PRECALCULUS: ALGEBRA AND TRIGONOMETRY (MATH 4 - FCC)



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Precalculus: Algebra and Trigonometry (Math 4 - FCC)

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Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: Solving Equations

1.1: Solving Linear Equations
1.2: Solving Absolute Value Equations
1.3: Solving Polynomial Equations
1.4: Solving Quadratic Equations
1.5: Solving Rational Equations
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1.1: Solving Linear Equations

Solve Equations Using the General Strategy

Until now we have dealt with solving one specific form of a linear equation. It is time now to lay out one overall strategy that can be used to solve any linear equation. Some equations we solve will not require all these steps to solve, but many will.

Beginning by simplifying each side of the equation makes the remaining steps easier.

Example 1.1.1: How to Solve Linear Equations Using the General Strategy

Solve: $-6(x+3)=24$.

Solution

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is simplified as much as possible.	$-6(x+3)=24$ $-6x-18=24$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do – all x's are on the left side.	
Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 18 to each side. Simplify.	$-6x-18+18=24+18$ $-6x=42$
Step 4. Make the coefficient of the variable term to equal to 1.	Divide each side by –6. Simplify.	$\frac{-6x}{-6}=\frac{42}{-6}$ $x=-7$
Step 5. Check the solution.	Let $x=-7$ Simplify. Multiply.	**Check:** $-6(x+3)=24$ $-6(-7+3)\stackrel{?}{=}24$ $-6(-4)\stackrel{?}{=}24$ $24=24$ ✓

Try It 1.1.2

Solve: $5(x+3)=35$

Answer

$x=4$

Try It 1.1.3

Solve: $6(y-4)=-18$

Answer

$y=1$

GENERAL STRATEGY FOR SOLVING LINEAR EQUATIONS.

1. **Simplify each side of the equation as much as possible.**
   Use the Distributive Property to remove any parentheses.
   Combine like terms.
2. **Collect all the variable terms on one side of the equation.**
   Use the Addition or Subtraction Property of Equality.
3. **Collect all the constant terms on the other side of the equation.**
   Use the Addition or Subtraction Property of Equality.
4. **Make the coefficient of the variable term to equal to 1.**
   Use the Multiplication or Division Property of Equality.
   State the solution to the equation.
5. **Check the solution.** Substitute the solution into the original equation to make sure the result is a true statement.

Example 1.1.4

Solve: $-(y+9)=8$

Solution

	$-(y+9)=8$
Simplify each side of the equation as much as possible by distributing.	$-y-9=8$
The only y term is on the left side, so all variable terms are on the left side of the equation.	
Add 9 to both sides to get all constant terms on the right side of the equation.	$-y-9+9=8+9$
Simplify.	$-y=17$
Rewrite $-y$ as $-1y$.	$-1y=17$
Make the coefficient of the variable term to equal to 1 by dividing both sides by -1.	$\frac{-1y}{-1}=\frac{17}{-1}$
Simplify.	$y=-17$
Check:	$-(y+9)=8$
Let $y=-17$.	$-(-17+9)\stackrel{?}{=}8$
	$-(-8)\stackrel{?}{=}8$
	$8=8\checkmark$

Try It 1.1.5

Solve: $-(y+8)=-2$

Answer

$y=-6$

Try It 1.1.6

Solve: $-(z+4)=-12$

Answer

$z=8$

Example 1.1.7

Solve: $5(a-3)+5=-10$

Solution

	$5(a-3)+5=-10$
Simplify each side of the equation as much as possible.	
Distribute.	$5a-15+5=-10$
Combine like terms.	$5a-10=-10$
The only a term is on the left side, so all variable terms are on one side of the equation.	
Add 10 to both sides to get all constant terms on the other side of the equation.	$5a-10+10=-10+10$
Simplify.	$5a=0$
Make the coefficient of the variable term to equal to 11 by dividing both sides by 55.	$\frac{5a}{5}=\frac{0}{5}$
Simplify.	$a=0$
Check:	$5(a-3)+5=-10$
Let a=0.	$5(0-3)+5\stackrel{?}{=}-10$
	$5(-3)+5\stackrel{?}{=}-10$
	$-15+5\stackrel{?}{=}-10$
	$-10=-10$ ✓

Try It 1.1.8

Solve: $2(m-4)+3=-1$

Answer

$m=2$

Try It 1.1.9

Solve:$7(n-3)-8=-15$

Answer

$n=2$

Example 1.1.10

Solve: $\frac{2}{3}(6m-3)=8-m$

Solution

	$\frac{2}{3}(6m-3)=8-m$
Distribute.	$4m-2=8-m$

Add m to get the variables only to the left.	$4m + m - 2 = 8 - m + m$
Simplify.	$5m - 2 = 8$
Add 2 to get constants only on the right.	$5m - 2 + 2 = 8 + 2$
Simplify.	$5m = 10$
Divide by 5.	$\frac{5m}{5} = \frac{10}{5}$
Simplify.	$m = 2$

Check:	$\frac{2}{3}(6m - 3) = 8 - m$
Let m=2.	$\frac{2}{3}(6 \cdot 2 - 3) \stackrel{?}{=} 8 - 2$
	$\frac{2}{3}(12 - 3) \stackrel{?}{=} 6$
	$\frac{2}{3}(9) \stackrel{?}{=} 6$
	$6 = 6$ ✓

Try It 1.1.11

Solve: $\frac{1}{3}(6u + 3) = 7 - u$

Answer

$u = 2$

Try It 1.1.12

Solve: $\frac{2}{3}(9x - 12) = 8 + 2x$

Answer

$x = 4$

Example 1.1.13

Solve: $8 - 2(3y + 5) = 0$

Solution

	$8 - 2(3y + 5) = 0$
Simplify—use the Distributive Property.	$8 - 6y - 10 = 0$
Combine like terms.	$-6y - 2 = 0$
Add 2 to both sides to collect constants on the right.	$-6y - 2 + 2 = 0 + 2$
Simplify.	$-6y = 2$
Divide both sides by $-6-6$.	$\frac{-6y}{-6} = \frac{2}{-6}$
Simplify.	$y = -\frac{1}{3}$

Check: Let y=−13.

$$8-2(3y+5)=0$$

$$8-2\left[3\left(-\frac{1}{3}\right)+5\right]=0$$

$$8-2(-1+5)\stackrel{?}{=}0$$

$$8-2(4)\stackrel{?}{=}0$$

$$8-8\stackrel{?}{=}0$$

$$0=0\checkmark$$

Try It 1.1.14

Solve: $12-3(4j+3)=-17$

Answer

$j=\frac{5}{3}$

Try It 1.1.15

Solve: $-6-8(k-2)=-10$

Answer

$k=\frac{5}{2}$

Example 1.1.16

Solve: $4(x-1)-2=5(2x+3)+6$

Solution

	$4(x-1)-2=5(2x+3)+6$
Distribute.	$4x-4-2=10x+15+6$
Combine like terms.	$4x-6=10x+21$
Subtract 4x to get the variables only on the right side since $10>4$.	$4x-4x-6=10x-4x+21$
Simplify.	$-6=6x+21$
Subtract 21 to get the constants on left.	$-6-21=6x+21-21$
Simplify.	$-27=6x$
Divide by 6.	$\frac{-27}{6}=\frac{6x}{6}$
Simplify.	$-\frac{9}{2}=x$
Check:	$4(x-1)-2=5(2x+3)+6$
Let $x=-\frac{9}{2}$.	$4\left(-\frac{9}{2}-1\right)-2\stackrel{?}{=}5\left[2\left(-\frac{9}{2}\right)+3\right]+6$
	$4\left(-\frac{11}{2}\right)-2\stackrel{?}{=}5(-9+3)+6$

	$-22-2\stackrel{?}{=}5(-6)+6$	
	$-24\stackrel{?}{=}-30+6$	
	$-24=-24$ ✓	

? Try It 1.1.17

Solve: $6(p-3)-7=5(4p+3)-12$

Answer

$p=-2$

? Try It 1.1.18

Solve: $8(q+1)-5=3(2q-4)-1$

Answer

$q=-8$

✓ Example 1.1.19

Solve: $10[3-8(2s-5)]=15(40-5s)$

Solution

	$10[3-8(2s-5)]=15(40-5s)$
Simplify from the innermost parentheses first.	$10[3-16s+40]=15(40-5s)$
Combine like terms in the brackets.	$10[43-16s]=15(40-5s)$
Distribute.	$430-160s=600-75s$
Add 160s to get the s's to the right.	$430-160s+160s=600-75s+160s$
Simplify.	$430=600+85s$
Subtract 600 to get the constants to the left.	$430-600=600+85s-600$
Simplify.	$-170=85s$
Divide.	$\frac{-170}{85}=\frac{85s}{85}$
Simplify.	$-2=s$

Check:	$10[3-8(2s-5)]=15(40-5s)$
Substitute s=−2.	$10[3-8(2(-2)-5)]\stackrel{?}{=}15(40-5(-2))$
	$10[3-8(-4-5)]\stackrel{?}{=}15(40+10)$
	$10[3-8(-9)]\stackrel{?}{=}15(50)$
	$10[3+72]\stackrel{?}{=}750$
	$10[75]\stackrel{?}{=}750$
	$750=750$ ✓

Try It 1.1.20

Solve: $6[4-2(7y-1)]=8(13-8y)$.

Answer

$y=-\frac{17}{5}$

Try It 1.1.21

Solve: $12[1-5(4z-1)]=3(24+11z)$.

Answer

$z=0$

Example 1.1.22

Solve: $0.36(100n+5)=0.6(30n+15)$.

Solution

	$0.36(100n + 5) = 0.6(30n + 15)$
Distribute.	$36n + 1.8 = 18n + 9$
Subtract 18n to get the variables to the left.	$36n - 18n + 1.8 = 18n - 18n + 9$
Simplify.	$18n + 1.8 = 9$
Subtract 1.8 to get the constants to the right.	$18n + 1.8 - 1.8 = 9 - 1.8$
Simplify.	$18n = 7.2$
Divide.	$\frac{18n}{18} = \frac{7.2}{18}$
Simplify.	$n = 0.4$
Check:	$0.36(100n + 5) = 0.6(30n + 15)$
Let n=0.4.	$0.36(100(0.4) + 5) \stackrel{?}{=} 0.6(30(0.4) + 15)$
	$0.36(40 + 5) \stackrel{?}{=} 0.6(12 + 15)$
	$0.36(45) \stackrel{?}{=} 0.6(27)$
	$16.2 = 16.2$ ✓

Try It 1.1.23

Solve: $0.55(100n+8)=0.6(85n+14)$.

Answer

$n=1$

Try It 1.1.24

Solve: $0.15(40m-120)=0.5(60m+12)$.

Answer

$m=-1$

Classify Equations

Consider the equation we solved at the start of the last section, 7x+8=−13. The solution we found was x=−3. This means the equation 7x+8=−13 is true when we replace the variable, *x*, with the value −3. We showed this when we checked the solution x=−3 and evaluated 7x+8=−13 for x=−3.

$$\begin{aligned} 7(-3)+8 &\stackrel{?}{=} -13 \\ -21+8 &\stackrel{?}{=} -13 \\ -13 &= -13 \checkmark \end{aligned}$$

If we evaluate 7x+8 for a different value of *x*, the left side will not be −13.

The equation 7x+8=−13 is true when we replace the variable, *x*, with the value −3, but not true when we replace *x* with any other value. Whether or not the equation 7x+8=−13 is true depends on the value of the variable. Equations like this are called conditional equations.

All the equations we have solved so far are conditional equations.

CONDITIONAL EQUATION

An equation that is true for one or more values of the variable and false for all other values of the variable is a **conditional equation**.

Now let's consider the equation 2y+6=2(y+3). Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for *y*.

	$2y+6=2(y+3)$
Distribute.	$2y+6=2y+6$
Subtract 2y to get the y's to one side.	$2y-2y+6=2y-2y+6$
Simplify—the y's are gone!	$6=6$

But 6=6 is true.

This means that the equation 2y+6=2(y+3) is true for any value of *y*. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable like this is called an **identity**.

IDENTITY

An equation that is true for any value of the variable is called an **identity**.

The solution of an identity is all real numbers.

What happens when we solve the equation 5z=5z−1?

	$5z=5z-1$
Subtract 5z to get the constant alone on the right.	$5z-5z=5z-5z-1$
Simplify—the z's are gone!	$0\neq -1$

But $0 \neq -1$.

Solving the equation 5z=5z−1 led to the false statement 0=−1. The equation 5z=5z−1 will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a **contradiction**.

CONTRADICTION

An equation that is false for all values of the variable is called a **contradiction**.

A contradiction has no solution.

Example 1.1.25

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$6(2n-1)+3=2n-8+5(2n+1)$

Solution

$6(2n-1)+3=2n-8+5(2n+1)$	
Distribute.	$12n-6+3=2n-8+10n+5$
Combine like terms.	$12n-3=12n-3$
Subtract 12n to get the nn's to one side.	$12n-12n-3=12n-12n-3$
Simplify.	$-3=-3$
This is a true statement.	The equation is an identity. The solution is all real numbers.

Try It 1.1.26

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$4+9(3x-7)=-42x-13+23(3x-2)$

Answer

identity; all real numbers

Try It 1.1.27

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$8(1-3x)+15(2x+7)=2(x+50)+4(x+3)+1$

Answer

identity; all real numbers

Example 1.1.28

Classify as a conditional equation, an identity, or a contradiction. Then state the solution.

$10+4(p-5)=0$

Solution

	$10 + 4(p - 5) = 0$
Distribute.	$10 + 4p - 20 = 0$
Combine like terms.	$4p - 10 = 0$
Add 10 to both sides.	$4p - 10 + 10 = 0 + 10$
Simplify.	$4p = 10$
Divide.	$\frac{4p}{4} = \frac{10}{4}$
Simplify.	$p = \frac{5}{2}$
The equation is true when $p = frac52$.	This is a conditional equation. The solution is $p = frac52$.

Try It 1.1.29

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11(q+3)-5=19$

Answer

conditional equation; \(q = \frac{9}{11}\

Try It 1.1.30

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6+14(k-8)=95$

Answer

conditional equation; $k=\frac{193}{14}$

Example 1.1.31

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$5m+3(9+3m)=2(7m-11)$

Solution

	$5m + 3(9 + 3m) = 2(7m - 11)$
Distribute.	$5m + 27 + 9m = 14m - 22$
Combine like terms.	$14m + 27 = 14m - 22$
Subtract 14m from both sides.	$14m + 27 - 14m = 14m - 22 - 14m$
Simplify.	$27 \neq -22$
But $27 \neq -22$.	The equation is a contradiction. It has no solution.

Try It 1.1.32

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$12c+5(5+3c)=3(9c-4)$

Answer

contradiction; no solution

? Try It 1.1.33

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$4(7d+18)=13(3d-2)-11d$

Answer

contradiction; no solution

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	**True** for any value of the variable	All real numbers
Contradiction	**False** for all values of the variable	No solution

Key Concepts

- **General Strategy for Solving Linear Equations**
  1. Simplify each side of the equation as much as possible.
     Use the Distributive Property to remove any parentheses.
     Combine like terms.
  2. Collect all the variable terms on one side of the equation.
     Use the Addition or Subtraction Property of Equality.
  3. Collect all the constant terms on the other side of the equation.
     Use the Addition or Subtraction Property of Equality.
  4. Make the coefficient of the variable term to equal to 1.
     Use the Multiplication or Division Property of Equality.
     State the solution to the equation.
  5. Check the solution.
     Substitute the solution into the original equation.

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1.2: Solving Absolute Value Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve absolute value equations
- Solve absolute value inequalities with “less than”
- Solve absolute value inequalities with “greater than”
- Solve applications with absolute value

Before you get started, take this readiness quiz.

1. Evaluate: $-|7|$.
   If you missed this problem, review [link].
2. Fill in $<, >, <, >$, or $=$ for each of the following pairs of numbers.
   ⓐ $|-8| \_\_\_ -|-8|$ ⓑ $12 \_\_\_ -|-12|$ ⓒ $|-6| \_\_\_ -6$ ⓓ $-(-15) \_\_\_ -|-15|$
   If you missed this problem, review [link].
3. Simplify: $14-2|8-3(4-1)|$.
   If you missed this problem, review [link].

Solve Absolute Value Equations

As we prepare to solve absolute value equations, we review our definition of **absolute value**.

ABSOLUTE VALUE

The absolute value of a number is its distance from zero on the number line.

The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.

Absolute values are always greater than or equal to zero.

We learned that both a number and its opposite are the same distance from zero on the number line. Since they have the same distance from zero, they have the same absolute value. For example:

- -5 is 5 units away from 0, so $|-5| = 5$.
- 5 is 5 units away from 0, so $|5| = 5$.

Figure $1.2.1$ illustrates this idea.

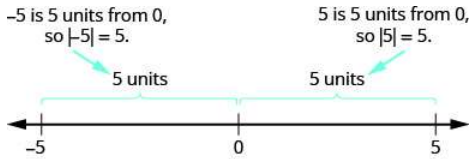


Figure 1.2.1: The numbers 5 and -5 are both five units away from zero.

For the equation |x|=5,|x|=5, we are looking for all numbers that make this a true statement. We are looking for the numbers whose distance from zero is 5. We just saw that both 5 and −5−5 are five units from zero on the number line. They are the solutions to the equation.

If $|x| = 5$
then $x = -5$ or $x = 5$

The solution can be simplified to a single statement by writing $x = \pm 5$. This is read, “x is equal to positive or negative 5”.

We can generalize this to the following property for absolute value equations.

ABSOLUTE VALUE EQUATIONS

For any algebraic expression, u, and any positive real number, a,

$$\begin{aligned} &\text{if} && |u| = a \\ &\text{then} && u = -a \text{ or } u = a \end{aligned}$$

Remember that an absolute value cannot be a negative number.

Example 1.2.1

Solve:

a. $|x| = 8$
b. $|y| = -6$
c. $|z| = 0$

Solution a

$$\begin{aligned} & && |x| = 8 \\ &\text{Write the equivalent equations.} && x = -8 \text{ or } x = 8 \\ & && x = \pm 8 \end{aligned}$$

Solution b

$$|y| = -6$$

No solution

Since an absolute value is always positive, there are no solutions to this equation.

Solution c

$$\begin{aligned} & && |z| = 0 \\ &\text{Write the equivalent equations.} && z = -0 \text{ or } z = 0 \\ &\text{Since } -0 = 0, && z = 0 \end{aligned}$$

Both equations tell us that z=0z=0 and so there is only one solution.

EXERCISE 1.2.2

Solve:

a. $|x| = 2$
b. $|y| = -4$
c. $|z| = 0$

Answer a

± 2

Answer b

no solution

Answer c

0

EXERCISE 1.2.3

Solve:

a. $|x| = 11$
b. $|y| = -5$

c. $|z| = 0$

Answer a

± 11

Answer b

no solution

Answer c

0

To solve an **absolute value equation**, we first isolate the absolute value expression using the same procedures we used to solve linear equations. Once we isolate the absolute value expression we rewrite it as the two equivalent equations.

How to Solve Absolute Value Equations

? Example 1.2.4

Solve $|5x - 4| - 3 = 8$.

Solution

Step 1. Isolate the absolute value expression.	Add 3 to both sides.	$\|5x-4\|-3=8$ $\|5x-4\|=11$
Step 2. Write the equivalent equations.		$5x-4=-11$ or $5x-4=11$
Step 3. Solve each equation.	Add 4 to each side. Divide each side by 5.	$5x=-7$ $\quad 5x=15$ $x=-\frac{7}{5}$ or $x=3$
Step 4. Check each solution.	Substitute 3 and $-\frac{7}{5}$ into the original equation.	$\|5x-4\|-3=8$ $x=3$ $\|5\cdot 3-4\|-3\stackrel{?}{=}8$ $\|15-4\|-3\stackrel{?}{=}8$ $\|11\|-3\stackrel{?}{=}8$ $11-3\stackrel{?}{=}8$ $8=8$ ✓ $\|5x-4\|-3=8$ $x=-\frac{7}{5}$ $\left\|5\left(-\frac{7}{5}\right)-4\right\|-3\stackrel{?}{=}8$ $\|-7-4\|-3\stackrel{?}{=}8$ $\|-11\|-3\stackrel{?}{=}8$ $11-3\stackrel{?}{=}8$ $8=8$ ✓

? EXERCISE 1.2.5

Solve: $|3x - 5| - 1 = 6$.

Answer

$x = 4,\ x = -\frac{2}{3}$

EXERCISE 1.2.6

Solve: $|4x - 3| - 5 = 2$.

Answer

$x = -1,\ x = \frac{5}{2}$

The steps for solving an absolute value equation are summarized here.

SOLVE ABSOLUTE VALUE EQUATIONS.

1. Isolate the absolute value expression.
2. Write the equivalent equations.
3. Solve each equation.
4. Check each solution.

Example 1.2.7

Solve $2|x - 7| + 5 = 9$.

Solution

	$2\|x - 7\| + 5 = 9$
Isolate the absolute value expression.	$2\|x - 7\| = 4$
	$\|x - 7\| = 2$
Write the equivalent equations.	$x - 7 = -2$ or $x - 7 = 2$
Solve each equation.	$x = 5$ or $x = 9$
Check: $2\|x-7\|+5=9$ $2\|5-7\|+5\stackrel{?}{=}9$ $2\|-2\|+5\stackrel{?}{=}9$ $2\cdot2+5\stackrel{?}{=}9$ $4+5\stackrel{?}{=}9$ $9=9$ ✓ $2\|x-7\|+5=9$ $2\|9-7\|+5\stackrel{?}{=}9$ $2\|2\|+5\stackrel{?}{=}9$ $2\cdot2+5\stackrel{?}{=}9$ $4+5\stackrel{?}{=}9$ $9=9$ ✓	

Exercise 1.2.8

Solve: $3|x - 4| - 4 = 8$.

Answer

$x = 8,\ x = 0$

Exercise 1.2.9

Solve: $2|x - 5| + 3 = 9$.

Answer

$x = 8,\ x = 2$

Remember, an absolute value is always positive!

Example 1.2.10

Solve: $|\frac{2}{3}x - 4| + 11 = 3$.

Solution

	$\|\frac{2}{3}x - 4\| = -8$
Isolate the absolute value term.	$\|\frac{2}{3}x - 4\| = -8$
An absolute value cannot be negative.	No solution

Exercise 1.2.11

Solve: $|\frac{3}{4}x - 5| + 9 = 4$.

Answer

No solution

Exercise 1.2.12

Solve: $|\frac{5}{6}x + 3| + 8 = 6$.

Answer

No solution

Some of our absolute value equations could be of the form $|u| = |v|$ where u and v are algebraic expressions. For example, $|x - 3| = |2x + 1|$.

How would we solve them? If two algebraic expressions are equal in absolute value, then they are either equal to each other or negatives of each other. The property for absolute value equations says that for any algebraic expression, u, and a positive real number, a, if $|u| = a$, then $u = -a$ or $u = a$.

This tell us that

if	$\|u\| = \|v\|$		
then	$\|u\| = v$	or	$\|u\| = -v$
and so	$u = v$ or $u = -v$	or	$u = -v$ or $u = -(-v)$

This leads us to the following property for equations with two absolute values.

EQUATIONS WITH TWO ABSOLUTE VALUES

For any algebraic expressions, u and v,

if	$\|u\| = \|v\|$
then	$u = -v$ or $u = v$

When we take the opposite of a quantity, we must be careful with the signs and to add parentheses where needed.

Example 1.2.13

Solve: $|5x-1|=|2x+3|$.

Solution

$$|5x-1|=|2x+3|$$

Write the equivalent equations.	$5x-1=-(2x+3)$	or	$5x-1=2x+3$
	$5x-1=-2x-3$	or	$3x-1=3$
Solve each equation.	$7x-1=-3$		$3x=4$
	$7x=-2$		$x=43$
	$x=-27$	or	$x=43$

Check.

We leave the check to you.

Exercise 1.2.14

Solve: $|7x-3|=|3x+7|$.

Answer

$x=-\frac{2}{5},\ x=\frac{5}{2}$

Exercise 1.2.15

Solve: $|6x-5|=|3x+4|$.

Answer

$x=3, x=19$

Solve Absolute Value Inequalities with "Less Than"

Let's look now at what happens when we have an **absolute value inequality**. Everything we've learned about solving inequalities still holds, but we must consider how the absolute value impacts our work. Again we will look at our definition of absolute value. The absolute value of a number is its distance from zero on the number line. For the equation $|x|=5$, we saw that both 5 and -5 are five units from zero on the number line. They are the solutions to the equation.

$$|x|=5$$
$$x=-5 \quad \text{or} \quad x=5$$

What about the inequality $|x|\leq 5$? Where are the numbers whose distance is less than or equal to 5? We know -5 and 5 are both five units from zero. All the numbers between -5 and 5 are less than five units from zero (Figure 1.2.2).

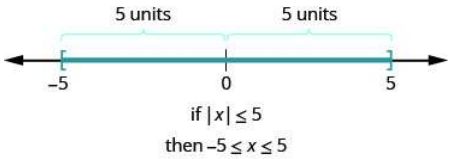


Figure 1.2.2.

In a more general way, we can see that if $|u|\leq a$, then $-a\leq u\leq a$ (Figure 1.2.3).

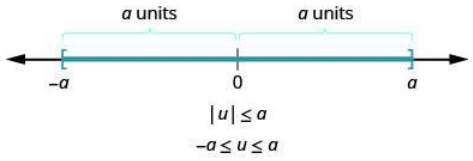


Figure 1.2.3.

This result is summarized here.

ABSOLUTE VALUE INEQUALITIES WITH $<$ OR $\leq$

For any algebraic expression, u, and any positive real number, a,

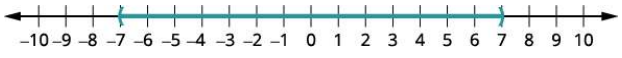
$$\text{if} \quad |u| < a, \quad \text{then} \; -a < u < a$$
$$\text{if} \quad |u| \leq a, \quad \text{then} \; -a \leq u \leq a$$

After solving an inequality, it is often helpful to check some points to see if the solution makes sense. The graph of the solution divides the number line into three sections. Choose a value in each section and substitute it in the original inequality to see if it makes the inequality true or not. While this is not a complete check, it often helps verify the solution.

Example 1.2.16

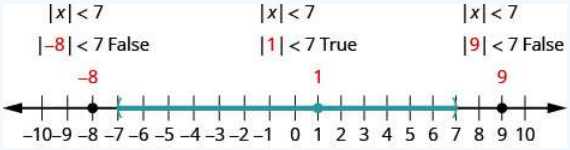
Solve $|x| < 7$. Graph the solution and write the solution in interval notation.

Solution

	$\lvert x\rvert < 7$
Write the equivalent inequality.	$-7 < x < 7$
Graph the solution.	
Write the solution using interval notation.	$(-7, 7)$

Check:

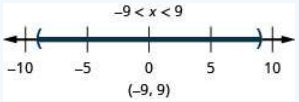
To verify, check a value in each section of the number line showing the solution. Choose numbers such as −8,−8, 1, and 9.



EXERCISE 1.2.17

Graph the solution and write the solution in interval notation: $|x| < 9$.

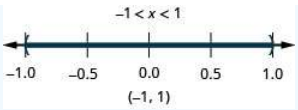
Answer



EXERCISE 1.2.18

Graph the solution and write the solution in interval notation: $|x| < 1$.

Answer



Example 1.2.19

Solve $|5x-6| \leq 4$. Graph the solution and write the solution in interval notation.

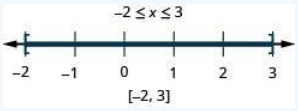
Solution

Step 1. Isolate the absolute value expression. It is isolated.	$\|5x-6\| \leq 4$
Step 2. Write the equivalent compound inequality.	$-4 \leq 5x-6 \leq 4$
Step 3. Solve the compound inequality.	$2 \leq 5x \leq 10$ $\frac{2}{5} \leq x \leq 2$
Step 4. Graph the solution.	
Step 5. Write the solution using interval notation.	$[\frac{2}{5}, 2]$
Check: The check is left to you.	

EXERCISE 1.2.20

Solve $|2x-1| \leq 5$. Graph the solution and write the solution in interval notation:

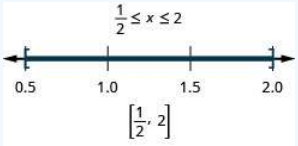
Answer



EXERCISE 1.2.21

Solve $|4x-5| \leq 3$. Graph the solution and write the solution in interval notation:

Answer



SOLVE ABSOLUTE VALUE INEQUALITIES WITH $<$ OR $\leq$

1. Isolate the absolute value expression.
2. Write the equivalent compound inequality.

$$|u| < a \quad \text{is equivalent to} \quad -a < u < a$$
$$|u| \leq a \quad \text{is equivalent to} \quad -a \leq u \leq a$$

3. Solve the compound inequality.
4. Graph the solution

5. Write the solution using interval notation.

Solve Absolute Value Inequalities with "Greater Than"

What happens for absolute value inequalities that have "greater than"? Again we will look at our definition of absolute value. The absolute value of a number is its distance from zero on the number line.

We started with the inequality $|x| \leq 5$. We saw that the numbers whose distance is less than or equal to five from zero on the number line were -5 and 5 and all the numbers between -5 and 5 (Figure 1.2.4).

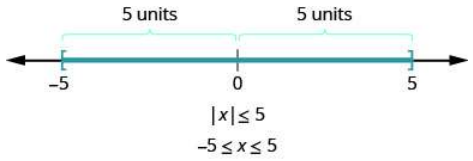


Figure 1.2.4.

Now we want to look at the inequality $|x| \geq 5$. Where are the numbers whose distance from zero is greater than or equal to five?

Again both -5 and 5 are five units from zero and so are included in the solution. Numbers whose distance from zero is greater than five units would be less than -5 and greater than 5 on the number line (Figure 1.2.5).

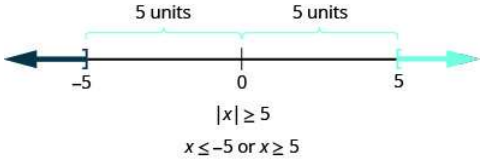


Figure 1.2.5.

In a more general way, we can see that if $|u| \geq a$, then $u \leq -a$ or $u \leq a$. See Figure.

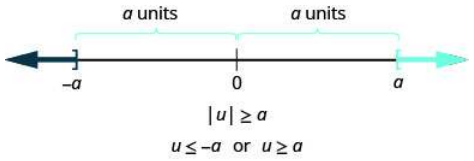


Figure 1.2.6.

This result is summarized here.

ABSOLUTE VALUE INEQUALITIES WITH $>$ OR $\geq$

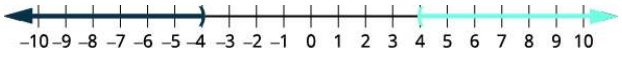
For any algebraic expression, *u*, and any positive real number, *a*,

$$\begin{aligned} &\text{if} \quad |u| > a, \quad &&\text{then } u < -a \text{ or } u > a \\ &\text{if} \quad |u| \geq a, \quad &&\text{then } u \leq -a \text{ or } u \geq a \end{aligned}$$

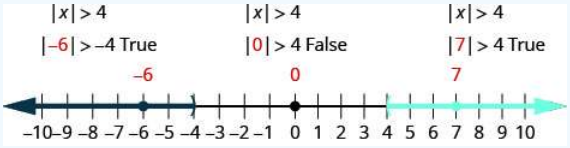
Example 1.2.22

Solve $|x| > 4$. Graph the solution and write the solution in interval notation.

Solution

	$\lvert x\rvert > 4$
Write the equivalent inequality.	$x < -4$ or $x > 4$
Graph the solution.	
Write the solution using interval notation.	$(-\inf, -4) \cup (4, \inf)$
Check:	

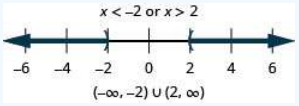
To verify, check a value in each section of the number line showing the solution. Choose numbers such as −6,−6, 0, and 7.



EXERCISE 1.2.23

Solve $|x| > 2$. Graph the solution and write the solution in interval notation.

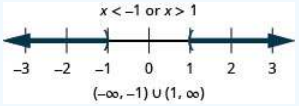
Answer



EXERCISE 1.2.24

Solve $|x| > 1$. Graph the solution and write the solution in interval notation.

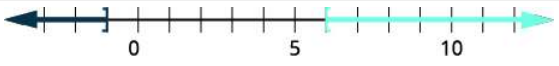
Answer



Example 1.2.25

Solve $|2x-3| \geq 5$. Graph the solution and write the solution in interval notation.

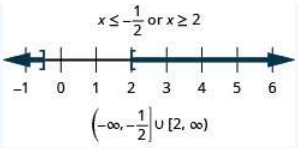
Solution

	$\|2x-3\| \geq 5$
Step 1. Isolate the absolute value expression. It is isolated.	
Step 2. Write the equivalent compound inequality.	$2x-3 \leq -5$ or $2x-3 \geq 5$
Step 3. Solve the compound inequality.	$2x \leq -2$ or $2x \geq 8$ $x \leq -1$ or $x \geq 4$
Step 4. Graph the solution.	
Step 5. Write the solution using interval notation.	$(-\inf, -1] \cup [4, \inf)$
Check: The check is left to you.	

EXERCISE 1.2.26

Solve $|4x-3| \geq 5$. Graph the solution and write the solution in interval notation.

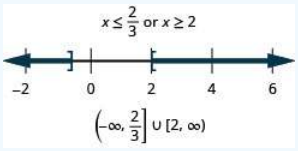
Answer



? EXERCISE 1.2.27

Solve $|3x - 4| \ge 2$. Graph the solution and write the solution in interval notation.

Answer



SOLVE ABSOLUTE VALUE INEQUALITIES WITH $>$ OR $\ge$.

1. Isolate the absolute value expression.
2. Write the equivalent compound inequality.

$$\begin{array}{llllll} |u| > a & \text{is equivalent to} & u < -a & \text{or} & u > a \\ |u| \ge a & \text{is equivalent to} & u \le -a & \text{or} & u \ge a \\ |u| > a & \text{is equivalent to} & u < -a & \text{or} & u > a \\ |u| \ge a & \text{is equivalent to} & u \le -a & \text{or} & u \ge a \end{array}$$

3. Solve the compound inequality.
4. Graph the solution
5. Write the solution using interval notation.

Solve Applications with Absolute Value

Absolute value inequalities are often used in the manufacturing process. An item must be made with near perfect specifications. Usually there is a certain *tolerance* of the difference from the specifications that is allowed. If the difference from the specifications exceeds the tolerance, the item is rejected.

$$|\text{actual-ideal}| \le \text{tolerance}$$

? Example 1.2.28

The ideal diameter of a rod needed for a machine is 60 mm. The actual diameter can vary from the ideal diameter by 0.075 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Solution

	Let $x =$ the actual measurement
Use an absolute value inequality to express this situation.	$\|\text{actual-ideal}\| \le \text{tolerance}$
	$\|x - 60\| \le 0.075$
Rewrite as a compound inequality.	$-0.075 \le x - 60 \le 0.075$
Solve the inequality.	$59.925 \le x \le 60.075$
Answer the question.	The diameter of the rod can be between $59.925mm$ and $60.075mm$.

EXERCISE 1.2.29

The ideal diameter of a rod needed for a machine is 80 mm. The actual diameter can vary from the ideal diameter by 0.009 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Answer

The diameter of the rod can be between 79.991 and 80.009 mm.

EXERCISE 1.2.30

The ideal diameter of a rod needed for a machine is 75 mm. The actual diameter can vary from the ideal diameter by 0.05 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Answer

The diameter of the rod can be between 74.95 and 75.05 mm.

Access this online resource for additional instruction and practice with solving linear absolute value equations and inequalities.

- Solving Linear Absolute Value Equations and Inequalities

Key Concepts

- **Absolute Value**
  The absolute value of a number is its distance from 0 on the number line.
  The absolute value of a number *n* is written as $|n|$ and $|n| \geq 0$ for all numbers.
  Absolute values are always greater than or equal to zero.
- **Absolute Value Equations**
  For any algebraic expression, *u*, and any positive real number, *a*,
  if $|u| = a$
  then $u = -a$ or $u = a$
  Remember that an absolute value cannot be a negative number.
- **How to Solve Absolute Value Equations**
  1. Isolate the absolute value expression.
  2. Write the equivalent equations.
  3. Solve each equation.
  4. Check each solution.
- **Equations with Two Absolute Values**
  For any algebraic expressions, *u* and *v*,
  if $|u| = |v|$
  then $u = -v$ or $u = v$
- **Absolute Value Inequalities with** $<$ or $\leq$
  For any algebraic expression, *u*, and any positive real number, *a*,
  if $|u| = a$ then $-a < u < a$
  if $|u| \leq a$ then $-a \leq u \leq a$
- **How To Solve Absolute Value Inequalities with** $<$ or $\leq$
  1. Isolate the absolute value expression.
  2. Write the equivalent compound inequality.
     $|u| < a$ is equivalent to $-a < u < a$
     $|u| \leq a$ is equivalent to $-a \leq u \leq a$
  3. Solve the compound inequality.
  4. Graph the solution
  5. Write the solution using interval notation

- **Absolute Value Inequalities with $>$ or $\geq$**
  For any algebraic expression, u, and any positive real number, a,
  if $|u| > a$, then $u < -a$ or $u > a$
  if $|u| \geq a$, then $u \leq -a$ or $u \geq a$
- **How To Solve Absolute Value Inequalities with $>$ or $\geq$**
  1. Isolate the absolute value expression.
  2. Write the equivalent compound inequality.
     $|u| > a$ is equivalent to $u < -a$ or $u > a$
     $|u| \geq a$ is equivalent to $u \leq -a$ or $u \geq a$
  3. Solve the compound inequality.
  4. Graph the solution
  5. Write the solution using interval notation

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1.3: Solving Polynomial Equations

Learning Objectives

- Review general strategies for factoring.
- Solve polynomial equations by factoring.

Reviewing General Factoring Strategies

We have learned various techniques for factoring polynomials with up to four terms. The challenge is to identify the type of polynomial and then decide which method to apply. The following outlines a general guideline for factoring polynomials.

general guidelines for factoring polynomials

Step 1: Check for common factors. If the terms have common factors, then factor out the greatest common factor (GCF).

Step 2: Determine the number of terms in the polynomial.

1. Factor four-term polynomials by grouping.
2. Factor trinomials (3 terms) using "trial and error" or the AC method.
3. Factor binomials (2 terms) using the following special products:
   - **Difference of squares**:$a^2-b^2=(a+b)(a-b)$
   - **Sum of squares**: a^2+b^2 no general formula
   - **Difference of cubes**: $a^3-b^3=(a-b)(a^2+ab+b^2)$
   - **Sum of cubes**: $a^3+b^3=(a+b)(a^2-ab+b^2)$

Step 3: Look for factors that can be factored further.

Step 4: Check by multiplying.

If a binomial is both a difference of squares and a difference cubes, then first factor it as difference of squares. This will result in a more complete factorization. In addition, not all polynomials with integer coefficients factor. When this is the case, we say that the polynomial is prime.

If an expression has a GCF, then factor this out first. Doing so is often overlooked and typically results in factors that are easier to work with. Furthermore, look for the resulting factors to factor further; many factoring problems require more than one step. A polynomial is completely factored when none of the factors can be factored further.

Example 1.3.1:

Factor $54x^4-36x^3-24x^2+16x$.

Solution

This four-term polynomial has a GCF of $2x$. Factor this out first.

$$54x^4-36x^3-24x^2+16x=2x\left(27x^3-18x^2-12x+8\right)$$

Now factor the resulting four-term polynomial by grouping and look for resulting factors to factor further.

$$
\begin{aligned}
54x^4-36x^3-24x^2+16x &= 2x\Big(\underbrace{27x^3-18x^2}_{\text{group}}\underbrace{-12x+8}_{\text{group}}\Big)\\
&= 2x\left(9x^2(3x-2)-4(3x-2)\right)\\
&= 2x(3x-2)\underbrace{\left(9x^2-4\right)}_{\substack{\text{difference}\\ \text{of squares}}}\\
&= 2x(3x-2)(3x-2)(3x+2)
\end{aligned}
$$

Figure 1.3.1

Answer

$2x(3x-2)^2(3x+2)$. The check is left to the reader.

Example 1.3.2:

Factor: x^4-3x^2-4 .

Solution

This trinomial does not have a GCF.

$$
\begin{aligned}
x^4-3x^2-4 &= \left(x^2\quad\right)\left(x^2\quad\right)\\
&= \left(x^2+1\right)\left(x^2-4\right) \quad \textit{Difference of squares}\\
&= \left(x^2+1\right)(x+2)(x-2)
\end{aligned}
$$

The factor $\left(x^2+1\right)$ is prime and the trinomial is completlely factored.

Answer:

$\left(x^2+1\right)(x+2)(x-2)$

Example 1.3.3:

Factor: x^6+6x^3-16 .

Solution

Begin by factoring $x^6=x^3\cdot x^3$ and look for the factors of 16 that add to 6.

$$
\begin{aligned}
x^6+6x^3-16 &= \left(x^3\quad\right)\left(x^3\quad\right)\\
&= \left(x^3-2\right)\left(x^3+8\right) \quad \textit{sum of cubes}\\
&= \left(x^3-2\right)(x+2)\left(x^2-2x+4\right)
\end{aligned}
$$

The factor $\left(x^3-2\right)$ cannot be factored any further using integers and the factorization is complete.

Answer:

$\left(x^3-2\right)(x+2)\left(x^2+2x+4\right)$

Exercise 1.3.1

Factor: $9x^4+17x^2-2$

Answer

$(3x+1)(3x-1)\left(x^2+2\right)$

www.youtube.com/v/lsAP8_UgUx0

Solving Polynomial Equations by Factoring

In this section, we will review a technique that can be used to solve certain polynomial equations. We begin with the *zero-product property*[20]:

$$a \cdot b = 0 \quad \text{if and only if} \quad a = 0 \quad \text{or} \quad b = 0$$

The zero-product property is true for any number of factors that make up an equation. In other words, if any product is equal to zero, then at least one of the variable factors must be equal to zero. If an expression is equal to zero and can be factored into linear factors, then we will be able to set each factor equal to zero and solve for each equation.

Example 1.3.4

Solve: $2x(x-4)(5x+3)=0$.

Solution

Set each variable factor equal to zero and solve.

$$\begin{aligned} 2x &= 0 \\ \frac{2x}{2} &= \frac{0}{2} \\ x &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x-4 &= 0 \\ x &= 4 \end{aligned} \quad \text{or} \quad \begin{aligned} 5x+3 &= 0 \\ \frac{5x}{5} &= \frac{-3}{5} \\ x &= -\frac{3}{5} \end{aligned}$$

To check that these are solutions we can substitute back into the original equation to see if we obtain a true statement. Note that each solution produces a zero factor. This is left to the reader.

Answer:

The solutions are $0, 4$, and $\frac{-3}{5}$.

Of course, most equations will not be given in factored form.

Example 1.3.5

Solve: $4x^3 - x^2 - 100x + 25 = 0$.

Solution

Begin by factoring the left side completely.

$$\begin{aligned} 4x^3 - x^2 - 100x + 25 &= 0 && \textit{Factor by grouping.} \\ x^2(4x-1) - 25(4x-1) &= 0 \\ (4x-1)\left(x^2-25\right) &= 0 && \textit{Factor as a difference of squares.} \\ (4x-1)(x+5)(x-5) &= 0 \end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{aligned} 4x-1 &= 0 \\ 4x &= 1 \\ x &= \tfrac{1}{4} \end{aligned} \quad \text{or} \quad \begin{aligned} x+5 &= 0 \\ x &= -5 \end{aligned} \quad \text{or} \quad \begin{aligned} x-5 &= 0 \\ x &= 5 \end{aligned}$$

Answer:

The solutions are $\frac{1}{4}, -5$, and 5.

Using the zero-product property after factoring an equation that is equal to zero is the key to this technique. However, the equation may not be given equal to zero, and so there may be some preliminary steps before factoring. The steps required to **solve by factoring**[21] are outlined in the following example.

Example 1.3.6

Solve: $15x^2+3x-8=5x-7$.

Solution

Step 1: Express the equation in standard form, equal to zero. In this example, subtract $5x$ from and add 7 to both sides.

$$15x^2+3x-8=5x-7$$
$$15x^2-2x-1=0$$

Step 2: Factor the expression.

$$(3x-1)(5x+1)=0$$

Step 3: Apply the zero-product property and set each variable factor equal to zero.

$$3x-1=0 \quad \text{or} \quad 5x+1=0$$

Step 4: Solve the resulting linear equations.

$$\begin{aligned} 3x-1&=0 \\ 3x&=1 \\ x&=\tfrac{1}{3} \end{aligned} \quad \text{or} \quad \begin{aligned} 5x+1&=0 \\ 5x&=-1 \\ x&=-\frac{1}{5} \end{aligned}$$

Answer:

The solutions are $\frac{1}{3}$ and $\frac{-1}{5}$. The check is optional.

Example 1.3.7:

Solve: $(3x+2)(x+1)=4$.

Solution

This quadratic equation appears to be factored; hence it might be tempting to set each factor equal to 4. However, this would lead to incorrect results. We must rewrite the equation equal to zero, so that we can apply the zero-product property.

$$(3x+2)(x+1)=4$$
$$3x^2+3x+2x+2=4$$
$$3x^2+5x+2=4$$
$$3x^2+5x-2=0$$

Once it is in standard form, we can factor and then set each factor equal to zero.

$$(3x-1)(x+2)=0$$
$$\begin{aligned} 3x-1&=0 \\ 3x&=1 \\ x&=\tfrac{1}{3} \end{aligned} \quad \text{or} \quad \begin{aligned} x+2&=0 \\ x&=-2 \end{aligned}$$

Answer:

The solutions are $\frac{1}{3}$ and -2.

Key Takeaways

- Factoring and the zero-product property allow us to solve equations.
- To solve a polynomial equation, first write it in standard form. Once it is equal to zero, factor it and then set each variable factor equal to zero. The solutions to the resulting equations are the solutions to the original.
- Not all polynomial equations can be solved by factoring. We will learn how to solve polynomial equations that do not factor later in the course.

Exercise 1.3.4

Factor completely.

1. $50x^2-18$
2. $12x^3-3x$
3. $10x^3+65x^2-35x$
4. $15x^4+7x^3-4x^2$
5. $6a^4b-15a^3b^2-9a^2b^3$
6. $8a^3b-44a^2b^2+20ab^3$
7. $36x^4-72x^3-4x^2+8x$
8. $20x^4+60x^3-5x^2-15x$
9. $3x^5+2x^4-12x^3-8x^2$
10. $10x^5-4x^4-90x^3+36x^2$
11. x^4-23x^2-50
12. $2x^4-31x^2-16$
13. $-2x^5-6x^3+8x$
14. $-36x^5+69x^3+27x$
15. $54x^5-78x^3+24x$
16. $4x^6-65x^4+16x^2$
17. x^6-7x^3-8
18. x^6-25x^3-54
19. $3x^6+4x^3+1$
20. $27x^6-28x^3+1$

Answer

1. $2(5x+3)(5x-3)$
3. $5x(x+7)(2x-1)$
5. $3a^2b(2a+b)(a-3b)$
7. $4x(x-2)(3x+1)(3x-1)$
9. $x^2(3x+2)(x+2)(x-2)$
11. $\left(x^2+2\right)(x+5)(x-5)$
13. $-2x\left(x^2+4\right)(x-1)(x+1)$
15. $6x(x+1)(x-1)(3x+2)(3x-2)$
17. $(x+1)\left(x^2-x+1\right)(x-2)\left(x^2+2x+4\right)$
19. $\left(3x^3+1\right)(x+1)\left(x^2-x+1\right)$

Exercise 1.3.5

Solve.

1. $(6x-5)(x+7)=0$
2. $(x+9)(3x-8)=0$
3. $5x(2x-5)(3x+1)=0$
4. $4x(5x-1)(2x+3)=0$
5. $(x-1)(2x+1)(3x-5)=0$
6. $(x+6)(5x-2)(2x+9)=0$
7. $(x+4)(x-2)=16$
8. $(x+1)(x-7)=9$
9. $(6x+1)(x+1)=6$

10. $(2x-1)(x-4)=39$
11. $x^2-15x+50=0$
12. $x^2+10x-24=0$
13. $3x^2+2x-5=0$
14. $2x^2+9x+7=0$
15. $\frac{1}{10}x^2-\frac{7}{15}x-\frac{1}{6}=0$
16. $\frac{1}{4}-\frac{4}{9}x^2=0$
17. $6x^2-5x-2=30x+4$
18. $6x^2-9x+15=20x-13$
19. $5x^2-23x+12=4(5x-3)$
20. $4x^2+5x-5=15(3-2x)$
21. $(x+6)(x-10)=4(x-18)$
22. $(x+4)(x-6)=2(x+4)$
23. $4x^3-14x^2-30x=0$
24. $9x^3+48x^2-36x=0$
25. $\frac{1}{3}x^3-\frac{3}{4}x=0$
26. $\frac{1}{2}x^3-\frac{1}{50}x=0$
27. $-10x^3-28x^2+48x=0$
28. $-2x^3+15x^2+50x=0$
29. $2x^3-x^2-72x+36=0$
30. $4x^3-32x^2-9x+72=0$
31. $45x^3-9x^2-5x+1=0$
32. $x^3-3x^2-x+3=0$
33. $x^4-5x^2+4=0$
34. $4x^4-37x^2+9=0$

Answer

1. $-7, \frac{5}{6}$
3. $0, \frac{5}{2}, -\frac{1}{3}$
5. $-\frac{1}{2}, 1, \frac{5}{3}$
7. $-6, 4$
9. $-\frac{5}{3}, \frac{1}{2}$
11. $5, 10$
13. $-\frac{5}{3}, 1$
15. $-\frac{1}{3}, 5$
17. $-\frac{1}{6}, 6$
19. $\frac{3}{5}, 8$
21. $2, 6$
23. $0, -\frac{3}{2}, 5$
25. $0, \pm\frac{3}{2}$
27. $-4, 0, \frac{6}{5}$
29. $\pm 6, \frac{1}{2}$
31. $\pm\frac{1}{3}, \frac{1}{5}$
33. $\pm 1, \pm 2$

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1.4: Solving Quadratic Equations

Solve Quadratic Equations Using the Zero Product Property

We will first solve some quadratic equations by using the Zero Product Property. The **Zero Product Property** says that if the product of two quantities is zero, it must be that at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

ZERO PRODUCT PROPERTY

If $a \cdot b = 0,$ then either $a = 0$ or $b = 0$ or both.

We will now use the **Zero Product Property**, to solve a quadratic equation.

Exercise 1.4.1: How to Use the Zero Product Property to Solve a Quadratic Equation

Solve: $(x+1)(x-4)=0$

Answer

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(x+1)(x-4)=0$ $x+1=0 \quad \text{or} \quad x-4=0$
Step 2. Solve the linear equations.	Solve each equation.	$x=-1 \quad \text{or} \quad x=4$
Step 3. Check.	Substitute each solution separately into the original equation.	$x=-1$ $(x+1)(x-4)=0$ $(-1+1)(-1-4) \stackrel{?}{=} 0$ $(0)(-5) \stackrel{?}{=} 0$ $0=0$ ✓ $x=4$ $(x+1)(x-4)=0$ $(4+1)(4-4) \stackrel{?}{=} 0$ $(5)(0) \stackrel{?}{=} 0$ $0=0$ ✓

Exercise 1.4.2

Solve: $(x-3)(x+5)=0$

Answer

$x=3, x=-5$

Exercise 1.4.3

Solve: $(y-6)(y+9)=0$

Answer

$y=6, y=-9$

We usually will do a little more work than we did in this last example to solve the linear equations that result from using the Zero Product Property.

Exercise 1.4.4

Solve: $(5n-2)(6n-1)=0$

Answer

	$(5n-2)(6n-1)=0$	
Use the Zero Product Property to set each factor to 0.	$5n-2=0$	$6n-1=0$
Solve the equations.	$n=\frac{2}{5}$	$n=\frac{1}{6}$
Check your answers.		

$n=\frac{2}{5}$	$n=\frac{1}{6}$
$(5n-2)(6n-1)=0$	$(5n-2)(6n-1)=0$
$\left(5\cdot\frac{2}{5}-2\right)\left(6\cdot\frac{2}{5}-1\right)\stackrel{?}{=}0$	$\left(5\cdot\frac{1}{6}-2\right)\left(6\cdot\frac{1}{6}-1\right)\stackrel{?}{=}0$
$(2-2)\left(\frac{12}{5}-\frac{5}{5}\right)\stackrel{?}{=}0$	$\left(\frac{5}{6}-\frac{12}{6}\right)(1-1)\stackrel{?}{=}0$
$(0)\left(\frac{7}{5}\right)\stackrel{?}{=}0$	$\left(-\frac{7}{6}\right)(0)\stackrel{?}{=}0$
$0=0$ ✓	$0=0$ ✓

Exercise 1.4.5

Solve: $(3m-2)(2m+1)=0$

Answer

$m=\frac{2}{3}, m=-\frac{1}{2}$

Exercise 1.4.6

Solve: $(4p+3)(4p-3)=0$

Answer

$p=-\frac{3}{4}, p=\frac{3}{4}$

Notice when we checked the solutions that each of them made just one factor equal to zero. But the product was zero for both solutions.

Exercise 1.4.7

Solve: $3p(10p+7)=0$

Answer

	$3p(10p+7)=0$	
Use the Zero Product Property to set each factor to 0.	3p=0	10p+7=0
Solve the equations.	p=0	10p=−7
		$p=-\frac{7}{10}$
Check your answers.		

$p=0$	$p=-\frac{7}{10}$
$3p(10p+7)=0$	$3p(10p+7)=0$
$3\cdot 0(10\cdot 0+7)\stackrel{?}{=}0$	$3\left(-\frac{7}{10}\right)10\left(-\frac{7}{10}\right)+7\stackrel{?}{=}0$
$0(0+7)\stackrel{?}{=}0$	$\left(-\frac{21}{10}\right)(-7+7)\stackrel{?}{=}0$
$0(7)\stackrel{?}{=}0$	$\left(-\frac{21}{10}\right)(0)\stackrel{?}{=}0$
$0=0$ ✓	$0=0$ ✓

Exercise 1.4.8

Solve: $2u(5u-1)=0$

Answer

$u=0, u=\frac{1}{5}$

Exercise 1.4.9

Solve: $w(2w+3)=0$

Answer

$w=0, w=-\frac{3}{2}$

It may appear that there is only one factor in the next example. Remember, however, that $(y-8)^2$ means $(y-8)(y-8)$.

Exercise 1.4.10

Solve: $(y-8)^2=0$

Answer

	$(y-8)^2=0$	
Rewrite the left side as a product.	(y−8)(y−8)=0	
Use the Zero Product Property and set each factor to 0.	y−8=0	y−8=0
Solve the equations.	y=8	y=8
When a solution repeats, we call it a double root.		
Check your answer.		
$y=8$ $(y-8)^2=0$ $(8-8)^2 \stackrel{?}{=} 0$ $(0)^2 \stackrel{?}{=} 0$ $0=0$ ✓		

Exercise 1.4.11

Solve: $(x+1)^2=0$

Answer

$x=1$

Exercise 1.4.12

Solve: $(v-2)^2=0$

Answer

$v=2$

Solve Quadratic Equations by Factoring

Each of the equations we have solved in this section so far had one side in factored form. In order to use the Zero Product Property, the quadratic equation must be factored, with zero on one side. So we be sure to start with the quadratic equation in standard form, $ax^2+bx+c=0$. Then we factor the expression on the left.

Exercise 1.4.13

Solve: $x^2+2x-8=0$

Answer

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.	The equation is already in standard form.	$x^2+2x-8=0$
Step 2. Factor the quadratic expression.	Factor x^2+2x-8 $(x+4)(x-2)$	$(x+4)(x-2)=0$

Step 3. Use the Zero Product Property.	Set each factor equal to zero.	$x+4=0$ or $x-2=0$
Step 4. Solve the linear equations.	We have two linear equations.	$x=-4$ or $x=2$
Step 5. Check.	Substitute each solution separately into the original equation.	$x^2+2x-8=0$ $x=-4$ $(-4)^2-2(-4)-8\stackrel{?}{=}0$ $16+(-8)-8\stackrel{?}{=}0$ $0=0$ ✓ $x^2+2x-8=0$ $x=2$ $2^2-2(2)-8\stackrel{?}{=}0$ $4+4-8\stackrel{?}{=}0$ $0=0$ ✓

? Exercise 1.4.14

Solve: $x^2-x-12=0$

Answer

$x=4, x=-3$

? Exercise 1.4.15

Solve: $b^2+9b+14=0$

Answer

$b=-2, b=-7$

SOLVE A QUADRATIC EQUATION BY FACTORING.

1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.
2. Factor the quadratic expression.
3. Use the Zero Product Property.
4. Solve the linear equations.
5. Check.

Before we factor, we must make sure the quadratic equation is in standard form.

? Exercise 1.4.16

Solve: $2y^2=13y+45$

Answer

	$2y^2=13y+45$	
Write the quadratic equation in standard form.	$2y^2-13y-45=0$	
Factor the quadratic expression.	$(2y+5)(y-9)=0$	
Use the Zero Product Property to set each factor to 0.	$2y+5=0$	$y-9=0$
Solve each equation.	$y=-\frac{5}{2}$	$y=9$
Check your answers.		

$y=-\frac{5}{2}$	$y=9$
$2y^2=13y+45$	$2y^2=13y+45$
$2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right)+45$	$2(9)^2 \stackrel{?}{=} 13(9)+45$
$2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right)+\frac{90}{2}$	$2(81) \stackrel{?}{=} 117+45$
$\frac{25}{2}=\frac{25}{2}$ ✓	$162=162$ ✓

Exercise 1.4.17

Solve: $3c^2 = 10c - 8$

Answer

$c = 0, c = \frac{4}{3}$

Exercise 1.4.18

Solve: $2d^2 - 5d = 3$

Answer

$d = 3, d = -\frac{1}{2}$

Exercise 1.4.19

Solve: $5x^2 - 13x = 7x$

Answer

	$5x^2 - 13x = 7x$	
Write the quadratic equation in standard form.	$5x^2 - 20x = 0$	
Factor the left side of the equation.	$5x(x-4) = 0$	
Use the Zero Product Property to set each factor to 0.	$5x = 0$	$x - 4 = 0$
Solve each equation.	$x = 0$	$x = 4$
Check your answers.		

$x=0$	$x=4$
$5x^2-13x=7x$	$5x^2-13x=7x$
$5(0)^2-13(0) \stackrel{?}{=} 7(0)$	$5(4)^2-13(4) \stackrel{?}{=} 7(4)$
$0-0 \stackrel{?}{=} 0$	$5(16)-52 \stackrel{?}{=} 28$
$0=0$ ✓	$28=28$ ✓

Exercise 1.4.20

Solve: $6a^2 + 9a = 3a$

Answer

$a = 0, a = -1$

Exercise 1.4.21

Solve: $45b^2 - 2b = -17b$

Answer

$b = 0, b = -\frac{1}{3}$

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

Exercise 1.4.22

Solve: $144q^2 = 25$

Answer

	$144q^2$	$=$	25				
Write the quadratic equation in standard form.	$144q^2-25$	$=$	0				
Factor. It is a difference of squares.	$(12q-5)(12q+5)$	$=$	0				
Use the Zero Product Property to set each factor to 0.	$12q-5$	$=$	0	$12q+5$	$=$	0	
Solve each equation.	$12q$	$=$	5	$12q$	$=$	-5	
	q	$=$	$\frac{5}{12}$	q	$=$	$-\frac{5}{12}$	
Check your answers.							

Exercise 1.4.23

Solve: $25p^2 = 49$

Answer

$p = \frac{7}{5}, p = -\frac{7}{5}$

Exercise 1.4.24

Solve: $36x^2 = 121$

Answer

$x = \frac{11}{6}, x = -\frac{11}{6}$

The left side in the next example is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

Exercise 1.4.25

Solve: $(3x-8)(x-1) = 3x$

Answer

	$(3x-8)(x-1) = 3x$
Multiply the binomials.	$3x^2-11x+8 = 3x$
Write the quadratic equation in standard form.	$3x^2-14x+8 = 0$
Factor the trinomial.	$(3x-2)(x-4) = 0$
Use the Zero Product Property to set each factor to 0.	$3x-2 = 0 \quad x-4 = 0$
Solve each equation.	$3x = 2 \quad x = 4$
	$x = \frac{2}{3}$
Check your answers.	The check is left to you!

Exercise 1.4.26

Solve: $(2m+1)(m+3) = 12m$

Answer

$m = 1, m = \frac{3}{2}$

Exercise 1.4.27

Solve: $(k+1)(k-1) = 8$

Answer

$k = 3, k = -3$

The Zero Product Property also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree more than two by using the Zero Product Property, just like we solved quadratic equations.

Exercise 1.4.28

Solve: $9m^3+100m=60m^2$

Answer

	$9m^3+100m$	$=$	$60m^2$					
Bring all the terms to one side so that the other side is zero.	$9m^3-60m^2+100m$	$=$	0					
Factor the greatest common factor first.	$m\left(9m^2-60m+100\right)$	$=$	0					
Factor the trinomial.	$m(3m-10)(3m-10)$	$=$	0					
Use the Zero Product Property to set each factor to 0.	m	$=$	0	$3m-10$	$=$	0	$3m-10$	$=$ 0
Solve each equation.	m	$=$	0	m	$=$	$\frac{10}{3}$	m	$=$ $\frac{10}{3}$
Check your answers.	The check is left to you.							

Exercise 1.4.29

Solve: $8x^3=24x^2-18x$

Answer

$x=0, x=\frac{3}{2}$

Exercise 1.4.30

Solve: $16y^2=32y^3+2y$

Answer

$y=0, y=\frac{1}{4}$

When we factor the quadratic equation in the next example we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Exercise 1.4.31

Solve: $4x^2=16x+84$

Answer

	$4x^2$	$=$	$16x+84$					
Write the quadratic equation in standard form.	$4x^2-16x-84$	$=$	0					
Factor the greatest common factor first.	$4\left(x^2-4x-21\right)$	$=$	0					
Factor the trinomial.	$4(x-7)(x+3)$	$=$	0					
Use the Zero Product Property to set each factor to 0.	4	$\neq$	0	$x-7$	$=$	0	$x+3$	$=$ 0
Solve each equation.	4	$\neq$	0	x	$=$	7	x	$=$ -3
Check your answers.	The check is left to you.							

Exercise 1.4.32

Solve: $18a^2-30=-33a$

Answer

$a=-\frac{5}{2}, a=\frac{2}{3}$

Exercise 1.4.33

Solve: $123b=-6-60b^2$

Answer

$b=2, b=\frac{1}{20}$

Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes'. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general', so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for x.

We start with the standard form of a quadratic equation and solve it for x by completing the square.

	$ax^2+bx+c=0, \quad a\neq 0$
Isolate the variable terms on one side.	$ax^2+bx \quad =-c$
Make the coefficient of x^2 equal to 1, by dividing by a.	$\frac{ax^2}{a}+\frac{b}{a}x \quad =-\frac{c}{a}$
Simplify.	$x^2+\frac{b}{a}x \quad =-\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ and add it to both sides of the equation.	
$\left(\frac{1}{2}\frac{b}{a}\right)^2=\frac{b^2}{4a^2}$	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c\cdot 4a}{a\cdot 4a}$
Simplify.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify the radical.	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

The final equation is called the "Quadratic Formula."

Definition 1.4.1: Quadratic Formula

The solutions to a **quadratic equation** of the form $ax^2+bx+c=0$, where $a\neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{1.4.1}$$

To use the **Quadratic Formula**, we substitute the values of a, b, and c from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the Quadratic Formula (Equation 1.4.1) is an equation. Make sure you use both sides of the equation.

Example 1.4.1 How to Solve a Quadratic Equation Using the Quadratic Formula

Solve by using the Quadratic Formula: $2x^2+9x-5=0$.

Solution:

Step 1: Write the quadratic equation in standard form. Identify the a, b, c values.	This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+9x-5=0$ $a=2, b=9, c=-5$

Step 2: Write the quadratic formula. Then substitute in the values of a, b, c.	Substitute in $a=2, b=9, c=-5$	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-9\pm\sqrt{9^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$
Step 3: Simplify the fraction, and solve for x.		$x=\frac{-9\pm\sqrt{81-(-40)}}{4}$ $x=\frac{-9\pm\sqrt{121}}{4}$ $x=\frac{-9\pm 11}{4}$ $x=\frac{-9+11}{4}$ $\quad x=\frac{-9-11}{4}$ $x=\frac{2}{4}$ $\quad x=\frac{-20}{4}$ $x=\frac{1}{2}$ $\quad x=-5$
Step 4: Check the solutions.	Put each answer in the original equation to check. Substitute $x=\frac{1}{2}$ and $x=-5$.	$2x^2+9x-5=0$ $2\left(\frac{1}{2}\right)^2+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+0\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $\frac{1}{2}+\frac{9}{2}-5\stackrel{?}{=}0$ $\frac{10}{2}-5\stackrel{?}{=}0$ $5-5\stackrel{?}{=}0$ $0=0$ $2x^2+9x-5=0$ $2(-5)^2+9(-5)-5\stackrel{?}{=}0$ $2\cdot 25-45-5\stackrel{?}{=}0$ $50-45-5\stackrel{?}{=}0$ $0=0$

? Exercise 1.4.1

Solve by using the Quadratic Formula: $3y^2-5y+2=0$.

Answer

$y=1, y=\frac{2}{3}$

? Exercise 1.4.2

Solve by using the Quadratic Formula: $4z^2+2z-6=0$.

Answer

$z=1, z=-\frac{3}{2}$

HowTo: Solve a Quadratic Equation Using the Quadratic Formula

1. Write the quadratic equation in standard form, $ax^2+bx+c=0$. Identify the values of a, b, and c.
2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.
3. Simplify.
4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic Formula is an EQUATION. Be sure you start with "$x=$".

Example 1.4.2

Solve by using the Quadratic Formula: $x^2 - 6x = -5$.

Solution:

	$x^2 - 6x = -5$
Write the equation in standard form by adding 5 to each side.	$x^2 - 6x + 5 = 0$
This equation is now in standard form.	$ax^2 + bx + c = 0$ $x^2 - 6x + 5 = 0$
Identify the values of a, b, c.	$a = 1, b = -6, c = 5$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{36 - 20}}{2}$ $x = \frac{6 \pm \sqrt{16}}{2}$ $x = \frac{6 \pm 4}{2}$
Rewrite to show two solutions.	$x = \frac{6+4}{2}, \quad x = \frac{6-4}{2}$
Simplify.	$x = \frac{10}{2}, \quad x = \frac{2}{2}$
	$x = 5, \quad x = 1$
Check: $x^2 - 6x + 5 = 0$ $5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0$ $25 - 30 + 5 \stackrel{?}{=} 0$ $0 = 0$ ✓ $x^2 - 6x + 5 = 0$ $1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0$ $1 - 6 + 5 \stackrel{?}{=} 0$ $0 = 0$ ✓	

Exercise 1.4.3

Solve by using the Quadratic Formula: $a^2 - 2a = 15$.

Answer

$a = -3, a = 5$

Exercise 1.4.4

Solve by using the Quadratic Formula: $b^2 + 24 = -10b$.

Answer

$b = -6, b = -4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the **Quadratic Formula**. If we get a **radical** as a solution, the final answer must have the radical in its simplified form.

Example 1.4.3

Solve by using the Quadratic Formula: $2x^2 + 10x + 11 = 0$.

Solution:

	$2x^2 + 10x + 11 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $2x^2 + 10x + 11 = 0$
Identify the values of a, b and c.	$a = 2, b = 10, c = 11$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \frac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$

Simplify.	$x=\frac{-10\pm\sqrt{100-88}}{4}$
	$x=\frac{-10\pm\sqrt{12}}{4}$
Simplify the radical.	$x=\frac{-10\pm2\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x=\frac{2(-5\pm\sqrt{3})}{4}$
Remove the common factors.	$x=\frac{-5\pm\sqrt{3}}{2}$
Rewrite to show two solutions.	$x=\frac{-5+\sqrt{3}}{2}, \quad x=\frac{-5-\sqrt{3}}{2}$
Check: We leave the check for you!	

? Exercise 1.4.5

Solve by using the Quadratic Formula: $3m^2+12m+7=0$.

Answer

$m=\frac{-6+\sqrt{15}}{3}, m=\frac{-6-\sqrt{15}}{3}$

? Exercise 1.4.6

Solve by using the Quadratic Formula: $5n^2+4n-4=0$.

Answer

$n=\frac{-2+2\sqrt{6}}{5}, n=\frac{-2-2\sqrt{6}}{5}$

When we substitute a, b, and c into the Quadratic Formula and the **radicand** is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

✓ Example 1.4.4

Solve by using the Quadratic Formula: $3p^2+2p+9=0$.

Solution:

Table 9.3.5

	$3p^2+2p+9=0$
This equation is in standard form.	$\underset{3p^2+2p+9=0}{ax^2+bx+c=0}$
Identify the values of a, b, c.	$a=3, b=2, c=9$
Write the Quadratic Formula.	$p=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$p=\frac{-(2)\pm\sqrt{(2)^2-4\cdot3\cdot(9)}}{2\cdot3}$
Simplify.	$p=\frac{-2\pm\sqrt{4-108}}{6}$
	$p=\frac{-2\pm\sqrt{-104}}{6}$
Simplify the radical using complex numbers.	$p=\frac{-2\pm\sqrt{104}\,i}{6}$
Simplify the radical.	$p=\frac{-2\pm2\sqrt{26}\,i}{6}$
Factor the common factor in the numerator.	$p=\frac{2(-1\pm\sqrt{26}\,i)}{6}$

Remove the common factors.	$p=\frac{-1\pm\sqrt{26}\,i}{3}$
Rewrite in standard $a+bi$ form.	$p=-\frac{1}{3}\pm\frac{\sqrt{26}\,i}{3}$
Write as two solutions.	$p=-\frac{1}{3}+\frac{\sqrt{26}\,i}{3},\quad p=-\frac{1}{3}-\frac{\sqrt{26}\,i}{3}$

Exercise 1.4.7

Solve by using the Quadratic Formula: $4a^2-2a+8=0$.

Answer

$$a=\frac{1}{4}+\frac{\sqrt{31}}{4}i,\quad a=\frac{1}{4}-\frac{\sqrt{31}}{4}i$$

Exercise 1.4.8

Solve by using the Quadratic Formula: $5b^2+2b+4=0$.

Answer

$$b=-\frac{1}{5}+\frac{\sqrt{19}}{5}i,\quad b=-\frac{1}{5}-\frac{\sqrt{19}}{5}i$$

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2+bx+c=0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

Example 1.4.5

Solve by using the Quadratic Formula: $x(x+6)+4=0$.

Solution:

Our first step is to get the equation in standard form.

Table 9.3.6

	$x(x+6)+4=0$
Distribute to get the equation in standard form.	$x^2+6x+4=0$
This equation is now in standard form.	$ax^2+bx+c=0$ $x^2+6x+4=0$
Identify the values of a, b, c.	$a=1,\ b=6,\ c=4$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(6)\pm\sqrt{(6)^2-4\cdot 1\cdot(4)}}{2\cdot 1}$
Simplify.	$x=\frac{-6\pm\sqrt{36-16}}{2}$
	$x=\frac{-6\pm\sqrt{20}}{2}$
Simplify the radical.	$x=\frac{-6\pm 2\sqrt{5}}{2}$
Factor the common factor in the numerator.	$x=\frac{2(-3\pm\sqrt{5})}{2}$
Remove the common factors.	$x=-3\pm 2\sqrt{5}$
Write as two solutions.	$x=-3+2\sqrt{5},\ \ x=-3-2\sqrt{5}$
Check: We leave the check for you!	

Exercise 1.4.9

Solve by using the Quadratic Formula: $x(x+2)-5=0$.

Answer

$x = -1 + \sqrt{6}, x = -1 - \sqrt{6}$

Exercise 1.4.10

Solve by using the Quadratic Formula: $3y(y-2) - 3 = 0$.

Answer

$y = 1 + \sqrt{2}, y = 1 - \sqrt{2}$

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

Example 1.4.6

Solve by using the Quadratic Formula: $\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$.

Solution:

Our first step is to clear the fractions.

Table 9.3.7

	$\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2 + \frac{2}{3}u\right) = 6\left(\frac{1}{3}\right)$
Multiply.	$3u^2 + 4u = 2$
Subtract 2 to get the equation in standard form.	$3u^2 + 4u - 2 = 0$
Identify the values of a, b, and c.	$a = 3, b = 4, c = -2$
Write the Quadratic Formula.	$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$u = \frac{-(4) \pm \sqrt{(4)^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3}$
Simplify.	$u = \frac{-4 \pm \sqrt{16 + 24}}{6}$
	$u = \frac{-4 \pm \sqrt{40}}{6}$
Simplify the radical.	$u = \frac{-4 \pm 2\sqrt{10}}{6}$
Factor the common factor in the numerator.	$u = \frac{2(-2 \pm \sqrt{10})}{6}$
Remove the common factors.	$u = \frac{-2 \pm \sqrt{10}}{3}$
Rewrite to show two solutions.	$u = \frac{-2 + \sqrt{10}}{3}, \quad u = \frac{-2 - \sqrt{10}}{3}$
Check: We leave the check for you!	

Exercise 1.4.11

Solve by using the Quadratic Formula: $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$.

Answer

$c = \frac{2 + \sqrt{7}}{3}, \quad c = \frac{2 - \sqrt{7}}{3}$

Exercise 1.4.12

Solve by using the Quadratic Formula: $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$.

Answer

$d = \frac{9+\sqrt{33}}{4}, d = \frac{9-\sqrt{33}}{4}$

Think about the equation $(x-3)^2 = 0$. We know from the **Zero Product Property** that this equation has only one solution, $x = 3$.

We will see in the next example how using the **Quadratic Formula** to solve an equation whose standard form is a perfect square **trinomial** equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0, which leads to only one solution.

Example 1.4.7

Solve by using the Quadratic Formula: $4x^2 - 20x = -25$.

Solution:

Table 9.3.8

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$\underset{ax^2+bx+c=0}{4x^2 - 20x + 25 = 0}$
Identify the values of a, b, and c.	$a = 4, b = -20, c = 25$
Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400-400}}{8}$
	$x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check: We leave the check for you!	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square trinomial. It is equivalent to $(2x-5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

Exercise 1.4.13

Solve by using the Quadratic Formula: $r^2 + 10r + 25 = 0$.

Answer

$r = -5$

Exercise 1.4.14

Solve by using the Quadratic Formula: $25t^2 - 40t = -16$.

Answer

$t = \frac{4}{5}$

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Definition 1.4.2

Discriminant

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,
the quantity $b^2 - 4ac$ is called the discriminant.

Figure 9.3.85

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Table 9.3.9

Quadratic Equation (in standard form)	Discriminat $b^2 - 4ac$	Value of the Discriminant	Number and Type of Solutions
$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2(-5)$ 121	$+$	2 real
$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \cdot 4 \cdot 25$ 0	0	1 real
$3p^2 + 2p + 9 = 0$	$2^2 - 4 \cdot 3 \cdot 9$ -104	$-$	2 complex

When the discriminant is positive, the quadratic equation has **2 real solutions**. $x = \frac{-b \pm \sqrt{+}}{2a}$

When the discriminant is zero, the quadratic equation has **1 real solution**. $x = \frac{-b \pm \sqrt{0}}{2a}$

When the discriminant is negative, the quadratic equation has **2 complex solutions**. $x = \frac{-b \pm \sqrt{-}}{2a}$

Figure 9.3.86

Using the Discriminant $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

Example 1.4.8

Determine the number of solutions to each quadratic equation.

a. $3x^2 + 7x - 9 = 0$
b. $5n^2 + n + 4 = 0$
c. $9y^2 - 6y + 1 = 0$

Solution:

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

a.

$$3x^2 + 7x - 9 = 0$$

The equation is in standard form, identify a, b, and c.

$$a = 3, \quad b = 7, \quad c = -9$$

Write the discriminant.

$$b^2 - 4ac$$

Substitute in the values of a, b, and c.

$$(7)^2 - 4 \cdot 3 \cdot (-9)$$

Simplify.

$$49 + 108$$
$$157$$

Since the discriminant is positive, there are 2 real solutions to the equation.

b.

$$5n^2 + n + 4 = 0$$

The equation is in standard form, identify a, b, and c.

$$a = 5, \quad b = 1, \quad c = 4$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(1)^2-4\cdot 5\cdot 4$$

Simplify.

$$1-80$$
$$-79$$

Since the discriminant is negative, there are 2 complex solutions to the equation.

c.

$$9y^2-6y+1=0$$

The equation is in standard form, identify a, b, and c.

$$a=9,\quad b=-6,\quad c=1$$

Write the discriminant.

$$b^2-4ac$$

Substitute in the values of a, b, and c.

$$(-6)^2-4\cdot 9\cdot 1$$

Simplify.

$$36-36$$
$$0$$

Since the discriminant is 0, there is 1 real solution to the equation.

? Exercise 1.4.15

Determine the number and type of solutions to each quadratic equation.

a. $8m^2-3m+6=0$
b. $5z^2+6z-2=0$
c. $9w^2+24w+16=0$

Answer

a. 2 complex solutions
b. 2 real solutions
c. 1 real solution

? Exercise 1.4.16

Determine the number and type of solutions to each quadratic equation.

a. $b^2+7b-13=0$
b. $5a^2-6a+10=0$
c. $4r^2-20r+25=0$

Answer

a. 2 real solutions
b. 2 complex solutions
c. 1 real solution

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2=k$ or $a(x-h)^2=k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic

Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

Identify the Most Appropriate Method to Solve a Quadratic Equation

1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.
2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.
3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

Example 1.4.9

Identify the most appropriate method to use to solve each quadratic equation.

a. $5z^2 = 17$
b. $4x^2 - 12x + 9 = 0$
c. $8u^2 + 6u = 11$

Solution:

a.

$$5z^2 = 17$$

Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

b.

$$4x^2 - 12x + 9 = 0$$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

c.

$$8u^2 + 6u = 11$$

Put the equation in standard form.

$$8u^2 + 6u - 11 = 0$$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

Exercise 1.4.17

Identify the most appropriate method to use to solve each quadratic equation.

a. $x^2 + 6x + 8 = 0$
b. $(n-3)^2 = 16$
c. $5p^2 - 6p = 9$

Answer

a. Factoring
b. Square Root Property
c. Quadratic Formula

Exercise 1.4.18

Identify the most appropriate method to use to solve each quadratic equation.

a. $8a^2 + 3a - 9 = 0$
b. $4b^2 + 4b + 1 = 0$
c. $5c^2 = 125$

Answer

a. Quadratic Formula
b. Factoring or Square Root Property
c. Square Root Property

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- Using the Quadratic Formula

- Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions
- Discriminant in Quadratic Formula

Key Concepts

- **Zero Product Property** If $a \cdot b = 0$, then either a=0 or b=0 or both. See Example.
- **Solve a quadratic equation by factoring** To solve a quadratic equation by factoring: See Example.
  1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
  2. Factor the quadratic expression.
  3. Use the Zero Product Property.
  4. Solve the linear equations.
  5. Check.
- **Use a problem solving strategy to solve word problems** See Example.
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Key Concepts

- Quadratic Formula
  - The solutions to a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.
  1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of a, b, c.
  2. Write the Quadratic Formula. Then substitute in the values of a, b, c.
  3. Simplify.
  4. Check the solutions.
- Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation
  - For a quadratic equation of the form $ax^2 + bx + c = 0, a \neq 0$,
    - If $b^2 - 4ac > 0$, the equation has 2 real solutions.
    - If $b^2 - 4ac = 0$, the equation has 1 real solution.
    - If $b^2 - 4ac < 0$, the equation has 2 complex solutions.
- Methods to Solve Quadratic Equations:
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to identify the most appropriate method to solve a quadratic equation.
  1. Try Factoring first. If the quadratic factors easily, this method is very quick.
  2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x - h)^2 = k$, it can easily be solved by using the Square Root Property.
  3. Use the **Quadratic Formula.** Any other quadratic equation is best solved by using the Quadratic Formula.

Glossary

quadratic equations

are equations in which the variable is squared.

Zero Product Property

The Zero Product Property states that, if the product of two quantities is zero, at least one of the quantities is zero.

Glossary

discriminant

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the discriminant.

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1.5: Solving Rational Equations

Learning Objectives

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

Before you get started, take this readiness quiz.

1. Solve: $\frac{1}{6}x+\frac{1}{2}=\frac{1}{3}$
   If you missed this problem, review Example 2.2.9.
2. Solve: $n^2-5n-36=0$
   If you missed this problem, review Example 6.6.4.
3. Solve the formula $5x+2y=10$ for y.
   If you missed this problem, review Example 2.4.10.

After defining the terms 'expression' and 'equation' earlier, we have used them throughout this book. We have simplified many kinds of expressions and solved many kinds of equations. We have simplified many rational expressions so far in this chapter. Now we will solve a **rational equation**.

Rational Equation

A **rational equation** is an equation that contains a rational expression.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to "clear" the fractions.

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve. But because the original equation may have a variable in a denominator, we must be careful that we don't end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an **extraneous solution to a rational equation**.

Extraneous Solution to a Rational Equation

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, c, by writing $x \neq c$ next to the equation.

Example 1.5.1: How to Solve a Rational Equation

Solve:

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

Solution

Step 1. Note any value of the variable that would make any denominator zero.

If $x=0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}, x \neq 0$$

Step 2. Find the least common denominator of all denominators in the equation.

Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$

The LCD is $6x$.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.

Multiply both sides of the equation by the LCD, $6x$.

$$6x \cdot \left(\frac{1}{x}+\frac{1}{3}\right)=6x \cdot \left(\frac{5}{6}\right)$$

Use the Distributive Property.

$$6x \cdot \frac{1}{x}+6x \cdot \frac{1}{3}=6x \cdot \left(\frac{5}{6}\right)$$

Simplify - and notice, no more fractions!

$$6+2x=5x$$

Step 4. Solve the resulting equation.

Simplify.

$$6=3x$$
$$2=x$$

Step 5. Check.

If any values found in Step 1 are algebraic solutions, discard them. Check any remaining solutions in the original equation.

We did not get 0 as an algebraic solution.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

We substitute $x=2$ into the original equation.

$$\frac{1}{2}+\frac{1}{3} \stackrel{?}{=} \frac{5}{6}$$
$$\frac{3}{6}+\frac{2}{6} \stackrel{?}{=} \frac{5}{6}$$
$$\frac{5}{6}=\frac{5}{6}\surd$$

The solution is $x=2$

? Exercise 1.5.1

Solve:

$$\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$$

Answer

$$y=-\frac{15}{7}$$

? Exercise 1.5.2

Solve:

$$\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$$

Answer

$$x=\frac{15}{3}$$

The steps of this method are shown.

how to Solve equations with rational expressions.

- Step 1. Note any value of the variable that would make any denominator zero.
- Step 2. Find the least common denominator of all denominators in the equation.
- Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
- Step 4. Solve the resulting equation.
- Step 5. Check:
  - If any values found in Step 1 are algebraic solutions, discard them.
  - Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

✓ Example 1.5.2: How to Solve a Rational Equation using the Zero Product Property

Solve:

$$1-\frac{5}{y}=-\frac{6}{y^2}$$

Solution

Note any value of the variable that would make any denominator zero.

$$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$$

Find the least common denominator of all denominators in the equation. The LCD is y^2.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Distribute.

$$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Multiply.

$$y^2-5y=-6$$

Solve the resulting equation. First write the quadratic equation in standard form.

$$y^2-5y+6=0$$

Factor.

$$(y-2)(y-3)=0$$

Use the Zero Product Property.

$$y-2=0 \text{ or } y-3=0$$

Solve.

$$y=2 \text{ or } y=3$$

Check. We did not get 0 as an algebraic solution.

Check $y=2$ and $y=3$ in the original equation.

$$1-\frac{5}{y}=-\frac{6}{y^2} \qquad 1-\frac{5}{y}=-\frac{6}{y^2}$$

$$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{2^2} \qquad 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{3^2}$$

$$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} \qquad 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$\frac{2}{2}-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} \qquad \frac{3}{3}-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$-\frac{3}{2}\stackrel{?}{=}-\frac{6}{4} \qquad -\frac{2}{3}\stackrel{?}{=}-\frac{6}{9}$$

$$-\frac{3}{2}=-\frac{3}{2}\surd \qquad -\frac{2}{3}=-\frac{2}{3}\surd$$

The solution is $y=2, y=3$

? Exercise 1.5.3

Solve:

$$1-\frac{2}{x}=\frac{15}{x^2}$$

Answer

$x=-3, x=5$

Exercise 1.5.4

Solve:

$$1-\frac{4}{y}=\frac{12}{y^2}$$

Answer

$y=-2, y=6$

In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

Example 1.5.3

Solve:

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$

Solution

Note any value of the variable that would make any denominator zero.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{(x+2)(x-2)}, x\neq -2, x\neq 2$$

Find the least common denominator of all denominators in the equation. The LCD is $(x+2)(x-2)$.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(x+2)(x-2)\left(\frac{2}{x+2}+\frac{4}{x-2}\right)=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Distribute.

$$(x+2)(x-2)\frac{2}{x+2}+(x+2)(x-2)\frac{4}{x-2}=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Remove common factors.

$$\cancel{(x+2)}(x-2)\frac{2}{\cancel{x+2}}+(x+2)\ \cancel{(x-2)}\frac{4}{\cancel{x-2}}=\ \cancel{(x+2)(x-2)}\left(\frac{x-1}{\cancel{x^2-4}}\right)$$

Simplify.

$$2(x-2)+4(x+2)=x-1$$

Distribute.

$$2x-4+4x+8=x-1$$

Solve.

$$\begin{aligned}6x+4&=x-1\\5x&=-5\\x&=-1\end{aligned}$$

Check: We did not get 2 or −2 as algebraic solutions.

Check $x=-1$ in the original equation.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$
$$\frac{2}{(-1)+2}+\frac{4}{(-1)-2}\stackrel{?}{=}\frac{(-1)-1}{(-1)^2-4}$$
$$\frac{2}{1}+\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$$
$$\frac{6}{3}-\frac{4}{3}\stackrel{?}{=}\frac{2}{3}$$
$$\frac{2}{3}=\frac{2}{3}\surd$$

The solution is $x=-1$.

? Exercise 1.5.5

Solve:

$$\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$$

Answer

$x=\frac{2}{3}$

? Exercise 1.5.6

Solve:

$$\frac{5}{y+3}+\frac{2}{y-3}=\frac{5}{y^2-9}$$

Answer

$y=2$

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

✓ Example 1.5.4

Solve:

$$\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$$

Solution

Note any value of the variable that would make any denominator zero. Use the factored form of the quadratic denominator.

$$\frac{m+11}{(m-4)(m-1)}=\frac{5}{m-4}-\frac{3}{m-1}, m\neq 4, m\neq 1$$

Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\left(\frac{5}{m-4}-\frac{3}{m-1}\right)$$

Distribute.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\frac{5}{m-4}-(m-4)(m-1)\frac{3}{m-1}$$

Remove common factors.

$$\cancel{(m-4)(m-1)}\left(\frac{m+11}{\cancel{(m-4)(m-1)}}\right)=\cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}}-(m-4)\ \cancel{(m-1)}\frac{3}{\cancel{m-1}}$$

Simplify.

$$m+11=5(m-1)-3(m-4)$$

Solve the resulting equation.

$$\begin{aligned} m+11 &= 5m-5-3m+12 \\ 4 &= m \end{aligned}$$

Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

? Exercise 1.5.7

Solve:

$$\frac{x+13}{x^2-7x+10}=\frac{6}{x-5}-\frac{4}{x-2}$$

Answer

There is no solution.

? Exercise 1.5.8

Solve:

$$\frac{y-6}{y^2+3y-4}=\frac{2}{y+4}+\frac{7}{y-1}$$

Answer

There is no solution.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

✓ Example 1.5.5

Solve:

$$\frac{y}{y+6}=\frac{72}{y^2-36}+4$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$$

Find the least common denominator. The LCD is $(y-6)(y+6)$

Clear the fractions.

$$(y-6)(y+6)\left(\frac{y}{y+6}\right)=(y-6)(y+6)\left(\frac{72}{(y-6)(y+6)}+4\right)$$

Simplify.

$$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$$

Simplify.

$$y(y-6)=72+4\left(y^2-36\right)$$

Solve the resulting equation.

$$\begin{aligned} y^2-6y &= 72+4y^2-144 \\ 0 &= 3y^2+6y-72 \\ 0 &= 3\left(y^2+2y-24\right) \\ 0 &= 3(y+6)(y-4) \\ y &= -6, y=4 \end{aligned}$$

Check.

$y=-6$ is an extraneous solution. Check $y=4$ in the original equation.

$$\begin{aligned} \frac{y}{y+6} &= \frac{72}{y^2-36}+4 \\ \frac{4}{4+6} &\stackrel{?}{=} \frac{72}{4^2-36}+4 \\ \frac{4}{10} &\stackrel{?}{=} \frac{72}{-20}+4 \\ \frac{4}{10} &\stackrel{?}{=} -\frac{36}{10}+\frac{40}{10} \\ \frac{4}{10} &= \frac{4}{10}\surd \end{aligned}$$

The solution is $y=4$.

Exercise 1.5.9

Solve:

$$\frac{x}{x+4}=\frac{32}{x^2-16}+5$$

Answer

$x=3$

Exercise 1.5.10

Solve:

$$\frac{y}{y+8}=\frac{128}{y^2-64}+9$$

Answer

$y=7$

In some cases, all the algebraic solutions are extraneous.

Example 1.5.6

Solve:

$$\frac{x}{2x-2}-\frac{2}{3x+3}=\frac{5x^2-2x+9}{12x^2-12}$$

Solution

We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.

$$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}$$

Note any value of the variable that would make any denominator zero.

$$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}, x\neq 1, x\neq -1$$

Find the least common denominator. The LCD is $12(x-1)(x+1)$.

Clear the fractions.

$$12(x-1)(x+1)\left(\frac{x}{2(x-1)}-\frac{2}{3(x+1)}\right)=12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$$

Simplify.

$$6(x+1)\cdot x-4(x-1)\cdot 2=5x^2-2x+9$$

Simplify.

$$6x(x+1)-4\cdot 2(x-1)=5x^2-2x+9$$

Solve the resulting equation.

$$\begin{aligned} 6x^2+6x-8x+8 &= 5x^2-2x+9 \\ x^2-1 &= 0 \\ (x-1)(x+1) &= 0 \\ x &= 1 \text{ or } x=-1 \end{aligned}$$

Check.

