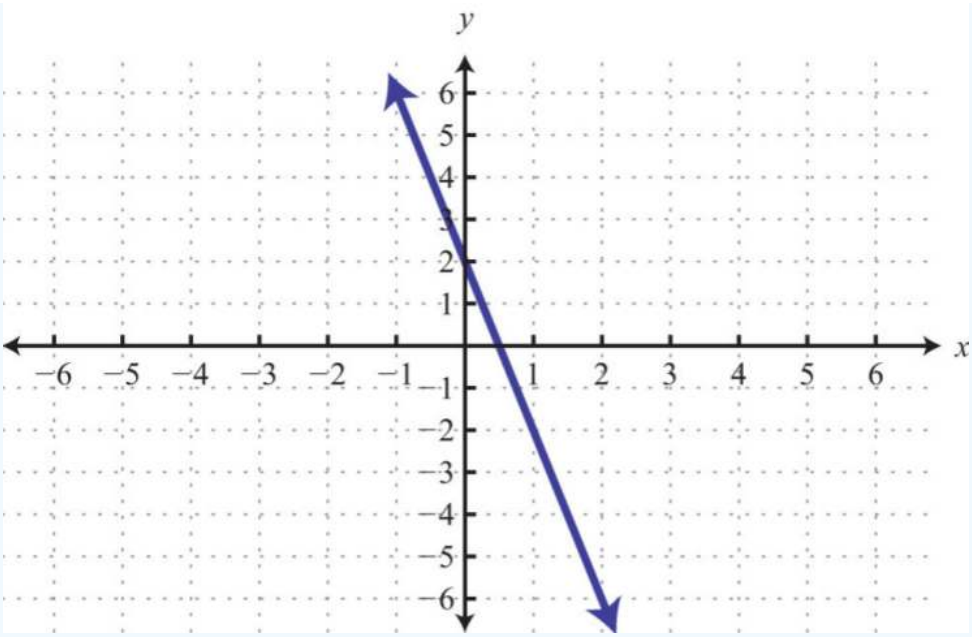


Figure 1.2.2.19

5.

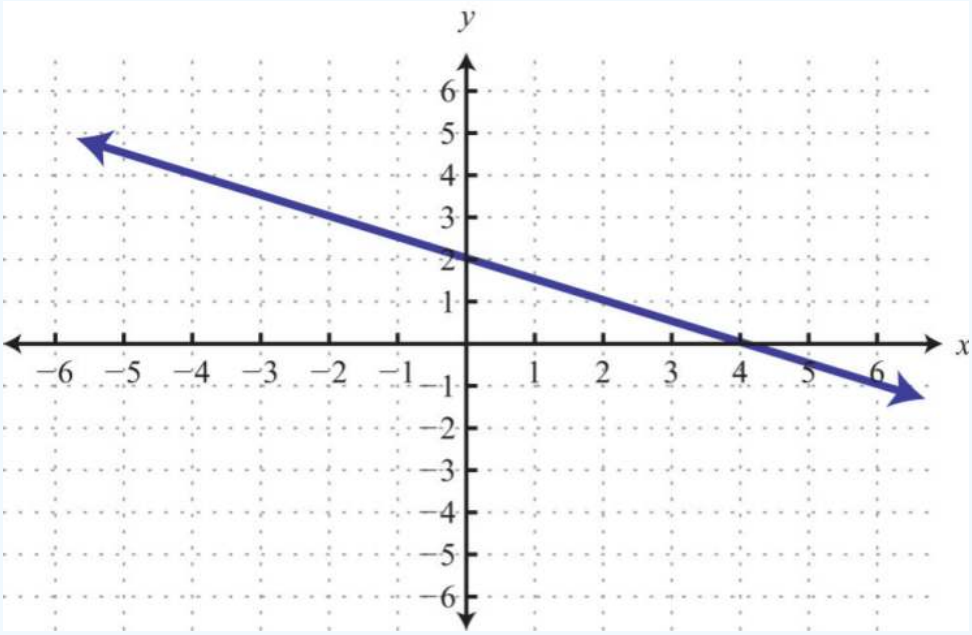


Figure 1.2.2.20

7.

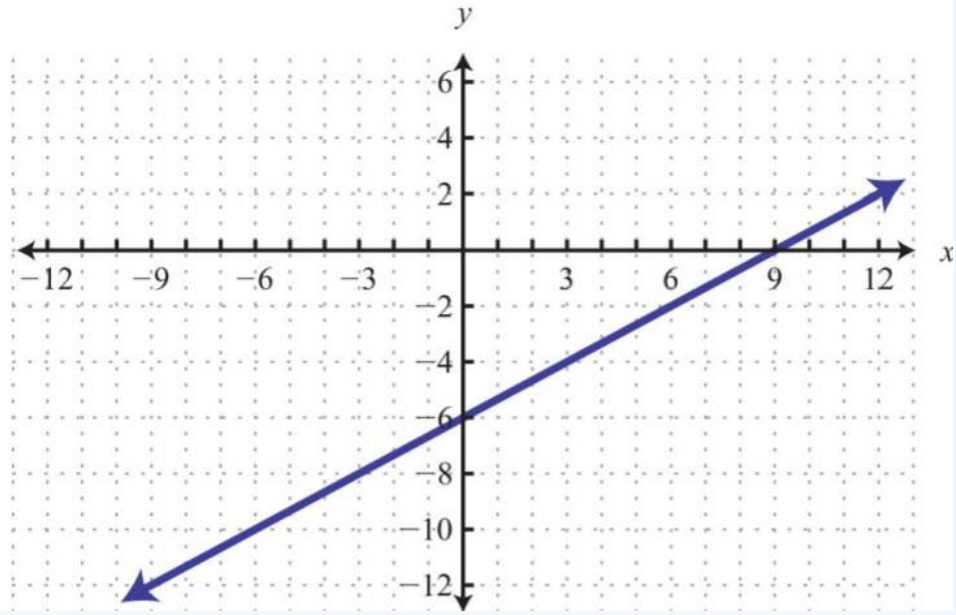


Figure 1.2.2.21

9.

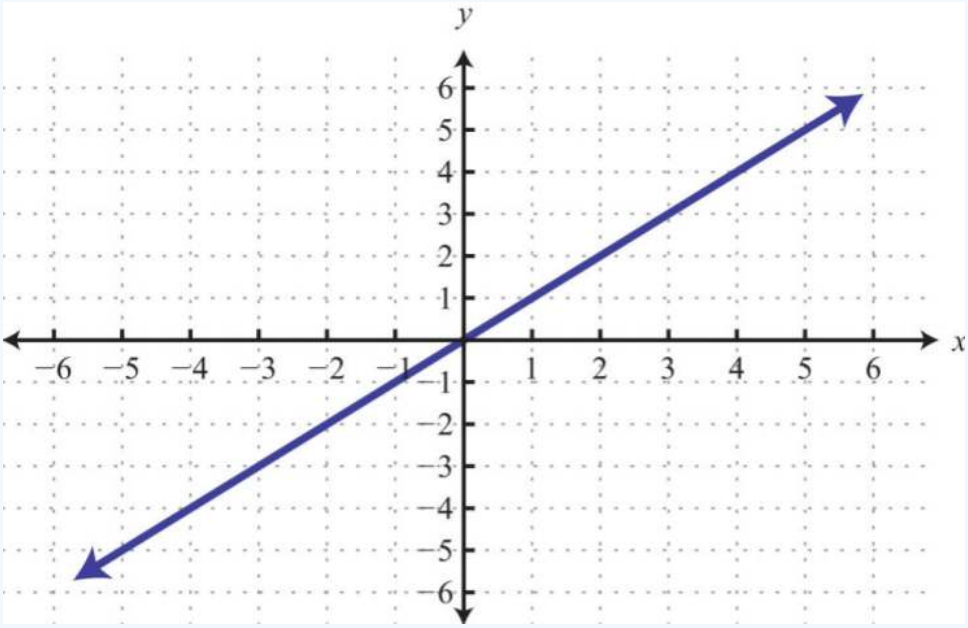


Figure 1.2.2.22

11.

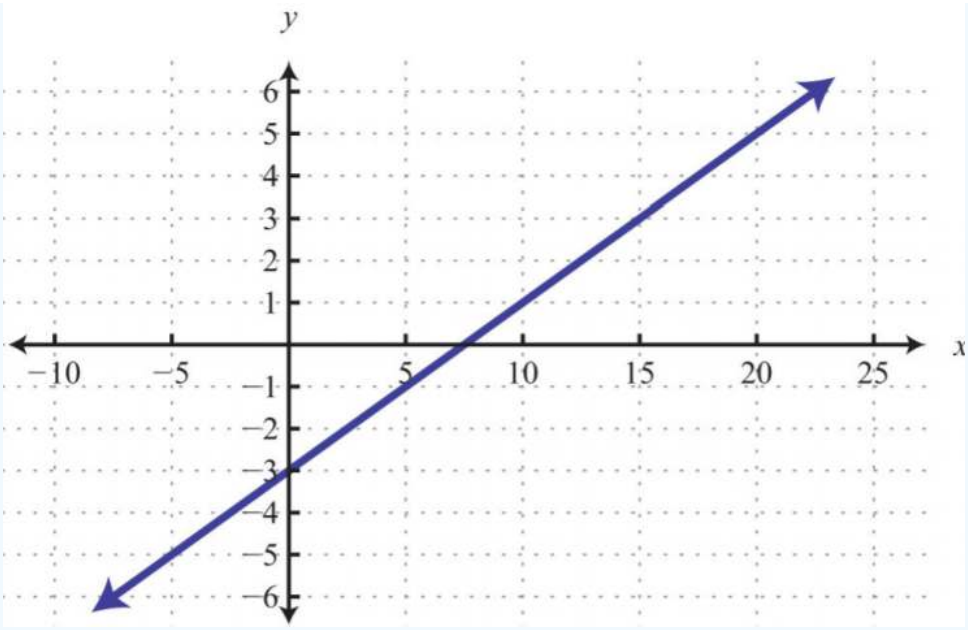


Figure 1.2.2.23

13.

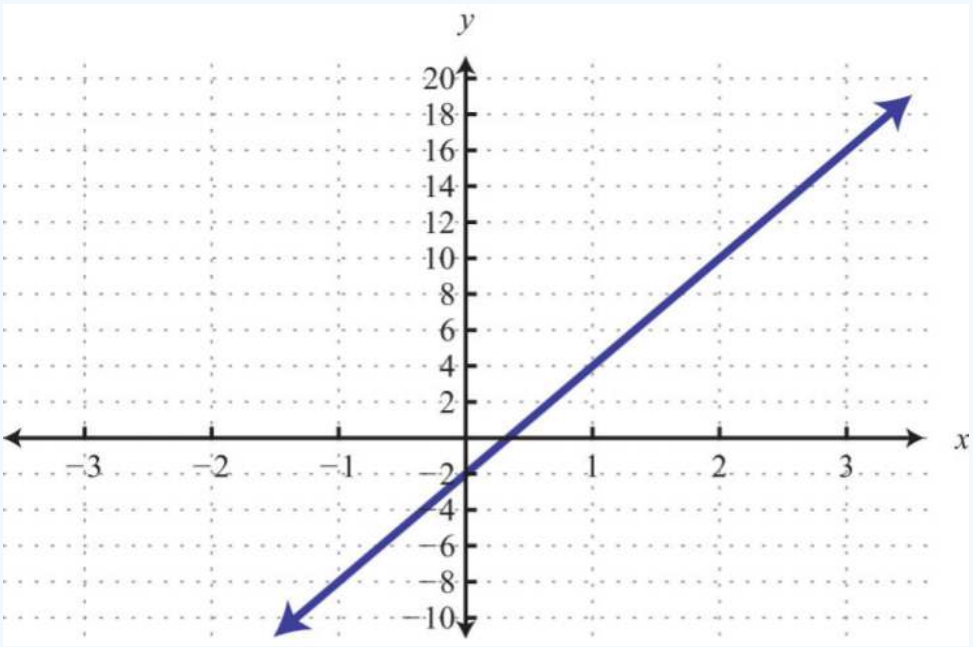


Figure 1.2.2.24

15.

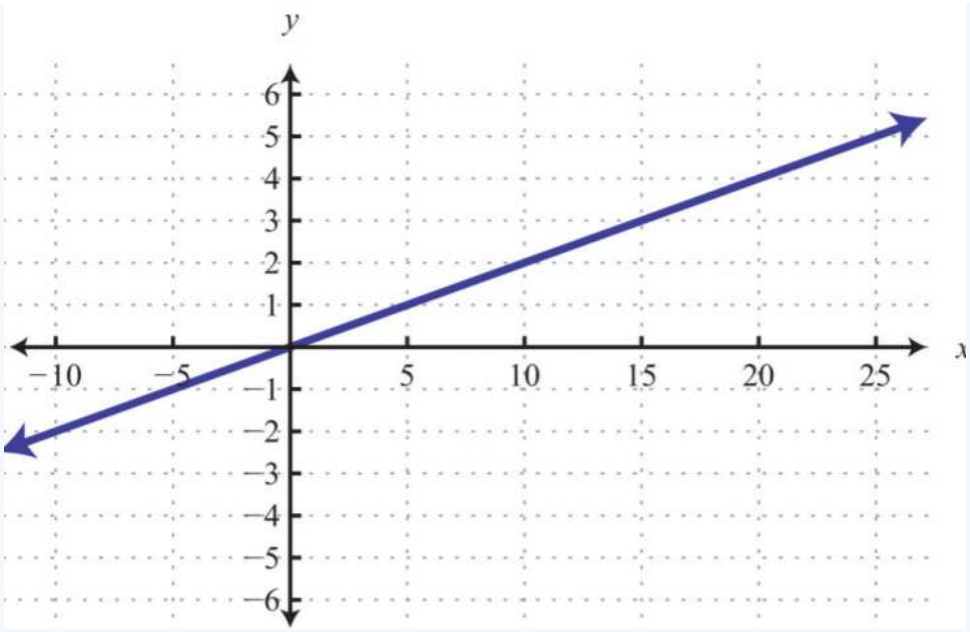


Figure 1.2.2.25

17.

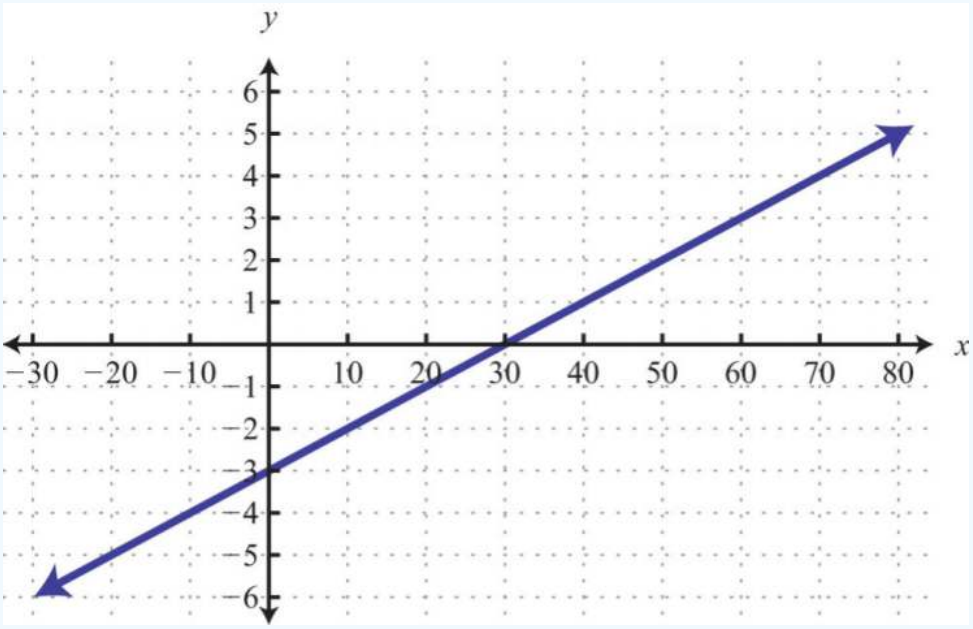


Figure 1.2.2.26

? EXERCISE 1.2.2.10 HORIZONTAL AND VERTICAL LINES

Find at least five ordered pair solutions and graph them.

1. $y = 4$
2. $y = -10$
3. $x = 4$
4. $x = -1$
5. $y = 0$
6. $x = 0$
7. $y = \frac{3}{4}$
8. $x = -\frac{5}{4}$

9. Graph the lines $y = -4$ and $x = 2$ on the same set of axes. Where do they intersect?
10. Graph the lines $y = 5$ and $x = -5$ on the same set of axes. Where do they intersect?
11. What is the equation that describes the x-axis?
12. What is the equation that describes the y-axis?

Answer

1.

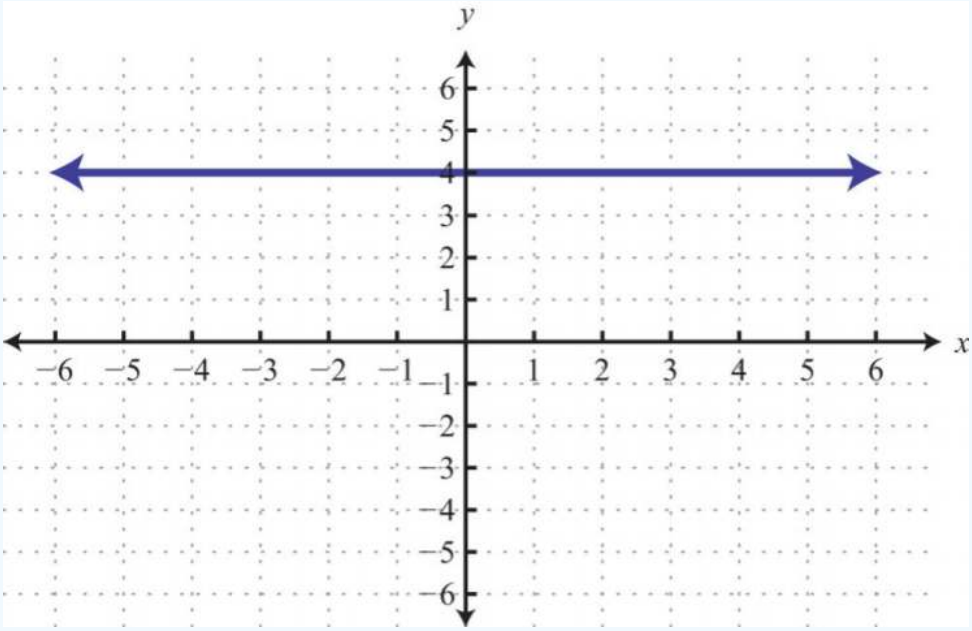


Figure 1.2.2.27

3.

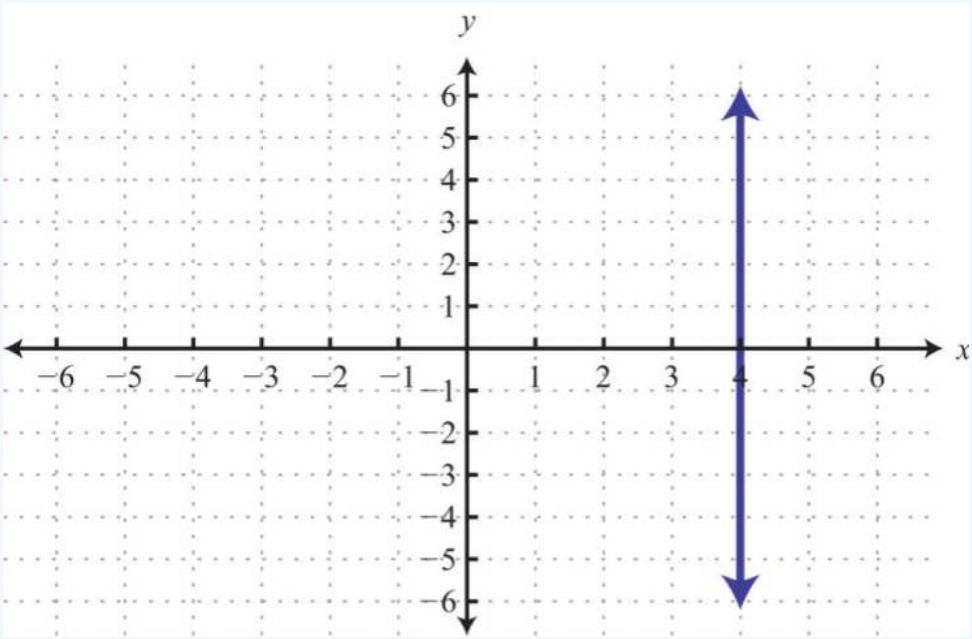


Figure 1.2.2.28

5.

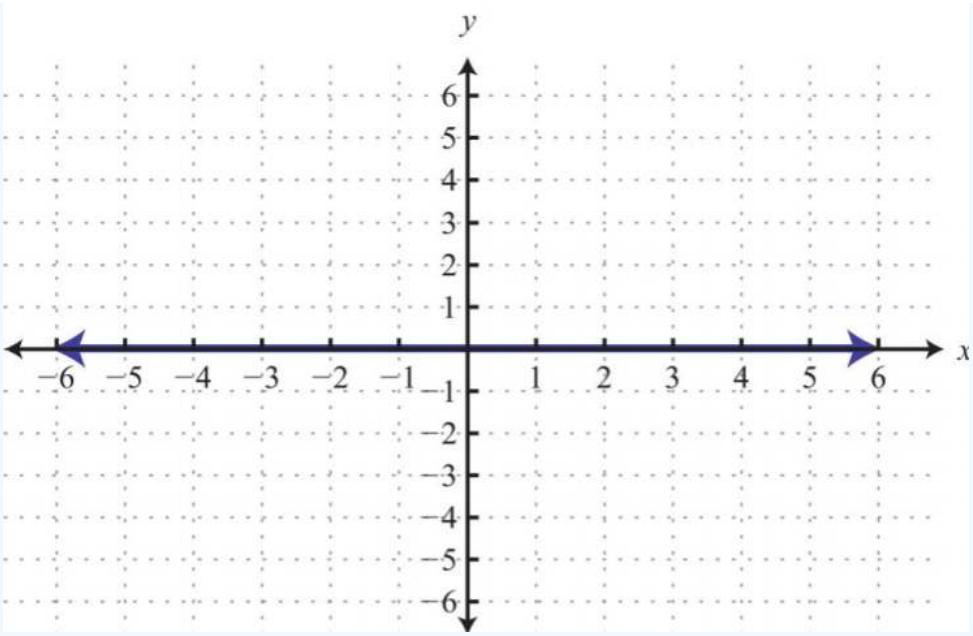


Figure 1.2.2.29

7.

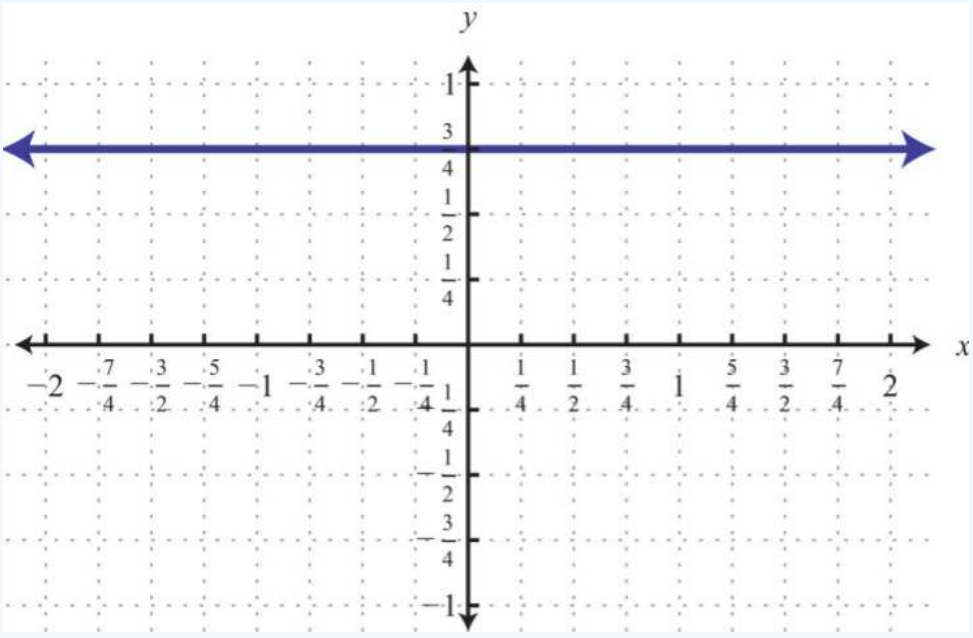


Figure 1.2.2.30

9.

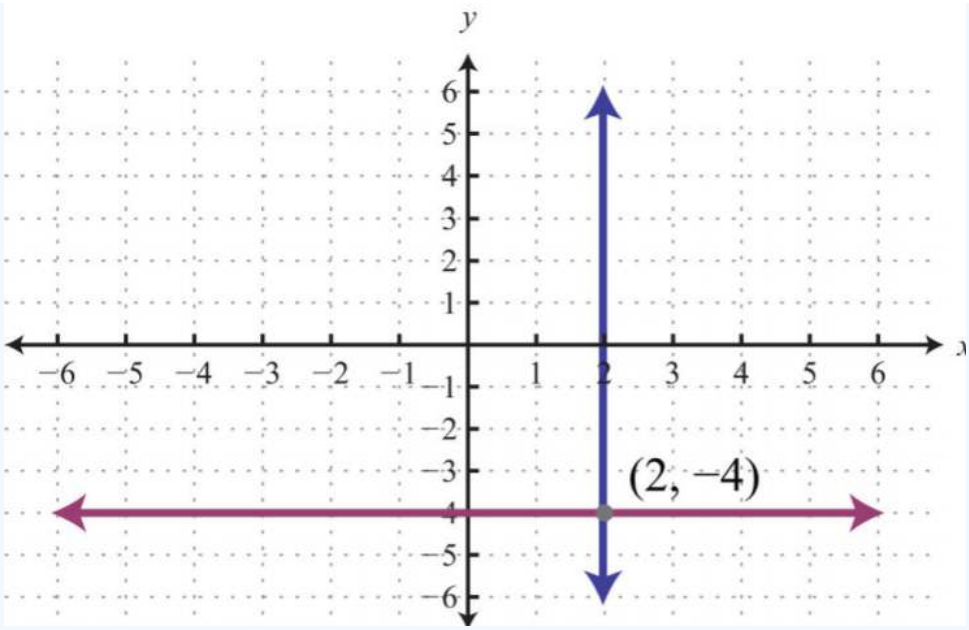


Figure 1.2.2.31

11. $y = 0$

EXERCISE 1.2.2.11 MIXED PRACTICE

Graph by plotting points.

1. $y = -\frac{3}{5}x + 6$
2. $y = \frac{3}{5}x - 3$
3. $y = -3$
4. $x = -5$
5. $3x - 2y = 6$
6. $-2x + 3y = -12$

Answer

1.

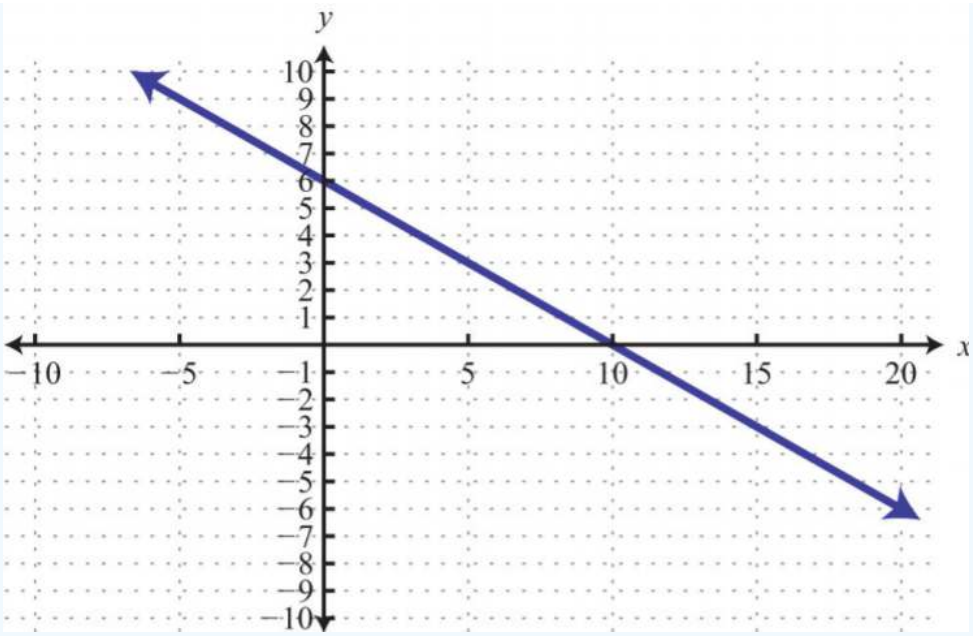


Figure 1.2.2.32

3.

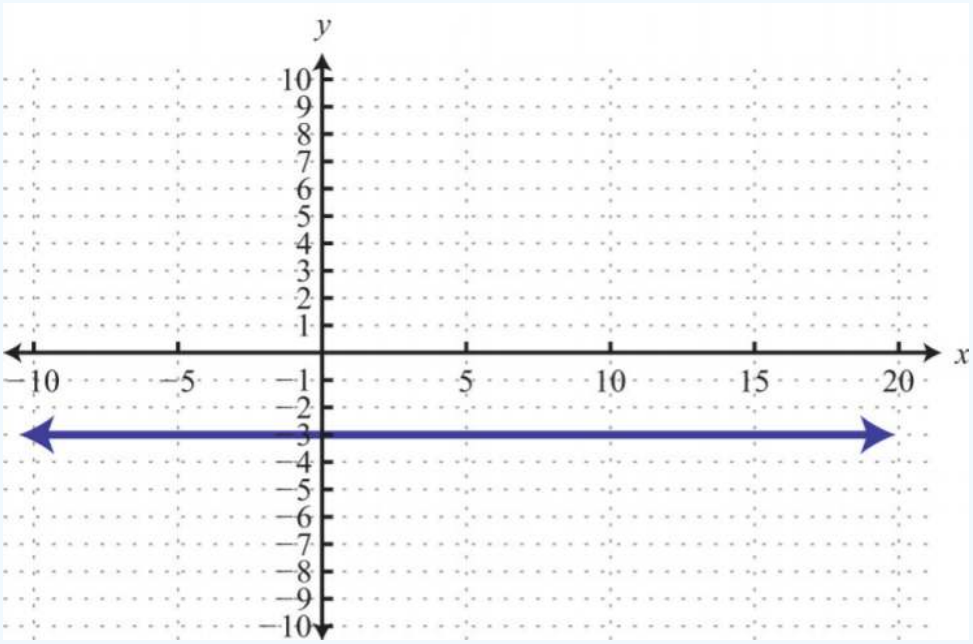


Figure 1.2.2.33

5.

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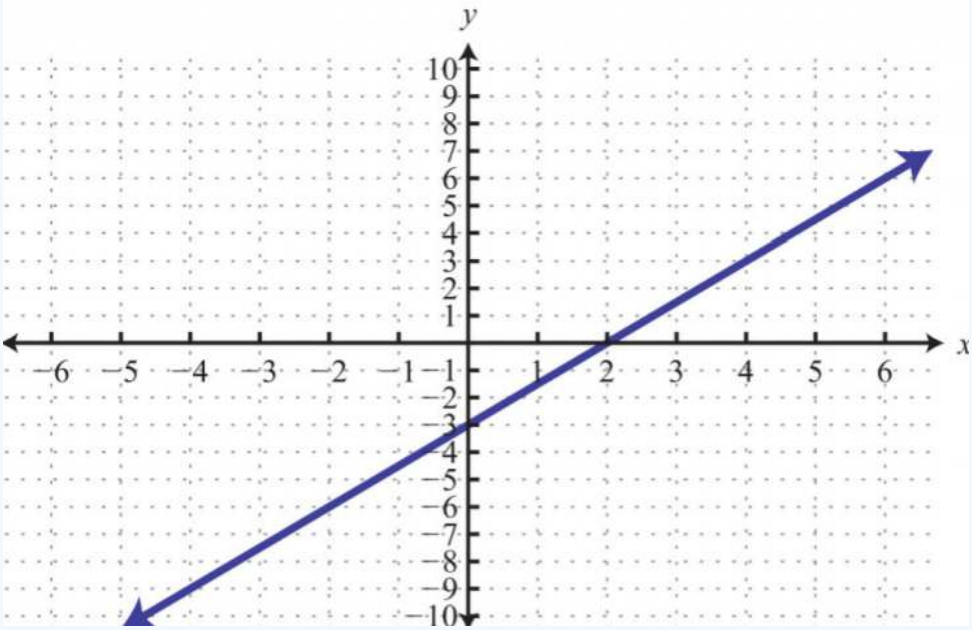


Figure 1.2.2.34

? EXERCISE 1.2.2.12 DISCUSSION BOARD TOPICS

1. Discuss the significance of the relationship between algebra and geometry in describing lines.
2. Give real-world examples relating two unknowns

Answer

1. Answers may vary

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1.2.3: GRAPH USING INTERCEPTS

LEARNING OBJECTIVES

- Identify and find x- and y-intercepts of a graph.
- Graph a line using x- and y-intercepts

DEFINITION OF x- AND y-INTERCEPTS

The ***x*-intercept** is the point where the graph of a line intersects the x-axis.

The ***y*-intercept** is the point where the graph of a line intersects the y-axis.

These points have the form $(x, 0)$ and $(0, y)$, respectively.

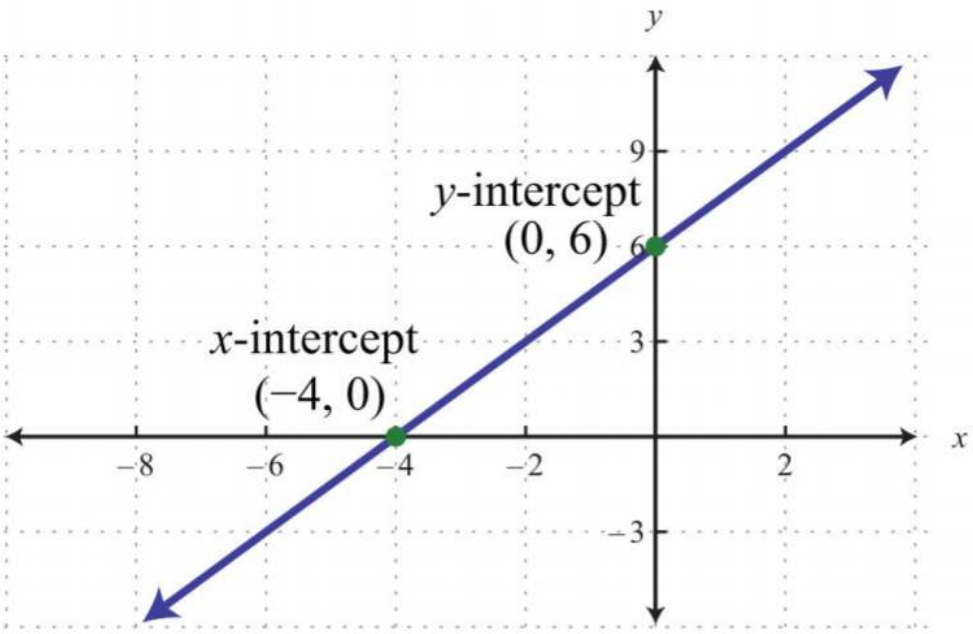


***Figure* 1.2.3.1**

To find the x- and y-intercepts algebraically, use the fact that all x-intercepts have a y-value of zero and all y-intercepts have an x-value of zero. To find the y-intercept, set $x = 0$ and determine the corresponding y-value. Similarly, to find the x-intercept, set $y = 0$ and determine the corresponding x-value.

EXAMPLE 1.2.3.1

Find the x- and y-intercepts:

$$-3x + 2y = 12.$$

Solution:

To find the x-intercept, set $y = 0$.

Therefore, the x-intercept is $(-4, 0)$. To find the y-intercept, set $x = 0$.

$$\begin{aligned} -3x + 2y &= 12 \qquad \textit{To find the } y-\textit{intercept,} \\ \downarrow & \qquad\qquad\quad \textit{set } x = 0. \\ -3(0) + 2y &= 12 \\ 2y &= 12 \\ y &= 6 \end{aligned}$$

Hence the y-intercept is $(0, 6)$. Note that this linear equation is graphed above.

Answer:

x-intercept: $(-4, 0)$; y-intercept: $(0, 6)$

EXAMPLE 1.2.3.2

Find the x- and y-intercepts:

$$y = -3x + 9$$

Solution:

Begin by finding the x-intercept.

$$\begin{aligned} y &= -3x + 9 && \textit{Set } y = 0. \\ &\downarrow \\ 0 &= -3x + 9 && \textit{Solve for } x. \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

The x-intercept is $(3, 0)$. Next, determine the y-intercept.

$$\begin{aligned} y &= -3x + 9 && \textit{Set } x = 0. \\ &\downarrow \\ y &= -3(0) + 9 && \textit{Solve for } y. \\ y &= 9 \end{aligned}$$

The y-intercept is $(0, 9)$.

Answer:

$$x\text{-intercept: } (3, 0);\ y\text{-intercept: } (0, 9)$$

Keep in mind that the intercepts are ordered pairs and not numbers. In other words, the x-intercept is not $x = 2$ but rather $(2, 0)$. In addition, not all graphs necessarily have both intercepts: for example,

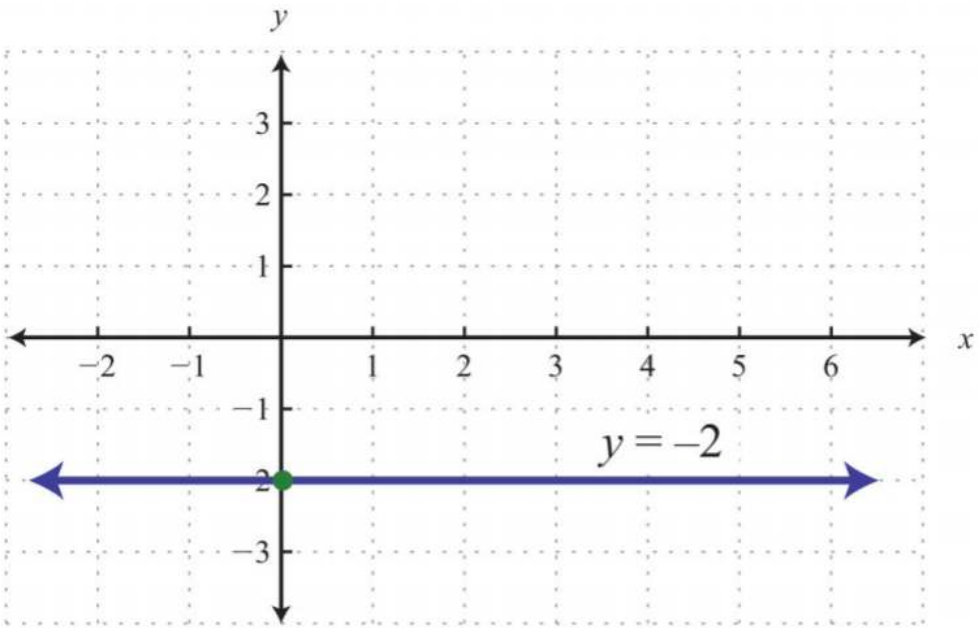


***Figure* 1.2.3.2**

The horizontal line graphed above has a y-intercept of $(0, -2)$ and no x-intercept.

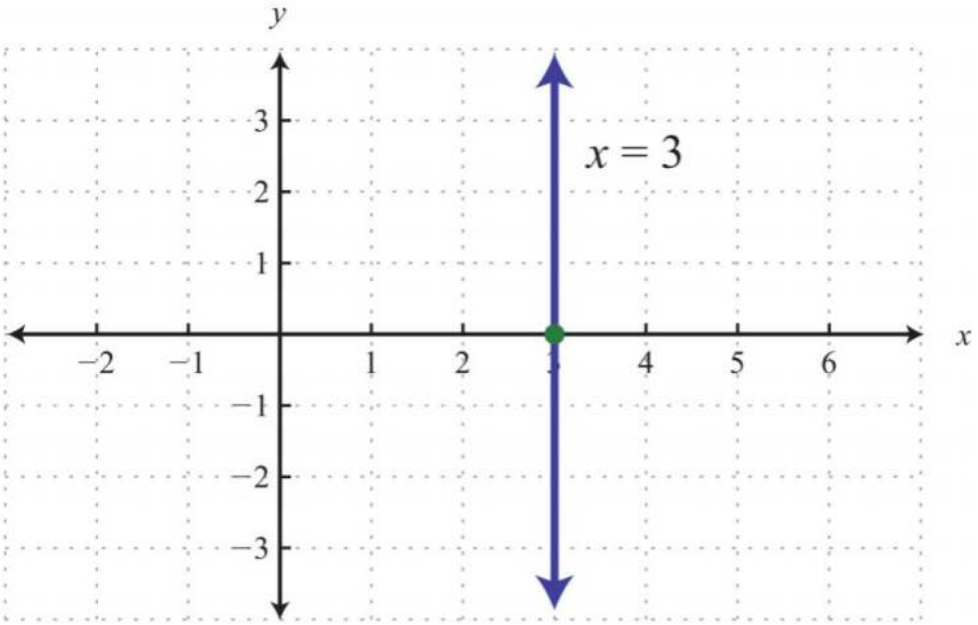


***Figure* 1.2.3.3**

The vertical line graphed above has an x-intercept $(3,0)$ and no y-intercept.

? EXERCISE 1.2.3.1

Find the x- and y-intercepts:

$$4x - y = 2\,.$$

Answer

x-intercept: $(\frac{1}{2},0)$; y-intercept: $(0,-2)$

GRAPHING LINES USING INTERCEPTS

Since two points determine a line, we can use the x- and y-intercepts to graph linear equations. We have just outlined an easy method for finding intercepts; now we outline the steps for graphing lines using the intercepts.

✓ EXAMPLE 1.2.3.3

Graph using intercepts:

$$2x - 3y = 12\,.$$

Solution:

Step 1: Find the x- and y-intercepts.

$To\ find\ the\ x-intercept,\ set\ y=0.$	$To\ find\ the\ y-intercept,\ set\ x=0.$
$2x-3y=12$	$2x-3y=12$
$2x-3(0)=12$	$2(0)-3y=12$
$2x=12$	$-3y=12$
$x=6$	$y=-4$
$x-intercept:\ (6,0)$	$y-intercept:\ (0,-4)$

Step 2: Plot the intercepts and draw the line through them. Use a straightedge to create a nice straight line. Add an arrow on either end to indicate that the line continues indefinitely in either direction.

Answer:

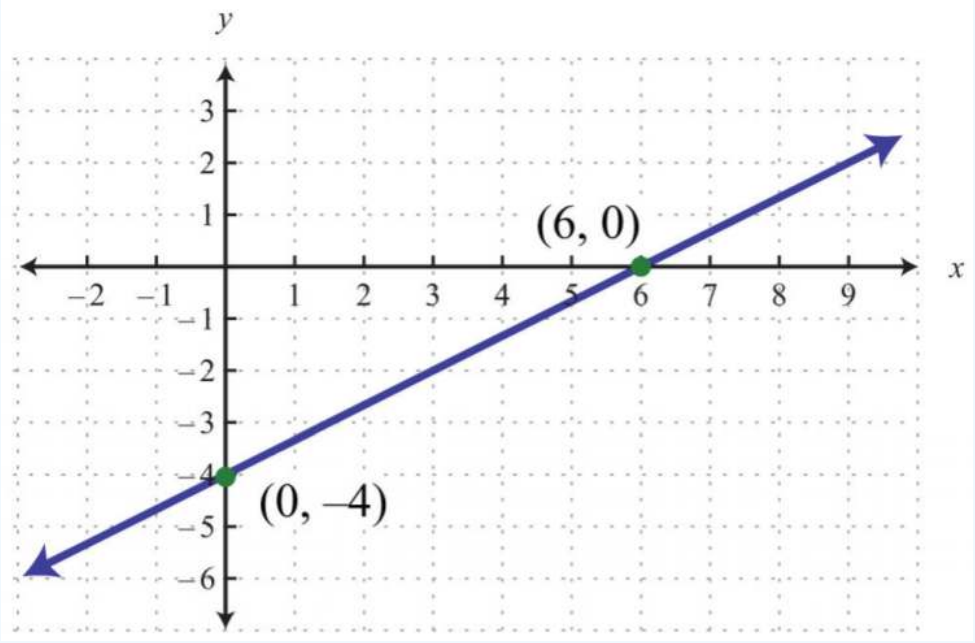


***Figure* 1.2.3.4**

EXAMPLE 1.2.3.4

Graph using intercepts:

$$y = -\tfrac{1}{5}x + 3.$$

Solution:

Begin by determining the x- and \(y\-intercepts.

$x-intercept$	$y-intercept$
$y = -\frac{1}{5}x + 3$	$y = -\frac{1}{5}x + 3$
$0 = -\frac{1}{5}x + 3$	$y = -\frac{1}{5}(0) + 3$
$\frac{1}{5}x = 3$	$y = 3$
$5 \cdot \frac{1}{5}x = 5 \cdot 3$	$y-intercept: (0,3)$
$x = 15$	
$x-intercept: (15,0)$	

Next, graph the two points and draw a line through them with a straighedge.

Answer:

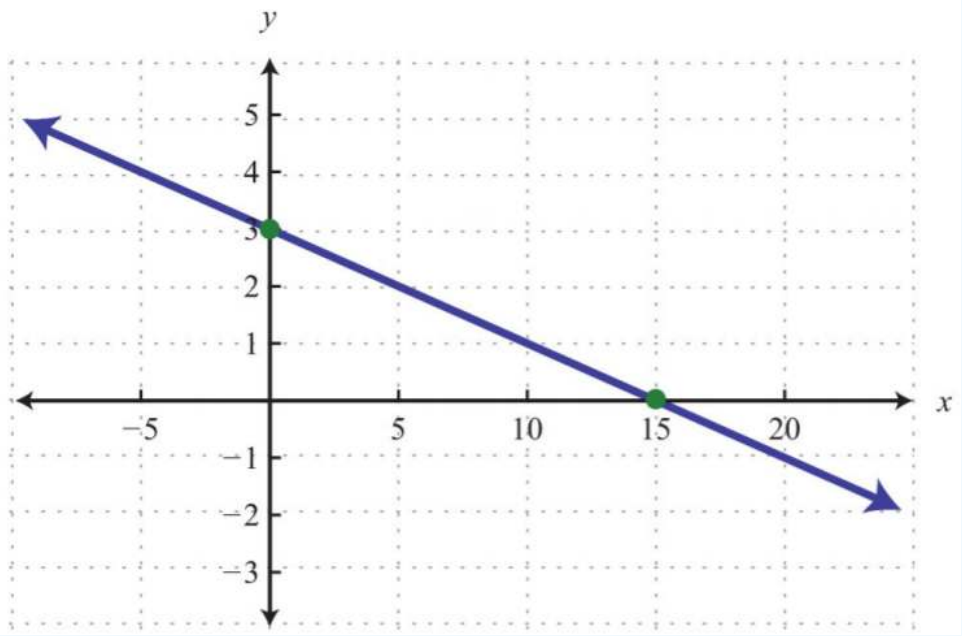


***Figure* 1.2.3.5**

EXAMPLE 1.2.3.5

Graph using intercepts:

$$y = -2x.$$

Solution:

$x-intercept$	$y-intercept$
$y = -2x$	$y = -2x$
$0 = -2x$	$y = -2(0)$
$\frac{0}{-2} = \frac{-2x}{-2}$	$y = 0$
$0 = x$	$y-intercept: (0,0)$
$x-intercept: (0,0)$	

Here the x- and y-intercepts are actually the same point, the origin. We will need at least one more point so that we can graph the line. Choose any value for x and determine the corresponding value for y.

x	y	
-1	2	$y = -2(-1) = 2 = 2$
1	-2	$y = -2(1) = -2$

***Figure* 1.2.3.6**

Use the ordered pair solutions $(0,0), (-1,2)$, and $(1,-2)$ to graph the line.

Answer:

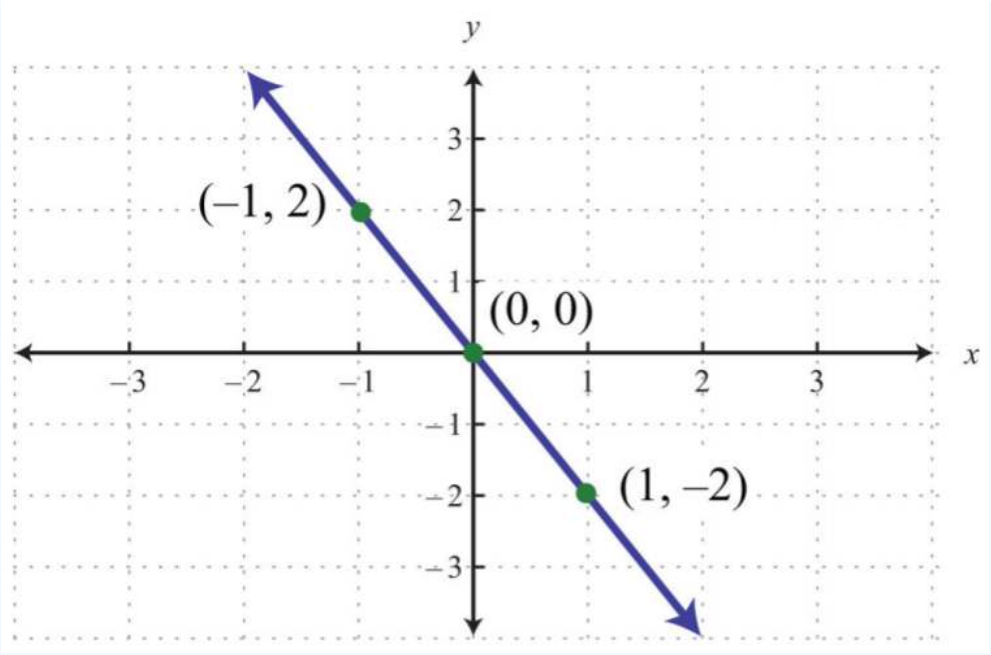


***Figure* 1.2.3.7**

To summarize, any linear equation can be graphed by finding two points and connecting them with a line drawn with a straightedge. Two important and useful points are the x- and y-intercepts; find these points by substituting $y = 0$ and $x = 0$, respectively. This method for finding intercepts will be used throughout our study of algebra.

? EXERCISE 1.2.3.2

Graph using intercepts:

$$3x - 5y = 15\,.$$

Answer

x-intercept: $(5, 0)$; y-intercept: $(0, -3)$

FINDING INTERCEPTS GIVEN THE GRAPH

The x- and y-intercepts are important points on any graph. This chapter will focus on the graphs of linear equations. However, at this point, we can use these ideas to determine intercepts of nonlinear graphs. Remember that intercepts are ordered pairs that indicate where the graph intersects the axes.

✔ EXAMPLE 1.2.3.6

Find the x- and y-intercepts given the following graph:

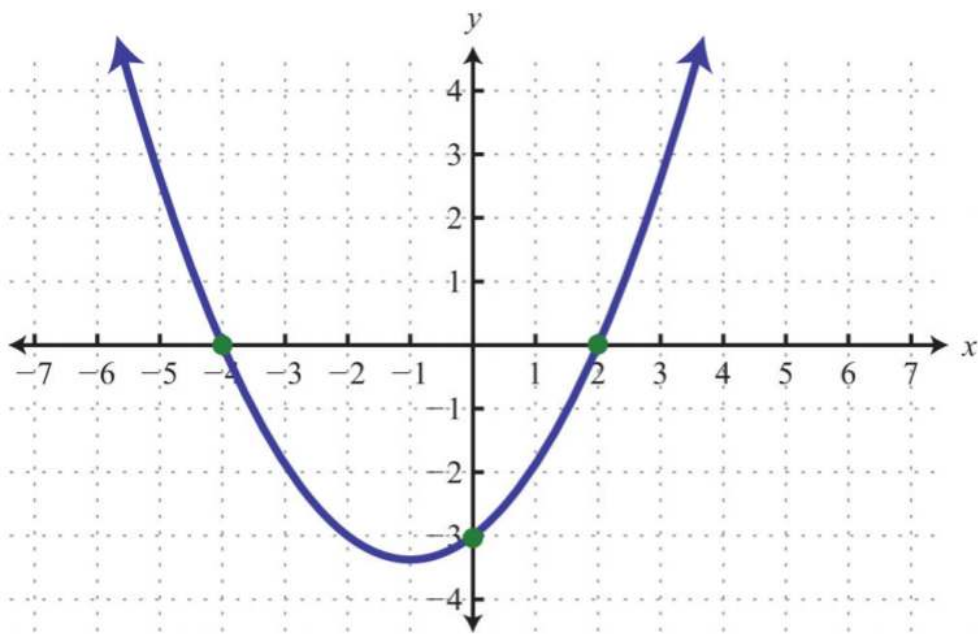


***Figure* 1.2.3.8**

Solution:

We see that the graph intersects the x-axis in two places. This graph has two x-intercepts, namely, $(-4,0)$ and $(2,0)$. Furthermore, the graph intersects the y-axis in one place. The only y-intercept is $(0,-3)$.

Answer:

$$x\text{-intercepts: } (-4,0),(2,0);\ y\text{-intercept: } (0,-3)$$

In our study of algebra, we will see that some graphs have many intercepts. Also, we will see that some graphs do not have any.

EXAMPLE 1.2.3.7

Given the following graph, find the x- and y-intercepts:

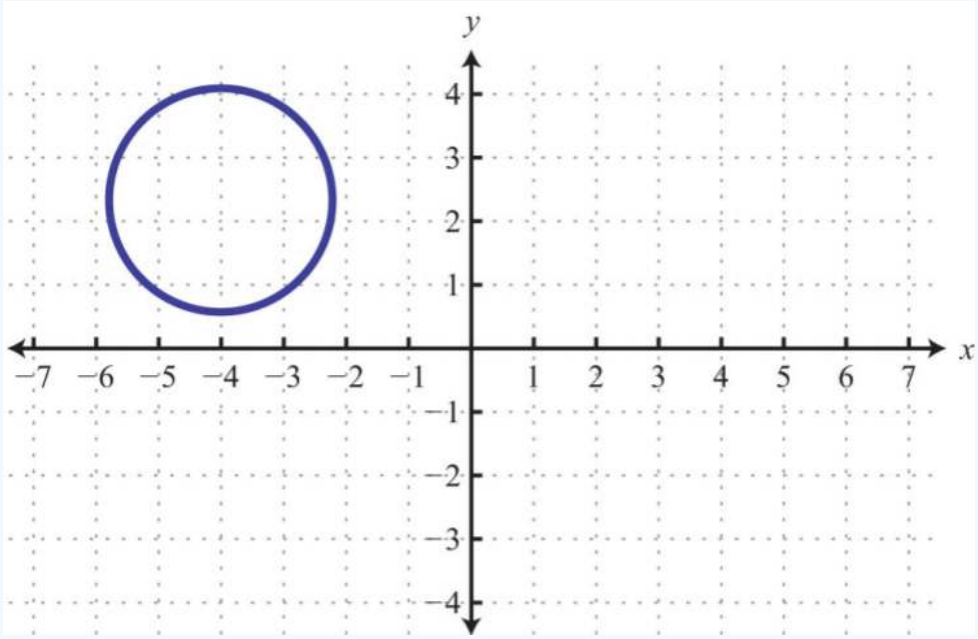


***Figure* 1.2.3.9**

Solution:

This is a graph of a circle; we can see that it does not intersect either axis. Therefore, this graph does not have any intercepts.

Answer:

None

KEY TAKEAWAYS

- Since two points determine any line, we can graph lines using the x- and y-intercepts.
- To find the x-intercept, set $y = 0$ and solve for x.
- To find the y-intercept, set $x = 0$ and solve for y.
- This method of finding x- and y-intercepts will be used throughout our study of algebra because it works for any equation.
- To graph a line, find the intercepts, if they exist, and draw a straight line through them. Use a straightedge to create the line and include arrows on either end to indicate that the line extends infinitely in either direction.
- Horizontal and vertical lines do not always have both x- and y-intercepts.

? EXERCISE 1.2.3.3 INTERCEPTS

Given the graph, find the x- and y-intercepts.

1.

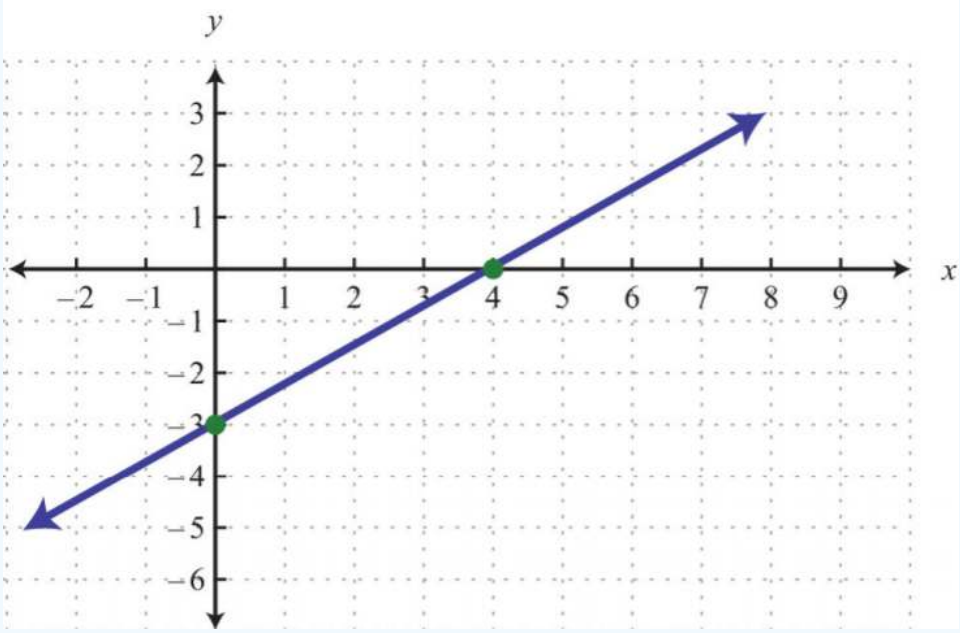


***Figure* 1.2.3.10**

2.

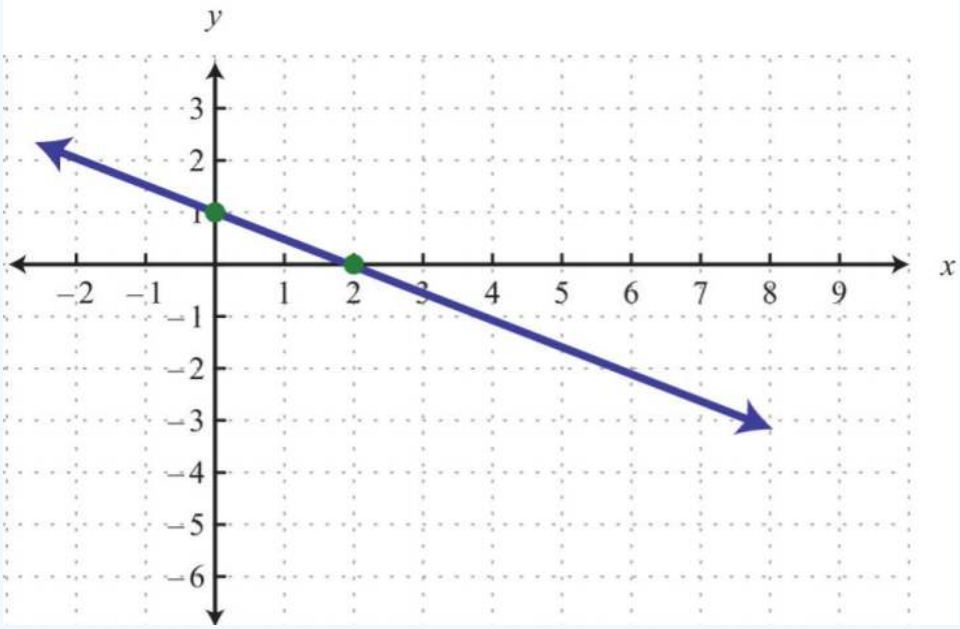


Figure **1.2.3.11**

3.

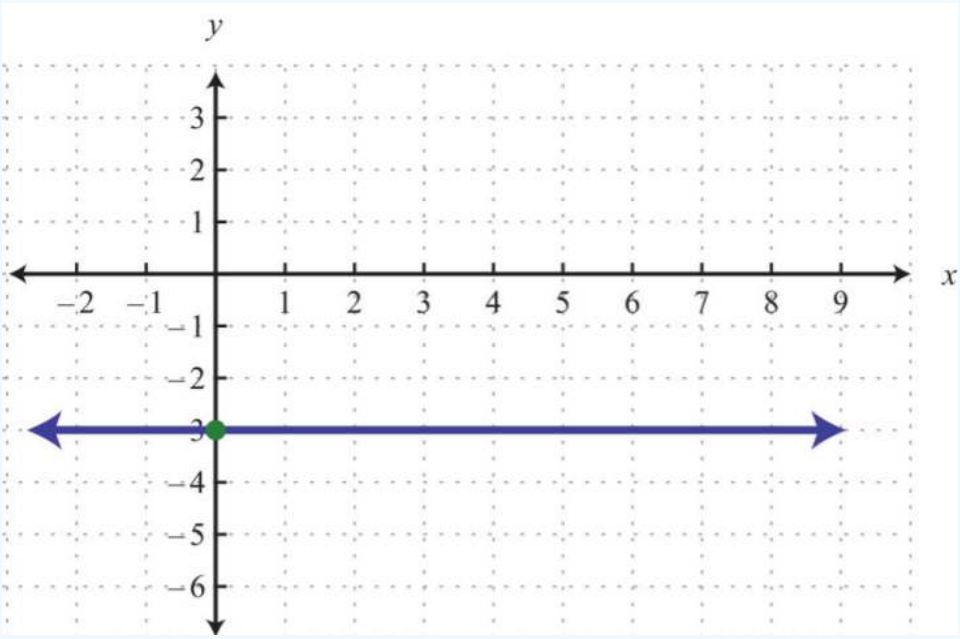


Figure **1.2.3.12**

4.

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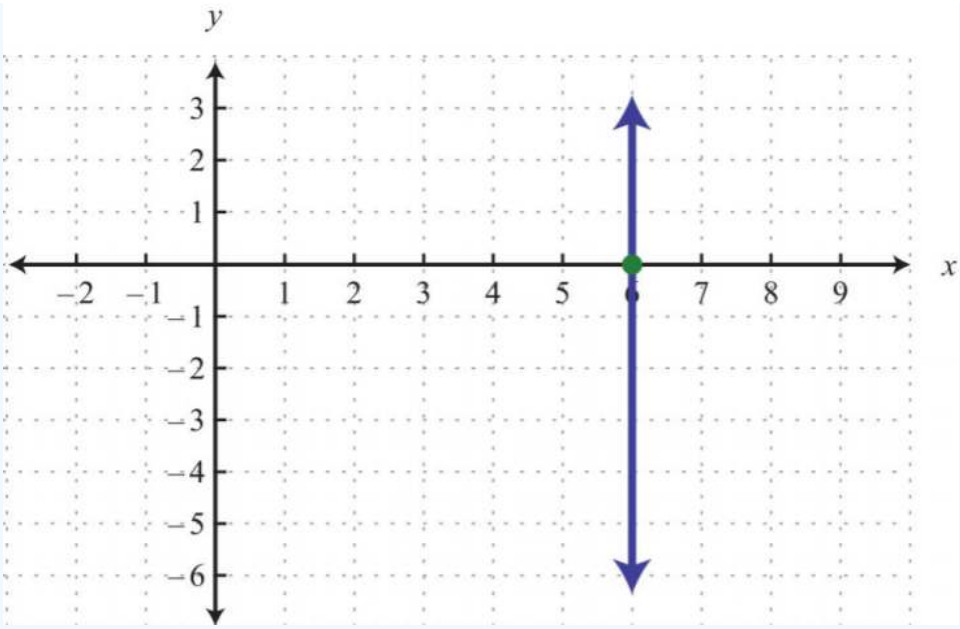


Figure **1.2.3.13**

5.

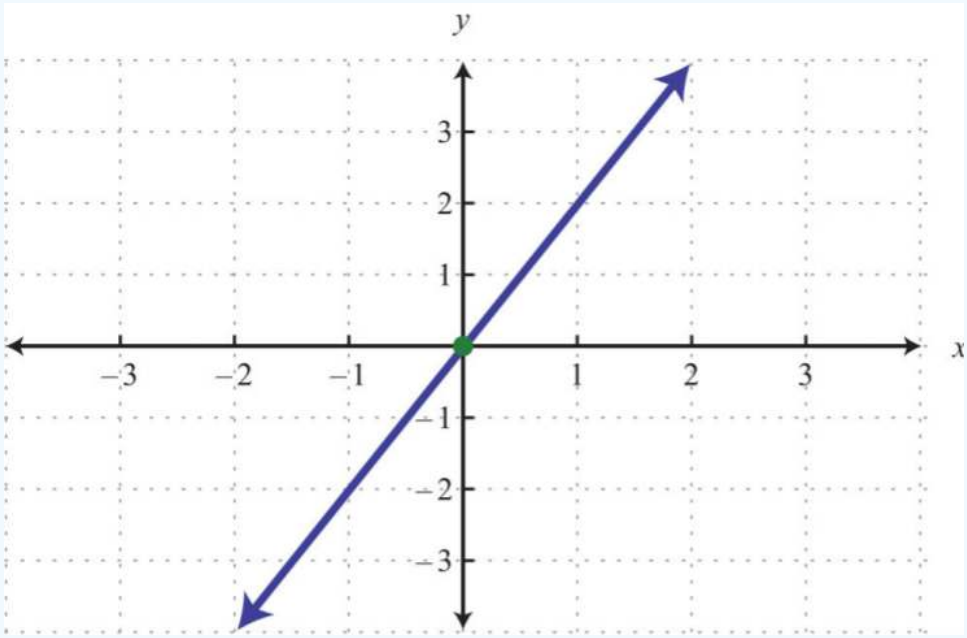


Figure **1.2.3.14**

6.

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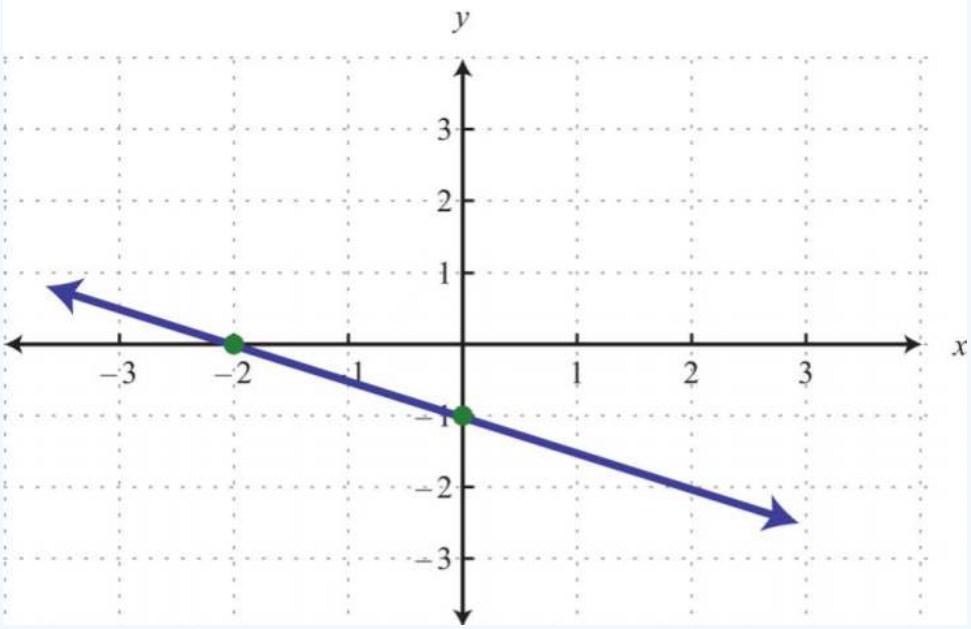


***Figure* 1.2.3.15**

Answer

1. y-intercept: $(0,-3)$; x-intercept: $(4,0)$
3. y-intercept: $(0,-3)$; x-intercept: none
5. y-intercept: $(0,0)$; x-intercept: $(0,0)$

? EXERCISE 1.2.3.4 INTERCEPTS

Find the x- and y-intercepts.

1. $5x-4y=20$
2. $-2x+7y=-28$
3. $x-y=3$
4. $-x+y=0$
5. $3x-4y=1$
6. $-2x+5y=3$
7. $\frac{1}{4}x-\frac{1}{3}y=1$
8. $-\frac{2}{5}x+\frac{3}{4}y=2$
9. $y=6$
10. $y=-3$
11. $x=2$
12. $x=-1$
13. $y=mx+b$
14. $ax+by=c$

Answer

1. x-intercept: $(4,0)$; y-intercept: $(0,-5)$
3. x-intercept: $(3,0)$; y-intercept: $(0,-3)$
5. x-intercept: $(\frac{1}{3},0)$; y-intercept: $(0,-\frac{1}{4})$
7. x-intercept: $(4,0)$; y-intercept: $(0,-3)$
9. x-intercept: none; y-intercept: $(0,6)$
11. x-intercept: $(2,0)$; y-intercept: none
13. x-intercept: $(-\frac{b}{m},0)$; y-intercept: $(0,b)$

EXERCISE 1.2.3.5 GRAPH USING INTERCEPTS

Find the intercepts and graph them.

1. $3x + 4y = 12$
2. $-2x + 3y = 6$
3. $5x - 2y = 10$
4. $-4x - 8y = 16$
5. $-\frac{1}{2}x + \frac{1}{3}y = 1$
6. $\frac{3}{4}x - \frac{1}{2}y = -3$
7. $2x - \frac{5}{2}y = 10$
8. $2x - \frac{7}{3}y = -14$
9. $4x - y = -8$
10. $6x - y = 6$
11. $-x + 2y = 1$
12. $3x + 4y = 6$
13. $2x + y = -1$
14. $-2x + 6y = 3$
15. $15x + 4y = -60$
16. $-25x + 3y = 75$
17. $4x + 2y = 0$
18. $3x - y = 0$
19. $-12x + 6y = -4$
20. $3x + 12y = -4$
21. $y = 2x + 4$
22. $y = -x + 3$
23. $y = \frac{1}{2}x + 1$
24. $y = \frac{2}{3}x - 3$
25. $y = -\frac{2}{5}x + 1$
26. $y = -\frac{5}{8}x - \frac{5}{4}$
27. $y = -\frac{7}{8}x - \frac{7}{2}$
28. $y = -x + \frac{3}{2}$
29. $y = 3$
30. $y = \frac{3}{2}$
31. $x = 5$
32. $x = -2$
33. $y = 5x$
34. $y = -x$

Answer

1.

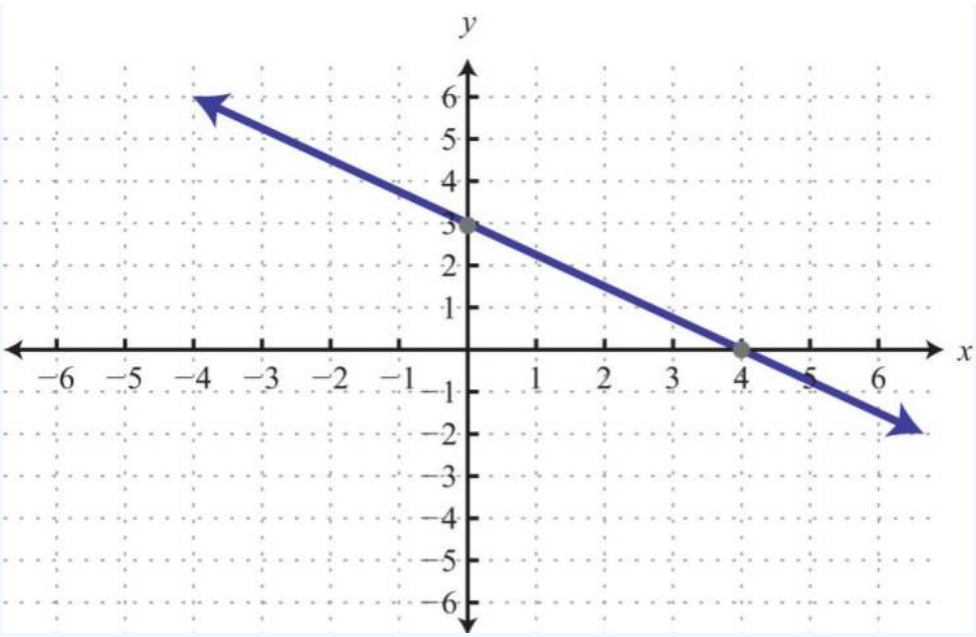


Figure 1.2.3.16

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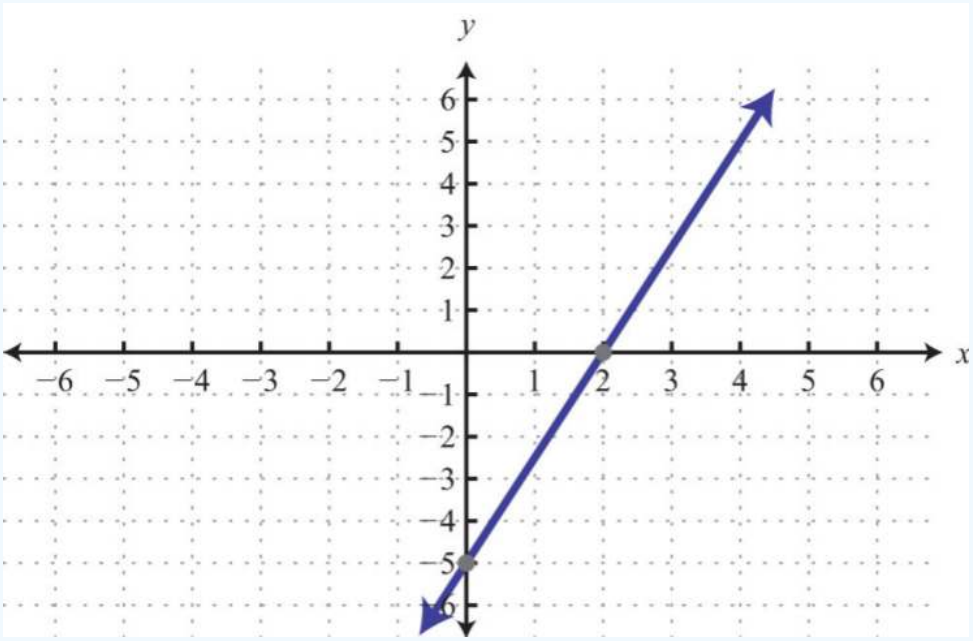


Figure 1.2.3.17

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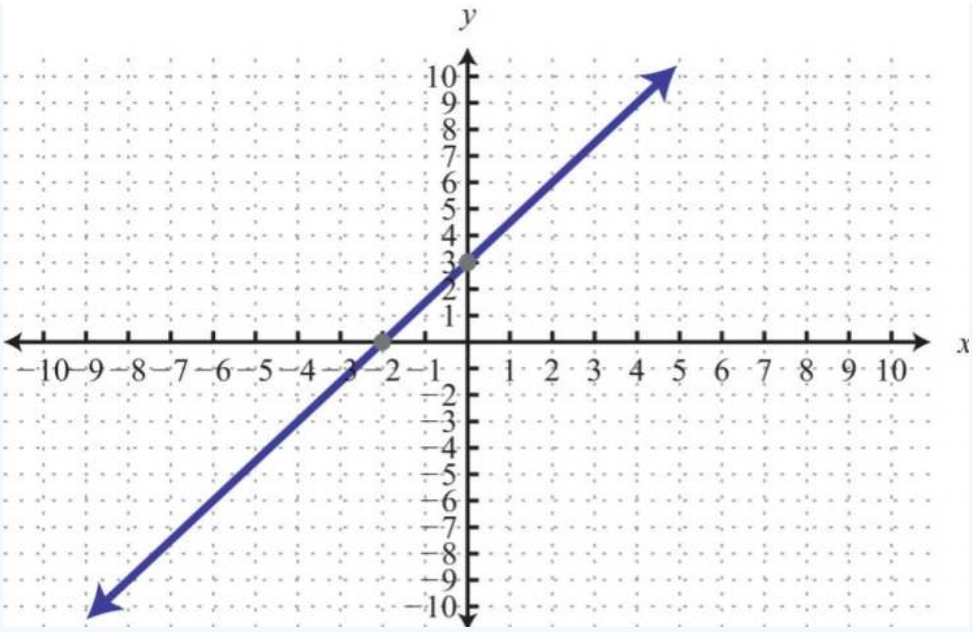


Figure 1.2.3.18

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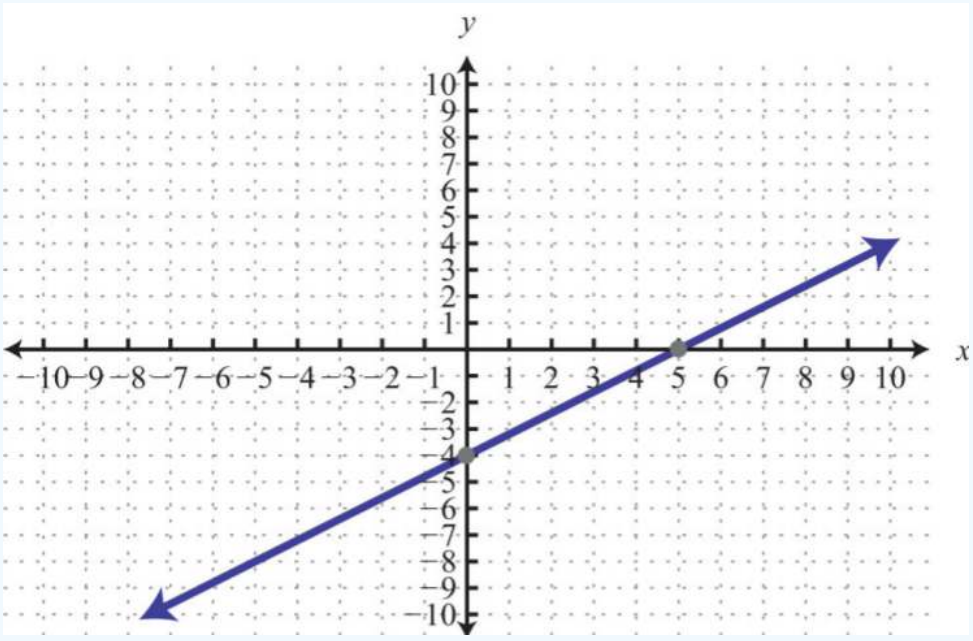


Figure 1.2.3.19

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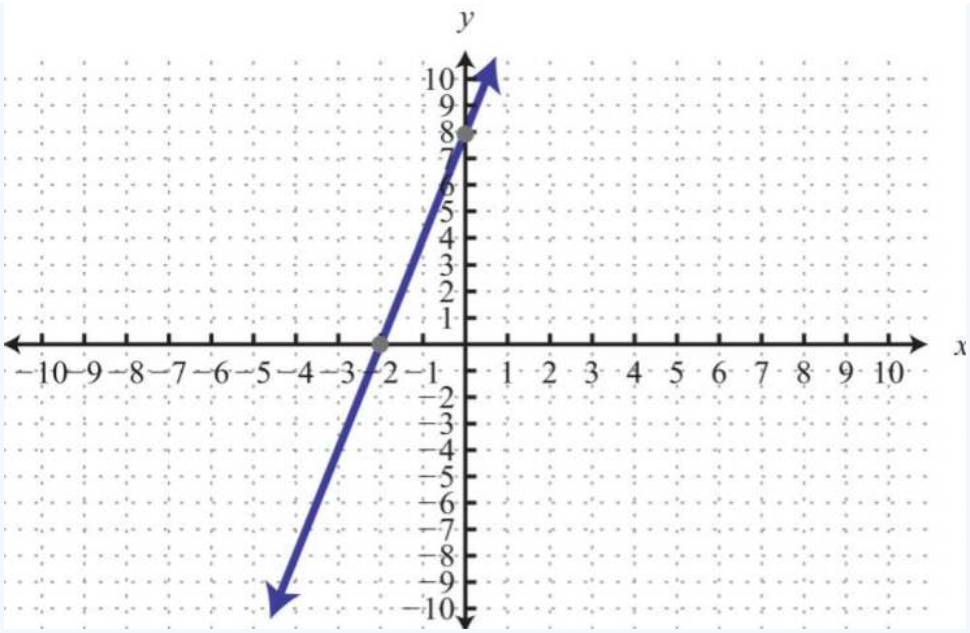


Figure 1.2.3.20

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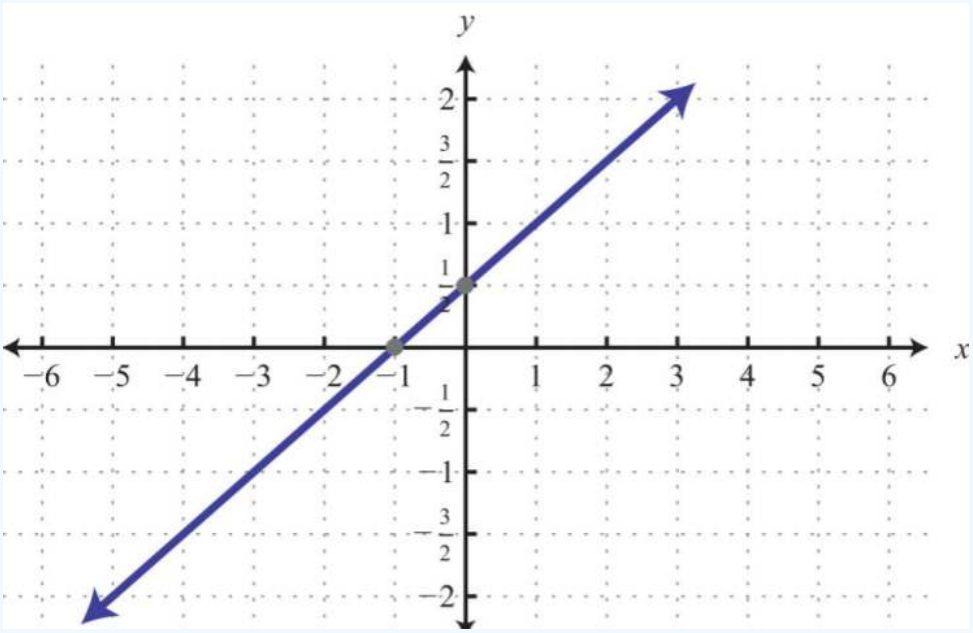


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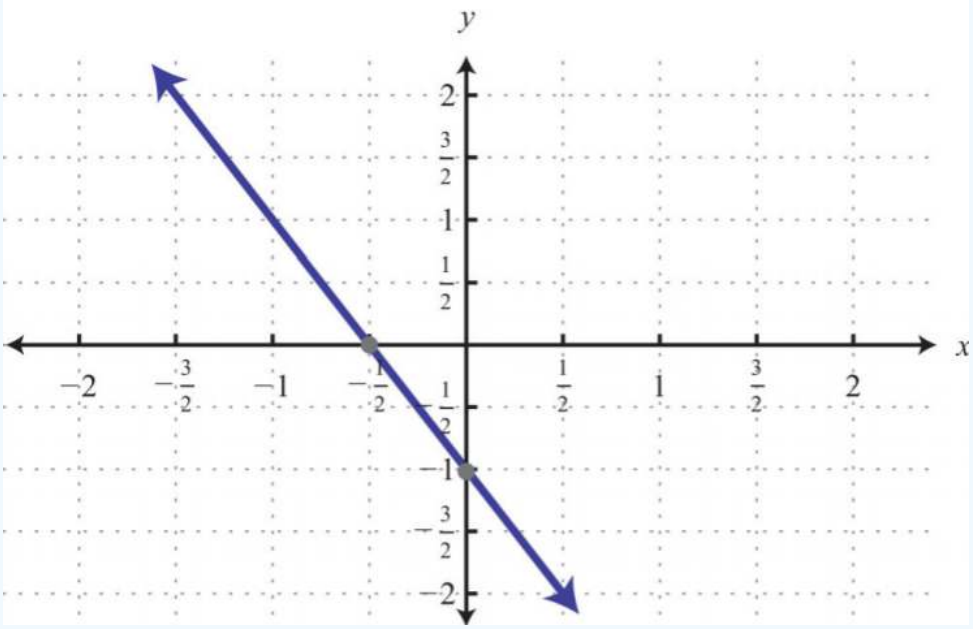


Figure 1.2.3.22

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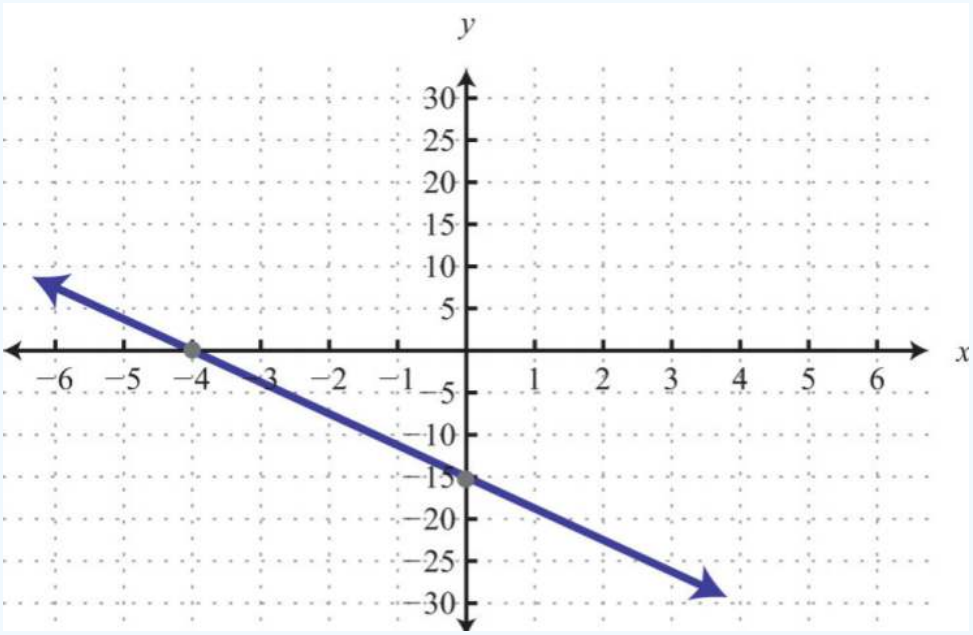


Figure 1.2.3.23

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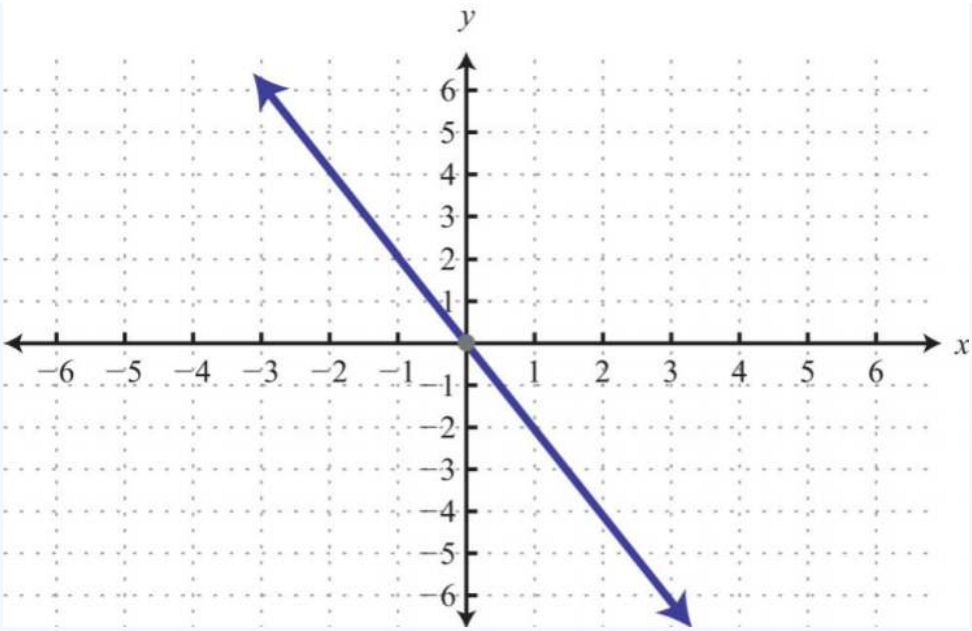


Figure 1.2.3.24

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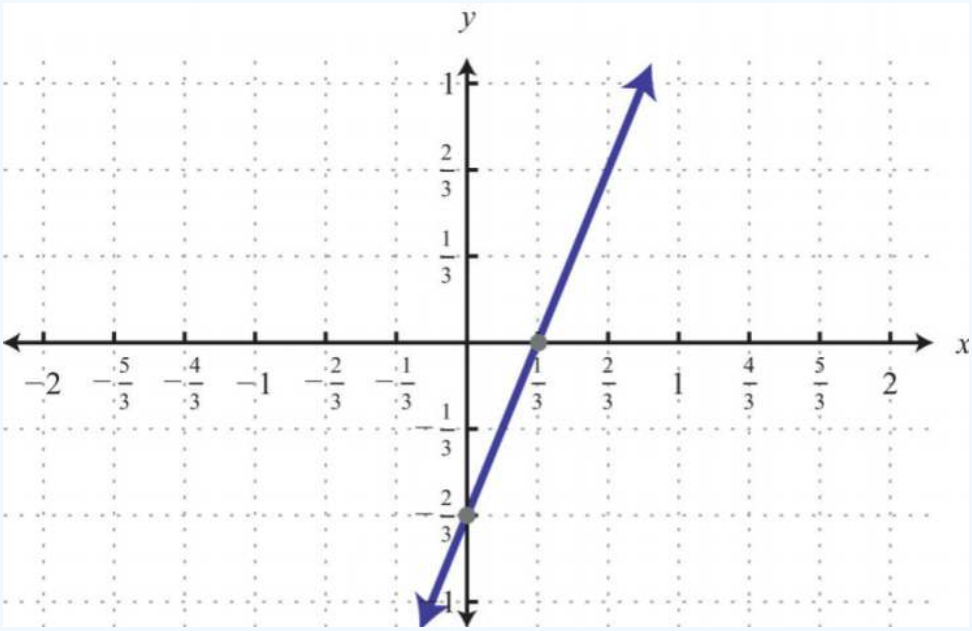


Figure 1.2.3.25

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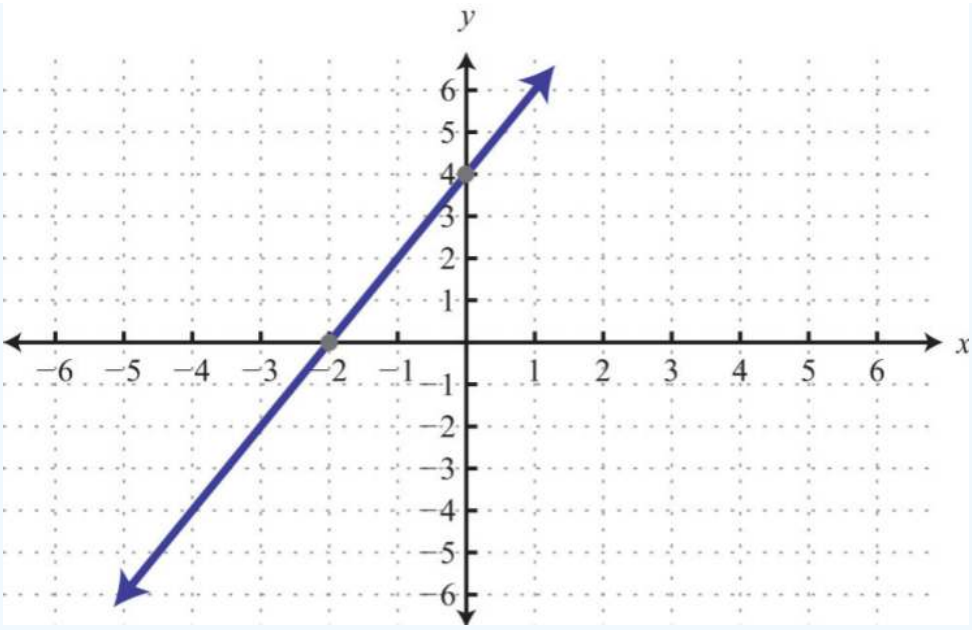


Figure 1.2.3.26

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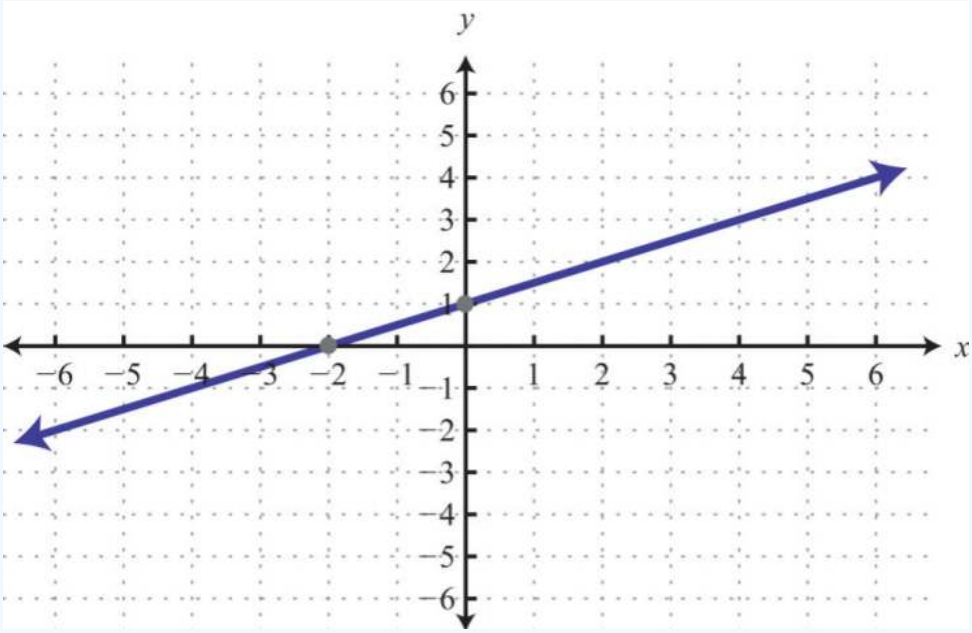


Figure 1.2.3.27

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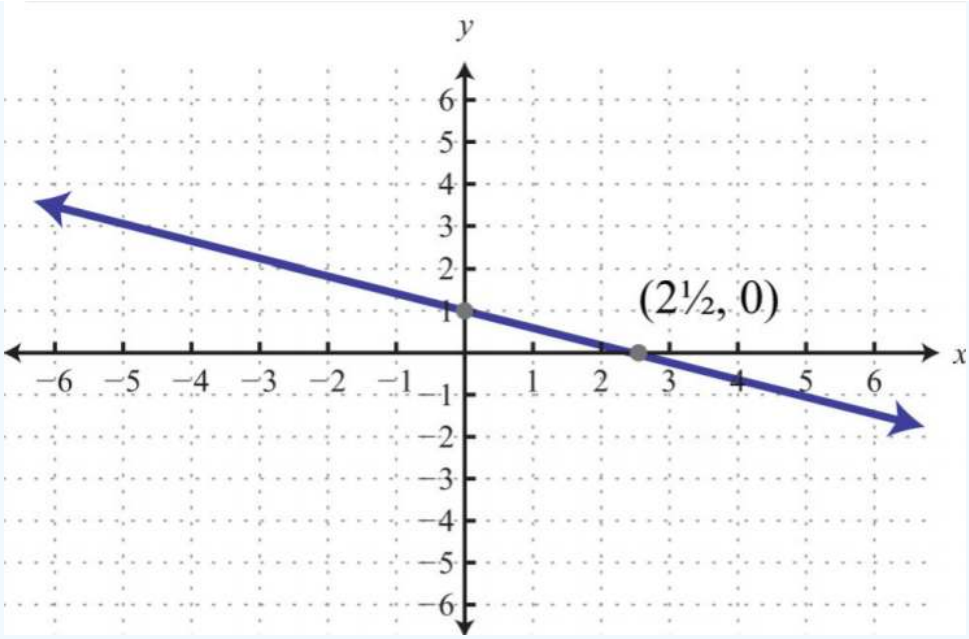


Figure 1.2.3.28

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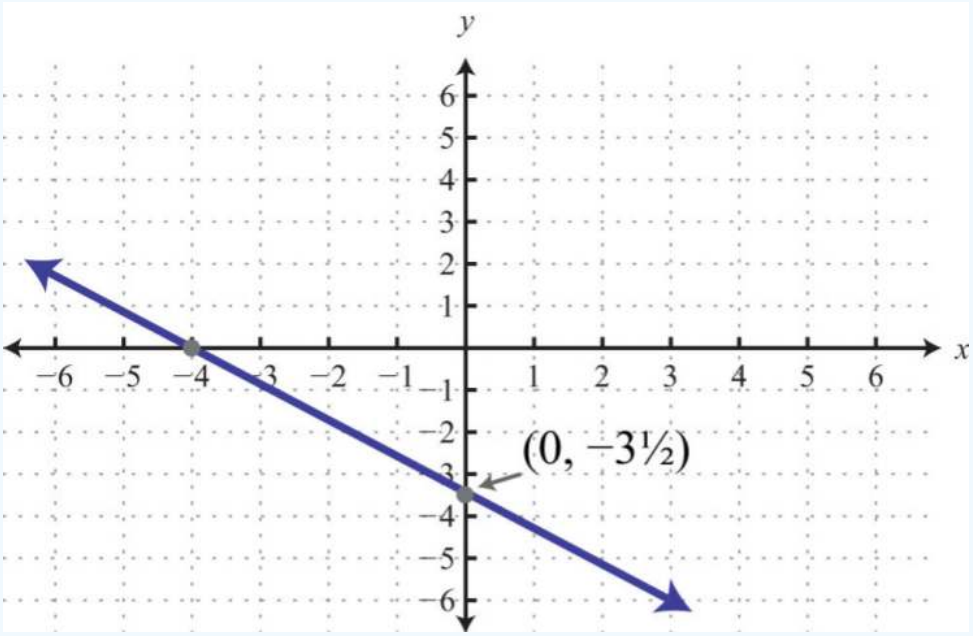


Figure 1.2.3.29

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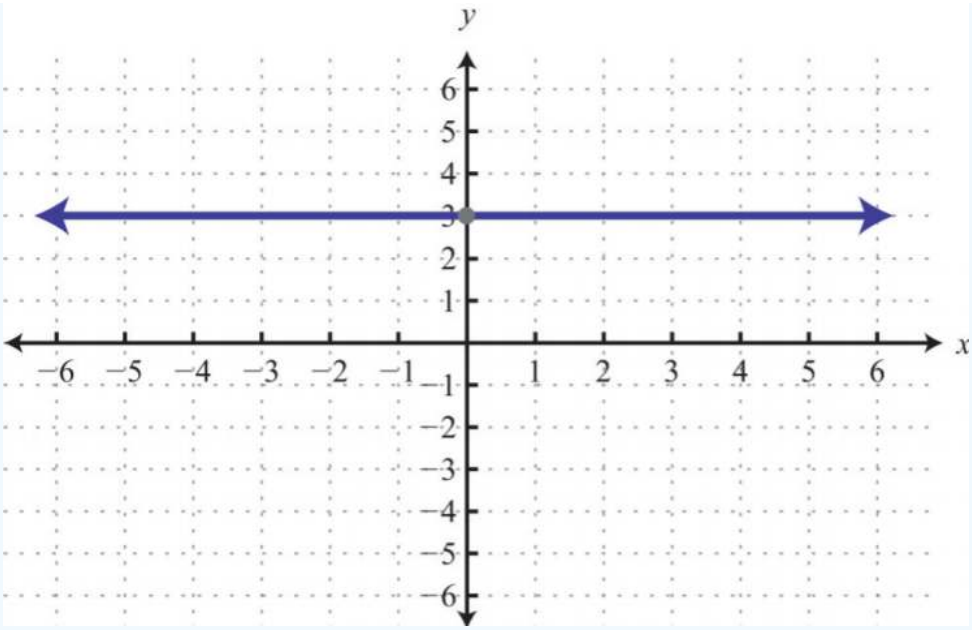


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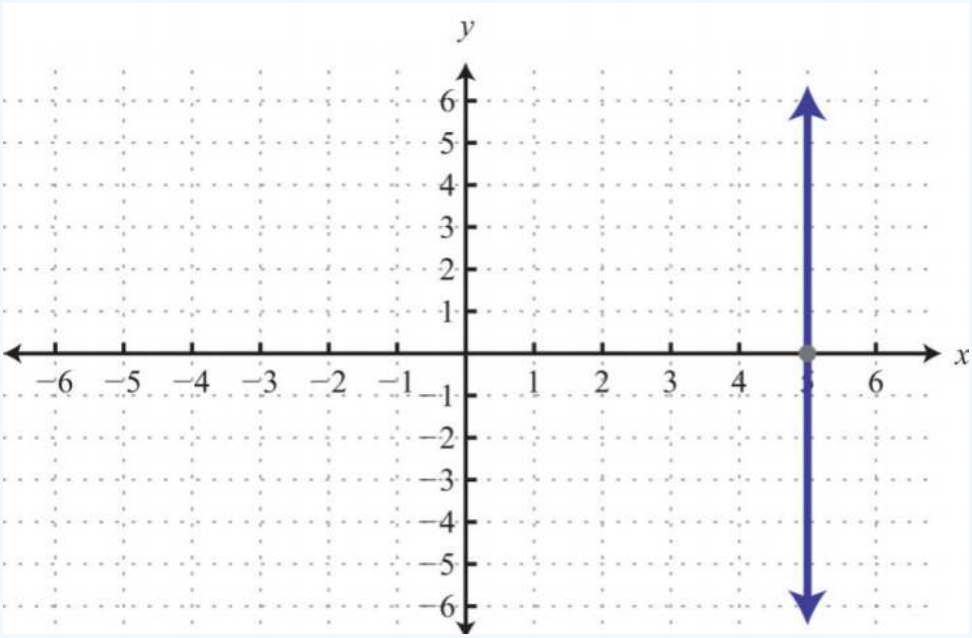


Figure 1.2.3.31

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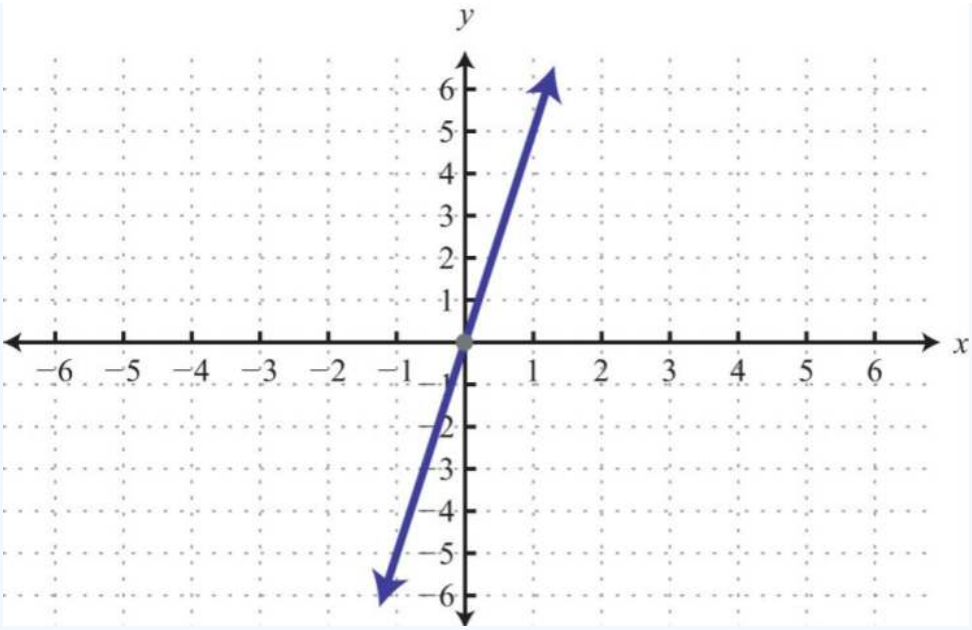


Figure 1.2.3.32

? EXERCISE 1.2.3.6 INTERCEPTS OF NONLINEAR GRAPHS

Given the graph find the x- and y-intercepts.

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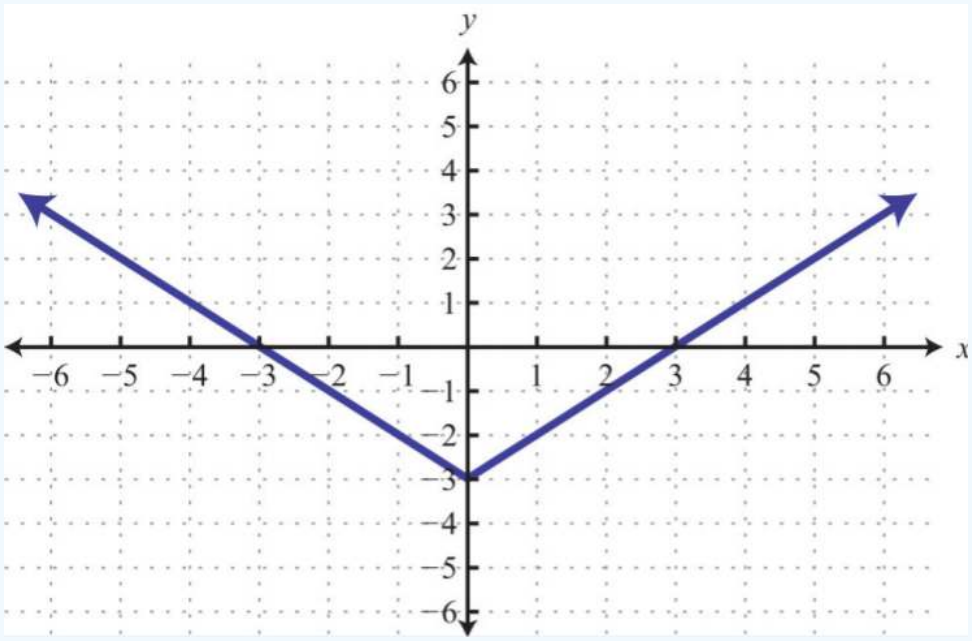


***Figure* 1.2.3.33**

2.

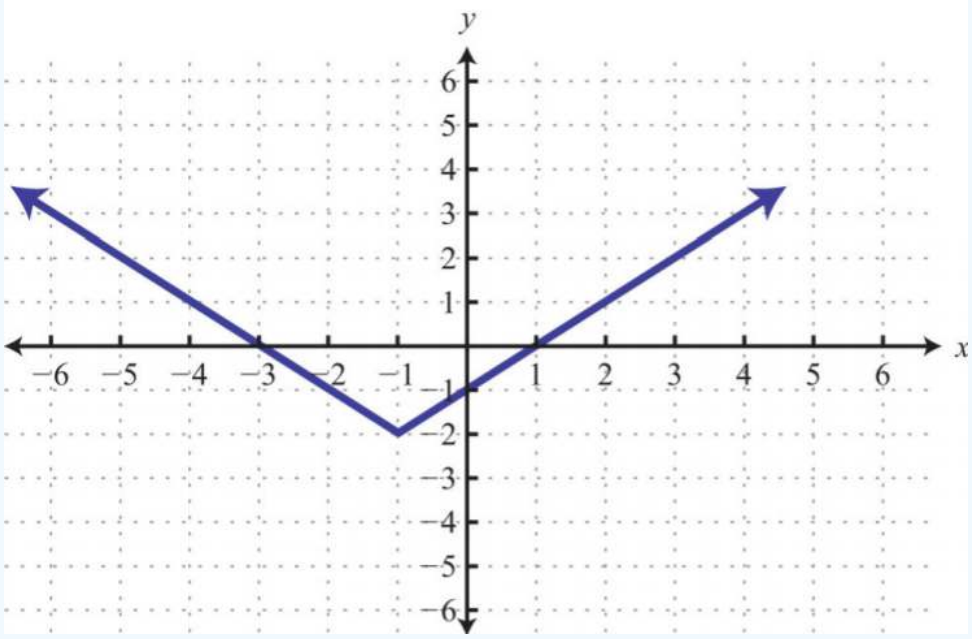


***Figure* 1.2.3.34**

3.

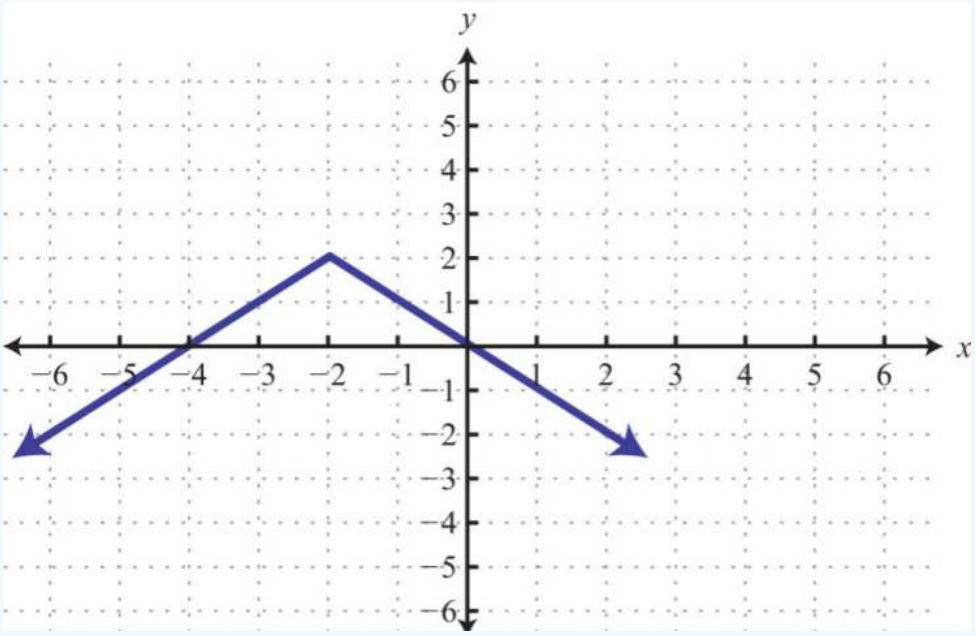


***Figure* 1.2.3.35**

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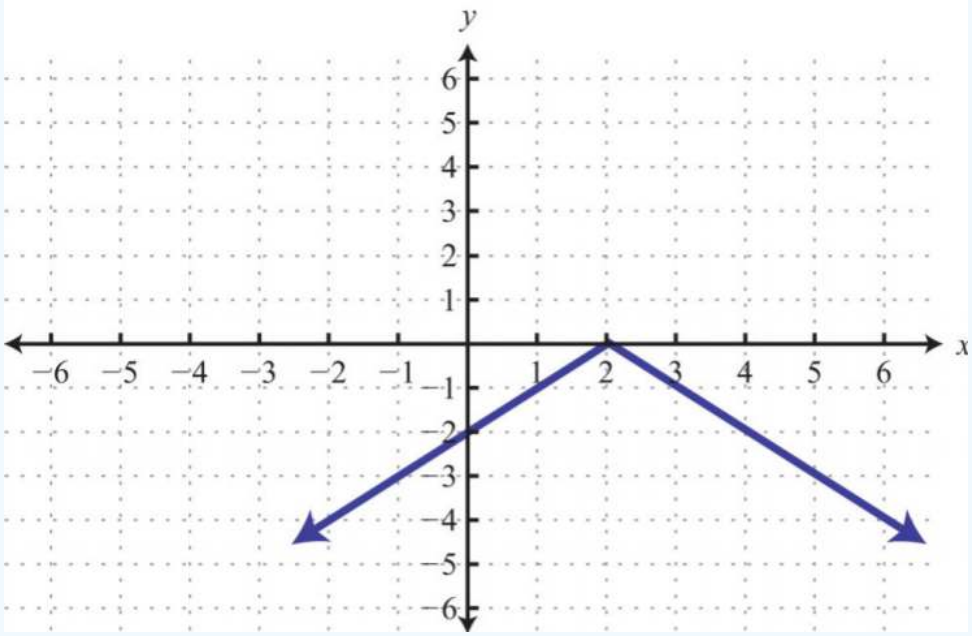


Figure **1.2.3.36**

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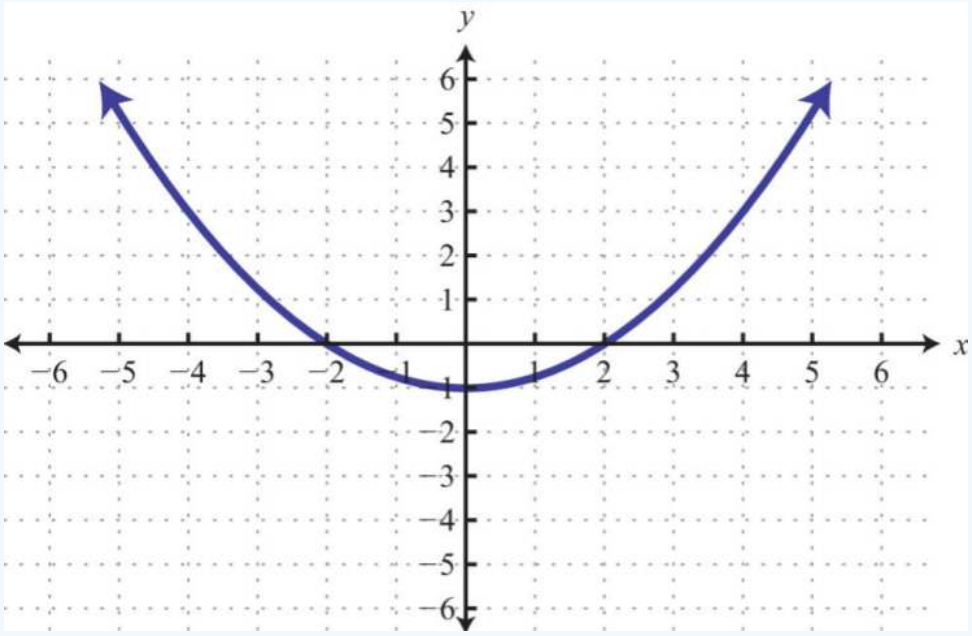


Figure **1.2.3.37**

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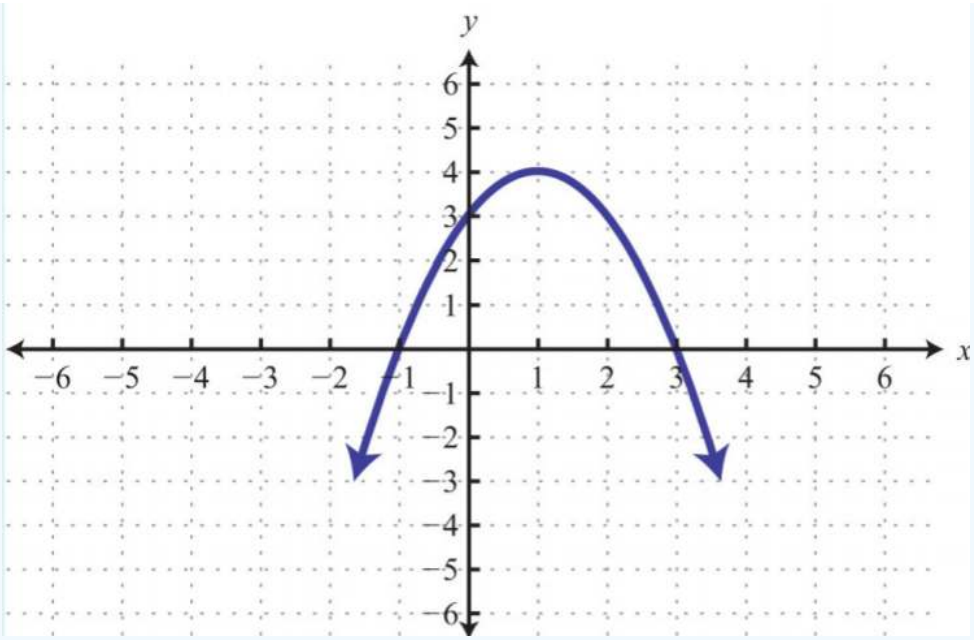


Figure **1.2.3.38**

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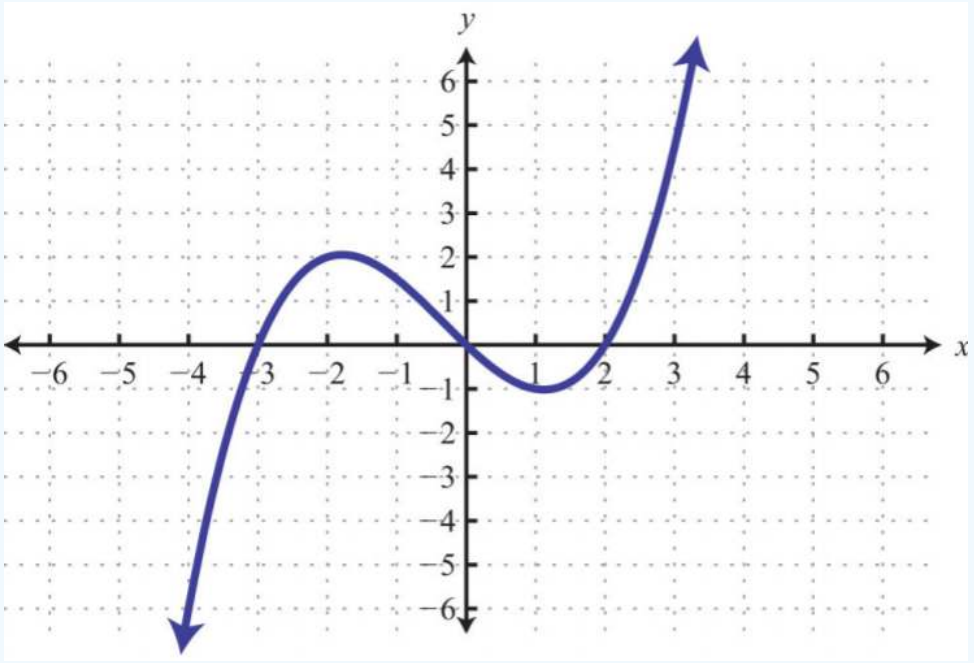


Figure **1.2.3.39**

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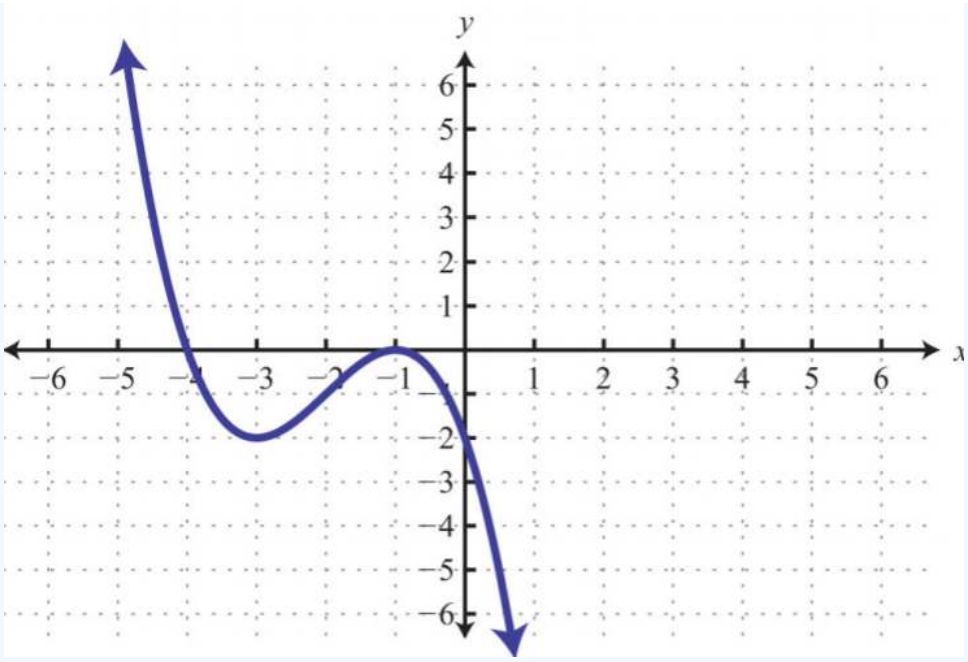


Figure **1.2.3.40**

9.