$x=1$ and $x=-1$ are extraneous solutions.

The equation has no solution.

Exercise 1.5.11

Solve:

$$\frac{y}{5y-10}-\frac{5}{3y+6}=\frac{2y^2-19y+54}{15y^2-60}$$

Answer

There is no solution.

Exercise 1.5.12

Solve:

$$\frac{z}{2z+8}-\frac{3}{4z-8}=\frac{3z^2-16z-16}{8z^2+2z-64}$$

Answer

There is no solution.

Example 1.5.7

Solve:

$$\frac{4}{3x^2-10x+3}+\frac{3}{3x^2+2x-1}=\frac{2}{x^2-2x-3}$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}=\frac{2}{(x-3)(x+1)}, x\neq -1, x\neq \frac{1}{3}, x\neq 3$$

Find the least common denominator. The LCD is $(3x-1)(x+1)(x-3)$.

Clear the fractions.

$$(3x-1)(x+1)(x-3)\left(\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}\right)=(3x-1)(x+1)(x-3)\left(\frac{2}{(x-3)(x+1)}\right)$$

Simplify.

$$4(x+1)+3(x-3)=2(3x-1)$$

Distribute.

$$4x+4+3x-9=6x-2$$

Simplify.

$$7x-5=6x-2$$

$$x=3$$

The only algebraic solution was $x=3$, but we said that $x=3$ would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

Exercise 1.5.13

Solve:

$$\frac{15}{x^2+x-6}-\frac{3}{x-2}=\frac{2}{x+3}$$

Answer

There is no solution.

Exercise 1.5.14

Solve:

$$\frac{5}{x^2+2x-3}-\frac{3}{x^2+x-2}=\frac{1}{x^2+5x+6}$$

Answer

There is no solution.

Use Rational Functions

Working with functions that are defined by rational expressions often lead to rational equations. Again, we use the same techniques to solve them.

Example 1.5.8

For rational function, $f(x)=\dfrac{2x-6}{x^2-8x+15}$:

a. Find the domain of the function
b. Solve $f(x)=1$
c. Find the points on the graph at this function value.

Solution

a. The domain of a rational function is all real numbers except those that make the rational expression undefined. So to find them, we will set the denominator equal to zero and solve.

$$\begin{aligned} x^2-8x+15 &= 0 \\ (x-3)(x-5) &= 0 \quad \text{Factor the trinomial.} \\ x-3 &= 0 \quad \text{Use the Zero Product Property.} \\ x-5 &= 0 \quad \text{Use the Zero Product Property.} \\ x=3 \quad x &= 5 \quad \text{Solve.} \end{aligned}$$

The domain is all real numbers except $x\neq 3, x\neq 5$

b. $$f(x)=1$$

Substitute in the rational expression.

$$\frac{2x-6}{x^2-8x+15}=1$$

Factor the denominator.

$$\frac{2x-6}{(x-3)(x-5)}=1$$

Multiply both sides by the LCD, $(x-3)(x-5)$

$$(x-3)(x-5)\left(\frac{2x-6}{(x-3)(x-5)}\right)=(x-3)(x-5)(1)$$

Simplify.

$$2x-6=x^2-8x+15$$

Solve.

$$0=x^2-10x+21$$

Factor.

$$0 = (x-7)(x-3)$$

Use the Zero Product Property.

$$x-7=0 \quad x-3=0$$

Solve.

$$x=7 \quad x=3$$

c. The value of the function is 1 when $x=7, x=3$. So the points on the graph of this function when $f(x)=1$, will be $(7,1),(3,1)$. However, now we need to check for extraneous solutions. Since $x=3$ is not in the domain of the original function, we must remove $(3,1)$, since it would be considered extraneous.

? Exercise 1.5.15

For rational function, $f(x)=\dfrac{8-x}{x^2-7x+12}$

a. Find the domain of the function.
b. Solve $f(x)=3$.
c. Find the points on the graph at this function value.

Answer

a. The domain is all real numbers except $x \neq 3$ and $x \neq 4$
b. $x=2, x=\dfrac{14}{3}$
c. $(2,3), \left(\dfrac{14}{3},3\right)$

? Exercise 1.5.16

For rational function, $f(x)=\dfrac{x-1}{x^2-6x+5}$

a. Solve $f(x)=4$.
b. Find the points on the graph at this function value.

Answer

a. The domain is all real numbers except $x \neq 1$ and $x \neq 5$
b. $x=\dfrac{21}{4}$
c. $\left(\dfrac{21}{4},4\right)$

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

When we developed the point-slope formula from our slope formula, we cleared the fractions by multiplying by the LCD.

$$
\begin{aligned}
m &= \frac{y-y_1}{x-x_1} \\
m\,(x-x_1) &= \left(\frac{y-y_1}{x-x_1}\right)(x-x_1) \quad \text{Multiply both sides of the equation by } x-x_1. \\
m\,(x-x_1) &= y-y_1 \quad \text{Simplify.} \\
y-y_1 &= m\,(x-x_1) \quad \text{Rewrite the equation with the y terms on the left.}
\end{aligned}
$$

In the next example, we will use the same technique with the formula for slope that we used to get the point-slope form of an equation of a line through the point $(2,3)$. We will add one more step to solve for y.

Example 1.5.9

Solve: $m = \dfrac{y-2}{x-3}$ for y.

Solution

$$m = \frac{y-2}{x-3}$$

Note any value of the variable that would make any denominator zero.

$$m = \frac{y-2}{x-3}, x \neq 3$$

Clear the fractions by multiplying both sides of the equation by the LCD, $x-3$.

$$(x-3)m = (x-3)\left(\frac{y-2}{x-3}\right)$$

Simplify.

$$xm-3m = y-2$$

Isolate the term with y.

$$xm-3m+2 = y$$

Exercise 1.5.17

Solve: $m = \dfrac{y-5}{x-4}$ for y.

Answer

$y = mx-4m+5$

Exercise 1.5.18

Solve: $m = \dfrac{y-1}{x+5}$ for y.

Answer

$y = mx+5m+1$

Remember to multiply both sides by the LCD in the next example.

Example 1.5.10

Solve: $\frac{1}{c}+\frac{1}{m}=1$ for c

Solution

$$\frac{1}{c}+\frac{1}{m}=1 \text{ for } c$$

Note any value of the variable that would make any denominator zero.

$$\frac{1}{c}+\frac{1}{m}=1, c\neq 0, m\neq 0$$

Clear the fractions by multiplying both sides of the equations by the LCD, cm.

$$cm\left(\frac{1}{c}+\frac{1}{m}\right)=cm(1)$$

Distribute.

$$cm\left(\frac{1}{c}\right)+cm\frac{1}{m}=cm(1)$$

Simplify.

$$m+c=cm$$

Collect the terms with c to the right.

$$m=cm-c$$

Factor the expression on the right.

$$m=c(m-1)$$

To isolate c, divide both sides by $m-1$.

$$\frac{m}{m-1}=\frac{c(m-1)}{m-1}$$

Simplify by removing common factors.

$$\frac{m}{m-1}=c$$

Notice that even though we excluded $c=0$ and $m=0$ from the original equation, we must also now state that $m\neq 1$.

Exercise 1.5.19

Solve: $\frac{1}{a}+\frac{1}{b}=c$ for a.

Answer

$$a=\frac{b}{cb-1}$$

Exercise 1.5.20

Solve: $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for y

Answer

$$y = \frac{3x}{x+6}$$

Media Access Additional Online Resources

Access this online resource for additional instruction and practice with equations with rational expressions.

- Equations with Rational Expressions

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1.6: Solving Radical Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Before you get started, take this readiness quiz.

1. Simplify: $(y-3)^2$.
   If you missed this problem, review Example 5.31.
2. Solve: $2x-5=0$.
   If you missed this problem, review Example 2.2.
3. Solve $n^2-6n+8=0$.
   If you missed this problem, review Example 6.45.

Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Definition: Radical Equation

An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index n to the nth power. This will eliminate the radical.

$$\text{For } a \geq 0, (\sqrt[n]{a})^n = a \text{ .}$$

Example 1.6.1 how to solve a radical equation

Solve: $\sqrt{5n-4}-9=0$.

Solution:

Table 8.6.1

Step 1: Isolate the radical on one side of the equation.	To isolate the radical, add 9 to both sides. Simplify.	$\sqrt{5n-4}-9=0$ $\sqrt{5n-4}-9+9=0+9$ $\sqrt{5n-4}=9$
Step 2: Raise both sides of the equation to the power of the index.	Since the index of a square root is 2, we square both sides.	$(\sqrt{5n-4})^2=(9)^2$
Step 3: Solve the new equation.	Remember, $(\sqrt{a})^2=a$.	$5n-4=81$ $5n=85$ $n=17$

Step 4: Check the answer in the original equation.		Check the answer. $\sqrt{5n-4}-9=0$ $\sqrt{5(17)-4}-9\stackrel{?}{=}0$ $\sqrt{85-4}-9\stackrel{?}{=}0$ $\sqrt{81}-9\stackrel{?}{=}0$ $9-9=0$ $0=0$ The solution is $n=17$.

Exercise 1.6.1

Solve: $\sqrt{3m+2}-5=0$.

Answer

$m=\frac{23}{3}$

Exercise 1.6.2

Solve: $\sqrt{10z+1}-2=0$.

Answer

$z=\frac{3}{10}$

Solve a Radical Equation With One Radical

1. Isolate the radical on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Solve the new equation.
4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

Example 1.6.2

Solve: $\sqrt{9k-2}+1=0$.

Solution:

Table 8.6.2

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 to both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$

Because the square root is equal to a negative number, the equation has no solution.

Exercise 1.6.3

Solve: $\sqrt{2r-3}+5=0$.

Answer

no solution

Exercise 1.6.4

Solve: $\sqrt{7s-3}+2=0$.

Answer

no solution

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Definition: Binomial Squares

$(a+b)^2=a^2+2ab+b^2$
$(a-b)^2=a^2-2ab+b^2$

Don't forget the middle term!

Example 1.6.3

Solve: $\sqrt{p-1}+1=p$.

Solution:

Table 8.6.3

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, using the Product of Binomial Squares Pattern on the right, Then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$
Use the Zero Product Property.	$0=p-1 \quad 0=p-2$
Solve each equation.	$p=1 \quad p=2$
Check the answers.	
$p=1$ $\sqrt{p-1}+1=p$ $\sqrt{1-1}+1\stackrel{?}{=}1$ $\sqrt{0}+1\stackrel{?}{=}1$ $1=1$ ✓ $p=2$ $\sqrt{p-1}+1=p$ $\sqrt{2-1}+1\stackrel{?}{=}2$ $\sqrt{1}+1\stackrel{?}{=}2$ $2=2$ ✓	

The solutions are $p=1, p=2$.

Exercise 1.6.5

Solve: $\sqrt{x-2}+2=x$.

Answer

$x = 2, x = 3$

Exercise 1.6.6

Solve: $\sqrt{y-5}+5=y$.

Answer

$y = 5, y = 6$

When the index of the radical is 3, we cube both sides to remove the radical.

$$(\sqrt[3]{a})^3 = a$$

Example 1.6.4

Solve: $\sqrt[3]{5x+1}+8=4$.

Solution:

Table 8.6.4

	$\sqrt[3]{5x+1}+8=4$
To isolate the radical, subtract 8 from both sides.	$\sqrt[3]{5x+1}=-4$
Cube both sides of the equation.	$(\sqrt[3]{5x+1})^3=(-4)^3$
Simplify.	$5x+1=-64$
Solve the equation.	$5x=-65$
	$x=-13$
Check the answer.	
$x=-13$ $\quad \sqrt[3]{5x+1}+8=4$ $\sqrt[3]{5(-13)+1}+8\stackrel{?}{=}4$ $\sqrt[3]{-64}+8\stackrel{?}{=}4$ $-4+8\stackrel{?}{=}4$ $4=4\checkmark$	
	The solution is $x=-13$.

Exercise 1.6.7

Solve: $\sqrt[3]{4x-3}+8=5$

Answer

$x = -6$

Exercise 1.6.8

Solve: $\sqrt[3]{6x-10}+1=-3$

Answer

$x = -9$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since $(a^m)^n = a^{m \cdot n}$, we have for example,

$$\left(x^{\frac{1}{2}}\right)^2 = x, \left(x^{\frac{1}{3}}\right)^3 = x$$

Remember, $x^{\frac{1}{2}} = \sqrt{x}$ and $x^{\frac{1}{3}} = \sqrt[3]{x}$.

Example 1.6.5

Solve: $(3x-2)^{\frac{1}{4}} + 3 = 5$.

Solution:

Table 8.6.5

	$(3x-2)^{\frac{1}{4}} + 3 = 5$
To isolate the term with the rational exponent, subtract 3 from both sides.	$(3x-2)^{\frac{1}{4}} = 2$
Raise each side of the equation to the fourth power.	$\left((3x-2)^{\frac{1}{4}}\right)^4 = (2)^4$
Simplify.	$3x - 2 = 16$
Solve the equation.	$3x = 18$
	$x = 6$
Check the answer.	
$x = 6 \quad (3x-2)^{\frac{1}{4}} + 3 = 5$ $(3 \cdot 6 - 2)^{\frac{1}{4}} + 3 \stackrel{?}{=} 5$ $(16)^{\frac{1}{4}} + 3 \stackrel{?}{=} 5$ $2 + 3 \stackrel{?}{=} 5$ $5 = 5$ ✓	
	The solution is $x = 6$.

Exercise 1.6.9

Solve: $(9x+9)^{\frac{1}{4}} - 2 = 1$

Answer

$x = 8$

Exercise 1.6.10

Solve: $(4x-8)^{\frac{1}{4}} + 5 = 7$

Answer

$x = 6$

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an **extraneous solution**!

Example 1.6.6

Solve: $\sqrt{r+4}-r+2=0$.

Solution:

Table 8.6.6

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$(\sqrt{r+4})^2=(r-2)^2$
Simplify and then solve the equation.	$r+4=r^2-4r+4$
If it is a quadratic equation, so get zero on one side.	$0=r^2-5r$
Factor the right side.	$0=r(r-5)$
Use the Zero Product Property.	$0=r \quad 0=r-5$
Solve the equation.	$r=0 \quad r=5$
Check your answer.	
$r=0$, $\sqrt{r+4}-r+2=0$ $\sqrt{0+4}-0+2\stackrel{?}{=}0$ $\sqrt{4}+2\stackrel{?}{=}0$ $4\neq 0$ $r=5$, $\sqrt{r+4}-r+2=0$ $\sqrt{5+4}-5+2\stackrel{?}{=}0$ $\sqrt{9}-3\stackrel{?}{=}0$ $0=0\checkmark$	The solution is $r=5$.
	$r=0$ is an extreneous solution.

Exercise 1.6.11

Solve: $\sqrt{m+9}-m+3=0$

Answer

$m=7$

Exercise 1.6.12

Solve: $\sqrt{n+1}-n+1=0$.

Answer

$n=3$

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

Example 1.6.7

Solve: $3\sqrt{3x-5}-8=4$.

Solution:

Table 8.6.7

	$3\sqrt{3x-5}-8=4$
Isolate the radical term.	$3\sqrt{3x-5}=12$

Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5}=4$
Square both sides of the equation.	$(\sqrt{3x-5})^2=(4)^2$
Simplify, then solve the new equation.	$3x-5=16$
	$3x=21$
Solve the equation.	$x=7$
Check the answer.	
$x=7$ $3\sqrt{3x-5}-8=4$ $3\sqrt{3(7)-5}-8\stackrel{?}{=}4$ $3\sqrt{21-5}-8\stackrel{?}{=}4$ $3\sqrt{16}-8\stackrel{?}{=}4$ $3(4)-8\stackrel{?}{=}4$ $4=4$ ✓	
	The solution is $x=7$.

Exercise 1.6.13

Solve: $2\sqrt{4a+4}-16=16$.

Answer

$a=63$

Exercise 1.6.14

Solve: $3\sqrt{2b+3}-25=50$

Answer

$b=311$

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

Example 1.6.8

Solve: $\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$.

Solution:

The radical terms are isolated.

$$\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$$

Since the index is 3, cube both sides of the equation.

$$(\sqrt[3]{4x-3})^3=(\sqrt[3]{3x+2})^3$$

Simplify, then solve the new equation.

$$\begin{aligned} 4x-3 &= 3x+2 \\ x-3 &= 2 \\ x &= 5 \end{aligned}$$

The solution is $x=5$.

Check the answer.

We leave it to you to show that 5 checks!

? Exercise 1.6.15

Solve: $\sqrt[3]{5x-4} = \sqrt[3]{2x+5}$.

Answer

$x=3$

? Exercise 1.6.16

Solve: $\sqrt[3]{7x+1} = \sqrt[3]{2x-5}$.

Answer

$x=-\frac{6}{5}$

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

✓ Example 1.6.9 how to solve a radical equation

Solve: $\sqrt{m}+1 = \sqrt{m+9}$.

Solution:

Table 8.6.8

Step 1: Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1 = \sqrt{m+9}$
Step 2: Raise both sides of the equation to the power of the index.	We square both sides. Simplify--be very careful as you multiply!	$(\sqrt{m}+1)^2 = (\sqrt{m+9})^2$
Step 3: Are there any more radicals? If yes, repeat Step 1 and Step 2 again. If no, solve the new equation.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term. Here, we can easily isolate the radical by dividing both sides by 2. Square both sides.	$\begin{aligned} m+2\sqrt{m}+1 &= m+9 \\ 2\sqrt{m} &= 8 \\ \sqrt{m} &= 4 \\ (\sqrt{m})^2 &= (4)^2 \\ m &= 16 \end{aligned}$
Step 4: Check the answer in the original equation.		$\begin{aligned} \sqrt{m}+1 &= \sqrt{m+9} \\ \sqrt{16}+1 &\stackrel{?}{=} \sqrt{16+9} \\ 4+1 &\stackrel{?}{=} 5 \\ 5 &= 5 \end{aligned}$ The solution is $m=16$.

Exercise 1.6.17

Solve: $3-\sqrt{x}=\sqrt{x-3}$.

Answer

$x=4$

Exercise 1.6.18

Solve: $\sqrt{x}+2=\sqrt{x+16}$.

Answer

$x=9$

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

Solve a Radical Equation

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
   If yes, repeat Step 1 and Step 2 again.
   If no, solve the new equation.
4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern in $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Example 1.6.10

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution:

Table 8.6.9

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right is isolated. Square both sides.	$\left(\sqrt{q-2}+3\right)^2=\left(\sqrt{4q+1}\right)^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation so we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides. It would not help to divide both sides by 6. Remember to square both the 6 and the $\sqrt{q-2}$.	$\left(6\sqrt{q-2}\right)^2=\left(3q-6\right)^2$ $a^2 - 2ab + b^2$ $6^2\left(\sqrt{q-2}\right)^2=(3q)^2-2\cdot 3q\cdot 6+6^2$
Simplify, then solve the new equation.	$36(q-2)=9q^2-36q+36$
Distribute.	$36q-72=9q^2-36q+36$
It is a quadratic equation, so get zero on one side.	$0=9q^2-72q+108$
Factor the right side.	$0=9(q^2-8q+12)$ $0=9(q-6)(q-2)$

Use the Zero Product Property.	$q-6=0 \quad q-2=0$ $q=6 \quad q=2$
The checks are left to you.	The solutions are $q=6$ and $q=2$.

Exercise 1.6.19

Solve: $\sqrt{x-1}+2=\sqrt{2x+6}$

Answer

$x=5$

Exercise 1.6.20

Solve: $\sqrt{x}+2=\sqrt{3x+4}$

Answer

$x=0x=4$

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

Use a Problem Solving Strategy for Applications With Formulas

1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Name** what we are looking for by choosing a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve the equation** using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of **gravity** on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Definition: Falling Objects

On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula

$$t=\frac{\sqrt{h}}{4}$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h=64$ into the formula.

Table 8.6.10

	$t=\frac{\sqrt{h}}{4}$
	$t=\frac{\sqrt{64}}{4}$

Take the square root of 64.	$t=\frac{8}{4}$
Simplify the fraction.	$t=2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Example 1.6.11

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution:

Table 8.6.11

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking.	The time it takes for the sunglasses to reach the river.
Step 3: **Name** what we are looking.	Let (t=\) time.
Step 4: **Translate** into an equation by writing the appropriate formula. Substitute in the given information.	$t=\frac{\sqrt{h}}{4}$, and $h=400$ $t=\frac{\sqrt{400}}{4}$
Step 5: **Solve the equation**.	$t=\frac{20}{4}$
	$t=5$
Step 6: **Check** the answer in the problem and make sure it makes sense.	$5\stackrel{?}{=}\frac{\sqrt{400}}{4}$ $5\stackrel{?}{=}\frac{20}{4}$ $5=5\checkmark$
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7: **Answer** the equation.	It will take 5 seconds for the sunglasses to reach the river.

Exercise 1.6.21

A helicopter dropped a rescue package from a height of $1,296$ feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

Answer

9 seconds

Exercise 1.6.22

A window washer dropped a squeegee from a platform 196 feet above the sidewalk. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Answer

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the **speed**, in miles per hour, a car was going before applying the brakes.

Definition: Skid Marks and Speed of a Car

If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula

$$s = \sqrt{24d}$$

Example 1.6.12

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

Table 8.6.12

Step 1: **Read** the problem.	
Step 2: **Identify** what we are looking for.	The speed of a car.
Step 3: **Name** what we are looking for.	Let $s =$ the speed.
Step 4: **Translate** into an equation by writing the appropriate formula. Substitute in the given information.	$s = \sqrt{24d}$, and $d = 190$ $s = \sqrt{24(190)}$
Step 5: **Solve the equation**.	$s = \sqrt{4{,}560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
	$67.5 \stackrel{?}{\approx} \sqrt{24(190)}$ $67.5 \stackrel{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277...$ ✓
	The speed of the car before the brakes were applied was 67.5 miles per hour.

Exercise 1.6.23

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer

42.7mph

Exercise 1.6.24

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

54.1mph

Access these online resources for additional instruction and practice with solving radical equations.

- Solving an Equation Involving a Single Radical
- Solving Equations with Radicals and Rational Exponents
- Solving Radical Equations
- Solve Radical Equations
- Radical Equation Application

Key Concepts

- **Binomial Squares**
  $(a+b)^2 = a^2 + 2ab + b^2$
  $(a-b)^2 = a^2 - 2ab + b^2$
- **Solve a Radical Equation**
  1. Isolate one of the radical terms on one side of the equation.
  2. Raise both sides of the equation to the power of the index.
  3. Are there any more radicals?
     If yes, repeat Step 1 and Step 2 again.
     If no, solve the new equation.
  4. Check the answer in the original equation.
- **Problem Solving Strategy for Applications with Formulas**
  1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  2. Identify what we are looking for.
  3. Name what we are looking for by choosing a variable to represent it.
  4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.
- **Falling Objects**
  - On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
  - If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

Glossary

radical equation

An equation in which a variable is in the radicand of a radical expression is called a radical equation.

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1.7: Solving Linear Inequalities

Learning Objectives

- Use interval notation.
- Use properties of inequalities.
- Solve inequalities in one variable algebraically.

It is not easy to make the honor role at most top universities. Suppose students were required to carry a course load of at least 12 credit hours and maintain a grade point average of 3.5 or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value inequalities.



Figure 1.7.1

Using Interval Notation

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.

- We can use a number line as shown in Figure $1.7.2$. The blue ray begins at $x = 4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.
- We can use **set-builder notation**: $\{x|x \geq 4\}$, which translates to "all real numbers x such that x is greater than or equal to 4." Notice that braces are used to indicate a set.
- The third method is **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $[4, \infty)$. This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

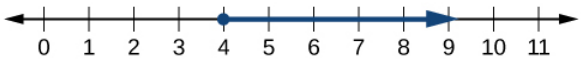


Figure 1.7.2

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled." A few examples of an interval, or a set of numbers in which a solution falls, are $[-2, 6)$, or all numbers between -2 and 6, including -2, but not including 6; $(-1, 0)$, all real numbers between, but not including -1 and 0; and $(-\infty, 1]$, all real numbers less than and including 1. Table $1.7.1$ outlines the possibilities.

a}; (a , infinity). The entries in the fourth row are: All real numbers less than b, but not including b; {x| x < b}; (negative infinity, b). The entries in the fifth row are: All real numbers greater than a, including a; {x| x a}; [a, infinity). The entries in the sixth row are: All real numbers less than b, including b; {x| x b}; (negative infinity, b]. The entries in the seventh row are: All real numbers between a and b, including a; {x| a x < b}; [a, b). The entries in the eighth row are: All real numbers between a and b, including b; {x| a < x b}; (a, b]. The entries in the ninth row are: All real numbers between a and b, including a and b; {x| a x b}; [a, b]. The entries in the tenth row are: all real numbers less than a and greater than b; {x| x < a and x > b}; (negative infinity, a) union (b, infinity). The entries in the eleventh row are: All real numbers; {x| x is all real numbers}; (negative infinity, infinity)."">

Table 1.7.1

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between a and b, but not including a or b	$\{x \mid a < x < b\}$	(a, b)
All real numbers greater than a, but not including a	$\{x \mid x > a\}$	(a, ∞)
All real numbers less than b, but not including b	$\{x \mid x < b\}$	$(-\infty, b)$
All real numbers greater than a, including a	$\{x \mid x \geq a\}$	$[a, \infty)$
All real numbers less than b, including b	$\{x \mid x \leq b\}$	$(-\infty, b]$
All real numbers between a and b, including a	$\{x \mid a \leq x < b\}$	$[a, b)$
All real numbers between a and b, including b	$\{x \mid a < x \leq b\}$	$(a, b]$
All real numbers between a and b, including a and b	$\{x \mid a \leq x \leq b\}$	$[a, b]$
All real numbers less than a or greater than b	$\{x \mid x < a \; and \; x > b\}$	$(-\infty, a) \cup (b, \infty)$
All real numbers	$\{x \mid x \; is \; all \; real \; numbers\}$	$(-\infty, \infty)$

Example 1.7.1: Using Interval Notation to Express All Real Numbers Greater Than or Equal to a

Use interval notation to indicate all real numbers greater than or equal to -2.

Solution

Use a bracket on the left of -2 and parentheses after infinity: $[-2, \infty)$. The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

Try It! 1.7.1

Use interval notation to indicate all real numbers between and including -3 and 5.

Answer

$[-3, 5]$

Example 1.7.2: Using Interval Notation to Express All Real Numbers Less Than or Equal to a or Greater Than or Equal to b

Write the interval expressing all real numbers less than or equal to -1 or greater than or equal to 1.

Solution

We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to 1. So, this interval begins at $-\infty$ and ends at -1, which is written as $(-\infty, -1]$.

The second interval must show all real numbers greater than or equal to 1, which is written as $[1, \infty)$. However, we want to combine these two sets. We accomplish this by inserting the union symbol, $\cup$, between the two intervals.

$$(-\infty, -1] \cup [1, \infty)$$

Try It! 1.7.2

Express all real numbers less than -2 or greater than or equal to 3 in interval notation.

Answer

$(-\infty, -2) \cup [3, \infty)$

Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the **addition property** and the **multiplication property** to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

PROPERTIES OF INEQUALITIES

Addition Property

- If $a < b$, then $a + c < b + c$.

Multiplication Property

- If $a < b$ and $c > 0$, then $ac < bc$.
- If $a < b$ and $c < 0$, then $ac > bc$.

These properties also apply to $a \leq b$, $a > b$, and $a \geq b$.

Example 1.7.3: Demonstrating the Addition Property

Illustrate the addition property for inequalities by solving each of the following:

a. (a) $x - 15 < 4$
b. (b) $6 \geq x - 1$
c. (c) $x + 7 > 9$

Solution

The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

a.

$$\begin{aligned} x - 15 &< 4 \\ x - 15 + 15 &< 4 + 15 \\ x &< 19 \end{aligned}$$

b.

$$\begin{aligned} 6 &\geq x - 1 \\ 6 + 1 &\geq x - 1 + 1 \\ 7 &\geq x \end{aligned}$$

c.

$$\begin{aligned} x + 7 &> 9 \\ x + 7 - 7 &> 9 - 7 \\ x &> 2 \end{aligned}$$

Try It! 1.7.3

Solve: $3x - 2 < 1$.

Answer

$x < 1$

Example 1.7.4: Demonstrating the Multiplication Property

Illustrate the multiplication property for inequalities by solving each of the following:

a. $3x < 6$
b. $-2x - 1 \geq 5$
c. $5 - x > 10$

Solution

a.

$$\begin{aligned} 3x &< 6 \\ \frac{1}{3}(3x) &< (6)\frac{1}{3} \\ x &< 2 \end{aligned}$$

b.

$$\begin{aligned} -2x - 1 &\geq 5 \\ -2x &\geq 6 \\ \left(-\frac{1}{2}\right)(-2) &\geq (6)\left(-\frac{1}{2}\right) && \text{Multiply by } \left(-\frac{1}{2}\right) \\ x &\leq -3 && \text{Reverse the inequality.} \end{aligned}$$

c.

$$\begin{aligned} 5 - x &> 10 \\ -x &> 5 \\ (-1)(-x) &> (5)(-1) && \text{Multiply by } -1 \\ x &< -5 && \text{Reverse the inequality.} \end{aligned}$$

Try It! 1.7.4

Solve: $4x + 7 \geq 2x - 3$.

Answer

$x \geq -5$

Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations; we combine like terms and perform operations. To solve, we isolate the variable.

Example 1.7.5: Solving an Inequality Algebraically

Solve the inequality: $13 - 7x \geq 10x - 4$.

Solution

Solving this inequality is similar to solving an equation up until the last step.

$$\begin{aligned} 13-7x &\geq 10x-4 & \\ 13-17x &\geq -4 & \text{Move variable terms to one side of the inequality} \\ -17x &\geq -17 & \text{Isolate the variable term} \\ x &\leq 1 & \text{Dividing both sides by -17 reverses the inequality.} \end{aligned}$$

The solution set is given by the interval $(-\infty, 1]$, or all real numbers less than and including 1.

? Try It! 1.7.5

Solve the inequality and write the answer using interval notation: $-x+4<\frac{1}{2}x+1$.

Answer

$(2, \infty)$

✓ Example 1.7.6: Solving an Inequality with Fractions

Solve the following inequality and write the answer in interval notation: $-\frac{3}{4}x \geq -\frac{5}{8}+\frac{2}{3}x$.

Solution

We begin solving in the same way we do when solving an equation.

$$\begin{aligned} -\frac{3}{4}x &\geq -\frac{5}{8}+\frac{2}{3}x & \\ -\frac{3}{4}x-\frac{2}{3}x &\geq -\frac{5}{8} & \text{Put variable terms on one side.} \\ -\frac{9}{12}x-\frac{8}{12}x &\geq -\frac{5}{8} & \text{Write fractions with common denominator.} \\ -\frac{17}{12}x &\geq -\frac{5}{8} & \\ x &\leq -\frac{5}{8}\left(-\frac{12}{17}\right) & \text{Multiplying by a negative number reverses the inequality.} \\ x &\leq \frac{15}{34} & \end{aligned}$$

The solution set is the interval $\left(-\infty, \frac{15}{34}\right]$.

? Try It! 1.7.6

Solve the inequality and write the answer in interval notation: $-\frac{5}{6}x \leq \frac{3}{4}+\frac{8}{3}x$.

Answer

$\left[-\frac{3}{14}, \infty\right)$

Understanding Compound Inequalities

A **compound inequality** includes two inequalities in one statement. A statement such as $4<x\leq 6$ means $4<x$ and $x\leq 6$. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

Example 1.7.7: Solving a Compound Inequality

Solve the compound inequality: $3 \leq 2x+2 < 6$

Solution

The first method is to write two separate inequalities: $3 \leq 2x+2$ and $2x+2<6$. We solve them independently.

$$\begin{aligned} 3 &\leq 2x+2 \quad \text{and} \quad 2x+2<6 \\ 1 &\leq 2x \qquad\qquad\quad 2x<4 \\ \tfrac{1}{2} &\leq x \qquad\qquad\quad x<2 \end{aligned}$$

Then, we can rewrite the solution as a compound inequality, the same way the problem began.

$$\frac{1}{2} \leq x < 2$$

In interval notation, the solution is written as $[\frac{1}{2},\ 2)$.

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.

$3 \leq 2x+2 < 6$

$1 \leq 2x < 4$ Isolate the variable term, and subtract 2 from all three parts.

$\frac{1}{2} \leq x < 2$ Divide through all three parts by 2.

We get the same solution: $[\frac{1}{2},\ 2)$.

Try It! 1.7.7

Solve the compound inequality: $4 < 2x-8 \leq 10$.

Answer

$6 < x \leq 9$ or $(6, 9]$

▼

Example 1.7.8: Solving a Compound Inequality with the Variable in All Three Parts

Solve the compound inequality with variables in all three parts: $3+x > 7x-2 > 5x-10$.

Solution

Let's try the first method. Write two inequalities:

$$\begin{aligned} 3+x &> 7x-2 \quad \text{and} \quad 7x-2 > 5x-10 \\ 3 &> 6x-2 \qquad\qquad 2x-2 > -10 \\ 5 &> 6x \qquad\qquad\quad 2x > -8 \\ \tfrac{5}{6} &> x \qquad\qquad\quad -4 < x \end{aligned}$$

The solution set is $-4 < x < \frac{5}{6}$ or in interval notation $\left(-4, \frac{5}{6}\right)$. Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right, as they appear on a number line. See Figure 1.7.3.



Figure 1.7.3

Try It! 1.7.8

Solve the compound inequality: $3y < 4 - 5y < 5 + 3y$.

Answer

$\left(-\frac{1}{8}, \frac{1}{2}\right)$

▼
▼

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CHAPTER OVERVIEW

2: Functions

2.1: Functions and Function Notation
2.2: Domain and Range
2.3: Rates of Change and Behavior of Graphs
2.4: Composition of Functions
2.5: Inverse Functions
2.6: Transformation of Functions

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2.1: Functions and Function Notation

Determining Whether a Relation Represents a Function

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the domain and the set of the second components of each ordered pair is called the range. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\} \tag{1.1.1}$$

The domain is $\{1,2,3,4,5\}$ The range is $\{2,4,6,8,10\}$

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an output value, or **dependent variable**, and is often labeled lowercase letter y.

A function f is a relation that assigns a single value in the range to each value in the domain. In other words, no x-values are repeated. For our example that relates the first five **natural numbers** to numbers double their values, this relation is a function because each element in the domain, {1, 2, 3, 4, 5}, is paired with exactly one element in the range, $\{2,4,6,8,10\}$

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\} \tag{1.1.2}$$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1,2,3,4,5\}$ For example, the term "odd" corresponds to three values from the range, $\{1,3,5\}$,and the term "even" corresponds to two values from the range, $\{2,4\}$. This violates the definition of a function, so this relation is not a function.

Figure 2.1.1 compares relations that are functions and not functions.

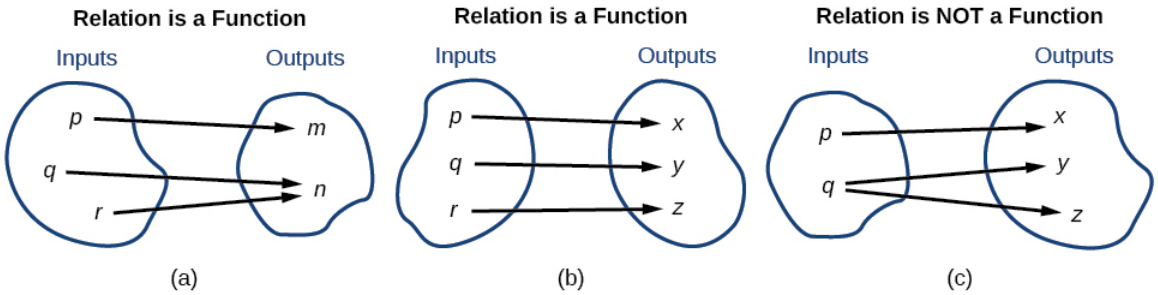


Figure 2.1.1: (a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."

The **input** values make up the **domain**, and the **output** values make up the **range**.

How To: Given a relationship between two quantities, determine whether the relationship is a function

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example 2.1.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in Figure 2.1.2 consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

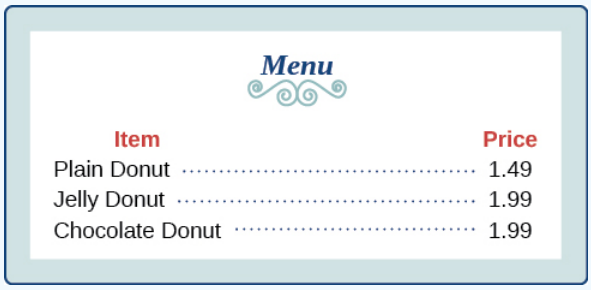


Figure 2.1.2: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Solution

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See Figure 2.1.3.



Figure 2.1.3: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Each item on the menu has only one price, so the price is a function of the item.

a. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See Figure 2.1.4.

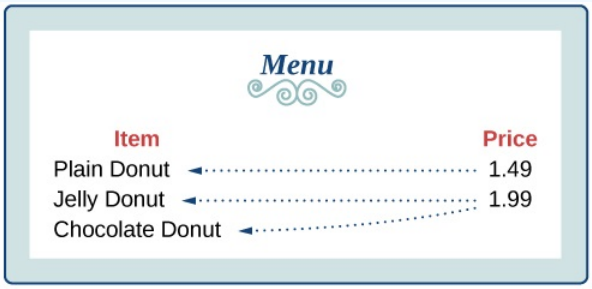


Figure 2.1.4: Association of the prices to the donuts.

Therefore, the item is a not a function of price.

Example 2.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 2.1.1 shows a possible rule for assigning grade points.

Table 2.1.1: Class grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Exercise 2.1.2

Table 2.1.2 lists the five greatest baseball players of all time in order of rank.

Table 2.1.2: Five greatest baseball players.

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Answer a

Yes

Answer b

yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters f, g,and h are often used to represent functions just as we use x, y,and z to represent numbers and A, B, and C to represent sets.

$$\begin{aligned} &h \text{ is } f \text{ of } a && \text{We name the function } f\text{; height is a function of age.} \\ &h = f(a) && \text{We use parentheses to indicate the function input.} \\ &f(a) && \text{We name the function } f \text{ ; the expression is read as "} f \text{ of } a\text{."} \end{aligned} \tag{2.1.1}$$

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means "first add a and b, and the result is the input for the function f." The operations must be performed in this order to obtain the correct result.

Function Notation

The notation $y=f(x)$ defines a function named f. This is read as "y is a function of x." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

Example 2.1.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Solution

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days}=f(\text{month})$ or $d=f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

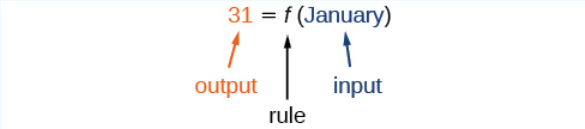


Figure 2.1.5: The function $31=f(January)$ where 31 is the output, f is the rule, and January is the input.

For example, $f(\text{March})=31$, because March has 31 days. The notation $d=f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 2.1.3B: Interpreting Function Notation

A function $N=f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005)=300$ represent?

Solution

When we read $f(2005)=300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N=f(y)$. The statement $f(2005)=300$ tells us that in the year 2005 there were 300 police officers in the town.

Exercise 2.1.3

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Answer

$w=f(d)$

Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

How To: Given the formula for a function, evaluate.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example 2.1.6A: Evaluating Functions at Specific Values

1. Evaluate $f(x) = x^2 + 3x - 4$ at
   a. 2
   b. a
   c. $a + h$
   d. Evaluate $\frac{f(a+h) - f(a)}{h}$

Solution

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h)-f(a)}{h} &= \frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h} \\ &= \frac{(2ah+h^2+3h)}{h} \\ &= \frac{h(2a+h+3)}{h} && \text{Factor out h.} \\ &= 2a+h+3 && \text{Simplify.} \end{aligned}$$

Example 2.1.6B: Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

Solution

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(p) &= p^2 + 2p \\ h(4) &= (4)^2 + 2(4) \\ &= 16 + 8 \\ &= 24 \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

Exercise 2.1.6

Given the function $g(m) = \sqrt{m-4}$, evaluate $g(5)$.

Answer

$g(5) = 1$

Example 2.1.7: Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

Solution

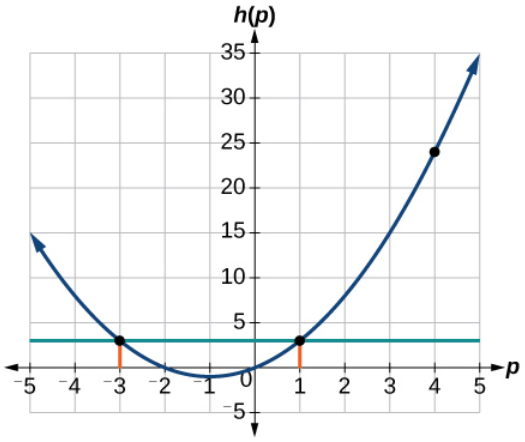
$$\begin{aligned} h(p) &= 3 \\ p^2 + 2p &= 3 && \text{Substitute the original function} \\ p^2 + 2p - 3 &= 0 && \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 && \text{Factor.} \end{aligned}$$

If $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3) = 0,\ p = -3$$

$$(p-1) = 0,\ p = 1$$

This gives us two solutions. The output $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing as in Figure 2.1.6. The graph verifies that $h(1) = h(-3) = 3$ and $h(4) = 24$.



p	−3	−2	0	1	4
h(p)	3	0	0	3	24

Figure 2.1.6: Graph of $h(p) = p^2 + 2p$

Exercise 2.1.7

Given the function $g(m) = \sqrt{m-4}$, solve $g(m) = 2$.

Answer

$m = 8$

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

How To: Given the formula for a function, evaluate.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example 2.1.6A: Evaluating Functions at Specific Values

1. Evaluate $f(x) = x^2 + 3x - 4$ at
   a. 2
   b. a
   c. $a + h$
   d. Evaluate $\frac{f(a+h) - f(a)}{h}$

Solution

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a+h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} \\ &= \frac{(2ah + h^2 + 3h)}{h} \\ &= \frac{h(2a + h + 3)}{h} && \text{Factor out h.} \\ &= 2a + h + 3 && \text{Simplify.} \end{aligned}$$

Example 2.1.6B: Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

Solution

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(p) &= p^2 + 2p \\ h(4) &= (4)^2 + 2(4) \\ &= 16 + 8 \\ &= 24 \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

Exercise 2.1.6

Given the function $g(m) = \sqrt{m-4}$, evaluate $g(5)$.

Answer

$g(5) = 1$

Example 2.1.7: Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

Solution

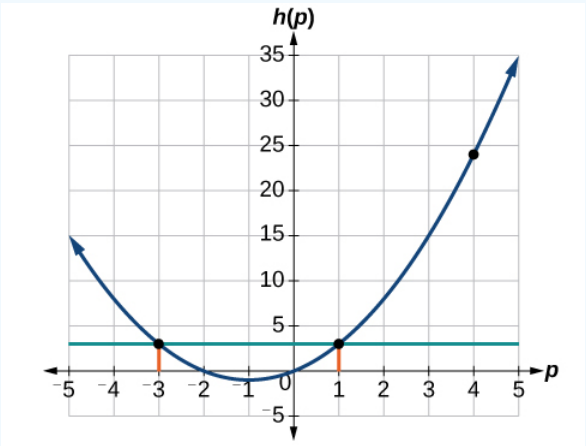
$$\begin{aligned} h(p) &= 3 \\ p^2 + 2p &= 3 && \text{Substitute the original function} \\ p^2 + 2p - 3 &= 0 && \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 && \text{Factor.} \end{aligned}$$

If $(p+3)(p-1)=0$, either $(p+3)=0$ or $(p-1)=0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3)=0,\ p=-3$$

$$(p-1)=0,\ p=1$$

This gives us two solutions. The output $h(p)=3$ when the input is either $p=1$ or $p=-3$. We can also verify by graphing as in Figure 2.1.6. The graph verifies that $h(1)=h(-3)=3$ and $h(4)=24$.



p	-3	-2	0	1	4
h(p)	3	0	0	3	24

Figure 2.1.6: Graph of $h(p)=p^2+2p$

Exercise 2.1.7

Given the function $g(m)=\sqrt{m-4}$, solve $g(m)=2$.

Answer

$m=8$

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in **equation** form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n+6p=12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

How to: Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves only the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example 2.1.8A: Finding an Equation of a Function

Express the relationship $2n+6p=12$ as a function $p=f(n)$, if possible.

Solution

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p=[\text{expression involving } n]$.

$$\begin{aligned} 2n+6p &= 12 \\ 6p &= 12-2n && \text{Subtract 2n from both sides.} \\ p &= \frac{12-2n}{6} && \text{Divide both sides by 6 and simplify.} \\ p &= \frac{12}{6}-\frac{2n}{6} \\ p &= 2-\frac{1}{3}n \end{aligned}$$

Therefore, p as a function of n is written as

$$p = f(n) = 2-\frac{1}{3}n$$

Analysis

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Example 2.1.8B: Expressing the Equation of a Circle as a Function

Does the equation $x^2+y^2=1$ represent a function with x as input and y as output? If so, express the relationship as a function $y=f(x)$.

Solution

First we subtract x^2 from both sides.

$$y^2 = 1-x^2$$

We now try to solve for y in this equation.

$$y = \pm\sqrt{1-x^2}$$

$$\text{so, } y = \sqrt{1-x^2} \text{ and } y = -\sqrt{1-x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y=f(x)$.

Exercise 2.1.8

If $x-8y^3=0$, express y as a function of x.

Answer

$$y = f(x) = \frac{\sqrt[3]{x}}{2}$$

Q & A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x=y+2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the

beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table (Table 2.1.10).

Table 2.1.10

Pet Memory	span in hours
Puppy	0.008
Adult Dog	0.083
Cat	3
Goldfish	2160
Beta Fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(goldfish) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

How To: Given a function represented by a table, identify specific output and input values

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Example 2.1.9: Evaluating and Solving a Tabular Function

Using Table 2.1.11,

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

Table 2.1.11

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

Solution

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. Table 2.1.12 shows two solutions: 2 and 4.

Table 2.1.12

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

Exercise 2.1.1

Using Table 2.1.12, evaluate $g(1)$.

Answer

$g(1) = 8$

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 2.1.10: Reading Function Values from a Graph

Given the graph in Figure 2.1.7,

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

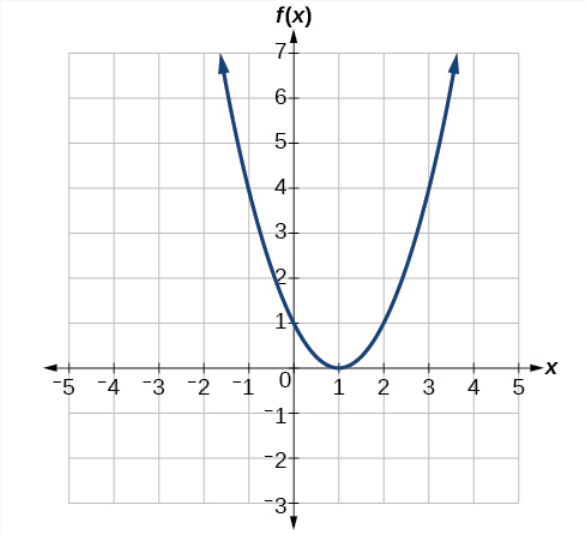
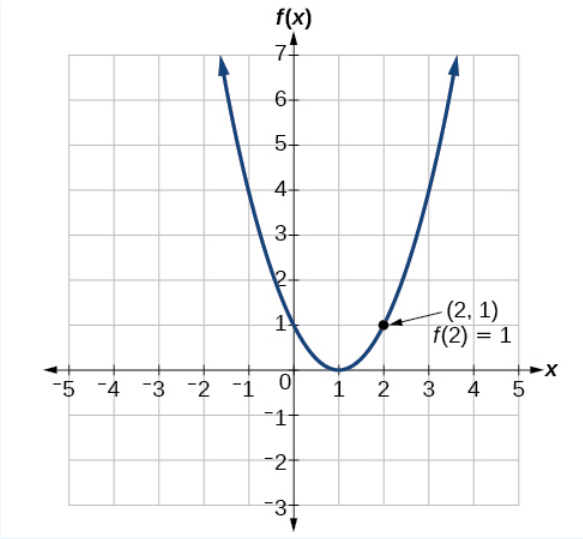


Figure 2.1.7: Graph of a positive parabola centered at $(1, 0)$.

Solution

To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure 2.1.8.



2.1.8: Graph of a positive parabola centered at $(1,0)$ with the labeled point $(2,1)$ where $f(2)=1$.

To solve $f(x)=4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y=4$, we locate two points of the curve with output value 4: $(-1,4)$ and $(3,4)$. These points represent the two solutions to $f(x)=4$: -1 or 3. This means $f(-1)=4$ and $f(3)=4$, or when the input is -1 or 3, the output is 4. See Figure 2.1.9.

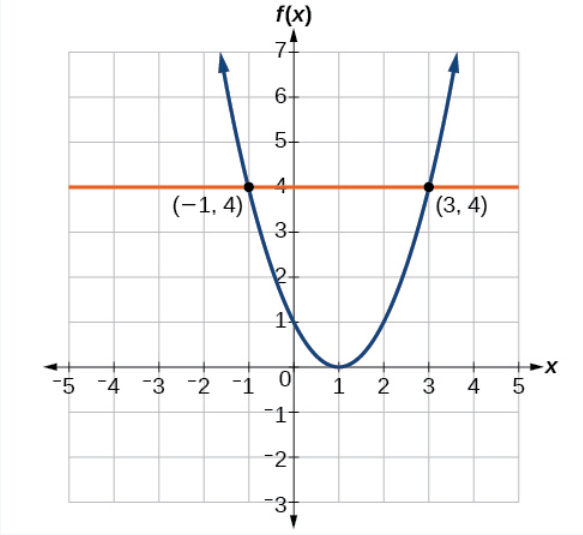


Figure 2.1.9: Graph of an upward-facing parabola with a vertex at $(0,1)$ and labeled points at $(-1,4)$ and $(3,4)$. A line at $y=4$ intersects the parabola at the labeled points.

Exercise 2.1.10

Given the graph in Figure 2.1.7, solve $f(x)=1$.

Answer

$x=0$ or $x=2$

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the Figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table 2.1.13

Table 2.1.13: Letter grades and decimal equivalents.

Letter Grade	Grade Point Average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figures 2.1.1a and 2.1.1b The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

One-to-One Functions

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

Example 2.1.11: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius$r$.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Exercise 2.1.11A

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Answer

a. yes, because each bank account has a single balance at any given time;

b. no, because several bank account numbers may have the same balance;

c. no, because the same output may correspond to more than one input.

Exercise 2.1.11B

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

Answer

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure 2.1.10 tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

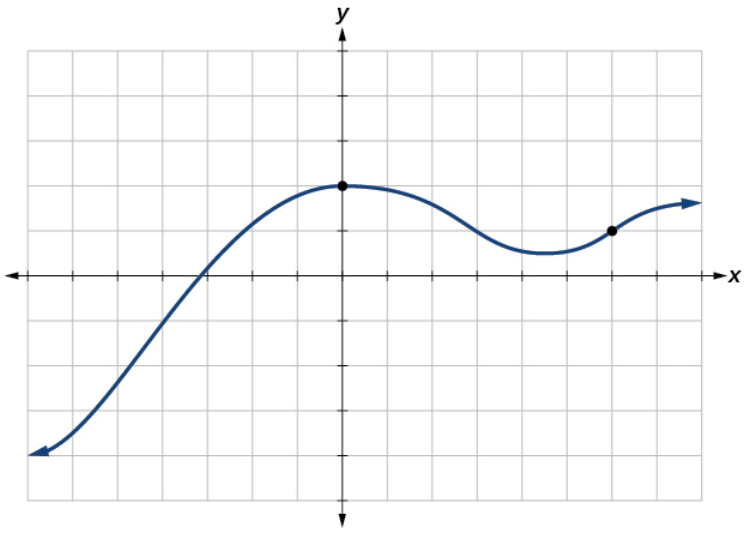


Figure 2.1.10: Graph of a polynomial.

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. See Figure 2.1.11.

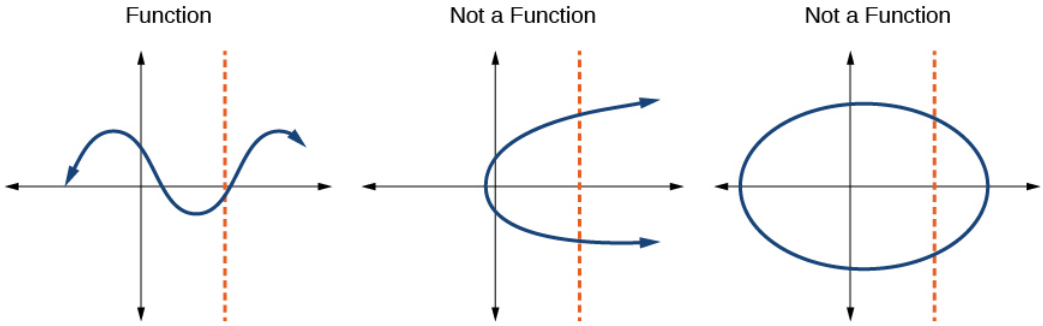


Figure 2.1.11: Three graphs visually showing what is and is not a function.

Howto: Given a graph, use the vertical line test to determine if the graph represents a function

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example 2.1.12: Applying the Vertical Line Test

Which of the graphs in Figure 2.1.12 represent(s) a function $y = f(x)$?

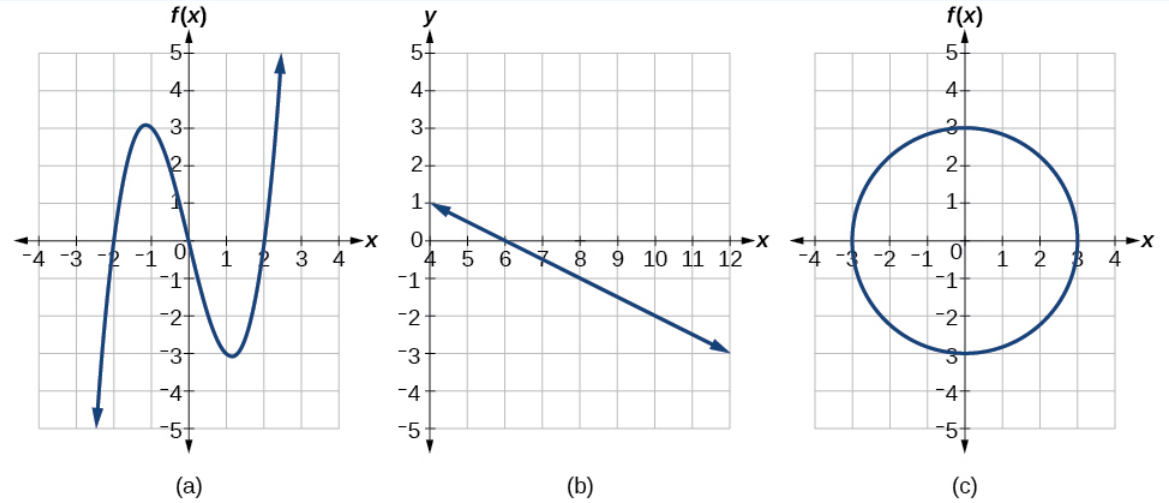


Figure 2.1.12: Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

Solution

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure 2.1.12. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in Figure 2.1.13.

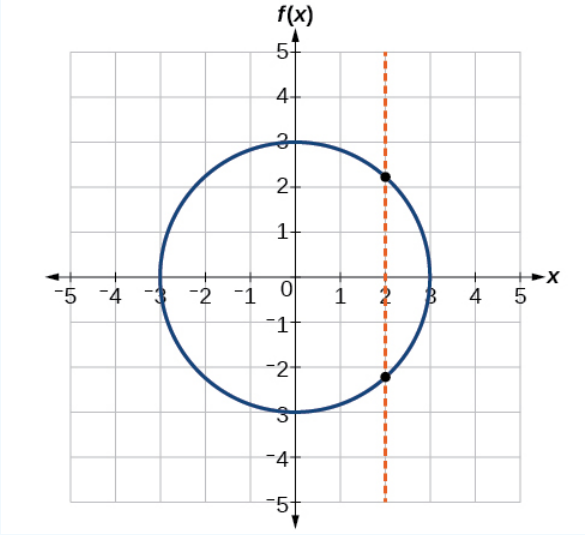


Figure 2.1.13: Graph of a circle.

Exercise 2.1.12

Does the graph in Figure 2.1.14 represent a function?

https://math.libretexts.org/@go/page/186131

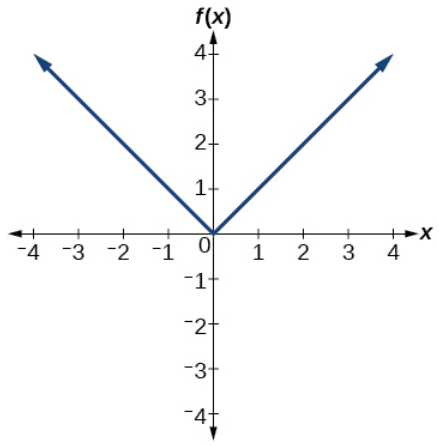


Figure 2.1.14: Graph of absolute value function.

Answer

yes

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Howto: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

Example 2.1.13: Applying the Horizontal Line Test

Consider the functions shown in Figure $2.1.12a$ and Figure $2.1.12b$ Are either of the functions one-to-one?

Solution

The function in Figure $2.1.12a$ is not one-to-one. The horizontal line shown in Figure $2.1.15$ intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

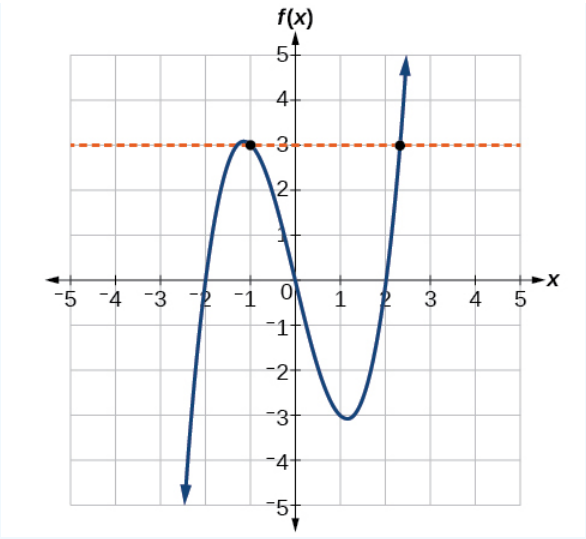


Figure 2.1.15: Graph of a polynomial with a horizontal line crossing through 2 points

The function in Figure $2.1.12b$is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Exercise 2.1.13

Is the graph shown in Figure $2.1.13$one-to-one?

Answer

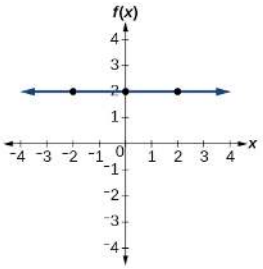
No, because it does not pass the horizontal line test.

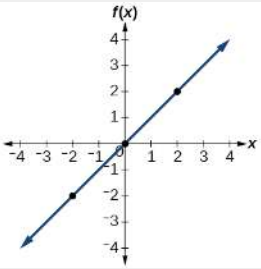
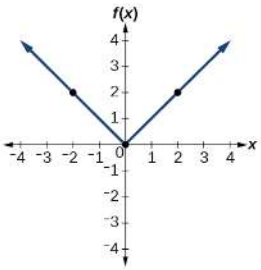
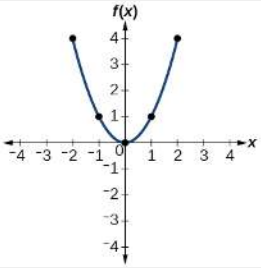
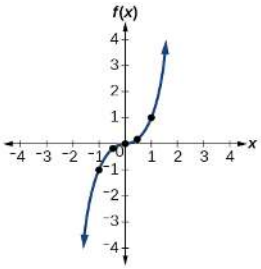
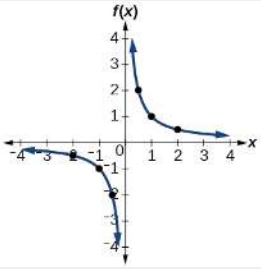
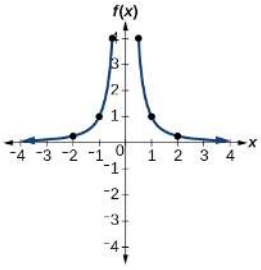
Identifying Basic Toolkit Functions

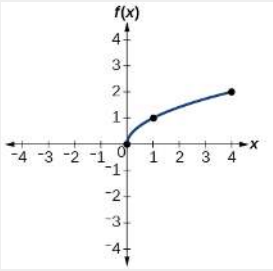
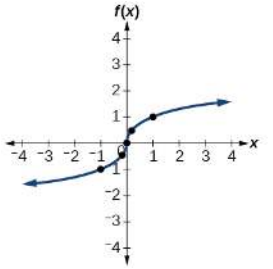
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table $2.1.14$

Table 2.1.14: Toolkit Functions

Name	Function	Graph
Constant	$f(x) = c$ where c is a constant	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>2</td></tr></table>

Name	Function	Graph
Identity	$f(x) = x$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table>
Absolute Value	$f(x) = \lvert x \rvert$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table>
Quadratic	$f(x) = x^2$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>4</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table>
Cubic	$f(x) = x^3$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.5</td><td>−0.125</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.5</td><td>0.125</td></tr><tr><td>1</td><td>1</td></tr></table>
reciprocal	$f(x) = \frac{1}{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−0.5</td></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.5</td><td>−2</td></tr><tr><td>0.5</td><td>2</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.5</td></tr></table>
Reciprocal squared	$f(x) = \frac{1}{x^2}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>0.25</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>−0.5</td><td>4</td></tr><tr><td>0.5</td><td>4</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.25</td></tr></table>

Name	Function	Graph
Square root	$f(x) = \sqrt{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>4</td><td>2</td></tr></table>
Cube root	$f(x) = \sqrt[3]{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.125</td><td>−0.5</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.125</td><td>0.5</td></tr><tr><td>1</td><td>1</td></tr></table>

Key Equations

- Constant function $f(x) = c$, where c is a constant
- Identity function $f(x) = x$
- Absolute value function $f(x) = |x|$
- Quadratic function $f(x) = x^2$
- Cubic function $f(x) = x^3$
- Reciprocal function $f(x) = \dfrac{1}{x}$
- Reciprocal squared function $f(x) = \frac{1}{x^2}$
- Square root function $f(x) = \sqrt{x}$
- Cube root function $f(x) = 3\sqrt{x}$

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output.
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$.
- In tabular form, a function can be represented by rows or columns that relate to input and output values.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value.
- To solve for a specific function value, we determine the input values that yield the specific output value.
- An algebraic form of a function can be written from an equation.
- Input and output values of a function can be identified from a table.
- Relating input values to output values on a graph is another way to evaluate a function.
- A function is one-to-one if each output value corresponds to only one input value.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point.
- The graph of a one-to-one function passes the horizontal line test.

Footnotes

1 http://www.baseball-almanac.com/lege.../lisn100.shtml. Accessed 3/24/2014.
2 www.kgbanswers.com/how-long-i...y-span/4221590. Accessed 3/24/2014.

Glossary

dependent variable
an output variable

domain
the set of all possible input values for a relation

function
a relation in which each input value yields a unique output value

horizontal line test
a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

independent variable
an input variable

input
each object or value in a domain that relates to another object or value by a relationship known as a function

one-to-one function
a function for which each value of the output is associated with a unique input value

output
each object or value in the range that is produced when an input value is entered into a function

range
the set of output values that result from the input values in a relation

relation
a set of ordered pairs

vertical line test
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

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2.2: Domain and Range

Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of **domain and range**. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products (Figure 2.2.2).

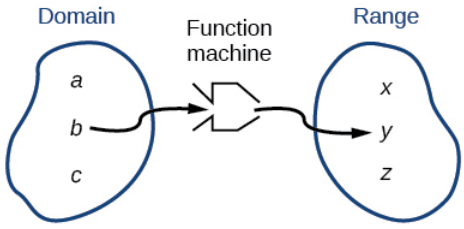


Figure 2.2.2: Diagram of how a function relates two relations

We can write the **domain and range** in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint is included, called inclusive.

See Figure 2.2.3 for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Figure 2.2.3: Summary of interval notation.

Example 2.2.1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2, 10), (3, 10), (4, 20), (5, 30), (6, 40)\}$

Solution

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2, 3, 4, 5, 6\}$$

Exercse 2.2.1

Find the domain of the function:

$$\{(-5, 4), (0, 0), (5, -4), (10, -8), (15, -12)\}$$

Answer

$\{-5, 0, 5, 10, 15\}$

How To: Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Example 2.2.2: Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$.

Solution

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty, \infty)$.

Exercse 2.2.2

Find the domain of the function:

$$f(x) = 5 - x + x^3$$

Answer

$(-\infty, \infty)$

Howto: Given a function written in an equation form that includes a fraction, find the domain

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Example 2.2.3: Finding the Domain of a Function Involving a Denominator

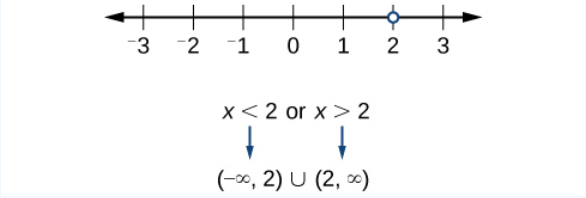
Find the domain of the function $f(x) = \dfrac{x+1}{2-x}$.

Solution

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2 - x &= 0 \\ -x &= -2 \\ x &= 2 \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty, 2) \cup (2, \infty)$.



In interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

Exercse 2.2.3

Find the domain of the function:

$$f(x) = \frac{1+4x}{2x-1}$$

Answer

$$(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$$

How To: Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example 2.2.4: Finding the Domain of a Function with an Even Root

Find the domain of the function:

$$f(x) = \sqrt{7-x}.$$

Solution

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7 - x &\geq 0 \\ -x &\geq -7 \\ x &\leq 7 \end{aligned}$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

Exercse 2.2.4

Find the domain of the function

$$f(x) = \sqrt{5+2x}.$$

Answer

$$[-2.5, \infty)$$

Q&A: Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function $f(x) = -\frac{1}{\sqrt{x}}$ *has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.*

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in set-builder notation. For example, $\{x|10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar $|$ is read as "such that," so we would read $\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure $2.2.4$ compares inequality notation, set-builder notation, and interval notation.

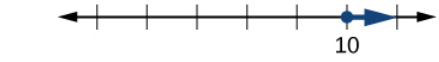
	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure 2.2.4: Summary of notations for inequalities, set-builder, and interval.

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$ It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x||x| \geq 3\} = (-\infty, -3] \cup [3, \infty) \tag{2.2.1}$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|$ statement about x$\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x|4 < x \leq 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

How To: Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol $\cup$ to combine all intervals into one set.

Example 2.2.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure 2.2.5 using inequality notation, set-builder notation, and interval notation.



Figure 2.2.5: Line graph of $1 \leq x \leq 3$ or $x > 5$.

Solution

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality

$$1 \leq x \leq 3 \text{ or } x > 5$$

Set-builder Notation

$$\{x | 1 \leq x \leq 3 \text{ or } x > 5\}$$

Interval notation

$$[1, 3] \cup (5, \infty)$$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Exercse 2.2.5

Given Figure 2.2.6, specify the graphed set in

a. words
b. set-builder notation
c. interval notation

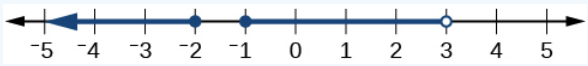


Figure 2.2.6: Line graph of $-2 \leq x$, $-1 \leq x < 3$.leq}x\), $-1 \leq x < 3$.

Answer a

Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;

Answer b

$\{x | x \leq -2 or -1 \leq x < 3\}$

Answer c

$(-\infty, -2] \cup [-1, 3)$

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure 2.2.7.

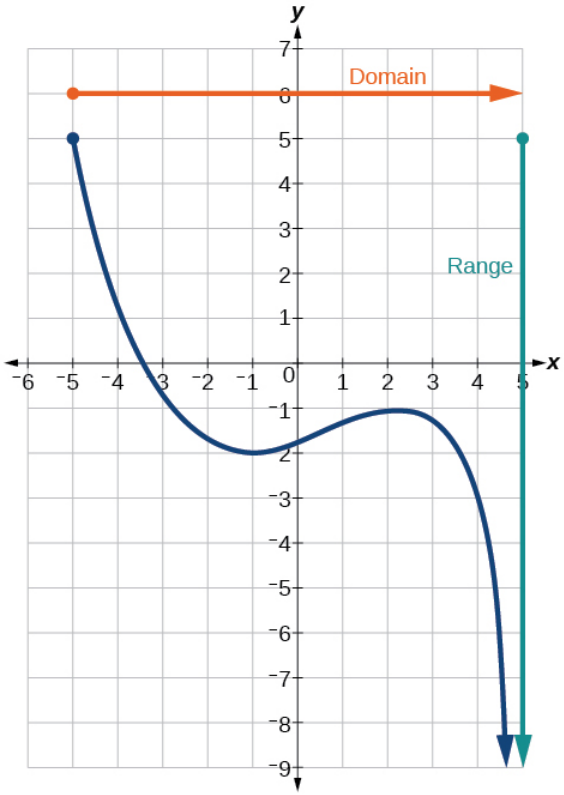


Figure 2.2.7: Graph of a polynomial that shows the x-axis is the domain and the y-axis is the range

We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example 2.2.6A: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 1.2.8.

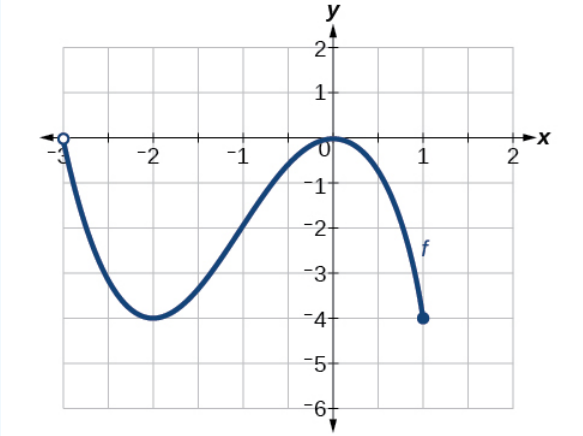


Figure 2.2.8: Graph of a function from (-3, 1].

Solution

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4, so the range is $[-4, 0)$. See Figure 2.2.9.

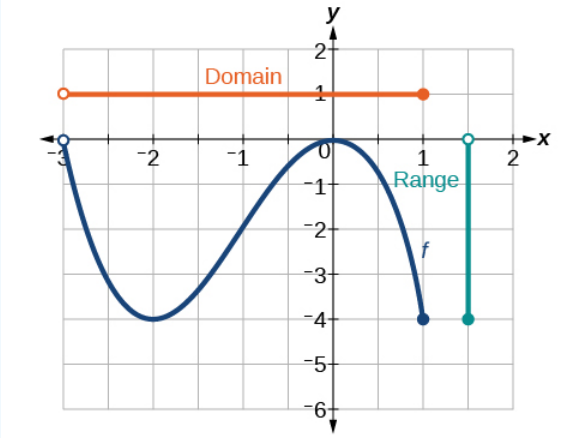


Figure 2.2.9: Graph of the previous function shows the domain and range

Example 2.2.6B: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in Figure 2.2.10.

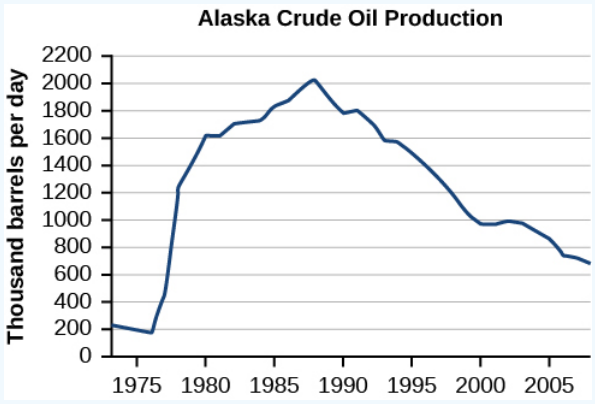


Figure 2.2.10: Graph of the Alaska Crude Oil Production where the y-axis is thousand barrels per day and the -axis is the years(credit: modification of work by the U.S. Energy Information Administration)

Solution

The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is $[1973, 2008]$ and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Exercse 2.2.6

Given Figure 2.2.11, identify the domain and range using interval notation.

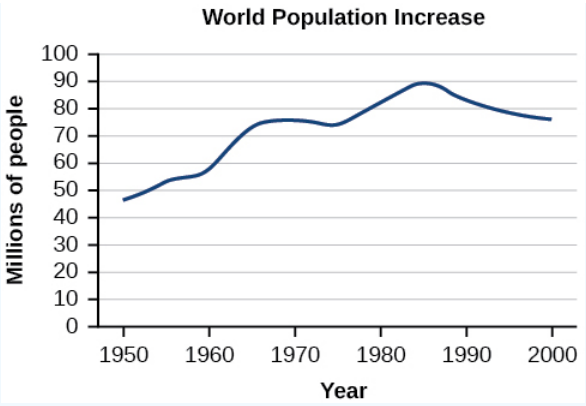


Figure 2.2.11: Graph of World Population Increase where the y-axis represents millions of people and the x-axis represents the year.

Answer

domain =$[1950, 2002]$

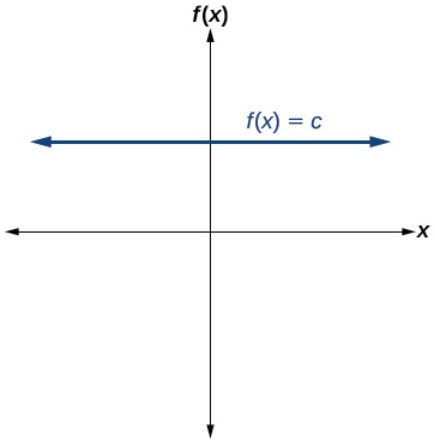
range = $[47,000,000, 89,000,000]$

Can a function's domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

Finding Domains and Ranges of the Toolkit Functions

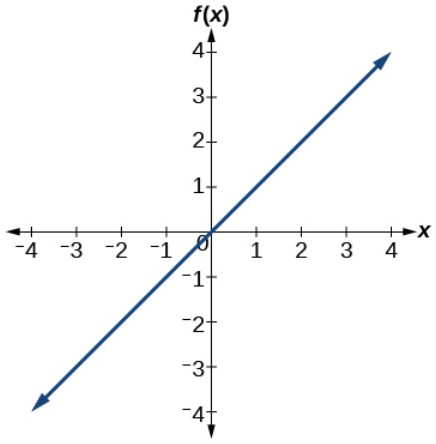
We will now return to our set of toolkit functions to determine the domain and range of each.



Domain: $(-\infty, \infty)$
Range: $[c,c]$

Figure 2.2.12: Constant function $f(x) = c$.

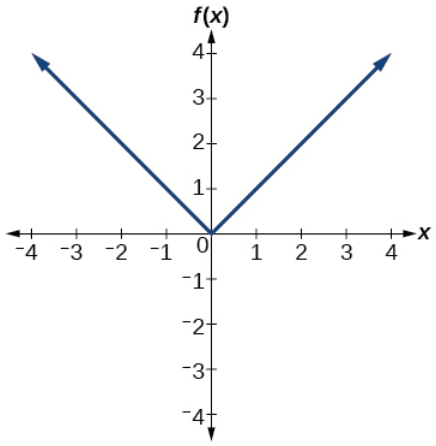
For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.2.13: *Identity function f(x)=x.*

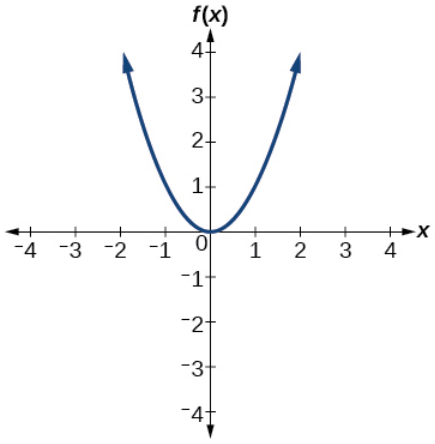
For the **identity function** $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 2.2.14: Absolute function $f(x) = |x|$.

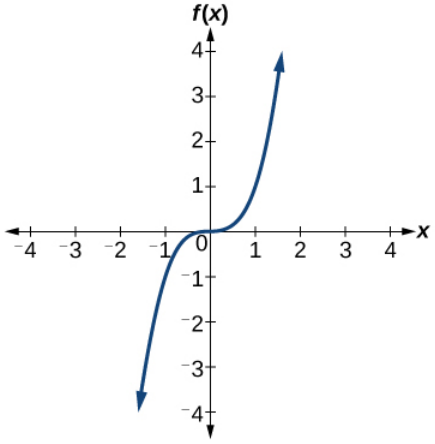
For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 2.2.15: Quadratic function $f(x) = x^2$.

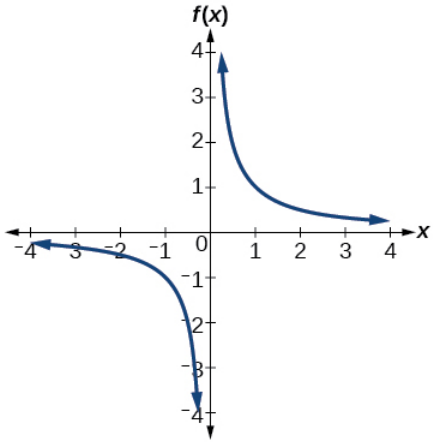
For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.2.16: Cubic function $f(x) - x^3$.

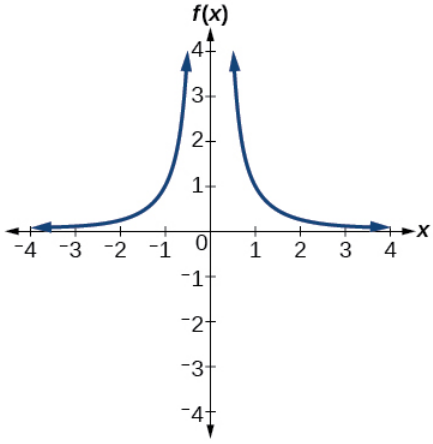
For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Figure 2.2.17: Reciprocal function $f(x) = \frac{1}{x}$.

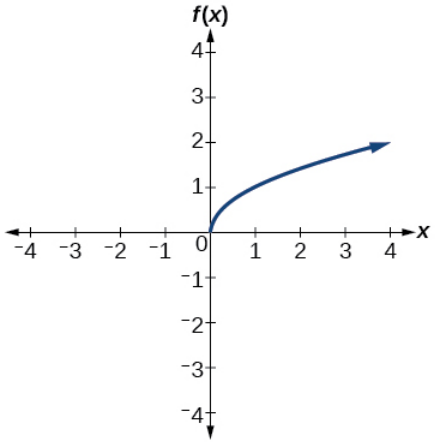
For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x | x \neq 0\}$,the set of all real numbers that are not zero.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

Figure 2.2.18: Reciprocal squared function $f(x) = \frac{1}{x^2}$

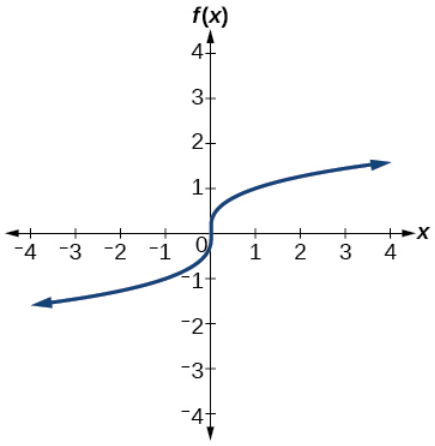
For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.



Domain: [0, ∞)
Range: [0, ∞)

Figure* 2.2.19*: *Square root function* $f(x)=\sqrt{(x)}$.

For the **square root function** $f(x)=\sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.2.20: Cube root function $f(x)=\sqrt[3]{x}$.

For the **cube root function** $f(x)=\sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x)=2x^3-x$.

Solution

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example 2.2.7B: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example 2.2.7C: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

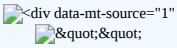
$x + 4 \geq 0$ when $x \geq -4$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

Analysis

Figure 2.2.19represents the function f.



" src="/@api/deki/files/892/CNX_Precalc_Figure_01_02_020.jpg" fileid="892">

Figure 2.2.19: Graph of a square root function at $(-4, 0)$.

Exercise 2.2.7

Find the domain and range of

$f(x) = \sqrt{-2 - x}$.

Answer

domain: $(-\infty, -2]$

range: $[0, \infty)$

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.
- An understanding of toolkit functions can be used to find the domain and range of related functions.
- A piecewise function is described by more than one formula.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain.

Footnotes

1 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014
2 www.eia.gov/dnav/pet/hist/Lea...s=MCRFPAK2&f=A.

Glossary

interval notation

a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

piecewise function

a function in which more than one formula is used to define the output

set-builder notation

a method of describing a set by a rule that all of its members obey; it takes the form {x| statement about x}

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2.3: Rates of Change and Behavior of Graphs

Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in Table 2.3.1 did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter Δ (delta) signifies the change in a quantity; we read the ratio as "delta-y over delta-x" or "the change in y divided by the change in x." Occasionally we write Δf instead of Δy, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by $1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7\text{years}} \approx 0.196 \text{ dollars per year.} \tag{2.3.1}$$

On average, the price of gas increased by about 19.6¢ each year. Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by $4,000 per quarter

Definition: Rate of Change

A **rate of change** describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{2.3.2}$$

How To...

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values x_1 and x_2.

1. Calculate the difference $y_2 - y_1 = \Delta y$.
2. Calculate the difference $x_2 - x_1 = \Delta x$.
3. Find the ratio $\frac{\Delta y}{\Delta x}$.

Example 2.3.1: Computing an Average Rate of Change

Using the data in Table 2.3.1, find the average rate of change of the price of gasoline between 2007 and 2009.

Solution

In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$2.41 - \$2.84}{2009 - 2007} \\ &= \frac{-\$0.43}{2 \text{ years}} \\ &= -\$0.22 \text{ per year}\end{aligned}$$

Analysis

Note that a decrease is expressed by a negative change or "negative increase." A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

Exercise 2.3.1

Using the data in Table 2.3.1, find the average rate of change between 2005 and 2010.

Solution

$$\frac{\$2.84 - \$2.315}{5 \text{ years}} = \frac{\$0.535}{5 \text{ years}} = \$0.106 \text{per year.}$$

Example 2.3.2: Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in Figure 2.3.1, find the average rate of change on the interval $[-1, 2]$.

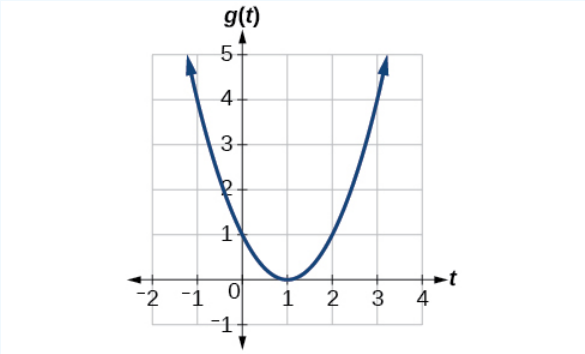


Figure 2.3.1: Graph of a parabola.

Solution

At $t = -1$, Figure 2.3.2 shows $g(-1) = 4$. At $t = 2$, the graph shows $g(2) = 1$.

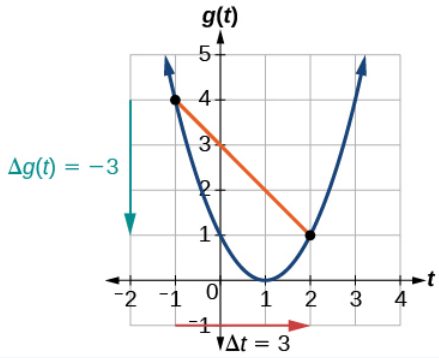


Figure 2.3.2: Graph of a parabola with a line from points (-1, 4) and (2, 1) to show the changes for g(t) and t.

The horizontal change $\Delta t = 3$ is shown by the red arrow, and the vertical change $\Delta g(t) = -3$ is shown by the turquoise arrow. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)} = \frac{-3}{3} = -1 \tag{2.3.3}$$

Analysis

Note that the order we choose is very important. If, for example, we use $\frac{y_2 - y_1}{x_1 - x_2}$, we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as (x_1, y_1) and (x_2, y_2).

Example 2.3.3: Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in Table 2.3.2. Find her average speed over the first 6 hours.

Table 2.3.2

t (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Solution

Here, the average speed is the average rate of change. She traveled 292 miles in 6 hours, for an average speed of

$$\begin{aligned} \frac{292-10}{6-0} &= \frac{282}{6} \\ &= 47 \end{aligned}$$

The average speed is about 47 miles per hour.

Analysis

Because the speed is not constant, the average speed depends on the interval chosen. For the interval $[2, 3]$, the average speed is 63 miles per hour.

Example 2.3.4: Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval $[2, 4]$.

Solution

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2 - \frac{1}{2}f(4) & &= 4^2 - \frac{1}{4} \\ &= 4 - \frac{1}{2} & &= 16 - \frac{1}{4} \\ &= 72 & &= \frac{63}{4} \end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned} \text{Average rate of change} &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{\frac{63}{4} - \frac{7}{2}}{4 - 2} \\ &= \frac{\frac{49}{4}}{2} \\ &= \frac{49}{8} \end{aligned}$$

Exercise 2.3.2

Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval $[1, 9]$.

Solution

$\frac{1}{2}$

Example 2.3.5: Finding the Average Rate of Change of a Force

The electrostatic force F, measured in newtons, between two charged particles can be related to the distance between the particles d,in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

Solution

We are computing the average rate of change of $F(d) = \dfrac{2}{d^2}$ on the interval $[2, 6]$.

$$\begin{aligned} \text{Average rate of change} &= \frac{F(6) - F(2)}{6 - 2} \\ &= \frac{\frac{2}{6^2} - \frac{2}{2^2}}{6 - 2} && \text{Simplify} \\ &= \frac{\frac{2}{36} - \frac{2}{4}}{4} \\ &= \frac{-\frac{16}{36}}{4} && \text{Combine numerator terms.} \\ &= -\frac{1}{9} && \text{Simplify} \end{aligned}$$

The average rate of change is $-\frac{1}{9}$ newton per centimeter.

Example 2.3.6: Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a.

Solution

We use the average rate of change formula.

$$\begin{aligned}
\text{Average rate of change} &= \frac{g(a) - g(0)}{a - 0} && \text{Evaluate.} \\
&= \frac{(a^2 + 3a + 1) - (0^2 + 3(0) + 1)}{a - 0} && \text{Simplify.} \\
&= \frac{a^2 + 3a + 1 - 1}{a} && \text{Simplify and factor.} \\
&= \frac{a(a+3)}{a} && \text{Divide by the common factor a.} \\
&= a + 3
\end{aligned}$$

This result tells us the average rate of change in terms of a between $t = 0$ and any other point $t = a$. For example, on the interval $[0, 5]$, the average rate of change would be $5 + 3 = 8$.

Exercise 2.3.3

Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$.

Solution

$a + 7$

Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. Figure 2.3.3 shows examples of increasing and decreasing intervals on a function.

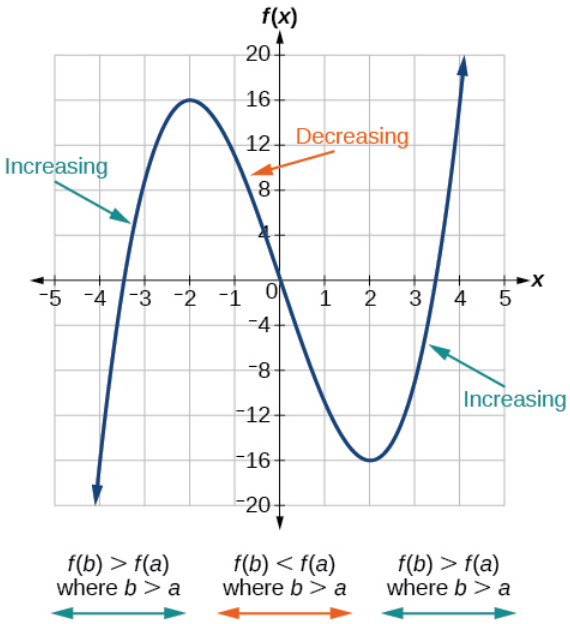


Figure 2.3.3: The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2, 2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is "local minima." Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is "extremum.") Often, the term local is replaced by the term relative. In this text, we will use the term local.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of local extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown in Figure 2.3.4, the local maximum is 16, and it occurs at $x = -2$. The local minimum is -16 and it occurs at $x = 2$.

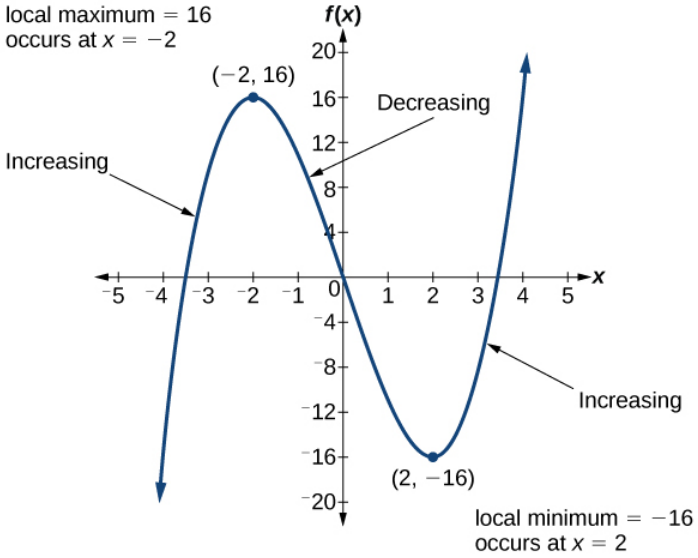


Figure 2.3.4: Graph of a polynomial that shows the increasing and decreasing intervals and local maximum.maximum

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. Figure 2.3.5 illustrates these ideas for a local maximum.

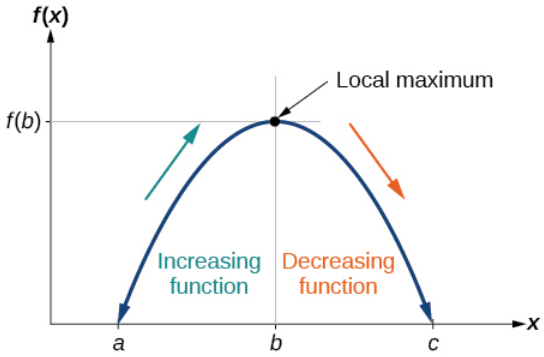


Figure 2.3.5: Definition of a local maximum

These observations lead us to a formal definition of local extrema.

Local Minima and Local Maxima

- A function f is an **increasing function** on an open interval if $f(b) > f(a)$ for every a, b interval where $b > a$.
- A function f is a **decreasing function** on an open interval if $f(b) < f(a)$ for every a, b interval where $b > a$.

A function f has a local maximum at a point b in an open interval (a, c) if $f(b)$ is greater than or equal to $f(x)$ for every point x (x does not equal b) in the interval. Likewise, f has a local minimum at a point b in (a, c) if $f(b)$ is less than or equal to $f(x)$ for every x (x does not equal b) in the interval.

Example 2.3.7 Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in Figure 2.3.6, identify the intervals on which the function appears to be increasing.

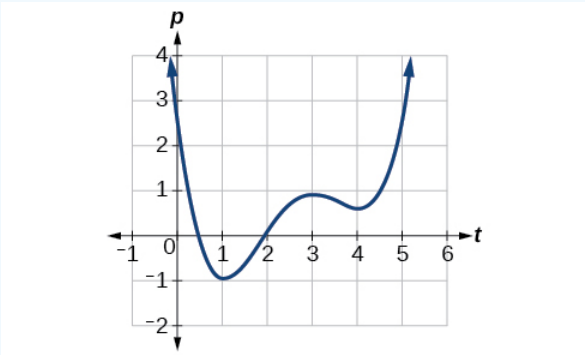


Figure 2.3.6: Graph of a polynomial.

Solution

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

Analysis

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

Example 2.3.8: Finding Local Extrema from a Graph

Graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

Solution

Using technology, we find that the graph of the function looks like that in Figure 2.3.7. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$

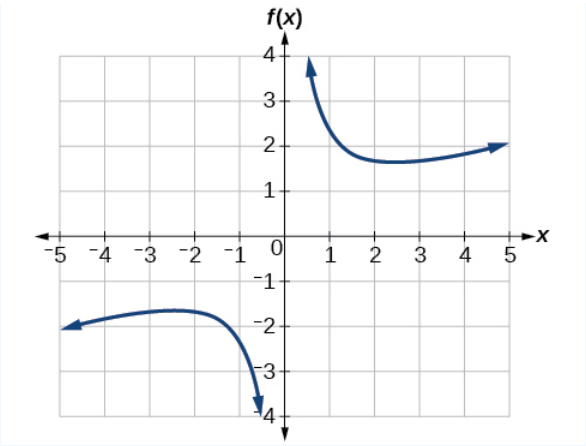


Figure 2.3.7: Graph of a reciprocal function.

Analysis

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Figure 2.3.8 provides screen images from two different technologies, showing the estimate for the local maximum and minimum.

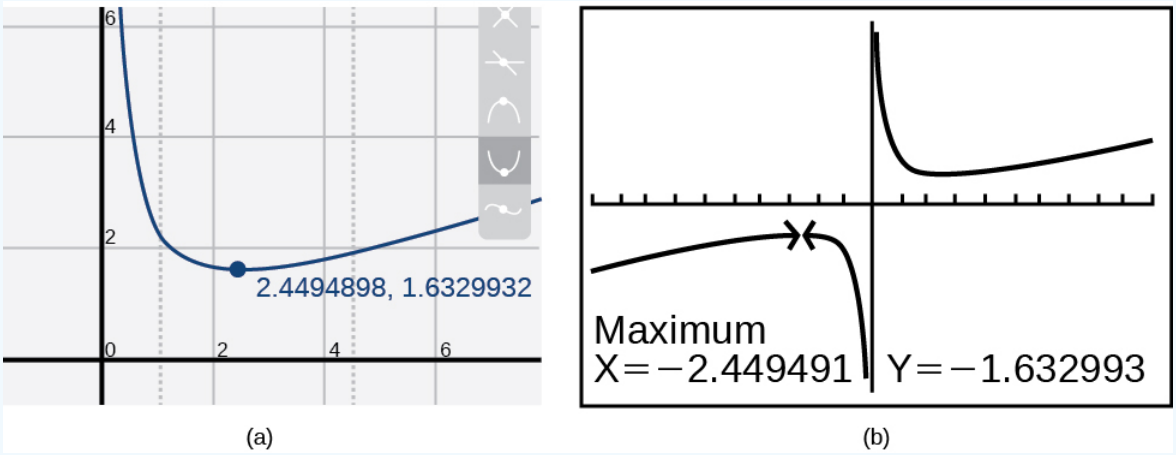


Figure 2.3.8: Graph of the reciprocal function on a graphing calculator.

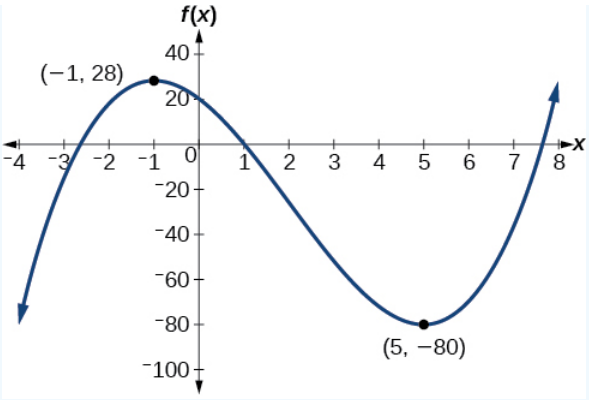
Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires calculus.)

Exercise 2.3.8

Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Solution

The local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.



Graph of a polynomial with a local maximum at (-1, 28) and local minimum at (5, -80).

Example 2.3.9: Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown in Figure 2.3.9, find all local maxima and minima.

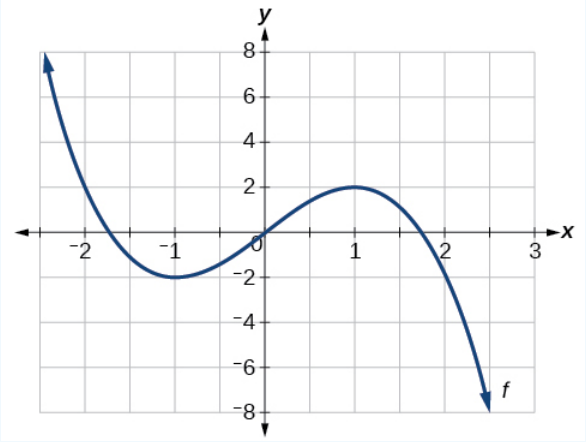


Figure 2.3.9: Graph of a polynomial.

Solution

Observe the graph of f. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$.The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is −2.

Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in Figure 2.3.10, Figure 2.3.11, and Figure 2.3.12

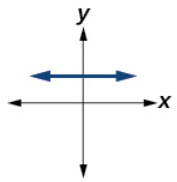
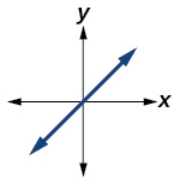
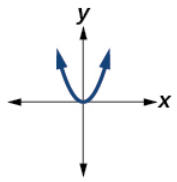
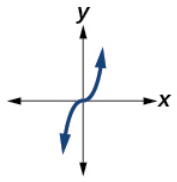
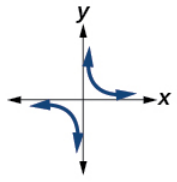
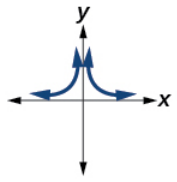
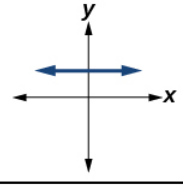
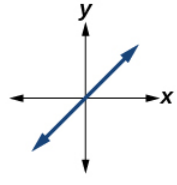
Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

Figure 2.3.10

Function	Increasing/Decreasing	Example
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	

.

Figure 2.3.11

Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	

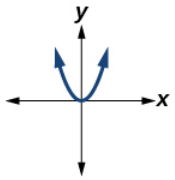
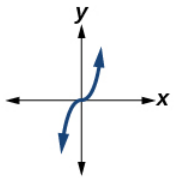
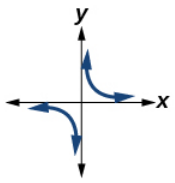
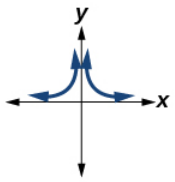
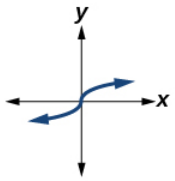
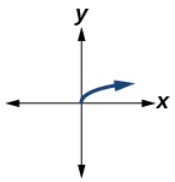
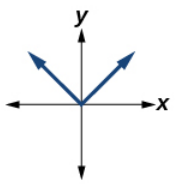
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \lvert x \rvert$	Increasing on $(0, \infty)$ Decreasing on $(\infty, 0)$	

***Figure* 2.3.12**

Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively. To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function (Figure 2.3.13).

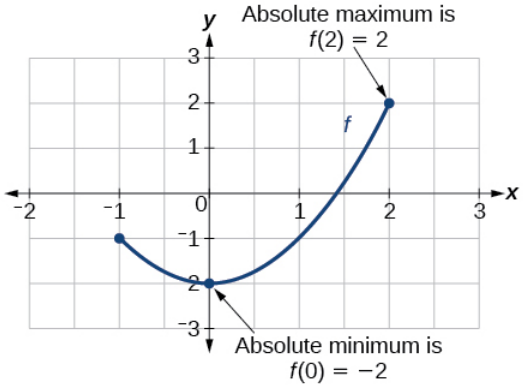


Figure 2.3.13: Graph of a segment of a parabola with an absolute minimum at (0, -2) and absolute maximum at (2, 2).

Not every function has an absolute maximum or minimum value. The toolkit function $f(x) = x^3$ is one such function.

Absolute Maxima and Minima

- The **absolute maximum** of f at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all x in the domain of f.
- The **absolute minimum** of f at $x = d$ is $f(d)$ where $f(d) \leq f(x)$ for all x in the domain of f.

Example 2.3.10: Finding Absolute Maxima and Minima from a Graph

For the function f shown in Figure 2.3.14, find all absolute maxima and minima.

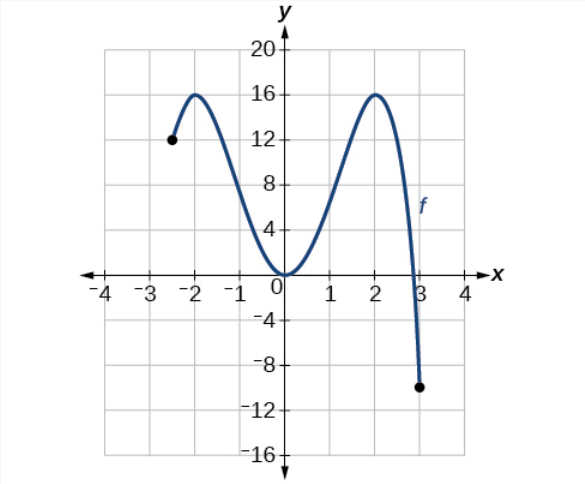


Figure 2.3.14: Graph of a polynomial.

Solution

Observe the graph of f. The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the y-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at x=3, because it is the lowest point on the domain of the function's graph. The absolute minimum is the y-coordinate at x=3,which is−10.

Key Equations

- Average rate of change: $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Key Concepts

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See Example.

- Identifying points that mark the interval on a graph can be used to find the average rate of change. See Example.
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See Example.
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See Example and Example.
- The average rate of change can sometimes be determined as an expression. See Example.
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See Example.
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.
- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See Example and Example.
- The highest and lowest points on a graph indicate the maxima and minima. See Example.

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2.4: Composition of Functions

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ by the relations.

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (f-g)(x) &= f(x)-g(x)\\ (fg)(x) &= f(x)g(x)\\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}\end{aligned}$$

Example 2.4.1: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x) = x-1$ and $g(x) = x^2-1$. Are they the same function?

Solution

Begin by writing the general form, and then substitute the given functions.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ (g-f)(x) &= x^2-1-(x-1)\\ &= x^2-x\\ &= x(x-1)\\ \left(\frac{g}{f}\right)(x) &= g(x)f(x)\\ \left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1}\\ &= \frac{(x+1)(x-1)}{x-1}\\ &= x+1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ is necessary because when $x = 1$, the denominator is equal to 0, which makes the function undefined.

? Exercise 2.4.1

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.

$$f(x) = x - 1$$

and

$$g(x) = x^2 - 1$$

Are they the same function?

Answer

$(fg)(x) = f(x)g(x) = (x-1)(x2-1) = x^3 - x^2 - x + 1$

$(f-g)(x) = f(x) - g(x) = (x-1) - (x^2-1) = x - x^2$

No, the functions are not the same.

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of **combining functions** so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$f \circ g(x) = f(g(x)) \tag{2.4.1}$$

We read the left-hand side as"f composed with g at x," and the right-hand side as"f of g of x."The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

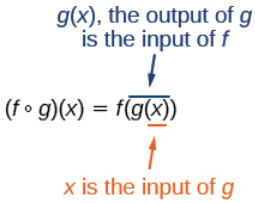


Figure 2.4.2: Explanation of the composite function.

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2+4x+4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2+2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x=-\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)) \tag{2.4.2}$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Example 2.4.2: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is **commutative**.

$$f(x) = 2x+1 \quad g(x) = 3-x$$

Solution

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= 2(3-x)+1 \\ &= 6-2x+1 \\ &= 7-2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= 3-(2x+1) \\ &= 3-2x-1 \\ &= 2-2x \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

Example 2.4.3: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t=3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example 2.4.4: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The function $y = f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles } = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons } = g(\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Question/Answer

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order

Exercise 2.4.2

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Answer

A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G(a(F))$ does not make sense.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 2.4.5: Using a Table to Evaluate a Composite Function

Using Table 2.4.1, evaluate $f(g(3))$ and $g(f(3))$.

Table 2.4.1

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

Solution

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g : $g(3) = 2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2) = 8$.

$$g(3) = 2$$

$$f(g(3)) = f(2) = 8$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for g, we can evaluate

$$g(f(3)) = g(3) = 2$$

Table 2.4.2 shows the composite functions $f \circ g$ and $g \circ f$ as tables.

Table 2.4.2

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$
3	2	8	3	2

Exercise 2.4.3

Using Table 2.4.1, evaluate $f(g(1))$ and $g(f(4))$.

Answer

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

How To ...

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

Example 2.4.6: Using a Graph to Evaluate a Composite Function

Using Figure 2.4.3, evaluate $f(g(1))$.

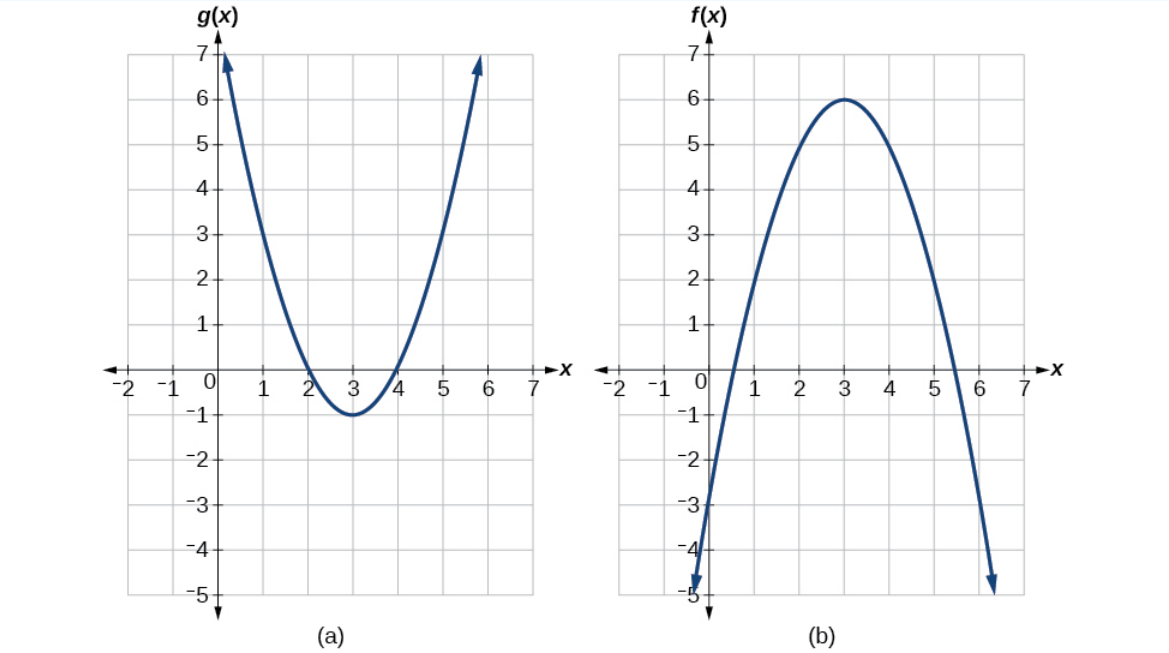


Figure 2.4.3: Two graphs of a positive and negative parabola.

Solution

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure 2.4.4.

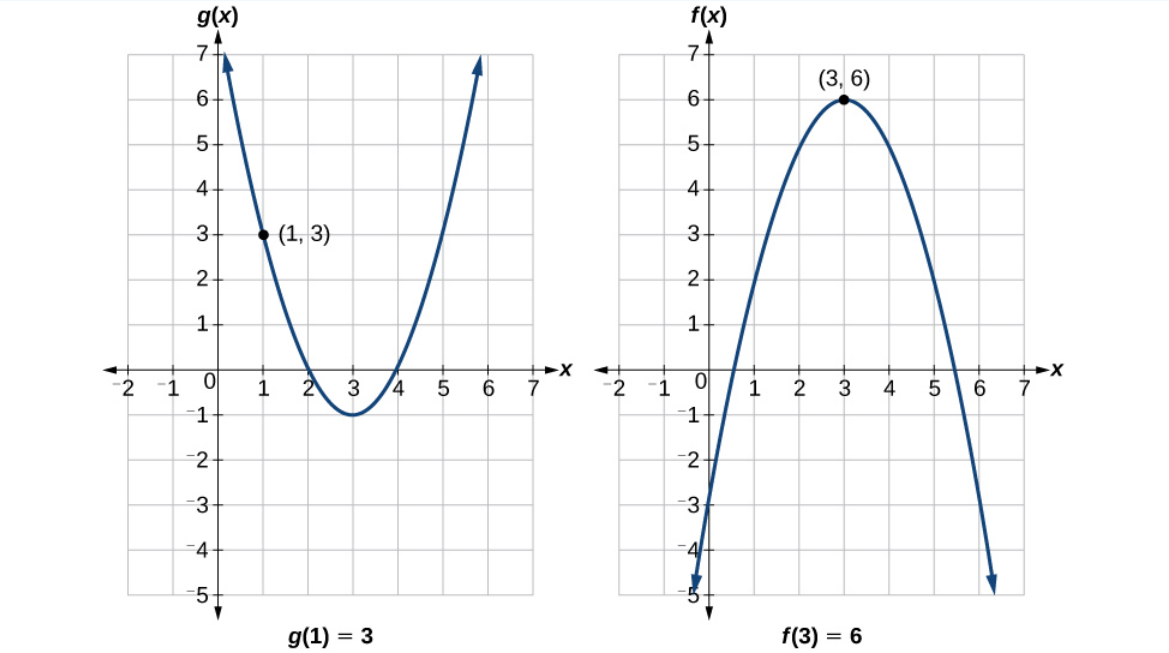


Figure 2.4.4: Two graphs of a positive parabola $g(x)$ and a negative parabola $f(x)$. The following points are plotted: $g(1) = 3$ and $f(3) = 6$.

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Analysis

Figure 2.4.5 shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

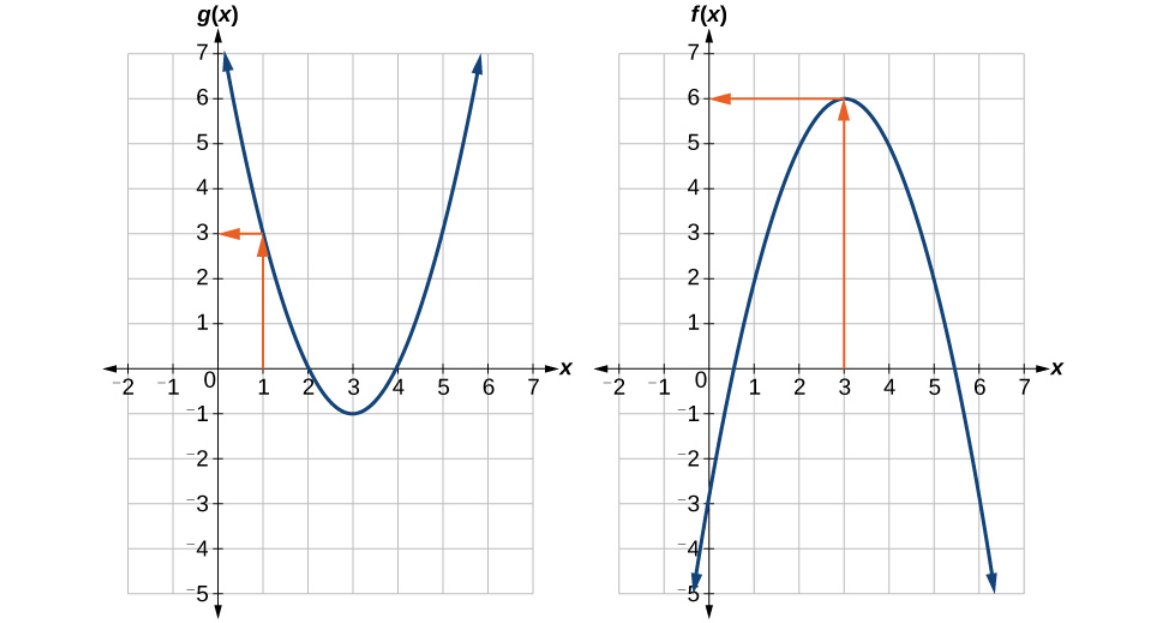


Figure 2.4.5: Two graphs of a positive and negative parabola.

? Exercise 2.4.4

Using Figure 2.4.3, evaluate $g(f(2))$.

Answer

$g(f(2)) = g(5) = 3$

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

How To...

Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

Example 2.4.7: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t)=t^2-t$ and $h(x)=3x+2$, evaluate $f(h(1))$.

Solution

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$h(1)=3(1)+2$$
$$h(1)=5$$

Then $f(h(1))=f(5)$, so we evaluate $f(t)$ at an input of 5.

$$f(h(1))=f(5)$$
$$f(h(1))=5^2-5$$
$$f(h(1))=20$$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Exercise 2.4.5

Given $f(t)=t^2-t$ and $h(x)=3x+2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Answer a

8

Answer b

20

Key Equation

- Composite function $(f\circ g)(x)=f(g(x))$

Key Concepts

- We can perform algebraic operations on functions. See Example.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See Example and Example.
- The order of function composition must be considered when interpreting the meaning of composite functions. See Example.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See Example.
- A composite function can be evaluated from a graph. See Example.
- A composite function can be evaluated from a formula. See Example.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See Example and Example.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See Example.

Glossary

composite function

the new function formed by function composition, when the output of one function is used as the input of another

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2.5: Inverse Functions

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the **Celsius** scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees **Fahrenheit** to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32) \tag{2.5.1}$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24^\circ \tag{2.5.2}$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure 2.5.2 for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

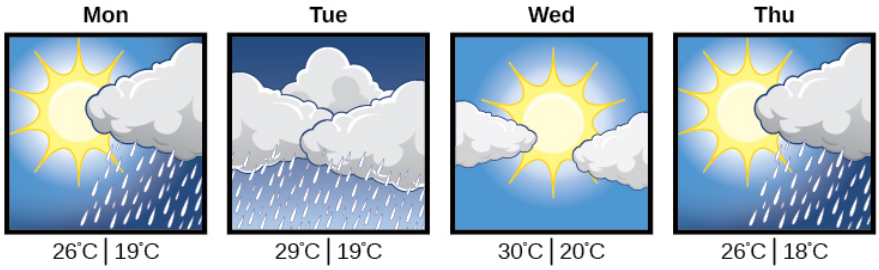


Figure 2.5.2: A forecast of Monday's through Thursday's weather.

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$26 = \frac{5}{9}(F - 32) \tag{2.5.3}$$

$$26 \cdot \frac{9}{5} = F - 32 \tag{2.5.4}$$

$$F = 26 \cdot \frac{9}{5} + 32 \approx 79 \tag{2.5.5}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "f inverse of x." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1 . In other words, $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1} \circ f$ equals the identity function, that is,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x \tag{2.5.6}$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a **reciprocal**, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically

equivalent (that is, if one is true, then so is the other.)

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x \tag{2.5.7}$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x \tag{2.5.8}$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Definition: Inverse Function

For any **one-to-one function** $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)} \tag{2.5.9}$$

and not all functions have inverses.

Example 2.5.1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution

The inverse function reverses the input and output quantities, so if

$$\begin{aligned} f(2) = 4, \text{ then } f^{-1}(4) = 2; \\ f(5) = 12, \text{ then } f^{-1}(12) = 5 \end{aligned} \tag{2.5.10}$$

.

Alternatively, if we want to name the inverse function g, then $g(4) = 2$ and $g(12) = 5$.

Analysis

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See Table 2.5.1.

Table 2.5.1

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Exercise 2.5.1

Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

Answer

$h(2) = 6$

How To: Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

1. Determine whether $f(g(x))=x$ or $g(f(x))=x$.
2. If both statements are true, then $g=f^{-1}$ and $f=g^{-1}$. If either statement is false, then both are false, and $g\neq f^{-1}$ and $f\neq g^{-1}$.

Example 2.5.2: Testing Inverse Relationships Algebraically

If $f(x)=\frac{1}{x+2}$ and $g(x)=\frac{1}{x}-2$, is $g=f^{-1}$?

Solution

$$g(f(x))=\frac{1}{(\frac{1}{x+2})-2} \tag{2.5.11}$$
$$=x+2-2 \tag{2.5.12}$$
$$=x \tag{2.5.13}$$

so

$$g=f^{-1} \text{ and } f=g^{-1} \tag{2.5.14}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$f(g(x))=\frac{1}{\frac{1}{x}-2+2} \tag{2.5.15}$$
$$=\frac{1}{\frac{1}{x}} \tag{2.5.16}$$
$$=x \tag{2.5.17}$$

Analysis

Notice the inverse operations are in reverse order of the operations from the original function.

Exercise 2.5.2

If $f(x)=x^3-4$ and $g(x)=\sqrt[3]{x+4}$, is $g=f^{-1}$?

Answer

Yes

Example 2.5.3: Determining Inverse Relationships for Power Functions

If $f(x)=x^3$ (the cube function) and $g(x)=\frac{1}{3}x$, is $g=f^{-1}$?

Solution

$$f(g(x))=\frac{x^3}{27}\neq x \tag{2.5.18}$$

No, the functions are not inverses.

Analysis

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x}=x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

Exercise 2.5.3

If $f(x)=(x-1)^3$ and $g(x)=\sqrt[3]{x}+1$, is $g=f^{-1}$?

Answer

Yes

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure 2.5.3.

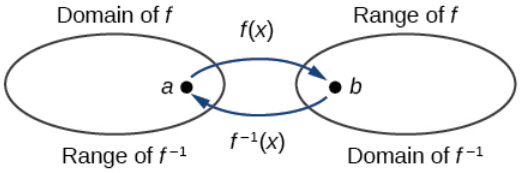


Figure 2.5.3: *Domain and range of a function and its inverse.*

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x)=\sqrt{x}$ is $f^{-1}(x)=x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0,\infty)$, since that is the range of $f(x)=\sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x)=x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x)=x^2$ with its range limited to $[0,\infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x)=(x-1)^2$ on $[1,\infty)$, then the inverse function is $f^{-1}(x)=\sqrt{x}+1$.

- The domain of f = range of $f^{-1}=[1,\infty)$.
- The domain of f^{-1} = range of $f=[0,\infty)$.

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g=h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Note: Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

How To: Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Example 2.5.4: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 2.5.2. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Table 2.5.2

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	Cube Root	Square Root	Absolute Value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Solution

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0, \infty)$,where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

Analysis

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 2.5.4. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.

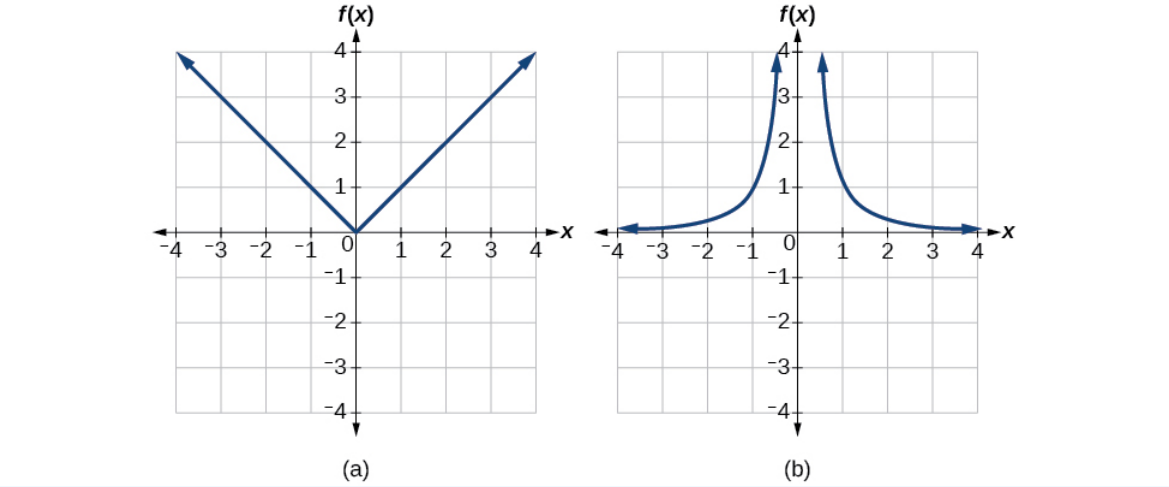


Figure 2.5.4: *(a) Absolute value (b) Reciprocal squared*

2.5.4: The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Solution

The domain of function f^{-1} is $(-\infty, -2)$ and the range of function f^{-1} is $(1, \infty)$.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example 2.5.5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table 2.5.3, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$

Table 2.5.3

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Exercise 2.5.5

Using Table 2.5.4, find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

Table 2.5.4

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Answer

$f(60) = 50$. In 60 minutes, 50 miles are traveled.
$f^{-1}(60) = 70$. To travel 60 miles, it will take 70 minutes.

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in **Functions and Function Notation** that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the y-axis of the given graph.
2. Read the inverse function's output from the x-axis of the given graph.

Example 2.5.6: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 2.5.5. Find $g(3)$ and $g^{-1}(3)$.

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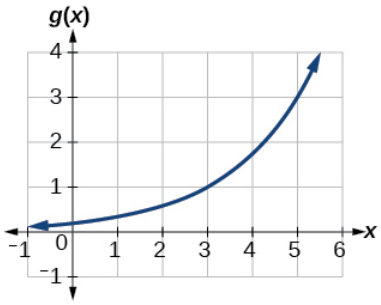


Figure 2.5.5: Graph of $g(x)$

Solution

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 2.5.6.

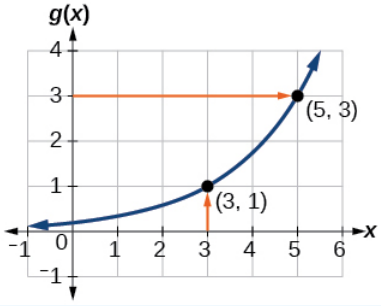


Figure 2.5.6: Graph of $g(x)$.

Exercise 2.5.6

Using the graph in Figure 2.5.6, (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

Answer a

3

Answer b

5.6

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula— for example, y as a function of x— we can often find the inverse function by solving to obtain x as a function of y.

How To: Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x
3. Interchange x and y.

Example 2.5.7: Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C = \frac{5}{9}(F - 32) \tag{2.5.19}$$

Solution

$$C = \frac{5}{9}(F-32) \tag{2.5.20}$$

$$C\cdot\frac{9}{5} = F-32 \tag{2.5.21}$$

$$F = \frac{9}{5}C+32 \tag{2.5.22}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F-32) \tag{2.5.23}$$

,

then

$$F = h^{-1}(C) = \frac{9}{5}C+32. \tag{2.5.24}$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Exercise 2.5.7

Solve for x in terms of y given $y = \frac{1}{3}(x-5)$

Answer

$x = 3y+5$

Example 2.5.8: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3}+4$.

Solution

$$y = \frac{2}{x-3+4} \qquad \text{Set up an equation.} \tag{2.5.25}$$

$$y-4 = \frac{2}{x-3} \qquad \text{Subtract 4 from both sides.} \tag{2.5.26}$$

$$x-3 = \frac{2}{y-4} \qquad \text{Multiply both sides by x−3 and divide by y−4.} \tag{2.5.27}$$

$$x = \frac{2}{y-4}+3 \qquad \text{Add 3 to both sides.} \tag{2.5.28}$$

So $f^{-1}(y) = \frac{2}{y-4}+3$ or $f^{-1}(x) = \frac{2}{x-4}+3$.

Analysis

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table 2.5.5.

Table 2.5.5

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example 2.5.9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x-4}$.

Solution

$$y = 2 + \sqrt{x-4} \quad (2.5.29)$$
$$(y-2)^2 = x - 4 \quad (2.5.30)$$
$$x = (y-2)^2 + 4 \quad (2.5.31)$$

So $f^{-1}(x) = (x-2)^2 + 4$.

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

Exercise 2.5.8

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Answer

$f^{-1}(x) = (2-x)^2$; domain of f: $[0, \infty)$; domain of f^{-1}: $(-\infty, 2]$

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

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t (minutes)	30	50	70	90
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Using Table 2.5.4, find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

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t (minutes)	30	50	60	70	90
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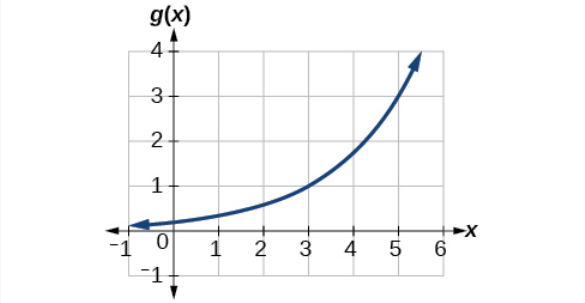


Figure 2.5.5: Graph of $g(x)$

Solution

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 2.5.6.

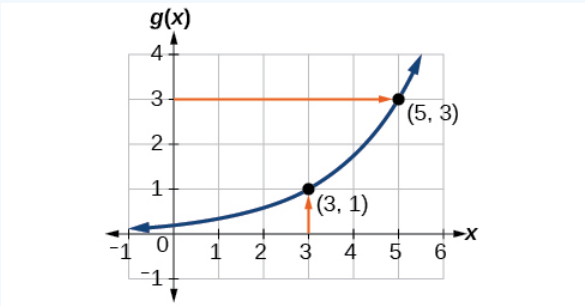


Figure 2.5.6: Graph of $g(x)$.

Exercise 2.5.6

Using the graph in Figure 2.5.6, (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

Answer a

3

Answer b

5.6

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$$C = \frac{5}{9}(F - 32) \tag{2.5.32}$$

Solution

$$C = \frac{5}{9}(F - 32) \tag{2.5.33}$$

$$C \cdot \frac{9}{5} = F - 32 \tag{2.5.34}$$

$$F = \frac{9}{5}C + 32 \tag{2.5.35}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32) \tag{2.5.36}$$

,

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32. \tag{2.5.37}$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Exercise 2.5.7

Solve for x in terms of y given $y = \frac{1}{3}(x - 5)$

Answer

$x = 3y + 5$

Example 2.5.8: Solving to Find an Inverse Function

Find the inverse of the function $f(x)=\frac{2}{x-3}+4$.

Solution

$$y=\frac{2}{x-3}+4 \quad \text{Set up an equation.} \quad (2.5.38)$$

$$y-4=\frac{2}{x-3} \quad \text{Subtract 4 from both sides.} \quad (2.5.39)$$

$$x-3=\frac{2}{y-4} \quad \text{Multiply both sides by x−3 and divide by y−4.} \quad (2.5.40)$$

$$x=\frac{2}{y-4}+3 \quad \text{Add 3 to both sides.} \quad (2.5.41)$$

So $f^{-1}(y)=\frac{2}{y-4}+3$ or $f^{-1}(x)=\frac{2}{x-4}+3$.

Analysis

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table 2.5.5.

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$f(x)$	3	2	5	y

Example 2.5.9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x)=2+\sqrt{x-4}$.

Solution

$$y=2+\sqrt{x-4} \quad (2.5.42)$$

$$(y-2)^2=x-4 \quad (2.5.43)$$

$$x=(y-2)^2+4 \quad (2.5.44)$$

So $f^{-1}(x)=(x-2)^2+4$.

The domain of f is $[4,\infty)$. Notice that the range of f is $[2,\infty)$, so this means that the domain of the inverse function f^{-1} is also $[2,\infty)$

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2,\infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

Exercise 2.5.8

What is the inverse of the function $f(x)=2-\sqrt{x}$? State the domains of both the function and the inverse function.

Answer

$f^{-1}(x)=(2-x)^2$; domain of f: $[0,\infty)$; domain of f^{-1}: $(-\infty,2]$

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x)=x^2$ restricted to the domain $[0,\infty)$, on which this function is one-to-one, and graph it as in Figure 2.5.7.

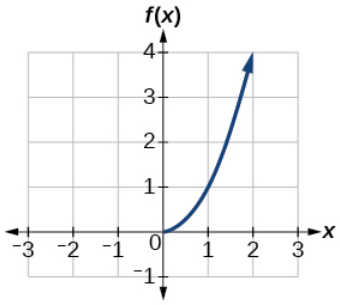


Figure 2.5.7: *Quadratic function with domain restricted to* $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure 2.5.8.

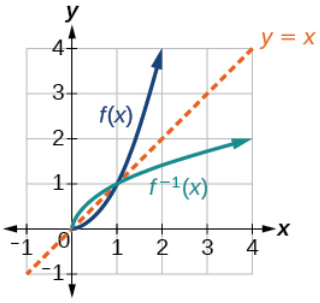


Figure 2.5.8: *Square and square-root functions on the non-negative domain*

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Example 2.5.10: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure 2.5.9, sketch a graph of $f^{-1}(x)$.

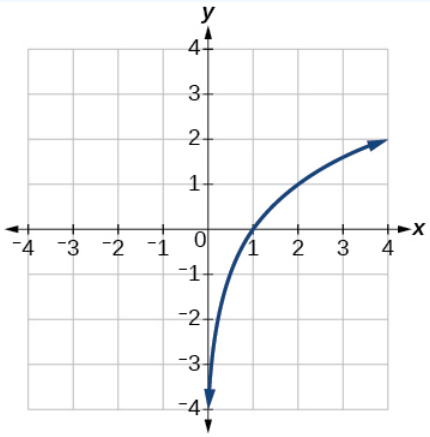


Figure 2.5.9: Graph of $f^(-1)(x)$.

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 2.5.10.

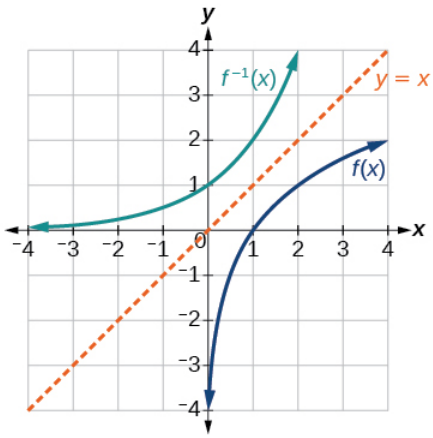


Figure 2.5.10: The function and its inverse, showing reflection about the identity line

Exercise 2.5.1

Draw graphs of the functions f and f^{-1} from Example 2.5.8.

Answer

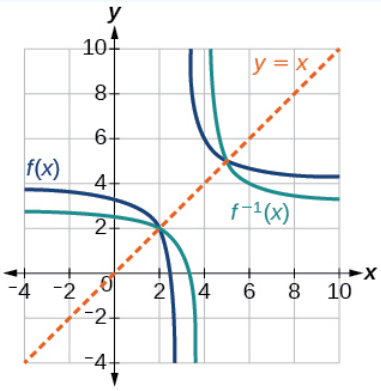


Figure 2.5.11: Graph of $f(x)$ and $f^(-1)(x)$.

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x \tag{2.5.45}$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Key Concepts

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x)) = f(g(x)) = x$.
- Each of the toolkit functions has an inverse.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse.
- The inverse of a function can be determined at specific points on its graph.
- To find the inverse of a formula, solve the equation $y = f(x)$ for x as a function of y. Then exchange the labels x and y.
- The graph of an inverse function is the reflection of the graph of the original function across the line $y = x$.

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2.6: Transformation of Functions

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See Figure 2.6.2 for an example.

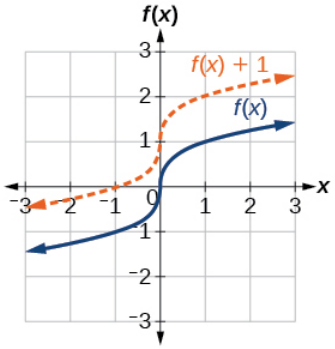


Figure 2.6.2: Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Definition: Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Example 2.6.1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 2.6.3 shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

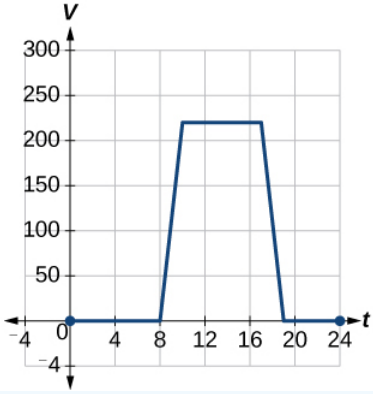


Figure 2.6.3

Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 2.6.4.

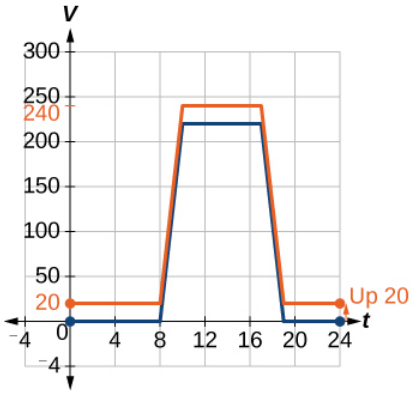


Figure 2.6.4

Notice that in Figure 2.6.4, for each input value, the output value has increased by 20, so if we call the new function $S(t)$,we could write

$$S(t) = V(t) + 20 \tag{2.6.1}$$

This notation tells us that, for any value of t,$S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 2.6.1.

Table 2.6.1

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

How To...

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Example 2.6.2: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 2.6.2. Create a table for the function $g(x) = f(x) - 3$.

Table 2.6.2

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example:

$$\begin{aligned} f(x) &= 1 && \text{Given} \\ g(x) &= f(x) - 3 && \text{Given Transformation} \\ g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 2.6.3.

Table 2.6.3

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	-2	0	4	8

Analysis

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

Exercise 2.6.1

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Answer

$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 2.6.4.

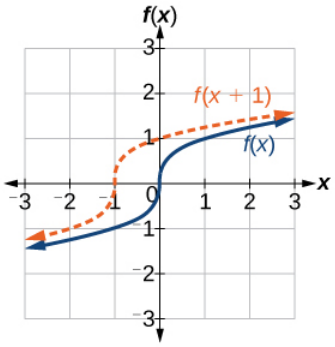


Figure 2.6.4: Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = -1$ shifts the graph to the left, that is, towards negative values of x.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Definition: Horizontal Shift

Given a function f, a new function $g(x) = f(x-h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example 2.6.4: Adding a Constant to an Input

Returning to our building airflow example from Figure 2.6.2, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{ the original venting plan}$$

$$F(t) = \text{ starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See Figure 2.6.5. Notice also that the vents first opened to 220ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220ft^2 at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t-(-2)) = V(t+2)$.

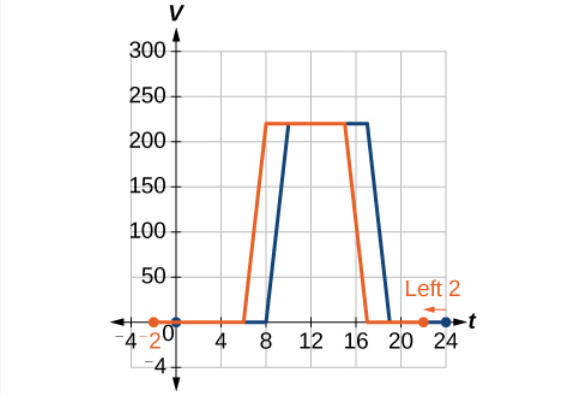


Figure 2.6.5

Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for F : $F(t) = V(t+2)$.

How To...

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

Example 2.6.5: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 2.6.4. Create a table for the function $g(x) = f(x-3)$.

Table 2.6.4

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 larger. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create Table 2.6.5.

Table 2.6.5

x	5	7	9	11
$x-3$	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Analysis

Figure 2.6.6 represents both of the functions. We can see the horizontal shift in each point.

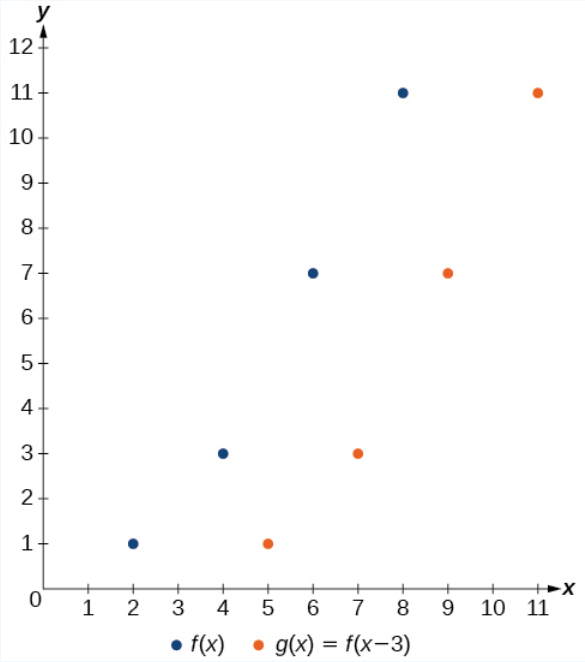


Figure 2.6.6: Graph of the points from the previous table for $f(x)$ and $g(x) = f(x-3)$

Example 2.6.6: Identifying a Horizontal Shift of a Toolkit Function

Figure 2.6.7 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

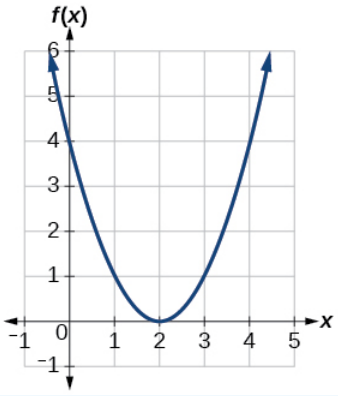


Figure 2.6.7: Graph of a parabola

Solution

Notice that the graph is identical in shape to the $f(x)=x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0,0)$, but now the vertex is at $(2,0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x)=f(x-2)$$

Notice how we must input the value $x=2$ to get the output value $y=0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$.

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x-2) \\ g(x) &= f(x-2)=(x-2)^2 \end{aligned}$$

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0)=0$. In our shifted function, $g(2)=0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2)=f(x-2)=f(0)=0$. For this to work, we will need to subtract 2 units from our input values.

Example 2.6.7: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m)+10$ and $G(m+10)$

Solution

$G(m)+10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m+10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Exercise 2.6.7

Given the function $f(x)=\sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x)=f(x+2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Answer

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

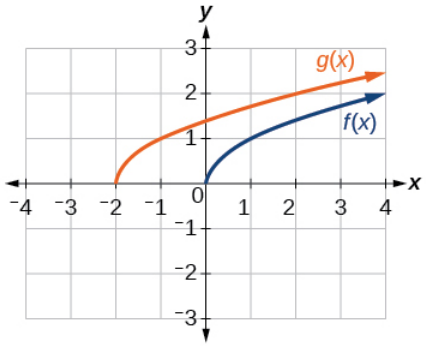


Figure 2.6.8

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output $(y-)$ axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input $(x-)$ axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

How To...

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Example 2.6.8: Graphing Combined Vertical and Horizontal Shifts

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x+1)-3$.

Solution

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 2.6.9.

Let us follow one point of the graph of $f(x)=|x|$.

- The point $(0,0)$ is transformed first by shifting left 1 unit:$(0,0)\rightarrow(-1,0)$
- The point $(-1,0)$ is transformed next by shifting down 3 units:$(-1,0)\rightarrow(-1,-3)$

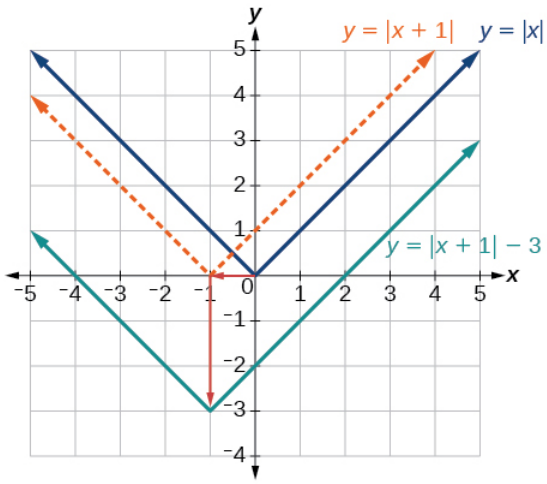


Figure 2.6.9: Graph of an absolute function, $y = |x|$, and how it was transformed to $y = |x + 1| - 3$

Figure $2.6.10$shows the graph of h.

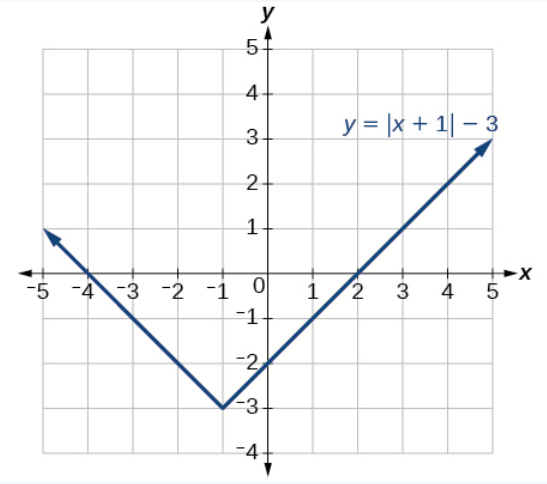


Figure 2.6.10: The final function $y = |x + 1| - 3$.

Exercise 2.6.8

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x - 2) + 4$.

Answer

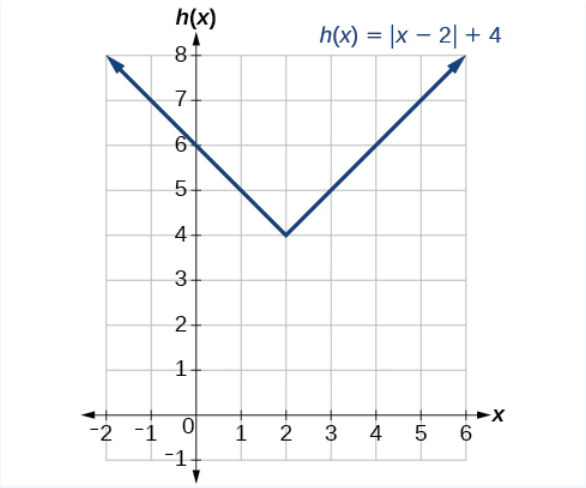


Figure 2.6.11

Example 2.6.9: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure 2.6.12, which is a transformation of the toolkit square root function.

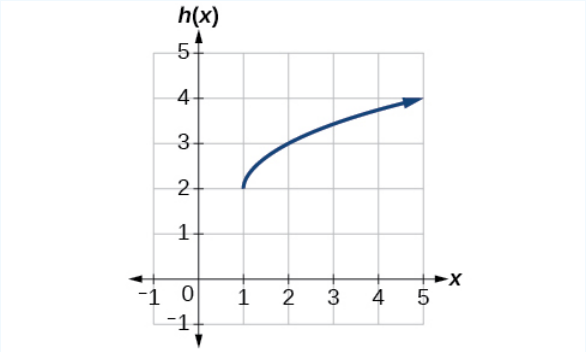


Figure 2.6.12: Graph of a square root function transposed right one unit and up 2.

Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1)+2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1}+2$$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

Exercise 2.6.9

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Answer

$$g(x) = \frac{1}{x-1}+1$$

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 2.6.13.

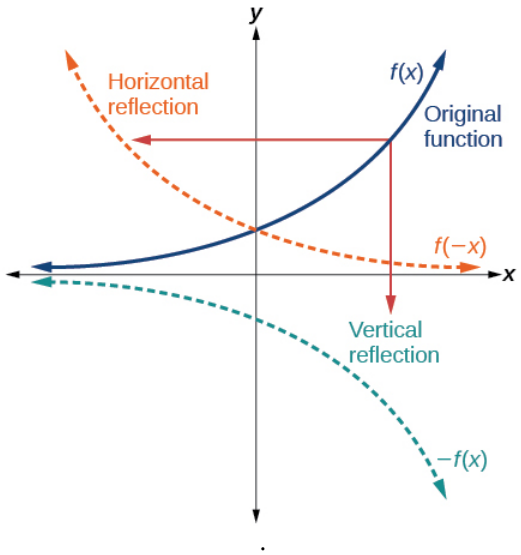


Figure 2.6.13: Graph of the vertical and horizontal reflection of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

Definitions: Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

How To...

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the y-axis.

Example 2.6.10: Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure 2.6.14

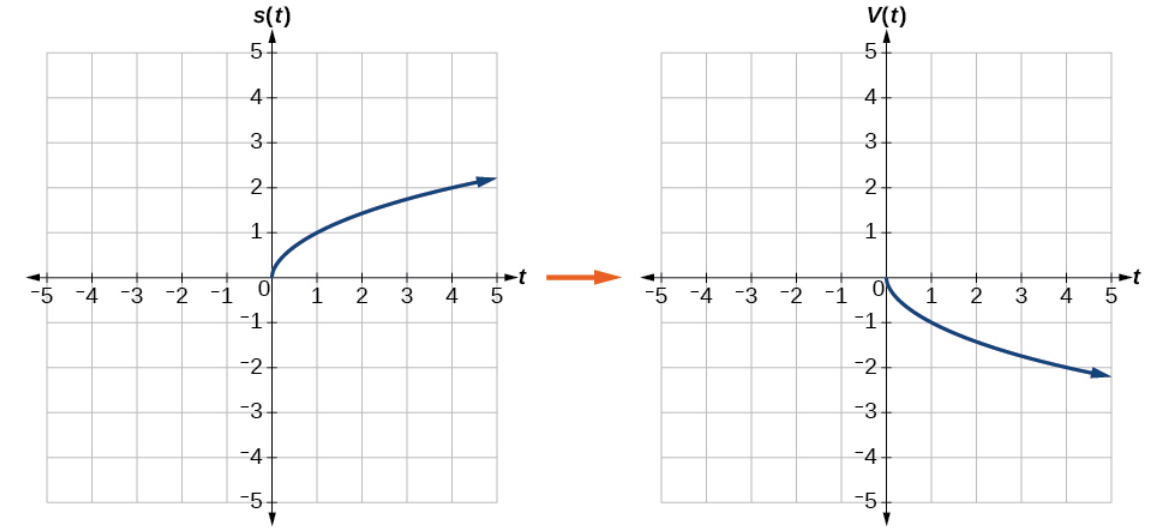


Figure 2.6.14: Graph of the vertical reflection of the square root function.

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 2.6.15.

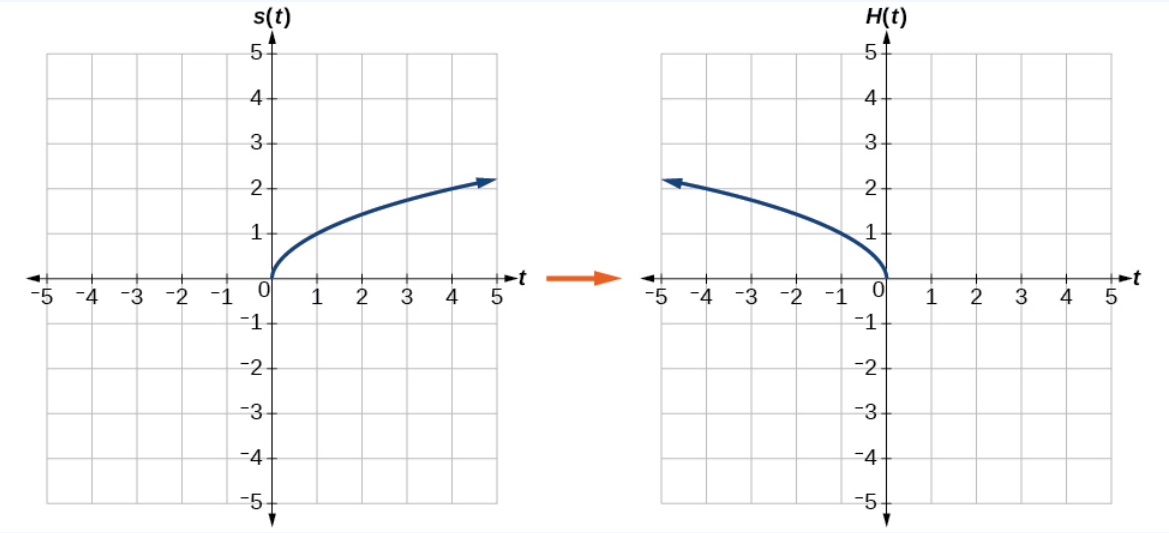


Figure 2.6.15: Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

Exercise 2.6.5

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

Answer

a.

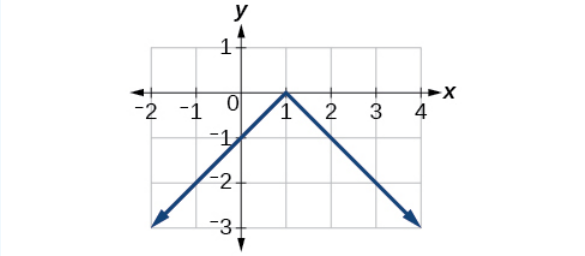


Figure 2.6.16: Graph of a vertically reflected absolute function.

b.

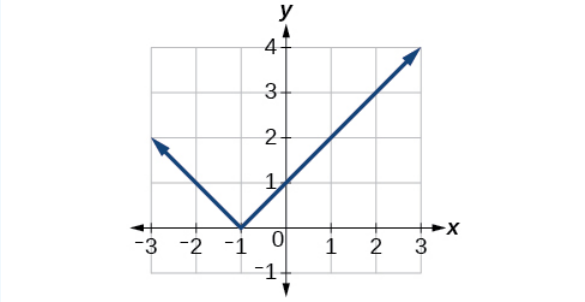


Figure 2.6.17: Graph of an absolute function translated one unit left.

Example 2.6.11: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table 2.6.6. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 2.6.6

x	2	4	6	8
$f(x)$	1	3	7	11

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 2.6.7.

Table 2.6.7

x	2	4	6	8
$g(x)$	-1	-3	-7	-11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 2.6.8.

Table 2.6.8

x	-2	-4	-6	-8
$h(x)$	1	3	7	11

Exercise 2.6.6

A function $f(x)$ is given as Table 2.6.9. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 2.6.9

x	-2	0	2	4
$f(x)$	5	10	15	20

Answer

a. $g(x) = -f(x)$

Table 2.6.10

x	-2	0	2	4
$g(x)$	-5	-10	-15	-20

b. $h(x) = f(-x)$

Table 2.6.11

x	-2	0	2	-4
$h(x)$	15	10	5	20

Example 2.6.12: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure 2.6.18. Sketch a graph of $k(t)$.

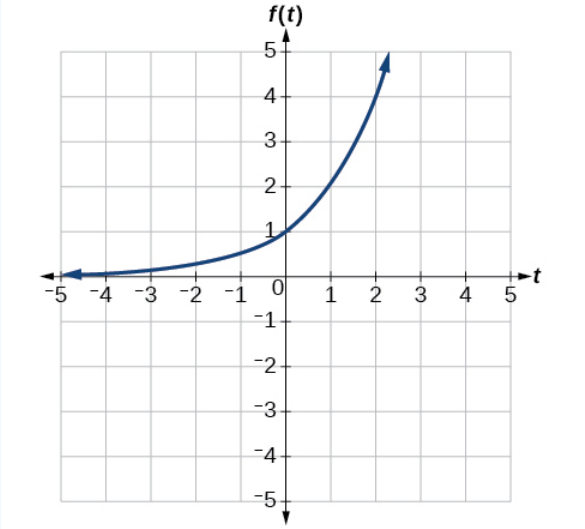


Figure 2.6.18: Graph of $k(t)$

Solution

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$

- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t)+1 = -2^{-t}+1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points $(0, 1)$ and $(1, 2)$.

- First, we apply a horizontal reflection: $(0, 1)$ $(-1, 2)$.
- Then, we apply a vertical reflection: $(0, -1)$ $(-1, -2)$.
- Finally, we apply a vertical shift: $(0, 0)$ $(-1, -1)$.

This means that the original points, $(0, 1)$ and $(1, 2)$ become $(0, 0)$ and $(-1, -1)$ after we apply the transformations.

In Figure 2.6.19, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

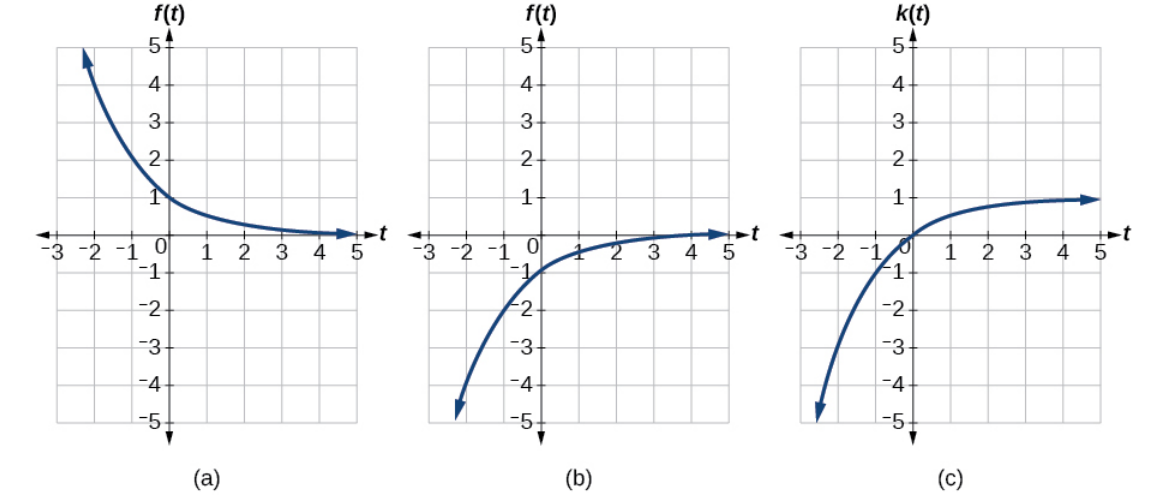


Figure 2.6.19: Graphs of all the transformations.

Analysis

As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

Exercise 2.6.7

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

Answer

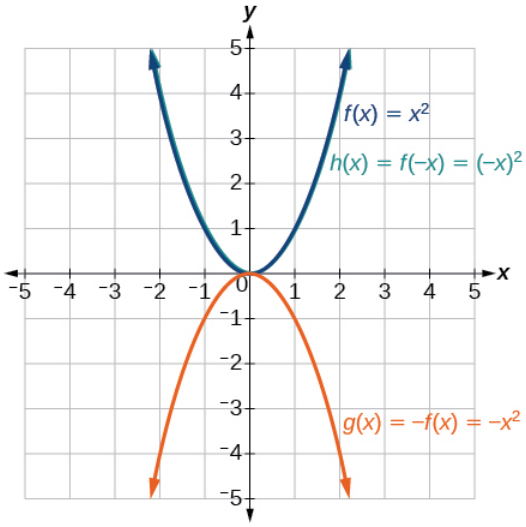


Figure 2.6.20: Graph of x^2 and its reflections.

Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. Functions whose graphs are symmetric about the y-axis are called **even functions**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph, as shown in Figure 2.6.21.

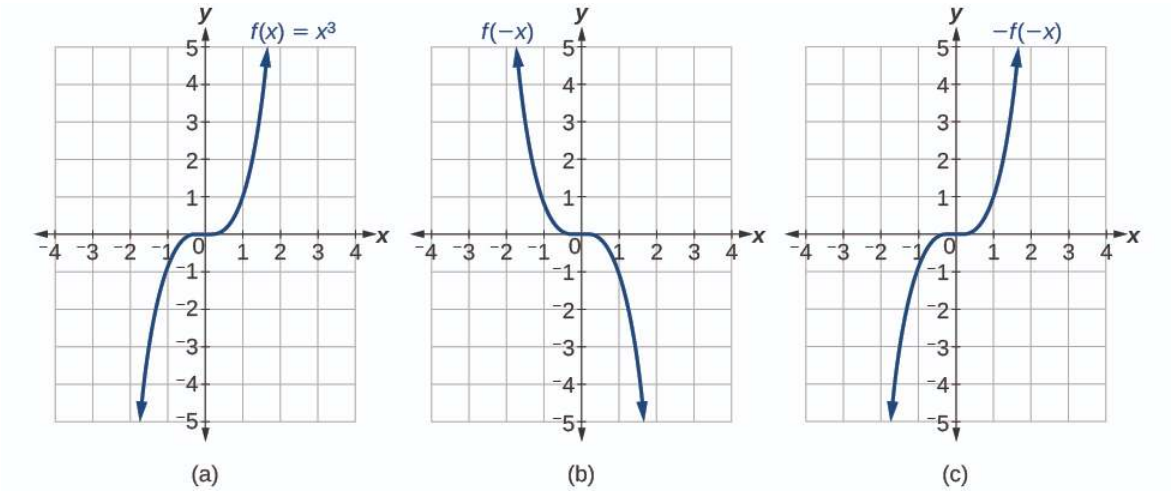


Figure 2.6.21: (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Definitions: Even and Odd Functions

A function is called an **even function** if for every input x

$f(x) = f(-x)$

The graph of an even function is symmetric about the y-axis.

A function is called an **odd function** if for every input x

$f(x) = -f(-x)$

The graph of an odd function is symmetric about the origin.

How To...

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Example 2.6.13: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

Analysis

Consider the graph of f in Figure 2.6.22 Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.

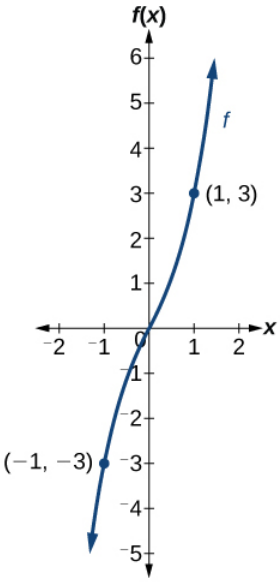


Figure 2.6.22: Graph of $f(x)$ with labeled points at $(1, 3)$ and $(-1, -3)$.

Exercise 2.6.8

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Answer

even

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 2.6.23 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

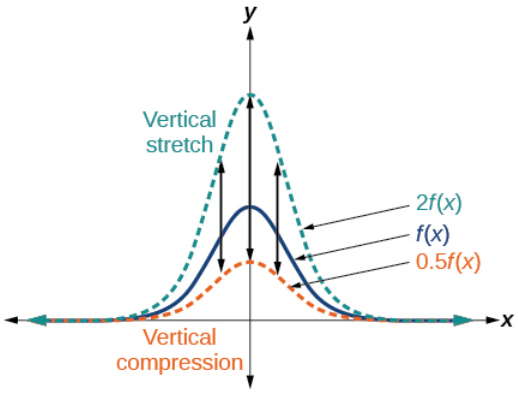


Figure 2.6.23: Vertical stretch and compression

Definitions: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How To...

Given a function, graph its vertical stretch.

a. Identify the value of a.
b. Multiply all range values by a
c. If $a > 1$, the graph is stretched by a factor of a.
d. If $0 < a < 1$, the graph is compressed by a factor of a.
e. If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

Example 1.5.14: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure $2.6.24$

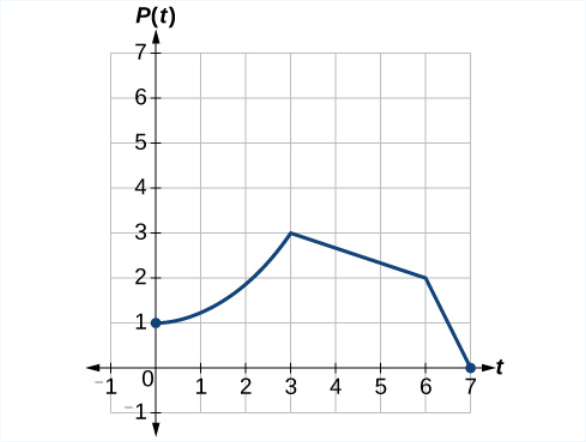


Figure 2.6.24: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure $2.6.25$

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$ and $(7, 0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \to (0, 2) \tag{2.6.2}$$
$$(3, 3) \to (3, 6) \tag{2.6.3}$$
$$(6, 2) \to (6, 4) \tag{2.6.4}$$
$$(7, 0) \to (7, 0) \tag{2.6.5}$$

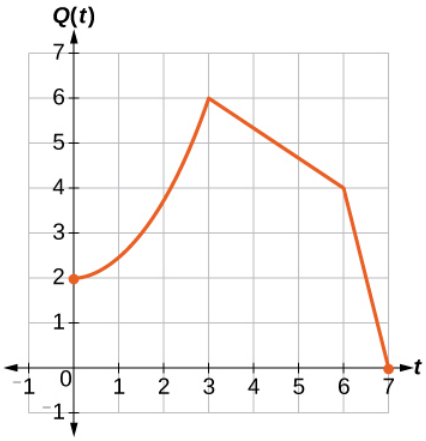


Figure 2.6.25: Graph of the population function doubled.

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

How To...

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

Example 2.6.15: Finding a Vertical Compression of a Tabular Function

A function f is given as Table 2.6.12 Create a table for the function $g(x) = \frac{1}{2}f(x)$.

Table 2.6.12

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 2.6.13

Table 2.6.13

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Analysis

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Exercise 2.6.9

A function f is given as Table 2.6.14 Create a table for the function $g(x) = \frac{3}{4} f(x)$.

Table 2.6.14

x	2	4	6	8
$f(x)$	12	16	20	0

Answer

Table 2.6.15

x	2	4	6	8
$g(x)$	9	12	15	0

Example 2.6.16: Recognizing a Vertical Stretch

The graph in Figure 2.6.26 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

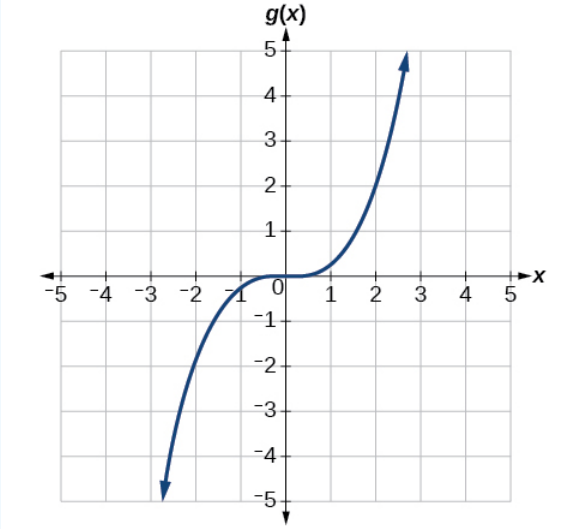


Figure 2.6.26: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4} f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4} f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4} f(x) = \frac{1}{4} x^3. \tag{2.6.6}$$

Exercise 2.6.1

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Answer

$g(x) = 3x - 2$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

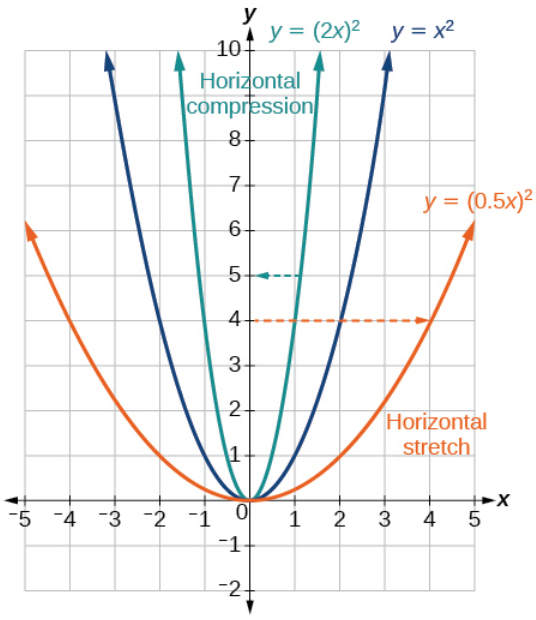


Figure 2.6.27: Graph of the vertical stretch and compression of x^2.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 2.6.27. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

Definitions: Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

How To...

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

Example 2.6.17: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$R(1) = P(2), \tag{2.6.7}$$
$$R(2) = P(4), \quad \text{and in general,} \tag{2.6.8}$$
$$R(t) = P(2t). \tag{2.6.9}$$

See Figure 2.6.28 for a graphical comparison of the original population and the compressed population.

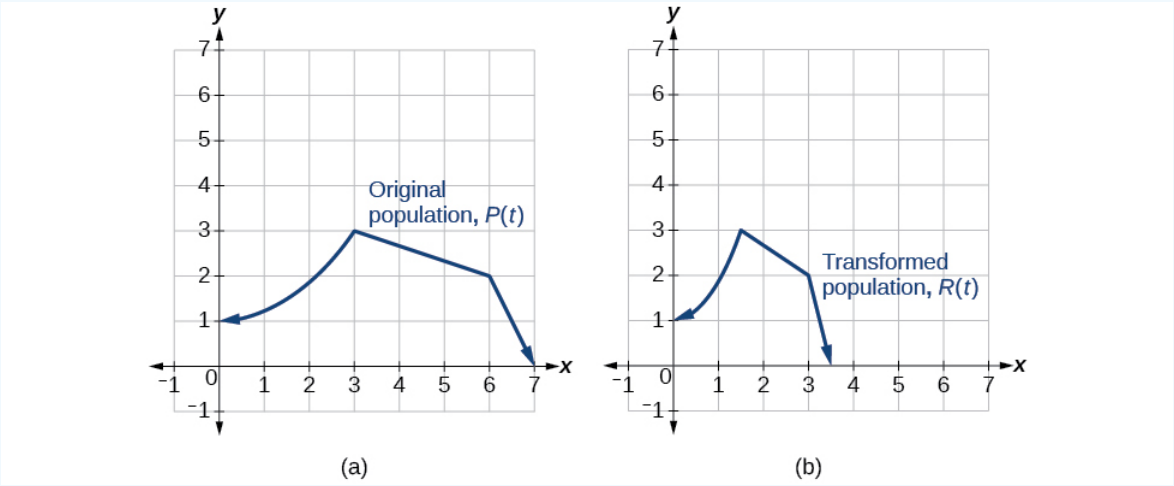


Figure 2.6.28: (a) Original population graph (b) Compressed population graph

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 2.6.18: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 2.6.16. Create a table for the function $g(x) = f(\frac{1}{2}x)$.

Table 2.6.16

x	2	4	6	8
$f(x)$	1	3	7	11

The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1 \tag{2.6.10}$$

We do the same for the other values to produce Table 2.6.17.

Table 2.6.17

x	4	8	12	16

$g(x)$	1	3	7	11

Figure 2.6.29 shows the graphs of both of these sets of points.

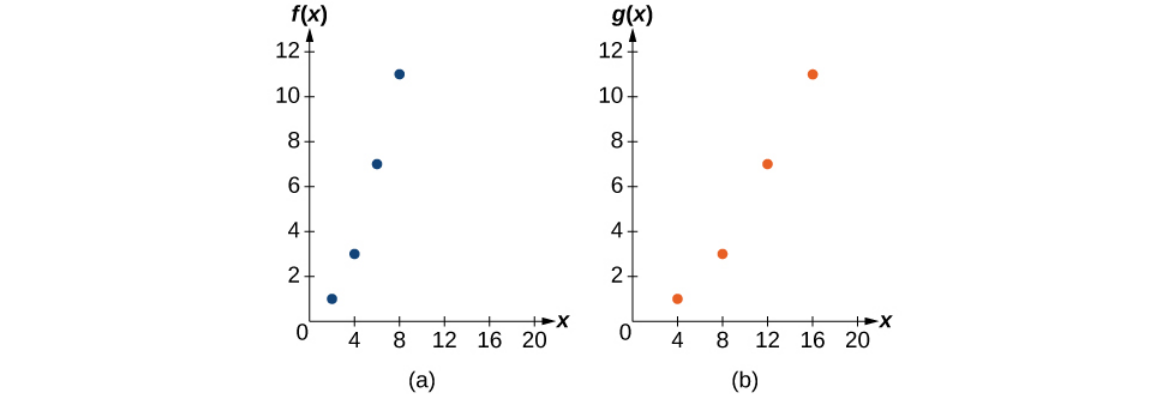


Figure 2.6.29: Graph of the previous table.

Analysis

Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Example 2.6.19: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 2.6.30.

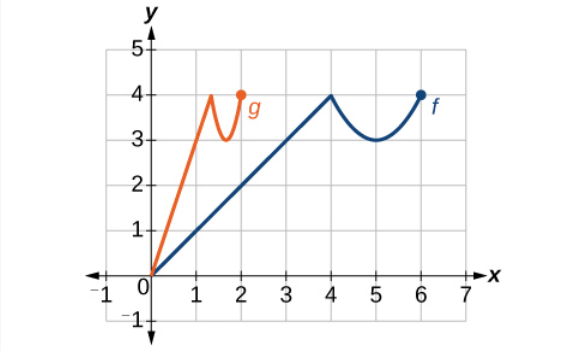


Figure 2.6.30: Graph of $f(x)$ being vertically compressed to $g(x)$.

Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Exercise 2.6.11

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Answer

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure $2.6.23$ shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

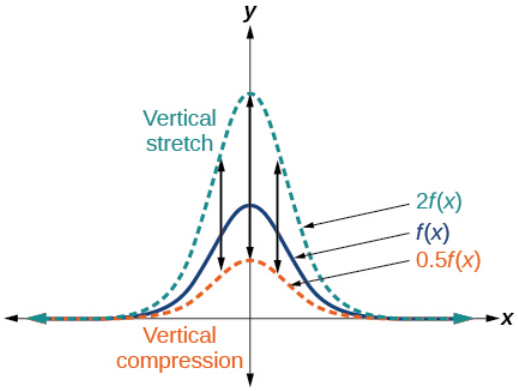


Figure 2.6.23: Vertical stretch and compression

Definitions: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How To...

Given a function, graph its vertical stretch.

a. Identify the value of a.
b. Multiply all range values by a
c. If $a > 1$, the graph is stretched by a factor of a.
d. If $0 < a < 1$, the graph is compressed by a factor of a.
e. If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

Example 1.5.14: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure $2.6.24$

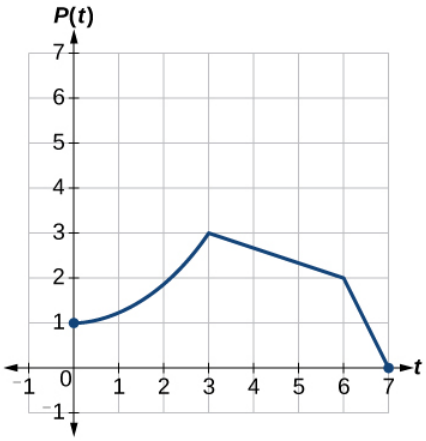


Figure 2.6.24: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure 2.6.25.

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$ and $(7, 0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \to (0, 2) \tag{2.6.11}$$

$$(3, 3) \to (3, 6) \tag{2.6.12}$$

$$(6, 2) \to (6, 4) \tag{2.6.13}$$

$$(7, 0) \to (7, 0) \tag{2.6.14}$$

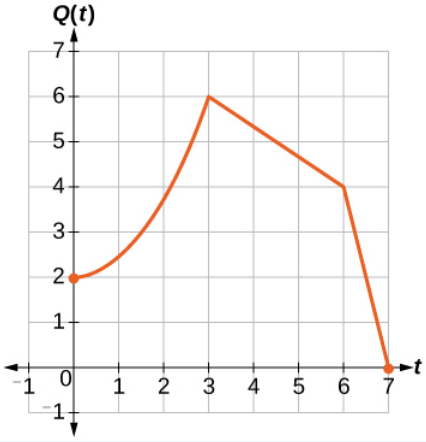


Figure 2.6.25: Graph of the population function doubled.

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

How To...

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

Example 2.6.15: Finding a Vertical Compression of a Tabular Function

A function f is given as Table 2.6.12 Create a table for the function $g(x) = \frac{1}{2} f(x)$.

Table 2.6.12

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2} f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2} f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 2.6.13

Table 2.6.13

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Analysis

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Exercise 2.6.9

A function f is given as Table 2.6.14 Create a table for the function $g(x) = \frac{3}{4} f(x)$.

Table 2.6.14

x	2	4	6	8
$f(x)$	12	16	20	0

Answer

Table 2.6.15

x	2	4	6	8
$g(x)$	9	12	15	0

Example 2.6.16: Recognizing a Vertical Stretch

The graph in Figure 2.6.26 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

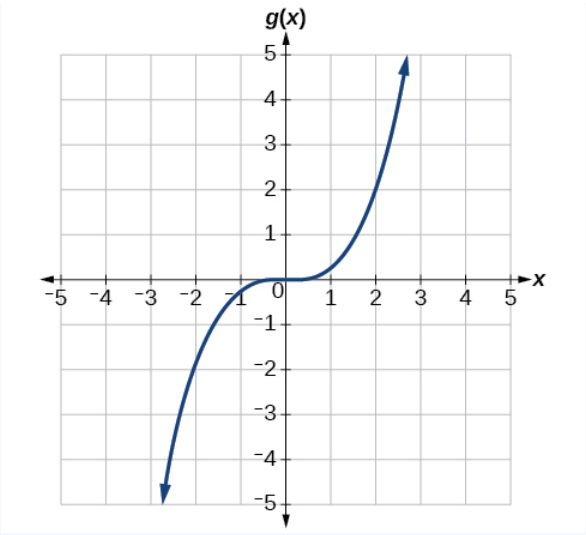


Figure 2.6.26: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3. \tag{2.6.15}$$

Exercise 2.6.1

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Answer

$g(x) = 3x - 2$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

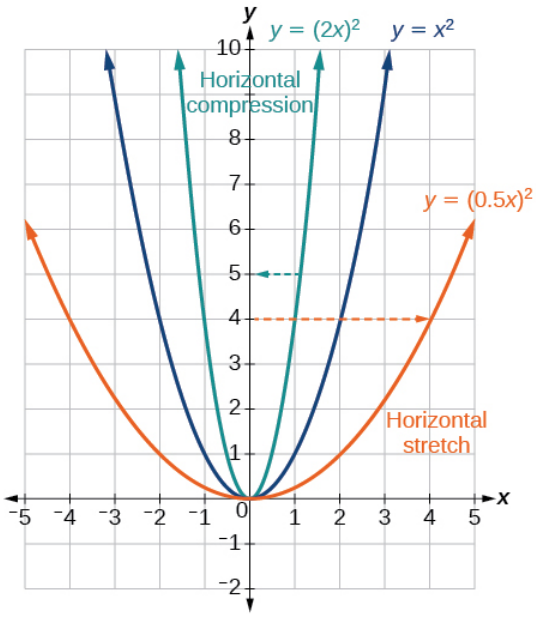


Figure 2.6.27: Graph of the vertical stretch and compression of x^2.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 2.6.27. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

Definitions: Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

How To...

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

Example 2.6.17: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$R(1) = P(2), \tag{2.6.16}$$
$$R(2) = P(4), \quad \text{and in general,} \tag{2.6.17}$$
$$R(t) = P(2t). \tag{2.6.18}$$

See Figure 2.6.28 for a graphical comparison of the original population and the compressed population.

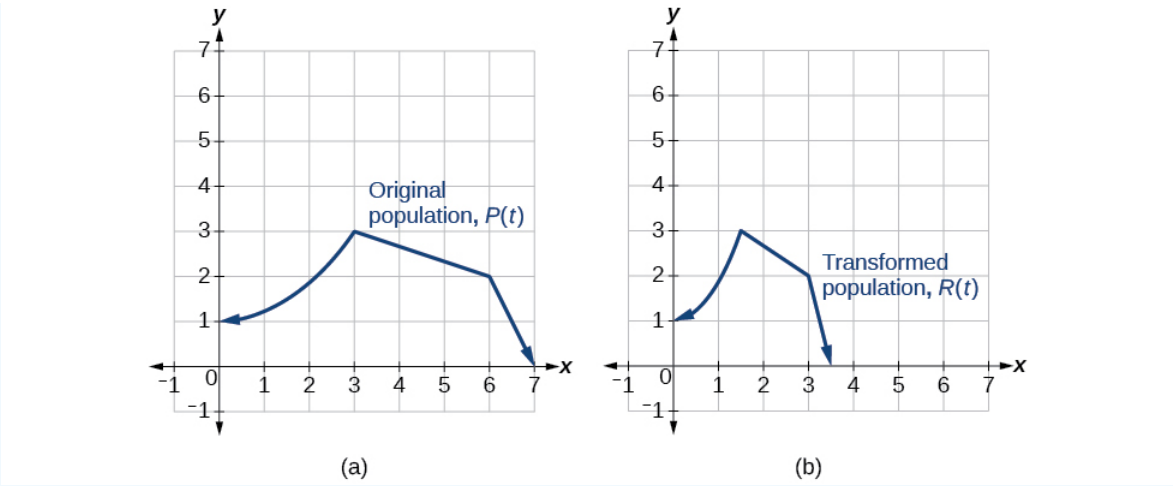


Figure 2.6.28: (a) Original population graph (b) Compressed population graph

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 2.6.18: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 2.6.16. Create a table for the function $g(x) = f(\frac{1}{2}x)$.

Table 2.6.16

x	2	4	6	8
$f(x)$	1	3	7	11

The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1 \tag{2.6.19}$$

We do the same for the other values to produce Table 2.6.17.

Table 2.6.17

x	4	8	12	16
$g(x)$	1	3	7	11

Figure 2.6.29 shows the graphs of both of these sets of points.

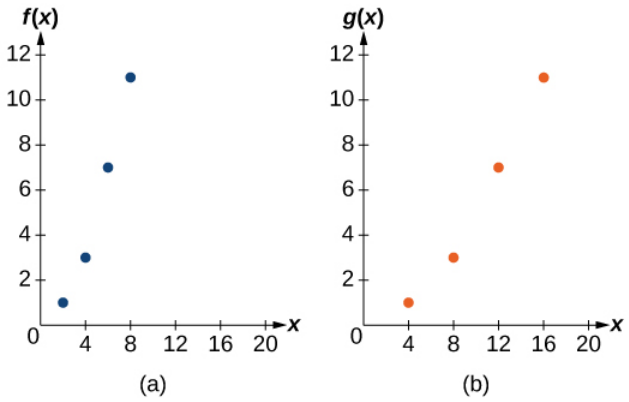


Figure 2.6.29: Graph of the previous table.

Analysis

Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Example 2.6.19: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 2.6.30

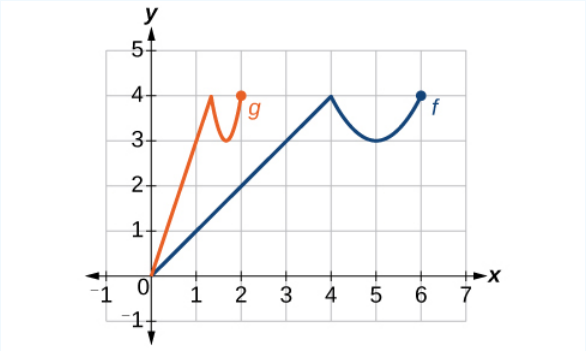


Figure 2.6.30: Graph of $f(x)$ being vertically compressed to $g(x)$.

Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Exercise 2.6.11

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Answer

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x)=f(2x+3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7)=12$. What input to g would produce that output? In other words, what value of x will allow $g(x)=f(2x+3)=12$? We would need $2x+3=7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p)=f(b(x+\frac{p}{b}))$$

Let's work through an example.

$$f(x)=(2x+4)^2$$

We can factor out a 2.

$$f(x)=(2(x+2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Combining Transformations

- When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by a and then vertically shift by k.
- When combining horizontal transformations written in the form $f(bx+h)$, first horizontally shift by h and then horizontally stretch by $\frac{1}{b}$.
- When combining horizontal transformations written in the form $f(b(x+h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h.
- Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example 2.6.20: Finding a Triple Transformation of a Tabular Function

Given Table 2.6.18 for the function $f(x)$, create a table of values for the function $g(x)=2f(3x)+1$.