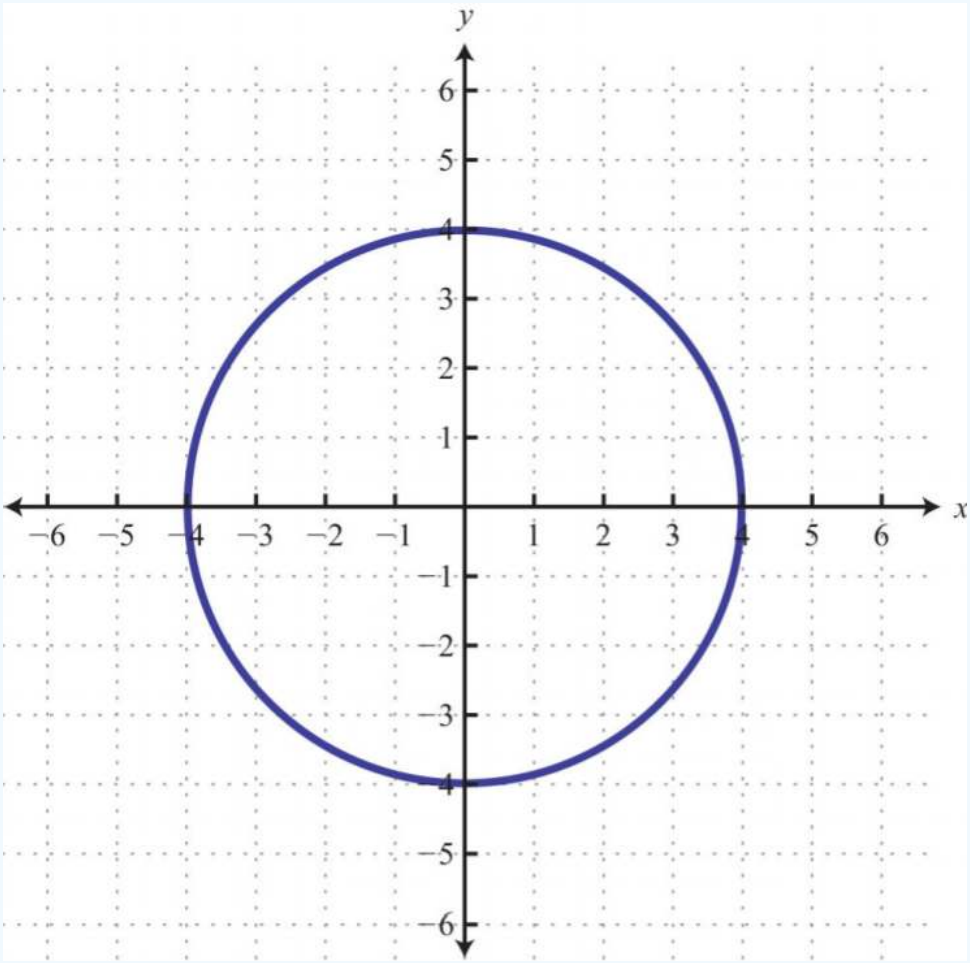


Figure **1.2.3.41**

10.

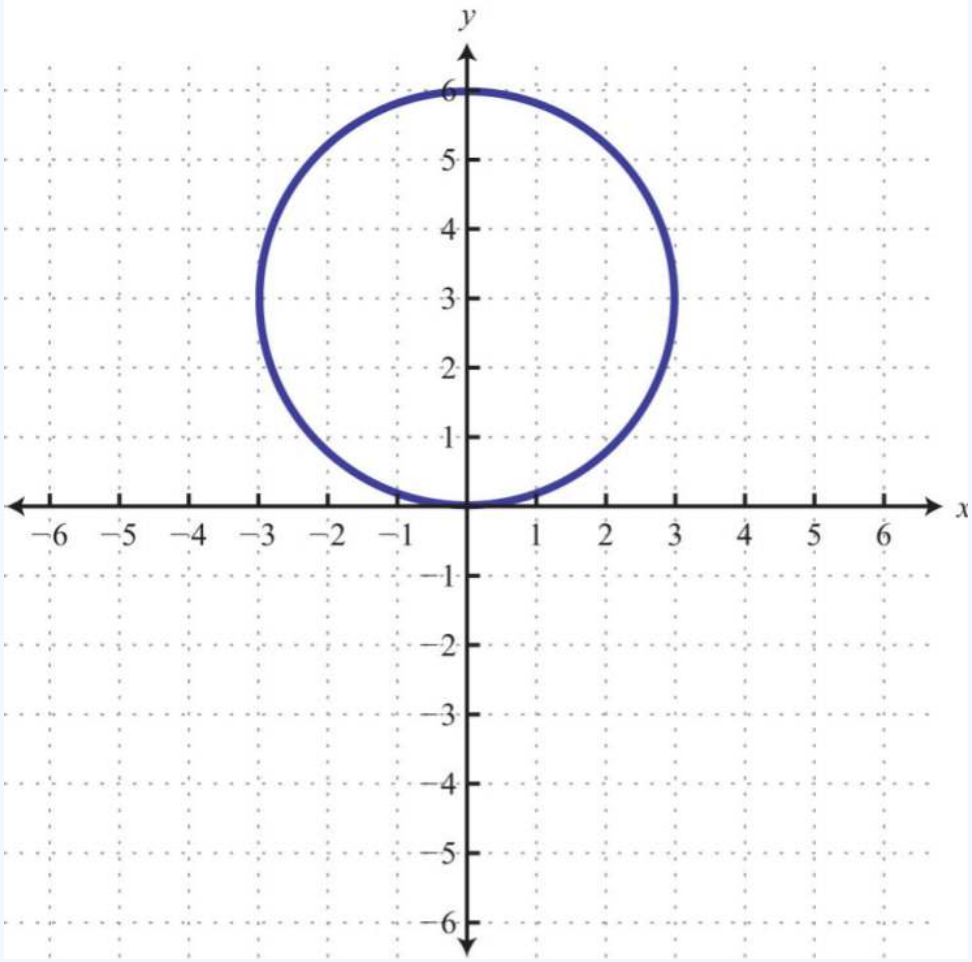


***Figure* 1.2.3.42**

Answer

1. x-intercepts: $(-3,0),(3,0)$; y-intercept: $(0,-3)$
3. x-intercepts: $(-4,0),(0,0)$; y-intercept: $(0,0)$
5. x-intercepts: $(-2,0),(2,0)$; y-intercept: $(0,-1)$
7. x-intercepts: $(-3,0),(0,0),(2,0)$; y-intercept: $(0,0)$
9. x-intercepts: $(-4,0),(4,0)$; y-intercepts: $(0,-4),(0,4)$

? EXERCISE 1.2.3.7 DISCUSSION BOARD TOPICS

1. What are the x-intercepts of the line $y=0$?
2. What are the y-intercepts of the line $x=0$?
3. Do all lines have intercepts?
4. How many intercepts can a circle have? Draw circles showing all possible numbers of intercepts.
5. Research and post the definitions of line segment, ray, and line. Why are the arrows important?

Answer

1. Answers may vary
3. Answers may vary
5. Answers may vary

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1.2.4: GRAPH USING THE Y-INTERCEPT AND SLOPE

LEARNING OBJECTIVES

- Identify and find the slope of a line.
- Graph a line using the slope and y-intercept.

SLOPE

The steepness of any incline can be measured as the ratio of the vertical change to the horizontal change. For example, a 5% incline can be written as $\frac{5}{100}$, which means that for every 100 feet forward, the height increases 5 feet.

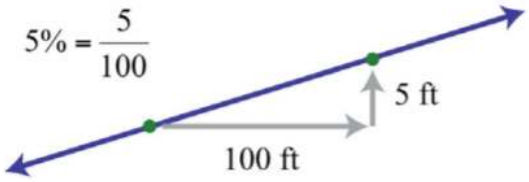


***Figure* 1.2.4.1**

In mathematics, we call the incline of a line the slope and use the letter m to denote it. The vertical change is called the rise and the horizontal change is called the run.

$$\textit{Slope} \quad m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{rise}{run} \tag{1.2.4.1}$$

The rise and the run can be positive or negative. A positive rise corresponds to a vertical change up and a negative rise corresponds to a vertical change down. A positive run denotes a horizontal change to the right and a negative run corresponds to a horizontal change to the left. Given the graph, we can calculate the slope by determining the vertical and horizontal changes between any two points.

EXAMPLE 1.2.4.1

Find the slope of the given line:

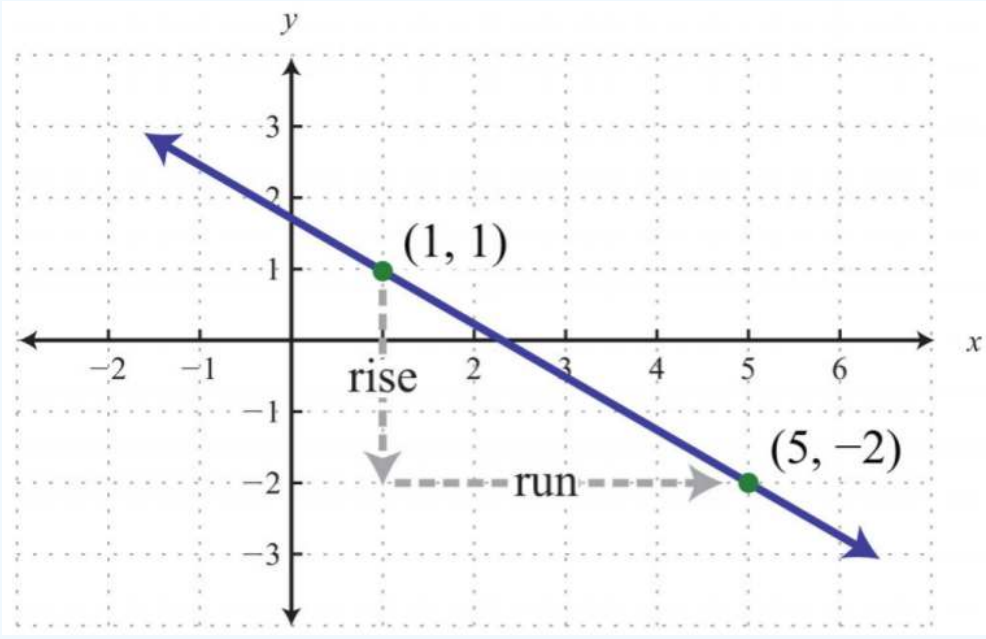


***Figure* 1.2.4.2**

Solution:

From the given points on the graph, count 3 units down and 4 units right.

$$m = \frac{rise}{run} = \frac{-3 \text{ units}}{4 \text{ units}} = -\frac{3}{4}$$

Answer:

$$m = -\frac{3}{4}$$

Here we have a negative slope, which means that for every 4 units of movement to the right, the vertical change is 3 units downward. There are four geometric cases for the value of the slope.

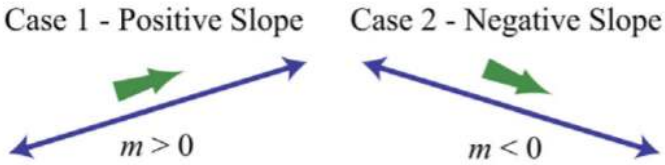


***Figure* 1.2.4.3**

Reading the graph from left to right, we see that lines with an upward incline have positive slopes and lines with a downward incline have negative slopes.

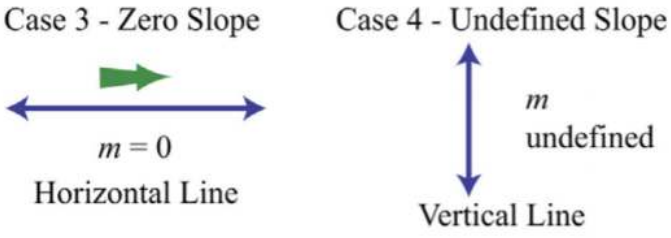


***Figure* 1.2.4.4**

If the line is horizontal, then the rise is 0:

$$m = \frac{rise}{run} = \frac{0}{run} = 0$$

The slope of a horizontal line is 0. If the line is vertical, then the run is 0:

$$m = \frac{rise}{run} = \frac{rise}{0} \quad Undefined$$

The slope of a vertical line is undefined.

? EXERCISE 1.2.4.1

Find the slope of the given line:

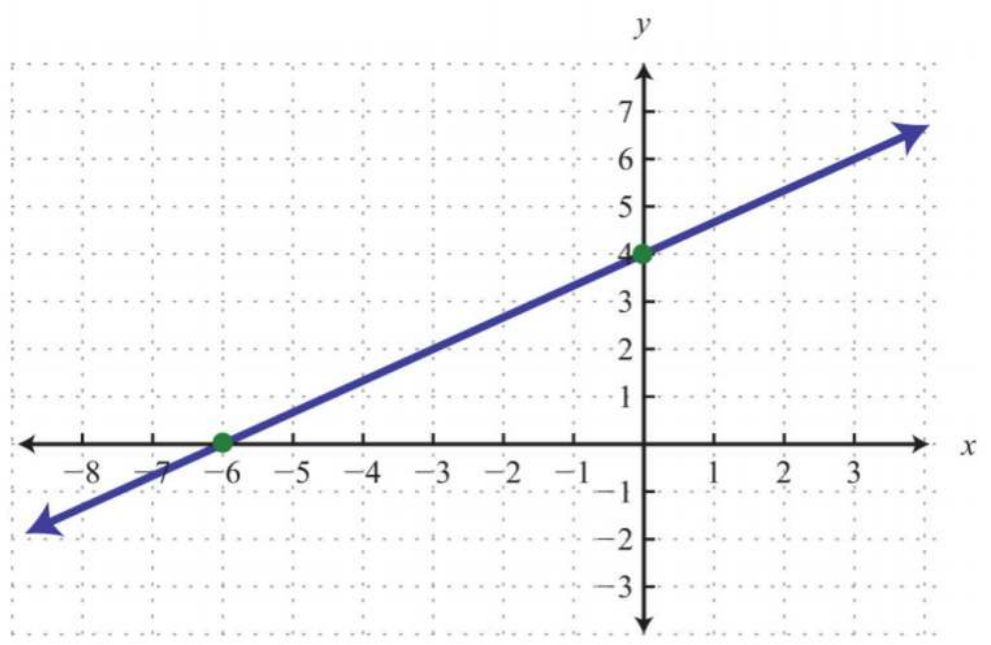


***Figure* 1.2.4.5**

Answer

$m = \frac{2}{3}$

Calculating the slope can be difficult if the graph does not have points with integer coordinates. Therefore, we next develop a formula that allows us to calculate the slope algebraically. Given any two points (x_1, y_1) and (x_2, y_2), we can obtain the rise and run by subtracting the corresponding coordinates.

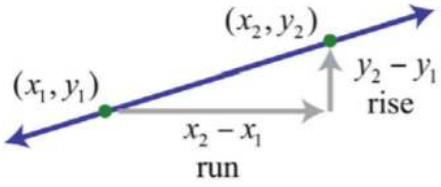


***Figure* 1.2.4.6**

This leads us to the slope formula. Given any two points (x_1, y_1) and (x_2, y_2), the slope is given by

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

EXAMPLE 1.2.4.2

Find the slope of the line passing through $(-3, -5)$ and $(2, 1)$.

Solution:

Given $(-3, -5)$ and $(2, 1)$, calculate the difference of the y-values divided by the difference of the x-values. Since subtraction is not commutative, take care to be consistent when subtracting the coordinates.

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-3, -5) & (2, 1) \end{array}$$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1-(-5)}{2-(-3)} \\ &= \frac{1+5}{2+3} \\ &= \frac{6}{5} \end{aligned}$$

Answer:

$$m = \tfrac{6}{5}$$

We can graph the line described in the previous example and verify that the slope is $\frac{6}{5}$.

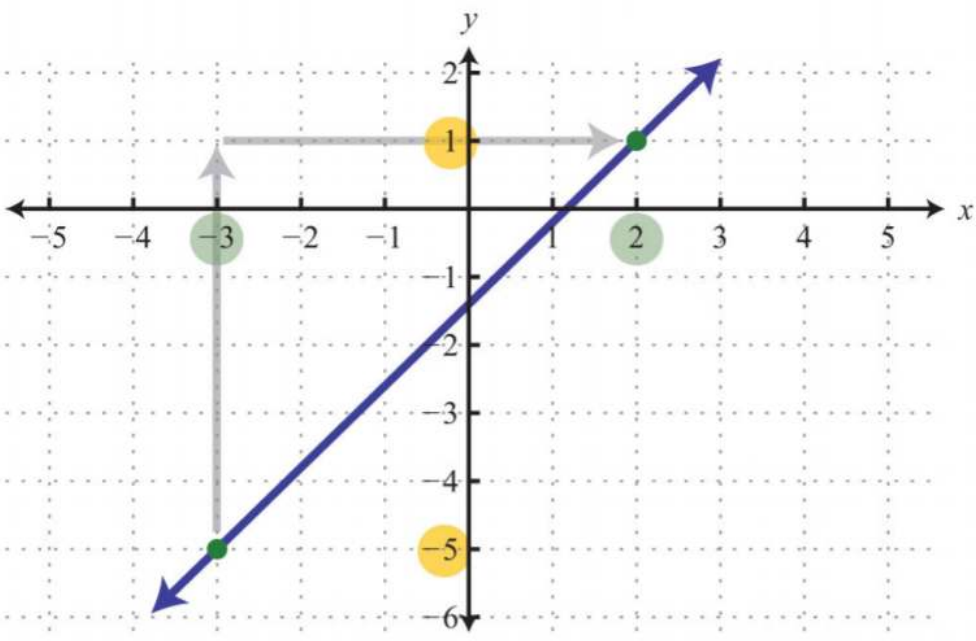


***Figure* 1.2.4.7**

Certainly the graph is optional; the beauty of the slope formula is that we can obtain the slope, given two points, using only algebra.

✔ EXAMPLE 1.2.4.3

Find the slope of the line passing through $(-4, 3)$ and $(-1, -7)$.

Solution:

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-4, 3) & (-1, -7) \end{array}$$

$$m = \tfrac{y_2 - y_1}{x_2 - x_1} = \tfrac{-7-(3)}{-1-(-4)} = \tfrac{-7-3}{-1+4} = \tfrac{-10}{3}$$

Answer:

$$m = -\tfrac{10}{3}$$

When using the slope formula, take care to be consistent since order does matter. You must subtract the coordinates of the first point from the coordinates of the second point for both the numerator and the denominator in the ***same order***.

✔ EXAMPLE 1.2.4.4

Find the slope of the line passing through $(7, -2)$ and $(-5, -2)$.

Solution:

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (7, -2) & (-5, -2) \end{array}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-(-2)}{-5-(7)} = \frac{-2+2}{-5-7} = \frac{0}{-12} = 0$$

Answer:

$m = 0$. As an exercise, plot the given two points and verify that they lie on a horizontal line.

EXAMPLE 1.2.4.5

Find the slope of the line passing through $(-4, -3)$ and $(-4, 5)$.

Solution:

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-4, -3) & (-4, 5) \end{array}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-3)}{-4-(-4)} = \frac{5+3}{-4+4} = \frac{8}{0} \quad Undefined$$

Answer:

The slope m is undefined. As an exercise, plot the given two points and verify that they lie on a vertical line.

EXERCISE 1.2.4.2

Calculate the slope of the line passing through $(-2, 3)$ and $(5, -5)$.

Answer

$m = -\frac{8}{7}$

When considering the slope as a rate of change it is important to include the correct units.

EXAMPLE 1.2.4.6

A Corvette Coupe was purchased new in 1970 for about $\$5,200$ and depreciated in value over time until it was sold in 1985 for $\$1,300$. At this point, the car was beginning to be considered a classic and started to increase in value. In the year 2000, when the car was 30 years old, it sold at auction for $\$10,450$. The following line graph depicts the value of the car over time.

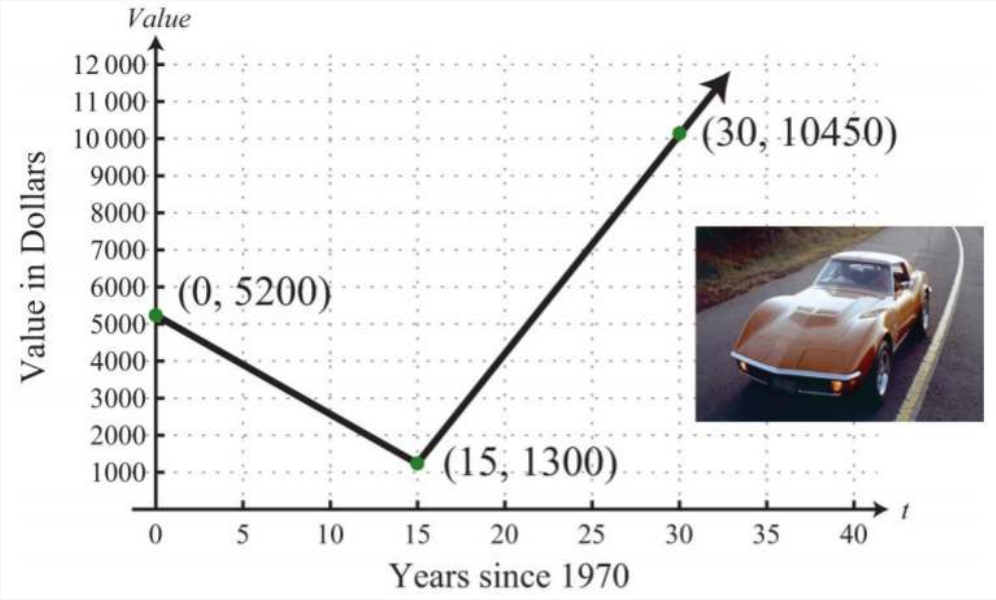


***Figure* 1.2.4.8**

a. Determine the rate at which the car depreciated in value from 1970 to 1985.
b. Determine the rate at which the car appreciated in value from 1985 to 2000.

Solution:

Notice that the value depends on the age of the car and that the slope measures the rate in dollars per year.

a. The slope of the line segment depicting the value for the first 15 years is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\$1{,}300 - \$5{,}200}{15 \text{ years} - 0 \text{ years}} = \frac{-\$3{,}900}{15 \text{ years}} = -\$260 \text{ per year}$$

b. The slope of the line segment depicting the value for the next 15 years is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\$10{,}450 - \$1{,}300}{30 \text{ years} - 15 \text{ years}} = \frac{\$9{,}150}{15 \text{ years}} = \$610 \text{ per year}$$

Answer:

a. The value of the car depreciated $260 per year from 1970 to 1985.
b. The value of the car appreciated $610 per year from 1985 to 2000.

SLOPE-INTERCEPT FORM OF A LINE

To this point, we have learned how to graph lines by plotting points and by using the x- and y-intercepts. In addition, we have seen that we need only two points to graph a line. In this section, we outline a process to easily determine two points using the y-intercept and the slope. The equation of any nonvertical line can be written in slope-intercept form $y = mx + b$. In this form, we can identify the slope, m, and the y-intercept, $(0, b)$.

✔ EXAMPLE 1.2.4.7

Determine the slope and y-intercept:

$$y = -\tfrac{4}{5}x + 7.$$

Solution:

In this form, the coefficient of x is the slope, and the constant is the y-value of the y-intercept. Therefore, by inspection, we have

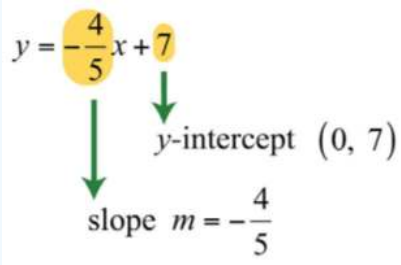


***Figure* 1.2.4.9**

Answer:

The y-intercept is $(0, 7)$, and the slope is $m = -\frac{4}{5}$.

It is not always the case that the linear equation is given in slope-intercept form. When it is given in standard form, you have to first solve for y to obtain slope-intercept form.

✔ EXAMPLE 1.2.4.8

Express $3x + 5y = 30$ in slope-intercept form and then identify the slope and y-intercept.

Solution:

Begin by solving for y. To do this, apply the properties of equality to first isolate $5y$ and then divide both sides by 5.

$$\begin{aligned} 3x + 5y &= 30 \\ 3x + 5y - 3x &= 30 - 3x \\ 5y &= -3x + 30 \\ \frac{5y}{5} &= \frac{-3x + 30}{5} \\ y &= \frac{-3x}{5} + \frac{30}{5} \\ y &= -\frac{3}{5}x + 6 \end{aligned}$$

Answer:

Slope-intercept form: $y = -\frac{3}{5}x + 6$; y-intercept: $(0, 6)$; slope: $m = -\frac{3}{5}$

Once the equation is in slope-intercept form, we immediately have one point to plot, the y-intercept. From the intercept, you can mark off the slope to plot another point on the line. From the previous example we have

$$y = -\frac{3}{5}x + 6$$

$$y\text{-intercept: } (0, 6)$$

$$\text{slope: } m = -\frac{3}{5} = \frac{-3}{5} = \frac{rise}{run}$$

Starting from the point $(0, 6)$, use the slope to mark another point 3 units down and 5 units to the right.

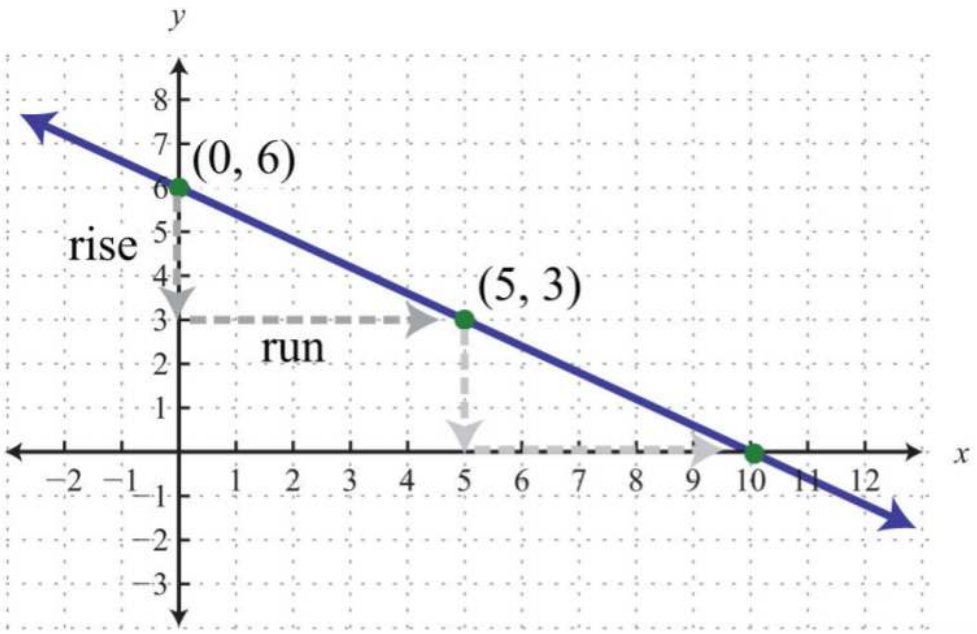


***Figure* 1.2.4.10**

It is not necessary to check that the second point, (5, 3), solves the original linear equation. However, we do it here for the sake of completeness.

$$\begin{aligned} 3x + 5y &= 30 \\ 3(5)+5(3) &= 30 \\ 15 + 15 &= 30 \\ 30 &= 30 \quad \checkmark \end{aligned}$$

Marking off the slope in this fashion produces as many ordered pair solutions as we desire. Notice that if we mark off the slope again, from the point $(5, 3)$, then we obtain the x-intercept, $(10, 0)$.

✔ EXAMPLE 1.2.4.9

Graph:

$$-x + 2y = 4.$$

Solution:

In this example, we outline the general steps for graphing a line using slope-intercept form.

Step 1: Solve for y to obtain slope-intercept form.

$$\begin{aligned} -x+2y &= 4 \\ -x+2y+x &= 4+x \\ 2y &= x+4 \\ \frac{2y}{2} &= \frac{x+4}{2} \\ y &= \frac{1x}{2}+\frac{4}{2} \\ y &= \frac{1}{2}x+2 \end{aligned}$$

Step 2: Identify the y-intercept and slope.

$$y\text{-intercept: } (0,2)$$
$$\text{slope: } m=\tfrac{1}{2}=\tfrac{rise}{run}$$

Step 3: Plot the y-intercept and use the slope to find another ordered pair solution. Starting from the y-intercept, mark off the slope and identify a second point. In this case, mark a point after a rise of 1 unit and a run of 2 units.

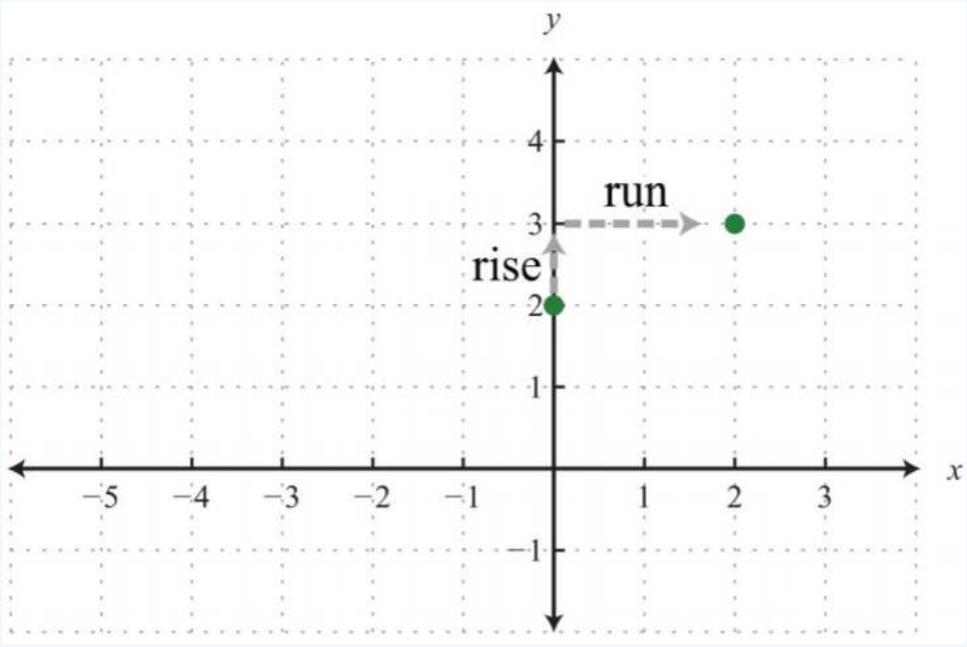


***Figure* 1.2.4.11**

Step 4: Draw the line through the two points with a straightedge.

Answer:

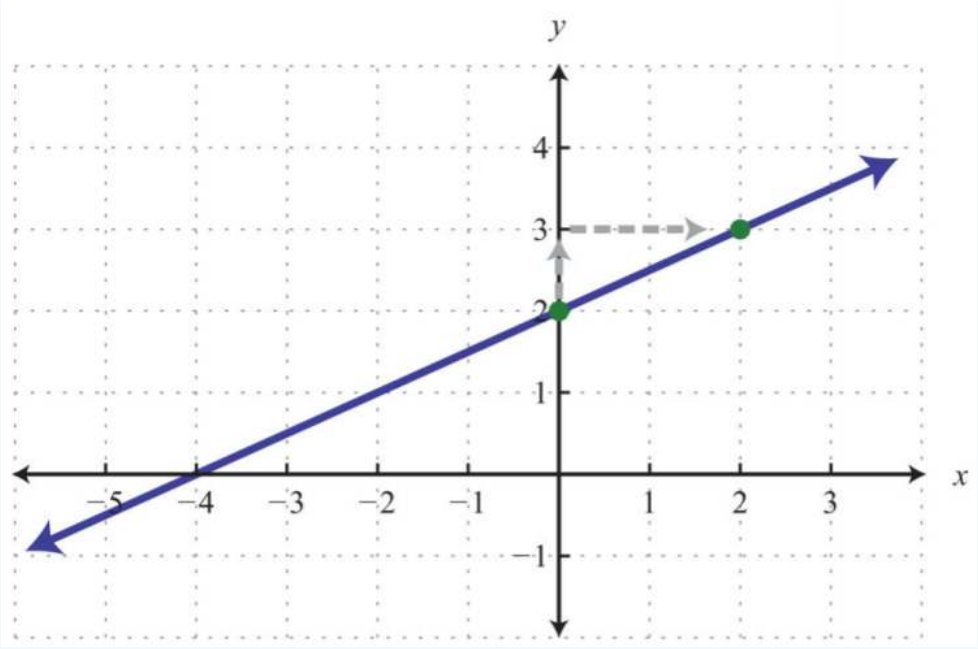


***Figure* 1.2.4.12**

In this example, we notice that we could get the x-intercept by marking off the slope in a different but equivalent manner. Consider the slope as the ratio of two negative numbers as follows:

$$m = \frac{1}{2} = \frac{-1}{-2} = \frac{rise}{run}$$

We could obtain another point on the line by marking off the equivalent slope down 1 unit and left 2 units. We do this twice to obtain the x-intercept, $(-4, 0)$.

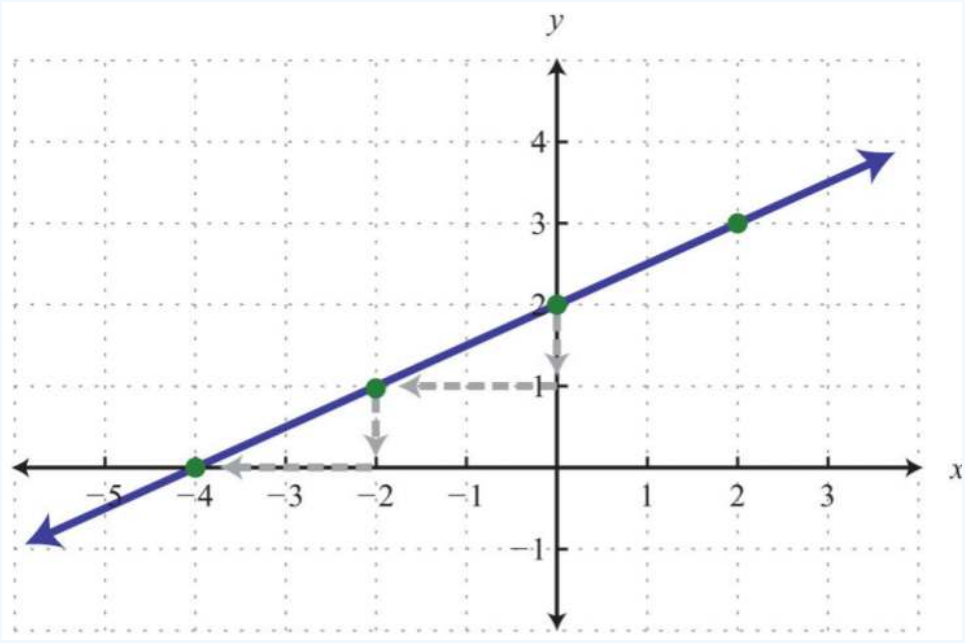


***Figure* 1.2.4.13**

Marking off the slope multiple times is not necessarily always going to give us the x-intercept, but when it does, we obtain a valuable point with little effort. In fact, it is a good practice to mark off the slope multiple times; doing so allows you to obtain more points on the line and produce a more accurate graph.

EXAMPLE 1.2.4.10

Graph and find the x-intercept:

$$y = \tfrac{3}{4}x - 2\,.$$

Solution:

The equation is given in slope-intercept form. Therefore, by inspection, we have the y-intercept and slope.

$$y\text{-intercept: } (0, -2)$$
$$\text{slope: } m = \tfrac{3}{4} = \tfrac{rise}{run}$$

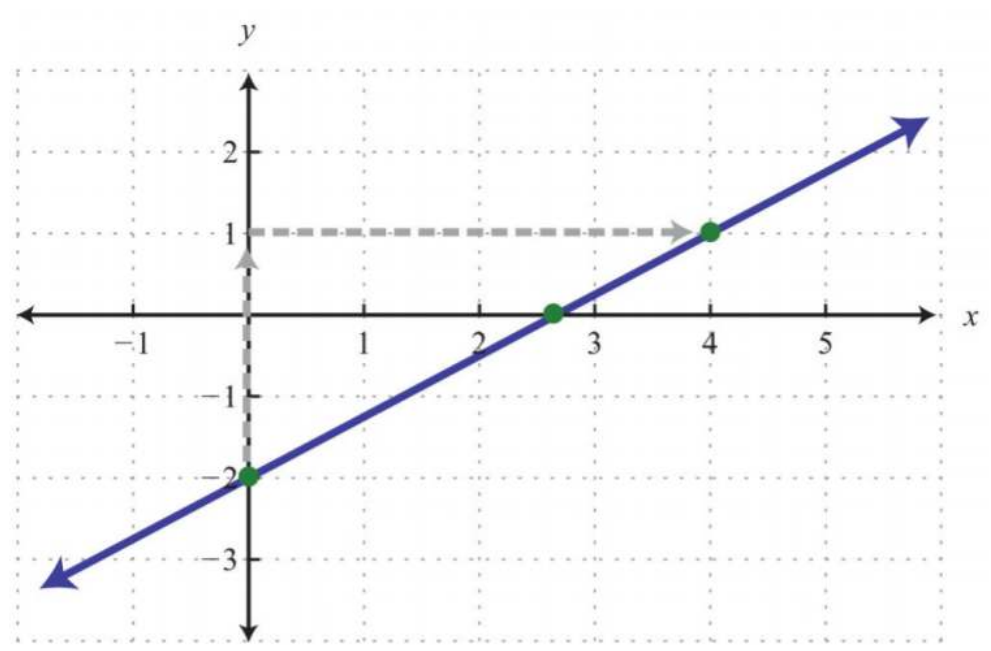


***Figure* 1.2.4.14**

We can see that the x-value of the x-intercept is a mixed number between 2 and 3. To algebraically find x-intercepts, recall that we must set $y = 0$ and solve for x.

Answer:

The x-intercept is $(2\tfrac{2}{3}, 0)$.

EXAMPLE 1.2.4.11

Graph:

$$x - y = 0\,.$$

Solution:

Begin by solving for y.

$$\begin{aligned} x - y &= 0 \\ x - y\color{cyan}{-x} &= 0\color{cyan}{-x} \\ -y &= -x \\ \color{cyan}{-1}\cdot(-y) &= \color{cyan}{-1}\cdot(-x) \\ y &= x \end{aligned}$$

The equation $y = x$ can be written $y = 1x + 0$, and we have

$$y\text{-intercept: } (0, 0)$$
$$\text{slope: } m = 1 = \tfrac{1}{1} = \tfrac{rise}{run}$$

Answer:

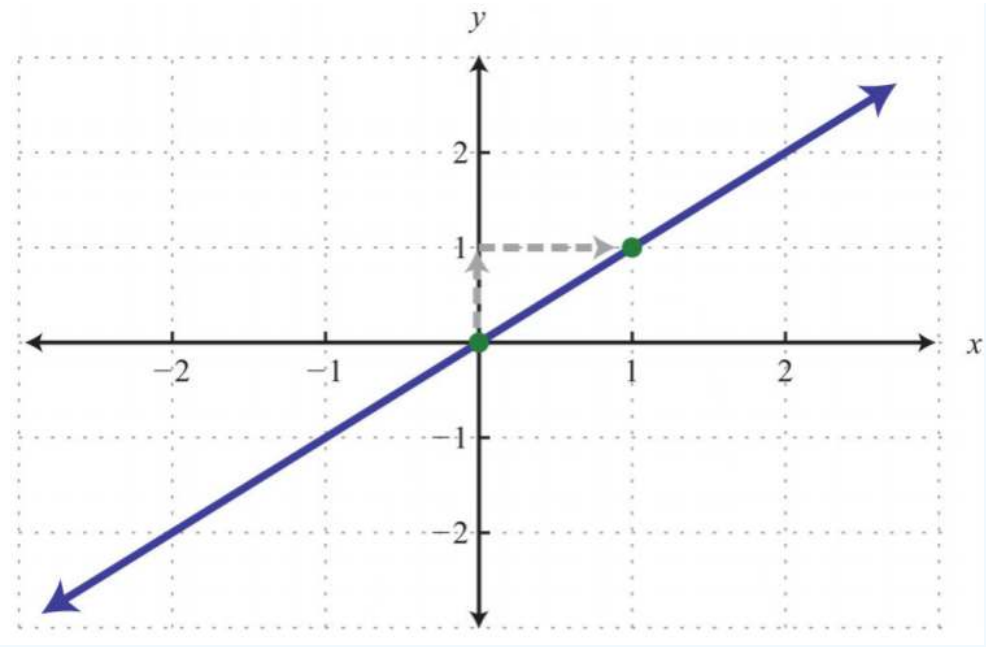


***Figure* 1.2.4.15**

? EXERCISE 1.2.4.3

Graph $-2x + 5y = 20$ and label the x-intercept.

Answer

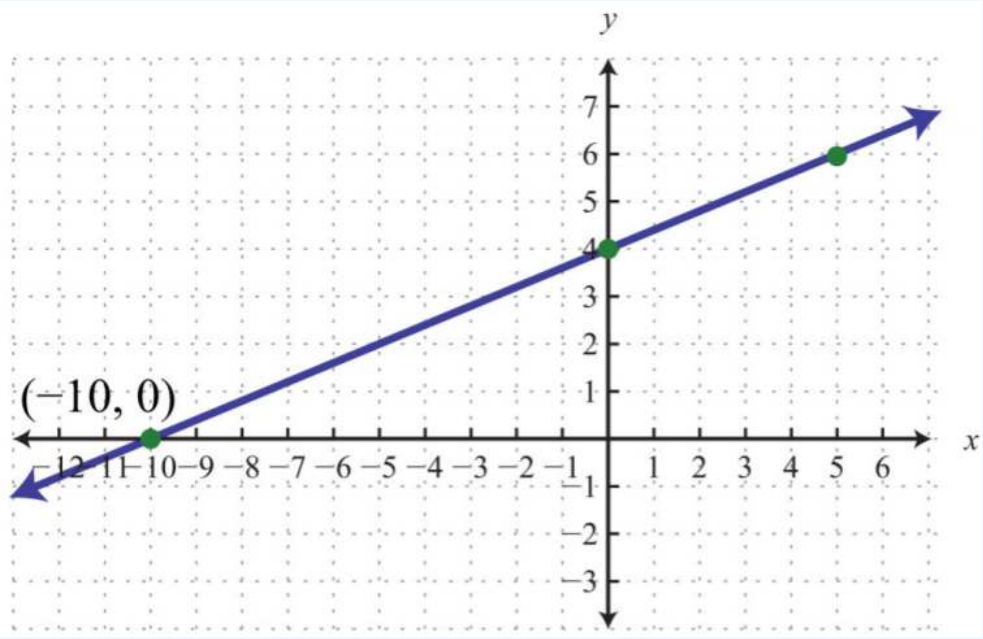


***Figure* 1.2.4.16**

KEY TAKEAWAYS

- Slope measures the steepness of a line as *rise over run*. A positive rise denotes a vertical change up, and a negative rise denotes a vertical change down. A positive run denotes a horizontal change right, and a negative run denotes a horizontal change left.
- Horizontal lines have a slope of zero, and vertical lines have undefined slopes.
- Given any two points on a line, we can algebraically calculate the slope using the slope formula, $m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$.

- Any nonvertical line can be written in slope-intercept form, $y = mx + b$, from which we can determine, by inspection, the slope m and y-intercept $(0, b)$.
- If we know the y-intercept and slope of a line, then we can easily graph it. First, plot the y-intercept, and from this point use the slope as rise over run to mark another point on the line. Finally, draw a line through these two points with a straightedge and add an arrow on either end to indicate that it extends indefinitely.
- We can obtain as many points on the line as we wish by marking off the slope multiple times.

? EXERCISE 1.2.4.4 SLOPE

Determine the slope and the y-intercept of the given graph.

1.

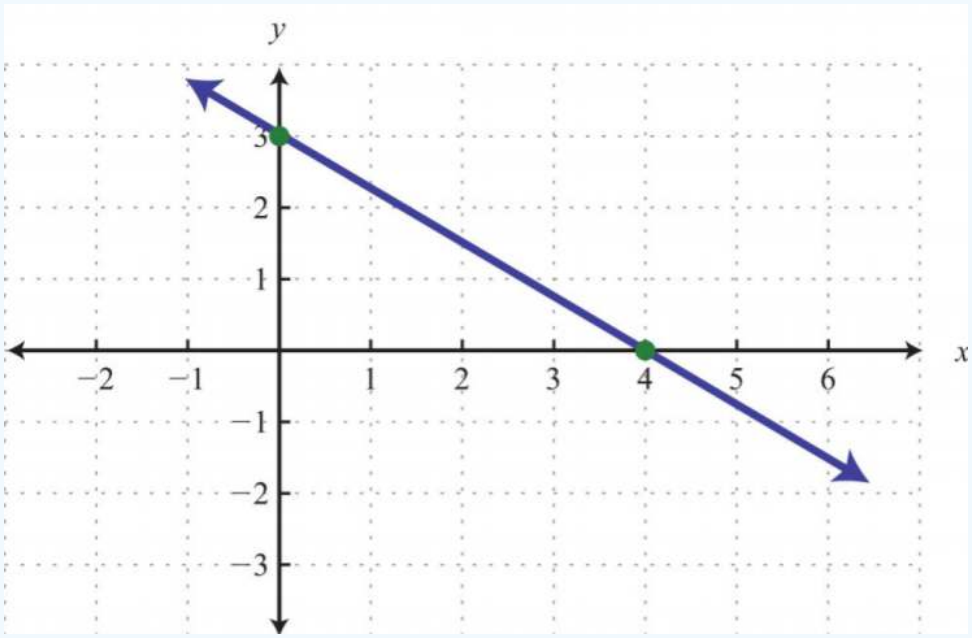


***Figure* 1.2.4.17**

2.

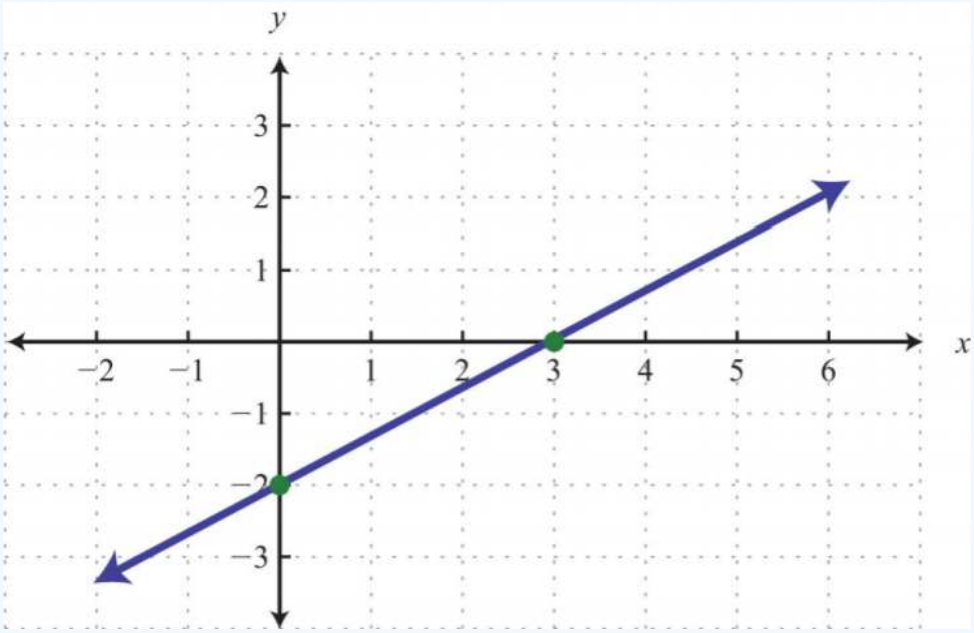


***Figure* 1.2.4.18**

3.

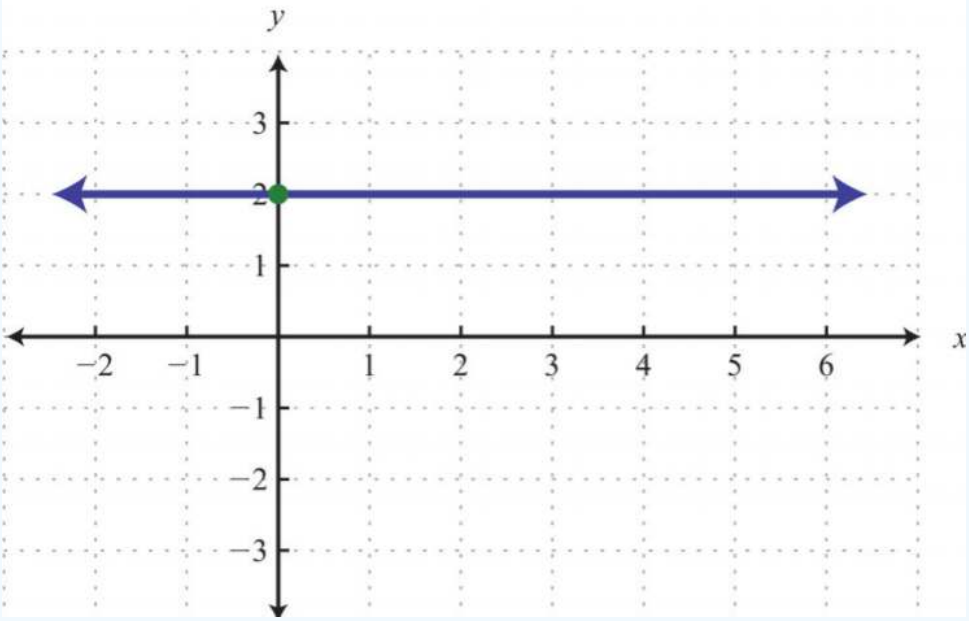


Figure **1.2.4.19**

4.

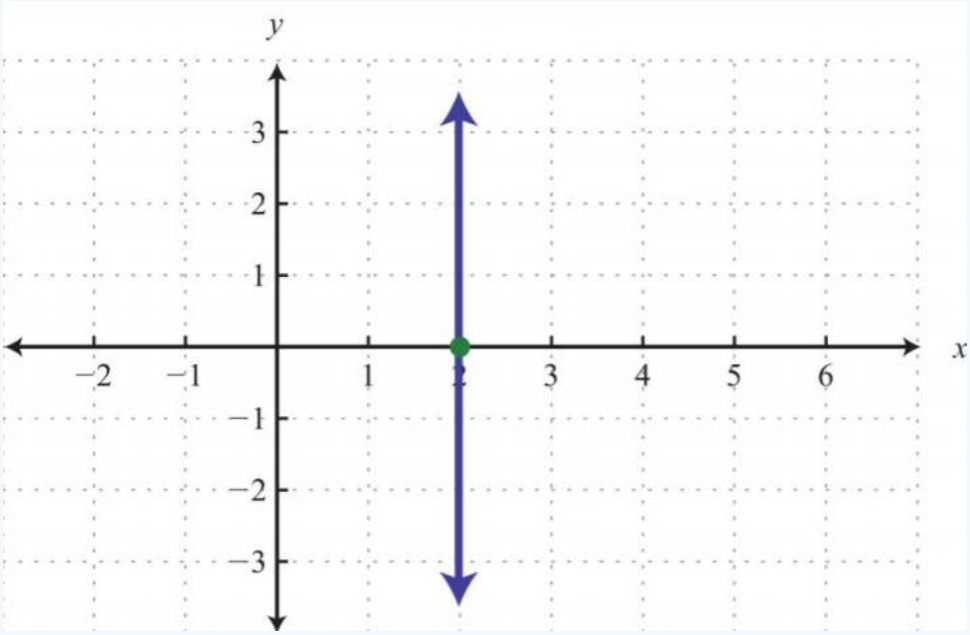


Figure **1.2.4.20**

5.

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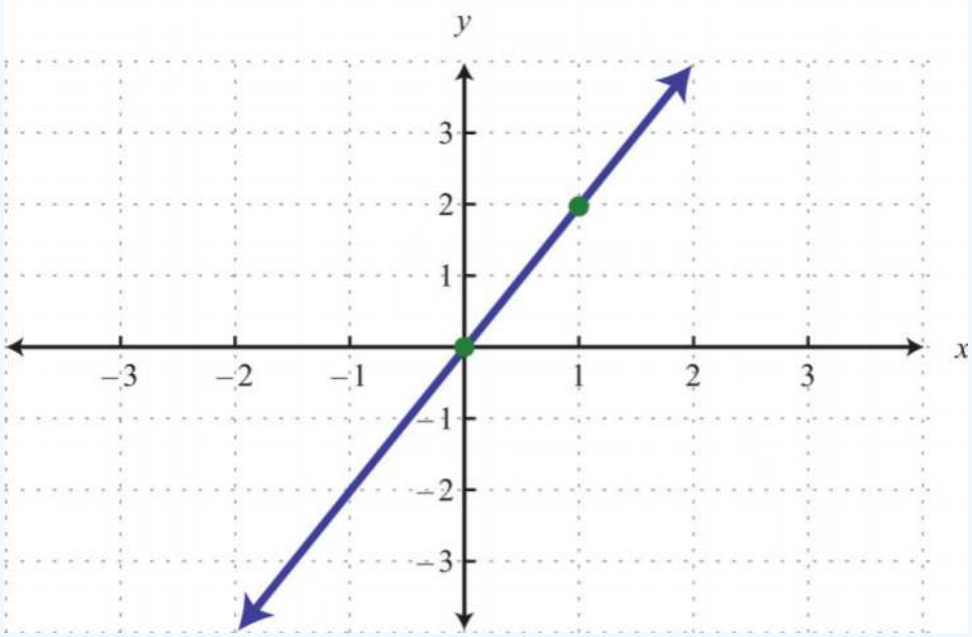


Figure **1.2.4.21**

6.

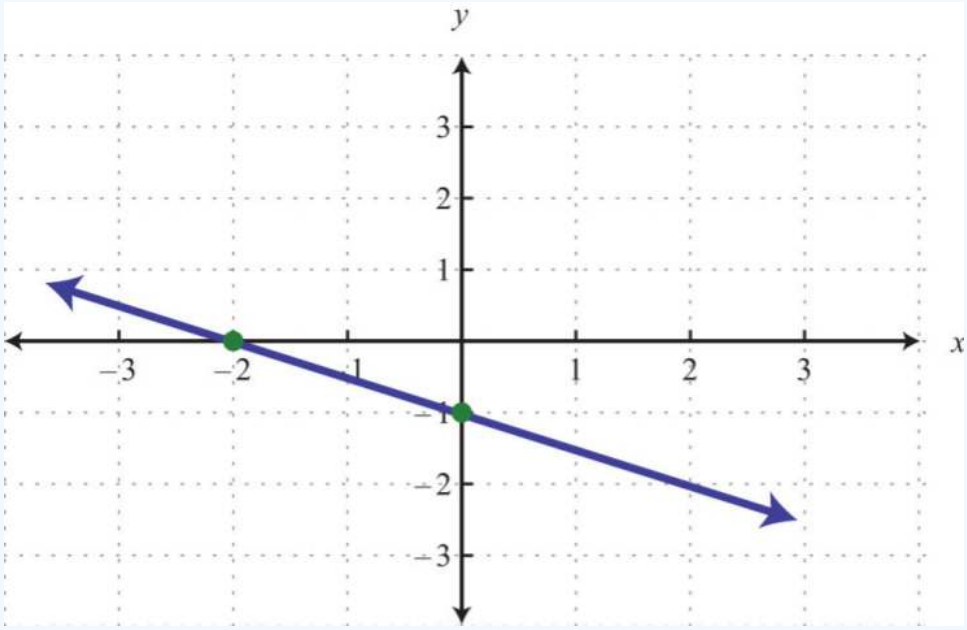


Figure **1.2.4.22**

Answer

1. y-intercept: $(0,3)$; slope: $m=-\frac{3}{4}$
3. y-intercept: $(0,2)$; slope: $m=0$
5. y-intercept: $(0,0)$; slope: $m=2$

? EXERCISE 1.2.4.5 SLOPE

Determine the slope, given two points.

1. $(3,2)$ and $(5,1)$
2. $(7,8)$ and $(-3,5)$

3. $(2,-3)$ and $(-3,2)$
4. $(-3,5)$ and $(7,-5)$
5. $(-1,-6)$ and $(3,2)$
6. $(5,3)$ and $(4,12)$
7. $(-9,3)$ and $(-6,-5)$
8. $(-22,4)$ and $(-8,-12)$
9. $(\frac{1}{2},-\frac{1}{3})$ and $(-\frac{1}{2},\frac{2}{3})$
10. $(-\frac{3}{4},\frac{3}{2})$ and $(\frac{1}{4},-\frac{1}{2})$
11. $(-\frac{1}{3},\frac{5}{8})$ and $(\frac{1}{2},-\frac{3}{4})$
12. $(-\frac{3}{5},-\frac{3}{2})$ and $(\frac{1}{10},\frac{4}{5})$
13. $(3,-5)$ and $(5,-5)$
14. $(-3,1)$ and $(-14,1)$
15. $(-2,3)$ and $(-2,-4)$
16. $(-4,-4)$ and $(5,5)$
17. A roof drops 4 feet for every 12 feet forward. Determine the slope of the roof.
18. A road drops 300 feet for every $5,280$ feet forward. Determine the slope of the road.
19. The following graph gives the US population of persons 65 years old and over. At what rate did this population increase from 2000 to 2008?

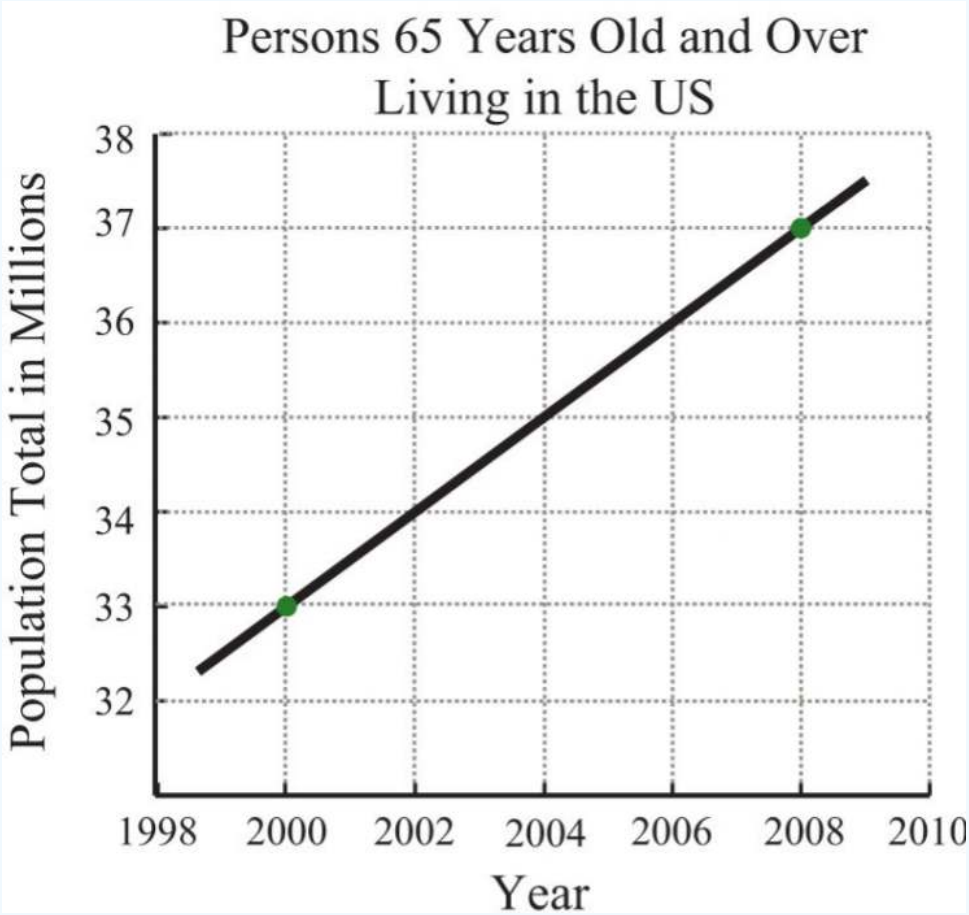


***Figure* 1.2.4.23** : *Source: US Census Bureau.*

20. The following graph gives total consumer credit outstanding in the United States. At what rate did consumer credit increase from 2002 to 2008?

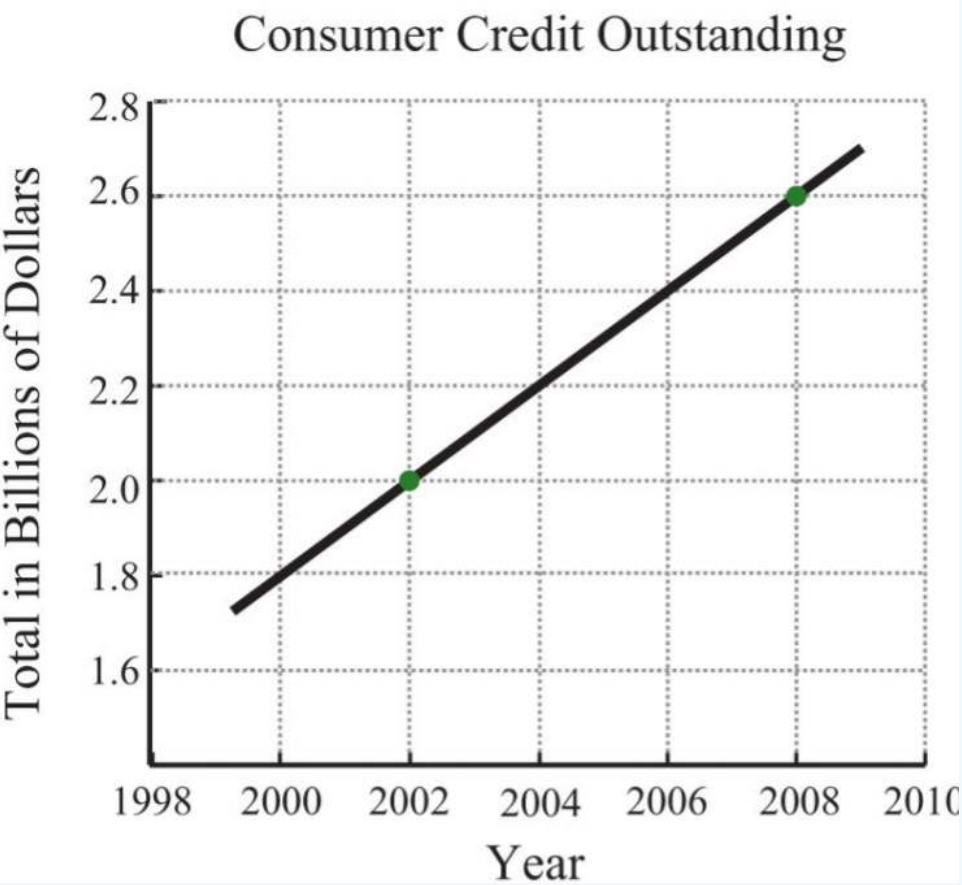


***Figure* 1.2.4.24** *: Source: US Census Bureau.*

21. A commercial van was purchased new for $\$20,000$ and is expected to be worth $\$4,000$ in 8 years. Determine the rate at which the van depreciates in value.

22. A commercial-grade copy machine was purchased new for $\$4,800$ and will be considered worthless in 6 years. Determine the rate at which the copy machine depreciates in value.

23. Find y if the slope of the line passing through $(-2,3)$ and $(4,y)$ is 12.

24. Find y if the slope of the line passing through $(5,y)$ and $(6,-1)$ is 10.

25. Find y if the slope of the line passing through $(5,y)$ and $(-4,2)$ is 0.

26. Find x if the slope of the line passing through $(-3,2)$ and $(x,5)$ is undefined.

Answer

1. $-\frac{1}{2}$
3. -1
5. 2
7. $-\frac{8}{3}$
9. -1
11. $-\frac{33}{20}$
13. 0
15. Undefined
17. $-\frac{1}{3}$
19. $\frac{1}{2}$ million per year
21. $\$2,000$ per year
23. 75

25. 2

EXERCISE 1.2.4.6 SLOPE-INTERCEPT FORM

Express the given linear equation in slope-intercept form and identify the slope and y-intercept.

1. $6x-5y=30$
2. $-2x+7y=28$
3. $9x-y=17$
4. $x-3y=18$
5. $2x-3y=0$
6. $-6x+3y=0$
7. $\frac{2}{3}x-\frac{5}{4}y=10$
8. $-\frac{4}{3}x+\frac{1}{5}y=-5$

Answer

1. $y=\frac{6}{5}x-6$; slope: $\frac{6}{5}$; y-intercept: $(0,-6)$
3. $y=9x-17$; slope: 9; y-intercept: $(0,-17)$
5. $y=\frac{2}{3}x$; slope: $\frac{2}{3}$; y-intercept: $(0,0)$
7. $y=\frac{8}{15}x-8$; slope: $\frac{8}{15}$; y-intercept: $(0,-8)$

EXERCISE 1.2.4.7 SLOPE-INTERCEPT FORM

Graph the line given the slope and the y-intercept.

1. $m=\frac{1}{3}$ and $(0,-2)$
2. $m=-\frac{2}{3}$ and $(0,4)$
3. $m=3$ and $(0,1)$
4. $m=-2$ and $(0,-1)$
5. $m=0$ and $(0,5)$
6. m undefined and $(0,0)$
7. $m=1$ and $(0,0)$
8. $m=-1$ and $(0,0)$
9. $m=-\frac{15}{3}$ and $(0,20)$
10. $m=-10$ and $(0,-5)$

Answer

1.

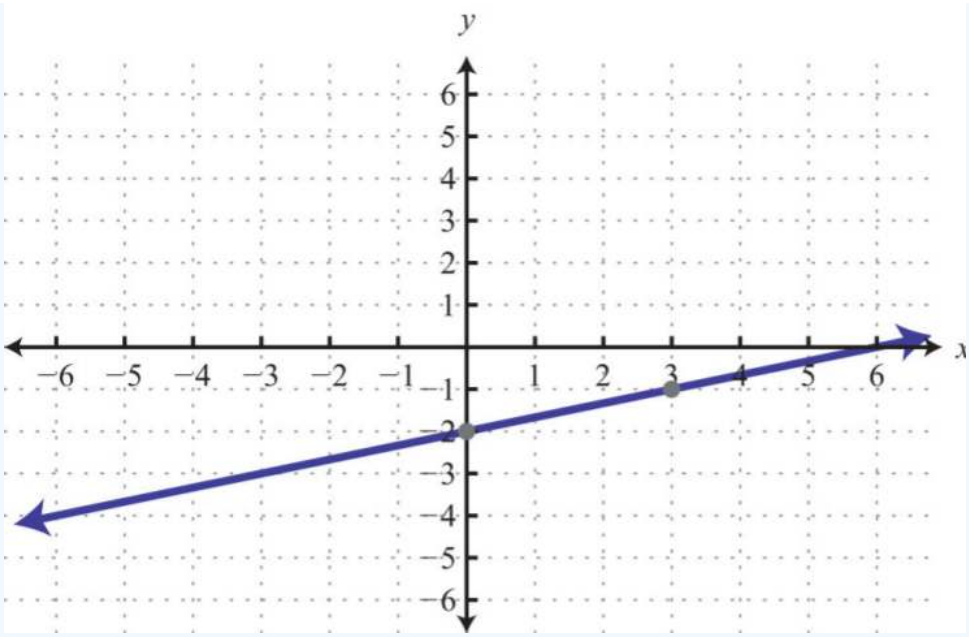


Figure 1.2.4.25

3.

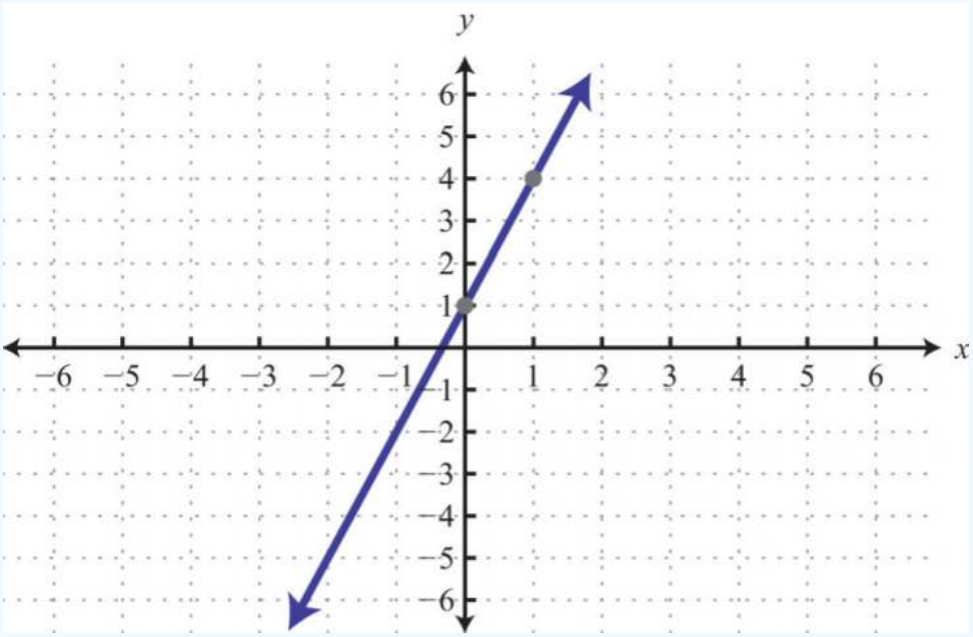


Figure 1.2.4.26

5.

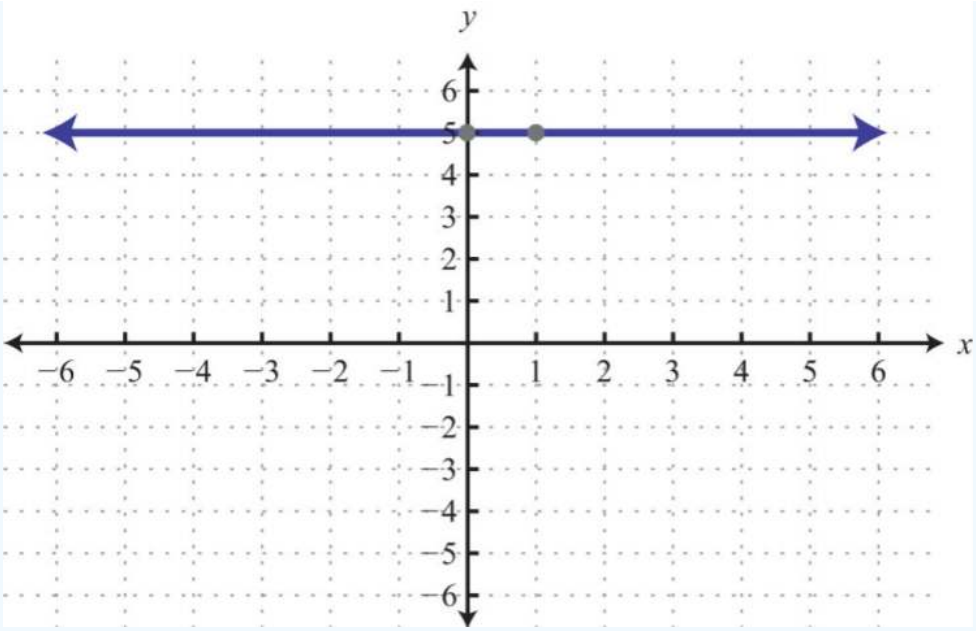


Figure 1.2.4.27

7.

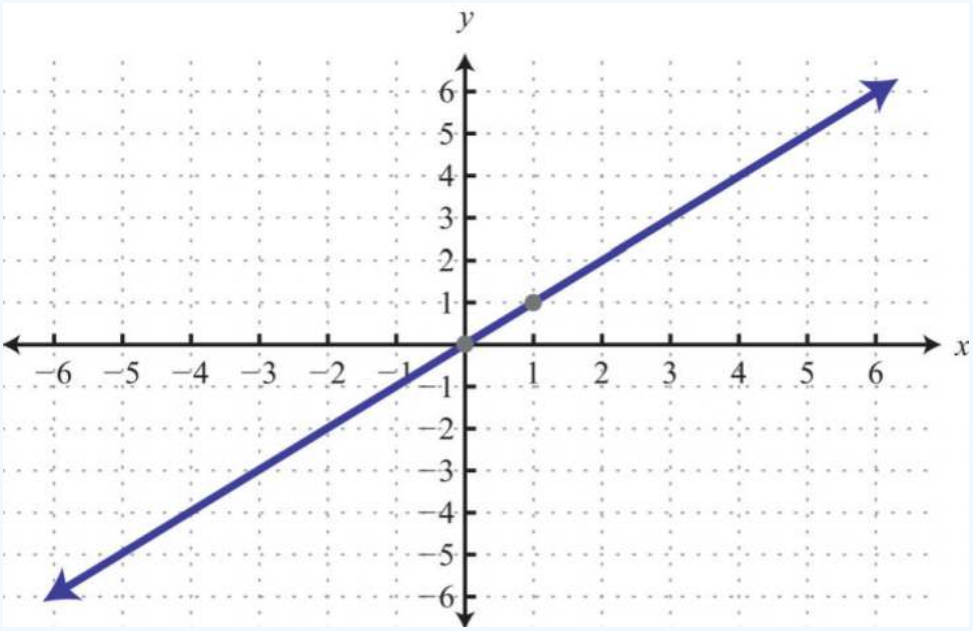


Figure 1.2.4.28

9.

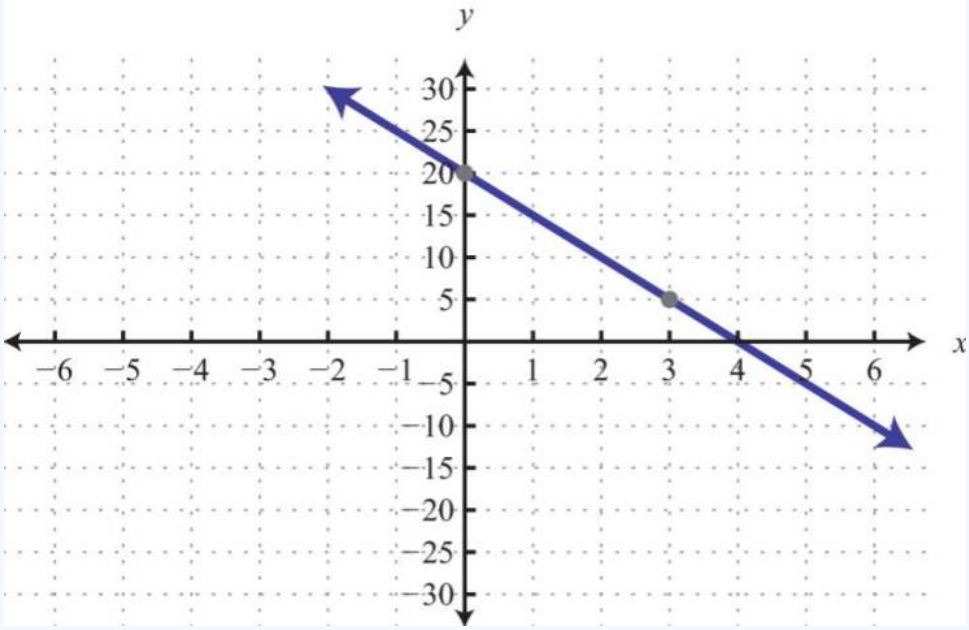


Figure 1.2.4.29

? EXERCISE 1.2.4.8 SLOPE-INTERCEPT FORM

Graph using the slope and y-intercept.

1. $y = \frac{2}{3}x - 2$
2. $y = -\frac{1}{3}x + 1$
3. $y = -3x + 6$
4. $y = 3x + 1$
5. $y = \frac{3}{5}x$
6. $y = -\frac{3}{7}x$
7. $y = -8$
8. $y = 7$
9. $y = -x + 2$
10. $y = x + 1$
11. $y = \frac{1}{2}x + \frac{3}{2}$
12. $y = -\frac{3}{4}x + \frac{5}{2}$
13. $4x + y = 7$
14. $3x - y = 5$
15. $5x - 2y = 10$
16. $-2x + 3y = 18$
17. $x - y = 0$
18. $x + y = 0$
19. $\frac{1}{2}x - \frac{1}{3}y = 1$
20. $-\frac{2}{3}x + \frac{1}{2}y = 2$
21. $3x + 2y = 1$
22. $5x + 3y = 1$
23. On the same set of axes, graph the three lines, where $y = \frac{3}{2}x + b$ and $b = \{-2, 0, 2\}$.
24. On the same set of axes, graph the three lines, where $y = mx + 1$ and $m = \{-\frac{1}{2}, 0, \frac{1}{2}\}$.

Answer

1.

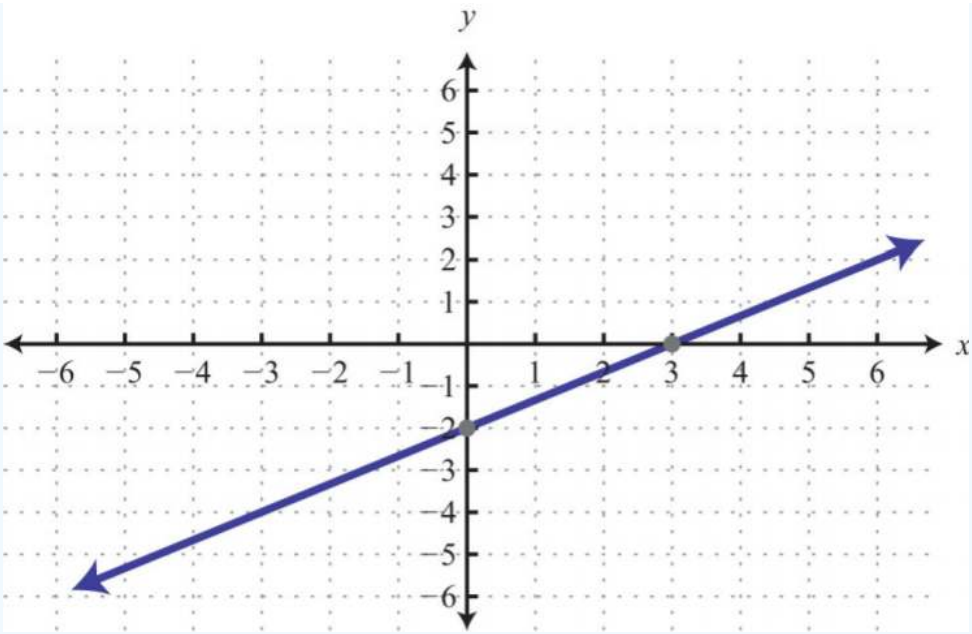


Figure 1.2.4.30

3.

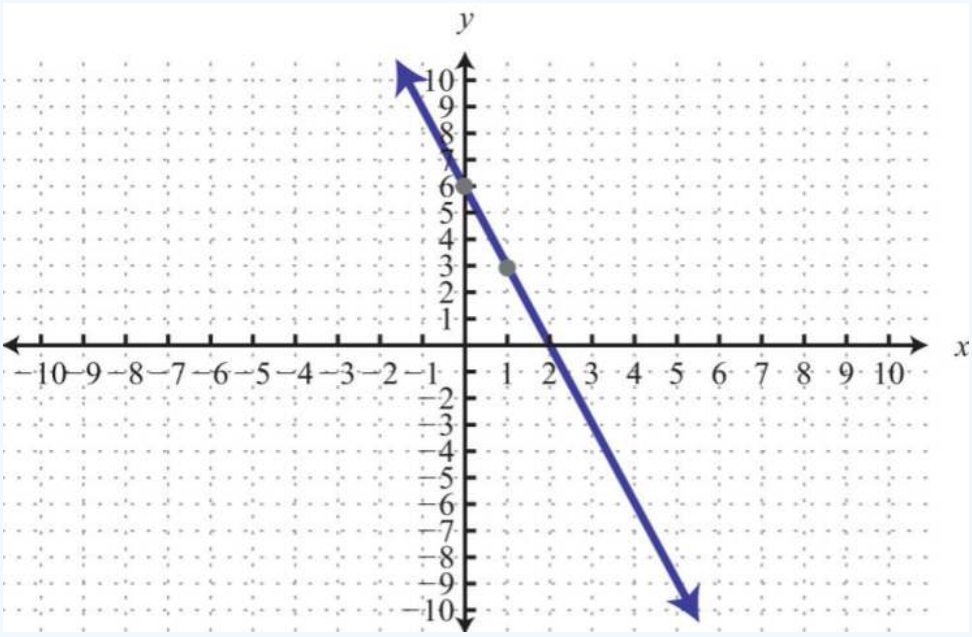


Figure 1.2.4.31

5.

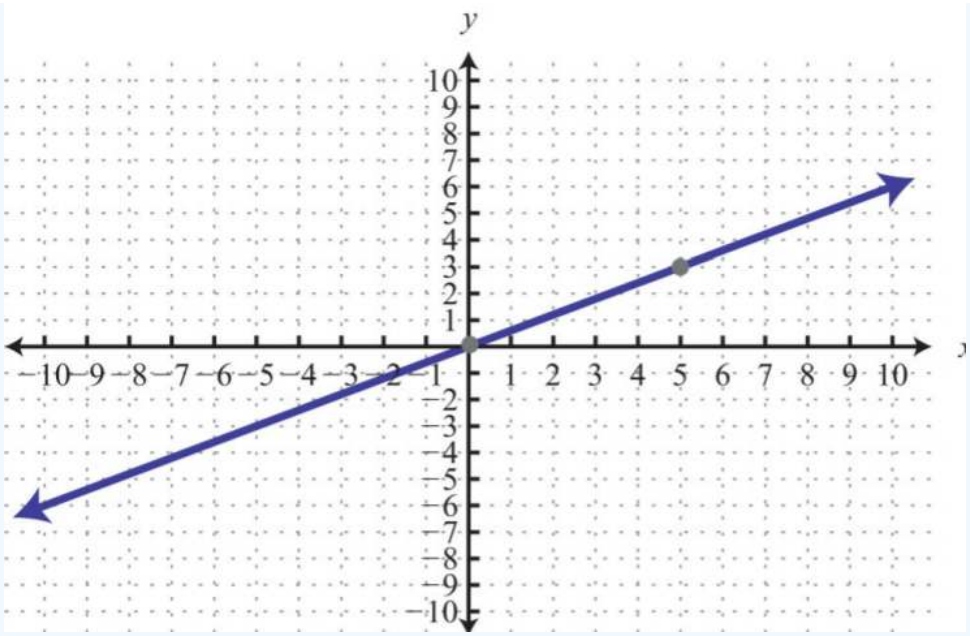


Figure 1.2.4.32

7.

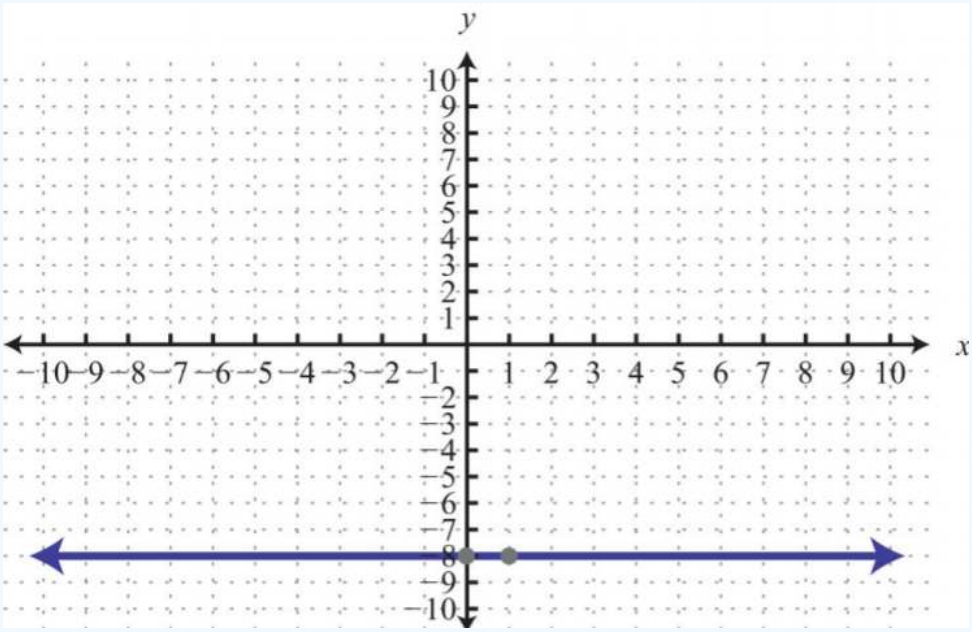


Figure 1.2.4.33

9.

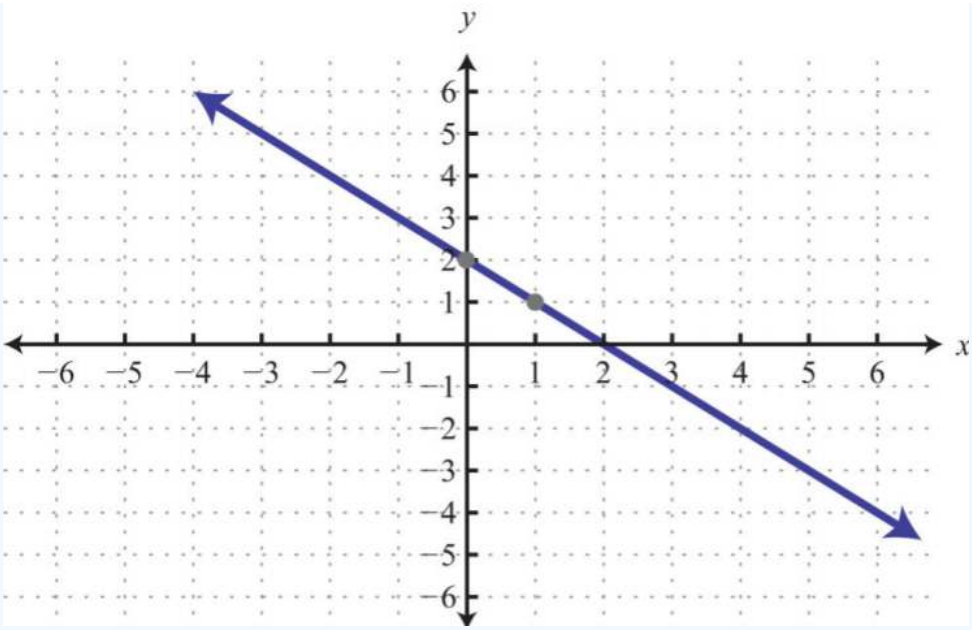


Figure 1.2.4.34

11.

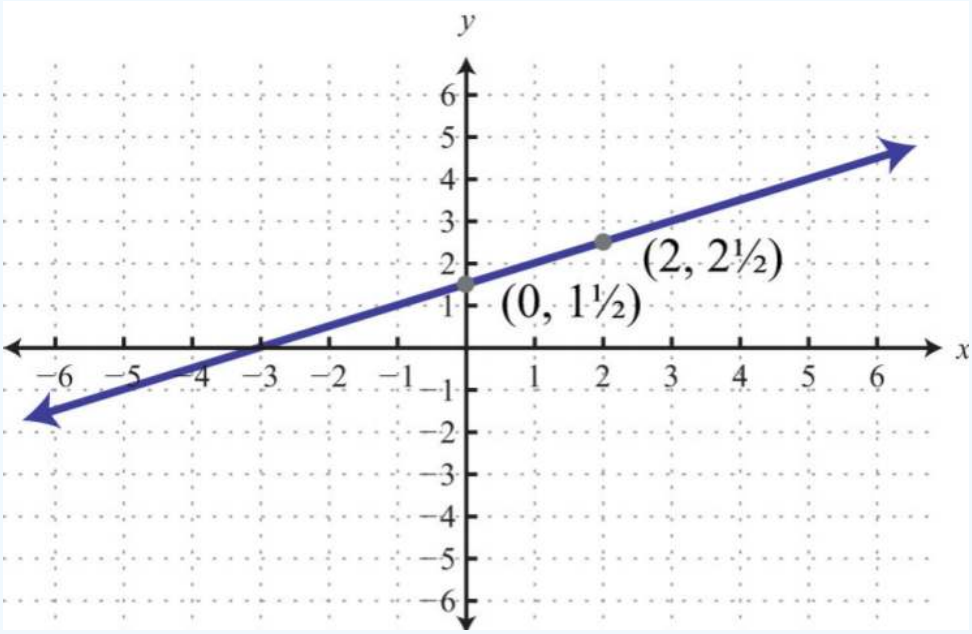


Figure 1.2.4.35

13.

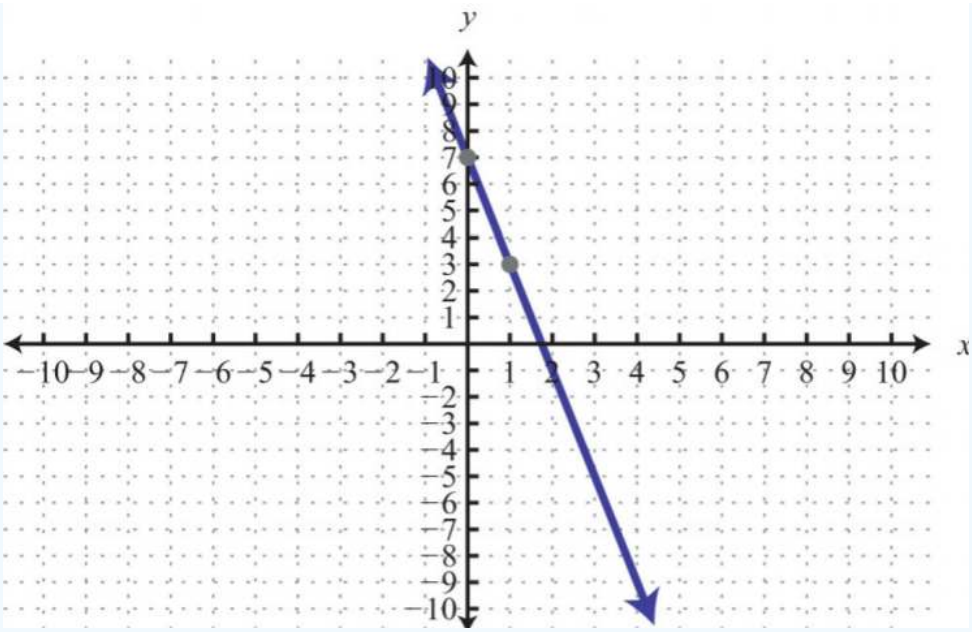


Figure 1.2.4.36

15.

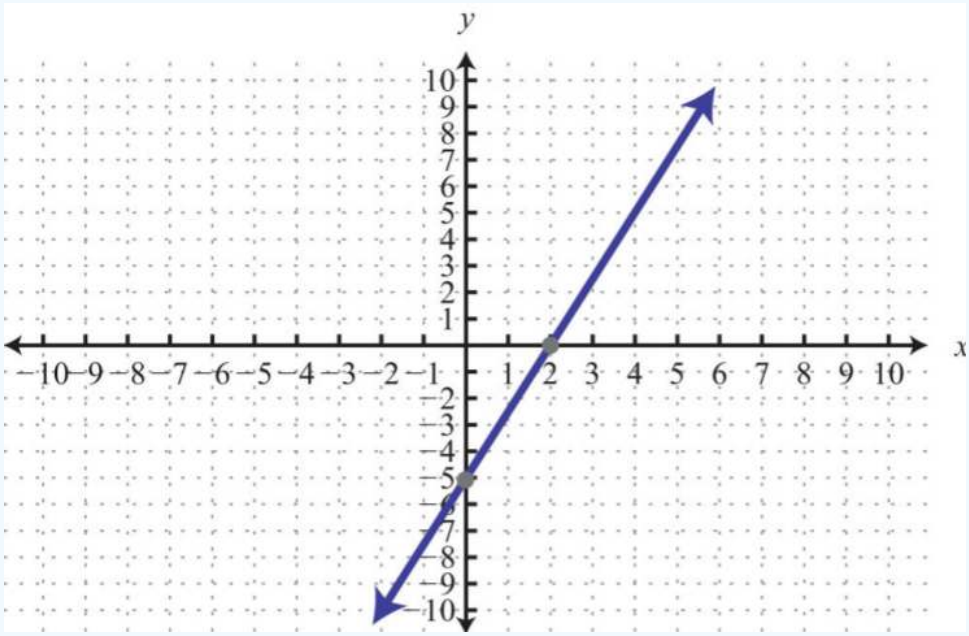


Figure 1.2.4.37

17.

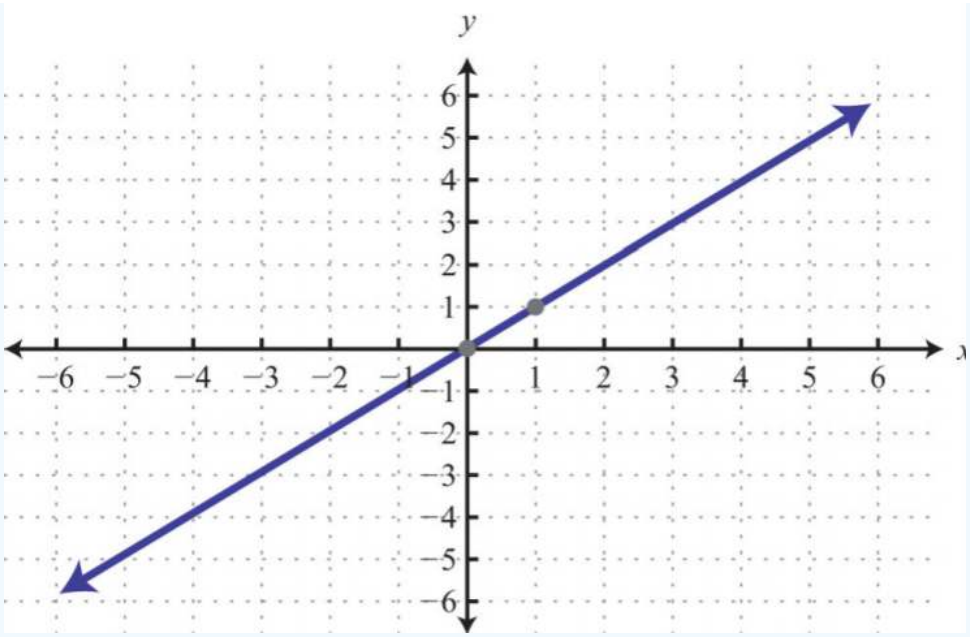


Figure 1.2.4.38

19.

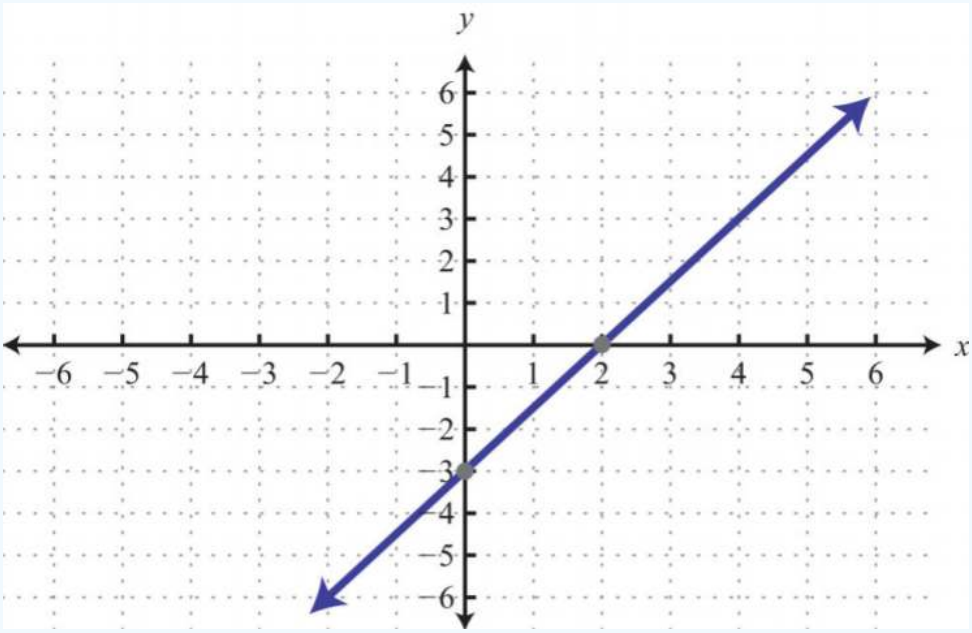


Figure 1.2.4.39

21.

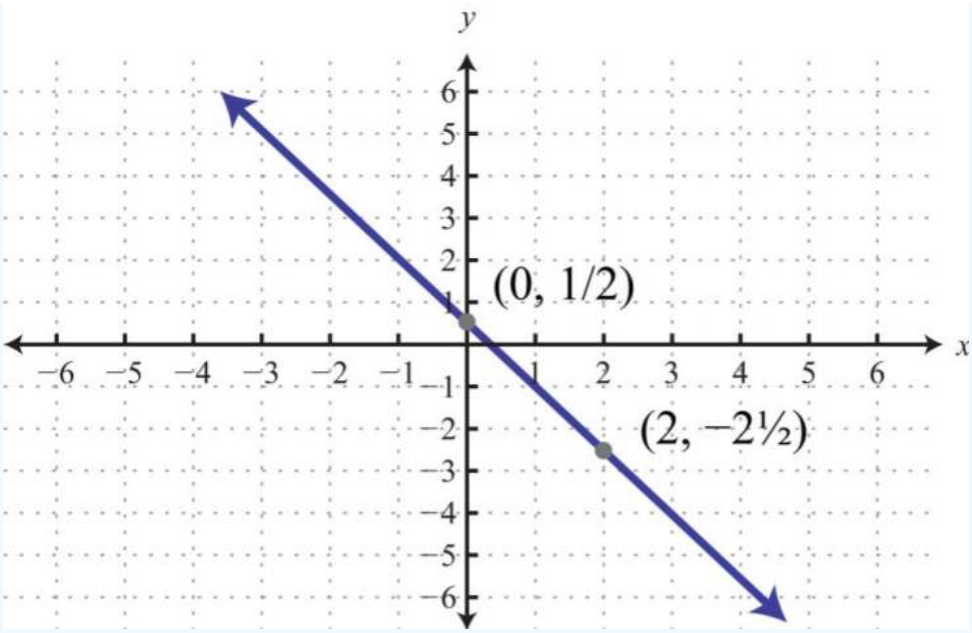


Figure 1.2.4.40

23.

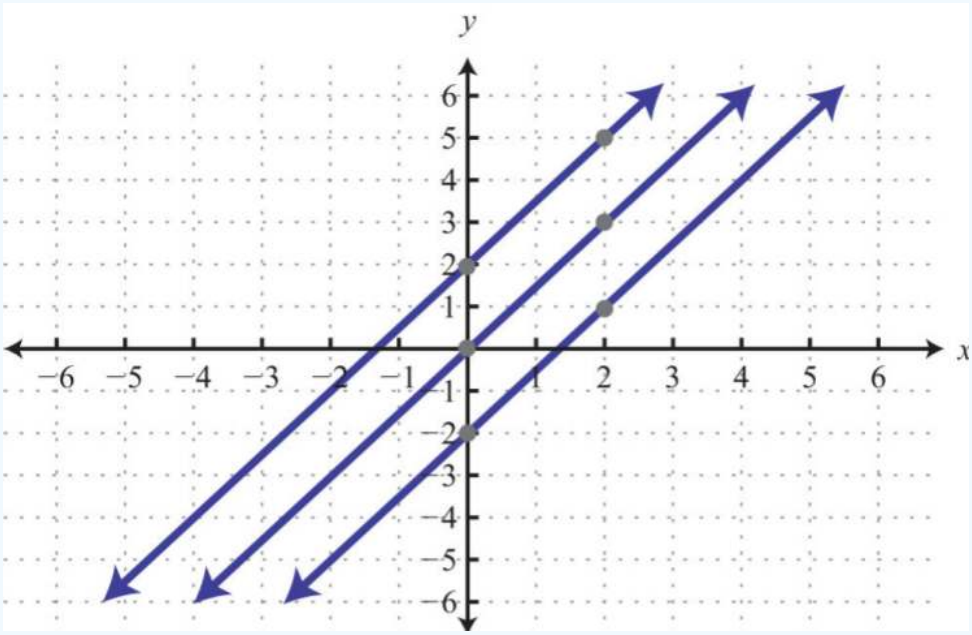


Figure 1.2.4.41

? EXERCISE 1.2.4.9 DISCUSSION BOARD TOPICS

1. Name three methods for graphing lines. Discuss the pros and cons of each method.
2. Choose a linear equation and graph it three different ways. Scan the work and share it on the discussion board.
3. Why do we use the letter m for slope?
4. How are equivalent fractions useful when working with slopes?
5. Can we graph a line knowing only its slope?
6. Research and discuss the alternative notation for slope: $m = \frac{\Delta y}{\Delta x}$.
7. What strategies for graphing lines should be brought to an exam? Explain

Answer

1. Answers may vary
3. Answers may vary
5. Answers may vary
7. Answers may vary

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1.2.E: REVIEW EXERCISES AND SAMPLE EXAM

REVIEW EXERCISES

? EXERCISE 1.2.E.1 RECTANGULAR COORDINATE SYSTEM

Graph the given set of ordered pairs.

1. $\{(-3,4),(-4,0),(0,3),(2,4)\}$
2. $\{(-5,5),(-3,-1),(0,0),(3,2)\}$
3. Graph the points $(-3,5),(-3,-3)$, and $(3,-3)$ on a rectangular coordinate plane. Connect the points and calculate the area of the shape.
4. Graph the points $(-4,1),(0,1),(0,-2)$, and $(-4,-2)$ on a rectangular coordinate plane. Connect the points and calculate the area of the shape.
5. Graph the points $(1,0),(4,0),(1,-5)$, and $(4,-5)$ on a rectangular coordinate plane. Connect the points and calculate the perimeter of the shape.
6. Graph the points $(-5,2),(-5,-3),(1,2)$, and $(1,-3)$ on a rectangular coordinate plane. Connect the points and calculate the perimeter of the shape.

Answer

1.

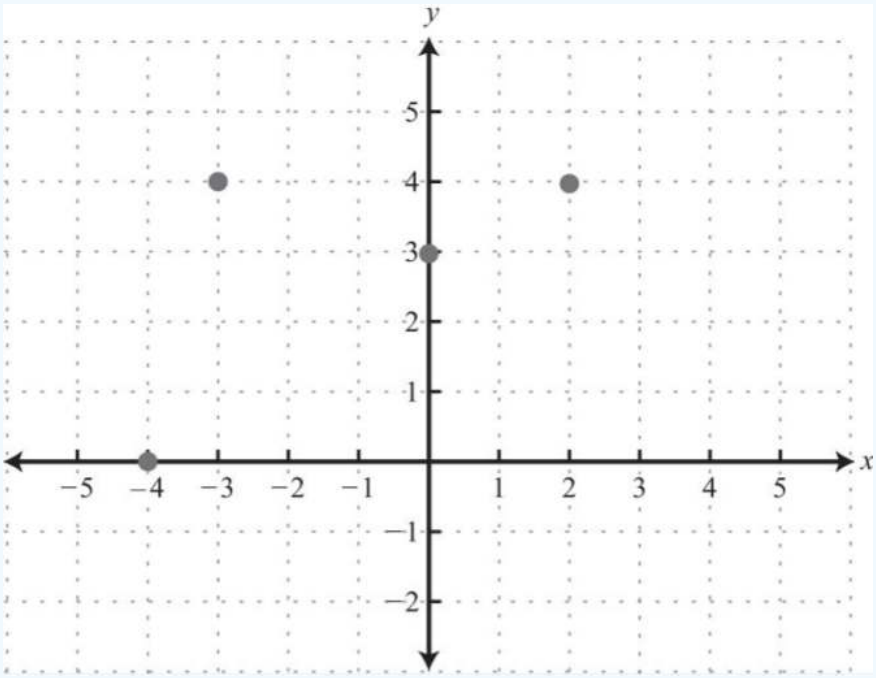


Figure 3.E.1

3. Area: 24 square units

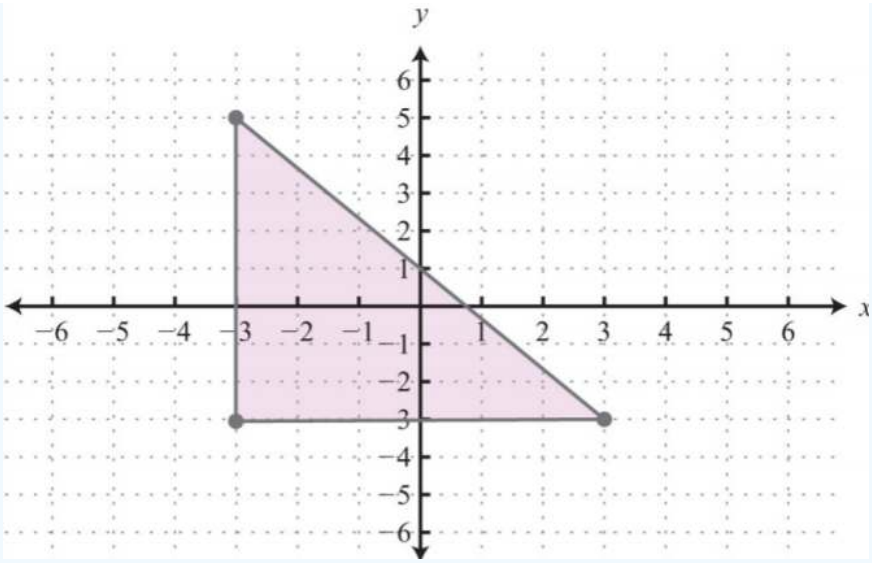


Figure 3.E.2

5. Perimeter: 16 units

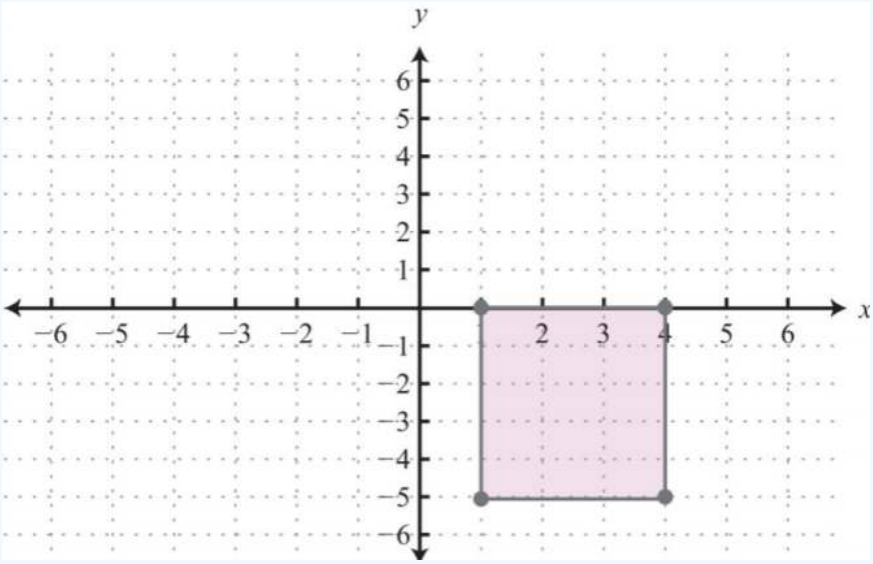


Figure 3.E.3

? EXERCISE 1.2.E.2 RECTANGULAR COORDINATE SYSTEM

Calculate the distance between the given two points.

1. $(-1,-2)$ and $(5,6)$
2. $(2,-5)$ and $(-2,-2)$
3. $(-9,-3)$ and $(-8,4)$
4. $(-1,3)$ and $(1,-3)$

Answer

1. 10 units
3. $5\sqrt{2}$ units

EXERCISE 1.2.E.3 RECTANGULAR COORDINATE SYSTEM

Calculate the midpoint between the given points.

1. $(-1,3)$ and $(5,-7)$
2. $(6,-3)$ and $(-8,-11)$
3. $(7,-2)$ and $(-6,-1)$
4. $(-6,0)$ and $(0,0)$
5. Show algebraically that the points $(-1,-1),(1,-3)$, and $(2,0)$ form an isosceles triangle.
6. Show algebraically that the points $(2,-1),(6,1)$, and $(5,3)$ form a right triangle.

Answer

1. $(2,-2)$
3. $(\frac{1}{2},-\frac{3}{2})$
5. Answers may vary

EXERCISE 1.2.E.4 GRAPH BY PLOTTING POINTS

Determine whether the given point is a solution.

1. $-5x+2y=7;\ (1,-1)$
2. $6x-5y=4;\ (-1,-2)$
3. $y=\frac{3}{4}x+1;\ (-\frac{2}{3},\frac{1}{2})$
4. $y=-\frac{3}{5}x-2;\ (10,-8)$

Answer

1. No
3. Yes

EXERCISE 1.2.E.5 GRAPH BY PLOTTING POINTS

Find at least five ordered pair solutions and graph.

1. $y=-x+2$
2. $y=2x-3$
3. $y=\frac{1}{2}x-2$
4. $y=-\frac{2}{3}x$
5. $y=3$
6. $x=-3$
7. $x-5y=15$
8. $2x-3y=12$

Answer

1.

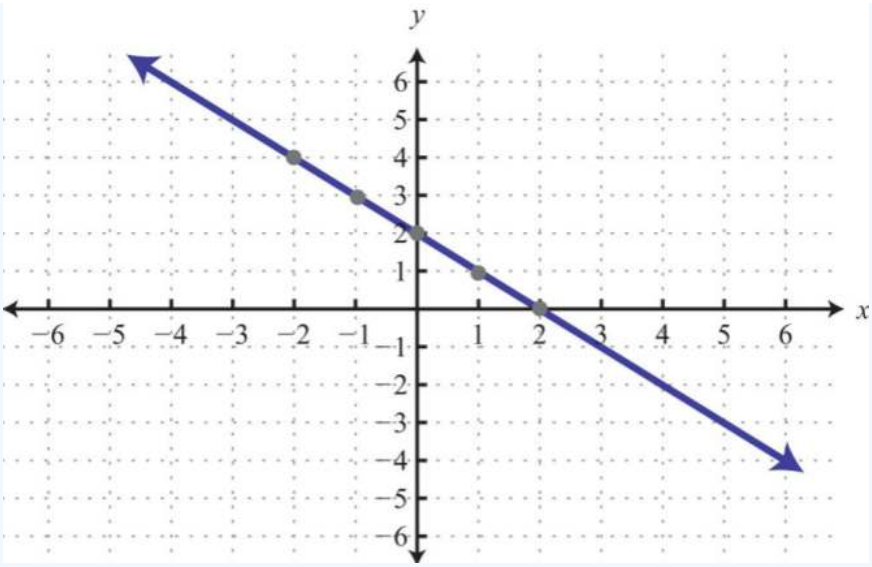


Figure 3.E.4

3.