Table 2.6.18

x	6	12	18	24
$f(x)$	10	14	15	17

Solution

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$. See Table 2.6.19

Table 2.6.19

x	2	4	6	8

$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 2.6.20

Table 2.6.20

x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 2.6.21

Table 2.6.21

x	2	4	6	8
$g(x)=2f(3x)+1+1$	21	29	31	35

Example 2.6.21: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 2.6.31 to sketch a graph of $k(x)=f\left(\frac{1}{2}x+1\right)-3$.

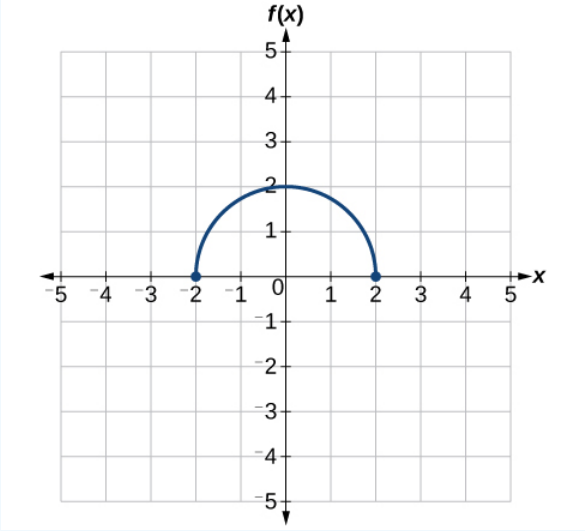


Figure 2.6.31: Graph of a half-circle.

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3 \tag{2.6.20}$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0,2)$ remains at $(0,2)$ while the point $(2,0)$ will stretch to $(4,0)$. See Figure 2.6.32

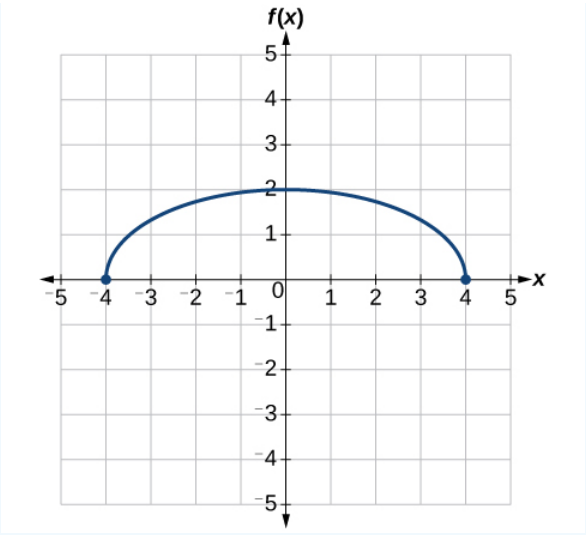


Figure 2.6.32: Graph of a vertically stretch half-circle.

Next, we horizontally shift left by 2 units, as indicated by $x+2$. See Figure 2.6.33

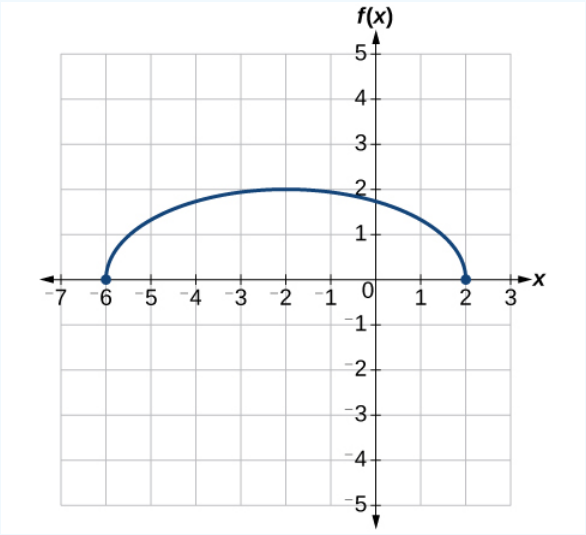


Figure 2.6.33: Graph of a vertically stretch and translated half-circle.

Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See Figure 2.6.34

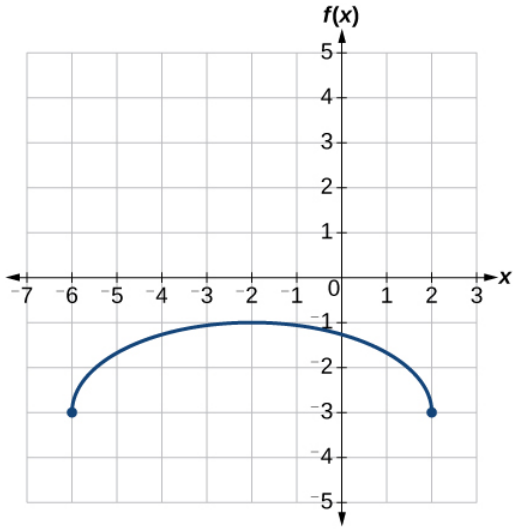


Figure 2.6.34: Graph of a vertically stretch and translated half-circle.

Key Equations

- Vertical shift $g(x) = f(x) + k$ (up for $k > 0$)
- Horizontal shift $g(x) = f(x - h)$ (right) for $h > 0$
- Vertical reflection $g(x) = -f(x)$
- Horizontal reflection $g(x) = f(-x)$
- Vertical stretch $g(x) = af(x)$ (a>0)
- Vertical compression $g(x) = af(x)$ (0<a<1)
- Horizontal stretch $g(x) = f(bx)(0 < b < 1)$
- Horizontal compression $g(x) = f(bx)$ (b>1)

Key Concepts

- A function can be shifted vertically by adding a constant to the output.
- A function can be shifted horizontally by adding a constant to the input.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts.
- Vertical and horizontal shifts are often combined.
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y-axis. A graph can be reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly.
- A function presented as an equation can be reflected by applying transformations one at a time.
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $f(x) = -f(-x)$.
- A function can be odd, even, or neither.
- A function can be compressed or stretched vertically by multiplying the output by a constant.
- A function can be compressed or stretched horizontally by multiplying the input by a constant.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order.

Glossary

even function

a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y-axis

horizontal compression
a transformation that compresses a function's graph horizontally, by multiplying the input by a constant b>1

horizontal reflection
a transformation that reflects a function's graph across the y-axis by multiplying the input by −1

horizontal shift
a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input

horizontal stretch
a transformation that stretches a function's graph horizontally by multiplying the input by a constant 0<b<1

odd function
a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x) = -f(-x)$, and is symmetric about the origin

vertical compression
a function transformation that compresses the function's graph vertically by multiplying the output by a constant 0<a<1

vertical reflection
a transformation that reflects a function's graph across the x-axis by multiplying the output by −1

vertical shift
a transformation that shifts a function's graph up or down by adding a positive or negative constant to the output

vertical stretch
a transformation that stretches a function's graph vertically by multiplying the output by a constant a>1

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CHAPTER OVERVIEW

3: Linear and Absolute Value Functions

Recall that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

3.1: Linear Functions

3.2: Modeling with Linear Functions

3.3: Absolute Value Functions

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3.1: Linear Functions

Learning Objectives

- Represent a linear function.
- Determine whether a linear function is increasing, decreasing, or constant.
- Interpret slope as a rate of change.
- Write and interpret an equation for a linear function.
- Graph linear functions.
- Determine whether lines are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train (maglev train in the world, the Shanghai MagLev Train (Figure 3.1.1). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes.



Figure 3.1.1: Shanghai MagLev Train (credit: “kanegen”/Flickr)

Suppose a maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train’s distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train’s distance from the station at a given point in time.maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train’s distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train’s distance from the station at a given point in time.

Representing Linear Functions

The function describing the train’s motion is a **linear function**, which is defined as a function with a constant rate of change, that is, a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train’s motion as a function using each method.

Representing a Linear Function in Word Form

Let’s begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train’s distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the form known as the slope-intercept form of a line, where x is the input value, m is the rate of change, and b is the initial value of the dependent variable.

$$\begin{aligned} &\text{Equation form} && y = mx + b \\ &\text{Equation notation} && f(x) = mx + b \end{aligned}$$

In the example of the train, we might use the notation $D(t)$ in which the total distance D is a function of the time t. The rate, m, is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

$$D(t) = 83t + 250 \tag{3.1.1}$$

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in Figure 3.1.2. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

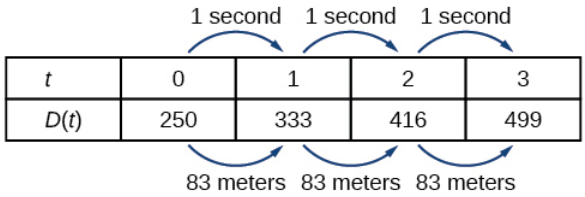


Figure 3.1.2: Tabular representation of the function D showing selected input and output values

Q/A? Can the input in the previous example be any real number?

No. The input represents time, so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250$, to draw a graph, represented in Figure 3.1.3. Notice the graph is a line. When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or **slope** of the line. The point at which the input value is zero is the vertical intercept, or **y-intercept**, of the line. We can see from the graph in Figure 3.1.3 that the y-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.

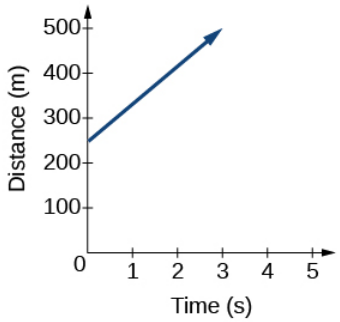


Figure 3.1.3: The graph of $D(t) = 83t + 250$. Graphs of linear functions are lines because the rate of change is constant.

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x + 1$. Ask yourself what numbers can be input to the function, that is, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

Definition: Linear Function

A **linear function** is a function whose graph is a line. Linear functions can be written in the slope-intercept form of a line

$$f(x) = mx + b \tag{3.1.2}$$

where b is the initial or starting value of the function (when input, $x = 0$), and m is the constant rate of change, or **slope** of the function. The **y-intercept** is at $(0, b)$.

Example 3.1.1: Using a Linear Function to Find the Pressure on a Diver

The pressure, P, in pounds per square inch (PSI) on the diver in Figure 3.1.4 depends upon her depth below the water surface, d, in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.



Figure 3.1.4: (credit: Ilse Reijs and Jan-Noud Hutten)Hutten)

To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Determining whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an **increasing function** has a positive slope. A line with a positive slope slants upward from left to right as in Figure 3.1.5(a). For a **decreasing function**, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in Figure 3.1.5(b). If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in Figure 3.1.5(c).

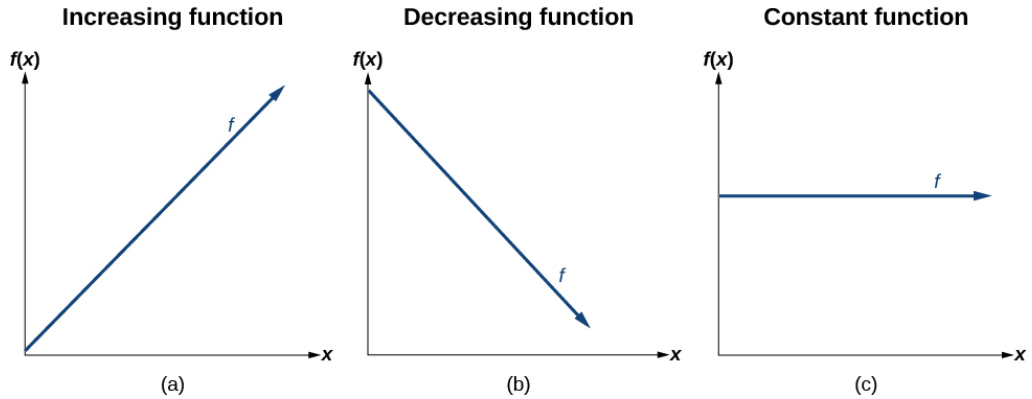


Figure 3.1.5: Three graphs depicting an increasing function, a decreasing function, and a constant function.]

Increasing and Decreasing Functions

Increasing and Decreasing Functions

The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a constant function.

- $f(x) = mx + b$ is an increasing function if $m > 0$.
- $f(x) = mx + b$ is an decreasing function if $m < 0$.
- $f(x) = mx + b$ is a constant function if $m = 0$.

Example 3.1.2: decide whether a Function is Increasing, Decreasing, or Constant

Some recent studies suggest that a teenager sends an average of 60 texts per day. For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant.

a. The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
b. A teen has a limit of 500 texts per month in his or her data plan. The input is the number of days, and output is the total number of texts remaining for the month.
c. A teen has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

Solution

Analyze each function.

a. The function can be represented as $f(x) = 60x$ where x is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
b. The function can be represented as $f(x) = 500 - 60x$ where x is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after x days.
c. The cost function can be represented as $f(x) = 50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

Calculating and Interpreting Slope

In the examples we have seen so far, we have had the slope provided for us. However, we often need to calculate the slope given input and output values. Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2—which can be represented by a set of points, (x_1, y_1) and (x_2, y_2)—we can calculate the slope m, as follows

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}}$$

$$= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where Δy is the vertical displacement and Δx is the horizontal displacement. Note in function notation two corresponding values for the output y_1 and y_2 for the function f, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Figure 3.1.6 indicates how the slope of the line between the points, (x_1, y_1) and (x_2, y_2), is calculated. Recall that the slope measures steepness. The greater the absolute value of the slope, the steeper the line is.

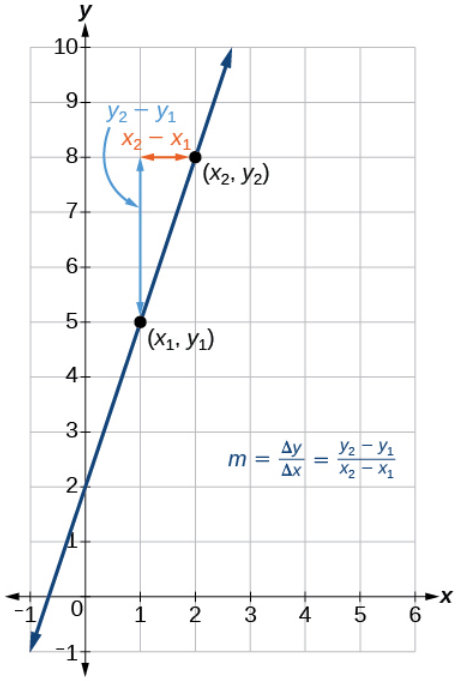


Figure 3.1.6: The slope of a function is calculated by the change in y divided by the change in x. It does not matter which coordinate is used as the (x_2, y_2) and which is the (x_1, y_1),as long as each calculation is started with the elements from the same coordinate pair.

Q & A

Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

Note: Calculate Slope

The slope or rate of change of a function, m, can be calculated according to the following:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{3.1.3}$$

where x_1 and x_2 are input values, y_1 and y_2 are output values.

How to: Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

Example 3.1.3: Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3,-2)$ and $(8,1)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution

The coordinate pairs are $(3,-2)$ and $(8,1)$. To find the rate of change, we divide the change in output by the change in input.

$$m=\frac{\text{change in output (rise)}}{\text{change in input (run)}}=\frac{1-(-2)}{8-3}=\frac{3}{5} \tag{3.1.4}$$

We could also write the slope as $m=0.6$. The function is increasing because $m>0$.

Analysis

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or *y*-coordinate, used corresponds with the first input value, or *x*-coordinate, used.

Try it! 3.1.1

If $f(x)$ is a linear function, and $(2,3)$ and $(0,4)$ are points on the line, find the slope. Is this function increasing or decreasing?

Answer

$m=\frac{4-3}{0-2}=\frac{1}{-2}=-\frac{1}{2}$; decreasing because $m<0$.

Example 3.1.4: Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

The rate of change relates the change in population to the change in time. The population increased by $27,800-23,400=4400$ people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

$$\frac{4,400 \text{ people}}{4 \text{ years}}=1,100\frac{\text{people}}{\text{year}} \tag{3.1.5}$$

So the population increased by 1,100 people per year.

Analysis

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

Try it! 3.1.2

The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

Answer

$m = \frac{1{,}868-1{,}442}{2{,}012-2{,}009} = \frac{426}{3} = 142$ people per year

Writing the Point-Slope Form of a Linear Equation

Up until now, we have been using the slope-intercept form of a linear equation to describe linear functions. Here, we will learn another way to write a linear function, the **point-slope form**.

$$y - y_1 = m(x - x_1) \tag{3.1.6}$$

The point-slope form is derived from the slope formula.

$$\begin{aligned} m &= \frac{y-y_1}{x-x_1} && \text{assuming } x \neq x_1 \\ m(x-x_1) &= \frac{y-y_1}{x-x_1}(x-x_1) && \text{Multiply both sides by } (x-x_1). \\ m(x-x_1) &= y-y_1 && \text{Simplify} \\ y-y_1 &= m(x-x_1) && \text{Rearrange} \end{aligned}$$

Keep in mind that the slope-intercept form and the point-slope form can be used to describe the same function. We can move from one form to another using basic algebra. For example, suppose we are given an equation in point-slope form, $y-4 = -\frac{1}{2}(x-6)$. We can convert it to the slope-intercept form as shown.

$$\begin{aligned} y-4 &= -\frac{1}{2}(x-6) \\ y-4 &= -\frac{1}{2}x+3 && \text{Distribute the } -\frac{1}{2}. \\ y &= -\frac{1}{2}x+7 && \text{Add 4 to each side.} \end{aligned}$$

Therefore, the same line can be described in slope-intercept form as $y = \frac{1}{2}x + 7$.

Note: Point-Slope Form of a Linear Equation

The **point-slope** form of a linear equation takes the form

$$y - y_1 = m(x - x_1) \tag{3.1.7}$$

where m is the slope, x_1 and y_1 are the x and y coordinates of a specific point through which the line passes.

Writing the Equation of a Line Using a Point and the Slope

The point-slope form is particularly useful if we know one point and the slope of a line. Suppose, for example, we are told that a line has a slope of 2 and passes through the point $(4, 1)$. We know that $m = 2$ and that $x_1 = 4$ and $y_1 = 1$. We can substitute these values into the general point-slope equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - 4) \end{aligned}$$

If we wanted to then rewrite the equation in slope-intercept form, we apply algebraic techniques.

$$\begin{aligned} y-1 &= 2(x-4) \\ y-1 &= 2x-8 && \text{Distribute the 2.} \\ y &= 2x-7 && \text{Add 1 to each side.} \end{aligned}$$

Both equations, $y-1=2(x-4)$ and $y=2x-7$, describe the same line. See Figure 3.1.7.

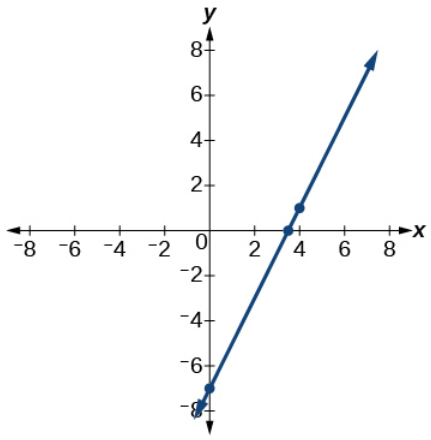


Figure 3.1.7

Example 3.1.5: Writing Linear Equations Using a Point and the Slope

Write the point-slope form of an equation of a line with a slope of 3 that passes through the point $(6,-1)$. Then rewrite it in the slope-intercept form.

Solution

Let's figure out what we know from the given information. The slope is 3, so $m=3$. We also know one point, so we know $x_1=6$ and $y_1=-1$. Now we can substitute these values into the general point-slope equation.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-(-1) &= 3(x-6) && \text{Substitute known values.} \\ y+1 &= 3(x-6) && \text{Distribute } -1 \text{ to find point-slope form.} \end{aligned}$$

Then we use algebra to find the slope-intercept form.

$$\begin{aligned} y+1 &= 3(x-6) \\ y+1 &= 3x-18 && \text{Distribute 3.} \\ y &= 3x-19 && \text{Simplify to slope-intercept form.} \end{aligned}$$

? Try it! 3.1.3

Write the point-slope form of an equation of a line with a slope of –2 that passes through the point $(-2,2)$. Then rewrite it in the slope-intercept form.

Answer

$y-2=-2(x+2)$; $y=-2x-2$

Writing the Equation of a Line Using Two Points

The point-slope form of an equation is also useful if we know any two points through which a line passes. Suppose, for example, we know that a line passes through the points $(0,1)$ and $(3,2)$. We can use the coordinates of the two points to find the slope.

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{2-1}{3-0} \\ &= \frac{1}{3} \end{aligned}$$

Now we can use the slope we found and the coordinates of one of the points to find the equation for the line. Let use $(0, 1)$ for our point.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{1}{3}(x - 0) \end{aligned}$$

As before, we can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y - 1 &= \frac{1}{3}(x - 0) && \\ y - 1 &= \frac{1}{3}x && \text{Distribute the } \frac{1}{3}. \\ y &= \frac{1}{3}x + 1 && \text{Add 1 to each side.} \end{aligned}$$

Both equations describe the line shown in Figure 3.1.8.

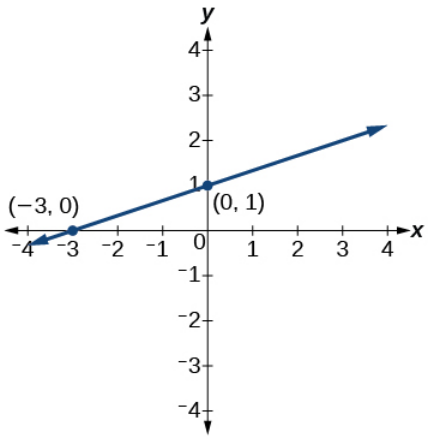


Figure 3.1.8

Example 3.1.6: Writing Linear Equations Using Two Points

Write the point-slope form of an equation of a line that passes through the points $(5, 1)$ and $(8, 7)$. Then rewrite it in the slope-intercept form.

Let's begin by finding the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{7 - 1}{8 - 5} \\ &= \frac{6}{3} \\ &= 2 \end{aligned}$$

So $m = 2$. Next, we substitute the slope and the coordinates for one of the points into the general point-slope equation. We can choose either point, but we will use $(5, 1)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - 5) \end{aligned}$$

The point-slope equation of the line is $y_2 - 1 = 2(x_2 - 5)$. To rewrite the equation in slope-intercept form, we use algebra.

$$\begin{aligned} y - 1 &= 2(x - 5) \\ y - 1 &= 2x - 10 \\ y &= 2x - 9 \end{aligned}$$

The slope-intercept equation of the line is $y = 2x - 9$.

? Try It! 3.1.4

Write the point-slope form of an equation of a line that passes through the points $(-1, 3)$ and $(0, 0)$. Then rewrite it in the slope-intercept form.

Answer

$y - 0 = -3(x - 0)$; $y = -3x$.

Writing and Interpreting an Equation for a Linear Function

Now that we have written equations for linear functions in both the slope-intercept form and the point-slope form, we can choose which method to use based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function f in Figure 3.1.9.

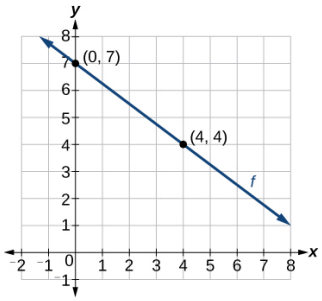


Figure 3.1.9: Graph depicting how to calculate the slope of a line.

We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let's choose $(0, 7)$ and $(4, 4)$. We can use these points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 7}{4 - 0} \\ &= -\frac{3}{4} \end{aligned}$$

Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= -\frac{3}{4}(x - 4) \end{aligned}$$

If we want to rewrite the equation in the slope-intercept form, we would find

$$\begin{aligned} y - 4 &= -\frac{3}{4}(x - 4) \\ y - 4 &= -\frac{3}{4}x + 3 \\ y &= -\frac{3}{4}x + 7 \end{aligned}$$

If we wanted to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the y-axis when the output value is 7. Therefore, $b = 7$. We now have the initial value b and the slope m so we can substitute m and b into the slope-intercept form of a line.

$$f(x) = mx + b$$

$$-\frac{3}{4} \qquad 7$$

$$f(x) = -\frac{3}{4}x + 7$$

Figure 3.1.10

So the function is $f(x) = -\frac{3}{4}x + 7$, and the linear equation would be $y = -\frac{3}{4}x + 7$.

How to: Given the graph of a linear function, write an equation to represent the function.

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the y-axis to identify the y-intercept by visual inspection.
4. Substitute the slope and y-intercept into the slope-intercept form of a line equation.

Example 3.1.7: Writing an Equation for a Linear Function

Write an equation for a linear function given a graph of f shown in Figure 3.1.11.

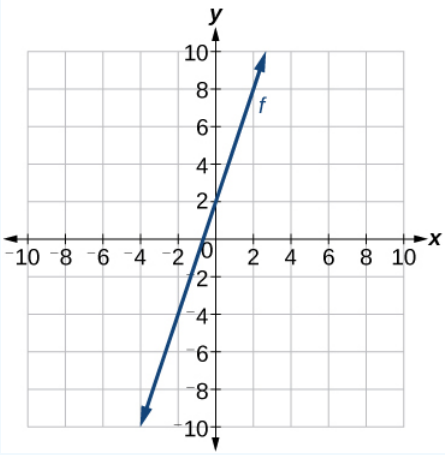


Figure 3.1.11

Solution

Identify two points on the line, such as $(0, 2)$ and $(-2, -4)$. Use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 2}{-2 - 0} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-4) &= 3(x - (-2)) \\ y + 4 &= 3(x + 2) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y+4 &= 3(x+2) \\ y+4 &= 3x+6 \\ y &= 3x+2 \end{aligned}$$

Analysis

This makes sense because we can see from Figure 3.1.12 that the line crosses the y-axis at the point $(0, 2)$, which is the y-intercept, so $b = 2$.

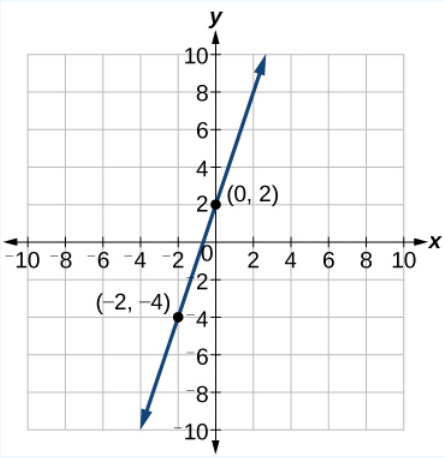


Figure 3.1.12: Graph of an increasing line with points at $(0, 2)$ and $(-2, -4)$.

Example 3.1.8: Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of $1,250 per month for the overhead, which includes his office rent. His production costs are $37.50 per item. Write a linear function C where $C(x)$ is the cost for x items produced in a given month.

Solution

The fixed cost is present every month, $1,250. The costs that can vary include the cost to produce each item, which is $37.50 for Ben. The variable cost, called the marginal cost, is represented by 37.5. The cost Ben incurs is the sum of these two costs, represented by $C(x) = 1250 + 37.5x$.

Analysis

If Ben produces 100 items in a month, his monthly cost is represented by

$$\begin{aligned} C(100) &= 1250 + 37.5(100) \\ &= 5000 \end{aligned}$$

So his monthly cost would be $5,000.

Example 3.1.9: Writing an Equation for a Linear Function Given Two Points

If f is a linear function, with $f(3) = -2$, and $f(8) = 1$, find an equation for the function in slope-intercept form.

Solution

We can write the given points using coordinates.

$$\begin{aligned} f(3) &= -2 \rightarrow (3, 2) \\ f(8) &= 1 \rightarrow (8, 1) \end{aligned}$$

We can then use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1-(-2)}{8-3} \\ &= \frac{3}{5} \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y-(-2) &= \frac{3}{5}(x-3) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y+2 &= \frac{3}{5}(x-3) \\ y+2 &= \frac{3}{5}x - \frac{9}{5} \\ y &= \frac{3}{5}x - \frac{19}{5} \end{aligned}$$

? Try It! 3.1.5

If $f(x)$ is a linear function, with $f(2) = -11$, and $f(4) = -25$, find an equation for the function in slope-intercept form.

Answer

$y = -7x + 3$

Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

How to: Given a linear function f and the initial value and rate of change, evaluate $f(c)$.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x) = mx + b$.
3. Evaluate the function at $x = c$.

Example 3.1.10: Using a Linear Function to Determine the Number of Songs in a Music Collection

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his collection as a function of time, t, the number of months. How many songs will he own in a year?

Solution

The initial value for this function is 200 because he currently owns 200 songs, so $N(0) = 200$, which means that $b = 200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m = 15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.

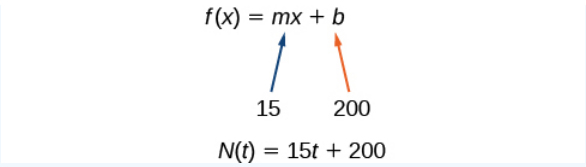


Figure 3.1.13

We can write the formula $N(t) = 15t + 200$.

With this formula, we can then predict how many songs Marcus will have in 1 year (12 months). In other words, we can evaluate the function at $t = 12$.

$$\begin{aligned} N(12) &= 15(12) + 200 \\ &= 180 + 200 \\ &= 380 \end{aligned}$$

Marcus will have 380 songs in 12 months.

Analysis

Notice that N is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

Example 3.1.11: Using a Linear Function to Calculate Salary Plus Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income, I,depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

Solution

The given information gives us two input-output pairs: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change.

$$\begin{aligned} m &= \frac{920 - 760}{5 - 3} \\ &= \frac{\$160}{2 \text{ policies}} \\ &= \$80 \text{ per policy} \end{aligned}$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy. Therefore, Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value.

$$\begin{aligned} I(n) &= 80n + b \\ 760 &= 80(3) + b \text{ when } n = 3, I(3) = 760 \\ 760 - 80(3) &= b \\ 520 &= b \end{aligned}$$

The value of b is the starting value for the function and represents Ilya's income when $n = 0$, or when no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

$$I(n) = 80n + 520$$

Our final interpretation is that Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold.

Example 3.1.12: Using Tabular Form to Write an Equation for a Linear Function

Table 3.1.1 relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

Table 3.1.1

w, number of weeks	0	2	4	6
P(w), number of rats	1000	1080	1160	1240

Solution

We can see from the table that the initial value for the number of rats is 1000, so $b = 1000$.

Rather than solving for m, we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

$$P(w) = 40w + 1000$$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2, 1080)$ and $(6, 1240)$

$$\begin{aligned} m &= \frac{1240 - 1080}{6 - 2} \\ &= \frac{160}{4} \\ &= 40 \end{aligned}$$

Is the initial value always provided in a table of values?

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x) = mx + b$, and solve for b.

Try It 3.1.6

A new plant food was introduced to a young tree to test its effect on the height of the tree. Table 3.1.2 shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$, where x is the number of months since the start of the experiment.

Table 3.1.2

x	0	2	4	8	12
$H(x)$	12.5	13.5	14.5	16.5	18.5

Answer

$H(x) = 0.5x + 12.5$

Key Equations

- slope-intercept form of a line: $f(x) = mx + b$

- slope: $m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$
- point-slope form of a line: $y - y_1 = m(x - x_1)$

Key Concepts

- The ordered pairs given by a linear function represent points on a line.
- Linear functions can be represented in words, function notation, tabular form, and graphical form.
- The rate of change of a linear function is also known as the slope.
- An equation in the slope-intercept form of a line includes the slope and the initial value of the function.
- The initial value, or y-intercept, is the output value when the input of a linear function is zero. It is the y-value of the point at which the line crosses the y-axis.
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope.
- A decreasing linear function results in a graph that slants downward from left to right and has a negative slope.
- A constant linear function results in a graph that is a horizontal line.
- Analyzing the slope within the context of a problem indicates whether a linear function is increasing, decreasing, or constant.
- The slope of a linear function can be calculated by dividing the difference between y-values by the difference in corresponding x-values of any two points on the line.
- The slope and initial value can be determined given a graph or any two points on the line.
- One type of function notation is the slope-intercept form of an equation.
- The point-slope form is useful for finding a linear equation when given the slope of a line and one point.
- The point-slope form is also convenient for finding a linear equation when given two points through which a line passes.
- The equation for a linear function can be written if the slope m and initial value b are known.
- A linear function can be used to solve real-world problems.
- A linear function can be written from tabular form.

Footnotes

1 www.chinahighlights.com/shang...glev-train.htm
2 www.cbsnews.com/8301-501465_1...ay-study-says/

Glossary

decreasing linear function

a function with a negative slope: If $f(x) = mx + b$, then $m < 0$.

increasing linear function

a function with a positive slope: If $f(x) = mx + b$, then $m > 0$.

linear function

a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

point-slope form

the equation for a line that represents a linear function of the form \(y−y_1=m(x−x_1)

slope

the ratio of the change in output values to the change in input values; a measure of the steepness of a line

slope-intercept form

the equation for a line that represents a linear function in the form $f(x) = mx + b$

y-intercept

the value of a function when the input value is zero; also known as initial value

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3.2: Modeling with Linear Functions

Identifying Steps to Model and Solve Problems

When **modeling** scenarios with linear functions and solving problems involving quantities with a **constant rate of change**, we typically follow the same problem strategies that we would use for any type of function. Let's briefly review them:

Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
Carefully read the problem to identify important information. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
Carefully read the problem to determine what we are trying to find, identify, solve, or interpret.
Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
When needed, write a formula for the function.
Solve or evaluate the function using the formula.
Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
Clearly convey your result using appropriate units, and answer in full sentences when necessary.

Building Linear Models

Now let's take a look at the student in Seattle. In her situation, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can use this information to define our variables, including units.

- Output: M, money remaining, in dollars
- Input: t, time, in weeks

So, the amount of money remaining depends on the number of weeks: $M(t)$

We can also identify the initial value and the rate of change.

- Initial Value: She saved $3,500, so $3,500 is the initial value for M.
- Rate of Change: She anticipates spending $400 each week, so –$400 per week is the rate of change, or slope.

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each week.

The **rate of change is constant**, so we can start with the linear model $M(t) = mt + b$. Then we can substitute the intercept and slope provided.

$$M(t) = mt + b$$

$$-400 \qquad 3500$$

$$M(t) = -400t + 3500$$

Figure 3.2.2

To find the x-intercept, we set the output to zero, and solve for the input.

$$\begin{aligned} 0 &= -400t + 3500 \\ t &= \frac{3500}{400} \\ &= 8.75 \end{aligned}$$

The x-intercept is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Emily will have no money left after 8.75 weeks.

When modeling any real-life scenario with functions, there is typically a limited domain over which that model will be valid—almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn't make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before she saved $3,500, but the

scenario discussed poses the question once she saved $3,500 because this is when her trip and subsequent spending starts. It is also likely that this model is not valid after the x-intercept, unless Emily will use a credit card and goes into debt. The domain represents the set of input values, so the reasonable domain for this function is $0 \le t \le 8.75$.

In the above example, we were given a written description of the situation. We followed the steps of modeling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given, and use it appropriately to build a linear model.

Using a Given Intercept to Build a Model

Some real-world problems provide the y-intercept, which is the constant or initial value. Once the y-intercept is known, the x-intercept can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is $1,000. She plans to pay $250 per month until her balance is $0. The y-intercept is the initial amount of her debt, or $1,000. The rate of change, or slope, is -$250 per month. We can then use the slope-intercept form and the given information to develop a linear model.

$$\begin{aligned} f(x) &= mx + b \\ &= -250x + 1000 \end{aligned}$$

Now we can set the function equal to 0, and solve for x to find the x-intercept.

$$\begin{aligned} 0 &= -250 + 1000 \\ 1000 &= 250x \\ 4 &= x \\ x &= 4 \end{aligned}$$

The x-intercept is the number of months it takes her to reach a balance of $0. The x-intercept is 4 months, so it will take Hannah four months to pay off her loan.

Using a Given Input and Output to Build a Model

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes instead be asked to evaluate the linear model at a given input or set the equation of the linear model equal to a specified output.

How To: Given a word problem that includes two pairs of input and output values, use the linear function to solve a problem.

1. Identify the input and output values.
2. Convert the data to two coordinate pairs.
3. Find the slope.
4. Write the linear model.
5. Use the model to make a prediction by evaluating the function at a given x-value.
6. Use the model to identify an x-value that results in a given y-value.
7. Answer the question posed.

Example 3.2.1: Using a Linear Model to Investigate a Town's Population

A town's population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. Assume this trend continues.

a. Predict the population in 2013.
b. Identify the year in which the population will reach 15,000.

Solution

The two changing quantities are the population size and time. While we could use the actual year value as the input quantity, doing so tends to lead to very cumbersome equations because the y-intercept would correspond to the year 0, more than 2000 years ago!

To make computation a little nicer, we will define our input as the number of years since 2004:

- Input: t, years since 2004
- Output: $P(t)$, the town's population

To predict the population in 2013 ($t = 9$), we would first need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the input that would provide an output of 15,000. To write an equation, we need the initial value and the rate of change, or slope.

To determine the rate of change, we will use the change in output per change in input.

$$m = \frac{\text{change in output}}{\text{change in input}} \tag{3.2.1}$$

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the y-intercept of the function. The year 2009 would correspond to $t = 5$, giving the point $(5, 8100)$.

The two coordinate pairs are $(0, 6200)$ and $(5, 8100)$. Recall that we encountered examples in which we were provided two points earlier in the chapter. We can use these values to calculate the slope.

$$\begin{aligned} m &= \frac{8100 - 6200}{5 - 0} \\ &= \frac{1900}{5} \\ &= 380 \text{ people per year} \end{aligned}$$

We already know the y-intercept of the line, so we can immediately write the equation:

$$P(t) = 380t + 6200 \tag{3.2.2}$$

To predict the population in 2013, we evaluate our function at $t = 9$.

$$\begin{aligned} P(9) &= 380(9) + 6,200 \\ &= 9,620 \end{aligned}$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.

$$\begin{aligned} 15000 &= 380t + 6200 \\ 8800 &= 380t \\ t &\approx 23.158 \end{aligned}$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

Exercise 3.2.1A

A company sells doughnuts. They incur a fixed cost of $25,000 for rent, insurance, and other expenses. It costs $0.25 to produce each doughnut.

a. Write a linear model to represent the cost C of the company as a function of x, the number of doughnuts produced.
b. Find and interpret the y-intercept.

Solution

a. $C(x) = 0.25x + 25,000$ b. The y-intercept is $(0, 25,000)$. If the company does not produce a single doughnut, they still incur a cost of $25,000.

Exercise 3.2.1B

A city's population has been growing linearly. In 2008, the population was 28,200. By 2012, the population was 36,800. Assume this trend continues.

a. Predict the population in 2014.
b. Identify the year in which the population will reach 54,000.

Solution

a. 41,100 b. 2020

Using a Diagram to Model a Problem

It is useful for many real-world applications to draw a picture to gain a sense of how the variables representing the input and output may be used to answer a question. To draw the picture, first consider what the problem is asking for. Then, determine the input and the output. The diagram should relate the variables. Often, geometrical shapes or figures are drawn. Distances are often traced out. If a right triangle is sketched, the Pythagorean Theorem relates the sides. If a rectangle is sketched, labeling width and height is helpful.

Example 3.2.2: Using a Diagram to Model Distance Walked

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio that has a range of 2 miles. How long after they start walking will they fall out of radio contact?

Solution

In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question:

"How long will it take them to be 2 miles apart?"

In this problem, our changing quantities are time and position, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define our input and output variables.

- Input: t, time in hours.
- Output: $A(t)$, distance in miles, and $E(t)$, distance in miles

Because it is not obvious how to define our output variable, we'll start by drawing a picture such as Figure 3.2.3.

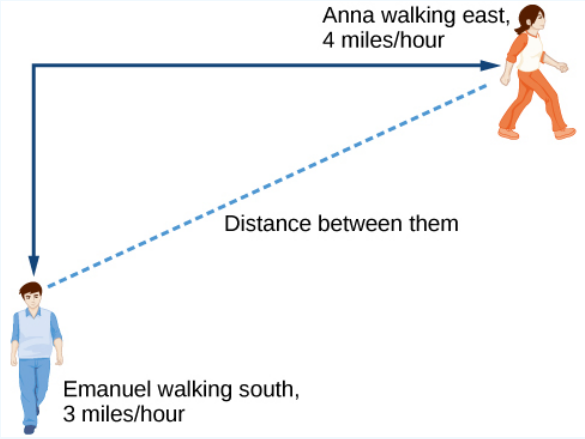


Figure 3.2.3

- Initial Value: They both start at the same intersection so when $t = 0$, the distance traveled by each person should also be 0. Thus the initial value for each is 0.
- Rate of Change: Anna is walking 4 miles per hour and Emanuel is walking 3 miles per hour, which are both rates of change. The slope for A is 4 and the slope for E is 3.

Using those values, we can write formulas for the distance each person has walked.

$$A(t) = 4t \tag{3.2.3}$$

$$E(t) = 3t \tag{3.2.4}$$

For this problem, the distances from the starting point are important. To notate these, we can define a coordinate system, identifying the "starting point" at the intersection where they both started. Then we can use the variable, A, which we introduced above, to represent Anna's position, and define it to be a measurement from the starting point in the eastward direction. Likewise, can use the variable, E, to represent Emanuel's position, measured from the starting point in the southward direction. Note that in defining the coordinate system, we specified both the starting point of the measurement and the direction of measure.

We can then define a third variable, D, to be the measurement of the distance between Anna and Emanuel. Showing the variables on the diagram is often helpful, as we can see from Figure 3.2.4.

Recall that we need to know how long it takes for D, the distance between them, to equal 2 miles. Notice that for any given input t, the outputs $A(t)$, $E(t)$, and $D(t)$ represent distances.

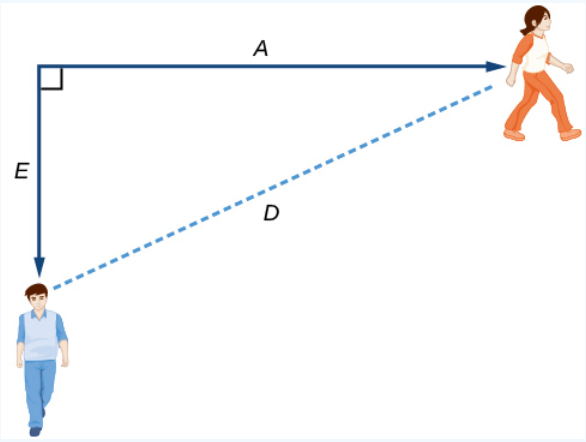


Figure 3.2.4: We can use the Pythagorean Theorem because we have drawn a right angle.

Using the Pythagorean Theorem, we get:

$$\begin{aligned} d(t)^2 &= A(t)^2 + E(t)^2 \\ &= (4t)^2 + (3t)^2 \\ &= 16t^2 + 9t^2 \\ &= 25t^2 \\ D(t) &= \pm\sqrt{25t^2} \qquad \text{Solve for } D(t) \text{ using the square root} \\ &= \pm 5|t| \end{aligned}$$

In this scenario we are considering only positive values of t, so our distance $D(t)$ will always be positive. We can simplify this answer to $D(t) = 5t$. This means that the distance between Anna and Emanuel is also a linear function. Because D is a linear function, we can now answer the question of when the distance between them will reach 2 miles. We will set the output $D(t) = 2$ and solve for t.

$$\begin{aligned} D(t) &= 2 \\ 5t &= 2 \\ t &= \frac{2}{5} = 0.4 \end{aligned}$$

They will fall out of radio contact in 0.4 hours, or 24 minutes.

Should I draw diagrams when given information based on a geometric shape?

Yes. Sketch the figure and label the quantities and unknowns on the sketch.

Example 3.2.3: Using a Diagram to Model Distance between Cities

There is a straight road leading from the town of Westborough to Agritown 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

Solution

It might help here to draw a picture of the situation. See Figure 3.2.5. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts Agritown at coordinates $(30, 10)$, and Eastborough at $(20, 0)$.

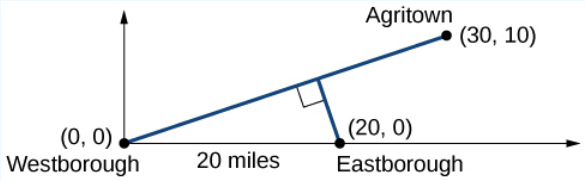


Figure 3.2.5

Using this point along with the origin, we can find the slope of the line from Westborough to Agritown:

$$m = \frac{10-0}{30-0} = \frac{1}{3} \tag{3.2.5}$$

The equation of the road from Westborough to Agritown would be

$$W(x) = \frac{1}{3}x \tag{3.2.6}$$

From this, we can determine the perpendicular road to Eastborough will have slope $m = -3$. Because the town of Eastborough is at the point $(20, 0)$, we can find the equation:

$$\begin{aligned} E(x) &= -3x + b \\ 0 &= -3(20) + b \qquad \text{Substitute in } (20, 0) \\ b &= 60 \\ E(x) &= -3x + 60 \end{aligned}$$

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,

$$\begin{aligned} \frac{1}{3}x &= -3x + 60 \\ \frac{10}{3}x &= 60 \\ 10x &= 180 \\ x &= 18 \qquad \text{Substituting this back into } W(x) \\ y &= W(18) \\ &= \frac{1}{3}(18) \\ &= 6 \end{aligned}$$

The roads intersect at the point $(18, 6)$. Using the distance formula, we can now find the distance from Westborough to the junction.

$$\begin{aligned} \text{distance} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(18-0)^2 + (6-0)^2} \\ &\approx 18.743 \text{miles} \end{aligned}$$

Analysis

One nice use of linear models is to take advantage of the fact that the graphs of these functions are lines. This means real-world applications discussing maps need linear functions to model the distances between reference points.

Exercise 3.2.2

There is a straight road leading from the town of Timpson to Ashburn 60 miles east and 12 miles north. Partway down the road, it junctions with a second road, perpendicular to the first, leading to the town of Garrison. If the town of Garrison is located 22 miles directly east of the town of Timpson, how far is the road junction from Timpson?

Solution

21.15 miles

Key Concepts

- We can use the same problem strategies that we would use for any type of function.
- When modeling and solving a problem, identify the variables and look for key values, including the slope and y-intercept.
- Draw a diagram, where appropriate.
- Check for reasonableness of the answer.
- Linear models may be built by identifying or calculating the slope and using the y-intercept.
- The x-intercept may be found by setting $y = 0$, which is setting the expression $mx + b$ equal to 0.
- The point of intersection of a system of linear equations is the point where the x- and y-values are the same.
- A graph of the system may be used to identify the points where one line falls below (or above) the other line.

Footnotes

1 Rates retrieved Aug 2, 2010 from www.budgettruck.com and http://www.uhaul.com/

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3.3: Absolute Value Functions

Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the **absolute value** function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

Absolute Value Function

The absolute value function can be defined as a piecewise function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{3.3.1}$$

Example 3.3.1: Determine a Number within a Prescribed Distance

Describe all values x within or including a distance of 4 from the number 5.

Solution

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as the one in , to represent the condition to be satisfied.

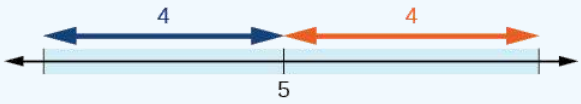


Figure 3.3.2: Number line describing the difference of the distance of 4 away from 5.

The distance from x to 5 can be represented using the absolute value as $|x-5|$. We want the values of x that satisfy the condition $|x-5| \leq 4$.

Analysis

Note that

$$\begin{aligned} -4 \leq x-5 \qquad & x-5 \leq 4 \\ 1 \leq x \qquad & x \leq 9 \end{aligned}$$

So $|x-5| \leq 4$ is equivalent to $1 \leq x \leq 9$.

However, mathematicians generally prefer absolute value notation.

Exercise 3.3.1

Describe all values x within a distance of 3 from the number 2.

Answer

$|x-2| \leq 3$

Example 3.3.2: Resistance of a Resistor

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often ±1%, ±5%, or ±10%.

Suppose we have a resistor rated at 680 ohms, ±5%. Use the absolute value function to express the range of possible values of the actual resistance.

Solution

5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,

$$|R-680| \leq 34$$

Exercise 3.3.2

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

Answer

Using the variable p for passing, $|p-80| \leq 20$

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in Figure 3.3.3.

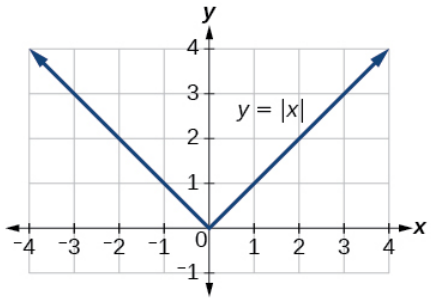


Figure 3.3.3: Graph of an absolute function.

Figure 3.3.3 shows the graph of $y = 2|x-3|+4$. The graph of $y = |x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3, 4)$ for this transformed function.

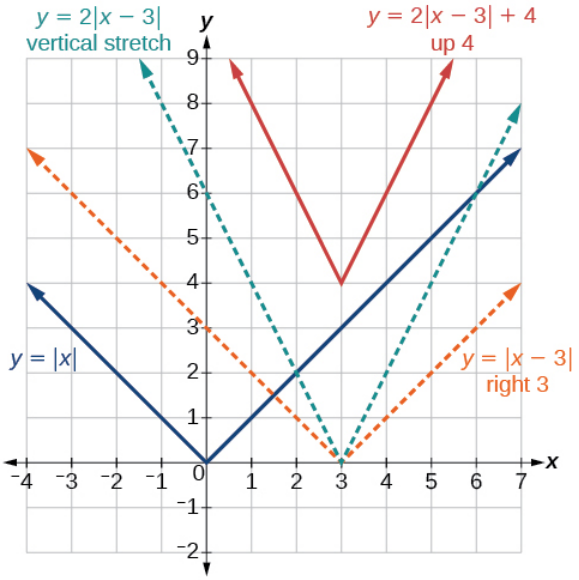


Figure 3.3.4: Graph of the different types of transformations for an absolute function.

Example 3.3.3: Writing an Equation for an Absolute Value Function

Write an equation for the function graphed in Figure 3.3.5.

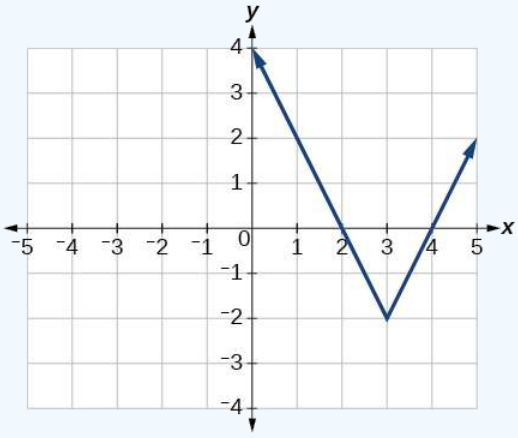


Figure 3.3.5: Graph of an absolute function.

Solution

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See Figure 3.3.6.

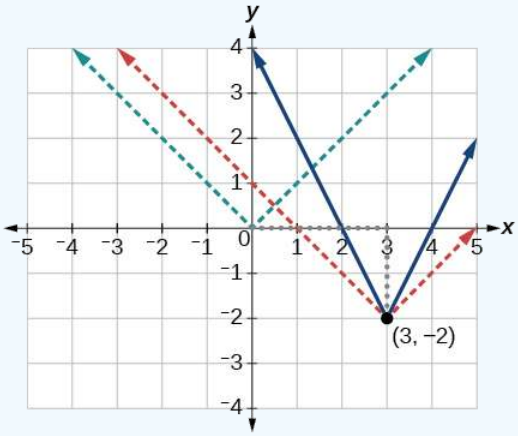


Figure 3.3.6: Graph of two transformations for an absolute function at $(3, -2)$.

We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in Figure 3.3.7.

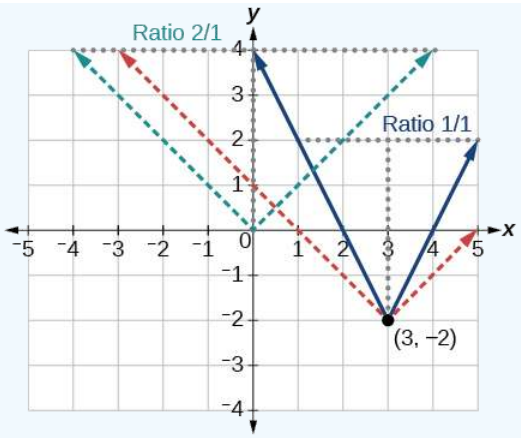


Figure 3.3.7: Graph of two transformations for an absolute function at $(3, -2)$ and the ratios between the two different transformations.

From this information we can write the equation

$$\begin{aligned} f(x) &= 2|x-3|-2, && \text{treating the stretch as a vertial stretch, or} \\ f(x) &= |2(x-3)|-2, && \text{treating the stretch as a horizontal compression.} \end{aligned}$$

Analysis

Note that these equations are algebraically equivalent—the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression.

Q & A

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

Answer

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for x and $f(x)$.

$$f(x) = a|x-3|-2$$

Now substituting in the point $(1, 2)$

$$\begin{aligned} 2 &= a|1-3|-2 \\ 4 &= 2a \\ a &= 2 \end{aligned}$$

Exercise 3.3.3

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

Answer

$f(x) = -|x+2|+3$

Q & A

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

Answer

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (Figure 3.3.8).

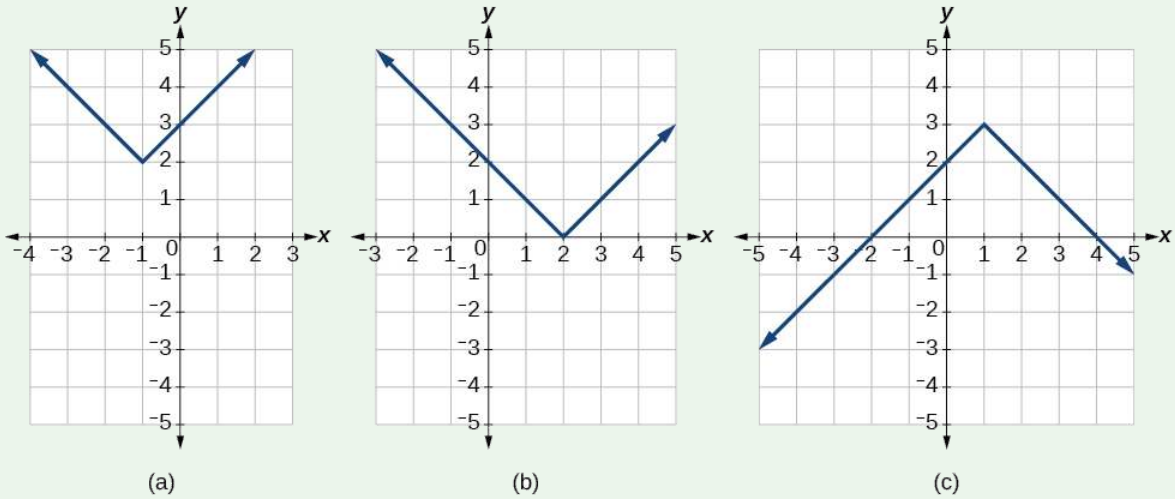


Figure 3.3.8: (a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

Solving an Absolute Value Equation

Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8 = |2x - 6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$2x - 6 = 8 \quad \text{or} \quad 2x - 6 = -8$$

$$\begin{aligned} 2x &= 14 & 2x &= -2 \\ x &= 7 & x &= -1 \end{aligned}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

An **absolute value equation** is an equation in which the unknown variable appears in absolute value bars. For example,

$$\begin{aligned} |x| &= 4, \\ |2x - 1| &= 3, \\ |5x + 2| - 4 &= 9. \end{aligned}$$

Solutions to Absolute Value Equations

For real numbers A and B, an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

How To ...

Given the formula for an absolute value function, find the horizontal intercepts of its graph.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $-A = B$, assuming $B > 0$.
3. Solve for x.

Example 3.3.4: Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x+1| - 7$, find the values of x such that $f(x) = 0$.

Solution

$$
\begin{aligned}
0 &= |4x+1| - 7 && && \text{Substitute 0 for f(x).}\\
7 &= |4x+1| && && \text{Isolate the absolute value on one side of the equation.}\\
7 &= 4x+1 && \text{or } -7 = 4x+1 && \text{Break into two separate equations and solve.}\\
6 &= 4x && -8 = 4x &&\\
x &= \frac{6}{4} = 1.5 && x = \frac{-8}{4} = -2 &&
\end{aligned}
$$

The function outputs 0 when $x = 1.5$ or $x = -2$ (Figure 3.3.9).

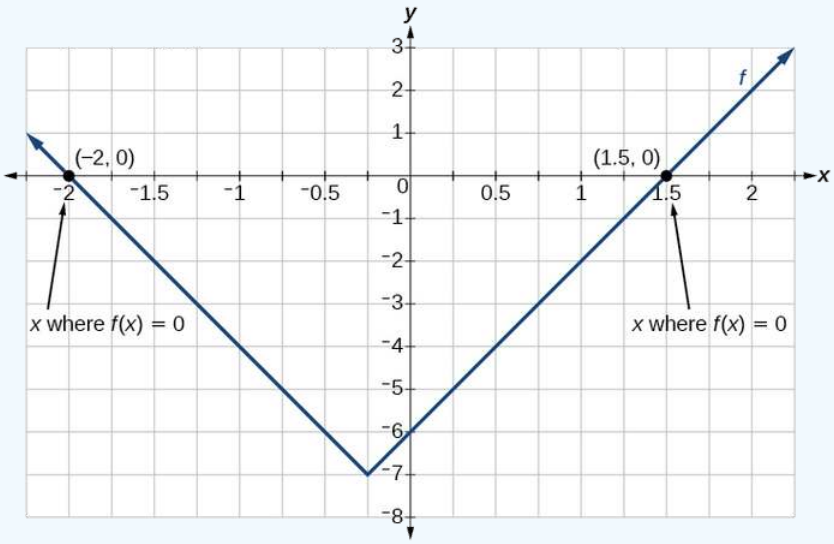


Figure 3.3.9: Graph of an absolute function with x-intercepts at -2 and 1.5.

Exercise 3.3.4

For the function $f(x) = |2x-1| - 3$, find the values of x such that $f(x) = 0$.

Solution

$x = -1$ or $x = 2$

Q & A

Should we always expect two answers when solving $|A| = B$?

Answer

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x-5| = 1$.

How To ...

Given an absolute value equation, solve it.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $A = -B$.
3. Solve for x.

Example 3.3.5: Solving an Absolute Value Equation

Solve $1 = 4|x-2|+2$.

Solution

Isolating the absolute value on one side of the equation gives the following.

$$\begin{aligned} 1 &= 4|x-2|+2 \\ -1 &= 4|x-2| \\ -\frac{1}{4} &= |x-2| \end{aligned}$$

The absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value. At this point, we notice that this equation has no solutions.

Q & A

In Example 3.3.5, if $f(x) = 1$ and $g(x) = 4|x-2|+2$ were graphed on the same set of axes, would the graphs intersect?

Answer

No. The graphs of f and g would not intersect, as shown in Figure 3.3.10. This confirms, graphically, that the equation $1 = 4|x-2|+2$ has no solution.

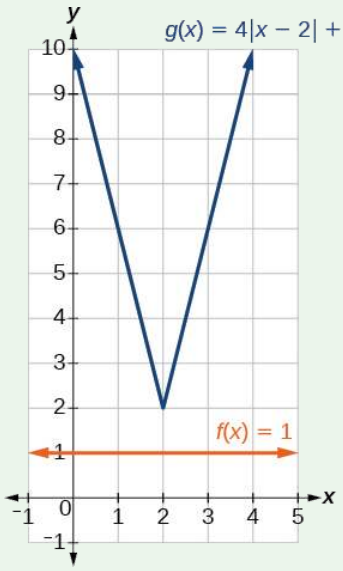


Figure 3.3.10: Graph of $g(x) = 4|x-2|+2$ and $f(x) = 1$.

Find where the graph of the function $f(x) = -|x+2|+3$ intersects the horizontal and vertical axes.

$f(0) = 1$, so the graph intersects the vertical axis at $(0, 1)$. $f(x) = 0$ when $x = -5$ and $x = 1$ so the graph intersects the horizontal axis at $(-5, 0)$ and $(1, 0)$.

Solving an Absolute Value Inequality

Absolute value equations may not always involve equalities. Instead, we may need to solve an equation within a range of values. We would use an **absolute value inequality** to solve such an equation. An absolute value inequality is an equation of the form

$$|A| < B,\ |A| \leq B, |A| > B,$$

or

$$|A| \geq B,$$

where an expression A (and possibly but not usually B) depends on a variable x. Solving the inequality means finding the set of all x that satisfy the inequality. Usually this set will be an interval or the union of two intervals.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two functions. The advantage of the algebraic approach is it yields solutions that may be difficult to read from the graph.

For example, we know that all numbers within 200 units of 0 may be expressed as

$$|x| < 200$$

or

$$-200 < x < 200$$

Suppose we want to know all possible returns on an investment if we could earn some amount of money within $200 of $600. We can solve algebraically for the set of values x such that the distance between x and 600 is less than 200. We represent the distance between x and 600 as $|x-600|$.

$$|x-600| < 200 \tag{3.3.2}$$

or

$$-200 < x-600 < 200 \tag{3.3.3}$$

$$-200+600 < x-600+600 < 200+600$$

$$400 < x < 800$$

This means our returns would be between $400 and $800.

Sometimes an absolute value inequality problem will be presented to us in terms of a shifted and/or stretched or compressed absolute value function, where we must determine for which values of the input the function's output will be negative or positive.

How To ...

Given an absolute value inequality of the form $|x-A| \leq B$ for real numbers a and b where b is positive, solve the absolute value inequality algebraically.

1. Find boundary points by solving $|x-A| = B$.
2. Test intervals created by the boundary points to determine where $|x-A| \leq B$.
3. Write the interval or union of intervals satisfying the inequality in interval, inequality, or set-builder notation.

Example 3.3.6: Solving an Absolute Value Inequality

Solve $|x-5| \leq 4$.

Solution

With both approaches, we will need to know first where the corresponding equality is true. In this case we first will find where $|x-5| = 4$. We do this because the absolute value is a function with no breaks, so the only way the function values can switch from being less than 4 to being greater than 4 is by passing through where the values equal 4. Solve $|x-5| = 4$.

$$\begin{aligned} x-5 &= 4 \quad \text{or} \quad x = 9 \\ x-5 &= -4 \quad\quad\;\; x = 1 \end{aligned}$$

After determining that the absolute value is equal to 4 at $x = 1$ and $x = 9$, we know the graph can change only from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals:

$$x < 1, \; 1 < x < 9, \text{ and } x > 9.$$

To determine when the function is less than 4, we could choose a value in each interval and see if the output is less than or greater than 4, as shown in Table 3.3.1.

Table 3.3.1

Interval test x	$f(x)$	<4 or >4	
$x<1$	0	$\lvert 0-5\rvert=5$	Greater than
$1<x<9$	6	$\lvert 6-5\rvert=1$	Less than
$x>9$	11	$\lvert 11-5\rvert=6$	Greater than

Because $1\leq x\leq 9$ is the only interval in which the output at the test value is less than 4, we can conclude that the solution to $|x-5|\leq 4$ is $1\leq x\leq 9$, or $[1,9]$.

To use a graph, we can sketch the function $f(x)=|x-5|$. To help us see where the outputs are 4, the line $g(x)=4$ could also be sketched as in Figure 3.3.11.

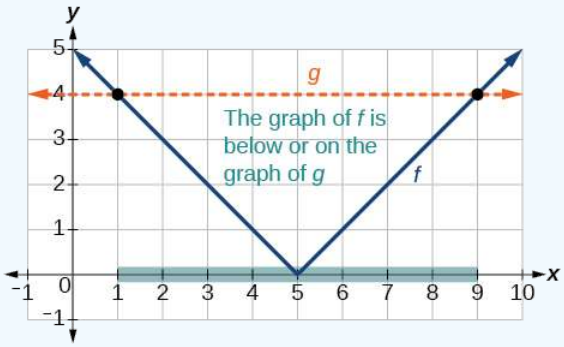


Figure 3.3.11: Graph to find the points satisfying an absolute value inequality.

We can see the following:

- The output values of the absolute value are equal to 4 at $x=1$ and $x=9$.
- The graph of f is below the graph of g on $1<x<9$. This means the output values of $f(x)$ are less than the output values of $g(x)$.
- The absolute value is less than or equal to 4 between these two points, when $1\leq x\leq 9$. In interval notation, this would be the interval $[1,9]$.

Analysis

For absolute value inequalities,

$$\begin{array}{ll} |x-A|<C, & |x-A|>C, \\ -C<x-A<C, & x-A<-C \text{ or } x-A>C. \end{array}$$

The $<$ or $>$ symbol may be replaced by $\leq$ or $\geq$.

So, for this example, we could use this alternative approach.

$$\begin{array}{ll} |x-5|\leq 4 & \\ -4\leq x-5\leq 4 & \text{Rewrite by removing the absolute value bars.} \\ -4+5\leq x-5+5\leq 4+5 & \text{Isolate the x.} \\ 1\leq x\leq 9 & \end{array}$$

Exercise 3.3.5

Solve $|x+2|\leq 6$.

Answer

$-8\leq x\leq 4$

How To ...

Given an absolute value function, solve for the set of inputs where the output is positive (or negative).

1. Set the function equal to zero, and solve for the boundary points of the solution set.
2. Use test points or a graph to determine where the function's output is positive or negative.

Example 3.3.7: Using a Graphical Approach to Solve Absolute Value Inequalities

Given the function $f(x) = -\frac{1}{2}|4x-5|+3$, determine the x-values for which the function values are negative.

Solution

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x-5|+3 < 0$. We begin by isolating the absolute value.

$$\begin{aligned} -\frac{1}{2}|4x-5| &< -3 \quad \text{Multiply both sides by } -2, \text{ and reverse the inequality.} \\ |4x-5| &> 6 \end{aligned}$$

Next we solve for the equality $|4x-5| = 6$.

$$\begin{aligned} 4x-5 &= 6 \qquad 4x-5 = -6 \\ 4x-6 &= 6 \end{aligned}$$

or

$$\begin{aligned} 4x &= -1 \\ x &= \frac{11}{4} \qquad x = -\frac{1}{4} \end{aligned}$$

Now, we can examine the graph of f to observe where the output is negative. We will observe where the branches are below the x-axis. Notice that it is not even important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$ and that the graph has been reflected vertically. See Figure 3.3.12

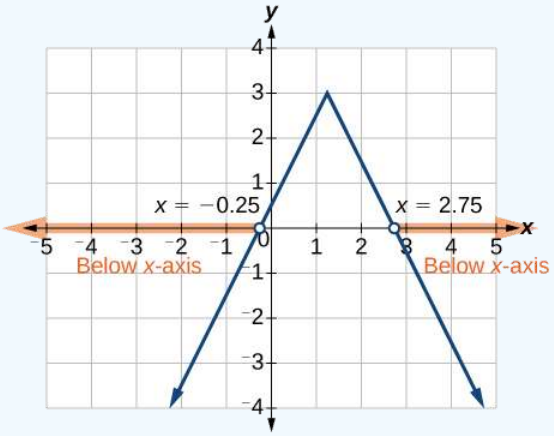


Figure 3.3.12: Graph of an absolute function with x-intercepts at -0.25 and 2.75.]

We observe that the graph of the function is below the x-axis left of $x = -\frac{1}{4}$ and right of $x = \frac{11}{4}$. This means the function values are negative to the left of the first horizontal intercept at $x = -\frac{1}{4}$, and negative to the right of the second intercept at $x = \frac{11}{4}$. This gives us the solution to the inequality.

$$x < -\frac{1}{4} \text{ or } x > 1\frac{1}{4}$$

In interval notation, this would be $(-\infty, -0.25) \cup (2.75, \infty)$.

Exercise 3.3.6

Solve $-2|k-4| \leq -6$.

Answer

$k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty, 1] \cup [7, \infty)$

Key Concepts

- The absolute value function is commonly used to measure distances between points.
- Applied problems, such as ranges of possible values, can also be solved using the absolute value function.
- The graph of the absolute value function resembles a letter V. It has a corner point at which the graph changes direction.
- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable.
- An absolute value equation may have one solution, two solutions, or no solutions.
- An absolute value inequality is similar to an absolute value equation but takes the form | A |<B, | A |≤B, | A |>B, or | A |≥B.It can be solved by determining the boundaries of the solution set and then testing which segments are in the set.
- Absolute value inequalities can also be solved graphically.

Glossary

absolute value equation
an equation of the form $|A| = B$, with $B \geq 0$; it will have solutions when $A = B$ or $A = -B$

absolute value inequality
a relationship in the form $|A| < B$, $|A| \leq B$, $|A| > B$, or $|A| \geq B$

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CHAPTER OVERVIEW

4: Polynomial Functions

In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.

4.1: Quadratic Functions
4.2: Power Functions and Polynomial Functions
4.3: Graphs of Polynomial Functions
4.4: Dividing Polynomials
4.5: Zeros of Polynomial Functions

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4.1: Quadratic Functions

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the **minimum value** of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 4.1.2.

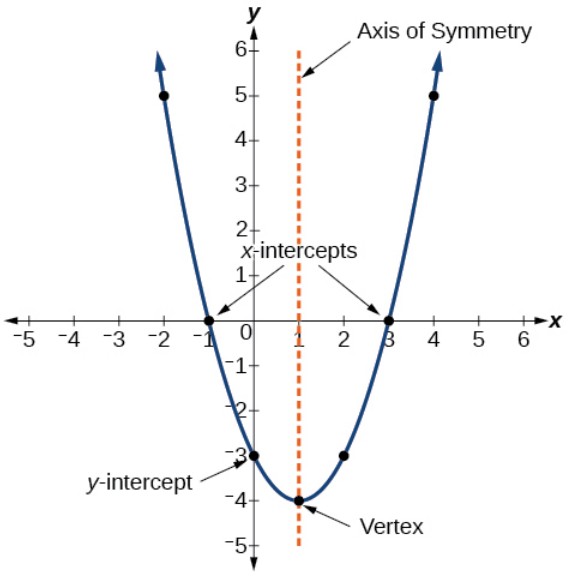


Figure 4.1.2: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are.

The y-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of x at which $y = 0$.

Example 4.1.1: Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure 4.1.3.

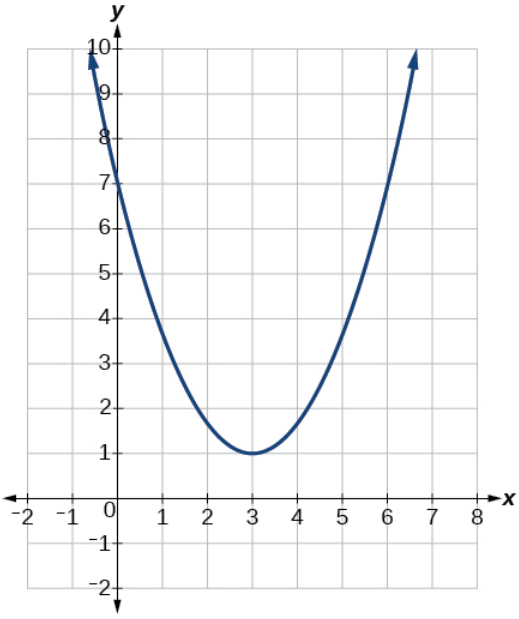


Figure 4.1.3.

Solution

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at $(0, 7)$ so this is the y-intercept.

Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The **general form of a quadratic function** presents the function in the form

$$f(x) = ax^2 + bx + c \tag{4.1.1}$$

where a, b, and c are real numbers and $a \neq 0$. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find the value of x halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure 4.1.4 represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

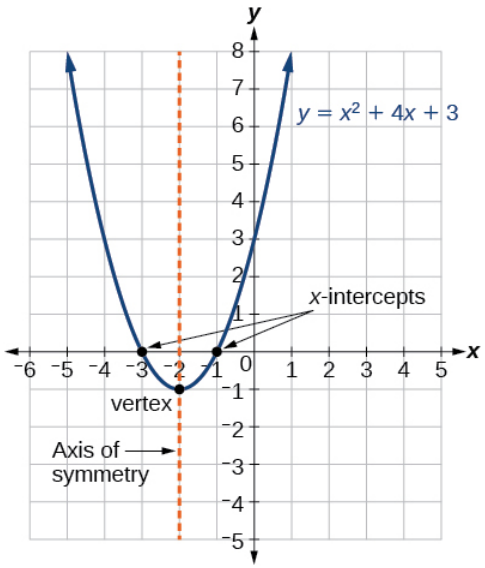


Figure 4.1.4: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are for the function $y = x^2 + 4x + 3$.

The **standard form of a quadratic function** presents the function in the form

$$f(x) = a(x - h)^2 + k \tag{4.1.2}$$

where (h, k) is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure 4.1.5 represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

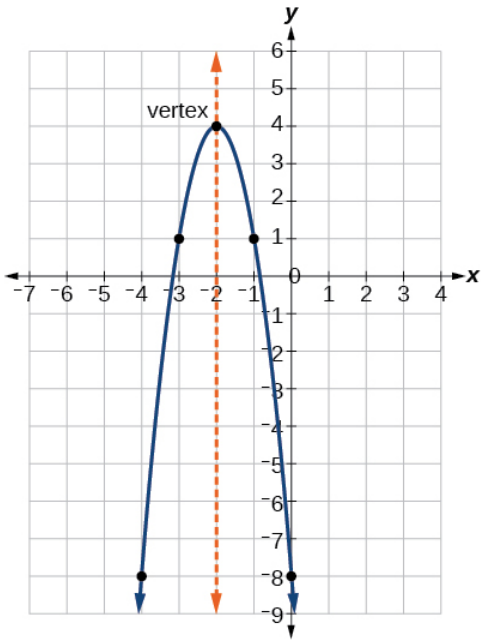


Figure 4.1.5: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are for the function $y = -3(x+2)^2 + 4$.

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. Figure 4.1.6 is the graph of this basic function.

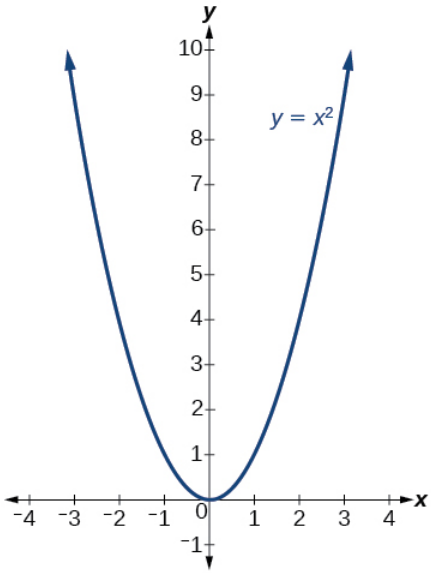


Figure 4.1.6: Graph of $y = x^2$.

If $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In Figure 4.1.5, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In Figure 4.1.5, $h < 0$, so the graph is shifted 2 units to the left. The magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In Figure 4.1.5, $|a| > 1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$\begin{aligned} a(x-h)^2+k &= ax^2+bx+c \\ ax^2-2ahx+(ah^2+k) &= ax^2+bx+c \end{aligned}$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah=b, \text{ so } h=-\frac{b}{2a}.$$

This is the **axis of symmetry** we defined earlier. Setting the constant terms equal:

$$\begin{aligned} ah^2+k &= c \\ k &= c-ah^2 \\ &= c-a\cdot\left(-\frac{b}{2a}\right)^2 \\ &= c-\frac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h)=k$.

Definitions: Forms of Quadratic Functions

A quadratic function is a function of degree two. The graph of a *quadratic function* is a parabola.

- The *general form* of a quadratic function is $f(x)=ax^2+bx+c$ where a, b, and c are real numbers and $a\neq 0$.
- The standard form of a quadratic function is $f(x)=a(x-h)^2+k$.
- The vertex (h,k) is located at

$$h=-\frac{b}{2a},\ k=f(h)=f(\frac{-b}{2a}). \tag{4.1.3}$$

HOWTO: Write a quadratic function in a general form

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is h. Identify the vertical shift of the parabola; this value is k.
2. Substitute the values of the horizontal and vertical shift for h and k. in the function $f(x)=a(x-h)^2+k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for x and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. If the parabola opens up, $a>0$. If the parabola opens down, $a<0$ since this means the graph was reflected about the x-axis.
6. Expand and simplify to write in general form.

Example 4.1.2: Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in Figure 4.1.7 as a transformation of $f(x)=x^2$, and then expand the formula, and simplify terms to write the equation in general form.

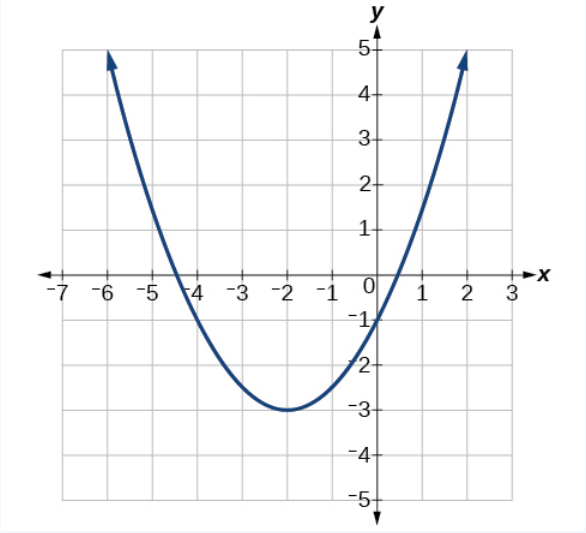


Figure 4.1.7: Graph of a parabola with its vertex at $(-2, -3)$.

Solution

We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$.

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$-1 = a(0+2)^2 - 3 \tag{4.1.4}$$
$$2 = 4a \tag{4.1.5}$$
$$a = \frac{1}{2} \tag{4.1.6}$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x+2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$g(x) = \frac{1}{2}(x+2)^2 - 3 \tag{4.1.7}$$
$$= \frac{1}{2}(x+2)(x+2) - 3 \tag{4.1.8}$$
$$= \frac{1}{2}(x^2 + 4x + 4) - 3 \tag{4.1.9}$$
$$= \frac{1}{2}x^2 + 2x + 2 - 3 \tag{4.1.10}$$
$$= \frac{1}{2}x^2 + 2x - 1 \tag{4.1.11}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Analysis

We can check our work using the table feature on a graphing utility. First enter $\mathrm{Y1} = \frac{1}{2}(\mathrm{x}+2)^2 - 3$. Next, select TBLSET, then use $\mathrm{TblStart} = -6$ and $\Delta\mathrm{Tbl} = 2$, and select TABLE. See Table 4.1.1

Table 4.1.1

x	-6	-4	-2	0	2
y	-5	-1	-3	-1	5

The ordered pairs in the table correspond to points on the graph.

Exercise 4.1.2

A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 4.1.8. Find an equation for the path of the ball. Does the shooter make the basket?

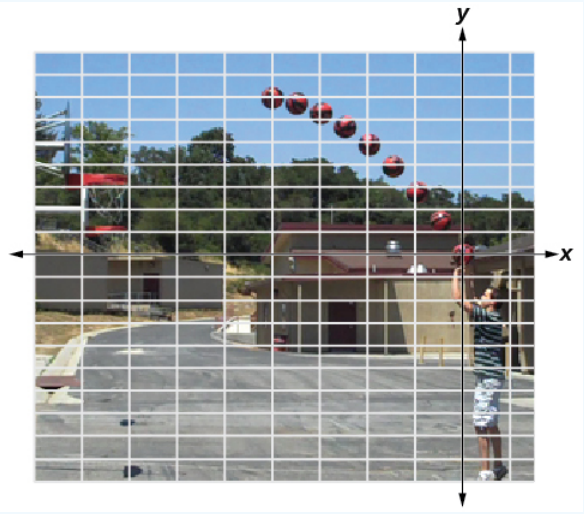


***Figure* 4.1.8:** *Stop motioned picture of a boy throwing a basketball into a hoop to show the parabolic curve it makes. (credit: modification of work by Dan Meyer)*

Answer

The path passes through the origin and has vertex at $(-4, 7)$, so $h(x) = -\frac{7}{16}(x+4)^2 + 7$. To make the shot, $h(-7.5)$ would need to be about 4 but $h(-7.5) \approx 1.64$; he doesn't make it.

How To: Given a quadratic function in general form, find the vertex of the parabola.

1. Identify a, b, and c.
2. Find h, the x-coordinate of the vertex, by substituting a and b into $h = -\frac{b}{2a}$.
3. Find k, the y-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

Example 4.1.3: Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} \tag{4.1.12}$$
$$= -\frac{-6}{2(2)} \tag{4.1.13}$$
$$= \frac{6}{4} \tag{4.1.14}$$
$$= \frac{3}{2} \tag{4.1.15}$$

The vertical coordinate of the vertex will be at

$$k = f(h) \tag{4.1.16}$$
$$= f\left(\frac{3}{2}\right) \tag{4.1.17}$$
$$= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \tag{4.1.18}$$
$$= \frac{5}{2} \tag{4.1.19}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.

$$f(x) = ax^2 + bx + c \tag{4.1.20}$$
$$f(x) = 2x^2 - 6x + 7$$

Using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2} \tag{4.1.21}$$

Analysis

One reason we may want to identify the vertex of the parabola is that this point will inform us what the maximum or minimum value of the function is, (k),and where it occurs, (h).

Exercise 4.1.3

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

Answer

$g(x) = x^2 - 6x + 13$ in general form; $g(x) = (x-3)^2 + 4$ in standard form.

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

Definition: Domain and Range of a Quadratic Function

The domain of any **quadratic function** is all real numbers.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $f(x) \geq f\left(-\frac{b}{2a}\right)$, or $[f(-\frac{b}{2a}), \infty)$; the range of a quadratic function written in general form with a negative a value is $f(x) \leq f(-\frac{b}{2a})$, or $(-\infty, f(-\frac{b}{2a})]$.

The range of a quadratic function written in standard form $f(x) = a(x-h)^2 + k$ with a positive a value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \leq k$.

How To: Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, k.
4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $(-\infty, k]$.

Example 4.1.4: Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$h = -\frac{b}{2a} \tag{4.1.22}$$
$$= -\frac{9}{2(-5)} \tag{4.1.23}$$
$$= \frac{9}{10} \tag{4.1.24}$$

The maximum value is given by $f(h)$.

$$f(\frac{9}{10}) = 5(\frac{9}{10})^2 + 9(\frac{9}{10}) - 1 \tag{4.1.25}$$
$$= \frac{61}{20} \tag{4.1.26}$$

The range is $f(x) \leq \frac{61}{20}$, or $\left(-\infty, \frac{61}{20}\right]$.

Exercise 4.1.4

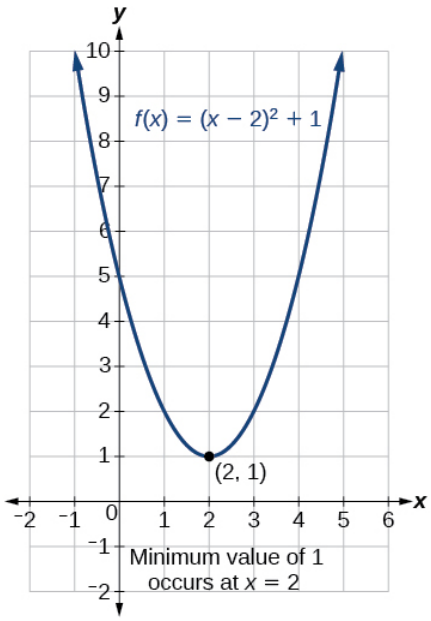
Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Answer

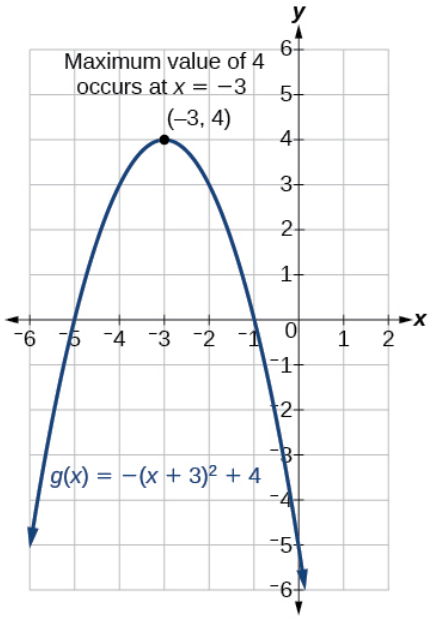
The domain is all real numbers. The range is $f(x) \geq \frac{8}{11}$, or $\left[\frac{8}{11}, \infty\right)$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the **parabola**. We can see the maximum and minimum values in Figure 4.1.9.



(a)



(b)

Figure 4.1.9: Minimum and maximum of two quadratic functions.

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Example 4.1.5: Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
b. What dimensions should she make her garden to maximize the enclosed area?

Solution

Let's use a diagram such as Figure 4.1.10 to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

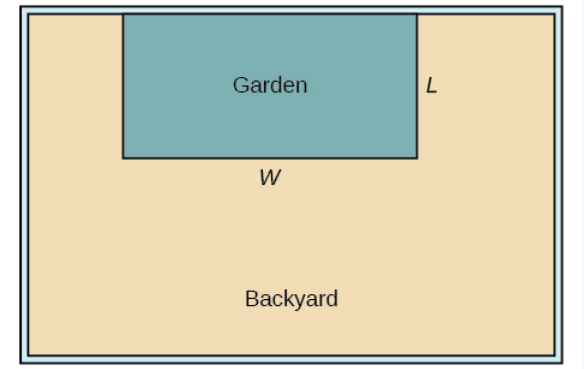


Figure 4.1.10: Diagram of the garden and the backyard.

a. We know we have only 80 feet of fence available, and $L+W+L=80$, or more simply, $2L+W=80$. This allows us to represent the width, W, in terms of L.

$$W=80-2L \tag{4.1.27}$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80-2L) \tag{4.1.28}$$
$$A(L) = 80L-2L^2 \tag{4.1.29}$$

This formula represents the area of the fence in terms of the variable length L. The function, written in general form, is

$$A(L)=-2L^2+80L \tag{4.1.30}$$

.

The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a=-2$, $b=80$, and $c=0$.

To find the vertex:

$$h=-\frac{80}{2(-2)} \qquad k=A(20) \tag{4.1.31}$$
$$=20 \qquad \text{and} \qquad =80(20)-2(20)^2 \tag{4.1.32}$$
$$=800 \tag{4.1.33}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L=20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 4.1.11.

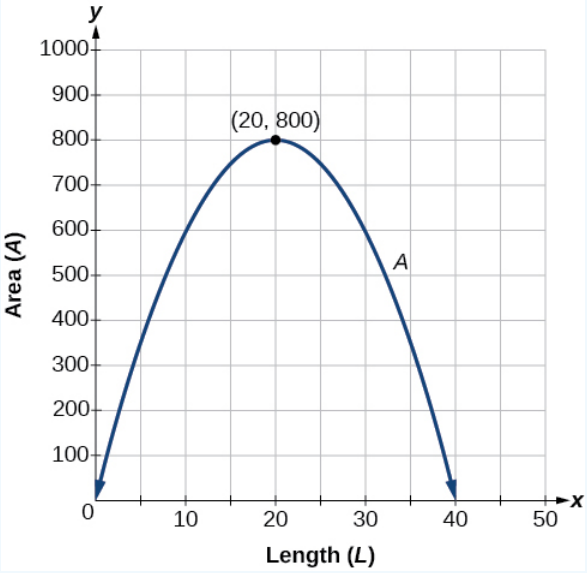


Figure 4.1.11: Graph of the parabolic function $A(L) = -2L^2 + 80L$

How To: Given an application involving revenue, use a quadratic equation to find the maximum.

1. Write a quadratic equation for revenue.
2. Find the vertex of the quadratic equation.
3. Determine the y-value of the vertex.

Example 4.1.6: Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30} \tag{4.1.34}$$

$$= -\frac{5,000}{2} \tag{4.1.35}$$

$$= -2,500 \tag{4.1.36}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$Q = -2500p + b \quad \text{Substitute in the point } Q = 84,000 \text{ and } p = 30 \tag{4.1.37}$$

$$84,000 = -2500(30) + b \quad \text{Solve for } b \tag{4.1.38}$$

$$b = 159,000 \tag{4.1.39}$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\text{Revenue} = pQ \tag{4.1.40}$$
$$\text{Revenue} = p(-2,500p + 159,000) \tag{4.1.41}$$
$$\text{Revenue} = -2,500p^2 + 159,000p \tag{4.1.42}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$h = -\frac{159,000}{2(-2,500)} \tag{4.1.43}$$
$$= 31.8 \tag{4.1.44}$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\text{maximum revenue} = -2,500(31.8)^2 + 159,000(31.8) \tag{4.1.45}$$
$$= 2,528,100 \tag{4.1.46}$$

Analysis

This could also be solved by graphing the quadratic as in Figure 4.1.12 We can see the maximum revenue on a graph of the quadratic function.

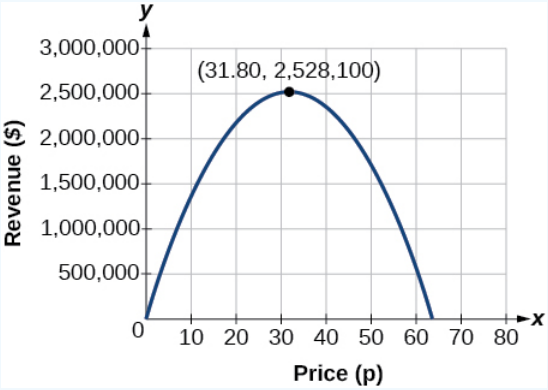
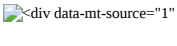


Figure 4.1.12: Graph of the parabolic function

Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure 4.1.13 that the number of x-intercepts can vary depending upon the location of the graph.



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Figure 4.1.13: Number of x-intercepts of a parabola.

How To: Given a quadratic function $f(x)$, find the y- and x-intercepts.

1. Evaluate $f(0)$ to find the y-intercept.
2. Solve the quadratic equation $f(x) = 0$ to find the x-intercepts.

Example 4.1.7: Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution

We find the y-intercept by evaluating $f(0)$.

$$f(0) = 3(0)^2 + 5(0) - 2 \quad (4.1.47)$$
$$= -2 \quad (4.1.48)$$

So the y-intercept is at $(0, -2)$.

For the x-intercepts, we find all solutions of $f(x) = 0$.

$$0 = 3x^2 + 5x - 2 \quad (4.1.49)$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = (3x - 1)(x + 2) \quad (4.1.50)$$
$$0 = 3x - 1 \qquad 0 = x + 2 \quad (4.1.51)$$
$$x = \frac{1}{3} \quad \text{or} \quad x = -2 \quad (4.1.52)$$

So the x-intercepts are at $(\frac{1}{3}, 0)$ and $(-2, 0)$.

Analysis

By graphing the function, we can confirm that the graph crosses the y-axis at $(0, -2)$. We can also confirm that the graph crosses the x-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$. See Figure 4.1.14

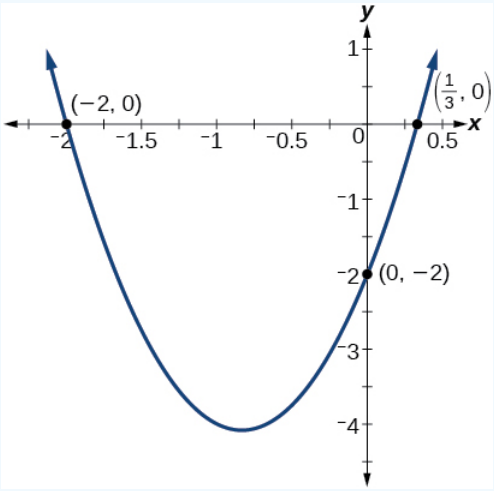


Figure 4.1.14: Graph of a parabola.

Rewriting Quadratics in Standard Form

In Example 4.1.7, the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

How To: Given a quadratic function, find the x-intercepts by rewriting in standard form.

1. Substitute a and b into $h = -\frac{b}{2a}$.
2. Substitute $x = h$ into the general form of the quadratic function to find k.
3. Rewrite the quadratic in standard form using h and k.
4. Solve for when the output of the function will be zero to find the x-intercepts.

Example 4.1.8: Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution

We begin by solving for when the output will be zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x-h)^2 + k$$

We know that $a = 2$. Then we solve for h and k.

$$\begin{aligned} h &= -\frac{b}{2a} & k &= f(-1) \\ &= -\frac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

$$f(x) = 2(x+1)^2 - 6$$

We can now solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x+1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has x-intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$.

Analysis

We can check our work by graphing the given function on a graphing utility and observing the x-intercepts. See Figure 4.1.15.

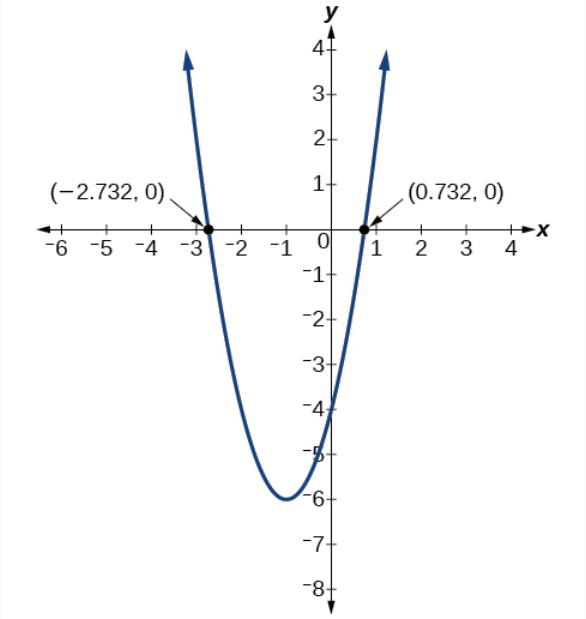


Figure 4.1.15: Graph of a parabola which has the following x-intercepts: $(-2.732, 0)$ and $(0.732, 0)$.

Exercise 4.1.1

In Try It 4.1.1, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the y- and x-intercepts (if any).

Answer

y-intercept at $(0, 13)$, No x-intercepts

Example 4.1.9: Solving a Quadratic Equation with the Quadratic Formula

Solve $x^2+x+2=0$.

Solution

Let's begin by writing the quadratic formula: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

When applying the quadratic formula, we identify the coefficients a, b and c. For the equation $x^2+x+2=0$, we have $a=1$, $b=1$, and $c=2$. Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-1\pm\sqrt{1^2-4\cdot 1\cdot (2)}}{2\cdot 1} \\ &= \frac{-1\pm\sqrt{1-8}}{2} \\ &= \frac{-1\pm\sqrt{-7}}{2} \\ &= \frac{-1\pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $x=\frac{-1+i\sqrt{7}}{2}$ and $x=\frac{-1-i\sqrt{7}}{2}$ or $x=-\frac{1}{2}+\frac{i\sqrt{7}}{2}$ and $x=\frac{-1}{2}-\frac{i\sqrt{7}}{2}$.

Example 4.1.10: Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t)=-16t^2+80t+40$.

When does the ball reach the maximum height?
What is the maximum height of the ball?
When does the ball hit the ground?

The ball reaches the maximum height at the vertex of the parabola.

$$\begin{aligned} h &= -\frac{80}{2(-16)} && (4.1.53)\\ &= \frac{80}{32} && (4.1.54)\\ &= \frac{5}{2} && (4.1.55)\\ &= 2.5 && (4.1.56) \end{aligned}$$

The ball reaches a maximum height after 2.5 seconds.

To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$\begin{aligned} k &= H(-\frac{b}{2a}) && (4.1.57)\\ &= H(2.5) && (4.1.58)\\ &= -16(2.5)^2+80(2.5)+40 && (4.1.59)\\ &= 140 && (4.1.60) \end{aligned}$$

The ball reaches a maximum height of 140 feet.

To find when the ball hits the ground, we need to determine when the height is zero, $H(t)=0$.

We use the quadratic formula.

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \tag{4.1.61}$$

$$= \frac{-80 \pm \sqrt{8960}}{-32} \tag{4.1.62}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458 \tag{4.1.63}$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 4.1.16.

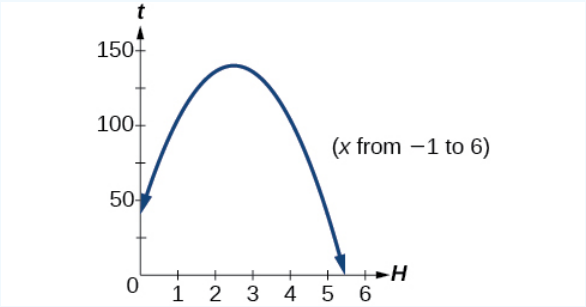


Figure 4.1.16

4.1.5: A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Solution

a. 3 seconds b. 256 feet c. 7 seconds

Key Equations

- general form of a quadratic function: $f(x) = ax^2 + bx + c$
- the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- standard form of a quadratic function: $f(x) = a(x - h)^2 + k$

Key Concepts

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x-intercepts, are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the y-axis.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph.
- The vertex can be found from an equation representing a quadratic function. .
- The domain of a quadratic function is all real numbers. The range varies with the function.
- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue.
- Some quadratic equations must be solved by using the quadratic formula.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems.

Glossary

axis of symmetry
a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

general form of a quadratic function
the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and a≠0.

standard form of a quadratic function
the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex.

vertex
the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

vertex form of a quadratic function
another name for the standard form of a quadratic function

zeros
in a given function, the values of x at which $y = 0$, also called roots

Learning Objectives

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.

Curved antennas, such as the ones shown in Figure 4.1.1, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

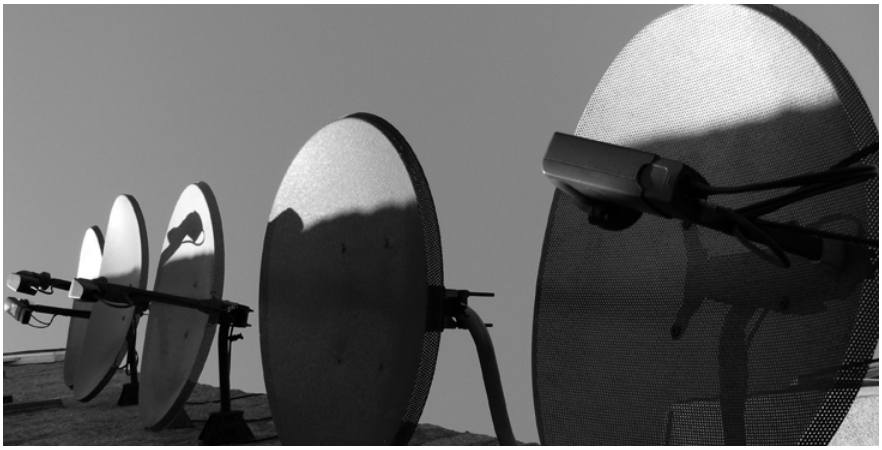


***Figure* 4.1.1:** *An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)*

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the **minimum value** of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 4.1.2.

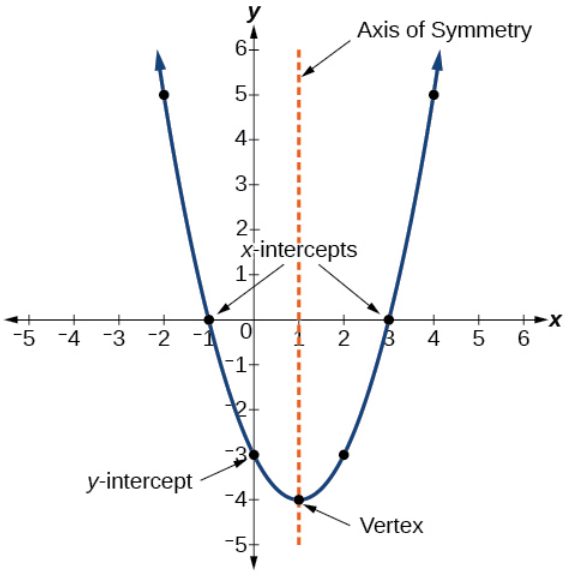


Figure 4.1.2: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are.

The y-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of x at which $y = 0$.

Example 4.1.1: Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure 4.1.3.

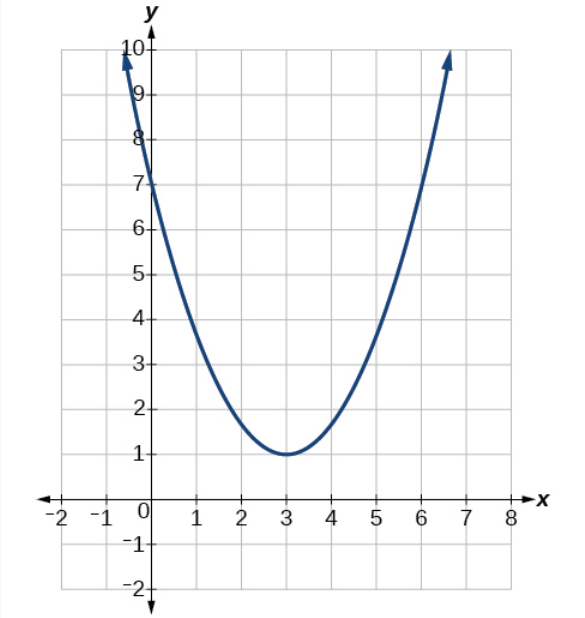


Figure 4.1.3.

Solution

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at $(0, 7)$ so this is the y-intercept.

Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The **general form of a quadratic function** presents the function in the form

$$f(x) = ax^2 + bx + c \tag{4.1.64}$$

where a, b, and c are real numbers and $a \neq 0$. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find the value of x halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure 4.1.4 represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

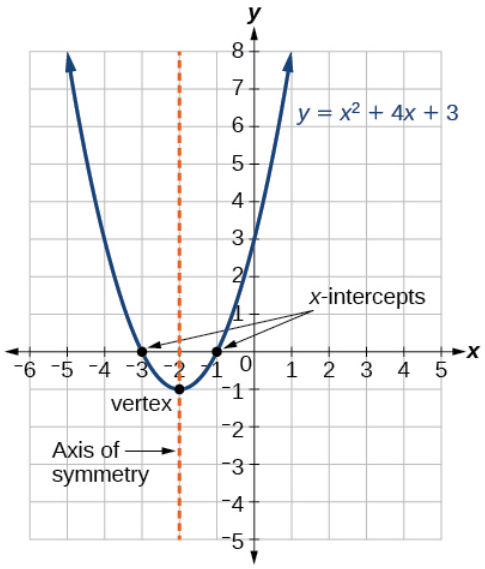


Figure 4.1.4: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are for the function $y = x^2 + 4x + 3$.

The **standard form of a quadratic function** presents the function in the form

$$f(x) = a(x - h)^2 + k \tag{4.1.65}$$

where (h, k) is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure 4.1.5 represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

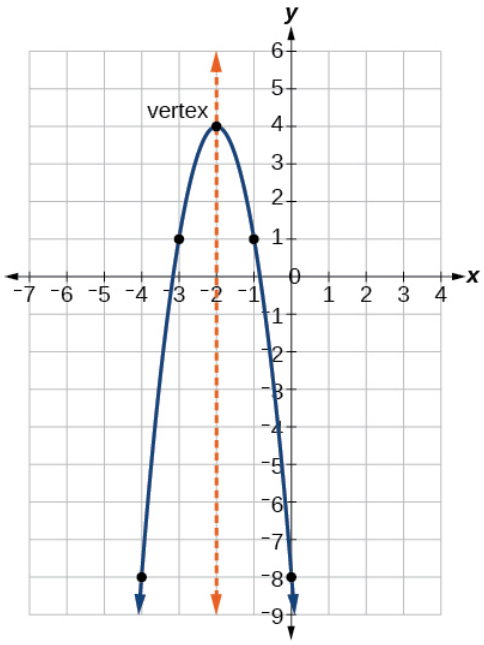


Figure 4.1.5: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are for the function $y=-3(x+2)^2+4$.

The standard form is useful for determining how the graph is transformed from the graph of $y=x^2$. Figure 4.1.6 is the graph of this basic function.

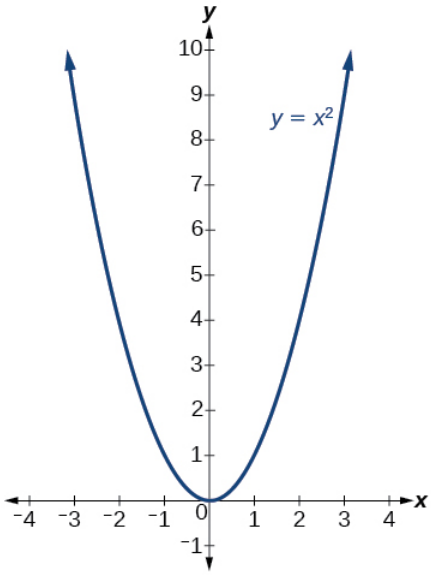


Figure 4.1.6: Graph of $y=x^2$.

If $k>0$, the graph shifts upward, whereas if $k<0$, the graph shifts downward. In Figure 4.1.5, $k>0$, so the graph is shifted 4 units upward. If $h>0$, the graph shifts toward the right and if $h<0$, the graph shifts to the left. In Figure 4.1.5, $h<0$, so the graph is shifted 2 units to the left. The magnitude of a indicates the stretch of the graph. If $|a|>1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a|<1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In Figure 4.1.5, $|a|>1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$\begin{aligned} a(x-h)^2+k &= ax^2+bx+c \\ ax^2-2ahx+(ah^2+k) &= ax^2+bx+c \end{aligned}$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah=b, \text{ so } h=-\frac{b}{2a}.$$

This is the **axis of symmetry** we defined earlier. Setting the constant terms equal:

$$\begin{aligned} ah^2+k &= c \\ k &= c-ah^2 \\ &= c-a\cdot\left(-\frac{b}{2a}\right)^2 \\ &= c-\frac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h)=k$.

Definitions: Forms of Quadratic Functions

A quadratic function is a function of degree two. The graph of a *quadratic function* is a parabola.

- The *general form* of a quadratic function is $f(x)=ax^2+bx+c$ where a, b, and c are real numbers and $a\neq 0$.
- The standard form of a quadratic function is $f(x)=a(x-h)^2+k$.
- The vertex (h,k) is located at

$$h=-\frac{b}{2a},\ k=f(h)=f(\frac{-b}{2a}). \tag{4.1.66}$$

HOWTO: Write a quadratic function in a general form

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is h. Identify the vertical shift of the parabola; this value is k.
2. Substitute the values of the horizontal and vertical shift for h and k. in the function $f(x)=a(x-h)^2+k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for x and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. If the parabola opens up, $a>0$. If the parabola opens down, $a<0$ since this means the graph was reflected about the x-axis.
6. Expand and simplify to write in general form.

Example 4.1.2: Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in Figure 4.1.7 as a transformation of $f(x)=x^2$, and then expand the formula, and simplify terms to write the equation in general form.

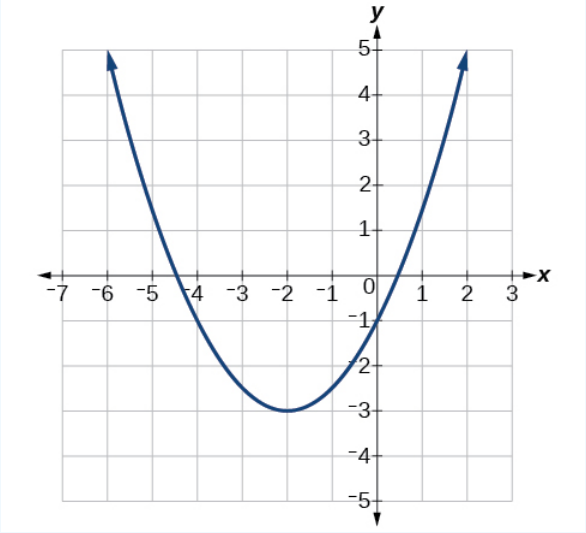


Figure 4.1.7: Graph of a parabola with its vertex at $(-2, -3)$.

Solution

We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$.

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$-1 = a(0+2)^2 - 3 \tag{4.1.67}$$
$$2 = 4a \tag{4.1.68}$$
$$a = \frac{1}{2} \tag{4.1.69}$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x+2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$g(x) = \frac{1}{2}(x+2)^2 - 3 \tag{4.1.70}$$
$$= \frac{1}{2}(x+2)(x+2) - 3 \tag{4.1.71}$$
$$= \frac{1}{2}(x^2 + 4x + 4) - 3 \tag{4.1.72}$$
$$= \frac{1}{2}x^2 + 2x + 2 - 3 \tag{4.1.73}$$
$$= \frac{1}{2}x^2 + 2x - 1 \tag{4.1.74}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Analysis

We can check our work using the table feature on a graphing utility. First enter $\mathrm{Y1} = \frac{1}{2}(\mathrm{x}+2)^2 - 3$. Next, select TBLSET, then use $\mathrm{TblStart} = -6$ and $\Delta\mathrm{Tbl} = 2$, and select TABLE. See Table 4.1.1

Table 4.1.1

x	-6	-4	-2	0	2
y	-5	-1	-3	-1	5

The ordered pairs in the table correspond to points on the graph.

Exercise 4.1.2

A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 4.1.8. Find an equation for the path of the ball. Does the shooter make the basket?

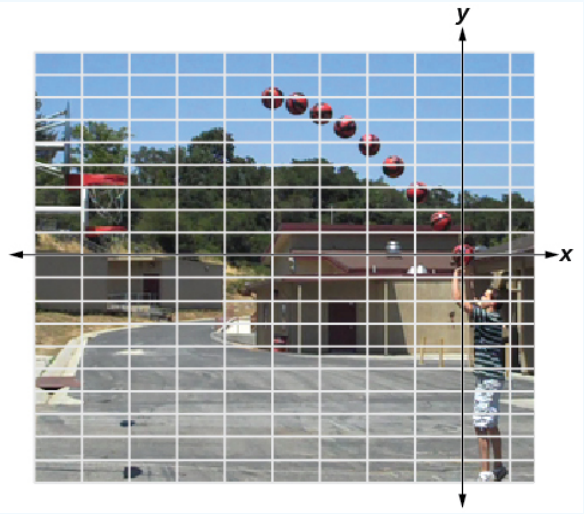


***Figure* 4.1.8:** *Stop motioned picture of a boy throwing a basketball into a hoop to show the parabolic curve it makes. (credit: modification of work by Dan Meyer)*

Answer

The path passes through the origin and has vertex at $(-4, 7)$, so $h(x) = -\frac{7}{16}(x+4)^2 + 7$. To make the shot, $h(-7.5)$ would need to be about 4 but $h(-7.5) \approx 1.64$; he doesn't make it.

How To: Given a quadratic function in general form, find the vertex of the parabola.

1. Identify a, b, and c.
2. Find h, the x-coordinate of the vertex, by substituting a and b into $h = -\frac{b}{2a}$.
3. Find k, the y-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

Example 4.1.3: Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} \tag{4.1.75}$$
$$= -\frac{-6}{2(2)} \tag{4.1.76}$$
$$= \frac{6}{4} \tag{4.1.77}$$
$$= \frac{3}{2} \tag{4.1.78}$$

The vertical coordinate of the vertex will be at

$$k = f(h) \tag{4.1.79}$$
$$= f\left(\frac{3}{2}\right) \tag{4.1.80}$$
$$= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \tag{4.1.81}$$
$$= \frac{5}{2} \tag{4.1.82}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.

$$f(x) = ax^2 + bx + c \tag{4.1.83}$$
$$f(x) = 2x^2 - 6x + 7$$

Using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2} \tag{4.1.84}$$

Analysis

One reason we may want to identify the vertex of the parabola is that this point will inform us what the maximum or minimum value of the function is, (k),and where it occurs, (h).

? Exercise 4.1.3

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

Answer

$g(x) = x^2 - 6x + 13$ in general form; $g(x) = (x-3)^2 + 4$ in standard form.

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

Definition: Domain and Range of a Quadratic Function

The domain of any **quadratic function** is all real numbers.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $f(x) \geq f\left(-\frac{b}{2a}\right)$, or $[f(-\frac{b}{2a}), \infty)$; the range of a quadratic function written in general form with a negative a value is $f(x) \leq f(-\frac{b}{2a})$, or $(-\infty, f(-\frac{b}{2a})]$.

The range of a quadratic function written in standard form $f(x) = a(x-h)^2 + k$ with a positive a value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \leq k$.

How To: Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, k.
4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $(-\infty, k]$.

Example 4.1.4: Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$h = -\frac{b}{2a} \tag{4.1.85}$$
$$= -\frac{9}{2(-5)} \tag{4.1.86}$$
$$= \frac{9}{10} \tag{4.1.87}$$

The maximum value is given by $f(h)$.

$$f(\frac{9}{10}) = 5(\frac{9}{10})^2 + 9(\frac{9}{10}) - 1 \tag{4.1.88}$$
$$= \frac{61}{20} \tag{4.1.89}$$

The range is $f(x) \leq \frac{61}{20}$, or $\left(-\infty, \frac{61}{20}\right]$.

Exercise 4.1.4

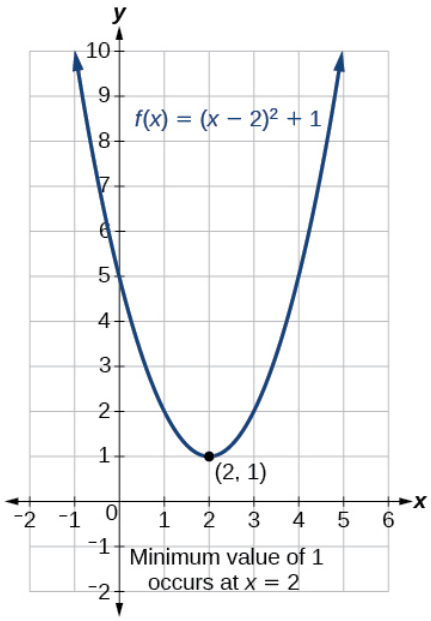
Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Answer

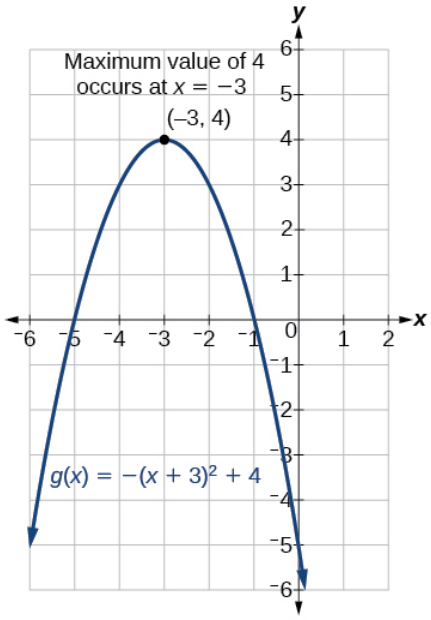
The domain is all real numbers. The range is $f(x) \geq \frac{8}{11}$, or $\left[\frac{8}{11}, \infty\right)$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the **parabola**. We can see the maximum and minimum values in Figure 4.1.9.



(a)



(b)

Figure 4.1.9: Minimum and maximum of two quadratic functions.

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Example 4.1.5: Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
b. What dimensions should she make her garden to maximize the enclosed area?

Solution

Let's use a diagram such as Figure 4.1.10 to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

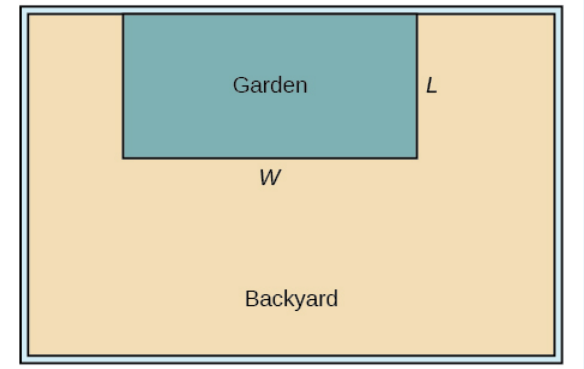


Figure 4.1.10: Diagram of the garden and the backyard.

a. We know we have only 80 feet of fence available, and $L+W+L=80$, or more simply, $2L+W=80$. This allows us to represent the width, W, in terms of L.

$$W=80-2L \tag{4.1.90}$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80-2L) \tag{4.1.91}$$
$$A(L) = 80L-2L^2 \tag{4.1.92}$$

This formula represents the area of the fence in terms of the variable length L. The function, written in general form, is

$$A(L)=-2L^2+80L \tag{4.1.93}$$

.

The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a=-2$, $b=80$, and $c=0$.

To find the vertex:

$$h=-\frac{80}{2(-2)} \qquad k=A(20) \tag{4.1.94}$$
$$=20 \qquad \text{and} \qquad =80(20)-2(20)^2 \tag{4.1.95}$$
$$=800 \tag{4.1.96}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L=20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 4.1.11.

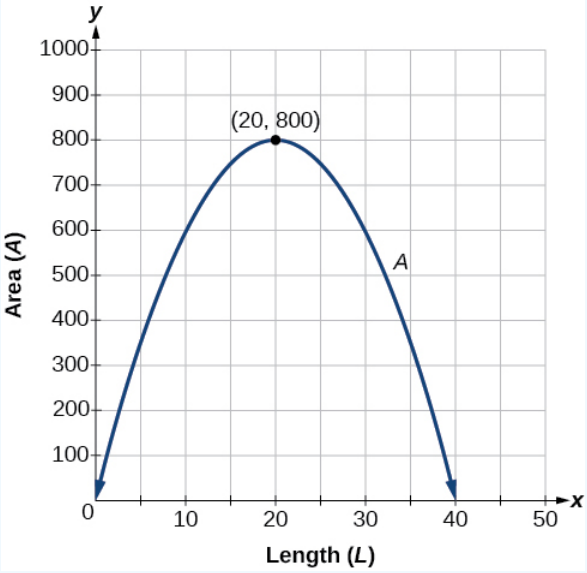


Figure 4.1.11: Graph of the parabolic function $A(L) = -2L^2 + 80L$

How To: Given an application involving revenue, use a quadratic equation to find the maximum.

1. Write a quadratic equation for revenue.
2. Find the vertex of the quadratic equation.
3. Determine the y-value of the vertex.

Example 4.1.6: Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30} \tag{4.1.97}$$

$$= -\frac{5,000}{2} \tag{4.1.98}$$

$$= -2,500 \tag{4.1.99}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$Q = -2500p + b \quad \text{Substitute in the point } Q = 84,000 \text{ and } p = 30 \tag{4.1.100}$$

$$84,000 = -2500(30) + b \quad \text{Solve for } b \tag{4.1.101}$$

$$b = 159,000 \tag{4.1.102}$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\text{Revenue} = pQ \tag{4.1.103}$$
$$\text{Revenue} = p(-2,500p + 159,000) \tag{4.1.104}$$
$$\text{Revenue} = -2,500p^2 + 159,000p \tag{4.1.105}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$h = -\frac{159,000}{2(-2,500)} \tag{4.1.106}$$
$$= 31.8 \tag{4.1.107}$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\text{maximum revenue} = -2,500(31.8)^2 + 159,000(31.8) \tag{4.1.108}$$
$$= 2,528,100 \tag{4.1.109}$$

Analysis

This could also be solved by graphing the quadratic as in Figure 4.1.12 We can see the maximum revenue on a graph of the quadratic function.



Figure 4.1.12: Graph of the parabolic function

Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure 4.1.13 that the number of x-intercepts can vary depending upon the location of the graph.

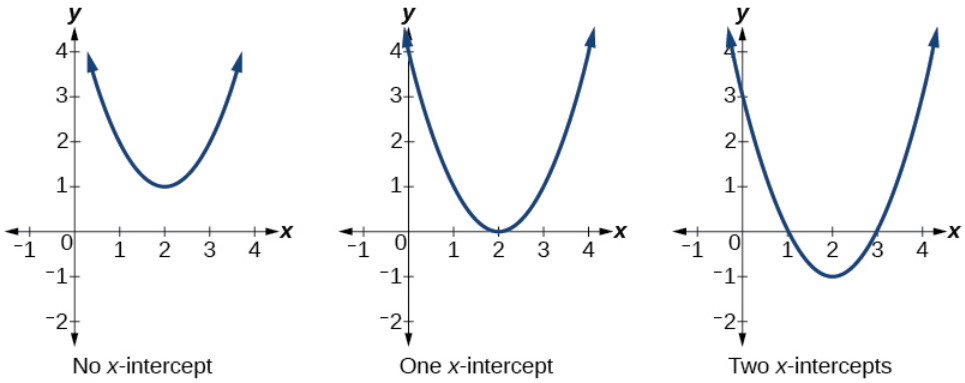


Figure 4.1.13: Number of x-intercepts of a parabola.

Given a quadratic function $f(x)$, find the y- and x-intercepts.

1. Evaluate $f(0)$ to find the y-intercept.
2. Solve the quadratic equation $f(x)=0$ to find the x-intercepts.

Example 4.1.7: Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x)=3x^2+5x-2$.

Solution

We find the y-intercept by evaluating $f(0)$.

$$f(0) = 3(0)^2+5(0)-2 \tag{4.1.110}$$
$$= -2 \tag{4.1.111}$$

So the y-intercept is at $(0,-2)$.

For the x-intercepts, we find all solutions of $f(x)=0$.

$$0=3x^2+5x-2 \tag{4.1.112}$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0=(3x-1)(x+2) \tag{4.1.113}$$
$$0=3x-1 \qquad\qquad 0=x+2 \tag{4.1.114}$$
$$x=\frac{1}{3} \qquad \text{or} \qquad x=-2 \tag{4.1.115}$$

So the x-intercepts are at $(\frac{1}{3},0)$ and $(-2,0)$.

Analysis

By graphing the function, we can confirm that the graph crosses the y-axis at $(0,-2)$. We can also confirm that the graph crosses the x-axis at $\left(\frac{1}{3},0\right)$ and $(-2,0)$. See Figure 4.1.14

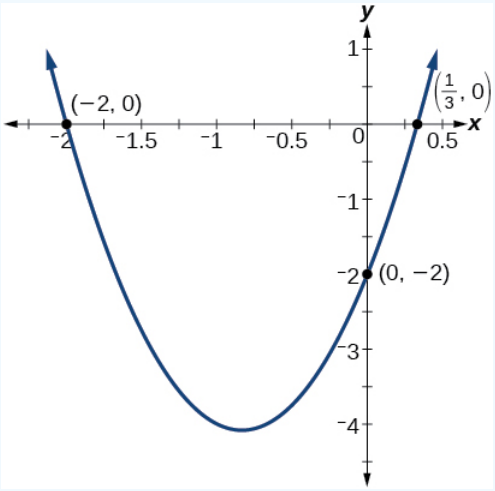


Figure 4.1.14: Graph of a parabola.

Rewriting Quadratics in Standard Form

In Example 4.1.7, the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

How to: Given a quadratic function, find the x-intercepts by rewriting in standard form.

1. Substitute a and b into $h=-\frac{b}{2a}$.
2. Substitute $x=h$ into the general form of the quadratic function to find k.
3. Rewrite the quadratic in standard form using h and k.

4. Solve for when the output of the function will be zero to find the x-intercepts.

Example 4.1.8: Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution

We begin by solving for when the output will be zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x-h)^2 + k$$

We know that $a = 2$. Then we solve for h and k.

$$\begin{aligned} h &= -\frac{b}{2a} & k &= f(-1) \\ &= -\frac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

$$f(x) = 2(x+1)^2 - 6$$

We can now solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x+1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has x-intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$.

Analysis

We can check our work by graphing the given function on a graphing utility and observing the x-intercepts. See Figure 4.1.15.

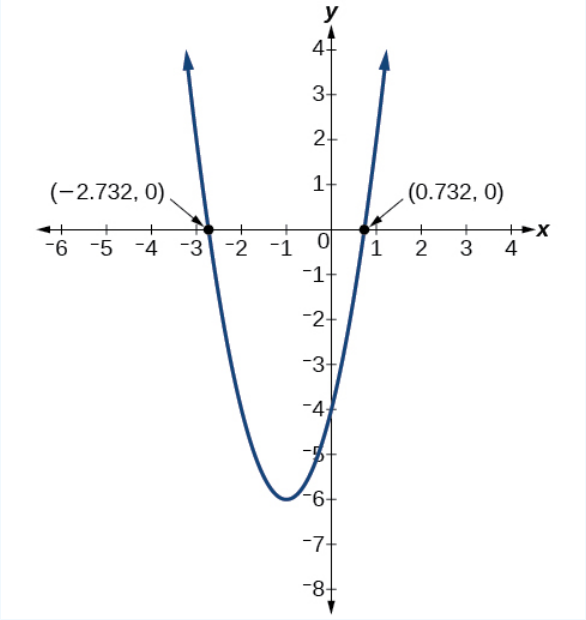


Figure 4.1.15: Graph of a parabola which has the following x-intercepts: $(-2.732, 0)$ and $(0.732, 0)$.

Exercise 4.1.1

In Try It 4.1.1, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the y- and x-intercepts (if any).

Answer

y-intercept at $(0, 13)$, No x-intercepts

Example 4.1.9: Solving a Quadratic Equation with the Quadratic Formula

Solve $x^2 + x + 2 = 0$.

Solution

Let's begin by writing the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

When applying the quadratic formula, we identify the coefficients a, b and c. For the equation $x^2 + x + 2 = 0$, we have $a = 1$, $b = 1$, and $c = 2$. Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (2)}}{2 \cdot 1} \\ &= \frac{-1 \pm \sqrt{1 - 8}}{2} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $x = \frac{-1 + i\sqrt{7}}{2}$ and $x = \frac{-1 - i\sqrt{7}}{2}$ or $x = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $x = \frac{-1}{2} - \frac{i\sqrt{7}}{2}$.

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Example 4.1.10: Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

When does the ball reach the maximum height?
What is the maximum height of the ball?
When does the ball hit the ground?

The ball reaches the maximum height at the vertex of the parabola.

$$h = -\frac{80}{2(-16)} \tag{4.1.116}$$
$$= \frac{80}{32} \tag{4.1.117}$$
$$= \frac{5}{2} \tag{4.1.118}$$
$$= 2.5 \tag{4.1.119}$$

The ball reaches a maximum height after 2.5 seconds.

To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$k = H(-\frac{b}{2a}) \tag{4.1.120}$$
$$= H(2.5) \tag{4.1.121}$$
$$= -16(2.5)^2 + 80(2.5) + 40 \tag{4.1.122}$$
$$= 140 \tag{4.1.123}$$

The ball reaches a maximum height of 140 feet.

To find when the ball hits the ground, we need to determine when the height is zero, $H(t) = 0$.

We use the quadratic formula.

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \tag{4.1.124}$$
$$= \frac{-80 \pm \sqrt{8960}}{-32} \tag{4.1.125}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458 \tag{4.1.126}$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 4.1.16.

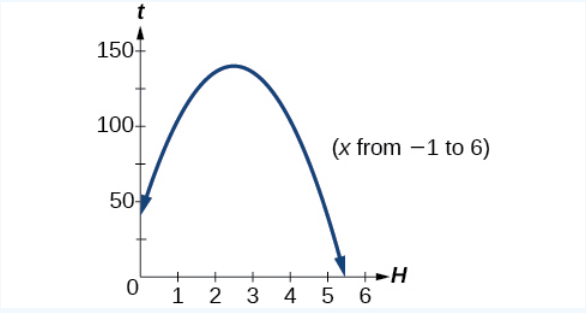


Figure 4.1.16

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4.1.5: A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Solution

a. 3 seconds b. 256 feet c. 7 seconds

Key Equations

- general form of a quadratic function: $f(x) = ax^2 + bx + c$
- the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- standard form of a quadratic function: $f(x) = a(x - h)^2 + k$

Key Concepts

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x-intercepts, are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the y-axis.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph.
- The vertex can be found from an equation representing a quadratic function. .
- The domain of a quadratic function is all real numbers. The range varies with the function.
- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue.
- Some quadratic equations must be solved by using the quadratic formula.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems.

Glossary

axis of symmetry
a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

general form of a quadratic function
the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and a≠0.

standard form of a quadratic function
the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex.

vertex
the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

vertex form of a quadratic function
another name for the standard form of a quadratic function

zeros
in a given function, the values of x at which $y = 0$, also called roots

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4.2: Power Functions and Polynomial Functions

Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function. A power function is a function with a single term that is the product of a real number, a coefficient, and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)

As an example, consider functions for area or volume. The function for the area of a circle with radius r is

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius r is

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a power.

Definition: Power Function

A *power function* is a function that can be represented in the form

$$f(x) = kx^p \tag{4.2.1}$$

where k and p are real numbers, and k is known as the *coefficient*.

Q&A: Is $f(x) = 2^x$ a power function?

No. A power function contains a variable base raised to a fixed power (Equation 4.2.1). This function has a constant base raised to a variable power. This is called an exponential function, not a power function. This function will be discussed later.

Example 4.2.1: Identifying Power Functions

Which of the following functions are power functions?

$f(x) = 1$	Constant function
$f(x) = x$	Identify function
$f(x) = x^2$	Quadratic function
$f(x) = x^3$	Cubic function
$f(x) = \frac{1}{x}$	Reciprocal function
$f(x) = \frac{1}{x^2}$	Reciprocal squared function
$f(x) = \sqrt{x}$	Square root function
$f(x) = \sqrt[3]{x}$	Cube root function

Solution

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The **reciprocal** and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and **cube root** functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Exercise 4.2.1

Which functions are power functions?

- $f(x) = 2x^2 \cdot 4x^3$
- $g(x) = -x^5 + 5x^3 - 4x$
- $h(x) = \frac{2x^5 - 1}{3x^2 + 4}$

Answer

$f(x)$ is a power function because it can be written as $f(x) = 8x^5$. The other functions are not power functions.

Identifying End Behavior of Power Functions

Figure 4.2.2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

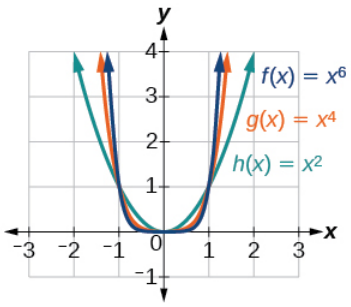


Figure 4.2.2: Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty, \ f(x) \to \infty$$

Figure 4.2.3 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

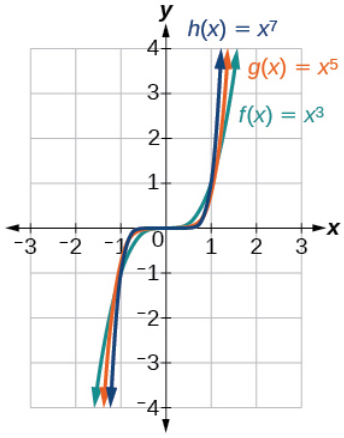


Figure 4.2.3: Odd-power function

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure $4.2.2$ we see that even functions of the form $f(x) = x^n$, n even, are symmetric about the y-axis. In Figure $4.2.3$ we see that odd functions of the form $f(x) = x^n$, n odd, are symmetric about the origin.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$
$$\text{as } x \to \infty,\ f(x) \to \infty$$

The behavior of the graph of a function as the input values get very small $(x \to -\infty)$ and get very large $x \to \infty$ is referred to as the *end behavior* of the function. We can use words or symbols to describe end behavior.

Figure $4.2.4$ shows the end behavior of power functions in the form $f(x) = kx^n$ where n is a non-negative integer depending on the power and the constant.

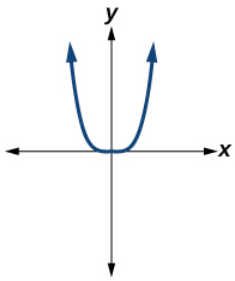
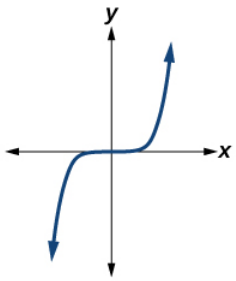
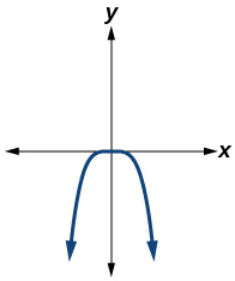
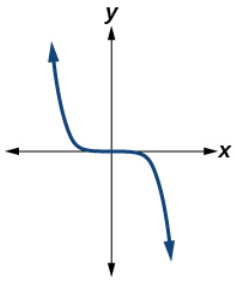
	Even power	Odd power
Positive constant $k > 0$	 $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to \infty$	 $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to \infty$
Negative constant $k < 0$	 $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to -\infty$	 $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to -\infty$

Figure 4.2.4

How To: Given a power function $f(x) = kx^n$ where n is a non-negative integer, identify the end behavior.

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use Figure $4.2.4$ to identify the end behavior.

Example 4.2.2: Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. We write as $x \to \infty,\ f(x) \to \infty$. As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty,\ f(x) \to \infty$. We can graphically represent the function as shown in Figure 4.2.5.

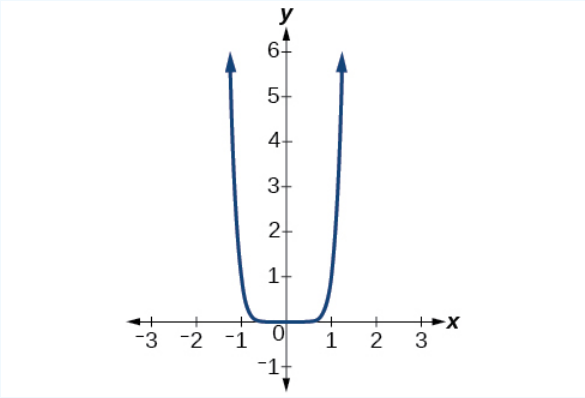


Figure 4.2.5: Graph of $f(x) = x^8$.

Example 4.2.3: Identifying the End Behavior of a Power Function.

Describe the end behavior of the graph of $f(x) = -x^9$.

Solution

The exponent of the power function is 9 (an odd number). Because the coefficient is –1 (negative), the graph is the reflection about the x-axis of the graph of $f(x) = x^9$. Figure 4.2.6 shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty,\ f(x) \to \infty$$
$$\text{as } x \to \infty,\ f(x) \to -\infty$$

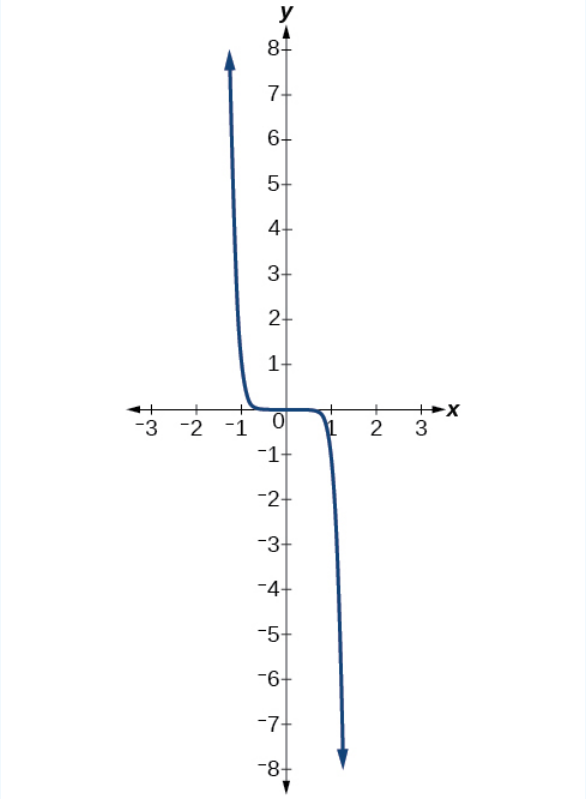


Figure 4.2.6: Graph of $f(x) = -x^9$.

Analysis

We can check our work by using the table feature on a graphing utility.

Table 4.2.2

x	$f(x)$
-10	1,000,000,000
-5	1,953,125
0	0
5	-1,953,125
10	-1,000,000,000

We can see from Table $4.2.2$ that, when we substitute very small values for x, the output is very large, and when we substitute very large values for x, the output is very small (meaning that it is a very large negative value).

Exercise 4.2.2

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

Answer

As x approaches positive or negative infinity, $f(x)$ decreases without bound: as $x \to \pm\infty$, $f(x) \to -\infty$ because of the negative coefficient.

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius r of the spill depends on the number of weeks w that have passed. This relationship is linear.

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area A of a circle.

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24 + 8w) \\ &= \pi(24 + 8w)^2 \end{aligned}$$

Multiplying gives the formula.

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a polynomial function. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

Definition: Polynomial Functions

Let n be a non-negative integer. A *polynomial function* is a function that can be written in the form

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \tag{4.2.2}$$

This is called the general form of a polynomial function. Each a_i is a coefficient and can be any real number. Each product $a_i x^i$ is a *term of a polynomial function.*

Example 4.2.4: Identifying Polynomial Functions

Which of the following are polynomial functions?

- $f(x) = 2x^3 \cdot 3x + 4$
- $g(x) = -x(x^2 - 4)$
- $h(x) = 5\sqrt{x} + 2$

Solution

The first two functions are examples of polynomial functions because they can be written in the form of Equation 4.2.2, where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The degree of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The leading term is the term containing the highest power of the variable, or the term with the highest degree. The leading coefficient is the coefficient of the leading term.

Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.

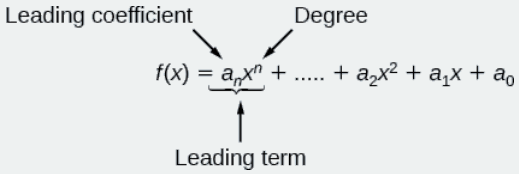


Figure 4.2.7

When a polynomial is written in this way, we say that it is in *general form*.

How To: Given a polynomial function, identify the degree and leading coefficient

1. Find the highest power of x to determine the degree function.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Example 4.2.5: Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$f(x) = 3 + 2x^2 - 4x^3$$
$$g(t) = 5t^5 - 2t^3 + 7t$$
$$h(p) = 6p - p^3 - 2$$

Solution

For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, -1.

Exercise 4.2.3

Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 6$.

Answer

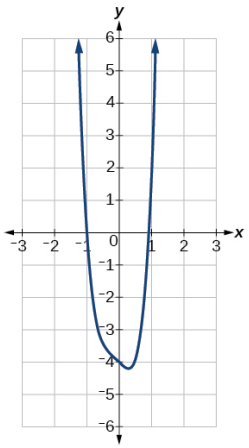
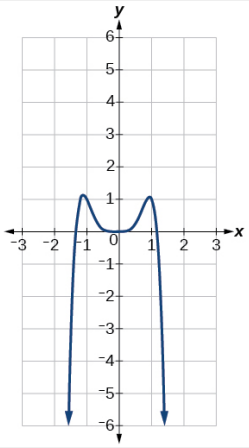
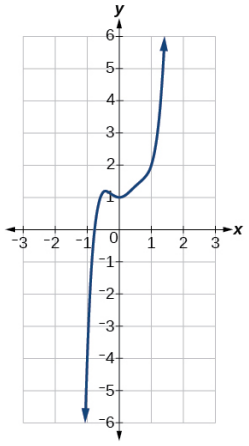
The degree is 6. The leading term is $-x^6$. The leading coefficient is -1.

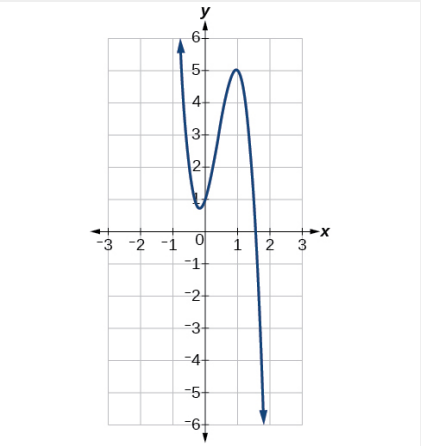
Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree (Table $4.2.3$).

Table 4.2.3

Polynomial Function	Leading Term	Graph of Polynomial Function

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$	
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$	
$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$	$3x^5$	

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = -6x^3 + 7x^2 + 3x + 1$	$-6x^3$	

Example 4.2.6: Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure 4.2.8.

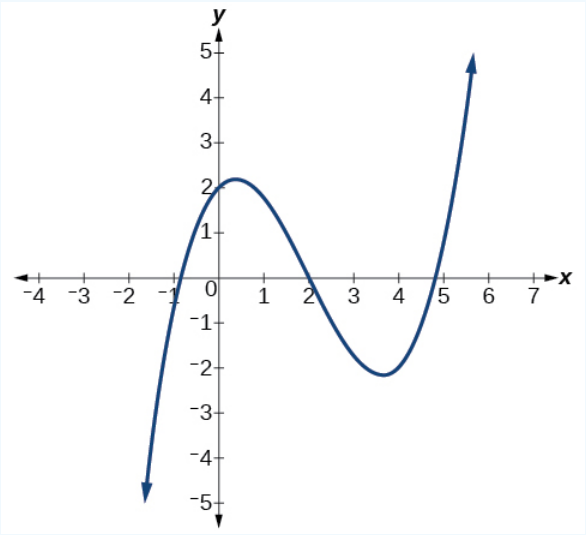


Figure 4.2.8.

Solution

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to \infty,\ f(x) \to \infty$$

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

Exercise 4.2.1

Describe the end behavior, and determine a possible degree of the polynomial function in Figure 4.2.9.

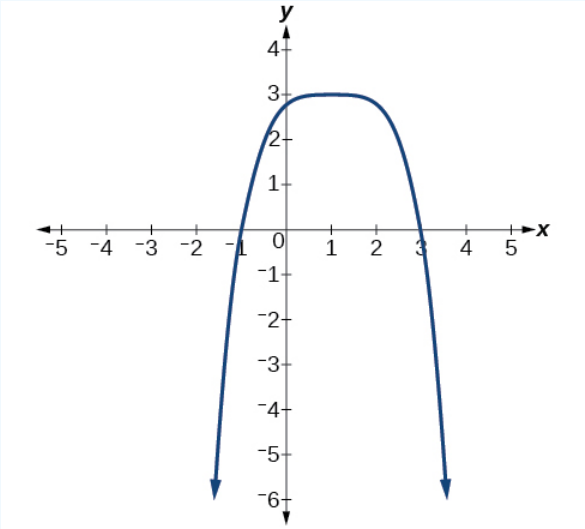


Figure 4.2.9

Answer

As $x \to \infty$, $f(x) \to -\infty$; as $x \to -\infty$, $f(x) \to -\infty$. It has the shape of an even degree power function with a negative coefficient.

Example 4.2.7: Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Solution

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2+3x-4) \\ &= -3x^4-9x^3+12x^2 \end{aligned}$$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

$$\text{as } x \to \infty,\ f(x) \to -\infty$$

Exercise 4.2.7

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

Answer

The leading term is $0.2x^3$, so it is a degree 3 polynomial. As x approaches positive infinity, $f(x)$ increases without bound; as x approaches negative infinity, $f(x)$ decreases without bound.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the y-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one y-intercept $(0, a_0)$. The x-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one x-intercept. See Figure $4.2.10$.

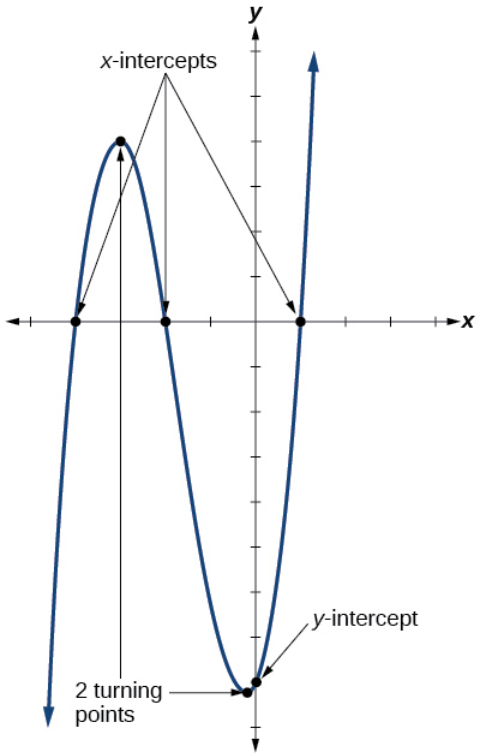


Figure 4.2.10

Defintion: Intercepts and Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The y-intercept is the point at which the function has an input value of zero. The x-intercepts are the points at which the output value is zero.

How To: Given a polynomial function, determine the intercepts.

1. Determine the y-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the x-intercepts by solving for the input values that yield an output value of zero.

Example 4.2.8: Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the y- and x-intercepts.

Solution

The y-intercept occurs when the input is zero, so substitute 0 for x.

$$\begin{aligned} f(0) &= (0-2)(0+1)(0-4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The y-intercept is $(0, 8)$.

The x-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$\begin{aligned} x-2 &= 0 \quad \text{or} \quad x+1 = 0 \quad \text{or} \quad x-4 = 0 \\ x &= 2 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = 4 \end{aligned}$$

The x-intercepts are $(2, 0)$,$(-1, 0)$, and $(4, 0)$.

We can see these intercepts on the graph of the function shown in Figure 4.2.11.

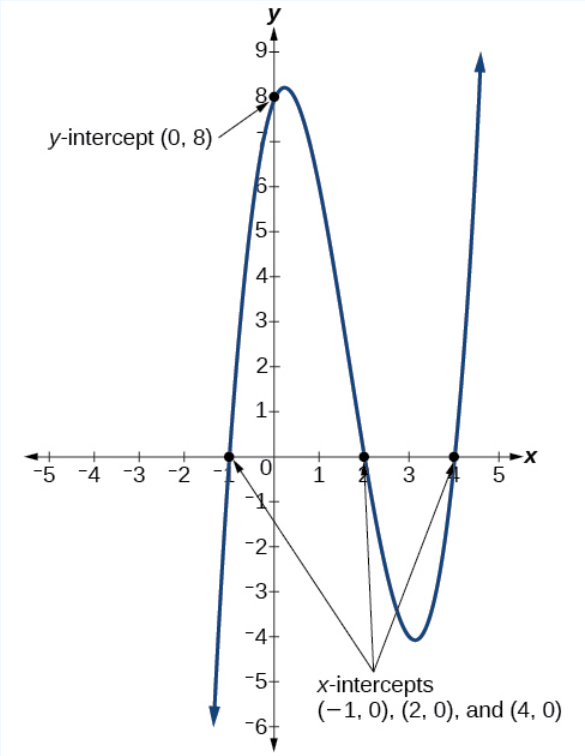


Figure 4.2.11: Graph of $f(x) = (x-2)(x+1)(x-4)$.

Example 4.2.9: Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the y- and x-intercepts.

Solution

The y-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^2 - 45 \\ &= -45 \end{aligned}$$

The y-intercept is $(0, -45)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$\begin{aligned} f(x) &= x^4 - 4x^2 - 45 \\ &= (x^2 - 9)(x^2 + 5) \\ &= (x-3)(x+3)(x^2+5) \end{aligned}$$

$$0 = (x-3)(x+3)(x^2+5)$$

$$\begin{array}{lllll} x-3=0 & \text{or} & x+3=0 & \text{or} & x^2+5=0 \\ x=3 & \text{or} & x=-3 & \text{or} & \text{(no real solution)} \end{array}$$

The x-intercepts are $(3,0)$ and $(-3,0)$.

We can see these intercepts on the graph of the function shown in Figure 4.2.12 We can see that the function is even because $f(x)=f(-x)$.

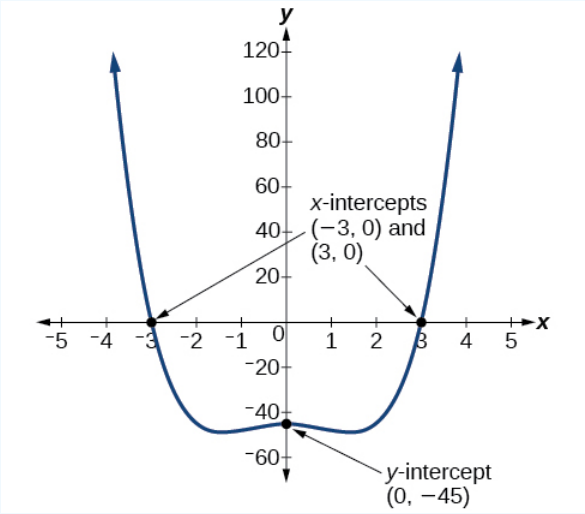


Figure 4.2.12: Graph of $f(x)=x^4-4x^2-45$.

Exercise 4.2.5

4.2.5: Given the polynomial function $f(x)=2x^3-6x^2-20x$, determine the y- and x-intercepts.

Solution

y-intercept $(0,0)$; x-intercepts $(0,0)$,$(-2,0)$, and $(5,0)$

Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of x-intercepts and the number of turning points. A polynomial function of nth degree is the product of n factors, so it will have at most n roots or zeros, or x-intercepts. The graph of the polynomial function of degree n must have at most $n-1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A smooth curve is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Intercepts and Turning Points of Polynomials

A polynomial of degree n will have, at most, n x-intercepts and $n-1$ turning points.

Example 4.2.10: Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of x-intercepts and turning points for $f(x)=-3x^{10}+4x^7-x^4+2x^3$.

Solution

The polynomial has a degree of **10**, so there are at most n x-intercepts and at most $n-1$ turning points.

Exercise 4.2.6

Without graphing the function, determine the maximum number of x-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$

Answer

There are at most 12 x-intercepts and at most 11 turning points.

Example 4.2.11: Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure 4.2.12 based on its intercepts and turning points?

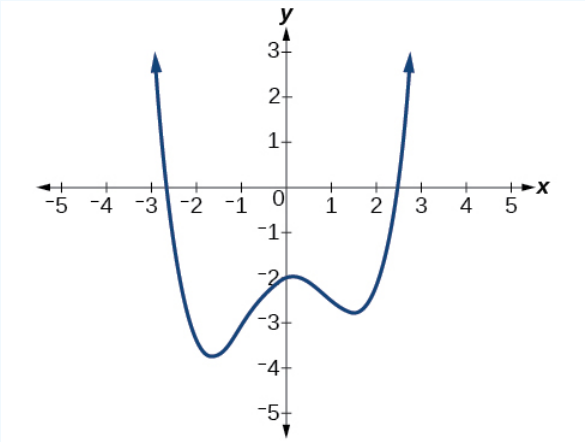


Figure 4.2.13.

Solution

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See Figure 4.2.14

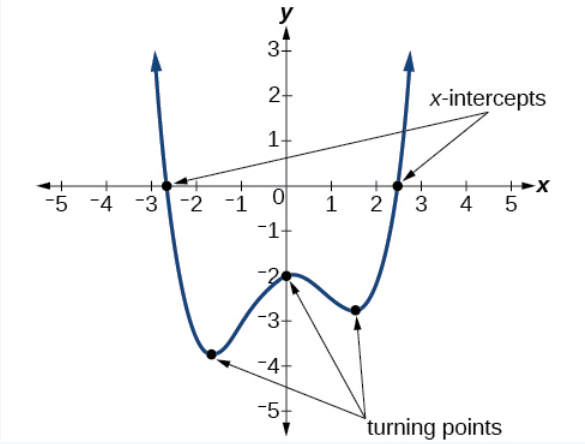


Figure 4.2.14: Graph of an even-degree polynomial.

The graph has 2 x-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

Exercise 4.2.7

What can we conclude about the polynomial represented by the graph shown in Figure 4.2.15 based on its intercepts and turning points?

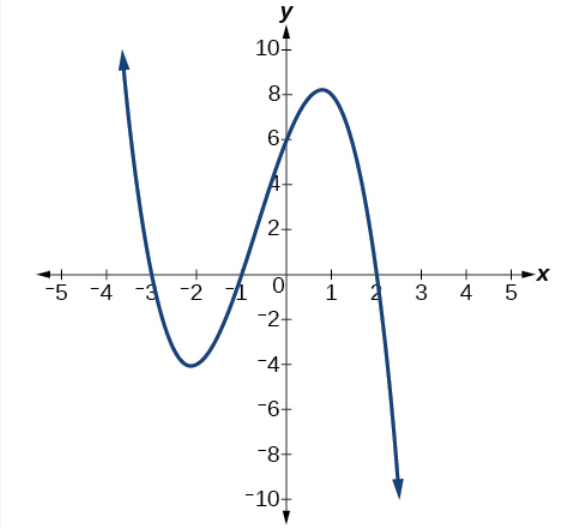


Figure 4.2.15.

Answer

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Solution

The end behavior indicates an odd-degree polynomial function; there are 3 x-intercepts and 2 turning points, so the degree is odd and at least 3. Because of the end behavior, we know that the lead coefficient must be negative.

Example 4.2.12: Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

Solution

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The y-intercept is $(0, 0)$.

The x-intercepts are found by determining the zeros of the function.

$$\begin{aligned} 0 &= -4x(x+3)(x-4) \\ x &= 0 \quad \text{or} \quad x+3 = 0 \quad \text{or} \quad x-4 = 0 \\ x &= 0 \quad \text{or} \quad x = -3 \quad \text{or} \quad x = 4 \end{aligned}$$

The x-intercepts are $(0, 0)$,$(-3, 0)$, and $(4, 0)$.

The degree is 3 so the graph has at most 2 turning points.

Exercise 4.2.8

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

Answer

The x-intercepts are $(2,0)$, $(-1,0)$, and $(5,0)$, the y-intercept is $(0,2)$, and the graph has at most 2 turning points.

Key Equations

- general form of a polynomial function: $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

Key Concepts

- A power function is a variable base raised to a number power.
- The behavior of a graph as the input decreases without bound and increases without bound is called the end behavior.
- The end behavior depends on whether the power is even or odd.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function.
- A polynomial of degree n will have at most n x-intercepts and at most $n-1$ turning points.

Glossary

coefficient

a nonzero real number that is multiplied by a variable raised to an exponent (only the number factor is the coefficient)

continuous function

a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

degree

the highest power of the variable that occurs in a polynomial

end behavior

the behavior of the graph of a function as the input decreases without bound and increases without bound

leading coefficient

the coefficient of the leading term

leading term

the term containing the highest power of the variable

polynomial function

a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

power function

a function that can be represented in the form $f(x) = kx^p$ where k is a constant, the base is a variable, and the exponent, p, is a constant

smooth curve

a graph with no sharp corners

term of a polynomial function

any $a_i x^i$ of a polynomial function in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

turning point

the location at which the graph of a function changes direction

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4.3: Graphs of Polynomial Functions

Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. Figure 4.3.1 shows a graph that represents a **polynomial function** and a graph that represents a function that is not a polynomial.

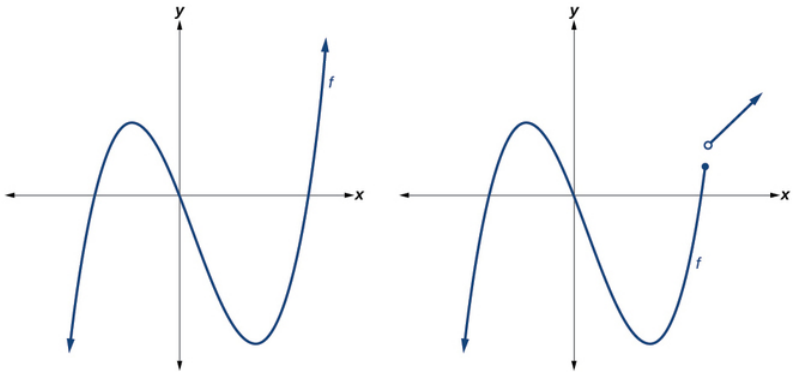


Figure 4.3.1: Graph of $f(x) = x^3 - 0.01x$.

Example 4.3.1: Recognizing Polynomial Functions

Which of the graphs in Figure 4.3.2 represents a polynomial function?

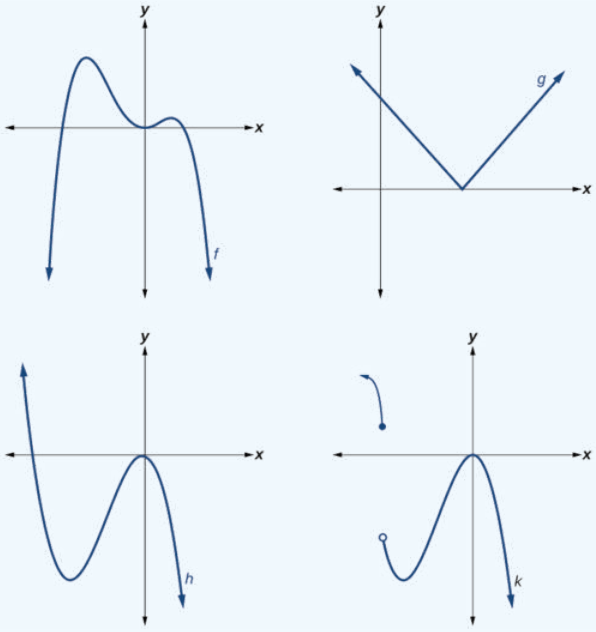


***Figure* 4.3.2**

Solution

- The graphs of f and h are graphs of polynomial functions. They are smooth and **continuous**.
- The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

Q&A

Do all polynomial functions have as their domain all real numbers?

- Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f(x)=0$ are called **zeros** of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
The polynomial is given in factored form.
Technology is used to determine the intercepts.

HOwTO: Given a polynomial function f, find the x-intercepts by factoring

1. Set $f(x)=0$.
2. If the polynomial function is not given in factored form:
   a. Factor out any common monomial factors.
   b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x-intercepts.

Example 4.3.2: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x)=x^6-3x^4+2x^2$.

Solution

We can attempt to factor this polynomial to find solutions for $f(x)=0$.

$$\begin{aligned} x^6-3x^4+2x^2 &= 0 && \text{Factor out the greatest common factor.}\\ x^2(x^4-3x^2+2) &= 0 && \text{Factor the trinomial.}\\ x^2(x^2-1)(x^2-2) &= 0 && \text{Set each factor equal to zero.}\end{aligned}$$

$$\begin{aligned} x^2&=0 \\ x^2&=0 \\ x&=0\end{aligned} \quad \text{or} \quad \begin{aligned} (x^2-1)&=0 \\ x^2&=1 \\ x&=\pm 1\end{aligned} \quad \text{or} \quad \begin{aligned} (x^2-2)&=0 \\ x^2&=2 \\ x&=\pm\sqrt{2}\end{aligned}$$

.

This gives us five x-intercepts: $(0,0)$, $(1,0)$, $(-1,0)$, $(\sqrt{2},0)$,and $(-\sqrt{2},0)$ (Figure 4.3.3). We can see that this is an even function.

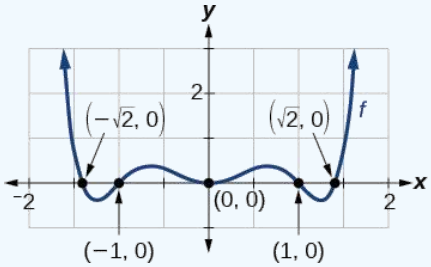


Figure 4.3.3.

Example 4.3.3: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

Solution

Find solutions for $f(x) = 0$ by factoring.

$$\begin{aligned} x^3 - 5x^2 - x + 5 &= 0 && \text{Factor by grouping.} \\ x^2(x-5) - (x-5) &= 0 && \text{Factor out the common factor.} \\ (x^2-1)(x-5) &= 0 && \text{Factor the difference of squares.} \\ (x+1)(x-1)(x-5) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x+1 &= 0 \quad \text{or} \quad x-1 = 0 \quad \text{or} \quad x-5 = 0 \\ x &= -1 \qquad\quad x = 1 \qquad\quad x = 5 \end{aligned}$$

There are three x-intercepts: $(-1, 0)$, $(1, 0)$, and $(5, 0)$ (Figure 4.3.4).

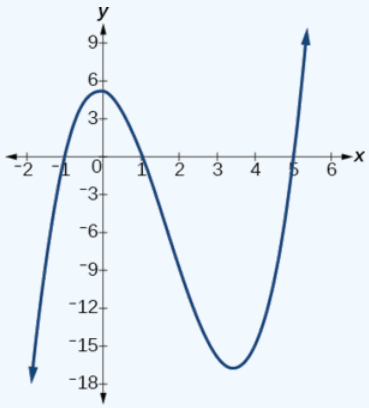


Figure 4.3.4: Graph of $f(x)$.

Example 4.3.4: Finding the y- and x-Intercepts of a Polynomial in Factored Form

Find the y- and x-intercepts of $g(x) = (x-2)^2(2x+3)$.

Solution

The y-intercept can be found by evaluating $g(0)$.

$$\begin{aligned} g(0) &= (0-2)^2(2(0)+3) \\ &= 12 \end{aligned}$$

So the y-intercept is $(0, 12)$.

The x-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3) = 0 \tag{4.3.1}$$

$$\begin{aligned} (x-2)^2 &= 0 \qquad\quad (2x+3) = 0 \\ x-2 &= 0 \quad \text{or} \quad x = -\frac{3}{2} \\ x &= 2 \end{aligned}$$

So the x-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

Analysis

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial as shown in Figure 4.3.5.

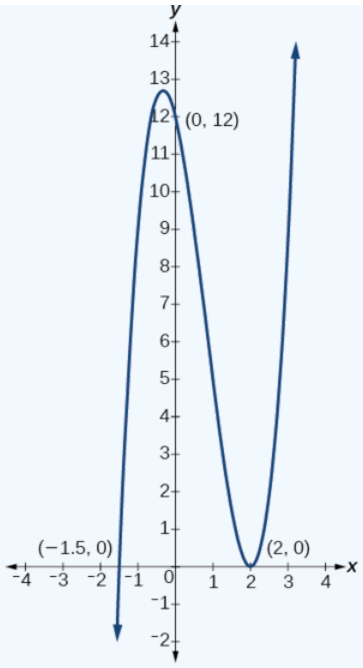


Figure 4.3.5: Graph of $g(x)$.

Example 4.3.5: Finding the x-Intercepts of a Polynomial Function Using a Graph

Find the x-intercepts of $h(x) = x^3 + 4x^2 + x - 6$.

Solution

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in Figure 4.3.6, it appears that there are x-intercepts at $x = -3, -2, \text{ and } 1$.

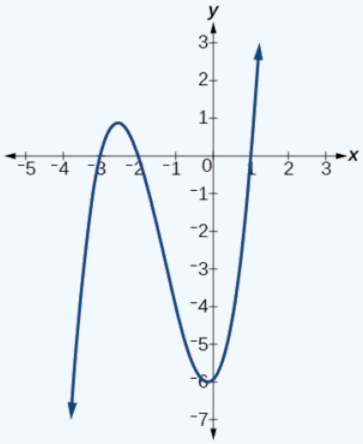


Figure 4.3.6: Graph of $h(x)$.

We can check whether these are correct by substituting these values for x and verifying that

$$h(-3) = h(-2) = h(1) = 0.$$

Since $h(x) = x^3 + 4x^2 + x - 6$, we have:

$$\begin{aligned} h(-3) &= (-3)^3+4(-3)^2+(-3)-6=-27+36-3-6=0 \\ h(-2) &= (-2)^3+4(-2)^2+(-2)-6=-8+16-2-6=0 \\ h(1) &= (1)^3+4(1)^2+(1)-6=1+4+1-6=0 \end{aligned}$$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$\begin{aligned} h(x) &= x^3+4x^2+x-6 \\ &= (x+3)(x+2)(x-1) \end{aligned}$$

Exercise 4.3.1

Find the y-and x-intercepts of the function $f(x)=x^4-19x^2+30x$.

Answer

- y-intercept $(0,0)$;
- x-intercepts $(0,0)$, $(-5,0)$, $(2,0)$, and $(3,0)$

Identifying Zeros and Their Multiplicities

Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and bounce off. Suppose, for example, we graph the function

$$f(x)=(x+3)(x-2)^2(x+1)^3. \tag{4.3.2}$$

Notice in Figure 4.3.7 that the behavior of the function at each of the x-intercepts is different.

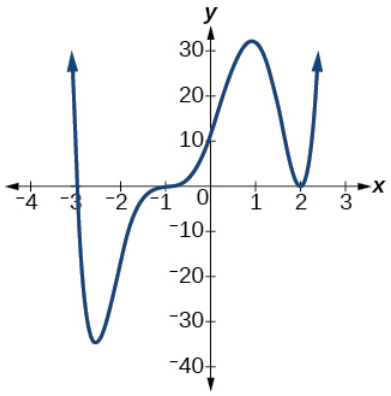


Figure 4.3.7: Identifying the behavior of the graph at an x-intercept by examining the multiplicity of the zero.

The x-intercept −3 is the solution of equation $(x+3)=0$. The graph passes directly through thex-intercept at $x=-3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

The x-intercept 2 is the repeated solution of equation $(x-2)^2=0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.

$$(x-2)^2=(x-2)(x-2) \tag{4.3.3}$$

The factor is repeated, that is, the factor $(x-2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x=2$, has multiplicity 2 because the factor $(x-2)$ occurs twice.

The x-intercept −1 is the repeated solution of factor $(x+1)^3=0$ .The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near

the intercept as the toolkit function $f(x) = x^3$. We call this a triple zero, or a zero with multiplicity 3.

For **zeros** with even multiplicities, the graphs touch or are tangent to the x-axis. For zeros with odd multiplicities, the graphs cross or intersect the x-axis. See Figure $4.3.8$ for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.

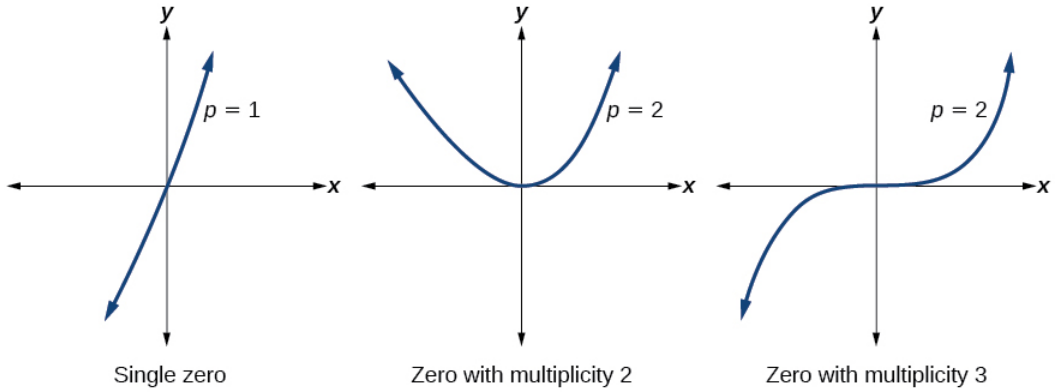


Figure $4.3.8$: Three graphs showing three different polynomial functions with multiplicity 1, 2, and 3.

For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

Graphical Behavior of Polynomials at x-Intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercepth is determined by the power p. We say that $x = h$ is a zero of **multiplicity** p.

The graph of a polynomial function will touch the x-axis at zeros with even multiplicities. The graph will cross the x-axis at zeros with odd multiplicities.

The sum of the multiplicities is the degree of the polynomial function.

HOWTO: Given a graph of a polynomial function of degree n, identify the zeros and their multiplicities

1. If the graph crosses the x-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the x-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is n.

Example $4.3.6$: Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in Figure $4.3.9$ to identify the zeros of the function and their possible multiplicities.

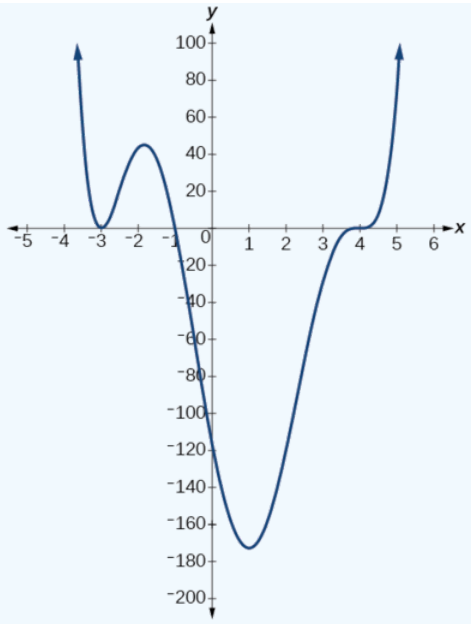


Figure 4.3.9: Graph of a polynomial function with degree 5.

Solution

The polynomial function is of degree n. The sum of the multiplicities must be n.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of −3 has multiplicity 2.

The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x = 4$.The graph crosses the x-axis, so the multiplicity of the zero must be odd. We know that the multiplicity is likely 3 and that the sum of the multiplicities is likely 6.

Exercise 4.3.2

Use the graph of the function of degree 5 in Figure 4.3.10to identify the zeros of the function and their multiplicities.

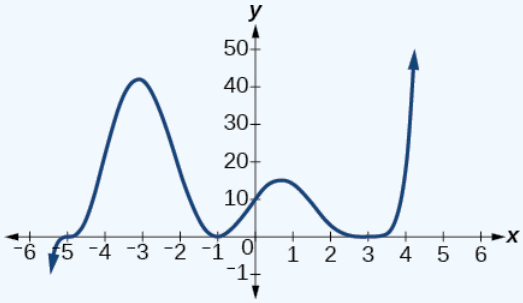


Figure* 4.3.10*: *Graph of a polynomial function with degree 5.*

Answer

The graph has a zero of –5 with multiplicity 1, a zero of –1 with multiplicity 2, and a zero of 3 with even multiplicity.

Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \tag{4.3.4}$$

will either ultimately rise or fall as x increases without bound and will either rise or fall as x decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the end behavior of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure $4.3.11$ summarizes all four cases.

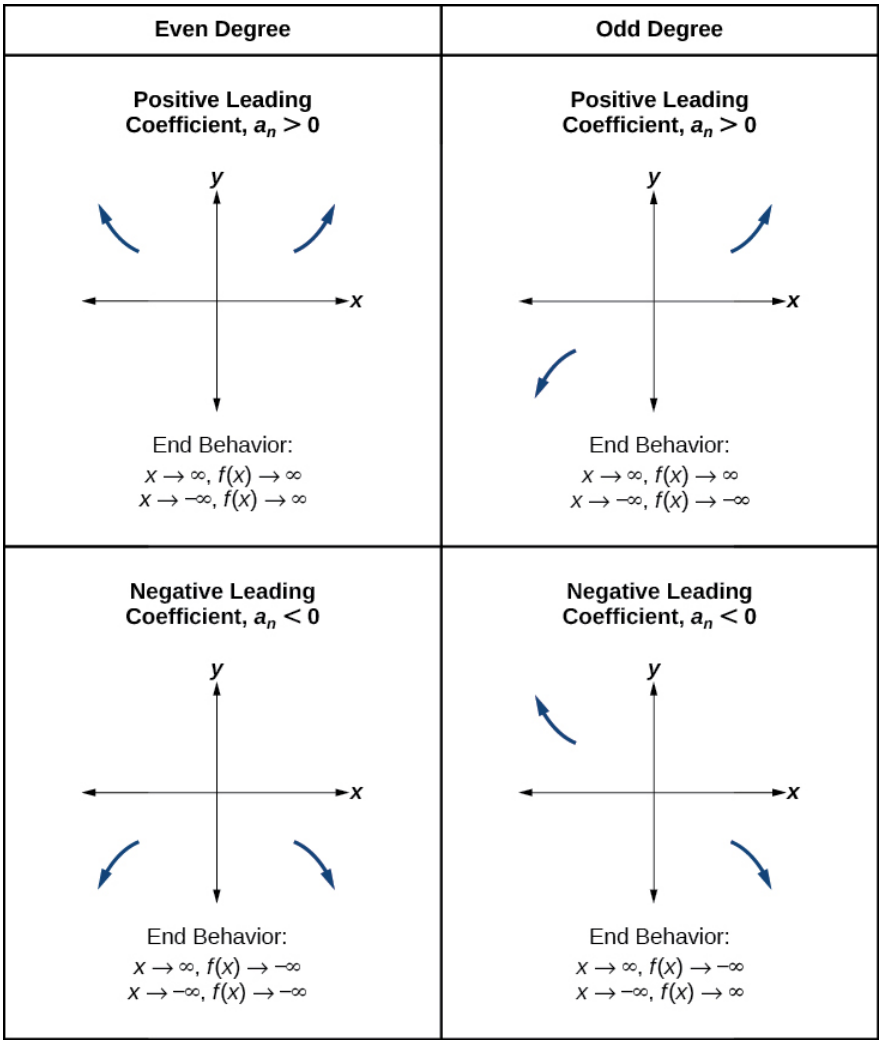


Figure 4.3.11.

Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure $4.3.12$ The graph has three turning points.

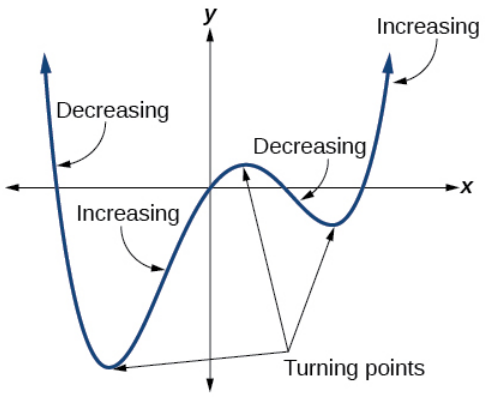


Figure 4.3.12: Graph of $f(x) = x^4 - x^3 - 4x^2 + 4x$

This function f is a 4th degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

Definition: Interpreting Turning Points

A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). A polynomial of degree n will have at most $n-1$ turning points.

Example 4.3.7: Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$
b. $f(x) = -(x-1)^2(1+2x^2)$

Solution

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$

First, rewrite the polynomial function in descending order: $f(x) = 4x^5 - x^3 - 3x^2 + 1$

Identify the degree of the polynomial function. This polynomial function is of degree 5.

The maximum number of turning points is $5 - 1 = 4$.

b. $f(x) = -(x-1)^2(1+2x^2)$

First, identify the leading term of the polynomial function if the function were expanded.

$$f(x) = -(x-1)^2\ (1+2x^2)$$
$$a_n = -(x^2)\ (2x^2) - 2x^4$$

Then, identify the degree of the polynomial function. This polynomial function is of degree 4.

The maximum number of turning points is $4 - 1 = 3$.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

Howto: Given a polynomial function, sketch the graph

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.

4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

Example 4.3.8: Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution

This graph has two x-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x = 5$,the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -2(0+3)^2(0-5) \\ &= -2 \cdot 9 \cdot (-5) \\ &= 90 \end{aligned}$$

The y-intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See Figure 4.3.13

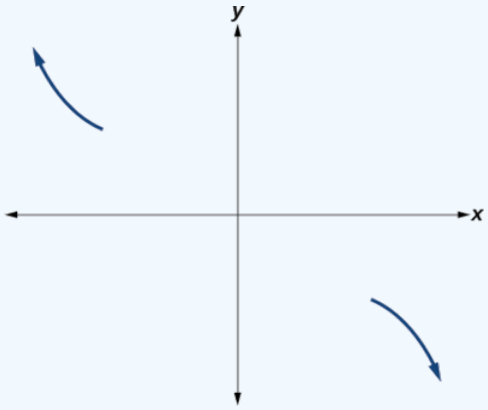


Figure 4.3.13: Showing the distribution for the leading term.

To sketch this, we consider that:

- As $x \to -\infty$ the function $f(x) \to \infty$,so we know the graph starts in the second quadrant and is decreasing toward the x-axis.
- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of thex-axis, so the function must start increasing.
- At $(0, 90)$, the graph crosses the y-axis at the y-intercept. See Figure 4.3.14

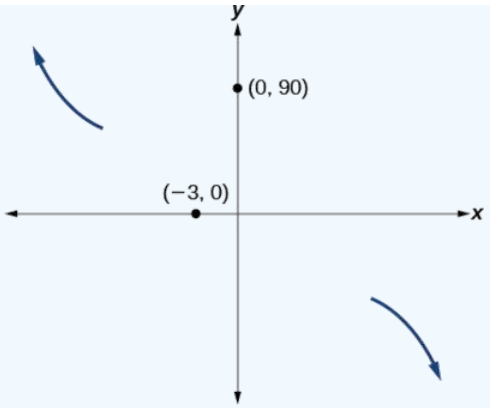


Figure 4.3.14: Graph of the end behavior and intercepts, $(-3,0)$ and $(0,90)$, for the function $f(x)=-2(x+3)^2(x-5)$.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5,0)$. See Figure 4.3.15.

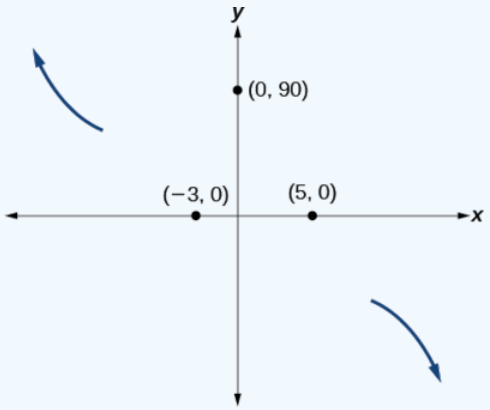


Figure 4.3.15: Graph of the end behavior and intercepts, $(-3,0)$, $(0,90)$ and $(5,0)$, for the function $f(x)=-2(x+3)^2(x-5)$.

As $x\to\infty$ the function $f(x)\to-\infty$,

so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure 4.3.16, and verify that the resulting graph looks like our sketch in Figure 4.3.15.

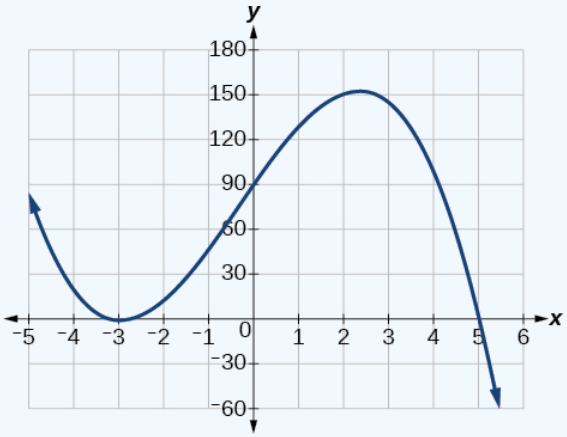


***Figure* 4.3.16:** *The complete graph of the polynomial function* $f(x)=-2(x+3)^2(x-5)$.

Exercise 4.3.8

Sketch a graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$.

Answer

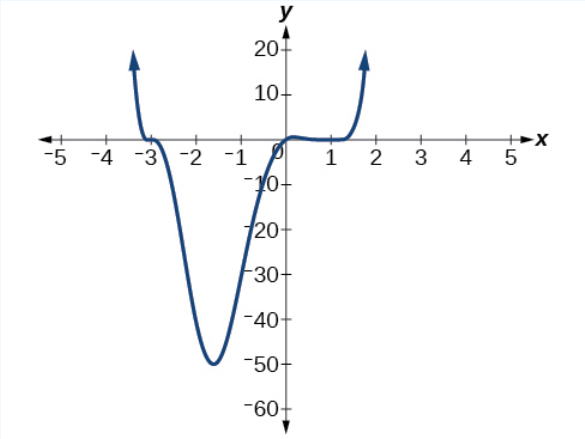


Figure 4.3.17: Graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$

Writing Formulas for Polynomial Functions

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a **polynomial function** written in factored form will have an x-intercept where each factor is equal to zero, we can form a function that will pass through a set of x-intercepts by introducing a corresponding set of factors.

Note: Factored Form of Polynomials

If a polynomial of lowest degree p has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, then the polynomial can be written in the factored form: $f(x) = a(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}$ where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the x-intercept.

How to: Given a graph of a polynomial function, write a formula for the function.

1. Identify the x-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the x-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the y-intercept may be easiest) to determine the stretch factor.