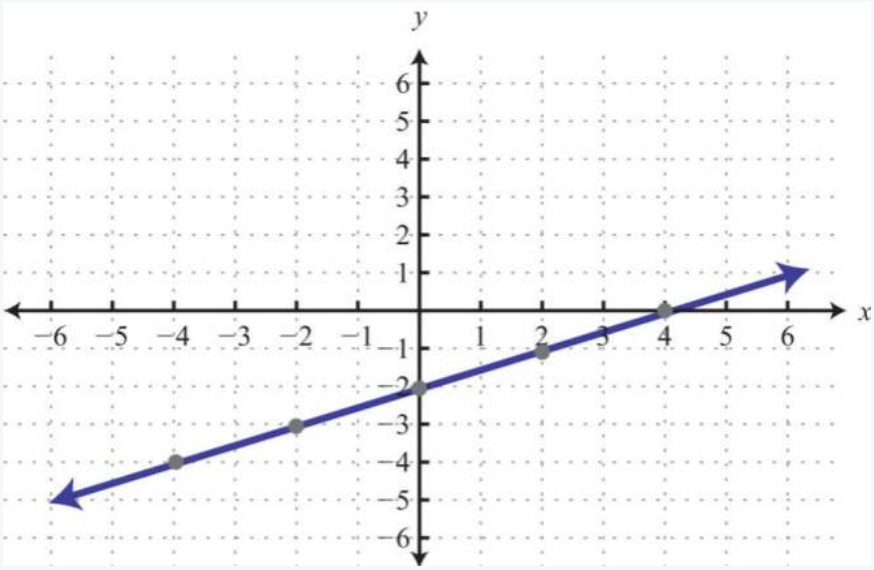


Figure 3.E.5

5.

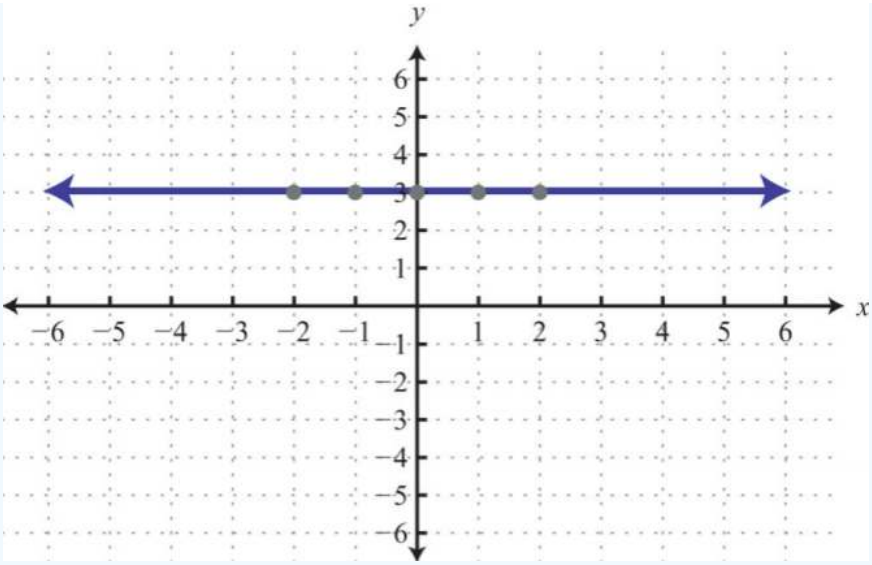


Figure 3.E.6

7.

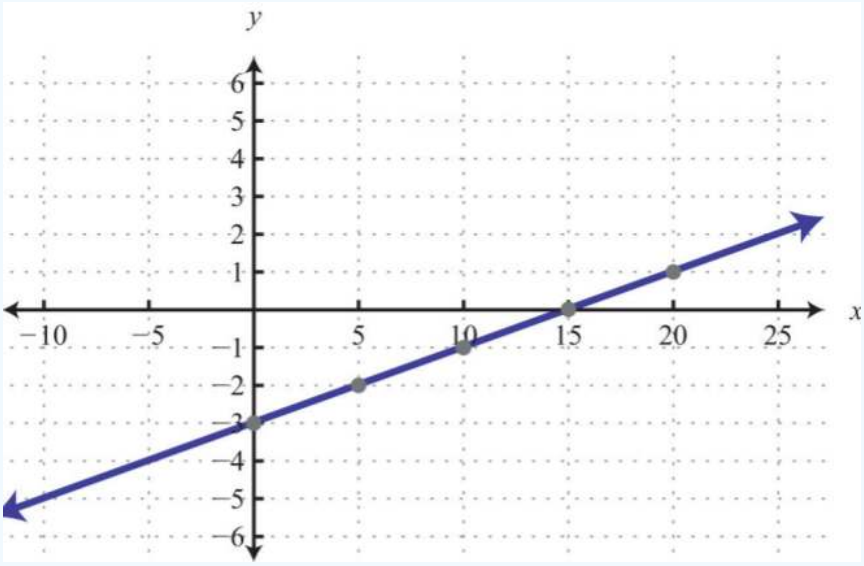


Figure 3.E.7

? EXERCISE 1.2.E. 6 GRAPH USING INTERCEPTS

Given the graph, find the x- and y- intercepts.

1.

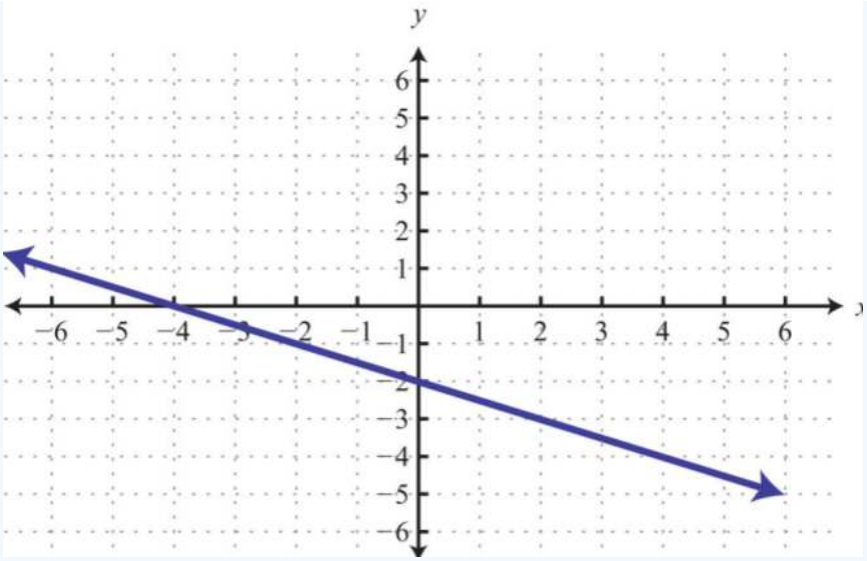


Figure 3.E.8

2.

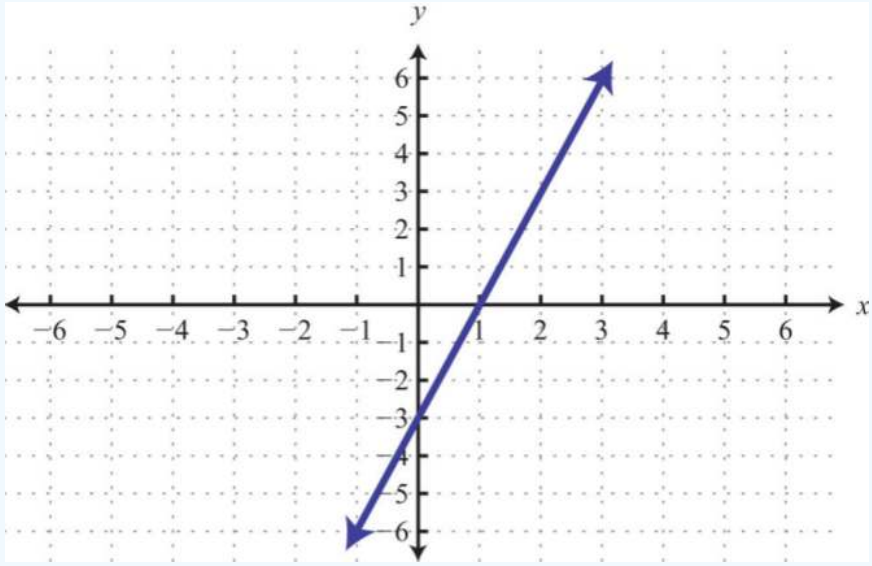


Figure 3.E.9

3.

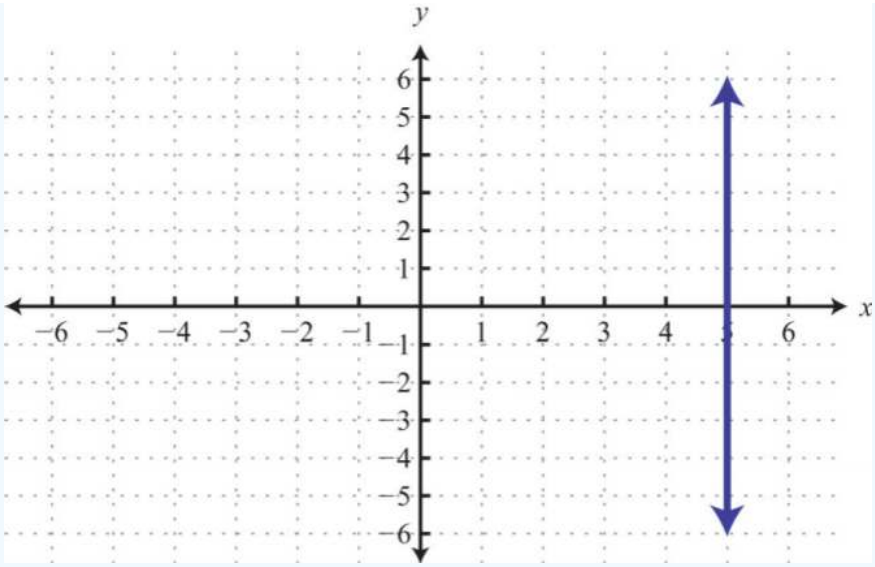


Figure 3.E.10

4.

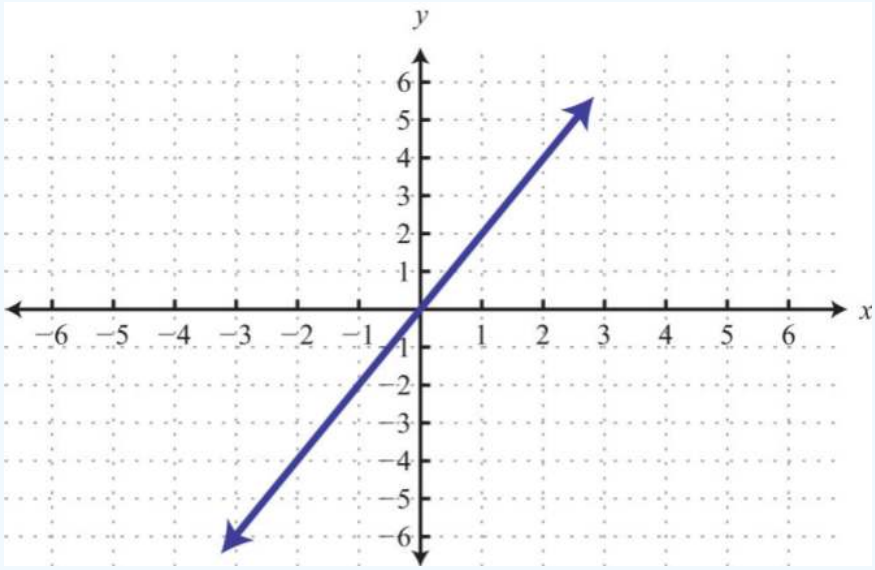


Figure 3.E.11

Answer

1. y-intercept: $(0,-2)$; x-intercept: $(-4,0)$
3. y-intercept: none; x-intercept: $(5,0)$

? EXERCISE 1.2.E.7 GRAPH USING INTERCEPTS

Find the intercepts and graph them.

1. $3x-4y=12$
2. $2x-y=-4$
3. $\frac{1}{2}x-\frac{1}{3}y=1$
4. $-\frac{1}{2}x+\frac{2}{3}y=2$
5. $y=-\frac{5}{3}x+5$
6. $y=-3x+4$

Answer

1.

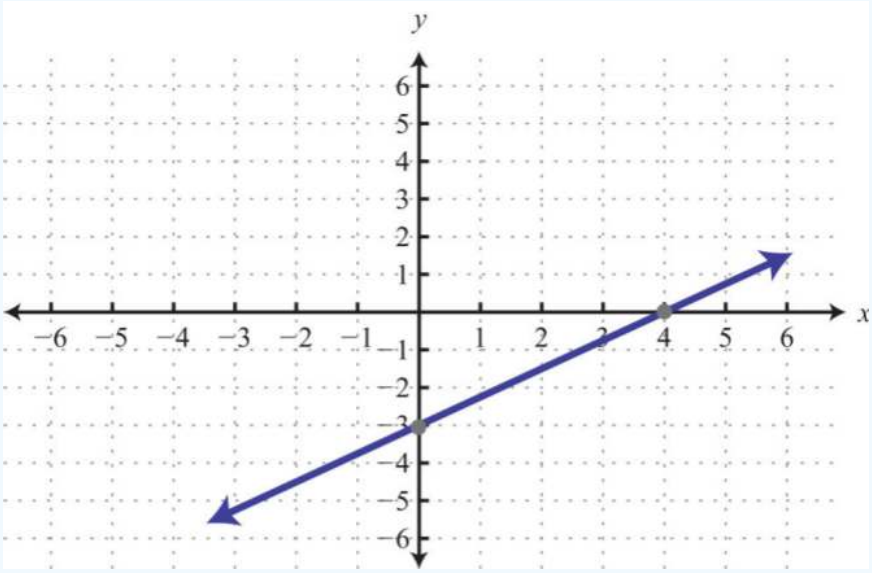


Figure 3.E.12

3.

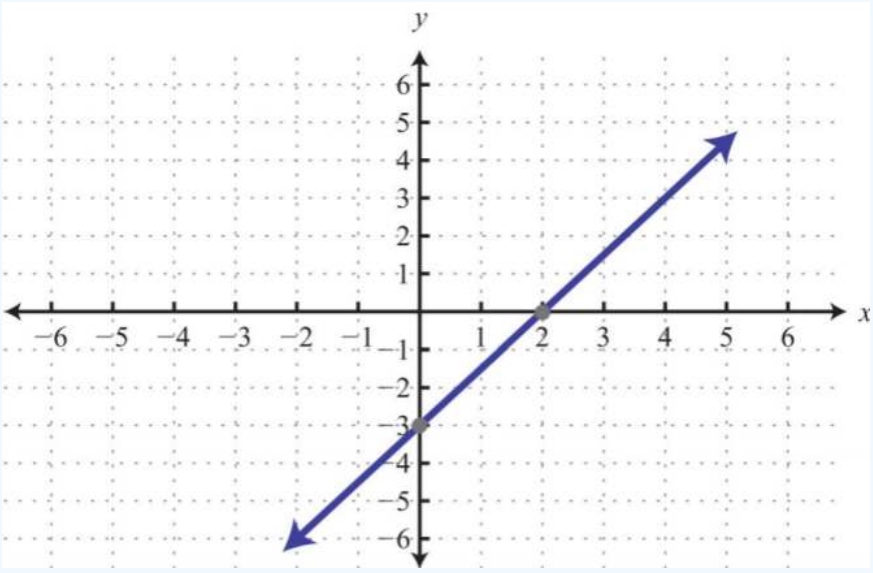


Figure 3.E.13

5.

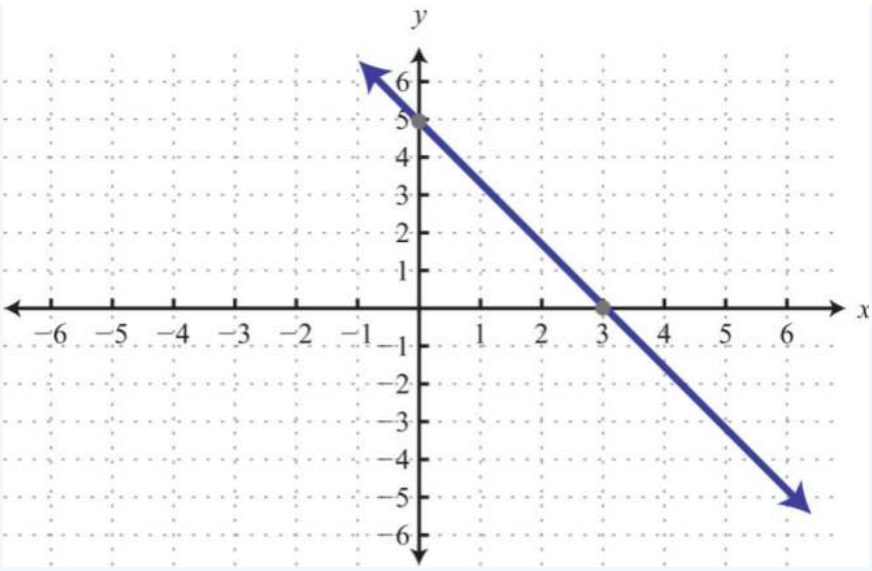


Figure 3.E.14

? EXERCISE 1.2.E. 8 GRAPH USING THE y-INTERCEPT AND SLOPE

Given the graph, determine the slope and y-intercept.

1.

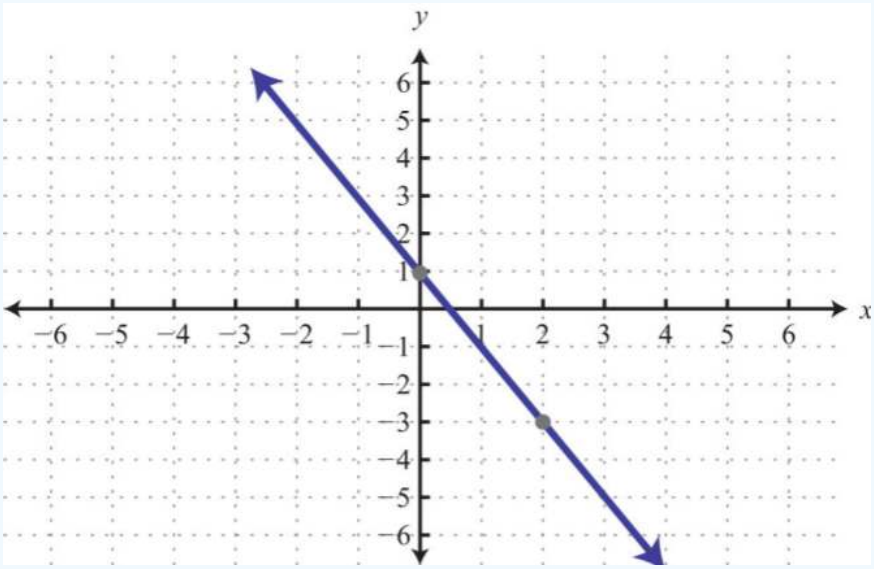


Figure 3.E.15

2.

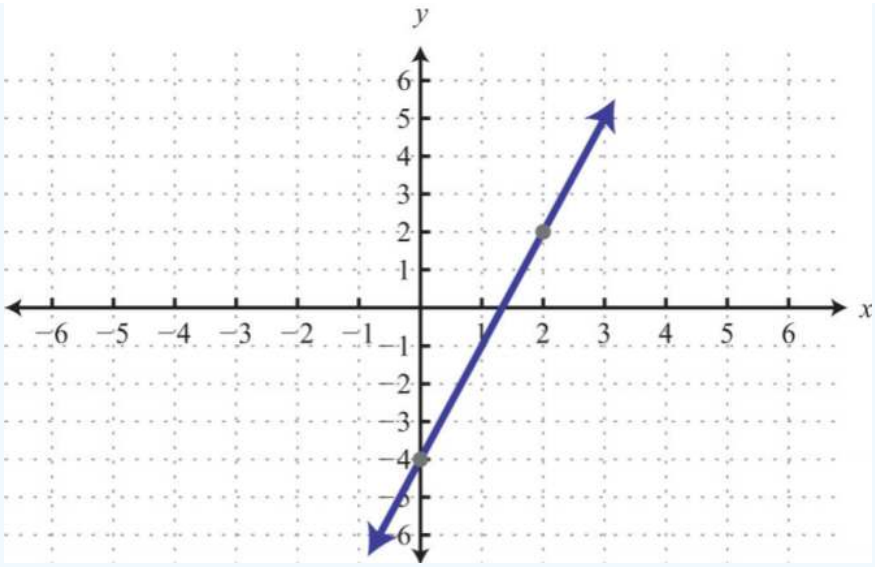


Figure 3.E.16

Answer

1. y-intercept: $(0,1)$; slope: -2

EXERCISE $1.2.E.9$ GRAPH USING THE y-INTERCEPT AND SLOPE

Determine the slope, given two points.

1. $(-3,8)$ and $(5,-6)$
2. $(0,-5)$ and $(-6,3)$
3. $(\frac{1}{2},-\frac{2}{3})$ and $(\frac{1}{4},-\frac{1}{3})$
4. $(5,-\frac{3}{4})$ and $(2,-\frac{3}{4})$

Answer

1. $-\frac{7}{4}$
3. $-\frac{4}{3}$

EXERCISE $1.2.E.10$ GRAPH USING THE y-INTERCEPT AND SLOPE

Express in slope-intercept form and identify the slope and y-intercept.

1. $12x-4y=8$
2. $3x-6y=24$
3. $-\frac{1}{3}x+\frac{3}{4}y=1$
4. $-5x+3y=0$

Answer

1. $y=3x-2$; slope: 3; y-intercept $(0,-2)$
3. $y=\frac{4}{9}x+\frac{4}{3}$; slope: $\frac{4}{9}$; y-intercept $(0,\frac{4}{3})$

EXERCISE $1.2.E.11$ GRAPH USING THE y-INTERCEPT AND SLOPE

1. $y=-x+3$
2. $y=4x-1$
3. $y=-2x$
4. $y=-\frac{5}{2}x+3$

5. $2x - 3y = 9$
6. $2x + \frac{3}{2}y = 3$
7. $y = 0$
8. $x - 4y = 0$

Answer

1.

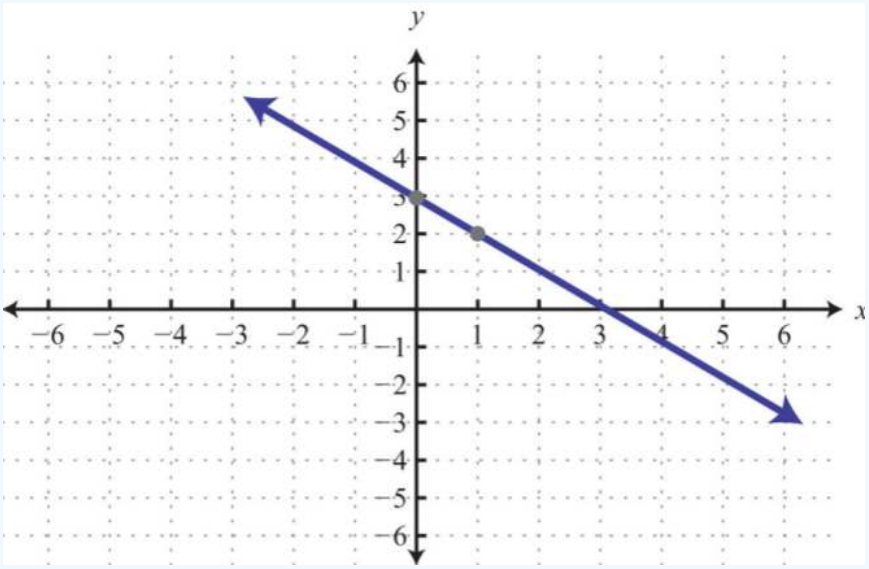


Figure 3.E.17

3.

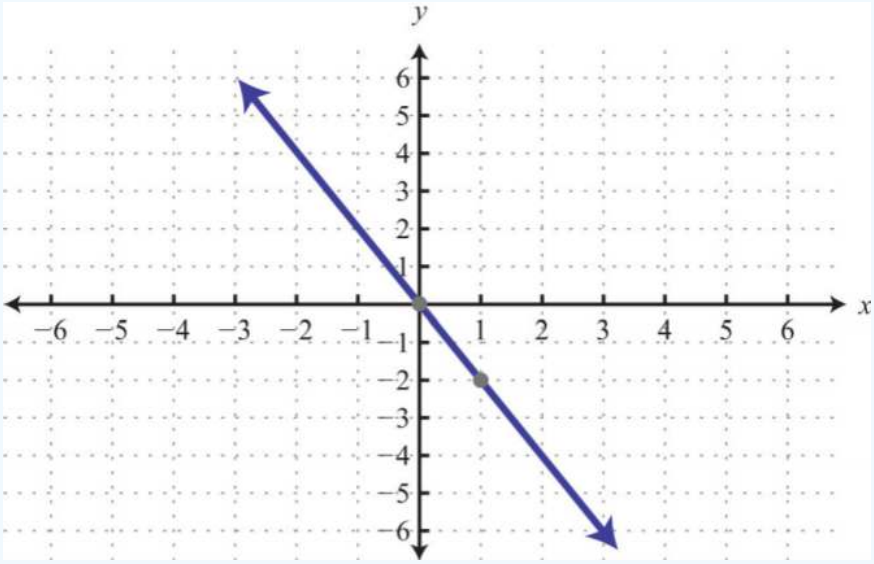


Figure 3.E.18

5.

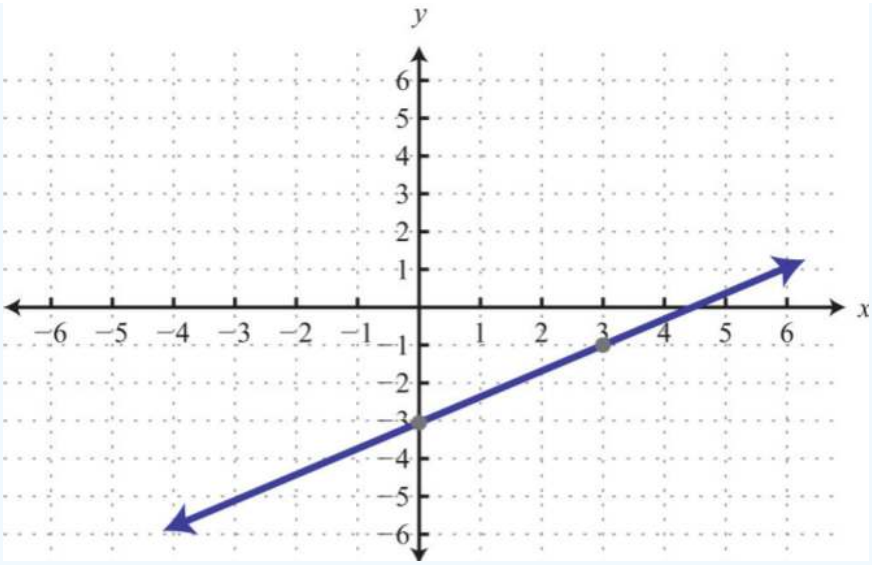


Figure 3.E.19

7.

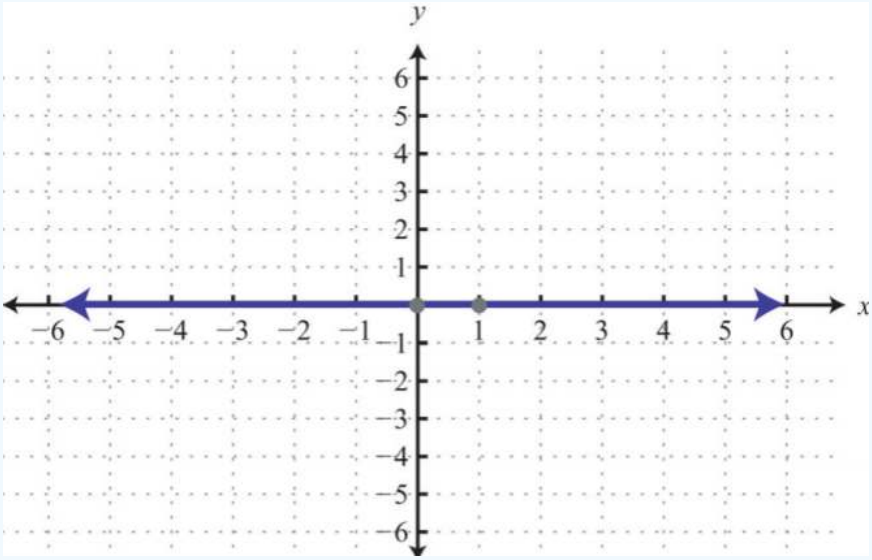


Figure 3.E.20

? EXERCISE 1.2.E. 12 FINDING LINEAR EQUATIONS

Given the graph, determine the equation of the line.

1.

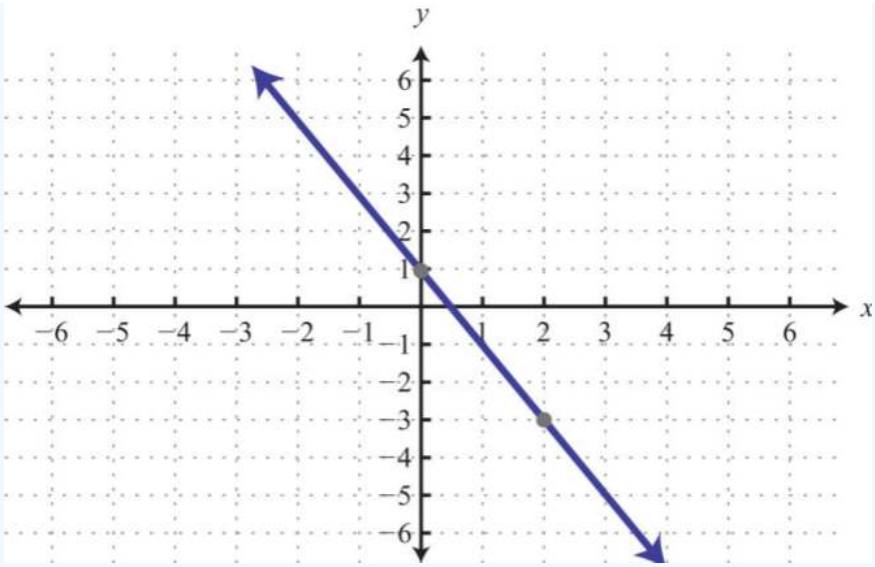


Figure 3.E.21

2.

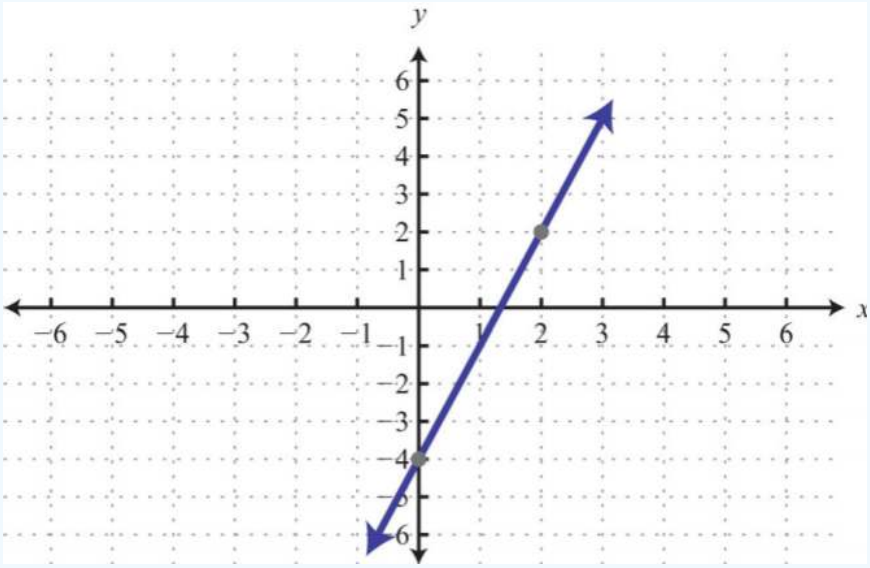


Figure 3.E.22

3.

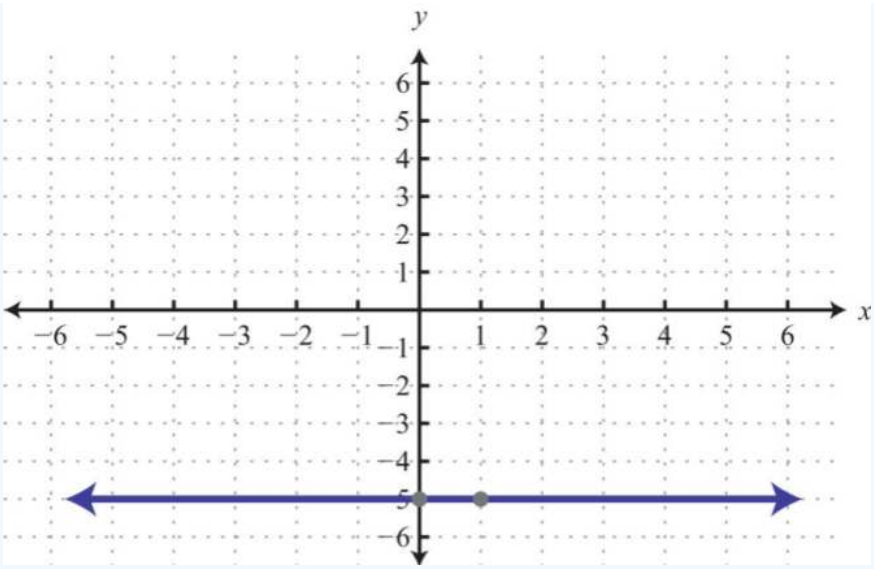


Figure 3.E.23

4.

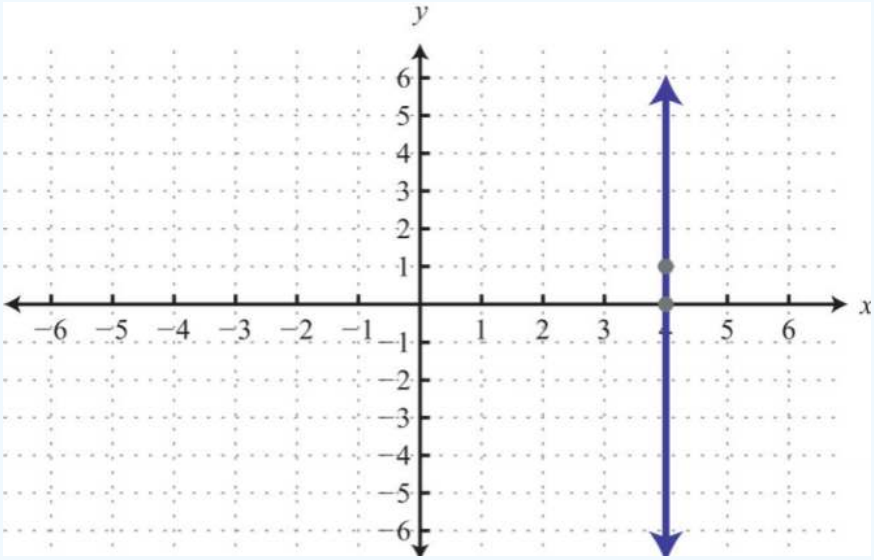


Figure 3.E.24

Answer

1. $y = -2x + 1$
3. $y = -5$

? EXERCISE 1.2.E. 13 FINDING LINEAR EQUATIONS

Find the equation of a line, given the slope and a point on the line.

1. $m = \frac{1}{2}; (-4, 8)$
2. $m = -\frac{1}{5}; (-5, -9)$
3. $m = \frac{2}{3}; (1, -2)$
4. $m = -\frac{3}{4}; (2, -3)$

Answer

1. $y = \frac{1}{2}x + 10$

3. $y = \frac{2}{3}x - \frac{8}{3}$

EXERCISE 1.2.E.14 FINDING LINEAR EQUATIONS

Find the equation of the line given two points on the line.

1. $(-5, -5)$ and $(10, 7)$
2. $(-6, 12)$ and $(3, -3)$
3. $(2, -1)$ and $(-2, 2)$
4. $(\frac{5}{2}, -2)$ and $(-5, \frac{5}{2})$
5. $(7, -6)$ and $(3, -6)$
6. $(10, 1)$ and $(10, -3)$

Answer

1. $y = \frac{4}{5}x - 1$
3. $y = -\frac{3}{4}x + \frac{1}{2}$
5. $y = -6$

EXERCISE 1.2.E.15 PARALLEL AND PERPENDICULAR LINES

Determine if the lines are parallel, perpendicular, or neither.

1. $\begin{cases} -3x + 7y = 14 \\ 6x - 14y = 42 \end{cases}$
2. $\begin{cases} 2x + 3y = 18 \\ 2x - 3y = 36 \end{cases}$
3. $\begin{cases} x + 4y = 2 \\ 8x - 2y = -1 \end{cases}$
4. $\begin{cases} y = 2 \\ x = 2 \end{cases}$

Answer

1. Parallel
3. Perpendicular

EXERCISE 1.2.E.16 PARALLEL AND PERPENDICULAR LINES

Find the equation of the line in slope-intercept form.

1. Parallel to $5x - y = 15$ and passing through $(-10, -1)$.
2. Parallel to $x - 3y = 1$ and passing through $(2, -2)$.
3. Perpendicular to $8x - 6y = 4$ and passing through $(8, -1)$.
4. Perpendicular to $7x + y = 14$ and passing through $(5, 1)$.
5. Parallel to $y = 1$ and passing through $(4, -1)$.
6. Perpendicular to $y = 1$ and passing through $(4, -1)$.

Answer

1. $y = 5x + 49$
3. $y = -\frac{3}{4}x + 5$
5. $y = -1$

EXERCISE 1.2.E.17 INTRODUCTION TO FUNCTIONS

Determine the domain and range and state whether it is a function or not.

1. $\{(-10,-1),(-5,2),(5,2)\}$
2. $\{(-12,4),(-1,-3),(-1,-2)\}$
3.

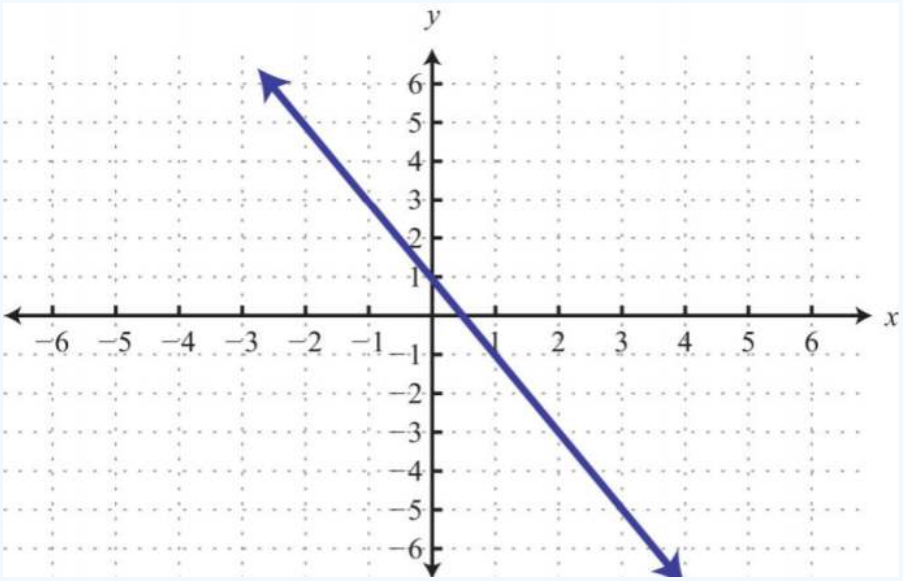


Figure 3.E.25

4.

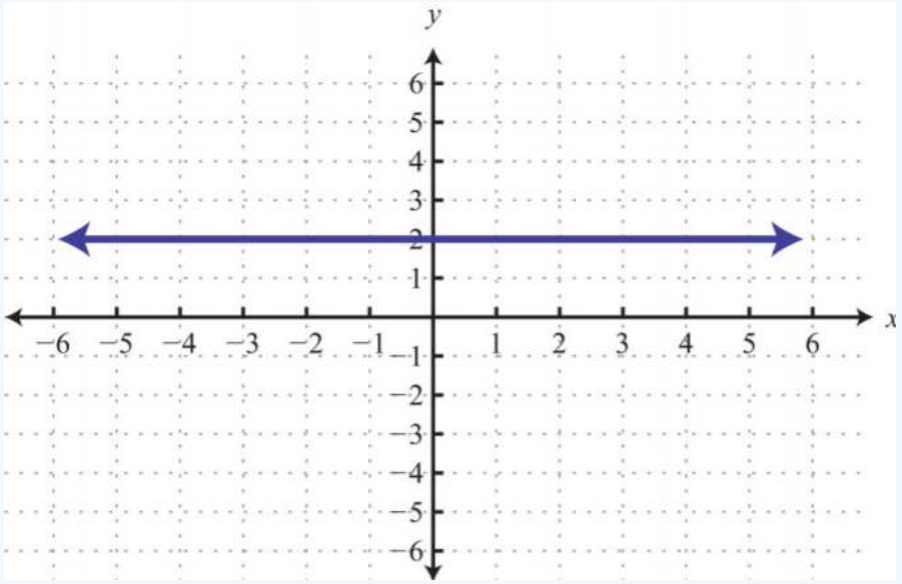


Figure 3.E.26

5.

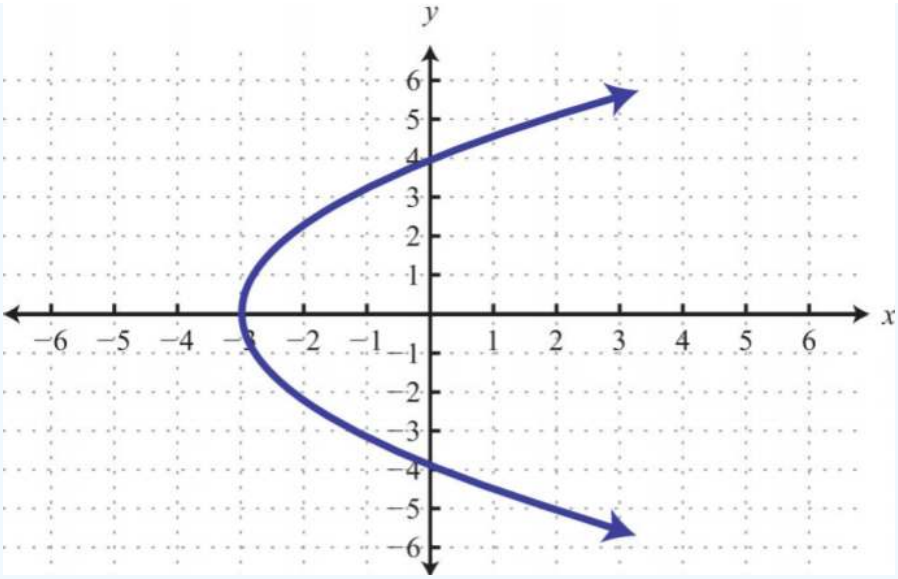


Figure 3.E.27

6.

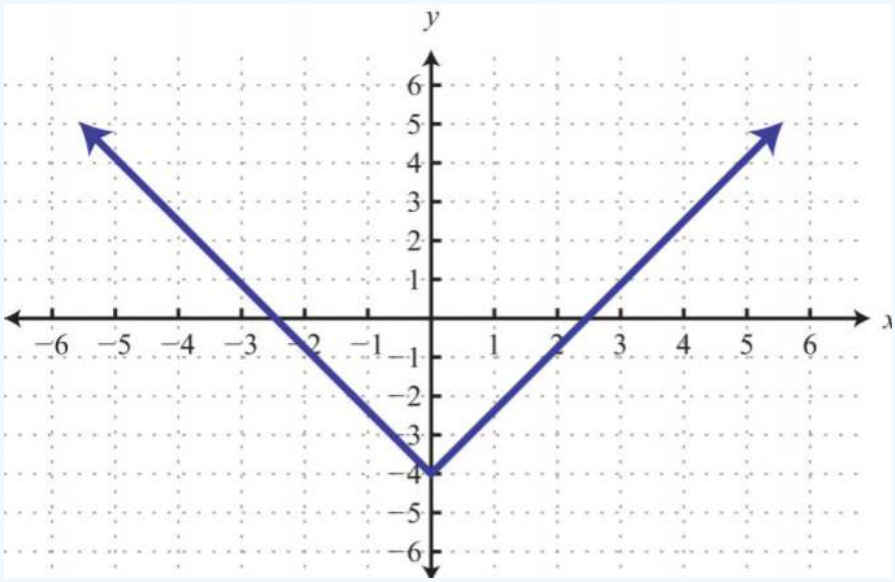


Figure 3.E.28

Answer

1. Domain: $\{-10,-5,5\}$; range: $\{-1,2\}$; function: yes
3. Domain: R; range: R; function: yes
5. Domain: $[-3,\infty)$; range: R; function: no

? EXERCISE 1.2.E.18 INTRODUCTION TO FUNCTIONS

Given the following,

1. $f(x)=9x-4$, find $f(-1)$.
2. $f(x)=-5x+1$, find $f(-3)$.
3. $g(x)=\frac{1}{2}x-\frac{1}{3}$, find $g(-\frac{1}{3})$.
4. $g(x)=-\frac{3}{4}x+\frac{1}{3}$, find $g(\frac{2}{3})$.
5. $f(x)=9x-4$, find x when $f(x)=0$.

6. $f(x) = -5x + 1$, find x when $f(x) = 2$.
7. $g(x) = \frac{1}{2}x - \frac{1}{3}$, find x when $g(x) = 1$.
8. $g(x) = -\frac{3}{4}x + \frac{1}{3}$, find x when $g(x) = -1$.

Answer

1. $f(-1) = -13$
3. $g(-\frac{1}{3}) = -\frac{1}{2}$
5. $x = \frac{4}{9}$
7. $x = \frac{8}{3}$

EXERCISE 1.2.E. 19 INTRODUCTION TO FUNCTIONS

Given the graph of a function $f(x)$, determine the following.

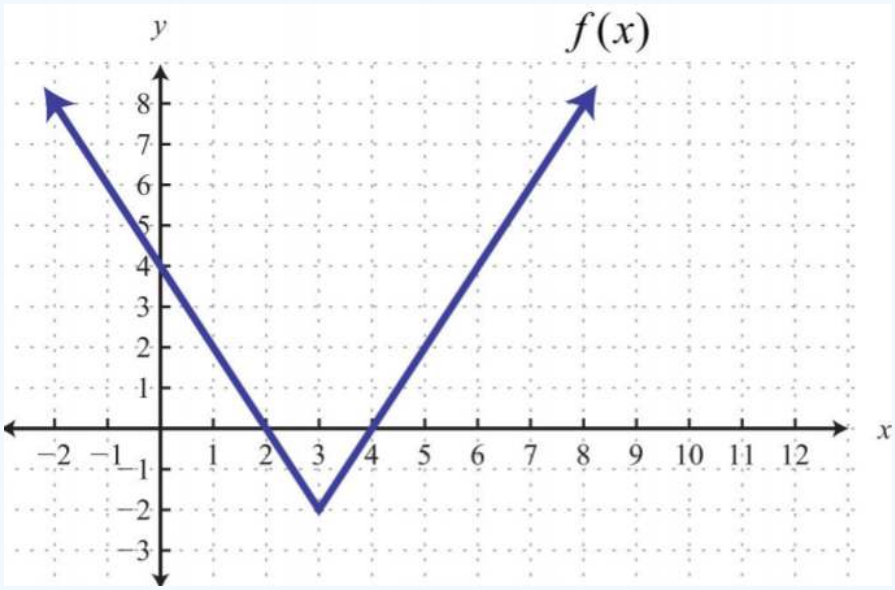


Figure 3.E.29

1. $f(3)$
2. x when $f(x) = 4$

Answer

1. $f(3) = -2$

EXERCISE 1.2.E. 20 LINEAR INEQUALITIES (TWO VARIABLES)

Is the ordered pair a solution to the given inequality?

1. $6x - 2y \leq 1$; $(-3, -7)$
2. $-3x + y > 2$; $(0, 2)$
3. $6x - 10y < -1$; $(5, -3)$
4. $x - \frac{1}{3}y > 0$; $(1, 4)$
5. $y > 0$; $(-3, -1)$
6. $x \leq -5$; $(-6, 4)$

Answer

1. Yes
3. No
5. Yes

EXERCISE 1.2.E. 21 LINEAR INEQUALITIES (TWO VARIABLES)

Graph the solution set.

1. $y \geq -2x + 1$
2. $y < 3x - 4$
3. $-x + y \leq 3$
4. $\frac{5}{2}x + \frac{1}{2}y \leq 2$
5. $3x - 5y > 0$
6. $y > 0$

Answer

1.

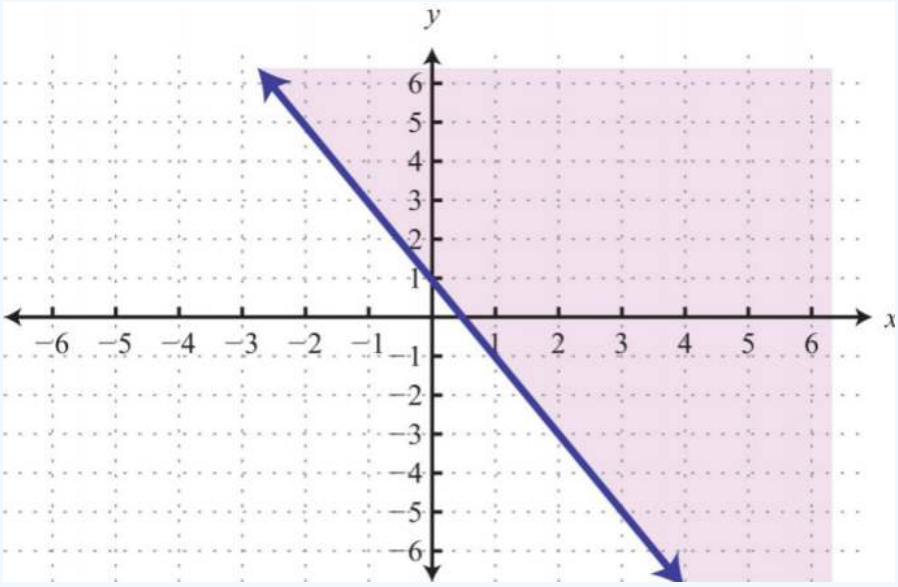


Figure 3.E.30

3.

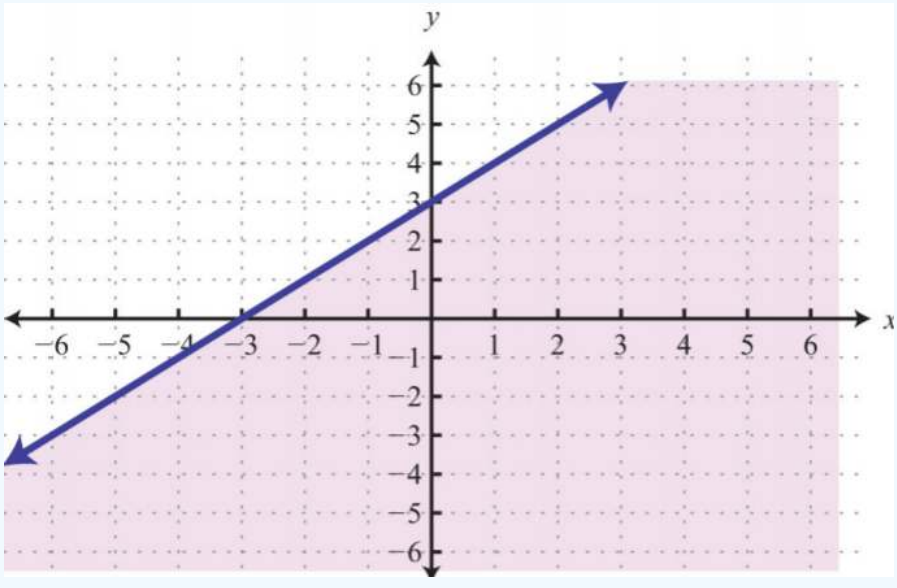


Figure 3.E.31

5.

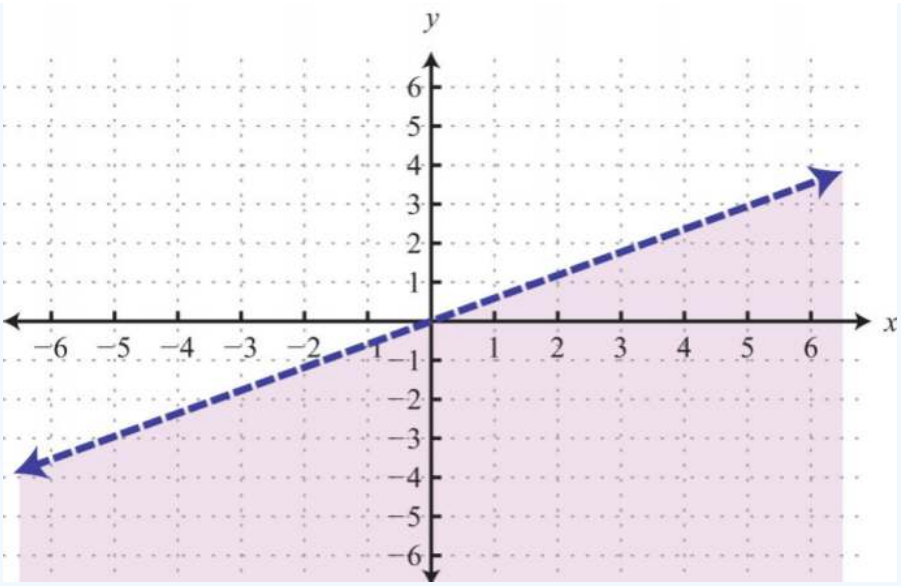


Figure 3.E.32

SAMPLE EXAM

? EXERCISE $1.2.E.22$

1. Graph the points $(-4,-2),(-4,1)$, and $(0,-2)$ on a rectangular coordinate plane. Connect the points and calculate the area of the shape.
2. Is $(-2,4)$ a solution to $3x-4y=-10$? Justify your answer.

Answer

1. Area: 6 square units

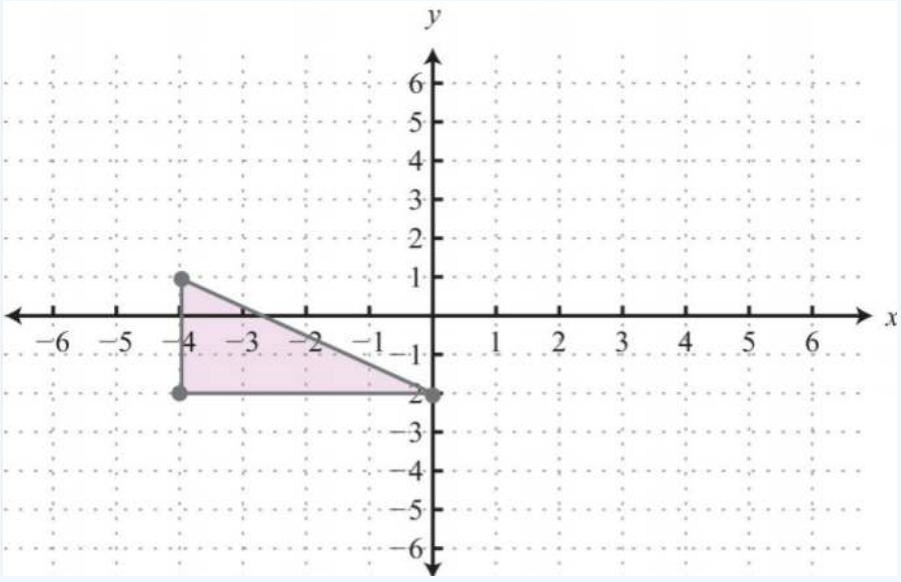


Figure 3.E.33

EXERCISE $1.2.E.23$

Given the set of x-values $\{-2,-1,0,1,2\}$, find the corresponding y-values and graph the following.

1. $y=x-1$
2. $y=-x+1$
3. On the same set of axes, graph $y=4$ and $x=-3$. Give the point where they intersect.

Answer

1.

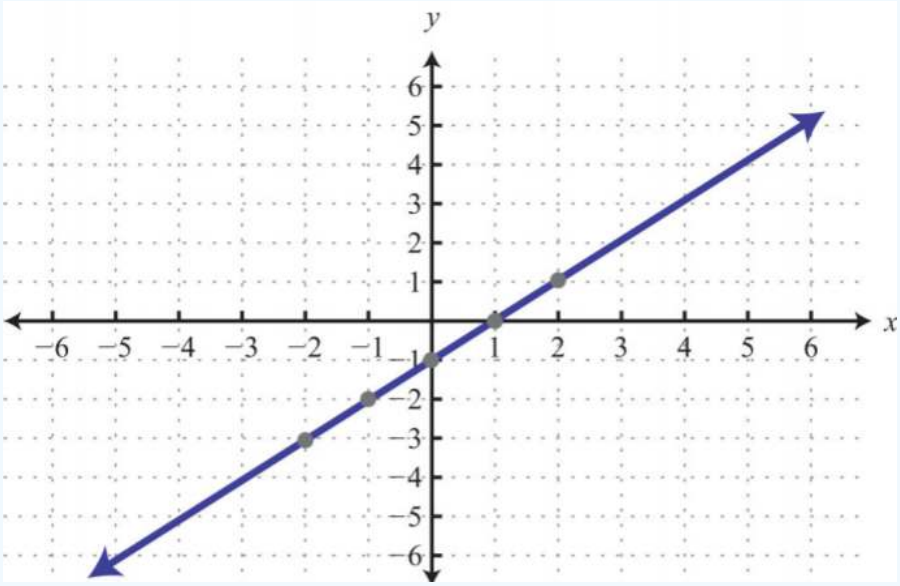


Figure 3.E.34

3. Intersection: $(-3,4)$

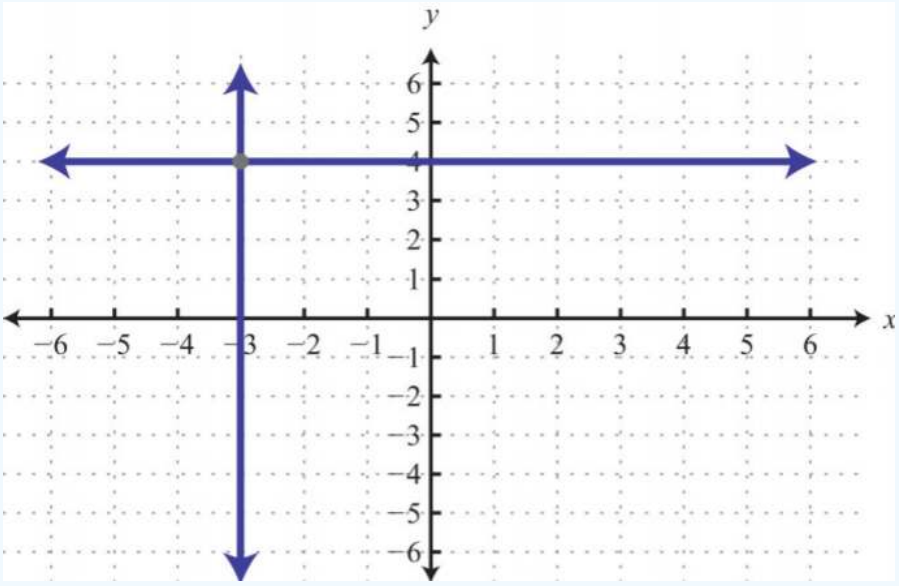


Figure 3.E.35

EXERCISE 1.2.E.24

Find the x- and y-intercepts and use those points to graph the following.

1. $2x - y = 8$
2. $12x + 5y = 15$
3. Calculate the slope of the line passing through $(-4, -5)$ and $(-3, 1)$.

Answer

2.

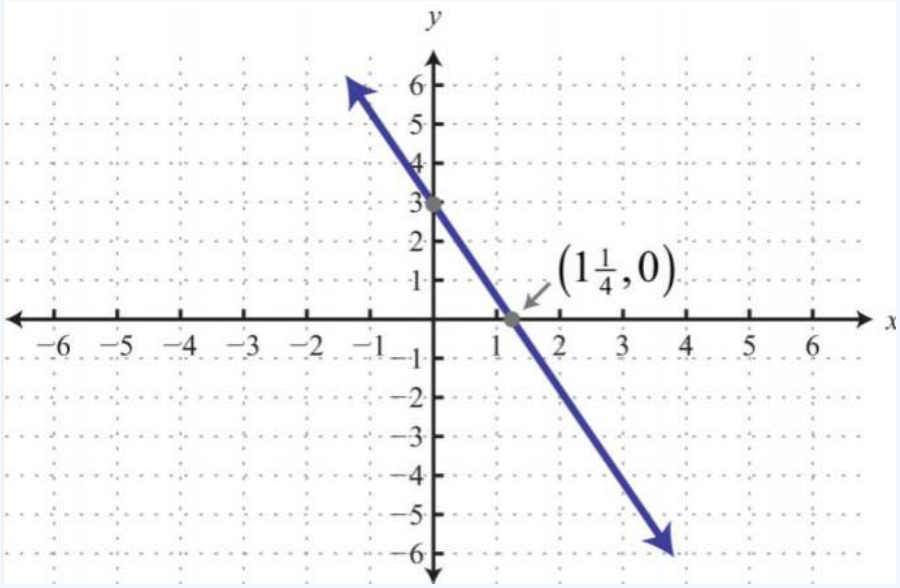


Figure 3.E.36

EXERCISE 1.2.E.25

Determine the slope and y-intercept. Use them to graph the following.

1. $y = -\frac{3}{2}x + 6$
2. $5x - 2y = 6$
3. Given $m = -3$, determine $m_{\perp}$.
4. Are the given lines parallel, perpendicular, or neither? $\begin{cases} -2x + 3y = -12 \\ 4x - 6y = 30 \end{cases}$
5. Determine the slope of the given lines.
   a. $y = -2$
   b. $x = \frac{1}{3}$
   c. Are these lines parallel, perpendicular, or neither?
6. Determine the equation of the line with slope $m = -\frac{3}{4}$ passing through $(8, 1)$.
7. Find the equation to the line passing through $(-2, 3)$ and $(4, 1)$.
8. Find the equation of the line parallel to $5x - y = 6$ passing through $(-1, -2)$.
9. Find the equation of the line perpendicular to $-x + 2y = 4$ passing through $(\frac{1}{2}, 5)$.

Answer

1. Slope: $-\frac{3}{2}$; y-intercept: $(0, 6)$

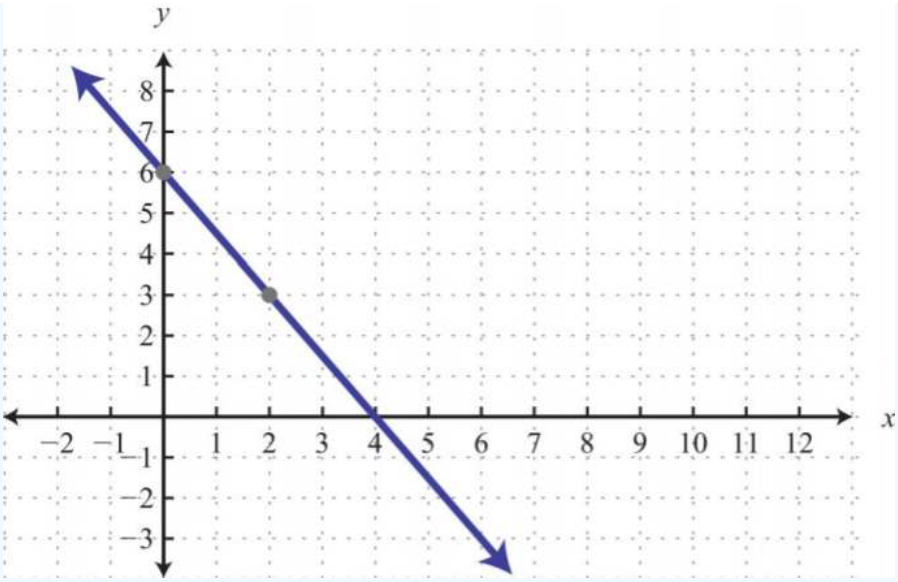


Figure 3.E.37

3. $m_{\perp} = \frac{1}{3}$
5. a. 0; b. Undefined; c. Perpendicular
6. $y = -\frac{3}{4}x + 7$
8. $y = 5x + 3$

? EXERCISE 1.2.E. 26

Given a linear function $f(x) = -\frac{4}{5}x + 2$, determine the following.

1. $f(10)$
2. x when $f(x) = 0$
3. Graph the solution set: $3x - 4y > 4$.
4. Graph the solution set: $y - 2x \geq 0$.
5. A rental car company charges $\$32.00$ plus $\$0.52$ per mile driven. Write an equation that gives the cost of renting the car in terms of the number of miles driven. Use the formula to determine the cost of renting the car and driving it 46 miles.
6. A car was purchased new for $\$12,000$ and was sold 5 years later for $\$7,000$. Write a linear equation that gives the value of the car in terms of its age in years.
7. The area of a rectangle is 72 square meters. If the width measures 4 meters, then determine the length of the rectangle.

Answer

1. $f(10) = -6$
3.

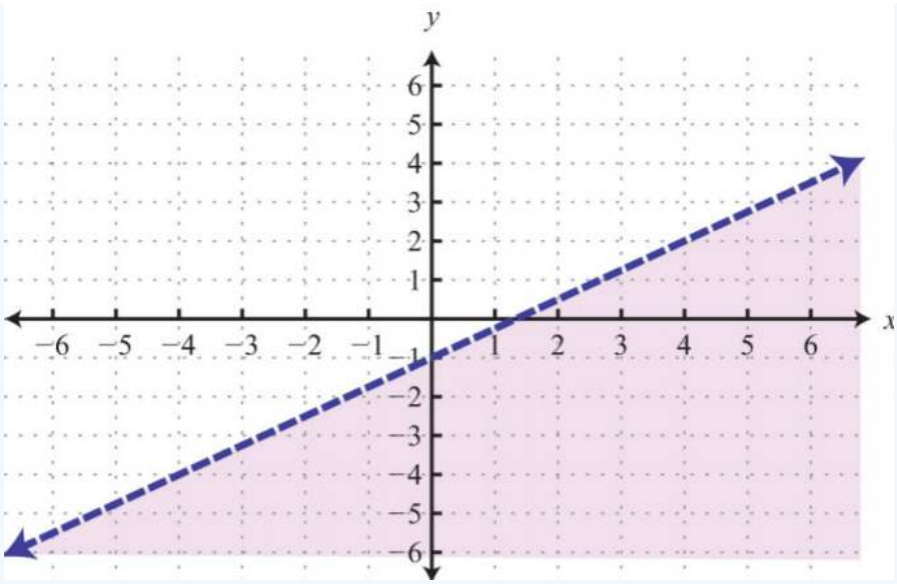


Figure 3.E.38

5. cost$= 0.52x + 32$; \$55.92

7. 18 meters

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1.3: SOLVING LINEAR SYSTEMS BY GRAPHING

LEARNING OBJECTIVES

- Check solutions to systems of linear equations.
- Solve linear systems using the graphing method.
- Identify dependent and inconsistent systems.

DEFINITION OF A LINEAR SYSTEM

Real-world applications are often modeled using more than one variable and more than one equation. A system of equations consists of a set of two or more equations with the same variables. In this section, we will study linear systems consisting of two linear equations each with two variables. For example,

A solution to a linear system, or simultaneous solution, to a linear system is an ordered pair (x, y) that solves both of the equations. In this case, $(3, 2)$ is the only solution. To check that an ordered pair is a solution, substitute the corresponding x-and y-values into each equation and then simplify to see if you obtain a true statement for both equations.

$Check: (3, 2)$

$$\begin{array}{l|l} Equation\,1: \ 2x-37=0 & Equation\,2 \ -4x+2y=-8 \\ 2(3)-3(2)=0 & -4(3)+2(2)=-8 \\ 6-6=0 & -12+4=-8 \\ 0=0 \quad \checkmark & -8=-8 \quad \checkmark \end{array}$$

EXAMPLE 1.3.1

Determine whether $(1, 0)$ is a solution to the system

$$\begin{cases} x-y=1 \\ -2x+3y=5 \end{cases}.$$

Solution:

Substitute the appropriate values into both equations.

$$Check: (1, 0)$$

$$\begin{array}{l|l} Equation\,1: \ x-y=1 & Equation\,2: \ -2x+3y=5 \\ (1)-(0)=1 & -2(1)+3(0)=5 \\ 1-0=1 & -2+0=5 \\ 1=1 \quad \checkmark & -2=5 \quad x \end{array}$$

Answer:

Since $(1, 0)$ does not satisfy both equations, it is not a solution.

EXERCISE 1.3.1

Is $(-2, 4)$ a solution to the system

$$\begin{array}{r} x-y=-6 \\ -2x+3y=16 \end{array}?$$

Answer

Yes

SOLVE BY GRAPHING

Geometrically, a linear system consists of two lines, where a solution is a point of intersection. To illustrate this, we will graph the following linear system with a solution of $(3, 2)$:

First, rewrite the equations in slope-intercept form so that we may easily graph them.

$2x-3y=0$	$-4x+2y=-8$
$2x-3y\color{#1E9BD7}{-2x}=0\color{#1E9BD7}{-2x}$	$-4x+2y\color{#1E9BD7}{+4x}=-8\color{#1E9BD7}{+4x}$
$-3y=-2x$	$2y=4x-8$
$\frac{-3y}{\color{#1E9BD7}{-3}}=\frac{-2x}{\color{#1E9BD7}{-3}}$	$\frac{2y}{\color{#1E9BD7}{2}}=\frac{4x-8}{\color{#1E9BD7}{2}}$
$y=\frac{2}{3}x$	$y=2x-4$

Next, replace these forms of the original equations in the system to obtain what is called an equivalent system. Equivalent systems share the same solution set.

Original system *Equivalent system*

If we graph both of the lines on the same set of axes, then we can see that the point of intersection is indeed $(3,2)$, the solution to the system.

Figure 1.3.1

To summarize, linear systems described in this section consist of two linear equations each with two variables. A solution is an ordered pair that corresponds to a point where the two lines in the rectangular coordinate plane intersect. Therefore, we can solve linear systems by graphing both lines on the same set of axes and determining the point where they cross. When graphing the lines, take care to choose a good scale and use a straightedge to draw the line through the points; accuracy is very important here. The steps for solving linear systems using the graphing method are outlined in the following example.

EXAMPLE 1.3.2

Solve by graphing:

$$\begin{cases} x-y=-4 \\ 2x+y=1 \end{cases}.$$

Solution:

Step 1: Rewrite the linear equations in slope-intercept form.

$$\begin{array}{c|c} x-y=-4 & 2x+y=1 \\ x-y\color{blue}{-x}=-4\color{blue}{-x} & 2x+y\color{blue}{-2x}=1\color{blue}{-2x} \\ -y=-x-4 & y=-2x+1 \\ \frac{-y}{\color{blue}{-1}}=\frac{-x-4}{\color{blue}{-1}} & \\ y=x+4 & \end{array}$$

Step 2: Write the equivalent system and graph the lines on the same set of axes.

$$\begin{array}{c|c} \color{blue}{Line\ 1}: y=x+4 & \color{blue}{Line\ 2}\ y=-2x+1 \\ y-intercept:\ (0,4) & y-intercept:\ (0,1) \\ slope:\ m=1=\frac{1}{1}=\frac{rise}{run} & slope:\ m=-2=\frac{-2}{1}=\frac{rise}{run} \end{array}$$

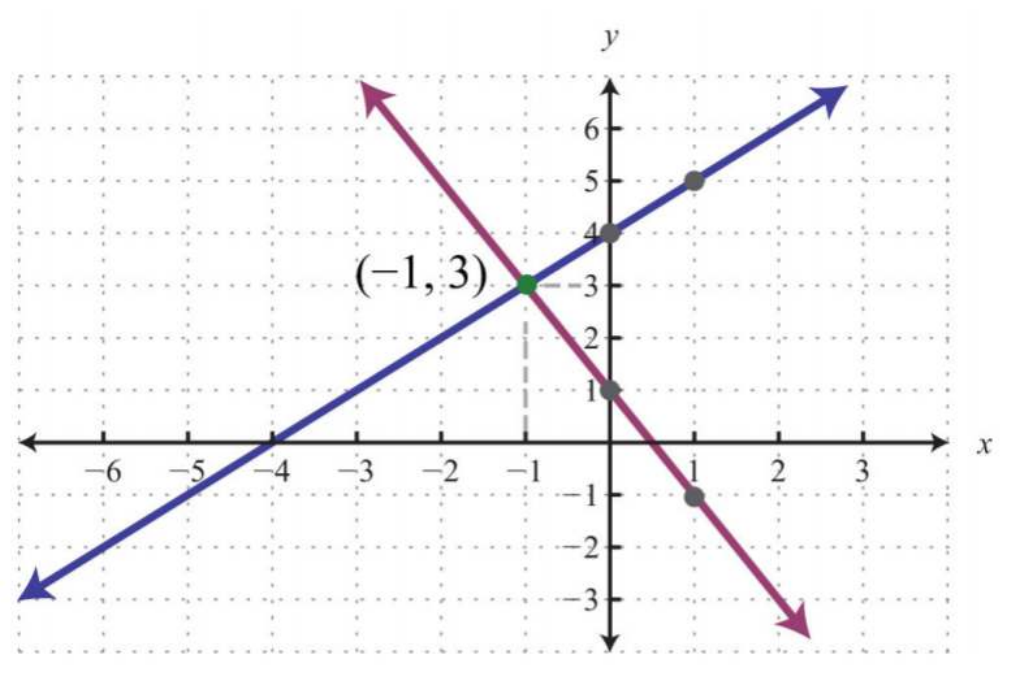


Figure 1.3.2

Step 3: Use the graph to estimate the point where the lines intersect and check to see if it solves the original system. In the above graph, the point of intersection appears to be $(-1,3)$.

$$\color{blue}{Check}: (-1,3)$$

$Line\ 1:\ x-y=-4$	$Line\ 2:\ 2x+y=1$
$(-1)-(3)=-4$	$2(-1)+(3)=1$
$-1-3=-4$	$-2+3=1$
$-4=-4$ ✓	$1=1$ ✓

Answer:

$$(-1,3)$$

EXAMPLE 1.3.3

Solve by graphing:

$$\begin{cases} 2x+y=2 \\ -2x+3y=-18 \end{cases}.$$

Solution:

We first solve each equation for y to obtain an equivalent system where the lines are in slope-intercept form. Graph the lines and determine the point of intersection.

Figure 1.3.3

$Check:\ (3,-4)$

$2x+y=2$	$-2x+3y=-18$
$2(3)+(-4)=2$	$-2(3)+3(-4)=-18$
$6-4=2$	$-6-12=-18$
$2=2$ ✓	$-18=-18$ ✓

Answer:

$$(3,-4)$$

EXAMPLE 1.3.4

Solve by graphing:

$$\begin{cases} 3x+y=6 \\ y=-3 \end{cases}$$

Solution:

Figure 1.3.4

$Check:\ (3,-3)$

$3x+y=6$	$y=-3$
$3(3)+(-3)=6$	$(-3)=-3$
$9-3=6$	$-3=-3$ ✓
$6=6$ ✓	

Answer:

$$(3,-3)$$

The graphing method for solving linear systems is not ideal when the solution consists of coordinates that are not integers. There will be more accurate algebraic methods in sections to come, but for now, the goal is to understand the geometry involved when solving systems. It is important to remember that the solutions to a system correspond to the point, or points, where the graphs of the equations intersect.

? EXERCISE 1.3.2

Solve by graphing:

$$\begin{cases} -x+y=6 \\ 5x+2y=-2 \end{cases}.$$

Answer

$(-2,4)$

DEPENDENT AND INCONSISTENT SYSTEMS

Systems with at least one solution are called consistent systems. Up to this point, all of the examples have been of consistent systems with exactly one ordered pair solution. It turns out that this is not always the case. Sometimes systems consist of two linear equations that are equivalent. If this is the case, the two lines are the same and when graphed will coincide. Hence the solution set consists of all the points on the line. This is a dependent system. Given a consistent linear system with two variables, there are two possible results:

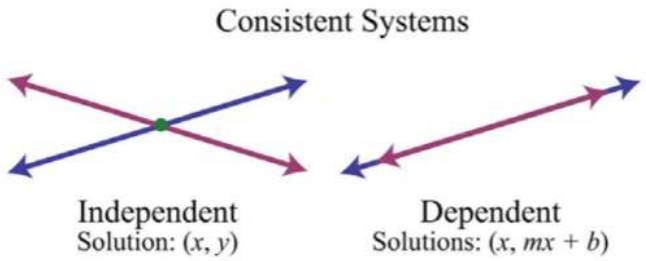


Figure 1.3.5

The solutions to independent systems are ordered pairs (x,y). We need some way to express the solution sets to dependent systems, since these systems have infinitely many solutions, or points of intersection. Recall that any line can be written in slope-intercept form, $y=mx+b$. Here, y depends on x. So we may express all the ordered pair solutions (x,y) in the form $(x,mx+b)$, where x is any real number.

EXAMPLE 1.3.5

Solve by graphing:

$$\begin{cases} -2x+3y=-9 \\ 4x-6y=18 \end{cases}.$$

Solution:

Determine slope-intercept form for each linear equation in the system.

$$\begin{array}{l|l} -2x+3y=-9 & 4x-6y=18 \\ -2x+3y+2x=-9+2x & 4x-6y-4x=18-4x \\ 3y=2x-9 & -6y=-4x+18 \\ \frac{3y}{3}=\frac{2x-9}{3} & \frac{-6y}{-6}=\frac{-4x+18}{-6} \\ y=\frac{2}{3}x-3 & y=\frac{-4}{-6}x+\frac{18}{-6} \\ & y=\frac{2}{3}x-3 \end{array}$$

In slope-intercept form, we can easily see that the system consists of two lines with the same slope and same y-intercept.

They are, in fact, the same line. And the system is dependent.

Figure 1.3.6

Answer:

$(x,\frac{2}{3}x-3)$

In this example, it is important to notice that the two lines have the same slope and same y-intercept. This tells us that the two equations are equivalent and that the simultaneous solutions are all the points on the line $y=\frac{2}{3}x-3$. This is a dependent system, and the infinitely many solutions are expressed using the form $(x,mx+b)$. Other resources may express this set using set notation, $\{(x,y)|y=\frac{2}{3}x-3\}$, which reads "the set of all ordered pairs (x,y) such that y equals two-thirds x minus 3." Sometimes the lines do not cross and there is no point of intersection. Such systems have no solution, $\varnothing$, and are called inconsistent systems.

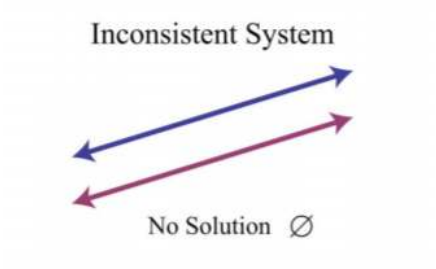


Figure 1.3.7

EXAMPLE 1.3.6

Solve by graphing:

$$\begin{cases} -2x+5y=-15 \\ -4x+10y=10 \end{cases}.$$

Solution:

Determine slope-intercept form for each linear equation.

$-2x+5y=-15$	$-4x+10y=10$
$-2x+5y+2x=-15+2x$	$-4x+10y+4x=10+4x$
$5y=2x-15$	$10y=4x+10$
$\frac{5y}{5}=\frac{2x-15}{5}$	$\frac{10y}{10}=\frac{4x+10}{10}$
$y=\frac{2}{5}x-3$	$y=\frac{2}{5}x+1$

In slope-intercept form, we can easily see that the system consists of two lines with the same slope and different y-intercepts.

Therefore, they are parallel and will never intersect.

Figure 1.3.8

Answer:

There is no simultaneous solution, Ø.

EXERCISE 1.3.3

Solve by graphing:

$$\begin{cases} x+y=-1 \\ -2x-2y=2 \end{cases}.$$

Answer

$(x, x-1)$

KEY TAKEAWAYS

- In this section, we limit our study to systems of two linear equations with two variables. Solutions to such systems, if they exist, consist of ordered pairs that satisfy both equations. Geometrically, solutions are the points where the graphs intersect.
- The graphing method for solving linear systems requires us to graph both of the lines on the same set of axes as a means to determine where they intersect.
- The graphing method is not the most accurate method for determining solutions, particularly when the solutions have coordinates that are not integers. It is a good practice to always check your solutions.
- Some linear systems have no simultaneous solution. These systems consist of equations that represent parallel lines with different y-intercepts and do not intersect in the plane. They are called inconsistent systems and the solution set is the empty set, Ø.
- Some linear systems have infinitely many simultaneous solutions. These systems consist of equations that are equivalent and represent the same line. They are called dependent systems and their solutions are expressed using the notation $(x, mx+b)$, where x is any real number.

EXERCISE 1.3.4 SOLUTIONS TO LINEAR SYSTEMS

Determine whether the given ordered pair is a solution to the given system.

1. $(3,-2);\begin{cases} x+y=-1 \\ -2x-2y=2 \end{cases}$
2. $(-5,0);\begin{cases} x+y=-1 \\ -2x-2y=2 \end{cases}$
3. $(-2,-6);\begin{cases} -x+y=-4 \\ 3x-y=-12 \end{cases}$
4. $(2,-7);\begin{cases} 3x+2y=-8 \\ -5x-3y=11 \end{cases}$
5. $(0,-3);\begin{cases} 5x-5y=15 \\ -13x+2y=-6 \end{cases}$
6. $(-12,14);\begin{cases} x+y=-14 \\ -2x-4y=0 \end{cases}$
7. $(\frac{3}{4},\frac{1}{4});\begin{cases} -x-y=-1 \\ -4x-8y=5 \end{cases}$
8. $(-3,4);\begin{cases} \frac{1}{3}x+\frac{1}{2}y=1 \\ \frac{2}{3}x-\frac{3}{2}y=-8 \end{cases}$
9. $(-5,-3);\begin{cases} y=-35x-10 \\ y=5 \end{cases}$
10. $(4,2);\begin{cases} x=4-7 \\ x+4y=8 \end{cases}$

Answer

1. No
3. No
5. Yes
7. No
9. Yes

EXERCISE 1.3.5 SOLUTIONS TO LINEAR SYSTEMS

Given the graph, determine the simultaneous solution.

1.

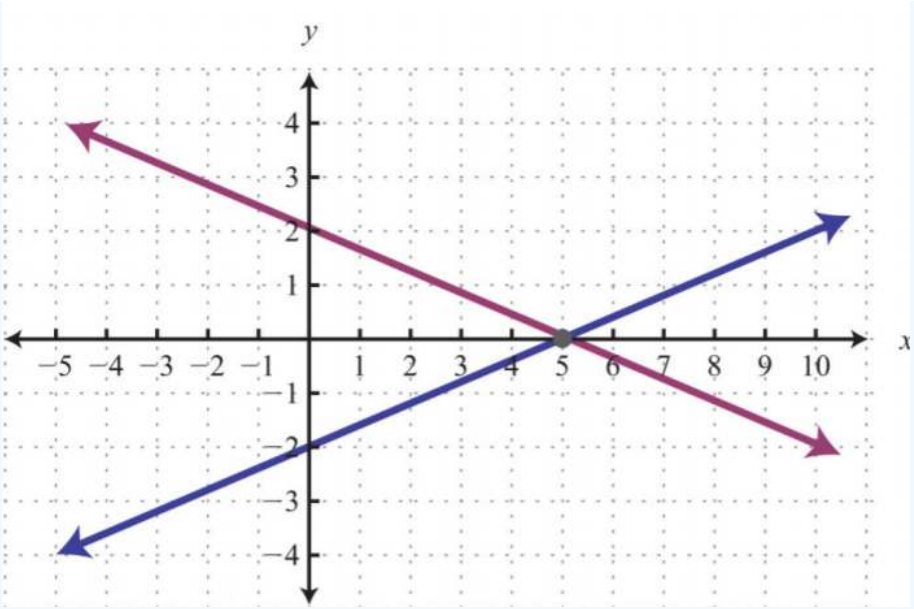


Figure 1.3.9

2.

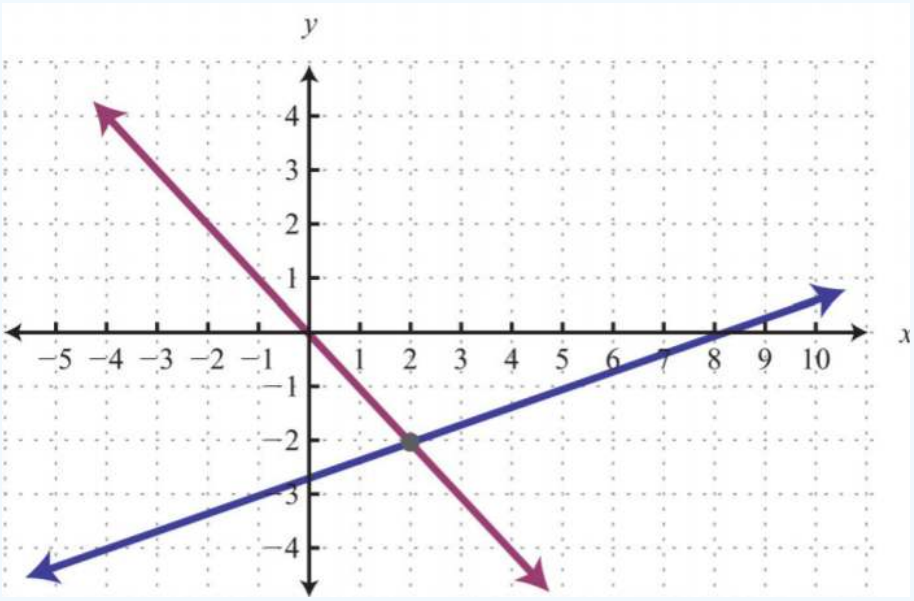


Figure 1.3.10

3.

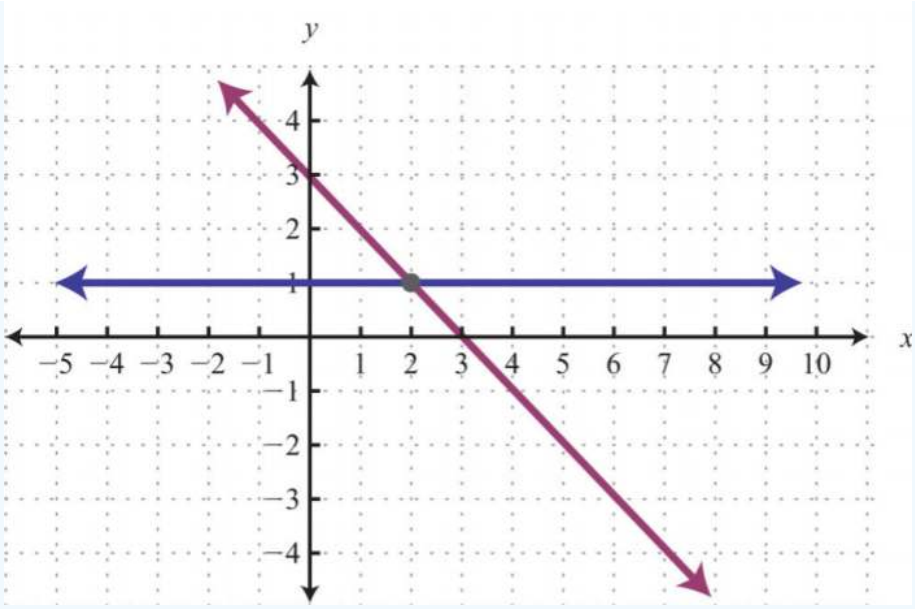


Figure 1.3.11

4.

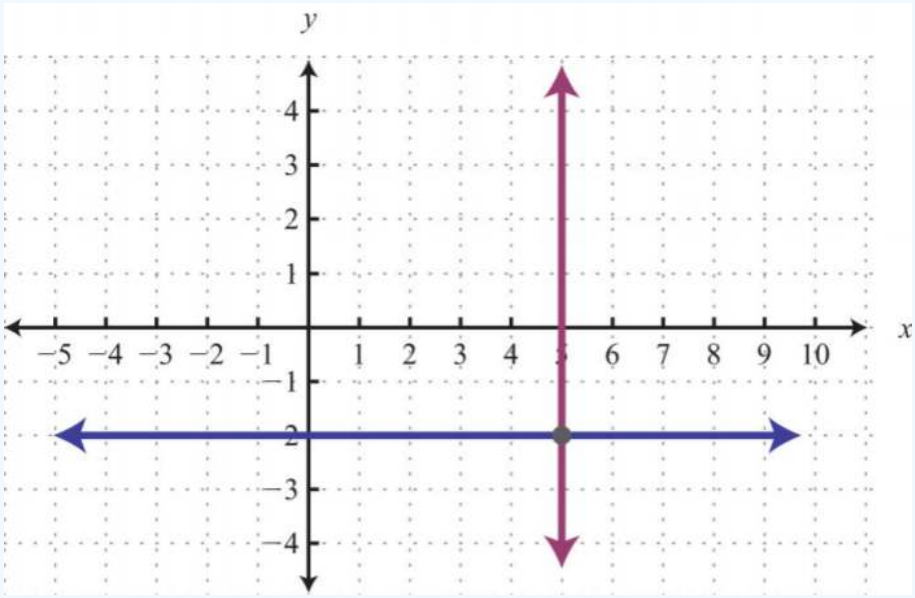


Figure 1.3.12

5.

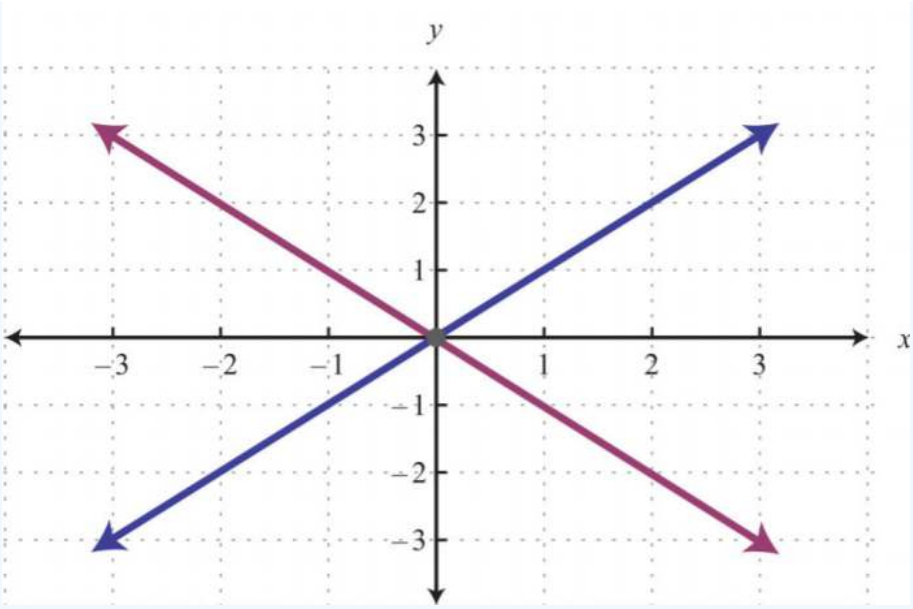


Figure 1.3.13

6.

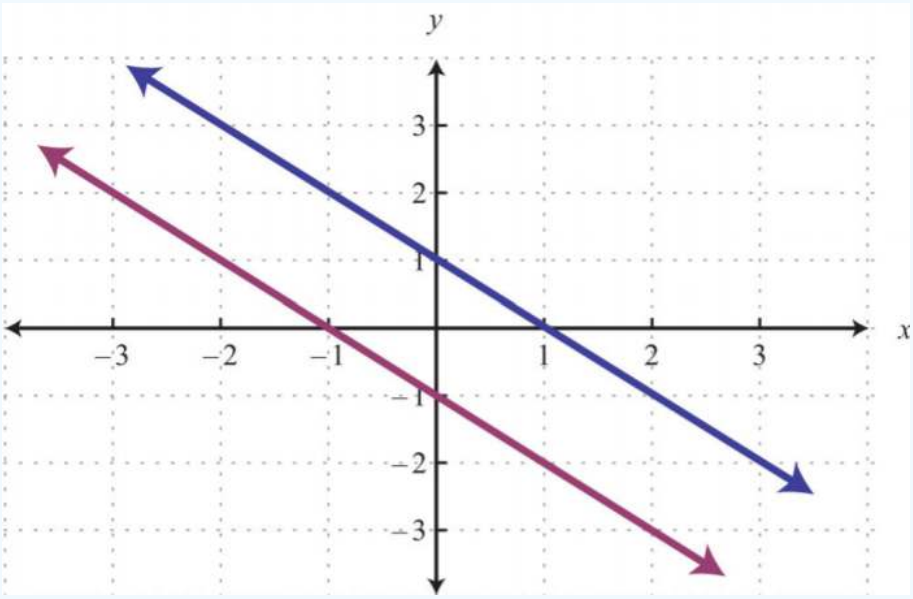


Figure 1.3.14

7.

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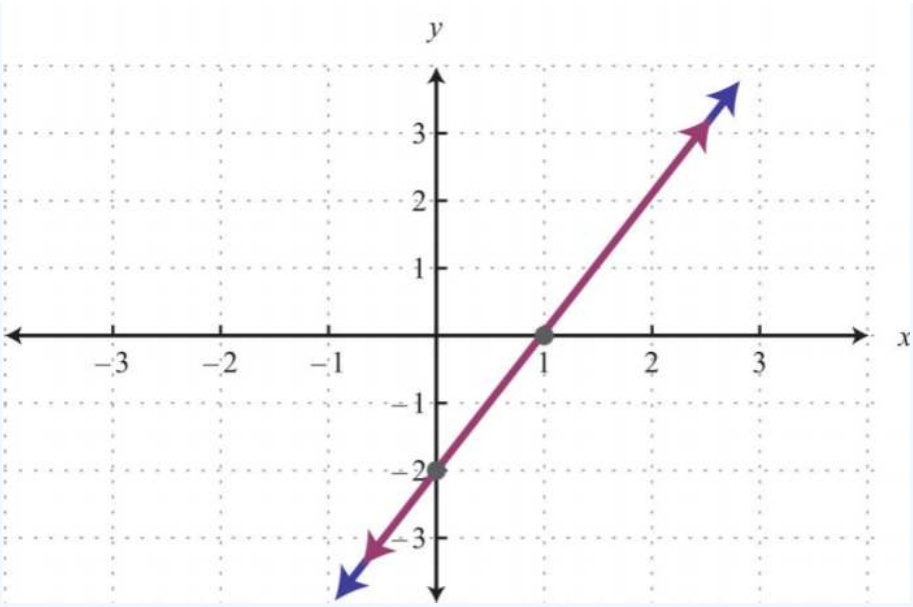


Figure 1.3.15

8.

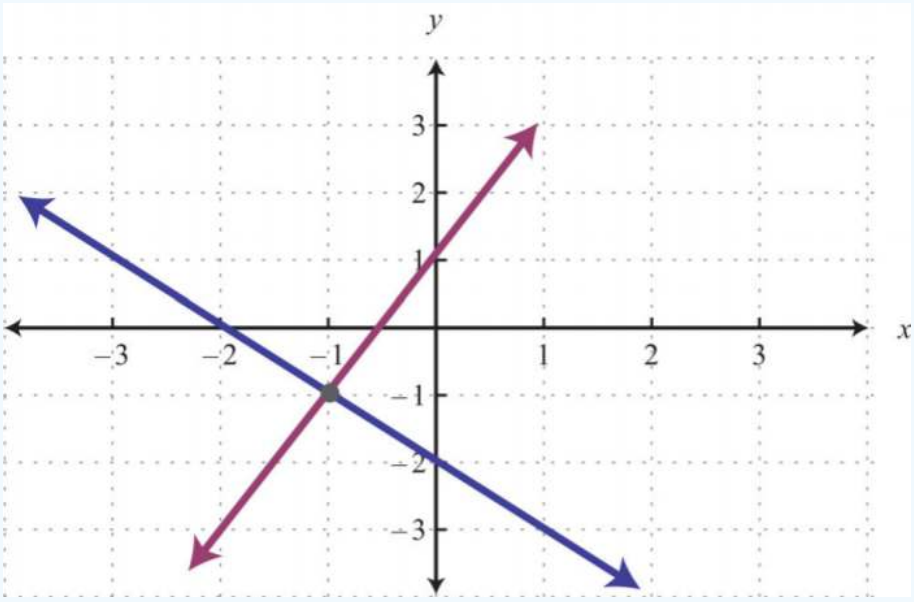


Figure 1.3.16

9.

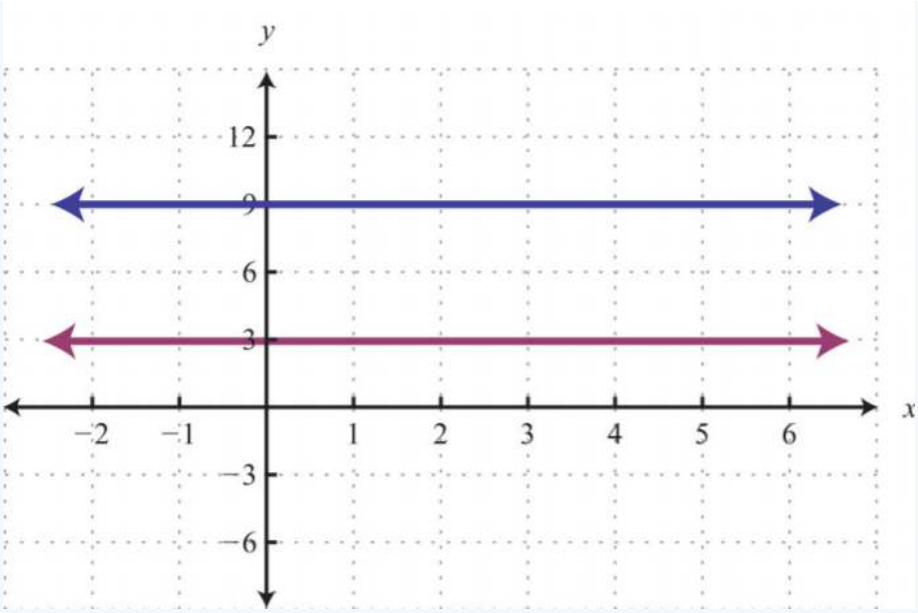


Figure 1.3.17

10.

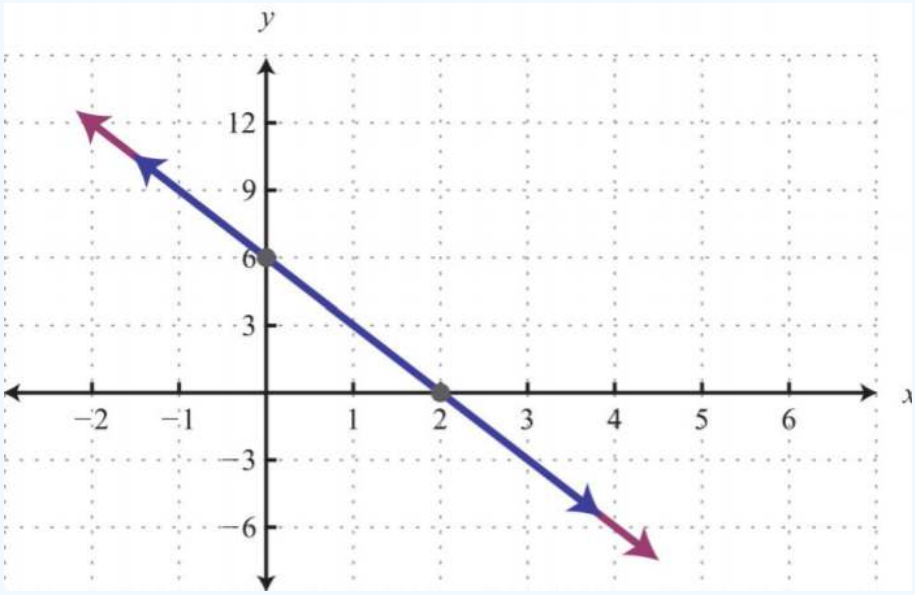


***Figure* 1.3.18**

Answer

1. $(5,0)$
3. $(2,1)$
5. $(0,0)$
7. $(x,2x-2)$
9. $\emptyset$

? EXERCISE 1.3.6 SOLVING LINEAR SYSTEMS

Solve by graphing.

1. $\begin{cases} y=\frac{3}{2}x+6 \\ y=-x+1 \end{cases}$

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2. $\begin{cases} y = \frac{3}{4}x + 2 \\ y = -\frac{1}{4}x - 2 \end{cases}$
3. $\begin{cases} y = x - 4 \\ y = -x + 2 \end{cases}$
4. $\begin{cases} y = -5x + 4 \\ y = 4x - 5 \end{cases}$
5. $\begin{cases} y = \frac{2}{5}x + 1 \\ y = \frac{3}{5}x \end{cases}$
6. $\begin{cases} y = -\frac{2}{5}x + 6 \\ y = \frac{2}{5}x + 10 \end{cases}$
7. $\begin{cases} y = -2 \\ y = x + 1 \end{cases}$
8. $\begin{cases} y = 3 \\ x = -3 \end{cases}$
9. $\begin{cases} y = 0 \\ y = \frac{2}{5}x - 4 \end{cases}$
10. $\begin{cases} x = 2 \\ y = 3x \end{cases}$
11. $\begin{cases} y = \frac{3}{5}x - 6 \\ y = \frac{3}{5}x - 3 \end{cases}$
12. $\begin{cases} y = -\frac{1}{2}x + 1 \\ y = -\frac{1}{2}x + 1 \end{cases}$
13. $\begin{cases} 2x + 3y = 18 \\ -6x + 3y = -6 \end{cases}$
14. $\begin{cases} -3x + 4y = 20 \\ 2x + 8y = 8 \end{cases}$
15. $\begin{cases} -2x + y = 1 \\ 2x - 3y = 9 \end{cases}$
16. $\begin{cases} x + 2y = -8 \\ 5x + 4y = -4 \end{cases}$
17. $\begin{cases} 4x + 6y = 36 \\ 2x - 3y = 6 \end{cases}$
18. $\begin{cases} 2x - 3y = 18 \\ 6x - 3y = -6 \end{cases}$
19. $\begin{cases} 3x + 5y = 30 \\ -6x - 10y = -10 \end{cases}$
20. $\begin{cases} -x + 3y = 3 \\ 5x - 15y = -15 \end{cases}$
21. $\begin{cases} x - y = 0 \\ -x + y = 0 \end{cases}$
22. $\begin{cases} y = x \\ y - x = 1 \end{cases}$
23. $\begin{cases} 3x + 2y = 0 \\ x = 2 \end{cases}$
24. $\begin{cases} 2x + \frac{1}{3}y = \frac{2}{3} \\ -3x + 12y = -2 \end{cases}$

25. $\begin{cases} \frac{1}{10}x+\frac{1}{5}y=2 \\ -\frac{1}{5}x+\frac{1}{5}y=-1 \end{cases}$
26. $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=1 \\ \frac{1}{3}x+\frac{1}{5}y=1 \end{cases}$
27. $\begin{cases} \frac{1}{9}x+\frac{1}{6}y=0 \\ \frac{1}{9}x+\frac{1}{4}y=\frac{1}{2} \end{cases}$
28. $\begin{cases} \frac{5}{16}x-\frac{1}{2}y=5 \\ -\frac{5}{16}x+\frac{1}{2}y=\frac{5}{2} \end{cases}$
29. $\begin{cases} \frac{1}{6}x-\frac{1}{2}y=\frac{9}{2} \\ -\frac{1}{18}x+\frac{1}{6}y=-\frac{3}{2} \end{cases}$
30. $\begin{cases} \frac{1}{2}x-\frac{1}{4}y=-\frac{1}{2} \\ \frac{1}{3}x-\frac{1}{2}y=3 \end{cases}$
31. $\begin{cases} y=4 \\ x=-5 \end{cases}$
32. $\begin{cases} y=-3 \\ x=2 \end{cases}$
33. $\begin{cases} y=0 \\ x=0 \end{cases}$
34. $\begin{cases} y=-2 \\ y=3 \end{cases}$
35. $\begin{cases} y=5 \\ y=-5 \end{cases}$
36. $\begin{cases} y=2 \\ y-2=0 \end{cases}$
37. $\begin{cases} x=-5 \\ x=1 \end{cases}$
38. $\begin{cases} y=x \\ x=0 \end{cases}$
39. $\begin{cases} 4x+6y=3 \\ -x+y=-2 \end{cases}$
40. $\begin{cases} -2x+20y=20 \\ 3x+10y=-10 \end{cases}$

Answer

1. $(-2,3)$
3. $(3,-1)$
5. $(5,3)$
7. $(-3,-2)$
9. $(10,0)$
11. $\emptyset$
13. $(3,4)$
15. $(-3,-5)$
17. $(6,2)$
19. $\emptyset$
21. (x,x)

23. $(2, -3)$
25. $(10, 5)$
27. $(-9, 6)$
29. $(x, 13x - 9)$
31. $(-5, 4)$
33. $(0, 0)$
35. $\emptyset$
37. $\emptyset$
39. $(\frac{3}{2}, -\frac{1}{2})$

EXERCISE 1.3.7 SOLVING LINEAR SYSTEMS

Set up a linear system of two equations and two variables and solve it using the graphing method.

1. The sum of two numbers is
2. The larger number is 10 less than five times the smaller.
3. The difference between two numbers is 12 and their sum is 4.
4. Where on the graph of $3x - 2y = 6$ does the x-coordinate equal the y-coordinate?
5. Where on the graph of $-5x + 2y = 30$ does the x-coordinate equal the y-coordinate?

Answer

1. The two numbers are 5 and 15.
3. $(6, 6)$

EXERCISE 1.3.8 SOLVING LINEAR SYSTEMS

A regional bottled water company produces and sells bottled water. The following graph depicts the supply and demand curves of bottled water in the region. The horizontal axis represents the weekly tonnage of product produced, Q. The vertical axis represents the price per bottle in dollars, P.

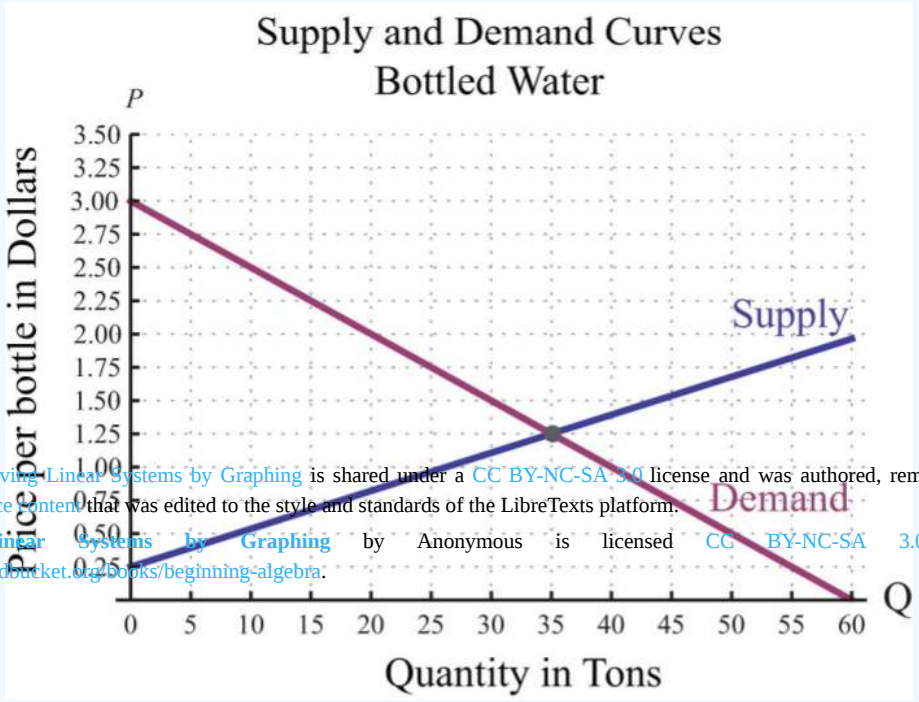


***Figure* 1.3.19**

Use the graph to answer the following questions.

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1. Determine the price at which the quantity demanded is equal to the quantity supplied.
2. If production of bottled water slips to 20 tons, then what price does the demand curve predict for a bottle of water?
3. If production of bottled water increases to 40 tons, then what price does the demand curve predict for a bottle of water?
4. If the price of bottled water is set at $\$2.50$ dollars per bottle, what quantity does the demand curve predict?

Answer

1. $\$1.25$
3. $\$1.00$

EXERCISE 1.3.9 DISCUSSION BOARD TOPICS

1. Discuss the weaknesses of the graphing method for solving systems.
2. Explain why the solution set to a dependent linear system is denoted by $(x, mx + b)$.

Answer

1. Answers may vary

1.4: SOLVING LINEAR SYSTEMS BY SUBSTITUTION

LEARNING OBJECTIVES

- Solve linear systems using the substitution method.

THE SUBSTITUTION METHOD

In this section, we will define a completely algebraic technique for solving systems. The idea is to solve one equation for one of the variables and substitute the result into the other equation. After performing this substitution step, we will be left with a single equation with one variable, which can be solved using algebra. This is called the substitution method, and the steps are outlined in the following example.

EXAMPLE 1.4.1

Solve by substitution:

Solution:

Step 1: Solve for either variable in either equation. If you choose the first equation, you can isolate y in one step.

$$\begin{aligned} 2x+y &= 7 \\ 2x+y-2x &= 7-2x \\ y &= -2x+7 \end{aligned}$$

Step 2: Substitute the expression $-2x+7$ for the y variable in the ***other*** equation.

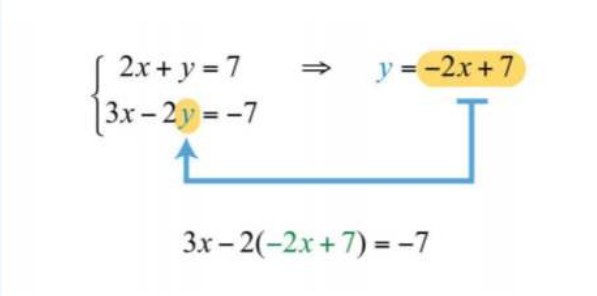


Figure 1.4.1

This leaves you with an equivalent equation with one variable, which can be solved using the techniques learned up to this point.

Step 3: Solve for the remaining variable. To solve for x, first distribute -2:

Step 4: Back substitute to find the value of the other coordinate. Substitute $x=1$ into either of the original equations or their equivalents. Typically, we use the equivalent equation that we found when isolating a variable in step 1.

$$\begin{aligned} y &= -2x+7 \\ &= -2(1)+7 \\ &= -2+7 \\ &= 5 \end{aligned}$$

The solution to the system is $(1,5)$. Be sure to present the solution as an ordered pair.

Step 5: Check. Verify that these coordinates solve both equations of the original system:

$$Check: (1,5)$$

$Equation\ 1:$	$Equation\ 2:$
$2x+y=7$	$3x-2y=-7$
$2(1)+(5)=7$	$3(1)-2(5)=-7$
$2+5=7$	$3-10=-7$
$7=7$ ✓	$-7=-7$ ✓

The graph of this linear system follows:

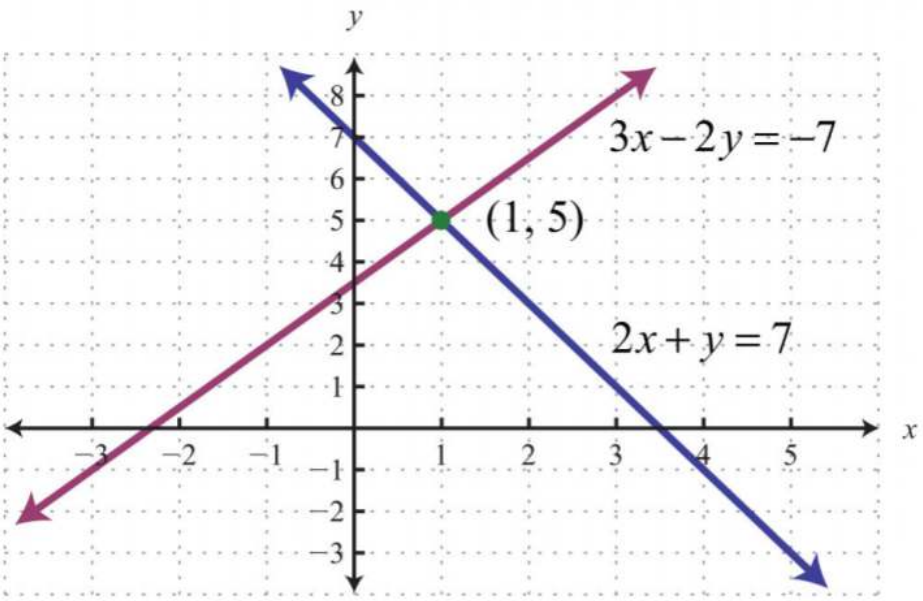


Figure 1.4.2

The substitution method for solving systems is a completely algebraic method. Thus graphing the lines is not required.

Answer:

$$(1,5)$$

EXAMPLE 1.4.2

Solve by substitution:

$$\begin{cases} 2x - y = 1 \\ 2x - y = 3 \end{cases}.$$

Solution:

In this example, we can see that x has a coefficient of 1 in the second equation. This indicates that it can be isolated in one step as follows:

$$\begin{aligned} x - y &= 3 \\ x - y + y &= 3 + y \\ x &= 3 + y \end{aligned}$$

$$\begin{cases} 2x - y = 12 \\ x - y = 3 \Rightarrow x = 3 + y \end{cases}$$

Substitute $3 + y$ for x in the first equation. Use parentheses and take care to distribute.

$$\begin{aligned} 2x - y &= 12 \\ 2(3 + y) - y &= 12 \\ 6 + 2y - y &= 12 \\ 6 + y &= 12 \\ 6 + y - 6 &= 12 - 6 \\ y &= 6 \end{aligned}$$

Use $x = 3 + y$ to find x.

$$\begin{aligned} x &= 3 + y \\ &= 3 + +6 \\ &= 9 \end{aligned}$$

Answer:

$(9,6)$. The check is left to the reader.

EXAMPLE 1.4.3

Solve by substitution:

$$\begin{cases} 3x-5y=1 \\ 7x=-1 \end{cases}.$$

Solution:

In this example, the variable x is already isolated. Hence we can substitute $x=-1$ into the first equation.

Answer:

$(-1,-4)$. It is a good exercise to graph this particular system to compare the substitution method to the graphing method for solving systems.

EXERCISE 1.4.1

Solve by substitution:

$$\begin{cases} 3x+y=4 \\ 8x+2y=10 \end{cases}.$$

Answer

$(1,1)$

Solving systems algebraically frequently requires work with fractions.

EXAMPLE 1.4.4

Solve by substitution:

$$\begin{cases} 2x+8y=5 \\ 24x-4y=-15 \end{cases}.$$

Solution:

Begin by solving for x in the first equation.

$$\begin{aligned} 2x+8y &= 5 \\ 2x+8y-8y &= 5-8y \\ \frac{2x}{2} &= \frac{-8y+5}{2} \\ x &= \frac{-8y}{2}+\frac{5}{2} \\ x &= -4y+\frac{5}{2} \end{aligned}$$

$$\begin{cases} 2x+8y=5 \Rightarrow x=-4y+\frac{5}{2} \\ 24x-4y=-15 \end{cases}$$

Next, substitute into the second equation and solve for y.

Back substitute into the equation used in the substitution step:

$$\begin{aligned} x &= -4y+\frac{5}{2} \\ &= -4\left(\frac{3}{4}\right)+\frac{5}{2} \\ &= -3+\frac{5}{2} \\ &= -\frac{6}{2}+\frac{5}{2} \\ &= -\frac{1}{2} \end{aligned}$$

Answer:

$$\left(-\tfrac{1}{2},\tfrac{3}{4}\right)$$

As we know, not all linear systems have only one ordered pair solution. Recall that some systems have infinitely many ordered pair solutions and some do not have any solutions. Next, we explore what happens when using the substitution method to solve a dependent system.

EXAMPLE 1.4.5

Solve by substitution:

$$\begin{cases} -5x+y=-1 \\ 10x-2y=2 \end{cases}.$$

Solution:

Since the first equation has a term with coefficient 1, we choose to solve for that first.

Next, substitute this expression in for y in the second equation.

$$\begin{aligned} 10x-2y &= 2 \\ 10x-2(5x-1) &= 2 \\ 10x-10x+2 &= 2 \\ 2 &= 2 \quad True \end{aligned}$$

This process led to a true statement; hence the equation is an identity and any real number is a solution. This indicates that the system is dependent. The simultaneous solutions take the form $(x, mx+b)$, or in this case, $(x, 5x-1)$, where x is any real number.

Answer:

$$(x, 5x-1)$$

To have a better understanding of the previous example, rewrite both equations in slope-intercept form and graph them on the same set of axes.

Figure 1.4.3

We can see that both equations represent the same line, and thus the system is dependent. Now explore what happens when solving an inconsistent system using the substitution method.

EXAMPLE 1.4.6

Solve by substitution:

$$\begin{cases} -7x+3y=3 \\ 14x-6y=-16 \end{cases}.$$

Solution:

Solve for y in the first equation.

$$\begin{aligned} -7x+3y &= 3 \\ -7x+3y+7x &= 3+7x \\ 3y &= 7x+3 \\ \frac{3y}{3} &= \frac{7x+3}{3} \\ y &= \frac{7}{3}x+1 \end{aligned}$$

$$\begin{cases} -7x+3y=3 \Rightarrow y=\frac{7}{3}x+1 \\ 14x-6y=-16 \end{cases}$$

Substitute into the second equation and solve.

Solving leads to a false statement. This indicates that the equation is a contradiction. There is no solution for x and hence no solution to the system.

Answer:

No solution, Ø

A false statement indicates that the system is inconsistent, or in geometric terms, that the lines are parallel and do not intersect. To illustrate this, determine the slope-intercept form of each line and graph them on the same set of axes.

Figure 1.4.4

In slope-intercept form, it is easy to see that the two lines have the same slope but different y-intercepts.

EXERCISE 1.4.2

Solve by substitution:

$$\begin{cases} 2x-5y=3 \\ 4x-10y=6 \end{cases}.$$

Answer

$(x, \frac{2}{5}x-\frac{3}{5})$

KEY TAKEAWAYS

- The substitution method is a completely algebraic method for solving a system of equations.
- The substitution method requires that we solve for one of the variables and then substitute the result into the other equation. After performing the substitution step, the resulting equation has one variable and can be solved using the techniques learned up to this point.
- When the value of one of the variables is determined, go back and substitute it into one of the original equations, or their equivalent equations, to determine the corresponding value of the other variable.
- Solutions to systems of two linear equations with two variables, if they exist, are ordered pairs (x, y).
- If the process of solving a system of equations leads to a false statement, then the system is inconsistent and there is no solution, Ø.
- If the process of solving a system of equations leads to a true statement, then the system is dependent and there are infinitely many solutions that can be expressed using the form $(x, mx+b)$.

EXERCISE 1.4.3 SUBSTITUTION METHOD

Solve by substitution.

1. $\begin{cases} y=4x-1 \\ -3x+y=1 \end{cases}$
2. $\begin{cases} y=3x-8 \\ 4x-y=2 \end{cases}$
3. $\begin{cases} x=2y-3 \\ x+3y=-8 \end{cases}$
4. $\begin{cases} x=-4y+12 \\ x+3y=12 \end{cases}$
5. $\begin{cases} y=3x-5 \\ x+2y=2 \end{cases}$
6. $\begin{cases} y=x \\ 2x+3y=10 \end{cases}$
7. $\begin{cases} y=4x+1 \\ -4x+y=2 \end{cases}$
8. $\begin{cases} y=-3x+5 \\ 3x+y=5 \end{cases}$
9. $\begin{cases} y=2x+3 \\ 2x-y=-3 \end{cases}$
10. $\begin{cases} y=5x-1 \\ x-2y=5 \end{cases}$
11. $\begin{cases} y=-7x+1 \\ 3x-y=4 \end{cases}$
12. $\begin{cases} x=6y+2 \\ 5x-2y=0 \end{cases}$
13. $\begin{cases} y=-2-2x \\ -y=-6 \end{cases}$
14. $\begin{cases} x=-3x-4 \\ y=-3 \end{cases}$
15. $\begin{cases} y=-\frac{1}{5}x+\frac{3}{7} \\ x-5y=9 \end{cases}$

16. $\begin{cases} y = \frac{2}{3}x - \frac{1}{6} \\ x - 9y = 0 \end{cases}$
17. $\begin{cases} y = \frac{1}{2}x + \frac{1}{3} \\ x - 6y = 4 \end{cases}$
18. $\begin{cases} y = -\frac{3}{8}x + \frac{1}{2} \\ 2x + 4y = 1 \end{cases}$
19. $\begin{cases} x + y = 6 \\ 2x + 3y = \frac{1}{6} \end{cases}$
20. $\begin{cases} x - y = 3 \\ -2x + 3y = -2 \end{cases}$
21. $\begin{cases} 2x + y = 2 \\ 3x - 2y = \frac{1}{7} \end{cases}$
22. $\begin{cases} x - 3y = -\frac{1}{13} \\ x + 5y = -5 \end{cases}$
23. $\begin{cases} x + 2y = -3 \\ 3x - 4y = -2 \end{cases}$
24. $\begin{cases} 5x - y = 12 \\ 9x - y = 10 \end{cases}$
25. $\begin{cases} x + 2y = -6 \\ -4x - 8y = 24 \end{cases}$
26. $\begin{cases} x + 3y = -6 \\ -2x - 6y = -12 \end{cases}$
27. $\begin{cases} -3x + y = -4 \\ 6x - 2y = -2 \end{cases}$
28. $\begin{cases} x - 5y = -10 \\ 2x - 10y = -20 \end{cases}$
29. $\begin{cases} 3x - y = 9 \\ 4x + 3y = -1 \end{cases}$
30. $\begin{cases} 2x - y = 5 \\ 4x + 2y = -2 \end{cases}$
31. $\begin{cases} -x + 4y = 0 \\ 2x - 5y = -6 \end{cases}$
32. $\begin{cases} 3y - x = 5 \\ 5x + 2y = -8 \end{cases}$
33. $\begin{cases} 2x - 5y = 1 \\ 4x + 10y = 2 \end{cases}$
34. $\begin{cases} 3x - 7y = -3 \\ 6x + 14y = 0 \end{cases}$
35. $\begin{cases} 10x - y = 3 \\ -5x + 12y = 1 \end{cases}$
36. $\begin{cases} -\frac{1}{3}x + \frac{1}{6}y = \frac{2}{3} \\ \frac{1}{2}x - \frac{1}{3}y = -\frac{3}{2} \end{cases}$
37. $\begin{cases} \frac{1}{3}x + \frac{2}{3}y = 1 \\ \frac{1}{4}x - \frac{1}{3}y = -\frac{1}{12} \end{cases}$
38. $\begin{cases} \frac{1}{7}x - y = \frac{1}{2} \\ \frac{1}{4}x + \frac{1}{2}y = 2 \end{cases}$

39. $\begin{cases} -\frac{3}{5}x+\frac{2}{5}y=\frac{1}{2} \\ \frac{1}{3}x-\frac{1}{12}y=-\frac{1}{3} \end{cases}$
40. $\begin{cases} \frac{1}{2}x=\frac{2}{3}y \\ x-\frac{2}{3}y=2 \end{cases}$
41. $\begin{cases} -\frac{1}{2}x+\frac{1}{2}y=\frac{5}{8} \\ \frac{1}{4}x+\frac{1}{2}y=\frac{1}{4} \end{cases}$
42. $\begin{cases} x-y=0 \\ -x+2y=3 \end{cases}$
43. $\begin{cases} y=3x \\ 2x-3y=0 \end{cases}$
44. $\begin{cases} 2x+3y=18 \\ -6x+3y=-6 \end{cases}$
45. $\begin{cases} -3x+4y=20 \\ 2x+8y=8 \end{cases}$
46. $\begin{cases} 5x-3y=-1 \\ 3x+2y=7 \end{cases}$
47. $\begin{cases} -3x+7y=2 \\ 2x+7y=1 \end{cases}$
48. $\begin{cases} y=3 \\ y=-3 \end{cases}$
49. $\begin{cases} x=5 \\ x=-2 \end{cases}$
50. $\begin{cases} y=4 \\ y=4 \end{cases}$

Answer

1. $(2,7)$
3. $(-5,-1)$
5. $(2,6)$
7. $\emptyset$
9. $(x,2x+3)$
11. $(\frac{1}{2},-\frac{5}{2})$
13. $(4,-2)$
15. $(3,\frac{12}{5})$
17. $(-3,-\frac{7}{6})$
19. $(2,4)$
21. $(3,-4)$
23. $(-\frac{8}{5},-\frac{7}{10})$
25. $(x,-\frac{1}{2}x-3)$
27. $\emptyset$
29. $(2,-3)$
31. $(-8,-2)$
33. $(\frac{1}{2},0)$
35. $\emptyset$
37. $(1,1)$
39. $(-\frac{11}{10},-\frac{2}{5})$

41. $(-\frac{1}{2}, \frac{3}{4})$
43. $(0, 0)$
45. $(-4, 2)$
47. $(-\frac{1}{5}, \frac{1}{5})$
49. $\emptyset$

EXERCISE 1.4.4 SUBSTITUTION METHOD

Set up a linear system and solve it using the substitution method.

1. The sum of two numbers is 19. The larger number is 1 less than three times the smaller.
2. The sum of two numbers is 15. The larger is 3 more than twice the smaller.
3. The difference of two numbers is 7 and their sum is 1.
4. The difference of two numbers is 3 and their sum is -7.
5. Where on the graph of $-5x + 3y = 30$ does the x-coordinate equal the y-coordinate?
6. Where on the graph of $\frac{1}{2}x - \frac{1}{3}y = 1$ does the x-coordinate equal the y-coordinate?

Answer

1. The two numbers are 5 and 14.
3. The two numbers are 4 and -3.
5. $(-\frac{1}{5}, -\frac{1}{5})$

EXERCISE 1.4.5 DISCUSSION BOARD TOPICS

1. Describe what drives the choice of variable to solve for when beginning the process of solving by substitution.
2. Discuss the merits and drawbacks of the substitution method.

Answer

1. Answers may vary

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1.5: SOLVING LINEAR SYSTEMS BY ELIMINATION

LEARNING OBJECTIVES

- Solve linear systems using the elimination method.
- Solve linear systems with fractions and decimals.
- Identify the weaknesses and strengths of each method for solving linear systems.

THE ELIMINATION METHOD

In this section, the goal is to develop another completely algebraic method for solving a system of linear equations. We begin by defining what it means to add equations together. In the following example, notice that if we add the expressions on both sides of the equal sign, we obtain another true statement.

$$\begin{array}{rl} & 2+3=5 \\ + & 1+7=8 \\ \hline & 3+10=13 \\ & 13=13 \quad \checkmark \end{array}$$

This is true in general: if A, B, C, and D are algebraic expressions, then we have the following addition property of equations:

$$\text{If } A=B \text{ and } C=D, \text{ then } A+C=B+D. \tag{1.5.1}$$

For the system

$$\begin{cases} x+y=5 \\ x-y=1 \end{cases}$$

we add the two equations together:

$$\begin{array}{rl} & x+y=5 \\ + & x-y=1 \\ \hline & 2x=6 \end{array}$$

The sum of y and $-y$ is zero and that term is eliminated. This leaves us with a linear equation with one variable that can be easily solved:

$$\begin{aligned} \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

At this point, we have the x coordinate of the simultaneous solution, so all that is left to do is back substitute to find the corresponding y-value.

$$\begin{aligned} x+y &= 5 \\ 3+y &= 5 \\ y &= 2 \end{aligned}$$

Hence the solution to the system is $(3,2)$. This process describes the elimination (or addition) method for solving linear systems. Of course, the variable is not always so easily eliminated. Typically, we have to find an equivalent system by applying the multiplication property of equality to one or both of the equations as a means to line up one of the variables to eliminate. The goal is to arrange that either the x terms or the y terms are opposites, so that when the equations are added, the terms eliminate. The steps for the elimination method are outlined in the following example.

EXAMPLE 1.5.1

Solve by elimination:

$$\begin{cases} 2x+y=7 \\ 3x-2y=-7 \end{cases}.$$

Solution:

Step 1: Multiply one, or both, of the equations to set up the elimination of one of the variables. In this example, we will eliminate the variable y by multiplying both sides of the first equation by 2. Take care to distribute.

$$\begin{aligned} 2(2x+y) &= 2(7) \\ 4x+2y &= 14 \end{aligned}$$

This leaves us with an equivalent system where the variable y is lined up to eliminate.

Step 2: Add the equations together to eliminate one of the variables.

$$\begin{array}{rl} & 4x+2y=14 \\ + & 3x-2y=-7 \\ \hline & 7x=7 \end{array}$$

Step 3: Solve for the remaining variable.

$$\begin{aligned} \frac{7x}{7} &= \frac{7}{7} \\ x &= 1 \end{aligned}$$

Step 3: Back substitute into either equation or its equivalent equation.

$$\begin{aligned} 2x+y &= 7 \\ 2(1)+y &= 7 \\ 2+y-2 &= 7-2 \\ y &= 5 \end{aligned}$$

Step 4: Check. Remember that the solution must solve both of the original equations.

$$Check: (1,5)$$

$Equation\ 1:$	$Equation\ 2:$
$2x+y=7$	$4x+2y=14$
$2(1)+(5)=7$	$4(1)+2(5)=14$
$2+5=7$	$4+10=14$
$7=7$ ✓	$14=14$ ✓

Answer:

$$(1,5)$$

Occasionally, we will have to multiply both equations to line up one of the variables to eliminate. We want the resulting equivalent equations to have terms with opposite coefficients.

EXAMPLE 1.5.2

Solve by elimination:

$$\begin{cases} 5x-3y=-1 \\ 3x+2y=7 \end{cases}.$$

Solution:

We choose to eliminate the terms with variable y because the coefficients have different signs. To do this, we first determine the least common multiple of the coefficients; in this case, the LCM$(3,2)$ is 6. Therefore, multiply both sides of both equations by the appropriate values to obtain coefficients of -6 and 6.

$$\begin{aligned} 2(5x-3y) &= 2(-1) \quad \textit{Multiply both sides of the first equation by 2.} \\ 10x-6y &= -2 \\ 3(3x+2y) &= 3(7) \quad \textit{Multiply both sides of the second equation by 3.} \\ 9x+6y &= 21 \end{aligned}$$

This results in the following equivalent system:

The y terms are now lined up to eliminate.

Back substitute.

$$\begin{aligned} 3x+2y &= 7 \\ 3(1)+2y &= 7 \\ 3+2y-3 &= 7-3 \\ 2y &= 4 \\ y &= 2 \end{aligned}$$

Answer:

$$(1,2)$$

Sometimes linear systems are not given in standard form. When this is the case, it is best to first rearrange the equations before beginning the steps to solve by elimination.

EXAMPLE 1.5.3

Solve by elimination:

$$\begin{cases} 5x+12y=11 \\ 3y=4x+1 \end{cases}.$$

Solution:

First, rewrite the second equation in standard form.

$$\begin{aligned} 3y &= 4x+1 \\ 3y-4x &= 4x+1-4x \\ -4x+3y &= 1 \end{aligned}$$

This results in the following equivalent system where like terms are aligned in columns:

$$\begin{cases} 5x+12y=11 \\ 3y=4x+1 \end{cases} \Rightarrow \begin{cases} 5x+12y=11 \\ -4x+3y=1 \end{cases}$$

We can eliminate the term with variable y if we multiply the second equation by -4.

Next, we add the equations together,

$$\begin{aligned} 5x12y &= 11 \\ +\quad 16x-12y &= -4 \\ \hline 21x &= 7 \\ \frac{21x}{21} &= \frac{7}{21} \\ x &= \frac{1}{3} \end{aligned}$$

Back substitute.

$$\begin{aligned} 3y &= 4x+1 \\ 3y &= 4\left(\frac{1}{3}\right)+1 \\ 3y &= \frac{4}{3}+\frac{3}{3} \\ 3y &= \frac{7}{3} \\ \frac{3y}{3} &= \frac{3}{3} \\ y &= \frac{7}{3}\cdot\frac{1}{3} \\ y &= \frac{7}{9} \end{aligned}$$

Answer:

$$(\tfrac{1}{3},\tfrac{7}{9})$$

EXERCISE 1.5.1

Solve by elimination:

$$\begin{cases} 2x+y=-3 \\ -3x-2y=4 \end{cases}.$$

Answer

$(-2,1)$

At this point, we explore what happens when solving dependent and inconsistent systems using the elimination method.

EXAMPLE 1.5.4

Solve by elimination:

$$\begin{cases} 3x - y = 7 \\ 6x - 2y = 14 \end{cases}.$$

Solution:

To eliminate the variable x, we could multiply the first equation by -2.

Now adding the equations we have

A true statement indicates that this is a dependent system. The lines coincide, and we need y in terms of x to present the solution set in the form $(x, mx + b)$. Choose one of the original equations and solve for y. Since the equations are equivalent, it does not matter which one we choose.

Answer:

$(x, 3x - 7)$

EXAMPLE 1.5.5

Solve by elimination:

$$\begin{cases} -x + 3y = 9 \\ 2x - 6y = 12 \end{cases}.$$

Solution:

We can eliminate x by multiplying the first equation by 2.

$$\begin{cases} -x + 3y = 9 \\ 2x - 6y = 12 \end{cases} \overset{\times 2}{\Rightarrow} \begin{cases} -2x + 6y = 18 \\ 2x - 6y = 12 \end{cases}$$

Now adding the equations we have

$$\begin{array}{r} -2x + 6y = 18 \\ \underline{2x - 6y = 12} \\ 0 = 30 \quad False \end{array}$$

A false statement indicates that the system is inconsistent. The lines are parallel and do not intersect.

Answer:

No solution, $\emptyset$

EXERCISE 1.5.2

Solve by elimination:

$$\begin{cases} 3x + 15y = -15 \\ 2x + 10y = 30 \end{cases}.$$

Answer

No solution, $\emptyset$

CLEARING FRACTIONS AND DECIMALS

Given a linear system where the equations have fractional coefficients, it is usually best to clear the fractions before beginning the elimination method.

EXAMPLE 1.5.6

Solve:

$$\begin{cases} -\frac{1}{10}x + \frac{1}{2}y = \frac{4}{5} \\ \frac{1}{7}x + \frac{1}{3}y = -\frac{2}{21} \end{cases}.$$

Solution:

Recall that we can clear fractions by multiplying both sides of an equation by the least common denominator (LCD). Take care to distribute and then simplify.

$$\begin{array}{c|c} \text{Equation 1:} & \text{Equation 2:} \\ 10\left(-\frac{1}{10}x+\frac{1}{2}y\right)=10\left(\frac{4}{5}\right) & 21\left(\frac{1}{7}x+\frac{1}{3}y\right)=21\left(-\frac{2}{21}\right) \\ 10\cdot\left(-\frac{1}{10}x\right)+10\cdot\frac{1}{2}y=10\cdot\frac{4}{5} & 21\cdot\frac{1}{7}x+21\cdot\frac{1}{3}y=21\left(-\frac{2}{21}\right) \\ -x+5y=8 & 3x+7y=-2 \end{array}$$

This results in an equivalent system where the equations have integer coefficients,

Solve using the elimination method.

$$\begin{aligned} -3x+15y &= 24 \\ +\quad 3x+7y &= -2 \\ \hline 22y &= 22 \\ \frac{22y}{22} &= \frac{22}{22} \\ y &= 1 \end{aligned}$$

Back substitute.

Answer:

$(-3,1)$

We can use a similar technique to clear decimals before solving.

EXAMPLE 1.5.7

Solve:

$$\begin{cases} 3x-0.6y=-0.9 \\ -0.5x+0.12y=0.16 \end{cases}.$$

Solution:

Multiply each equation by the lowest power of 10 necessary to result in integer coefficients. In this case, multiply the first equation by 10 and the second equation by 100.

$$\begin{array}{c|c} \text{Equation 1:} & \text{Equation 2:} \\ 10(3x-0.6y)=10(-0.9) & 100(-0.5x+0.12y)=100(0.16) \\ 30x-6y=-9 & -50x+12y=16 \end{array}$$

This results in an equivalent system where the equations have integer coefficients:

Solve using the elimination method.

Back substitute.

$$\begin{aligned} 3x-0.6y &= -0.9 \\ 3(-0.2)-0.6y &= -0.9 \\ -0.6-0.6y+0.6 &= -0.9+0.6 \\ \frac{-0.6y}{-0.6} &= \frac{-0.3}{-0.6} \\ y &= 0.5 \end{aligned}$$

Answer:

$$(-0.2,0.5)$$

EXERCISE 1.5.3

Solve using elimination:

$$\begin{cases} \frac{1}{3}x-\frac{2}{3}y=3 \\ \frac{1}{3}x-\frac{1}{2}y=\frac{8}{3} \end{cases}$$

Answer

$(5, -2)$

SUMMARY OF THE METHODS FOR SOLVING LINEAR SYSTEMS

We have developed three methods for solving linear systems of two equations with two variables. In this section, we summarize the strengths and weaknesses of each method.

The graphing method is useful for understanding what a system of equations is and what the solutions must look like. When the equations of a system are graphed on the same set of axes, we can see that the solution is the point where the graphs intersect. The graphing is made easy when the equations are in slope-intercept form. For example

$$\begin{cases} y = 5x + 15 \\ y = -5x + 5 \end{cases}$$

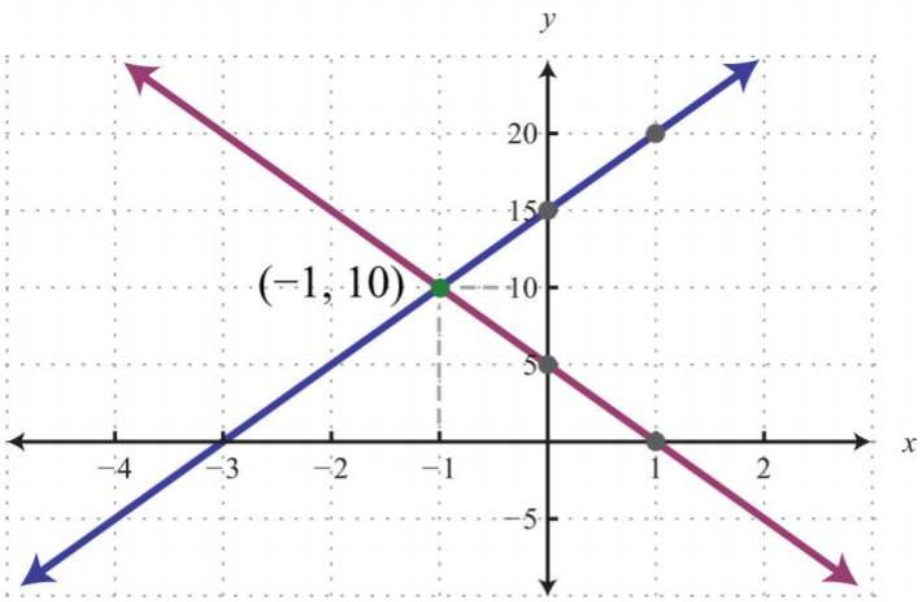


Figure 1.5.1

The simultaneous solution $(-1, 10)$ corresponds to the point of intersection. One drawback of this method is that it is very inaccurate. When the coordinates of the solution are not integers, the method is practically unusable. If we have a choice, we typically avoid this method in favor of the more accurate algebraic techniques.

The substitution method, on the other hand, is a completely algebraic method. It requires you to solve for one of the variables and substitute the result into the other equation. The resulting equation has one variable for which you can solve.

This method is particularly useful when there is a variable within the system with coefficient of 1. For example

$$\begin{cases} 10x + y = 20 \\ 7x + 5y = 14 \end{cases} \quad \textit{Choose the substitution method.}$$

In this case, it is easy to solve for y in the first equation and then substitute the result into the other equation. One drawback of this method is that it often leads to equivalent equations with fractional coefficients, which are tedious to work with. If there is not a coefficient of 1, then it usually is best to choose the elimination method.

The elimination method is a completely algebraic method that makes use of the addition property of equations. We multiply one or both of the equations to obtain equivalent equations where one of the variables is eliminated if we add them together. For example,

Here we multiply both sides of the first equation by 5 and both sides of the second equation by -2. This results in an equivalent system where the variable x is eliminated when we add the equations together. Of course, there are other combinations of numbers that achieve the same result. We could even choose to eliminate the variable y. No matter which variable is eliminated first, the solution will be the same. Note that the substitution method, in this case, would require tedious calculations with fractional coefficients. One weakness of the elimination method, as we will see later in our study of algebra, is that it does not always work for nonlinear systems.

KEY TAKEAWAYS

- The elimination method is a completely algebraic method for solving a system of equations.

- Multiply one or both of the equations in a system by certain numbers to obtain an equivalent system consisting of like terms with opposite coefficients. Adding these equivalent equations together eliminates a variable, and the resulting equation has one variable for which you can solve.
- It is a good practice to first rewrite the equations in standard form before beginning the elimination method.
- When the value of one of the variables is determined, back substitute into one of the original equations, or their equivalent equations, and determine the corresponding value of the other variable.

Solve by elimination.

1. $\begin{cases} x+y=3 \\ 2x-y=9 \end{cases}$
2. $\begin{cases} x-y=-6 \\ 5x+y=-18 \end{cases}$
3. $\begin{cases} x+3y=5 \\ -x-2y=0 \end{cases}$
4. $\begin{cases} -x+4y=4 \\ x-y=-7 \end{cases}$
5. $\begin{cases} -x+y=2 \\ x-y=-3 \end{cases}$
6. $\begin{cases} 3x-y=-2 \\ 6x+4y=2 \end{cases}$
7. $\begin{cases} 5x+2y=-3 \\ 10x-y=4 \end{cases}$
8. $\begin{cases} -2x+14y=28 \\ x-7y=21 \end{cases}$
9. $\begin{cases} -2x+y=4 \\ 12x-6y=-24 \end{cases}$
10. $\begin{cases} x+8y=3 \\ 3x+12y=6 \end{cases}$
11. $\begin{cases} 2x-3y=15 \\ 4x+10y=14 \end{cases}$
12. $\begin{cases} 4x+3y=-10 \\ 3x-9y=15 \end{cases}$
13. $\begin{cases} -4x-5y=-3 \\ 8x+3y=-15 \end{cases}$
14. $\begin{cases} -2x+7y=56 \\ 4x-2y=-112 \end{cases}$
15. $\begin{cases} -9x-15y=-15 \\ 3x+5y=-10 \end{cases}$
16. $\begin{cases} 6x-7y=4 \\ 2x+6y=-7 \end{cases}$
17. $\begin{cases} 4x+2y=4 \\ -5x-3y=-7 \end{cases}$
18. $\begin{cases} 5x-3y=-1 \\ 3x+2y=7 \end{cases}$
19. $\begin{cases} 7x+3y=9 \\ 2x+5y=-14 \end{cases}$
20. $\begin{cases} 9x-3y=3 \\ 7x+2y=-15 \end{cases}$
21. $\begin{cases} 5x-3y=-7 \\ -7x+6y=11 \end{cases}$
22. $\begin{cases} 2x+9y=8 \\ 3x+7y=-1 \end{cases}$
23. $\begin{cases} 2x+2y=5 \\ 3x+3y=-5 \end{cases}$

24. $\begin{cases} -3x+6y=-12 \\ 2x-4y=8 \end{cases}$
25. $\begin{cases} 25x+15y=-1 \\ 15x+10y=-1 \end{cases}$
26. $\begin{cases} 2x-3y=2 \\ 18x-12y=5 \end{cases}$
27. $\begin{cases} y=-2x-3 \\ -3x-2y=4 \end{cases}$
28. $\begin{cases} 28x+6y=9 \\ 6y=4x-15 \end{cases}$
29. $\begin{cases} y=5x+15 \\ y=-5x+5 \end{cases}$
30. $\begin{cases} 2x-3y=9 \\ 5x-8y=-16 \end{cases}$
31. $\begin{cases} \frac{1}{2}x-\frac{1}{3}y=\frac{1}{6} \\ \frac{5}{2}x+y=\frac{7}{2} \end{cases}$
32. $\begin{cases} \frac{1}{4}x-\frac{1}{9}y=1 \\ x+y=\frac{3}{4} \end{cases}$
33. $\begin{cases} \frac{1}{2}x-\frac{1}{4}y=\frac{1}{3} \\ \frac{1}{4}x+\frac{1}{2}y=-\frac{19}{6} \end{cases}$
34. $\begin{cases} -\frac{14}{3}x+2y=4 \\ -\frac{1}{3}x+\frac{1}{7}y=\frac{4}{21} \end{cases}$
35. $\begin{cases} 0.025x+0.1y=0.5 \\ 0.11x+0.04y=-0.2 \end{cases}$
36. $\begin{cases} 1.3x+0.1y=0.35 \\ 0.5x+y=-2.75 \end{cases}$
37. $\begin{cases} x+y=5 \\ 0.02x+0.03y=0.125 \end{cases}$
38. $\begin{cases} x+y=30 \\ 0.05x+0.1y=2.4 \end{cases}$

Answer

1. $(4,-1)$
3. $(-10,5)$
5. $\emptyset$
7. $(\frac{1}{5},-2)$
9. $(x,2x+4)$
11. $(6,-1)$
13. $(-3,3)$
15. $\emptyset$
17. $(-1,4)$
19. $(3,-4)$
21. $(-1,\frac{2}{3})$
23. $\emptyset$
25. $(\frac{1}{5},-\frac{2}{5})$
27. $(-2,1)$
29. $(-1,10)$

31. $(1,1)$
33. $(-2,-\frac{16}{3})$
35. $(-4,6)$
37. $(2.5,2.5)$

EXERCISE 1.5.5 ELIMINATION METHOD

Set up a linear system and solve it using the elimination method.

1. The sum of two numbers is 14. The larger number is 1 less than two times the smaller.
2. The sum of two numbers is 30. The larger is 2 more than three times the smaller.
3. The difference of two numbers is 13 and their sum is 11.
4. The difference of two numbers is 2 and their sum is -12.

Answer

1. The two numbers are 5 and 9.
3. The two numbers are 12 and -1.

EXERCISE 1.5.6 MIXED EXERCISES

Solve using any method.

1. $\begin{cases} y = 2x - 3 \\ 3x + y = 12 \end{cases}$
2. $\begin{cases} x + 3y = -5 \\ y = 13x + 5 \end{cases}$
3. $\begin{cases} x = -1 \\ y = 3 \end{cases}$
4. $\begin{cases} y = 1 \\ 2x + 9 = 0 \end{cases}$
5. $\begin{cases} y = x \\ -x + y = 1 \end{cases}$
6. $\begin{cases} y = 5x \\ y = -10 \end{cases}$
7. $\begin{cases} 3y = 2x - 24 \\ 3x + 4y = 2 \end{cases}$
8. $\begin{cases} y = -32x + 1 \\ -2y + 2 = 3x \end{cases}$
9. $\begin{cases} 7y = -2x - 1 \\ 7x = 2y + 23 \end{cases}$
10. $\begin{cases} 5x + 9y - 14 = 0 \\ 3x + 2y - 5 = 0 \end{cases}$
11. $\begin{cases} y = -\frac{5}{16}x + 10 \\ y = \frac{5}{16}x - 10 \end{cases}$
12. $\begin{cases} y = -\frac{6}{5}x + 1 \\ 2x = 6 \end{cases}$
13. $\begin{cases} 2(x-3) + y = 0 \\ 3(2x + y - 1) = 15 \end{cases}$
14. $\begin{cases} 3 - 2(x - y) = -3 \\ 4x - 3(y + 1) = 8 \end{cases}$
15. $\begin{cases} 2(x+1) = 3(2y-1) - 21 \\ 3(x+2) = 1 - (3y - 2) \end{cases}$

16. $\begin{cases} \frac{x}{2}-\frac{y}{3}=-7 \\ \frac{x}{3}-\frac{y}{2}=-8 \end{cases}$
17. $\begin{cases} \frac{x}{4}-\frac{y}{2}=\frac{3}{4} \\ \frac{x}{3}+\frac{y}{6}=\frac{1}{6} \end{cases}$
18. $\begin{cases} \frac{1}{3}x-\frac{2}{3}y=3 \\ \frac{1}{3}x-\frac{1}{2}y=\frac{8}{3} \end{cases}$
19. $\begin{cases} -\frac{1}{10}x+\frac{1}{2}y=\frac{4}{5} \\ \frac{1}{7}x+\frac{1}{3}y=-\frac{2}{21} \end{cases}$
20. $\begin{cases} y=-\frac{5}{3}x+\frac{1}{2} \\ \frac{1}{3}x+\frac{1}{5}y=\frac{1}{10} \end{cases}$
21. $\begin{cases} -\frac{1}{7}x+y=-\frac{2}{3} \\ -\frac{1}{14}x+\frac{1}{2}y=\frac{1}{3} \end{cases}$
22. $\begin{cases} \frac{1}{15}x-\frac{1}{12}y=\frac{1}{3} \\ -\frac{3}{10}x+\frac{3}{8}y=-\frac{3}{2} \end{cases}$
23. $\begin{cases} x+y=4,200 \\ 0.03x+0.0525y=193.5 \end{cases}$
24. $\begin{cases} x+y=350 \\ 0.2x+0.1y=52.5 \end{cases}$
25. $\begin{cases} 0.2x-0.05y=0.43 \\ 0.3x+0.1y=-0.3 \end{cases}$
26. $\begin{cases} 0.1x+0.3y=0.3 \\ 0.05x-0.5y=-0.63 \end{cases}$
27. $\begin{cases} 0.15x-0.25y=-0.3 \\ -0.75x+1.25y=-4 \end{cases}$
28. $\begin{cases} -0.15x+1.25y=0.4 \\ -0.03x+0.25y=0.08 \end{cases}$

Answer

1. $(3,3)$
3. $(-1,3)$
5. $\varnothing$
7. $(6,-4)$
9. $(3,-1)$
11. $(32,0)$
13. $(x,-2x+6)$
15. $(-4,3)$
17. $(1,-1)$
19. $(-3,1)$
21. $\emptyset$
23. $(1,200,3,000)$
25. $(0.8,-5.4)$
27. $\varnothing$

EXERCISE 1.5.7 DISCUSSION BOARD TOPICS

1. How do we choose the best method for solving a linear system?
2. What does it mean for a system to be dependent? How can we tell if a given system is dependent?

Answer

1. Answers may vary

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1.6: APPLICATIONS OF LINEAR SYSTEMS

LEARNING OBJECTIVES

- Set up and solve applications involving relationships between numbers.
- Set up and solve applications involving interest and money.
- Set up and solve mixture problems.
- Set up and solve uniform motion problems (distance problems).

PROBLEMS INVOLVING RELATIONSHIPS BETWEEN REAL NUMBERS

We now have the techniques needed to solve linear systems. For this reason, we are no longer limited to using one variable when setting up equations that model applications. If we translate an application to a mathematical setup using two variables, then we need to form a linear system with two equations.

EXAMPLE 1.6.1

The sum of two numbers is 40 and their difference is 8. Find the numbers.

Solution:

Identify variables:

Let x represent one of the unknown numbers.

Let y represent the other unknown number.

Set up equations:

When using two variables, we need to set up two equations. The first key phrase, "the ***sum*** of the two numbers is 40," translates as follows:

$$x+y=40$$

And the second key phrase, "the ***difference*** is 8," leads us to the second equation:

$$x-y=8$$

Therefore, our algebraic setup consists of the following system:

$$\begin{cases} x+y=40 \\ x-y=8 \end{cases}$$

Solve:

We can solve the resulting system using any method of our choosing. Here we choose to solve by elimination. Adding the equations together eliminates the variable y.

$$\begin{aligned} x+y &= 40 \\ +\quad x-y &= 8 \\ \hline 2x &= 48 \\ x &= 24 \end{aligned}$$

Once we have x, back substitute to find y.

$$\begin{aligned} x+y &= 40 \\ 24+y &= 40 \\ 24+y-24 &= 40-24 \\ y &= 16 \end{aligned}$$

Check:

The sum of the two numbers should be 42 and their difference 8.

$$\begin{aligned} 24+16 &= 40 \\ 24-16 &= 8 \end{aligned}$$

Answer:

The two numbers are 24 and 16.

EXAMPLE 1.6.2

The sum of 9 times a larger number and twice a smaller is 6. The difference of 3 times the larger and the smaller is 7. Find the numbers.

Solution:

Begin by assigning variables to the larger and smaller number.

Let x represent the larger number.

Let y represent the smaller number.

The first sentence describes a sum and the second sentence describes a difference.

$$\overbrace{9x}^{\text{9 times a larger}} + \overbrace{2y}^{\text{twice a smaller}} = 6$$

$$\overbrace{3x}^{\text{3 times the larger}} - \overbrace{y}^{\text{the smaller}} = 7$$

Figure 1.6.1

This leads to the following system:

$$\begin{cases} 9x + 2y = 6 \\ 3x - y = 7 \end{cases}$$

Solve using the elimination method. Multiply the second equation by 2 and add.

$$\begin{cases} 9x + 2y = 6 \\ 3x - y = 7 \end{cases} \overset{\times 2}{\Rightarrow} \begin{cases} 9x + 2y = 6 \\ 6x - 2y = 14 \end{cases}$$

$$\begin{array}{rl} & 9x + 2y = 6 \\ + & 6x - 2y = 14 \\ \hline & 15x = 20 \\ & x = \dfrac{20}{15} \\ & x = \dfrac{4}{3} \end{array}$$

Back substitute to find y.

Answer:

The larger number is $\frac{4}{3}$ and the smaller number is -3.

EXERCISE 1.6.1

The sum of two numbers is 3. When twice the smaller number is subtracted from 6 times the larger the result is 22. Find the numbers.

Answer

The two numbers are $-\frac{1}{2}$ and $\frac{7}{2}$.

INTEREST AND MONEY PROBLEMS

In this section, the interest and money problems should seem familiar. The difference is that we will be making use of two variables when setting up the algebraic equations.

EXAMPLE 1.6.3

A roll of 32 bills contains only \$5 bills and \$10 bills. If the value of the roll is \$220, then how many of each bill are in the roll?

Solution:

Begin by identifying the variables.

Let x represent the number of \$5 bills.

Let y represent the number of \$10 bills.

When using two variables, we need to set up two equations. The first equation is created from the fact that there are 32 bills.

$$x+y=32$$

The second equation sums the value of each bill: the total value is \$220.

$$\$5\cdot x+\$10\cdot y=\$220$$

Present both equations as a system; this is our algebraic setup.

$$\begin{cases} x+y=32 \\ 5x+10y=220 \end{cases}$$

Here we choose to solve by elimination, although substitution would work just as well. Eliminate x by multiplying the first equation by -5.

Now add the equations together:

Once we have y, the number of \$10 bills, back substitute to find x.

$$\begin{aligned} x+y &= 32 \\ x+12 &= 32 \\ x+12-12 &= 32-12 \\ x &= 20 \end{aligned}$$

Answer:

There are twenty \$5 bills and twelve \$10 bills. The check is left to the reader.

EXAMPLE 1.6.4

A total of $\$6,300$ was invested in two accounts. Part was invested in a CD at a $4\frac{1}{2}\%$ annual interest rate and part was invested in a money market fund at a $3\frac{3}{4}\%$ annual interest rate. If the total simple interest for one year was $\$267.75$, then how much was invested in each account?

Solution:

Let x represent the amount invested at $4\frac{1}{2}\%=4.5\%=0.045$

Let y represent the amount invested at $3\frac{3}{4}\%=3.75\%=0.0375$

The total amount in both accounts can be expressed as

$$x+y=6,300$$

To set up a second equation, use the fact that the total interest was $\$267.75$. Recall that the interest for one year is the interest rate times the principal $(I=prt=pr\cdot 1=pr)$. Use this to add the interest in both accounts. Be sure to use the decimal equivalents for the interest rates given as percentages.

$interest\ from\ the\ CD$	$+$	$interest\ from\ the\ fund$	$=$	$total\ interest$
$0.045x$	$+$	$0.375y$	$=$	267.75

These two equations together form the following linear system:

$$\begin{cases} x+y=6,300 \\ 0.045x+0.0375y=267.75 \end{cases}$$

Eliminate y by multiplying the first equation by -0.0375.

$$\begin{cases} x+y=6,300 \\ 0.045x+0.0375y=267.75 \end{cases} \overset{\times(-0.0375)}{\Rightarrow} \begin{cases} -0.0375x-0.0375y=-236.25 \\ 0.045x+0.0375y=267.75 \end{cases}$$

Next, add the equations together to eliminate the variable y.

$$\begin{aligned} -0.0375x-0.0375y &= -236.25 \\ +\quad 0.045x+0.0375y &= 267.75 \\ \hline 0.0075x &= 31.5 \\ \frac{0.0075x}{0.0075} &= \frac{31.5}{0.0075} \\ x &= 4,200 \end{aligned}$$

Back substitute.

$$\begin{aligned} x+y &= 6,300 \\ 4,200+y &= 6,300 \\ 4,200+y-4,200 &= 6,300-4,200 \\ y &= 2,100 \end{aligned}$$

Answer:

$\$4,200$ was invested at $4\frac{1}{2}\%$ and $\$2,100$ was invested at $3\frac{3}{4}\%$

At this point, we should be able to solve these types of problems in two ways: with one variable and now with two variables. Setting up word problems with two variables often simplifies the entire process, particularly when the relationships between the variables are not so clear.

? EXERCISE 1.6.2

On the first day of a two-day meeting, 10 coffees and 10 doughnuts were purchased for a total of $\$20.00$. Since nobody drank the coffee and all the doughnuts were eaten, the next day only 2 coffees and 14 doughnuts were purchased for a total of $\$13.00$. How much did each coffee and each doughnut cost?

Answer

Coffee: $\$1.25$; doughnut: $\$0.75$

MIXTURE PROBLEMS

Mixture problems often include a percentage and some total amount. It is important to make a distinction between these two types of quantities. For example, if a problem states that a 20-ounce container is filled with a 2% saline (salt) solution, then this means that the container is filled with a mixture of salt and water as follows:

Table 1.6.1

	Percentage	Amount
Salt	$2\% = 0.02$	$0.02(20\,\text{ounces}) = 0.4$ ounces
Water	$98\% = 0.98$	$0.98(20\,\text{ounces}) = 19.6$ ounces

In other words, we multiply the percentage times the total to get the amount of each part of the mixture.

✓ EXAMPLE 1.6.5

A 2% saline solution is to be combined and mixed with a 5% saline solution to produce 72 ounces of a 2.5% saline solution. How much of each is needed?

Solution:

Let x represent the amount of 2% saline solution needed.

Let y represent the amount of 5% saline solution needed.

The total amount of saline solution needed is 72 ounces. This leads to one equation,

$$x+y=72$$

The second equation adds up the amount of salt in the correct percentages. The amount of salt is obtained by multiplying the percentage times the amount, where the variables x and y represent the amounts of the solutions.

$$\begin{array}{ccccc} \textit{salt in 2\% solution} & + & \textit{salt in 5\% solution} & = & \textit{salt in the end solution} \\ 0.02x & + & 0.05y & = & 0.025(72) \end{array}$$

Solve.

$$\begin{cases} x+y=72 \\ 0.02x+0.05y=0.025(72) \end{cases} \overset{\times(-0.02)}{\Rightarrow} \begin{cases} -0.02x-0.02y=-1.44 \\ 0.02x+0.05y=1.8 \end{cases}$$

$$\begin{array}{r} -0.02x-0.02y=-1.44 \\ +\quad 0.02x+0.05y=1.8 \\ \hline 0.03y=0.36 \end{array}$$

$$\frac{0.03y}{0.03}=\frac{0.36}{0.03}$$

$$y=12$$

Back substitute.

$$\begin{aligned} x+y &= 72 \\ x+12 &= 72 \\ x+12-12 &= 72-12 \\ x &= 60 \end{aligned}$$

Answer:

We need 60 ounces of the 2% saline solution and 12 ounces of the 5% saline solution.

EXAMPLE 1.6.6

A 50% alcohol solution is to be mixed with a 10% alcohol solution to create an 8-ounce mixture of a 32% alcohol solution. How much of each is needed?

Solution:

Let x represent the amount of 50% alcohol solution needed.

Let y represent the amount of 10% alcohol solution needed.

The total amount of the mixture must be 8 ounces.

$$x+y=8$$

The second equation adds up the amount of alcohol from each solution in the correct percentages. The amount of alcohol in the end result is 32% of 8 ounces, or $0.032(8)$.

$$\begin{array}{ccccc} \textit{alcohol in 50\% solution} & + & \textit{alcohol in 10\% solution} & = & \textit{alcohol in the end solution} \\ 0.50x & + & 0.10y & = & 0.32(8) \end{array}$$

Now we can form a system of two linear equations and two variables as follows:

$$\begin{cases} x+y=8 \\ 0.50x+0.10y=0.32(8) \end{cases}$$

In this example, multiply the second equation by 100 to eliminate the decimals. In addition, multiply the first equation by -10 to line up the variable y to eliminate.

$$\begin{array}{c|c} \textit{Equation 1}: & \textit{Equation 2}: \\ -10(x+y)=-10(8) & 100\ 0.50x+0.10y=100(0.32)(8) \\ -10x-10y=-80 & 50x+10y=256 \end{array}$$

We obtain the following equivalent system:

Add the equations and then solve for x:

Back substitute.

$$\begin{aligned} x+y &= 8 \\ 4.4+y &= 8 \\ 4.4+y-4.4 &= 8-4.4 \\ x &= 3.6 \end{aligned}$$

Answer:

To obtain 8 ounces of a 32% alcohol mixture we need to mix 4.4 ounces of the 50% alcohol solution and 3.6 ounces of the 10% solution.

EXERCISE 1.6.3

A 70% antifreeze concentrate is to be mixed with water to produce a 5-gallon mixture containing 28% antifreeze. How much water and antifreeze concentrate is needed?

Answer

We need to mix 3 gallons of water with 2 gallons of antifreeze concentrate.

UNIFORM MOTION PROBLEMS (DISTANCE PROBLEMS)

Recall that the distance traveled is equal to the average rate times the time traveled at that rate, $D = r \cdot t$.

These uniform motion problems usually have a lot of data, so it helps to first organize that data in a chart and then set up a linear system. In this section, you are encouraged to use two variables.

EXAMPLE 1.6.7

An executive traveled a total of 8 hours and 1,930 miles by car and by plane. Driving to the airport by car, she averaged 60 miles per hour. In the air, the plane averaged 350 miles per hour. How long did it take her to drive to the airport?

Solution:

We are asked to find the time it takes her to drive to the airport; this indicates that time is the unknown quantity.

Let x represent the time it took to drive to the airport.

Let y represent the time spent in the air.

	Distance =	***Rate*** ×	***Time***
Travel by car		60 mph	x
Travel by air		350 mph	y
Total	1,930 mi		8 hours

Figure 1.6.2

Use the formula $D = r \cdot t$ to fill in the unknown distances.

$$\textit{Distance traveled in the car}: \quad D = r \cdot t = 60 \cdot x$$
$$\textit{Distance traveled in the air}: \quad D = r \cdot t = 350 \cdot y$$

	Distance =	***Rate*** ×	***Time***
Travel by car	$60x$	60 mph	x
Travel by air	$350y$	350 mph	y
Total	1,930 mi		8 hours

Figure 1.6.3

The distance column and the time column of the chart help us to set up the following linear system.

	Distance =	***Rate*** ×	***Time***
Travel by car	$60x$	60 mph	x
Travel by air	$350y$	350 mph	y
Total	1,930 mi		8 hours

$60x + 350y = 1{,}930$ $\qquad$ $x + y = 8$

Figure 1.6.4

$$\begin{cases} x + y = 8 & \leftarrow \textit{total time traveled} \\ 6 - x + 350y = 1{,}930 & \leftarrow \textit{total distance traveled} \end{cases}$$

Solve.

Now back substitute to find the time it took to drive to the airport x:

$$\begin{aligned} x+y &= 8 \\ x+5 &= 8 \\ x &= 3 \end{aligned}$$

Answer:

It took her 3 hours to drive to the airport.

It is not always the case that time is the unknown quantity. Read the problem carefully and identify what you are asked to find; this defines your variables.

EXAMPLE 1.6.8

Flying with the wind, an airplane traveled $1,365$ miles in 3 hours. The plane then turned against the wind and traveled another 870 miles in 2 hours. Find the speed of the airplane and the speed of the wind.

Solution:

There is no obvious relationship between the speed of the plane and the speed of the wind. For this reason, use two variables as follows:

Let x represent the speed of the airplane.

Let w represent the speed of the wind.

Use the following chart to organize the data:

	Distance =	Rate ×	Time
Flight with wind	1,365 mi		3 hrs
Flight against wind	870 mi		2 hrs
Total			

Figure 1.6.5

With the wind, the airplane's total speed is $x+w$. Flying against the wind, the total speed is $x-w$.

	Distance =	Rate ×	Time
Flight with wind	1,365 mi	$x+w$	3 hrs
Flight against wind	870 mi	$x-w$	2 hrs
Total	2,235 mi		5 hrs

Figure 1.6.6

Use the rows of the chart along with the formula $D = r \cdot t$ to construct a linear system that models this problem. Take care to group the quantities that represent the rate in parentheses.

	Distance =	Rate ×	Time	
Flight with wind	1,365 mi	$x+w$	3 hrs	$1,365 = (x+w)\cdot 3$
Flight against wind	870 mi	$x-w$	2 hrs	$870 = (x-w)\cdot 2$
Total	2,235 mi		5 hrs	

Figure 1.6.7

$$\begin{cases} 1,365 = (x+w)\cdot 3 & \leftarrow \textit{distance traveled with the wind} \\ 870 = (x-w)\cdot 2 & \leftarrow \textit{distance traveled against the wind} \end{cases}$$

If we divide both sides of the first equation by 3 and both sides of the second equation by 2, then we obtain the following equivalent system:

$$\begin{cases} 1,365 = (x+w)\cdot 3 \\ 870 = (x-w)\cdot 2 \end{cases} \overset{\div 3}{\underset{\div 2}{\Rightarrow}} \begin{cases} 455 = x+w \\ 435 = x-w \end{cases}$$

$$\begin{aligned} x+w &= 455 \\ +\quad x-w &= 435 \\ \hline 2x &= 890 \\ \frac{2x}{2} &= \frac{890}{2} \\ x &= 455 \end{aligned}$$

Back substitute.

$$\begin{aligned} x+w &= 455 \\ 455+w &= 455 \\ w &= 10 \end{aligned}$$

Answer:

The speed of the airplane is 445 miles per hour and the speed of the wind is 10 miles per hour.

EXERCISE 1.6.4

A boat traveled 24 miles downstream in 2 hours. The return trip, which was against the current, took twice as long. What are the speeds of the boat and of the current?

Answer

The speed of the boat is 9 miles per hour and the speed of the current is 3 miles per hour.

KEY TAKEAWAYS

- Use two variables as a means to simplify the algebraic setup of applications where the relationship between unknowns is unclear.
- Carefully read the problem several times. If two variables are used, then remember that you need to set up two linear equations in order to solve the problem.
- Be sure to answer the question in sentence form and include the correct units for the answer.

EXERCISE 1.6.5 APPLICATIONS INVOLVING NUMBERS

Set up a linear system and solve.

1. The sum of two integers is 54 and their difference is 10. Find the integers.
2. The sum of two integers is 50 and their difference is 24. Find the integers.
3. The sum of two positive integers is 32. When the smaller integer is subtracted from twice the larger, the result is 40. Find the two integers.
4. The sum of two positive integers is 48. When twice the smaller integer is subtracted from the larger, the result is 12. Find the two integers.
5. The sum of two integers is 74. The larger is 26 more than twice the smaller. Find the two integers.
6. The sum of two integers is 45. The larger is 3 less than three times the smaller. Find the two integers.
7. The sum of two numbers is zero. When 4 times the smaller number is added to 8 times the larger, the result is 1. Find the two numbers.
8. The sum of a larger number and 4 times a smaller number is 5. When 8 times the smaller is subtracted from twice the larger, the result is -2. Find the numbers.
9. The sum of 12 times the larger number and 11 times the smaller is -36. The difference of 12 times the larger and 7 times the smaller is 36. Find the numbers.
10. The sum of 4 times the larger number and 3 times the smaller is 7. The difference of 8 times the larger and 6 times the smaller is 10. Find the numbers.

Answer

1. The integers are 22 and 32.
3. The integers are 8 and 24.
5. The integers are 16 and 58.

7. The two numbers are $-\frac{1}{4}$ and $\frac{1}{4}$.

9. The smaller number is -4 and the larger is $\frac{2}{3}$.

EXERCISE 1.6.6 INTEREST AND MONEY PROBLEMS

Set up a linear system and solve.

1. A $\$7,000$ principal is invested in two accounts, one earning 3% interest and another earning 7% interest. If the total interest for the year is $\$262$, then how much is invested in each account?
2. Mary has her total savings of $\$12,500$ in two different CD accounts. One CD earns 4.4% interest and another earns 3.2% interest. If her total interest for the year is $\$463$, then how much does she have in each CD account?
3. Sally's $\$1,800$ savings is in two accounts. One account earns 6% annual interest and the other earns 3%. Her total interest for the year is $\$93$. How much does she have in each account?
4. Joe has two savings accounts totaling $\$4,500$. One account earns $3\frac{3}{4}\%$ annual interest and the other earns $2\frac{5}{8}\%$. If his total interest for the year is $\$141.75$, then how much is in each account?
5. Millicent has $\$10,000$ invested in two accounts. For the year, she earns $\$535$ more in interest from her 7% mutual fund account than she does from her 4% CD. How much does she have in each account?
6. A small business has $\$85,000$ invested in two accounts. If the account earning 3% annual interest earns $\$825$ more in interest than the account earning 4.5% annual interest, then how much is invested in each account?
7. Jerry earned a total of $\$284$ in simple interest from two separate accounts. In an account earning 6% interest, Jerry invested $\$1,000$ more than twice the amount he invested in an account earning 4%. How much did he invest in each account?
8. James earned a total of $\$68.25$ in simple interest from two separate accounts. In an account earning 2.6% interest, James invested one-half as much as he did in the other account that earned 5.2%. How much did he invest in each account?
9. A cash register contains $\$10$ bills and $\$20$ bills with a total value of $\$340$. If there are 23 bills total, then how many of each does the register contain?
10. John was able to purchase a pizza for $\$10.80$ with quarters and dimes. If he uses 60 coins to buy the pizza, then how many of each did he have?
11. Dennis mowed his neighbor's lawn for a jar of dimes and nickels. Upon completing the job, he counted the coins and found that there were 4 less than twice as many dimes as there were nickels. The total value of all the coins is $\$6.60$. How many of each coin did he have?
12. Two families bought tickets for the big football game. One family ordered 2 adult tickets and 3 children's tickets for a total of $\$26.00$. Another family ordered 3 adult tickets and 4 children's tickets for a total of $\$37.00$. How much did each adult ticket cost?
13. Two friends found shirts and shorts on sale at a flea market. One bought 5 shirts and 3 shorts for a total of $\$51.00$. The other bought 3 shirts and 7 shorts for a total of $\$80.00$. How much was each shirt and each pair of shorts?
14. On Monday Joe bought 10 cups of coffee and 5 doughnuts for his office at a cost of $\$16.50$. It turns out that the doughnuts were more popular than the coffee. Therefore, on Tuesday he bought 5 cups of coffee and 10 doughnuts for a total of $\$14.25$. How much was each cup of coffee?

Answer

1. $\$5,700$ at 3% and $\$1,300$ at 7%

3. $\$1,300$ at 6% and $\$500$ at 3%

5. $\$8,500$ at 7% and $\$1,500$ at 4%

7. $\$1,400$ at 4% and $\$3,800$ at 6%

9. 12 tens and 11 twenties

11. 52 dimes and 28 nickels

13. Shirts: $\$4.50$; shorts: $\$9.50$

EXERCISE 1.6.7 MIXTURE PROBLEMS

Set up a linear system and solve.

1. A 15% acid solution is to be mixed with a 25% acid solution to produce 12 gallons of a 20% acid solution. How much of each is needed?
2. One alcohol solution contains 12% alcohol and another contains 26% alcohol. How much of each should be mixed together to obtain 5 gallons of a 14.8% alcohol solution?

3. A nurse wishes to obtain 40 ounces of a 1.2% saline solution. How much of a 1% saline solution must she mix with a 2% saline solution to achieve the desired result?
4. A customer ordered 20 pounds of fertilizer that contains 15% nitrogen. To fill the customer's order, how much of the stock 30% nitrogen fertilizer must be mixed with the 10% nitrogen fertilizer?
5. A customer ordered 2 pounds of a mixed peanut product containing 15% cashews. The inventory consists of only two mixes containing 10% and 30% cashews. How much of each type must be mixed to fill the order?
6. How many pounds of pure peanuts must be combined with a 20% peanut mix to produce 10 pounds of a 32% peanut mix?
7. How much cleaning fluid with 20% alcohol content, must be mixed with water to obtain a 24-ounce mixture with 10% alcohol content?
8. A chemist wishes to create a 32-ounce solution with 12% acid content. He uses two types of stock solutions, one with 30% acid content and another with 10% acid content. How much of each does he need?
9. A concentrated cleaning solution that contains 50% ammonia is mixed with another solution containing 10% ammonia. How much of each is mixed to obtain 8 ounces of a 32% ammonia cleaning formula?
10. A 50% fruit juice concentrate can be purchased wholesale. Best taste is achieved when water is mixed with the concentrate in such a way as to obtain a 12% fruit juice mixture. How much water and concentrate is needed to make a 50-ounce fruit juice drink?
11. A 75% antifreeze concentrate is to be mixed with water to obtain 6 gallons of a 25% antifreeze solution. How much water is needed?
12. Pure sugar is to be mixed with a fruit salad containing 10% sugar to produce 48 ounces of a salad containing 16% sugar. How much pure sugar is required?

Answer

1. 6 gallons of each
3. 32 ounces of the 1% saline solution and 8 ounces of the 2% saline solution
5. 1.5 pounds of the 10% cashew mix and 0.5 pounds of the 30% cashew mix
7. 12 ounces of cleaning fluid
9. 4.4 ounces of the 50% ammonia solution and 3.6 ounces of the 10% ammonia solution
11. 4 gallons

? EXERCISE 1.6.8 UNIFORM MOTION PROBLEMS

Set up a linear system and solve.

1. An airplane averaged 460 miles per hour on a trip with the wind behind it and 345 miles per hour on the return trip against the wind. If the total round trip took 7 hours, then how long did the airplane spend on each leg of the trip?
2. The two legs of a 330-mile trip took 5 hours. The average speed for the first leg of the trip was 70 miles per hour and the average speed for the second leg of the trip was 60 miles per hour. How long did each leg of the trip take?
3. An executive traveled 1,200 miles, part by helicopter and part by private jet. The jet averaged 320 miles per hour while the helicopter averaged 80 miles per hour. If the total trip took $4\frac{1}{2}$ hours, then how long did she spend in the private jet?
4. Joe took two buses on the 463-mile trip from San Jose to San Diego. The first bus averaged 50 miles per hour and the second bus was able to average 64 miles per hour. If the total trip took 8 hours, then how long was spent in each bus?
5. Billy canoed downstream to the general store at an average rate of 9 miles per hour. His average rate canoeing back upstream was 4 miles per hour. If the total trip took $6\frac{1}{2}$ hours, then how long did it take Billy to get back on the return trip?
6. Two brothers drove the 2,793 miles from Los Angeles to New York. One of the brothers, driving in the day, was able to average 70 miles per hour, and the other, driving at night, was able to average 53 miles per hour. If the total trip took 45 hours, then how many hours did each brother drive?
7. A boat traveled 24 miles downstream in 2 hours. The return trip took twice as long. What was the speed of the boat and the current?
8. A helicopter flying with the wind can travel 525 miles in 5 hours. On the return trip, against the wind, it will take 7 hours. What are the speeds of the helicopter and of the wind?
9. A boat can travel 42 miles with the current downstream in 3 hours. Returning upstream against the current, the boat can only travel 33 miles in 3 hours. Find the speed of the current.
10. A light aircraft flying with the wind can travel 180 miles in $1\frac{1}{2}$ hours. The aircraft can fly the same distance against the wind in 2 hours. Find the speed of the wind.

Answer

1. The airplane flew 3 hours with the wind and 4 hours against the wind.

3. 3.5 hours
5. 4.5 hours
7. Boat: 9 miles per hour; current: 3 miles per hour
9. 1.5 miles per hour

? EXERCISE 1.6.9 DISCUSSION BOARD

1. Compose a number or money problem that can be solved with a system of equations of your own and share it on the discussion board.
2. Compose a mixture problem that can be solved with a system of equations of your own and share it on the discussion board.
3. Compose a uniform motion problem that can be solved with a system of equations of your own and share it on the discussion board

Answer

1. Answers may vary
3. Answers may vary

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1.7: SIMPLIFYING ALGEBRAIC EXPRESSIONS

LEARNING OBJECTIVES

- Apply the distributive property to simplify an algebraic expression.
- Identify and combine like terms.

DISTRIBUTIVE PROPERTY

The properties of real numbers are important in our study of algebra because a variable is simply a letter that represents a real number. In particular, the distributive property states that given any real numbers $a, b,$ and c,

$$a(b+c) = ab+ac \tag{1.7.1}$$

This property is applied when simplifying algebraic expressions. To demonstrate how it is used, we simplify $2(5-3)$ in two ways, and observe the same correct result.

Certainly, if the contents of the parentheses can be simplified, do that first. On the other hand, when the contents of parentheses cannot be simplified, multiply every term within the parentheses by the factor outside of the parentheses using the distributive property. Applying the distributive property allows you to multiply and remove the parentheses.

Simplify:

$$5(7y+2).$$

Solution:

Multiply 5 times each term inside the parentheses.

$$\begin{aligned} 5(7y+2) &= 5\cdot 7y+5\cdot 2 \\ &= 35y+10 \end{aligned}$$

Answer:

$$35y+10$$

EXAMPLE 1.7.2

Simplify:

$$-3(2x^2+5x+1).$$

Solution:

Multiply -3 times each of the coefficients of the terms inside the parentheses.

Answer:

$-6x^2-15x-3$

EXAMPLE 1.7.3

Simplify:

$$5(-2a+5b)-2c.$$

Solution:

Apply the distributive property by multiplying only the terms grouped within the parentheses by 5.

$$\begin{aligned} &5(-2a+5b)-2c \\ &= 5\cdot(-2a)+5\cdot 5b-2c \\ &= -10a+25b-2c \end{aligned}$$

***Figure* 1.7.1**

Answer:

$$-10a+25b-2c$$

Because multiplication is commutative, we can also write the distributive property in the following manner:

$$(b+c)a = ba+ca \tag{1.7.2}$$

EXAMPLE 1.7.4

Simplify:

$$(3x-4y+1)\cdot 3\,.$$

Solution:

Multiply each term within the parentheses by 3.

$$\begin{aligned}(3x-4y+1)\cdot 3 &= 3x\cdot 3-4y\cdot 3+1\cdot 3\\ &= 9x-12y+3\end{aligned}$$

Answer:

$$9x-12y+3$$

Division in algebra is often indicated using the fraction bar rather than with the symbol (÷). And sometimes it is useful to rewrite expressions involving division as products:

$$\frac{x}{5} = \frac{1x}{5} = \frac{1}{5}\cdot x$$
$$\frac{3ab}{7} = \frac{3}{7}\cdot\frac{ab}{1} = \frac{3}{7}\cdot ab$$
$$\frac{x+y}{3} = \frac{1}{3}\cdot\frac{(x+y)}{1} = \frac{1}{3}\cdot(x+y)$$

Rewriting algebraic expressions as products allows us to apply the distributive property.

EXAMPLE 1.7.5

Divide:

\(\frac{25x^{2}-5x+10}{5}.

Solution:

First, treat this as $\frac{1}{5}$ times the expression in the numerator and then distribute.

$$\begin{aligned}\frac{25x^2-5x+10}{5} &= \frac{1}{5}\cdot\frac{(25x^2-5x+10)}{1}\\ &= \frac{1}{5}\cdot(25x^2-5x+10) && \textit{Multiply each term by } \frac{1}{5}.\\ &= \frac{1}{5}\cdot 25x^2-\frac{1}{5}\cdot 5x+\frac{1}{5}\cdot 10 && \textit{Simplify.}\\ &= 5x^2-x+2\end{aligned}$$

Alternate Solution:

Think of 5 as a common denominator and divide each of the terms in the numerator by 5:

$$\begin{aligned}\frac{25x^2-5x+10}{5} &= \frac{25x^2}{5}-\frac{5x}{5}+\frac{10}{5}\\ &= 5x^2-x+2\end{aligned}$$

Answer:

$$5x^2-x+2$$

We will discuss the division of algebraic expressions in more detail as we progress through the course.

EXERCISE 1.7.1

Simplify:

$$\tfrac{1}{3}(-9x+27y-3)\,.$$

Answer

$-3x+9y-1$

COMBINING LIKE TERMS

Terms with the same variable parts are called like terms, or similar terms. Furthermore, constant terms are considered to be like terms. If an algebraic expression contains like terms, apply the distributive property as follows:

$$2a+3a=(2+3)a=5a$$
$$7xy-5xy=(7-5)xy=2xy$$
$$10x^2+4x^2-6x^2=(10+4-6)x^2=8x^2$$

In other words, if the variable parts of terms are ***exactly the same***, then we may add or subtract the coefficients to obtain the coefficient of a single term with the same variable part. This process is called combining like terms. For example,

$$3a^2b+2a^2b=5a^2b$$

Notice that the variable factors and their exponents do not change. Combining like terms in this manner, so that the expression contains no other similar terms, is called simplifying the expression. Use this idea to simplify algebraic expressions with multiple like terms.

EXAMPLE 1.7.6

Simplify:

$$3a+2b-4a+9b\ .$$

Solution:

Identify the like terms and combine them.

$$\begin{aligned} 3a+2b-4a+9b &= 3a-4a+2b+9b && \textit{Commutative property of addition} \\ &= -1a+11b && \textit{Combine like terms.} \\ &= -a+11b \end{aligned}$$

Answer:

$$-a+11b$$

In the previous example, rearranging the terms is typically performed mentally and is not shown in the presentation of the solution.

EXAMPLE 1.7.7

Simplify:

$$x^2+3x+2+4x^2-5x-7\ .$$

Solution:

Identify the like terms and add the corresponding coefficients.

$$\begin{aligned} &\underline{1x^2}+\underline{\underline{3x}}+\underline{\underline{\underline{2}}}+\underline{4x^2}-\underline{\underline{5x}}-\underline{\underline{\underline{7}}} && \textit{Identify like terms.} \\ &= 5x^2-2x-5 && \textit{Combine like terms.} \end{aligned}$$

Answer:

$$5x^2-2x-5$$

EXAMPLE 1.7.8

Simplify:

$$5x^2y-3xy^2+4x^2y-2xy^2\ .$$

Solution:

Remember to leave the variable factors and their exponents unchanged in the resulting combined term.

$$\begin{aligned} &\underline{5x^2y}-\underline{\underline{3xy^2}}+\underline{4x^2y}-\underline{\underline{2xy^2}} \\ &= 9x^2y-5xy^2 \end{aligned}$$

Answer:

$$9x^2y-5xy^2$$

EXAMPLE 1.7.9

Simplify:

$$\tfrac{1}{2}a - \tfrac{1}{3}b + \tfrac{3}{4}a + b\,.$$

To add the fractional coefficients, use equivalent coefficients with common denominators for each like term.

$$\begin{aligned}\frac{1}{2}a - \frac{1}{3}b + \frac{3}{4}a + 1b &= \frac{1}{2}a + \frac{3}{4}a - \frac{1}{3}b + 1b \\ &= \frac{2}{4}a + \frac{3}{4}a - \frac{1}{3}b + \frac{3}{3}b \\ &= \frac{5}{4}a + \frac{2}{3}b\end{aligned}$$

Answer:

$$\tfrac{5}{4}a + \tfrac{2}{3}b$$

EXAMPLE 1.7.10

Simplify:

$$-12x(x+y)^3 + 26x(x+y)^3\,.$$

Solution:

Consider the variable part to be $x(x+y)^3$. Then this expression has two like terms with coefficients -12 and 26.

$$\begin{aligned}&-12x(x+y)^3 + 26x(x+y)^3 \qquad \textit{Add the coefficients.} \\ &= 14x(x+y)^3\end{aligned}$$

Answer:

$$14x(x+y)^3$$

EXERCISE 1.7.2

Simplify:

$$-7x + 8y - 2x - 3y\,.$$

Answer

$-9x + 5y$

DISTRIBUTIVE PROPERTY AND LIKE TERMS

When simplifying, we will often have to combine like terms after we apply the distributive property. This step is consistent with the order of operations: multiplication before addition.

EXAMPLE 1.7.11

Simplify:

$$2(3a - b) - 7(-2a + 3b).$$

Solution:

Distribute 2 and -7 and then combine like terms.

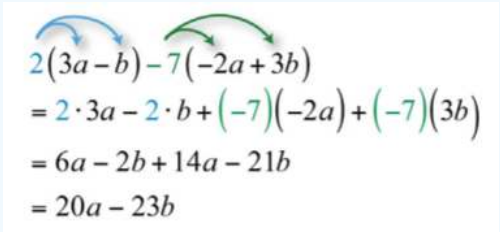


***Figure* 1.7.2**

Answer:

$$20a - 23b$$

In the example above, it is important to point out that you can remove the parentheses and collect like terms because you multiply the second quantity by -7, not just by 7. To correctly apply the distributive property, think of this as adding -7 times the given quantity, $2(3a - b) + (-7)(-2a + 3b)$.

? EXERCISE 1.7.3

Simplify:

$$-5(2x - 3) + 7x.$$

Answer

$-3x + 15$

Often we will encounter algebraic expressions like $+(a + b)$ or $-(a + b)$. As we have seen, the coefficients are actually implied to be $+1$ and -1, respectively, and therefore, the distributive property applies using $+1$ or -1 as the factor. Multiply each term within the parentheses by these factors:

$$+(a + b) = +1(a + b) = (+1)a + (+1)b = a + b \tag{1.7.3}$$

$$-(a + b) = -1(a + b) = (-1)a + (-1)b = -a - b \tag{1.7.4}$$

This leads to two useful properties,

$$+(a + b) = a + b \tag{1.7.5}$$

$$-(a + b) = -a - b \tag{1.7.6}$$

✓ EXAMPLE 1.7.12

Simplify:

$$5x - (-2x^2 + 3x - 1)\,.$$

Solution:

Multiply each term within the parentheses by -1 and then combine like terms.

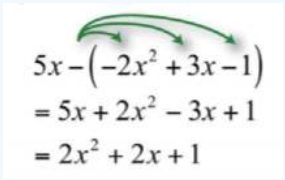


***Figure* 1.7.3**

Answer:

$$2x^2 + 2x + 1$$

When distributing a negative number, all of the signs within the parentheses will change. Note that $5x$ in the example above is a separate term; hence the distributive property does not apply to it.

✓ EXAMPLE 1.7.13

Simplify:

$$5 - 2(x^2 - 4x - 3)\,.$$

Solution:

The order of operations requires that we multiply before subtracting. Therefore, distribute -2 and then combine the constant terms. Subtracting $5 - 2$ first leads to an incorrect result, as illustrated below:

Incorrect!	*Correct!*
$5-2(x^2-4x-3)$	$5-2(x^2-4x-3)$
$=3(x^2-4x-3)$	$=5-2x^2+8x+6$
$=3x^2-12x-9$ x	$=-2x^2+8x+11$ ✓

Answer:

$$-2x^2+8x+11$$

NOTE

It is worth repeating that you must ***follow the order of operations***: multiply and divide before adding and subtracting!

EXERCISE 1.7.4

Simplify:

$$8-3(-x^2+2x-7)\ .$$

Answer

$3x^2-6x+29$

EXAMPLE 1.7.14

Subtract $3x-2$ from twice the quantity $-4x^2+2x-8$.

Solution:

First, group each expression and treat each as a quantity:

$$(3x-2) \quad \text{and} \quad (-4x^2+2x-8)$$

Next, identify the key words and translate them into a mathematical expression.

$$\overbrace{2\cdot(-4x^2+2x-8)}^{\textit{Twice the quantity...}} \overbrace{-(3x-2)}^{\textit{Subtract from...}}$$

***Figure* 1.7.4**

Finally, simplify the resulting expression.

Answer:

$-8x^2+x-14$

KEY TAKEAWAYS

- The properties of real numbers apply to algebraic expressions, because variables are simply representations of unknown real numbers.
- Combine like terms, or terms with the same variable part, to simplify expressions.
- Use the distributive property when multiplying grouped algebraic expressions, $a(b+c)=ab+ac$.
- It is a best practice to apply the distributive property only when the expression within the grouping is completely simplified.
- After applying the distributive property, eliminate the parentheses and then combine any like terms.
- Always use the order of operations when simplifying.

EXERCISE 1.7.5 DISTRIBUTIVE PROPERTY

Multiply.

1. $3(3x-2)$
2. $12(-5y+1)$
3. $-2(x+1)$
4. $5(a-b)$
5. $\frac{5}{8}(8x-16)$
6. $-\frac{3}{5}(10x-5)$

7. $(2x+3)\cdot 2$
8. $(5x-1)\cdot 5$
9. $(-x+7)(-3)$
10. $(-8x+1)(-2)$
11. $-(2a-3b)$
12. $-(x-1)$
13. $\frac{1}{3}(2x+5)$
14. $-\frac{3}{4}(y-2)$
15. $-3(2a+5b-c)$
16. $-(2y^2-5y+7)$
17. $5(y^2-6y-9)$
18. $-6(5x^2+2x-1)$
19. $7x^2-(3x-11)$
20. $-(2a-3b)+c$
21. $3(7x^2-2x)-3$
22. $\frac{1}{2}(4a^2-6a+4)$
23. $-\frac{1}{3}(9y^2-3y+27)$
24. $(5x^2-7x+9)(-5)$
25. $6(\frac{1}{3}x^2-\frac{1}{6}x+\frac{1}{2})$
26. $-2(3x^3-2x^2+x-3)$
27. $\frac{20x+30y-10z}{10}$
28. $\frac{-4a+20b-8c}{4}$
29. $\frac{3x^2-9x+81}{-3}$
30. $\frac{15y^2+20y-5}{5}$

Answer

1. $9x-6$
3. $-2x-2$
5. $5x-10$
7. $4x+6$
9. $3x-21$
11. $-2a+3b$
13. $\frac{2}{3}x+\frac{5}{3}$
15. $-6a-15b+3c$
17. $5y^2-30y-45$
19. $7x^2-3x+11$
21. $21x^2-6x-3$
23. $-3y^2+y-9$
25. $2x^2-x+3$
27. $2x+3y-z$
29. $-x^2+3x-27$

? EXERCISE 1.7.6 DISTRIBUTIVE PROPERTY

Translate the following sentences into algebraic expressions and then simplify.

1. Simplify two times the expression $25x^2-9$.
2. Simplify the opposite of the expression $6x^2+5x-1$.
3. Simplify the product of 5 and x^2-8.
4. Simplify the product of -3 and $-2x^2+x-8$.

Answer

1. $50x^2 - 18$
3. $5x^2 - 40$

EXERCISE 1.7.7 COMBINING LIKE TERMS

Simplify.

1. $2x - 3x$
2. $-2a + 5a - 12a$
3. $10y - 30 - 15y$
4. $\frac{1}{3}x + \frac{5}{12}x$
5. $-\frac{1}{4}x + \frac{4}{5} + \frac{3}{8}x$
6. $2x - 4x + 7x - x$
7. $-3y - 2y + 10y - 4y$
8. $5x - 7x + 8y + 2y$
9. $-8\alpha + 2\beta - 5\alpha - 6\beta$
10. $-6\alpha + 7\beta - 2\alpha + \beta$
11. $3x + 5 - 2y + 7 - 5x + 3y$
12. $-y + 8x - 3 + 14x + 1 - y$
13. $4xy - 6 + 2xy + 8$
14. $-12ab - 3 + 4ab - 20$
15. $\frac{1}{3}x - \frac{2}{5}y + \frac{2}{3}x - \frac{3}{5}y$
16. $\frac{3}{8}a - \frac{2}{7}b - \frac{1}{4}a + \frac{3}{14}b$
17. $-4x^2 - 3xy + 7 + 4x^2 - 5xy - 3$
18. $x^2 + y^2 - 2xy - x^2 + 5xy - y^2$
19. $x^2 - y^2 + 2x^2 - 3y$
20. $\frac{1}{2}x^2 - \frac{2}{3}y^2 - \frac{1}{8}x^2 + \frac{1}{5}y^2$
21. $\frac{3}{16}a^2 - \frac{4}{5} + \frac{1}{4}a^2 - \frac{1}{4}$
22. $\frac{1}{5}y^2 - \frac{3}{4} + \frac{7}{10}y^2 - \frac{1}{2}$
23. $6x^2y - 3xy^2 + 2x^2y - 5xy^2$
24. $12x^2y^2 + 3xy - 13x^2y^2 + 10xy$
25. $-ab^2 + a^2b - 2ab^2 + 5a^2b$
26. $m^2n^2 - mn + mn - 3m^2n + 4m^2n^2$
27. $2(x + y)^2 + 3(x + y)^2$
28. $\frac{1}{5}(x + 2)^3 - \frac{2}{3}(x + 2)^3$
29. $-3x(x^2 - 1) + 5x(x^2 - 1)$
30. $5(x - 3) - 8(x - 3)$
31. $-14(2x + 7) + 6(2x + 7)$
32. $4xy(x + 2)^2 - 9xy(x + 2)^2 + xy(x + 2)^2$

Answer

1. $-x$
3. $-5y - 30$
5. $\frac{1}{8}x + \frac{4}{5}$
7. y
9. $-13\alpha - 4\beta$
11. $-2x + y + 12$
13. $6xy + 2$
15. $x - y$
17. $-8xy + 4$
19. $3x^2 - y^2 - 3y$
21. $\frac{7}{16}a^2 - \frac{21}{20}$

23. $8x^2y - 8xy^2$
25. $6a^2b - 3ab^2$
27. $5(x+y)^2$
29. $2x(x^2-1)$
31. $-8(2x+7)$

EXERCISE 1.7.8 MIXED PRACTICE

Simplify.

1. $5(2x-3)+7$
2. $-2(4y+2)-3y$
3. $5x-2(4x-5)$
4. $3-(2x+7)$
5. $2x-(3x-4y-1)$
6. $(10y-8)-(40x+20y-7)$
7. $\frac{1}{2}y-\frac{3}{4}x-(\frac{2}{3}y-\frac{1}{5}x)$
8. $\frac{1}{5}a-\frac{3}{4}b+\frac{3}{15}a-\frac{1}{2}b$
9. $\frac{2}{3}(x-y)+x-2y$
10. $-\frac{1}{3}(6x-1)+\frac{1}{2}(4y-1)-(-2x+2y-\frac{1}{6})$
11. $(2x^2-7x+1)+(x^2+7x-5)$
12. $6(-2x^2+3x-1)+10x^2-5x$
13. $-(x^2-3x+8)+x^2-12$
14. $2(3a-4b)+4(-2a+3b)$
15. $-7(10x-7y)-6(8x+4y)$
16. $10(6x-9)-(80x-35)$
17. $10-5(x^2-3x-1)$
18. $4+6(y^2-9)$
19. $\frac{3}{4}x-(\frac{1}{2}x^2+\frac{2}{3}x-\frac{7}{5})$
20. $-\frac{7}{3}x^2+(-\frac{1}{6}x^2+7x-1)$
21. $(2y^2-3y+1)-(5y^2+10y-7)$
22. $(-10a^2-b^2+c)+(12a^2+b^2-4c)$
23. $-4(2x^2+3x-2)+5(x^2-4x-1)$
24. $2(3x^2-7x+1)-3(x^2+5x-1)$
25. $x^2y+3xy^2-(2x^2y-xy^2)$
26. $3(x^2y^2-12xy)-(7x^2y^2-20xy+18)$
27. $3-5(ab-3)+2(ba-4)$
28. $-9-2(xy+7)-(yx-1)$
29. $-5(4\alpha-2\beta+1)+10(\alpha-3\beta+2)$
30. $\frac{1}{2}(100\alpha^2-50\alpha\beta+2\beta^2)-\frac{1}{5}(50\alpha^2+10\alpha\beta-5\beta^2)$

Answer

1. $10x-8$
3. $-3x+10$
5. $-x+4y+1$
7. $-\frac{11}{20}x-\frac{1}{6}y$
9. $\frac{5}{3}x-\frac{8}{3}y$
11. $3x^2-4$
13. $3x-20$
15. $-118x+25y$
17. $-5x^2+15x+15$
19. $-\frac{1}{2}x^2+\frac{1}{12}x+\frac{7}{5}$

21. $-3y^2-13y+8$
23. $-3x^2-32x+3$
25. $-x^2y+4xy^2$
27. $-3ab+10$
29. $-10\alpha-20\beta+15$

EXERCISE 1.7.9 MIXED PRACTICE

Translate the following sentences into algebraic expressions and then simplify.

1. What is the difference of $3x-4$ and $-2x+5$?
2. Subtract $2x-3$ from $5x+7$.
3. Subtract $4x+3$ from twice the quantity $x-2$.
4. Subtract three times the quantity $-x+8$ from $10x-9$.

Answer

1. $5x-9$
3. $-2x-7$

EXERCISE 1.7.10 DISCUSSION BOARD TOPICS

1. Do we need a distributive property for division, $(a+b)\div c$? Explain.
2. Do we need a separate distributive property for three terms, $a(b+c+d)$? Explain.
3. Explain how to subtract one expression from another. Give some examples and demonstrate the importance of the order in which subtraction is performed.
4. Given the algebraic expression $8-5(3x+4)$, explain why subtracting $8-5$ is not the first step.
5. Can you apply the distributive property to the expression $5(abc)$? Explain why or why not and give some examples.
6. How can you check to see if you have simplified an expression correctly? Give some examples.

Answer

1. Answers may vary
3. Answers may vary
5. Answers may vary

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1.8: SOLVING LINEAR EQUATIONS- PART 1

LEARNING OBJECTIVES

- Identify linear equations with one variable and verify their solutions.
- Use the properties of equality to solve basic linear equations.
- Use multiple steps to solve linear equations by isolating the variable.
- Solve linear equations where the coefficients are fractions or decimals.

LINEAR EQUATIONS WITH ONE VARIABLE AND THEIR SOLUTIONS

Learning how to solve various algebraic equations is one of the main goals in algebra. This section introduces the basic techniques used for solving linear equations with one variable.

An equation is a statement indicating that two algebraic expressions are equal. A linear equation with one variable, x, is an equation that can be written in the general form $ax+b=0$, where a and b are real numbers and $a \neq 0$. Here are some examples of linear equations, all of which are solved in this section:

$$x+3=-5 \qquad 2x-5=15 \qquad \tfrac{5}{3}x+2=-8$$

A solution to a linear equation is any value that can replace the variable to produce a true statement. The variable in the linear equation $2x+3=13$ is x, and the solution is $x=5$. To verify this, substitute the value 5 for x and check that you obtain a true statement.

$$\begin{aligned} 2x+13 &= 13 \\ 2(5)+3 &= 13 \\ 10+3 &= 13 \\ 13 &= 13 \quad \checkmark \end{aligned}$$

Alternatively, when an equation is equal to a constant, we can verify a solution by substituting the value for the variable and show that the result is equal to that constant. In this sense, we say that solutions satisfy the equation.

EXAMPLE 1.8.1

Is $x=3$ a solution to $-2x-3=-9$?

Solution:

$$-2x-3=-2(3)-3=-6-3=-9 \quad \checkmark$$

Answer:

Yes, it is a solution, because $x=3$ satisfies the equation.

EXAMPLE 1.8.2

Is $a=-\frac{1}{2}$ a solution to $-10a+5=25$?

Solution:

$$-10a+5=-10\left(-\tfrac{1}{2}\right)+5=5+5=10 \neq 25 \quad x$$

Answer:

No, it is not a solution, because $a=-\frac{1}{2}$ does not satisfy the equation.

Recall that when evaluating expressions, it is a good practice to first replace all variables with parentheses, then substitute the appropriate values. By making use of parentheses we avoid some common errors when working the order of operations.

EXAMPLE 1.8.3

Is $y=-3$ a solution to $2y-5=-y-14$?

Solution:

Answer:

Yes, it is a solution, because $y=-3$ produces a true statement.

? EXERCISE 1.8.1

Is $x = -3$ a solution to $-2x + 5 = -1$?

Answer

No

SOLVING BASIC LINEAR EQUATIONS

We begin by defining equivalent equations as equations with the same solution set. Consider the following two linear equations and check to see if the solution is $x = 7$.

$$\begin{array}{r|l} 3x - 5 = 16 & 3x = 21 \\ 3(7) - 5 = 16 & 3(7) = 21 \\ 21 - 5 = 16 & 21 = 21 \quad \checkmark \\ 16 = 16 \quad \checkmark & \end{array}$$

Here we can see that the two linear equations $3x - 5 = 16$ and $3x = 21$ are equivalent because they share the same solution set, namely, $\{7\}$. The goal is to develop a systematic process to find equivalent equations until the variable is isolated:

$$\left.\begin{array}{r} 3x - 5 = 16 \\ 3x = 21 \\ x = 7 \end{array}\right\} \quad \textit{Equivalent equations}$$

To do this, use the properties of equality. Given algebraic expressions A and B, where c is a real number, we have the following:

$$\begin{array}{ll} \textbf{Addition Property of Equality}: & \textit{If } A = B, \textit{ then } A + c = B + c \\ \textbf{Subtraction Property of Equality:} & \textit{If } A = B, \textit{ then } A - c = B - c \\ \textbf{Multiplication Property of Equality:} & \textit{If } A = B, \textit{ then } cA = cB \\ \textbf{Division Property of Equality:} & \textit{If } A = B, \textit{ then } \frac{A}{c} = \frac{B}{c} \; c \neq 0 \end{array}$$

NOTE

Multiplying or dividing both sides of an equation by 0 is carefully avoided. Dividing by 0 is undefined and multiplying both sides by 0 results in the equation $0 = 0$.

To summarize, equality is retained and you obtain an equivalent equation if you add, subtract, multiply, or divide ***both sides*** of an equation by any nonzero real number. The technique for solving linear equations involves applying these properties in order to ***isolate the variable*** on one side of the equation. If the linear equation has a constant term, then we add to or subtract it from both sides of the equation to obtain an equivalent equation where the variable term is isolated.

✓ EXAMPLE 1.8.4

Solve:

$$x + 3 = -5.$$

Solution:

To isolate the variable x on the left side, subtract 3 from both sides.

Answer:

The solution is $x = -8$.

To check that this is true, substitute -8 into the original equation and simplify to see that it is satisfied: $x + 3 = -8 + 3 = -5 \quad \checkmark$.

In the previous example, after subtracting 3 from both sides, you get $x + 0 = -8$. By the additive identity property of real numbers, this is equivalent to $x = -8$. This step is often left out in the presentation of the solution.

If the ***variable term*** of the equation (including the coefficient) is isolated, then apply the multiplication or division property of equality to obtain an equivalent equation with the ***variable*** isolated. In other words, our goal is to obtain an equivalent equation with x or $1x$ isolated on one side of the equal sign.

EXAMPLE 1.8.5

Solve:

$$-5x = -35.$$

Solution:

The coefficient of x is -5, so divide both sides by -5.

Answer:

The solution is $x = 7$. Perform the check mentally by substituting 7 for x in the original equation.

In the previous example, after dividing both sides by $-5, x$ is left with a coefficient of 1, because $\frac{-5}{-5} = 1$. In fact, when we say "isolate the variable," we mean to change the coefficient of the variable to 1, because $1x = 7$ is equivalent to $x = 7$. This step is often left out of the instructional examples even though its omission is sometimes a source of confusion.

Another important property is the symmetric property: for any algebraic expressions A and B,

$$\text{If } A = B, \text{ then } B = A \tag{1.8.1}$$

The equation $2 = x$ is equivalent to $x = 2$. It does not matter on which side we choose to isolate the variable.

EXAMPLE 1.8.6

Solve:

$$2 = 5 + x\,.$$

Solution:

Isolate the variable x by subtracting 5 from both sides of the equation.

Answer:

The solution is -3, and checking the solution shows that $2 = 5 - 3\,.$

EXERCISE 1.8.2

Solve:

$$6 = x - 4$$

Answer

$x = 10$

ISOLATING THE VARIABLE IN TWO STEPS

A linear equation of the form $ax + b = c$ takes two steps to solve. First, use the appropriate equality property of addition or subtraction to isolate the variable term. Next, isolate the variable using the equality property of multiplication or division. Checking solutions in the following examples is left to the reader.

EXAMPLE 1.8.7

Solve:

$$2x - 5 = 15\,.$$

Solution:

$$\begin{aligned} 2x - 5 &= 15 \\ 2x - 5{\color{cyan}+5} &= 15{\color{cyan}+5} && \textit{Add } 5 \textit{ to both sides.} \\ 2x &= 20 \\ \frac{2x}{\color{cyan}2} &= \frac{20}{\color{cyan}2} && \textit{Divide both sides by } 2. \\ x &= 10 \end{aligned}$$

Answer:

The solution is 10.

EXAMPLE 1.8.8

Solve:

$$-3x-2=9.$$

Solution:

$$\begin{aligned} -3x-2 &= 9 \\ -3x-2+2 &= 9+2 && \textit{Add 2 to both sides.} \\ -3x &= 11 \\ \frac{-3x}{-3} &= \frac{11}{-3} && \textit{Divide both sides by } -3. \\ x &= -\frac{11}{3} \end{aligned}$$

Answer:

The solution is $-\frac{11}{3}$.

EXAMPLE 1.8.9

Solve:

$$6-5y=-14.$$

Solution:

When no sign precedes the term, it is understood to be positive. In other words, think of this as $+6-5y=-14$. Begin by subtracting 6 from both sides of the equal sign.

Answer:

The solution is 4.

EXAMPLE 1.8.10

Solve:

$$3x+\tfrac{1}{2}=\tfrac{2}{3}.$$

Solution:

$$\begin{aligned} 3x+\frac{1}{2} &= \frac{2}{3} \\ 3x+\frac{1}{2}-\frac{1}{2} &= \frac{2}{3}-\frac{1}{2} && \textit{Subtract } \frac{1}{2} \textit{ from both sides.} \\ 3x &= \frac{2\cdot 2}{3\cdot 2}-\frac{1\cdot 3}{2\cdot 3} && \textit{Obtain equivalent fractions with a common denominator.} \\ 3x &= \frac{4}{6}-\frac{3}{6} \\ 3x &= \frac{1}{6} \\ \frac{3x}{3} &= \frac{6}{3} && \textit{Divide both sides by 3.} \\ x &= \frac{1}{6}\div 3=\frac{1}{6}\cdot\frac{1}{3}=\frac{1}{18} \end{aligned}$$

Answer:

The solution is $\frac{1}{18}$.

EXAMPLE 1.8.11

Solve:

$$3-y=1.$$

Solution:

Recall that $-y$ is equivalent to $-1y$; divide both sides of the equation by -1.

$$\begin{aligned} -y &= -2 \\ \frac{-1y}{-1} &= \frac{-2}{-1} \\ y &= 2 \end{aligned}$$

Alternatively, multiply both sides of $-y = -2$ by -1 and achieve the same result:

Answer:

The solution is 2.

In summary, to retain equivalent equations, we must perform the same operation on both sides of the equation. First, apply the addition or subtraction property of equality to isolate the variable term and then apply the multiplication or division property of equality to isolate the variable on one side of the equation.

? EXERCISE 1.8.3

Solve:

$$-7x + 6 = 27.$$

Answer

$x = -3$

MULTIPLYING BY THE RECIPROCAL

To solve an equation like $\frac{3}{4}x = 1$, we can isolate the variable by dividing both sides by the coefficient. For example

$$\begin{aligned} \frac{3}{4}x &= 1 \\ \frac{\frac{3}{4}x}{\frac{3}{4}} &= \frac{1}{\frac{3}{4}} \\ x &= 1 \cdot \frac{4}{3} \\ x &= \frac{4}{3} \end{aligned}$$

On the left side of the equal sign, the fraction cancels. On the right side, we have a complex fraction and multiply by the reciprocal of the coefficient. You can save a step by recognizing this and start by multiplying both sides of the equation by the reciprocal of the coefficient.

$$\begin{aligned} \frac{3}{4}x &= 1 \\ \frac{4}{3} \cdot \frac{3}{4}x &= \frac{4}{3} \cdot 1 \\ x &= \frac{4}{3} \end{aligned}$$

Recall that the product of reciprocals is 1, in this case $\frac{4}{3} \cdot \frac{3}{4} = 1$, leaving the variable isolated.

✓ EXAMPLE 1.8.12

Solve:

$$\frac{5}{3}x + 2 = -8.$$

Solution:

Isolate the variable term using the addition property of equality and then multiply both sides of the equation by the reciprocal of the coefficient $\frac{5}{3}$.

Answer:

The solution is -6.

EXAMPLE 1.8.13

Solve:

$$-\tfrac{4}{5}x - 5 = 15.$$

Solution:

$$\begin{aligned} -\frac{4}{5}x - 5 &= 15 \\ -\frac{4}{5}x - 5{\color{cyan}+5} &= 15{\color{cyan}+5} \\ -\frac{4}{5}x &= 20 \end{aligned}$$

The reciprocal of $-\frac{4}{5}$ is $-\frac{5}{4}$ because $(-\frac{5}{4})(-\frac{4}{5}) = +\frac{20}{20} = 1$. Therefore, to isolate the variable x, multiply both sides by $-\frac{5}{4}$.

Answer:

The solution is -25.

EXERCISE 1.8.4

Solve:

$$\tfrac{2}{3}x - 9 = -4.$$

Answer

$x = \frac{15}{2}$

KEY TAKEAWAYS

- Linear equations with one variable can be written in the form $ax + b = 0$, where a and b are real numbers and $a \neq 0$.
- To "solve a linear equation" means to find a numerical value that can replace the variable and produce a true statement.
- The properties of equality provide tools for isolating the variable and solving equations.
- To solve a linear equation, first isolate the *variable term* by adding the opposite of the constant term to both sides of the equation. Then isolate the *variable* by dividing both sides of the equation by its coefficient.
- After isolating a variable term with a fraction coefficient, solve by multiplying both sides by the reciprocal of the coefficient.

EXERCISE 1.8.5 SOLUTIONS TO LINEAR EQUATIONS

Is the given value a solution to the linear equation?

1. $x - 6 = 20; x = 26$
2. $y + 7 = -6; y = -13$
3. $-x + 5 = 17; x = 12$
4. $-2y = 44; y = 11$
5. $4x = -24; x = -6$
6. $5x - 1 = 34; x = -7$
7. $-2a - 7 = -7; a = 0$
8. $-\frac{1}{3}x - 4 = -5; x = -3$
9. $-\frac{1}{2}x + \frac{2}{3} = -\frac{1}{4}; x = \frac{11}{6}$
10. $-8x - 33 = 3x; x = 3$
11. $3y - 5 = -2y - 15; y = -2$
12. $3(2x + 1) = -4x - 3; x = -\frac{1}{2}$
13. $\frac{1}{2}y - \frac{1}{3} = \frac{1}{3}y + \frac{1}{6}; y = 3$
14. $-\frac{4}{3}y + \frac{1}{9} = -\frac{2}{3}y - \frac{1}{9}; y = \frac{1}{3}$

Answer

1. Yes
3. No
5. Yes

7. Yes
9. Yes
11. Yes
13. Yes

EXERCISE 1.8.6 SOLVING BASIC LINEAR EQUATIONS

Solve.

1. $x+3=13$
2. $y-4=22$
3. $-6+x=12$
4. $9+y=-4$
5. $x-\frac{1}{2}=\frac{1}{3}$
6. $x+\frac{2}{3}=-\frac{1}{5}$
7. $x+2\frac{1}{2}=3\frac{1}{3}$
8. $-37+x=-37$
9. $4x=-44$
10. $-9x=63$
11. $-y=13$
12. $-x=-10$
13. $-9x=0$
14. $-3a=-33$
15. $27=18y$
16. $14=-7x$
17. $31.5.6a=-39.2$
18. $-1.2y=3.72$
19. $\frac{1}{3}x=-\frac{1}{2}$
20. $-\frac{t}{12}=\frac{1}{4}$
21. $-\frac{7}{3}x=\frac{1}{2}$
22. $\frac{x}{5}=-3$
23. $\frac{4}{9}y=-\frac{2}{3}$
24. $-\frac{5}{8}y=-\frac{5}{2}$

Answer

1. 10
3. 18
5. $\frac{5}{6}$
7. $\frac{5}{6}$
9. -11
11. -13
13. 0
15. $\frac{3}{2}$
17. -7
19. $-\frac{3}{2}$
21. $-\frac{3}{14}$
23. $-\frac{3}{2}$

EXERCISE 1.8.7 SOLVING LINEAR EQUATIONS

Solve.

1. $5x + 7 = 32$
2. $4x - 3 = 21$
3. $3a - 7 = 23$
4. $12y + 1 = 1$
5. $21x - 7 = 0$
6. $-3y + 2 = -13$
7. $-5x + 9 = 8$
8. $22x - 55 = -22$
9. $4.5x - 2.3 = 6.7$
10. $1.4 - 3.2x = 3$
11. $9.6 - 1.4y = -10.28$
12. $4.2y - 3.71 = 8.89$
13. $3 - 2y = -11$
14. $-4 - 7a = 24$
15. $-10 = 2x - 5$
16. $24 = 6 - 12y$
17. $\frac{5}{6}x - \frac{1}{2} = \frac{2}{3}$
18. $\frac{1}{2}x + \frac{1}{3} = \frac{2}{5}$
19. $4a - \frac{2}{3} = -\frac{1}{6}$
20. $\frac{3}{5}x - \frac{1}{2} = \frac{1}{10}$
21. $-\frac{4}{5}y + \frac{1}{3} = \frac{1}{15}$
22. $-\frac{9}{16}x + \frac{4}{3} = \frac{4}{3}$
23. $-x + 5 = 14$
24. $-y - 7 = -12$
25. $75 - a = 200$
26. $15 = 5 - x$
27. $-8 = 4 - 2x$
28. $33 - x = 33$
29. $18 = 6 - y$
30. $-12 = -2x + 3$
31. $-3 = 3.36 - 1.2a$
32. $0 = -3.1a + 32.55$
33. $\frac{1}{4} = -\frac{3}{8} + 10x$
34. $70 = 50 - \frac{1}{2}y$

Answer

1. 5
3. 10
5. $\frac{1}{3}$
7. $\frac{1}{5}$
9. 2
11. 14.2
13. 7
15. $-\frac{5}{2}$
17. $\frac{7}{5}$
19. $\frac{1}{8}$
21. $\frac{1}{3}$
23. -9

25. -125
27. 6
29. -12
31. 5.3
33. $\frac{1}{16}$

EXERCISE 1.8.8 SOLVING LINEAR EQUATIONS

Translate the following sentences into linear equations and then solve.

1. The sum of $2x$ and 5 is equal to 15.
2. The sum of $-3x$ and 7 is equal to 14.
3. The difference of $5x$ and 6 is equal to 4.
4. Twelve times x is equal to 36.
5. A number n divided by 8 is 5.
6. Six subtracted from two times a number x is 12.
7. Four added to three times a number n is 25.
8. Three-fourths of a number x is 9.
9. Negative two-thirds times a number x is equal to 20.
10. One-half of a number x plus 3 is equal to 10.

Answer

1. $2x+5=15; x=5$
3. $5x-6=4; x=2$
5. $\frac{n}{8}=5; n=40$
7. $3n+4=25; n=7$
9. $-\frac{2}{3}x=20; x=-30$

EXERCISE 1.8.9 SOLVING LINEAR EQUATIONS

Find a linear equation of the form $ax+b=0$ with the given solution, where a and b are integers. (Answers may vary.)

1. $x=2$
2. $x=-3$
3. $x=-\frac{1}{2}$
4. $x=\frac{2}{3}$

Answer

1. $x-2=0$
3. $2x+1=0$

EXERCISE 1.8.10 DISCUSSION BOARD TOPICS

1. How many steps are needed to solve any equation of the form $ax+b=c$? Explain.
2. Instead of dividing by 6 when $6x=12$, could you multiply by the reciprocal of 6? Does this always work?

Answer

1. Answers may vary

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1.9: SOLVING LINEAR EQUATIONS- PART II

LEARNING OBJECTIVES

- Solve general linear equations.
- Identify and solve conditional equations, identities, and contradictions.
- Clear decimals and fractions from equations.
- Solve literal equations or formulas for a given variable.

COMBINING LIKE TERMS AND SIMPLIFYING

Linear equations typically are not given in standard form, so solving them requires additional steps. These additional steps include simplifying expressions on each side of the equal sign using the order of operations.

SAME-SIDE LIKE TERMS

We will often encounter linear equations where the expressions on each side of the equal sign can be simplified. Typically, this involves combining same-side like terms. If this is the case, then it is best to simplify each side first before solving.

EXAMPLE 1.9.1

Solve:

$$-4a+2-a=3-2\ .$$

Solution:

First, combine the like terms on each side of the equal sign.

Answer:

The solution is $\frac{1}{5}$.

OPPOSITE-SIDE LIKE TERMS

Given a linear equation in the form $ax+b=cx+d$, we begin by combining like terms on opposite sides of the equal sign. To combine opposite-side like terms, use the addition or subtraction property of equality to effectively "move terms" from one side to the other so that they can be combined.

EXAMPLE 1.9.2

Solve:

$$-2y-3=5y+11\ .$$

Solution:

To "move" the term $5y$ to the left side, subtract it on both sides.

$$\begin{aligned} -2y-3 &= 5y+11 \\ -2y-3\color{#0088cc}{-5y} &= 5y+11\color{#0088cc}{-5y} \qquad \textit{Subtract } 5y \textit{ from both sides.} \\ -7y-3 &= 11 \end{aligned}$$

From here, solve using the techniques developed previously.

Always check to see that the solution is correct by substituting the solution back into the original equation and simplifying to see if you obtain a true statement.

$$\begin{aligned} -2y-3 &= 5y+11 \\ -2(\color{#0088cc}{-2}\color{black})-3 &= 5(\color{#0088cc}{-2}\color{black})+1 \\ 4-3 &= -10+11 \\ 1 &= 1 \quad \checkmark \end{aligned}$$

Answer:

The solution is -2.

GENERAL GUIDELINES FOR SOLVING LINEAR EQUATIONS

When solving linear equations, the goal is to determine what value, if any, will produce a true statement when substituted in the original equation. Do this by isolating the variable using the following steps:

Step 1: Simplify both sides of the equation using the order of operations and combine all same-side like terms.

Step 2: Use the appropriate properties of equality to combine opposite-side like terms with the variable term on one side of the equation and the constant term on the other.

Step 3: Divide or multiply as needed to isolate the variable.

Step 4: Check to see if the answer solves the original equation.

EXAMPLE 1.9.3

Solve:

$$-\tfrac{1}{2}(10y-2)+3=14$$

Solution:

Simplify the linear expression on the left side before solving.

To check,

$$\begin{aligned} -\frac{1}{2}(10(-2)-2)+3 &= 14 \\ -\frac{1}{2}(-20-2)+3 &= 14 \\ -\frac{1}{2}(-22)+3 &= 14 \\ 11+3 &= 14 \\ 14 &= 14 \quad \checkmark \end{aligned}$$

Answer:

The solution is -2.

EXAMPLE 1.9.4

Solve:

$$5(3x+2)-2=-2(1-7x)\ .$$

Solution:

First, simplify the expressions on both sides of the equal sign.

Answer:

The solution is -10. The check is left as an exercise.

EXERCISE 1.9.1

Solve:

$$6-3(4x-1)=4x-7\ .$$

Answer

$x=1$

CONDITIONAL EQUATIONS, IDENTITIES, AND CONTRADICTIONS

There are three different types of equations. Up to this point, we have been solving conditional equations. These are equations that are true for particular values. An identity is an equation that is true for all possible values of the variable. For example,

$$x=x \quad \textit{Identity}$$

has a solution set consisting of all real numbers, R. A contradiction is an equation that is never true and thus has no solutions. For example,

$$x+1=x \quad \textit{Contradiction}$$

has no solution. We use the empty set, $\emptyset$, to indicate that there are no solutions.

If the end result of solving an equation is a true statement, like $0 = 0$, then the equation is an identity and any real number is a solution. If solving results in a false statement, like $0 = 1$, then the equation is a contradiction and there is no solution.

EXAMPLE 1.9.5

Solve:

$$4(x+5)+6=2(2x+3)\ .$$

Solution:

$$\begin{aligned} 4(x+5)+6 &= 2(2x+3) && \textit{Distribute.}\\ 4x+20+6 &= 4x+6 && \textit{Combine same - side like terms.}\\ 4x+26 &= 4x+6 && \textit{Combine opposite - side like terms.}\\ 4x+26-4x &= 4x+6-4x\\ 26 &= 6 \quad x && \textit{False} \end{aligned}$$

Answer:

$\emptyset$. Solving leads to a false statement; therefore, the equation is a contradiction and there is no solution.

EXAMPLE 1.9.6

Solve:

$$3(3y+5)+5=10(y+2)-y\ .$$

Solution:

$$\begin{aligned} 3(3y+5)+5 &= 10(y+2)-y && \textit{Distribute.}\\ 9y+15+5 &= 10y+20-y && \textit{Combine same - side like terms.}\\ 9y+20 &= 9y+20 && \textit{Combine opposite - side like terms.}\\ 9y+20-9y &= 9y+20-9y\\ 20 &= 20 \quad \checkmark && \textit{True} \end{aligned}$$

Answer:

R. Solving leads to a true statement; therefore, the equation is an identity and any real number is a solution.

If it is hard to believe that any real number is a solution to the equation in the previous example, then choose your favorite real number, and substitute it in the equation to see that it leads to a true statement. Choose $x = 7$ and check:

Solve:

$$-2(3x+1)-(x-3)=-7x+1\ .$$

Answer

R

CLEARING DECIMALS AND FRACTIONS

The coefficients of linear equations may be any real number, even decimals and fractions. When decimals and fractions are used, it is possible to use the multiplication property of equality to clear the coefficients in a single step. If given decimal coefficients, then multiply by an appropriate power of 10 to clear the decimals. If given fractional coefficients, then multiply both sides of the equation by the least common multiple of the denominators (LCD).

EXAMPLE 1.9.7

Solve:

$$2.3x+2.8=-1.2x+9.8\ .$$

Solution:

Notice that all decimal coefficients are expressed with digits in the tenths place; this suggests that we can clear the decimals by multiplying both sides by 10. Take care to distribute 10 to each term on both sides of the equation.

$$
\begin{aligned}
10\cdot(2.3x+2.8) &= 10\cdot -1.2x+9.8 && \textit{Multiply both sides by } 10.\\
10\cdot 2.3x+10\cdot 2.8 &= 10\cdot(-1.2x)+10\cdot 9.8 &&\\
23x+28 &= -12x+98 && \textit{Integer coefficients}\\
23x+28+12x &= -12x+98+12x && \textit{Solve.}\\
35x+28 &= 98 &&\\
35x+28-28 &= 98-28 &&\\
35x &= 70 &&\\
\frac{35x}{35} &= \frac{70}{35} &&\\
x &= 2 &&
\end{aligned}
$$

Answer:

The solution is 2.

EXAMPLE 1.9.8

Solve:

$$\tfrac{1}{3}x+\tfrac{1}{5}=\tfrac{1}{5}x-1\,.$$

Solution:

Clear the fractions by multiplying both sides by the least common multiple of the given denominators. In this case, the LCM $(3,5)=15$.

Answer:

The solution is -9.

It is important to know that these techniques only work for equations. ***Do not try to clear fractions when simplifying expressions***. As a reminder

Expression	*Equation*
$\frac{1}{2}x+\frac{5}{3}$	$\frac{1}{2}x+\frac{5}{3}=0$

Solve equations and simplify expressions. If you multiply an expression by 6, you will change the problem. However, if you multiply both sides of an equation by 6, you obtain an equivalent equation.

Incorrect	*Correct*
$\frac{1}{2}x+\frac{5}{3}$	$\frac{1}{2}x+\frac{5}{3}=0$
$\neq 6\cdot\left(\frac{1}{2}x+\frac{5}{3}\right)$	$6\cdot\left(\frac{1}{2}x+\frac{5}{3}\right)=6\cdot 0$
$=3x+10$ x	$3x+10=10$ ✓

LITERAL EQUATIONS (LINEAR FORMULAS)

Algebra lets us solve whole classes of applications using literal equations, or formulas. Formulas often have more than one variable and describe, or model, a particular real-world problem. For example, the familiar formula $D=rt$ describes the distance traveled in terms of the average rate and time; given any two of these quantities, we can determine the third. Using algebra, we can solve the equation for any one of the variables and derive two more formulas.

$$
\begin{aligned}
D &= rt\\
\frac{D}{r} &= \frac{rt}{r} && \textit{Divide both sides by } r.\\
\frac{D}{r} &= t &&
\end{aligned}
$$

If we divide both sides by r, we obtain the formula $t=Dr$. Use this formula to find the time, given the distance and the rate

$$
\begin{aligned}
D &= rt\\
\frac{D}{t} &= \frac{rt}{t} && \textit{Divide both sides by } t.\\
\frac{D}{t} &= r &&
\end{aligned}
$$

If we divide both sides by t, we obtain the formula $r=Dt$. Use this formula to find the rate, given the distance traveled and the time it takes to travel that distance. Using the techniques learned up to this point, we now have three equivalent formulas relating distance, average

rate, and time:

$$D = rt \qquad t = \frac{D}{r} \qquad r = \frac{D}{t}$$

When given a literal equation, it is often necessary to solve for one of the variables in terms of the others. Use the properties of equality to isolate the indicated variable.

EXAMPLE 1.9.9

Solve for a:

$$P = 2a + b\,.$$

Solution:

The goal is to isolate the variable a.

$$\begin{aligned} P &= 2a + b \\ P - b &= 2a + b - b \qquad \textit{Subtract } b \textit{ from both sides.} \\ P - b &= 2a \\ \frac{P-b}{2} &= \frac{2a}{2} \qquad \textit{Divide both sides by } 2. \\ \frac{P-b}{2} &= a \end{aligned}$$

Answer:

$$a = \tfrac{P-b}{2}$$

EXAMPLE 1.9.10

Solve for y:

$$z = \tfrac{x+y}{2}\,.$$

Solution:

The goal is to isolate the variable y.

$$\begin{aligned} z &= \frac{x+y}{2} \\ 2 \cdot z &= 2 \cdot \frac{x+y}{2} \qquad \textit{Multiply both sides by } 2. \\ 2z &= x + y \\ 2z - x &= x + y - x \qquad \textit{Subtract } x \textit{ from both sides.} \\ 2z - x &= y \end{aligned}$$

Answer:

$$y = 2z - x$$

EXERCISE 1.9.3

Solve for b:

$$2a - 3b = c\,.$$

Answer

$b = \tfrac{2a-c}{3}$

KEY TAKEAWAYS

- Solving general linear equations involves isolating the variable, with coefficient 1, on one side of the equal sign.
- The steps for solving linear equations are:
  - Simplify both sides of the equation and combine all same-side like terms.
  - Combine opposite-side like terms to obtain the variable term on one side of the equal sign and the constant term on the other.
  - Divide or multiply as needed to isolate the variable.
  - Check the answer.

- Most linear equations that you will encounter are conditional and have one solution.
- If solving a linear equation leads to a true statement like $0 = 0$, then the equation is an identity and the solution set consists of all real numbers, R.
- If solving a linear equation leads to a false statement like $0 = 5$, then the equation is a contradiction and there is no solution, $\emptyset$.
- Clear fractions by multiplying both sides of a linear equation by the least common multiple of all the denominators. Distribute and multiply all terms by the LCD to obtain an equivalent equation with integer coefficients.
- Given a formula, solve for any variable using the same techniques for solving linear equations. This works because variables are simply representations of real numbers.

? EXERCISE 1.9.4 CHECKING FOR SOLUTIONS

Is the given value a solution to the linear equation?

1. $2(3x+5)-6=3x-8; x=-4$
2. $-x+17-8x=9-x; x=-1$
3. $4(3x-7)-3(x+2)=-1; x=\frac{1}{3}$
4. $-5-2(x-5)=-(x+3); x=-8$
5. $7-2(\frac{1}{2}x-6)=x-1; x=10$
6. $3x-\frac{2}{3}(9x-2)=0; x=\frac{4}{9}$

Answer

1. Yes
3. No
5. Yes

? EXERCISE 1.9.5 SOLVING LINEAR EQUATIONS

Solve.