Example 4.3.10: Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in Figure 4.3.20

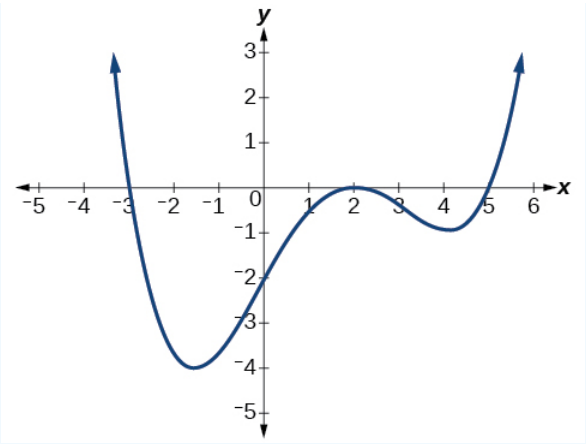


Figure 4.3.20.

Solution

This graph has three x-intercepts: $x=-3,\ 2,$ and 5. The y-intercept is located at $(0,2)$.At $x=-3$ and $x=5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x=2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

$$f(x)=a(x+3)(x-2)^2(x-5) \tag{4.3.5}$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0,-2)$, to solve for a.

$$\begin{aligned} f(0) &= a(0+3)(0-2)^2(0-5) \\ -2 &= a(0+3)(0-2)^2(0-5) \\ -2 &= -60a \\ a &= \frac{1}{30} \end{aligned}$$

The graphed polynomial appears to represent the function $f(x)=\frac{1}{30}(x+3)(x-2)^2(x-5)$.

Exercise 4.3.5

Given the graph shown in Figure $4.3.21$, write a formula for the function shown.

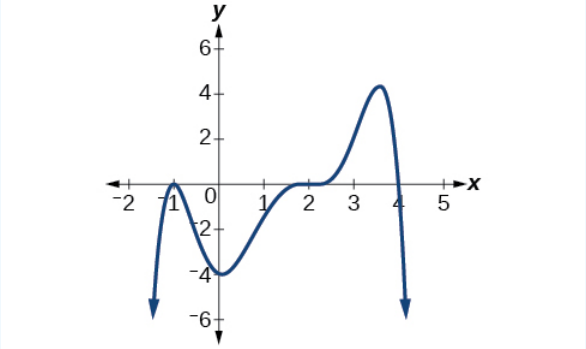


Figure 4.3.21.

Answer

$f(x)=-\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

Key Concepts

- Polynomial functions of degree 2 or more are smooth, continuous functions.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero.
- Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x-axis.
- The multiplicity of a zero determines how the graph behaves at the x-intercepts.
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree n has at most $n-1$ turning points.
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n-1$ turning points.
- Graphing a polynomial function helps to estimate local and global extremas.
- The Intermediate Value Theorem tells us that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c)=0$.

Glossary

global maximum
highest turning point on a graph; $f(a)$ where $f(a) \geq f(x)$ for all x.

global minimum
lowest turning point on a graph; $f(a)$ where $f(a) \leq f(x)$ for all x.

Intermediate Value Theorem
for two numbers a and b in the domain of f, if $a<b$ and $f(a) \neq f(b)$, then the functionf takes on every value between $f(a)$ and $f(b)$; specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis

multiplicity
the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x-h)^p$, $x=h$ is a zero of multiplicity p.

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4.4: Dividing Polynomials

Using Long Division to Divide Polynomials

We are familiar with the **long division** algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

Long Division

$$\begin{array}{r} 59 \\ 3\overline{)178} \\ \underline{-15\downarrow} \\ 28 \\ \underline{-27} \\ 1 \end{array}$$

Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$
Step 2: Bring down the 8
Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$

Answer: $59\,R\,1$ or $59\frac{1}{3}$

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \text{dividend} &= (\text{divisor}\cdot\text{quotient}) + \text{remainder} \\ 178 &= (3\cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm, it would look like this:

$$x+2\overline{)2x^3-3x^2+4x+5}$$ Set up the division problem.

$$\begin{array}{r} 2x^2 \\ x+2\overline{)2x^3-3x^2+4x+5} \end{array}$$ $2x^3$ divided by x is $2x^2$.

$$\begin{array}{r} 2x^2 \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \end{array}$$ Multiply $x+2$ by $2x^2$ and subtract.
Bring down the next term.

$$\begin{array}{r} 2x^2-7x \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \end{array}$$ $-7x^2$ divided by x is $-7x$.

$$\begin{array}{r} 2x^2-7x \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \\ \underline{-\left(-7x^2-14x\right)} \\ 18x+5 \end{array}$$ Multiply $x+2$ by $-7x$.
Subtract and bring down the next term.

$$\begin{array}{r} 2x^2-7x+18 \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \\ \underline{-\left(-7x^2-14x\right)} \\ 18x+5 \\ \underline{-(18x+36)} \\ -31 \end{array}$$ $18x$ divided by x is 18.
Multiply $x+2$ by 18.
Subtract.

We have found

$$\frac{2x^3-3x^2+4x+5}{x+2}=2x^2-7x+18-\frac{31}{x+2}$$

or

$$2x^3-3x^2+4x+5=(x+2)(2x^2-7x+18)-31$$

We can identify the dividend, the **divisor**, the **quotient**, and the **remainder**.

Writing the result in this manner illustrates the Division Algorithm.

The Division Algorithm

The Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x)=d(x)q(x)+r(x) \tag{4.4.1}$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

If $r(x)=0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

Given a polynomial and a binomial, use long division to divide the polynomial by the binomial

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom **binomial** from the top binomial.

5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

Example 4.4.1: Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2+3x-2$ by $x+1$.

Solution

$$
\begin{array}{rl}
x+1\,\overline{)\,5x^2+3x-2} & \text{Set up division problem.}\\
\quad 5x & \\
x+1\,\overline{)\,5x^2+3x-2} & 5x^2 \text{ divided by } x \text{ is } 5x.\\
\quad 5x & \\
x+1\,\overline{)\,5x^2+3x-2} & \text{Multiply } x+1 \text{ by } 5x.\\
\quad -(5x^2+5x) & \\
\hline
\quad -2x-2 & \text{Subtract.}\\
\quad 5x-2 & \text{Bring down the next term.}\\
x+1\,\overline{)\,5x^2+3x-2} & -2x \text{ divided by } x \text{ is } -2.\\
\quad -(5x^2+5x) & \\
\hline
\quad -2x-2 & \\
\quad -(-2x-2) & \text{Multiply } x+1 \text{ by } -2.\\
\hline
\quad 0 & \text{Subtract.}
\end{array}
$$

The quotient is $5x-2$. The remainder is 0. We write the result as

$$\frac{5x^2+3x-2}{x+1}=5x-2$$

or

$$5x^2+3x-2=(x+1)(5x-2)$$

Analysis

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

Example 4.4.2: Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3+11x^2-31x+15$ by $3x-2$.

Solution

$$
\begin{array}{rl}
\quad 2x^2+\ 5x-\ 7 & \\
3x-2\,\overline{)\,6x^3+11x^2-31x+15} & 6x^3 \text{ divided by } 3x \text{ is } 2x^2.\\
\quad -\left(6x^3-4x^2\right) & \text{Multiply } 3x-2 \text{ by } 2x^2.\\
\hline
\quad 15x^2-31x & \text{Subtract. Bring down next term. } 15x^2 \text{ divided by } 3x \text{ is } 5x.\\
\quad -\left(15x^2-10x\right) & \text{Multiply } 3x-2 \text{ by } 5x.\\
\hline
\quad -21x+15 & \text{Subtract. Bring down next term. } -21x \text{ divided by } 3x \text{ is } -7.\\
\quad -(-21x+14) & \text{Multiply } 3x-2 \text{ by } -7.\\
\hline
\quad 1 & \text{Subtract. The remainder is 1.}
\end{array}
$$

There is a remainder of 1. We can express the result as:

$$\frac{6x^3+11x^2-31x+15}{3x-2}=2x^2+5x-7+\frac{1}{3x-2}$$

Analysis

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x-2)(2x^2+5x-7)+1=6x^3+11x^2-31x+15$$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is 1

Try It! 4.4.2

Divide $16x^3-12x^2+20x-3$ by $4x+5$.

Solution

$$4x^2-8x+15-\frac{78}{4x+5}$$

Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3-3x^2+4x+5$ by $x+2$ using the long division algorithm.

The final form of the process looked like this:

$$\begin{array}{r} 2x^2-7x+18 \\ x+2\overline{)\,2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \\ \underline{-\left(-7x^2-14x\right)} \\ 18x+5 \\ \underline{-(18x+36)} \\ -31 \end{array}$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & 5 \\ & -2 & -4 & & \\ \hline & & -7 & 14 & \\ & & & 18 & -36 \\ \hline & & & & -31 \end{array}$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & 14 & -36 \\ \hline & 2 & -7 & 18 & -31 \end{array}$$

We then multiply it by the "divisor" and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2-7x+18$ and the remainder is -31. The process will be made more clear in Example 4.4.3.

Synthetic Division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x-k$. In *synthetic division*, only the coefficients are used in the division process.

Given two polynomials, use synthetic division to divide

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

Example 4.4.3: Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2-3x-36$ by $x-3$.

Solution

Begin by setting up the synthetic division. Write k and the coefficients.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ \hline & & & \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by k.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by k.Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x+12$. The remainder is 0. So $x-3$ is a factor of the original polynomial.

Analysis

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x-3)(5x+12)+0=5x^2-3x-36$$

Example 4.4.4: Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3+10x^2-6x-20$ by $x+2$.

Solution

The binomial divisor is $x+2$ so $k=-2$. Add each column, multiply the result by –2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -2 & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The result is $4x^2+2x-10$.

The remainder is 0. Thus, $x+2$ is a factor of $4x^3+10x^2-6x-20$.

Analysis

The graph of the polynomial function $f(x)=4x^3+10x^2-6x-20$ in Figure 4.4.2 shows a zero at $x=k=-2$. This confirms that $x+2$ is a factor of $4x^3+10x^2-6x-20$.

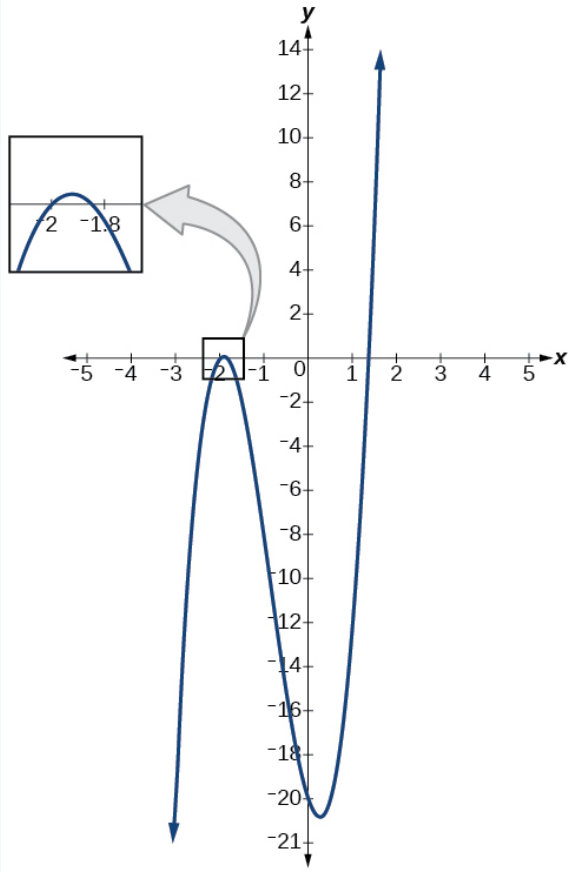


Figure 4.4.2

Example 4.4.5: Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4+10x^3+7x^2-6$ by $x-1$.

Solution

Notice there is no x-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3+x^2+8x+8+\dfrac{2}{x-1}$.

Try It! 4.4.3

Use synthetic division to divide $3x^4+18x^3-3x+40$ by $x+7$.

Solution

$$3x^3-3x^2+21x-150+\frac{1090}{x+7}$$

Key Equations

Division Algorithm $f(x)=d(x)q(x)+r(x)$ where $q(x)\neq 0$

Key Concepts

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form $x-k$.
- Polynomial division can be used to solve application problems, including area and volume.

Footnotes

[1] National Park Service. "Lincoln Memorial Building Statistics." www.nps.gov/linc/historycultu...statistics.htm. Accessed 4/3/2014

Glossary

Division Algorithm

given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x)=d(x)q(x)+r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

synthetic division

a shortcut method that can be used to divide a polynomial by a binomial of the form $x-k$

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4.5: Zeros of Polynomial Functions

Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by $x-k$, the remainder may be found quickly by evaluating the polynomial function at k, that is, $f(k)$. Let's walk through the proof of the theorem.

Recall that the Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$,there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x)=d(x)q(x)+r(x)$$

If the divisor, $d(x)$, is $x-k$, this takes the form

$$f(x)=(x-k)q(x)+r$$

Since the divisor $x-k$

is linear, the remainder will be a constant, r. And, if we evaluate this for $x=k$, we have

$$\begin{aligned} f(k) &= (k-k)q(k)+r \\ &= 0\cdot q(k)+r \\ &= r \end{aligned}$$

In other words, $f(k)$ is the remainder obtained by dividing $f(x)$by $x-k$.

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x-k$,then the remainder is the value $f(k)$.

How To: Given a polynomial function f, evaluate $f(x)$ at $x=k$ using the Remainder Theorem.

1. Use synthetic division to divide the polynomial by $x-k$.
2. The remainder is the value $f(k)$.

Example 4.5.1: Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate $f(x)=6x^4-x^3-15x^2+2x-7$ at $x=2$.

Solution

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x-2$.

$$\begin{array}{r|rrrrr} 2 & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array} \tag{4.5.1}$$

The remainder is 25. Therefore, $f(2)=25$.

Analysis

We can check our answer by evaluating $f(2)$.

$$\begin{aligned} f(x) &= 6x^4-x^3-15x^2+2x-7 \\ f(2) &= 6(2)^4-(2)^3-15(2)^2+2(2)-7 \\ &= 25 \end{aligned}$$

Exercise 4.5.1

se the Remainder Theorem to evaluate $f(x)=2x^5-3x^4-9x^3+8x^2+2$ at $x=-3$.

Answer

$f(-3)=-412$

Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm.

$$f(x)=(x-k)q(x)+r \tag{4.5.2}$$

If k is a zero, then the remainder r is $f(k)=0$ and $f(x)=(x-k)q(x)+0$ or $f(x)=(x-k)q(x)$.

Notice, written in this form, $x-k$ is a factor of $f(x)$. We can conclude if k is a zero of $f(x)$, then $x-k$ is a factor of $f(x)$.

Similarly, if $x-k$ is a factor of $f(x)$, then the remainder of the Division Algorithm $f(x)=(x-k)q(x)+r$ is 0. This tells us that k is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree n in the complex number system will have n zeros. We can use the Factor Theorem to completely factor a polynomial into the product of n factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

THE FACTOR THEOREM

According to the Factor Theorem, k is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$.

How to: Given a factor and a third-degree polynomial, use the Factor Theorem to factor the polynomial

1. Use synthetic division to divide the polynomial by $(x-k)$.
2. Confirm that the remainder is 0.
3. Write the polynomial as the product of $(x-k)$ and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

Example 4.5.2: Using the Factor Theorem to Solve a Polynomial Equation

Show that $(x+2)$ is a factor of x^3-6x^2-x+30. Find the remaining factors. Use the factors to determine the zeros of the **polynomial**.

Solution

We can use synthetic division to show that $(x+2)$ is a factor of the polynomial.

$$\begin{array}{r|cccc} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array} \tag{4.5.3}$$

The remainder is zero, so $(x+2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$$(x+2)(x^2-8x+15) \tag{4.5.4}$$

We can factor the quadratic factor to write the polynomial as

$$(x+2)(x-3)(x-5) \tag{4.5.5}$$

By the Factor Theorem, the zeros of x^3-6x^2-x+30 are –2, 3, and 5.

Exercise 4.5.2

Use the Factor Theorem to find the zeros of $f(x) = x^3 + 4x^2 - 4x - 16$ given that $(x - 2)$ is a factor of the polynomial.

Answer

The zeros are 2, –2, and –4.

Using the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading **coefficient** of the polynomial

Consider a quadratic function with two zeros, $x = \frac{2}{5}$ and $x = \frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0, and then construct the original quadratic function absent its stretching factor.

$x - \frac{2}{5} = 0$ or $x - \frac{3}{4} = 0$	Set each factor equal to 0.
$5x - 2 = 0$ or $4x - 3 = 0$	Multiply both sides of the equation to eliminate fractions.
$f(x) = (5x - 2)(4x - 3)$	Create the quadratic function, multiplying the factors.
$f(x) = 20x^2 - 23x + 6$	Expand the polynomial.
$f(x) = (5 \cdot 4)x^2 - 23x + (2 \cdot 3)$	

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

THE RATIONAL ZERO THEOREM

The Rational Zero Theorem states that, if the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ has integer coefficients, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

How to: Given a polynomial function $f(x)$, use the Rational Zero Theorem to find rational zeros.

1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of $f(\frac{p}{q})$.

Example 4.5.3: Listing All Possible Rational Zeros

List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$.

Solution

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, –4, and the factors of the leading coefficient, 2.

The constant term is –4; the factors of –4 are $p = \pm 1, \pm 2, \pm 4$.

The leading coefficient is 2; the factors of 2 are $q = \pm 1, \pm 2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of –4 divided by one of the factors of 2.

$$\frac{p}{q}=\pm\frac{1}{1},\pm\frac{1}{2} \qquad \frac{p}{q}=\pm\frac{2}{1},\pm\frac{2}{2} \qquad \frac{p}{q}=\pm\frac{4}{1},\pm\frac{4}{2}$$

Note that $\frac{2}{2}=1$ and $\frac{4}{2}=2$, which have already been listed. So we can shorten our list.

$$\frac{p}{q}=\frac{\text{Factors of the last}}{\text{Factors of the first}}=\pm1,\pm2,\pm4,\pm\frac{1}{2}$$

Example 4.5.4: Using the Rational Zero Theorem to Find Rational Zeros

Use the Rational Zero Theorem to find the rational zeros of $f(x)=2x^3+x^2-4x+1$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 1 and q is a factor of 2.

$$\frac{p}{q}=\frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$

$$=\frac{factor\ of\ 1}{factor\ of\ 2}$$

The factors of 1 are ±1 and the factors of 2 are ±1 and ±2. The possible values for $\frac{p}{q}$ are ±1 and $\pm\frac{1}{2}$. These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by substituting these values for x in $f(x)$.

$$f(-1)=2(-1)^3+(-1)^2-4(-1)+1=4 \tag{4.5.6}$$

$$f(1)=1(1)^3+(1)^2-4(1)+1=0 \tag{4.5.7}$$

$$f(-\frac{1}{2})=2(-\frac{1}{2})^3+(-\frac{1}{2})^2-4(-\frac{1}{2})+1=3 \tag{4.5.8}$$

$$f(\frac{1}{2})=2(\frac{1}{2})^3+(\frac{1}{2})^2-4(\frac{1}{2})+1=-\frac{1}{2} \tag{4.5.9}$$

Of those, $-1,-\frac{1}{2}$, and $\frac{1}{2}$ are not zeros of $f(x)$. 1 is the only rational zero of $f(x)$.

Exercise 4.5.3

Use the Rational Zero Theorem to find the rational zeros of $f(x)=x^3-5x^2+2x+1$.

Answer

There are no rational zeros.

Finding the Zeros of Polynomial Functions

The Rational Zero Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use **synthetic division** repeatedly to determine all of the **zeros** of a polynomial function.

How to: Given a polynomial function f, use synthetic division to find its zeros.

1. Use the Rational Zero Theorem to list all possible rational zeros of the function.
2. Use synthetic division to evaluate a given possible zero by synthetically dividing the candidate into the polynomial. If the remainder is 0, the candidate is a zero. If the remainder is not zero, discard the candidate.
3. Repeat step two using the quotient found with synthetic division. If possible, continue until the quotient is a quadratic.

4. Find the zeros of the quadratic function. Two possible methods for solving quadratics are factoring and using the quadratic formula.

Example 4.5.5: Finding the Zeros of a Polynomial Function with Repeated Real Zeros

Find the zeros of $f(x) = 4x^3 - 3x - 1$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of –1 and q is a factor of 4.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$
$$= \frac{factor\ of\ -1}{factor\ of\ 4}$$

The factors of –1 are ±1 and the factors of 4 are ±1,±2, and ±4. The possible values for $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}$, and $\pm\frac{1}{4}$. These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 1.

Dividing by $(x-1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

$$(x-1)(4x^2+4x+1)$$

The quadratic is a perfect square. $f(x)$ can be written as

$$(x-1)(2x+1)^2$$

We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

$$2x+1=0$$
$$x = -\frac{1}{2}$$

The zeros of the function are 1 and $-\frac{1}{2}$ with multiplicity 2.

Analysis

Look at the graph of the function f in Figure 4.5.1. Notice, at $x = -0.5$, the graph bounces off the x-axis, indicating the even multiplicity (2,4,6…) for the zero −0.5. At $x = 1$, the graph crosses the x-axis, indicating the odd multiplicity (1,3,5…) for the zero $x = 1$.

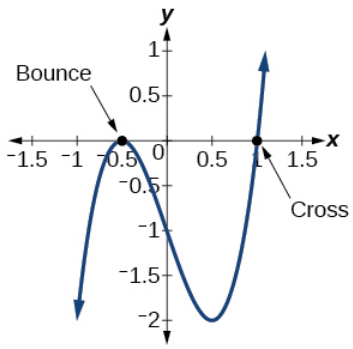


Figure 4.5.1.

Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one

complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose f is a polynomial function of degree four, and $f(x)=0$. The Fundamental Theorem of Algebra states that there is at least one complex solution, call it c_1. By the Factor Theorem, we can write $f(x)$ as a product of $x-c_1$ and a polynomial quotient. Since $x-c_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it c_2. So we can write the polynomial quotient as a product of $x-c_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of $f(x)$.

THE FUNDAMENTAL THEOREM OF ALGEBRA

The *Fundamental Theorem of Algebra* states that, if $f(x)$ is a polynomial of degree $n>0$, then $f(x)$ has at least one complex zero.

We can use this theorem to argue that, if $f(x)$ is a polynomial of degree $n>0$, and a is a non-zero real number, then $f(x)$ has exactly n linear factors

$$f(x)=a(x-c_1)(x-c_2)\ldots(x-c_n) \tag{4.5.10}$$

where $c_1, c_2,...,c_n$ are complex numbers. Therefore, $f(x)$ has n roots if we allow for multiplicities.

Q&A: Does every polynomial have at least one imaginary zero?

No. Real numbers are a subset of complex numbers, but not the other way around. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

Example 4.5.6: Finding the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of $f(x)=3x^3+9x^2+x+3$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 3 and q is a factor of 3.

$$\frac{p}{q}=\frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$
$$=\frac{factor\ of\ 3}{factor\ of\ 3}$$

The factors of 3 are ±1 and ±3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are ±3,±1, and $\pm\frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with –3.

Dividing by $(x+3)$ gives a remainder of 0, so –3 is a zero of the function. The polynomial can be written as

$$(x+3)(3x^2+1)$$

We can then set the quadratic equal to 0 and solve to find the other zeros of the function.

$$3x^2+1=0$$
$$x^2=-\frac{1}{3}$$
$$x=\pm-\sqrt{\frac{1}{3}}$$
$$=\pm\frac{i\sqrt{3}}{3}$$

The zeros of $f(x)$ are -3 and $\pm\frac{i\sqrt{3}}{3}$.

Analysis

Look at the graph of the function f in Figure 4.5.2. Notice that, at $x=-3$, the graph crosses the x-axis, indicating an odd multiplicity (1) for the zero $x=-3$. Also note the presence of the two turning points. This means that, since there is a 3rd degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the x-intercepts for the function are shown. So either the multiplicity of $x=-3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x=-3$ is three. Either way, our result is correct.

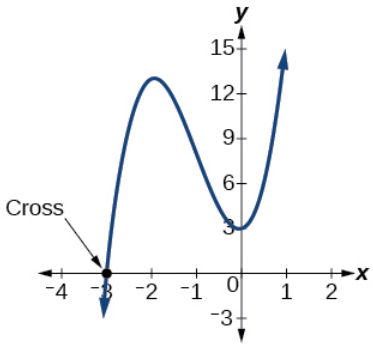


Figure 4.5.2.

4.5.4

Find the zeros of $f(x)=2x^3+5x^2-11x+4$.

Solution

The zeros are -4, $\frac{1}{2}$, and 1.

Using the Linear Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the **Fundamental Theorem of Algebra**, as we stated above, is that a polynomial function of degree n will have n zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into n factors. The **Linear Factorization Theorem** tells us that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x-c)$, where c is a complex number.

Let f be a polynomial function with real coefficients, and suppose $a+bi$, $b\neq 0$, is a zero of $f(x)$. Then, by the Factor Theorem, $x-(a+bi)$ is a factor of $f(x)$. For f to have real coefficients, $x-(a-bi)$ must also be a factor of $f(x)$. This is true because any factor other than $x-(a-bi)$, when multiplied by $x-(a+bi)$, will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function f with real coefficients has a complex zero $a+bi$, then the complex conjugate $a-bi$ must also be a zero of $f(x)$. This is called the **Complex Conjugate Theorem**.

COMPLEX CONJUGATE THEOREM

According to the Linear Factorization Theorem, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where c is a complex number.

If the polynomial function f has real coefficients and a complex zero in the form $a+bi$, then the complex conjugate of the zero, $a-bi$, is also a zero.

How to

Given the zeros of a polynomial function f and a point $(c,f(c))$ on the graph of f, use the Linear Factorization Theorem to find the polynomial function.

1. Use the zeros to construct the linear factors of the polynomial.

2. Multiply the linear factors to expand the polynomial.
3. Substitute $(c, f(c))$ into the function to determine the leading coefficient.
4. Simplify.

Example 4.5.7: Using the Linear Factorization Theorem to Find a Polynomial with Given Zeros

Find a fourth degree polynomial with real coefficients that has zeros of -3, 2, i, such that $f(-2) = 100$.

Solution

Because $x = i$ is a zero, by the Complex Conjugate Theorem $x = -i$ is also a zero. The polynomial must have factors of $(x+3), (x-2), (x-i)$, and $(x+i)$. Since we are looking for a degree 4 polynomial, and now have four zeros, we have all four factors. Let's begin by multiplying these factors.

$$f(x) = a(x+3)(x-2)(x-i)(x+i) \tag{4.5.11}$$
$$f(x) = a(x^2+x-6)(x^2+1) \tag{4.5.12}$$
$$f(x) = a(x^4+x^3-5x^2+x-6) \tag{4.5.13}$$

We need to find a to ensure $f(-2) = 100$. Substitute $x = -2$ and $f(-2) = 100$ into $f(x)$.

$$100 = a((-2)^4+(-2)^3-5(-2)^2+(-2)-6) \tag{4.5.14}$$
$$100 = a(-20) \tag{4.5.15}$$
$$-5 = a \tag{4.5.16}$$

So the polynomial function is

$$f(x) = -5(x^4+x^3-5x^2+x-6) \tag{4.5.17}$$

or

$$f(x) = -5x^4-5x^3+25x^2-5x+30 \tag{4.5.18}$$

Analysis

We found that both i and $-i$ were zeros, but only one of these zeros needed to be given. If i is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of i.

Q&A

If $2+3i$ were given as a zero of a polynomial with real coefficients, would $2-3i$ also need to be a zero?

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

4.5.5

Find a third degree polynomial with real coefficients that has zeros of 5 and $-2i$ such that $f(1) = 10$.

Solution

$f(x) = -\frac{1}{2}x^3+\frac{5}{2}x^2-2x+10$

Using Descartes' Rule of Signs

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign changes in $f(x)$ and the number of positive real zeros. For example, the polynomial function below has one sign change.

$$f(x) = x^4 + x^3 + x^2 + x - 1$$

This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in $f(-x)$ and the number of negative real zeros.

$$f(-x) = (-x)^4 + (-x)^3 + (-x)^2 + (-x) - 1$$
$$f(-x) = +x^4 - x^3 + x^2 - x - 1$$

In this case, $f(-x)$ has 3 sign changes. This tells us that $f(x)$ could have 3 or 1 negative real zeros.

DESCARTES' RULE OF SIGNS

According to *Descartes' Rule of Signs*, if we let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial function with real coefficients:

- The number of positive real zeros is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.

Example 4.5.8: Using Descartes' Rule of Signs

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for $f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$.

Solution

Begin by determining the number of sign changes.

There are two sign changes, so there are either 2 or 0 positive real roots. Next, we examine $f(-x)$ to determine the number of negative real roots.

$$f(-x) = -(-x)^4 - 3(-x)^3 + 6(-x)^2 - 4(-x) - 12 \quad (4.5.19)$$
$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12 \quad (4.5.20)$$

$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12$$

Again, there are two sign changes, so there are either 2 or 0 negative real roots.

There are four possibilities, as we can see in Table 4.5.1.

Table 4.5.1

Positive Real Zeros	Negative Real Zeros	Complex Zeros	Total Zeros
2	2	0	4
2	0	2	4
0	2	2	4
0	0	4	4

Analysis

We can confirm the numbers of positive and negative real roots by examining a graph of the function. See Figure 4.5.3. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.

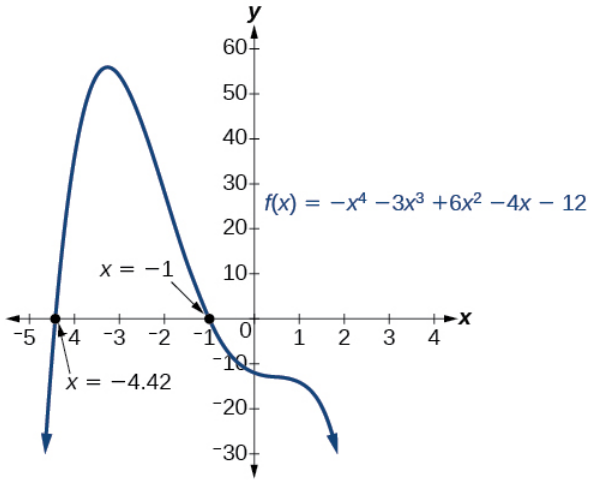


Figure 4.5.3.

4.5.6

Use Descartes' Rule of Signs to determine the maximum possible numbers of positive and negative real zeros for $f(x) = 2x^4 - 10x^3 + 11x^2 - 15x + 12$. Use a graph to verify the numbers of positive and negative real zeros for the function.

Solution

There must be 4, 2, or 0 positive real roots and 0 negative real roots. The graph shows that there are 2 positive real zeros and 0 negative real zeros.

Key Concepts

- To find $f(k)$, determine the remainder of the polynomial $f(x)$ when it is divided by $x - k$. This is known as the Remainder Theorem. See Example 4.5.1.
- According to the Factor Theorem, k is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$. See Example4.5.2.
- According to the Rational Zero Theorem, each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient. See Example 4.5.3 and Example 4.5.4.
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function. See Example 4.5.5.
- According to the Fundamental Theorem, every polynomial function with degree greater than 0 has at least one complex zero. See Example 4.5.6.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form $(x - c)$, where c is a complex number. See Example 4.5.7.
- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of $f(-x)$ or less than the number of sign changes by an even integer. See Example 4.5.8.
- Polynomial equations model many real-world scenarios. Solving the equations is easiest done by synthetic division. See Example 4.5.9.

Glossary

Descartes' Rule of Signs

a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $f(x)$ and $f(-x)$

Factor Theorem

k is a zero of polynomial function $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$

Fundamental Theorem of Algebra

a polynomial function with degree greater than 0 has at least one complex zero

Linear Factorization Theorem

allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where c is a complex number

Rational Zero Theorem

the possible rational zeros of a polynomial function have the form $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading coefficient.

Remainder Theorem

if a polynomial $f(x)$ is divided by $x-k$,then the remainder is equal to the value $f(k)$

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CHAPTER OVERVIEW

5: Rational and Radical Functions

5.1: Rational Functions
5.2: Radical Functions

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5.1: Rational Functions

Using Arrow Notation

We have seen the graphs of the basic **reciprocal function** and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure 5.1.1, and notice some of their features.

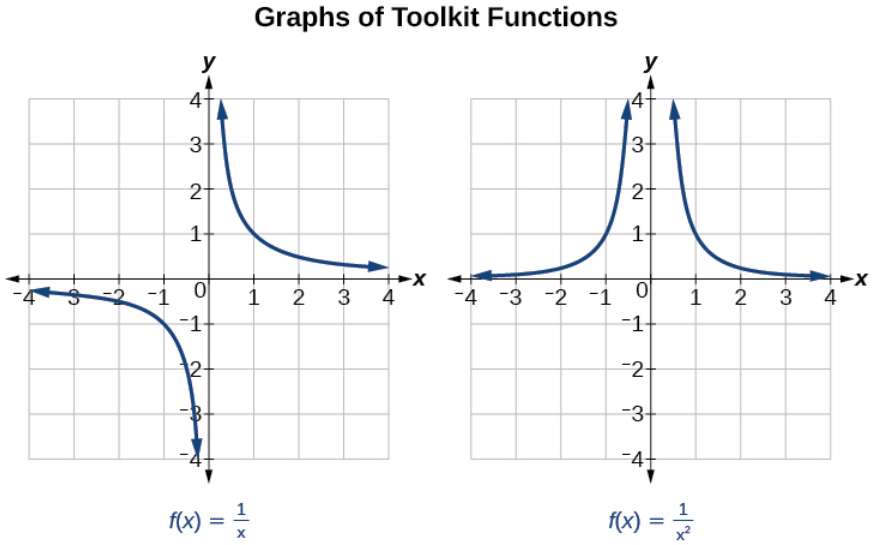


Figure 5.1.1

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

- On the left branch of the graph, the curve approaches the x-axis $(y = 0)$ as $x \to -\infty$.
- As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.
- Finally, on the right branch of the graph, the curves approaches the x-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that x or $f(x)$ is approaching a particular value (Table 5.1.1).

Table 5.1.1

Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Finding the Domains of Rational Functions

A *vertical asymptote* represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Definition: DOMAIN OF A RATIONAL FUNCTION

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

How To: Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

Example 5.1.4: Finding the Domain of a Rational Function

Find the domain of $f(x) = \dfrac{x+3}{x^2-9}$.

Solution

Begin by setting the denominator equal to zero and solving.

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$.

Analysis

A graph of this function, as shown in Figure 5.1.9, confirms that the function is not defined when $x = \pm 3$.

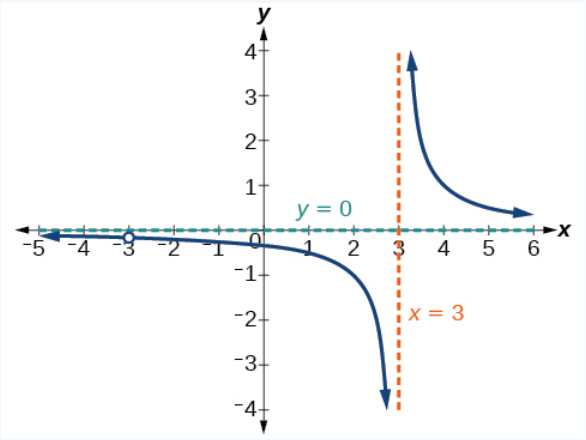


Figure 5.1.9.

There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

Exercise 5.1.4

Find the domain of $f(x) = \dfrac{4x}{5(x-1)(x-5)}$.

Answer

The domain is all real numbers except $x = 1$ and $x = 5$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

How To: Given a rational function, identify any vertical asymptotes of its graph

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or "holes."

Example 5.1.5: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \dfrac{5+2x^2}{2-x-x^2}$.

Solution

First, factor the numerator and denominator.

$$k(x) = \frac{5+2x^2}{2-x-x^2}$$
$$= \frac{5+2x^2}{(2+x)(1-x)}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2+x)(1-x) = 0$$
$$x = -2,\ x = 1$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 5.1.10 confirms the location of the two vertical asymptotes.

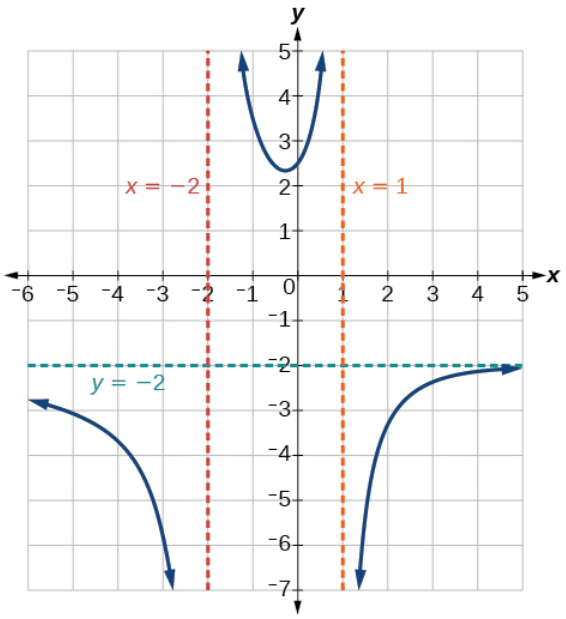


***Figure* 5.1.10.**

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a *removable discontinuity*. For example, the function $f(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x = -1$, is the location of the removable discontinuity. Notice also that $(x-3)$ is not a factor in both the numerator and denominator. The zero of this factor, $x = 3$, is the vertical asymptote. See Figure 5.1.11. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

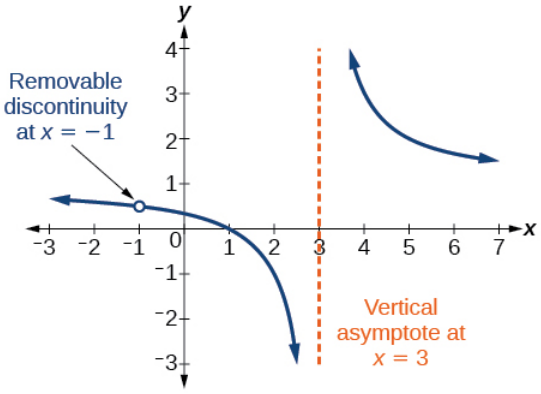


Figure 5.1.11.

REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x = a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the

multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Example 5.1.6: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \dfrac{x-2}{x^2-4}$.

Solution

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x-2$. The zero for this factor is $x=2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x+2$. The zero for this factor is $x=-2$. The vertical asymptote is $x=-2$. See Figure 5.1.12

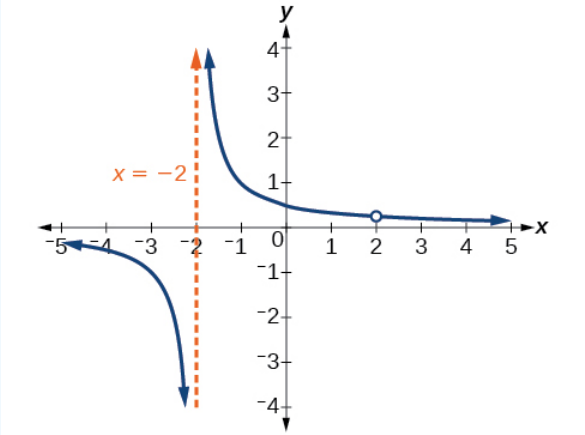


Figure 5.1.12.

The graph of this function will have the vertical asymptote at $x=-2$, but at $x=2$ the graph will have a hole.

Exercise 5.1.5

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \dfrac{x^2-25}{x^3-6x^2+5x}$.

Answer

Removable discontinuity at $x=5$.

Vertical asymptotes: $x=0$, $x=1$.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

How To: Given a rational function, identify any vertical asymptotes of its graph

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.

4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or "holes."

Example 5.1.5: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \dfrac{5+2x^2}{2-x-x^2}$.

Solution

First, factor the numerator and denominator.

$$k(x) = \frac{5+2x^2}{2-x-x^2}$$
$$= \frac{5+2x^2}{(2+x)(1-x)}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2+x)(1-x) = 0$$
$$x = -2,\ x = 1$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 5.1.10 confirms the location of the two vertical asymptotes.

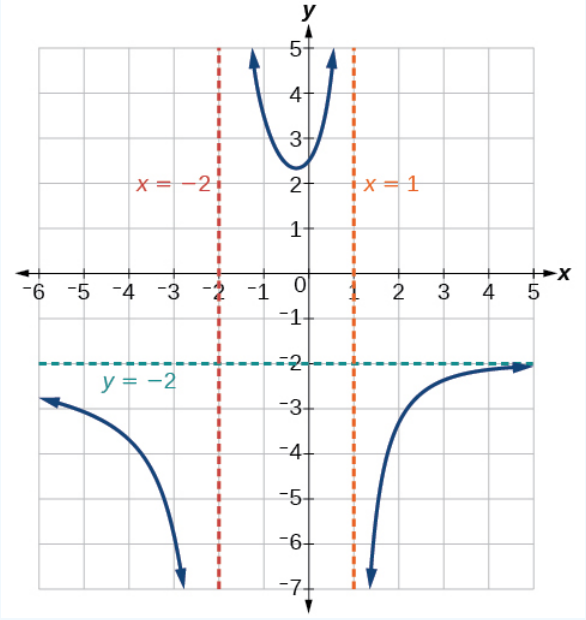


***Figure* 5.1.10.**

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a *removable discontinuity*. For example, the function $f(x) = \dfrac{x^2-1}{x^2-2x-3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x=-1$, is the location of the removable discontinuity. Notice also that $(x-3)$ is not a factor in both the numerator and denominator. The zero of this factor, $x=3$, is the vertical asymptote. See Figure 5.1.11. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

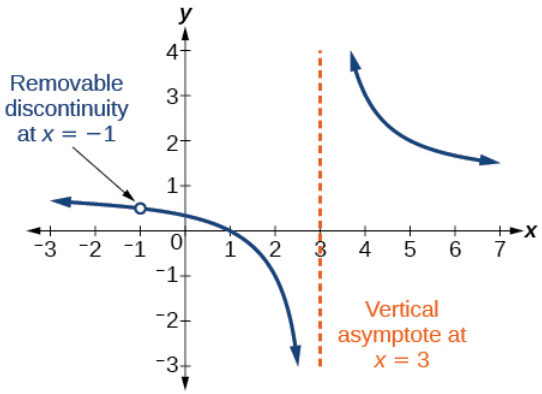


Figure 5.1.11.

REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x=a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Example 5.1.6: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x)=\dfrac{x-2}{x^2-4}$.

Solution

Factor the numerator and the denominator.

$$k(x)=\frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x-2$. The zero for this factor is $x=2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x+2$. The zero for this factor is $x=-2$. The vertical asymptote is $x=-2$. See Figure 5.1.12

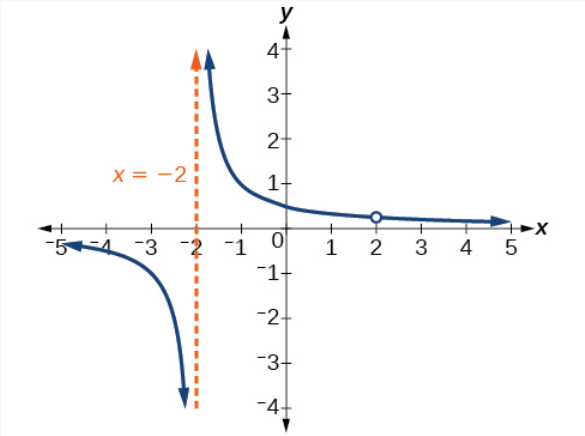


Figure 5.1.12.

The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole.

Exercise 5.1.5

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \dfrac{x^2 - 25}{x^3 - 6x^2 + 5x}$.

Answer

Removable discontinuity at $x = 5$.

Vertical asymptotes: $x = 0$, $x = 1$.

Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are three distinct outcomes when checking for horizontal asymptotes:

Case 1: If the degree of the denominator > degree of the numerator, there is a **horizontal asymptote** at $y = 0$.

Example: $f(x) = \dfrac{4x + 2}{x^2 + 4x - 5}$

In this case, the end behavior is $f(x) \approx \dfrac{4x}{x^2} = \dfrac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = \dfrac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y = 0$. See **Figure** 5.1.13. Note that this graph crosses the horizontal asymptote.

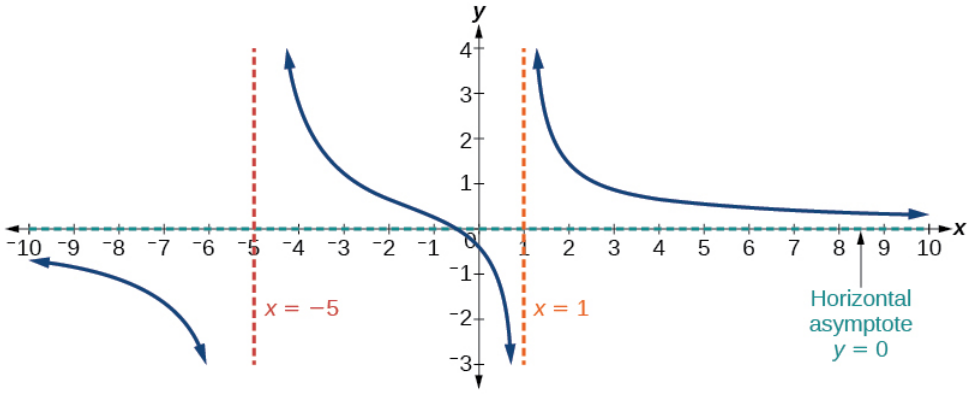


Figure 5.1.13: Horizontal asymptote $y = 0$ when $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of p < degree of q.

Case 2: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

Example: $f(x) = \dfrac{3x^2 - 2x + 1}{x - 1}$

In this case, the end behavior is $f(x) \approx \dfrac{3x^2}{x} = 3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x) = 3x$ looks like a diagonal line, and since f will behave similarly to g, it will approach a line close to $y = 3x$. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\frac{3x^2-2x+1}{x-1}$. The quotient is $3x+1$, and the remainder is 2. The slant asymptote is the graph of the line $g(x)=3x+1$. See Figure 5.1.14

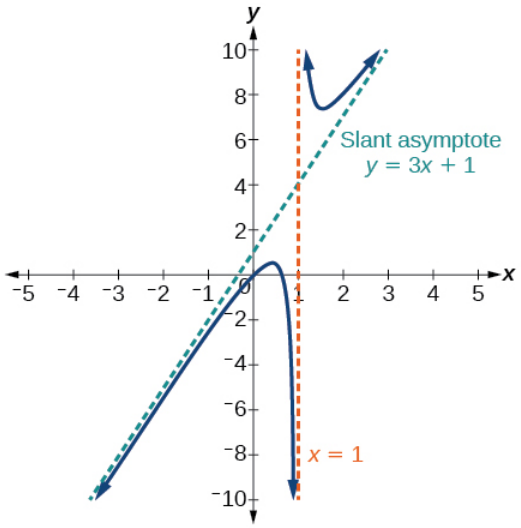


Figure 5.1.14: Slant asymptote when $f(x)=\frac{p(x)}{q(x)}$, $q(x)\neq 0$ where degree of p >degree of q by 1.

Case 3: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y=\frac{a_n}{b_n}$, where a_n and b_n are respectively the leading coefficients of $p(x)$ and $q(x)$ for $f(x)=\frac{p(x)}{q(x)}$, $q(x)\neq 0$.

Example: $f(x)=\frac{3x^2+2}{x^2+4x-5}$

In this case, the end behavior is $f(x)\approx\frac{3x^2}{x^2}=3$. This tells us that as the inputs grow large, this function will behave like the function $g(x)=3$, which is a horizontal line. As $x\to\pm\infty$, $f(x)\to 3$, resulting in a horizontal asymptote at $y=3$. See Figure 5.1.15. Note that this graph crosses the horizontal asymptote.

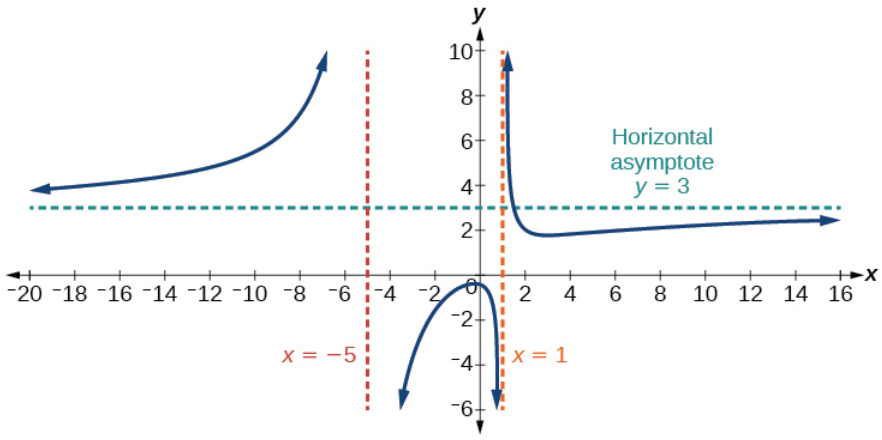


Figure 5.1.15: Horizontal asymptote when $f(x)=\frac{p(x)}{q(x)}$, $q(x)\neq 0$ where degree of p = degree of q.

Notice that, while the graph of a rational function will never cross a **vertical asymptote**, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the **end behavior** of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x + 3}$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \to \pm\infty, f(x) \to \infty$$

HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator is less than degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

Example 5.1.7: Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$
2. $h(x) = \dfrac{x^2 - 4x + 1}{x + 2}$
3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$

Solution

For these solutions, we will use $f(x) = \dfrac{p(x)}{q(x)},\ q(x) \neq 0$.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$: The degree of $p =$ degree of $q = 3$, so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.
2. $h(x) = \dfrac{x^2 - 4x + 1}{x + 2}$: The degree of $p = 2$ and degree of $q = 1$. Since $p > q$ by 1, there is a slant asymptote found at $\dfrac{x^2 - 4x + 1}{x + 2}$.

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$: The degree of $p = 2 <$ degree of $q = 3$, so there is a horizontal asymptote $y = 0$.

Example 5.1.8 Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \dfrac{5 + t}{100 + 10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

Solution

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is 10t, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty,\ C(t) \to \tfrac{1}{10}$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Example 5.1.9: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

Solution

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1, -2,$and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure 5.1.16.

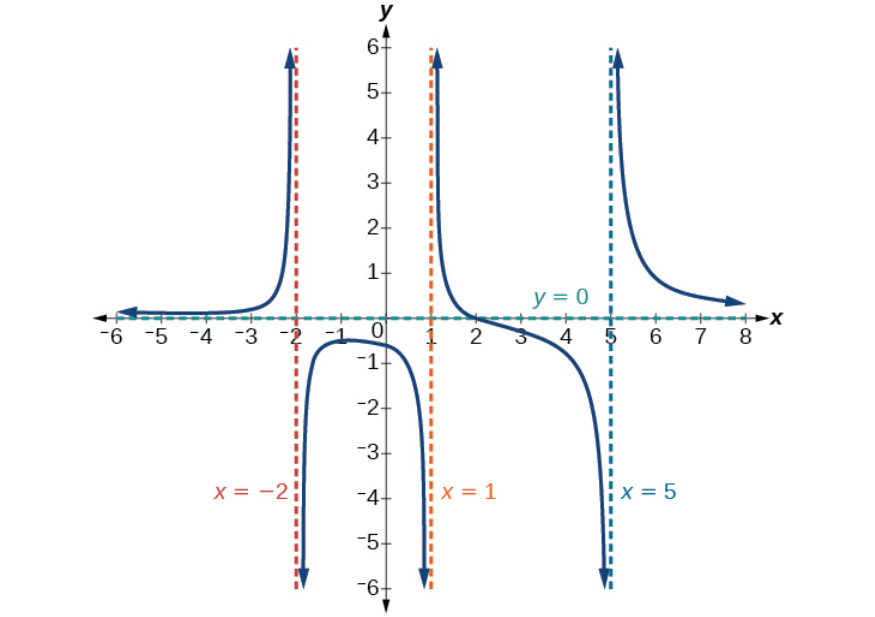


Figure 5.1.16.

Exercise 5.1.6

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Answer

Vertical asymptotes at $x = 2$ and $x = -3$

horizontal asymptote at $y = 4$.

INTERCEPTS OF RATIONAL FUNCTIONS

A *rational function* will have a y-intercept at $f(0)$, if the function is defined at zero. A rational function will not have a y-intercept if the function is not defined at zero.

Likewise, a rational function will have x-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

Example 5.1.10: Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution

We can find the y-intercept by evaluating the function at zero

$$\begin{aligned} f(0) &= \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} \\ &= -\frac{6}{10} \\ &= -\frac{3}{5} \\ &= -0.6 \end{aligned}$$

The x-intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \text{ This is zero when the numerator is zero.}$$

$$0 = (x-2)(x+3)$$

$$x = 2, x = -3$$

The y-intercept is $(0, -0.6)$, the x-intercepts are $(2, 0)$ and $(-3, 0)$. See Figure 5.1.17.

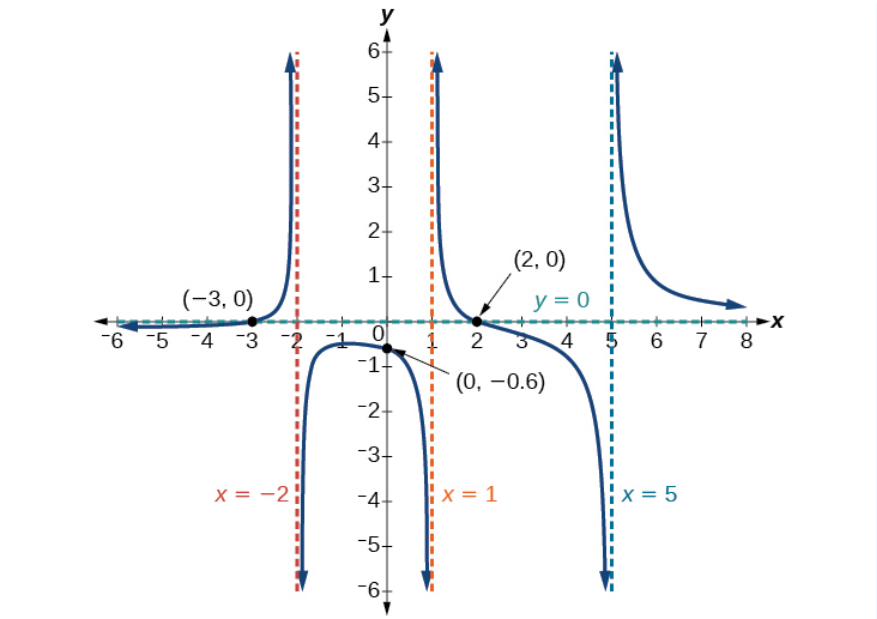


Figure 5.1.17.

Exercise 5.1.7

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

Answer

For the transformed reciprocal squared function, we find the rational form.

$$f(x)=\frac{1}{(x-3)^2}-4=\frac{1-4(x-3)^2}{(x-3)^2}=\frac{1-4(x^2-6x+9)}{(x-3)(x-3)}=\frac{-4x^2+24x-35}{x^2-6x+9}$$

Because the numerator is the same degree as the denominator we know that as $x\to\pm\infty$, $f(x)\to -4$; so $y=-4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x=3$, because as $x\to 3$, $f(x)\to\infty$. We then set the numerator equal to 0 and find the x-intercepts are at $(2.5,0)$ and $(3.5,0)$. Finally, we evaluate the function at 0 and find the y-intercept to be at $(0,-\frac{35}{9})$.

Graphing Rational Functions

In Example 5.1.10, we see that the numerator of a rational function reveals the *x*-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure 5.1.18

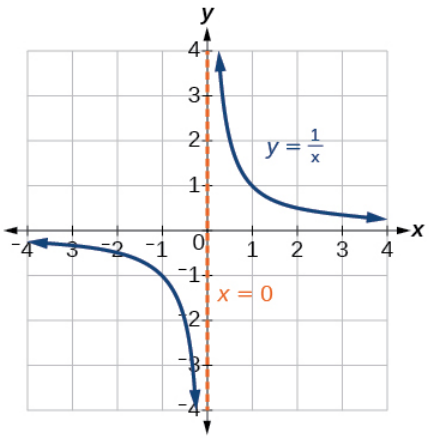


Figure 5.1.18.

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure 5.1.19

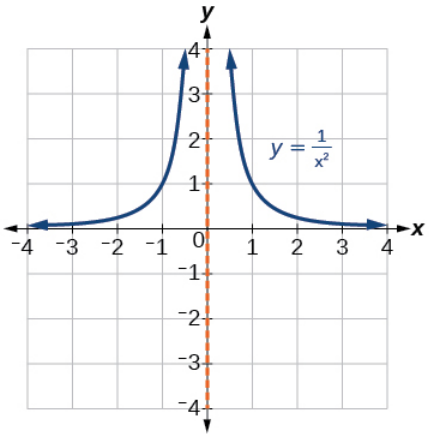


Figure 5.1.19.

For example, the graph of $f(x) = \dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure 5.1.20

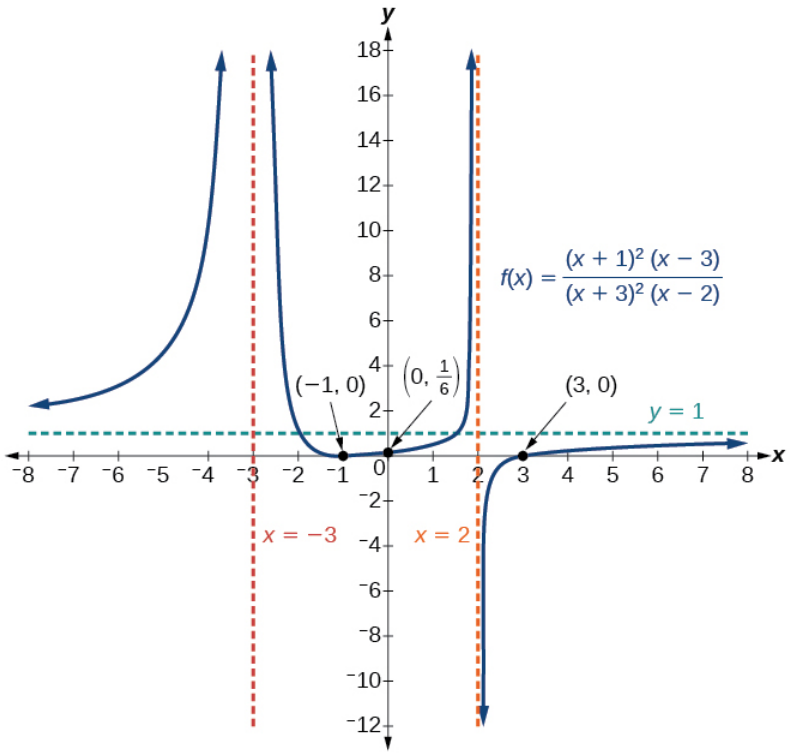


Figure 5.1.20.

- At the x-intercept $x=-1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.
- At the x-intercept $x=3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x=-3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x)=\dfrac{1}{x^2}$.
- At the vertical asymptote $x=2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function $f(x)=\dfrac{1}{x}$.

Howto: Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Example 5.1.11: Graphing a Rational Function

Sketch a graph of $f(x)=\frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the y-intercept:

$$f(0)=\frac{(0+2)(0-3)}{(0+1)^2(0-2)}=3$$

To find the x-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find x-intercepts at $x=-2$ and $x=3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a y-intercept at $(0,3)$ and x-intercepts at $(-2,0)$ and $(3,0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x+1=0$ and when $x-2=0$, giving us vertical asymptotes at $x=-1$ and $x=2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y=0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no x-intercepts between the vertical asymptotes, and the y-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure 5.1.21.

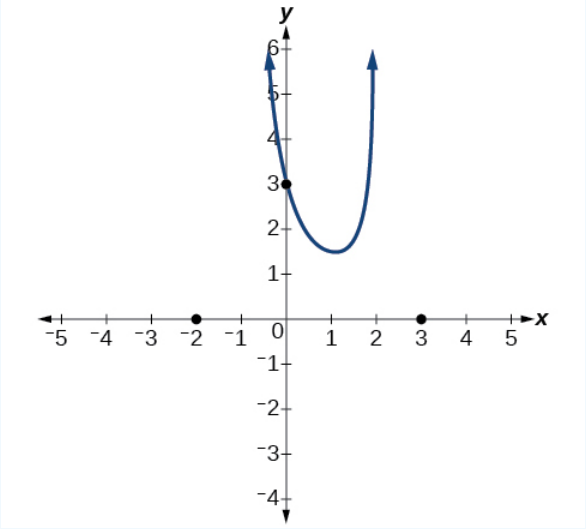


Figure 5.1.21.

The factor associated with the vertical asymptote at $x=-1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x=2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure 5.1.22 After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

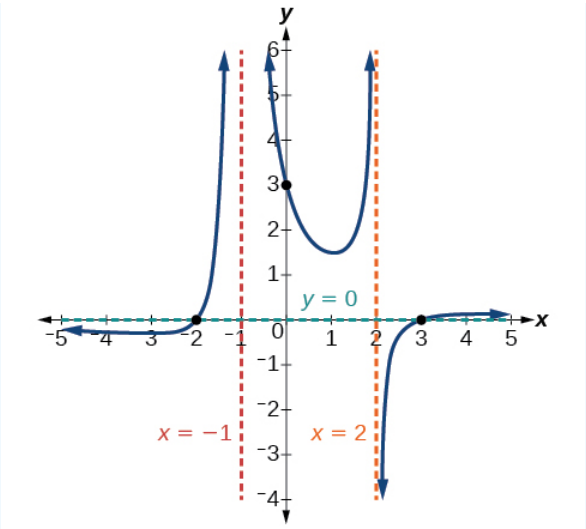


Figure 5.1.22.

Exercise 5.1.8

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Answer

Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1$ and $x = 3$. y-intercept at $(0, \frac{4}{3})$.

x-intercepts at $(2, 0)$ and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity 2, and the graph bounces off the x-axis at this point. $(2, 0)$ is a single zero and the graph crosses the axis at this point.

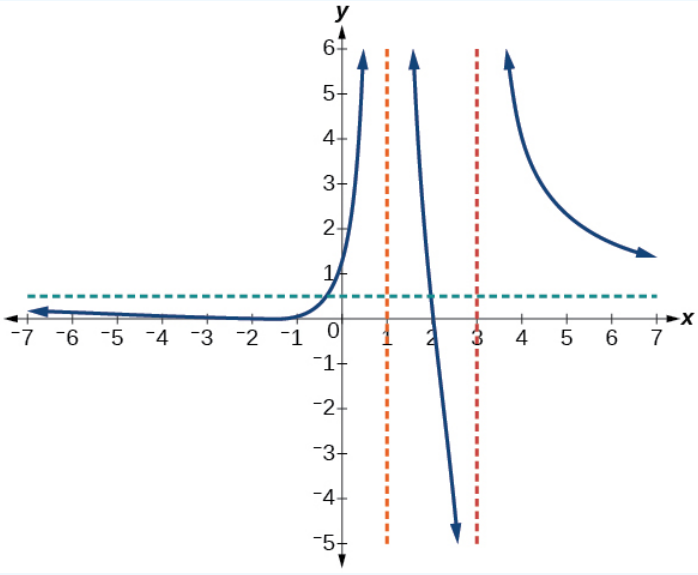


Figure 5.1.23.

Key Equations

Rational Function $$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0},\ Q(x) \neq 0$$

Key Concepts

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$. See Example 5.1.1.
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See Example.
- Application problems involving rates and concentrations often involve rational functions. See Example.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See Example.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See Example.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See Example.
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See Example, Example, Example, and Example.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See Example.
- If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any v_j, then the function can be written in the form

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5.2: Radical Functions

Evaluate a Radical Function

In this section we will extend our previous work with functions to include radicals. If a function is defined by a radical expression, we call it a **radical function**.

- The square root function is $f(x) = \sqrt{x}$.
- The cube root function is $f(x) = \sqrt[3]{x}$.

Definition 5.2.1: radical function

A **radical function** is a function that is defined by a radical expression.

To evaluate a radical function, we find the value of $f(x)$ for a given value of x just as we did in our previous work with functions.

Example 5.2.1

For the function $f(x) = \sqrt{2x-1}$, find

a. $f(5)$
b. $f(-2)$

Solution:

a.

$$f(x) = \sqrt{2x-1}$$

To evaluate $f(5)$, substitute 5 for x.

$$f(5) = \sqrt{2 \cdot 5 - 1}$$

Simplify.

$$f(5) = \sqrt{9}$$

Take the square root.

$$f(5) = 3$$

b.

$$f(x) = \sqrt{2x-1}$$

To evaluate $f(-2)$, substitute -2 for x.

$$f(-2) = \sqrt{2(-2)-1}$$

Simplify.

$$f(-2) = \sqrt{-5}$$

Since the square root of a negative number is not a real number, the function does not have a value at $x = -2$.

Exercise 5.2.1

For the function $f(x) = \sqrt{3x-2}$, find

a. $f(6)$
b. $f(0)$

Answer

a. $f(6) = 4$
b. no value at $x = 0$

Exercise 5.2.2

For the function $g(x) = \sqrt{5x+5}$, find

a. $g(4)$
b. $g(-3)$

Answer

a. $g(4) = 5$
b. no value at $f(-3)$

We follow the same procedure to evaluate cube roots.

Example 5.2.2

For the function $g(x) = \sqrt[3]{x-6}$, find

a. $g(14)$
b. $g(-2)$

Solution:

a.

$$g(x) = \sqrt[3]{x-6}$$

To evaluate $g(14)$, substitute 14 for x.

$$g(14) = \sqrt[3]{14-6}$$

Simplify.

$$g(14) = \sqrt[3]{8}$$

Take the cube root.

$$g(14) = 2$$

b.

$$g(x) = \sqrt[3]{x-6}$$

To evaluate $g(-2)$, substitute -2 for x.

$$g(-2) = \sqrt[3]{-2-6}$$

Simplify.

$$g(-2) = \sqrt[3]{-8}$$

Take the cube root.

$$g(-2) = -2$$

Exercise 5.2.3

For the function $g(x) = \sqrt[3]{3x-4}$, find

a. $g(4)$
b. $g(1)$

Answer

a. $g(4) = 2$
b. $g(1) = -1$

Exercise 5.2.4

For the function $h(x) = \sqrt[3]{5x-2}$, find

a. $h(2)$
b. $h(-5)$

Answer

a. $h(2) = 2$
b. $h(-5) = -3$

The next example has fourth roots.

Example 5.2.3

For the function $f(x) = \sqrt[4]{5x-4}$, find

a. $f(4)$
b. $f(-12)$

Solution:

a.

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(4)$, substitute 4 for x.

$$f(4) = \sqrt[4]{5 \cdot 4 - 4}$$

Simplify.

$$f(4) = \sqrt[4]{16}$$

Take the fourth root.

$$f(4) = 2$$

b.

$$f(x) = \sqrt[4]{5x-4}$$

To evaluate $f(-12)$, substitute -12 for x.

$$f(-12) = \sqrt[4]{5(-12)-4}$$

Simplify.

$$f(-12) = \sqrt[4]{-64}$$

Since the fourth root of a negative number is not a real number, the function does not have a value at $x = -12$.

Exercise 5.2.5

For the function $f(x) = \sqrt[4]{3x+4}$, find

a. $f(4)$
b. $f(-1)$

Answer

a. $f(4) = 2$
b. $f(-1) = 1$

Exercise 5.2.6

For the function $g(x) = \sqrt[4]{5x+1}$, find

a. $g(16)$
b. $g(3)$

Answer

a. $g(16) = 3$
b. $g(3) = 2$

Find the Domain of a Radical Function

To find the **domain** and **range** of radical functions, we use our properties of radicals. For a radical with an even index, we said the radicand had to be greater than or equal to zero as even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

Properties of $\sqrt[n]{a}$

When n is an **even** number and:

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.

So, to find the domain of a radical function with even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number.

Domain of a Radical Function

When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.

When the **index** of the radical is **odd**, the radicand can be any real number.

Example 5.2.4

Find the domain of the function, $f(x) = \sqrt{3x-4}$. Write the domain in interval notation.

Solution:

Since the function, $f(x) = \sqrt{3x-4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0. We set the radicand to be greater than or equal to 0 and then solve to find the domain.

Solve.

$$\begin{aligned} 3x-4 &\geq 0 \\ 3x &\geq 4 \\ x &\geq \frac{4}{3} \end{aligned}$$

The domain of $f(x) = \sqrt{3x-4}$ is all values $x \geq \frac{4}{3}$ and we write it in interval notation as $\left[\frac{4}{3}, \infty\right)$.

Exercise 5.2.7

Find the domain of the function, $f(x) = \sqrt{6x-5}$. Write the domain in interval notation.

Answer

$\left[\frac{5}{6}, \infty\right)$

Exercise 5.2.8

Find the domain of the function, $f(x)=\sqrt{4-5x}$. Write the domain in interval notation.

Answer

$\left(-\infty, \frac{4}{5}\right]$

Example 5.2.5

Find the domain of the function, $g(x)=\sqrt{\frac{6}{x-1}}$. Write the domain in interval notation.

Solution:

Solve the function, $g(x)=\sqrt{\frac{6}{x-1}}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0.

The radicand cannot be zero since the numerator is not zero.

For $\frac{6}{x-1}$ to be greater than zero, the denominator must be positive since the numerator is positive. We know a positive divided by a positive is positive.

We set $x-1>0$ and solve.

$$x-1>0$$

Solve.

$$x>1$$

Also, since the radicand is a fraction, we must realize that the denominator cannot be zero.

We solve $x-1=0$ to find the value that must be eliminated from the domain.

$$x-1=0$$

Solve.

$$x=1 \text{ so } x/neq1 \text{ in the domain.}$$

Putting this together we get the domain is $x>1$ and we write it as $(1,\infty)$.

Exercise 5.2.9

Find the domain of the function, $f(x)=\sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

Answer

$(-3,\infty)$

Exercise 5.2.10

Find the domain of the function, $h(x)=\sqrt{\frac{9}{x-5}}$. Write the domain in interval notation.

Answer

$(5,\infty)$

The next example involves a cube root and so will require different thinking.

Example 5.2.6

Find the domain of the function, $f(x)=\sqrt[3]{2x^2+3}$. Write the domain in interval notation.

Solution:

Since the function, $f(x)=\sqrt[3]{2x^2+3}$ has a radical with an index of 3, which is odd, we know the radicand can be any real number. This tells us the domain is any real number. In interval notation, we write $(-\infty,\infty)$.

The domain of $f(x)=\sqrt[3]{2x^2+3}$ is all real numbers and we write it in interval notation as $(-\infty,\infty)$.

Exercise 5.2.11

Find the domain of the function, $f(x)=\sqrt[3]{3x^2-1}$. Write the domain in interval notation.

Answer

$(-\infty,\infty)$

Exercise 5.2.12

Find the domain of the function, $g(x)=\sqrt[3]{5x-4}$. Write the domain in interval notation.

Answer

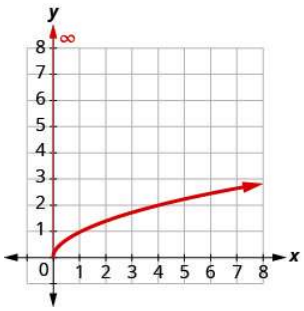
$(-\infty,\infty)$

Graph Radical Functions

Before we graph any radical function, we first find the **domain** of the function. For the function, $f(x)=\sqrt{x}$, the index is even, and so the radicand must be greater than or equal to 0.

This tells us the domain is $x\geq 0$ and we write this in interval notation as $[0,\infty)$.

Previously we used point plotting to graph the function, $f(x)=\sqrt{x}$. We chose x-values, substituted them in and then created a chart. Notice we chose points that are perfect squares in order to make taking the square root easier.



x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

Figure 8.7.1

Once we see the graph, we can find the range of the function. The y-values of the function are greater than or equal to zero. The range then is $[0,\infty)$.

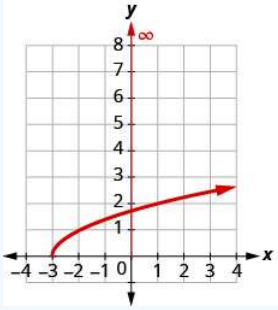
Example 5.2.7

For the function $f(x)=\sqrt{x+3}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Solution:

a. Since the radical has index 2, we know the radicand must be greater than or equal to zero. If $x+3 \geq 0$, then $x \geq -3$. This tells us the domain is all values $x \geq -3$ and written in interval notation as $[-3, \infty)$.
b. To graph the function, we choose points in the interval $[-3, \infty)$ that will also give us a radicand which will be easy to take the square root.



x	$f(x)=\sqrt{x+3}$	$(x, f(x))$
–3	0	(–3, 0)
–2	1	(–2, 1)
1	2	(1, 2)
6	3	(6, 3)

Figure 8.7.2

c. Looking at the graph, we see the y-values of the function are greater than or equal to zero. The range then is $[0, \infty)$.

Exercise 5.2.13

For the function $f(x)=\sqrt{x+2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[-2, \infty)$
b.

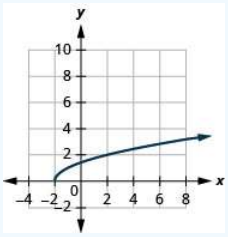


Figure 8.7.3

c. range: $[0, \infty)$

Exercise 5.2.14

For the function $f(x)=\sqrt{x-2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $[2, \infty)$

b.

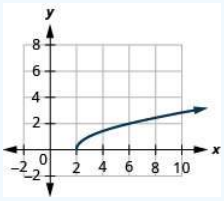


Figure 8.7.4

c. range: $[0, \infty)$

In our previous work graphing functions, we graphed $f(x) = x^3$ but we did not graph the function $f(x) = \sqrt[3]{x}$. We will do this now in the next example.

Example 5.2.8

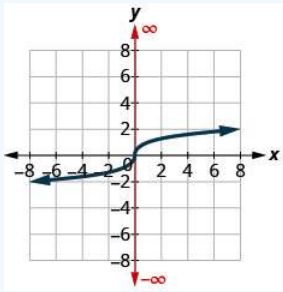
For the function, $f(x) = \sqrt[3]{x}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Solution:

a. Since the radical has index 3, we know the radicand can be any real number. This tells us the domain is all real numbers and written in interval notation as $(-\infty, \infty)$

b. To graph the function, we choose points in the interval $(-\infty, \infty)$ that will also give us a radicand which will be easy to take the cube root.



x	$f(x) = \sqrt[3]{x}$	$(x, f(x))$
–8	–2	(–8, –2)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
8	2	(8, 2)

Figure 8.7.5

c. Looking at the graph, we see the y-values of the function are all real numbers. The range then is $(-\infty, \infty)$.

Exercise 5.2.15

For the function $f(x) = -\sqrt[3]{x}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $(-\infty, \infty)$

b.

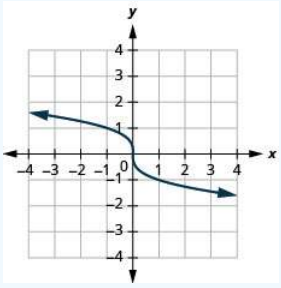


Figure 8.7.6

c. range: $(-\infty, \infty)$

? Exercise 5.2.16

For the function $f(x) = \sqrt[3]{x-2}$,

a. find the domain
b. graph the function
c. use the graph to determine the range

Answer

a. domain: $(-\infty, \infty)$
b.

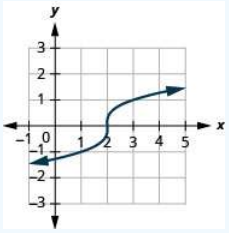


Figure 8.7.7

c. range: $(-\infty, \infty)$

Access these online resources for additional instruction and practice with radical functions.

- Domain of a Radical Function
- Domain of a Radical Function 2
- Finding Domain of a Radical Function

Key Concepts

- **Properties of $\sqrt[n]{a}$**
  - When n is an **even** number and:
    $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
    $a < 0$, then $\sqrt[n]{a}$ is not a real number.
  - When n is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Domain of a Radical Function**
  - When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
  - When the **index** of the radical is **odd**, the radicand can be any real number.

Glossary

radical function

A radical function is a function that is defined by a radical expression.

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6: Exponential and Logarithmic Functions

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6.1: Evaluate and Graph Exponential Functions

Graph Exponential Functions

The functions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential functions.

An **exponential function** is a function of the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$.

Definition 6.1.1

An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form

$$f(x) = a^x$$

Notice that in this function, the variable is the exponent. In our functions so far, the variables were the base.

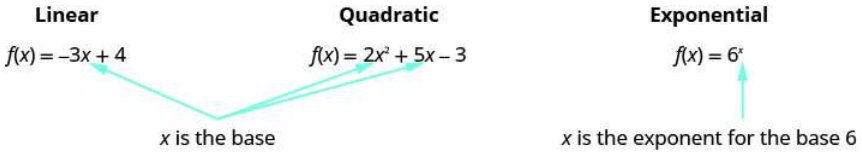


Figure 10.2.1

Our definition says $a \neq 1$. If we let $a = 1$, then $f(x) = a^x$ becomes $f(x) = 1^x$. Since $1^x = 1$ for all real numbers, $f(x) = 1$. This is the constant function.

Our definition also says $a > 0$. If we let a base be negative, say -4, then $f(x) = (-4)^x$ is not a real number when $x = \frac{1}{2}$.

$$f(x) = (-4)^x$$
$$f\left(\frac{1}{2}\right) = (-4)^{\frac{1}{2}}$$
$$f\left(\frac{1}{2}\right) = \sqrt{-4} \text{ not a real number}$$

In fact, $f(x) = (-4)^x$ would not be a real number any time x is a fraction with an even denominator. So our definition requires $a > 0$.

By graphing a few exponential functions, we will be able to see their unique properties.

Example 6.1.1

On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 3^x$.

Solution:

We will use point plotting to graph the functions.

x	$f(x) = 2^x$	$(x, f(x))$	$g(x) = 3^x$	$(x, g(x))$
–2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
–1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$2^0 = 1$	(0, 1)	$3^0 = 1$	(0, 1)
1	$2^1 = 2$	(1, 2)	$3^1 = 3$	(1, 3)
2	$2^2 = 4$	(2, 4)	$3^2 = 9$	(2, 9)
3	$2^3 = 8$	(3, 8)	$3^3 = 27$	(3, 27)

Figure 10.2.2

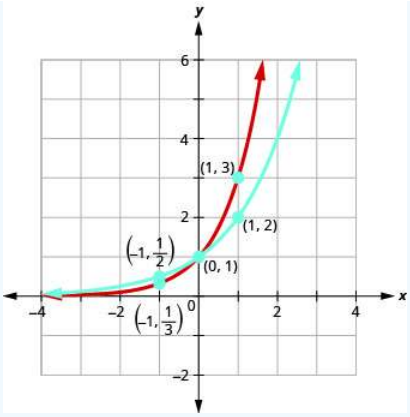


Figure 10.2.3

? Exercise 6.1.1

Graph: $f(x) = 4^x$.

Answer

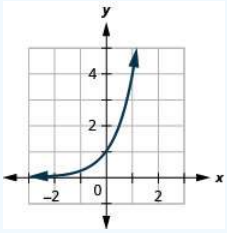


Figure 10.2.4

? Exercise 6.1.2

Graph: $g(x) = 5^x$

Answer

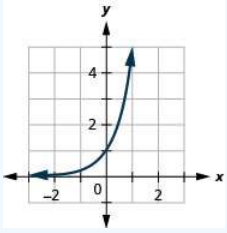


Figure 10.2.5

If we look at the graphs from the previous Example 10.2.1 and Exercises 10.2.1 and 10.2.2, we can identify some of the properties of exponential functions.

The graphs of $f(x) = 2^x$ and $g(x) = 3^x$, as well as the graphs of $f(x) = 4^x$ and $g(x) = 5^x$, all have the same basic shape. This is the shape we expect from an exponential function where $a > 1$.

We notice, that for each function, the graph contains the point $(0, 1)$. This make sense because $a^0 = 1$ for any a.

The graph of each function, $f(x) = a^x$ also contains the point $(1, a)$. The graph of $f(x) = 2^x$ contained $(1, 2)$ and the graph of $g(x) = 3^x$ contained $(1, 3)$. This makes sense as $a^1 = a$.

Notice too, the graph of each function $f(x) = a^x$ also contains the point $(-1, \frac{1}{a})$. The graph of $f(x) = 2^x$ contained $(-1, \frac{1}{2})$ and the graph of $g(x) = 3^x$ contained $(-1, \frac{1}{3})$.This makes sense as $a^{-1} = \frac{1}{a}$.

What is the domain for each function? From the graphs we can see that the domain is the set of all real numbers. There is no restriction on the domain. We write the domain in interval notation as $(-\infty, \infty)$.

Look at each graph. What is the range of the function? The graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

Whenever a graph of a function approaches a line but never touches it, we call that line an **asymptote**. For the exponential functions we are looking at, the graph approaches the x-axis very closely but will never cross it, we call the line $y = 0$, the x-axis, a horizontal asymptote.

Properties of the Graph of $f(x) = a^x$ when $a > 1$

Table 10.2.1

Domain	$(-\infty, \infty)$
Range	$(0, \infty)$
x-intercept	None
y-intercept	$(0, 1)$
Contains	$(1, a), \left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$

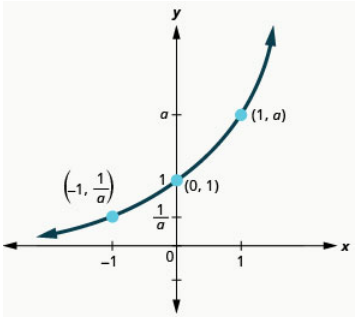


Figure 10.2.6

Our definition of an exponential function $f(x) = a^x$ says $a > 0$, but the examples and discussion so far has been about functions where $a > 1$. What happens when $0 < a < 1$ The next example will explore this possibility.

Example 6.1.2

On the same coordinate system, graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.

Solution:

We will use point plotting to graph the functions.

x	$f(x)=\left(\frac{1}{2}\right)^x$	$(x, f(x))$	$g(x)=\left(\frac{1}{3}\right)^x$	$(x, g(x))$
-2	$\left(\frac{1}{2}\right)^{-2}=2^2=4$	$(-2, 4)$	$\left(\frac{1}{3}\right)^{-2}=3^2=9$	$(-2, 9)$
-1	$\left(\frac{1}{2}\right)^{-1}=2^1=2$	$(-1, 2)$	$\left(\frac{1}{3}\right)^{-1}=3^1=3$	$(-1, 3)$
0	$\left(\frac{1}{2}\right)^0=1$	$(0, 1)$	$\left(\frac{1}{3}\right)^0=1$	$(0, 1)$
1	$\left(\frac{1}{2}\right)^1=\frac{1}{2}$	$\left(1, \frac{1}{2}\right)$	$\left(\frac{1}{3}\right)^1=\frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
2	$\left(\frac{1}{2}\right)^2=\frac{1}{4}$	$\left(2, \frac{1}{4}\right)$	$\left(\frac{1}{3}\right)^2=\frac{1}{9}$	$\left(2, \frac{1}{9}\right)$
3	$\left(\frac{1}{2}\right)^3=\frac{1}{8}$	$\left(3, \frac{1}{8}\right)$	$\left(\frac{1}{3}\right)^3=\frac{1}{27}$	$\left(3, \frac{1}{27}\right)$

Figure 10.2.7

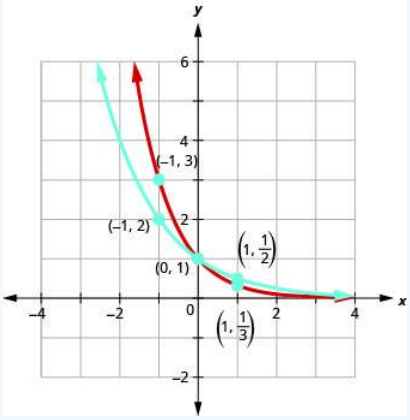


Figure 10.2.8

Exercise 6.1.3

Graph: $f(x)=\left(\frac{1}{4}\right)^x$.

Answer

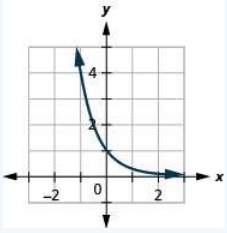


Figure 10.2.9

Exercise 6.1.4

Graph: $g(x)=\left(\frac{1}{5}\right)^x$.

Answer

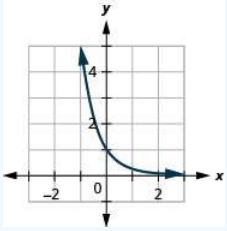


Figure 10.2.10

Now let's look at the graphs from the previous Example 10.2.2 and Exercises 10.2.3 and 10.2.4 so we can now identify some of the properties of exponential functions where $0 < a < 1$.

The graphs of $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$ as well as the graphs of $f(x) = \left(\frac{1}{4}\right)^x$ and $g(x) = \left(\frac{1}{5}\right)^x$ all have the same basic shape. While this is the shape we expect from an exponential function where $0 < a < 1$, the graphs go down from left to right while the previous graphs, when $a > 1$, went from up from left to right.

We notice that for each function, the graph still contains the point $(0, 1)$. This make sense because $a^0 = 1$ for any a.

As before, the graph of each function, $f(x) = a^x$, also contains the point $(1, a)$. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ contained $\left(1, \frac{1}{2}\right)$ and the graph of $g(x) = \left(\frac{1}{3}\right)^x$ contained $\left(1, \frac{1}{3}\right)$. This makes sense as $a^1 = a$.

Notice too that the graph of each function, $f(x) = a^x$, also contains the point $\left(-1, \frac{1}{a}\right)$. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ contained $(-1, 2)$ and the graph of $g(x) = \left(\frac{1}{3}\right)^x$ contained $(-1, 3)$. This makes sense as $a^{-1} = \frac{1}{a}$.

What is the domain and range for each function? From the graphs we can see that the domain is the set of all real numbers and we write the domain in interval notation as $(-\infty, \infty)$. Again, the graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

We will summarize these properties in the chart below. Which also include when $a > 1$.

Properties of the Graph of $f(x) = a^x$

Table 10.2.2

When $a > 1$		When $0 < a < 1$	
1\)">Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
1\)">Range	$(0, \infty)$	Range	$(0, \infty)$
1\)">x-intercept	none	x-intercept	none
1\)">y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
1\)">Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$	Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
1\)">Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
1\)">Basic Shape	increasing	Basic Shape	decreasing

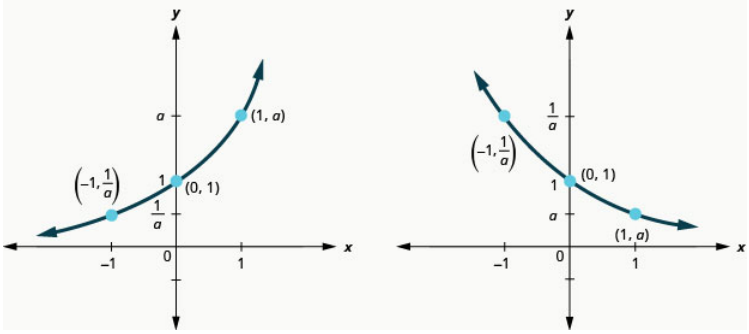


Figure 10.2.11

It is important for us to notice that both of these graphs are one-to-one, as they both pass the horizontal line test. This means the exponential function will have an inverse. We will look at this later.

When we graphed quadratic functions, we were able to graph using translation rather than just plotting points. Will that work in graphing exponential functions?

Example 6.1.3

On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 2^{x+1}$.

Solution:

We will use point plotting to graph the functions.

x	$f(x) = 2^x$	$(x, f(x))$	$g(x) = 2^{x+1}$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$2^{-2+1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-2, \frac{1}{2}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$2^{-1+1} = 2^0 = 1$	$(-1, 1)$
0	$2^0 = 1$	$(0, 1)$	$2^{0+1} = 2^1 = 2$	$(0, 2)$
1	$2^1 = 2$	$(1, 2)$	$2^{1+1} = 2^2 = 4$	$(1, 4)$
2	$2^2 = 4$	$(2, 4)$	$2^{2+1} = 2^3 = 8$	$(2, 8)$
3	$2^3 = 8$	$(3, 8)$	$2^{3+1} = 2^4 = 16$	$(3, 16)$

Figure 10.2.12

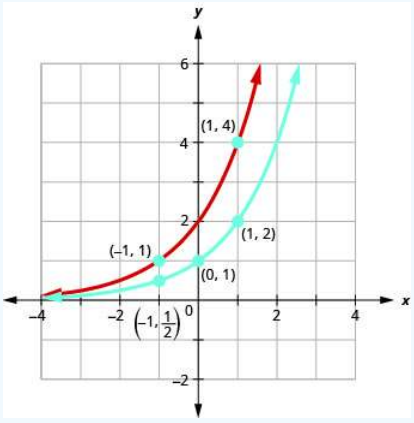


Figure 10.2.13

Exercise 6.1.5

On the same coordinate system, graph: $f(x) = 2^x$ and $g(x) = 2^{x-1}$.

Answer

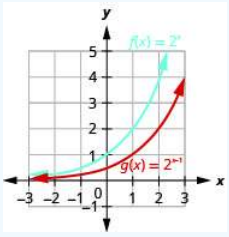


Figure 10.2.14

Exercise 6.1.6

On the same coordinate system, graph $f(x) = 3^x$ and $g(x) = 3^{x+1}$.

Answer

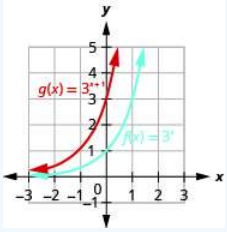


Figure 10.2.15

Looking at the graphs of the functions $f(x) = 2^x$ and $g(x) = 2^{x+1}$ in the last example, we see that adding one in the exponent caused a horizontal shift of one unit to the left. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

Let's now consider another situation that might be graphed more easily by translation, once we recognize the pattern.

Example 6.1.4

On the same coordinate system graph $f(x) = 3^x$ and $g(x) = 3^x - 2$.