1. $4x-7=7x+5$
2. $-5x+3=-8x-9$
3. $3x-5=2x-17$
4. $-2y-52=3y+13$
5. $-4x+2=7x-20$
6. $4x-3=6x-15$
7. $9x-25=12x-25$
8. $12y+15=-6y+23$
9. $1.2x-0.7=3x+4.7$
10. $2.1x+6.1=-1.3x+4.4$
11. $2.02x+4.8=14.782-1.2x$
12. $-3.6x+5.5+8.2x=6.5+4.6x$
13. $\frac{1}{2}x-\frac{2}{3}=x+\frac{1}{5}$
14. $\frac{1}{3}x-\frac{1}{2}=-\frac{1}{4}x-\frac{1}{3}$
15. $-\frac{1}{10}y+\frac{2}{5}=\frac{1}{5}y+\frac{3}{10}$
16. $x-\frac{20}{3}=\frac{5}{2}x+\frac{5}{6}$
17. $\frac{2}{3}y+\frac{1}{2}=\frac{5}{8}y+\frac{37}{24}$
18. $\frac{1}{3}+\frac{4}{3}x=\frac{10}{7}x+\frac{1}{3}-\frac{2}{21}x$
19. $\frac{8}{9}-\frac{11}{18}x=\frac{7}{6}-12x$
20. $\frac{1}{3}-9x=\frac{4}{9}+\frac{1}{2}x$
21. $12x-5+9x=44$
22. $10-6x-13=12$
23. $-2+4x+9=7x+8-2x$
24. $20x-5+12x=6-x+7$
25. $3a+5-a=2a+7$
26. $-7b+3=2-5b+1-2b$
27. $7x-2+3x=4+2x-2$

28. $-3x+8-4x+2=10$
29. $6x+2-3x=-2x-13$
30. $3x-0.75+0.21x=1.24x+7.13$
31. $-x-2+4x=5+3x-7$
32. $-2y-5=8y-6-10y$
33. $\frac{1}{10}x-\frac{1}{3}=\frac{1}{30}-\frac{1}{15}x-\frac{7}{15}$
34. $\frac{5}{8}-\frac{4}{3}x+\frac{1}{3}=-\frac{3}{9}x-\frac{1}{4}+\frac{1}{3}x$

Answer

1. -4
3. -12
5. 2
7. 0
9. -3
11. 3.1
13. $-\frac{26}{15}$
15. $\frac{1}{3}$
17. 25
19. $-\frac{5}{2}$
21. $\frac{7}{3}$
23. -1
25. $\emptyset$
27. $\frac{1}{2}$
29. -3
31. R
33. $-\frac{3}{5}$

? EXERCISE 1.9.6 SOLVING LINEAR EQUATIONS INVOLVING PARENTHESES

Solve.

1. $-5(2y-3)+2=12$
2. $3(5x+4)+5x=-8$
3. $4-2(x-5)=-2$
4. $10-5(3x+1)=5(x-4)$
5. $9-(x+7)=2(x-1)$
6. $-5(2x-1)+3=-12$
7. $3x-2(x+1)=x+5$
8. $5x-3(2x-1)=2(x-3)$
9. $-6(x-1)-3x=3(x+8)$
10. $-\frac{3}{5}(5x+10)=\frac{1}{2}(4x-12)$
11. $3.1(2x-3)+0.5=22.2$
12. $4.22-3.13(x-1)=5.2(2x+1)-11.38$
13. $6(x-2)-(7x-12)=14$
14. $-9(x-3)-3x=-3(4x+9)$
15. $3-2(x+4)=-3(4x-5)$
16. $12-2(2x+1)=4(x-1)$
17. $3(x+5)-2(2x+3)=7x+9$
18. $3(2x-1)-4(3x-2)=-5x+10$
19. $-3(2a-3)+2=3(a+7)$
20. $-2(5x-3)-1=5(-2x+1)$

21. $\frac{1}{2}(2x+1)-\frac{1}{4}(8x+2)=3(x-4)$
22. $-\frac{2}{3}(6x-3)-\frac{1}{2}=\frac{3}{2}(4x+1)$
23. $\frac{1}{2}(3x-1)+\frac{1}{3}(2x-5)=0$
24. $\frac{1}{3}(x-2)+\frac{1}{5}=\frac{1}{9}(3x+3)$
25. $-2(2x-7)-(x+3)=6(x-1)$
26. $10(3x+5)-5(4x+2)=2(5x+20)$
27. $2(x-3)-6(2x+1)=-5(2x-4)$
28. $5(x-2)-(4x-1)=-2(3-x)$
29. $6(3x-2)-(12x-1)+4=0$
30. $-3(4x-2)-(9x+3)-6x=0$

Answer

1. $\frac{1}{2}$
3. 8
5. $\frac{4}{3}$
7. $\emptyset$
9. $-\frac{3}{2}$
11. 5
13. -14
15. 2
17. 0
19. $-\frac{10}{9}$
21. 3
23. 1
25. $\frac{17}{11}$
27. $\emptyset$
29. $\frac{7}{6}$

? EXERCISE 1.9.7 LITERAL EQUATIONS

Solve for the indicated variable.

1. Solve for w: $A=l\cdot w$.
2. Solve for a: $F=ma$.
3. Solve for w: $P=2l+2w$.
4. Solve for r: $C=2\pi r$.
5. Solve for b: $P=a+b+c$.
6. Solve for C: $F=\frac{9}{5}C+32$.
7. Solve for h: $A=\frac{1}{2}bh$.
8. Solve for t: $I=Prt$.
9. Solve for y: $ax+by=c$.
10. Solve for h: $S=2\pi r^2+2\pi rh$.
11. Solve for x: $z=\frac{2x+y}{5}$.
12. Solve for c: $a=3b-\frac{2c}{3}$.
13. Solve for b: $y=mx+b$.
14. Solve for m: $y=mx+b$.
15. Solve for y: $3x-2y=6$.
16. Solve for y: $-5x+2y=12$.
17. Solve for y: $\frac{x}{3}-\frac{y}{5}=1$.
18. Solve for y: $\frac{3}{4}x-\frac{1}{5}y=\frac{1}{2}$.

Answer

1. $w = \frac{A}{l}$
3. $w = \frac{P-2l}{2}$
5. $b = P - a - c$
7. $h = \frac{2A}{b}$
9. $y = \frac{-ax+c}{b}$
11. $x = \frac{5z-y}{2}$
13. $b = y - mx$
15. $y = \frac{3x-6}{2}$
17. $y = \frac{5x-15}{3}$

? EXERCISE 1.9.8 LITERAL EQUATIONS

Translate the following sentences into linear equations and then solve.

1. The sum of $3x$ and 5 is equal to the sum of $2x$ and 7.
2. The sum of $-5x$ and 6 is equal to the difference of $4x$ and 2.
3. The difference of $5x$ and 25 is equal to the difference of $3x$ and 51.
4. The sum of $\frac{1}{2}x$ and $\frac{3}{4}$ is equal to $\frac{2}{3}x$.
5. A number n divided by 5 is equal to the sum of twice the number and 3.
6. Negative ten times a number n is equal to the sum of three times the number and 13.

Answer

1. $3x + 5 = 2x + 7$; $x = 2$
3. $5x - 25 = 3x - 51$; $x = -13$
5. $\frac{n}{5} = 2n + 3$; $n = -\frac{5}{3}$

? EXERCISE 1.9.9 DISCUSSION BOARD TOPICS

1. What is the origin of the word algebra?
2. What is regarded as the main business of algebra?
3. Why is solving equations such an important algebra topic?
4. Post some real-world linear formulas not presented in this section.
5. Research and discuss the contributions of Diophantus of Alexandria.
6. Create an identity or contradiction of your own and share on the discussion board. Provide a solution and explain how you found it.

Answer

1. Answers may vary
3. Answers may vary
5. Answers may vary

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1.E: REVIEW EXERCISES AND SAMPLE EXAM

REVIEW EXERCISES

EXERCISE 1.E.1 SOLVING LINEAR SYSTEMS BY GRAPHING

Determine whether the given ordered pair is a solution to the given system.

1. $(1,-3)$; $\begin{cases} 5x-y=8 \\ -3x+y=-6 \end{cases}$
2. $(-3,-4)$; $\begin{cases} 4x-12y=-10 \\ 6x-5y=-2 \end{cases}$
3. $(-1,\frac{1}{5})$; $\begin{cases} \frac{3}{5}x-\frac{1}{3}y=-\frac{2}{3} \\ -\frac{1}{5}x-\frac{1}{2}y=\frac{1}{10} \end{cases}$
4. $(\frac{1}{2},-1)$; $\begin{cases} x+\frac{3}{4}y=-\frac{1}{4} \\ \frac{2}{3}x-y=\frac{4}{3} \end{cases}$

Answer

1. Yes
3. Yes

EXERCISE 1.E.2 SOLVING LINEAR SYSTEMS BY GRAPHING

Given the graph, determine the simultaneous solution.

1.

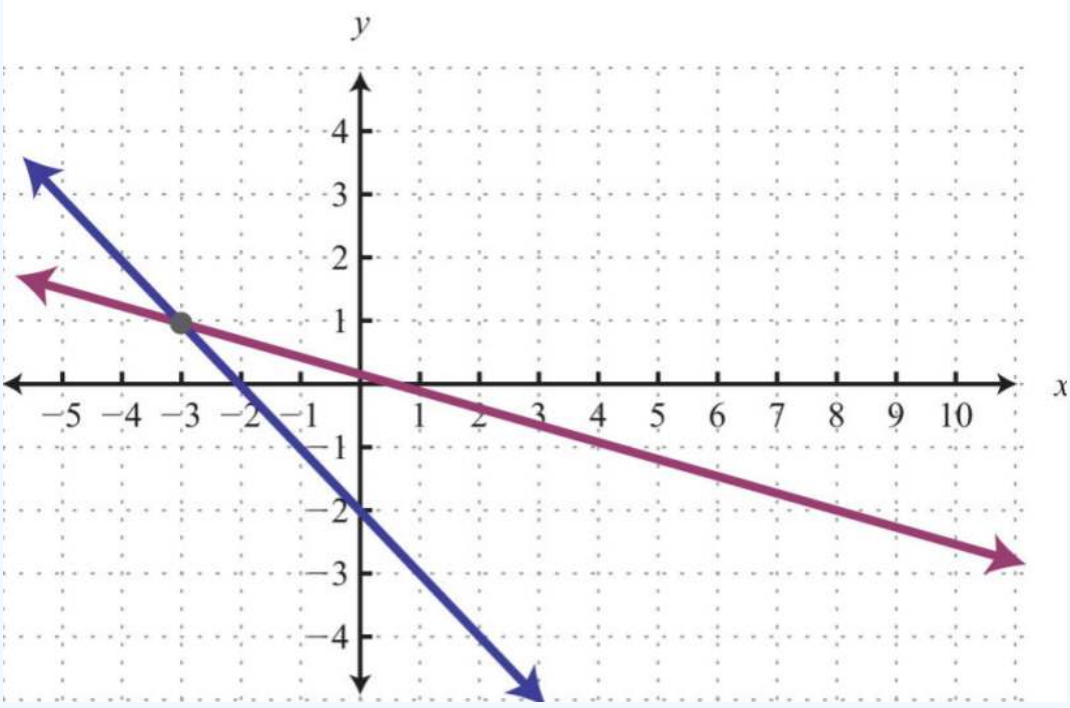


Figure 4.E.1

2.

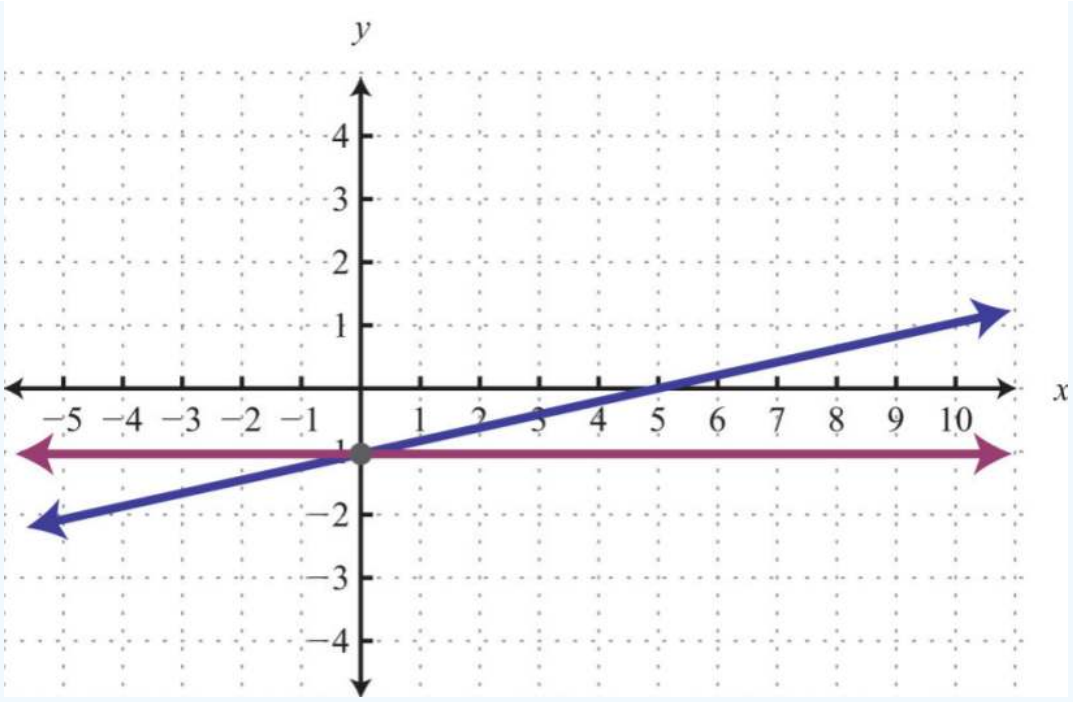


Figure 4.E.2

3.

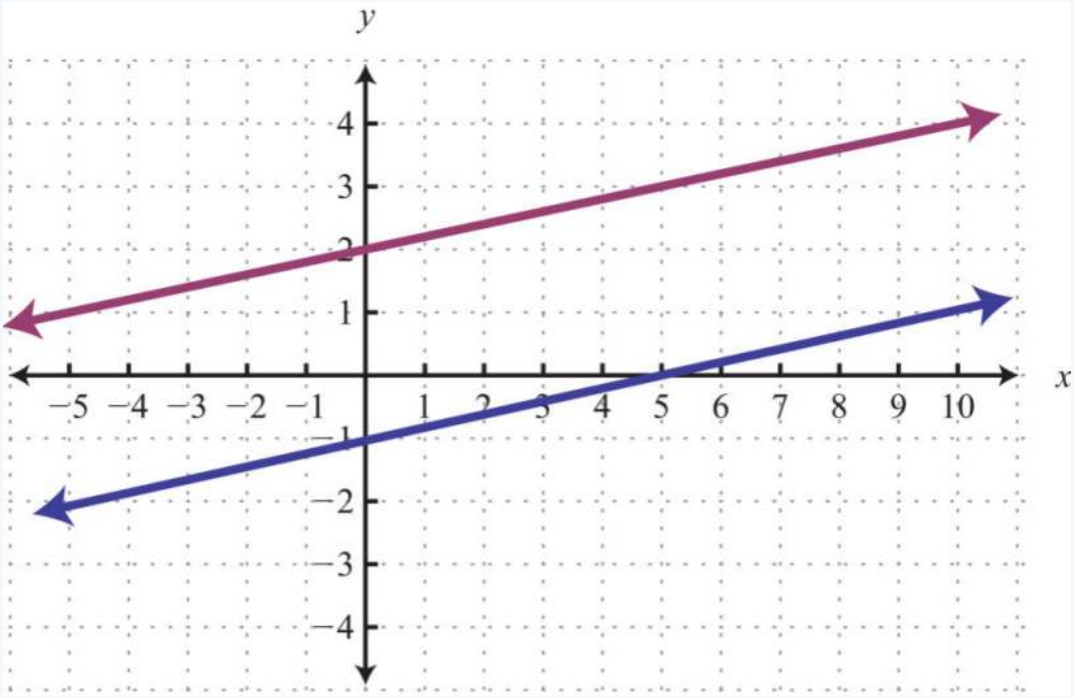


Figure 4.E.3

4.

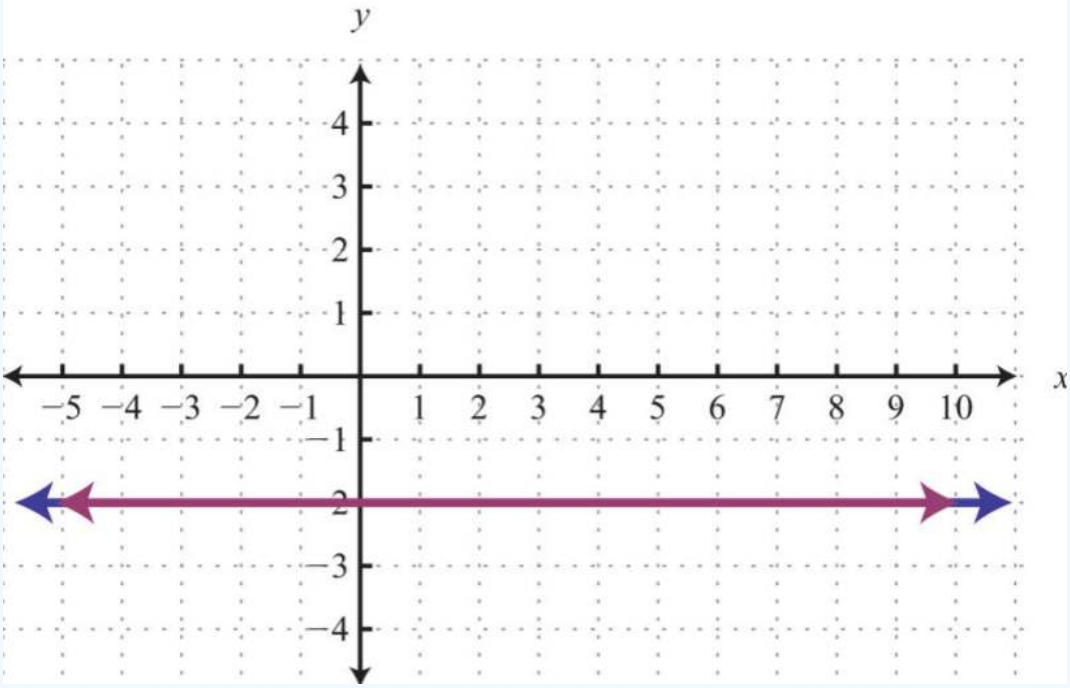


Figure 4.E.4

Answer

1. $(-3, 1)$
3. Ø

EXERCISE 1.E.3 SOLVING LINEAR SYSTEMS BY GRAPHING

Solve by graphing.

1. $\begin{cases} y = 12x - 3 \\ y = -\frac{3}{4}x + 2 \end{cases}$
2. $\begin{cases} y = 5 \\ y = -\frac{4}{5}x + 1 \end{cases}$
3. $\begin{cases} x - 2y = 0 \\ 2x - 3y = 3 \end{cases}$
4. $\begin{cases} 5x - y = -11 \\ -4x + 2y = 16 \end{cases}$
5. $\begin{cases} \frac{5}{2}x + 2y = 6 \\ 5x + 4y = 12 \end{cases}$
6. $\begin{cases} 6x - 10y = -2 \\ 3x - 5y = 5 \end{cases}$

Answer

1. $(4, -1)$
3. $(6, 3)$
5. $(x, -\frac{5}{4}x + 3)$

EXERCISE 1.E.4 SOLVING LINEAR SYSTEMS BY SUBSTITUTION

Solve by substitution.

1. $\begin{cases} y = 7x - 2 \\ x + y = 6 \end{cases}$
2. $\begin{cases} 2x - 4y = 10 \\ x = -2y - 1 \end{cases}$
3. $\begin{cases} x - y = 0 \\ 5x - 7y = -8 \end{cases}$
4. $\begin{cases} 9x + 2y = -41 \\ -x + y = 7 \end{cases}$
5. $\begin{cases} 6x - 3y = 4 \\ 2x - 9y = 4 \end{cases}$
6. $\begin{cases} 8x - y = 7 \\ 12x + 3y = 6 \end{cases}$
7. $\begin{cases} 20x - 4y = -3 \\ -5x + y = -12 \end{cases}$
8. $\begin{cases} 3x - y = 6 \\ x - 13y = 2 \end{cases}$
9. $\begin{cases} x = -1 \\ 8x - 4y = -10 \end{cases}$
10. $\begin{cases} y = -7 \\ 14x - 4y = 0 \end{cases}$

Answer

1. $(1, 5)$
3. $(4, 4)$
5. $(\frac{1}{2}, -\frac{1}{3})$
7. Ø
9. $(-1, \frac{1}{2})$

EXERCISE 1.E.5 SOLVING LINEAR SYSTEMS BY ELIMINATION

Solve by elimination.

1. $\begin{cases} x - y = 5 \\ 3x - 8y = 5 \end{cases}$
2. $\begin{cases} 7x + 2y = -10 \\ 9x + 4y = -30 \end{cases}$
3. $\begin{cases} 9x - 6y = -6 \\ 2x - 5y = 17 \end{cases}$
4. $\begin{cases} 4x - 2y = 30 \\ 3x + 7y = 14 \end{cases}$
5. $\begin{cases} \frac{5}{2}x - 2y = -\frac{1}{14} \\ \frac{1}{6}x - \frac{1}{3}y = -\frac{1}{3} \end{cases}$
6. $\begin{cases} 2x - \frac{3}{2}y = 20 \\ \frac{3}{32}x - \frac{1}{3}y = \frac{1}{16} \end{cases}$
7. $\begin{cases} 0.1x - 0.3y = 0.17 \\ 0.6x + 0.5y = -0.13 \end{cases}$
8. $\begin{cases} -1.25x - 0.45y = -12.23 \\ 0.5x - 1.5y = 5.9 \end{cases}$

9. $\begin{cases} 6x-52y=-5 \\ -12x+5y=10 \end{cases}$
10. $\begin{cases} 27x+12y=-2 \\ 9x+4y=3 \end{cases}$
11. $\begin{cases} 6x-5y=0 \\ 4x-3y=2 \end{cases}$
12. $\begin{cases} 5x=1 \\ 10x+3y=6 \end{cases}$
13. $\begin{cases} 8y=-2x+6 \\ 3x=6y-18 \end{cases}$
14. $\begin{cases} 6y=3x+1 \\ 9x-27y-3=0 \end{cases}$

Answer

1. $(7,2)$
3. $(-4,-5)$
5. $(-\frac{1}{2},\frac{3}{4})$
7. $(0.2,-0.5)$
9. $(x,\frac{12}{5}x+2)$
11. $(5,6)$
13. $(-3,\frac{3}{2})$

EXERCISE 1.E.6 APPLICATIONS OF LINEAR SYSTEMS

Set up a linear system and solve.

1. The sum of two numbers is 74 and their difference is 38. Find the numbers.
2. The sum of two numbers is 34. When the larger is subtracted from twice the smaller, the result is 8. Find the numbers.
3. A jar full of 40 coins consisting of dimes and nickels has a total value of $2.90. How many of each coin are in the jar?
4. A total of $9,600 was invested in two separate accounts earning 5.5% and 3.75% annual interest. If the total simple interest earned for the year was $491.25, then how much was invested in each account?
5. A 1% saline solution is to be mixed with a 3% saline solution to produce 6 ounces of a 1.8% saline solution. How much of each is needed?
6. An 80% fruit juice concentrate is to be mixed with water to produce 10 gallons of a 20% fruit juice mixture. How much of each is needed?
7. An executive traveled a total of $4\frac{1}{2}$ hours and 435 miles to a conference by car and by light aircraft. Driving to the airport by car, he averaged 50 miles per hour. In the air, the light aircraft averaged 120 miles per hour. How long did it take him to drive to the airport?
8. Flying with the wind, an airplane traveled 1,065 miles in 3 hours. On the return trip, against the wind, the airplane traveled 915 miles in 3 hours. What is the speed of the wind?

Answer

1. 18 and 56
3. 18 dimes and 22 nickels
5. 3.6 ounces of the 1% saline solution and 2.4 ounces of the 3% saline solution
7. It took him $1\frac{1}{2}$ hours to drive to the airport

EXERCISE 1.E.7 SYSTEMS OF LINEAR INEQUALITIES (TWO VARIABLES)

Determine whether the given point is a solution to the system of linear inequalities.

1. $(5,-2)$; $\begin{cases} 5x-y>8 \\ -3x+y\leq -6 \end{cases}$

2. $(2,3)$; $\begin{cases} 2x-3y>-10 \\ -5x+y>1 \end{cases}$
3. $(2,-10)$; $\begin{cases} y<-10x \\ -y\geq 0 \end{cases}$
4. $(0,-2)$; $\begin{cases} y>12x-4 \\ y<-\frac{3}{4}x+2 \end{cases}$

Answer

1. Yes
3. No

EXERCISE 1.E.8 SYSTEMS OF LINEAR INEQUALITIES (TWO VARIABLES)

Graph the solution set.

1. $\begin{cases} 8x+3y\leq 24 \\ 2x+3y<12 \end{cases}$
2. $\begin{cases} x+y\geq 7 \\ 4x-y\geq 0 \end{cases}$
3. $\begin{cases} x-3y>-12 \\ -2x+6y>-6 \end{cases}$
4. $\begin{cases} y\leq 7 \\ x-y>0 \end{cases}$
5. $\begin{cases} y<4 \\ y\geq \frac{4}{3}x+1 \\ y>-x-1 \end{cases}$
6. $\begin{cases} x-y\geq -3 \\ x-y\leq 3 \\ x+y<1 \end{cases}$

Answer

1.

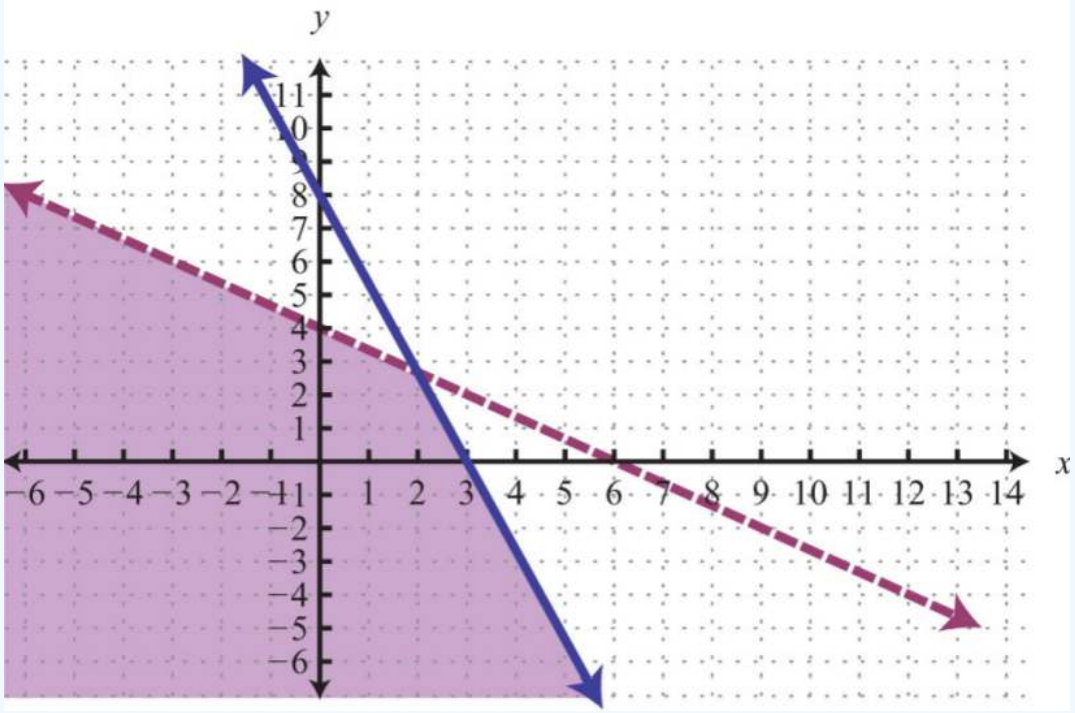


Figure 4.E.5

3.

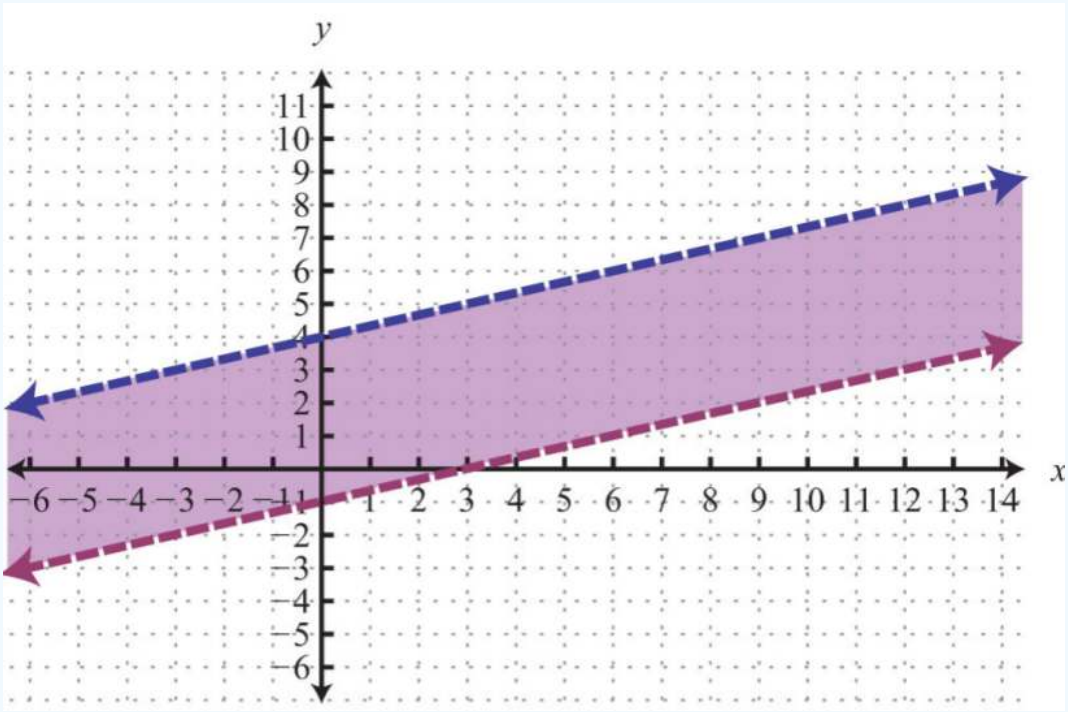


Figure 4.E.6

5.

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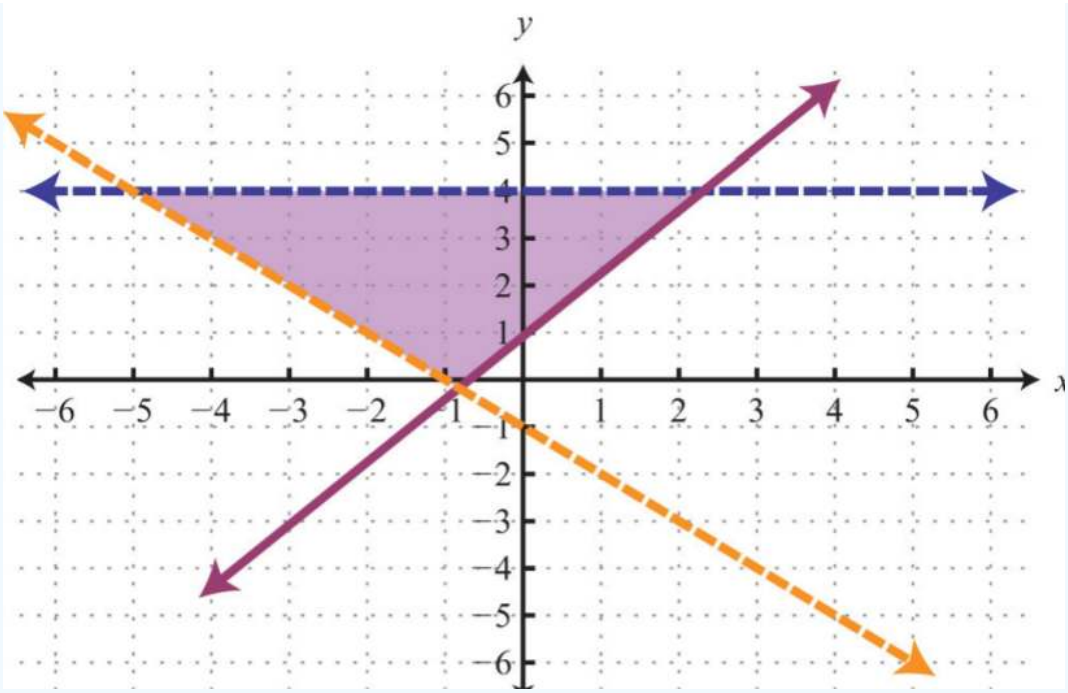


Figure 4.E.7

SAMPLE EXAM

EXERCISE $1.E.9$

1. Is $(-3, 2)$ a solution to the system $\begin{cases} 2x - 3y = -12 \\ -4x + y = 14 \end{cases}$?
2. Is $(-2, 9)$ a solution to the system $\begin{cases} x + y \geq 7 \\ 4x - y < 0 \end{cases}$?

Answer

1. Yes

EXERCISE $1.E.10$

Given the graph, determine the simultaneous solution.

1.

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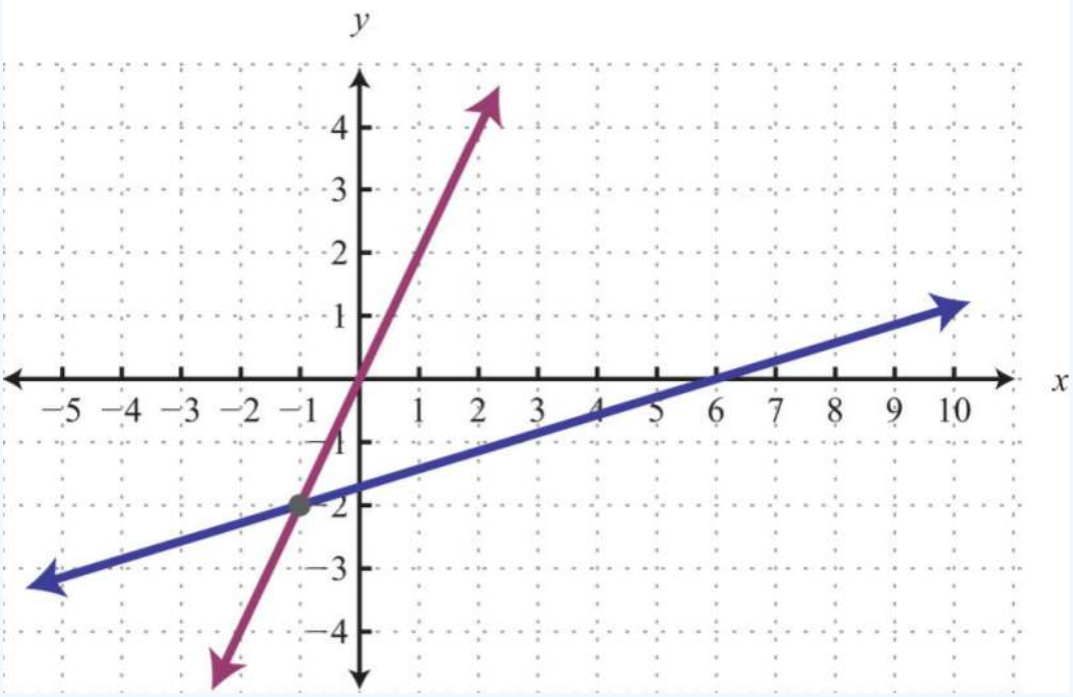


Figure 4.E.8

2.

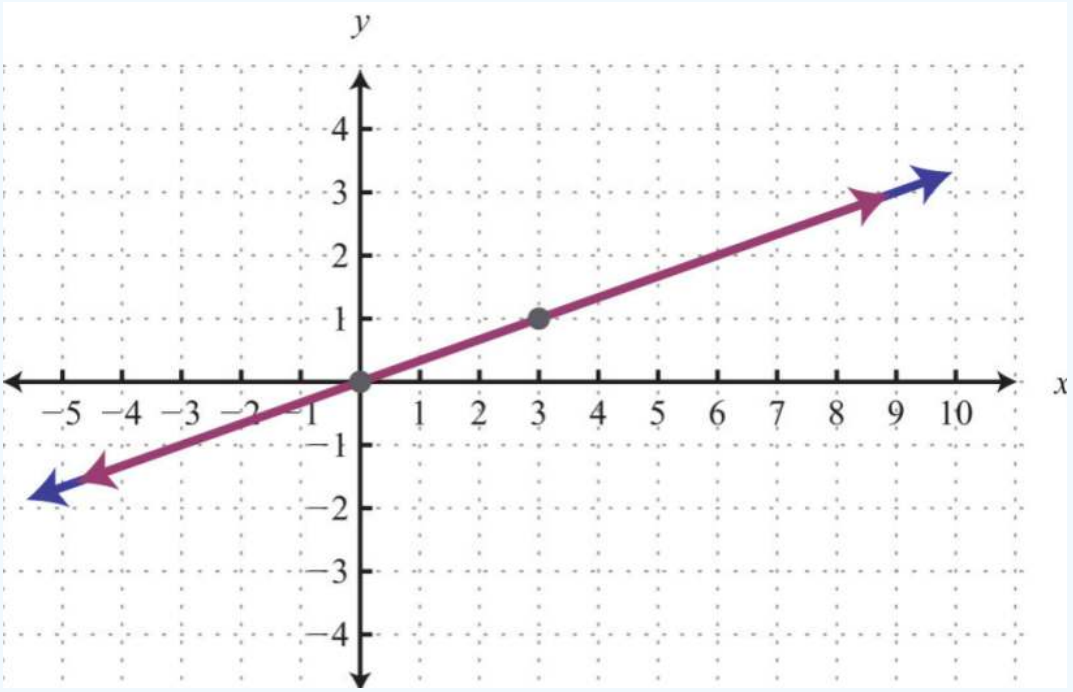


Figure 4.E.9

Answer

1. $(-1,-2)$

EXERCISE 1.E. 11

Solve using the graphing method.

1. $\begin{cases} y = x - 3 \\ y = -12x + 3 \end{cases}$

2. $\begin{cases} 2x+3y=6 \\ -x+6y=-18 \end{cases}$
3. $\begin{cases} y=2 \\ x+y=3 \end{cases}$
4. $\begin{cases} y=x \\ x=-5 \end{cases}$

Answer

1. $(4,1)$
3. $(1,2)$

EXERCISE 1.E.12

Solve using the substitution method.

1. $\begin{cases} 5x+y=-14 \\ 2x-3y=-9 \end{cases}$
2. $\begin{cases} 4x-3y=1 \\ x-2y=2 \end{cases}$
3. $\begin{cases} 5x+y=1 \\ 10x+2y=4 \end{cases}$
4. $\begin{cases} x-2y=4 \\ 3x-6y=12 \end{cases}$

Answer

1. $(-3,1)$
3. Ø

EXERCISE 1.E.13

Solve using the elimination method.

1. $\begin{cases} 4x-y=13 \\ -5x+2y=-17 \end{cases}$
2. $\begin{cases} 7x-3y=-23 \\ 4x+5y=7 \end{cases}$
3. $\begin{cases} -3x+18y=1 \\ 8x-6y=6 \end{cases}$
4. $\begin{cases} -4x+3y=-3 \\ 8x-6y=6 \end{cases}$
5. $\begin{cases} \frac{1}{2}x+\frac{3}{4}y=\frac{7}{4} \\ 4x-\frac{1}{3}y=\frac{4}{3} \end{cases}$
6. $\begin{cases} 0.2x-0.1y=-0.24 \\ -0.3x+0.5y=0.08 \end{cases}$

Answer

1. $(3,-1)$
3. Ø
5. $(\frac{1}{2},2)$

EXERCISE 1.E.14

Graph the solution set.

1. $\begin{cases} 3x + 4y < 2 \\ 4x - 4y < 8 \end{cases}$
2. $\begin{cases} x \leq 8 \\ 3x - 8y \leq 0 \end{cases}$

Answer

1.

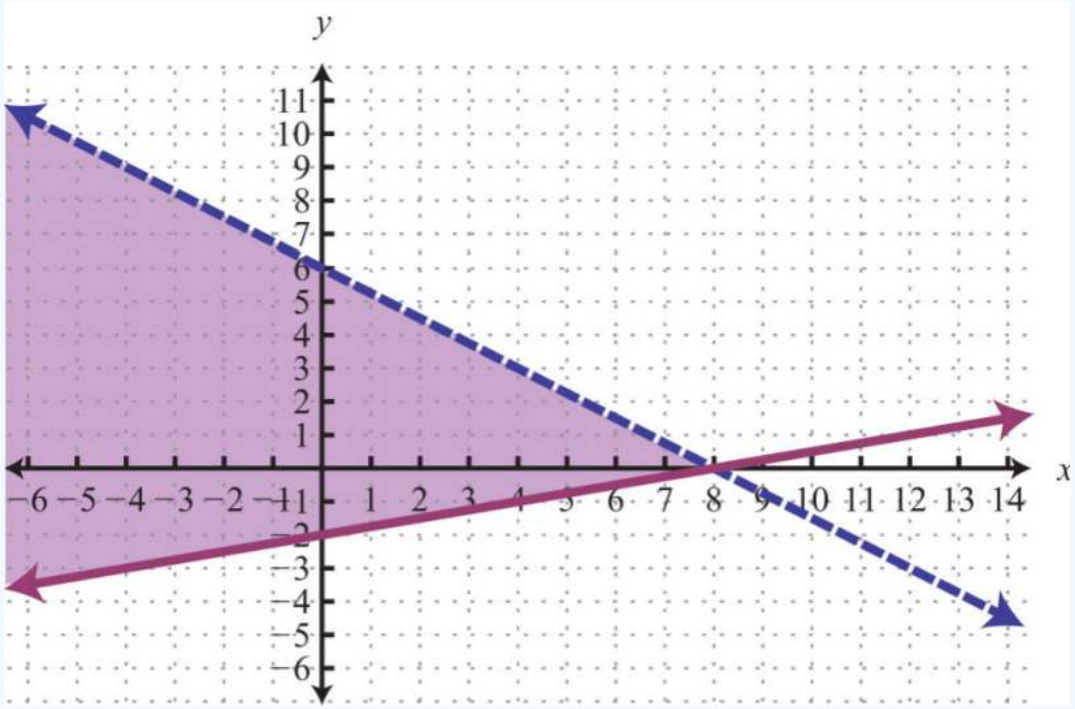


Figure 4.E.10

EXERCISE 1.E.15

Set up a linear system of two equations and two variables and solve it using any method.

1. The sum of two integers is 23. If the larger integer is one less than twice the smaller, then find the two integers.
2. James has $\$2,400$ saved in two separate accounts. One account earns 3% annual interest and the other earns 4%. If his interest for the year totals \$88, then how much is in each account?
3. Mary drives 110 miles to her grandmother's house in a total of 2 hours. On the freeway, she averages 62 miles per hour. In the city she averages 34 miles per hour. How long does she spend on the freeway?
4. A 15% acid solution is to be mixed with a 35% acid solution to produce 12 ounces of a 22% acid solution. How much of each is needed?
5. Joey has bag full of 52 dimes and quarters with a total value of \$8.35. How many of each coin does Joey have?

Answer

1. 8 and 15
3. She drives $1\frac{1}{2}$ hours on the freeway.
5. 21 quarters and 31 dimes

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CHAPTER OVERVIEW

2: POLYNOMIALS AND THEIR OPERATIONS

2.1: Introduction to Functions
2.2: Rules of Exponents
2.3: Introduction to Polynomials
2.4: Adding and Subtracting Polynomials
2.5: Multiplying Polynomials
2.6: Dividing Polynomials
2.7: Negative Exponents
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2.1: INTRODUCTION TO FUNCTIONS

LEARNING OBJECTIVES

- Identify a function.
- State the domain and range of a function.
- Use function notation.

RELATIONS, FUNCTIONS, DOMAIN, AND RANGE

Relationships between sets occur often in everyday life. For example, for each month in Cape Canaveral, we can associate an average amount of rainfall. In this case, the amount of precipitation depends on the month of the year, and the data can be written in tabular form or as a set of ordered pairs.

Table 2.1.1

Month	Precipitation	Ordered pairs
January	2.4 in	(January, 2.4)
February	3.3 in	(February, 3.3)
March	3.1 in	(March, 3.1)
April	2.0 in	(April, 2.0)
May	3.8 in	(May, 3.8)
June	6.8 in	(June, 6.8)
July	8.1 in	(July, 8.1)
August	7.6 in	(August, 7.6)
September	7.3 in	(September, 7.3)
October	4.1 in	(October, 4.1)
November	3.3 in	(November, 3.3)
December	2.4 in	(December, 2.4)

We define a relation as any set of ordered pairs. Usually we write the independent component of the relation in the first column and the dependent component in the second column. In the opening example, notice that it makes sense to relate the average amount of precipitation as dependent on the month of year. The set of all elements in the first column of a relation is called the domain. The set of all elements that compose the second column is called the range. In this example, the domain consists of the set of all months of the year, and the range consists of the values that represent the average rainfall for each month.

In the context of algebra, the relations of interest are sets of ordered pairs (x, y) in the rectangular coordinate plane. In this case, the x-values define the domain and the y-values define the range. Of special interest are relations where every x-value corresponds to exactly one y-value; these relations are called functions.

EXAMPLE 2.1.1

Determine the domain and range of the following relation and state whether or not it is a function:

$$\{(-1, 4), (0, 7), (2, 3), (3, 3), (4, -2)\}$$

Solution:

Here we separate the domain and range and depict the correspondence between the values with arrows.

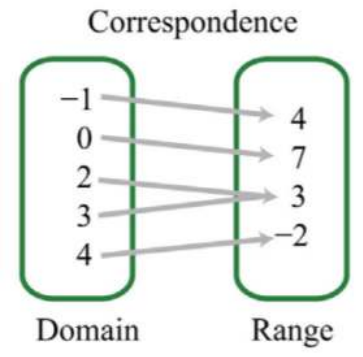


***Figure* 2.1.1**

Answer:

The domain is $\{-1, 0, 2, 3, 4\}$, and the range is $\{-2, 3, 4, 7\}$. The relation is a function because each x-value corresponds to exactly one y-value.

EXAMPLE 2.1.2

Determine the domain and range of the following relation and state whether or not it is a function:

$$\{(-4, -3), (-2, 6), (0, 3), (3, 5), (3, 7)\}.$$

Solution:

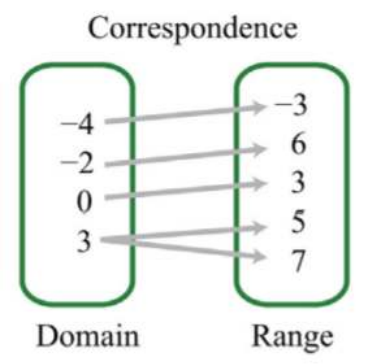


***Figure* 2.1.2**

Answer:

The domain is $\{-4, -2, 0, 3\}$, and the range is $\{-3, 3, 5, 6, 7\}$. This relation is not a function because the x-value 3 has two corresponding y-values.

In the previous example, the relation is not a function because it contains ordered pairs with the same x-value, $(3, 5)$ and $(3, 7)$. We can recognize functions as relations where no x-values are repeated.

In algebra, equations such as $y = \frac{3}{4}x - 2$ define relations. This linear equation can be graphed as follows:

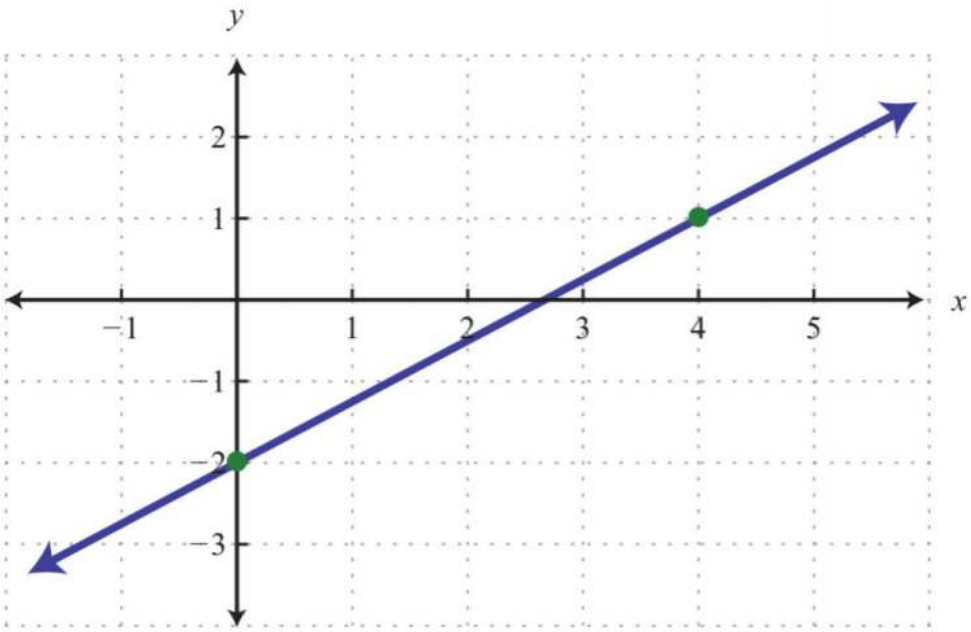


***Figure* 2.1.3**

The graph is a relation since it represents the infinite set of ordered pair solutions to $y = \frac{3}{4}x - 2$. The domain is the set of all x-values, and in this case consists of all real numbers. The range is the set of all possible y-values, and in this case also consists of all real numbers.

Furthermore, the graph is a function because for each x-value there is only one corresponding y-value. In fact, any nonvertical or nonhorizontal line is a function with domain and range consisting of all real numbers.

Any graph is a set of ordered pairs and thus defines a relation. Consider the following graph of a circle:

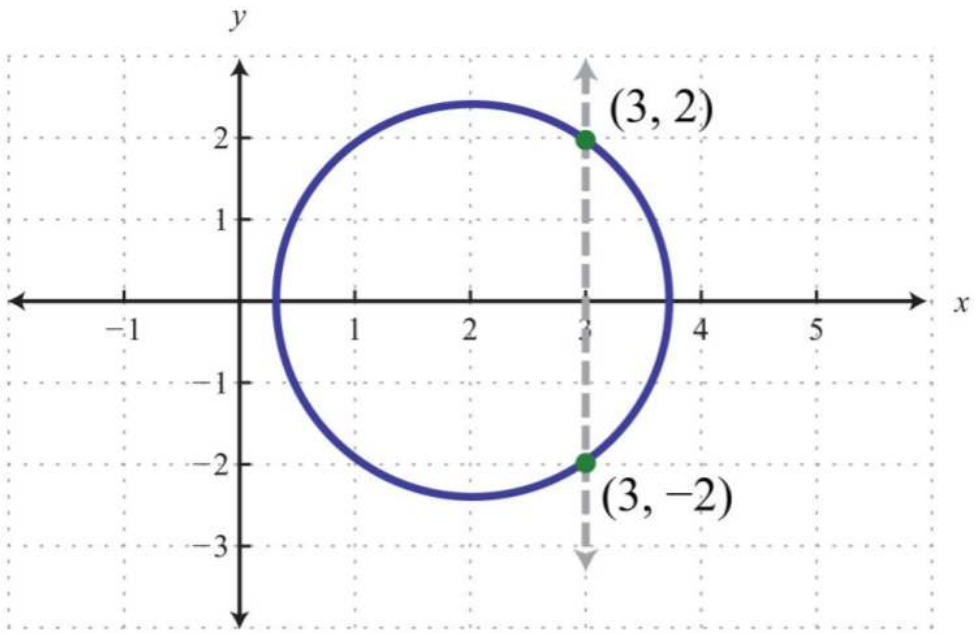


***Figure* 2.1.4**

Here the graph represents a relation where many x-values in the domain correspond to two y-values. If we draw a vertical line, as illustrated, we can see that $(3, 2)$ and $(3, -2)$ are two ordered pairs with the same x-value. Therefore, the x-value 3 corresponds to two y-values; hence the graph does not represent a function. The illustration suggests that if any vertical line intersects a graph more than once, then the graph does not represent a function. This is called the vertical line test.

EXAMPLE 2.1.3

Given the following graph, determine the domain and range and state whether or not it is a function.

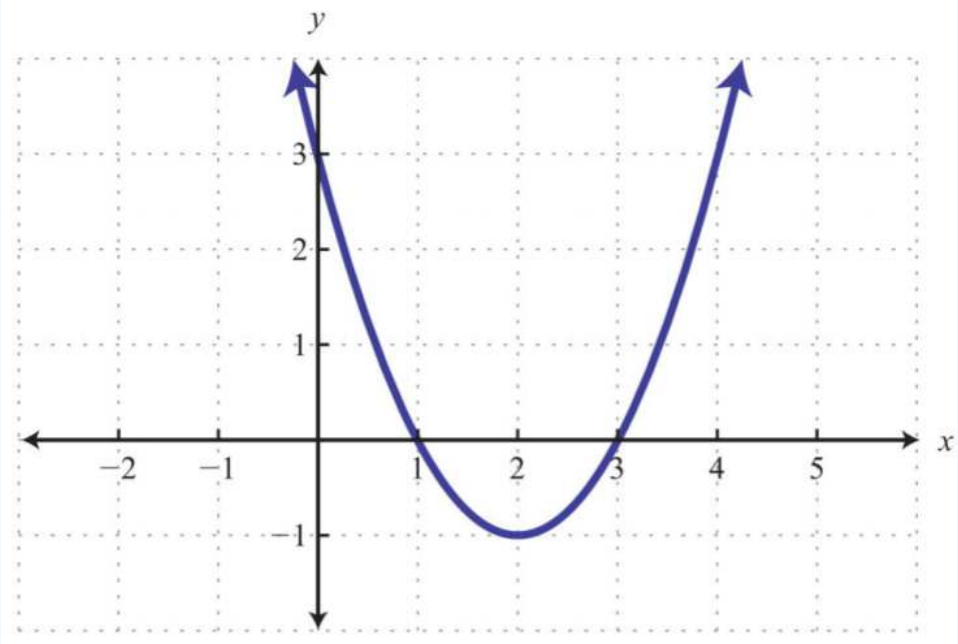


***Figure* 2.1.5**

Solution:

The given shape is called a parabola and extends indefinitely to the left and right as indicated by the arrows. This suggests that if we choose any x-value, then we will be able to find a corresponding point on the graph; therefore, the domain consists of all real numbers. Furthermore, the graph shows that -1 is the minimum y-value, and any y-value greater than that is represented in the relation. Hence the range consists of all y-values greater than or equal to -1, or in interval notation, $[-1, \infty)$.

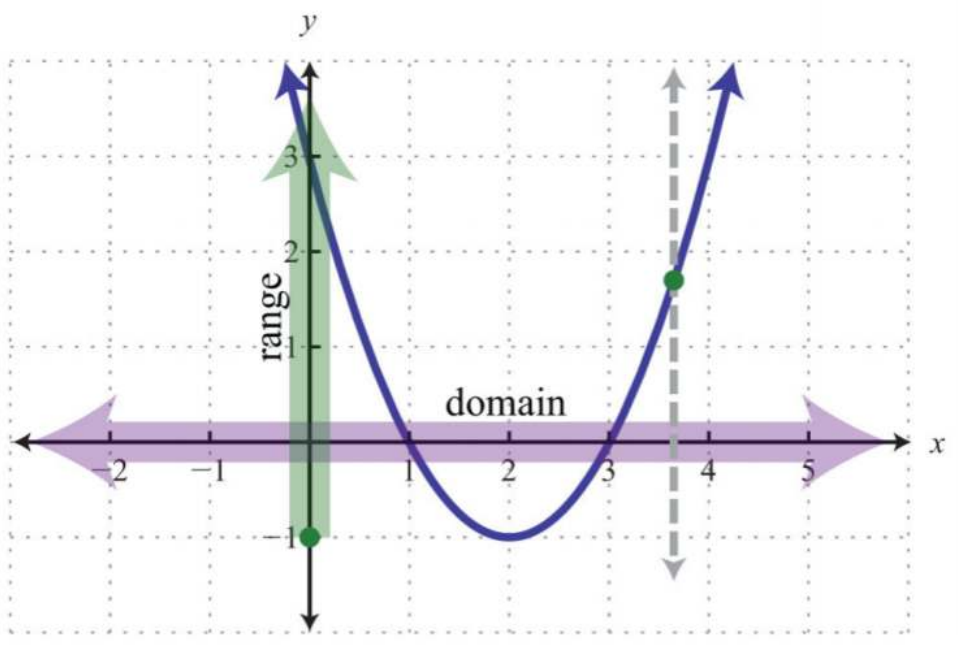


***Figure* 2.1.6**

Lastly, any vertical line will intersect the graph only once; therefore, it is a function.

Answer:

The domain is all real numbers $R = (-\infty, \infty)$, and the range is $[-1, \infty)$. The graph represents a function because it passes the vertical line test.

? EXERCISE 2.1.1

Given the graph, determine the domain and range and state whether or not it is a function:

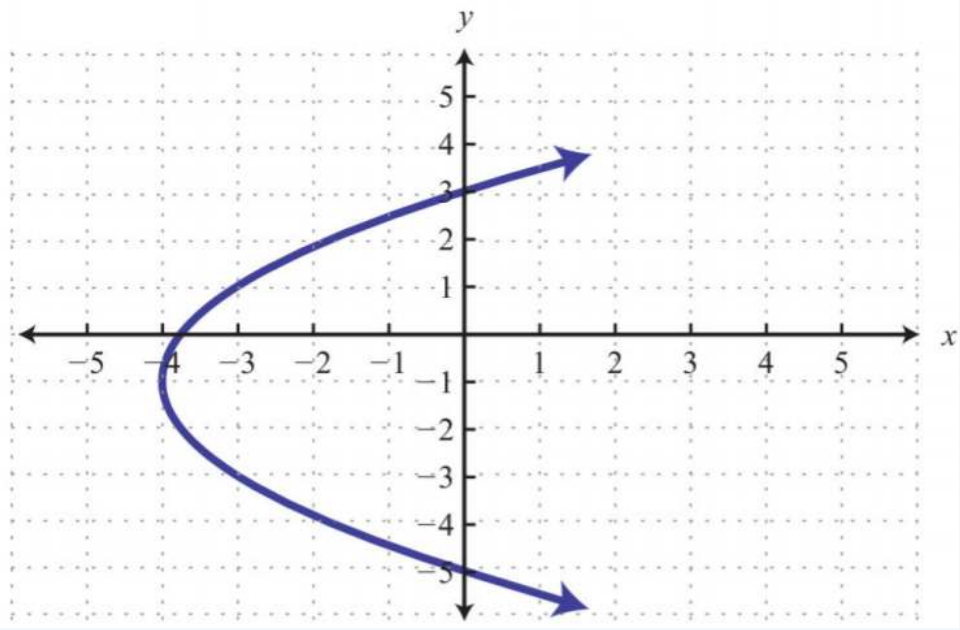


***Figure* 2.1.7**

Answer

Domain: $[-4, \infty)$; range: $(-\infty, \infty)$; function: no

FUNCTION NOTATION AND LINEAR FUNCTIONS

With the definition of a function comes special notation. If we consider each x-value to be the input that produces exactly one output, then we can use the notation

$$f(x) = y \tag{2.1.1}$$

The notation $f(x)$ reads "f of x" and should not be confused with multiplication. Most of our study of algebra involves functions, so the notation becomes very useful when performing common tasks. Functions can be named with different letters; some common names for functions are $g(x), h(x), C(x)$, and $R(x)$. First, consider nonvertical lines that we know can be expressed using slope-intercept form, $y = mx + b$. For any real numbers m and b, the equation defines a function, and we can replace y with the new notation $f(x)$ as follows:

$$y = mx + b \tag{2.1.2}$$

$$f(x) = mx + b \tag{2.1.3}$$

Therefore, a linear function is any function that can be written in the form $f(x) = mx + b$. In particular, we can write the following:

The notation also shows values to evaluate in the equation. If the value for x is given as 8, then we know that we can find the corresponding y-value by substituting 8 in for x and simplifying. Using function notation, this is denoted $f(8)$ and can be interpreted as follows:

Finally, simplify:

We have $f(8) = 4$. This notation tells us that when $x = 8$ (***the input***), the function results in 4 (***the output***).

Given the linear function $f(x) = -5x + 7$, find $f(-2)$.

Solution:

In this case, $f(-2)$ indicates that we should evaluate when $x = -2$.

$$\begin{aligned} f(x) &= -5x + 7 \\ f(-2) &= -5(-2) + 7 \qquad \textit{Replace } x \textit{ with } -2. \\ &= 10 + 7 \\ &= 17 \end{aligned}$$

Answer:

$$f(-2) = 17$$

EXAMPLE 2.1.5

Given the linear function $f(x) = -5x + 7$, find x when $f(x) = 10$.

Solution:

In this case, $f(x) = 10$ indicates that the function should be set equal to 10.

$$\begin{aligned} f(x) &= -5x + 7 \\ 10 &= -5x + 7 \qquad \textit{Replace } f(x) \textit{ with } 10. \\ 10 - 7 &= -5x + 7 - 7 \qquad \textit{Solve for } x. \\ 3 &= -5x \\ \frac{3}{-5} &= \frac{-5x}{-5} - \frac{3}{5} \qquad = x \end{aligned}$$

Answer:

Here $x = -\frac{3}{5}$, and we can write $f(-\frac{3}{5}) = 10$.

EXAMPLE 2.1.6

Given the graph of a linear function $g(x)$,

a. Find $g(2)$.
b. Find x when $g(x) = 3$.

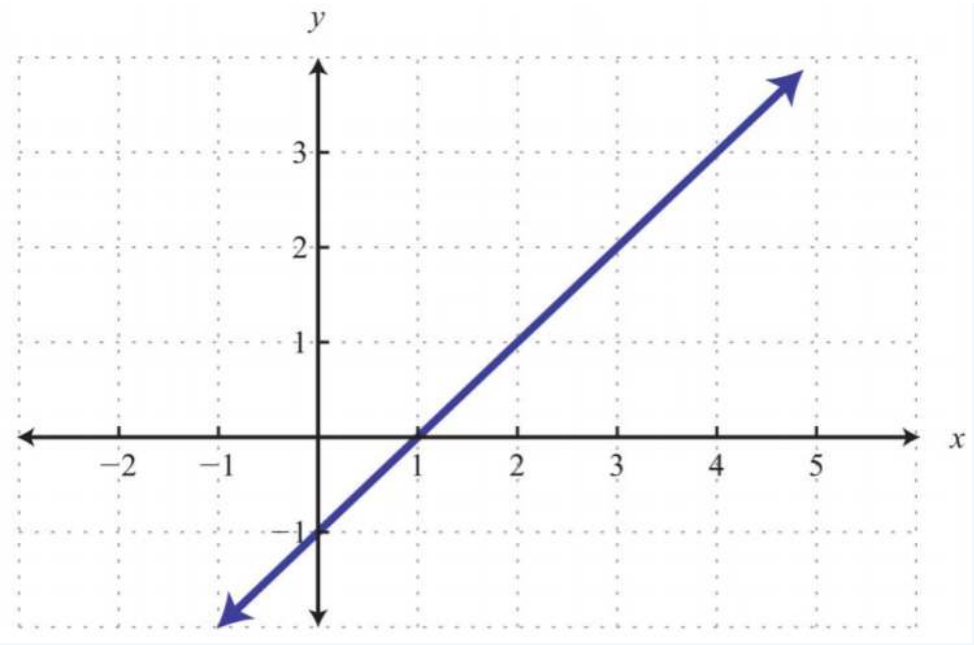


***Figure* 2.1.8**

Solution:

a. The notation $g(2)$ implies that $x = 2$. Use the graph to determine the corresponding y-value.

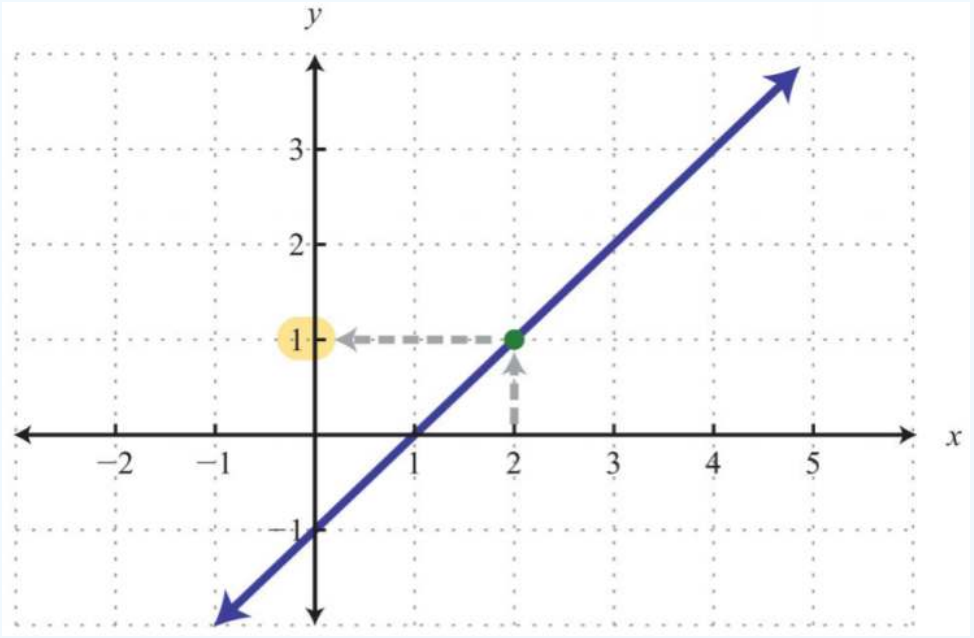


***Figure* 2.1.9**

b. The notation $g(x) = 3$ implies that the y-value is given as 3. Use the graph to determine the corresponding x-value.

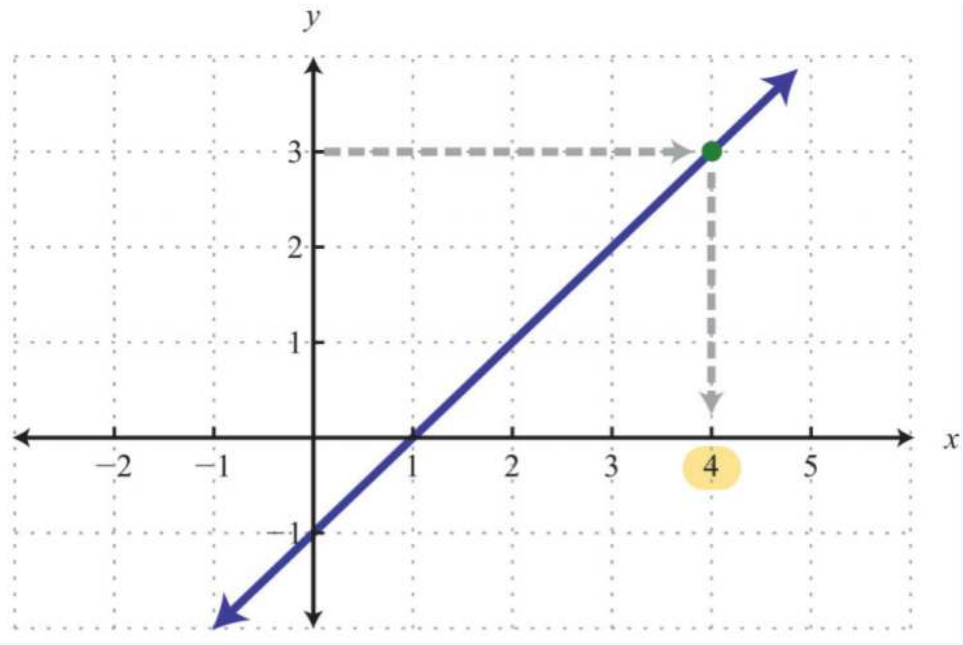


***Figure* 2.1.10**

Answer:

a. $g(2) = 1$
b. $x = 4$

EXAMPLE 2.1.7

Graph the linear function $f(x) = -\frac{5}{3}x + 6$ and state the domain and range.

Solution:

From the function, we see that $b = 6$ and thus the y-intercept is $(0, 6)$. Also, we can see that the slope is $m = \frac{-5}{3} = -\frac{5}{3} = \frac{rise}{run}$. Starting from the y-intercept, mark a second point down 5 units and right 3 units.

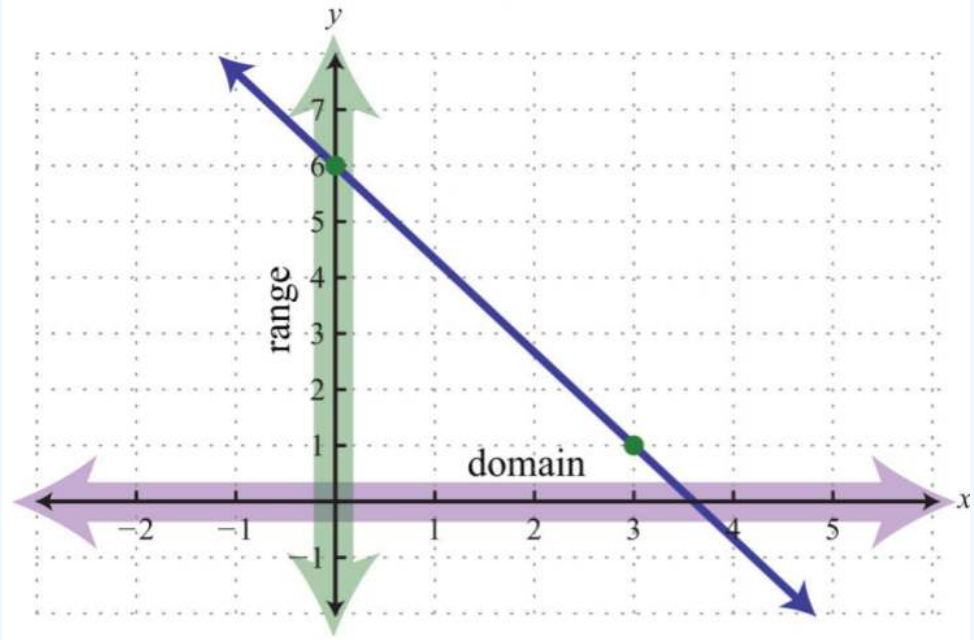


***Figure* 2.1.11**

Given any coordinate on the x-axis, we can find a corresponding point on the graph; the domain consists of all real numbers. Also, for any coordinate on the y-axis, we can find a point on the graph; the range consists of all real numbers.

Answer:

Both the domain and range consist of all real numbers R.

? EXERCISE 2.1.2

Given the linear function $g(x) = -x + 5$,

a. Find $g(-\frac{1}{2})$.

b. Find x when $g(x) = 18$.

Answer

a. $g(-\frac{1}{2}) = \frac{1}{12}$

b. $x = -13$

KEY TAKEAWAYS

- A relation is any set of ordered pairs. However, in the context of this course, we will be working with sets of ordered pairs (x, y) in the rectangular coordinate system. The set of x-values defines the domain and the set of y-values defines the range.
- Special relations where every x-value (*input*) corresponds to exactly one y-value (*output*) are called functions.
- We can easily determine whether an equation represents a function by performing the vertical line test on its graph. If any vertical line intersects the graph more than once, then the graph does not represent a function. In this case, there will be more than one point with the same x-value.
- Any nonvertical or nonhorizontal line is a function and can be written using function notation $f(x) = mx + b$. Both the domain and range consist of all real numbers.
  - If asked to find $f(a)$, we substitute a in for the variable and then simplify.
  - If asked to find x when $f(x) = a$, we set the function equal to a and then solve for x.

? EXERCISE 2.1.3 FUNCTIONS

For each problem below, does the correspondence represent a function?

1. Algebra students to their scores on the first exam.
2. Family members to their ages.
3. Lab computers to their users.
4. Students to the schools they have attended.
5. People to their citizenships.
6. Local businesses to their number of employees.

Answer

1. Yes

3. No

5. No

? EXERCISE 2.1.4 FUNCTIONS

Determine the domain and range and state whether the relation is a function or not.

1. $\{(3, 2), (5, 3), (7, 4)\}$
2. $\{(-5, -3), (0, 0), (5, 0)\}$
3. $\{(-10, 2), (-8, 1), (-8, 0)\}$
4. $\{(9, 12), (6, 6), (6, 3)\}$

5.

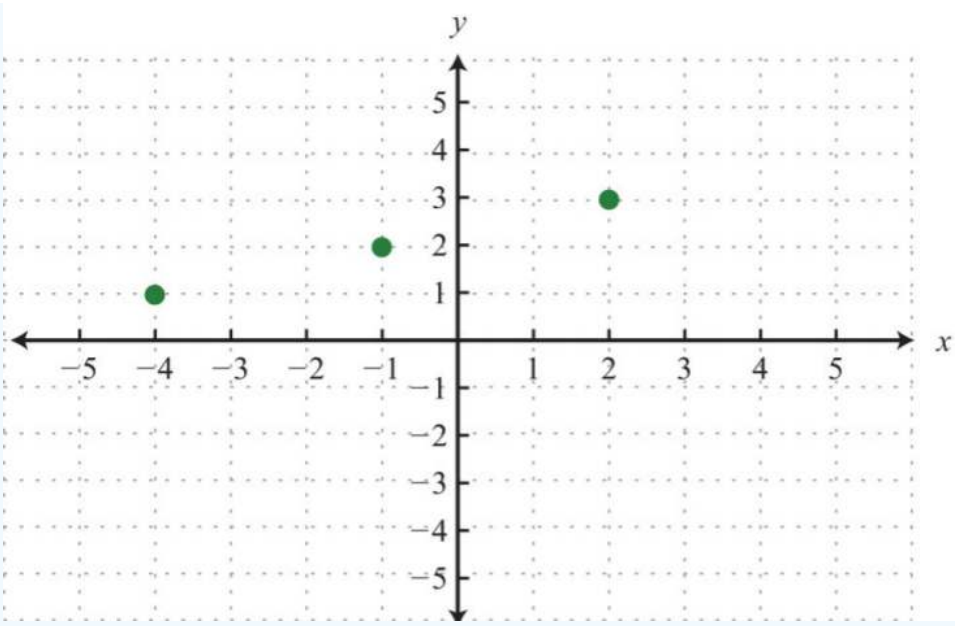


***Figure* 2.1.12**

6.

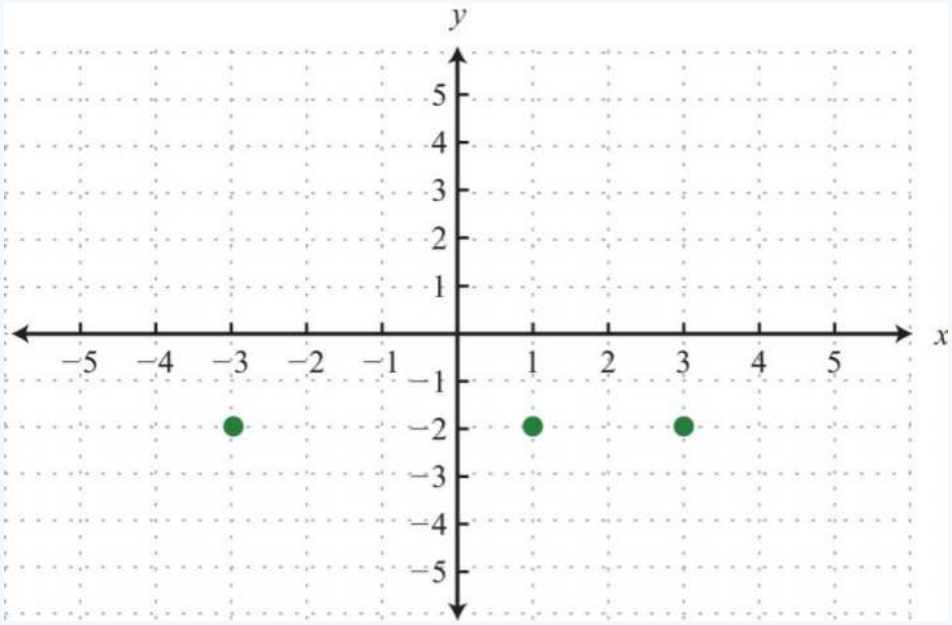


***Figure* 2.1.13**

7.

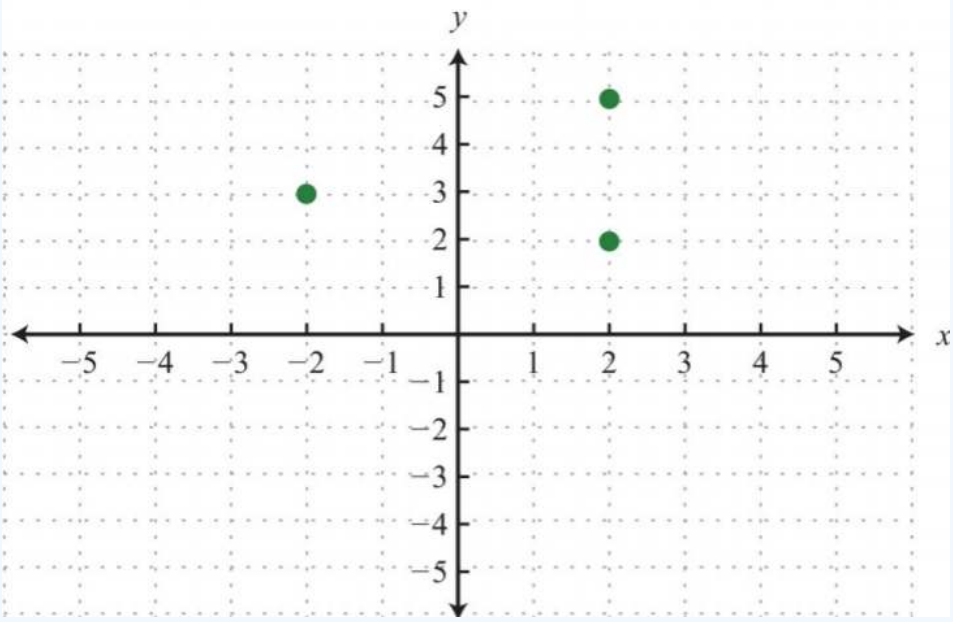


Figure **2.1.14**

8.

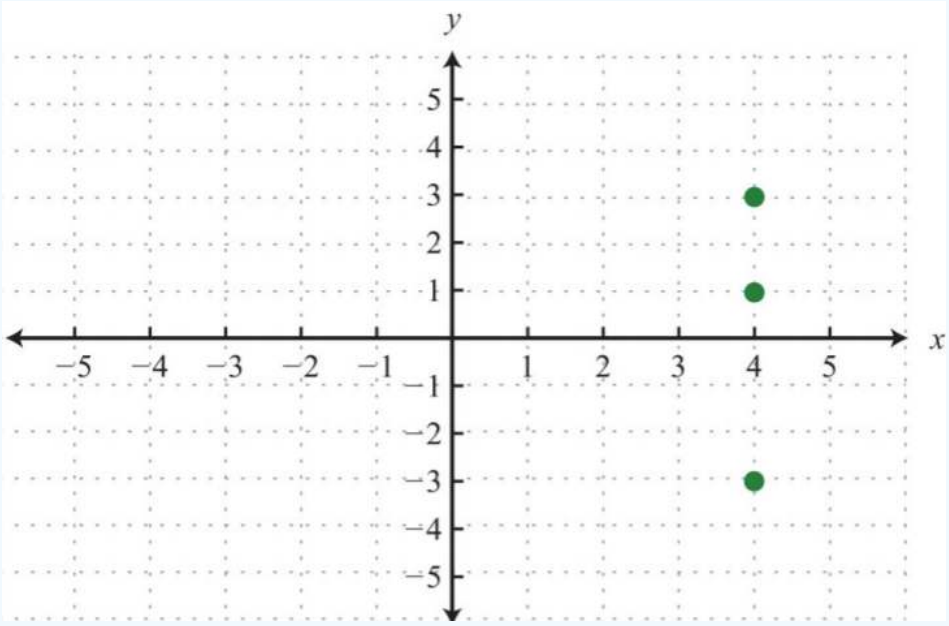


Figure **2.1.15**

9.

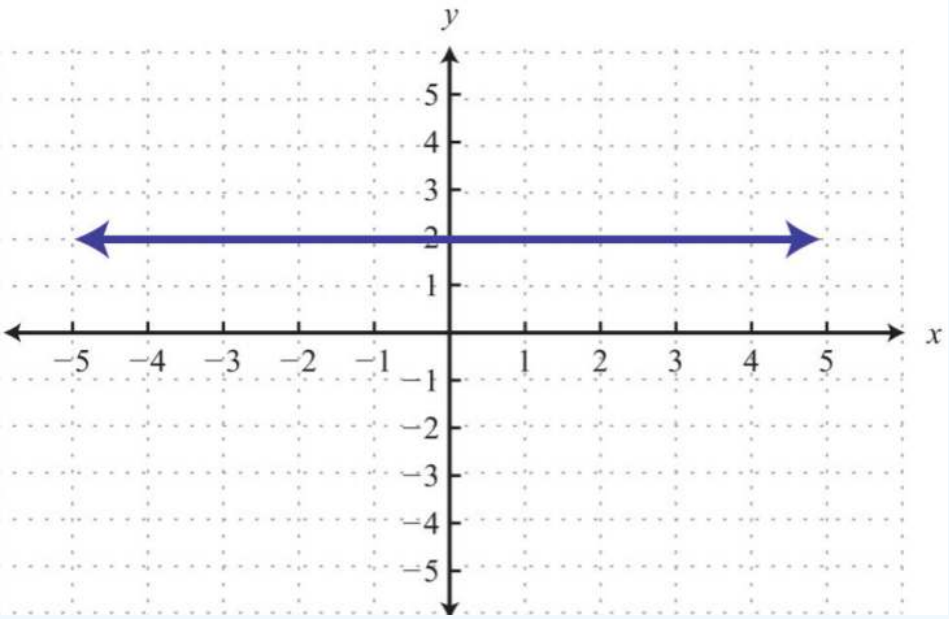


***Figure* 2.1.16**

10.

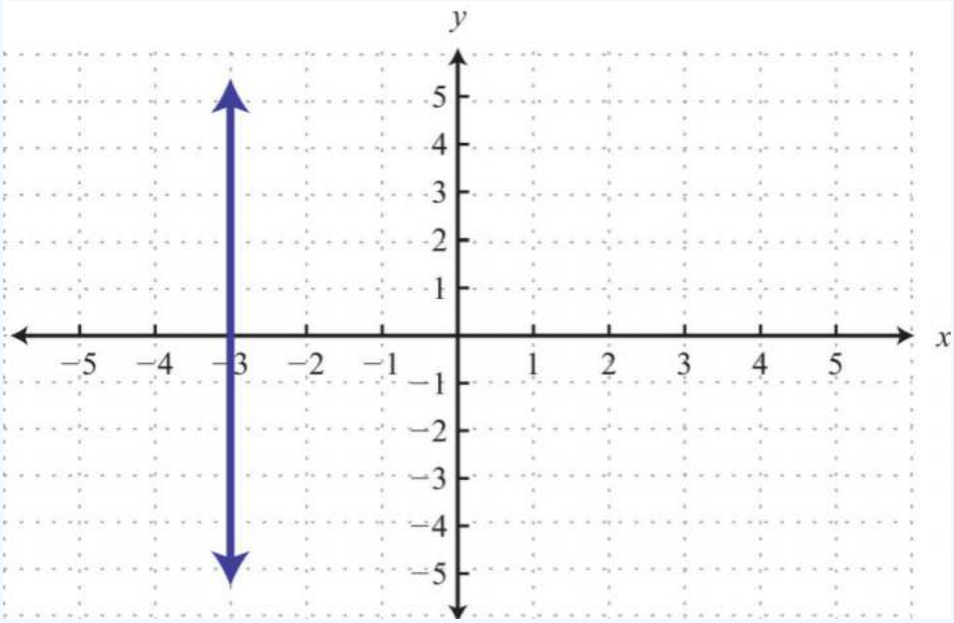


***Figure* 2.1.17**

11.

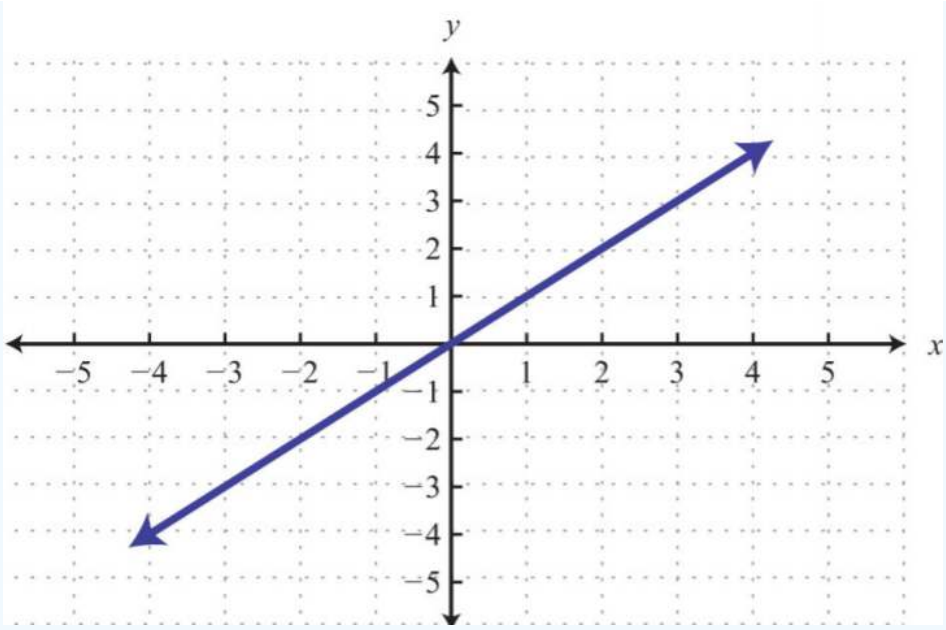


***Figure* 2.1.18**

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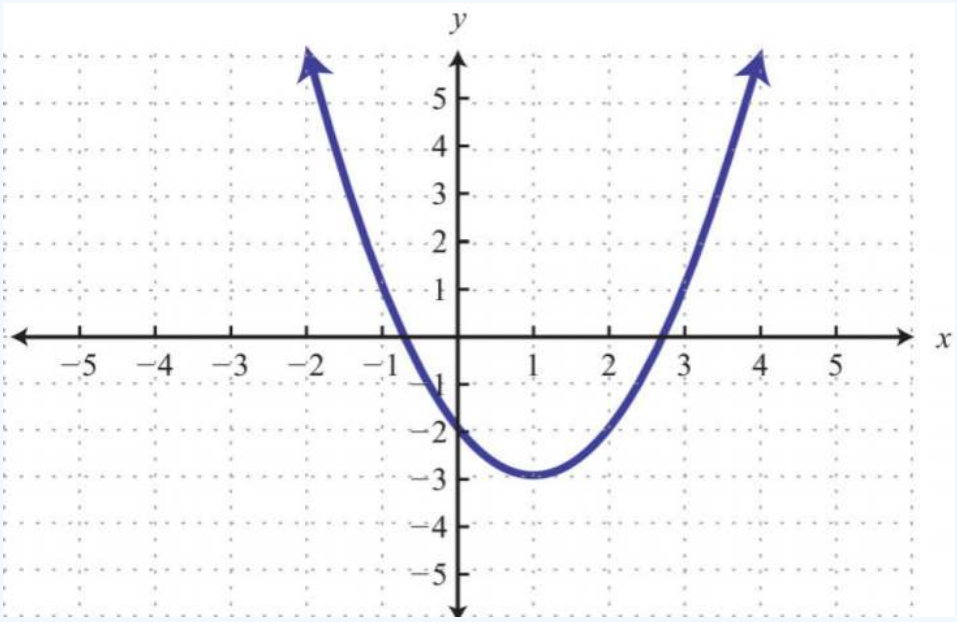


***Figure* 2.1.19**

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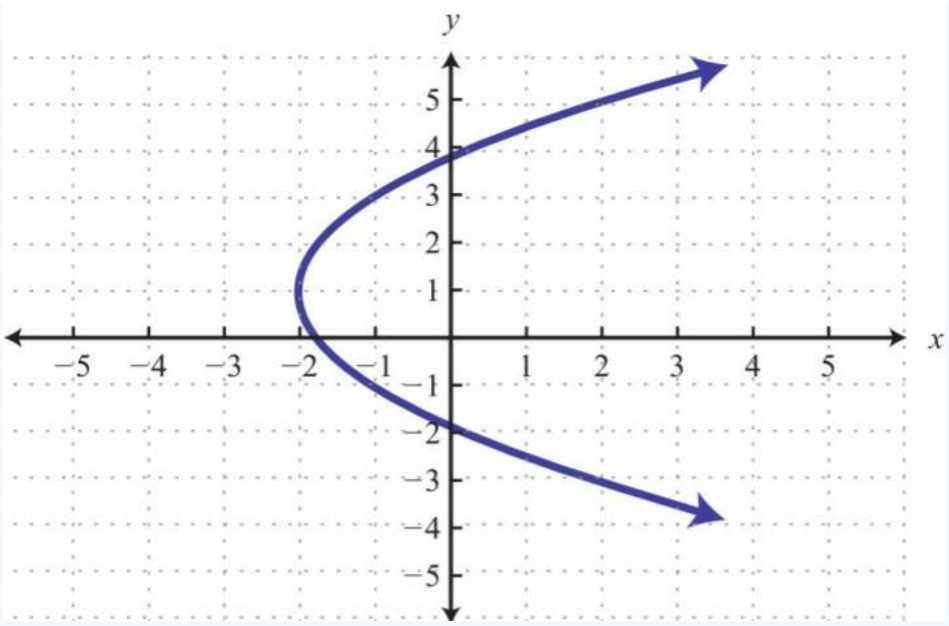


***Figure* 2.1.20**

14.

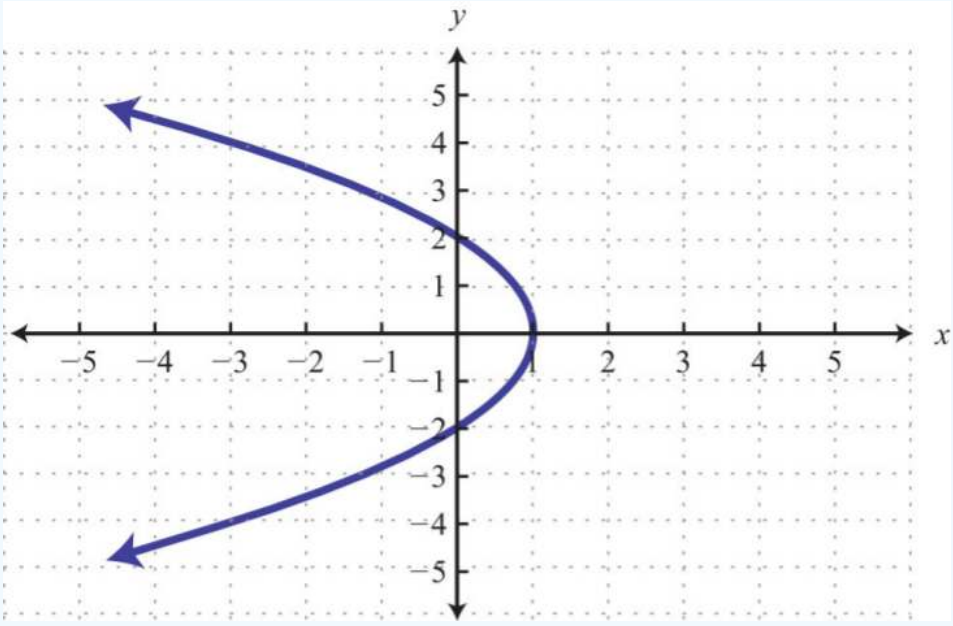


***Figure* 2.1.21**

15.

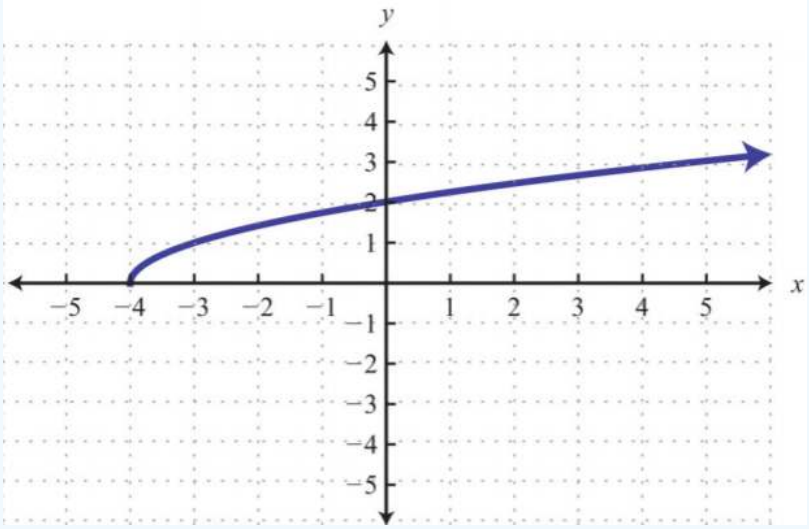


***Figure* 2.1.22**

16.

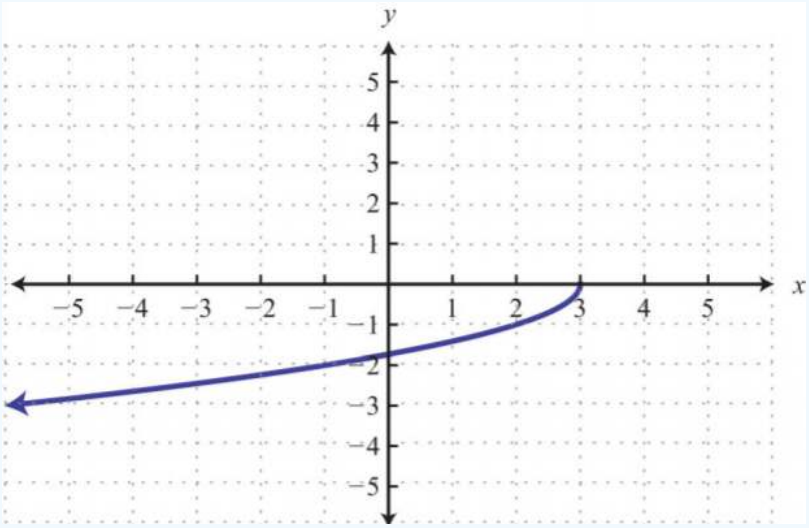


***Figure* 2.1.23**

17.

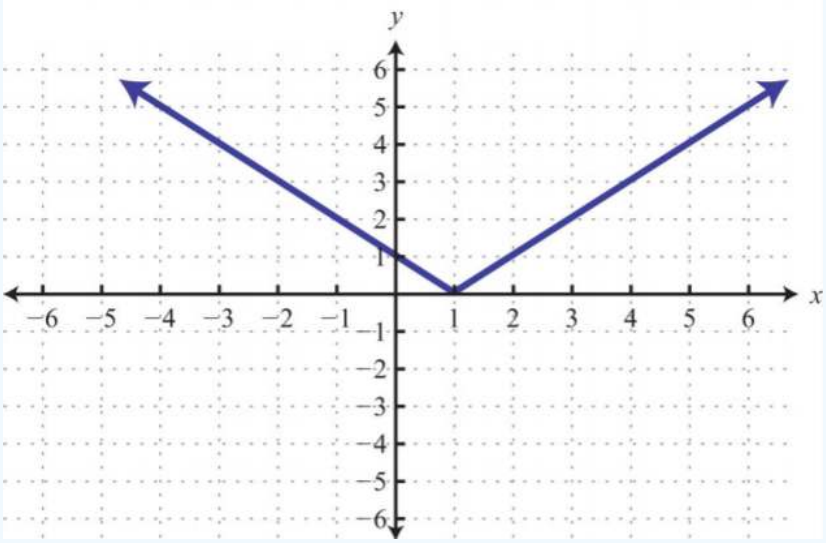


Figure **2.1.24**

18.

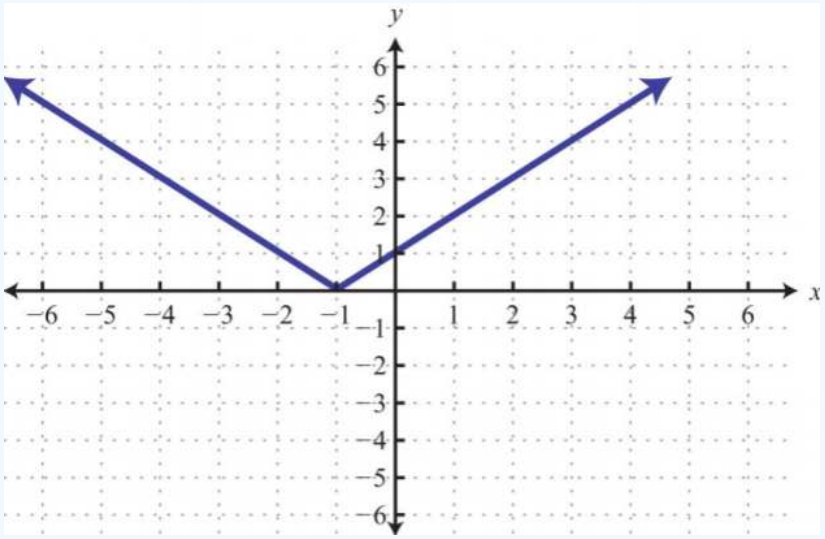


Figure **2.1.25**

19.

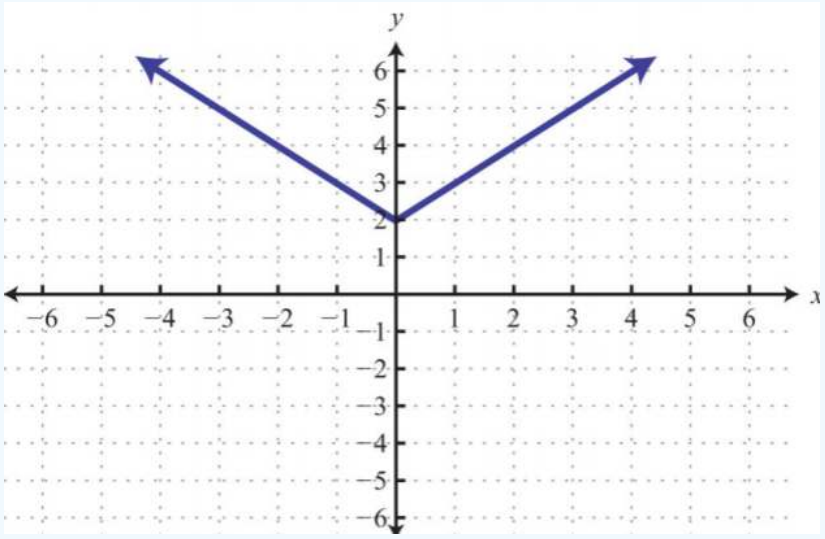


Figure **2.1.26**

20.

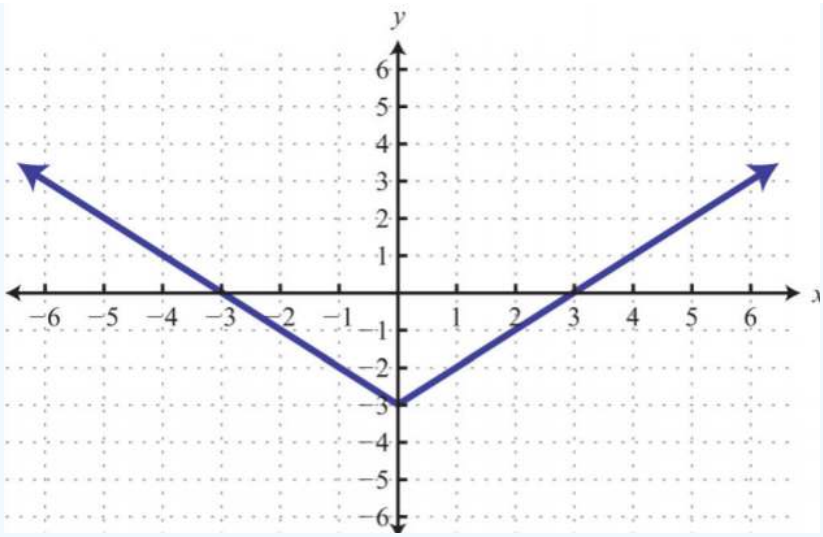


***Figure* 2.1.27**

Answer

1. Domain: $\{3,5,7\}$; range: $\{2,3,4\}$; function: yes
3. Domain: $\{-10,-8\}$; range: $\{0,1,2\}$; function: no
5. Domain: $\{-4,-1,2\}$; range: $\{1,2,3\}$; function: yes
7. Domain: $\{-2,2\}$; range: $\{2,3,5\}$; function: no
9. Domain: $(-\infty,\infty)$; range: $\{2\}$; function: yes
11. Domain: $(-\infty,\infty)$; range: $(-\infty,\infty)$; function: yes
13. Domain: $[-2,\infty)$; range: $(-\infty,\infty)$; function: no
15. Domain: $[-4,\infty)$; range: $[0,\infty)$; function: yes
17. Domain: $(-\infty,\infty)$; range: $[0,\infty)$; function: yes
19. Domain: $(-\infty,\infty)$; range: $[2,\infty)$; function: yes

? EXERCISE 2.1.5 FUNCTION NOTATION

Given the following functions, find the function values.

1. $f(x)=3x$, find $f(-2)$.
2. $f(x)=-5x+1$, find $f(-1)$.
3. $f(x)=\frac{3}{5}x-4$, find $f(15)$.
4. $f(x)=\frac{2}{5}x-\frac{1}{5}$, find $f(3)$.
5. $f(x)=\frac{5}{2}x-\frac{1}{3}$, find $f(-\frac{1}{3})$.
6. $f(x)=-6$, find $f(7)$.
7. $g(x)=5$, find $g(-4)$.
8. $g(x)=-5x$, find $g(-3)$.
9. $g(x)=-\frac{1}{8}x+\frac{5}{8}$, find $g(\frac{5}{8})$.
10. $g(x)=\frac{5}{3}x-5$, find $g(3)$.
11. $f(x)=5x-9$, find x when $f(x)=1$.
12. $f(x)=-7x+2$, find x when $f(x)=0$.
13. $f(x)=-\frac{7}{5}x-2$, find x when $f(x)=-9$.
14. $f(x)=-x-4$, find x when $f(x)=12$.
15. $g(x)=x$, find x when $g(x)=12$.
16. $g(x)=-x+1$, find x when $g(x)=\frac{2}{3}$.
17. $g(x)=-5x+\frac{1}{3}$, find x when $g(x)=-\frac{1}{2}$.
18. $g(x)=-\frac{5}{8}x+3$, find x when $g(x)=3$.

Answer

1. $f(-2) = -6$
3. $f(15) = 5$
5. $f(-\frac{1}{3}) = -\frac{7}{6}$
7. $g(-4) = 5$
9. $g(\frac{5}{8}) = \frac{35}{64}$
11. $x = 2$
13. $x = 5$
15. $x = 12$
17. $x = \frac{1}{6}$

EXERCISE 2.1.6 FUNCTION NOTATION

Given $f(x) = \frac{2}{3}x - 1$ and $g(x) = -3x + 2$ calculate the following.

1. $f(6)$
2. $f(-\frac{1}{2})$
3. $f(0)$
4. $f(1)$
5. $g(\frac{2}{3})$
6. $g(0)$
7. $g(-1)$
8. $g(-\frac{1}{2})$
9. Find x when $f(x) = 0$.
10. Find x when $f(x) = -3$.
11. Find x when $g(x) = -1$.
12. Find x when $g(x) = 0$.

Answer

1. $f(6) = 3$
3. $f(0) = -1$
5. $g(\frac{2}{3}) = 0$
7. $g(-1) = 5$
9. $x = \frac{3}{2}$
11. $x = 1$

EXERCISE 2.1.7 FUNCTION NOTATION

Given the graph, find the function values.

1. Given the graph of $f(x)$, find $f(-4), f(-1), f(0)$, and $f(2)$.

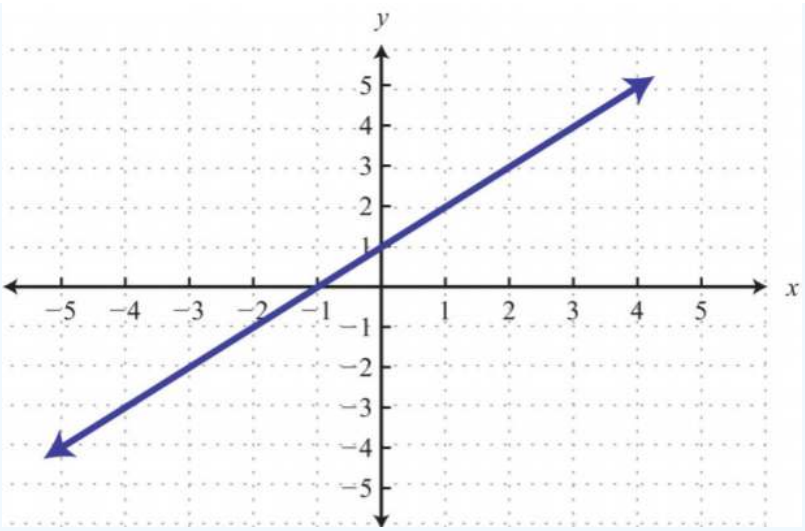


***Figure* 2.1.28**

2. Given the graph of $g(x)$, find $g(-3), g(-1), g(0)$, and $g(1)$.

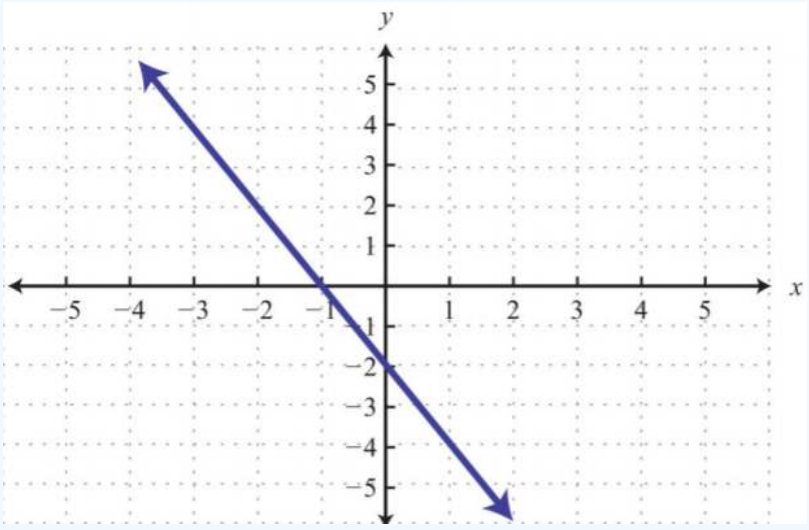


***Figure* 2.1.29**

3. Given the graph of $f(x)$, find $f(-4), f(-1), f(0)$, and $f(2)$.

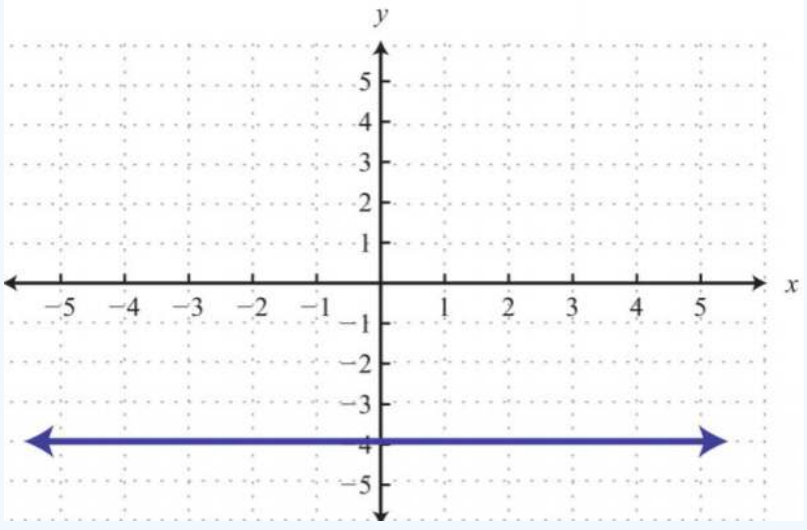


Figure **2.1.30**

4. Given the graph of $g(x)$, find $g(-4), g(-1), g(0)$, and $g(2)$.

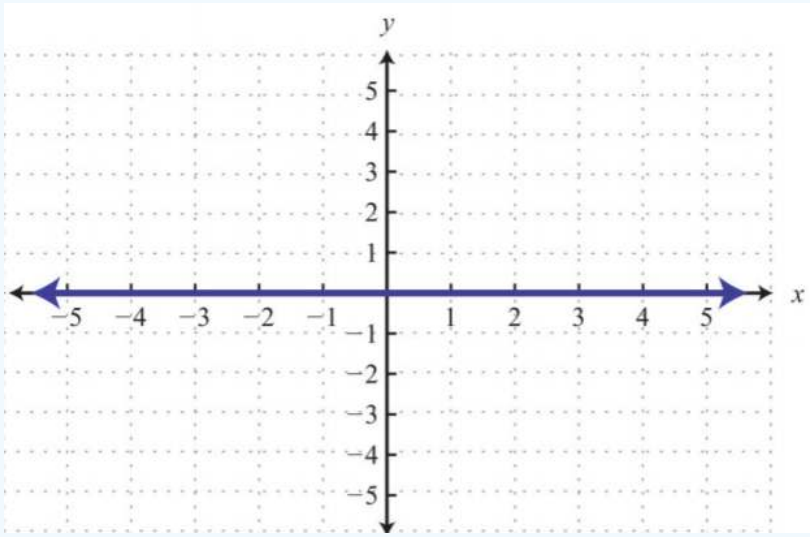


Figure **2.1.31**

5. Given the graph of $f(x)$, find $f(-1), f(0), f(1)$, and $f(3)$.

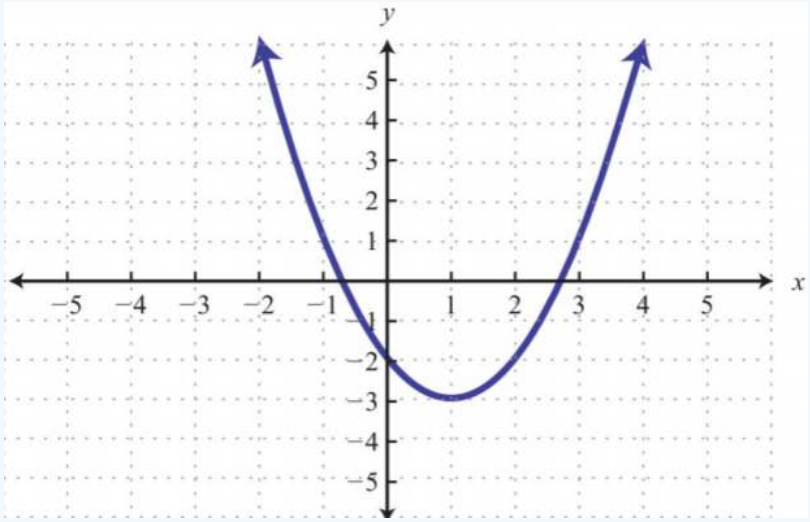


Figure **2.1.32**

6. Given the graph of $g(x)$, find $g(-2), g(0), g(2)$, and $g(6)$.

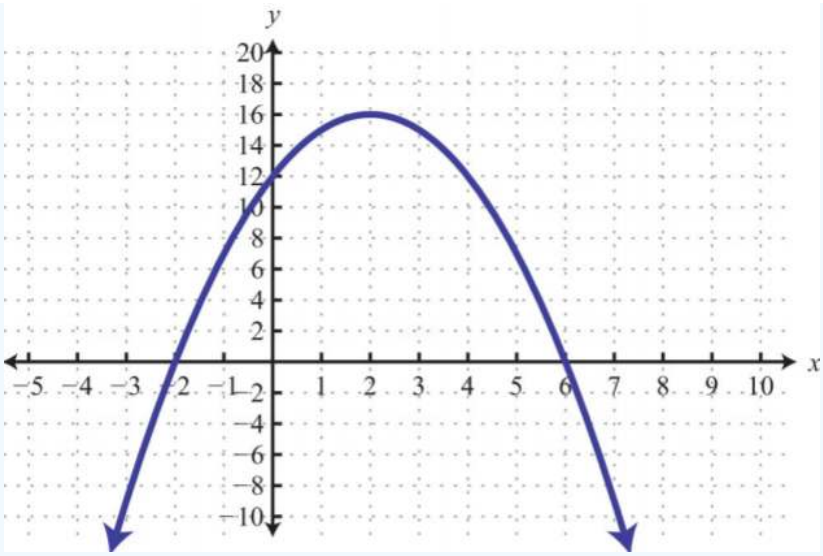


***Figure* 2.1.33**

7. Given the graph of $g(x)$, find $g(-4), g(-3), g(0)$, and $g(4)$.

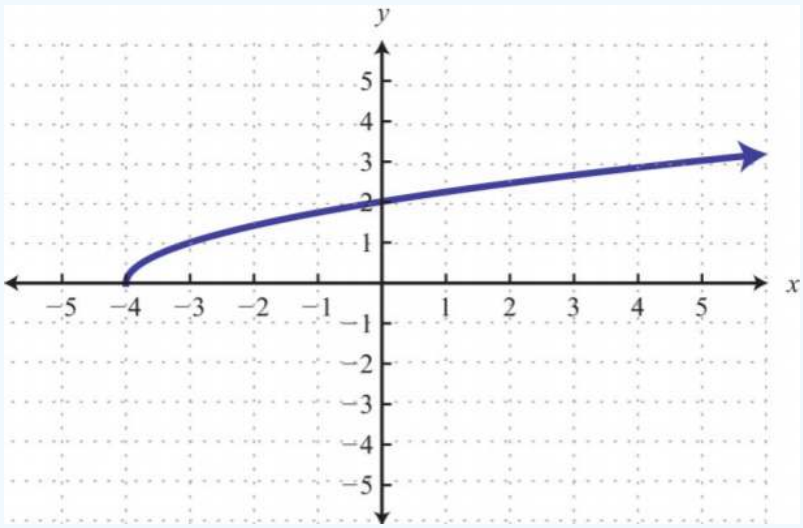


***Figure* 2.1.34**

8. Given the graph of $f(x)$, find $f(-4), f(0), f(1)$, and $f(3)$.

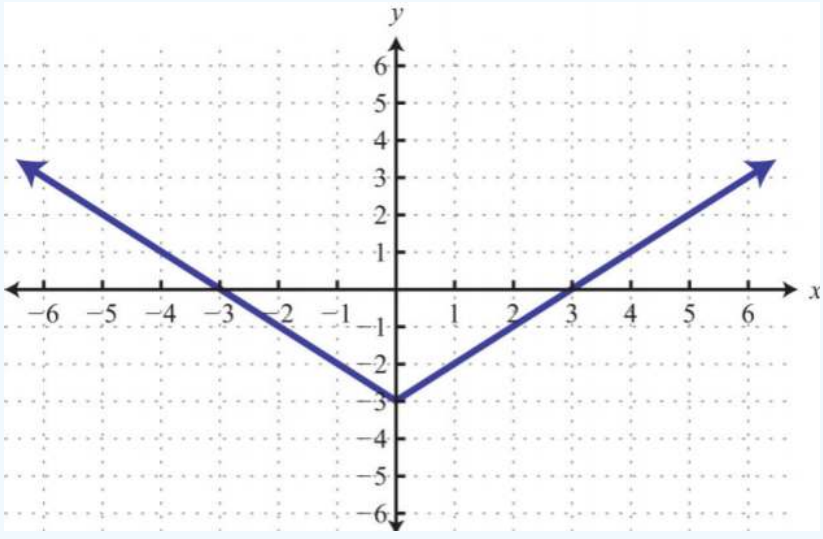


Figure **2.1.35**

Answer

1. $f(-4)=-3, f(-1)=0, f(0)=1,$ and $f(2)=3$
3. $f(-4)=-4, f(-1)=-4, f(0)=-4,$ and $f(2)=-4$
5. $f(-1)=1, f(0)=-2, f(1)=-3,$ and $f(3)=1$
7. $g(-4)=0, g(-3)=1, g(0)=2,$ and $g(4)=3$

? EXERCISE 2.1.8 FUNCTION NOTATION

Given the graph, find the x-values.

1. Given the graph of $f(x)$, find x when $f(x)=3, f(x)=1,$ and $f(x)=-3$.

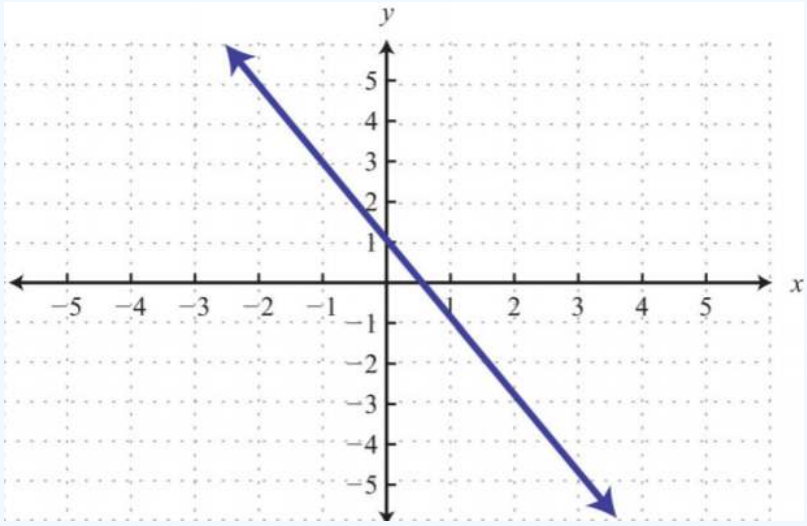


Figure **2.1.36**

2. Given the graph of $g(x)$, find x when $g(x)=-1, g(x)=0,$ and $g(x)=1$.

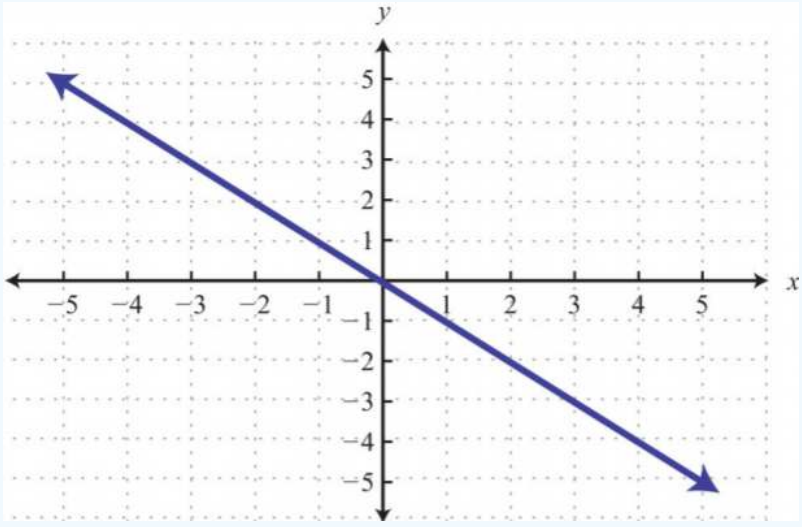


Figure **2.1.37**

3. Given the graph of $f(x)$, find x when $f(x)=3$.

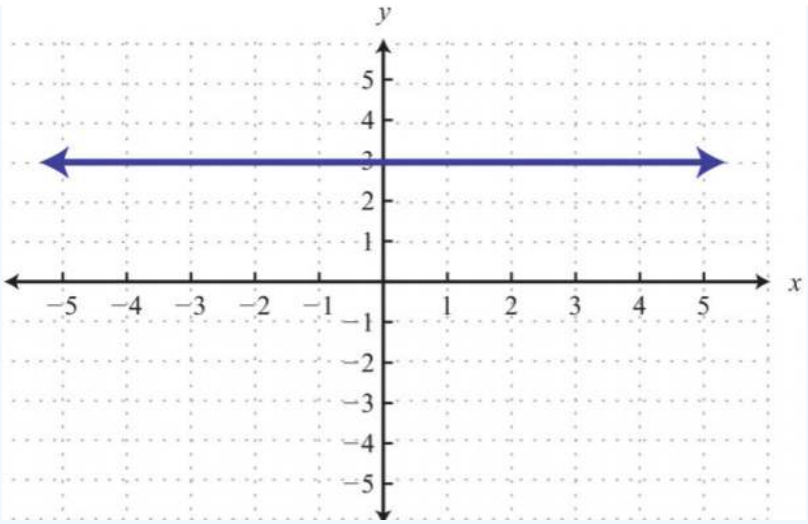


***Figure* 2.1.38**

4. Given the graph of $g(x)$, find x when $g(x) = -2, g(x) = 0$, and $g(x) = 4$.

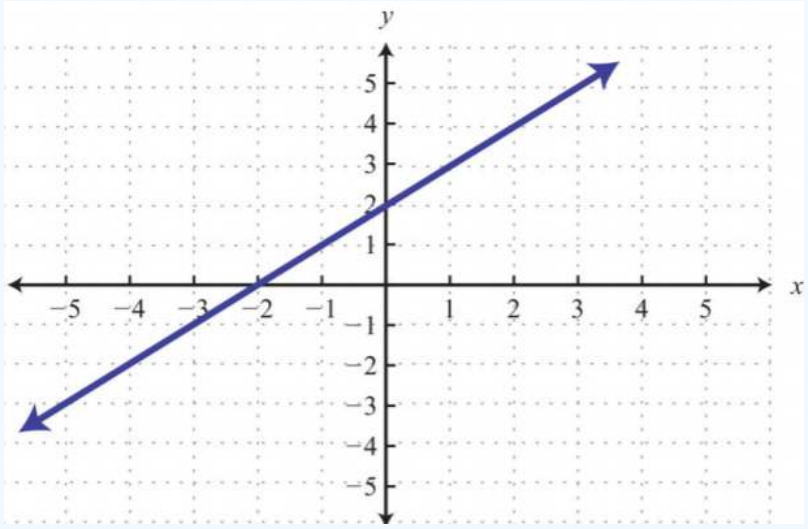


***Figure* 2.1.39**

5. Given the graph of $f(x)$, find x when $f(x) = -16, f(x) = -12$, and $f(x) = 0$.

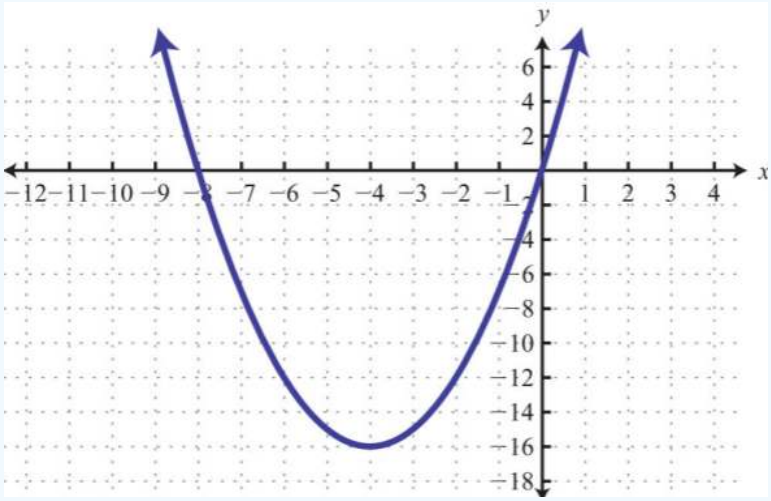


***Figure* 2.1.40**

6. Given the graph of $g(x)$, find x when $g(x) = -3, g(x) = 0$, and $g(x) = 1$.

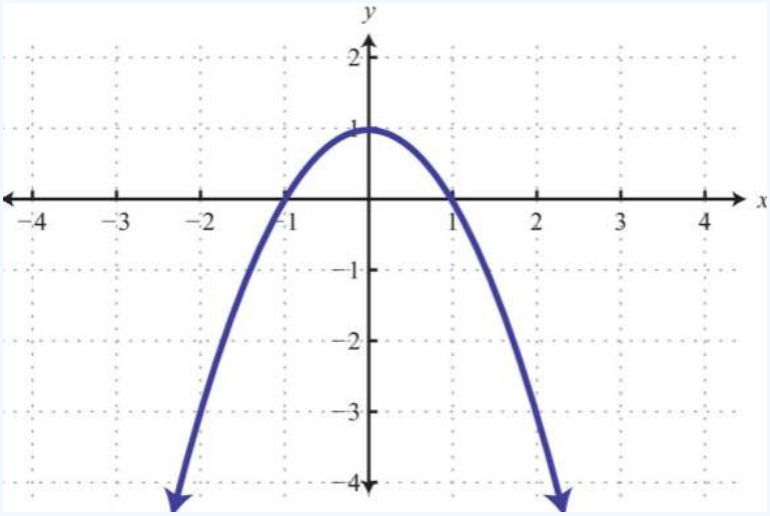


***Figure* 2.1.41**

7. Given the graph of $f(x)$, find x when $f(x) = -4, f(x) = 0$, and $f(x) = -2$.

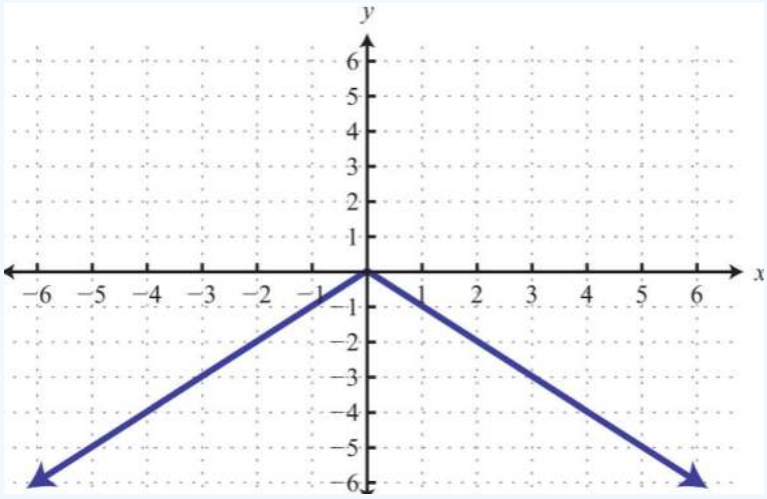


***Figure* 2.1.42**

8. Given the graph of $g(x)$, find x when $g(x) = 5, g(x) = 3$, and $g(x) = 2$.

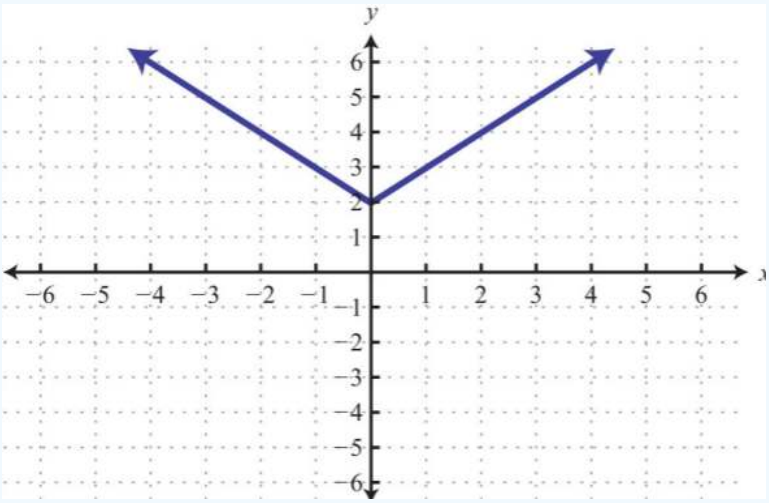


***Figure* 2.1.43**

9. The cost in dollars of producing pens with a company logo is given by the function $C(x) = 1.65x + 120$, where x is the number of pens produced. Use the function to calculate the cost of producing 200 pens.

10. The revenue in dollars from selling sweat shirts is given by the function $R(x) = 29.95x$, where x is the number of sweat shirts sold. Use the function to determine the revenue if 20 sweat shirts are sold.

11. The value of a new car in dollars is given by the function $V(t) = -2,500t + 18,000$, where t represents the age of the car in years. Use the function to determine the value of the car when it is 5 years old. What was the value of the car when new?

12. The monthly income in dollars of a commissioned car salesman is given by the function $I(n) = 550n + 1,250$, where n represents the number of cars sold in the month. Use the function to determine the salesman's monthly income if he sells 3 cars this month. What is his income if he does not sell any cars in a month?

13. The perimeter of an isosceles triangle with a base measuring 10 centimeters is given by the function $P(x) = 2x + 10$, where x represents the length of each of the equal sides. Find the length of each side if the perimeter is 40 centimeters.

14. The perimeter of a square depends on the length of each side s and is modeled by the function $P(s) = 4s$. If the perimeter of a square measures 140 meters, then use the function to calculate the length of each side.

15. A certain cellular phone plan charges \$18 per month and \$0.10 per minute of usage. The cost of the plan is modeled by the function $C(x) = 0.10x + 18$, where x represents the number of minutes of usage per month. Determine the minutes of usage if the cost for the month was \$36.

16. The monthly revenue generated by selling subscriptions to a tutoring website is given by the function $R(x) = 29x$, where x represents the number of subscription sales per month. How many subscriptions were sold if the revenues for the month totaled \$ $1,508$?

Answer

1. $f(-1) = 3, f(0) = 1,$ and $f(2) = -3$
3. $f(1) = 3$ (answers may vary)
5. $f(-4) = -16$; $f(-6) = -12$ and $f(-2) = -12$; $f(-8) = 0$ and $f(0) = 0$
7. $f(-4) = -4$ and $f(4) = -4$; $f(0) = 0$; $f(-2) = -2$ and $f(2) = -2$
9. \$450
11. New: \$18,000; 5 years old: \$5,500
13. 15 centimeters
15. 180 minutes

? EXERCISE 2.1.9 FUNCTION NOTATION

Graph the linear function and state the domain and range.

1. $f(x) = -\frac{5}{2}x + 10$
2. $f(x) = \frac{3}{5}x - 10$
3. $g(x) = 6x + 2$
4. $g(x) = -4x + 6$
5. $h(t) = \frac{1}{2}t - 3$
6. $h(t) = -\frac{3}{4}t + 3$
7. $C(x) = 100 + 50x$
8. $C(x) = 50 + 100x$

Answer

1. Domain and range: R

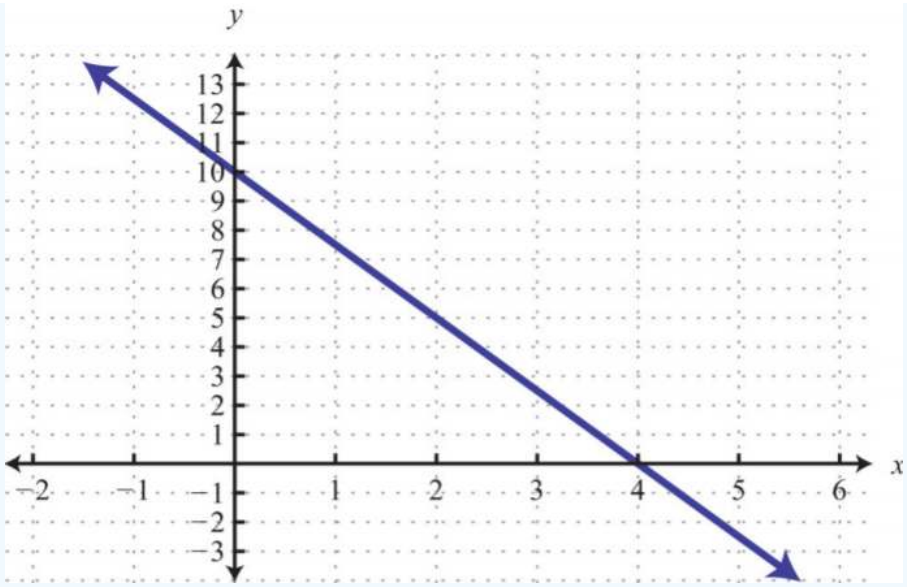


Figure 2.1.44

3. Domain and range: R

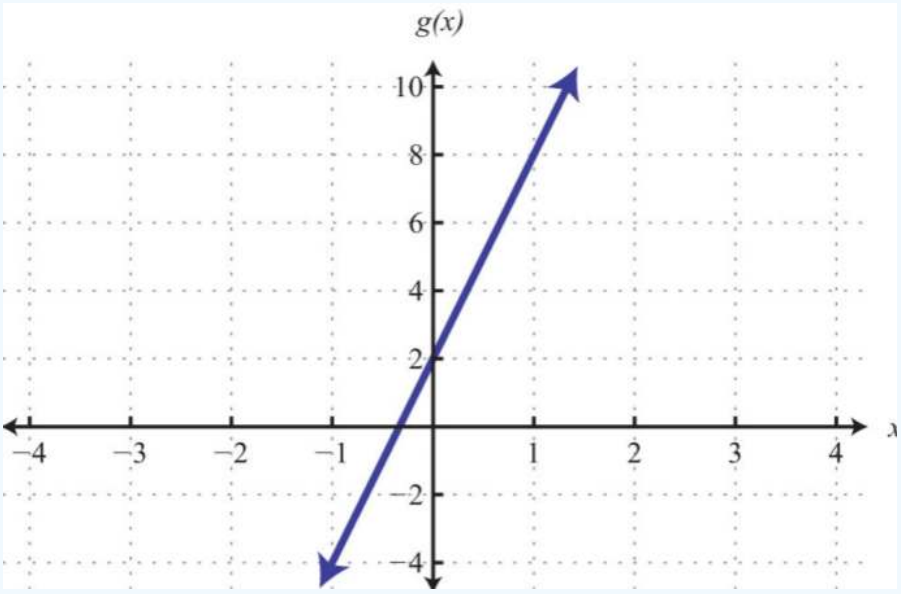


Figure 2.1.45

5. Domain and range: R

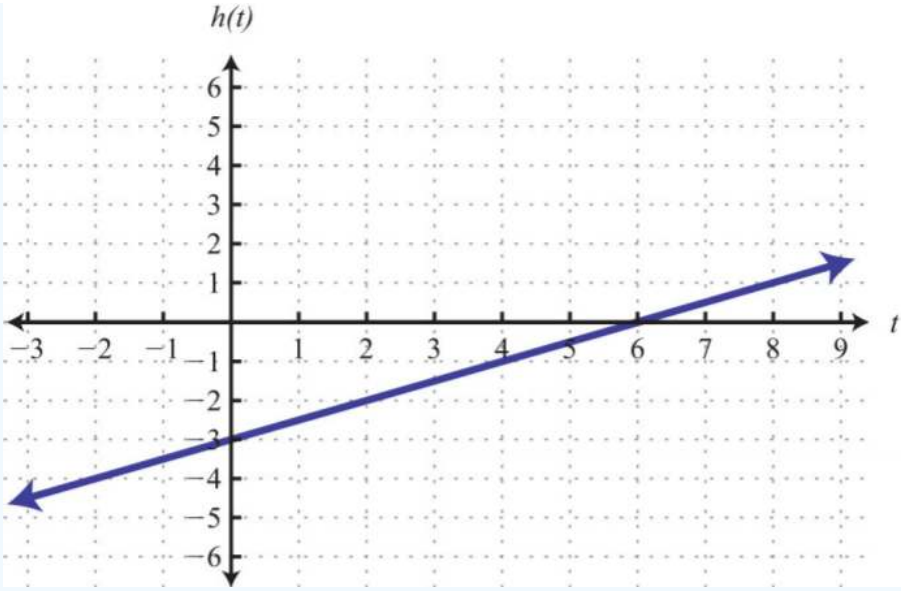


Figure 2.1.46

7. Domain and range: R

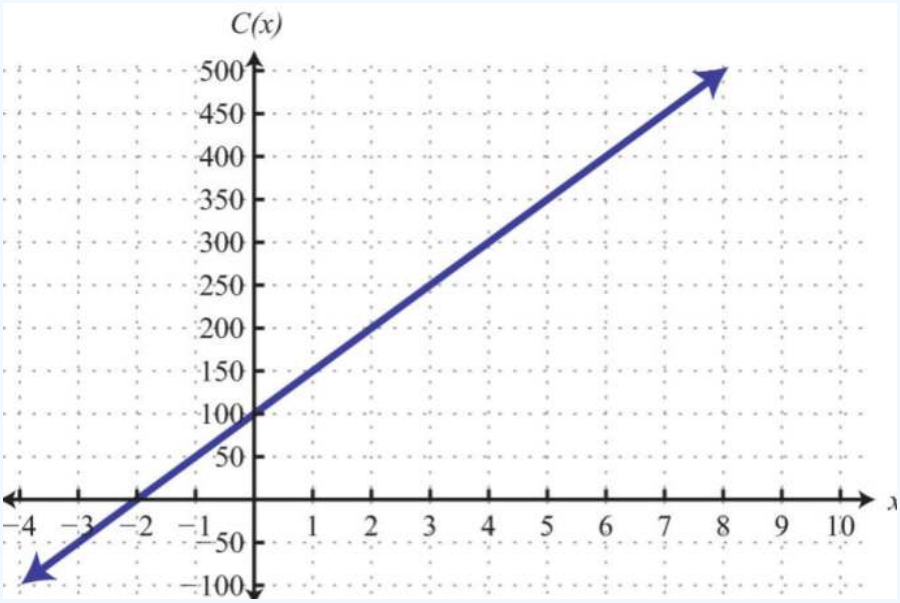


Figure 2.1.47

? EXERCISE 2.1.10 DISCUSSION BOARD TOPICS

1. Is a vertical line a function? What are the domain and range of a vertical line?
2. Is a horizontal line a function? What are the domain and range of a horizontal line?
3. Come up with your own correspondence between real-world sets. Explain why it does or does not represent a function.
4. Can a function have more than one y-intercept? Explain.

Answer

1. Answers may vary
3. Answers may vary

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2.2: RULES OF EXPONENTS

LEARNING OBJECTIVES

- Simplify expressions using the rules of exponents.
- Simplify expressions involving parentheses and exponents.
- Simplify expressions involving 0 as an exponent.

PRODUCT, QUOTIENT, AND POWER RULE FOR EXPONENTS

If a factor is repeated multiple times, then the product can be written in exponential form x_n. The positive integer **exponent** n indicates the number of times the **base** x is repeated as a factor

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ times}}$$

For example,

$$5^4 = 5 \cdot 5 \cdot 5 \cdot 5$$

Here the base is 5 and the exponent is 4. Exponents are sometimes indicated with the caret (^) symbol found on the keyboard: 5^ $4 = 5 * 5 * 5 * 5$.

Next consider the product of 2^3 and 2^5,

$$2^3 \cdot 2^5 = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{8 \text{ factors of } 2} = 2^8$$

Expanding the expression using the definition produces multiple factors of the base, which is quite cumbersome, particularly when n is large. For this reason, we will develop some useful rules to help us simplify expressions with exponents. In this example, notice that we could obtain the same result by adding the exponents.

$$2^3 \cdot 2^5 = 2^{3+5} = 2^8$$

In general, this describes the product rule for exponents. If m and n are positive integers, then

$$x^m \cdot x^n = x^{m+n} \tag{2.2.1}$$

In other words, when multiplying two expressions with the same base, add the exponents.

EXAMPLE 2.2.1

Simplify: $10^5 \cdot 10^{18}$.

Solution:

$$\begin{aligned} 10^5 \cdot 10^{18} &= 10^{5+18} \\ &= 10^{23} \end{aligned}$$

Answer:

$$10^{23}$$

In the previous example, notice that we did not multiply the base 10 times itself. When applying the product rule, add the exponents and leave the base unchanged.

EXAMPLE 2.2.2

Simplify: $x^6 \cdot x^{12} \cdot x$.

Solution:

Recall that the variable x is assumed to have an exponent of 1 : $x = x^1$.

$$\begin{aligned} x^6 \cdot x^{12} \cdot x &= x^6 \cdot x^{12} \cdot x^1 \\ &= x^{6+12+1} \\ &= x^{19} \end{aligned}$$

Answer:

$$x^{19}$$

The base could be any algebraic expression.

EXAMPLE 2.2.3

Simplify: $(x+y)^9(x+y)^{13}$.

Solution:

Treat the expression $(x+y)$ as the base.

$$\begin{aligned} (x+y)^9(x+y)^{13} &= (x+y)^{9+13} \\ &= (x+y)^{22} \end{aligned}$$

Answer:

$$(x+y)^{22}$$

The commutative property of multiplication allows us to use the product rule for exponents to simplify factors of an algebraic expression.

EXAMPLE 2.2.4

Simplify: $2x^8y \cdot 3x^4y^7$.

Solution:

Multiply the coefficients and add the exponents of variable factors with the same base.

$$\begin{aligned} 2x^8y \cdot 3x^4y^7 &= 2 \cdot 3 \cdot x^8 \cdot x^4 \cdot y^1 \cdot y^7 && \textit{Commutative property} \\ &= 6 \cdot x^{8+4} \cdot y^{1+7} && \textit{Power rule for exponents} \\ &= 6x^{12}y^8 \end{aligned}$$

Answer:

$$6x^{12}y^8$$

Next, we will develop a rule for division by first looking at the quotient of 2^7 and 2^3.

$$\frac{2^7}{2^3} = \frac{\overbrace{2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2}^{7 \text{ factors of } 2}}{\underbrace{2\cdot2\cdot2}_{3 \text{ factors of } 2}} = \frac{\cancel{2\cdot2\cdot2}^{1}\cdot2\cdot2\cdot2\cdot2}{\cancel{2\cdot2\cdot2}_{1}} = \frac{2\cdot2\cdot2\cdot2}{1} = 2^4$$

Figure 2.2.3

Here we can cancel factors after applying the definition of exponents. Notice that the same result can be obtained by subtracting the exponents.

$$\frac{2^7}{2^3} = 2^{7-3} = 2^4$$

This describes the quotient rule for exponents. If m and n are positive integers and $x \neq 0$, then

$$\frac{x^m}{x^n} = x^{m-n}$$

In other words, when you divide two expressions with the same base, subtract the exponents.

EXAMPLE 2.2.5

Simplify: $\frac{12y^{15}}{4y^7}$.

Solution:

Divide the coefficients and subtract the exponents of the variable y.

$$\begin{aligned}\frac{12y^{15}}{4y^7} &= \frac{12}{4}\cdot y^{15-7}\\ &= 3y^8\end{aligned}$$

Answer:

$$3y^8$$

EXAMPLE 2.2.6

Simplify: $\frac{20x^{10}(x+5)^6}{10x^9(x+5)^2}$

Solution:

$$\begin{aligned}\frac{20x^{10}(x+5)^6}{10x^9(x+5)^2} &= \frac{20}{10}\cdot x^{10-9}\cdot(x+5)^{6-2}\\ &= 2x^1(x+5)^4\end{aligned}$$

Answer:

$$2x(x+5)^4$$

Now raise 2^3 to the fourth power as follows:

$$\begin{aligned}\left(2^3\right)^4 &= \underbrace{2^3\cdot 2^3\cdot 2^3\cdot 2^3}_{4\ factors\ of\ 2^3}\\ &= \underbrace{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2}_{12\ factors\ of\ 2}\\ &= 2^{12}\end{aligned}$$

After writing the base 2^3 as a factor four times, expand to obtain 12 factors of 2. We can obtain the same result by multiplying the exponents.

$$\left(2^3\right)^4 = 2^{3\cdot 4} = 2^{12}$$

In general, this describes the power rule for exponents. Given positive integers m and n, then

$$\left(x^m\right)^n = x^{m\cdot n} \tag{2.2.2}$$

In other words, when raising a power to a power, multiply the exponents.

EXAMPLE 2.2.7

Simplify: $\left(y^6\right)^7 = y^{6\cdot 7}$

Solution:

$$\begin{aligned}\left(y^6\right)^7 &= y^{6\cdot 7}\\ &= y^{42}\end{aligned}$$

Answer:

$$y^{42}$$

To summarize, we have developed three very useful rules of exponents that are used extensively in algebra. If given positive integers m and n, then

- Product rule:

$$x^m\cdot x^n = x^{m+n} \tag{2.2.3}$$

- Quotient rule:

$$\frac{x^m}{x^n} = x^{m-n}, x \neq 0 \tag{2.2.4}$$

- Power rule:

$$(x^m)^n = x^{m \cdot n} \tag{2.2.5}$$

? EXERCISE 2.2.1

Simplify: $y^5 \cdot (y^4)^6$.

Answer

y^{29}

POWER RULES FOR PRODUCTS AND QUOTIENTS

Now we consider raising grouped products to a power. For example,

$$\begin{aligned}(xy)^4 &= xy \cdot xy \cdot xy \cdot xy \\ &= x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \quad \textit{Commutative property} \\ &= x^4 \cdot y^4\end{aligned}$$

After expanding, we have four factors of the product xy. This is equivalent to raising each of the original factors to the fourth power. In general, this describes the power rule for a product. If n is a positive integer, then

$$(xy)^n = x^n y^n \tag{2.2.6}$$

✓ EXAMPLE 2.2.8

Simplify: $(2ab)^7 = 2^7 a^7 b^7$.

Solution:

We must apply the exponent 7 to all the factors, including the coefficient, 2.

$$\begin{aligned}(2ab)^7 &= 2^7 a^7 b^7 \\ &= 128 a^7 b^7\end{aligned}$$

If a coefficient is raised to a relatively small power, then present the real number equivalent, as we did in this example: $2^7 = 128$.

Answer:

$$128 a^7 b^7$$

In many cases, the process of simplifying expressions involving exponents requires the use of several rules of exponents.

✓ EXAMPLE 2.2.9

Simplify: $(3xy^3)^4$.

Solution:

$$\begin{aligned}(3xy^3)^4 &= 3^4 \cdot x^4 \cdot (y^3)^4 \quad \textit{Power rule for products} \\ &= 3^4 x^4 y^{3 \cdot 4} \quad \textit{Power rule for exponents} \\ &= 81 x^4 y^{12}\end{aligned}$$

Answer:

$$81 x^4 y^{12}$$

✓ EXAMPLE 2.2.10

Simplify: $(4x^2 y^5 z)^3$.

Solution:

$$\begin{aligned}(4x^2 y^5 z)^3 &= 4^3 \cdot (x^2)^3 \cdot (y^5)^3 \cdot z^3 \\ &= 64 x^6 y^{15} z^3\end{aligned}$$

Answer:

$$64x^6y^{15}z^3$$

EXAMPLE 2.2.11

Simplify: $[5(x+y)^3]^3$

Solution:

$$\begin{aligned} {}^3 &= 5^3 \cdot (x+y)^9 \\ &= 125(x+y)^9 \end{aligned}$$

Answer:

$$125(x+y)^9$$

Next, consider a quotient raised to a power.

$$\begin{aligned} \left(\frac{x}{y}\right)^4 &= \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \\ &= \frac{x \cdot x \cdot x \cdot x}{y \cdot y \cdot y \cdot y} \\ &= \frac{x^4}{y^4} \end{aligned}$$

Here we obtain four factors of the quotient, which is equivalent to the numerator and the denominator both raised to the fourth power. In general, this describes the power rule for a quotient. If n is a positive integer and $y \neq 0$, then

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \tag{2.2.7}$$

In other words, given a fraction raised to a power, we can apply that exponent to the numerator and the denominator. This rule requires that the denominator is nonzero. We will make this assumption for the remainder of the section.

EXAMPLE 2.2.12

Simplify: $\left(\frac{3a}{b}\right)^3$

Solution:

First, apply the power rule for a quotient and then the power rule for a product.

$$\begin{aligned} \left(\frac{3a}{b}\right)^3 &= \frac{(3a)^3}{b^3} && \textit{Power rule for a quotient} \\ &= \frac{3^3 \cdot a^3}{b^3} && \textit{Power rule for a product} \\ &= \frac{27a^3}{b^3} \end{aligned}$$

Answer:

$$\frac{27a^3}{b^3}$$

In practice, we often combine these two steps by applying the exponent to all factors in the numerator and the denominator.

EXAMPLE 2.2.13

Simplify: $\left(\frac{ab^2}{2c^3}\right)^5$

Solution:

Apply the exponent 5 to all of the factors in the numerator and the denominator.

$$\left(\frac{ab^2}{2c^3}\right)^5 = \frac{a^5(b^2)^5}{2^5(c^3)^5}$$
$$= \frac{a^5b^{10}}{32c^{15}}$$

Answer:

$$\frac{a^5b^{10}}{32c^{15}}$$

EXAMPLE 2.2.14

Simplify: $\left(\frac{5x^5(2x-1)^4}{3y^7}\right)^2$

Solution:

$$\left(\frac{5x^5(2x-1)^4}{3y^7}\right)^2 = \frac{(5x^5(2x-1)^4)^2}{(3y^7)^2} \qquad \textit{Power rule for a quotient}$$
$$= \frac{5^2\cdot(x^5)^2\cdot[(2x-1)^4]^2}{3^2\cdot(y^7)^2} \qquad \textit{Power rule for products}$$
$$= \frac{25x^{10}(2x-1)^8}{9y^{14}} \qquad \textit{Power rule for exponents}$$

Answer:

$$\frac{25x^{10}(2x-1)^8}{9y^{14}}$$

It is a good practice to simplify within parentheses before using the power rules; this is consistent with the order of operations.

EXAMPLE 2.2.15

Simplify: $\left(\frac{-2x^3y^4z}{xy^2}\right)^4$

Solution:

$$\left(\frac{-2x^3y^4z}{xy^2}\right)^4 = (-2\cdot x^{3-1}\cdot y^{4-2}\cdot z)^4 \qquad \textit{Simplify within the parentheses first.}$$
$$= (-2\cdot x^2\cdot y^2\cdot z)^4 \qquad \textit{Apply the power rule for a product.}$$
$$= (-2)^4\cdot(x^2)^4\cdot(y^2)^4\cdot z^4 \qquad \textit{Apply the power rule for exponents.}$$
$$= 16x^8y^8z^4$$

Answer:

$$16x^8y^8z^4$$

To summarize, we have developed two new rules that are useful when grouping symbols are used in conjunction with exponents. If given a positive integer n, where y is a nonzero number, then

- Power rule for a product:

$$(xy)^n = x^ny^n \tag{2.2.8}$$

- Power rule for a quotient:

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \tag{2.2.9}$$

EXERCISE 2.2.2

Simplify: $\left(\frac{4x^2(x-y)^3}{3yz^5}\right)^3$

Answer

$\frac{64x^6(x-y)^9}{27y^3z^{15}}$

ZERO AS AN EXPONENT

Using the quotient rule for exponents, we can define what it means to have 0 as an exponent. Consider the following calculation:

\(\color{Cerulean}{1}\color{black}{=\frac{8}{8}=\frac{2^{3}}{2^{3}}=2^{3-3}=}\color{Cerulean}{2^{0}}

Eight divided by 8 is clearly equal to 1, and when the quotient rule for exponents is applied, we see that a 0 exponent results. This leads us to the definition of zero as an exponent, where $x \neq 0$:

$$x^0 = 1 \tag{2.2.10}$$

It is important to note that 0^0 is undefined. If the base is negative, then the result is still $+1$. In other words, any nonzero base raised to the 0 power is defined to be 1. In the following examples, assume all variables are nonzero.

EXAMPLE 2.2.16

Simplify:

a. $(-5)^0$
b. -5^0

Solution:

a. Any nonzero quantity raised to the 0 power is equal to 1.

$$(-5)^0 = 1$$

b.In the example -5^0, the base is 5, not -5.

Answer:

1. 1
2. -1

EXAMPLE 2.2.17

Simplify:

$$(5x^3y^0z^2)^2.$$

Solution:

It is good practice to simplify within the parentheses first.

$$\begin{aligned}(5x^3y^0z^2)^2 &= (5x^3\cdot 1\cdot z^2)^2\\ &= (5x^3z^2)^2\\ &= 5^2x^{3\cdot 2}z^{2\cdot 2}\\ &= 25x^6z^4\end{aligned}$$

Answer:

$$25x^6z^4$$

EXAMPLE 2.2.18

Simplify:

$$\left(-\frac{8a^{10}b^5}{5c^{12}d^{14}}\right)^0.$$

Solution:

$$\left(-\frac{8a^{10}b^5}{5c^{12}d^{14}}\right)^0 = 1$$

Answer:

$$1$$

EXERCISE 2.2.3

Simplify:

$$5x^0 \text{ and } (5x)^0$$

Answer

$5x^0 = 5$ and $(5x)^0 = 1$

KEY TAKEAWAYS

- The rules of exponents allow you to simplify expressions involving exponents.
- When multiplying two quantities with the same base, add exponents: $x^m \cdot x^n = x^{m+n}$.
- When dividing two quantities with the same base, subtract exponents: $\frac{x^m}{x^n} = x^{m-n}$.
- When raising powers to powers, multiply exponents: $(x^m)^n = x^{m \cdot n}$.
- When a grouped quantity involving multiplication and division is raised to a power, apply that power to all of the factors in the numerator and the denominator: $(xy)^n = x^n y^n$ and $(\frac{x}{y})^n = \frac{x^n}{y^n}$.
- Any nonzero quantity raised to the 0 power is defined to be equal to 1 : $x^0 = 1$.

EXERCISE 2.2.4 PRODUCT, QUOTIENT, AND POWER RULE FOR EXPONENTS

Write each expression using exponential form.

1. $(2x)(2x)(2x)(2x)(2x)$
2. $(-3y)(-3y)(-3y)$
3. $-10 \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$
4. $12 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y$
5. $-6 \cdot (x-1)(x-1)(x-1)$
6. $(9ab)(9ab)(9ab)(a^2-b)(a^2-b)$

Answer

1. $(2x)^5$
3. $-10a^7$
5. $-6(x-1)^3$

EXERCISE 2.2.5 PRODUCT, QUOTIENT, AND POWER RULE FOR EXPONENTS

Simplify.

1. $2^7 \cdot 2^5$
2. $3^9 \cdot 3$
3. -2^4
4. $(-2)^4$
5. -3^3
6. $(-3)^4$
7. $10^{13} \cdot 10^5 \cdot 10^4$
8. $10^8 \cdot 10^7 \cdot 10$
9. $\frac{5^{12}}{5^2}$
10. $\frac{10^7}{10^{10}}$
11. $\frac{10^{12}}{10^9}$
12. $(7^3)^5$
13. $(4^8)^4$
14. $10^6 \cdot (10^5)^4$

Answer

1. 2^{12}
3. -16

5. -27
7. 10^{22}
9. 5^{10}
11. 10^{3}
13. 4^{32}

EXERCISE 2.2.6 PRODUCT, QUOTIENT, AND POWER RULE FOR EXPONENTS

Simplify.

1. $(-x)^6$
2. $a^5 \cdot (-a)^2$
3. $x^3 \cdot x^5 \cdot x$
4. $y^5 \cdot y^4 \cdot y^2$
5. $(a^5)^2 \cdot (a^3)^4 \cdot a$
6. $(x+1)^4 (y^5)^4 \cdot y^2$
7. $(x+1)^5 (x+1)^8$
8. $(2a-b)^{12} (2a-b)^9$
9. $\frac{(3x-1)^5}{(3x-1)^2}$
10. $\frac{(a-5)^{37}}{(a-5)^{13}}$
11. $xy^2 \cdot x^2 y$
12. $3x^2 y^3 \cdot 7xy^5$
13. $-8a^2 b \cdot 2ab$
14. $-3ab^2 c^3 \cdot 9a^4 b^5 c^6$
15. $2a^2 b^4 c(-3abc)$
16. $5a^2 (b^3)^3 c^3 \cdot (-2) 2a^3 (b^2)^4$
17. $2x^2 (x+y)^5 \cdot 3x^5 (x+y)^4$
18. $-5xy^6 (2x-1)^6 \cdot x^5 y (2x-1)^3$
19. $x^2 y \cdot xy^3 \cdot x^5 y^5$
20. $-2x^{10} y \cdot 3x^2 y^{12} \cdot 5xy^3$
21. $3^2 x^4 y^2 z \cdot 3xy^4 z^4$
22. $(-x^2)^3 (x^3)^2 (x^4)^3$
23. $a^{10} \cdot \frac{(a^6)^3}{a^3}$
24. $\frac{10x^9 (x^3)^5}{2x^5}$
25. $\frac{a^6 b^3}{a^2 b^2}$
26. $\frac{m^{10} n^7}{m^3 n^4}$
27. $\frac{20x^5 y^{12} z^3}{10x^2 y^{10} z}$
28. $\frac{-24a^{16} b^{12} c^3}{6a^6 b^{11} c}$
29. $\frac{16x^4 (x+2)^3}{4x(x+2)}$
30. $\frac{50y^2 (x+y)^{20}}{10y(x+y)^{17}}$

Answer

1. x^6
3. x^9
5. a^{23}
7. $(x+1)^{13}$
9. $(3x-1)^3$
11. $x^3 y^3$
13. $-16a^3 b^2$

15. $-6a^3b^5c^2$
17. $6x^7(x+y)^9$
19. x^8y^9
21. $27x^5y^6z^5$
23. a^{25}
25. a^4b
27. $2x^3y^2z^2$
29. $4x^3(x+2)^2$

? EXERCISE 2.2.7 POWER RULES FOR PRODUCTS AND QUOTIENTS

Simplify.

1. $(2x)^5$
2. $(-3y)^4$
3. $(-xy)^3$
4. $(5xy)^3$
5. $(-4abc)^2$
6. $\left(\frac{7}{2x}\right)^2$
7. $-\left(\frac{5}{3y}\right)^3$
8. $(3abc)^3$
9. $\left(\frac{-2xy}{3z}\right)^4$
10. $\left(\frac{5y}{(2x-1)x}\right)^3$
11. $\left(3x^2\right)^3$
12. $\left(-2x^3\right)^2$
13. $\left(xy^5\right)^7$
14. $\left(x^2y^{10}\right)^2$
15. $\left(\frac{3x^2}{y}\right)^3$
16. $\left(2x^2y^3z^4\right)^5$
17. $\left(\frac{-7ab^4}{c^2}\right)^2$
18. $[x^5y^4(x+y)^4]^5$
19. $[2y(x+1)^5]^3$
20. $\left(ab^3\right)^3$
21. $\left(\frac{5a^2}{3b}\right)^4$
22. $\left(\frac{-2x^3}{3y^2}\right)^2$
23. $\left(\frac{-x^2}{y^3}\right)^3$
24. $\left(\frac{ab^2}{3c^3d^2}\right)^4$
25. $\left(\frac{2x^7y}{(x-1)^3z^5}\right)^6$
26. $\left(2x^4\right)^3\cdot\left(x^5\right)^2$
27. $\left(x^3y\right)^2\cdot\left(xy^4\right)^3$
28. $\left(-2a^2b^3\right)^2\cdot\left(2a^5b\right)^4$
29. $(-a^2b)^3(3ab^4)^4$
30. $(2x^3(x+y)^4)^5\cdot(2x^4(x+y)^2)^3$
31. $\left(\frac{-3x^5y^4}{xy^2}\right)^3$
32. $\left(\frac{-3x^5y^4}{xy^2}\right)^2$

33. $\left(\frac{-25x^{10}y^{15}}{5x^5y^{10}}\right)^3$
34. $\left(\frac{10x^3y^5}{5xy^2}\right)^2$
35. $\left(\frac{-24ab^3}{6bc}\right)^5$
36. $\left(\frac{-2x^3y^{16}}{x^2y}\right)^2$
37. $\left(\frac{30ab^3}{3abc}\right)^3$
38. $\left(\frac{3s^3t^2}{2s^2t}\right)^3$
39. $\left(\frac{6xy^5(x+y)^6}{3y^2z(x+y)^2}\right)^5$
40. $\left(\frac{-64a^5b^{12}c^2(2ab-1)^{14}}{32a^2b^{10}c^2(2ab-1)^7}\right)^4$
41. The probability of tossing a fair coin and obtaining n heads in a row is given by the formula $P=(12)^n$. Determine the probability, as a percent, of tossing 5 heads in a row.
42. The probability of rolling a single fair six-sided die and obtaining n of the same faces up in a row is given by the formula $P=(16)^n$. Determine the probability, as a percent, of obtaining the same face up two times in a row.
43. If each side of a square measures $2x^3$ units, then determine the area in terms of the variable x.
44. If each edge of a cube measures $5x^2$ units, then determine the volume in terms of the variable x.

Answer

1. $32x^5$
3. $-x^3y^3$
5. $16a^2b^2c^2$
7. $-\frac{125}{27y^3}$
9. $\frac{16x^4y^4}{81z^4}$
11. $27x^6$
13. x^7y^{35}
15. $\frac{27x^6}{y^3}$
17. $\frac{49a^2b^8}{c^4}$
19. $8y^3(x+1)^{15}$
21. $\frac{625a^8}{81b^4}$
23. $-\frac{x^6}{y^9}$
25. $\frac{64x^{42}y^6}{(x-1)^{18}z^{30}}$
27. x^9y^{14}
29. $-81a^{10}b^{19}$
31. $-27x^{12}y^6$
33. $-125x^{15}y^{15}$
35. $\frac{-1024a^5b^{10}}{c^5}$
37. $\frac{1000b^6}{c^3}$
39. $\frac{32x^5y^{15}(x+y)^{20}}{z^5}$
41. $3\frac{1}{8}\%$
43. $A=4x^6$

EXERCISE 2.2.8 ZERO EXPONENTS

Simplify. (Assume variables are nonzero.)

1. 7^0
2. $(-7)^0$
3. -10^0
4. $-3^0 \cdot (-7)^0$
5. 8675309^0
6. $5^2 \cdot 3^0 \cdot 2^3$
7. $-3^0 \cdot (-2)^2 \cdot (-3)^0$
8. $\frac{5x^0}{y^2}$
9. $(-3)^2 x^2 y^0 z^5$
10. $-3^2 (x^3)^2 y^2 (z^3)^0$
11. $2x^3 y^0 z \cdot 3x^0 y^3 z^5$
12. $-3ab^2 c^0 \cdot 3a^2 (b^3 c^2)^0$
13. $(-8xy^2)^0$
14. $(2x^2 y^3)^0$
15. $\frac{9x^0 y^4}{3y^3}$

Answer

1. 1
3. -1
5. 1
7. -4
9. $9x^2 z^5$
11. $6x^3 y^3 z^6$
13. 1
15. $3y$

EXERCISE 2.2.9 DISCUSSION BOARD TOPICS

1. René Descartes (1637) established the usage of exponential form: a^2, a^3, and so on. Before this, how were exponents denoted?
2. Discuss the accomplishments accredited to Al-Karismi.
3. Why is 0^0 undefined?
4. Explain to a beginning student why $3^4 \cdot 3^2 \neq 9^6$.

Answer

1. Answers may vary
3. Answers may vary

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2.3: INTRODUCTION TO POLYNOMIALS

LEARNING OBJECTIVES

- Identify a polynomial and determine its degree.
- Evaluate a polynomial for given values of the variables.
- Evaluate a polynomial using function notation.

DEFINITIONS

A polynomial is a special algebraic expression with terms that consist of real number coefficients and variable factors with whole number exponents.

Examples of polynomials :

$$3x^2 \quad 7xy+5 \quad \tfrac{3}{2}x^3+3x^2-\tfrac{1}{2}x+1 \quad 6x^2y-4xy^3-4xy^3+7$$

Polynomials do not have variables in the denominator of any term.

Examples that are not polynomials :

$$\tfrac{2x^2}{y} \quad 5\sqrt{x}+5 \quad 5x^2+3x^{-2}+7 \quad \tfrac{2}{x}-\tfrac{5}{y}=3$$

The degree of a term in a polynomial is defined to be the exponent of the variable, or if there is more than one variable in the term, the degree is the sum of their exponents. Recall that $x^0=1$; any constant term can be written as a product of x^0 and itself. Hence the degree of a constant term is 0.

Table 2.3.1

Term	Degree
$3x^2$	2
$6x^2y$	$2+1=3$
$7a^2b^3$	$2+3=5$
8	0, since $8=8x^0$
$2x$	1, since $x=x^1$

The degree of a polynomial is the largest degree of all of its terms.

Table 2.3.2

Polynomial	Degree
$4x^5-3x^3+2x-1$	5
$6x^2y-5xy^3+7$	4, because $5xy^3$ has degree 4.
$12x+54$	1, because $x=x^1$

We classify polynomials by the number of terms and the degree as follows:

Table 2.3.3

Expression	Classification	Degree
$5x^7$	Monomial (one term)	7
$8x^6-1$	Binomial (two terms)	6
$-3x^2+x-1$	Trinomial (three terms)	2
$5x^3-2x^2+3x-6$	Polynomial (many terms)	3

In this text, we will call polynomials with four or more terms simply polynomials.

EXAMPLE 2.3.1

Classify and state the degree:

$$7x^2-4x^5-1\,.$$

Solution:

Here there are three terms. The highest variable exponent is 5. Therefore, this is a trinomial of degree 5.

Answer:

Trinomial; degree 5

EXAMPLE 2.3.2

Classify and state the degree:

$$12a^5bc^3.$$

Solution:

Since the expression consists of only multiplication, it is one term, a monomial. The variable part can be written as $a^5b^1c^3$; hence its degree is $5+1+3=9$.

Answer:

Monomial; degree 9

EXAMPLE 2.3.3

Classify and state the degree:

$$4x^2y-6xy^4+5x^3y^3+4\ .$$

Solution:

The term $4x^2y$ has degree 3; $-6xy^4$ has degree 5; $5x^3y^3$ has degree 6; and the constant term 4 has degree 0. Therefore, the polynomial has 4 terms with degree 6.

Answer:

Polynomial; degree 6

Of particular interest are polynomials with one variable, where each term is of the form a_nx^n. Here a_n is any real number and n is any whole number. Such polynomials have the standard form

$$a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0 \tag{2.3.1}$$

Typically, we arrange terms of polynomials in descending order based on the degree of each term. The leading coefficient is the coefficient of the variable with the highest power, in this case, a_n.

EXAMPLE 2.3.4

Write in standard form:

$$3x-4x^2+5x^3+7-2x^4\ .$$

Solution:

Since terms are separated by addition, write the following:

$$\begin{aligned}&3x-4x^2+5x^3+7-2x^4\\&=3x+(-4x^2)+5x^3+7+(-2x^4)\end{aligned}$$

In this form, we can see that the subtraction in the original corresponds to negative coefficients. Because addition is commutative, we can write the terms in descending order based on the degree of each term as follows:

$$\begin{aligned}&=(-2x^4)+5x^3+(-4x^2)+3x+7\\&=-2x^4+5x^3-4x^2+3x+7\end{aligned}$$

Answer:

$$-2x^4+5x^3-4x^2+3x+7$$

We can further classify polynomials with one variable by their degree as follows:

Table 2.3.4

Polynomial	Name
5	**Constant** (degree 0)
$2x+1$	**Linear** (degree 1)
$3x^2+5x-3$	**Quadratic** (degree 2)
x^3+x^2+x+1	**Cubic** (degree 3)
$7x^4+3x^3-7x+8$	Fourth-degree polynomial

In this text, we call any polynomial of degree $n \geq 4$ an nth-degree polynomial. In other words, if the degree is 4, we call the polynomial a fourth-degree polynomial. If the degree is 5, we call it a fifth-degree polynomial, and so on.

EVALUATING POLYNOMIALS

Given the values for the variables in a polynomial, we can substitute and simplify using the order of operations.

EXAMPLE 2.3.5

Evaluate:

$$3x - 1, \text{ where } x = -\tfrac{3}{2}.$$

Solution:

First, replace the variable with parentheses and then substitute the given value.

Answer:

$-\frac{11}{2}$

EXAMPLE 2.3.6

Evaluate:

$$3x^2 + 2x - 1 \text{ , where } x = -1.$$

Solution:

Answer:

0

EXAMPLE 2.3.7

Evaluate:

$$-2a^2b + ab^2 - 7 \text{, where } a = 3 \text{ and } b = -2.$$

Solution:

Answer:

41

EXAMPLE 2.3.8

The volume of a sphere in cubic units is given by the formula $V = \frac{4}{3}\pi r^3$, where r is the radius. Calculate the volume of a sphere with radius $r = \frac{3}{2}$ meters.

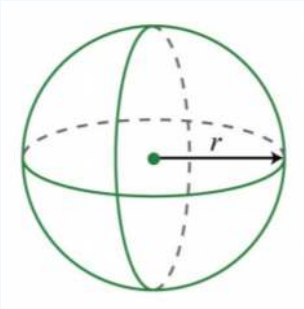


Figure 2.3.1

Solution:

$$
\begin{aligned}
V &= \frac{4}{3}\pi r^3 \\
&= \frac{4}{3}\pi\left(\frac{3}{2}\right)^3 \\
&= \frac{4}{3}\pi\frac{3^3}{2^3} \\
&= \frac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{3}}}\pi\frac{\overset{9}{\cancel{27}}}{\underset{2}{\cancel{8}}} \\
&= \frac{9}{2}\pi
\end{aligned}
$$

Answer:

$\frac{9}{2}\pi$ cubic meters

? EXERCISE 2.3.1

Evaluate:

$$x^3 - x^2 + 4x - 2 \text{ , where } x = -3.$$

Answer

-50

POLYNOMIAL FUNCTIONS

Polynomial functions with one variable are functions that can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 \tag{2.3.2}$$

,

where a_n is any real number and n is any whole number. Some examples of the different classes of polynomial functions are listed below:

Table 2.3.5

Polynomial function	Name
$f(x) = 5$	Constant function (degree 0)
$f(x) = -2x + 1$	Linear function (degree 1)
$f(x) = 5x^2 + 4x - 3$	Quadratic function (degree 2)
$f(x) = x^3 - 1$	Cubic function (degree 3)
$f(x) = 4x^5 + 3x^4 - 7$	**Polynomial function**

Since there are no restrictions on the values for x, the domain of any polynomial function consists of all real numbers.

✓ EXAMPLE 2.3.9

Calculate:

$$f(5), \text{ given } f(x) = -2x^2 + 5x + 10\,.$$

Solution:

Recall that the function notation $f(5)$ indicates we should evaluate the function when $x = 5$. Replace every instance of the variable x with the value 5.

Answer:

$f(5) = -15$

✓ EXAMPLE 2.3.10

Calculate:

$$f(-1), \text{ given } f(x) = -x^3 + 2x^2 - 4x + 1\,.$$

Solution:

Replace the variable x with -1.

$$\begin{aligned} f(-1) &= -(-1)^3+2(-1)^2-4(-1)+1 \\ &= -(-1)+2\cdot 1+4+1 \\ &= 1+2+4+1 \\ &= 8 \end{aligned}$$

Answer:

$$f(-1) = 8$$

EXERCISE 2.3.2

Given $g(x) = x^3 - 2x^2 - x - 4$, calculate $g(-1)$.

Answer

$g(-1) = -6$

KEY TAKEAWAYS

- Polynomials are special algebraic expressions where the terms are the products of real numbers and variables with whole number exponents.
- The degree of a polynomial with one variable is the largest exponent of the variable found in any term.
- The terms of a polynomial are typically arranged in descending order based on the degree of each term.
- When evaluating a polynomial, it is a good practice to replace all variables with parentheses and then substitute the appropriate values.
- All polynomials are functions.

EXERCISE 2.3.3 DEFINITIONS

Classify the given polynomial as linear, quadratic, or cubic.

1. $2x + 1$
2. $x^2 + 7x + 2$
3. $2 - 3x^2 + x$
4. $4x$
5. $x^2 - x^3 + x + 1$
6. $5 - 10x^3$

Answer

1. Linear
3. Quadratic
5. Cubic

EXERCISE 2.3.4 DEFINITIONS

Classify the given polynomial as a monomial, binomial, or trinomial and state the degree.

1. $x^3 - 1$
2. x^2y^2
3. $x - x^5 + 1$
4. $x^2 + 3x - 1$
5. $5ab^4$
6. $13x - 12$
7. $-5x^3 + 2x + 1$
8. $8x^2 - 9$
9. $4x^5 - 5x^3 + 6x$
10. $8x^4 - x^5 + 2x - 3$
11. $9x + 7$
12. $x^5 + x^4 + x^3 + x^2 - x + 1$
13. $6x - 1 + 5x^4 - 8$

14. $4x - 3x^2 + 3$
15. 7
16. x^2
17. $4x^2y - 3x^3y^3 + xy^3$
18. $a^3b^2 - 6ab$
19. a^3b^3
20. $x^2y - y^2x$
21. $xy - 3$
22. $a^5bc^2 + 3a^9 - 5a^4b^3c$
23. $-3x^{10}y^2z - xy^{12}z + 9x^{13} + 30$
24. $7x^0$

Answer

1. Binomial; degree 3
3. Trinomial; degree 5
5. Monomial; degree 5
7. Trinomial; degree 3
9. Trinomial; degree 5
11. Binomial; degree 1
13. Not a polynomial
15. Monomial; degree 0
17. Trinomial; degree 6
19. Monomial; degree 6
21. Binomial; degree 2
23. Polynomial; degree 14

EXERCISE 2.3.5 DEFINITIONS

Write the following polynomials in standard form.

1. $1 - 6x + 7x^2$
2. $x - 9x^2 - 8$
3. $7 - x^3 + x^7 - x^2 + x - 5x^5$
4. $a^3 - a^9 + 6a^5 - a + 3 - a^4$

Answer

1. $7x^2 - 6x + 1$
3. $x^7 - 5x^5 - x^3 - x^2 + x + 7$

EXERCISE 2.3.6 EVALUATING POLYNOMIALS

1. Fill in the following chart:

x	x^2
-3	
-2	
0	
2	
3	

***Figure* 2.3.2**

2. Fill in the following chart:

x	x^3
-3	
-2	
0	
2	
3	

***Figure* 2.3.3**

Answer

1.

x	x^2
-3	9
-2	4
0	0
2	4
3	9

Figure 2.3.4

EXERCISE 2.3.7 EVALUATING POLYNOMIALS

Evaluate.

1. $2x-3$, where $x=3$
2. x^2-3x+5, where $x=-2$
3. $-12x+13$, where $x=-13$
4. $-x^2+5x-1$, where $x=-12$
5. $-2x^2+3x-5$, where $x=0$
6. $8x^5-27x^3+81x-17$, where $x=0$
7. y^3-2y+1, where $y=-2$
8. y^4+2y^2-32, where $y=2$
9. a^3+2a^2+a-3, where $a=-3$
10. x^3-x^2, where $x=5$
11. $34x^2-12x+36$, where $x=-23$
12. $58x^2-14x+12$, where $x=4$
13. x^2y+xy^2, where $x=2$ and $y=-3$
14. $2a^5b-ab^4+a^2b^2$, where $a=-1$ and $b=-2$
15. a^2-b^2, where $a=5$ and $b=-6$
16. a^2-b^2, where $a=34$ and $b=-14$
17. a^3-b^3, where $a=-2$ and $b=3$
18. a^3+b^3, where $a=5$ and $b=-5$

Answer

1. 3
3. $\frac{1}{2}$
5. -5
7. -3
9. -15
11. $\frac{7}{6}$
13. 6
15. -11
17. -35

EXERCISE 2.3.8 EVALUATING POLYNOMIALS

For each problem, evaluate b^2-4ac, given the following values.

1. $a=-1, b=2$, and $c=-1$
2. $a=2, b=-2$, and $c=12$
3. $a=3, b=-5, c=0$
4. $a=1, b=0$, and $c=-4$
5. $a=14, b=-4$, and $c=2$
6. $a=1, b=5$, and $c=6$

Answer

1. 0
3. 25
5. 14

EXERCISE 2.3.9 EVALUATING POLYNOMIALS

The volume of a sphere in cubic units is given by the formula $V = \frac{4}{3}\pi r^3$, where r is the radius. For each problem, calculate the volume of a sphere given the following radii.

1. $r = 3$ centimeters
2. $r = 1$ centimeter
3. $r = \frac{1}{2}$ feet
4. $r = \frac{3}{2}$ feet
5. $r = 0.15$ in
6. $r = 1.3$ inches

Answer

1. 36π cubic centimeters
3. $\frac{\pi}{6}$ cubic feet
5. 0.014 cubic inches

EXERCISE 2.3.10 EVALUATING POLYNOMIALS

The height in feet of a projectile launched vertically from the ground with an initial velocity v_0 in feet per second is given by the formula $h = -16t^2 + v_0 t$, where t represents time in seconds. For each problem, calculate the height of the projectile given the following initial velocity and times.

1. $v_0 = 64$ feet/second, at times $t = 0, 1, 2, 3, 4$ seconds
2. $v_0 = 80$ feet/second, at times $t = 0, 1, 2, 2.5, 3, 4, 5$ seconds

Answer

1.

Table 2.3.6

Time	Height
$t = 0$ seconds	$h = 0$ feet
$t = 1$ second	$h = 48$ feet
$t = 2$ seconds	$h = 64$ feet
$t = 3$ seconds	$h = 48$ feet
$t = 4$ seconds	$h = 0$ feet

EXERCISE 2.3.11 EVALUATING POLYNOMIALS

The stopping distance of a car, taking into account an average reaction time, can be estimated with the formula $d = 0.05v^2 + 1.5$, where d is in feet and v is the speed in miles per hour. For each problem, calculate the stopping distance of a car traveling at the given speeds.

1. 20 miles per hour
2. 40 miles per hour
3. 80 miles per hour
4. 100 miles per hour

Answer

1. 21.5 feet
3. 321.5 feet

EXERCISE 2.3.12 POLYNOMIAL FUNCTIONS

Given the linear function $f(x) = \frac{2}{3}x + 6$, evaluate each of the following.

1. $f(-6)$

2. $f(-3)$
3. $f(0)$
4. $f(3)$
5. Find x when $f(x) = 10$.
6. Find x when $f(x) = -4$.

Answer

1. 2
3. 6
5. $x = 6$

EXERCISE 2.3.13 POLYNOMIAL FUNCTIONS

Given the quadratic function $f(x) = 2x^2 - 3x + 5$, evaluate each of the following.

1. $f(-2)$
2. $f(-1)$
3. $f(0)$
4. $f(2)$

Answer

1. 19
3. 5

EXERCISE 2.3.14 POLYNOMIAL FUNCTIONS

Given the cubic function $g(x) = x^3 - x^2 + x - 1$, evaluate each of the following.

1. $g(-2)$
2. $g(-1)$
3. $g(0)$
4. $g(1)$

Answer

1. -15
3. -1

EXERCISE 2.3.15 POLYNOMIAL FUNCTIONS

The height in feet of a projectile launched vertically from the ground with an initial velocity of 128 feet per second is given by the function $h(t) = -16t^2 + 128t$, where t is in seconds. Calculate and interpret the following.

1. $h(0)$
2. $h(12)$
3. $h(1)$
4. $h(3)$
5. $h(4)$
6. $h(5)$
7. $h(7)$
8. $h(8)$

Answer

1. The projectile is launched from the ground.
3. The projectile is 112 feet above the ground 1 second after launch.
5. The projectile is 256 feet above the ground 4 seconds after launch.
7. The projectile is 112 feet above the ground 7 seconds after launch.

EXERCISE 2.3.16 DISCUSSION BOARD TOPICS

1. Find and share some graphs of polynomial functions.
2. Explain how to convert feet per second into miles per hour.
3. Find and share the names of fourth-degree, fifth-degree, and higher polynomials.

Answer

1. Answers may vary
3. Answers may vary

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2.4: ADDING AND SUBTRACTING POLYNOMIALS

LEARNING OBJECTIVES

- Add polynomials.
- Subtract polynomials.
- Add and subtract polynomial functions

ADDING POLYNOMIALS

Recall that we combine like terms, or terms with the same variable part, as a means to simplify expressions. To do this, add the coefficients of the terms to obtain a single term with the same variable part. For example,

$$5x^2 + 8x^2 = 13x^2 \tag{2.4.1}$$

Notice that the variable part, x^2, does not change. This, in addition to the commutative and associative properties of addition, allows us to add polynomials.

EXAMPLE 2.4.1

Add:

$$3x + (4x - 5).$$

Solution:

The property $+(a+b) = a+b$, which was derived using the distributive property, allows us to remove the parentheses so that we can add like terms.

Answer:

$7x - 5$

EXAMPLE 2.4.2

Add:

$$(3x^2 + 3x + 5) + (2x^2 - x - 2)\ .$$

Solution:

Remove the parentheses and then combine like terms.

Answer:

$5x^2 + 2x + 3$

EXAMPLE 2.4.3

Add:

$$(-5x^2y - 2xy^2 + 7xy) + (4x^2y + 7xy^2 - 3xy)\ .$$

Solution:

Remember that the variable parts have to be exactly the same before we can add the coefficients.

$$\begin{aligned}&(-5x^2y - 2xy^2 + 7xy) + (4x^2y + 7xy^2 - 3xy)\\ &= \underline{-5x^2y} - \underline{\underline{2xy^2}} + \underline{\underline{\underline{7xy}}} + \underline{4x^2y} + \underline{\underline{7xy^2}} - \underline{\underline{\underline{3xy}}}\\ &= -x^2y + 5xy^2 + 4xy\end{aligned}$$

Answer:

$$-x^2y + 5xy^2 + 4xy$$

It is common practice to present the terms of the simplified polynomial expression in descending order based on their degree. In other words, we typically present polynomials in standard form, with terms in order from highest to lowest degree.

EXAMPLE 2.4.4

Add:

$$(a-4a^3+a^5-8)+(-9a^5+a^4-7a+5+a^3)\ .$$

Solution:

Answer:

$-8a^5+a^4-3a^3-6a-3$

EXERCISE 2.4.1

Add:

$$(6-5x^3+x^2-x)+(x^2+x+6x^3-1)\ .$$

Answer

x^3+2x^2+5

SUBTRACTING POLYNOMIALS

When subtracting polynomials, we see that the parentheses become very important. Recall that the distributive property allowed us to derive the following:

$$-(a+b)=-a-b \tag{2.4.2}$$

In other words, when subtracting an algebraic expression, we remove the parentheses by subtracting each term.

EXAMPLE 2.4.5

Subtract:

$$10x-(3x+5)\,.$$

Solution:

Subtract each term within the parentheses and then combine like terms.

Answer:

$7x-5$

Subtracting a quantity is equivalent to multiplying it by -1.

EXAMPLE 2.4.6

Subtract:

$$(3x^2+3x+5)-(2x^2-x-2)\ .$$

Solution:

Distribute the -1, remove the parentheses, and then combine like terms.

$$\begin{aligned}&(3x^2+3x+5)-(2x^2-x-2)\\&=3x^2+3x+5-2x^2+x+2\\&=x^2+4x+7\end{aligned}$$

Figure 2.4.1

Answer:

$$x^2+4x+7$$

Multiplying the terms of a polynomial by -1 changes all the signs.

EXAMPLE 2.4.7

Subtract:

$$(-5x^3 - 2x^2 + 7) - (4x^3 + 7x^2 - 3x + 2) \ .$$

Solution:

Distribute the -1, remove the parentheses, and then combine like terms.

Answer:

$-9x^3 - 9x^2 + 3x + 5$

EXAMPLE 2.4.8

Subtract $6x^2 - 3x - 1$ from $2x^2 + 5x - 2$.

Solution

Since subtraction is not commutative, we must take care to set up the difference correctly. First, write the quantity $(2x^2 + 5x - 2)$; from this, subtract the quantity $(6x^2 - 3x - 1)$.

Answer:

$-4x^2 + 8x - 1$

EXAMPLE 2.4.9

Simplify:

$$(2x^2 - 3x + 5) - (x^2 - 3x + 1) + (5x^2 - 4x - 8) \ .$$

Solution:

Apply the distributive property, remove the parentheses, and then combine like terms.

Answer:

$6x^2 - 4x - 4$

EXERCISE 2.4.2

Subtract:

$$(8x^2y - 5xy^2 + 6) - (x^2y + 2xy^2 - 1) \ .$$

Answer

$7x^2y - 7xy^2 + 7$

ADDING AND SUBTRACTING POLYNOMIAL FUNCTIONS

We use function notation to indicate addition and subtraction of functions as follows:

Table 2.4.1

Addition of functions:	$(f+g)(x) = f(x) + g(x)$
Subtraction of functions:	$(f-g)(x) = f(x) - g(x)$

When using function notation, be careful to group the entire function and add or subtract accordingly.

EXAMPLE 2.4.10

Calculate:

$$(f+g)(x), \text{ given } f(x) = -x^2 - 3x + 5 \text{ and } g(x) = 3x^2 + 2x + 1 \ .$$

Solution:

The notation $(f+g)(x)$ indicates that you should add the functions $f(x) + g(x)$ and collect like terms.

$$\begin{aligned} f(x)+g(x) &= (-x^2-3x+5)+(3x^2+2x+1) \\ &= -x^2-3x+5+3x^2+2x+1 \\ &= 2x^2-x+6 \end{aligned}$$

Answer:

$$(f+g)(x) = 2x^2-x+6$$

EXAMPLE 2.4.11

Calculate:

$$(f-g)(x), \text{ given } f(x)=2x-3 \text{ and } g(x)=-2x^2+2x+5\,.$$

Solution:

The notation $(f-g)(x)$ indicates that you should subtract the functions $f(x)-g(x)$:

Answer:

$(f-g)(x)=2x^2-8$

We may be asked to evaluate the sum or difference of two functions. We have the option to first find the sum or difference and use the resulting function to evaluate for the given variable, or to first evaluate each function and then find the sum or difference.

EXAMPLE 2.4.12

Calculate:

$$(f-g)(5), \text{ given } f(x)=x^2+x-7 \text{ and } g(x)=4x+10.$$

Solution:

First, find $(f-g)(x)=f(x)-g(x)$.

Therefore,

$(f-g)(x)=x^2-3x-17$

Next, substitute 5 for the variable x.

Answer:

$(f-g)(5)=-7$

Alternate Solution:

Since $(f-g)(5)=f(5)-g(5)$, we can find $f(5)$ and $g(5)$ and then subtract the results.

$f(x)=x^2+x-7$	$g(x)=4x+10$
$f(5)=(5)^2+(5)-7$	$g(5)=4(5)+10$
$=25+5-7$	$=20+10$
$=23$	$=30$

Therefore, we have

Answer:

$(f-g)(5)=-7$

KEY TAKEAWAYS

- When adding polynomials, remove the associated parentheses and then combine like terms.
- When subtracting polynomials, distribute the -1 and subtract all the terms before removing the parentheses and combining like terms.
- The notation $(f+g)(x)$ indicates that you add the functions.
- The notation $(f-g)(x)$ indicates that you subtract the functions

EXERCISE 2.4.3 ADDITION OF POLYNOMIALS

Add.

1. $(2x+1)+(-x+7)$
2. $(-6x+5)+(3x-1)$
3. $(\frac{2}{3}x+\frac{1}{2})+(\frac{1}{3}x-2)$

4. $(\frac{1}{3}x-\frac{3}{4})+(\frac{5}{6}x+\frac{1}{8})$
5. $(2x+1)+(x-3)+(5x-2)$
6. $(2x-8)+(-3x^2+7x-5)$
7. $(x^2-3x+7)+(3x^2-8x-5)$
8. $(-5x^2-1+x)+(-x+7x^2-9)$
9. $(\frac{1}{2}x^2-\frac{1}{3}x+\frac{1}{6})+(-\frac{3}{2}x^2+\frac{2}{3}x-1)$
10. $(-\frac{3}{5}x2+\frac{1}{4}x-6)+(2x^2-\frac{3}{8}x+\frac{5}{2})$
11. $(x^2+5)+(3x^2-2x+1)+(x^2+x-3)$
12. $(a^3-a^2+a-8)+(a^3+a^2+6a-2)$
13. $(a^3-8)+(-3a^3+5a^2-2)$
14. $(4a^5+5a^3-a)+(3a^4-2a^2+7)$
15. $(2x^2+5x-12)+(7x-5)$
16. $(3x+5)+(x^2-x+1)+(x^3+2x^2-3x+6)$
17. $(6x^5-7x^3+x^2-15)+(x^4+2x^3-6x+12)$
18. $(1+7x-5x^3+4x^4)+(-3x^3+5-x^2+x)$
19. $(x^2y^2-7xy+7)+(4x^2y^2-3xy-8)$
20. $(x^2+xy-y^2)+(7x^2-5xy+2y^2)$
21. $(2x^2+3xy-7y^2)+(-5x^2-3xy+8y^2)$
22. $(a^2b^2-100)+(2a^2b^2-3ab+20)$
23. $(ab^2-3a^2b+ab-3)+(-2a^2b+ab^2-7ab-1)$
24. $(10a^2b-7ab+8ab^2)+(6a^2b-ab+5ab^2)$
25. Find the sum of $2x+8$ and $7x-1$.
26. Find the sum of $13x-15$ and $16x+110$.
27. Find the sum of $x^2-10x+8$ and $5x^2-2x-6$.
28. Find the sum of $a^2-5a+10$ and $-9a^2+7a-11$.
29. Find the sum of x^2y^2-xy+6 and x^2y^2+xy-7.
30. Find the sum of $x^2-9xy+7y^2$ and $-3x^2-3xy+7y^2$.

Answer

1. $x+8$
3. $x-\frac{3}{2}$
5. $8x-4$
7. $4x^2-11x+2$
9. $-x^2+\frac{1}{3}x-\frac{5}{6}$
11. $5x^2-x+3$
13. $-2a^3+5a^2-10$
15. $2x^2+12x-17$
17. $6x^5+x^4-5x^3+x^2-6x-3$
19. $5x^2y^2-10xy-1$
21. $-3x^2+y^2$
23. $-5a^2b+2ab^2-6ab-4$
25. $9x+7$
27. $6x^2-12x+2$
29. $2x^2y^2-1$

? EXERCISE 2.4.4 SUBTRACTION OF POLYNOMIALS

Subtract.

1. $(5x-3)-(2x-1)$
2. $(-4x+1)-(7x+10)$
3. $(\frac{1}{4}x-\frac{3}{4})-(\frac{3}{4}x+\frac{1}{8})$

4. $(-\frac{3}{5}x+\frac{3}{7})-(\frac{2}{5}x-\frac{3}{2})$
5. $(x^2+7x-5)-(4x^2-5x+1)$
6. $(-6x^2+3x-12)-(-6x^2+3x-12)$
7. $(-3x^3+4x-8)-(-x^2+4x+10)$
8. $(\frac{1}{2}x^2+\frac{1}{3}x-\frac{3}{4})-(\frac{3}{2}x^2-\frac{1}{6}x+\frac{1}{2})$
9. $(\frac{5}{9}x^2+\frac{1}{5}x-\frac{1}{3})-(\frac{1}{3}x^2+\frac{3}{10}x+\frac{5}{9})$
10. $(a^3-4a^2+3a-7)-(7a^3-2a^2-6a+9)$
11. $(3a^3+5a^2-2)-(a^3-a+8)$
12. $(5x^5+4x^3+x^2-6)-(4x^4-3x^3-x+3)$
13. $(3-5x-x^3+5x^4)-(-5x^3+2-x^2-7x)$
14. $(x^5-6x^3+9x)-(4x^4+2x^2-5)$
15. $(2x^2y^2-4xy+9)-(3x^2y^2-3xy-5)$
16. $(x^2+xy-y^2)-(x^2+xy-y^2)$
17. $(2x^2+3xy-7y^2)-(-5x^2-3xy+8y^2)$
18. $(ab^2-3a^2b+ab-3)-(-2a^2b+ab^2-7ab-1)$
19. $(10a^2b-7ab+8ab^2)-(6a^2b-ab+5ab^2)$
20. $(10a^2b^2+5ab-6)-(5a^2b^2+5ab-6)$
21. Subtract $3x+1$ from $5x-9$.
22. Subtract $x^2-5x+10$ from x^2+5x-5.
23. Find the difference of $3x-7$ and $8x+6$.
24. Find the difference of $2x^2+3x-5$ and x^2-9.
25. The cost in dollars of producing customized coffee mugs with a company logo is given by the formula $C=150+0.10x$, where x is the number of cups produced. The revenue from selling the cups in the company store is given by $R=10x-0.05x^2$, where x is the number of units sold.
    a. Find a formula for the profit. (profit = revenue − cost)
    b. Find the profit from producing and selling 100 mugs in the company store.
26. The cost in dollars of producing sweat shirts is given by the formula $C=10q+1200$, where C is the cost and q represents the quantity produced. The revenue generated by selling the sweat shirts for $\$37$ each is given by $R=37q$, where q represents the quantity sold. Determine the profit generated if 125 sweat shirts are produced and sold.
27. The outer radius of a washer is 3 times the radius of the hole.

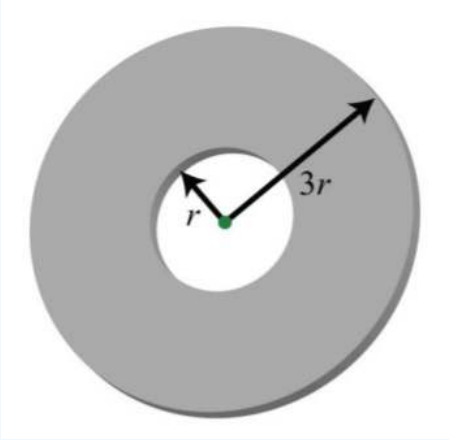


***Figure* 2.4.2**

a. Derive a formula for the area of the face of the washer.
b. What is the area of the washer if the hole has a diameter of 10 millimeters?

28. Derive a formula for the surface area of the following rectangular solid.

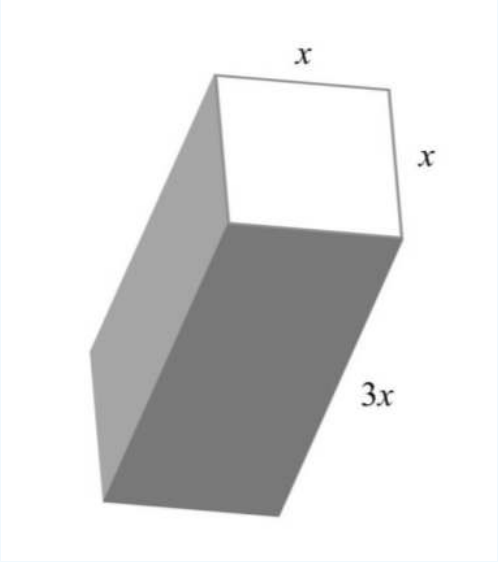


***Figure* 2.4.3**

Answer

1. $3x-2$
3. $-\frac{1}{2}x-\frac{7}{8}$
5. $-3x^2+12x-6$
7. $-3x^3+x^2-18$
9. $\frac{2}{9}x^2-\frac{1}{10}x-\frac{8}{9}$
11. $2a^3+5a^2+a-10$
13. $5x^4+4x^3+x^2+2x+1$
15. $-x^2y^2-xy+14$
17. $7x^2+6xy-15y^2$
19. $4a^2b+3ab^2-6ab$
21. $2x-10$
23. $-5x-13$
25. a. $P=-0.05x^2+9.9x-150$; b. \$340
27. a. $A=8\pi r^2$; b. 628.32 square millimeters

? EXERCISE 2.4.5 ADDITION AND SUBTRACTION OF POLYNOMIAL

Simplify.

1. $(2x+3)-(5x-8)+(x-7)$
2. $(3x-5)-(7x-11)-(5x+2)$
3. $(3x-2)-(4x-1)+(x+7)$
4. $(5x-3)-(2x+1)-(x-1)$
5. $(5x^2-3x+2)-(x^2+x-4)+(7x^2-2x-6)$
6. $(-2x^3+x^2-8)-(3x^2+x-6)-(2x-1)$
7. $(2x-7)-(x^2+3x-7)+(6x-1)$
8. $(6x^2-10x+13)+(4x^2-9)-(9-x^2)$
9. $(a^2-b^2)-(2a^2+3ab-4b^2)+(5ab-1)$
10. $(a^2-3ab+b^2)-(a^2+\ b^2)-(3ab-5)$
11. $(\frac{1}{2}x^2-\frac{3}{4}x+\frac{1}{4})-(\frac{3}{2}x-\frac{3}{4})+(\frac{5}{4}x-\frac{1}{2})$
12. $(\frac{9}{5}x^2-\frac{1}{3}x+2)-(\frac{3}{10}x^2-\frac{4}{5})-(x+\frac{5}{2})$

Answer

1. $-2x+4$
3. 6
5. $11x^2-6x$
7. $-x^2+5x-1$
9. $-a^2+2ab+3b^2-1$
11. $12x^2-x+12$

EXERCISE 2.4.6 ADDITION AND SUBTRACTION OF POLYNOMIAL FUNCTIONS

Find $(f+g)(x)$ and $(f-g)(x)$, given the following functions.

1. $f(x)=4x-1$ and $g(x)=-3x+1$
2. $f(x)=-x+5$ and $g(x)=2x-3$
3. $f(x)=3x^2-5x+7$ and $g(x)=-2x^2+5x-1$
4. $f(x)=x^3+2x^2-6x+2$ and $g(x)=2x^3+2x^2-5x-1$
5. $f(x)=\frac{1}{2}x+\frac{1}{3}$ and $g(x)=\frac{1}{5}x^2-\frac{3}{2}x+\frac{1}{6}$
6. $f(x)=x^2-5x+\frac{1}{3}$ and $g(x)=\frac{2}{3}x^2-x-\frac{1}{2}$

Answer

1. $(f+g)(x)=x$ and $(f-g)(x)=7x-2$
3. $(f+g)(x)=x^2+6$ and $(f-g)(x)=5x^2-10x+8$
5. $(f+g)(x)=\frac{1}{5}x^2-x+\frac{1}{2}$ and $(f-g)(x)=-\frac{1}{5}x^2+2x+\frac{1}{6}$

EXERCISE 2.4.7 ADDITION AND SUBTRACTION OF POLYNOMIAL FUNCTIONS

Given $f(x)=2x-3$ and $g(x)=x^2+3x-1$, find the following.

1. $(f+g)(x)$
2. $(g+f)(x)$
3. $(f-g)(x)$
4. $(g-f)(x)$
5. $(g+g)(x)$
6. $(f+g)(3)$
7. $(f+g)(-2)$
8. $(f+g)(0)$
9. $(f-g)(0)$
10. $(f-g)(-2)$
11. $(g-f)(-2)$
12. $(g-f)(12)$

Answer

1. $(f+g)(x)=x^2+5x-4$
3. $(f-g)(x)=-x^2-x-2$
5. $(g+g)(x)=2x^2+6x-2$
7. $(f+g)(-2)=-10$
9. $(f-g)(0)=-2$
11. $(g-f)(-2)=4$

EXERCISE 2.4.8 ADDITION AND SUBTRACTION OF POLYNOMIAL FUNCTIONS

Given $f(x) = 5x^2 - 3x + 2$ and $g(x) = 2x^2 + 6x - 4$, find the following.

1. $(f+g)(x)$
2. $(g+f)(x)$
3. $(f-g)(x)$
4. $(g-f)(x)$
5. $(f+g)(-2)$
6. $(f-g)(-2)$
7. $(f+g)(0)$
8. $(f-g)(0)$

Answer

1. $(f+g)(x) = 7x^2 + 3x - 2$
3. $(f-g)(x) = 3x^2 - 9x + 6$
5. $(f+g)(-2) = 20$
7. $(f+g)(0) = -2$

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2.5: MULTIPLYING POLYNOMIALS

LEARNING OBJECTIVES

- Multiply a polynomial by a monomial.
- Multiply a polynomial by a binomial.
- Multiply a polynomial by any size polynomial.
- Recognize and calculate special products.
- Multiply polynomial functions.

MULTIPLYING BY A MONOMIAL

Recall the product rule for exponents: if m and n are positive integers, then

$$x^m \cdot x^n = x^{m+n} \tag{2.5.1}$$

In other words, when multiplying two expressions with the same base, add the exponents. This rule applies when multiplying a monomial by a monomial. To find the product of monomials, multiply the coefficients and add the exponents of variable factors with the same base. For example,

$$\begin{aligned} 3x \cdot 5x^2 &= 3 \cdot 5 \cdot x^1 \cdot x^2 && \textit{Commutative property} \\ &= 15x^{1+2} && \textit{Product rule for exponents} \\ &= 15x^3 \end{aligned}$$

To multiply a polynomial by a monomial, apply the distributive property and then simplify each term.

EXAMPLE 2.5.1

Multiply:

$$-5x(4x-2).$$

Solution:

In this case, multiply the monomial, $-5x$, by the binomial, $4x-2$. Apply the distributive property and then simplify.

$$\begin{aligned} &-5x(4x-2) \\ &= (-5x)\cdot 4x + (-5x)\cdot(-2) \\ &= -20x^2 + 10x \end{aligned}$$

Figure 2.5.1

Answer:

$$-20x^2 + 10x$$

EXAMPLE 2.5.2

Multiply:

$$2x^2(3x^2 - 5x + 1)\,.$$

Solution:

Apply the distributive property and then simplify.

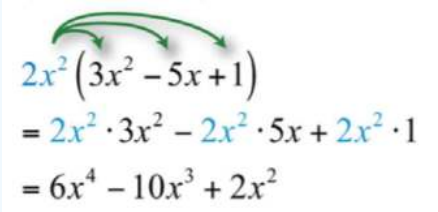


Figure 2.5.2

Answer:

$$6x^4 - 10x^3 + 2x^2$$

EXAMPLE 2.5.3

Multiply:

$$-3ab^2(a^2b^3 + 2a^3b - 6ab - 4)\ .$$

Solution:

Answer:

To summarize, multiplying a polynomial by a monomial involves the distributive property and the product rule for exponents. Multiply all of the terms of the polynomial by the monomial. For each term, multiply the coefficients and add exponents of variables where the bases are the same.

EXERCISE 2.5.1

Multiply:

$$-5x^2y(2xy^2 - 3xy + 6x^2y - 1)\ .$$

Answer

$-10x^3y^3 + 15x^3y^2 - 30x^4y^2 + 5x^2y$

MULTIPLYING BY A BINOMIAL

In the same way that we used the distributive property to find the product of a monomial and a binomial, we will use it to find the product of two binomials.

$$\begin{aligned}(a+b)(c+d) &= (a+b)\cdot c + (a+b)\cdot d \\ &= ac + bc + ad + bd \\ &= ac + ad + bc + bd\end{aligned}$$

Here we apply the distributive property multiple times to produce the final result. This same result is obtained in one step if we apply the distributive property to a and b separately as follows:

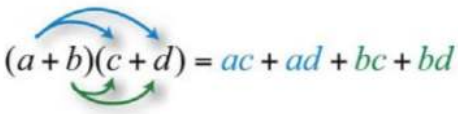


Figure 2.5.3

This is often called the FOIL method. We add the products of the first terms of each binomial ac, the *o*uter terms ad, the *i*nner terms bc, and finally the last terms bd. This mnemonic device only works for products of binomials; hence it is best to just remember that the distributive property applies.

EXAMPLE 2.5.4

Multiply:

$$(2x+3)(5x-2).$$

Solution:

Distribute $2x$ and then distribute 3.

Simplify by combining like terms.

$= 10x^2 + 11x - 6$

Answer:

$$10x^2 + 11x - 6$$

EXAMPLE 2.5.5

Multiply:

$$(\tfrac{1}{2}x - \tfrac{1}{4})(\tfrac{1}{2}x + \tfrac{1}{4}).$$

Solution:

Distribute $\frac{1}{2}x$ and then distribute $-\frac{1}{4}$.

$$\begin{aligned}(\frac{1}{2}x - \frac{1}{4})(\frac{1}{2}x + \frac{1}{4}) &= \frac{1}{2}x\frac{1}{2}x + \frac{1}{2}x\cdot\frac{1}{4} + \left(-\frac{1}{4}\right)\cdot\frac{1}{2}x + \left(-\frac{1}{4}\right)\cdot\frac{1}{4} \\ &= \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{8}x - \frac{1}{16} \\ &= \frac{1}{4}x^2 - \frac{1}{16}\end{aligned}$$

Answer:

$$\tfrac{1}{4}x^2 - \tfrac{1}{16}$$

EXAMPLE 2.5.6

Multiply:

$$(3y^2 - 1)(2y + 1)\,.$$

Solution:

Answer:

$6y^3 + 3y^2 - 2y - 1$

After applying the distributive property, combine any like terms.

EXAMPLE 2.5.7

Multiply:

$$(x^2 - 5)(3x^2 - 2x + 2)\,.$$

Solution:

After multiplying each term of the trinomial by x^2 and -5, simplify.

Answer:

$3x^4 - 2x^3 - 13x^2 + 10x - 10$

EXAMPLE 2.5.8

Multiply:

$$(2x - 1)^3.$$

Solution:

Perform one product at a time.

$$
\begin{aligned}
(2x-1)^3 &= (2x-1)\underbrace{(2x-1)(2x-1)}_{\text{multiply first}} \\
&= (2x-1)\underbrace{(4x^2-2x-2x+1)}_{\text{simplify}} \\
&= \underbrace{(2x-1)(4x^2-4x+1)}_{\text{multiply second}} \\
&= 8x^3-8x^2+2x-4x^2+4x-1 \\
&= 8x^3-12x^2+6x-1
\end{aligned}
$$

Figure 2.5.4

Answer:

$$8x^3 - 12x^2 + 6x - 1$$

At this point, it is worth pointing out a common mistake:

$$(2x-1)^3 \neq (2x)^3 - (1)^3$$

The confusion comes from the product to a power rule of exponents, where we apply the power to all factors. Since there are two terms within the parentheses, that rule does not apply. Care should be taken to understand what is different in the following two examples:

$$(xy)^2 = x^2y^2 \quad \checkmark$$
$$(x+y)^2 \neq x^2+y^2 \quad x$$

? EXERCISE 2.5.2

Multiply:

$$(2x-3)(7x^2-5x+4)\,.$$

Answer

$14x^3 - 31x^2 + 23x - 12$

PRODUCT OF POLYNOMIALS

When multiplying polynomials, we apply the distributive property many times. Multiply all of the terms of each polynomial and then combine like terms.

✔ EXAMPLE 2.5.9

Multiply:

$$(2x^2+x-3)(x^2-2x+5)\,.$$

Solution:

Multiply each term of the first trinomial by each term of the second trinomial and then combine like terms.

Aligning like terms in columns, as we have here, aids in the simplification process

Answer:

$2x^4 - 3x^3 + 5x^2 + 11x - 15$

Notice that when multiplying a trinomial by a trinomial, we obtain nine terms before simplifying. In fact, when multiplying an n-term polynomial by an m-term polynomial, we will obtain $n \times m$ terms. In the previous example, we were asked to multiply and found that

$$(2x^2+x-3)(x^2-2x+5) = 2x^4 - 3x^3 + 5x^2 + 11x - 15$$

Because it is easy to make a small calculation error, it is a good practice to trace through the steps mentally to verify that the operations were performed correctly. Alternatively, we can check by evaluating any value for x in both expressions to verify that the results are the

same. Here we choose $x = 2$:

$$\begin{aligned}(2x^2+x-3)(x^2-2x+5) &= (2(2)^2+(2)-3)((2)^2-2(2)+5)\\ &= (8+2-3)(4-4+5)\\ &= (7)(5)\\ &= 35\end{aligned}$$

Because the results could coincidentally be the same, a check by evaluating does not necessarily prove that we have multiplied correctly. However, after verifying a few values, we can be fairly confident that the product is correct.

Multiply:

$$(x^2-2x-3)^2\,.$$

Answer

$x^4-4x^3-2x^2+12x+9$

SPECIAL PRODUCTS

In this section, the goal is to recognize certain special products that occur often in our study of algebra. We will develop three formulas that will be very useful as we move along. The three should be memorized. We begin by considering the following two calculations:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+ab+ba+b^2\\ &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2-ab-ba+b^2\\ &= a^2-ab-ab+b^2\\ &= a^2-2ab+b^2\end{aligned}$$

This leads us to two formulas that describe perfect square trinomials:

$$(a+b)^2 = a^2+2ab+b^2 \tag{2.5.2}$$

$$(a-b)^2 = a^2-2ab+b^2 \tag{2.5.3}$$

We can use these formulas to quickly square a binomial.

EXAMPLE 2.5.10

Multiply:

$$(3x+5)^2.$$

Solution:

Here $a = 3x$ and $b = 5$. Apply the formula:

$$\begin{array}{rcl}(a+b)^2 = & a^2 & +2\quad a\quad b\quad +\quad b^2\\ & \downarrow & \quad\downarrow\quad\downarrow\quad\quad\downarrow\\ (3x+5)^2 &= (3x)^2+2\cdot(3x)(5)+(5)^2\\ &= 9x^2+30x+25\end{array}$$

Answer:

$$9x^2+30x+25$$

This process should become routine enough to be performed mentally.

EXAMPLE 2.5.11

Multiply:

$$(x-4)^2.$$

Solution:

Here $a = x$ and $b = 4$. Apply the appropriate formula as follows:

$$\begin{array}{rl} (a-b)^2 = & a^2 \quad -2 \quad a \quad b \quad + b^2 \\ & \downarrow \qquad \downarrow \; \downarrow \qquad \downarrow \\ (x-4)^2 & = (x)^2 - 2\cdot(x)(4) + (4)^2 \\ & = x^2 - 8x + 16 \end{array}$$

Answer:

$$x^2 - 8x + 16$$

Our third special product follows:

$$\begin{aligned} (a+b)(a-b) &= a^2 - ab + ba - b^2 \\ &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2 \end{aligned}$$

This product is called difference of squares:

$$(a+b)(a-b) = a^2 - b^2 \tag{2.5.4}$$

The binomials $(a+b)$ and $(a-b)$ are called conjugate binomials. Therefore, when conjugate binomials are multiplied, the middle term eliminates, and the product is itself a binomial.

EXAMPLE 2.5.12

Multiply:

$$(7x+4)(7x-4)\,.$$

Solution:

Answer:

$49x^2 - 16$

EXERCISE 2.5.4

Multiply:

$$(-5x+2)^2\,.$$

Answer

$25x^2 - 20x + 4$

MULTIPLYING POLYNOMIAL FUNCTIONS

We use function notation to indicate multiplication as follows:

Table 2.5.1

Multiplication of functions:	$(f\cdot g)(x) = f(x)\cdot g(x)$

EXAMPLE 2.5.13

Calculate:

$$(f\cdot g)(x), \text{ given } f(x) = 5x^2 \text{ and } g(x) = -x^2 + 2x - 3\,.$$

Solution:

Multiply all terms of the trinomial by the monomial function $f(x)$.

$$\begin{aligned} (f\cdot g)(x) &= f(x)\cdot g(x) \\ &= 5x^2\cdot(-x^2 + 2x - 3) \\ &= -5x^4 + 10x^3 - 15x^2 \end{aligned}$$

Answer:

$$(f\cdot g)(x) = -5x^4 + 10x^3 - 15x^2$$

EXAMPLE 2.5.14

Calculate:

$$(f\cdot g)(-1),\text{ given } f(x)=-x+3 \text{ and } g(x)=4x^2-3x+6\,.$$

Solution:

First, determine $(f\cdot g)(x)$.

$$\begin{aligned}(f\cdot g)(x) &= f(x)\cdot g(x)\\ &= (-x+3)(4x^2-3x+6)\\ &= -4x^3+3x^2-6x+12x^2-9x+18\\ &= -4x^3+15x^2-15x+18\end{aligned}$$

We have

\((f\cdot g)(x) = -4x^{3}+15x^{2}-15x+18

Next, substitute -1 for the variable x.

$$\begin{aligned}(f\cdot g)(-1) &= -4(-1)^3+15(-1)^2-15(-1)+18\\ &= -4\cdot(-1)+15\cdot 1+15+18\\ &= 4+15+15+18\\ &= 52\end{aligned}$$

Answer:

$$(f\cdot g)(-1)=52$$

Because $(f\cdot g)(-1)=f(-1)\cdot g(-1)$, we could alternatively calculate $f(-1)$ and $g(-1)$ separately and then multiply the results (try this as an exercise). However, if we were asked to evaluate multiple values for the function $(f\cdot g)(x)$, it would be best to first determine the general form, as we have in the previous example.

KEY TAKEAWAYS

- To multiply a polynomial by a monomial, apply the distributive property and then simplify each of the resulting terms.
- To multiply polynomials, multiply each term in the first polynomial with each term in the second polynomial. Then combine like terms.
- The product of an n-term polynomial and an m-term polynomial results in an $m\times n$ term polynomial before like terms are combined.
- Check results by evaluating values in the original expression and in your answer to verify that the results are the same.
- Use the formulas for special products to quickly multiply binomials that occur often in algebra.

EXERCISE 2.5.5 PRODUCT OF A MONOMIAL AND A POLYNOMIAL

Multiply.

1. $5x(-3x^2y)$
2. $(-2x^3y^2)(-3xy^4)$
3. $\frac{1}{2}(4x-3)$
4. $-\frac{3}{4}(\frac{2}{3}x-6)$
5. $3x(5x-2)$
6. $-4x(2x-1)$
7. $x^2(3x+2)$
8. $-6x^2(5x+3)$
9. $2ab(4a-2b)$
10. $5a^2b(a^2-b^2)$
11. $6x^2y^3(-3x^3y+xy^2)$
12. $3ab^3(-5ab^3+6a^2b)$
13. $-\frac{1}{2}x^2y(4xy-10)$
14. $-3x^4y^2(3x^8y^3)$
15. $2x^2(-5x^3)(3x^4)$
16. $4ab(a^2b^3c)(a^4b^2c^4)$
17. $-2(5x^2-3x+4)$
18. $\frac{4}{5}(25x^2-50xy+5y^2)$
19. $3x(5x^2-2x+3)$

20. $-x(x^2+x-1)$
21. $x^2(3x^2-5x-7)$
22. $x^3(-4x^2-7x+9)$
23. $\frac{1}{4}x^4(8x^3-2x^2+\frac{1}{2}x-5)$
24. $-\frac{1}{3}x^3(\frac{3}{2}x^5-\frac{2}{3}x^3+\frac{9}{2}x-1)$
25. $a^2b(a^2-3ab+b^2)$
26. $6a^2bc^3(2a-3b+c^2)$
27. $\frac{2}{3}xy^2(9x^3y-27xy+3xy^3)$
28. $-3x^2y^2(12x^2-10xy-6y^2)$
29. Find the product of $3x$ and $2x^2-3x+5$.
30. Find the product of $-8y$ and $y^2-2y+12$.
31. Find the product of $-4x$ and $x^4-3x^3+2x^2-7x+8$.
32. Find the product of $3xy^2$ and $-2x^2y+4xy-xy^2$.

Answer

1. $-15x^3y$
3. $2x-\frac{3}{2}$
5. $15x^2-6x$
7. $3x^3+2x^2$
9. $8a^2b-4ab^2$
11. $-18x^5y^4+6x^3y^5$
13. $-2x^3y^2+5x^2y$
15. $-30x^9$
17. $-10x^2+6x-8$
19. $15x^3-6x^2+9x$
21. $3x^4-5x^3-7x^2$
23. $2x^7-\frac{1}{2}x^6+\frac{1}{8}x^5-\frac{5}{4}x^4$
25. $a^4b-3a^3b^2+a^2b^3$
27. $6x^4y^3-18x^2y^3+2x^2y^5$
29. $6x^3-9x^2+15x$
31. $-4x^5+12x^4-8x^3+28x^2-32x$

? EXERCISE 2.5.6 PRODUCT OF A BINOMIAL AND A POLYNOMIAL

Multiply.

1. $(3x-2)(x+4)$
2. $(x+2)(x-3)$
3. $(x-1)(x+1)$
4. $(3x-1)(3x+1)$
5. $(2x-5)(x+3)$
6. $(5x-2)(3x+4)$
7. $(-3x+1)(x-1)$
8. $(x+5)(-x+1)$
9. $(y-\frac{2}{3})(y+\frac{2}{3})$
10. $(\frac{1}{2}x+\frac{1}{3})(\frac{3}{2}x-\frac{2}{3})$
11. $(\frac{3}{4}x+\frac{1}{5})(\frac{1}{4}x+\frac{2}{5})$
12. $(\frac{1}{5}x+\frac{3}{10})(\frac{3}{5}x-\frac{5}{2})$
13. $(y^2-2)(y+2)$
14. $(y^3-1)(y^2+2)$
15. $(a^2-b^2)(a^2+b^2)$
16. $(a^2-3b)^2$

17. $(x-5)(2x^2+3x+4)$
18. $(3x-1)(x^2-4x+7)$
19. $(2x-3)(4x^2+6x+9)$
20. $(5x+1)(25x^2-5x+1)$
21. $(x-\frac{1}{2})(3x^2+4x-1)$
22. $(\frac{1}{3}x-\frac{1}{4})(3x^2+9x-3)$
23. $(x+3)^3$
24. $(x-2)^3$
25. $(3x-1)^3$
26. $(2x+y)^3$
27. $(5x-2)(2x^3-4x^2+3x-2)$
28. $(x^2-2)(x^3-2x^2+x+1)$

Answer

1. $3x^2+10x-8$
3. x^2-1
5. $2x^2+x-15$
7. $-3x^2+4x-1$
9. $y^2-\frac{4}{9}$
11. $\frac{3}{16}x^2+\frac{7}{20}x+\frac{2}{25}$
13. y^3+2y^2-2y-4
15. a^4-b^4
17. $2x^3-7x^2-11x-20$
19. $8x^3-27$
21. $3x^3+\frac{5}{2}x^2-3x+12$
23. $x^3+9x^2+27x+27$
25. $27x^3-27x^2+9x-1$
27. $10x^4-24x^3+23x^2-16x+4$

? EXERCISE 2.5.7 PRODUCT OF POLYNOMIALS

Multiply.

1. $(x^2-x+1)(x^2+2x+1)$
2. $(3x^2-2x-1)(2x^2+3x-4)$
3. $(2x^2-3x+5)(x^2+5x-1)$
4. $(a+b+c)(a-b-c)$
5. $(a+2b-c)^2$
6. $(x+y+z)^2$
7. $(x-3)^4$
8. $(x+y)^4$
9. Find the volume of a rectangular solid with sides measuring $x, x+2$, and $x+4$ units.
10. Find the volume of a cube where each side measures $x-5$ units.

Answer

1. x^4+x^3+x+1
3. $2x^4+7x^3-12x^2+28x-5$
5. $a^2+4ab-2ac+4b^2-4bc+c^2$
7. $x^4-12x^3+54x^2-108x+81$
9. x^3+6x^2+8x

EXERCISE 2.5.8 SPECIAL PRODUCTS

Multiply.

1. $(x+2)^2$
2. $(x-3)^2$
3. $(2x+5)^2$
4. $(3x-7)^2$
5. $(-x+2)^2$
6. $(-9x+1)^2$
7. $(a+6)^2$
8. $(2a-3b)^2$
9. $(\frac{2}{3}x+\frac{3}{4})^2$
10. $(\frac{1}{2}x-\frac{3}{5})^2$
11. $(x^2+2)^2$
12. $(x^2+y^2)^2$
13. $(x+4)(x-4)$
14. $(2x+1)(2x-1)$
15. $(5x+3)(5x-3)$
16. $(\frac{1}{5}x-\frac{1}{3})(\frac{1}{5}x+\frac{1}{3})$
17. $(\frac{3}{2}x+\frac{2}{5})(\frac{3}{2}x-\frac{2}{5})$
18. $(2x-3y)(2x+3y)$
19. $(4x-y)(4x+y)$
20. $(a^3-b^3)(a^3+b^3)$
21. A box is made by cutting out the corners and folding up the edges of a square piece of cardboard. A template for a cardboard box with a height of 2 inches is given. Find a formula for the volume, if the initial piece of cardboard is a square with sides measuring x inches.

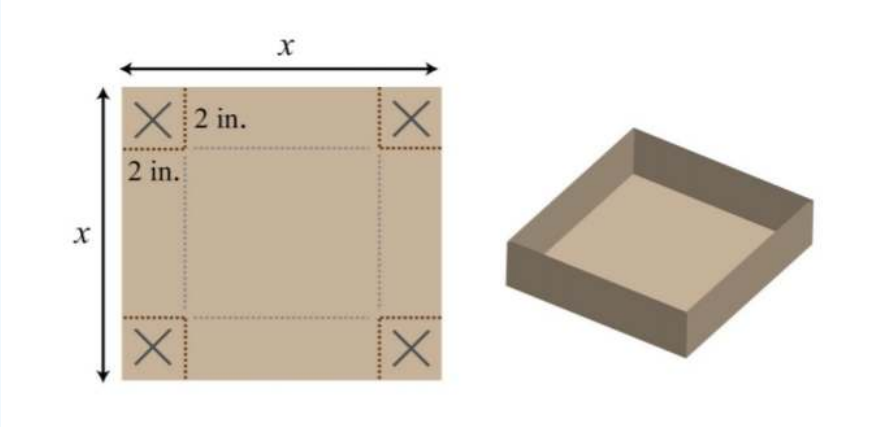


***Figure* 2.5.5**

22. A template for a cardboard box with a height of x inches is given. Find a formula for the volume, if the initial piece of cardboard is a square with sides measuring 12 inches.

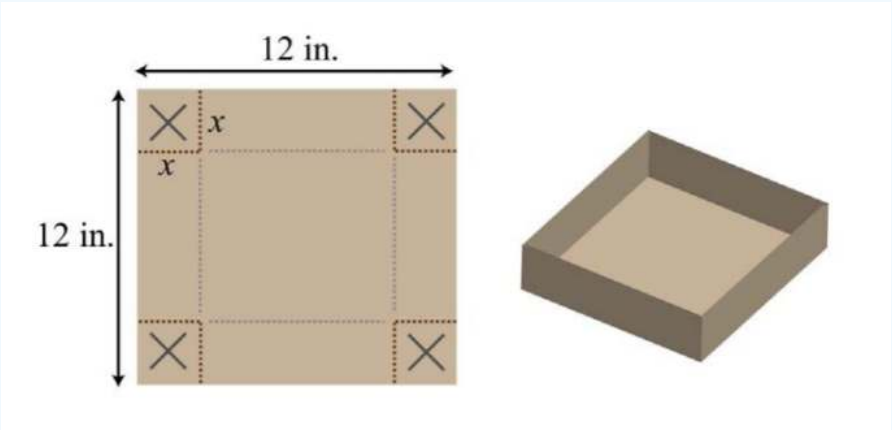


***Figure* 2.5.6**

Answer

1. x^2+4x+4
3. $4x^2+20x+25$
5. x^2-4x+4
7. $a^2+12a+36$
9. $\frac{4}{9}x^2+x+\frac{9}{16}$
11. x^4+4x^2+4
13. x^2-16
15. $25x^2-9$
17. $\frac{9}{4}x^2-\frac{4}{25}$
19. $16x^2-y^2$
21. $V=2x^2-16x+32$ cubic inches

? EXERCISE 2.5.9 MULTIPLYING POLYNOMIAL FUNCTIONS

For each problem, calculate $(f\cdot g)(x)$, given the functions.

1. $f(x)=8x$ and $g(x)=3x-5$
2. $f(x)=x^2$ and $g(x)=-5x+1$
3. $f(x)=x-7$ and $g(x)=6x-1$
4. $f(x)=5x+3$ and $g(x)=x^2+2x-3$
5. $f(x)=x^2+6x-3$ and $g(x)=2x^2-3x+5$
6. $f(x)=3x^2-x+1$ and $g(x)=-x^2+2x-1$

Answer

1. $(f\cdot g)(x)=24x^2-40x$
3. $(f\cdot g)(x)=6x^2-43x+7$
5. $(f\cdot g)(x)=2x^4+9x^3-19x^2+39x-15$

? EXERCISE 2.5.10 MULTIPLYING POLYNOMIAL FUNCTIONS

Given $f(x)=2x-3$ and $g(x)=3x-1$, find the following

1. $(f\cdot g)(x)$
2. $(g\cdot f)(x)$
3. $(f\cdot g)(0)$

4. $(f \cdot g)(-1)$
5. $(f \cdot g)(1)$
6. $(f \cdot g)(\frac{1}{2})$

Answer

1. $(f \cdot g)(x) = 6x^2 - 11x + 3$
3. $(f \cdot g)(0) = 3$
5. $(f \cdot g)(1) = -2$

? EXERCISE 2.5.11 MULTIPLYING POLYNOMIAL FUNCTIONS

Given $f(x) = 5x - 1$ and $g(x) = 2x^2 - 4x + 5$, find the following.

1. $(f \cdot g)(x)$
2. $(g \cdot f)(x)$
3. $(f \cdot g)(0)$
4. $(f \cdot g)(-1)$
5. $(f \cdot g)(1)$
6. $(f \cdot g)(\frac{1}{2})$
7. $(f \cdot f)(x)$
8. $(g \cdot g)(x)$

Answer

1. $(f \cdot g)(x) = 10x^3 - 22x^2 + 29x - 5$
3. $(f \cdot g)(0) = -5$
5. $(f \cdot g)(1) = 12$
7. $(f \cdot f)(x) = 25x^2 - 10x + 1$

? EXERCISE 2.5.12 DISCUSSION BOARD TOPICS

1. Explain why $(x + y)^2 \neq x^2 + y2$.
2. Explain how to quickly multiply a binomial with its conjugate. Give an example.
3. What are the advantages and disadvantages of using the mnemonic device FOIL?

Answer

1. Answers may vary
3. Answers may vary

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2.6: DIVIDING POLYNOMIALS

LEARNING OBJECTIVES

- Divide by a monomial.
- Divide by a polynomial using the division algorithm.
- Divide polynomial functions.

DIVIDING BY A POLYNOMIAL

Recall the quotient rule for exponents: if x is nonzero and m and n are positive integers, then

$$\frac{x^m}{x^n} = x^{m-n} \tag{2.6.1}$$

In other words, when dividing two expressions with the same base, subtract the exponents. This rule applies when dividing a monomial by a monomial. In this section, we will assume that all variables in the denominator are nonzero.

EXAMPLE 2.6.1

Divide:

$$\frac{28y^3}{7y}.$$

Solution:

Divide the coefficients and subtract the exponents of the variable y.

$$\begin{aligned}\frac{28y^3}{7y} &= \frac{28}{7}y^{3-1} \\ &= 4y^2\end{aligned}$$

Answer:

$$4y^2$$

EXAMPLE 2.6.2

Divide:

$$\frac{24x^7y^5}{8x^3y^2}.$$

Solution:

Divide the coefficients and apply the quotient rule by subtracting the exponents of the like bases.

$$\begin{aligned}\frac{24x^7y^5}{8x^3y^2} &= \frac{24}{8}x^{7-3}y^{5-2} \\ &= 3x^4y^3\end{aligned}$$

Answer:

$$3x^4y^3$$

When dividing a polynomial by a monomial, we may treat the monomial as a common denominator and break up the fraction using the following property:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \tag{2.6.2}$$

Applying this property results in terms that can be treated as quotients of monomials.

EXAMPLE 2.6.3

Divide:

$$\frac{-5x^4+25x^3-15x^2}{5x^2}.$$

Solution:

Break up the fraction by dividing each term in the numerator by the monomial in the denominator and then simplify each term.

Answer:

$-x^2+5x-3\cdot 1$

Check your division by multiplying the answer, the quotient, by the monomial in the denominator, the divisor, to see if you obtain the original numerator, the dividend.

$$\frac{dividend}{divisor}=quotient \tag{2.6.3}$$

or

$$dividend=divisor\cdot quotient \tag{2.6.4}$$

$$\begin{aligned} 5x^2\cdot(-x^2+5x-3) &= 5x^2\cdot(-x^2)+5x^2\cdot 5x-5x^2\cdot 3 \\ &= -5x^4+25x^3-15x^2 \quad \checkmark \end{aligned}$$

EXAMPLE 2.6.4

Divide:

$$\frac{9a^4b-7a^3b^2+3a^2b}{-3a^2b}.$$

Solution:

Answer:

$-3a^2+\frac{7}{3}ab-1$. The check is optional and is left to the reader.

EXERCISE 2.6.1

$(16x^5-8x^4+5x^3+2x^2)\div(2x^2)$.

Answer

$8x^3-4x^2+\frac{5}{2}x+1$

DIVIDING BY A POLYNOMIAL

The same technique outlined for dividing by a monomial ***does not*** work for polynomials with two or more terms in the denominator. In this section, we will outline a process called polynomial long division, which is based on the division algorithm for real numbers. For the sake of clarity, we will assume that all expressions in the denominator are nonzero.

EXAMPLE 2.6.5

Divide:

$$\frac{x^3+3x^2-8x-4}{x-2}.$$

Solution:

Here $x-2$ is the divisor and x^3+3x^2-8x-4 is the dividend.

Step 1: To determine the first term of the quotient, divide the leading term of the dividend by the leading term of the divisor.

$$x-2\overline{)\,x^3+3x^2-8x-4}\qquad \begin{matrix} ? \end{matrix}$$

Begin by dividing the leading terms.

$$x^3\div x=x^2$$

Figure 2.6.1

Step 2: Multiply the first term of the quotient by the divisor, remembering to distribute, and line up like terms with the dividend.

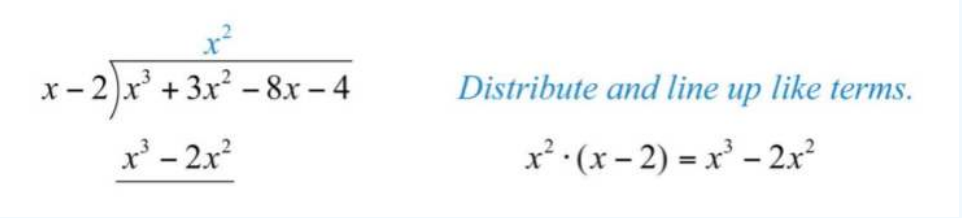


Figure 2.6.2

Step 3: Subtract the resulting quantity from the dividend. Take care to subtract both terms.

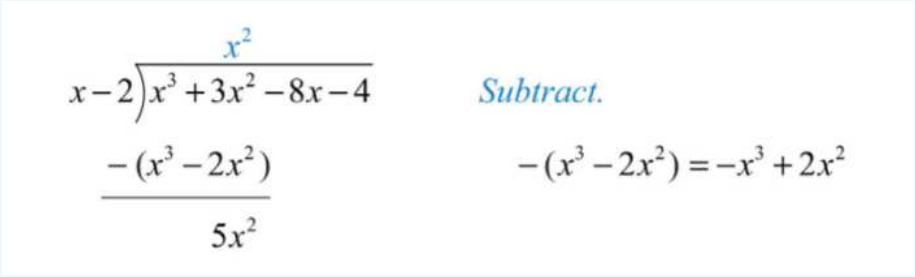


Figure 2.6.3

Step 4: Bring down the remaining terms and repeat the process from step 1.