Solution:

We will use point plotting to graph the functions.

x	$f(x) = 3^x$	$(x, g(x))$	$g(x) = 3^x - 2$	$(x, g(x))$
–2	$3^{-2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$	$3^{-2} - 2 = \frac{1}{9} - 2 = -\frac{17}{9}$	$\left(-2, -\frac{17}{9}\right)$
–1	$3^{-1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$	$3^{-1} - 2 = \frac{1}{3} - 2 = -\frac{5}{3}$	$\left(-1, -\frac{5}{3}\right)$
0	$3^0 = 1$	(0, 1)	$3^0 - 2 = 1 - 2 = -1$	(0, –1)
1	$3^1 = 3$	(1, 3)	$3^1 - 2 = 3 - 2 = 1$	(1, 1)
2	$3^2 = 9$	(2, 9)	$3^2 - 2 = 9 - 2 = 7$	(2, 8)

Figure 10.2.16

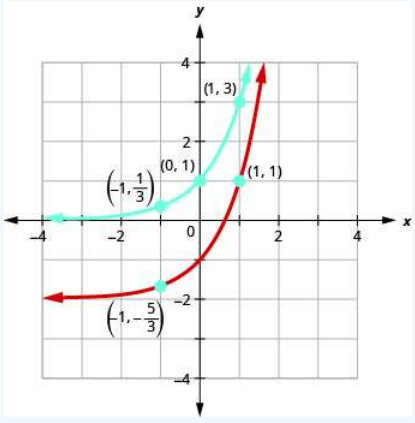


Figure 10.2.17

Exercise 6.1.7

On the same coordinate system, graph $f(x) = 3^x$ and $g(x) = 3^x + 2$.

Answer

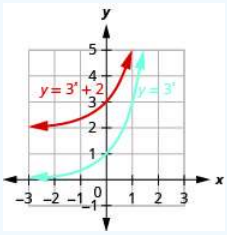


Figure 10.2.18

Exercise 6.1.8

On the same coordinate system, graph $f(x) = 4^x$ and $g(x) = 4^x - 2$.

Answer

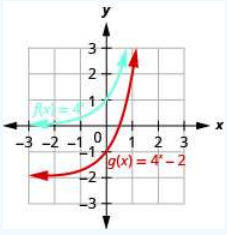


Figure 10.2.19

Looking at the graphs of the functions $f(x) = 3^x$ and $g(x) = 3^x - 2$ in the last example, we see that subtracting 2 caused a vertical shift of down two units. Notice that the horizontal asymptote also shifted down 2 units. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

All of our exponential functions have had either an integer or a rational number as the base. We will now look at an exponential function with an irrational number as the base.

Before we can look at this exponential function, we need to define the irrational number, e. This number is used as a base in many applications in the sciences and business that are modeled by exponential functions. The number is defined as the value of $\left(1 + \frac{1}{n}\right)^n$ as n gets larger and larger. We say, as n approaches infinity, or increases without bound. The table shows the value of $\left(1 + \frac{1}{n}\right)^n$ for several values of n.

Table 10.2.3

n	$\left(1 + \frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374246
100	2.704813829 . . .
1, 000	2.716923932 . . .
10, 000	2.718145927 . . .
100, 000	2.718268237 . . .

n	$\left(1+\frac{1}{n}\right)^n$
$1,000,000$	$2.718280469\ldots$
$1,000,000,000$	$2.718281827\ldots$

$$e \approx 2.718281827$$

The number e is like the number π in that we use a symbol to represent it because its decimal representation never stops or repeats. The irrational number e is called the **natural base**.

Definition 6.1.2

Natural Base e

The number e is defined as the value of $\left(1+\frac{1}{n}\right)^n$, as n increases without bound. We say, as n approaches infinity,

$$e \approx 2.718281827$$

The exponential function whose base is e, $f(x)=e^x$ is called the **natural exponential function**.

Definition 6.1.3

Natural Exponential Function

The natural exponential function is an exponential function whose base is e

$$f(x)=e^x$$

The domain is $(-\infty,\infty)$ and the range is $(0,\infty)$.

Let's graph the function $f(x)=e^x$ on the same coordinate system as $g(x)=2^x$ and $h(x)=3^x$.

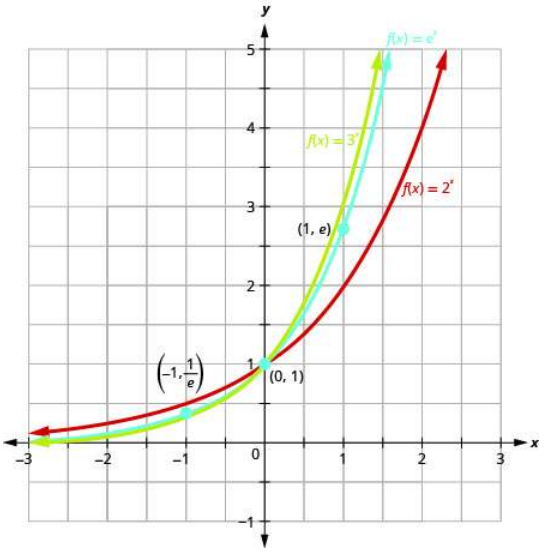


Figure 10.2.20

Notice that the graph of $f(x)=e^x$ is "between" the graphs of $g(x)=2^x$ and $h(x)=3^x$.Does this make sense as $2<e<3$?

Use Exponential Models in Applications

Exponential functions model many situations. If you own a bank account, you have experienced the use of an exponential function. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal, P, is invested at an interest rate, r, for t years, the new balance, A, will depend on how often the interest is compounded. If the interest is

compounded n times a year we use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, we use the formula $A = Pe^{rt}$. These are the formulas for **compound interest**.

Definition 6.1.5

Compound Interest

For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is:

$$\begin{aligned} A &= P\left(1 + \tfrac{r}{n}\right)^{nt} && \text{when compounded } n \text{ times a year.} \\ A &= Pe^{rt} && \text{when compounded continuously.} \end{aligned}$$

As you work with the Interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

Example 6.1.7

A total of $\$10,000$ was invested in a college fund for a new grandchild. If the interest rate is 5%, how much will be in the account in 18 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

Solution:

Identify the values of each variable in the formulas. Remember to express the percent as a decimal.

$$\begin{aligned} A &= ? \\ P &= \$10,000 \\ r &= 0.05 \\ t &= 18 \text{ years} \end{aligned}$$

a. For quarterly compounding, $n = 4$. There are 4 quarters in a year.

$$A = P\left(1 + \tfrac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10,000\left(1 + \tfrac{0.05}{4}\right)^{4 \cdot 18}$$

Compute the amount. Be careful to consider the order of operations as you enter the expression into your calculator.

$$A = \$24,459.20$$

b. For monthly compounding, $n = 12$.There are 12 months in a year.

$$A = P\left(1 + \tfrac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10,000\left(1 + \tfrac{0.05}{12}\right)^{12 \cdot 18}$$

Compute the amount.

$$A = \$24,550.08$$

c. For compounding continuously,

$$A = Pe^{rt}$$

Substitute the values in the formula.

$$A = 10,000e^{0.05 \cdot 18}$$

Compute the amount.

$$A = \$24,596.03$$

Exercise 6.1.13

Angela invested $\$15,000$ in a savings account. If the interest rate is 4%, how much will be in the account in 10 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

Answer

a. $\$22,332.96$
b. $\$22,362.49$
c. $\$22,377.37$

Exercise 6.1.14

Allan invested $\$10,000$ in a mutual fund. If the interest rate is 5%, how much will be in the account in 15 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

Answer

a. $\$21,071.81$
b. $\$21,137.04$
c. $\$21,170.00$

Other topics that are modeled by exponential functions involve growth and decay. Both also use the formula $A = Pe^{rt}$ we used for the growth of money. For growth and decay, generally we use A_0, as the original amount instead of calling it P, the principal. We see that **exponential growth** has a positive rate of growth and **exponential decay** has a negative rate of growth.

Definition 6.1.6

Exponential Growth and Decay

For an original amount, A_0, that grows or decays at a rate, r, for a certain time, t, the final amount, A, is:

$$A = A_0 e^{rt}$$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

Example 6.1.8

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behavior of a new and dangerous virus. He starts his experiment with 100 of the virus that grows at a rate of 25% per hour. He will check on the virus in 24 hours. How many viruses will he find?

Solution:

Identify the values of each variable in the formulas. Be sure to put the percent in decimal form. Be sure the units match--the rate is per hour and the time is in hours.

$$A = ?$$
$$A_0 = 100$$
$$r = 0.25/\text{ hour}$$
$$t = 24 \text{ hours}$$

Substitute the values in the formula: $A = A_0 e^{rt}$.

$$A = 100e^{0.25 \cdot 24}$$

Compute the amount.

$$A = 40,342.88$$

Round to the nearest whole virus.

$$A = 40,343$$

The researcher will find $40,343$ viruses.

? Exercise 6.1.15

Another researcher at the Center for Disease Control and Prevention, Lisa, is studying the growth of a bacteria. She starts his experiment with 50 of the bacteria that grows at a rate of 15% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

Answer

She will find 166 bacteria.

? Exercise 6.1.16

Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of 10% per hour. She will check on the virus in 24 hours. How many viruses will she find?

Answer

She will find $1,102$ viruses.

Access these online resources for additional instruction and practice with evaluating and graphing exponential functions.

- Graphing Exponential Functions
- Solving Exponential Equations
- Applications of Exponential Functions
- Continuously Compound Interest
- Radioactive Decay and Exponential Growth

Key Concepts

- **Properties of the Graph of $f(x) = a^x$:**

Table 10.2.2

When $a > 1$		When $0 < a < 1$	
1\)">Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
1\)">Range	$(0, \infty)$	Range	$(0, \infty)$
1\)">x-intercept	none	x-intercept	none
1\)">y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
1\)">Contains	$(1, a), \left(-1, \frac{1}{a}\right)$	Contains	$(1, a), \left(-1, \frac{1}{a}\right)$

When $a > 1$		When $0 < a < 1$	
1\)">Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
1\)">Basic Shape	increasing	Basic Shape	decreasing

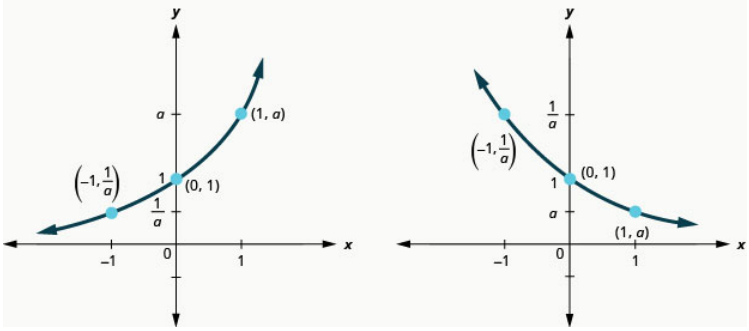


Figure 10.2.11

- **One-to-One Property of Exponential Equations:**
  For $a > 0$ and $a \neq 1$,
  $$A = A_0 e^{rt}$$
- **How to Solve an Exponential Equation**
  1. Write both sides of the equation with the same base, if possible.
  2. Write a new equation by setting the exponents equal.
  3. Solve the equation.
  4. Check the solution.
- **Compound Interest:** For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is
  $A = P\left(1 + \frac{r}{n}\right)^{nt}$ when compounded n times a year.
  $A = Pe^{rt}$ when compounded continuously.
- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, for a certain time t, the final amount, A, is $A = A_0 e^{rt}$.

Glossary

asymptote

A line which a graph of a function approaches closely but never touches.

exponential function

An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form $f(x) = a^x$.

natural base

The number e is defined as the value of $(1 + \frac{1}{n})^n$, as n gets larger and larger. We say, as n increases without bound, $e \approx 2.718281827\ldots$

natural exponential function

The natural exponential function is an exponential function whose base is e: $f(x) = e^x$. The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

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6.2: Evaluate and Graph Logarithmic Functions

Definition 6.2.1: Logarithmic Function

The function $f(x) = \log_a x$ is the **logarithmic function** with base a, where $a > 0, x > 0$, and $a \neq 1$.

$$y = \log_a x \text{ is equivalent to } x = a^y$$

Convert Between Exponential and Logarithmic Form

Since the equations $y = \log_a x$ and $x = a^y$ are equivalent, we can go back and forth between them. This will often be the method to solve some exponential and logarithmic equations. To help with converting back and forth let's take a close look at the equations. See *Figure 10.3.1*. Notice the positions of the exponent and base.

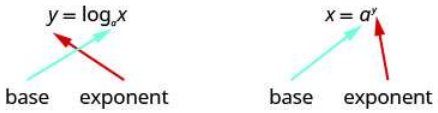


Figure 10.3.1

If we realize the logarithm is the exponent it makes the conversion easier. You may want to repeat, "base to the exponent give us the number."

Example 6.2.1

Convert to logarithmic form:

a. $2^3 = 8$
b. $5^{\frac{1}{2}} = \sqrt{5}$
c. $\left(\frac{1}{2}\right)^x = \frac{1}{16}$

Solution:

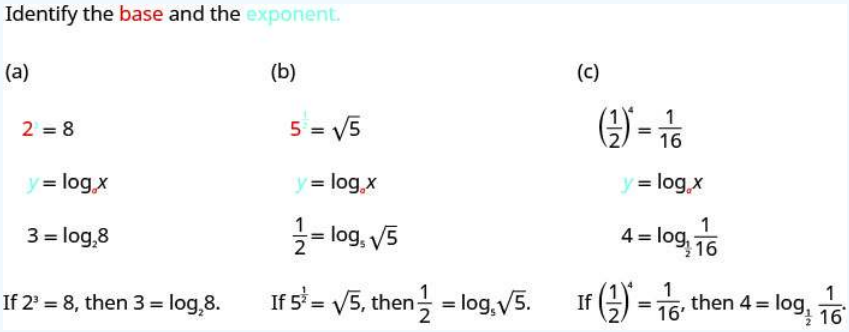


Figure 10.3.2

Exercise 6.2.1

Convert to logarithmic form:

a. $3^2 = 9$
b. $7^{\frac{1}{2}} = \sqrt{7}$
c. $\left(\frac{1}{3}\right)^x = \frac{1}{27}$

Answer

a. $\log_3 9 = 2$
b. $\log_7 \sqrt{7} = \frac{1}{2}$
c. $\log_{\frac{1}{3}} \frac{1}{27} = x$

Exercise 6.2.2

Convert to logarithmic form:

a. $4^3 = 64$
b. $4^{\frac{1}{3}} = \sqrt[3]{4}$
c. $\left(\frac{1}{2}\right)^x = \frac{1}{32}$

Answer

a. $\log_4 64 = 3$
b. $\log_4 \sqrt[3]{4} = \frac{1}{3}$
c. $\log_{\frac{1}{2}} \frac{1}{32} = x$

In the next example we do the reverse—convert logarithmic form to exponential form.

Example 6.2.2

Convert to exponential form:

a. $2 = \log_8 64$
b. $0 = \log_4 1$
c. $-3 = \log_{10} \frac{1}{1000}$

Solution:

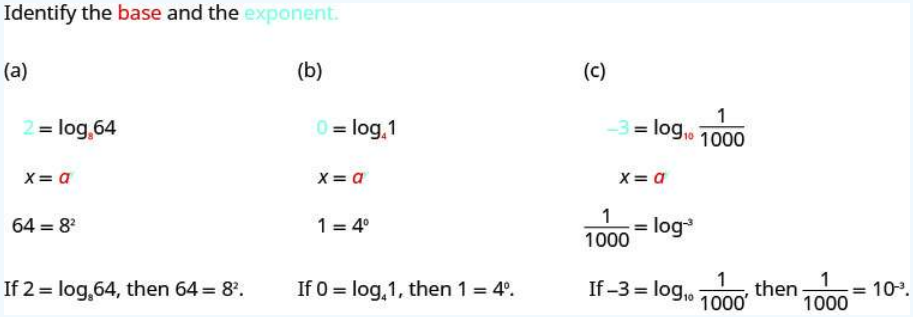


Figure 10.3.3

Exercise 6.2.3

Convert to exponential form:

a. $3 = \log_4 64$
b. $0 = \log_x 1$
c. $-2 = \log_{10} \frac{1}{100}$

Answer

a. $64 = 4^3$
b. $1 = x^0$
c. $\frac{1}{100} = 10^{-2}$

Exercise 6.2.4

Convert to exponential form:

a. $3 = \log_3 27$
b. $0 = \log_x 1$

c. $-1 = \log_{10} \frac{1}{10}$

Answer

a. $27 = 3^3$
b. $1 = x^0$
c. $\frac{1}{10} = 10^{-1}$

Evaluate Logarithmic Functions

We can solve and evaluate logarithmic equations by using the technique of converting the equation to its equivalent exponential equation.

Example 6.2.3

Find the value of x:

a. $\log_x 36 = 2$
b. $\log_4 x = 3$
c. $\log_{\frac{1}{2}} \frac{1}{8} = x$

Solution:

a.

$$\log_x 36 = 2$$

Convert to exponential form.

$$x^2 = 36$$

Solve the quadratic.

$$x = 6, \quad x = \cancel{-6}$$

The base of a logarithmic function must be positive, so we eliminate $x = -6$.

$$x = 6 \quad \text{Therefore, } \log_6 36 = 2$$

b.

$$\log_4 x = 3$$

Convert to exponential form.

$$4^3 = x$$

Simplify.

$$x = 64 \quad \text{Therefore , } \log_4 64 = 3$$

c.

$$\log_{\frac{1}{2}} \frac{1}{8} = x$$

Convert to exponential form.

$$\left(\frac{1}{2}\right)^x = \frac{1}{8}$$

Rewrite $\frac{1}{8}$ as $\left(\frac{1}{2}\right)^3$.

$$\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^3$$

With the same base, the exponents must be equal.

$$x = 3 \quad \text{Therefore } \log_{\frac{1}{2}} \frac{1}{8} = 3$$

Exercise 6.2.5

Find the value of x:

a. $\log_x 64 = 2$
b. $\log_5 x = 3$
c. $\log_{\frac{1}{2}} \frac{1}{4} = x$

Answer

a. $x = 8$
b. $x = 125$
c. $x = 2$

Exercise 6.2.6

Find the value of x:

a. $\log_x 81 = 2$
b. $\log_3 x = 5$
c. $\log_{\frac{1}{3}} \frac{1}{27} = x$

Answer

a. $x = 9$
b. $x = 243$
c. $x = 3$

When see an expression such as $log_3 27$, we can find its exact value two ways. By inspection we realize it means "3 to what power will be 27"? Since $3^3 = 27$, we know $log_3 27 = 3$. An alternate way is to set the expression equal to x and then convert it into an exponential equation.

Example 6.2.4

Find the exact value of each logarithm without using a calculator:

a. $\log_5 25$
b. $\log_9 3$
c. $\log_2 \frac{1}{16}$

Solution:

a.

$$\log_5 25$$

5 to what power will be 25?

$$\log_5 25 = 2$$

Or

Set the expression equal to x.

$$\log_5 25 = x$$

Change to exponential form.

$$5^x = 25$$

Rewrite 25 as 5^2.

$$5^x = 5^2$$

With the same base the exponents must be equal.

$$x = 2 \quad \text{Therefore , } \log_5 25 = 2.$$

b.

$$\log_9 3$$

Set the expression equal to x.

$$\log_9 3 = x$$

Change to exponential form.

$$9^x = 3$$

Rewrite 9 as 3^2.

$$\left(3^2\right)^x = 3^1$$

Simplify the exponents.

$$3^{2x} = 3^1$$

With the same base the exponents must be equal.

$$2x = 1$$

Solve the equation.

$$x = \tfrac{1}{2} \quad \text{Therefore , } \log_9 3 = \tfrac{1}{2}.$$

c.

$$\log_2 \tfrac{1}{16}$$

Set the expression equal to x.

$$\log_2 \tfrac{1}{16} = x$$

Change to exponential form.

$$2^x = \tfrac{1}{16}$$

Rewrite 16 as 2^4.

$$2^x = \tfrac{1}{2^4}$$

$$2^x = 2^{-4}$$

With the same base the exponents must be equal.

$$x = -4 \quad \text{Therefore , } \log_2 \tfrac{1}{16} = -4.$$

Exercise 6.2.7

Find the exact value of each logarithm without using a calculator:

a. $\log_{12} 144$
b. $\log_4 2$
c. $\log_2 \frac{1}{32}$

Answer

a. 2
b. $\frac{1}{2}$
c. -5

Exercise 6.2.8

Find the exact value of each logarithm without using a calculator:

a. $\log_9 81$
b. $\log_8 2$
c. $\log_3 \frac{1}{9}$

Answer

a. 2
b. $\frac{1}{3}$
c. -2

Graph Logarithmic Functions

To graph a logarithmic function $y = log_a x$, it is easiest to convert the equation to its exponential form, $x = a^y$. Generally, when we look for ordered pairs for the graph of a function, we usually choose an x-value and then determine its corresponding y-value. In this case you may find it easier to choose y-values and then determine its corresponding x-value.

Example 6.2.5

Graph $y = \log_2 x$.

Solution:

To graph the function, we will first rewrite the logarithmic equation, $y = \log_2 x$, in exponential form, $2^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of y and then get x.

Table 10.3.1

y	$2^y = x$	(x, y)
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$(\frac{1}{4}, 2)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$(\frac{1}{2}, -1)$
0	$2^0 = 1$	$(1, 0)$
1	$2^1 = 2$	$(2, 1)$
2	$2^2 = 4$	$(4, 2)$
3	$2^3 = 8$	$(8, 3)$

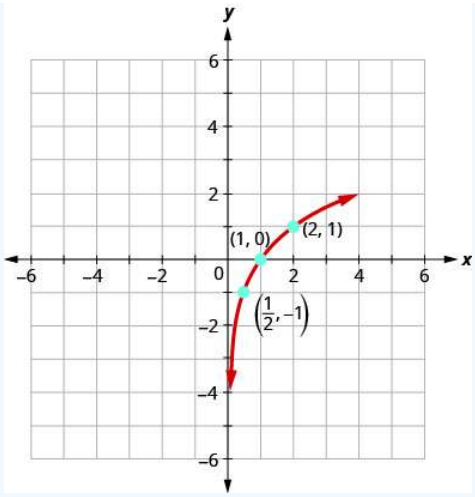


Figure 10.3.4

? Exercise 6.2.9

Graph: $y = \log_3 x$.

Answer

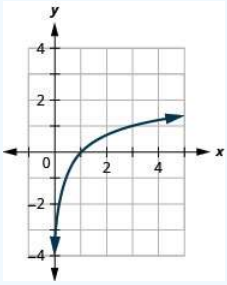


Figure 10.3.5

? Exercise 6.2.10

Graph: $y = \log_5 x$.

Answer

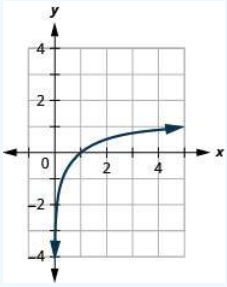


Figure 10.3.6

The graphs of $y = \log_2 x, y = \log_3 x$, and $y = \log_5 x$ are the shape we expect from a logarithmic function where $a > 1$.

We notice that for each function the graph contains the point $(1, 0)$. This make sense because $0 = log_a 1$ means $a^0 = 1$ which is true for any a.

The graph of each function, also contains the point $(a, 1)$. This makes sense as $1 = \log_a a$ means $a^1 = a$. which is true for any a.

Notice too, the graph of each function $y = \log_a x$ also contains the point $\left(\frac{1}{a}, -1\right)$. This makes sense as $-1 = \log_a \frac{1}{a}$ means $a^{-1} = \frac{1}{a}$, which is true for any a.

Look at each graph again. Now we will see that many characteristics of the logarithm function are simply 'mirror images' of the characteristics of the corresponding exponential function.

What is the domain of the function? The graph never hits the y-axis. The domain is all positive numbers. We write the domain in interval notation as $(0, \infty)$.

What is the range for each function? From the graphs we can see that the range is the set of all real numbers. There is no restriction on the range. We write the range in interval notation as $(-\infty, \infty)$.

When the graph approaches the y-axis so very closely but will never cross it, we call the line $x = 0$, the y-axis, a vertical asymptote.

Table 10.3.2: Properties of the Graph of $y = \log_a x$ When $a > 1$

Domain	$(0, \infty)$
Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$
y-intercept	None
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	y-axis

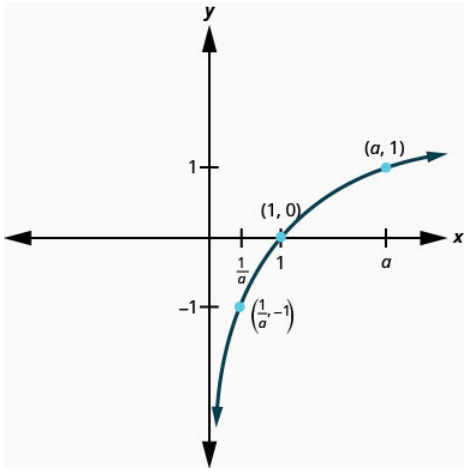


Figure 10.3.7

Our next example looks at the graph of $y = log_a x$ when $0 < a < 1$.

Example 6.2.6

Graph $y = \log_{\frac{1}{3}} x$.

Solution:

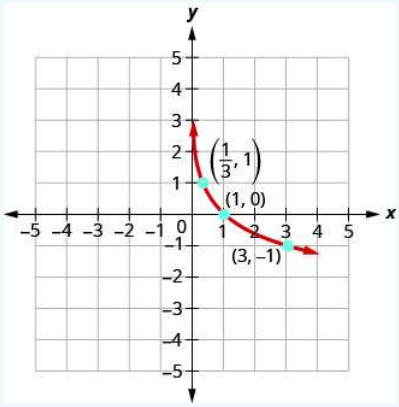
To graph the function, we will first rewrite the logarithmic equation, $y = \log_{\frac{1}{3}} x$, in exponential form, $\left(\frac{1}{3}\right)^y = x$.

We will use point plotting to graph the function. It will be easier to start with values of y and then get x.

Table 10.3.3

y	$\left(\frac{1}{3}\right)^y = x$	(x, y)
-2	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(9, -2)$

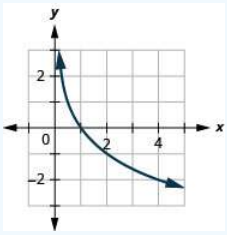
y	$\left(\frac{1}{3}\right)^y = x$	(x, y)
-1	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(3, -1)$
0	$\left(\frac{1}{3}\right)^0 = 1$	$(1, 0)$
1	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(\frac{1}{3}, 1\right)$
2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(\frac{1}{9}, 2\right)$
3	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(\frac{1}{27}, 3\right)$



Exercise 6.2.11

Graph: $y = \log_{\frac{1}{2}} x$.

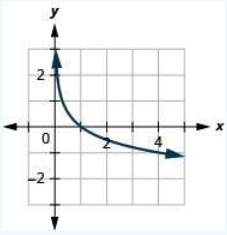
Answer



Exercise 6.2.12

Graph: $y = \log_{\frac{1}{4}} x$.

Answer



Now, let's look at the graphs $y = \log_{\frac{1}{2}} x, y = \log_{\frac{1}{3}} x$ and $y = \log_{\frac{1}{4}} x$, so we can identify some of the properties of logarithmic functions where $0 < a < 1$.

The graphs of all have the same basic shape. While this is the shape we expect from a logarithmic function where $0 < a < 1$.

We notice, that for each function again, the graph contains the points,$(1, 0), (a, 1), \left(\frac{1}{a}, -1\right)$. This make sense for the same reasons we argued above.

We notice the domain and range are also the same—the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$. The y-axis is again the vertical asymptote.

We will summarize these properties in the chart below. Which also include when $a > 1$.

Table 10.3.4: Properties of the Graph of $y = \log_a x$

When $a > 1$		**When $0 < a < 1$**	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	x-intercept	$(1, 0)$
y-intercept	None	y-intercept	None
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	y -axis	Asymptote	y -axis
Basic shape	Increasing	Basic shape	Decreasing

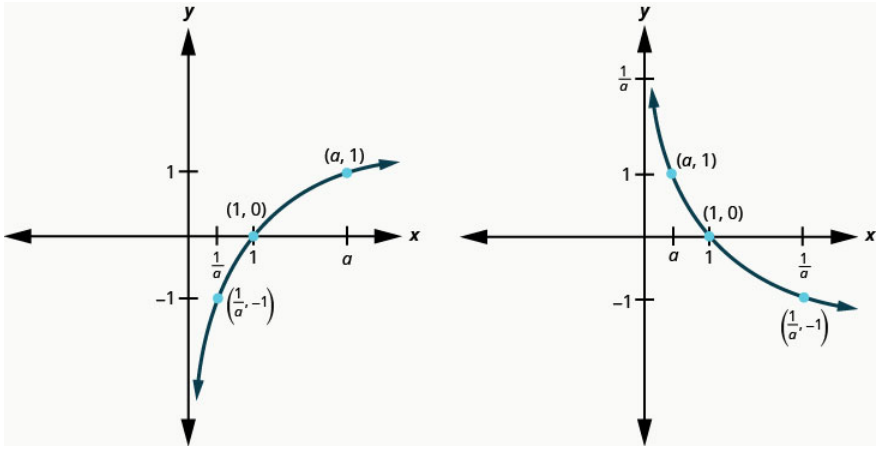


Figure 10.3.11

We talked earlier about how the logarithmic function $f^{-1}(x) = \log_a x$ is the inverse of the exponential function $f(x) = a^x$. The graphs in *Figure 10.3.12* show both the exponential (blue) and logarithmic (red) functions on the same graph for both $a > 1$ and $0 < a < 1$.

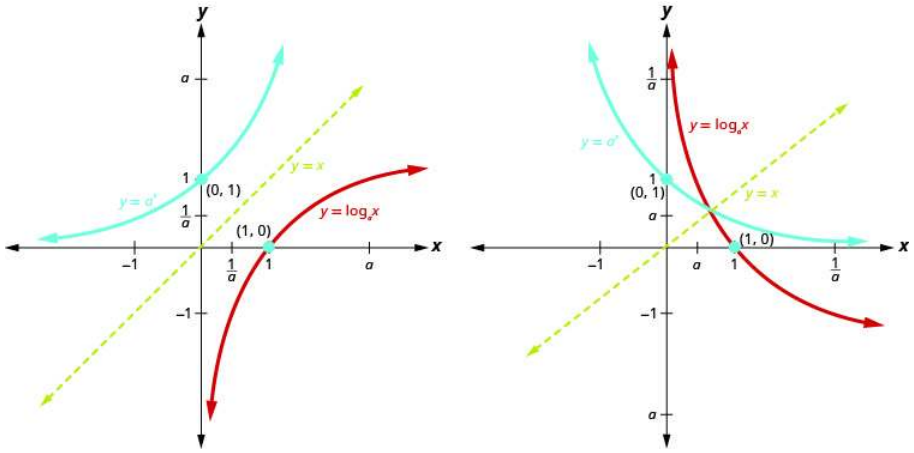


Figure 10.3.12

Notice how the graphs are reflections of each other through the line $y = x$. We know this is true of inverse functions. Keeping a visual in your mind of these graphs will help you remember the domain and range of each function. Notice the x-axis is the horizontal asymptote for the exponential functions and the y-axis is the vertical asymptote for the logarithmic functions.

Use Logarithmic Models in Applications

There are many applications that are modeled by logarithmic equations. We will first look at the logarithmic equation that gives the decibel (dB) level of sound. Decibels range from 0, which is barely audible to 160, which can rupture an eardrum. The 10^{-12} in the formula represents the intensity of sound that is barely audible.

Definition 6.2.4

Decibel Level of Sound

The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is

$$D = 10 \log\left(\frac{I}{10^{-12}}\right)$$

Example 6.2.9

Extended exposure to noise that measures 85 dB can cause permanent damage to the inner ear which will result in hearing loss. What is the decibel level of music coming through ear phones with intensity 10^{-2} watts per square inch?

Solution:

	$D = 10 \log\left(\frac{I}{10^{-12}}\right)$
Substitute in the intensity level, I.	$D = 10 \log\left(\frac{10^{-2}}{10^{-12}}\right)$
Simplify.	$D = 10 \log(10^{10})$
Since $\log 10^{10} = 10$.	$D = 10 \cdot 10$
Multiply.	$D = 100$
	The decibel level of music coming through earphones is 100 dB.

Exercise 6.2.17

What is the decibel level of one of the new quiet dishwashers with intensity 10^{-7} watts per square inch?

Answer

The quiet dishwashers have a decibel level of 50 dB.

Exercise 6.2.18

What is the decibel level heavy city traffic with intensity 10^{-3} watts per square inch?

Answer

The decibel level of heavy traffic is 90 dB.

The magnitude R of an earthquake is measured by a logarithmic scale called the Richter scale. The model is $R = \log I$, where I is the intensity of the shock wave. This model provides a way to measure **earthquake intensity**.

Definition 6.2.5: Earthquake Intensity

The magnitude R of an earthquake is measured by$R = \log I$, where I is the intensity of its shock wave.

Example 6.2.10

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. Over 80% of the city was destroyed by the resulting fires. In 2014, Los Angeles experienced a moderate earthquake that measured 5.1 on the Richter scale and caused $108 million dollars of damage. Compare the intensities of the two earthquakes.

Solution:

To compare the intensities, we first need to convert the magnitudes to intensities using the log formula. Then we will set up a ratio to compare the intensities.

Convert the magnitudes to intensities.

$$R = \log I$$

1906 earthquake

$$7.8 = \log I$$

Convert to exponential form.

$$I = 10^{7.8}$$

2014 earthquake

$$5.1 = \log I$$

Convert to exponential form.

$$I = 10^{5.1}$$

Form a ratio of the intensities.

$$\frac{\text{Intensity for 1906}}{\text{Intensity for 2014}}$$

Substitute in the values.

$$\frac{10^{7.8}}{10^{5.1}}$$

Divide by subtracting the exponents.

$$10^{2.7}$$

Evaluate.

$$501$$

The intensity of the 1906 earthquake was about 501 times the intensity of the 2014 earthquake.

Exercise 6.2.19

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. In 1989, the Loma Prieta earthquake also affected the San Francisco area, and measured 6.9 on the Richter scale. Compare the intensities of the two earthquakes.

Answer

The intensity of the 1906 earthquake was about 8 times the intensity of the 1989 earthquake.

Exercise 6.2.20

In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Answer

The intensity of the earthquake in Chile was about $1,259$ times the intensity of the earthquake in Los Angeles.

Access these online resources for additional instruction and practice with evaluating and graphing logarithmic functions.

- Re-writing logarithmic equations in exponential form
- Simplifying Logarithmic Expressions
- Graphing logarithmic functions
- Using logarithms to calculate decibel levels

Key Concepts

- **Properties of the Graph of** $y = \log_a x$:

When $a > 1$		**When $0 < a < 1$**	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
x-intercept	$(1, 0)$	x-intercept	$(1, 0)$
y-intercept	None	y-intercept	None
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	y -axis	Asymptote	y -axis
Basic shape	Increasing	Basic shape	Decreasing

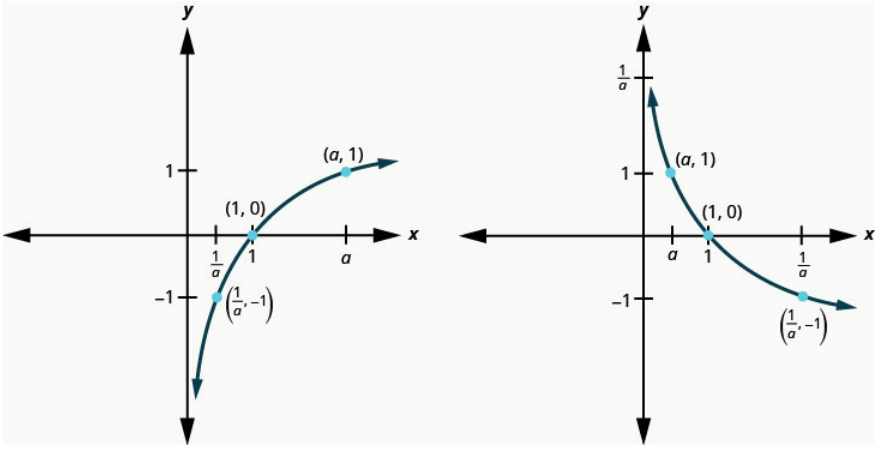


Figure 10.3.11

- **Decibel Level of Sound:** The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is $D = 10 \log\left(\frac{I}{10^{-12}}\right)$.
- **Earthquake Intensity:** The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.

Glossary

common logarithmic function

The function $f(x) = \log x$ is the common logarithmic function with base 10, where $x > 0$.

$$y = \log x \text{ is equivalent to } x = 10^y$$

logarithmic function

The function $f(x) = \log_a x$ is the logarithmic function with base a, where $a > 0, x > 0$, and $a \neq 1$.

$$y = \log_a x \text{ is equivalent to } x = a^y$$

natural logarithmic function

The function $f(x) = \ln x$ is the natural logarithmic function with base e, where $x > 0$.

$$y = \ln x \text{ is equivalent to } x = e^y$$

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6.3: Properties of the Logarithm

Logarithms and Their Inverse Properties

Recall the definition of the base-b logarithm: given $b > 0$ where $b \neq 1$,

$$y = \log_b x \text{ if and only if } x = b^y$$

Use this definition to convert logarithms to exponential form. Doing this, we can derive a few properties:

$$\log_b 1 = 0 \text{ because } b^0 = 1$$
$$\log_b b = 1 \text{ because } b^1 = b$$
$$\log_b\left(\frac{1}{b}\right) = -1 \text{ because } b^{-1} = \frac{1}{b}$$

Example 6.3.1

Evaluate:

a. $\log 1$
b. $\ln e$
c. $\log_5\left(\frac{1}{5}\right)$

Solution

a. When the base is not written, it is assumed to be 10. This is the common logarithm,

$$\log 1 = \log_{10} 1 = 0$$

b. The natural logarithm, by definition, has base e,

$$\ln e = \log_e e = 1$$

c. Because $5^{-1} = \frac{1}{5}$ we have,

$$\log_5\left(\frac{1}{5}\right) = -1$$

Furthermore, consider fractional bases of the form $1/b$ where $b > 1$.

$$\log_{1/b} b = -1 \quad \text{because } \left(\tfrac{1}{b}\right)^{-1} = \frac{1^{-1}}{b^{-1}} = \frac{b}{1} = b$$

Example 6.3.2

Evaluate:

a. $\log_{1/4} 4$
b. $\log_{2/3}\left(\frac{3}{2}\right)$

Solution

a. $\log_{1/4} 4 = -1$ because $\left(\frac{1}{4}\right)^{-1} = 4$
b. $\log_{2/3}\left(\frac{3}{2}\right) = -1$ because $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$

Given an exponential function defined by $f(x) = b^x$, where $b > 0$ and $b \neq 1$, its inverse is the base-b logarithm, $f^{-1}(x) = log_b x$. And because $f\left(f^{-1}(x)\right) = x$ and $f^{-1}(f(x)) = x$, we have the following **inverse properties of the logarithm**[11]:

$$f^{-1}(f(x)) = \log_b b^x = x \text{ and}$$
$$f\left(f^{-1}(x)\right) = b^{\log_b x} = x, x > 0$$

Since $f^{-1}(x) = \log_b x$ has a domain consisting of positive values $(0, \infty)$, the property $b^{\log_b x} = x$ is restricted to values where $x > 0$.

Example 6.3.3

Evaluate

a. $\log_5 625$
b. $5^{\log_5 3}$
c. $e^{\ln 5}$

Solution

Apply the inverse properties of the logarithm.

a. $\log_5 625 = \log_5 5^4 = 4$
b. $5^{\log_5 3} = 3$
c. $e^{\ln 5} = 5$

In summary, when $b > 0$ and $b \neq 1$, we have the following properties:

Table 6.3.1

$\log_b 1 = 0$	$\log_b b = 1$
$\log_{1/b} b = -1$	$\log_b \left(\frac{1}{b}\right) = -1$
$\log_b b^x = x$	$b^{\log_b x} = x, x > 0$

Exercise 6.3.1

Evaluate: $\log 0.00001$

Answer

-5

www.youtube.com/v/YFsZxKiMCTg

Product, Quotient, and Power Properties of Logarithms

In this section, three very important properties of the logarithm are developed. These properties will allow us to expand our ability to solve many more equations. We begin by assigning u and v to the following logarithms and then write them in exponential form:

$$\log_b x = u \Longrightarrow b^u = x$$
$$\log_b y = v \Longrightarrow b^v = y$$

Substitute $x = b^u$ and $y = b^v$ into the logarithm of a product $log_b(xy)$ and the logarithm of a quotient $\log_b \left(\frac{x}{y}\right)$.Then simplify using the rules of exponents and the inverse properties of the logarithm.

Table 6.3.2

Logarithm of a Product	Logarithm of a Quotient
$\begin{aligned}\log_b(xy) &= \log_b(b^u b^v) \\ &= \log_b b^{u+v} \\ &= u+v \\ &= \log_b x + \log_b y\end{aligned}$	$\begin{aligned}\log_b\left(\frac{x}{y}\right) &= \log_b\left(\frac{b^u}{b^v}\right) \\ &= \log_b b^{u-v} \\ &= u-v \\ &= \log_b x - \log_b y\end{aligned}$

This gives us two essential properties: the **product property of logarithms**[12],

$$\log_b(xy) = \log_b x + \log_b y$$

and the **quotient property of logarithms**[13],

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

In words, the logarithm of a product is equal to the sum of the logarithm of the factors. Similarly, the logarithm of a quotient is equal to the difference of the logarithm of the numerator and the logarithm of the denominator.

Example 6.3.4

Write as a sum $\log_2(8x)$.

Solution

Apply the product property of logarithms and then simplify.

$$\begin{aligned}\log_2(8x) &= \log_2 8 + \log_2 x \\ &= \log_2 2^3 + \log_2 x \\ &= 3 + \log_2 x\end{aligned}$$

Answer

$$3 + \log_2 x$$

Example 6.3.5

Write as a difference $\log\left(\frac{x}{10}\right)$.

Solution

Apply the quotient property of logarithms and then simplify.

$$\begin{aligned}\log\left(\frac{x}{10}\right) &= \log x - \log 10 \\ &= \log x - 1\end{aligned}$$

Answer

$$\log x - 1$$

Next we begin with $log_b x = u$ and rewrite it in exponential form. After raising both sides to the nth power, convert back to logarithmic form, and then back substitute.

$$\begin{aligned}\log_b x = u &\Longrightarrow b^u = x \\ (b^u)^n &= (x)^n \\ \log_b x^n = nu &\Longleftarrow b^{nu} = x^n \\ \log_b x^n &= n\log_b x\end{aligned}$$

This leads us to the **power property of logarithms**[14],

$$\log_b x^n = n\log_b x$$

In words, the logarithm of a quantity raised to a power is equal to that power times the logarithm of the quantity.

Example 6.3.6

Write as a product:

a. $\log_2 x^4$
b. $\log_5(\sqrt{x})$

Solution

a. Apply the power property of logarithms.

$$\log_2 x^4 = 4\log_2 x \tag{6.3.1}$$

b. Recall that a square root can be expressed using rational exponents, $\sqrt{x} = x^{1/2}$. Make this replacement and then apply the power property of logarithms.

$$\begin{aligned}\log_5(\sqrt{x}) &= \log_5 x^{1/2} \\ &= \frac{1}{2}\log_5 x\end{aligned}$$

In summary,

Table 6.3.3

Product Property of Logarithms	$\log_b(xy) = \log_b x + \log_b y$
Quotient Property of Logarithms	$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
Power Property of Logarithms	$\log_b x^n = n\log_b x$

We can use these properties to expand logarithms involving products, quotients, and powers using sums, differences and coefficients. A logarithmic expression is completely expanded when the properties of the logarithm can no further be applied.

Caution

It is important to point out the following:

$$\log(xy) \neq \log x \cdot \log y \tag{6.3.2}$$

and

$$\log\left(\frac{x}{y}\right) \neq \frac{\log x}{\log y} \tag{6.3.3}$$

Example 6.3.7

Expand completely: $\ln\left(2x^3\right)$.

Solution

Recall that the natural logarithm is a logarithm base e, $\ln x = \log_e x$. Therefore, all of the properties of the logarithm apply.

$$\begin{aligned}\ln\left(2x^3\right) &= \ln 2 + \ln x^3 \quad \textit{Product rule for logarithms} \\ &= \ln 2 + 3\ln x \quad \textit{Power rule for logarithms}\end{aligned}$$

Answer

$$\ln 2 + 3\ln x$$

Example 6.3.8:

Expand completely: $\log\sqrt[3]{10xy^2}$.

Solution

Begin by rewriting the cube root using the rational exponent $\frac{1}{3}$ and then apply the properties of the logarithm.

$$\begin{aligned}\log\sqrt[3]{10xy^2} &= \log\left(10xy^2\right)^{1/3} \\ &= \frac{1}{3}\log\left(10xy^2\right) \\ &= \frac{1}{3}\left(\log 10 + \log x + \log y^2\right) \\ &= \frac{1}{3}\left(1 + \log x + 2\log y\right) \\ &= \frac{1}{3} + \frac{1}{3}\log x + \frac{2}{3}\log y\end{aligned}$$

Answer

$$\frac{1}{3}+\frac{1}{3}\log x+\frac{2}{3}\log y$$

Example 6.3.9:

Expand completely: $\log_2\left(\frac{(x+1)^2}{5y}\right)$.

Solution

When applying the product property to the denominator, take care to distribute the negative obtained from applying the quotient property.

$$\begin{aligned}\log_2\left(\frac{(x+1)^2}{5y}\right) &= \log_2(x+1)^2-\log_2(5y)\\ &= \log_2(x+1)^2-(\log_2 5+\log_2 y) \quad \textit{Distribute.}\\ &= \log_2(x+1)^2-\log_2 5-\log_2 y\\ &= 2\log_2(x+1)-\log_2 5-\log_2 y\end{aligned}$$

Answer

$$2\log_2(x+1)-\log_2 5-\log_2 y$$

Caution

There is no rule that allows us to expand the logarithm of a sum or difference. In other words,

$$\log(x\pm y)\neq \log x\pm\log y$$

Exercise 6.3.2

Expand completely: $\ln\left(\frac{5y^4}{\sqrt{x}}\right)$

Answer

$\ln 5+4\ln y-\frac{1}{2}\ln x$

www.youtube.com/v/h_Zu-SkATl0

Example 6.3.10:

Given that $\log_2 x=a$, $\log_2 y=b$, and that $\log_2 z=c$, write the following in terms of a, b, and c:

a. $\log_2\left(8x^2y\right)$
b. $\log_2\left(\frac{2x^4}{\sqrt{z}}\right)$

Solution

a. Begin by expanding using sums and coefficients and then replace a and b with the appropriate logarithm.

$$\begin{aligned}\log_2\left(8x^2y\right) &= \log_2 8+\log_2 x^2+\log_2 y\\ &= \log_2 8+2\log_2 x+\log_2 y\\ &= 3+2a+b\end{aligned}$$

b. Expand and then replace a, b, and c where appropriate.

$$\begin{aligned}\log_2\left(\frac{2x^4}{\sqrt{z}}\right) &= \log_2\left(2x^4\right) - \log_2 z^{1/2} \\ &= \log_2 2 + \log_2 x^4 - \log_2 z^{1/2} \\ &= \log_2 2 + 4\log_2 x - \frac{1}{2}\log_2 z \\ &= 1 + 4a - \frac{1}{2}b\end{aligned}$$

Next we will condense logarithmic expressions. As we will see, it is important to be able to combine an expression involving logarithms into a single logarithm with coefficient 1. This will be one of the first steps when solving logarithmic equations.

Example 6.3.11:

Write as a single logarithm with coefficient 1: $3\log_3 x - \log_3 y + 2\log_3 5$.

Solution

Begin by rewriting all of the logarithmic terms with coefficient 1. Use the power rule to do this. Then use the product and quotient rules to simplify further.

$$\begin{aligned}3\log_3 x - \log_3 y + 2\log_3 5 &= \left\{\log_3 x^3 - \log_3 y\right\} + \log_3 5^2 \quad \textit{quotient property} \\ &= \left\{\log_3\left(\frac{x^3}{y}\right) + \log_3 25\right\} \quad \textit{product property} \\ &= \log_3\left(\frac{x^3}{y}\cdot 25\right) \\ &= \log_3\left(\frac{25x^3}{y}\right)\end{aligned}$$

Answer

$$\log_3\left(\frac{25x^3}{y}\right)$$

Example 6.3.12:

Write as a single logarithm with coefficient 1: $\frac{1}{2}\ln x - 3\ln y - \ln z$.

Solution

Begin by writing the coefficients of the logarithms as powers of their argument, after which we will apply the quotient rule twice working from left to right.

$$\begin{aligned}\frac{1}{2}\ln x - 3\ln y - \ln z &= \ln x^{1/2} - \ln y^3 - \ln z \\ &= \ln\left(\frac{x^{1/2}}{y^3}\right) - \ln z \\ &= \ln\left(\frac{x^{1/2}}{y^3} \div z\right) \\ &= \ln\left(\frac{x^{1/2}}{y^3}\cdot\frac{1}{z}\right) \\ &= \ln\left(\frac{x^{1/2}}{y^3 z}\right) \quad \text{or} \quad = \ln\left(\frac{\sqrt{x}}{y^3 z}\right)\end{aligned}$$

Answer

$$\ln\left(\frac{\sqrt{x}}{y^3 z}\right)$$

? Exercise 6.3.3

Write as a single logarithm with coefficient 1: $3\log(x+y)-6\log z+2\log 5$

Answer

$\log\left(\frac{25(x+y)^3}{z^6}\right)$

www.youtube.com/v/o1AVbmS9AU8

Key Takeaways

- Given any base $b>0$ and $b\neq 1$, we can say that $log_b 1=0$, $log_b b=1$, $log_{1/b} b=-1$ and that $log_b(\frac{1}{b})=-1$.
- The inverse properties of the logarithm are $log_b b^x=x$ and $b^{log_b x}=x$ where $x>0$.
- The product property of the logarithm allows us to write a product as a sum: $\log_b(xy)=\log_b x+\log_b y$.
- The quotient property of the logarithm allows us to write a quotient as a difference: $\log_b\left(\frac{x}{y}\right)=\log_b x-\log_b y$.
- The power property of the logarithm allows us to write exponents as coefficients: $\log_b x^n=n\log_b x$.
- Since the natural logarithm is a base-e logarithm, $\ln x=\log_e x$, all of the properties of the logarithm apply to it.
- We can use the properties of the logarithm to expand logarithmic expressions using sums, differences, and coefficients. A logarithmic expression is completely expanded when the properties of the logarithm can no further be applied.
- We can use the properties of the logarithm to combine expressions involving logarithms into a single logarithm with coefficient 1. This is an essential skill to be learned in this chapter.

Footnotes

[11]Given $b>0$ we have $\log_b b^x=x$ and $b^{\log_b x}=x$ when $x>0$.

[12]$\log_b(xy)=\log_b x+\log_b y$; the logarithm of a product is equal to the sum of the logarithm of the factors.

[13]$\log_b\left(\frac{x}{y}\right)=\log_b x-\log_b y$; the logarithm of a quotient is equal to the difference of the logarithm of the numerator and the logarithm of the denominator.

[14]$\log_b x^n=n\log_b x$; the logarithm of a quantity raised to a power is equal to that power times the logarithm of the quantity.]

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6.4: Solving Exponential and Logarithmic Equations

Solving Exponential Equations

An **exponential equation**[15] is an equation that includes a variable as one of its exponents. In this section we describe two methods for solving exponential equations. First, recall that exponential functions defined by $f(x) = b^x$ where $b > 0$ and $b \neq 1$, are one-to-one; each value in the range corresponds to exactly one element in the domain. Therefore, $f(x) = f(y)$ implies $x = y$. The converse is true because f is a function. This leads to the very important **one-to-one property of exponential functions**[16]:

$$b^x = b^y \quad \text{if and only if} \quad x = y$$

Use this property to solve special exponential equations where each side can be written in terms of the same base.

Example 6.4.1:

Solve $3^{2x-1} = 27$.

Solution

Begin by writing 27 as a power of 3.

$$\begin{aligned} 3^{2x-1} &= 27 \\ 3^{2x-1} &= 3^3 \end{aligned}$$

Next apply the one-to-one property of exponential functions. In other words, set the exponents equal to each other and then simplify.

$$\begin{aligned} 2x - 1 &= 3 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

Answer:

2

Example 6.4.2:

Solve: $16^{1-3x} = 2$.

Solution

Begin by writing 16 as a power of 2 and then apply the power rule for exponents.

$$\begin{aligned} 16^{1-3x} &= 2 \\ \left(2^4\right)^{1-3x} &= 2 \\ 2^{4(1-3x)} &= 2^1 \end{aligned}$$

Now that the bases are the same we can set the exponents equal to each other and simplify.

$$\begin{aligned} 4(1-3x) &= 1 \\ 4 - 12x &= 1 \\ -12x &= -3 \\ x &= \frac{-3}{-12} = \frac{1}{4} \end{aligned}$$

Answer:

$\frac{1}{4}$

? Exercise 6.4.1

Solve: $25^{2x+3} = 125$.

Answer

$-\frac{3}{4}$

www.youtube.com/v/yU_tJg6Unnc

In many cases we will not be able to equate the bases. For this reason we develop a second method for solving exponential equations. Consider the following equations:

$$3^2 = 9$$
$$3^{?} = 12$$
$$3^3 = 27$$

We can see that the solution to $3^x = 12$ should be somewhere between 2 and 3. A graphical interpretation follows.

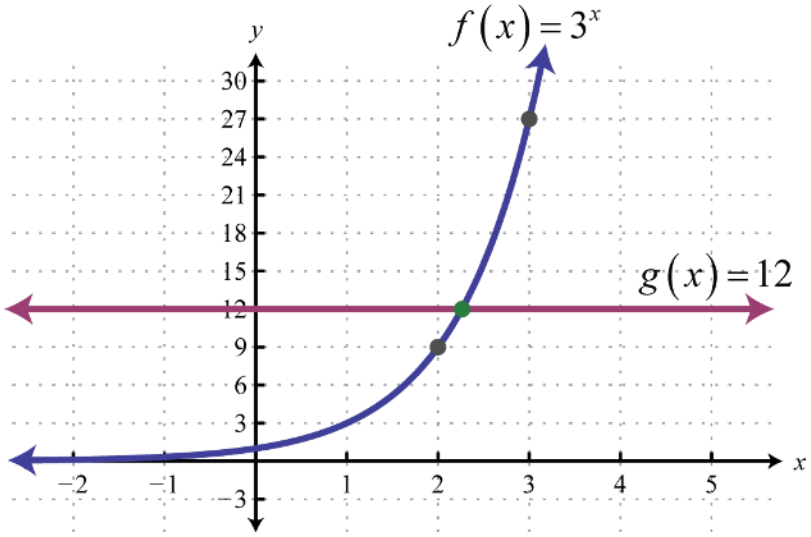


Figure 6.4.1

To solve this we make use of fact that logarithms are one-to-one functions. Given $x, y > 0$ the **one-to-one property of logarithms**[17] follows:

$$\log_b x = \log_b y \quad \text{if and only if} \quad x = y$$

This property, as well as the properties of the logarithm, allows us to solve exponential equations. For example, to solve $3^x = 12$ apply the common logarithm to both sides and then use the properties of the logarithm to isolate the variable.

$$\begin{aligned} 3^x &= 12 \\ \log 3^x &= \log 12 \quad \textit{One-to-one property of logarithms} \\ x \log 3 &= \log 12 \quad \textit{Power rule for logarithms} \\ x &= \frac{\log 12}{\log 3} \end{aligned}$$

Approximating to four decimal places on a calculator.

$$x = \log(12)/\log(3) \approx 2.2619$$

An answer between 2 and 3 is what we expected. Certainly we can check by raising 3 to this power to verify that we obtain a good approximation of 12.

$$3 \wedge 2.2618 \approx 12 \ \checkmark$$

Note that we are *not* multiplying both sides by "log"; we are applying the one-to-one property of logarithmic functions — which is often expressed as "*taking the log of both sides*." The general steps for solving exponential equations are outlined in the following example.

Example 6.4.3:

Solve: $5^{2x-1}+2=9$.

Solution

Step 1: Isolate the exponential expression.

$$\begin{aligned} 5^{2x-1}+2 &= 9 \\ 5^{2x-1} &= 7 \end{aligned}$$

Step 2: Take the logarithm of both sides. In this case, we will take the common logarithm of both sides so that we can approximate our result on a calculator.

$$\log 5^{2x-1} = \log 7$$

Step 3: Apply the power rule for logarithms and then solve.

$$\begin{aligned} \log 5^{2x-1} &= \log 7 \\ (2x-1)\log 5 &= \log 7 \qquad \textit{Distribute.} \\ 2x\log 5-\log 5 &= \log 7 \\ 2x\log 5 &= \log 5+\log 7 \\ x &= \frac{\log 5+\log 7}{2\log 5} \end{aligned}$$

This is an irrational number which can be approximated using a calculator. Take care to group the numerator and the product in the denominator when entering this into your calculator. To do this, make use of the parenthesis buttons (and) :

$$x = (\log 5+\log(7))/(2*\log(5)) \approx 1.1045$$

Answer:

$\frac{\log 5+\log 7}{2\log 5} \approx 1.1045$

Example 6.4.4:

Solve: $e^{5x+3}=1$.

Solution

The exponential function is already isolated and the base is e. Therefore, we choose to apply the natural logarithm to both sides.

$$\begin{aligned} e^{5x+3} &= 1 \\ \ln e^{5x+3} &= \ln 1 \end{aligned}$$

Apply the power rule for logarithms and then simplify.

$$\begin{aligned} \ln e^{5x+3} &= \ln 1 \\ (5x+3)\ln e &= \ln 1 \qquad \textit{Recall } \ln e=1 \textit{ and } \ln 1=0. \\ (5x+3)\cdot 1 &= 0 \\ 5x+3 &= 0 \\ x &= -\frac{3}{5} \end{aligned}$$

Answer:

$-\frac{3}{5}$

On most calculators there are only two logarithm buttons, the common logarithm LOG and the natural logarithm LN. If we want to approximate $\log_3 10$ we have to somehow change this base to 10 or e. The idea begins by rewriting the logarithmic function $y = \log_a x$, in exponential form.

$$\log_a x = y \Longrightarrow x = a^y$$

Here $x > 0$ and so we can apply the one-to-one property of logarithms. Apply the logarithm base b to both sides of the function in exponential form.

$$x = a^y$$
$$\log_b x = \log_b a^y$$

And then solve for y.

$$\log_b x = y \log_b a$$
$$\frac{\log_b x}{\log_b a} = y$$

Replace y into the original function and we have the very important **change of base formula**[18]:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

We can use this to approximate $\log_3 10$ as follows.

$$\log_3 10 = \frac{\log 10}{\log 3} \approx 2.0959 \text{ or } \log_3 10 = \frac{\ln 10}{\ln 3} \approx 2.0959$$

Notice that the result is independent of the choice of base. In words, we can approximate the logarithm of any given base on a calculator by dividing the logarithm of the argument by the logarithm of that given base.

Example 6.4.5:

Approximate $\log_7 120$ the nearest hundredth.

Solution

Apply the change of base formula and use a calculator.

$$\log_7 120 = \frac{\log 120}{\log 7}$$

On a calculator,

$$\log(120) / \log(7) \approx 2.46$$

Answer:

2.46

Exercise 6.4.2

Solve: $2^{3x+1} - 4 = 1$. Give the exact and approximate answer rounded to four decimal places.

Answer

$\frac{\log 5 - \log 2}{3 \log 2} \approx 0.4406$

www.youtube.com/v/kQf1-cEQUg8

Solving Logarithmic Equations

A **logarithmic equation**[19] is an equation that involves a logarithm with a variable argument. Some logarithmic equations can be solved using the one-to-one property of logarithms. This is true when a single logarithm with the same base can be obtained on both sides of the equal sign.

Example 6.4.6:

Solve: $\log_2(2x-5)-\log_2(x-2)=0$.

Solution

We can obtain two equal logarithms base 2 by adding $\log_2(x-2)$ to both sides of the equation.

$$\begin{aligned}\log_2(2x-5)-\log_2(x-2) &= 0\\ \log_2(2x-5) &= \log_2(x-2)\end{aligned}$$

Here the bases are the same and so we can apply the one-to-one property and set the arguments equal to each other.

$$\begin{aligned}\log_2(2x-5) &= \log_2(x-2)\\ 2x-5 &= x-2\\ x &= 3\end{aligned}$$

Checking $x=3$ in the original equation:

$$\begin{aligned}\log_2(2(3)-5) &= \log_2((3)-2)\\ \log_2 1 &= \log_2 1\\ 0 &= 0\ \checkmark\end{aligned}$$

Answer:

3

When solving logarithmic equations the check is very important because extraneous solutions can be obtained. The properties of the logarithm only apply for values in the domain of the given logarithm. And when working with variable arguments, such as $\log(x-2)$, the value of x is not known until the end of this process. The logarithmic expression $\log(x-2)$ is only defined for values $x>2$.

Example 6.4.7:

Solve: $\log(3x-4)=\log(x-2)$.

Solution

Apply the one-to-one property of logarithms (set the arguments equal to each other) and then solve for x.

$$\begin{aligned}\log(3x-4) &= \log(x-2)\\ 3x-4 &= x-2\\ 2x &= 2\\ x &= 1\end{aligned}$$

When performing the check we encounter a logarithm of a negative number:

$$\begin{aligned}\log(x-2) &= \log(1-2)\\ &= \log(-1) \quad \textit{Undefined}\end{aligned}$$

Try this on a calculator, what does it say? Here $x=1$ is not in the domain of $\log(x-2)$. Therefore our only possible solution is extraneous and we conclude that there are no solutions to this equation.

Answer

No solution, $\emptyset$.

Caution: Solving logarithmic equations sometimes leads to extraneous solutions — we must check our answers.

Exercise 6.4.3

Solve: $\ln\left(x^2-15\right)-\ln(2x)=0$.

Answer

5

www.youtube.com/v/-kBkk-q2pNs

In many cases we will not be able to obtain two equal logarithms. To solve such equations we make use of the definition of the logarithm. If $b>0$, where $b\neq 1$, then $\log_b x=y$ implies that $b^y=x$. Consider the following common logarithmic equations (base 10),

$$\log x=0 \Longrightarrow = 1 \quad \textit{Because } 10^{10}=1.$$
$$\log x=0.5 \Longrightarrow = ?$$
$$\log x=1 \Longrightarrow = 10 \quad \textit{Because } 10^{1}=10.$$

We can see that the solution to $\log x=0.5$ will be somewhere between 1 and 10. A graphical interpretation follows.

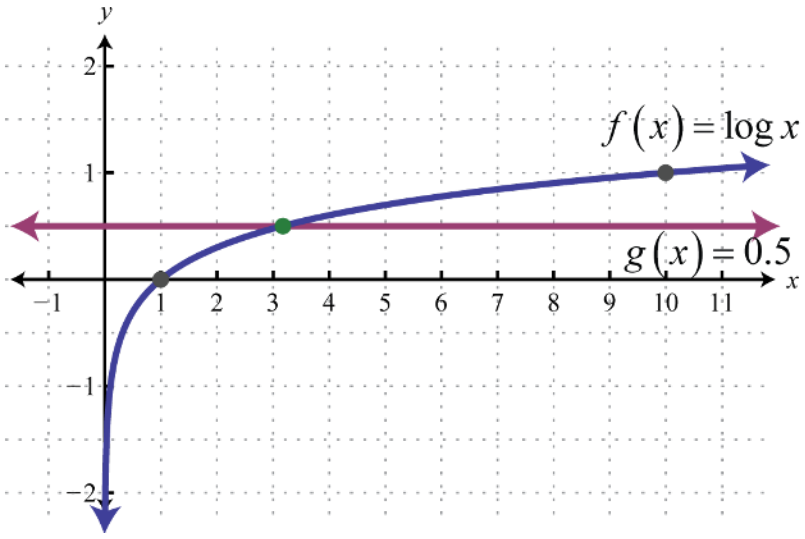


Figure 6.4.2

To find x we can apply the definition as follows.

$$\log_{10} x=0.5 \Longrightarrow 10^{0.5}=x$$

This can be approximated using a calculator,

$$x=10^{0.5}=10\text{^}0.5\approx 3.1623$$

An answer between 1 and 10 is what we expected. Check this on a calculator.

$$\log 3.1623\approx 5 \ \checkmark$$

Example 6.4.8:

Solve: $\log_3(2x-5)=2$.

Solution

Apply the definition of the logarithm.

$$\log_3(2x-5)=2 \Longrightarrow 2x-5=3^2$$

Solve the resulting equation.

$$\begin{aligned} 2x-5 &= 9 \\ 2x &= 14 \\ x &= 7 \end{aligned}$$

Check.

$$\log_3(2(7)-5) \stackrel{?}{=} 2$$
$$\log_3(9) = 2 \ \checkmark$$

Answer:

7

In order to apply the definition, we will need to rewrite logarithmic expressions as a single logarithm with coefficient 1. The general steps for solving logarithmic equations are outlined in the following example.

Example 6.4.9:

Solve: $\log_2(x-2)+\log_2(x-3)=1$.

Solution

Step 1: Write all logarithmic expressions as a single logarithm with coefficient 1. In this case, apply the product rule for logarithms.

$$\begin{aligned}\log_2(x-2)+\log_2(x-3) &= 1\\ \log_2[(x-2)(x-3)] &= 1\end{aligned}$$

Step 2: Use the definition and rewrite the logarithm in exponential form,

$$\log_2[(x-2)(x-3)]=1 \Longrightarrow (x-2)(x-3)=2^1$$

Step 3: Solve the resulting equation. Here we can solve by factoring.

$$\begin{aligned}(x-2)(x-3) &= 2\\ x^2-5x+6 &= 2\\ x^2-5x+4 &= 0\\ (x-4)(x-1) &= 0\\ x-4=0 \quad &\text{or} \quad x-1=0\\ x=4 \quad &\quad\ \ x=1\end{aligned}$$

Step 4: Check. This step is required.

Table 6.4.1

Check $x=4$	Check $x=1$
$\log_2(x-2)+\log_2(x-3) = 1$ $\log_2(4-2)+\log_2(4-3) = 1$ $\log_2(2)+\log_2(1) = 1$ $1+0=1$ ✓	\(\begin{aligned}\log _{2}(x-2)+\log _{2}(x-3)&=1 \\ \log _{2}(\color{Cerulean}{1}\color{black}{-}2)+\log _{2}(\color{Cerulean}{1}\color{black}{-}3)&=1 \\ \log _{2}(-1)+\log _{2}(-2)&=1\\N/A&\neq1\:\:\color{red}{✗}\end{aligned}\)

In this example, $x=1$ is not in the domain of the given logarithmic expression and is extraneous. The only solution is $x=4$.

Answer:

4

Example 6.4.10:

Solve: $\log(x+15)-1=\log(x+6)$

Solution

Begin by writing all logarithmic expressions on one side and constants on the other.

$$\begin{aligned}\log(x+15)-1 &= \log(x+6)\\ \log(x+15)-\log(x+6) &= 1\end{aligned}$$

Apply the quotient rule for logarithms as a means to obtain a single logarithm with coefficient 1.

$$\log(x+15)-\log(x+6)=1$$
$$\log\left(\tfrac{x+15}{x+6}\right)=1$$

This is a common logarithm; therefore use 10 as the base when applying the definition.

$$\begin{aligned}\frac{x+15}{x+6} &= 10^1\\ x+15 &= 10(x+6)\\ x+15 &= 10x+60\\ -9x &= 45\\ x &= -5\end{aligned}$$

Check.

$$\begin{aligned}\log(x+15)-1 &= \log(x+6)\\ \log(-5+15)-1 &= \log(-5+6)\\ \log 10-1 &= \log 1\\ 1-1 &= 0\\ 0 &= 0\ \checkmark\end{aligned}$$

Answer:

-5

Exercise 6.4.4

Solve: $\log_2(x)+\log_2(x-1)=1$.

Answer

2

www.youtube.com/v/Lzwh1jkj-EY

Example 6.4.11:

Find the inverse: $f(x)=\log_2(3x-4)$.

Solution

Begin by replacing the function notation $f(x)$ with y.

$$\begin{aligned}f(x) &= \log_2(3x-4)\\ y &= \log_2(3x-4)\end{aligned}$$

Interchange x and y and then solve for y.

$$\begin{aligned}x=\log_2(3y-4)\Longrightarrow 3y-4 &= 2^x\\ 3y &= 2^x+4\\ y &= \frac{2^x+4}{3}\end{aligned}$$

The resulting function is the inverse of f. Present the answer using function notation.

Answer:

$f^{-1}(x)=\frac{2^x+4}{3}$

Key Takeaways

- If each side of an exponential equation can be expressed using the same base, then equate the exponents and solve.
- To solve a general exponential equation, first isolate the exponential expression and then apply the appropriate logarithm to both sides. This allows us to use the properties of logarithms to solve for the variable.
- The change of base formula allows us to use a calculator to calculate logarithms. The logarithm of a number is equal to the common logarithm of the number divided by the common logarithm of the given base.
- If a single logarithm with the same base can be isolated on each side of an equation, then equate the arguments and solve.
- To solve a general logarithmic equation, first isolate the logarithm with coefficient 1 and then apply the definition. Solve the resulting equation.
- The steps for solving logarithmic equations sometimes produce extraneous solutions. Therefore, the check is required.

Footnotes

[15] An equation which includes a variable as an exponent.

[16] Given $b > 0$ and $b \neq 1$ we have $b^x = b^y$ if and only if $x = y$.

[17] Given $b > 0$ and $b \neq 1$ where $x, y > 0$ we have $log_b x = log_b y$ if and only if $x = y$.

[18] $\log_a x = \frac{\log_b x}{\log_b a}$; we can write any base-$a$ logarithm in terms of base-b logarithms using this formula.

[19] An equation that involves a logarithm with a variable argument.

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6.5: Applications of Exponential and Logarithmic Functions

Compound and Continuous Interest Formulas

Recall that compound interest occurs when interest accumulated for one period is added to the principal investment before calculating interest for the next period. The amount A accrued in this manner over time t is modeled by the compound interest formula:

$$A(t) = P\left(1 + \tfrac{r}{n}\right)^{nt}$$

Here the initial principal P is accumulating compound interest at an annual rate r where the value n represents the number of times the interest is compounded in a year.

Example 6.5.1:

Susan invested $\$500$ in an account earning $4\frac{1}{2}\%$ annual interest that is compounded monthly.

a. How much will be in the account after 3 years?

b. How long will it take for the amount to grow to $\$750$?

Solution

In this example, the principal $P = \$500$, the interest rate $r = 4\frac{1}{2}\% = 0.045$, and because the interest is compounded monthly, $n = 12$. The investment can be modeled by the following function:

$$A(t) = 500\left(1 + \tfrac{0.045}{12}\right)^{12t}$$
$$A(t) = 500(1.00375)^{12t}$$

a. Use this model to calculate the amount in the account after $t = 3$ years.

$$\begin{aligned} A(3) &= 500(1.00375)^{12(3)} \\ &= 500(1.00375)^{36} \\ &\approx 572.12 \end{aligned}$$

Rounded off to the nearest cent, after 3 years, the amount accumulated will be $\$572.12$

b. To calculate the time it takes to accumulate $\$750$, set $A(t) = 750$ and solve for t.

$$A(t) = 500(1.00375)^{12t}$$
$$750 = 500(1.00375)^{12t}$$

This results in an exponential equation that can be solved by first isolating the exponential expression.

$$750 = 500(1.00375)^{12t}$$
$$\frac{750}{500} = (1.00375)^{12t}$$
$$1.5 = (1.00375)^{12t}$$

At this point take the common logarithm of both sides, apply the power rule for logarithms, and then solve for t.

$$\log(1.5) = \log(1.00375)^{12t}$$
$$\log(1.5) = 12t\log(1.00375)$$
$$\frac{\log(1.5)}{12\log(1.00375)} = \frac{12t\ \log(1.00375)}{12\ \log(1.00375)}$$
$$\frac{\log(1.5)}{12\ \log(1.00375)} = t$$

Using a calculator we can approximate the time it takes.

$$t = \log(1.5)/(12 * \log(1.00375)) \approx 9\text{years}$$

Answer:

a. $572.12

b. Approximately 9 years

The period of time it takes a quantity to double is called the **doubling time**[20]. We next outline a technique for calculating the time it takes to double an initial investment earning compound interest.

Example 6.5.2:

Mario invested $1000 in an account earning 6.3% annual interest, that is compounded semi-annually. How long will it take the investment to double?

Solution

Here the principal $P = \$1,000$, the interest rate $r = 6.3\% = 0.063$, and because the interest is compounded semi-annually $n = 2$. This investment can be modeled as follows:

$$A(t) = 1,000\left(1 + \tfrac{0.063}{2}\right)^{2t}$$
$$A(t) = 1,000(1.0315)^{2t}$$

Since we are looking for the time it takes to double $1,000, substitute $2,000 for the resulting amount $A(t)$ and then solve for t.

$$2,000 = 1,000(1.0315)^{2t}$$
$$\frac{2,000}{1,000} = (1.0315)^{2t}$$
$$2 = (1.0315)^{2t}$$

At this point we take the common logarithm of both sides.

$$2 = (1.0315)^{2t}$$
$$\log 2 = \log(1.0315)^{2t}$$
$$\log 2 = 2t\log(1.0315)$$
$$\frac{\log 2}{2\log(1.0315)} = t$$

Using a calculator we can approximate the time it takes:

$$t = \log(2)/(2 * \log(1.0315)) \approx 11.17 \text{years}$$

Answer:

Approximately 11.17 years to double at 6.3%.

If the investment in the previous example was one million dollars, how long would it take to double? To answer this we would use $P = \$1,000,000$ and $A(t) = \$2,000,000$

$$A(t) = 1,000(1.0315)^{2t}$$
$$2,000,000 = 1,000,000(1.0315)^{2t}$$

Dividing both sides by $1,000,000$ we obtain the same exponential function as before.

$$2 = (1.0315)^{2t}$$

Hence, the result will be the same, about 11.17 years. In fact, doubling time is independent of the initial investment P.

Interest is typically compounded semi-annually $(n = 2)$, quarterly $(n = 4)$, monthly $(n = 12)$, or daily $(n = 365)$. However if interest is compounded every instant we obtain a formula for continuously compounding interest:

$$A(t) = Pe^{rt}$$

Here P represents the initial principal amount invested, r represents the annual interest rate, and t represents the time in years the investment is allowed to accrue continuously compounded interest.

Example 6.5.3:

Mary invested \$200 in an account earning $5\frac{3}{4}\%$ annual interest that is compounded continuously. How long will it take the investment to grow to \$350?

Solution

Here the principal $P = \$200$ and the interest rate $r = 5\frac{3}{4}\% = 5.75\% = 0.0575$. Since the interest is compounded continuously, use the formula $A(t) = Pe^{rt}$. Hence, the investment can be modeled by the following,

$$A(t) = 200e^{0.0575t}$$

To calculate the time it takes to accumulate to \$350, set $A(t) = 350$ and solve for t.

$$\begin{aligned} A(t) &= 200e^{0.0575t} \\ 350 &= 200e^{0.0575t} \end{aligned}$$

Begin by isolating the exponential expression.

$$\begin{aligned} \frac{350}{200} &= e^{0.0575t} \\ \frac{7}{4} &= e^{0.0575t} \\ 1.75 &= e^{0.0575t} \end{aligned}$$

Because this exponential has base e, we choose to take the natural logarithm of both sides and then solve for t.

$$\begin{aligned} \ln(1.75) &= \ln e^{0.0575t} && \textit{Apply the power rule for logarithms.} \\ \ln(1.75) &= 0.0575t \ln e && \textit{Recall that } \ln e = 1. \\ \ln(1.75) &= 0.0575t \cdot 1 \\ \tfrac{\ln(1.75)}{0.0575} &= t \end{aligned}$$

Using a calculator we can approximate the time it takes:

$$t = \ln(1.75)/0.0575 \approx 9.73 \quad years$$

Answer:

It will aprroximately 9.73 years.

When solving applications involving compound interest, look for the keyword “continuous,” or the keywords that indicate the number of annual compoundings. It is these keywords that determine which formula to choose.

Exercise 6.5.1

Mario invested $\$1,000$ in an account earning 6.3% annual interest that is compounded continuously. How long will it take the investment to double?

Answer

Approximately 11 years.

www.youtube.com/v/Z_ilJHIQvWQ

Modeling Exponential Growth and Decay

In the sciences, when a quantity is said to grow or decay exponentially, it is specifically meant to be modeled using the **exponential growth/decay formula**[21]:

$$P(t) = P_0 e^{kt}$$

Here P_0, read “P naught,” or “P zero,” represents the initial amount, k represents the growth rate, and t represents the time the initial amount grows or decays exponentially. If k is negative, then the function models exponential decay. Notice that the function

looks very similar to that of continuously compounding interest formula. We can use this formula to model population growth when conditions are optimal.

Example 6.5.4:

It is estimated that the population of a certain small town is $93,000$ people with an annual growth rate of 2.6%. If the population continues to increase exponentially at this rate:

a. Estimate the population in 7 years' time.
b. Estimate the time it will take for the population to reach 120,000 people.

Solution

We begin by constructing a mathematical model based on the given information. Here the initial population $P_0 = 93,000$ people and the growth rate $r = 2.6\% = 0.026$. The following model gives population in terms of time measured in years:

$$P(t) = 93,000e^{0.026t}$$

a. Use this function to estimate the population in $t = 7$ years.

$$\begin{aligned} P(t) &= 93,000e^{0006(7)} \\ &= 93,000e^{0.182} \\ &\approx 111,564 \quad people \end{aligned}$$

b. Use the model to determine the time it takes to reach $P(t) = 120,000$ people.

$$\begin{aligned} P(t) &= 93,000e^{0.026t} \\ 120,000 &= 93,000e^{0.026t} \\ \frac{120,000}{93,000} &= e^{0.026t} \\ \frac{40}{31} &= e^{0.026t} \end{aligned}$$

Take the natural logarithm of both sides and then solve for t.

$$\begin{aligned} \ln\left(\tfrac{40}{31}\right) &= \ln e^{0.026t} \\ \ln\left(\tfrac{40}{31}\right) &= 0.026t \ln e \\ \ln\left(\tfrac{40}{31}\right) &= 0.026t \cdot 1 \\ \frac{\ln\left(\frac{40}{31}\right)}{0.026} &= t \end{aligned}$$

Using a calculator,

$$t = \ln(40/31)/0.026 \approx 9.8 \quad years$$

Answer:

a. $111,564$ people
b. 9.8 years

Often the growth rate k is not given. In this case, we look for some other information so that we can determine it and then construct a mathematical model. The general steps are outlined in the following example.

Example 6.5.5:

Under optimal conditions *Escherichia coli (E. coli)* bacteria will grow exponentially with a doubling time of 20 minutes. If $1,000$ *E. coli* cells are placed in a Petri dish and maintained under optimal conditions, how many *E. coli* cells will be present in 2 hours?

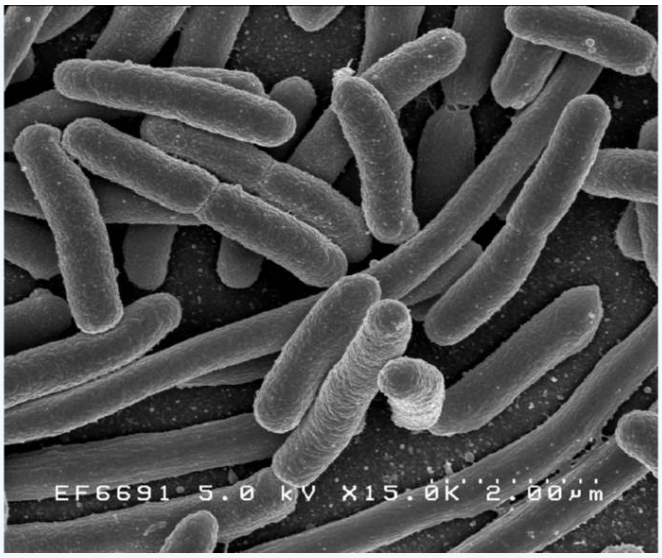


***Figure* 6.5.1** *: Escherichia coli (E. coli)*

Solution

The goal is to use the given information to construct a mathematical model based on the formula $P(t) = P_0 e^{kt}$.

Step 1: Find the growth rate k. Use the fact that the initial amount, $P_0 = 1,000$ cells, doubles in 20 minutes. That is, $P(t) = 2,000$ cells when $t = 20$ minutes.

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ 2,000 &= 1,000 e^{k20} \end{aligned}$$

Solve for the only variable k.

$$\begin{aligned} 2,000 &= 1,000 e^{k20} \\ \frac{2,000}{1,000} &= e^{k20} \\ 2 &= e^{k20} \\ \ln(2) &= \ln e^{k20} \\ \ln(2) &= k20 \ln e \\ \ln(2) &= k20 \cdot 1 \\ \frac{\ln(2)}{20} &= k \end{aligned}$$

Step 2: Write a mathematical model based on the given information. Here $k \approx 0.0347$, which is about 3.5% growth rate per minute. However, we will use the exact value for k in our model. This will allow us to avoid round-off error in the final result. Use $P_0 = 1,000$ and $k = \ln(2)/20$:

$$P(t) = 1,000 e^{(\ln(2)/20)t}$$

This equation models the number of *E. coli* cells in terms of time in minutes.

Step 3: Use the function to answer the questions. In this case, we are asked to find the number of cells present in 2 hours. Because time is measured in minutes, use $t = 120$ minutes to calculate the number of *E. coli* cells.

$$\begin{aligned} P(120) &= 1,000 e^{(\ln(2)/20)(120)} \\ &= 1,000 e^{\ln(2) \cdot 6} \\ &= 1,000 e^{\ln 2^6} \\ &= 1,000 \cdot 2^6 \\ &= 64,000 \text{ cells} \end{aligned}$$

Answer:

In two hours $64,000$ cells will be present.

When the growth rate is negative the function models exponential decay. We can describe decreasing quantities using a **half-life**[22], or the time it takes to decay to one-half of a given quantity.

Example 6.5.6:

Due to radioactive decay, caesium-137 has a half-life of 30 years. How long will it take a 50-milligram sample to decay to 10 milligrams?

Solution

Use the half-life information to determine the rate of decay k. In $t = 30$ years the initial amount $P_0 = 50$ milligrams will decay to half $P(30) = 25$ milligrams.

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ 25 &= 50e^{k30} \end{aligned}$$

Solve for the only variable, k.

$$\begin{aligned} 25 &= 50e^{130} \\ \frac{25}{50} &= e^{30k} \\ \ln\left(\frac{1}{2}\right) &= \ln e^{30k} \\ \ln\left(\frac{1}{2}\right) &= 30k \ln e \\ \frac{\ln 1 - \ln 2}{30} &= k \qquad \textit{Recall that } \ln 1 = 0. \\ -\frac{\ln 2}{30} &= k \end{aligned}$$

Note that $k = -\ln \frac{2}{30} \approx -0.0231$ is negative. However, we will use the exact value to construct a model that gives the amount of cesium-137 with respect to time in years.

$$P(t) = 50e^{(-\ln 2/30)t}$$

Use this model to find t when $P(t) = 10$ milligrams.

$$\begin{aligned} 10 &= 50e^{(-\ln 2/30)t} \\ \frac{10}{50} &= e^{(-\ln 2/30)t} \\ \ln\left(\frac{1}{5}\right) &= \ln e^{(-\ln 2/30)t} \\ \ln 1 - \ln 5 &= \left(-\frac{\ln 2}{30}\right) t \ln e \quad \textit{Recall that } \ln e = 1. \\ -30\frac{(\ln 1 - \ln 5)}{\ln 2} &= t \\ -\frac{30(0 - \ln 5)}{\ln 2} &= t \\ \frac{30 \ln 5}{\ln 2} &= t \end{aligned}$$

Answer:

Using a calculator, it will take $t \approx 69.66$ years to decay to 10 milligrams.

Radiocarbon dating is a method used to estimate the age of artifacts based on the relative amount of carbon-14 present in it. When an organism dies, it stops absorbing this naturally occurring radioactive isotope, and the carbon-14 begins to decay at a known rate. Therefore, the amount of carbon-14 present in an artifact can be used to estimate the age of the artifact.

Example 6.5.7:

An ancient bone tool is found to contain 25% of the carbon-14 normally found in bone. Given that carbon-14 has a half-life of $5,730$ years, estimate the age of the tool.

Solution

Begin by using the half-life information to find k. Here the initial amount P_0 of carbon-14 is not given, however, we know that in $t = 5,730$ years, this amount decays to half, $\frac{1}{2}P_0$.

$$P(t) = P_0 e^{kt}$$
$$\tfrac{1}{2}P_0 = P_0 e^{k5,730}$$

Dividing both sides by P_0 leaves us with an exponential equation in terms of k. This shows that half-life is independent of the initial amount.

$$\tfrac{1}{2} = e^{k5,730}$$

Solve for k.

$$\begin{aligned} \ln\left(\frac{1}{2}\right) &= \ln e^{k5,730} \\ \ln 1 - \ln 2 &= 5,730k \ln e \\ \frac{0 - \ln 2}{5,730} &= k \\ -\frac{\ln 2}{5,730} &= k \end{aligned}$$

Therefore we have the model,

$$P(t) = P_0 e^{(-\ln 2/5,730)t}$$

Next we wish to the find time it takes the carbon-14 to decay to 25% of the initial amount, or $P(t) = 0.25P_0$

$$.0.25P_0 = P_0 e^{(-\ln 2/5,730)t}$$

Divide both sides by P_0 and solve for t.

$$\begin{aligned} 0.25 &= e^{(-\ln 2/5,730)t} \\ \ln(0.25) &= \ln e^{(-\ln 2/5,730)t} \\ \ln(0.25) &= \left(-\frac{\ln 2}{5,730}\right)_{t \ln e} \\ -\frac{5,730 \ln(0.25)}{\ln 2} &= t \\ 11,460 &\approx t \end{aligned}$$

Answer:

The tool is approximately $11,460$ years old.

Exercise 6.5.2

The half-life of strontium-90 is about 28 years. How long will it take a 36 milligram sample of strontium-90 to decay to 30 milligrams?

Answer

7.4 years

www.youtube.com/v/Oto0ihiyvBc

Key Takeaways

- When interest is compounded a given number of times per year use the formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.
- When interest is to be compounded continuously use the formula $A(t) = Pe^{rt}$.
- Doubling time is the period of time it takes a given amount to double. Doubling time is independent of the principal.
- When amounts are said to be increasing or decaying exponentially, use the formula $P(t) = P_0 e^{kt}$.
- Half-life is the period of time it takes a given amount to decrease to one-half. Half-life is independent of the initial amount.
- To model data using the exponential growth/decay formula, use the given information to determine the growth/decay rate k. Once k is determined, a formula can be written to model the problem. Use the formula to answer the questions.

Footnotes

[20] The period of time it takes a quantity to double.

[21] A formula that models exponential growth or decay: $P(t) = P_0 e^{kt}$.

[22] The period of time it takes a quantity to decay to one-half of the initial amount.

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CHAPTER OVERVIEW

7: Systems of Equations

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions.

7.1: Systems of Linear Equations - Two Variables
7.2: Systems of Linear Equations - Three Variables
7.3: Partial Fractions
7.4: Solve Systems of Equations Using Matrices
7.5: Graphing Systems of Linear Inequalities

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7.1: Systems of Linear Equations - Two Variables

Learning Objectives

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.



Figure 7.1.1

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. **A system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$\begin{aligned} 2x+y &= 15 \\ 3x-y &= 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair $(4,7)$ is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

$$\begin{aligned} 2(4)+(7) &= 15 \text{ True} \\ 3(4)-(7) &= 5 \text{ True} \end{aligned}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A consistent system of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same y-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different y-intercepts. There are no points common to both lines; hence, there is no solution to the system.

TYPES OF LINEAR SYSTEMS

There are three types of systems of linear equations in two variables, and three types of solutions.

- An independent system has exactly one solution pair (x, y). The point where the two lines intersect is the only solution.
- An inconsistent system has no solution. Notice that the two lines are parallel and will never intersect.
- A dependent system has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

Figure $7.1.2$ compares graphical representations of each type of system.

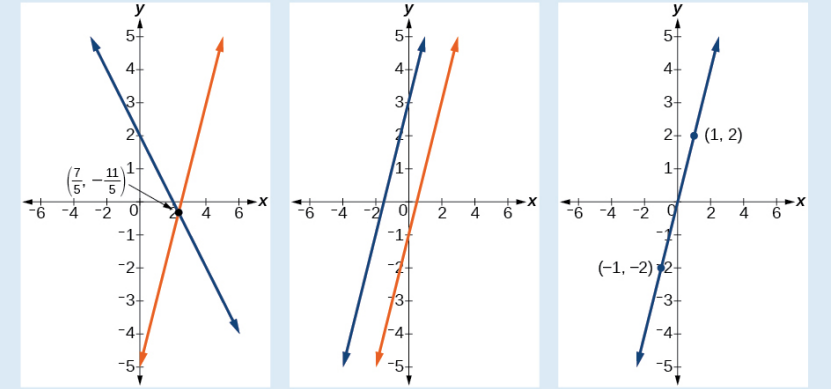


Figure 7.1.2

Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Example 7.1.1: Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

$$\begin{aligned} x + 3y &= 8 \\ 2x - 9 &= y \end{aligned}$$

Solution

Substitute the ordered pair $(5, 1)$ into both equations.

$$\begin{aligned} (5) + 3(1) &= 8 \\ 8 &= 8 \text{ True} \\ 2(5) - 9 &= (1) \\ 1 &= 1 \text{ True} \end{aligned}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

Analysis

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure $7.1.3$.

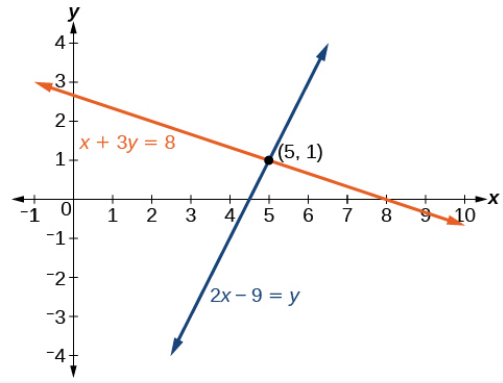


Figure 7.1.3

? Exercise 7.1.1

Determine whether the ordered pair $(8, 5)$ is a solution to the following system.

$$\begin{aligned} 5x - 4y &= 20 \\ 2x + 1 &= 3y \end{aligned}$$

Answer

Not a solution.

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

✔ Example 7.1.2: Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

Solution

Solve the first equation for y.

$$\begin{aligned} 2x + y &= -8 \\ y &= -2x - 8 \end{aligned}$$

Solve the second equation for y.

$$\begin{aligned} x - y &= -1 \\ y &= x + 1 \end{aligned}$$

Graph both equations on the same set of axes as in Figure 7.1.4.