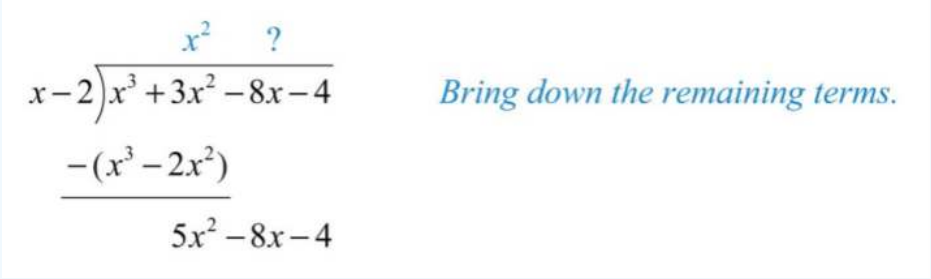


Figure 2.6.4

Notice that the leading term is eliminated and that the result has a degree that is one less than the dividend. The complete process is illustrated below:

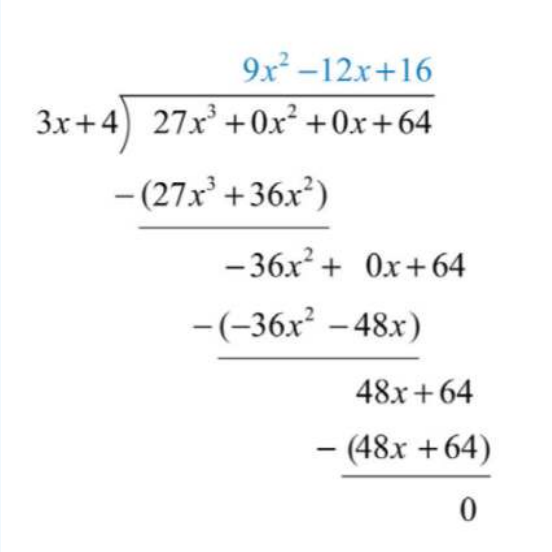


Figure 2.6.5

Polynomial long division ends when the degree of the remainder is less than the degree of the divisor. Here the remainder is 0. Therefore, the binomial divides the polynomial evenly and the answer is the quotient shown above the division line.

$$\frac{x^3+3x^2-8x-4}{x-2} = x^2 + 5x + 2$$

To check the answer, multiply the divisor by the quotient to see if you obtain the dividend:

$$x^3+3x^2-8x-4=(x-2)(x^2+5x+2)$$

Answer:

$$x^2+5x+2$$

Next, we demonstrate the case where there is a nonzero remainder.

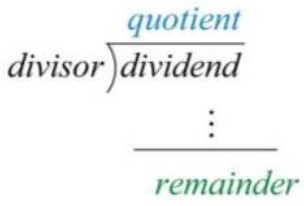


Figure 2.6.6

Just as with real numbers, the final answer adds the fraction where the remainder is the numerator and the divisor is the denominator to the quotient. In general, when dividing we have

$$\frac{dividend}{divisor}=quotient+\frac{remainder}{divisor} \tag{2.6.5}$$

If we multiply both sides by the divisor we obtain

$$dividend=quotient\times divisor+remainder \tag{2.6.6}$$

EXAMPLE 2.6.6

Divide:

$$\frac{6x^2-5x+3}{2x-1}$$

Solution:

Since the denominator is a binomial, begin by setting up polynomial long division.

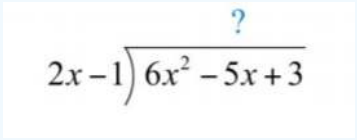


Figure 2.6.7

To start, determine what monomial times $2x-1$ results in a leading term $6x^2$. This is the quotient of the given leading terms: $(6x^2)\div(2x)=3x$. Multiply $3x$ times the divisor $2x-1$ and line up the result with like terms of the dividend.

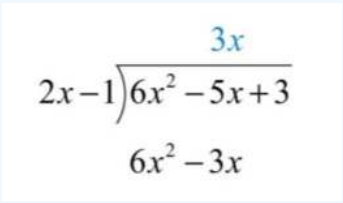


Figure 2.6.8

Subtract the result from the dividend and bring down the constant term $+3$.

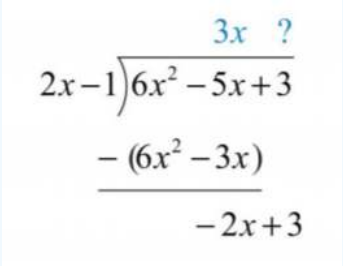


Figure 2.6.9

Subtracting eliminates the leading term and $-5x-(-3x)=-5x+3x=-2x$. The quotient of $-2x$ and $2x$ is -1. Multiply $2x-1$ by -1 and line up the result.

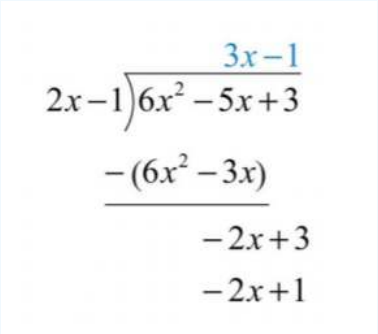


Figure 2.6.10

Subtract again and notice that we are left with a remainder.

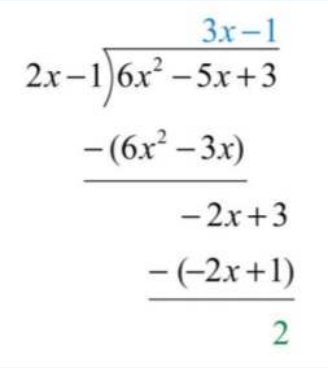


Figure 2.6.11

The constant term 2 has degree 0, and thus the division ends. We may write

$$\frac{6x^2-5x+3}{2x-1}=3x-1+\frac{2}{2x-1}$$

Answer:

$3x-1+\frac{2}{2x-1}$. To check that this result is correct, we multiply as follows:

$$\begin{aligned} quotient\times divisor+remainder &= (3x-1)(2x-1)+2 \\ &= 6x^2-3x-2x+1+2 \\ &= 6x^2-5x+3=dividend \quad \checkmark \end{aligned}$$

Occasionally, some of the powers of the variables appear to be missing within a polynomial. This can lead to errors when lining up like terms. Therefore, when first learning how to divide polynomials using long division, fill in the missing terms with zero coefficients, called placeholders.

EXAMPLE 2.6.7

Divide:

$$\frac{27x^3+64}{3x+4}.$$

Solution:

Notice that the binomial in the numerator does not have terms with degree 2 or 1. The division is simplified if we rewrite the expression with placeholders:

$$27x^3+64=27x^3+0x^2+0x+64$$

Set up polynomial long division:

$$3x+4\overline{)27x^3+0x^2+0x+64}$$

Figure 2.6.12

We begin with $27x^3 \div 3x = 9x^2$ and work the rest of the division algorithm.

$$\begin{array}{r} 9x^2-12x+16 \\ 3x+4\overline{)\ 27x^3+0x^2+0x+64} \\ \underline{-(27x^3+36x^2)} \\ -36x^2+\ 0x+64 \\ \underline{-(-36x^2-48x)} \\ 48x+64 \\ \underline{-(48x+64)} \\ 0 \end{array}$$

Figure 2.6.13

Answer:

$$9x^2-12x+16$$

EXAMPLE 2.6.8

Divide:

$$\frac{3x^4-2x^3+6x^2+23x-7}{x^2-2x+5}.$$

Solution:

$$x^2-2x+5\overline{)3x^4-2x^3+6x^2+23x-7}$$

Figure 2.6.14

Begin the process by dividing the leading terms to determine the leading term of the quotient $3x^4 \div x^2 = 3x^2$. Take care to distribute and line up the like terms. Continue the process until the remainder has a degree less than 2.

$$
\begin{array}{r}
3x^2+4x-1 \\
x^2-2x+5\overline{)\,3x^4-2x^3+6x^2+23x-7} \\
\underline{-(3x^4-6x^3+15x^2)} \\
4x^3-9x^2+23x-7 \\
\underline{-(4x^3-8x^2+20x)} \\
-x^2+3x-7 \\
\underline{-(-x^2+2x-5)} \\
x-2
\end{array}
$$

Figure 2.6.15

The remainder is $x-2$. Write the answer with the remainder:

$$\frac{3x^4-2x^3+6x^2+23x-7}{x^2-2x+5} = 3x^2+4x-1+\frac{x-2}{x^2-2x+5}$$

Answer:

$$3x^2+4x-1+\frac{x-2}{x^2-2x+5}$$

Polynomial long division takes time and practice to master. Work lots of problems and remember that you may check your answers by multiplying the quotient by the divisor (and adding the remainder if present) to obtain the dividend.

? EXERCISE 2.6.2

Divide:

$$\frac{20x^4-32x^3+7x^2+8x-10}{5x-3}.$$

Answer

$4x^3-4x^2-x+1-\frac{7}{5}x-3$

DIVIDING POLYNOMIAL FUNCTIONS

We may use function notation to indicate division as follows:

Table 2.6.1

Division of functions:	$(f/g)(x)=\frac{f(x)}{g(x)}$

The quotient of two polynomial functions does not necessarily have a domain of all real numbers. The values for x that make the function in the denominator 0 are restricted from the domain. This will be discussed in more detail at a later time. For now, assume all functions in the denominator are nonzero.

✓ EXAMPLE 2.6.9

Calculate:

$$(f/g)(x) \text{ given } f(x)=6x^5-36x^4+12x^3-6x^2 \text{ and } g(x)=-6x^2.$$

Solution:

The notation indicates that we should divide:

$$
\begin{aligned}
(f/g)(x) &= \frac{f(x)}{g(x)} \\
&= \frac{6x^5 - 36x^4 + 12x^3 - 6x^2}{-6x^2} \\
&= \frac{6x^5}{-6x^2} - \frac{36x^4}{-6x^2} + \frac{12x^3}{-6x^2} - \frac{6x^2}{-6x^2} \\
&= -1x^{5-2} + 6x^{4-2} - 2x^{3-2} + 1x^{2-2} \\
&= -x^3 + 6x^2 - 2x + 1
\end{aligned}
$$

Answer:

$$(f/g)(x) = -x^3 + 6x^2 - 2x + 1$$

EXAMPLE 2.6.10

Calculate:

$$(f/g)(-1), \text{ given } f(x) = -3x^3 + 7x^2 - 11x - 1 \text{ and } g(x) = 3x - 1.$$

Solution:

First, determine $(f/g)(x)$.

$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{-3x^3+7x^2-11x-1}{3x-1}$$

$$
\begin{array}{r}
-x^2+2x-3 \\
3x-1\overline{)-3x^3+7x^2-11x-1} \\
-(-3x^3+x^2) \\
\hline
6x^2-11x-1 \\
-(6x^2-2x) \\
\hline
-9x-1 \\
-(-9x+3) \\
\hline
-4
\end{array}
$$

Figure 2.6.16

Therefore,

$$(f/g)(x) = -x^2 + 2x - 3 - \frac{4}{3x-1}$$

Substitute -1 for the variable x.

Answer:

$(f/g)(-1) = -5$

KEY TAKEAWAYS

- When dividing by a monomial, divide all terms in the numerator by the monomial and then simplify each term. To simplify each term, divide the coefficients and apply the quotient rule for exponents.
- When dividing a polynomial by another polynomial, apply the division algorithm.
- To check the answer after dividing, multiply the divisor by the quotient and add the remainder (if necessary) to obtain the dividend.
- It is a good practice to include placeholders when performing polynomial long division.

EXERCISE 2.6.3 DIVIDING BY A MONOMIAL

Divide.

1. $\frac{81y^5}{9y^2}$
2. $\frac{36y^9}{9y^3}$
3. $\frac{52x^2y}{4xy}$
4. $\frac{24xy^5}{2xy^4}$
5. $\frac{25x^2y^5z^3}{5xyz}$
6. $\frac{-77x^4y^9z^2}{2x^3y^3z}$
7. $\frac{125a^3b^2c}{-10abc}$
8. $\frac{36a^2b^3c^5}{-6a^2b^2c^3}$
9. $\frac{9x^2+27x-3}{3}$
10. $\frac{10x^3-5x^2+40x-15}{5}$
11. $\frac{20x^3-10x^2+30x}{2x}$
12. $\frac{10x^4+8x^2-6x}{24x}$
13. $\frac{-6x^5-9x^3+3x}{-3x}$
14. $\frac{36a^{12}-6a^9+12a^5}{-12a^5}$
15. $\frac{-12x^5+18x^3-6x^2}{-6x^2}$
16. $\frac{-49a^8+7a^5-21a^3}{7a^3}$
17. $\frac{9x^7-6x^4+12x^3-x^2}{3x^2}$
18. $\frac{8x^9+16x^7-24x^4+8x^3}{-8x^3}$
19. $\frac{16a^7-32a^6+20a^5-a^4}{4a^4}$
20. $\frac{5a^6+2a^5+6a^3-12a^2}{3a^2}$
21. $\frac{-4x^2y^3+16x^7y^8-8x^2y^5}{-4x^2y^3}$
22. $\frac{100a^{10}b^{30}c^5-50a^{20}b^5c^{40}+20a^5b^{20}c^{10}}{10a^5b^5c^5}$
23. Find the quotient of $-36x^9y^7$ and $2x^8y^5$.
24. Find the quotient of $144x^3y^{10}z^2$ and $-12x^3y^5z$.
25. Find the quotient of $3a^4-18a^3+27a^2$ and $3a^2$.
26. Find the quotient of $64a^2bc^3-16a^5bc^7$ and $4a^2bc^3$.

Answer

1. $9y^3$
3. $13x$
5. $5xy^4z^2$
7. $-\frac{25}{2}a^2b$
9. $3x^2+9x-1$
11. $10x^2-5x+15$
13. $2x^4+3x^2-1$
15. $2x^3-3x+1$
17. $3x^5-2x^2+4x-\frac{1}{3}$
19. $4a^3-8a^2+5a-\frac{1}{4}$
21. $-4x^5y^5+2y^2+1$
23. $-18xy^2$
25. a^2-6a+9

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EXERCISE 2.6.4 DIVIDING BY A POLYNOMIAL

Divide.

1. $(2x^2-5x-3)\div(x-3)$
2. $(3x^2+5x-2)\div(x+2)$
3. $(6x^2+11x+3)\div(3x+1)$
4. $(8x^2-14x+3)\div(2x-3)$
5. $\frac{x^3-x^2-2x-12}{x-3}$
6. $\frac{2x^3+11x^2+4x-5}{x+5}$
7. $\frac{2x^3-x^2-4x+3}{2x+3}$
8. $\frac{-15x^3-14x^2+23x-6}{5x-2}$
9. $\frac{14x^4-9x^3+22x^2+4x-1}{7x-1}$
10. $\frac{8x^5+16x^4-8x^3-5x^2-21x+10}{2x+5}$
11. $\frac{x^2+8x+17}{x+5}$
12. $\frac{2x^2-5x+5}{x-2}$
13. $\frac{6x^2-13x+9}{-2x+1}$
14. $\frac{-12x^2+x+1}{3x+2}$
15. $\frac{x^3+9x^2+19x+1}{x+4}$
16. $\frac{2x^3-13x^2+17x-11}{x-5}$
17. $\frac{9x^3-12x^2+16x-15}{3x-2}$
18. $\frac{3x^4-8x^3+5x^2-5x+9}{x-2}$
19. $(6x^5-13x^4+4x^3-3x^2+13x-2)\div(3x+1)$
20. $(8x^5-22x^4+19x^3-20x^2+23x-3)\div(2x-3)$
21. $\frac{5x^5+12x^4+12x^3-7x^2-19x+3}{x^2+2x+3}$
22. $\frac{6x^5-17x^4+5x^3+16x^2-7x-3}{2x^2-3x-1}$
23. $\frac{x^5+7x^4-x^3-7x^2-49x+9}{x^2+7x-1}$
24. $\frac{5x^6-6x^4-4x^2+x+2}{5x^2-1}$
25. $\frac{x^3-27}{x-3}$
26. $\frac{8x^3+125}{2x+5}$
27. $(15x^5-9x^4-20x^3+12x^2+15x-9)\div(5x-3)$
28. $(2x^6-5x^5-4x^4+10x^3+6x^2-17x+5)\div(2x-5)$
29. $\frac{x^5-2x^3+3x-1}{x-1}$
30. $\frac{x^4-3x^2+5x-1}{3x+2}$
31. $\frac{a^2-4}{a+2}$
32. $\frac{a^5+1}{a^5+1}$
33. $\frac{a^6-1}{a-1}$
34. $\frac{x^5-1}{x-1}$
35. $\frac{x^5+x^4+6x^3+12x^2-4}{x^2+x-1}$
36. $\frac{50x^6-30x^5-5x^4+15x^3-5x+1}{5x^2-3x+2}$
37. $\frac{5x^5-15x^3+25x^2-5}{5x}$
38. $\frac{-36x^6+12x^4-6x^2}{6x^2}$
39. $\frac{150x^5y^2z^{15}-10x^3y^6z^5+4x^3y^2z^4}{10x^3y^2z^5}$
40. $\frac{27m^6+9m^4-81m^2+1}{9m^2}$
41. Divide $3x^6-2x^5+27x^4-18x^3-6x^2+7x-10$ by $3x-2$.
42. Divide $8x^6+4x^5-14x^4-5x^3+x^2-2x-3$ by $2x+1$.

Answer

1. $2x+1$
3. $2x+3$
5. x^2+2x+4
7. x^2-2x+1
9. $2x^3-x^2+3x+1$
11. $x+3+\frac{2}{x+5}$
13. $-3x+5+\frac{4}{-2x+1}$
15. $x^2+5x-1+\frac{5}{x+4}$
17. $3x^2-2x+4-\frac{7}{3x-2}$
19. $2x^4-5x^3+3x^2-2x+5-\frac{7}{3x+1}$
21. $5x^3+2x^2-7x+1$
23. $x^3-7+\frac{2}{x^2+7x-1}$
25. x^2+3x+9
27. $3x^4-4x^2+3$
29. $x^4+x^3-x^2-x+2+\frac{1}{x-1}$
31. $a-2$
33. $a^5+a^4+a^3+a^2+a+1$
35. $x^3+7x+5+\frac{2x+1}{x^2+x-1}$
37. $x^4-3x^2+5x-\frac{1}{x}$
39. $15x^2z^{10}-y^4+\frac{2}{5z}$
41. $x^5+9x^3-2x+1-\frac{8}{3x-2}$

? EXERCISE 2.6.5 DIVIDING POLYNOMIAL FUNCTIONS

Calculate $(f/g)(x)$, given the functions.

1. $f(x)=40x^8$ and $g(x)=10x^5$
2. $f(x)=54x^5$ and $g(x)=9x^3$
3. $f(x)=12x^2+24x-15$ and $g(x)=2x+5$
4. $f(x)=-8x^2+30x-7$ and $g(x)=2x-7$
5. $f(x)=18x^2-36x+5$ and $g(x)=3x-5$
6. $f(x)=-7x^2+29x-6$ and $g(x)=7x-1$
7. $f(x)=10x^3-9x^2+27x-10$ and $g(x)=5x-2$
8. $f(x)=15x^3+28x^2-11x+56$ and $g(x)=3x+8$
9. $f(x)=2x^4+5x^3-11x^2-19x+20$ and $g(x)=x^2+x-5$
10. $f(x)=4x^4-12x^3-20x^2+26x-3$ and $g(x)=2x^2+2x-3$

Answer

1. $(f/g)(x)=4x^3$
3. $(f/g)(x)=6x-3$
5. $(f/g)(x)=6x-2-\frac{5}{3x-5}$
7. $(f/g)(x)=2x^2-x+5$
9. $(f/g)(x)=2x^2+3x-4$

EXERCISE 2.6.6 DIVIDING POLYNOMIAL FUNCTIONS

Given $f(x)=6x^3+4x^2-11x+3$ and $g(x)=3x-1$, find the following.

1. $(f/g)(x)$
2. $(f/g)(-1)$
3. $(f/g)(0)$
4. $(f/g)(1)$

Answer

1. $(f/g)(x)=2x^2+2x-3$
3. $(f/g)(0)=-3$

EXERCISE 2.6.7 DIVIDING POLYNOMIAL FUNCTIONS

Given $f(x)=5x^3-13x^2+7x+3$ and $g(x)=x-2$, find the following.

1. $(f/g)(x)$
2. $(f/g)(-3)$
3. $(f/g)(0)$
4. $(f/g)(7)$

Answer

1. $(f/g)(x)=5x^2-3x+1+\frac{5}{x-2}$
3. $(f/g)(0)=-\frac{3}{2}$

EXERCISE 2.6.8 DISCUSSION BOARD TOPICS

1. How do you use the distributive property when dividing a polynomial by a monomial?
2. Compare long division of real numbers with polynomial long division. Provide an example of each.

Answer

1. Answers may vary

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2.7: NEGATIVE EXPONENTS

LEARNING OBJECTIVES

- Simplify expressions with negative integer exponents.
- Work with scientific notation.

NEGATIVE EXPONENTS

In this section, we define what it means to have negative integer exponents. We begin with the following equivalent fractions:

$$\frac{1}{8} = \frac{4}{32}$$

Notice that $4, 8$, and 32 are all powers of 2. Hence we can write $4 = 2^2, 8 = 2^3, and 32 = 2^5$.

$$\frac{1}{2^3} = \frac{1}{8} = \frac{4}{32} = \frac{2^2}{2^5}$$

If the exponent of the term in the denominator is larger than the exponent of the term in the numerator, then the application of the quotient rule for exponents results in a negative exponent. In this case, we have the following:

$$\frac{1}{2^3} = \frac{1}{8} = \frac{4}{32} = \frac{2^2}{2^5} = 2^{2-5} = 2^{-3}$$

We conclude that $2^{-3} = \frac{1}{2}^3$. This is true in general and leads to the definition of negative exponents. Given any integer n and $x \neq 0$, then

$$x^{-n} = \frac{1}{x^n} \tag{2.7.1}$$

Here $x \neq 0$ because \frac{1}{0}\) is undefined. For clarity, in this section, assume all variables are nonzero.

Simplifying expressions with negative exponents requires that we rewrite the expression with positive exponents.

EXAMPLE 2.7.1

Simplify:

$$10^{-2}.$$

Solution:

$$\begin{aligned} 10^{-2} &= \frac{1}{10^2} \\ &= \frac{1}{100} \end{aligned}$$

Answer:

$$\frac{1}{100}$$

EXAMPLE 2.7.2

Simplify:

$$(-3)^{-1}.$$

Solution:

$$\begin{aligned} (-3)^{-1} &= \frac{1}{(-3)^1} \\ &= -\frac{1}{3} \end{aligned}$$

Answer:

$$-\frac{1}{3}$$

EXAMPLE 2.7.3

Simplify:

$$\frac{1}{y^{-3}}.$$

Solution:

$$\begin{aligned}\frac{1}{y^{-3}} &= \frac{1}{\frac{1}{y^3}} \\ &= 1 \cdot \frac{y^3}{1} \\ &= y^3\end{aligned}$$

Answer:

$$y^3$$

At this point we highlight two very important examples,

Simplify $(2x)^{-1}$	*Simplify* $2x^{-1}$
$(2x)^{-1} = \frac{1}{(2x)^1}$	$2x^{-1} = 2 \cdot \frac{1}{x^1}$
$= \frac{1}{2x}$	$= \frac{2}{x}$

Figure 2.7.1

If the grouped quantity is raised to a negative exponent, then apply the definition and write the entire grouped quantity in the denominator. If there is no grouping, then apply the definition only to the base preceding the exponent.

EXAMPLE 2.7.4

Simplify:

$$(2ab)^{-3}.$$

Solution:

First, apply the definition of -3 as an exponent and then apply the power of a product rule.

$$\begin{aligned}(2ab)^{-3} &= \frac{1}{(2ab)^3} && \textit{Apply the negative exponent.} \\ &= \frac{1}{2^3a^3b^3} && \textit{Apply the power rule for a product.} \\ &= \frac{1}{8a^3b^3}\end{aligned}$$

Answer:

$$\frac{1}{8a^3b^3}$$

EXAMPLE 2.7.5

Simplify:

$$(-3xy^3)^{-2}.$$

Solution:

$$\begin{aligned}(-3xy^3)^{-2} &= \frac{1}{(-3xy^3)^2} \\ &= \frac{1}{(-3)^2x^2(y^3)^2} \\ &= \frac{1}{9x^2y^6}\end{aligned}$$

Answer:

$$\frac{1}{9x^2y^6}$$

EXAMPLE 2.7.6

Simplify:

$$\frac{x^{-3}}{y^{-4}}.$$

Solution:

$$\frac{x^{-3}}{y^{-4}} = \frac{\frac{1}{x^3}}{\frac{1}{y^4}} = \frac{1}{x^3} \cdot \frac{y^4}{1} = \frac{y^4}{x^3}$$

Answer:

$$\frac{y^4}{x^3}$$

The previous example suggests a property of quotients with negative exponents. If given any integers m and n, where $x \neq 0$ and $y \neq 0$, then

$$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n} \tag{2.7.2}$$

In other words, negative exponents in the numerator can be written as positive exponents in the denominator, and negative exponents in the denominator can be written as positive exponents in the numerator.

EXAMPLE 2.7.7

Simplify:

$$\frac{-2x^{-5}y^3}{z^{-2}}.$$

Solution:

Take care with the coefficient -2; recognize that this is the base and that the exponent is actually $+1: \ -2 = (-2)^1$. Hence the rules of negative exponents do not apply to this coefficient; leave it in the numerator.

$$\begin{aligned} \frac{-2x^{-5}y^3}{z^{-2}} &= \frac{-2x^{-5}y^3}{z^{-2}} \\ &= \frac{-2y^3z^2}{x^5} \end{aligned}$$

Answer:

$$\frac{-2y^3z^2}{x^5}$$

EXAMPLE 2.7.8

Simplify:

$$\frac{(-3x^{-4})^{-3}}{y^{-2}}.$$

Solution:

$$\begin{aligned} \frac{(-3x^{-4})^{-3}}{y^{-2}} &= \frac{(-3)^{-3}(x^{-4})^{-3}}{y^{-2}} && \textit{Apply the product to a power rule.} \\ &= \frac{(-3)^{-3}x^{12}}{y^{-2}} && \textit{Power rule} \\ &= \frac{x^{12}y^2}{(-3)^3} && \textit{Negative exponents} \\ &= \frac{x^{12}y^2}{-27} \\ &- \frac{x^{12}y^2}{27} \end{aligned}$$

Answer:

$$-\frac{x^{12}y^2}{27}$$

EXAMPLE 2.7.9

Simplify:

$$\frac{(3x^2)^{-4}}{(-2y^{-1}z^3)^{-2}}.$$

Solution:

$$\begin{aligned}\frac{(3x^2)^{-4}}{(-2y^{-1}z^3)^{-2}} &= \frac{3^{-4}(x^2)^{-4}}{(-2)^{-2}(y^{-1})^{-2}(z^3)^{-2}} && \textit{Product to a power rule}\\ &= \frac{3^{-4}x^{-8}}{(-2)^{-2}y^2z^{-6}} && \textit{Power rule}\\ &= \frac{(-2)^2z^6}{3^4x^8y^2} && \textit{Negative exponents}\\ &= \frac{4z^6}{81x^8y^2}\end{aligned}$$

Answer:

$$\frac{4z^6}{81x^8y^2}$$

EXAMPLE 2.7.10

Simplify:

$$\frac{(5x^2y)^3}{x^{-5}y^{-3}}.$$

Solution:

First, apply the power of a product rule and then the quotient rule.

$$\frac{(5x^2)^3}{x^{-5}y^{-3}} = \frac{5^3x^6y^3}{x^{-5}y^{-3}} = 5^3x^{6-(-5)}y^{3-(-3)} = 5^3x^{6+5}y^{3+3} = 125x^{11}y^6$$

Answer:

$$125x^{11}y^6$$

To summarize, we have the following rules for negative integer exponents with nonzero bases:

Table 2.7.1

Negative exponents:	$x^{-n} = \frac{1}{x^n}$
Quotients with negative exponents:	$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$

EXERCISE 2.7.1

Simplify:

$$\frac{(-5xy^{-3})^{-2}}{5x^4y^{-4}}.$$

Answer

$$\frac{y^{10}}{125x^6}$$

SCIENTIFIC NOTATION

Real numbers expressed in scientific notation have the form

$$a \times 10^n$$

where n is an integer and $1 \le a < 10$. This form is particularly useful when the numbers are very large or very small. For example,

$$\begin{aligned}9,460,000,000,000,000m &= 9.46 \times 10^{15}m && \textit{One light year}\\ 0.000000000025m &= 2.5 \times 10^{-11}m && \textit{Radius of a hydrogen atom}\end{aligned}$$

It is cumbersome to write all the zeros in both of these cases. Scientific notation is an alternative, compact representation of these numbers. The factor 10^n indicates the power of 10 to multiply the coefficient by to convert back to decimal form:

$$9.46\times10^{15} = 9.46\times1,\overbrace{000,000,000,000,000}^{15\ zeros} = 9,460,000,000,000,000$$

Figure 2.7.2

This is equivalent to moving the decimal in the coefficient fifteen places to the right. A negative exponent indicates that the number is very small:

$$2.5\times10^{-11} = 2.5\times\frac{1}{10^{11}} = \frac{2.5}{\underbrace{100,000,000,000}_{11\ zeros}} = 0.000000000025$$

Figure 2.7.3

This is equivalent to moving the decimal in the coefficient eleven places to the left.

Converting a decimal number to scientific notation involves moving the decimal as well. Consider all of the equivalent forms of 0.00563 with factors of 10 that follow:

$$\begin{aligned} 0.00563 &= 0.0563\times10^{-1} \\ &= 0.563\times10^{-2} \\ &= 5.63\times10^{-3} \\ &= 56.3\times10^{-4} \\ &= 563\times10^{-5} \end{aligned}$$

While all of these are equal, 5.63×10^{-3} is the only form considered to be expressed in scientific notation. This is because the coefficient 5.63 is between 1 and 10 as required by the definition. Notice that we can convert 5.63×10^{-3} back to decimal form, as a check, by moving the decimal to the left three places.

EXAMPLE 2.7.11

Write $1,075,000,000,000$ using scientific notation.

Solution:

Here we count twelve decimal places to the left of the decimal point to obtain the number 1.075.

$$1,075,000,000,000 = 1.075\times10^{12}$$

Answer:

$$1.075\times10^{12}$$

EXAMPLE 2.7.12

Write 0.000003045 using scientific notation.

Solution:

Here we count six decimal places to the right to obtain 3.045.

$$0.000003045 = 3.045\times10^{-6}$$

Answer:

$$3.045\times10^{-6}$$

Often we will need to perform operations when using numbers in scientific notation. All the rules of exponents developed so far also apply to numbers in scientific notation.

EXAMPLE 2.7.13

Multiply:

$$(4.36\times10^{-5})(5.3\times10^{12})\,.$$

Solution:

Use the fact that multiplication is commutative and apply the product rule for exponents.

$$\begin{aligned}(4.36\times10^{-5})(5.3\times10^{12}) &= (4.36\cdot5.30)\times(10^{-5}\cdot10^{12})\\ &= 23.108\times10^{-5+12}\\ &= 2.3108\times10^{1}\times10^{7}\\ &= 2.3108\times10^{1+7}\\ &= 2.3108\times10^{8}\end{aligned}$$

Answer:

$$2.3108\times10^{8}$$

EXAMPLE 2.7.14

Divide:

$$(3.24\times10^{8})\div(9.0\times10^{-3})\ .$$

Solution:

$$\begin{aligned}\frac{(3.24\times10^{8})}{(9.0\times10^{-3})} &= \left(\frac{3.24}{9.0}\right)\times\left(\frac{10^{8}}{10^{-3}}\right)\\ &= 0.36\times10^{8-(-3)}\\ &= 0.36\times10^{8+3}\\ &= 3.6\times10^{-1}\times10^{11}\\ &= 3.6\times10^{-1+11}\\ &= 3.6\times10^{10}\end{aligned}$$

Answer:

$$3.6\times10^{10}$$

EXAMPLE 2.7.15

The speed of light is approximately 6.7×10^{8} miles per hour. Express this speed in miles per second.

Solution:

A unit analysis indicates that we must divide the number by $3,600$.

$$\begin{aligned}6.7\times10^{8}\ mph &= \frac{6.7\times10^{8}miles}{1\ \cancel{hour}}\cdot\left(\frac{1\ \cancel{hour}}{60\ \cancel{minutes}}\right)\cdot\left(\frac{1\ \cancel{minutes}}{60seconds}\right)\\ &= \frac{6.7\times10^{8}miles}{3600seconds}\\ &= \left(\frac{6.7}{3600}\right)\times10^{8}\\ &\approx 0.0019\times10^{8} \qquad Rounded\ to\ two\ significant\ digits\\ &= 1.9\times10^{-3}\times10^{8}\\ &= 1.9\times10^{-3+8}\\ &= 1.9\times10^{5}\end{aligned}$$

Answer:

The speed of light is approximately 1.9×10^{5} miles per second.

EXAMPLE 2.7.16

By what factor is the radius of the sun larger than the radius of earth?

$$\begin{aligned}6,300,000m &= 6.3\times10^{6}m \qquad Radius\ of\ Earth\\ 700,000,000m &= 7.0\times10^{8}m \qquad Radius\ of\ the\ Sun\end{aligned}$$

Solution:

We want to find the number that when multiplied times the radius of earth equals the radius of the sun.

$$n \cdot \textit{radius of the Earth} = \textit{radius of the Sun}$$
$$n = \frac{\textit{radius of the Sun}}{\textit{radius of the Earth}}$$

Therefore,

$$\begin{aligned} n &= \frac{7.0 \times 10^8 m}{6.3 \times 10^6 m} \\ &= \frac{7.0}{6.3} \times \frac{10^8}{10^6} \\ &\approx 1.1 \times 10^{8-6} \\ &= 1.1 \times 10^2 \\ &= 110 \end{aligned}$$

? EXERCISE 2.7.2

Divide:

$$(6.75 \times 10^{-8}) \div (9 \times 10^{-17}) \, .$$

Answer

7.5×10^8

KEY TAKEAWAYS

- Expressions with negative exponents in the numerator can be rewritten as expressions with positive exponents in the denominator.
- Expressions with negative exponents in the denominator can be rewritten as expressions with positive exponents in the numerator.
- Take care to distinguish negative coefficients from negative exponents.
- Scientific notation is particularly useful when working with numbers that are very large or very small.

? EXERCISE 2.7.3 NEGATIVE EXPONENTS

Simplify. (Assume variables are nonzero.)

1. 5^{-1}
2. 5^{-2}
3. $(-7)^{-1}$
4. -7^{-1}
5. $\frac{1}{2}^{-3}$
6. $\frac{5}{3}^{-2}$
7. $(\frac{3}{5})^{-2}$
8. $(\frac{1}{2})^{-5}$
9. $(-\frac{2}{3})^{-4}$
10. $(-\frac{1}{3})^{-3}$
11. x^{-4}
12. y^{-1}
13. $3x^{-5}$
14. $(3x)^{-5}$
15. $\frac{1}{y^{-3}}$
16. $\frac{5}{2}x^{-1}$
17. $\frac{x^{-1}}{y^{-2}}$
18. $\frac{1}{(x-y)^{-4}}$
19. $\frac{x^2y^{-3}}{z^{-5}}$
20. $\frac{x}{y^{-3}}$
21. $(ab)^{-1}$
22. $\frac{1}{(ab)^{-1}}$
23. $-5x^{-3}y^2z^{-4}$

24. $\frac{3}{-2x^3y^{-5}z}$
25. $3x^{-4}y^2 \cdot 2x^{-1}y^3$
26. $-10a^2b^3 \cdot 2a^{-8}b^{-10}$
27. $(2a^{-3})^{-2}$
28. $(-3x^2)^{-1}$
29. $(5a^2b^{-3}c)^{-2}$
30. $(7r^3s^{-5}t)^{-3}$
31. $(-2r^2s^0t^{-3})^{-1}$
32. $(2xy^{-3}z^2)^{-3}$
33. $(-5a^2b^{-3}c^0)^4$
34. $(-x^{-2}y^3z^{-4})^{-7}$
35. $(\frac{1}{2}x^{-3})^{-5}$
36. $(2xy^2)^{-2}$
37. $(x^2y^{-1})^{-4}$
38. $(-3a^2bc^5)^{-5}$
39. $(\frac{20x^{-3}y^2}{5yz^{-1}})^{-1}$
40. $(\frac{4r^5s^{-3}t^4}{2r^3st^0})^{-3}$
41. $(\frac{2xy^3z^{-1}}{y^2z^3})^{-3}$
42. $(-\frac{3a^2bc}{ab^0c^4})^2$
43. $(\frac{-xyz}{x^4y^{-2}z^3})^{-4}$
44. $(-\frac{125x^{-3}y^4z^{-5}}{5x^2y^4(x+y)^3})^0$
45. $(x^n)^{-2}$
46. $(x^ny^n)^{-2}$

Answer

1. $\frac{1}{5}$
3. $-\frac{1}{7}$
5. 8
7. $\frac{25}{9}$
9. $\frac{81}{16}$
11. $\frac{1}{x^4}$
13. $3x^5$
15. y^3
17. $\frac{y^2}{x}$
19. $\frac{x^2z^5}{y^3}$
21. $\frac{1}{ab}$
23. $\frac{-5y^2}{x^3z^4}$
25. $\frac{6y^5}{x^5}$
27. $\frac{a^6}{4}$
29. $\frac{b^6}{25a^4c^2}$
31. $-\frac{t^3}{2r^2}$
33. $\frac{625a^8}{b^{12}}$
35. $32x^{15}$
37. $\frac{y^4}{x^8}$

39. $\frac{x^3}{4yz}$
41. $\frac{z^{12}}{8x^3y^3}$
43. $\frac{x^{12}z^8}{y^{12}}$
45. $\frac{1}{x^{2n}}$

EXERCISE 2.7.4 NEGATIVE EXPONENTS

The value in dollars of a new MP3 player can be estimated by using the formula $V = 100(t+1)^{-1}$, where t is the number of years after purchase.

1. How much was the MP3 player worth new?
2. How much will the MP3 player be worth in 1 year?
3. How much will the MP3 player be worth in 4 years?
4. How much will the MP3 player be worth in 9 years?
5. How much will the MP3 player be worth in 99 years?
6. According to the formula, will the MP3 ever be worthless? Explain.

Answer

1. \$100
3. \$20
5. \$1

EXERCISE 2.7.5 SCIENTIFIC NOTATION

Convert to a decimal number.

1. 9.3×10^9
2. 1.004×10^4
3. 6.08×10^{10}
4. 3.042×10^7
5. 4.01×10^{-7}
6. 1.0×10^{-10}
7. 9.9×10^{-3}
8. 7.0011×10^{-5}

Answer

1. $9,300,000,000$
3. $60,800,000,000$
5. 0.000000401
7. 0.0099

EXERCISE 2.7.6 SCIENTIFIC NOTATION

Rewrite using scientific notation.

1. $500,000,000$
2. $407,300,000,000,000$
3. $9,740,000$
4. $100,230$
5. 0.0000123
6. 0.000012
7. 0.000000010034
8. 0.99071

Answer

1. 5×10^{8}
3. 9.74×10^{6}
5. 1.23×10^{-5}
7. 1.0034×10^{-8}

? EXERCISE 2.7.7 SCIENTIFIC NOTATION

Perform the indicated operations.

1. $(3 \times 10^{5})(9 \times 10^{4})$
2. $(8 \times 10^{-22})(2 \times 10^{-12})$
3. $(2.1 \times 10^{-19})(3.0 \times 10^{8})$
4. $(4.32 \times 10^{7})(1.50 \times 10^{-18})$
5. $9.12 \times 10^{-9} 3.2 \times 10^{10}$
6. $1.15 \times 10^{9} 2.3 \times 10^{-11}$
7. $1.004 \times 10^{-8} 2.008 \times 10^{-14}$
8. $3.276 \times 10^{25} 5.2 \times 10^{15}$
9. $59,000,000,000,000 \times 0.000032$
10. $0.0000000000432 \times 0.0000000000673$
11. $1,030,000,000,000,000,000 \div 2,000,000$
12. $6,045,000,000,000,000 \div 0.00000005$
13. The population density of earth refers to the number of people per square mile of land area. If the total land area on earth is 5.751×10^{7} square miles and the population in 2007 was estimated to be 6.67×10^{9} people, then calculate the population density of earth at that time.
14. In 2008 the population of New York City was estimated to be 8.364 million people. The total land area is 305 square miles. Calculate the population density of New York City.
15. The mass of earth is 5.97×10^{24} kilograms and the mass of the moon is 7.35×10^{22} kilograms. By what factor is the mass of earth greater than the mass of the moon?
16. The mass of the sun is 1.99×10^{30} kilograms and the mass of earth is 5.97×10^{24} kilograms. By what factor is the mass of the sun greater than the mass of earth? Express your answer in scientific notation.
17. The radius of the sun is 4.322×10^{5} miles and the average distance from earth to the moon is 2.392×10^{5} miles. By what factor is the radius of the sun larger than the average distance from earth to the moon?
18. One light year, 9.461×10^{15} meters, is the distance that light travels in a vacuum in one year. If the distance to the nearest star to our sun, Proxima Centauri, is estimated to be 3.991×10^{16} meters, then calculate the number of years it would take light to travel that distance.
19. It is estimated that there are about 1 million ants per person on the planet. If the world population was estimated to be 6.67 billion people in 2007, then estimate the world ant population at that time.
20. The sun moves around the center of the galaxy in a nearly circular orbit. The distance from the center of our galaxy to the sun is approximately $26,000$ light years. What is the circumference of the orbit of the sun around the galaxy in meters?
21. Water weighs approximately 18 grams per mole. If one mole is about 6×10^{23} molecules, then approximate the weight of each molecule of water.
22. A gigabyte is 1×10^{9} bytes and a megabyte is 1×10^{6} bytes. If the average song in the MP3 format consumes about 4.5 megabytes of storage, then how many songs will fit on a 4-gigabyte memory card?

Answer

1. 2.7×10^{10}
3. 6.3×10^{-11}
5. 2.85×10^{-19}
7. 5×10^{5}
9. 1.888×10^{9}
11. 5.15×10^{11}
13. About 116 people per square mile
15. 81.2

17. 1.807
19. 6.67×10^{15} ants
21. 3×10^{-23} grams

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2.E: REVIEW EXERCISES AND SAMPLE EXAM

REVIEW EXERCISES

EXERCISE 2.E.1 RULES OF EXPONENTS

Simplify.

1. $7^3 \cdot 7^6$
2. $5^9 5^6$
3. $y^5 \cdot y^2 \cdot y^3$
4. $x^3 y^2 \cdot xy^3$
5. $-5a^3b^2c \cdot 6a^2bc^2$
6. $\frac{55x^2yz}{55xyz^2}$
7. $(\frac{-3a^2b^4}{2c^3})^2$
8. $(-2a^3b^4c^4)^3$
9. $-5x^3y^0(z^2)^3 \cdot 2x^4(y^3)^2z$
10. $(-25x^6y^5z)^0$
11. Each side of a square measures $5x^2$ units. Find the area of the square in terms of x.
12. Each side of a cube measures $2x^3$ units. Find the volume of the cube in terms of x.

Answer

1. 7^9
3. y^{10}
5. $-30a^5b^3c^3$
7. $\frac{9a^4b^8}{4c^6}$
9. $-10x^7y^6z^7$
11. $A = 25x^4$

EXERCISE 2.E.2 INTRODUCTION TO POLYNOMIALS

Classify the given polynomial as a monomial, binomial, or trinomial and state the degree.

1. $8a^3 - 1$
2. $5y^2 - y + 1$
3. $-12ab^2$
4. 10

Answer

1. Binomial; degree 3
3. Monomial; degree 3

EXERCISE 2.E.3 INTRODUCTION TO POLYNOMIALS

Write the following polynomials in standard form.

1. $7 - x^2 - 5x$
2. $5x^2 - 1 - 3x + 2x^3$

Answer

1. $-x^2 - 5x + 7$

EXERCISE 2.E.4 INTRODUCTION TO POLYNOMIALS

Evaluate.

1. $2x^2 - x + 1$, where $x = -3$
2. $\frac{1}{2}x - \frac{3}{4}$, where $x = \frac{1}{3}$
3. $b^2 - 4ac$, where $a = -\frac{1}{2}, b = -3$, and $c = -\frac{3}{2}$
4. $a^2 - b^2$, where $a = -\frac{1}{2}$ and $b = -\frac{1}{3}$
5. $a^3 - b^3$, where $a = -2$ and $b = -1$
6. $xy^2 - 2x^2y$, where $x = -3$ and $y = -1$
7. Given $f(x) = 3x^2 - 5x + 2$, find $f(-2)$.
8. Given $g(x) = x^3 - x^2 + x - 1$, find $g(-1)$.
9. The surface area of a rectangular solid is given by the formula $SA = 2lw + 2wh + 2lh$, where l, w, and h represent the length, width, and height, respectively. If the length of a rectangular solid measures 2 units, the width measures 3 units, and the height measures 5 units, then calculate the surface area.
10. The surface area of a sphere is given by the formula $SA = 4\pi r^2$, where r represents the radius of the sphere. If a sphere has a radius of 5 units, then calculate the surface area.

Answer

1. 22
3. 6
5. -7
7. $f(-2) = 24$
9. 62 square units

EXERCISE 2.E.5 ADDING AND SUBTRACTING POLYNOMIALS

Perform the operations.

1. $(3x - 4) + (9x - 1)$
2. $(13x - 19) + (16x + 12)$
3. $(7x^2 - x + 9) + (x^2 - 5x + 6)$
4. $(6x^2y - 5xy^2 - 3) + (-2x^2y + 3xy^2 + 1)$
5. $(4y + 7) - (6y - 2) + (10y - 1)$
6. $(5y^2 - 3y + 1) - (8y^2 + 6y - 11)$
7. $(7x^2y^2 - 3xy + 6) - (6x^2y^2 + 2xy - 1)$
8. $(a^3 - b^3) - (a^3 + 1) - (b^3 - 1)$
9. $(x^5 - x^3 + x - 1) - (x^4 - x^2 + 5)$
10. $(5x^3 - 4x^2 + x - 3) - (5x^3 - 3) + (4x^2 - x)$
11. Subtract $2x - 1$ from $9x + 8$.
12. Subtract $3x^2 - 10x - 2$ from $5x^2 + x - 5$.
13. Given $f(x) = 3x^2 - x + 5$ and $g(x) = x^2 - 9$, find $(f + g)(x)$.
14. Given $f(x) = 3x^2 - x + 5$ and $g(x) = x^2 - 9$, find $(f - g)(x)$.
15. Given $f(x) = 3x^2 - x + 5$ and $g(x) = x^2 - 9$, find $(f + g)(-2)$.
16. Given $f(x) = 3x^2 - x + 5$ and $g(x) = x^2 - 9$, find $(f - g)(-2)$.

Answer

1. $12x - 5$
3. $8x^2 - 6x + 15$
5. $8y + 8$
7. $x^2y^2 - 5xy + 7$
9. $x^5 - x^4 - x^3 + x^2 + x - 6$
11. $7x + 9$
13. $(f + g)(x) = 4x^2 - x - 4$

15. $(f+g)(-2)=14$

EXERCISE 2.E.6 MULTIPLYING POLYNOMIALS

Multiply.

1. $6x^2(-5x^4)$
2. $3ab^2(7a^2b)$
3. $2y(5y-12)$
4. $-3x(3x^2-x+2)$
5. $x^2y(2x^2y-5xy^2+2)$
6. $-4ab(a^2-8ab+b^2)$
7. $(x-8)(x+5)$
8. $(2y-5)(2y+5)$
9. $(3x-1)^2$
10. $(3x-1)^3$
11. $(2x-1)(5x^2-3x+1)$
12. $(x^2+3)(x^3-2x-1)$
13. $(5y+7)^2$
14. $(y^2-1)^2$
15. Find the product of x^2-1 and x^2+1.
16. Find the product of $32x^2y$ and $10x-30y+2$.
17. Given $f(x)=7x-2$ and $g(x)=x^2-3x+1$, find $(f\cdot g)(x)$.
18. Given $f(x)=x-5$ and $g(x)=x^2-9$, find $(f\cdot g)(x)$.
19. Given $f(x)=7x-2$ and $g(x)=x^2-3x+1$, find $(f\cdot g)(-1)$.
20. Given $f(x)=x-5$ and $g(x)=x^2-9$, find $(f\cdot g)(-1)$.

Answer

1. $-30x^6$
3. $10y^2-24y$
5. $2x^4y^2-5x^3y^3+2x^2y$
7. $x^2-3x-40$
9. $9x^2-6x+1$
11. $10x^3-11x^2+5x-1$
13. $25y^2+70y+49$
15. x^4-1
17. $(f\cdot g)(x)=7x^3-23x^2+13x-2$
19. $(f\cdot g)(-1)=-45$

EXERCISE 2.E.7 DIVIDING POLYNOMIALS

Divide.

1. $\frac{7y^2-14y+28}{7}$
2. $\frac{12x^5-30x^3+6x}{6x}$
3. $\frac{4a^2b-16ab^2-4ab}{-4ab}$
4. $\frac{6a^6-24a^4+5a^2}{3a^2}$
5. $(10x^2-19x+6)\div(2x-3)$
6. $(2x^3-5x^2+5x-6)\div(x-2)$
7. $\frac{10x^4-21x^3-16x^2+23x-20}{2x-5}$
8. $\frac{x^5-3x^4-28x^3+61x^2-12x+36}{x-6}$
9. $\frac{10x^3-55x^2+72x-4}{2x-7}$

10. $\frac{3x^4+19x^3+3x^2-16x-11}{3x+1}$
11. $\frac{5x^4+4x^3-5x^2+21x+21}{5x+4}$
12. $\frac{x^4-4}{x-4}$
13. $\frac{2x^4+10x^3-23x^2-15x+30}{2x^2-3}$
14. $\frac{7x^4-17x^3+17x^2-11x+2}{x^2-2x+1}$
15. Given $f(x)=x^3-4x+1$ and $g(x)=x-1$, find $(f/g)(x)$.
16. Given $f(x)=x^5-32$ and $g(x)=x-2$, find $(f/g)(x)$.
17. Given $f(x)=x^3-4x+1$ and $g(x)=x-1$, find $(f/g)(2)$.
18. Given $f(x)=x^5-32$ and $g(x)=x-2$, find $(f/g)(0)$.

Answer

1. y^2-2y+4
3. $-a+4b+1$
5. $5x-2$
7. $5x^3+2x^2-3x+4$
9. $5x^2-10x+1+\frac{3}{2x-7}$
11. $x^3-x+5+\frac{1}{5x+4}$
13. $x^2+5x-10$
15. $(f/g)(x)=x^2+x-3-\frac{2}{x-1}$
17. $(f/g)(2)=1$

EXERCISE 2.E.8 NEGATIVE EXPONENTS

Simplify.

1. $(-10)^{-2}$
2. -10^{-2}
3. $5x^{-3}$
4. $(5x)^{-3}$
5. $\frac{1}{7y^{-3}}$
6. $3x^{-4}y^{-2}$
7. $\frac{-2a^2b^{-5}}{c^{-8}}$
8. $(-5x^2yz^{-1})^{-2}$
9. $(-2x^{-3}y^0z^2)^{-3}$
10. $\left(\frac{-10a^5b^3c^2}{5ab^2c^2}\right)^{-1}$
11. $\left(\frac{a^2b^{-4}c^0}{2a^4b^{-3}c}\right)^{-3}$

Answer

1. $\frac{1}{100}$
3. $\frac{5}{x^3}$
5. $\frac{y^3}{7}$
7. $\frac{-2a^2c^8}{b^5}$
9. $\frac{-x^9}{8z^6}$
11. $8a^6b^3c^3$

EXERCISE 2.E.9 NEGATIVE EXPONENTS

The value in dollars of a new laptop computer can be estimated by using the formula $V = 1200(t+1)^{-1}$, where t represents the number of years after the purchase.

1. Estimate the value of the laptop when it is $1\frac{1}{2}$ years old.
2. What was the laptop worth new?

Answer

2. $\$1,200$

EXERCISE 2.E.10 NEGATIVE EXPONENTS

Rewrite using scientific notation.

1. $2,030,000,000$
2. 0.00000004011

Answer

2. 5.796×10^{19}

EXERCISE 2.E.11 NEGATIVE EXPONENTS

Perform the indicated operations.

1. $(5.2 \times 10^{12})(1.8 \times 10^{-3})$
2. $(9.2 \times 10^{-4})(6.3 \times 10^{22})$
3. $\frac{4 \times 10^{16}}{8 \times 10^{-7}}$
4. $\frac{9 \times 10^{-30}}{4 \times 10^{-10}}$
5. $5,000,000,000,000 \times 0.0000023$
6. $\frac{0.0003}{120,000,000,000,000}$

Answer

2. 5.796×10^{19}
4. 2.25×10^{-20}
6. 2.5×10^{-18}

SIMPLE EXAM

EXERCISE 2.E.12

Simplify.

1. $-5x^3(2x^2y)$
2. $(x^2)^4 \cdot x^3 \cdot x$
3. $\frac{(-2x^2y^3)^2}{x^2y}$
4. a. $(-5)^0$
   b. -5^0

Answer

1. $-10x^5y$
3. $4x^2y^5$

EXERCISE 2.E.13

Evaluate.

1. $2x^2 - x + 5$, where $x = -5$
2. $a^2 - b^2$, where $a = 4$ and $b = -3$

Answer

1. 60

EXERCISE 2.E.14

Perform the operations.

1. $(3x^2 - 4x + 5) + (-7x^2 + 9x - 2)$
2. $(8x^2 - 5x + 1) - (10x^2 + 2x - 1)$
3. $(\frac{3}{5}a - \frac{1}{2}) - (\frac{2}{3}a^2 + \frac{2}{3}a - \frac{2}{9}) + (\frac{1}{15}a - \frac{5}{18})$
4. $2x^2(2x^3 - 3x^2 - 4x + 5)$
5. $(2x - 3)(x + 5)$
6. $(x - 1)^3$
7. $\frac{81x^5y^2z}{-3x^3yz}$
8. $\frac{10x^9 - 15x^5 + 5x^2}{-5x^2}$
9. $\frac{x^3 - 5x^2 + 7x - 2}{x - 2}$
10. $\frac{6x^4 - x^3 - 13x^2 - 2x - 1}{2x - 1}$

Answer

1. $-4x^2 + 5x + 3$
3. $-\frac{2}{3}a^2 - \frac{5}{9}$
5. $2x^2 + 7x - 15$
7. $-27x^2y$
9. $x^2 - 3x + 1$

EXERCISE 2.E.15

Simplify.

1. 2^{-3}
2. $-5x^{-2}$
3. $(2x^4y^{-3}z)^{-2}$
4. $(\frac{-2a^3b^{-5}c^{-2}}{ab^{-3}c^2})^{-3}$
5. Subtract $5x^2y - 4xy^2 + 1$ from $10x^2y - 6xy^2 + 2$.
6. If each side of a cube measures $4x4$ units, calculate the volume in terms of x.
7. The height of a projectile in feet is given by the formula $h = -16t^2 + 96t + 10$, where t represents time in seconds. Calculate the height of the projectile at $1\frac{1}{2}$ seconds.
8. The cost in dollars of producing custom t-shirts is given by the formula $C = 120 + 3.50x$, where x represents the number of t-shirts produced. The revenue generated by selling the t-shirts for $\$6.50$ each is given by the formula $R = 6.50x$, where x represents the number of t-shirts sold.
   a. Find a formula for the profit. (profit = revenue − cost)
   b. Use the formula to calculate the profit from producing and selling 150 t-shirts.
9. The total volume of water in earth's oceans, seas, and bays is estimated to be 4.73×10^{19} cubic feet. By what factor is the volume of the moon, 7.76×10^{20} cubic feet, larger than the volume of earth's oceans? Round to the nearest tenth.

Answer

1. $\frac{1}{8}$

3. $\frac{y^6}{4x^8z^2}$

5. $5x^2y - 2xy^2 + 1$

7. 118 feet

9. 16.4

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CHAPTER OVERVIEW

3: FACTORING AND SOLVING BY FACTORING

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3.1: INTRODUCTION TO FACTORING

LEARNING OBJECTIVES

- Determine the greatest common factor (GCF) of natural numbers.
- Determine the GCF of monomials.
- Factor out the GCF of a polynomial.
- Factor a four-term polynomial by grouping.

GCF OF NATURAL NUMBERS

The process of writing a number or expression as a product is called factoring. If we write $60 = 5 \cdot 12$, we say that the product $5 \cdot 12$ is a factorization of 60 and that 5 and 12 are factors. Typically, there are many ways to factor a number. For example,

$$\begin{aligned} 60 &= 6 \cdot 10 \\ 60 &= 2 \cdot 30 \qquad \textit{Factorizations of } 60 \\ 60 &= 4 \cdot 3 \cdot 5 \end{aligned}$$

Recall that a prime number is defined as a natural number with exactly two natural number factors, 1 and itself. The first ten prime numbers follow:

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ \ldots$$

Any natural number greater than 1 can be uniquely written as a product of prime numbers. This product is called the prime factorization. The prime factorization of 60 can be determined by continuing to factor until only a product of prime numbers remains.

$$\begin{aligned} 60 &= 2 \cdot 30 \\ &= 2 \cdot 2 \cdot 15 \\ &= 2 \cdot 2 \cdot 3 \cdot 5 \end{aligned}$$

Since the prime factorization is unique, it does not matter how we choose to initially factor the number; the end result will be the same. The prime factorization of 60 follows:

$$60 = 2^2 \cdot 3 \cdot 5 \qquad \textit{The prime factorization of } 60$$

Recall that the greatest common factor (GCF) of any two natural numbers is the product of all the common prime factors.

EXAMPLE 3.1.1

Find the GCF of 60 and 140.

Solution:

First, determine the prime factorizations of both integers.

$$\begin{array}{l|l} 140 = 10 \cdot 14 & 60 = 6 \cdot 10 \\ \quad = 2 \cdot 5 \cdot 2 \cdot 7 & \quad = 2 \cdot 3 \cdot 2 \cdot 5 \\ \quad = 2^2 \cdot 5 \cdot 7 & \quad = 2^2 \cdot 3 \cdot 5 \end{array}$$

The product of the common prime factors is $2^2 \cdot 5$; hence the $\text{GCF}(60, 140) = 2^2 \cdot 5 = 20$. To see that it is the greatest common factor, we can write the following:

$$\begin{aligned} 140 &= 20 \cdot 7 \\ 60 &= 20 \cdot 3 \end{aligned}$$

Answer:

The greatest common factor of 60 and 140 is 20.

EXAMPLE 3.1.2

Find the GCF of 504 and 1,080.

Solution:

First, determine the prime factorizations of both integers.

$$\begin{aligned} 504 &= 9 \cdot 56 \\ &= 3 \cdot 3 \cdot 7 \cdot 8 \\ &= 3 \cdot 3 \cdot 7 \cdot 2 \cdot 2 \cdot 2 \\ &= 2^3 \cdot 3^2 \cdot 7 \end{aligned} \quad \left| \quad \begin{aligned} 1080 &= 10 \cdot 108 \\ &= 2 \cdot 5 \cdot 9 \cdot 12 \\ &= 2 \cdot 5 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 3 \\ &= 2^3 \cdot 3^3 \cdot 5 \end{aligned} \right.$$

The product of the common prime factors is $2^3 \cdot 3^2$.

The GCF$(504, 1080) = 2^3 \cdot 3^2 = 72$. Note that we multiplied the common prime factors with the smallest exponent.

$$\begin{aligned} 5040 &= 72 \cdot 7 \\ 1080 &= 72 \cdot 15 \end{aligned}$$

The numbers 7 and 15 share no common natural number factor other than 1; we say that they are relatively prime.

Answer:

The greatest common factor of 504 and $1,080$ is 72.

GCF OF MONOMIALS

We next consider factorizations of monomials. For example, $6x$ and x^4 are factors of $6x^5$ because $6x^5 = 6x \cdot x^4$. Typically, there are many ways to factor a monomial. Some factorizations of $6x^5$ follow:

$$\begin{aligned} 6x^5 &= 2x^3 \cdot 3x^2 \\ 6x^5 &= 6x^4 \cdot x \qquad \textit{Factorizations of } 6x^5 \\ 6x^5 &= 2x \cdot 3x \cdot x^3 \end{aligned}$$

Given two or more monomials, it will be useful to find the greatest common monomial factor of each. For example, consider $6x^5y^3z$ and $8x^2y^3z^2$. The variable part of these two monomials look very much like the prime factorization of natural numbers and, in fact, can be treated the same way.

Steps for finding the GCF of monomials are outlined in the following example.

EXAMPLE 3.1.3

Find the GCF of $6x^5y^3z$ and $8x^2y^3z^2$.

Solution:

Step 1: Find the GCF of the coefficients.

$$6x^5y^3z \qquad \text{and} \qquad 8x^2y^3z^2$$

In this case, the GCF$(6, 8) = 2$.

Step 2: Determine the common variable factors with smallest exponents.

$$6x^5y^3z \qquad \text{and} \qquad 8x^2y^3z^2$$

In this case, the common variables with the smallest exponents are $x^2, y^3, and z^1$.

Step 3: The GCF of the monomials is the product of the common variable factors and the GCF of the coefficients. Therefore,

$$\text{GCF}\left(6x^5y^3z, 8x^2y^3z^2\right) = 2 \cdot x^2 \cdot y^3 \cdot z$$

Answer:

$$2x^2y^3z$$

It is worth pointing out that the GCF in the previous example divides both expressions evenly:

$$\frac{6x^5y^3z}{2x^2y^3z} = 3x^3 \qquad \text{and} \qquad \frac{8x^2y^3z^2}{2x^2y^3z} = 4z$$

Furthermore, we can write the following:

$$6x^5y^3z = 2x^2y^3z \cdot 3x^3 \qquad \text{and} \qquad 8x^2y^3z^2 = 2x^2y^3z \cdot 4z$$

The factors $3x^3$ and $4z$ share no common monomial factors other than 1; they are relatively prime.

EXAMPLE 3.1.4

Determine the GCF of the following expressions:

$$30x^6y \text{ and } 18x^4y^2z.$$

Solution:

The prime factorizations of the coefficients are

$$30 = 2{\cdot}3{\cdot}5$$
$$18 = 2{\cdot}3{\cdot}3$$

Thus the GCF$(30, 18) = 2 \cdot 3 = 6$. Next, consider the variable part:

$$30x^6y \quad \text{and} \quad 18x^4y^2z$$

The variable factors in common are x^4 and y. The factor z is not in common and we have

$$\text{GCF} = 6 \cdot x^4 \cdot y$$

Answer:

$$6x^4y$$

EXAMPLE 3.1.5

Determine the GCF of the following three expressions:

$$12a^5b^2(a+b)^5 \text{ , } 60a^4b^3c(a+b)^3 \text{ , and } 24a^2b^7c^3(a+b)^2.$$

Solution:

First, determine the GCF of the coefficients.

$$12 = 2^2 \cdot 3$$
$$60 = 2^2 \cdot 3 \cdot 5$$
$$24 = 2^3 \cdot 3$$

The GCF $(12, 60, 24) = 2^2 \cdot 3 = 12$. Next, determine the common factors of the variable part:

$$12a^5b^2(a+b)^5 \quad \text{and} \quad 60a^4b^3c(a+b)^3 \quad \text{and} \quad 24a^2b^7c^3(a+b)^2$$

The variable factors in common are a^2, b^2, and $(a+b)^2$. Therefore,

$$\text{GCF} = 12 \cdot a^2 \cdot b^2 \cdot (a+b)^2$$

Answer:

$12a^2b^2(a+b)^2$. Note that the variable c is not common to all three expressions and thus is not included in the GCF.

EXERCISE 3.1.1

Determine the GCF of the following:

$$60x^4y^3(x+2y)^7, 45x^2y^5(x+2y)^4 \text{ , and } 30x^7y^7(x+2y)^3.$$

Answer

$15x^2y^3(x+2y)^3$

FACTORING OUT THE GCF

We have seen that application of the distributive property is the key to multiplying polynomials. The process of factoring a polynomial involves using the distributive property in reverse to write each polynomial as a product of polynomial factors.

$$a(b+c) = ab+ac \quad \textit{Multiplying}$$
$$ab+ac = a(b+c) \quad \textit{Factoring}$$

To demonstrate this idea, we multiply and factor side by side. Factoring utilizes the GCF of the terms.

Table 3.1.1

Multiplying	Factoring
$3(5x+1) = 3{\cdot}5x+3{\cdot}1$ $= 15x+3$	$15x+3 = 3{\cdot}5x+3{\cdot}1$ $= 3(5x+1)$
$2x^2(3x^3+4) = 2x^2{\cdot}3x^3+2x^2{\cdot}4$ $= 6x^5+8x^2$	$6x^5+8x^2 = 2x^2{\cdot}3x^3+2x^2{\cdot}4$ $= 2x^2(3x^3+4)$

In the previous example, we see that the distributive property allows us to write the polynomial $6x^5+8x^2$ as a product of the two factors $2x^2$ and $(3x^3+4)$. Note that in this case, $2x^2$ is the GCF of the terms of the polynomial:

$$\text{GCF } 6x^5, 8x^2) = 2x^2$$

Factoring out the GCF involves rewriting a polynomial as a product where a factor is the GCF of all of its terms:

$$15x+3=3(5x+1) \qquad \textit{Factoring out the GCF}$$
$$6x^5+8x^2=2x^2(3x^3+4)$$

The steps for factoring out the GCF of a polynomial are outlined in the following example.

EXAMPLE 3.1.6

Factor out the GCF:

$$7x^4+21x^3-14x^2\,.$$

Solution:

Step 1: Identify the GCF of all the terms. In this case, the $\text{GCF}(7,21,14)=7$, and the common variable factor with the smallest exponent is x^2. The GCF of the polynomial is $7x^2$.

$$7x^4+21x^3-14x^2=7x^2(\quad ?\quad)$$

Step 2: Determine the terms of the missing factor by dividing each term of the original expression by the GCF. (**This step is usually performed mentally**.)

$$\tfrac{7x^4}{7x^2}=x^2 \qquad \tfrac{21x^3}{7x^2}=3x \qquad \tfrac{-14x^2}{7x^2}=-2$$

Step 3: Apply the distributive property (in reverse) using the terms found in the previous step.

$$7x^4+21x^3-14x^2=7x^2(x^2+3x-2)$$

Step 4: As a check, multiply using the distributive property to verify that the product equals the original expression. (**This step is optional and can be performed mentally**.)

$$\begin{aligned}7x^2(x^2+3x-2) &= 7x^2\cdot x^2+7x^2\cdot 3x-7x^2\cdot 2\\ &= 7x^4+21x^3-14x^2 \quad \checkmark\end{aligned}$$

Answer:

$$7x^2(x^2+3x-2)$$

EXAMPLE 3.1.7

Factor out the GCF:

$$48a-16b+4c$$

Solution:

There are no variable factors in common and the $\text{GCF}(48,16,4)=4$.

$$\begin{aligned}48a-16b+4c &= 4(\quad ?\quad)\\ &= 4(12a-4b+c)\end{aligned}$$

Answer:

$$4(12a-4b+c)$$

EXAMPLE 3.1.8

Factor out the GCF:

$$25x^3+15x^2+5x$$

Solution:

The $\text{GCF}(25,15,5)=5$, and the common variable factor with smallest exponents is x^1. The GCF of all the terms is $5x$.

$$\begin{aligned}25x^3+15x^2+5x &= 5x(\quad ?\quad)\\ &= 5x(5x^2+3x+1)\end{aligned}$$

Answer:

$$5x(5x^2+3x+1)$$

If the GCF is the same as one of the terms, then, after the GCF is factored out, a constant term 1 will remain. In the previous example, we can see that $\frac{5x}{5x}=1$. The importance of remembering the constant term becomes clear when performing the check using the distributive property:

$$\begin{aligned} 5x(5x^2+3x+1) &= 5x\cdot 5x^2+5x\cdot 3x+5x\cdot 1 \\ &= 25x^3+15x^2+5x \quad \checkmark \end{aligned}$$

The constant term 1 allows us to obtain the same original expression after we distribute.

EXAMPLE 3.1.9

Factor out the GCF:

$$15x^6y^4+10x^5y^3z^2-20xy^6z^3\ .$$

Solution:

The GCF$(10,15,20)=5$, and the common variables with smallest exponent are x^1 and y^3. Therefore, the GCF of the terms is $5xy^3$. The first term does not have a variable factor of z and thus cannot be a part of the greatest common factor. If we divide each term by $5xy^3$, we obtain

$$\frac{15x^6y^4}{5xy^3}=3x^5y \qquad \frac{10x^5y^3z^2}{5xy^3}=2x^4z^2 \qquad \frac{-20xy^6z^3}{5xy^3}=-4y^3z^3$$

and can write

$$\begin{aligned} 15x^6y^4+10x^5y^3z^2-20xy^6z^3 &= 5xy^3(\quad ?\quad) \\ &= 5xy^3(3x^5y+2x^4z^2-4y^3z^3) \end{aligned}$$

Answer:

$$5xy^3(3x^5y+2x^4z^2-4y^3z^3)$$

EXAMPLE 3.1.10

Factor out the GCF:

$$24a^6b^2c^5+8a^7b^5c$$

Solution:

The GCF$(24,8)=8$, and the variable factors with smallest exponents are a^6, b^2, and c. Therefore, the GCF of all the terms is $8a^6b^2c$.

$$\begin{aligned} 24a^6b^2c^5+8a^7b^5c &= 8a^6b^2c(\quad ?\quad) \\ &= 8a^6b^2c(3c^4+ab^3) \end{aligned}$$

Answer:

$$8a^6b^2c(3c^4+ab^3)$$

Of course, not all polynomials with integer coefficients can be factored as a product of polynomials with integer coefficients other than 1 and itself. If this is the case, then we say that it is a prime polynomial.

EXAMPLE 3.1.11

Factor:

$$3x-5$$

Solution:

Prime: there are no polynomial factors other than 1 and itself.

Answer:

Prime

EXERCISE 3.1.2

Factor out the GCF:

$$16x^4y^3-8x^2y^5-4x^2y\ .$$

Answer

$4x^2y(4x^2y^2-2y^4-1)$

FACTOR BY GROUPING

In this section, we outline a technique for factoring polynomials with four terms. First, review some preliminary examples where the terms have a common binomial factor.

EXAMPLE 3.1.12

Factor:

$$5x(x-3)+2(x-3)\,.$$

Solution:

This expression is a binomial with terms $5x(x-3)$ and $2(x-3)$. In this case, $(x-3)$ is a common factor. Begin by factoring this common factor out:

$$5x(x-3)+2(x-3)=(x-3)(\quad ?\quad)$$

To determine the terms of the remaining factor, divide each term by $(x-3)$:

$$\frac{5x(x-3)}{(x-3)}=5x \qquad \frac{2(x-3)}{(x-3)}=2$$

This step is typically performed mentally. We have

$$\begin{aligned} 5x(x-3)+2(x-3) &= (x-3)(\quad ?\quad) \\ &= (x-3)(5x+2) \end{aligned}$$

Answer:

$$(x-3)(5x+2)$$

Recall that 1 is always a common factor. If the GCF is the same as a term, then the factor 1 remains after we factor out that GCF.

EXAMPLE 3.1.13

Factor:

$$3x(4x+1)-(4x+1)\,.$$

Solution:

Rewrite the second term $-(4x+1)$ as $-1(4x+1)$ and then factor out the common binomial factor $(4x+1)$.

$$\begin{aligned} 3x(4x+1)-(4x+1) &= 3x(4x+1)-1(4x+1) \\ &= (4x+1)(\quad ?\quad) \\ &= (4x+1)(3x-1) \end{aligned}$$

Answer:

$$(4x+1)(3x-1)$$

Remember that the goal for this section is to develop a technique that enables us to factor polynomials with four terms into a product of binomials. The intermediate step of this process looks like the previous two examples. For example, we wish to factor

$$5x^2-15x+2x-6$$

Begin by grouping the first two terms and the last two terms. Then factor out the GCF of each grouping:

$$\underbrace{5x^2-15x}_{group} \quad + \quad \underbrace{2x-6}_{group}$$

$$=5x(x-3)+2(x-3)$$

In this form, it is a binomial with a common binomial factor, (x−3).

$$\begin{aligned} &= (x-3)(\quad ?\quad) \\ &= (x-3)(5x+2) \end{aligned}$$

The steps that follow outline a technique for factoring four-term polynomials called factor by grouping.

EXAMPLE 3.1.14

Factor:

$$2x^3+4x^2+3x+6\,.$$

Solution:

Group terms in such a way as to obtain a binomial with common factors.

Step 1: Group the first two and last two terms and then factor out the GCF of each.

$$\underbrace{2x^3+4x^2}_{group} \quad + \quad \underbrace{3x+6}_{group}$$

The GCF of the first two terms is $2x^2$, and the GCF of the second two terms is 3.

$$\begin{aligned} 2x^3+4x^2+3x+6 = & \quad \underbrace{2x^3+4x^2}_{group} \quad + \quad \underbrace{3x+6}_{group} \\ & = 2x^2(\quad ? \quad)+3(\quad ? \quad) \\ & = 2x^2(x+2)+3(x+2) \end{aligned}$$

Step 2: At this point, the polynomial is a binomial. Factor out any factors common to both terms. Here $(x+2)$ is a common factor.

$$\begin{aligned} & = 2x^2(x+2)+3(x+2) \\ & = (x+2)(\quad ? \quad) \\ & = (x+2)(2x^2+3) \end{aligned}$$

Step 3: Optional check: multiply to verify that we obtain the original expression.

$$\begin{aligned} (x+2)(2x^2+3) & = 2x^3+3x+4x^2+6 \\ & = 2x^3+4x^2+3x+6 \quad \checkmark \end{aligned}$$

Answer:

$$(x+2)(2x^2+3)$$

EXAMPLE 3.1.15

Factor:

$$2a^3-3a^2+2a-3\ .$$

Solution:

The GCF of the first two terms is a^2 and the GCF of the second two terms is 1.

$$\begin{aligned} 2a^3-3a^2+2a-3 = & \quad \underbrace{2a^3-3a^2}_{group} \quad + \quad \underbrace{2a-3}_{group} \\ & = a^2(\quad ? \quad)+1(\quad ? \quad) \\ & = a^2(2a-3)+1(2a-3) \\ & = (2a-3)(\quad ? \quad) \\ & = (2a-3)(a^2+1) \end{aligned}$$

Answer:

$2a-3)(a^2+1)$. The check is left to the reader.

EXAMPLE 3.1.16

Factor:

$$6x^4-24x^3-5x+20\ .$$

Solution:

The GCF for the first group is $6x^3$. We have to choose 5 or -5 to factor out of the second group.

$$\begin{aligned} & \underbrace{6x^4-24x^3}_{group} \quad - \quad \underbrace{5x+20}_{group} \\ & = 6x^3(x-4)+5(-x+4) \qquad x \\ & = 6x^3(x-4)-5(x-4) \qquad \checkmark \end{aligned}$$

Factoring out a $+5$ does not result in a common binomial factor. If we choose to factor out -5, then we obtain a common binomial factor and can proceed. Note that when factoring out a negative number, we change the signs of the factored terms.

$$6x^4-24x^3-5x+20=\underbrace{6x^4-24x^3}_{group}-\underbrace{5x+20}_{group}$$
$$=6x^3(\quad ?\quad)-5(\quad ?\quad)$$
$$=6x^3(x-4)-5(x-4)$$
$$=(x-4)(\quad ?\quad)$$
$$=(x-4)(6x^3-5)$$

Answer:

$(x-4)(6x^3-5)$. The check is left to the reader.

NOTE

The sign of the leading coefficient in the second grouping usually indicates whether or not to factor out a negative factor. If that coefficient is positive, factor out a positive factor. If it is negative, factor out a negative factor.

When all the terms of a polynomial have a GCF other than 1, it is a best practice to factor that out before factoring by grouping.

EXAMPLE 3.1.17

Factor:

$$3y^4+9y^2-6y^3-18y\ .$$

Solution:

Here we notice that the greatest common factor of all the terms is $3y$. Begin by factoring out the GCF and then factor the result by grouping.

$$\begin{aligned}&3y^4+9y^2-6y^3-18y\\&=3y[y^3+3y-2y^2-6] && \textit{Factor out the GCF.}\\&=3y[y(y^2+3)-2(y^2+3)] && \textit{Factor by grouping.}\\&=3y[(y^2+3)(y-2)]\\&=3y(y^2+3)(y-2)\end{aligned}$$

Answer:

$$3y(y^2+3)(y-2)$$

Sometimes we must first rearrange the terms in order to obtain a common factor.

EXAMPLE 3.1.18

Factor:

$$ab-2a^2b+a^3-2b^3\ .$$

Solution:

Simply factoring the GCF out of the first group and last group does not yield a common binomial factor.

$$\underbrace{ab-2a^2b}_{group}+\underbrace{a^3-2b^3}_{group}$$
$$=ab(1-2a)+1(a^3-2b^3)$$

We must rearrange the terms, searching for a grouping that produces a common factor. In this example, we have a workable grouping if we switch the terms a^3 and ab.

$$ab-2a^2b+a^3-2b^3=\underbrace{a^3-2a^2b}_{group}+\underbrace{ab-2b^3}_{group}$$
$$=a^2(a-2b)+b(a-2b)$$
$$=(a-2b)(a^2+b)$$

Answer:

$$(a-2b)(a^2+b)$$

Not all factorable four-term polynomials can be factored with this technique. For example,

$$3x^3+5x^2-x+2$$

This four-term polynomial cannot be grouped in any way as to produce a common binomial factor. Despite this, the polynomial is not prime and can be written as a product of polynomials. It can be factored as follows:

$$3x^3+5x^2-x+2=(x+2)(3x^2-x+1)$$

Factoring such polynomials is something that we will learn to do as we move further along in our study of algebra. For now, we will limit our attempt to factor four-term polynomials to using the factor by grouping technique.

? EXERCISE 3.1.3

Factor:

$$x^3-x^2y-xy+y^2$$

Answer

$(x-y)(x^2-y)$

KEY TAKEAWAYS

- To find the greatest common factor (GCF) of any collection of natural numbers, first find the prime factorization of each. The GCF is the product of all the common prime factors.
- The GCF of two or more monomials is the product of the GCF of the coefficients and the common variable factors with the smallest power.
- If the terms of a polynomial have a greatest common factor, then factor out that GCF using the distributive property. Divide each term of the polynomial by the GCF to determine the terms of the remaining factor.
- Some four-term polynomials can be factored by grouping the first two terms and the last two terms. Factor out the GCF of each group and then factor out the common binomial factor.
- When factoring by grouping, you sometimes have to rearrange the terms to find a common binomial factor. After factoring out the GCF, the remaining binomial factors must be the same for the technique to work.
- Not all polynomials can be factored as the product of polynomials with integer coefficients. In this case, we call it a prime polynomial.

? EXERCISE 3.1.4 GCF OF NATURAL NUMBERS

Give the prime factorization of each number and determine the GCF.

1. $18,24$
2. $45,75$
3. $72,60$
4. $168,175$
5. $144,245$
6. $15,50,60$
7. $14,63,70$
8. $12,48,125$
9. $60,72,900$
10. $252,336,360$

Answer

1. $18=2\cdot 32, 24=23\cdot 3,\ \text{GCF}=6$
3. $72=23\cdot 32, 60=22\cdot 3\cdot 5,\ \text{GCF}=12$
5. $144=24\cdot 32, 245=5\cdot 72,\ \text{GCF}=1$
7. $14=2\cdot 7, 63=32\cdot 7, 70=2\cdot 5\cdot 7,\ \text{GCF}=7$
9. $60=22\cdot 3\cdot 5, 72=23\cdot 32, 900=22\cdot 32\cdot 52,\ \text{GCF}=12$

EXERCISE 3.1.5 GCF OF VARIABLE EXPRESSIONS

Determine the GCF of all the terms.

1. $15x, 30$
2. $14x, 21$
3. $45x^4, 8x^3$
4. $36x^5, 35y^2$
5. $6x, 27x, 36x$
6. $12x^3, 4x^2, 6x$
7. $12x^2y, 60xy^3$
8. $7ab^2, 2a^2b, 3a^3b^3$
9. $6a^2b^2, 18a^3b^2, 9ab^2$
10. $15x(x+2), 9(x+2)$
11. $20x(2x-1), 16(2x-1)$
12. $20x^3(x+y)^5, 10x^5(x+y)^2$

Answer

1. 15
3. x^3
5. $3x$
7. $12xy$
9. $3ab^2$
11. $4(2x-1)$

EXERCISE 3.1.6 FACTORING OUT THE GCF

Given the GCF, determine the missing factor.

1. $25x^2+10x = 5x(\quad ?\quad)$
2. $12y^5+7y^2 = y^2(\quad ?\quad)$
3. $22x^4-121x^2+11x = 11x(\quad ?\quad)$
4. $30y^3-45y^2-3y = 3y(\quad ?\quad)$
5. $36a^5b^7-60a^6b^5 = 12a^5b^5(\quad ?\quad)$
6. $24x^2y+48xy^2-12xy = 12xy(\quad ?\quad)$

Answer

1. $(5x+2)$
3. $(2x^3-11x+1)$
5. $(3b^2-5a)$

EXERCISE 3.1.7 FACTORING OUT THE GCF

Factor out the GCF.

1. $4x-8$
2. $27x-9$
3. $3x-18$
4. $5x-10$
5. $25x-16$
6. $72x-35$
7. $15x^2+30x$
8. $14a^2-7a$
9. $30a^5-10a^2$
10. $8x^4-16x^2$
11. $5x^6+x^3$

12. $3x^7 - 9x^5$
13. $18a^2 + 30a - 6$
14. $24a^2 - 36a - 12$
15. $27x^3 - 6x^2 + 3x$
16. $8x^3 - 12x^2 + 2x$
17. $9x^4 + 18x^3 - 3x^2$
18. $12y^4 - 16y^3 + 20y^2$
19. $7x^5 - 21x^3 - 14x^2 + 28x$
20. $36y^{10} + 12y^8 - 18y^4 - 6y^3$
21. $12x^5y^2 - 8x^3y$
22. $125a^8b^4c^3 - 25a^2b^3c^3$
23. $6x^4y^3 - 4x^3y^2 + 8x^2y$
24. $15x^4y^2 - 30x^3y^3 + 15x^2y^4$
25. $81x^7y^6z^2 - 18x^2y^8z^4 + 9x^2y^5z^2$
26. $4x^5y^4z^9 + 26x^5y^3z^4 - 14x^6y^8z^5$
27. $2x(x-3) + 5(x-3)$
28. $3x(2x+1) - 4(2x+1)$
29. $5x(5x+2) - (5x+2)$
30. $2x(3x+4) + (3x+4)$
31. $x^2(4x-7) - 5(4x-7)$
32. $(x+6) - 3x^2(x+6)$
33. $(a+b)^2 - 3a(a+b)^2$
34. $(ab+2)^3 + 3ab(ab+2)^3$
35. $7x(x+7)^5 + 14x^2(x+7)^5$
36. $36x^5(3x+2)^4 - 12x^3(3x+2)^4$

Answer

1. $4(x-2)$
3. $3(x-6)$
5. Prime
7. $15x(x+2)$
9. $10a^2(3a^3-1)$
11. $x^3(5x^3+1)$
13. $6(3a^2+5a-1)$
15. $3x(9x^2-2x+1)$
17. $3x^2(3x^2+6x-1)$
19. $7x(x^4-3x^2-2x+4)$
21. $4x^3y(3x^2y-2)$
23. $2x^2y(3x^2y^2-2xy+4)$
25. $9x^2y^5z^2(9x^5y-2y^3z^2+1)$
27. $(x-3)(2x+5)$
29. $(5x+2)(5x-1)$
31. $(4x-7)(x^2-5)$
33. $(a+b)^2(1-3a)$
35. $7x(x+7)^5(1+2x)$

? EXERCISE 3.1.8 FACTORING OUT THE GCF

Are the following factored correctly? Check by multiplying.

1. $4x^2 - 16x = 4x(x-4)$
2. $3a^3 - 3a = 3a(a^2)$

3. $3x^3 - 5x^6 = x^3(3 - x^2)$
4. $5x^3 - 10x^4 + 15x^5 = 5x^3(1 - 2x + 3x^2)$
5. $x^3 - x^2 + x = x(x^2 - x)$
6. $12x^4y^3 - 16x^5y^2 + 8x^6y^7 = 4x^4y^2(3y - 4x + 2x^2y^5)$

Answer

1. Yes
3. No
5. No

? EXERCISE 3.1.9 FACTORING OUT THE GCF

Use polynomial long division to show that the given factor divides the polynomial evenly.

1. Show that $(x - 1)$ is a factor of $(2x^3 - 5x^2 + 4x - 1)$.
2. Show that $(x + 3)$ is a factor of $(3x^3 + 7x^2 - 4x + 6)$.
3. Show that $(3x - 2)$ is a factor of $(3x^3 + 4x^2 - 7x + 2)$.
4. Show that $(2x + 1)$ is a factor of $(2x^3 - 5x^2 + x + 2)$.
5. The height in feet of an object tossed into the air is given by the function $h(t) = -16t^2 + 32t$, where t is the time in seconds after it is tossed. Write the function in factored form.
6. The height in feet of an object dropped from a 16-foot ladder is given by the function $h(t) = -16t^2 + 16$, where t is the time in seconds after it is tossed. Write the function in factored form.
7. The surface area of a cylinder is given by the formula $SA = 2\pi r^2 + 2\pi rh$, where r represents the radius of the base and h is the height of the cylinder. Express this formula in factored form.

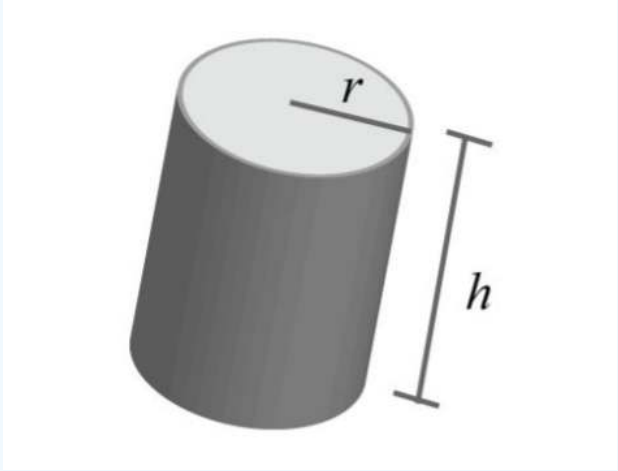


***Figure* 3.1.1**

8. The surface area of a cone is given by the formula $SA = \pi r^2 + \pi rs$, where r represents the radius of the base and s represents the slant height. Express this formula in factored form.

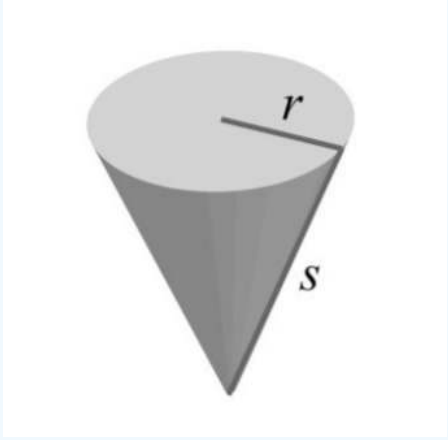


Figure **3.1.2**

Answer

5. $h(t) = -16t(t-2)$
7. $SA = 2\pi r(r+h)$

EXERCISE 3.1.10 FACTOR BY GROUPING

Factor by grouping.

1. $x^2 - 10x + 2x - 20$
2. $x^2 - 6x - 3x + 18$
3. $x^3 + 2x^2 + 2x + 4$
4. $x^3 - 3x^2 + 5x - 15$
5. $x^3 + 7x^2 - 2x - 14$
6. $2x^3 + 2x^2 - x - 1$
7. $x^3 - 5x^2 + 4x - 20$
8. $6x^3 - 3x^2 + 2x - 1$
9. $9x^3 - 6x^2 - 3x + 2$
10. $2x^4 - x^3 - 6x + 3$
11. $x^5 + x^3 + 2x^2 + 2$
12. $6x^5 - 4x^3 - 9x^2 + 6$
13. $3a^3b + 3ab^2 + 2a^2 + 2b$
14. $2a^3 + 2ab^3 - 3a^2b - 3b^4$
15. $2a^3 - a^2b^2 - 2ab + b^3$
16. $a^4 - 3a^3b^2 + ab^2 - 3b^4$
17. $3a^2b^4 - 6b^3 - a^2b + 2$
18. $3x^3 + 2y^3 + x^3y^3 + 6$
19. $-3x^3 - 5y^3 + x^3y^3 + 15$
20. $2x^3y^3 + 2 - y^3 - 4x^3$
21. $3x^2 - y^3 + xy^2 - 3xy$
22. $2x^2 + y^3 - 2xy - xy^2$

Answer

1. $(x-10)(x+2)$
3. $(x+2)(x^2+2)$
5. $(x+7)(x^2-2)$
7. $(x-5)(x^2+4)$
9. $(3x-2)(3x^2-1)$
11. $(x^2+1)(x^3+2)$
13. $(a^2+b)(3ab+2)$
15. $(a^2-b)(2a-b^2)$
17. $(3b^3-1)(a^2b-2)$
19. $(x^3-5)(y^3-3)$
21. $(x-y)(3x+y^2)$

EXERCISE 3.1.11 FACTOR BY GROUPING

Factor out the GCF first and then factor by grouping.

1. $5x^2 - 35x - 15x + 105$
2. $12x^2 - 30x - 12x + 30$
3. $2x^3 + 6x^2 - 10x - 30$
4. $6x^3 - 3x^2 - 42x + 21$

5. $4x^4+4x^3-12x^2-12x$
6. $-9x^4+6x^3-45x^2+30x$
7. $-12x^5+4x^4+6x^3-2x^2$
8. $24x^5-36x^4+8x^3-12x^2$
9. $24a^3b^2-60a^3b+40ab^2-100ab$
10. $a^4b^2-2a^3b^3+a^2b^3-2ab^4$

Answer

1. $5(x-7)(x-3)$
3. $2(x+3)(x^2-5)$
5. $4x(x+1)(x^2-3)$
7. $-2x^2(3x-1)(2x^2-1)$
9. $4ab(3a^2+5)(2b-5)$

? EXERCISE 3.1.12 DISCUSSION BOARD TOPICS

1. Research the Euclidean algorithm for finding the GCF of two natural numbers. Give an example that illustrates the steps.
2. Research and discuss the contributions of Euclid of Alexandria.
3. Explain what factoring is and give an example.
4. Is $5x(x+2)-3(x+2)$ fully factored? Explain.
5. Make up a factoring problem of your own and provide the answer. Post the problem and the solution on the discussion board.

Answer

1. Answers may vary
3. Answers may vary
5. Answers may vary

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3.2: FACTORING TRINOMIALS OF THE FORM X²+BX+C

LEARNING OBJECTIVES

- Factor trinomials of the form $x^2 + bx + c$.
- Factor trinomials using the AC method.

FACTORING TRINOMIALS OF THE FORM $x^2 + bx + c$

Some trinomials of the form $x^2 + bx + c$ can be factored as a product of binomials. For example,

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

We can verify this factorization by multiplying:

$$\begin{aligned}(x + 2)(x + 5) &= x^2 + 5x + 2x + 10\\ &= x^2 + 7x + 10 \quad \checkmark\end{aligned}$$

Factoring trinomials requires that we work the distributive process in reverse. Notice that the product of the first terms of each binomial is equal to the first term of the trinomial.

$$x \cdot x = x^2$$

The middle term of the trinomial, $7x$, is the sum of the products of the outer and inner terms of the binomials:

$$5x + 2x = 7x$$

And the product of the last terms of each binomial is equal to the last term of the trinomial.

$$2 \cdot 5 = 10$$

This can be visually interpreted as follows:

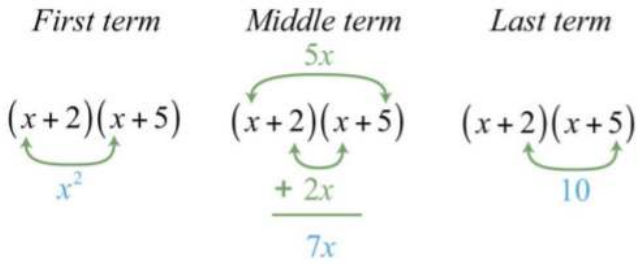


Figure 3.2.1

If a trinomial of this type factors, then these relationships will be true:

$$\begin{aligned}x^2 + bx + c &= (x + m)(x + n)\\ &= x^2 + nx + mx + mn\\ &= x^2 + (n + m)x + mn\end{aligned}$$

This gives us

$$b = n + m \quad \text{and} \quad c = mn \tag{3.2.1}$$

In short, if the leading coefficient of a factorable trinomial is one, then the factors of the last term must add up to the coefficient of the middle term. This observation is the key to factoring trinomials using the technique known astrial and error (or guess and check). The steps are outlined in the following example.

EXAMPLE 3.2.1

Factor:

$$x^2 + 7x + 12$$

Solution:

Note that the polynomial to be factored has three terms; it is a trinomial with a leading coefficient of 1. Use trial and error to factor as follows:

Step 1: Write two sets of blank parentheses. If a trinomial of this form factors, then it will factor into two linear binomial factors.

$$x^2+7x+12=(\quad)(\quad)$$

Step 2: Write the factors of the first term in the first space of each set of parentheses. In this case, factor $x^2=x\cdot x$.

$$x^2+7x+12=(x\quad)(x\quad)$$

Step 3: Determine the factors of the last term whose sum equals the coefficient of the middle term. To do this, list all of the factorizations of 12 and search for factors whose sum equals the coefficient of the middle term, 7.

$$\begin{aligned} 12&=1\cdot 12 &&\rightarrow 1+12=13\\ &=2\cdot 6 &&\rightarrow 2+6=8\\ &=3\cdot 4 &&\rightarrow 3+4=7 \end{aligned}$$

Choose $12=3\cdot 4$ because $3+4=7$.

Step 4: Write in the last term of each binomial using the factors determined in the previous step.

$$x^2+7x+12=(x+3)(x+4)$$

Step 5: Check by multiplying the two binomials.

$$\begin{aligned} (x+3)(x+4)&=x^2+4x+3x+12\\ &=x^2+7x+12\quad\checkmark \end{aligned}$$

Answer:

$$(x+3)(x+4)$$

Since multiplication is commutative, the order of the factors does not matter.

$$\begin{aligned} x^2+7x+12&=(x+3)(x+4)\\ &=(x+4)(x+3) \end{aligned}$$

If the last term of the trinomial is positive, then either both of the constant factors must be negative or both must be positive. Therefore, when looking at the list of factorizations of the last term, we are searching for sums that are equal to the coefficient of the middle term.

EXAMPLE 3.2.2

Factor:

$$x^2-9x+20\,.$$

Solution:

First, factor $x^2=x\cdot x$.

$$x^2-9x+20=(x\quad ?)(x\quad ?)$$

Next, determine which factors of 20 add up to -9:

$$\begin{aligned} 20&=1\cdot 20\\ &-2\cdot 10\\ &=4\cdot 5 \end{aligned}$$

In this case, choose -4 and -5 because $(-4)(-5)=+20$ and $-4+(-5)=-9$.

$$x^2-9x+20=(x-4)(x-5)$$

Check.

$$\begin{aligned} (x-4)(x-5)&=x^2-5x-4x+20\\ &=x^2-9x+20\quad\checkmark \end{aligned}$$

Answer:

$$(x-4)(x-5)$$

If the last term of the trinomial is negative, then one of its factors must be negative. In this case, search the list of factorizations of the last term for differences that equal the coefficient of the middle term.

EXAMPLE 3.2.3

Factor:

$$x^2-4x-12\,.$$

Solution:

Begin by factoring the first term $x^2 = x \cdot x$.

$$x^2 - 4x - 12 = (x \quad ?)(x \quad ?)$$

The factors of 12 are listed below. In this example, we are looking for factors whose difference is -4.

Here choose the factors 2 and -6 because the coefficient of the middle term, -4, is obtained if we add $2 + (-6)$.

$x^2 - 4x - 12 = (x + 2)(x - 6)$

Multiply to check.

Answer:

$(x + 2)(x - 6)$

Often our first guess will not produce a correct factorization. This process may require repeated trials. For this reason, the check is very important and is not optional.

EXAMPLE 3.2.4

Factor:

$$x^2 + 5x - 6\,.$$

Solution:

The first term of this trinomial x^2 factors as $x \cdot x$.

\(x^{2}+5x-6=(x\quad ?)(x\quad ?)

Consider the factors of 6:

$$\begin{aligned} 6 &= 1 \cdot 6 \\ &= 2 \cdot 3 \end{aligned}$$

Suppose we choose the factors 2 and 3 because $2 + 3 = 5$, the coefficient of the middle term. Then we have the following incorrect factorization:

$$x^2 + 5x - 6 \stackrel{?}{=} (x + 2)(x + 3)$$

When we multiply to check, we find the error.

$$\begin{aligned} (x + 2)(x + 3) &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6 \qquad x \end{aligned}$$

In this case, the middle term is correct but the last term is not. Since the last term in the original expression is negative, we need to choose factors that are opposite in sign. Therefore, we must try again. This time we choose the factors -1 and 6 because $-1 + 6 = 5$.

$$x^2 + 5x - 6 = (x - 1)(x + 6)$$

Now the check shows that this factorization is correct.

Answer:

$(x - 1)(x + 6)$

If we choose the factors wisely, then we can reduce much of the guesswork in this process. However, if a guess is not correct, do not get discouraged; just try a different set of factors.

EXAMPLE 3.2.5

Factor:

$$x^2 + 3x + 20\,.$$

Solution:

$$\begin{aligned} x^2 + 3x + 20 &= (x \quad ?)(x \quad ?) \\ 20 &= 1 \cdot 20 \\ &= 2 \cdot 10 \\ &= 4 \cdot 5 \end{aligned}$$

Here there are no factors of 20 whose sum is 3. Therefore, the original trinomial cannot be factored as a product of two binomials. This trinomial is prime.

Answer:

Prime

? EXERCISE 3.2.1

Factor:

$$x^2 - 13x - 30\,.$$

Answer

$(x+2)(x-15)$

The techniques described can also be used to factor trinomials with more than one variable.

✔ EXAMPLE 3.2.6

Factor:

$$x^2 - 14xy - 72y^2$$

Solution:

The first term x^2 factors as $x \cdot x$.

$$x^2 + xy - 72y^2 = (x \quad ?)(x \quad ?)$$

Next, look for factors of the coefficient of the last term, 72, whose sum is -14.

$$\begin{aligned} 72 &= 1 \cdot 72 \\ &= 2 \cdot 36 \\ &= 3 \cdot 24 \\ &= 4 \cdot 18 \qquad \rightarrow 4 + (-18) = -14 \\ &= 6 \cdot 12 \\ &= 8 \cdot 9 \end{aligned}$$

Therefore, the coefficient of the last term can be factored $-72 = 4(-18)$, where $4 + (-18) = -14$. Because the last term has a variable factor of y^2, factor $72y^2$ as $4y(-18y)$ and try the following factorization:

$$x^2 - 14xy - 72y^2 = (x + 4y)(x - 18y)$$

Multiply to check.

$$\begin{aligned} (x+4y)(x-18y) &= x^2 - 18xy + 4xy - 72y^2 \\ &= x^2 - 14xy - 72y^2 \quad \checkmark \end{aligned}$$

Visually, we have the following:

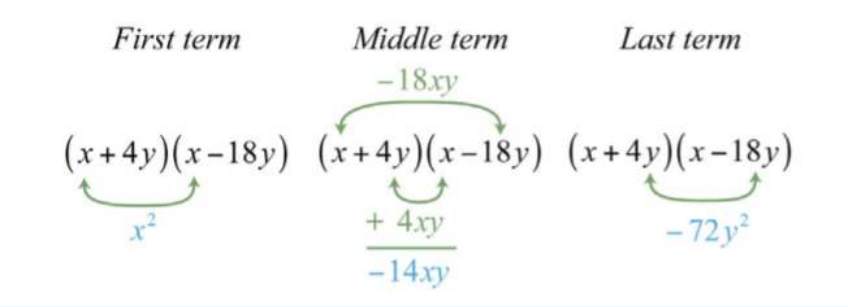


Figure 3.2.2

Answer:

$$(x + 4y)(x - 18y)$$

? EXERCISE 3.2.2

Factor:

$$x^2y^2 + 9xy - 10\,.$$

Answer

$(xy-1)(xy+10)$

FACTORING USING THE AC METHOD

An alternate technique for factoring trinomials, called the AC method, makes use of the grouping method for factoring four-term polynomials. If a trinomial in the form ax^2+bx+c can be factored, then the middle term, bx, can be replaced with two terms with coefficients whose sum is b and product ac. This substitution results in an equivalent expression with four terms that can be factored by grouping. The steps are outlined in the following example.

EXAMPLE 3.2.7

Factor using the AC method:

$$x^2-x-30\,.$$

Solution:

In this example $a=1, b=-1$, and $c=-30$.

Step 1: Determine the product ac.

Step 2: Find factors of ac whose sum equals the coefficient of the middle term, b.

$$\begin{aligned} -30 &= 1(-30)\\ &= 2(-15)\\ &= 3(-10)\\ &= 5(-5) \qquad \rightarrow 5+(-6)=-1 \end{aligned}$$

We can see that the sum of the factors 5 and -6 is equal to the coefficient of the middle term, -1.

Step 3: Use the factors as coefficients for the terms that replace the middle term. Here $-x=-6x+5x$. Write

$$x^2-x-30=x^2-6x+5x-30$$

Step 4: Factor the equivalent expression by grouping.

$$\begin{aligned} x^2-x-30 &= \underbrace{x^2-6x}_{group} + \underbrace{5x-20}_{group}\\ &= x(x-6)+5(x-6)\\ &= (x-6)(x+5) \end{aligned}$$

Answer:

$$(x-6)(x+5)$$

Notice that the AC method is consistent with the trial and error method. Both methods require that $b=m+n$, where $c=mn$. In the example above, $-30=(-6)(5)$ and $-1=(-6)+5$. The only difference between the methods, when the leading coefficient is 1, is in the process used to obtain the final factorization.

EXAMPLE 3.2.8

Factor:

$$y^2-14x+48\,.$$

Solution:

Here $ac=48$ and we search for factors whose sum is -14.

$$\begin{aligned} 48 &= 1(48)\\ &= 2(24)\\ &= 3(16)\\ &= 4(12)\\ &= 6(8) \qquad \rightarrow -6+(-8)=-14 \end{aligned}$$

Therefore, $-14x=-6x-8x$. Substitute the new terms and factor by grouping.

$$\begin{aligned} y^2-14x+48 &= x^2-6x-8x+48\\ &= x(x-6)-8(x-6)\\ &= (x-6)(x-8) \end{aligned}$$

Answer:

$(x-6)(x-8)$. The check is left to the reader.

At this point, it is recommended that the reader stop and factor as many trinomials of the form x^2+bx+c as time allows before moving on to the next section. Factoring trinomials is one of the more important skills that we learn in this course and should be mastered.

KEY TAKEAWAYS

- Factor a trinomial by systematically guessing what factors give two binomials whose product is the original trinomial.
- If a trinomial of the form x^2+bx+c factors into the product of two binomials, then the coefficient of the middle term is the sum of factors of the last term.
- Not all trinomials can be factored as the product of binomials with integer coefficients. In this case, we call it a prime trinomial.
- Factoring is one of the more important skills required in algebra. For this reason, you should practice working as many problems as it takes to become proficient.

? EXERCISE 3.2.3 FACTORING TRINOMIALS WITH LEADING COEFFICIENT 1

Are the following factored correctly? Check by multiplying.

1. $x^2+5x-6=(x+2)(x+3)$
2. $x^2+6x+16=(x+8)(x-2)$
3. $y^2+2y-8=(y+4)(y-2)$
4. $y^2-10y+21=(y-3)(y-7)$
5. $a^2-10a+25=(a-5)^2$
6. $a^2+6a+9=(a-3)^2$
7. $x^2+10x-25=(x+5)(x-5)$
8. $x^2+5x+14=(x-2)(x+7)$
9. $y^2+50y-600=(y+60)(y-10)$
10. $y^2-3y+2=(y-2)(y-1)$

Answer

1. No
3. Yes
5. Yes
7. No
9. Yes

? EXERCISE 3.2.4 FACTORING TRINOMIALS WITH LEADING COEFFICIENT 1

Factor.

1. x^2+6x+8
2. x^2+4x+3
3. x^2+3x+2
4. x^2+x-2
5. $x^2+3x-10$
6. $x^2-2x-35$
7. $x^2-13x+12$
8. $x^2-15x+36$
9. $x^2-12x+36$
10. $x^2+18x+81$
11. x^2-2x+1
12. $x^2-18x+81$
13. x^2+5x+5
14. x^2-4x+6
15. $x^2-20x+91$
16. $x^2+20x+91$

17. $x^2 - 2x - 48$
18. $x^2 + 16x + 48$
19. $x^2 + 22x + 48$
20. $x^2 + 22x - 48$
21. $y^2 + 7y + 12$
22. $y^2 + 8y - 20$
23. $y^2 - 16y + 60$
24. $y^2 - 31y - 32$
25. $a^2 - 11a - 42$
26. $a^2 - 14a - 51$
27. $a^2 + 26a + 25$
28. $a^2 - 22a + 120$
29. $a^2 + 4a - 1$
30. $a^2 - 6a + 2$
31. $y^2 - 14x + 40$
32. $y^2 - 4y - 96$
33. $x^2 - 2xy + y^2$
34. $x^2 + 2xy + y^2$
35. $x^2 - 16xy + 15y^2$
36. $x^2 - 4xy - 32y^2$
37. $x^2 + 2xy - 15y^2$
38. $x^2 - 12xy + 32y^2$
39. $x^2y^2 - 6xy + 9$
40. $x^2y^2 + 25xy - 26$
41. $a^2 + 4ab + 4b^2$
42. $a^2 - 19ab - 20b^2$
43. $a^2 - ab - 12b^2$
44. $a^2 - 10ab - 56b^2$
45. $a^2b^2 - 2ab - 15$
46. $a^2b^2 - 10ab + 24$
47. The area of a square is given by the function $A(x) = x^2 - 14x + 49$, where x is measured in meters. Rewrite this function in factored form.
48. The area of a square is given by the function $A(x) = x^2 + 16x + 64$, where x is measured in meters. Rewrite this function in factored form.

Answer

1. $(x + 2)(x + 4)$
3. $(x + 1)(x + 2)$
5. $(x - 2)(x + 5)$
7. $(x - 1)(x - 12)$
9. $(x - 6)^2$
11. $(x - 1)^2$
13. Prime
15. $(x - 7)(x - 13)$
17. $(x + 6)(x - 8)$
19. Prime
21. $(y + 3)(y + 4)$
23. $(y - 6)(y - 10)$
25. $(a + 3)(a - 14)$
27. $(a + 1)(a + 25)$
29. Prime
31. $(y - 10)(y - 4)$

33. $(x-y)^2$
35. $(x-15y)(x-y)$
37. $(x+5y)(x-3y)$
39. $(xy-3)^2$
41. $(a+2b)^2$
43. $(a+3b)(a-4b)$
45. $(ab+3)(ab-5)$
45. $A(x)=(x-7)^2$

EXERCISE 3.2.5 FACTOR USING THE AC METHOD

Factor using the AC method.

1. $x^2+5x-14$
2. $x^2+2x-48$
3. x^2-9x+8
4. $x^2-14x+24$
5. x^2-x-72
6. x^2-x-90
7. $y^2-8y+16$
8. $y^2+16y+64$
9. $x^2+4x+12$
10. x^2+5x-8
11. $x^2+3xy-18y2$
12. $x^2-15xy+50y^2$

Answer

1. $(x+7)(x-2)$
3. $(x-8)(x-1)$
5. $(x-9)(x+8)$
7. $(y-4)^2$
9. Prime
11. $(x+6y)(x-3y)$

EXERCISE 3.2.6 DISCUSSION BOARD TOPICS

1. Create your own trinomial of the form x^2+bx+c that factors. Share it along with the solution on the discussion board.
2. Write out your own list of steps for factoring a trinomial of the form x^2+bx+c and share your steps on the discussion board.
3. Create a trinomial that does not factor and share it along with an explanation of why it does not factor.

Answer

1. Answers may vary
3. Answers may vary

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3.3: FACTORING TRINOMIALS OF THE FORM AX²+BX+C

LEARNING OBJECTIVES

- Factor trinomials of the form ax^2+bx+c.
- Factor trinomials with a common factor.

FACTORING TRINOMIALS OF THE FORM ax^2+bx+c

Factoring trinomials of the form ax^2+bx+c can be challenging because the middle term is affected by the factors of both a and c. To illustrate this, consider the following factored trinomial:

$$10x^2+17x+3=(2x+3)(5x+1)$$

We can multiply to verify that this is the correct factorization.

$$\begin{aligned}(2x+3)(5x+1) &= 10x^2+2x+15x+3\\ &= 10x^2+17x+3 \quad \checkmark\end{aligned}$$

As we have seen before, the product of the first terms of each binomial is equal to the first term of the trinomial. The middle term of the trinomial is the sum of the products of the outer and inner terms of the binomials. The product of the last terms of each binomial is equal to the last term of the trinomial. Visually, we have the following:

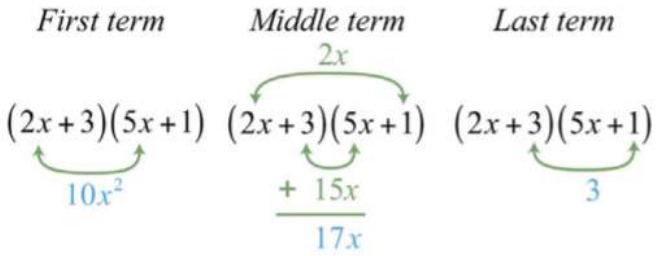


Figure 3.3.1

In general,

$$\begin{aligned}ax^2+bx+c &= (px+m)(qx+n)\\ &= pqx^2+pnx+qmx+mn\\ &= pqx^2+(pn+qm)x+mn\end{aligned}$$

This gives us,

$$a=pq \quad \text{and} \quad b=pn+qm, \quad \text{where} \quad c=mn \tag{3.3.1}$$

In short, when the leading coefficient of a trinomial is something other than 1, there will be more to consider when determining the factors using the trial and error method. The key lies in the understanding of how the middle term is obtained. Multiply $(2x+5)(3x+7)$ and carefully follow the formation of the middle term.

$$\begin{aligned}(2x+5)(3x+7) &= 2x\cdot 3x \quad \underbrace{+2x\cdot 7+5\cdot 3x}_{middle\ term} \quad +5\cdot 7\\ &= 6x^2+14x+15x+35\\ &= 6x^2+29x+35\end{aligned}$$

If we think of the FOIL method for multiplying binomials, then the middle term results from the sum of the inner product and the outer product. In this case, $14x+15x=29x$, as illustrated below:

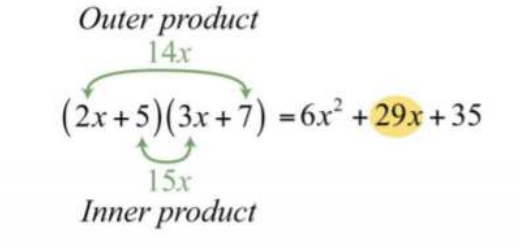


Figure 3.3.2

For this reason, we need to look for products of the factors of the first and last terms whose sum is equal to the coefficient of the middle term. For example, to factor $6x^2 + 29x + 35$, look at the factors of 6 and 35.

$$\begin{aligned} 6 &= 1 \cdot 6 & 35 &= 1 \cdot 35 \\ &= 2 \cdot 3 & &= 5 \cdot 7 \end{aligned}$$

The combination that produces the coefficient of the middle term is $2 \cdot 7 + 3 \cdot 5 = 14 + 15 = 29$. Make sure that the outer terms have coefficients 2 and 7, and that the inner terms have coefficients 5 and 3. Use this information to factor the trinomial:

$$\begin{aligned} 6x^2 + 29x + 35 &= (2x \quad ?)(3x \quad ?) \\ &= (2x + 5)(3x + 7) \end{aligned}$$

EXAMPLE 3.3.1

Factor:

$$3x^2 + 7x + 2\,.$$

Solution:

Since the leading coefficient and the last term are both prime, there is only one way to factor each.

$$3 = 1 \cdot 3 \quad \text{and} \quad 2 = 1 \cdot 2$$

Begin by writing the factors of the first term, $3x^2$, as follows:

$$3x^2 + 7x + 2 = (x \quad ?)(3x \quad ?)$$

The middle and last term are both positive; therefore, the factors of 2 are chosen as positive numbers. In this case, the only choice is in which grouping to place these factors.

$$(x + 1)(3x + 2) \quad \text{or} \quad (x + 2)(3x + 1)$$

Determine which grouping is correct by multiplying each expression.

$$\begin{aligned} (x + 1)(3x + 2) &= 3x^2 + 2x + 3x + 2 \\ &= 3x^2 + 5x + 2 \quad x \\ (x + 2)(3x + 1) &= 3x^2 + x + 6x + 2 \\ &= 3x^2 + 7x + 2 \quad \checkmark \end{aligned}$$

Notice that these products differ only in their middle terms. Also, notice that the middle term is the sum of the inner and outer product, as illustrated below:

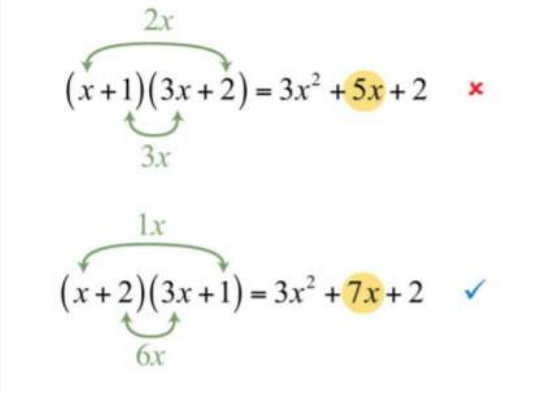


Figure 3.3.3

Answer:

$$(x+2)(3x+1)$$

EXAMPLE 3.3.2

Factor:

$$12x^2+38x+20\,.$$

Solution:

First, consider the factors of the first and last terms.

$$\begin{aligned} 12 &= 1\cdot 12 & 20 &= 1\cdot 20\\ &= 2\cdot 6 & &= 2\cdot 10\\ &= 3\cdot 4 & &= 4\cdot 5 \end{aligned}$$

We search for products of factors whose sum equals the coefficient of the middle term, 38. For brevity, the thought process is illustrated starting with the factors 2 and 6. Factoring begins at this point with the first term.

$$12x^2+38x+20 = (2x \quad ?)(6x \quad ?)$$

We search for factors of 20 that along with the factors of 12 produce a middle term of 38x

Factors of 20	*Possible*	*factorization*
$1\cdot 20$	$(2x+1)(6x+20)$	*middle term* $\Rightarrow 46x$
$1\cdot 20$	$(2x+20)(6x+1)$	*middle term* $\Rightarrow 122x$
$2\cdot 10$	$(2x+2)(6x+10)$	*middle term* $\Rightarrow 32x$
$2\cdot 10$	$(2x+10)(6x+2)$	*middle term* $\Rightarrow 64x$
$4\cdot 5$	$(2x+4)(6x+5)$	*middle term* $\Rightarrow 34x$
$4\cdot 5$	$(2x+5)(6x+4)$	*middle term* $\Rightarrow 38x$ ✓

Here the last combination produces a middle term of $38x$.

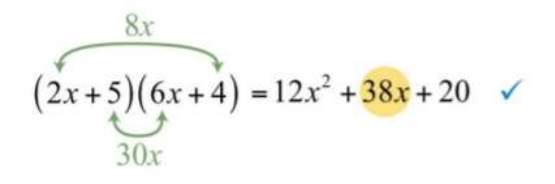


Figure 3.3.4

Answer:

$$(2x+5)(6x+4)$$

EXAMPLE 3.3.3

Factor:

$$10x^2-23x+6\,.$$

Solution

First, consider the factors of the first and last terms.

$$\begin{aligned} 10 &= 1\cdot 10 & 6 &= 1\cdot 6\\ &= 2\cdot 5 & &= 2\cdot 3 \end{aligned}$$

We are searching for products of factors whose sum equals the coefficient of the middle term, -23. Factoring begins at this point with two sets of blank parentheses:

$$10x^2-23x+6 = (\quad)(\quad)$$

Since the last term is positive and the middle term is negative, we know that both factors of the last term must be negative. Here we list all possible combinations with the factors of $10x^2 = 2x\cdot 5x$.

$$10x^2-23x+6 = (2x \quad ?)(5x \quad ?)$$

$$(2x-1)(5x-6) \quad middle\ term \Rightarrow -17x$$
$$(2x-6)(5x-1) \quad middle\ term \Rightarrow -32x$$
$$(2x-2)(5x-3) \quad middle\ term \Rightarrow -16x$$
$$(2x-3)(5x-2) \quad middle\ term \Rightarrow -19x$$

There is no combination that produces a middle term of $-23x$. We then move on to the factors of $10x^2 = 10x \cdot x$ and list all possible combinations:

$$10x^2 - 23x + 6 = (10x \quad ?)(x \quad ?)$$
$$(10x-1)(x-6) \quad middle\ term \Rightarrow -61x$$
$$(10x-6)(x-1) \quad middle\ term \Rightarrow -162x$$
$$(10x-2)(x-3) \quad middle\ term \Rightarrow -32x$$
$$(10x-3)(x-2) \quad middle\ term \Rightarrow -23x \quad \checkmark$$

And we can write

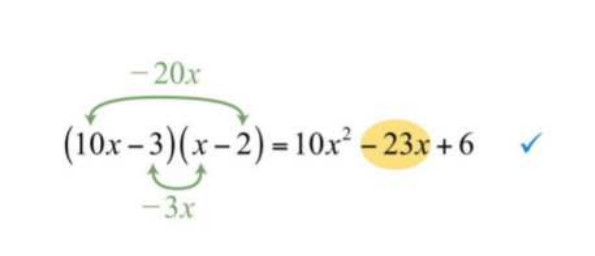


Figure 3.3.5

Answer:

$(10x-3)(x-2)$. The complete check is left to the reader.

We can reduce much of the guesswork involved in factoring trinomials if we consider all of the factors of the first and last terms and their products.

EXAMPLE 3.3.4

Factor:

$$5x^2 + 38x - 16\,.$$

Solution:

We begin with the factors of 5 and 16.

$$5 = 1 \cdot 5 \qquad \begin{aligned} 16 &= 1 \cdot 16 \\ &= 2 \cdot 8 \\ &= 4 \cdot 4 \end{aligned}$$

Since the leading coefficient is prime, we can begin with the following:

$$5x^2 + 38x - 16 = (x \quad ?)(5x \quad ?)$$

We look for products of the factors of 5 and 16 that could possibly add to 38.

Factors of 16	*Possible*	*products*	
$1 \cdot 16$	$1 \cdot 1$ *and* $5 \cdot 16$	*products* $\Rightarrow$ 1 *and* 80	
$1 \cdot 16$	$1 \cdot 16$ *and* $5 \cdot 1$	*products* $\Rightarrow$ 16 *and* 5	
$2 \cdot 8$	$1 \cdot 2$ *and* $5 \cdot 8$	*products* $\Rightarrow$ 2 *and* 40	$\checkmark$
$2 \cdot 8$	$1 \cdot 8$ *and* $5 \cdot 2$	*products* $\Rightarrow$ 8 *and* 10	
$4 \cdot 4$	$1 \cdot 4$ *and* $5 \cdot 4$	*products* $\Rightarrow$ 4 *and* 20	

Since the last term is negative, we must look for factors with opposite signs. Here we can see that the products 2 and 40 add up to 38 if they have opposite signs:

$$1 \cdot (-2) + 5 \cdot 8 = -2 + 40 = 38$$

Therefore, use -2 and 8 as the factors of 16, making sure that the inner and outer products are $-2x$ and $40x$:

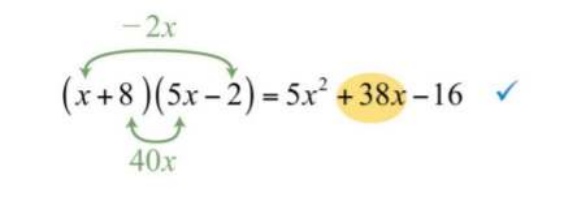


Figure 3.3.6

Answer:

$(x+8)(5x-2)$. The complete check is left to the reader.

After lots of practice, the process described in the previous example can be performed mentally.

? EXERCISE 3.3.1

Factor:

$$12x^2-31x-30$$

Answer

$(3x-10)(4x+3)$

When given trinomials with multiple variables, the process is similar.

✓ EXAMPLE 3.3.5

Factor:

$$9x^2+30xy+25y^2\ .$$

Solution:

Search for factors of the first and last terms such that the sum of the inner and outer products equals the middle term.

$$\begin{aligned} 9x^2 &= 1x\cdot 9x & 25y^2 &= 1y\cdot 25y \\ &= 3x\cdot 3x & &= 5y\cdot 5y \end{aligned}$$

Add the following products to obtain the middle term: $3x\cdot 5y+3x\cdot 5y=30xy$.

$$\begin{aligned} 9x^2+30xy+25y^2 &= (3x\quad)(3x\quad) \\ &= (3x+5y)(3x+5y) \\ &= (3x+5y)^2 \end{aligned}$$

In this example, we have a perfect square trinomial. Check.

$$\begin{aligned} (3x+5y)^2 &= 9x^2+2\cdot 3x\cdot 5y+25y^2 \\ &= 9x^2+30xy+25y^2 \quad ✓ \end{aligned}$$

Answer:

$$(3x+5y)^2$$

? EXERCISE 3.3.2

Factor:

$$16x^2-24xy+9y^2\ .$$

Answer

$(4x-3y)^2$

FACTORING TRINOMIALS WITH COMMON FACTORS

It is a good practice to first factor out the GCF, if there is one. Doing this produces a trinomial factor with smaller coefficients. As we have seen, trinomials with smaller coefficients require much less effort to factor. This commonly overlooked step is worth identifying early.

EXAMPLE 3.3.6

Factor:

$$12x^2 - 27x + 6\,.$$

Solution:

Begin by factoring out the GCF.

$$12x^2 - 27x + 6 = 3(4x^2 - 9x + 2)$$

After factoring out 3, the coefficients of the resulting trinomial are smaller and have fewer factors.

$$\begin{aligned} 4 &= 1\cdot 4 \quad 2 = 1\cdot 2 \\ &= 2\cdot 2 \end{aligned}$$

After some thought, we can see that the combination that gives the coefficient of the middle term is $4(-2) + 1(-1) = -8 - 1 = -9$.

$$\begin{aligned} 3(4x^2 - 9x + 2) &= 3(4x \quad ?)(x \quad ?) \\ &= 3(4x - 1)(x - 2) \end{aligned}$$

Check.

$$\begin{aligned} 3(4x - 1)(x - 2) &= 3(4x^2 - 8x - x + 2) \\ &= 3(4x^2 - 9x + 2) \\ &= 12x^2 - 27x + 6 \quad \checkmark \end{aligned}$$

The factor 3 is part of the factored form of the original expression; be sure to include it in the answer.

Answer:

$$3(4x - 1)(x - 2)$$

It is a good practice to consistently work with trinomials where the leading coefficient is positive.

EXAMPLE 3.3.7

Factor:

$$-x^2 + 2x + 15\,.$$

Solution

In this example, the leading coefficient is -1. Before beginning the factoring process, factor out the -1:

$$-x^2 + 2x + 15 = -1(x^2 - 2x - 15)$$

At this point, factor the remaining trinomial as usual, remembering to write the -1 as a factor in your final answer. Because $3 + (-5) = -2$, use 3 and 5 as the factors of 15.

$$\begin{aligned} -x^2 + 2x = 15 &= -1(x^2 - 2x - 15) \\ &= -1(x \quad)(x \quad) \\ &= -(x + 3)(x - 5) \end{aligned}$$

Answer:

$-1(x + 3)(x - 5)$. The check is left to the reader.

EXAMPLE 3.3.8

Factor:

$$-60a^2 - 5a + 30$$

Solution

The GCF of all the terms is 5. However, in this case factor out -5 because this produces a trinomial factor where the leading coefficient is positive.

$$-60a^2 - 5a + 30 = -5(12a^2 + a - 6)$$

Focus on the factors of 12 and 6 that combine to give the middle coefficient, 1.

$$\begin{aligned} 12 &= 1\cdot 12 \quad 6 = 1\cdot 6 \\ &= 2\cdot 6 \quad\quad = 2\cdot 3 \\ &= 3\cdot 4 \end{aligned}$$

After much thought, we find that $3 \cdot 3 - 4 \cdot 2 = 9 - 8 = 1$. Factor the remaining trinomial.

$$\begin{aligned} -60a^2 - 5a + 30 &= -5(12a^2 + a - 6) \\ &= -5(4a \quad)(3a \quad) \\ &= -5(4a + 3)(3a - 2) \end{aligned}$$

Answer:

$-5(4a + 3)(3a - 2)$. The check is left to the reader.

? EXERCISE 3.3.3

Factor:

$$24 + 2x - x^2 .$$

Answer

$-1(x - 6)(x + 4)$

FACTORING USING THE AC METHOD

In this section, we factor trinomials of the form $ax^2 + bx + c$ using the AC method described previously.

✔ EXAMPLE 3.3.9

Factor using the AC method:

$$18x^2 - 21x + 5 .$$

Solution:

Here $a = 18, b = -21$, and $c = 5$.

$$\begin{aligned} ac &= 18(5) \\ &= 90 \end{aligned}$$

Factor 90 and search for factors whose sum is -21.

$$\begin{aligned} 90 &= 1(90) \\ &= 2(45) \\ &= 3(30) \\ &= 5(18) \\ &= 6(15) \quad \checkmark \\ &= 9(10) \end{aligned}$$

In this case, the sum of the factors -6 and -15 equals the middle coefficient, -21. Therefore, $-21x = -6x - 15x$, and we can write

$$18x^2 - 21x + 5 = 18x^2 - 6x - 15x + 5$$

Factor the equivalent expression by grouping.

$$\begin{aligned} 18x^2 - 21x + 5 &= 18x^2 - 6x - 15x + 5 \\ &= 6x(3x - 1) - 5(3x - 1) \\ &= (3x - 1)(6x - 5) \end{aligned}$$

Answer:

$$(3x - 1)(6x - 5)$$

✔ EXAMPLE 3.3.10

Factor using the AC method: $9x^2 - 61x - 14$.

Solution:

Here $a = 9, b = -61$, and $c = -14$.

We factor -126 as follows:

$$
\begin{aligned}
-126 &= 1(-126) \\
&= 2(-63) \quad \checkmark \\
&= 3(-42) \\
&= 6(-21) \\
&= 7(-18) \\
&= 9(-14)
\end{aligned}
$$

The sum of factors 2 and -63 equals the middle coefficient, -61. Replace $-61x$ with $2x - 63x$:

$$
\begin{aligned}
9x^2 - 61x - 14 &= 9x^2 + 2x - 63x - 14 \quad \textit{Rearrange the terms.} \\
&= 9x^2 - 63x + 2x - 14 \quad \textit{Factor by grouping.} \\
&= 9x(x - 7) + 2(x - 7) \\
&= (x - 7)(9x + 2)
\end{aligned}
$$

Answer:

$(x - 7)(9x + 2)$. The check is left to the reader.

KEY TAKEAWAYS

- If a trinomial of the form $ax^2 + bx + c$ factors into the product of two binomials, then the coefficient of the middle term will be the sum of certain products of factors of the first and last terms.
- If the trinomial has a greatest common factor, then it is a best practice to first factor out the GCF before attempting to factor it into a product of binomials.
- If the leading coefficient of a trinomial is negative, then it is a best practice to factor that negative factor out before attempting to factor the trinomial.
- Factoring trinomials of the form $ax^2 + bx + c$ takes lots of practice and patience. It is extremely important to take the time to become proficient by working lots of exercises.

? EXERCISE 3.3.4 FACTORING TRINOMIALS

Factor.

1. $3x^2 - 14x - 5$
2. $5x^2 + 7x + 2$
3. $2x^2 + 5x - 3$
4. $2x^2 + 13x - 7$
5. $2x^2 + 9x - 5$
6. $7x^2 + 20x - 3$
7. $7x^2 - 46x - 21$
8. $3x^2 + x - 2$
9. $5x^2 + 34x - 7$
10. $5x^2 - 28x - 12$
11. $9x^2 - 12x + 4$
12. $4x^2 - 20x + 25$
13. $49x^2 + 14x + 1$
14. $25x^2 - 10x + 1$
15. $2x^2 + 7x + 16$
16. $6x^2 - 19x - 10$
17. $27x^2 + 66x - 16$
18. $12x^2 - 88x - 15$
19. $12y^2 - 8y + 1$
20. $16y^2 - 66y - 27$
21. $9x^2 - 12xy + 4y^2$
22. $25x^2 + 40x + 16$
23. $15x^2 - 26xy + 8y^2$
24. $12a^2 - 4ab - 5b^2$
25. $4x^2y^2 + 16xy - 9$
26. $20x^2y^2 + 4xy - 7$

27. The area of a rectangle is given by the function $A(x) = 3x^2 - 10x + 3$, where x is measured in meters. Rewrite this function in factored form.
28. The area of a rectangle is given by the function $A(x) = 10x^2 - 59x - 6$, where x is measured in meters. Rewrite this function in factored form.

Answer

1. $(x-5)(3x+1)$
3. $(x+3)(2x-1)$
5. $(x+5)(2x-1)$
7. $(x-7)(7x+3)$
9. $(x+7)(5x-1)$
11. $(3x-2)^2$
13. $(7x+1)^2$
15. Prime
17. $(3x+8)(9x-2)$
19. $(6y-1)(2y-1)$
21. $(3x-2y)^2$
23. $(3x-4y)(5x-2y)$
25. $(2xy-1)(2xy+9)$
27. $A(x) = (3x-1)(x-3)$

? EXERCISE 3.3.5 FACTORING TRINOMIALS WITH COMMON FACTORS

Factor.

1. $6x^2 - 20x - 16$
2. $45x^2 + 27x - 18$
3. $20x^2 - 20x + 5$
4. $3x^2 + 39x - 90$
5. $16x^2 + 26x - 10$
6. $54x^2 - 15x + 6$
7. $45x^2 - 45x - 20$
8. $90x^2 + 300x + 250$
9. $40x^2 - 36xy + 8y^2$
10. $24a^2b^2 + 18ab - 81$
11. $6x^2y^2 + 46xy + 28$
12. $2x^5 + 44x^4 + 144x^3$
13. $5x^3 - 65x^2 + 60x$
14. $15a^4b^2 - 25a^3b - 10a^2$
15. $6a^4b + 2a^3b^2 - 4a^2b^3$
16. $20a^3b^2 - 60a^2b^3 + 45ab^4$

Answer

1. $2(x-4)(3x+2)$
3. $5(2x-1)^2$
5. $2(8x^2 + 13x - 5)$
7. $5(3x-4)(3x+1)$
9. $4(5x-2y)(2x-y)$
11. $2(xy+7)(3xy+2)$
13. $5x(x-12)(x-1)$
15. $2a^2b(3a-2b)(a+b)$

EXERCISE 3.3.6 FACTORING TRINOMIALS WITH COMMON FACTORS

Factor out -1 and then factor further.

1. $-x^2-4x+21$
2. $-x^2+x+12$
3. $-x^2+15x-56$
4. $-x^2+x+72$
5. $-y^2+10y-25$
6. $-y^2-16y-64$
7. $36-9a-a^2$
8. $72-6a-a^2$
9. $32+4x-x^2$
10. $200+10x-x^2$

Answer

1. $-1(x-3)(x+7)$
3. $-1(x-7)(x-8)$
5. $-1(y-5)^2$
7. $-1(a-3)(a+12)$
9. $-1(x-8)(x+4)$

EXERCISE 3.3.7 FACTORING TRINOMIALS WITH COMMON FACTORS

Factor out a negative common factor first and then factor further if possible.

1. $-8x^2+6x+9$
2. $-4x^2+28x-49$
3. $-18x^2-6x+4$
4. $2+4x-30x^2$
5. $15+39x-18x^2$
6. $90+45x-10x^2$
7. $-2x^2+26x+28$
8. $-18x^3-51x^2+9x$
9. $-3x^2y^2+18xy^2-24y^2$
10. $-16a^4+16a^3b-4a^2b^2$
11. The height in feet of a projectile launched from a tower is given by the function $h(t)=-16t^2+64t+80$, where t represents the number of seconds after launch. Rewrite the given function in factored form.
12. The height in feet of a projectile launched from a tower is given by the function $h(t)=-16t^2+64t+192$, where t represents the number of seconds after launch. Rewrite the given function in factored form.

Answer

1. $-(2x-3)(4x+3)$
3. $-2(3x-1)(3x+2)$
5. $-3(2x-5)(3x+1)$
7. $-2(x-14)(x+1)$
9. $-3y^2(x-4)(x-2)$
11. $h(t)=-16(t+1)(t-5)$

EXERCISE 3.3.8 FACTORING USING THE AC METHOD

Factor using the AC method.

1. $2x^2+5x-7$
2. $3x^2+7x-10$

3. $4x^2 - 25x + 6$
4. $16x^2 - 38x - 5$
5. $6x^2 + 23x - 18$
6. $8x^2 + 10x - 25$
7. $4x^2 + 28x + 40$
8. $-6x^2 - 3x + 30$
9. $12x^2 - 56xy + 60y^2$
10. $20x^2 + 80xy + 35y^2$

Answer

1. $(x - 1)(2x + 7)$
3. $(x - 6)(4x - 1)$
5. $(2x + 9)(3x - 2)$
7. $4(x + 2)(x + 5)$
9. $4(x - 3y)(3x - 5y)$

? EXERCISE 3.3.9 DISCUSSION BOARD TOPICS

1. Create your own trinomial of the form $ax^2 + bx + c$ that factors. Share it, along with the solution, on the discussion board.
2. Write out your own list of steps for factoring a trinomial of the form $ax^2 + bx + c$ and share it on the discussion board.
3. Create a trinomial of the form $ax^2 + bx + c$ that does not factor and share it along with the reason why it does not factor.

Answer

1. Answers may vary
3. Answers may vary

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3.4: FACTORING SPECIAL BINOMIALS

LEARNING OBJECTIVES

- Factor binomials that are differences of squares.
- Factor binomials that are sums and differences of cubes.

DIFFERENCE OF SQUARES

A binomial is a polynomial with two terms. We begin with our first special binomial called difference of squares:

$$a^2 - b^2 = (a+b)(a-b) \tag{3.4.1}$$

To verify the above formula, multiply:

$$\begin{aligned}(a+b)(a-b) &= a^2 - ab + ba - b^2 \\ &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2\end{aligned}$$

We use this formula to factor certain special binomials.

EXAMPLE 3.4.1

Factor:

$$x^2 - 16.$$

Solution:

Step 1: Identify the binomial as difference of squares and determine the square factors of each term.

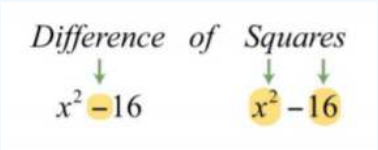


Figure 3.4.1

Here we can write

$$x^2 - 16 = (x)^2 - (4)^2$$

The terms are squares of x and 4. Hence $a = x$ and $b = 4$.

Step 2: Substitute into the difference of squares formula.

$$\begin{aligned}a^2 - b^2 &= (a+b)(a-b) \\ &\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ x^2 - 16 &= (x+4)(x-4)\end{aligned}$$

Step 3: Multiply to check. This step is optional.

Answer:

$(x+4)(x-4)$

It is worth taking some extra time at this point to review all of the squares of integers from 1 to 12.

$$\begin{aligned} 1^2 &= 1 & 7^2 &= 49 \\ 2^2 &= 4 & 8^2 &= 64 \\ 3^2 &= 9 & 9^2 &= 81 \\ 4^2 &= 16 & 10^2 &= 100 \\ 5^2 &= 25 & 11^2 &= 121 \\ 6^2 &= 36 & 12^2 &= 144 \end{aligned}$$

Recognizing these perfect square integers helps speed the factoring process.

EXAMPLE 3.4.2

Factor:

$$9x^2 - 121.$$

Solution:

The subtraction indicates that this is a difference. Furthermore, we recognize that the terms are squares.

$$9x^2 - 121 = (3x)^2 - (11)^2$$

In this case, $a = 3x$ and $b = 11$. Substitute into the formula for difference of squares.

$$\begin{aligned} a^2 - b^2 &= (a+b)(a-b) \\ 9x^2 - 121 &= (3x+11)(3x-11) \end{aligned}$$

Answer:

$$(3x+11)(3x-11)$$

It may be the case that the terms of the binomial have a common factor. If so, it will be difficult to identify the perfect squares until we first factor out the GCF.

EXAMPLE 3.4.3

Factor:

$$12y^2 - 75.$$

Solution:

The terms are not perfect squares. However, notice that they do have a common factor. First, factor out the GCF, 3.

$$12y^2 - 75 = 3(4y^2 - 25)$$

The resulting binomial factor is a difference of squares with $a = 2y$ and $b = 5$.

$$\begin{aligned} 12y^2 - 75 &= 3(4y^2 - 25) \\ &= 3(2y+5)(2y-5) \end{aligned}$$

Answer:

$$3(2y+5)(2y-5)$$

EXAMPLE 3.4.4

Factor:

$$49x^2 - 100y^2.$$

Solution:

Here we have a binomial with two variables and recognize that it is a difference of squares.

$$49x^2 - 100y^2 = (7x)^2 - (10y)^2$$

Therefore, $a = 7x$ and $b = 10y$. Substitute into the formula for difference of squares.

$$\begin{aligned} a^2 - b^2 &= (a+b)(a-b) \\ 49x^2 - 100y^2 &= (7x+10y)(7x-10y) \end{aligned}$$

Answer:

$$(7x+10y)(7x-10y)$$

EXERCISE 3.4.1

Factor:

$$36x^2 - 1.$$

Answer

$(6x+1)(6x-1)$

Given any real number b, a polynomial of the form x^2+b^2 is prime.

Furthermore, the sum of squares a^2+b^2 does not have a general factored equivalent. Care should be taken not to confuse this with a perfect square trinomial:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+ab+ba+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

Therefore,

$$(a+b)^2 \neq a^2+b^2 \tag{3.4.2}$$

When the degree of the special binomial is greater than two, we may need to apply the difference of squares formula multiple times. A polynomial is completely factored when none of the factors can be factored any further.

EXAMPLE 3.4.5

Factor completely:

$$x^4-16.$$

Solution:

First, identify what is being squared:

$$x^4-16=(\quad)^2-(\quad)^2$$

To do this, recall the power rule for exponents, $(x^m)^n=x^{mn}$. When exponents are raised to a power, multiply them. With this in mind, determine that $(x^2)^2=x^4$ and write

$$x^4-16=(x^2)^2-(4)^2$$

Therefore, $a=x^2$ and $b=4$. Substitute into the formula for difference of squares.

$$x^4-16=(x^2+4)(x^2-4)$$

At this point, notice that the factor (x^2-4) is itself a difference of two squares and thus can be further factored using $a=x$ and $b=2$. The factor (x^2+4) is a sum of squares, which cannot be factored using real numbers.

$$\begin{aligned}x^4-16 &= (x^2+4)(x^2-4)\\ &= (x^2+4)(x+2)(x-2)\end{aligned}$$

Answer:

$$(x^2+4)(x+2)(x-2)$$

EXERCISE 3.4.2

Factor completely:

$$81x^4-1.$$

Answer

$(9x^2+1)(3x+1)(3x-1)$

SUM AND DIFFERENCE OF CUBES

Two other special binomials of interest are the sum and difference of cubes:

$$a^3+b^3=(a+b)(a^2-ab+b^2) \tag{3.4.3}$$

$$a^3-b^3=(a-b)(a^2+ab+b^2) \tag{3.4.4}$$

We can verify these formulas by multiplying:

$$\begin{aligned}(a+b)(a^2-ab+b^2) &= a^3-a^2b+ab^2-ab^2+b^3\\ &= a^3+b^3 \quad \checkmark\\ (a-b)(a^2+ab+b^2) &= a^3+a^2b+ab^2=a^2b-ab^2-b^3\\ &= a^3-b^3 \quad \checkmark\end{aligned}$$

The process for factoring the sum and difference of cubes is very similar to that for the difference of squares. We first identify a and b and then substitute into the appropriate formula. The separate formulas for sum and difference of cubes allow us to always choose a and b to be positive.

EXAMPLE 3.4.6

Factor:

$$x^3+8.$$

Solution:

The plus sign and the fact that the terms are cubes indicate to us that this is a sum of cubes.

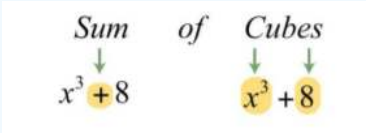


Figure 3.4.2

Next, identify what is being cubed.

$$x^3+8=(x)^3+(2)^3$$

In this case, $a=x$ and $b=2$. Substitute into the sum of cubes formula.

$$\begin{aligned} a^3+b^3 &= (a+b)(a^2-a\ b+\ b^2) \\ x^3+8 &= (x+2)(x^2-x\cdot 2+2^2) \\ &= (x+2)(x^2-2x+4) \end{aligned}$$

The resulting trinomial is prime and the factoring is complete. We can check this factorization by multiplying.

$$\begin{aligned} (x+2)(x^2-2x+4) &= x^3-2x^2+4x+2x^2-4x+8 \\ &= x^3-\ 2x^2+\ 4x+\ 2x^2-\ 4x+8 \\ &= x^3+8 \quad \checkmark \end{aligned}$$

Answer:

$$(x-2)(x^2-2x+4)$$

It is helpful to review the perfect cubes of integers from 1 to 12. This will aid you in identifying sums and differences of cubes.

$$\begin{aligned} 1^3&=1 & y^3&=343 \\ 2^3&=8 & 8^3&=512 \\ 3^3&=27 & 9^3&=729 \\ 4^3&=64 & 10^3&=1000 \\ 5^3&=125 & 11^3&=1331 \\ 6^3&=216 & 12^3&=1728 \end{aligned}$$

EXAMPLE 3.4.7

Factor:

$$y^3-125.$$

Solution:

In this case, we have a difference of cubes.

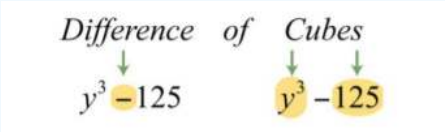


Figure 3.4.3

We can write

$$y^3-125=(y)^3-(5)^3$$

Substitute $a=y$ and $b=5$ into the formula for difference of cubes.

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2 + a\ b + \ b^2) \\ &\quad\ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \downarrow\ \ \downarrow \\ y^3 - 1258 &= (y-5)(y^2 + y\cdot 5 + 5^2) \\ &= (y-5)(y^2 + 5y + 25) \end{aligned}$$

Answer:

$$(y-5)(y^2+5y+25)$$

Always look for common factors when factoring. If the terms of the binomial have a GCF other than 1, then factor that out first.

EXAMPLE 3.4.8

Factor:

$$54x^4 + 128x\,.$$

Solution:

Begin by factoring out the GCF $2x$.

$$54x^4 + 128x = 2x(27x^3 + 64)$$

The resulting binomial factor is a sum of cubes, where $a = 3x$ and $b = 4$.

$$\begin{aligned} 54x^4 + 128x &= 2x(27x^3 + 64) \\ &= 2x(3x+4)((3x)^2 - (3x)(4) + 4^2) \\ &= 2x(3x+4)(9x^2 - 12x + 16) \end{aligned}$$

Answer:

$$2x(3x+4)(9x^2-12x+16)$$

EXAMPLE 3.4.9

Factor:

$$x^3y^3 - 1\,.$$

Solution

This binomial is a difference of cubes with two variables. Identify what is being cubed.

$$x^3y^3 - 1 = (xy)^3 - (1)^3$$

Here $a = xy$ and $b = 1$. Substitute into the appropriate formula and simplify.

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \\ x^3y^3 - 1 &= (xy-1)((xy)^2 + xy\cdot 1 + 1^2) \\ &= (xy-1)(x^2y^2 + xy + 1) \end{aligned}$$

Answer:

$$(xy-1)(x^2y^2+xy+1)$$

EXERCISE 3.4.3

Factor:

$$8x^3 + 343\,.$$

Answer

$(2x+7)(4x^2 - 14x + 49)$

When factoring, always look for resulting factors to factor further.

EXAMPLE 3.4.10

Factor completely:

$$x^6 - 64.$$

Solution:

When confronted with a binomial that is both a difference of squares and cubes, as this is, make it a rule to factor using difference of squares first.

$$x^6 - 64 = \left(x^3\right)^2 - (8)^2$$

Therefore, $a = x^3$ and $b = 8$. Substitute into the difference of squares formula.

$$x^6 - 64 = (x^3 + 8)(x^3 - 8)$$

The resulting two binomial factors are a sum and a difference of cubes. Each can be factored further.

$$x^3 + 8 = (x + 2)(x^2 - 2x + 2^2)$$
$$x^3 - 8 = (x - 2)(x^2 + 2x + 2^2)$$

Therefore, we have

$$\begin{aligned} x^6 - 64 &= \overbrace{(x^3 + 8)} \cdot \overbrace{(x^3 - 8)} \\ &= \overbrace{(x + 2)(x^2 - 2x + 4)} \cdot \overbrace{(x - 2)(x^2 + 2x + 4)} \end{aligned}$$

The trinomial factors are prime and the expression is completely factored.

Answer:

$$(x + 2)(x^2 - 2x + 4)(x - 2)(x^2 + 2x + 4)$$

As an exercise, factor the previous example as a difference of cubes first and then compare the results. Why do you think we make it a rule to factor using difference of squares first?

EXERCISE 3.4.4

Factor:

$$x^6 - y^6.$$

Answer

$(x + y)(x^2 - xy + y2)(x - y)(x^2 + xy + y2)$

KEY TAKEAWAYS

- When factoring special binomials, the first step is to identify it as a sum or difference. Once we identify the binomial, we then determine the values of a and b and substitute into the appropriate formula.
- The formulas for all of the special binomials should be memorized. In addition, to help facilitate the identification of special binomials, memorize the squares and cubes of integers up to at least 12.
- If a binomial is both a difference of squares and cubes, then first factor it as a difference of squares.

EXERCISE 3.4.5 DIFFERENCE OF SQUARES

Factor completely.

1. $x^2 - 9$
2. $x^2 - 100$
3. $y^2 - 36$
4. $y^2 - 144$
5. $x^2 + 4$
6. $x^2 - 5$
7. $m^2 - 81$
8. $m^2 - 64$
9. $16x^2 - 9$
10. $25x^2 - 4$
11. $144x^2 - 1$

12. $9x^2 - 121$
13. $4y^2 - 81$
14. $100y^2 - 49$
15. $9 - 4x^2$
16. $100 - x^2$
17. $1 - y^2$
18. $25 - 9y^2$
19. $-3x^2 + 75$
20. $-16x^2 + 25$
21. $2x^2 - 72$
22. $20x^3 - 45x$
23. $-48x + 27x^3$
24. $36x^2 - 100$
25. $x^2 - y^2$
26. $25x^2 - 9y^2$
27. $a^2 - 4b^2$
28. $a^2b^2 - 36$
29. $4x^2y^2 - 1$
30. $x^2y^2 - 25$
31. $2a^3 - 8ab^2$
32. $3a^3b^4 - 75ab^2$
33. $-100xy^3 + 4x^3y$
34. $-18x^3y^3 + 32xy$
35. $(x + 1)^2 - y^2$
36. $x^2 - (y - 2)^2$
37. $(x - 3)^2 - (y + 3)^2$
38. $(x^2 + 2)^2 - (x - 1)^2$
39. $(x^2 - 1)^2 - (2x + 3)^2$
40. $x^4 - 1$
41. $x^4 - y^4$
42. $16x^4 - 81$
43. $a^4b^4 - 16$
44. $a^4 - 16b^4$
45. $x^8 - 1$
46. $25x^8 - 1$
47. $a^8 - b^2$
48. $a^4 - 9$
49. $x^8 - y^8$
50. $81x^8 - 1$
51. The height of a projectile dropped from a 64-foot tower is given by the function $h(t) = -16t^2 + 64$, where t represents the time in seconds after it is dropped. Rewrite this function in factored form. (Hint: Factor out -16 first.)
52. The height of a projectile dropped from a 36-foot tower is given by the function $h(t) = -16t^2 + 36$, where t represents the time in seconds after it is dropped. Rewrite this function in factored form.

Answer

1. $(x + 3)(x - 3)$
3. $(y + 6)(y - 6)$
5. Prime
7. $(m + 9)(m - 9)$
9. $(4x + 3)(4x - 3)$
11. $(12x + 1)(12x - 1)$
13. $(2y + 9)(2y - 9)$
15. $(3 + 2x)(3 - 2x)$
17. $(1 + y)(1 - y)$

19. $-3(x+5)(x-5)$
21. $2(x+6)(x-6)$
23. $3x(3x+4)(3x-4)$
25. $(x+y)(x-y)$
27. $(a+2b)(a-2b)$
29. $(2xy+1)(2xy-1)$
31. $2a(a+2b)(a-2b)$
33. $4xy(x+5y)(x-5y)$
35. $(x+1+y)(x+1-y)$
37. $(x+y)(x-y-6)$
39. $(x^2+2x+2)(x^2-2x-4)$
41. $(x^2+y^2)(x+y)(x-y)$
43. $(a^2b^2+4)(ab+2)(ab-2)$
45. $(x^4+1)(x^2+1)(x+1)(x-1)$
47. $(a^4+b)(a^4-b)$
49. $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$
51. $h(t)=-16(t+2)(t-2)$

? EXERCISE 3.4.6 SUM AND DIFFERENCE OF CUBES

Factor completely.

1. x^3-1
2. x^3+1
3. y^3-27
4. y^3-8
5. $8y^3+1$
6. $27y^3-1$
7. $64a^3-125$
8. $8a^3+27$
9. a^3+216
10. a^3-125
11. x^3-1000
12. $343m^3-1$
13. $512n^3+1$
14. $8x^3+343$
15. $40y^3-135$
16. $27y^3+729$
17. $27y^3-64$
18. x^3+3
19. $5x^3+1$
20. $1-y^3$
21. $27-1,000y^3$
22. $343+125a^3$
23. x^3-y^3
24. x^3+y^3
25. x^3y^3+125
26. $8x^3y^3-27$
27. $27a^3-8b^3$
28. $16x^3-250y^3$
29. $128x^3+2y^3$
30. $54x^3-2y^3$

31. $3a^4b - 24ab^4$
32. $a^3b^3c^3 - 1$
33. $(x+1)^3 - 8$
34. $8x^3 - (x-5)^3$
35. $(x-2)^3 + (x+2)^3$
36. $(a+3)^3 + (b-3)^3$
37. $x^6 - 1$
38. $x^6 + 1$
39. $64a^6 - 1$
40. $a^6 - 81b^6$
41. $x^6 - y^6$
42. $x^6 + y^6$

Answer

1. $(x-1)(x^2+x+1)$
3. $(y-3)(y^2+3y+9)$
5. $(2y+1)(4y^2-2y+1)$
7. $(4a-5)(16a^2+20a+25)$
9. $(a+6)(a^2-6a+36)$
11. $(x-10)(x^2+10x+100)$
13. $(8n+1)(64n^2-8n+1)$
15. $5(2y-3)(4y^2+6y+9)$
17. $(3y-4)(9y^2+12y+16)$
19. Prime
21. $(3-10y)(9+30y+100y^2)$
23. $(x-y)(x^2+xy+y^2)$
25. $(xy+5)(x^2y^2-5xy+25)$
27. $(3a-2b)(9a^2+6ab+4b^2)$
29. $2(4x+y)(16x^2-4xy+y^2)$
31. $3ab(a-2b)(a^2+2ab+4b^2)$
33. $(x-1)(x^2+4x+7)$
35. $2x(x^2+12)$
37. $(x+1)(x^2-x+1)(x-1)(x^2+x+1)$
39. $(2a+1)(4a^2-2a+1)(2a-1)(4a^2+2a+1)$
41. $(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$

? EXERCISE 3.4.7 DISCUSSION BOARD TOPICS

1. If a binomial falls into both categories, difference of squares and difference of cubes, which would be best to factor it as, and why? Create an example that illustrates this situation and factor it using both formulas.
2. What can be said about the degrees of the factors of a polynomial? Give an example.
3. Make up your own difference of squares factoring exercise and provide the answer. Explain how you solved it.
4. Make up your own sum or difference of cubes factoring exercise and provide the answer. Explain how you solved it.

Answer

1. Answers may vary
3. Answers may vary

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3.5: GENERAL GUIDELINES FOR FACTORING POLYNOMIALS

LEARNING OBJECTIVES

- Develop a general strategy for factoring polynomials.

GENERAL FACTORING STRATEGY

We have learned various techniques for factoring polynomials with up to four terms. The challenge is to identify the type of polynomial and then decide which method to apply. The following outlines a **general guideline for factoring** polynomials:

1. Check for common factors. If the terms have common factors, then factor out the greatest common factor (GCF) and look at the resulting polynomial factors to factor further.
2. Determine the number of terms in the polynomial.
   a. Factor four-term polynomials by grouping.
   b. Factor trinomials (three terms) using "trial and error" or the AC method.
   c. Factor binomials (two terms) using the following special products:

Table 3.5.1

Difference of Squares:	$a^2-b^2+(a+b)(a-b)$
Sum of Squares:	a^2+b^2 no general formula
Difference of Cubes:	$a^3-b^3=(a-b)(a^2+ab+b^2)$
Sum of Cubes:	$a^3+b^3=(a+b)(a^2-ab+b^2)$

3. Look for factors that can be factored further.
4. Check by multiplying.

NOTE

- If a binomial is both a difference of squares and a difference of cubes, then first factor it as difference of squares and then as a sum and difference of cubes to obtain a more complete factorization.
- Not all polynomials with integer coefficients factor. When this is the case, we say that the polynomial is prime.

If an expression has a GCF, then factor this out first. Doing so is often overlooked and typically results in factors that are easier to work with. Also, look for the resulting factors to factor further; many factoring problems require more than one step. A polynomial is completely factored when none of the factors can be factored further.

EXAMPLE 3.5.1

Factor:

$$6x^4-3x^3-24x^2+12x\ .$$

Solution:

This four-term polynomial has a GCF of $3x$. Factor this out first.

$$6x^4-3x^3-24x^2+12x=3x(2x^3-x^2-8x+4)$$

Now factor the resulting four-term polynomial by grouping.

$$6x^4-3x^3-24x^2+12x=3x\quad\underbrace{(2x^3-x^2}_{group}\quad\underbrace{-8x+4)}_{group}$$

$$=3x(x^2(2x-1)-4(2x-1))$$
$$=3x((2x-1)(x^2-4))$$
$$=3x(2x-1)(x^2-4)$$

The factor (x^2-4) is a difference of squares and can be factored further.

$$=3x(2x-1)(x^2-4)$$
$$=3x(2x-1)(x+2)(x-2)$$

Answer:

$$3x(2x-1)(x+2)(x-2)$$

EXAMPLE 3.5.2

Factor:

$$18x^3y-60x^2y+50xy\,.$$

Solution:

This trinomial has a GCF of $2xy$. Factor this out first.

$$18x^3y-60x^2y+50xy=2xy(9x^2-30x+25)$$

The trinomial factor can be factored further using the trial and error method. Use the factors $9=3\cdot 3$ and $25=(-5)\cdot(-5)$. These combine to generate the correct coefficient for the middle term: $3(-5)+3(-5)=-15-15=-30$.

$$\begin{aligned}18x^3y-60x^2y+50xy &= 2xy(9x^2-30x+25)\\ &= 2xy(3x\quad ?)(3x\quad ?)\\ &= 2xy(3x-5)(3x-5)\\ &= 2xy(3x-5)^2\end{aligned}$$

Check.

$$\begin{aligned}2xy(3x-5)^2 &= 2xy(9x^2-2\cdot 3x\cdot 5+5^2)\\ &= 2xy(9x^2-30x+25)\\ &= 18x^3y-60x^2y+50xy\quad\checkmark\end{aligned}$$

Answer:

$$2xy(3x-5)^2$$

EXAMPLE 3.5.3

Factor:

$$5a^3b^4+10a^2b^3-75ab^2\,.$$

Solution:

This trinomial has a GCF of 5ab2. Factor this out first.

$$5a^3b^4+10a^2b^3-75ab^2=5ab^2(a^2b^2+2ab-15)$$

The resulting trinomial factor can be factored as follows:

$$\begin{aligned}5a^3b^4+10a^2b^3-75ab^2 &= 5ab^2(a^2b^2+2ab-15)\\ &= 5ab^2(ab\quad ?)(ab\quad ?)\\ &= 5ab^2(ab+5)(ab-3)\end{aligned}$$

Answer:

$$5ab^2(ab+5)(ab-3)$$

? EXERCISE 3.5.1

Factor:

$$3x^3y-12x^2y^2+12xy^3\,.$$

Answer

$3xy(x-2y)^2$

EXAMPLE 3.5.4

Factor:

$$16y^4-1\,.$$

Solution:

This binomial does not have a GCF. Therefore, begin factoring by identifying it as a difference of squares.

$$16y^4 - 1 = (4y^2)^2 - (1)^2$$

Here $a = 4y^2$ and $b = 1$. Substitute into the formula for difference of squares.

The factor $(4y^2 + 1)$ is a sum of squares and is prime. However, $(4y^2 - 1)$ is a difference of squares and can be factored further.

$$\begin{aligned} 16y^4 - 1 &= (4y^2 + 1)(4y^2 - 1) \\ &= (4y^2 + 1)(2y + 1)(2y - 1) \end{aligned}$$

Answer:

$$(4y^2 + 1)(2y + 1)(2y - 1)$$

EXAMPLE 3.5.5

Factor:

$$x^6 - 64y^6 .$$

Solution:

This binomial is a difference of squares and a difference of cubes. When this is the case, first factor it as a difference of squares.

$$x^6 - 64y^6 = (x^3)^2 - (8y^3)^2$$

We can write

$$x^6 - 64y^6 = (x^3 + 8y^3)(x^3 - 8y^3)$$

Each factor can be further factored either as a sum or difference of cubes, respectively.

$$\begin{aligned} x^3 + 8y^3 &= (x)^3 + (2y)^3 = (x + 2y)(x^2 - 2xy + 4y^2) \\ x^3 - 8y^3 &= (x)^3 - (2y)^3 = (x - 2y)(x^2 + 2xy + 4y^2) \end{aligned}$$

Therefore,

$$\begin{aligned} x^6 - 64y^6 &= (x^3 + 8y^3)(x^3 - 8y^3) \\ &= (x + 2y)(x^2 - 2xy + 4y^2)(x - 2y)(x^2 + 2xy + 4y^2) \end{aligned}$$

Answer:

$$(x + 2y)(x^2 - 2xy + 4y^2)(x - 2y)(x^2 + 2xy + 4y^2)$$

EXAMPLE 3.5.6

First, identify this expression as a difference of squares.

$$x^2 - (2x - 1)^2 = (x)^2 - (2x - 1)^2$$

Solution:

Here use $a = x$ and $b = 2x - 1$ in the formula for a difference of squares.

$$\begin{aligned} a^2 - b^2 &= (a + b)(a - b) \\ x^2 - (2x - 1)^2 &= [(x + (2x - 1))][(x - (2x - 1))] \\ &= (x + 2x - 1)(x - 2x + 1) \\ &= (3x - 1)(-x + 1) \end{aligned}$$

Answer:

$$(3x - 1)(-x + 1)$$

EXERCISE 3.5.2

Factor:

$$x^4 + 2x^3 + 27x + 54 .$$

Answer

$(x + 2)(x + 3)(x^2 - 3x + 9)$

KEY TAKEAWAYS

- Use the polynomial type to determine the method used to factor it.
- It is a best practice to look for and factor out the greatest common factor (GCF) first. This will facilitate further factoring and simplify the process. Be sure to include the GCF as a factor in the final answer.
- Look for resulting factors to factor further. It is often the case that factoring requires more than one step.
- If a binomial can be considered as both a difference of squares and a difference of cubes, then first factor it as a difference of squares. This results in a more complete factorization.

? EXERCISE 3.5.3 MIXED FACTORING

Factor completely.

1. $2x^5y^2 - 12x^4y^3$
2. $18x^5y^3 - 6x^4y^5$
3. $5x^2 + 20x - 25$
4. $4x^2 + 10x - 6$
5. $24x^3 - 30x^2 - 9x$
6. $30x^3 - 65x^2 + 10x$
7. $6x^3 + 27x^2 - 9x$
8. $21x^3 + 49x^2 - 28x$
9. $5x^3 - 30x^2 - 15x + 90$
10. $6x^4 + 24x^3 - 2x^2 - 8x$
11. $x^4 - 6x^3 + 8x - 48$
12. $x^4 - 5x^3 + 27x - 135$
13. $4x^3 - 4x^2 - 9x + 9$
14. $50x^3 + 25x^2 - 32x - 16$
15. $2x^3 + 250$
16. $3x^5 - 81x^2$
17. $2x^5 - 162x^3$
18. $4x^4 - 36$
19. $x^4 + 16$
20. $x^3 + 9$
21. $72 - 2x^2$
22. $5x^4 - 25x^2$
23. $7x^3 - 14x$
24. $36x^2 - 12x + 1$
25. $25x^2 + 10x + 1$
26. $250x^3 + 200x^4 + 40x^5$
27. $-7x^2 + 19x + 6$
28. $-8x^4 + 40x^3 - 50x^2$
29. $a^4 - 16$
30. $16a^4 - 81b^4$
31. $y^5 + y^4 - y - 1$
32. $4y^5 + 2y^4 - 4y^2 - 2y$
33. $3x^8 - 192x^2$
34. $4x^7 + 4x$
35. $4x^2 - 19xy + 12y^2$
36. $16x^2 - 66xy - 27y^2$
37. $5x^5 - 3x^4 - 5x^3 + 3x^2$
38. $4a^2b^2 - 4a^2 - 9b^2 + 9$
39. $15a^2 - 4ab - 4b^2$
40. $6a^2 - 25ab + 4b^2$
41. $6x^2 + 5xy + 6y^2$
42. $9x^2 + 5xy - 12y^2$
43. $(3x - 1)^2 - 64$
44. $(x - 5)^2 - (x - 2)^2$
45. $(x + 1)^3 + 8$

46. $(x-4)^3-27$
47. $(2x-1)^2-(2x-1)-12$
48. $(x-4)^2+5(x-4)+6$
49. $a^3b-10a^2b^2+25ab^3$
50. $2a^3b^2-12a^2b+18a$
51. $15a^2b^2-57ab-12$
52. $-60x^3+4x^2+24x$
53. $-24x^5+78x^3-54x$
54. $9y^6-13y^4+4y^2$
55. $36-15a-6a^2$
56. $60ab^2+5a^2b^2-5a^3b^2$
57. x^4-1
58. $16x^4-64$
59. x^8-1
60. $81x^8-1$
61. $x^{16}-1$
62. $x^{12}-1$
63. $54x^6-216x^4-2x^3+8x$
64. $4a^4-4a^2b^2-a^2+b^2$
65. $32y^3+32y^2-18y-18$
66. $3a^3+a^2b-12ab-4b^2$
67. $18m^2-21mn-9n^2$
68. $5m^2n^2+10mn-15$
69. The volume of a certain rectangular solid is given by the function $V(x)=x^3-2x^2-3x$. Write the function in its factored form.
70. The volume of a certain right circular cylinder is given by the function $V(x)=4\pi x^3-4\pi x^2+\pi x$. Write the function in its factored form.

Answer

1. $2x^4y^2(x-6y)$
3. $5(x-1)(x+5)$
5. $3x(2x-3)(4x+1)$
7. $3x(2x^2+9x-3)$
9. $5(x-6)(x^2-3)$
11. $(x-6)(x+2)(x^2-2x+4)$
13. $(x-1)(2x-3)(2x+3)$
15. $2(x+5)(x^2-5x+25)$
17. $2x^3(x+9)(x-9)$
19. Prime
21. $2(6+x)(6-x)$
23. $7x(x^2-2)$
25. $(5x+1)^2$
27. $-(x-3)(7x+2)$
29. $(a^2+4)(a+2)(a-2)$
31. $(y^2+1)(y-1)(y+1)^2$
33. $3x^2(x+2)(x^2-2x+4)(x-2)(x^2+2x+4)$
35. $(x-4y)(4x-3y)$
37. $x^2(5x-3)(x+1)(x-1)$
39. $(3a-2b)(5a+2b)$
41. Prime
43. $3(x-3)(3x+7)$
45. $(x+3)(x^2+3)$

47. $2(x+1)(2x-5)$
49. $ab(a-5b)^2$
51. $3(ab-4)(5ab+1)$
53. $-6x(x+1)(x-1)(2x+3)(2x-3)$
55. $-3(a+4)(2a-3)$
57. $(x^2+1)(x+1)(x-1)$
59. $(x^4+1)(x^2+1)(x+1)(x-1)$
61. $(x^8+1)(x^4+1)(x^2+1)(x+1)(x-1)$
63. $2x(x+2)(x-2)(3x-1)(9x^2+3x+1)$
65. $2(y+1)(4y-3)(4y+3)$
67. $3(2m-3n)(3m+n)$
69. $V(x)=x(x+1)(x-3)$

? EXERCISE 3.5.4 DISCUSSION BOARD

1. First, factor the trinomial $24x^2-28x-40$. Then factor out the GCF. Discuss the significance of factoring out the GCF first. Do you obtain the same result?
2. Discuss a plan for factoring polynomial expressions on an exam. What should you be looking for and what should you be expecting?

Answer

1. Answer may vary

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3.6: SOLVING EQUATIONS BY FACTORING

LEARNING OBJECTIVES

- Verify solutions to quadratic equations.
- Solve quadratic equations by factoring.
- Determine a quadratic equation with given solutions.
- Solve polynomial equations by factoring.

SOLVING QUADRATIC EQUATIONS BY FACTORING

Learning how to solve equations is one of our main goals in algebra. Up to this point, we have solved linear equations, which are of degree 1. In this section, we will learn a technique that can be used to solve certain equations of degree 2. A quadratic equation is any equation that can be written in the standard form

$$ax^2 + bx + c = 0 \tag{3.6.1}$$

,

where a, b, and c are real numbers and $a \neq 0$. The following are some examples of quadratic equations, all of which will be solved in this section:

$$x^2 + x - 6 = 0$$
$$4x^2 - 9 = 0$$
$$2x^2 + 10x + 20 = -3x + 5$$

A solution of a quadratic equation in standard form is called a root. Quadratic equations can have two real solutions, one real solution, or no real solution. The quadratic equation $x^2 + x - 6 = 0$ has two solutions, namely, $x = -3$ and $x = 2$.

EXAMPLE 3.6.1

Verify that $x = -3$ and $x = 2$ are solutions to $x^2 + x - 6 = 0$.

Solution:

To verify solutions, substitute the values for x and then simplify to see if a true statement results.

$Check: x = -3$	$Check: x = 2$
$x^2 + x - 6 = 0$	$x^2 + x - 6 = 0$
$(-3)^2 + (-3) - 6 = 0$	$(2)^2 + (2) - 6 = 0$
$9 - 3 - 6 = 0$	$4 + 2 - 6 = 0$
$0 = 0$ ✓	$0 = 0$ ✓

Answer:

Both values produce true statements. Therefore, they are both solutions to the equation.

Our goal is to develop algebraic techniques for finding solutions to quadratic equations. The first technique requires the zero-product property:

If $a \cdot b = 0$, then $a = 0$ or $b = 0$

In other words, if any product is equal to zero, then one or both of the variable factors must be equal to zero.

EXAMPLE 3.6.2

Solve:

$$(x - 8)(x + 7) = 0.$$

Solution:

This equation consists of a product of two quantities equal to zero; therefore, the zero-product property applies. One or both of the quantities must be zero.

$$(x-8)=0 \quad \text{or} \quad (x+7)=0$$
$$x-8+8=0+8 \qquad x+7-7=0-7$$
$$x=8 \qquad x=-7$$

To verify that these are solutions, substitute them for the variable x.

$Check: x=8$	$Check: x=-7$
$(x-8)(x+7)=0$	$(x-8)(x+7)=0$
$(8-8)(8+7)=0$	$(-7+8)(-7+7)=0$
$(0)(15)=0$	$(1)(0)=0$
$0=0$ ✓	$0=0$ ✓

Notice that each solution produces a factor that is equal to zero.

Answer:

The solutions are 8 and -7.

The quadratic equation may not be given in its factored form.

EXAMPLE 3.6.3

Solve:

$$x^2+3x-10=0\ .$$

Solution:

The goal is to produce a product that is equal to zero. We can do that by factoring the trinomial on the left side of the equation.

$$x^2+3x-10=0$$
$$(x \quad ?)(x \quad ?)=0 \qquad -10=5(-2)$$
$$and\ 3=5+(-2)$$
$$(x+5)(x-2)=0$$

Next, apply the zero-product property and set each factor equal to zero.

$$x+5=0 \quad \text{or} \quad x-2=0$$

This leaves us with two linear equations, each of which can be solved for x.

$$x+5-5=0-5 \qquad x-2+2=0+2$$
$$x=-5 \qquad x=2$$

Check the solutions by substituting into the original equation to verify that we obtain true statements.

$Check: x=-5$	$Check: x=2$
$x^2+3x-10=0$	$x^2+3x-10=0$
$(-5)^2+3(-5)-10=0$	$(2)^2+3(2)-10=0$
$25-15-10=0$	$4+6-10=0$
$0=0$ ✓	$0=0$ ✓

Answer:

The solutions are -5 and 2.

Using the zero-product property after factoring a quadratic equation in standard form is the key to this technique. However, the quadratic equation may not be given in standard form, and so there may be some preliminary steps before factoring. The steps required to solve by factoring are outlined in the following example.

EXAMPLE 3.6.4

Solve:

$$2x^2+10x+20=-3x+5\ .$$

Solution:

Step 1: Express the quadratic equation in standard form. For the zero-product property to apply, the quadratic expression must be equal to zero. Use the addition and subtraction properties of equality to combine opposite-side like terms and obtain zero on one side of the

equation. In this example, add $3x$ to and subtract 5 from both sides.

$$\begin{aligned} 2x^2+10x+20+3x &= -3x+5+3x \\ 2x^2+13x+20 &= 5 \\ 2x^2+13x+20-5 &= 5-5 \\ 2x^2+13x+15 &= 0 \end{aligned}$$

Step 2: Factor the quadratic expression.

$$(2x+3)(x+5)=0$$

Step 3: Apply the zero-product property and set each variable factor equal to zero.

$$2x+3=0 \quad \text{or} \quad x+5=0$$

Step 4: Solve the resulting linear equations.

$$\begin{aligned} 2x+3=0 \quad &\text{or} \quad x+5=0 \\ 2x=-3 \quad &\qquad x=-5 \\ \tfrac{2x}{2}=\tfrac{-3}{2} \quad & \\ x=-\tfrac{3}{2} \quad & \end{aligned}$$

Answer:

The solutions are -5 and $-\frac{3}{2}$. The check is optional.

EXAMPLE 3.6.5

Solve:

$$9x^2+1=6x\,.$$

Solution:

Write this in standard form by subtracting \(6x\ from both sides.

$$\begin{aligned} 9x^2+1-6x &= 6x-6x \\ 9x^2-6x+1 &= 0 \end{aligned}$$

Once the equation is in standard form, equal to zero, factor.

$$(3x-1)(3x-1)=0$$

This is a perfect square trinomial. Hence setting each factor equal to zero results in a repeated solution.

$$\begin{aligned} 3x-1=0 \quad &\text{or} \quad 3x-1=0 \\ 3x=1 \quad &\qquad 3x=1 \\ x=\tfrac{1}{3} \quad &\qquad x=\tfrac{1}{3} \end{aligned}$$

A repeated solution is called a double root and does not have to be written twice.

Answer:

The solution is $\frac{1}{3}$.

EXERCISE 3.6.1

Solve:

$$x^2-3x=28\,.$$

Answer

$x=-4$ or $x=7$

Not all quadratic equations in standard form are trinomials. We often encounter binomials.

EXAMPLE 3.6.6

Solve:

$$x^2-9=0\,.$$

Solution:

This quadratic equation is given in standard form, where the binomial on the left side is a difference of squares. Factor as follows:

$$\begin{aligned} x^2 - 9 &= 0 \\ (x+3)(x-3) &= 0 \end{aligned}$$

Next, set each factor equal to zero and solve.

$$\begin{aligned} x+3=0 \quad &\text{or} \quad x-3=0 \\ x=-3 \quad &\qquad x=3 \end{aligned}$$

Answer:

The solutions are 3 and -3, which can also be written as ± 3.

EXAMPLE 3.6.7

Solve:

$$5x^2 = 15x$$

Solution:

By inspection, we see that $x = 0$ is a solution to this quadratic equation. Since dividing by zero is undefined, we want to avoid dividing both sides of this equation by x. In general, we wish to avoid dividing both sides of any equation by a variable or an expression containing a variable. We will discuss this in more detail later. The first step is to rewrite this equation in standard form with zero on one side.

$$\begin{aligned} 5x^2 &= 15x \\ 5x^2 - 15x &= 15x - 15x \quad \textit{Subtract } 15x \textit{ from both sides.} \\ 5x^2 - 15x &= 0 \end{aligned}$$

Next, factor the expression. Notice that the binomial on the left has a GCF of $5x$.

$$\begin{aligned} 5x^2 - 15x &= 0 \\ 5x(x-3) &= 0 \end{aligned}$$

Set each factor to equal to zero.

$$\begin{aligned} 5x=0 \quad &\text{or} \quad x-3=0 \\ \tfrac{5x}{5} = \tfrac{0}{5} \quad &\qquad x=3 \\ x=0 \quad & \end{aligned}$$

Answer:

The solutions are 0 and 3.

EXAMPLE 3.6.8

Solve:

$$(2x+1)(x+5) = 11\,.$$

Solution:

This quadratic equation appears to be factored; hence it might be tempting to set each factor equal to 11. However, this would lead to incorrect results. We must rewrite the equation in standard form, equal to zero, so that we can apply the zero-product property.

$$\begin{aligned} (2x+1)(x+5) &= 11 \\ 2x^2 + 10x + x + 5 &= 11 \\ 2x^2 + 11x + 5 - 11 &= 11 - 11 \\ 2x^2 + 11x - 6 &= 0 \end{aligned}$$

Once it is in standard form, we can factor and then set each factor equal to zero.

$$\begin{aligned} (2x-1)(x+6) &= 0 \\ 2x-1=0 \quad &\text{or} \quad x+6=0 \\ 2x=1 \quad &\qquad x=-6 \\ x=\tfrac{1}{2} \quad & \end{aligned}$$

Answer:

The solutions are $\frac{1}{2}$ and -6.

EXAMPLE 3.6.9

Solve:

$$15x^2 - 25x + 10 = 0\ .$$

Solution:

We begin by factoring out the GCF of 5. Then factor the resulting trinomial.

$$\begin{aligned} 15x^2 - 25x + 10 &= 0 \\ 5(3x^2 - 5x + 2) &= 0 \\ 5(3x - 2)(x - 1) &= 0 \end{aligned}$$

Next, we set each ***variable factor*** equal to zero and solve for x.

$$\begin{aligned} 3x - 2 = 0 \quad &\text{or} \quad x - 1 = 0 \\ 3x = 2 \quad &\qquad\quad x = 1 \\ x = \tfrac{2}{3} \quad & \end{aligned}$$

Notice that the factor 5 is not a variable factor and thus did not contribute to the solution set.

Answer:

The solutions are $\frac{2}{3}$ and 1.

EXAMPLE 3.6.10

Factor:

$$52x^2 + 76x - 13 = 0\ .$$

Solution:

Clear the fractions by multiplying both sides of the equation by the LCD, which is equal to 6.

$$\begin{aligned} \frac{5}{2}x^2 + \frac{7}{6}x - \frac{1}{3} &= 0 \\ 6\cdot\left(\frac{5}{2}x^2 + \frac{7}{6}x - \frac{1}{3}\right) &= 6\cdot(0) \\ 15x^2 + 7x - 2 &= 0 \end{aligned}$$

At this point, we have an equivalent equation with integer coefficients and can factor as usual. Begin with the factors of 15 and 2.

$$\begin{aligned} 15 &= 1\cdot 15 \quad 2 = 1\cdot 2 \\ &= 3\cdot 5 \end{aligned}$$

The coefficient of the middle term is $7 = 3(-1) + 5(2)$. Factor as follows:

$$\begin{aligned} 15x^2 + 7x - 2 &= 0 \\ (3x + 2)(5x - 1) &= 0 \end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{aligned} 3x + 2 = 0 \quad &\text{or} \quad 5x - 1 = 0 \\ 3x = -2 \quad &\qquad\quad 5x = 1 \\ x = -\tfrac{2}{3} \quad &\qquad\quad x = \tfrac{1}{5} \end{aligned}$$

Answer:

The solutions are $-\frac{2}{3}$ and $\frac{1}{5}$.

EXERCISE 3.6.2

Solve:

$$4x^2 - 9 = 0\ .$$

Answer

$-\frac{3}{2}$ and $\frac{3}{2}$

FINDING EQUATIONS WITH GIVEN SOLUTIONS

The zero-product property states,

$$\text{If } a \cdot b = 0 \text{ , then } a = 0 \text{ or } b = 0$$

And, in fact, the converse is true as well:

$$\text{If } a = 0 \text{ or } b = 0, \text{ then } ab = 0$$

When this is the case, we can write the following:

$$a \cdot b = 0 \text{ , if and only if } a = 0 \text{ or } b = 0$$

We use this property to find equations, given the solutions. To do this, the steps for solving by factoring are performed in reverse.

EXAMPLE 3.6.11

Find a quadratic equation with solutions -7 and 2.

Solution:

Given the solutions, we can determine two linear factors.

$$\begin{array}{ccc} x = -7 & \text{or} & x = 2 \\ x + 7 = 0 & & x - 2 = 0 \end{array}$$

The product of these linear factors is equal to zero when $x = -7$ or $x = 2$:

$$(x + 7)(x - 2) = 0$$

Multiply the binomials and present the equation in standard form.

$$\begin{aligned} x^2 - 2x + 7x - 14 &= 0 \\ x^2 + 5x - 14 &= 0 \end{aligned}$$

Answer:

$$x^2 + 5x - 14 = 0 \text{ .}$$

We may check our equation by substituting the given answers to see if we obtain a true statement. Also, the equation found aboveis not unique and so the check becomes essential when our equation looks different from someone else's. This is left as an exercise.

EXAMPLE 3.6.12

Find a quadratic equation with integer coefficients, given solutions $\frac{1}{2}$ and $-\frac{3}{4}$.

Solution:

To avoid fractional coefficients, we first clear the fractions by multiplying both sides by the denominator.

$$\begin{array}{ccc} x = \frac{1}{2} & \text{or} & x = -\frac{3}{4} \\ 2 \cdot x = 2 \cdot \frac{1}{2} & & 4 \cdot x = -\frac{3}{4} \cdot 4 \\ 2x = 1 & & 4x = -3 \\ 2x - 1 = 0 & & 4x + 3 = 0 \end{array}$$

Apply the zero-product property and multiply.

$$\begin{aligned} (2x - 1)(4x + 3) &= 0 \\ 8x^2 + 6x - 4x - 3 &= 0 \\ 8x^2 + 2x - 3 &= 0 \end{aligned}$$

Answer:

$$8x^2 + 2x - 3 = 0$$

EXERCISE 3.6.3

Find a quadratic equation with integer coefficients, given solutions -1 and $\frac{2}{3}$.

Answer

$3x^2 + x - 2 = 0$

SOLVING POLYNOMIAL EQUATIONS BY FACTORING

The zero-product property is true for any number of factors that make up an equation. If an expression is equal to zero and can be factored into linear factors, then we will be able to set each factor equal to zero and solve for each equation.

EXAMPLE 3.6.13

Solve:

$$3x(x-5)(3x-2)=0\,.$$

Solution:

Set each variable factor equal to zero and solve.

$$\begin{array}{ccccc} 3x=0 & \text{or} & x-5=0 & \text{or} & 3x-2=0 \\ \frac{3x}{3}=\frac{0}{3} & & x=5 & & \frac{3x}{3}=\frac{2}{3} \\ x=0 & & & & x=\frac{2}{3} \end{array}$$

Answer:

The solutions are $0, 5$, and $\frac{2}{3}$.

Of course, we cannot expect the equation to be given in factored form.

EXAMPLE 3.6.14

Solve:

$$x^3+2x^2-9x-18=0\ .$$

Solution

Begin by factoring the left side completely.

$$\begin{aligned} x^3+2x^2-9x-18&=0 && \textit{Factor by grouping.} \\ x^2(x+2)-9(x+2)&=0 && \\ (x+2)(x^2-9)&=0 && \textit{Factor as a difference of squares.} \\ (x+2)(x+3)(x-3)&=0 && \end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{array}{ccccc} x+2=0 & \text{or} & x+3=0 & \text{or} & x-3=0 \\ x=-2 & & x=-3 & & x=3 \end{array}$$

Answer:

The solutions are $-2, -3$, and 3.

Notice that the degree of the polynomial is 3 and we obtained three solutions. In general, for any polynomial equation with one variable of degree n, the fundamental theorem of algebra guarantees n real solutions or fewer. We have seen that many polynomials do not factor. This does not imply that equations involving these unfactorable polynomials do not have real solutions. In fact, many polynomial equations that do not factor do have real solutions. We will learn how to solve these types of equations as we continue in our study of algebra.

EXERCISE 3.6.4

Solve:

$$-10x^3-18x^2+4x=0\,.$$

Answer

$-2, 0, \frac{1}{5}$

KEY TAKEAWAYS

- A polynomial can have at most a number of solutions equal to its degree. Therefore, quadratic equations can have up to two real solutions.
- To solve a quadratic equation, first write it in standard form. Once the quadratic expression is equal to zero, factor it and then set each variable factor equal to zero. The solutions to the resulting linear equations are the solutions to the quadratic equation.

- Not all quadratic equations can be solved by factoring. We will learn how to solve quadratic equations that do not factor later in the course.
- To find a quadratic equation with given solutions, perform the process of solving by factoring in reverse.
- If any polynomial is factored into linear factors and is set to zero, then we can determine the solutions by setting each variable factor equal to zero and solving each individually.

EXERCISE 3.6.5 SOLUTIONS TO QUADRATIC EQUATIONS

Determine whether the given set of values are solutions to the quadratic equation.

1. $-3, 5; x^2 - 2x - 15 = 0$
2. $7, -1; x^2 - 6x - 7 = 0$
3. $-\frac{1}{2}, \frac{1}{2}; x^2 - 14 = 0$
4. $-\frac{3}{4}, \frac{3}{4}; x^2 - 916 = 0$
5. $-3, 2; x^2 - x - 6 = 0$
6. $-5, 1; x^2 - 4x - 5 = 0$

Answer

1. Yes
3. Yes
5. No

EXERCISE 3.6.6 SOLUTIONS TO QUADRATIC EQUATIONS

Solve.

1. $(x-3)(x+2) = 0$
2. $(x+5)(x+1) = 0$
3. $(2x-1)(x-4) = 0$
4. $(3x+1)(3x-1) = 0$
5. $(x-2)^2 = 0$
6. $(5x+3)^2 = 0$
7. $7x(x-5) = 0$
8. $-2x(2x-3) = 0$
9. $(x-12)(x+34) = 0$
10. $(x+58)(x-38) = 0$
11. $(14x+12)(16x-23) = 0$
12. $(15x-3)^2 = 0$
13. $-5(x+1)(x-2) = 0$
14. $12(x-7)(x-6) = 0$
15. $(x+5)(x-1) = 0$
16. $(x+5)(x+1) = 0$
17. $-2(3x-2)(2x+5) = 0$
18. $5(7x-8)^2 = 0$

Answer

1. $-2, 3$
3. $\frac{1}{2}, 4$
5. 2
7. $0, 5$
9. $-\frac{3}{4}, \frac{1}{2}$
11. $-2, 4$
13. $-1, 2$
15. $-5, 1$
17. $-\frac{5}{2}, \frac{2}{3}$

EXERCISE 3.6.7 SOLVE BY FACTORING

Solve.

1. $x^2 - x - 6 = 0$
2. $x^2 + 3x - 10 = 0$
3. $y^2 - 10y + 24 = 0$
4. $y^2 + 6y - 27 = 0$
5. $x^2 - 14x + 40 = 0$
6. $x^2 + 14x + 49 = 0$
7. $x^2 - 10x + 25 = 0$
8. $3x^2 + 2x - 1 = 0$
9. $5x^2 - 9x - 2 = 0$
10. $7y^2 + 20y - 3 = 0$
11. $9x^2 - 42x + 49 = 0$
12. $25x^2 + 30x + 9 = 0$
13. $2y^2 + y - 3 = 0$
14. $7x^2 - 11x - 6 = 0$
15. $2x^2 = -15x + 8$
16. $8x - 5 = 3x^2$
17. $x^2 - 36 = 0$
18. $x^2 - 100 = 0$
19. $4x^2 - 81 = 0$
20. $49x^2 - 4 = 0$
21. $x^2 = 4$
22. $9y^2 = 1$
23. $16y^2 = 25$
24. $36x^2 = 25$
25. $4x^2 - 36 = 0$
26. $2x^2 - 18 = 0$
27. $10x^2 + 20x = 0$
28. $-3x^2 + 6x = 0$
29. $25x^2 = 50x$
30. $x^2 = 0$
31. $(x + 1)^2 - 25 = 0$
32. $(x - 2)^2 - 36 = 0$
33. $5x(x - 4) = -4 + x$
34. $(x - 1)(x - 10) = 22$
35. $(x - 3)(x - 5) = 24$
36. $-2x(x - 9) = x + 21$
37. $(x + 1)(6x + 1) = 2x$
38. $(x - 2)(x + 12) = 15x$
39. $(x + 1)(x + 2) = 2(x + 16)$
40. $(x - 9)(2x + 3) = 2(x - 9)$

Answer

1. $-2, 3$
3. $4, 6$
5. $4, 10$
7. 5
9. $-\frac{1}{5}, 2$
11. $\frac{7}{3}$
13. $-\frac{3}{2}, 1$

15. $-8, \frac{1}{2}$
17. $-6, 6$
19. $-\frac{9}{2}, \frac{9}{2}$
21. $-2, 2$
23. $-\frac{5}{4}, \frac{5}{4}$
25. $-3, 3$
27. $-2, 0$
29. $0, 2$
31. $-6, 4$
33. $\frac{1}{5}, 4$
35. $-1, 9$
37. $-\frac{1}{2}, -\frac{1}{3}$
39. $-6, 5$

EXERCISE 3.6.8 SOLVE BY FACTORING

Clear the fractions by first multiplying both sides by the LCD and then solve.

1. $115x^2 + 13x + 25 = 0$
2. $114x^2 - 12x + 37 = 0$
3. $32x^2 - 23 = 0$
4. $52x^2 - 110 = 0$
5. $314x^2 - 212 = 0$
6. $13x^2 - 15x = 0$
7. $132x^2 - 12x + 2 = 0$
8. $13x^2 + 56x - 12 = 0$
9. The sides of a square measure $x + 3$ units. If the area is 25 square units, then find x.
10. The height of a triangle is 2 units more than its base. If the area is 40 square units, then find the length of the base.
11. The sides of a right triangle have measures that are consecutive integers. Find the length of the hypotenuse. (Hint: The hypotenuse is the longest side. Apply the Pythagorean theorem.)
12. The profit in dollars generated by producing and selling x custom lamps is given by the function $P(x) = -10x^2 + 800x - 12000$. How many lamps must be sold and produced to break even? (Hint: We break even when the profit is zero.)

Answer

1. $-3, -2$
3. $-\frac{2}{3}, \frac{2}{3}$
5. ± 7
7. 8
9. 2 units
11. 5 units

EXERCISE 3.6.9 SOLVE BY FACTORING

Assuming dry road conditions and average reaction times, the safe stopping distance, d in feet of an average car is given using the formula $d = 120v^2 + v$, where v represents the speed of the car in miles per hour. For each problem below, given the stopping distance, determine the safe speed.

1. 15 feet
2. 40 feet
3. 75 feet
4. 120 feet

Answer

1. 10 miles per hour
3. 30 miles per hour

EXERCISE 3.6.10 FINDING EQUATIONS WITH GIVEN SOLUTIONS

Find a quadratic equation with integer coefficients, given the following solutions.

1. $-3, 1$
2. $-5, 3$
3. $-10, -3$
4. $-7, -4$
5. $-1, 0$
6. $0, \frac{3}{5}$
7. $-2, 2$
8. $-\frac{1}{2}, \frac{1}{2}$
9. $-4, \frac{1}{3}$
10. $\frac{2}{3}, \frac{2}{5}$
11. $-\frac{1}{5}, -\frac{2}{3}$
12. $-\frac{3}{2}, \frac{3}{4}$
13. 3, double root
14. -5, double root

Answer

1. $x^2 + 2x - 3 = 0$
3. $x^2 + 13x + 30 = 0$
5. $x^2 + x = 0$
7. $x^2 - 4 = 0$
9. $3x^2 + 11x - 4 = 0$
11. $15x^2 + 13x + 2 = 0$
13. $x^2 - 6x + 9 = 0$

EXERCISE 3.6.11 SOLVING POLYNOMIAL EQUATIONS

Solve.

1. $7x(x + 5)(x - 9) = 0$
2. $(x - 1)(x - 2)(x - 3) = 0$
3. $-2x(x - 10)(x - 1) = 0$
4. $8x(x - 4)^2 = 0$
5. $4(x + 3)(x - 2)(x + 1) = 0$
6. $-2(3x + 1)(3x - 1)(x - 1)(x + 1) = 0$
7. $x^3 - x^2 - 2x = 0$
8. $2x^3 + 5x^2 - 3x = 0$
9. $5x^3 - 15x^2 + 10x = 0$
10. $-2x^3 + 2x^2 + 12x = 0$
11. $3x^3 - 27x = 0$
12. $-2x^3 + 8x = 0$
13. $x^3 + x^2 - x - 1 = 0$
14. $x^3 + 2x^2 - 16x - 32 = 0$
15. $8x^3 - 4x^2 - 18x + 9 = 0$
16. $12x^3 = 27x$

Answer

1. $-5, 0, 9$
3. $0, 1, 10$
5. $-3, -1, 2$
7. $-1, 0, 2$
9. $0, 1, 2$
11. $-3, 0, 3$
13. $-1, 1$
15. $-\frac{3}{2}, \frac{1}{2}, \frac{3}{2}$

EXERCISE 3.6.12 DISCUSSION BOARD TOPICS

1. Explain why $2(x+5)(x-5)=0$ has two solutions and $2x(x+5)(x-5)=0$ has three solutions.
2. Make up your own quadratic equation and post it and the solutions on the discussion board.
3. Explain, in your own words, how to solve a quadratic equation in standard form.

Answer

1. Answers may vary
3. Answers may vary

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3.7: APPLICATIONS INVOLVING QUADRATIC EQUATIONS

LEARNING OBJECTIVES

- Set up and solve applications involving relationships between real numbers.
- Set up and solve applications involving geometric relationships involving area and the Pythagorean theorem.
- Set up and solve applications involving the height of projectiles.

NUMBER PROBLEMS

The algebraic setups of the word problems that we have previously encountered led to linear equations. When we translate the applications to algebraic setups in this section, the setups lead to quadratic equations. Just as before, we want to avoid relying on the "guess and check" method for solving applications. Using algebra to solve problems simplifies the process and is more reliable.

EXAMPLE 3.7.1

One integer is 4 less than twice another integer, and their product is 96. Set up an algebraic equation and solve it to find the two integers.

Solution:

First, identify the variables. Avoid two variables by using the relationship between the two unknowns.

Let n represent one of the integers.

$$\underbrace{2n}_{\text{twice}} \quad \underbrace{-\ 4}_{\text{4 less than}}$$

Let $2n-4$ represent the other integer.

Figure 3.7.1

The key phrase, "their product is 96," indicates that we should multiply and set the product equal to 96.

$$n\cdot(2n-4)=96$$

Once we have the problem translated to a mathematical equation, we then solve. In this case, we can solve by factoring. The first step is to write the equation in standard form:

$$\begin{aligned} n\cdot(2n-4)&=96 && \textit{Distribute } n.\\ 2n^2-4n&=96 && \textit{Subtract 96 from both sides.}\\ 2n^2-4n-96&=0 \end{aligned}$$

Next, factor completely and set each variable factor equal to zero.

$$\begin{aligned} 2n^2-4n-96&=0 && \textit{Factor out the GCF, 2.}\\ 2(n^2-2n-48)&=0 && \textit{Factor the resulting trinomial.}\\ 2(n+6)(n-8)&=0 && \textit{Set each variable factor equal to zero.}\\ n+6=0 \quad &\text{or} \quad n-8=0\\ n=-6 \quad &\quad\ \ n=8 \end{aligned}$$

The problem calls for two integers whose product is $+96$. The product of two positive numbers is positive and the product of two negative numbers is positive. Hence we can have two sets of solutions. Use $2n-4$ to determine the other integers.

$$\begin{aligned} n&=-6 & n&=8\\ 2n-4&=2(-6)-4 & 2n-4&=2(8)-4\\ &=-12-4 & &=16-4\\ &=-16 & &=12 \end{aligned}$$

Answer:

Two sets of integers solve this problem: $\{8, 12\}$ and $\{-6, -16\}$. Notice that $(8)(12) = 96$ and $(-6)(-16) = 96$; our solutions check out.

With quadratic equations, we often obtain two solutions for the identified unknown. Although it may be the case that both are solutions to the equation, they may not be solutions to the problem. If a solution does not solve the original application, then we disregard it.

Recall that consecutive odd and even integers both are separated by two units.

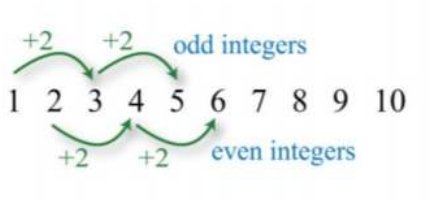


Figure 3.7.2

EXAMPLE 3.7.2

The product of two consecutive positive odd integers is 99. Find the integers.

Solution:

Let n represent the first positive odd integer.

Let $n+2$ represent the next positive odd integer.

The key phase, "product…is 99," indicates that we should multiply and set the product equal to 99.

$$n \cdot (n+2) = 99$$

Rewrite the quadratic equation in standard form and solve by factoring.

$$\begin{aligned} n^2 + 2n &= 99 \\ n^2 + 2n - 99 &= 0 \\ (n-9)(n+11) &= 0 \end{aligned}$$

$$n - 9 = 0 \quad \text{or} \quad n + 11 = 0$$

$$n = 9 \qquad\qquad n = -11$$

Because the problem asks for positive integers, $n = 9$ is the only solution. Back substitute to determine the next odd integer.

$$\begin{aligned} n+2 &= 9+2 \\ &= 11 \end{aligned}$$

Answer:

The consecutive positive odd integers are 9 and 11.

EXAMPLE 3.7.3

Given two consecutive positive odd integers, the product of the larger and twice the smaller is equal to 70. Find the integers.

Solution:

Let n represent the smaller positive odd integer.

Let $n+2$ represent the next positive odd integer.

The key phrase "twice the smaller" can be translated to $2n$. The phrase "product…is 70" indicates that we should multiply this by the larger odd integer and set the product equal to 70.

$$(n+2) \cdot 2n = 70$$

Solve by factoring.

$$\begin{aligned} (n+2) \cdot 2n &= 70 && \textit{Distribute.} \\ 2n^2 + 4n &= 70 && \textit{Subtract 70 from both sides.} \\ 2n^2 + 4n - 70 &= 0 && \textit{Factor out the GCF, 2.} \\ 2(n^2 + 2n - 35) &= 0 && \textit{Factor the resulting trinomial.} \\ 2(n-5)(n+7) &= 0 && \textit{Set each variable factor equal to zero.} \end{aligned}$$

$$n-5=0 \quad \text{or} \quad n+7=0$$
$$n=5 \qquad\qquad n=-7$$

Because the problem asks for positive integers, $n=5$ is the only solution.

Back substitute into $n+2$ to determine the next odd integer.

$$\begin{aligned} n+2 &= 5+2 \\ &= 7 \end{aligned}$$

Answer:

The positive odd integers are 5 and 7.

EXERCISE 3.7.1

The product of two consecutive positive even integers is 168. Find the integers.

Answer

The positive even integers are 12 and 14.

GEOMETRY PROBLEMS

When working with geometry problems, it is helpful to draw a picture. Below are some area formulas that you are expected to know. (Recall that $\pi \approx 3.14$.)

Table 3.7.1

Area of a rectangle:	$A=l\cdot w$
Area of a square:	$A=s^2$
Area of a triangle:	$A=\frac{1}{2}bh$
Area of a circle:	$A=\pi r^2$

EXAMPLE 3.7.4

The floor of a rectangular room has a length that is 4 feet more than twice its width. If the total area of the floor is 240 square feet, then find the dimensions of the floor.

Solution:

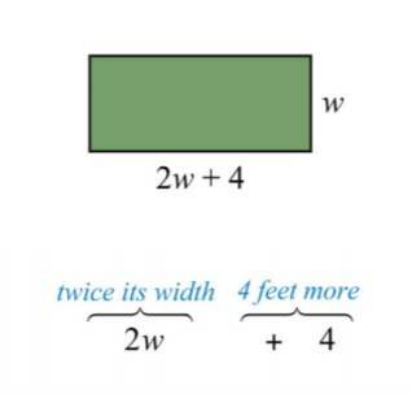


Figure 3.7.3

Use the formula $A=l\cdot w$ and the fact that the area is 240 square feet to set up an algebraic equation.

$$\begin{aligned} A &= l\cdot w \\ 240 &= (2w+4)\cdot w \end{aligned}$$

Solve by factoring.

$$w-10=0 \quad \text{or} \quad w+12=0$$
$$w=10 \qquad\qquad w=-12$$

At this point we have two possibilities for the width of the rectangle. However, since a negative width is not defined, choose the positive solution, $w=10$. Back substitute to find the length.

$$\begin{aligned} 2w+4 &= 2(10)+4 \\ &= 20+4 \\ &= 24 \end{aligned}$$

Answer:

The width is 10 feet and the length is 24 feet.

It is important to include the correct units in the final presentation of the answer. In the previous example, it would not make much sense to say the width is 10. Make sure to indicate that the width is 10 feet.

EXAMPLE 3.7.5

The height of a triangle is 3 inches less than twice the length of its base. If the total area of the triangle is 7 square inches, then find the lengths of the base and height.

Solution:

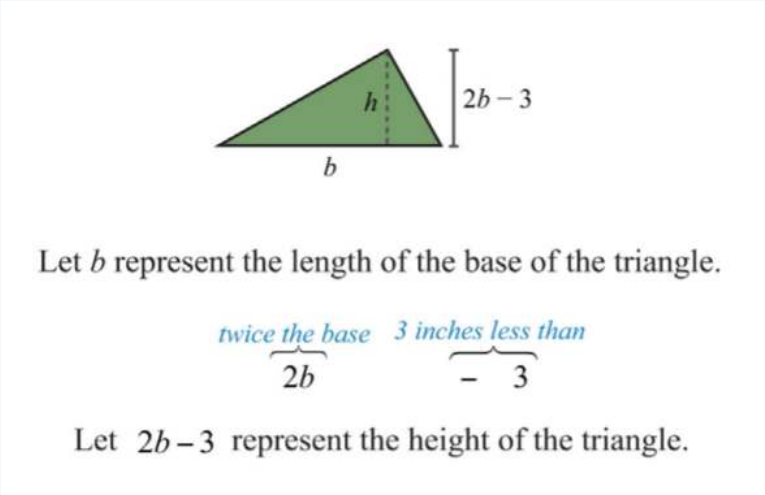


Figure 3.7.4

Use the formula $A = \frac{1}{2}bh$ and the fact that the area is 7 square inches to set up an algebraic equation.

$$\begin{aligned} A &= \frac{1}{2}b \cdot h \\ 7 &= \frac{1}{2}b(2b-3) \end{aligned}$$

To avoid fractional coefficients, multiply both sides by 2 and then rewrite the quadratic equation in standard form.

Factor and then set each factor equal to zero.

$$\begin{aligned} 2b-7=0 \quad &\text{or} \quad b+2=0 \\ 2b=7 \quad &\quad\quad b=-2 \\ b=\tfrac{7}{2} \quad &\end{aligned}$$

In this case, disregard the negative answer; the length of the base is $\frac{7}{2}$ inches long. Use $2b-3$ to determine the height of the triangle.

Answer:

The base measures $\frac{7}{2} = 3\frac{1}{2}$ inches and the height is 4 inches.

EXERCISE 3.7.1

The base of a triangle is 5 units less than twice the height. If the area is 75 square units, then what is the length of the base and height?

Answer

The height is 10 units and the base is 15 units.

Recall that a right triangle is a triangle where one of the angles measures 90°. The side opposite of the right angle is the longest side of the triangle and is called the hypotenuse. The Pythagorean theorem gives us a relationship between the legs and hypotenuse of any right

triangle, where a and b are the lengths of the legs and c is the length of the hypotenuse:

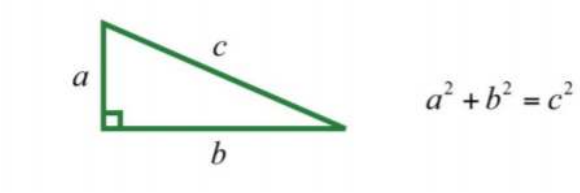


Figure 3.7.5

Given certain relationships, we use this theorem when determining the lengths of sides of right triangles.

EXAMPLE 3.7.6

The hypotenuse of a right triangle is 10 inches. If the short leg is 2 inches less than the long leg, then find the lengths of the legs.

Solution:

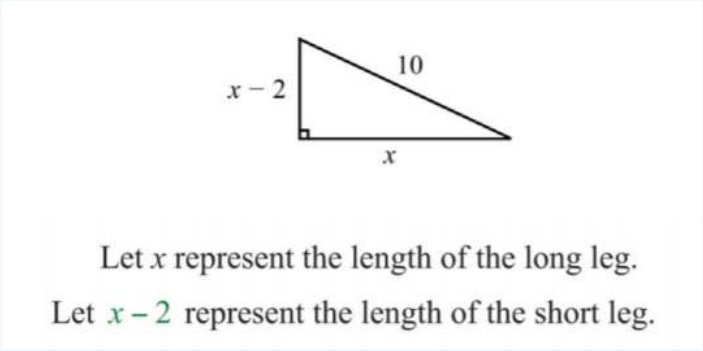


Figure 3.7.6

Given that the hypotenuse measures 10 inches, substitute its value into the Pythagorean theorem and obtain a quadratic equation in terms of x.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (x-2)^2 + x^2 &= 10^2 \end{aligned}$$

Multiply and rewrite the equation in standard form.

$$\begin{aligned} (x-2)^2 + x^2 &= 10^2 \\ x^2 - 4x + 4 + x^2 &= 100 \\ 2x^2 - 4x - 96 &= 0 \end{aligned}$$

Once it is in standard form, factor and set each variable factor equal to zero.

$$\begin{aligned} 2x^2 - 4x - 96 &= 0 \\ 2(x^2 - 2x - 48) &= 0 \\ 2(x+6)(x-8) &= 0 \\ x + 6 = 0 \quad \text{or} \quad & x - 8 = 0 \\ x = -6 \qquad\quad & x = 8 \end{aligned}$$

Because lengths cannot be negative, disregard the negative answer. In this case, the long leg measures 8 inches. Use $x - 2$ to determine the length of the short leg.

$$\begin{aligned} x - 2 &= 8 - 2 \\ &= 6 \end{aligned}$$

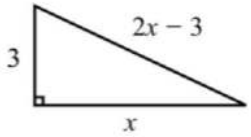
Answer:

The short leg measures 6 inches and the long leg measures 8 inches.

EXAMPLE 3.7.7

One leg of a right triangle measures 3 centimeters. The hypotenuse of the right triangle measures 3 centimeters less than twice the length of the unknown leg. Find the measure of all the sides of the triangle.

Solution:



Let x represent the length of the unknown leg.

Let $2x-3$ represent the length of the unknown leg.

Figure 3.7.7

To set up an algebraic equation, we use the Pythagorean theorem.

$$a^2+b^2 = c^2$$
$$3^2+x^2 = (2x-3)^2$$

Solve by factoring.

$$\begin{aligned} 3^2+x^2 &= (2x-3)^2 \\ 9+x^2 &= 4x^2-12x+9 \\ 0 &= 3x^2-12x \\ 0 &= 3x(x-4) \\ 3x=0 \quad &\text{or} \quad x-4=0 \\ x=0 \quad &\quad\quad x=4 \end{aligned}$$

Disregard 0. The length of the unknown leg is 4 centimeters. Use $2x-3$ to determine the length of the hypotenuse.

Answer:

The sides of the triangle measure 3 centimeters, 4 centimeters, and 5 centimeters.

? EXERCISE 3.7.2

The hypotenuse of a right triangle measures 13 units. If one leg is 2 units more than twice that of the other, then find the length of each leg.

Answer

The two legs measure 5 units and 12 units.

PROJECTILE PROBLEMS

The height of an object launched upward, ignoring the effects of air resistance, can be modeled with the following formula:

$$\text{height} = -\frac{1}{2}gt^2+v_0t+s_0 \tag{3.7.1}$$

Using function notation, which is more appropriate, we have

$$h(t) = -\frac{1}{2}gt^2+v_0t+s_0 \tag{3.7.2}$$

With this formula, the height can be calculated at any given time t after the object is launched. The coefficients represent the following:

Table 3.7.2

$-\frac{1}{2}g$	The letter g represents the acceleration due to gravity.
v_0	"v-naught" represents the initial velocity of the object.
s_0	"s-naught" represents the initial height from which the object is launched.

We consider only problems where the acceleration due to gravity can be expressed as $g = 32$ ft/sec^2. Therefore, in this section time will be measured in seconds and the height in feet. Certainly though, the formula is valid using units other than these.

EXAMPLE 3.7.8

The height of a projectile launched upward at a speed of 32 feet/second from a height of 128 feet is given by the function $h(t) = -16t^2 + 32t + 128$. How long does it take to hit the ground?

Solution:

An inefficient method for finding the time to hit the ground is to simply start guessing at times and evaluating. To do this, construct a chart.

t	h		*Points*
0	128	$h(0) = -16(0)^2 + 32(0) + 128 = 128$	(0, 128)
1	144	$h(1) = -16(1)^2 + 32(1) + 128 = 144$	(1, 144)
2	128	$h(2) = -16(2)^2 + 32(2) + 128 = 128$	(2, 128)
3	80	$h(3) = -16(3)^2 + 32(3) + 128 = 80$	(3, 80)
4	0	$h(4) = -16(4)^2 + 32(4) + 128 = 0$	(4, 0)

Figure 3.7.8

Use the table to sketch the height of the projectile over time.

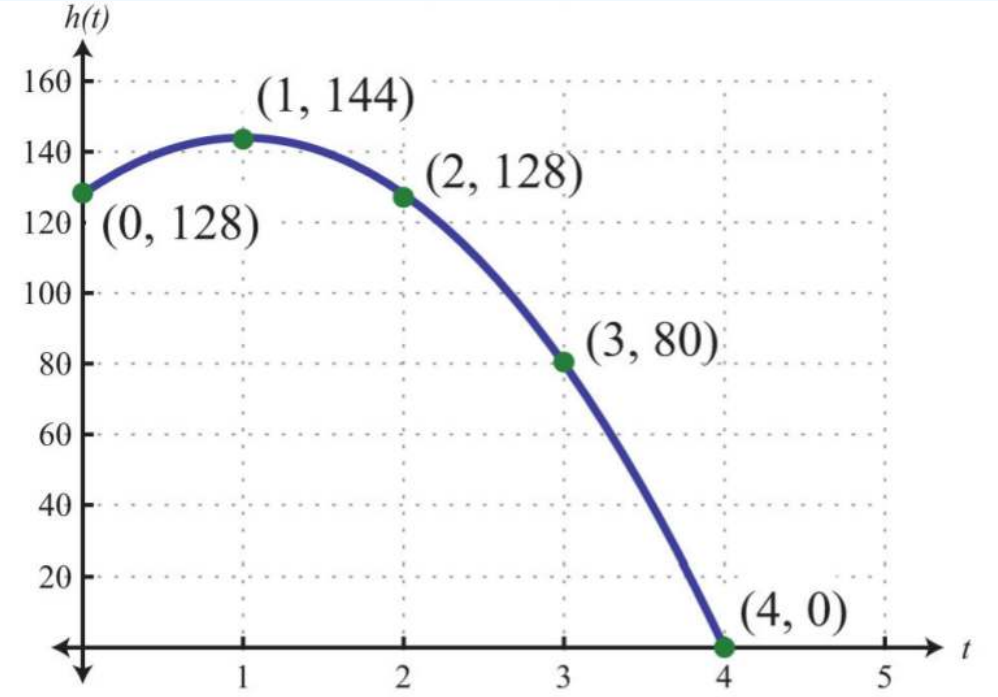


Figure 3.7.9

We see that at 4 seconds, the projectile hits the ground. Note that when this occurs, the height is equal to 0. Now we need to solve this problem algebraically. To find the solution algebraically, use the fact that the height is 0 when the object hits the ground. We need to find the time, t, when $h(t) = 0$.

$$h(t) = -16t^2 + 32t + 128$$
$$\downarrow$$
$$0 = -16t^2 + 32t + 128$$

Solve the equation by factoring

$$0 = -16t^2 + 32t + 128$$
$$0 = -16(t^2 - 2t - 8)$$
$$0 = -16(t - 4)(t + 2)$$

Now set each variable factor to zero.

$$t-4=0 \quad \text{or} \quad t+2=0$$
$$t=4 \qquad\qquad t=-2$$

As expected, the projectile hits the ground at $t=4$ seconds. Disregard -2 as a solution because negative time is not defined.

Answer:

The projectile hits the ground 4 seconds after it is launched.

EXAMPLE 3.7.9

The height of a certain book dropped from the top of a 144 foot building is given by $h(t)=-16t^2+144$. How long does it take to hit the ground?

Solution:

Find the time t when the height $h(t)=0$.

$$0=-16t^2+144$$
$$0=-16(t^2-9)$$
$$0=-16(t+3)(t-3)$$
$$t+3=0 \quad \text{or} \quad t-3=0$$
$$t=-3 \qquad\qquad t=3$$

Answer:

The book takes 3 seconds to hit the ground when dropped from the top of a 144-foot building.

EXERCISE 3.7.3

The height of a projectile, shot straight up into the air from the ground, is given by $h(t)=-16t^2+80t$. How long does it take to come back down to the ground?

Answer

It will take 5 seconds to come back down to the ground.

KEY TAKEAWAYS

- It is best to translate a word problem to a mathematical setup and then solve using algebra. Avoid using the "guess and check" method of solving applications in this section.
- When solving applications, check that your solutions make sense in the context of the question. For example, if you wish to find the length of the base of a triangle, then you would disregard any negative solutions.
- It is important to identify each variable and state in a sentence what each variable represents. It is often helpful to draw a picture.

EXERCISE 3.7.4 NUMBER PROBLEMS

Set up an algebraic equation and then solve.

1. One integer is five times another. If the product of the two integers is 80, then find the integers.
2. One integer is four times another. If the product of the two integers is 36, then find the integers.
3. An integer is one more than four times another. If the product of the two integers is 39, then find the integers.
4. An integer is 3 more than another. If the product of the two integers is 130, then find the integers.
5. An integer is 2 less than twice another. If the product of the two integers is 220, then find the integers.
6. An integer is 3 more than twice another. If the product of the two integers is 90, then find the integers.
7. One integer is 2 units more than another. If the product of the two integers is equal to five times the larger, then find the two integers.
8. A positive integer is 1 less than twice another. If the product of the two integers is equal to fifteen times the smaller, then find the two integers.
9. A positive integer is 3 more than twice a smaller positive integer. If the product of the two integers is equal to six times the larger, then find the integers.
10. One positive integer is 3 more than another. If the product of the two integers is equal to twelve times the smaller, then find the integers.

11. An integer is 3 more than another. If the product of the two integers is equal to 2 more than four times their sum, then find the integers.
12. An integer is 5 more than another. If the product of the two integers is equal to 2 more than twice their sum, then find the integers.
13. The product of two consecutive positive even integers is 120. Find the integers.
14. The product of two consecutive positive odd integers is 99. Find the integers.
15. The product of two consecutive positive integers is 110. Find the integers.
16. The product of two consecutive positive integers is 42. Find the integers.
17. The product of two consecutive positive odd integers is equal to 1 less than seven times the sum of the integers. Find the integers.
18. The product of two consecutive positive even integers is equal to 22 more than eleven times the sum of the integers. Find the integers.
19. The sum of the squares of two consecutive positive odd integers is 74. Find the integers.
20. The sum of the squares of two consecutive positive even integers is 100. Find the integers.
21. The sum of the squares of two consecutive positive integers is 265. Find the integers.
22. The sum of the squares of two consecutive positive integers is 181. Find the integers.
23. For two consecutive positive odd integers, the product of twice the smaller and the larger is 126. Find the integers.
24. For two consecutive positive even integers, the product of the smaller and twice the larger is 160. Find the integers.

Answer

1. $\{4, 20\}$ or $\{-4, -20\}$
3. $3, 13$
5. $\{11, 20\}$ or $\{-22, -10\}$
7. $\{5, 7\}$ or $\{-2, 0\}$
9. $6, 15$
11. $\{7, 10\}$ or $\{-2, 1\}$
13. $10, 12$
15. $10, 11$
17. $13, 15$
19. $5, 7$
21. $11, 12$
23. $7, 9$

? EXERCISE 3.7.5 GEOMETRY PROBLEMS

Set up an algebraic equation and then solve.

1. The width of a rectangle is 7 feet less than its length. If the area of the rectangle is 170 square feet, then find the length and width.
2. The length of a rectangle is 2 feet more than its width. If the area of the rectangle is 48 square feet, then find the length and width.
3. The width of a rectangle is 3 units less than the length. If the area is 70 square units, then find the dimensions of the rectangle.
4. The width of a rectangle measures one half of the length. If the area is 72 square feet, then find the dimensions of the rectangle.
5. The length of a rectangle is twice that of its width. If the area of the rectangle is 72 square inches, then find the length and width.
6. The length of a rectangle is three times that of its width. If the area of the rectangle is 75 square centimeters, then find the length and width.
7. The length of a rectangle is 2 inches more than its width. The area of the rectangle is equal to 12 inches more than three times the perimeter. Find the length and width of the rectangle.
8. The length of a rectangle is 3 meters more than twice the width. The area of the rectangle is equal to 10 meters less than three times the perimeter. Find the length and width of the rectangle.
9. A uniform border is to be placed around an 8-inch-by-10-inch picture. If the total area including the border must be 224 square inches, then how wide should the border be?

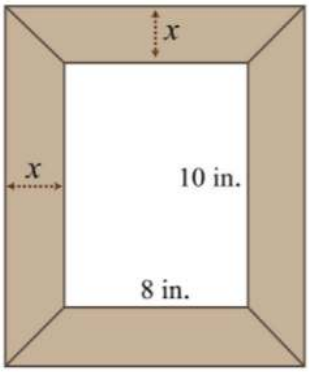


Figure 3.7.10

10. A 2-foot brick border is constructed around a square cement slab. If the total area, including the border, is 121 square feet, then what are the dimensions of the slab?

11. The area of a picture frame including a 2-inch wide border is 99 square inches. If the width of the inner area is 2 inches more than its length, then find the dimensions of the inner area.

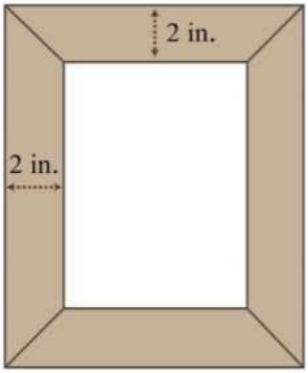


Figure 3.7.11

12. A box can be made by cutting out the corners and folding up the edges of a square sheet of cardboard. A template for a cardboard box with a height of 2 inches is given. What is the length of each side of the cardboard sheet if the volume of the box is to be 50 cubic inches?

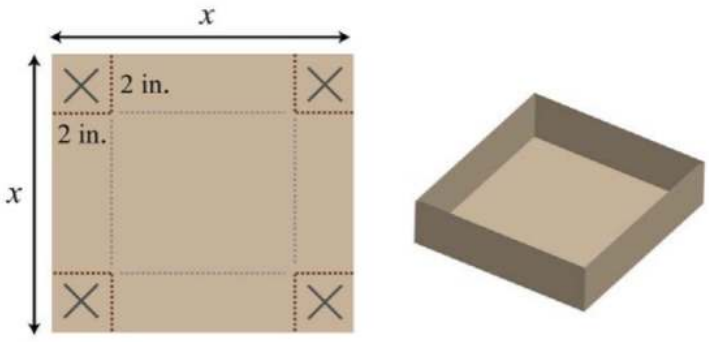


Figure 3.7.12

13. The height of a triangle is 3 inches more than the length of its base. If the area of the triangle is 44 square inches, then find the length of its base and height.

14. The height of a triangle is 4 units less than the length of the base. If the area of the triangle is 48 square units, then find the length of its base and height.

15. The base of a triangle is twice that of its height. If the area is 36 square centimeters, then find the length of its base and height.

16. The height of a triangle is three times the length of its base. If the area is $73\frac{1}{2}$ square feet, then find the length of the base and height.

17. The height of a triangle is 1 unit more than the length of its base. If the area is 5 units more than four times the height, then find the length of the base and height of the triangle.

18. The base of a triangle is 4 times that of its height. If the area is 3 units more than five times the height, then find the length of the base and height of the triangle.

19. The diagonal of a rectangle measures 5 inches. If the length is 1 inch more than its width, then find the dimensions of the rectangle.

20. The diagonal of a rectangle measures 10 inches. If the width is 2 inches less than the length, then find the area of the rectangle.

21. If the sides of a right triangle are consecutive even integers, then what are their measures?

22. The hypotenuse of a right triangle is 13 units. If the length of one leg is 2 more than twice the other, then what are their lengths?

23. The shortest leg of a right triangle measures 9 centimeters and the hypotenuse measures 3 centimeters more than the longer leg. Find the length of the hypotenuse.

24. The long leg of a right triangle measures 24 centimeters and the hypotenuse measures 4 centimeters more three times the short leg. Find the length of the hypotenuse.

Answer

1. Length: 17 feet; width: 10 feet
3. Length: 10 units; width: 7 units
5. Length: 12 inches; width: 6 inches
7. Length: 14 inches; width: 12 inches
9. 3 inches
11. 5 inches by 7 inches
13. Base: 8 inches; height: 11 inches
15. Base: 12 centimeters; height: 6 centimeters
17. Base: 9 units; height: 10 units
19. 3 inches by 4 inches
21. 6 units, 8 units, and 10 units
23. 15 centimeters

? EXERCISE 3.7.6 PROJECTILE PROBLEMS

Set up an algebraic equation and then solve.

1. The height of a projectile launched upward at a speed of 32 feet/second from a height of 48 feet is given by the function \(h(t)=−16t^{2}+32t+48. How long will it take the projectile to hit the ground?
2. The height of a projectile launched upward at a speed of 16 feet/second from a height of 192 feet is given by the function $h(t)=-16t^2+16t+192$. How long will it take to hit the ground?
3. An object launched upward at a speed of 64 feet/second from a height of 80 feet. How long will it take the projectile to hit the ground?
4. An object launched upward at a speed of 128 feet/second from a height of 144 feet. How long will it take the projectile to hit the ground?
5. The height of an object dropped from the top of a 64-foot building is given by $h(t)=-16t^2+64$. How long will it take the object to hit the ground?
6. The height of an object dropped from an airplane at $1,600$ feet is given by $h(t)=-16t^2+1,600$. How long will it take the object to hit the ground?
7. An object is dropped from a ladder at a height of 16 feet. How long will it take to hit the ground?
8. An object is dropped from a 144-foot building. How long will it take to hit the ground?
9. The height of a projectile, shot straight up into the air from the ground at 128 feet/second, is given by $h(t)=-16t^2+128t$. How long does it take to come back down to the ground?
10. A baseball, tossed up into the air from the ground at 32 feet/second, is given by $h(t)=-16t^2+32t$. How long does it take to come back down to the ground?
11. How long will it take a baseball thrown into the air at 48 feet/second to come back down to the ground?

12. A football is kicked up into the air at 80 feet/second. Calculate how long will it hang in the air.

Answer

1. 3 seconds
3. 5 seconds
5. 2 seconds
7. 1 second
9. 8 seconds
11. 3 seconds

? EXERCISE 3.7.7 DISCUSSION BOARD TOPICS

1. Research and discuss the life of Pythagoras.
2. If the sides of a square are doubled, then by what factor is the area increased? Why?
3. Design your own geometry problem involving the area of a rectangle or triangle. Post the question and a complete solution on the discussion board.
4. Write down your strategy for setting up and solving word problems. Share your strategy on the discussion board.

Answer

1. Answers may vary
3. Answers may vary

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3.E: REVIEW EXERCISES AND SAMPLE EXAM

REVIEW EXERCISES

EXERCISE $3.E.1$ INTRODUCTION TO FACTORING

Determine the missing factor.

1. $12x^3 - 24x^2 + 4x = 4x($? $)$
2. $10y^4 - 35y^3 - 5y^2 = 5y^2($? $)$
3. $-18a^5 + 9a^4 - 27a^3 = -9a^3($? $)$
4. $-21x^2y + 7xy^2 - 49xy = -7xy($? $)$

Answer

1. $(3x^2 - 6x + 1)$
3. $(2a^2 - a + 3)$

EXERCISE $3.E.2$ INTRODUCTION TO FACTORING

Factor out the GCF.

1. $22x^2 + 11x$
2. $15y^4 - 5y^3$
3. $18a^3 - 12a^2 + 30a$
4. $12a^5 + 20a^3 - 4a$
5. $9x^3y^2 - 18x^2y^2 + 27xy^2$
6. $16a^5b^5c - 8a^3b^6 + 24a^3b^2c$

Answer

1. $11x(2x + 1)$
3. $6a(3a^2 - 2a + 5)$
5. $9xy2(x^2 - 2x + 3)$

EXERCISE $3.E.3$ INTRODUCTION TO FACTORING

Factor by grouping.

1. $x^2 + 2x - 5x - 10$
2. $2x^2 - 2x - 3x + 3$
3. $x^3 + 5x^2 - 3x - 15$
4. $x^3 - 6x^2 + x - 6$
5. $x^3 - x^2y - 2x + 2y$
6. $a^2b^2 - 2a^3 + 6ab - 3b^3$

Answer

1. $(x + 2)(x - 5)$
3. $(x + 5)(x^2 - 3)$
5. $(x - y)(x^2 - 2)$

EXERCISE $3.E.4$ FACTORING TRINOMIALS OF THE FORM $x^2 + bx + c$

Are the following factored correctly? Check by multiplying.

1. $x^2 + 5x + 6 = (x + 6)(x - 1)$
2. $x^2 + 3x - 10 = (x + 5)(x - 2)$
3. $x^2 + 6x + 9 = (x + 3)^2$

4. $x^2 - 6x - 9 = (x-3)(x+3)$

Answer

1. No
3. Yes

EXERCISE 3.E.5 FACTORING TRINOMIALS OF THE FORM $x^2 + bx + c$

Factor.

1. $x^2 - 13x - 14$
2. $x^2 + 13x + 12$
3. $y^2 + 10y + 25$
4. $y^2 - 20y + 100$
5. $a^2 - 8a - 48$
6. $b^2 - 18b + 45$
7. $x^2 + 2x + 24$
8. $x^2 - 10x - 16$
9. $a^2 + ab - 2b^2$
10. $a^2b^2 + 5ab - 50$

Answer

1. $(x-14)(x+1)$
3. $(y+5)^2$
5. $(a-12)(a+4)$
7. Prime
9. $(a-b)(a+2b)$

EXERCISE 3.E.6 FACTORING TRINOMIALS OF THE FORM $ax^2 + bx + c$

Factor.

1. $5x^2 - 27x - 18$
2. $3x^2 - 14x + 8$
3. $4x^2 - 28x + 49$
4. $9x^2 + 48x + 64$
5. $6x^2 - 29x - 9$
6. $8x^2 + 6x + 9$
7. $60x^2 - 65x + 15$
8. $16x^2 - 40x + 16$
9. $6x^3 - 10x^2y + 4xy^2$
10. $10x^3y - 82x^2y^2 + 16xy^3$
11. $-y^2 + 9y + 36$
12. $-a^2 - 7a + 98$
13. $16 + 142x - 18x^2$
14. $45 - 132x - 60x^2$

Answer

1. $(5x+3)(x-6)$
3. $(2x-7)^2$
5. Prime
7. $5(3x-1)(4x-3)$
9. $2x(3x-2y)(x-y)$
11. $-1(y-12)(y+3)$

13. $-2(9x+1)(x-8)$

EXERCISE 3.E.7 FACTORING SPECIAL BINOMIALS

Factor completely.

1. x^2-81
2. $25x^2-36$
3. $4x^2-49$
4. $81x^2-1$
5. x^2-64y^2
6. $100x^2y^2-1$
7. $16x^4-y^4$
8. x^4-81y^4
9. $8x^3-125$
10. $27+y^3$
11. $54x^4y-2xy^4$
12. $3x^4y^2+24xy^5$
13. $64x^6-y^6$
14. x^6+1

Answer

1. $(x+9)(x-9)$
3. $(2x+7)(2x-7)$
5. $(x+8y)(x-8y)$
7. $(4x^2+y^2)(2x+y)(2x-y)$
9. $(2x-5)(4x^2+10x+25)$
11. $2xy(3x-y)(9x^2+3xy+y^2)$
13. $(2x+y)(4x^2-2xy+y^2)(2x-y)(4x^2+2xy+y^2)$

EXERCISE 3.E.8 GENERAL GUIDELINES FOR FACTORING POLYNOMIALS

Factor completely.

1. $8x^3-4x^2+20x$
2. $50a^4b^4c+5a^3b^5c^2$
3. x^3-12x^2-x+12
4. $a^3-2a^2-3ab+6b$
5. $-y^2-15y+16$
6. $x^2-18x+72$
7. $144x^2-25$
8. $3x^4-48$
9. $20x^2-41x-9$
10. $24x^2+14x-20$
11. $a^4b-343ab^4$
12. $32x^7y^2+4xy^8$

Answer

1. $4x(2x^2-x+5)$
3. $(x-12)(x+1)(x-1)$
5. $-1(y+16)(y-1)$
7. $(12x+5)(12x-5)$
9. $(4x-9)(5x+1)$
11. $ab(a-7b)(a^2+7ab+49b^2)$

EXERCISE $3.E.9$ SOLVING EQUATIONS BY FACTORING

Solve.

1. $(x-9)(x+10)=0$
2. $-3x(x+8)=0$
3. $6(x+1)(x-1)=0$
4. $(x-12)(x+4)(2x-1)=0$
5. $x^2+5x-50=0$
6. $3x^2-13x+4=0$
7. $3x^2-12=0$
8. $16x^2-9=0$
9. $(x-2)(x+6)=20$
10. $2(x-2)(x+3)=7x-9$
11. $52x^2-203x=0$
12. $23x^2-512x+124=0$

Answer

1. $9,-10$
3. $-1,1$
5. $-10,5$
7. ± 2
9. $-8,4$
11. $0,\frac{8}{3}$

EXERCISE $3.E.10$ SOLVING EQUATIONS BY FACTORING

Find a quadratic equation with integer coefficients, given the following solutions.

1. $-7,6$
2. $0,-10$
3. $-\frac{1}{9},\frac{1}{2}$
4. $\pm\frac{3}{2}$

Answer

1. $x^2+x-42=0$
3. $18x^2-7x-1=0$

EXERCISE $3.E.11$ APPLICATIONS INVOLVING QUADRATIC EQUATIONS

Set up an algebraic equation and then solve the following.

1. An integer is 4 less than twice another. If the product of the two integers is 96, then find the integers.
2. The sum of the squares of two consecutive positive even integers is 52. Find the integers.
3. A 20-foot ladder leaning against a wall reaches a height that is 4 feet more than the distance from the wall to the base of the ladder. How high does the ladder reach?
4. The height of an object dropped from the top of a 196-foot building is given by $h(t)=-16t^2+196$, where t represents the number of seconds after the object has been released. How long will it take the object to hit the ground?
5. The length of a rectangle is 1 centimeter less than three times the width. If the area is 70 square centimeters, then find the dimensions of the rectangle.
6. The base of a triangle is 4 centimeters more than twice the height. If the area of the triangle is 80 square centimeters, then find the measure of the base.

Answer

1. $\{8,12\}$ or $\{-6,-16\}$

3. 16 feet

5. Length: 14 centimeters; width: 5 centimeters

SAMPLE EXAM

EXERCISE 3.E.12

1. Determine the GCF of the terms $25a^2b^2c, 50ab^4$, and $35a^3b^3c^2$.
2. Determine the missing factor: $24x^2y^3 - 16x^3y^2 + 8x^2y = 8x^2y($? $)$.

Answer

1. $5ab^2$

EXERCISE 3.E.13

Factor.

1. $12x^5 - 15x^4 + 3x^2$
2. $x^3 - 4x^2 - 2x + 8$
3. $x^2 - 7x + 12$
4. $9x^2 - 12x + 4$
5. $x^2 - 81$
6. $x^3 + 27y^3$

Answer

1. $3x^2(4x^3 - 5x^2 + 1)$
3. $(x - 4)(x - 3)$
5. $(x + 9)(x - 9)$

EXERCISE 3.E.14

Factor completely.

1. $x^3 + 2x^2 - 4x - 8$
2. $x^4 - 1$
3. $-6x^3 + 20x^2 - 6x$
4. $x^6 - 1$

Answer

1. $(x + 2)^2(x - 2)$
3. $-2x(3x - 1)(x - 3)$

EXERCISE 3.E.15

Solve.

1. $(2x + 1)(x - 7) = 0$
2. $3x(4x - 3)(x + 1) = 0$
3. $x^2 - 64 = 0$
4. $x^2 + 4x - 12 = 0$
5. $23x^2 + 89x - 16 = 0$
6. $(x - 5)(x - 3) = -1$
7. $3x(x + 3) = 14x + 2$
8. $(3x + 1)(3x + 2) = 9x + 3$

Answer

1. $-\frac{1}{2}, 7$
3. ± 8
5. $-\frac{3}{2}, \frac{1}{6}$
7. $-\frac{1}{3}, 2$

EXERCISE $3.E.16$

For each problem, set up an algebraic equation and then solve.

1. An integer is 4 less than twice another. If the product of the two integers is 70, then find the integers.
2. The sum of the squares of two consecutive positive odd integers is 130. Find the integers.
3. The length of a rectangle is 4 feet more than twice its width. If the area is 160 square feet, then find the dimensions of the rectangle.
4. The height of a triangle is 6 centimeters less than four times the length of its base. If the area measures 27 square centimeters, then what is the height of the triangle?
5. The height of a projectile launched upward at a speed of 64 feet/second from a height of 36 feet is given by the function $h(t) = -16t^2 + 64t + 36$. How long will it take the projectile to hit the ground?

Answer

1. $\{7, 10\}$ or $\{-14, -5\}$
3. Width: 8 feet; length: 20 feet
5. $4\frac{1}{2}$ sec

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