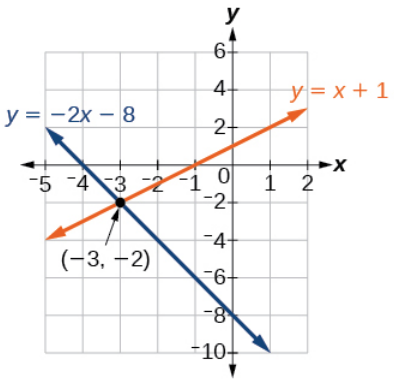


Figure 7.1.4

The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned} 2(-3)+(-2) &= -8 \\ -8 &= -8 \text{ True} \\ (-3)-(-2) &= -1 \\ -1 &= -1 \text{ True} \end{aligned}$$

The solution to the system is the ordered pair $(-3,-2)$,so the system is independent.

Exercise 7.1.2

Solve the following system of equations by graphing.

$$\begin{aligned} 2x-5y &= -25 \\ -4x+5y &= 35 \end{aligned}$$

Answer

The solution to the system is the ordered pair $(-5,3)$.

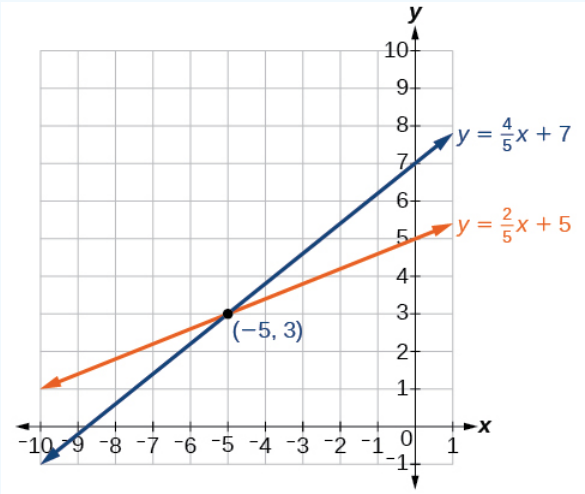


Figure 7.1.5

Q&A

Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent

system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a **system of linear equations** that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

How to: Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Example 7.1.3: Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$\begin{aligned} -x+y &= -5 \\ 2x-5y &= 1 \end{aligned}$$

Solution

First, we will solve the first equation for y.

$$\begin{aligned} -x+y &= -5 \\ y &= x-5 \end{aligned}$$

Now we can substitute the expression $x-5$ for y in the second equation.

$$\begin{aligned} 2x-5y &= 1 \\ 2x-5(x-5) &= 1 \\ 2x-5x+25 &= 1 \\ -3x &= -24 \\ x &= 8 \end{aligned}$$

Now, we substitute $x=8$ into the first equation and solve for y.

$$\begin{aligned} -(8)+y &= -5 \\ y &= 3 \end{aligned}$$

Our solution is $(8,3)$.

Check the solution by substituting $(8,3)$ into both equations.

$$\begin{aligned} -x+y &= -5 \\ -(8)+(3) &= -5 \text{ True} \\ 2x-5y &= 1 \\ 2(8)-5(3) &= 1 \text{ True} \end{aligned}$$

Exercise 7.1.3

Solve the following system of equations by substitution.

$$x = y + 3$$
$$4 = 3x - 2y$$

Answer

$(-2, -5)$

Q&A

Can the substitution method be used to solve any linear system in two variables?

Yes, but the method works best if one of the equations contains a coefficient of 1 *or* -1 *so that we do not have to deal with fractions.*

Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the addition method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

How to: Given a system of equations, solve using the addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Example 7.1.4: Solving a System by the Addition Method

Solve the given system of equations by addition.

$$x + 2y = -1$$
$$-x + y = 3$$

Solution

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, -1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{aligned} x + 2y &= -1 \\ \underline{-x + y} &= \underline{3} \\ 3y &= 2 \end{aligned}$$

Now that we have eliminated x, we can solve the resulting equation for y.

$$3y = 2$$
$$y = \frac{2}{3}$$

Then, we substitute this value for y into one of the original equations and solve for x.

$$\begin{aligned} -x+y &= 3 \\ -x+\frac{2}{3} &= 3 \\ -x &= 3-\frac{2}{3} \\ -x &= \frac{7}{3} \\ x &= -\frac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned} x+2y &= -1 \\ \left(-\frac{7}{3}\right)+2\left(\frac{2}{3}\right) &= \\ -\frac{7}{3}+\frac{4}{3} &= -\frac{3}{3} \\ -1 &= -1 \qquad \text{True} \end{aligned}$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure 7.1.6 to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

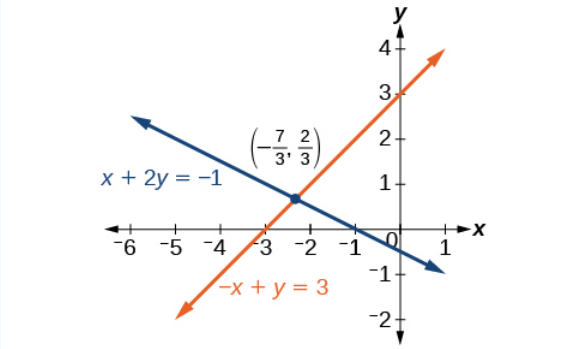


Figure 7.1.6

Example 7.1.5: Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

$$\begin{aligned} 3x+5y &= -11 \\ x-2y &= 11 \end{aligned}$$

Solution

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by -3,the x-terms will add to zero.

$$\begin{aligned} x-2y &= 11 \\ -3(x-2y) &= -3(11) \qquad \text{Multiply both sides by } -3. \\ -3x+6y &= -33 \qquad \text{Use the distributive property.} \end{aligned}$$

Now, let's add them.

$$\begin{aligned} 3x+5y &= -11 \\ \underline{-3x+6y} &= \underline{-33} \\ 11y &= -44 \\ y &= -4 \end{aligned}$$

For the last step, we substitute $y=-4$ into one of the original equations and solve for x.

$$\begin{aligned} 3x+5y &= -11 \\ 3x+5(-4) &= -11 \\ 3x-20 &= -11 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Our solution is the ordered pair $(3,-4)$. See Figure 7.1.7. Check the solution in the original second equation.

$$\begin{aligned} x-2y &= 11 \\ (3)-2(-4) &= 3+8 \\ &= 11 \qquad \text{True} \end{aligned}$$

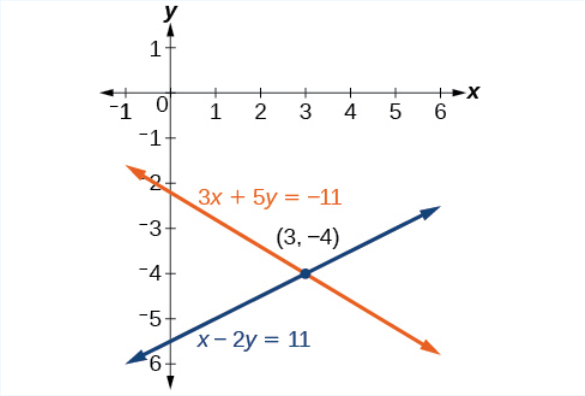


Figure 7.1.7

Exercise 7.1.4

Solve the system of equations by addition.

$$\begin{aligned} 2x-7y &= 2 \\ 3x+y &= -20 \end{aligned}$$

Answer

$(-6,-2)$

Example 7.1.6: Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.

$$\begin{aligned} 2x+3y &= -16 \\ 5x-10y &= 30 \end{aligned}$$

Solution

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.

$$\begin{aligned} -5(2x+3y) &= -5(-16) \\ -10x-15y &= 80 \\ 2(5x-10y) &= 2(30) \\ 10x-20y &= 60 \end{aligned}$$

Then, we add the two equations together.

$$\begin{aligned} -10x-15y &= 80 \\ \underline{10x-20y} &= \underline{60} \\ -35y &= 140 \\ y &= -4 \end{aligned}$$

Substitute $y=-4$ into the original first equation.

$$\begin{aligned} 2x+3(-4) &= -16 \\ 2x-12 &= -16 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

The solution is $(-2,-4)$. Check it in the other equation.

$$\begin{aligned} 5x-10y &= 30 \\ 5(-2)-10(-4) &= 30 \\ -10+40 &= 30 \\ 30 &= 30 \end{aligned}$$

See Figure 7.1.8.

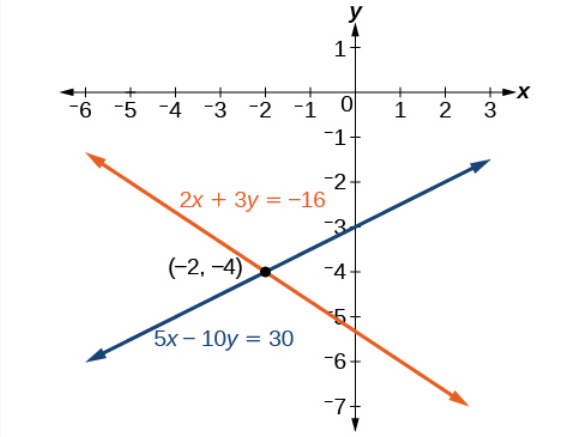


Figure 7.1.8

Example 7.1.7: Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.

$$\begin{aligned} \frac{x}{3}+\frac{y}{6} &= 3 \\ \frac{x}{2}-\frac{y}{4} &= 1 \end{aligned}$$

Solution

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$$6\left(\frac{x}{3}+\frac{y}{6}\right)=6(3)$$
$$2x+y=18$$
$$4\left(\frac{x}{2}-\frac{y}{4}\right)=4(1)$$
$$2x-y=4$$

Now multiply the second equation by -1 so that we can eliminate the x-variable.

$$-1(2x-y)=-1(4)$$
$$-2x+y=-4$$

Add the two equations to eliminate the x-variable and solve the resulting equation.

$$2x+y=18$$
$$-2x+y=-4$$
$$2y=14$$
$$y=7$$

Substitute $y=7$ into the first equation.

$$2x+(7)=18$$
$$2x=11$$
$$x=\frac{11}{2}$$
$$=7.5$$

The solution is $\left(\frac{11}{2},7\right)$. Check it in the other equation.

$$\frac{x}{2}-\frac{y}{4}=1$$
$$\frac{\frac{11}{2}}{2}-\frac{7}{4}=1$$
$$\frac{11}{4}-\frac{7}{4}=1$$
$$\frac{4}{4}=1$$

? Exercise 7.1.5

Solve the system of equations by addition.

$$2x+3y=8$$
$$3x+5y=10$$

Answer

$(10,-4)$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12=0$.

Example 7.1.8: Solving an Inconsistent System of Equations

Solve the following system of equations.

$$\begin{aligned} x &= 9-2y \\ x+2y &= 13 \end{aligned}$$

Solution

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$\begin{aligned} x+2y &= 13 \\ (9-2y)+2y &= 13 \\ 9+0y &= 13 \\ 9 &= 13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$\begin{aligned} x &= 9-2y \\ 2y &= -x+9 \\ y &= -\frac{1}{2}x+\frac{9}{2} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

$$\begin{aligned} x+2y &= 13 \\ 2y &= -x+13 \\ y &= -\frac{1}{2}x+\frac{13}{2} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different y-intercepts. Therefore, the lines are parallel and do not intersect.

$$\begin{aligned} y &= -\frac{1}{2}x+\frac{9}{2} \\ y &= -\frac{1}{2}x+\frac{13}{2} \end{aligned}$$

Analysis

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in Figure 7.1.9.

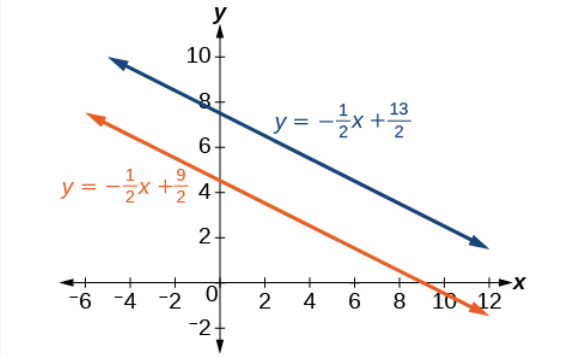


Figure 7.1.9

? Exercise 7.1.6

Solve the following system of equations in two variables.

$$\begin{aligned} 2y-2x &= 2 \\ 2y-2x &= 6 \end{aligned}$$

Answer

No solution. It is an inconsistent system.

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0=0$.

✓ Example 7.1.9: Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the **addition method**.

$$\begin{aligned} x+3y &= 2 \\ 3x+9y &= 6 \end{aligned}$$

Solution

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable.

$$\begin{aligned} x+3y &= 2 \\ (-3)(x+3y) &= (-3)(2) \\ -3x-9y &= -6 \end{aligned}$$

Now add the equations.

$$\begin{aligned} -3x-9y &= -6 \\ \underline{+\,3x+9y} &= \underline{6} \\ 0 &= 0 \end{aligned}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.

$$\begin{aligned} x+3y &= 2 \\ 3y &= -x+2 \\ y &= -\frac{1}{3}x+\frac{2}{3} \\ 3x+9y &= 6 \\ 9y &= -3x+6 \\ y &= -\frac{3}{9}x+\frac{6}{9} \\ y &= -\frac{1}{3}x+\frac{2}{3} \end{aligned}$$

See Figure 7.1.10. Notice the results are the same. The general solution to the system is $\left(x, -\frac{1}{3}x+\frac{2}{3}\right)$.

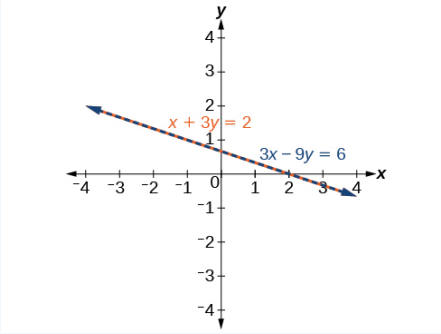


Figure 7.1.10

Exercise 7.1.7

Solve the following system of equations in two variables.

$$\begin{aligned} y-2x &= 5 \\ -3y+6x &= -15 \end{aligned}$$

Answer

The system is dependent so there are infinite solutions of the form $(x, 2x+5)$.

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's revenue function is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where x=quantity and p=price. The revenue function is shown in orange in Figure 7.1.11.

The cost function is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in Figure 7.1.11. The x-axis represents quantity in hundreds of units. The y-axis represents either cost or revenue in hundreds of dollars.

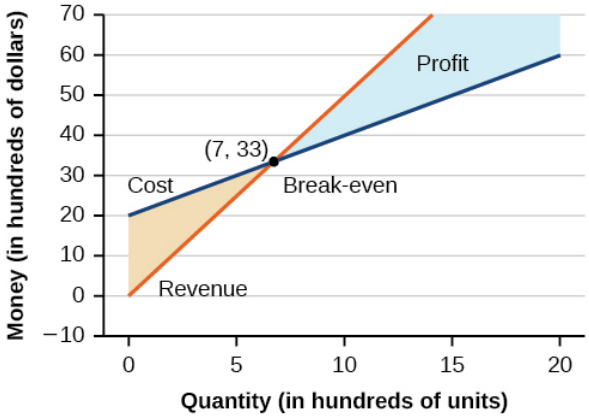


Figure 7.1.11

The point at which the two lines intersect is called the break-even point. We can see from the graph that if 700 units are produced, the cost is $\$3,300$ and the revenue is also $\$3,300$. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The profit function is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

Example 7.1.10: Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35,000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

Solution

Write the system of equations using y to replace function notation.

$$\begin{aligned} y &= 0.85x + 35,000 \\ y &= 1.55x \end{aligned}$$

Substitute the expression $0.85x + 35,000$ from the first equation into the second equation and solve for x.

$$\begin{aligned} 0.85x + 35,000 &= 1.55x \\ 35,000 &= 0.7x \\ 50,000 &= x \end{aligned}$$

Then, we substitute $x = 50,000$ into either the cost function or the revenue function.

$$1.55(50,000) = 77,500$$

The break-even point is $(50,000, 77,500)$.

The profit function is found using the formula $P(x) = R(x) - C(x)$.

$$\begin{aligned} P(x) &= 1.55x - (0.85x + 35,000) \\ &= 0.7x - 35,000 \end{aligned}$$

The profit function is $P(x) = 0.7x - 35,000$.

Analysis

The cost to produce $50,000$ units is $\$77,500$, and the revenue from the sales of $50,000$ units is also $\$77,500$. To make a profit, the business must produce and sell more than $50,000$ units. See Figure 7.1.12.

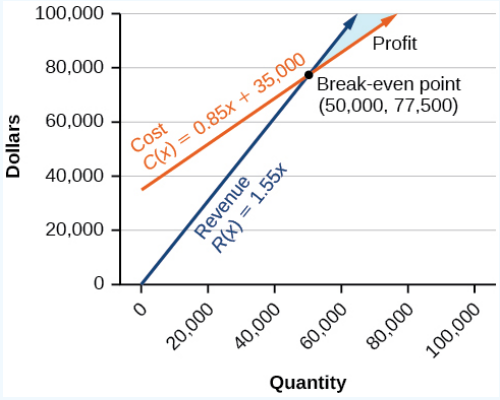


Figure 7.1.12

We see from the graph in Figure 7.1.13 that the profit function has a negative value until $x = 50,000$, when the graph crosses the x-axis. Then, the graph emerges into positive y-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.

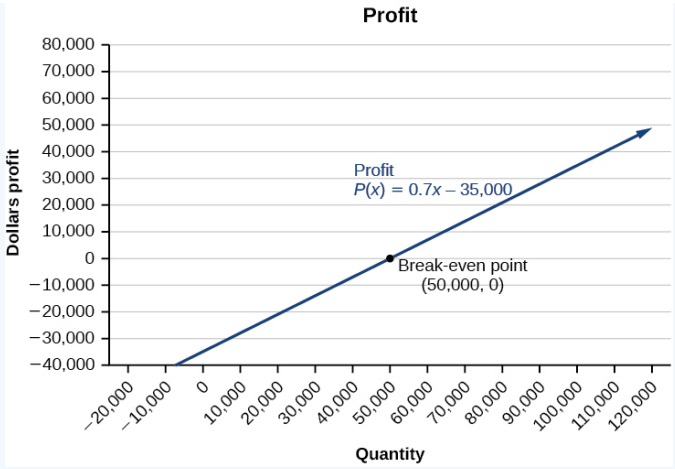


Figure 7.1.13

Example 7.1.11: Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2,000$ and the total gate revenue is $\$70,000$. How many children and how many adults bought tickets?

Solution

Let c = the number of children and a = the number of adults in attendance.

The total number of people is $2,000$. We can use this to write an equation for the number of people at the circus that day.

$$c+a=2,000$$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c$. The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a$. The total revenue is $\$70,000$. We can use this to write an equation for the revenue.

$$25c+50a=70,000$$

We now have a system of linear equations in two variables.

$$c+a=2,000$$
$$25c+50a=70,000$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.

$$\begin{aligned} c+a &= 2,000 \\ a &= 2,000-c \end{aligned}$$

Substitute the expression $2,000-c$ in the second equation fora a and solve for c.

$$\begin{aligned} 25c+50(2,000-c) &= 70,000 \\ 25c+100,000-50c &= 70,000 \\ -25c &= -30,000 \\ c &= 1,200 \end{aligned}$$

Substitute $c=1,200$ into the first equation to solve for a.

$$\begin{aligned} 1,200+a &= 2,000 \\ a &= 800 \end{aligned}$$

We find that $1,200$ children and 800 adults bought tickets to the circus that day.

Exercise 7.1.8

Meal tickets at the circus cost $\$4.00$ for children and $\$12.00$ for adults. If $1,650$ meal tickets were bought for a total of, $\$14,200$, how many children and how many adults bought meal tickets?

Answer

700 children, 950 adults

Media

Access these online resources for additional instruction and practice with systems of linear equations.

- Solving Systems of Equations Using Substitution
- Solving Systems of Equations Using Elimination
- Applications of Systems of Equations

Key Concepts

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See Example 7.1.1.
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See Example 7.1.2.
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See Example 7.1.3.
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See Example 7.1.4.
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See Example 7.1.5, Example 7.1.6, and Example 7.1.7.
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See Example 7.1.8.
- The solution to a system of dependent equations will always be true because both equations describe the same line. See Example 7.1.9.
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See Example 7.1.10 and Example 7.1.11.

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7.2: Systems of Linear Equations - Three Variables

Learning Objectives

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

John received an inheritance of $\$12,000$ that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $\$4,000$ more in municipal funds than in municipal bonds. He earned $\$670$ in interest the first year. How much did John invest in each type of fund?



Figure 7.2.1: (credit: "Elembis," Wikimedia Commons)

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution (x, y, z), which we call an ordered triple. A system in upper triangular form looks like the following:

$$\begin{aligned} Ax + By + Cz &= D \\ Ey + Fz &= G \\ Hz &= K \end{aligned}$$

The third equation can be solved for z,and then we back-substitute to find y and x. To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

The **solution set** to a three-by-three system is an ordered triple (x, y, z). Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

NUMBER OF POSSIBLE SolutionS

Figure 7.2.2 and Figure 7.2.3 illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a solution set consisting of an ordered triple (x, y, z). Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.

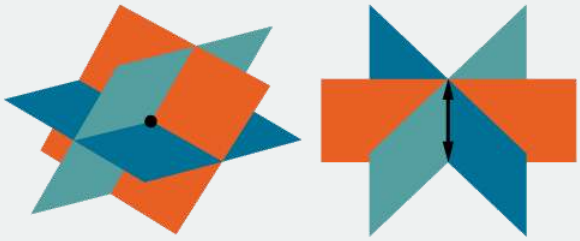


Figure 7.2.2: (a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.



Figure 7.2.3: All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

Example 7.2.1: Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$\begin{aligned} x + y + z &= 2 \\ 6x - 4y + 5z &= 31 \\ 5x + 2y + 2z &= 13 \end{aligned}$$

Solution

We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$x + y + z = 2$	$6x - 4y + 5z = 31$	$5x + 2y + 2z = 13$
$(3) + (-2) + (1) = 2$	$6(3) - 4(-2) + 5(1) = 31$	$5(3) + 2(-2) + 2(1) = 13$
True	$18 + 8 + 5 = 31$	$15 - 4 + 2 = 13$
	True	True

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

How to: Given a linear system of three equations, solve for three unknowns

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.

4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Example 7.2.2: Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$\begin{aligned} x-2y+3z &= 9 \quad (1)\\ -x+3y-z &= -6 \quad (2)\\ 2x-5y+5z &= 17 \quad (3) \end{aligned}$$

Solution

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).

$$\begin{aligned} x-2y+3z &= 9 \quad (1)\\ \underline{-x+3y-z} &\underline{= -6 \quad (2)}\\ y+2z &= 3 \quad (4) \end{aligned}$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x.

$$\begin{aligned} -2x+4y-6z &= -18 \quad (1) \quad \text{multiplied by } -2\\ \underline{2x-5y+5z} &\underline{= 17 \quad (3)}\\ -y-z &= -1 \quad (5) \end{aligned}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for z by adding the two equations.

$$\begin{aligned} y+2z &= 3 \quad (4)\\ \underline{-y-z} &\underline{= -1 \quad (5)}\\ z &= 2 \quad (6) \end{aligned}$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$\begin{aligned} x-2y+3z &= 9 \quad (1)\\ y+2z &= 3 \quad (4)\\ z &= 2 \quad (6) \end{aligned}$$

Next, we back-substitute $z=2$ into equation (4) and solve for y.

$$\begin{aligned} y+2(2) &= 3\\ y+4 &= 3\\ y &= -1 \end{aligned}$$

Finally, we can back-substitute $z=2$ and $y=-1$ into equation (1). This will yield the solution for x.

$$\begin{aligned} x-2(-1)+3(2) &= 9\\ x+2+6 &= 9\\ x &= 1 \end{aligned}$$

The solution is the ordered triple $(1,-1,2)$. See Figure 7.2.4.

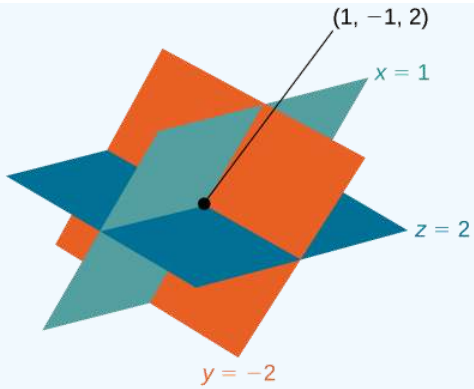


Figure 7.2.4

Example 7.2.3: Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of $\$12,000$ in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. John invested $\$4,000$ more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $\$670$. How much did he invest in each type of fund?

Solution

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$x = \text{amount invested in money-market fund}$$
$$y = \text{amount invested in municipal bonds}$$
$$z = \text{amount invested in mutual funds}$$

The first equation indicates that the sum of the three principal amounts is $\$12,000$.

$$x + y + z = 12,000$$

We form the second equation according to the information that John invested $\$4,000$ more in mutual funds than he invested in municipal bonds.

$$z = y + 4,000$$

The third equation shows that the total amount of interest earned from each fund equals $\$670$.

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.

$$\begin{aligned} x + y + z &= 12,000 \\ -y + z &= 4,000 \\ 0.03x + 0.04y + 0.07z &= 670 \end{aligned}$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,

$$\begin{aligned} x + y + z &= 12,000 \quad (1) \\ -y + z &= 4,000 \quad (2) \\ 3x + 4y + 7z &= 67,000 \quad (3) \end{aligned}$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.

$$\begin{aligned} x+y+z &= 12,000 \\ 3x+4y+7z &= 67,000 \\ -y+z &= 4,000 \end{aligned}$$

Step 2. Multiply equation (1) by -3 and add to equation (2). Write the result as row 2.

$$\begin{aligned} x+y+z &= 12,000 \\ y+4z &= 31,000 \\ -y+z &= 4,000 \end{aligned}$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).

$$\begin{aligned} x+y+z &= 12,000 \\ y+4z &= 31,000 \\ 5z &= 35,000 \end{aligned}$$

Step 4. Solve for z in equation (3). Back-substitute that value in equation (2) and solve for y. Then, back-substitute the values for z and y into equation (1) and solve for x.

$$\begin{aligned} 5z &= 35,000 \\ z &= 7,000 \end{aligned}$$

$$\begin{aligned} y+4(7,000) &= 31,000 \\ y &= 3,000 \end{aligned}$$

$$\begin{aligned} x+3,000+7,000 &= 12,000 \\ x &= 2,000 \end{aligned}$$

John invested $\$2,000$ in a money-market fund, $\$3,000$ in municipal bonds, and $\$7,000$ in mutual funds.

? Exercise 7.2.1

Solve the system of equations in three variables.

$$\begin{aligned} 2x+y-2z &= -1 \\ 3x-3y-z &= 5 \\ x-2y+3z &= 6 \end{aligned}$$

Answer

$(1,-1,1)$

Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3=7$ or some other contradiction.

✓ Example 7.2.4: Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.

$$x-3y+z=4 \quad (7.2.1)$$
$$-x+2y-5z=3 \quad (7.2.2)$$
$$5x-13y+13z=8 \quad (7.2.3)$$

Solution

Looking at the coefficients of x, we can see that we can eliminate x by adding Equation 7.2.1 to Equation 7.2.2.

$$\begin{aligned} x-3y+z&=4\,(1)\\ -x+2y-5z&=3\,(2)\\ \hline -y-4z&=7\,(4)\end{aligned}$$

Next, we multiply equation (1) by -5 and add it to equation (3).

$$\begin{aligned} -5x+15y-5z&=-20\,(1) \quad \text{multiplied by } -5\\ 5x-13y+13z&=8\,(3)\\ \hline 2y+8z&=-12\,(5)\end{aligned}$$

Then, we multiply equation (4) by 2 and add it to equation (5).

$$\begin{aligned} -2y-8z&=14\,(4) \quad \text{multiplied by } 2\\ 2y+8z&=-12\,(5)\\ \hline 0&=2\end{aligned}$$

The final equation $0=2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

? Exercise 7.2.2

Solve the system of three equations in three variables.

$$x+y+z=2$$
$$y-3z=1$$
$$2x+y+5z=0$$

Answer

No solution.

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

✓ Example 7.2.5: Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$$\begin{aligned} 2x+y-3z &= 0 \quad (1) \\ 4x+2y-6z &= 0 \quad (2) \\ x-y+z &= 0 \quad (3) \end{aligned}$$

Solution

First, we can multiply equation (1) by -2 and add it to equation (2).

$$\begin{aligned} -4x-2y+6z &= 0 \ (1) \quad \text{multiplied by } -2 \\ \underline{4x+2y-6z} &= \underline{0 \ (2)} \\ 0 &= 0 \end{aligned}$$

We do not need to proceed any further. The result we get is an identity, $0=0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2, and adding it to equation (1). We then perform the same steps as above and find the same result, $0=0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have

$$\begin{aligned} 2x+y-3z &= 0 \\ x-y+z &= 0 \\ 3x-2z &= 0 \end{aligned}$$

We then solve the resulting equation for z.

$$\begin{aligned} 3x-2z &= 0 \\ z &= \frac{3}{2}x \end{aligned}$$

We back-substitute the expression for z into one of the equations and solve for y.

$$\begin{aligned} 2x+y-3\left(\frac{3}{2}x\right) &= 0 \\ 2x+y-\frac{9}{2}x &= 0 \\ y &= \frac{9}{2}x-2x \\ y &= \frac{5}{2}x \end{aligned}$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are dependent on the value selected for x.

Analysis

As shown in Figure 7.2.5, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.

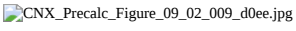


Figure 7.2.5

Q&A: Does the generic solution to a dependent system always have to be written in terms of x?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x and if needed x and y.

Exercise 7.2.3:

Solve the following system.

$$\begin{aligned} x+y+z &= 7 \\ 3x-2y-z &= 4 \\ x+6y+5z &= 24 \end{aligned}$$

Answer

Infinite number of solutions of the form $(x, 4x-11, -5x+18)$.

Media

Access these online resources for additional instruction and practice with systems of equations in three variables.

- Ex 1: System of Three Equations with Three Unknowns Using Elimination
- Ex. 2: System of Three Equations with Three Unknowns Using Elimination

Key Concepts

- A solution set is an ordered triple {(x,y,z)} that represents the intersection of three planes in space. See Example 7.2.1.
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See Example 7.2.2.
- Systems of three equations in three variables are useful for solving many different types of real-world problems. See Example 7.2.3.
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See Example 7.2.4.
- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See Example 7.2.5.
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

Contributors

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7.3: Partial Fractions

Learning Objectives

Decompose $\frac{P(x)}{Q(x)}$, where

- $Q(x)$ has only nonrepeated linear factors.
- $Q(x)$ has repeated linear factors.
- $Q(x)$ has a nonrepeated irreducible quadratic factor.
- $Q(x)$ has a repeated irreducible quadratic factor.

Earlier in this chapter, we studied systems of two equations in two variables, systems of three equations in three variables, and nonlinear systems. Here we introduce another way that systems of equations can be utilized—the decomposition of rational expressions. Fractions can be complicated; adding a variable in the denominator makes them even more so. The methods studied in this section will help simplify the concept of a rational expression.

Decomposing $\frac{P(x)}{Q(x)}$ where $Q(x)$ Has Only Nonrepeated Linear Factors

Recall the algebra regarding adding and subtracting rational expressions. These operations depend on finding a common denominator so that we can write the sum or difference as a single, simplified rational expression. In this section, we will look at **partial fraction decomposition**, which is the undoing of the procedure to add or subtract rational expressions. In other words, it is a return from the single simplified **rational expression** to the original expressions, called the **partial fractions**.

For example, suppose we add the following fractions:

$$\frac{2}{x-3}+\frac{-1}{x+2}$$

We would first need to find a common denominator: $(x+2)(x-3)$.

Next, we would write each expression with this common denominator and find the sum of the terms.

$$\begin{aligned}\frac{2}{x-3}\left(\frac{x+2}{x+2}\right)+\frac{-1}{x+2}\left(\frac{x-3}{x-3}\right) &= \frac{2x+4-x+3}{(x+2)(x-3)}\\ &= \frac{x+7}{x^2-x-6}\end{aligned}$$

Partial fraction **decomposition** is the reverse of this procedure. We would start with the solution and rewrite (decompose) it as the sum of two fractions.

$$\underbrace{\frac{x+7}{x^2-x-6}}_{\text{Simplified sum}}=\underbrace{\frac{2}{x-3}+\frac{-1}{x+2}}_{\text{Partial fraction decomposition}}$$

We will investigate rational expressions with linear factors and quadratic factors in the denominator where the degree of the numerator is less than the degree of the denominator. Regardless of the type of expression we are decomposing, the first and most important thing to do is factor the denominator.

When the denominator of the simplified expression contains distinct linear factors, it is likely that each of the original rational expressions, which were added or subtracted, had one of the linear factors as the denominator. In other words, using the example above, the factors of x^2-x-6 are $(x-3)(x+2)$, the denominators of the decomposed rational expression. So we will rewrite the simplified form as the sum of individual fractions and use a variable for each numerator. Then, we will solve for each numerator using one of several methods available for partial fraction decomposition.

PARTIAL FRACTION DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS NONREPEATED LINEAR FACTORS

The *partial fraction decomposition* of $\dfrac{P(x)}{Q(x)}$ when $Q(x)$ has nonrepeated linear factors and the degree of $P(x)$ is less than the degree of $Q(x)$ is

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(a_1x+b_1)}+\frac{A_2}{(a_2x+b_2)}+\frac{A_3}{(a_3x+b_3)}+\cdots+\frac{A_n}{(a_nx+b_n)} \tag{7.3.1}$$

How to: Given a rational expression with distinct linear factors in the denominator, decompose it

1. Use a variable for the original numerators, usually A, B, or C, depending on the number of factors, placing each variable over a single factor. For the purpose of this definition, we use A_n for each numerator
$$\frac{P(x)}{Q(x)}=\frac{A_1}{(a_1x+b_1)}+\frac{A_2}{(a_2x+b_2)}+\frac{A_3}{(a_3x+b_3)}+\cdots+\frac{A_n}{(a_nx+b_n)}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 7.3.1: Decomposing a Rational Function with Distinct Linear Factors

Decompose the given rational expression with distinct linear factors.

$$\frac{3x}{(x+2)(x-1)}$$

Solution

We will separate the denominator factors and give each numerator a symbolic label, like A, B ,or C.

$$\frac{3x}{(x+2)(x-1)}=\frac{A}{(x+2)}+\frac{B}{(x-1)}$$

Multiply both sides of the equation by the common denominator to eliminate the fractions:

$$(x+2)(x-1)\left[\frac{3x}{(x+2)(x-1)}\right]=(x+2)(x-1)\left[\frac{A}{(x+2)}\right]+(x+2)(x-1)\left[\frac{B}{(x-1)}\right]$$

The resulting equation is

$$3x=A(x-1)+B(x+2)$$

Expand the right side of the equation and collect like terms.

$$\begin{aligned}3x&=Ax-A+Bx+2B\\3x&=(A+B)x-A+2B\end{aligned}$$

Set up a system of equations associating corresponding coefficients.

$$\begin{aligned}3&=A+B\\0&=-A+2B\end{aligned}$$

Add the two equations and solve for B.

$$\begin{aligned}3&=A+B\\\underline{0}&=\underline{-A+2B}\\3&=0+3B\\1&=B\end{aligned}$$

Substitute $B=1$ into one of the original equations in the system.

$$\begin{aligned} 3 &= A+1 \\ 2 &= A \end{aligned}$$

Thus, the partial fraction decomposition is

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Another method to use to solve for A or B is by considering the equation that resulted from eliminating the fractions and substituting a value for x that will make either the $A-$ or $B-$term equal 0. If we let $x=1$, the

$A-$ term becomes 0 and we can simply solve for B.

$$\begin{aligned} 3x &= A(x-1)+B(x+2) \\ 3(1) &= A[(1)-1]+B[(1)+2] \\ 3 &= 0+3B \\ 1 &= B \end{aligned}$$

Next, either substitute $B=1$ into the equation and solve for A, or make the $B-$term 0 by substituting $x=-2$ into the equation.

$$\begin{aligned} 3x &= A(x-1)+B(x+2) \\ 3(-2) &= A[(-2)-1]+B[(-2)+2] \\ -6 &= -3A+0 \\ \frac{-6}{-3} &= A \\ 2 &= A \end{aligned}$$

We obtain the same values for A and B using either method, so the decompositions are the same using either method.

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Although this method is not seen very often in textbooks, we present it here as an alternative that may make some partial fraction decompositions easier. It is known as the **Heaviside method**, named after Charles Heaviside, a pioneer in the study of electronics.

? Exercise 7.3.1

Find the partial fraction decomposition of the following expression.

$$\frac{x}{(x-3)(x-2)}$$

Answer

$$\frac{3}{x-3} - \frac{2}{x-2}$$

Decomposing $\frac{P(x)}{Q(x)}$ where $Q(x)$ Has Repeated Linear Factors

Some fractions we may come across are special cases that we can decompose into partial fractions with repeated linear factors. We must remember that we account for repeated factors by writing each factor in increasing powers.

PARTIAL FRACTION DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS REPEATED LINEAR FACTORS

The partial fraction decomposition of $\dfrac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated linear factor occurringn n times and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{7.3.2}$$

Write the denominator powers in increasing order.

How to: decompose a rational expression with repeated linear factors

1. Use a variable like A, B, or C for the numerators and account for increasing powers of the denominators.

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{7.3.3}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 7.3.2: Decomposing with Repeated Linear Factors

Decompose the given rational expression with repeated linear factors.

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}$$

Solution

The denominator factors are $x(x-2)^2$. To allow for the repeated factor of $(x-2)$, the decomposition will include three denominators: x, $(x-2)$, and $(x-2)^2$. Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

Next, we multiply both sides by the common denominator.

$$x(x-2)^2\left[\frac{-x^2+2x+4}{x(x-2)^2}\right] = \left[\frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}\right] x(x-2)^2$$

$$-x^2+2x+4 = A(x-2)^2 + Bx(x-2) + Cx$$

On the right side of the equation, we expand and collect like terms.

$$\begin{aligned} -x^2+2x+4 &= A(x^2-4x+4) + B(x^2-2x) + Cx \\ &= Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx \\ &= (A+B)x^2 + (-4A-2B+C)x + 4A \end{aligned}$$

Next, we compare the coefficients of both sides. This will give the system of equations in three variables:

$$-x^2+2x+4 = (A+B)x^2 + (-4A-2B+C)x + 4A \tag{7.3.4}$$

$$A+B = -1 \tag{7.3.5}$$

$$-4A-2B+C = 2 \tag{7.3.6}$$

$$4A = 4 \tag{7.3.7}$$

Solving for A in Equation 7.3.7, we have

$$\begin{aligned} 4A &= 4 \\ A &= 1 \end{aligned}$$

Substitute $A = 1$ into Equation 7.3.5.

$$\begin{aligned} A + B &= -1 \\ (1) + B &= -1 \\ B &= -2 \end{aligned}$$

Then, to solve for C, substitute the values for A and B into Equation 7.3.6.

$$\begin{aligned} -4A - 2B + C &= 2 \\ -4(1) - 2(-2) + C &= 2 \\ -4 + 4 + C &= 2 \\ C &= 2 \end{aligned}$$

Thus,

$$\frac{-x^2 + 2x + 4}{x^3 - 4x^2 + 4x} = \frac{1}{x} - \frac{2}{(x-2)} + \frac{2}{(x-2)^2}$$

? Exercise 7.3.2

Find the partial fraction decomposition of the expression with repeated linear factors.

$$\frac{6x - 11}{(x-1)^2}$$

Answer

$$\frac{6}{x-1} - \frac{5}{(x-1)^2}$$

Decomposing $\frac{P(x)}{Q(x)}$, where $Q(x)$ Has a Nonrepeated Irreducible Quadratic Factor

So far, we have performed partial fraction decomposition with expressions that have had linear factors in the denominator, and we applied numerators A, B, or C representing constants. Now we will look at an example where one of the factors in the denominator is a quadratic expression that does not factor. This is referred to as an irreducible quadratic factor. In cases like this, we use a linear numerator such as $Ax + B$, $Bx + C$, etc.

DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS A NONREPEATED IRREDUCIBLE QUADRATIC FACTOR

The partial fraction decomposition of $\dfrac{P(x)}{Q(x)}$ such that $Q(x)$ has a nonrepeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$ is written as

$$\frac{P(x)}{Q(x)} = \frac{A_1 x + B_1}{(a_1 x^2 + b1_x + c_1)} + \frac{A_2 x + B_2}{(a_2 x^2 + b_2 x + c_2)} + \cdots + \frac{A_n x + B_n}{(a_n x^2 + b_n x + c_n)} \tag{7.3.8}$$

The decomposition may contain more rational expressions if there are linear factors. Each linear factor will have a different constant numerator: A, B, C, and so on.

Howto: decompose a rational expression where the factors of the denominator are distinct, irreducible quadratic factors

1. Use variables such as A, B, or C for the constant numerators over linear factors, and linear expressions such as A_1x+B_1, A_2x+B_2, etc., for the numerators of each quadratic factor in the denominator.

$$\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{A_1x+B_1}{(a_1x^2+b1_x+c_1)}+\frac{A_2x+B_2}{(a_2x^2+b_2x+c_2)}+\cdots+\frac{A_nx+B_n}{(a_nx^2+b_nx+c_n)}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 7.3.3: Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Contains a Nonrepeated Irreducible Quadratic Factor

Find a partial fraction decomposition of the given expression.

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}$$

Solution

We have one linear factor and one irreducible quadratic factor in the denominator, so one numerator will be a constant and the other numerator will be a linear expression. Thus,

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}=\frac{A}{(x+3)}+\frac{Bx+C}{(x^2+x+2)}$$

We follow the same steps as in previous problems. First, clear the fractions by multiplying both sides of the equation by the common denominator.

$$\begin{aligned}(x+3)(x^2+x+2)\left[\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}\right] &= \left[\frac{A}{(x+3)}+\frac{Bx+C}{(x^2+x+2)}\right](x+3)(x^2+x+2)\\ 8x^2+12x-20 &= A(x^2+x+2)+(Bx+C)(x+3)\end{aligned}$$

Notice we could easily solve for A by choosing a value for x that will make the $Bx+C$ term equal 0. Let $x=-3$ and substitute it into the equation.

$$\begin{aligned}8x^2+12x-20 &= A(x^2+x+2)+(Bx+C)(x+3)\\ 8(-3)^2+12(-3)-20 &= A((-3)^2+(-3)+2)+(B(-3)+C)((-3)+3)\\ 16 &= 8A\\ A &= 2\end{aligned}$$

Now that we know the value of A, substitute it back into the equation. Then expand the right side and collect like terms.

$$\begin{aligned}8x^2+12x-20 &= 2(x^2+x+2)+(Bx+C)(x+3)\\ 8x^2+12x-20 &= 2x^2+2x+4+Bx^2+3B+Cx+3C\\ 8x^2+12x-20 &= (2+B)x^2+(2+3B+C)x+(4+3C)\end{aligned}$$

Setting the coefficients of terms on the right side equal to the coefficients of terms on the left side gives the system of equations.

$$2+B=8 \tag{7.3.9}$$

$$2+3B+C=12 \tag{7.3.10}$$

$$4+3C=-20 \tag{7.3.11}$$

Solve for B using Equation 7.3.9

\[

$$2+B=8$$
$$B=6$$

and solve for C using Equation 7.3.11.

$$4+3C=-20$$
$$3C=-24$$
$$C=-8$$

Thus, the partial fraction decomposition of the expression is

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}=\frac{2}{(x+3)}+\frac{6x-8}{(x^2+x+2)}$$

Q&A: Could we have just set up a system of equations to solve the example above?

Yes, we could have solved it by setting up a system of equations without solving for A first. The expansion on the right would be:

$$8x^2+12x-20 = Ax^2+Ax+2A+Bx^2+3B+Cx+3C$$
$$8x^2+12x-20 = (A+B)x^2+(A+3B+C)x+(2A+3C)$$

So the system of equations would be:

$$A+B=8$$
$$A+3B+C=12$$
$$2A+3C=-20$$

? Exercise 7.3.3

Find the partial fraction decomposition of the expression with a nonrepeating irreducible quadratic factor.

$$\frac{5x^2-6x+7}{(x-1)(x^2+1)}$$

Answer

$$\frac{3}{x-1}+\frac{2x-4}{x^2+1}$$

Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Has a Repeated Irreducible Quadratic Factor

Now that we can decompose a simplified rational expression with an irreducible quadratic factor, we will learn how to do partial fraction decomposition when the simplified rational expression has repeated irreducible quadratic factors. The decomposition will consist of partial fractions with linear numerators over each irreducible quadratic factor represented in increasing powers.

DECOMPOSITION OF $\frac{P(x)}{Q(x)}$ WHEN $Q(X)$ HAS A REPEATED IRREDUCIBLE QUADRATIC FACTOR

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{(ax^2+bx+c)^n}=\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n} \quad (7.3.12)$$

Write the denominators in increasing powers.

How to: decompose a rational expression that has a repeated irreducible factor

1. Use variables like A, B, or C for the constant numerators over linear factors, and linear expressions such as A_1x+B_1, A_2x+B_2, etc., for the numerators of each quadratic factor in the denominator written in increasing powers, such as
$$\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 7.3.4: Decomposing a Rational Function with a Repeated Irreducible Quadratic Factor in the Denominator

Decompose the given expression that has a repeated irreducible factor in the denominator.
$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}$$

Solution

The factors of the denominator are x, (x^2+1), and $(x^2+1)^2$. Recall that, when a factor in the denominator is a quadratic that includes at least two terms, the numerator must be of the linear form $Ax+B$. So, let's begin the decomposition.
$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{(x^2+1)}+\frac{Dx+E}{(x^2+1)^2}$$

We eliminate the denominators by multiplying each term by $x(x^2+1)^2$. Thus,
$$\begin{aligned} x^4+x^3+x^2-x+1 &= A(x^2+1)^2+(Bx+C)(x)(x^2+1)+(Dx+E)(x) \\ x^4+x^3+x^2-x+1 &= A(x^4+2x^2+1)+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex \qquad \text{Expand the right side.} \\ &= Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex \end{aligned}$$

Now we will collect like terms.
$$x^4+x^3+x^2-x+1=(A+B)x^4+(C)x^3+(2A+B+D)x^2+(C+E)x+A$$

Set up the system of equations matching corresponding coefficients on each side of the equal sign.
$$\begin{aligned} A+B &= 1 \\ C &= 1 \\ 2A+B+D &= 1 \\ C+E &= -1 \\ A &= 1 \end{aligned}$$

We can use substitution from this point. Substitute $A=1$ into the first equation.
$$\begin{aligned} 1+B &= 1 \\ B &= 0 \end{aligned}$$

Substitute $A=1$ and $B=0$ into the third equation.
$$\begin{aligned} 2(1)+0+D &= 1 \\ D &= -1 \end{aligned}$$

Substitute $C = 1$ into the fourth equation.

$$1 + E = -1$$
$$E = -2$$

Now we have solved for all of the unknowns on the right side of the equal sign. We have $A = 1$, $B = 0$, $C = 1$, $D = -1$, and $E = -2$. We can write the decomposition as follows:

$$\frac{x^4 + x^3 + x^2 - x + 1}{x(x^2+1)^2} = \frac{1}{x} + \frac{1}{(x^2+1)} - \frac{x+2}{(x^2+1)^2}$$

Exercise 7.3.4

Find the partial fraction decomposition of the expression with a repeated irreducible quadratic factor.

$$\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2}$$

Answer

$$\frac{x-2}{x^2-2x+3} + \frac{2x+1}{(x^2-2x+3)^2}$$

Media

Access these online resources for additional instruction and practice with partial fractions.

- Partial Fraction Decomposition
- Partial Fraction Decomposition With Repeated Linear Factors
- Partial Fraction Decomposition With Linear and Quadratic Factors

Key Concepts

- Decompose $\frac{P(x)}{Q(x)}$ by writing the partial fractions as

  $$\frac{A}{a_1 x + b_1} + \frac{B}{a_2 x + b_2}.$$

  Solve by clearing the fractions, expanding the right side, collecting like terms, and setting corresponding coefficients equal to each other, then setting up and solving a system of equations (see Example 7.3.1).
- The decomposition of $\frac{P(x)}{Q(x)}$ with repeated linear factors must account for the factors of the denominator in increasing powers (see Example 7.3.2).
- The decomposition of $\frac{P(x)}{Q(x)}$ with a nonrepeated irreducible quadratic factor needs a linear numerator over the quadratic factor, as in $\frac{A}{x} + \frac{Bx + C}{(ax^2 + bx + c)}$ (see Example 7.3.3).
- In the decomposition of $\frac{P(x)}{Q(x)}$, where $Q(x)$ has a repeated irreducible quadratic factor, when the irreducible quadratic factors are repeated, powers of the denominator factors must be represented in increasing powers as

  $$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n}$$

  See Example 7.3.4.

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7.4: Solve Systems of Equations Using Matrices

Summary

By the end of this section, you will be able to:

- Write the augmented matrix for a system of equations
- Use row operations on a matrix
- Solve systems of equations using matrices

Before you get started, take this readiness quiz.

1. Solve: $3(x+2)+4=4(2x-1)+9$.
   If you missed this problem, review [link].
2. Solve: $0.25p+0.25(x+4)=5.20$.
   If you missed this problem, review [link].
3. Evaluate when $x=-2$ and $y=3$: $2x^2-xy+3y^2$.
   If you missed this problem, review [link].

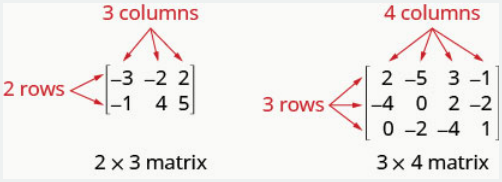
Write the Augmented Matrix for a System of Equations

Solving a system of equations can be a tedious operation where a simple mistake can wreak havoc on finding the solution. An alternative method which uses the basic procedures of elimination but with notation that is simpler is available. The method involves using a **matrix**. A matrix is a rectangular array of numbers arranged in rows and columns.

MATRIX

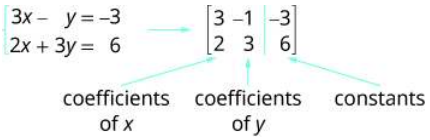
A **matrix** is a rectangular array of numbers arranged in rows and columns.

A matrix with *m* rows and *n* columns has order $m \times n$. The matrix on the left below has 2 rows and 3 columns and so it has order 2×3. We say it is a 2 by 3 matrix.



Each number in the matrix is called an element or entry in the matrix.

We will use a matrix to represent a system of linear equations. We write each equation in standard form and the coefficients of the variables and the constant of each equation becomes a row in the matrix. Each column then would be the coefficients of one of the variables in the system or the constants. A vertical line replaces the equal signs. We call the resulting matrix the augmented matrix for the system of equations.



Notice the first column is made up of all the coefficients of *x*, the second column is the all the coefficients of *y*, and the third column is all the constants.

Example 7.4.1

ⓐ $\begin{cases} 5x-3y=-1 \\ y=2x-2 \end{cases}$ ⓑ $\begin{cases} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{cases}$

Answer

ⓐ The second equation is not in standard form. We rewrite the second equation in standard form.

$$y=2x-2$$
$$-2x+y=-2$$

We replace the second equation with its standard form. In the augmented matrix, the first equation gives us the first row and the second equation gives us the second row. The vertical line replaces the equal signs.

$$\begin{array}{l} \\ 5x-3y=-1 \\ 2x-\ y=\ 2 \end{array} \qquad \begin{array}{c} x\ \ y \\ \left[\begin{array}{cc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \end{array}$$

ⓑ All three equations are in standard form. In the augmented matrix the first equation gives us the first row, the second equation gives us the second row, and the third equation gives us the third row. The vertical line replaces the equal signs.

$$\begin{array}{l} \\ 6x-5y+2z=\ 3 \\ 2x+\ y-4z=\ 5 \\ 3x-3y+\ z=-1 \end{array} \qquad \begin{array}{c} x\ \ y\ \ z \\ \left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \end{array}$$

Example 7.4.2

Write each system of linear equations as an augmented matrix:

ⓐ $\begin{cases} 3x+8y=-3 \\ 2x=-5y-3 \end{cases}$ ⓑ $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$

Answer

ⓐ $\left[\begin{array}{ccc} 3 & 8 & -3 \\ 2 & 5 & -3 \end{array}\right]$

ⓑ $\left[\begin{array}{cccc} 2 & 3 & 1 & -5 \\ -1 & 3 & 3 & 4 \\ 2 & 8 & 7 & -3 \end{array}\right]$

Example 7.4.3

Write each system of linear equations as an augmented matrix:

ⓐ $\begin{cases} 11x=-9y-5 \\ 7x+5y=-1 \end{cases}$ ⓑ $\begin{cases} 5x-3y+2z=-5 \\ 2x-y-z=4 \\ 3x-2y+2z=-7 \end{cases}$

Answer

ⓐ $\begin{bmatrix} 11 & 9 & -5 \\ 7 & 5 & -1 \end{bmatrix}$

ⓑ $\begin{bmatrix} 5 & -3 & 2 & -5 \\ 2 & -1 & -1 & 4 \\ 3 & -2 & 2 & -7 \end{bmatrix}$

It is important as we solve systems of equations using matrices to be able to go back and forth between the system and the matrix. The next example asks us to take the information in the matrix and write the system of equations.

Example 7.4.4

Write the system of equations that corresponds to the augmented matrix:

$$\left[\begin{array}{ccc|c} 4 & -3 & 3 & -1 \\ 1 & 2 & -1 & 2 \\ -2 & -1 & 3 & -4 \end{array}\right].$$

Answer

We remember that each row corresponds to an equation and that each entry is a coefficient of a variable or the constant. The vertical line replaces the equal sign. Since this matrix is a 4×3, we know it will translate into a system of three equations with three variables.

$$\begin{array}{ccc} x & y & z \\ \left[\begin{array}{ccc|c} 4 & -3 & 3 & -1 \\ 1 & 2 & -1 & 2 \\ -2 & -1 & 3 & -4 \end{array}\right] & \longrightarrow & \begin{array}{r} 4x-3y+3z=-1 \\ x+2y-z=2 \\ -2x-y+3z=-4 \end{array} \end{array}$$

Example 7.4.5

Write the system of equations that corresponds to the augmented matrix: $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & -2 & 1 \\ 4 & -1 & 2 & 0 \end{bmatrix}$.

Answer

$$\begin{cases} x-y+2z=3 \\ 2x+y-2z=1 \\ 4x-y+2z=0 \end{cases}$$

Example 7.4.6

Write the system of equations that corresponds to the augmented matrix: $\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 3 & -1 & 8 \\ 1 & 1 & -1 & 3 \end{bmatrix}$.

Answer

$$\begin{cases} x+y+z=4 \\ 2x+3y-z=8 \\ x+y-z=3 \end{cases}$$

Use Row Operations on a Matrix

Once a system of equations is in its augmented matrix form, we will perform operations on the rows that will lead us to the solution.

To solve by elimination, it doesn't matter which order we place the equations in the system. Similarly, in the matrix we can interchange the rows.

When we solve by elimination, we often multiply one of the equations by a constant. Since each row represents an equation, and we can multiply each side of an equation by a constant, similarly we can multiply each entry in a row by any real number except 0.

In elimination, we often add a multiple of one row to another row. In the matrix we can replace a row with its sum with a multiple of another row.

These actions are called row operations and will help us use the matrix to solve a system of equations.

ROW OPERATIONS

In a matrix, the following operations can be performed on any row and the resulting matrix will be equivalent to the original matrix.

1. Interchange any two rows.
2. Multiply a row by any real number except 0.
3. Add a nonzero multiple of one row to another row.

Performing these operations is easy to do but all the arithmetic can result in a mistake. If we use a system to record the row operation in each step, it is much easier to go back and check our work.

We use capital letters with subscripts to represent each row. We then show the operation to the left of the new matrix. To show interchanging a row:

$$\left[\begin{array}{cc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{\substack{R_2 \\ R_1}} \left[\begin{array}{cc|c} 2 & -1 & 2 \\ 5 & -3 & -1 \end{array}\right]$$

To multiply row 2 by -3:

$$\left[\begin{array}{cc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{-3R_2} \left[\begin{array}{cc|c} 5 & -3 & -1 \\ -6 & 3 & -6 \end{array}\right]$$

To multiply row 2 by -3 and add it to row 1:

$$\left[\begin{array}{cc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{-3R_2 + R_1} \left[\begin{array}{cc|c} -1 & 0 & -7 \\ 2 & -1 & 2 \end{array}\right]$$

Example 7.4.7

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 2 and 3.

ⓑ Multiply row 2 by 5.

ⓒ Multiply row 3 by $-2-2$ and add to row 1.

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

Answer

ⓐ We interchange rows 2 and 3.

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{\substack{R_3 \\ R_2}} \left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 3 & -3 & 1 & -1 \\ 2 & 1 & -4 & 5 \end{array}\right]$$

ⓑ We multiply row 2 by 5.

$$\left[\begin{array}{rrr|r} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{5R_2} \left[\begin{array}{rrr|r} 6 & -5 & 2 & 3 \\ 10 & 5 & -20 & 25 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

ⓒ We multiply row 3 by -2 and add to row 1.

$$\left[\begin{array}{rrr|r} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{-2R_3+R_1} \left[\begin{array}{rrrr} (-2\cdot 3)+6 & (-2\cdot(-3))+(-5) & (-2\cdot 1)+2 & (-2\cdot(-1))+3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-2R_3+R_1} \left[\begin{array}{rrr|r} 0 & 1 & 0 & 5 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

Example 7.4.8

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 1 and 3.

ⓑ Multiply row 3 by 3.

ⓒ Multiply row 3 by 2 and add to row 2.

$$\left[\begin{array}{rrr|r} 5 & -2 & -2 & -2 \\ 4 & -1 & -4 & 4 \\ -2 & 3 & 0 & -1 \end{array}\right]$$

Answer

ⓐ $\left[\begin{array}{rrrr} -2 & 3 & 0 & -2 \\ 4 & -1 & -4 & 4 \\ 5 & -2 & -2 & -2 \end{array}\right]$

ⓑ $\left[\begin{array}{rrrr} -2 & 3 & 0 & -2 \\ 4 & -1 & -4 & 4 \\ 15 & -6 & -6 & -6 \end{array}\right]$

ⓒ $\left[\begin{array}{rrrrr} -2 & 3 & 0 & 2 & \\ 3 & 4 & -13 & -16 & -8 \\ 15 & -6 & -6 & -6 & \end{array}\right]$

Example 7.4.9

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 1 and 2,

ⓑ Multiply row 1 by 2,

ⓒ Multiply row 2 by 3 and add to row 1.

$$\left[\begin{array}{rrr|r} 2 & -3 & -2 & -4 \\ 4 & 1 & -3 & 2 \\ 5 & 0 & 4 & -1 \end{array}\right]$$

Answer

ⓐ $\begin{bmatrix} 4 & 1 & -3 & 2 \\ 2 & -3 & -2 & -4 \\ 5 & 0 & 4 & -1 \end{bmatrix}$

ⓑ $\begin{bmatrix} 8 & 2 & -6 & 4 \\ 2 & -3 & -2 & -4 \\ 5 & 0 & 4 & -1 \end{bmatrix}$

ⓒ $\begin{bmatrix} 14 & -7 & -12 & -8 \\ 2 & -3 & -2 & -4 \\ 5 & 0 & 4 & -1 \end{bmatrix}$

Now that we have practiced the row operations, we will look at an augmented matrix and figure out what operation we will use to reach a goal. This is exactly what we did when we did elimination. We decided what number to multiply a row by in order that a variable would be eliminated when we added the rows together.

Given this system, what would you do to eliminate x?

$$\begin{matrix} x-y=2 \\ 4x-8y=0 \end{matrix} \xrightarrow{\text{multiply the first equation by } -4} \begin{matrix} -4x+4y=-8 \\ 4x-8y=0 \end{matrix} \xrightarrow{\text{then add}} \begin{matrix} -4x+4y=-8 \\ 4x-8y=0 \\ \hline -4y=-8 \end{matrix}$$

This next example essentially does the same thing, but to the matrix.

Example 7.4.10

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 4 & -8 & 0 \end{array}\right]$

Answer

To make the 4 a 0, we could multiply row 1 by -4 and then add it to row 2.

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 4 & -8 & 0 \end{array}\right] \xrightarrow{-4R_1+R_2} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & -4 & -8 \end{array}\right]$$

Example 7.4.11

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 3 & -6 & 2 \end{array}\right]$

Answer

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -3 & -4 \end{bmatrix}$$

Example 7.4.12

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $\left[\begin{array}{cc|c} 1 & -1 & 3 \\ -2 & -3 & 2 \end{array}\right]$

Answer

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & -5 & 8 \end{bmatrix}$$

Solve Systems of Equations Using Matrices

To solve a system of equations using matrices, we transform the augmented matrix into a matrix in **row-echelon form** using row operations. For a consistent and independent system of equations, its **augmented matrix** is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

ROW-ECHELON FORM

For a consistent and independent system of equations, its augmented matrix is in **row-echelon form** when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \end{bmatrix}$ $\begin{bmatrix} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{bmatrix}$ a, b, c, d, e, f are real numbers

Once we get the augmented matrix into row-echelon form, we can write the equivalent system of equations and read the value of at least one variable. We then substitute this value in another equation to continue to solve for the other variables. This process is illustrated in the next example.

Example 7.4.13: How to Solve a System of Equations Using a Matrix

Solve the system of equations using a matrix: $\begin{cases} 3x+4y=5 \\ x+2y=1 \end{cases}$

Answer

Step 1. Write the augmented matrix for the system of equations.		$\begin{aligned} 3x+4y&=5 \\ x+2y&=1 \end{aligned}$ $\left[\begin{array}{cc\|c} 3 & 4 & 5 \\ 1 & 2 & 1 \end{array}\right]$
Step 2. Using row operations get the entry in row 1, column 1 to be 1.	Interchange the rows, so 1 will be in row 1, column 1.	$\left[\begin{array}{cc\|c} 1 & 2 & 1 \\ 3 & 4 & 5 \end{array}\right]$ R_2, R_1
Step 3. Using row operations, get zeros in column 1 below the 1.	Multiply row 1 by –3 and add it to row 2.	$\left[\begin{array}{cc\|c} 1 & 2 & 1 \\ 0 & -2 & 2 \end{array}\right]$ $-3R_1+R_2$
Step 4. Using row operations, get the entry in row 2, column 2 to be 1.	Multiply row 2 by $-\frac{1}{2}$.	$\left[\begin{array}{cc\|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right]$ $-\frac{1}{2}R_2$
Step 5. Continue the process until the matrix is in row-echelon form.	The matrix is now in row-echelon form.	$\left[\begin{array}{cc\|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right]$
Step 6. Write the corresponding system of equations.	$\begin{matrix} x & y \end{matrix}$ $\left[\begin{array}{cc\|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right]$	$\begin{aligned} x+2y&=1 \\ y&=-1 \end{aligned}$
Step 7. Use substitution to find the remaining variables.	Substitute $y=-1$ into $x+2y=1$.	$\begin{aligned} y&=-1 \\ x+2y&=1 \\ x+2(-1)&=1 \\ x-2&=1 \\ x&=3 \end{aligned}$

Step 8. Write the solution as an ordered pair or triple.		(3, –1)
Step 9. Check that the solution makes the original equations true.		We leave the check to you.

? Example 7.4.14

Solve the system of equations using a matrix: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Answer

The solution is $(4,-1)$.

? Example 7.4.15

Solve the system of equations using a matrix: $\begin{cases} 2x+y=-4 \\ x-y=-2 \end{cases}$

Answer

The solution is $(-2,0)$.

The steps are summarized here.

SOLVE A SYSTEM OF EQUATIONS USING MATRICES.

1. Write the augmented matrix for the system of equations.
2. Using row operations get the entry in row 1, column 1 to be 1.
3. Using row operations, get zeros in column 1 below the 1.
4. Using row operations, get the entry in row 2, column 2 to be 1.
5. Continue the process until the matrix is in row-echelon form.
6. Write the corresponding system of equations.
7. Use substitution to find the remaining variables.
8. Write the solution as an ordered pair or triple.
9. Check that the solution makes the original equations true.

Here is a visual to show the order for getting the 1's and 0's in the proper position for row-echelon form.

2 × 3 matrix

Step 1 $\begin{bmatrix} 1 & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$ **Step 2** $\begin{bmatrix} 1 & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare \end{bmatrix}$ **Step 3** $\begin{bmatrix} 1 & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare \end{bmatrix}$

3 × 4 matrix

Step 1 $\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$ **Step 2** $\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$ **Step 3** $\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$ **Step 4** $\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$

Step 5 $\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare \end{bmatrix}$ **Step 6** $\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$

We use the same procedure when the system of equations has three equations.

Example 7.4.16

Solve the system of equations using a matrix: $\begin{cases} 3x+8y+2z=-5 \\ 2x+5y-3z=0 \\ x+2y-2z=-1 \end{cases}$

Answer

	$\begin{cases} 3x+8y+2z=-5 \\ 2x+5y-3z=0 \\ x+2y-2z=-1 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{ccc\|c} 3 & 8 & 2 & -5 \\ 2 & 5 & -3 & 0 \\ 1 & 2 & -2 & -1 \end{array}\right]$
Interchange row 1 and 3 to get the entry in row 1, column 1 to be 1.	$\begin{matrix} R_3 \\ \\ R_1 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 2 & 5 & -3 & 0 \\ 3 & 8 & 2 & -5 \end{array}\right]$
Using row operations, get zeros in column 1 below the 1.	$\begin{matrix} \\ -2R_1+R_2 \\ \\ \end{matrix}\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 3 & 8 & 2 & -5 \end{array}\right]$
	$\begin{matrix} \\ \\ -3R_1+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 8 & -2 \end{array}\right]$
The entry in row 2, column 2 is now 1.	
Continue the process until the matrix is in row-echelon form.	$\begin{matrix} \\ \\ -2R_2+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6 & -6 \end{array}\right]$
	$\begin{matrix} \\ \\ \frac{1}{6}R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$
The matrix is now in row-echelon form.	$\left[\begin{array}{ccc\|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$
Write the corresponding system of equations.	$\begin{cases} x+2y-2z=-1 \\ y+z=2 \\ z=-1 \end{cases}$
Use substitution to find the remaining variables.	$\begin{aligned} y+z&=2 \\ y+(-1)&=2 \\ y&=3 \end{aligned}$
	$\begin{aligned} x+2y-2z&=-1 \\ x+2(3)-2(-1)&=-1 \\ x+6+2&=-1 \\ x&=-9 \end{aligned}$
Write the solution as an ordered pair or triple.	$(-9, 3, -1)$
Check that the solution makes the original equations true.	We leave the check for you.

Example 7.4.17

Solve the system of equations using a matrix: $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$

Answer

$(6,-1,-3)$

Example 7.4.18

Solve the system of equations using a matrix: $\begin{cases} -3x+y+z=-4 \\ -x+2y-2z=1 \\ 2x-y-z=-1 \end{cases}$

Answer

$(5,7,4)$

So far our work with matrices has only been with systems that are consistent and independent, which means they have exactly one solution. Let's now look at what happens when we use a matrix for a dependent or inconsistent system.

Example 7.4.19

Solve the system of equations using a matrix: $\begin{cases} x+y+3z=0 \\ x+3y+5z=0 \\ 2x+4z=1 \end{cases}$

Answer

	$\begin{cases} x+y+3z=0 \\ x+3y+5z=0 \\ 2x+4z=1 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 1 & 3 & 5 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right]$
The entry in row 1, column 1 is 1.	
Using row operations, get zeros in column 1 below the 1.	$\begin{matrix} \\ -1R_1+R_2 \\ \\ \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right]$
	$\begin{matrix} \\ \\ -2R_1+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right]$
Continue the process until the matrix is in row-echelon form.	$\begin{matrix} \\ \frac{1}{2}R_2 \\ \\ \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right]$
Multiply row 2 by 2 and add it to row 3.	$\begin{matrix} \\ \\ 2R_2+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$
At this point, we have all zeros on the left of row 3.	

Write the corresponding system of equations.	$\begin{cases} x+y+3z=0 \\ y+z=0 \\ 0\neq 1 \end{cases}$

Since $0 \neq 1$ we have a false statement. Just as when we solved a system using other methods, this tells us we have an inconsistent system. There is no solution.

Example 7.4.20

Solve the system of equations using a matrix: $\begin{cases} x-2y+2z=1 \\ -2x+y-z=2 \\ x-y+z=5 \end{cases}$

Answer

no solution

Example 7.4.21

Solve the system of equations using a matrix: $\begin{cases} 3x+4y-3z=-2 \\ -2x+3y-z=-1 \\ 2x+y-2z=6 \end{cases}$

Answer

no solution

The last system was inconsistent and so had no solutions. The next example is dependent and has infinitely many solutions.

Example 7.4.22

Solve the system of equations using a matrix: $\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$

Answer

	$\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$
Write the augmented matrix for the equations.	$\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 1 \\ 1 & 1 & -3 & 7 \\ 3 & -4 & 5 & 7 \end{array}\right]$
The entry in row 1, column 1 is 1.	
Using row operations, get zeros in column 1 below the 1.	$\begin{matrix} \\ -1R_1+R_2 \\ \\ \end{matrix}\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 1 \\ 0 & 3 & -6 & 6 \\ 3 & -4 & 5 & 7 \end{array}\right]$
	$\begin{matrix} \\ \\ -3R_1+R_3 \end{matrix}\left[\begin{array}{ccc\|c} 1 & -2 & 3 & 1 \\ 0 & 3 & -6 & 6 \\ 0 & 2 & -4 & 4 \end{array}\right]$

Continue the process until the matrix is in row-echelon form.	$\frac{1}{3}R_2 \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -4 & 4 \end{bmatrix}$
Multiply row 2 by -2 and add it to row 3.	$-2R_2+R_3 \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
At this point, we have all zeros in the bottom row.	
Write the corresponding system of equations.	$\begin{cases} x-2y+3z=1 \\ y-2z=2 \\ 0=0 \end{cases}$
Since $0=0$ we have a true statement. Just as when we solved by substitution, this tells us we have a dependent system. There are infinitely many solutions.	
Solve for y in terms of z in the second equation.	$y-2z=2$ $y=2z+2$
Solve the first equation for x in terms of z.	$x-2y+3z=1$
Substitute $y=2z+2$.	$x-2(2z+2)+3z=1$
Simplify.	$x-4z-4+3z=1$
Simplify.	$x-z-4=1$
Simplify.	$x=z+5$
The system has infinitely many solutions (x,y,z), where $x=z+5;\ y=2z+2;\ z$ is any real number.	

? Example 7.4.23

Solve the system of equations using a matrix: $\begin{cases} x+y-z=0 \\ 2x+4y-2z=6 \\ 3x+6y-3z=9 \end{cases}$

Answer

infinitely many solutions (x,y,z), where $x=z-3;\ y=3;\ z$ is any real number.

? Example 7.4.24

Solve the system of equations using a matrix: $\begin{cases} x-y-z=1 \\ -x+2y-3z=-4 \\ 3x-2y-7z=0 \end{cases}$

Answer

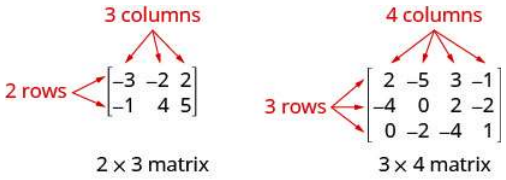
infinitely many solutions (x,y,z), where $x=5z-2;\ y=4z-3;\ z$ is any real number.

Access this online resource for additional instruction and practice with Gaussian Elimination.

- Gaussian Elimination

Key Concepts

- **Matrix:** A matrix is a rectangular array of numbers arranged in rows and columns. A matrix with *m* rows and *n* columns has *order* $m \times n$. The matrix on the left below has 2 rows and 3 columns and so it has order 2×3. We say it is a 2 by 3 matrix.



Each number in the matrix is called an *element* or *entry* in the matrix.

- **Row Operations:** In a matrix, the following operations can be performed on any row and the resulting matrix will be equivalent to the original matrix.
  - Interchange any two rows
  - Multiply a row by any real number except 0
  - Add a nonzero multiple of one row to another row
- **Row-Echelon Form:** For a consistent and independent system of equations, its augmented matrix is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \end{bmatrix}$ $\begin{bmatrix} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{bmatrix}$ a, b, c, d, e, f are real numbers

- **How to solve a system of equations using matrices.**
  1. Write the augmented matrix for the system of equations.
  2. Using row operations get the entry in row 1, column 1 to be 1.
  3. Using row operations, get zeros in column 1 below the 1.
  4. Using row operations, get the entry in row 2, column 2 to be 1.
  5. Continue the process until the matrix is in row-echelon form.
  6. Write the corresponding system of equations.
  7. Use substitution to find the remaining variables.
  8. Write the solution as an ordered pair or triple.
  9. Check that the solution makes the original equations true.

Glossary

matrix

A matrix is a rectangular array of numbers arranged in rows and columns.

row-echelon form

A matrix is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

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7.5: Graphing Systems of Linear Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of linear inequalities
- Solve a system of linear inequalities by graphing
- Solve applications of systems of inequalities

Before you get started, take this readiness quiz.

1. Solve the inequality $2a < 5a + 12$.
   If you missed this problem, review [link].
2. Determine whether the ordered pair $(3, 12)$ is a solution to the system $y > 2x + 3$.
   If you missed this problem, review [link].

Determine whether an ordered pair is a solution of a system of linear inequalities

The definition of a **system of linear inequalities** is very similar to the definition of a system of linear equations.

Definition: System of Linear Inequalities

Two or more linear inequalities grouped together form a system of linear inequalities.

A system of linear inequalities looks like a system of linear equations, but it has inequalities instead of equations. A system of two linear inequalities is shown here.

$$\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs (x, y) that make both inequalities true.

Solutions of a System of Linear Inequalities

Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true.

The solution of a system of linear inequalities is shown as a shaded region in the xy-coordinate system that includes all the points whose ordered pairs make the inequalities true.

To determine if an ordered pair is a solution to a system of two inequalities, we substitute the values of the variables into each inequality. If the ordered pair makes both inequalities true, it is a solution to the system.

Example 7.5.1

Determine whether the ordered pair is a solution to the system $\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$

a. $(-2, 4)$ b. $(3, 1)$

Solution:

a. Is the ordered pair $(-2, 4)$ a solution?

We substitute $x = -2$ and $y = 4$ into both inequalities.

$x + 4y \geq 10$ | $3x - 2y < 12$

$-2 + 4(4) \stackrel{?}{\geq} 10$ | $3(-2) - 2(4) \stackrel{?}{<} 12$

$14 \geq 10$ true | $-14 < 12$ true

The ordered pair $(-2, 4)$ made both inequalities true. Therefore $(-2, 4)$ is a solution to this system.

b. Is the ordered pair $(3, 1)$ a solution?

We substitute $x = 3$ and $y = 1$ into both inequalities.

$x + 4y \geq 10$ | $3x - 2y < 12$

$3 + 4(1) \stackrel{?}{\geq} 10$ | $3(3) - 2(1) \stackrel{?}{<} 12$

$7 \geq 10$ false | $7 < 12$ true

The ordered pair $(3, 1)$ made one inequality true, but the other one false. Therefore $(3, 1)$ is not a solution to this system.

Try It! 7.5.1

Determine whether the ordered pair is a solution to the system: $\begin{cases} x - 5y > 10 \\ 2x + 3y > -2 \end{cases}$

a. $(3, -1)$ b. $(6, -3)$

Answer

a. no
b. yes

Try It! 7.5.2

Determine whether the ordered pair is a solution to the system: $\begin{cases} y > 4x - 2 \\ 4x - y < 20 \end{cases}$

a. $(-2, 1)$ b. $(4, -1)$

Answer

a. yes
b. no

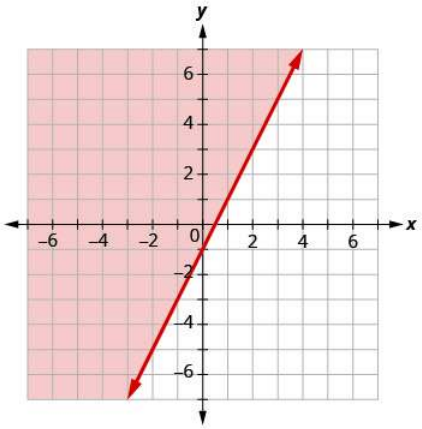
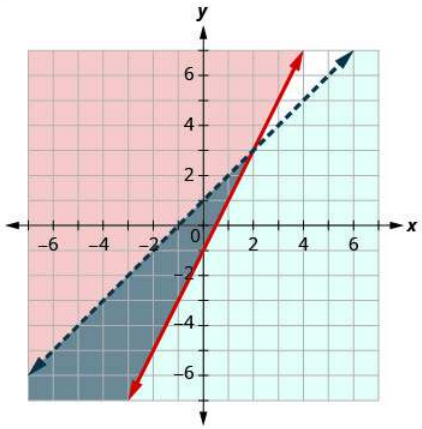
Solve a System of Linear Inequalities by Graphing

The solution to a single linear inequality is the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this region, we will graph each inequality separately and then locate the region where they are both true. The solution is always shown as a graph.

Example 7.5.2: How to Solve a System of Linear Inequalities by Graphing

Solve the system by graphing: $\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$

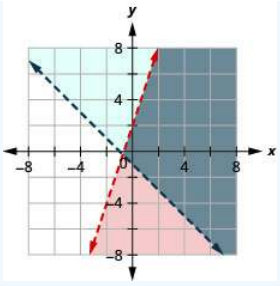
Solution:

Step 1. Graph the first inequality. • Graph the boundary line. • Shade in the side of the boundary line where the inequality is true.	We will graph $y \geq 2x - 1$. We graph the line $y = 2x - 1$. It is a solid line because the inequality sign is $\geq$. We choose (0, 0) as a test point. It is a solution to $y \geq 2x - 1$, so we shade in above the boundary line.	$\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$ 
Step 2. On the same grid, graph the second inequality. • Graph the boundary line. • Shade in the side of that boundary line where the inequality is true.	We will graph $y < x + 1$ on the same grid. We graph the line $y = x + 1$. It is a dashed line because the inequality sign is $<$. Again, we use (0, 0) as a test point. It is a solution so we shade in that side of the line $y = x + 1$.	
Step 3. The solution is the region where the shading overlaps.	The point where the boundary lines intersect is not a solution because it is not a solution to $y < x + 1$.	The solution is all points in the area shaded twice—which appears as the darkest shaded region.
Step 4. Check by choosing a test point.	We'll use (–1, –1) as a test point.	Is (–1, –1) a solution to $y \geq 2x - 1$? $-1 \stackrel{?}{\geq} 2(-1) - 1$ $-1 \geq -3$ true Is (–1, –1) a solution to $y < x + 1$? $-1 \stackrel{?}{<} -1 + 1$ $-1 < 0$ true The region containing (–1, –1) is the solution to this system.

Try It! 7.5.3

Solve the system by graphing: $\begin{cases} y < 3x + 2 \\ y > -x - 1 \end{cases}$

Answer

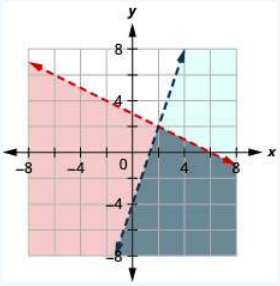


The solution is the grey region.

Try It! 7.5.4

Solve the system by graphing: $\begin{cases} y < -12x + 3 \\ y < 3x - 4 \end{cases}$

Answer



The solution is the grey region.

Solve a System of Linear Inequalities by Graphing

1. Graph the first inequality.
   - Graph the boundary line.
   - Shade in the side of the boundary line where the inequality is true.
2. On the same grid, graph the second inequality.
   - Graph the boundary line.
   - Shade in the side of that boundary line where the inequality is true.
3. The solution is the region where the shading overlaps.
4. Check by choosing a test point.

Example 7.5.3

Solve the system by graphing: $\begin{cases} x - y > 3 \\ y < -15x + 4 \end{cases}$

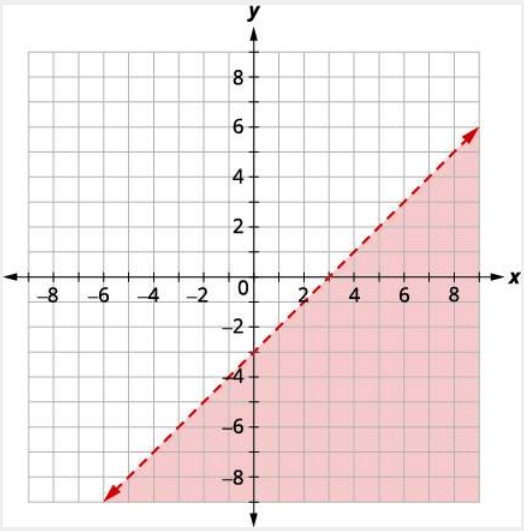
Solution:

	$\begin{cases} x - y > 3 \\ y < -15x + 4 \end{cases}$

Graph $x - y > 3$, by graphing $x - y = 3$ and testing a point.

The intercepts are $x = 3$ and $y = -3$ and the boundary line will be dashed.

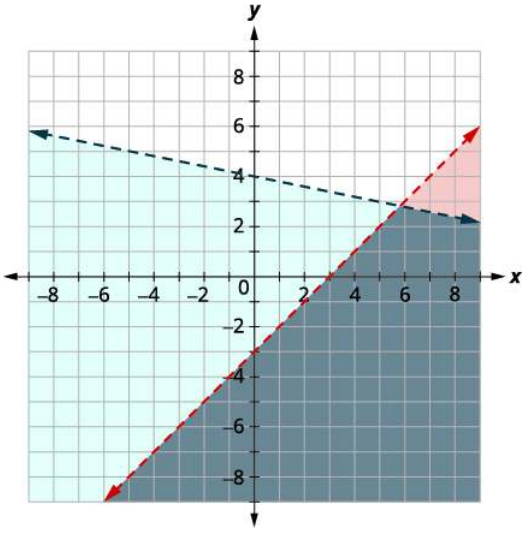
Test $(0, 0)$ which makes the inequality false so shade (red) the side that does not contain $(0, 0)$.



Graph $y < -15x + 4$ by graphing $y = -15x + 4$ using the slope $m = -15$ and y-intercept $b = 4$. The boundary line will be dashed

Test $(0, 0)$ which makes the inequality true, so shade (blue) the side that contains $(0, 0)$.

Choose a test point in the solution and verify that it is a solution to both inequalties.

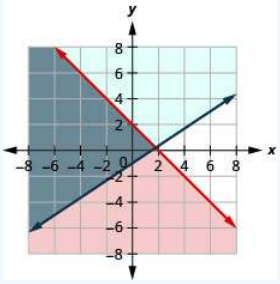


The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice—which appears as the darkest shaded region.

? Try It! 7.5.5

Solve the system by graphing: $\begin{cases} x + y \leq 2 \\ y \geq \frac{2}{3}x - 1 \end{cases}$

Answer

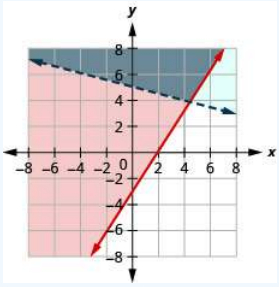


The solution is the grey region.

Try It! 7.5.6

Solve the system by graphing: $\begin{cases} 3x-2y \le 6 \\ y > -\frac{1}{4}x+5 \end{cases}$

Answer

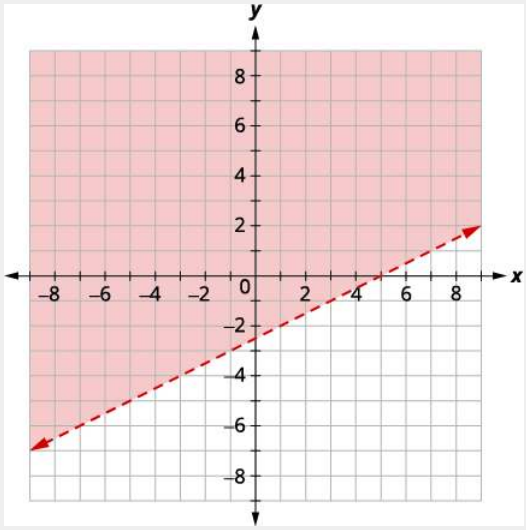


The solution is the grey region.

Example 7.5.4

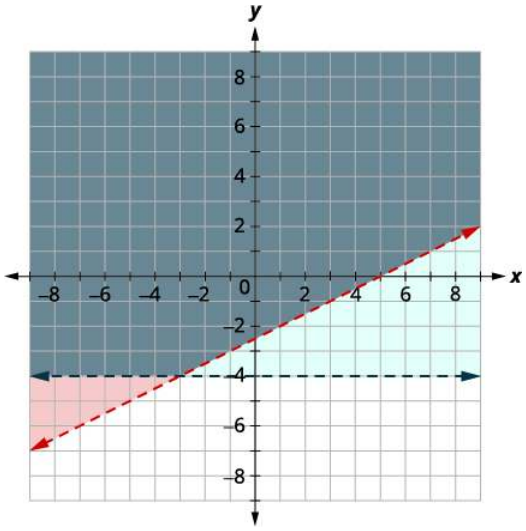
Solve the system by graphing: $\begin{cases} x-2y < 5 \\ y > -4 \end{cases}$

Solution:

	$\begin{cases} x-2y < 5 \\ y > -4 \end{cases}$
Graph $x-2y<5$, by graphing $x-2y=5$ and testing a point. The intercepts are $x=5$ and $y=-2.5$ and the boundary line will be dashed. Test $(0,0)$ which makes the inequality true, so shade (red) the side that contains $(0,0)$.	

Graph $y > -4$, by graphing $y = -4$ and recognizing that it is a horizontal line through $y = -4$. The boundary line will be dashed.

Test $(0, 0)$ which makes the inequality true so shade (blue) the side that contains $(0, 0)$.



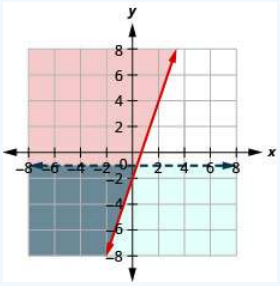
The point $(0, 0)$ is in the solution and we have already found it to be a solution of each inequality. The point of intersection of the two lines is not included as both boundary lines were dashed.

The solution is the area shaded twice—which appears as the darkest shaded region.

Try It! 7.5.7

Solve the system by graphing: $\begin{cases} y \geq 3x - 2 \\ y < -1 \end{cases}$

Answer

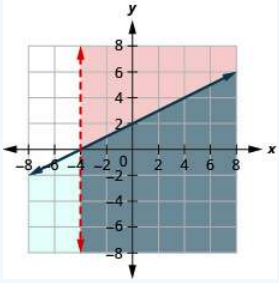


The solution is the grey region.

Try It! 7.5.8

Solve the system by graphing: $\begin{cases} x > -4x - 2 \\ y \geq -4 \end{cases}$

Answer



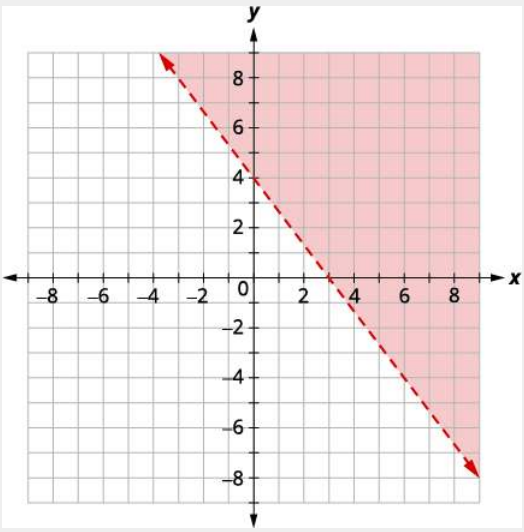
The solution is the grey region.

Systems of linear inequalities where the boundary lines are parallel might have no solution. We'll see this in the next example.

Example 7.5.5

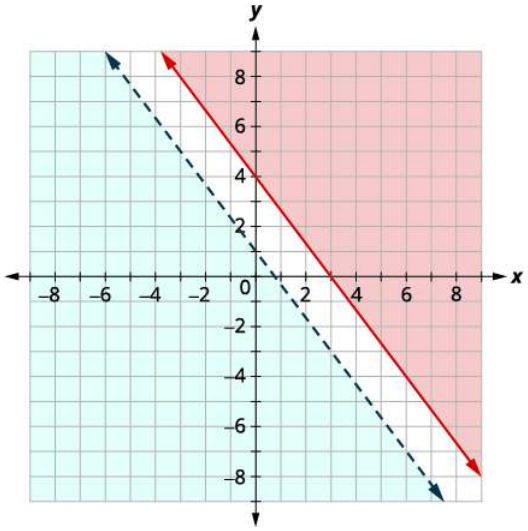
Solve the system by graphing: $\begin{cases} 4x+3y \geq 12 \\ y < -\frac{4}{3}x+1 \end{cases}$

Solution:

	$\begin{cases} 4x+3y \geq 12 \\ y < -\frac{4}{3}x+1 \end{cases}$
Graph $4x+3y \geq 12$, by graphing $4x+3y=12$ and testing a point. The intercepts are $x=3$ and $y=4$ and the boundary line will be solid. Test $(0,0)$ which makes the inequality false, so shade (red) the side that does not contain $(0,0)$.	

Graph $y < -\frac{4}{3}x + 1$ by graphing $y = -\frac{4}{3}x + 1$ using the slope $m = -\frac{4}{3}$ and y-intercept $b = 1$. The boundary line will be dashed.

Test $(0, 0)$ which makes the inequality true, so shade (blue) the side that contains $(0, 0)$.

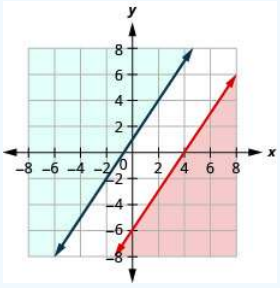


There is no point in both shaded regions, so the system has no solution.

Try It! 7.5.9

Solve the system by graphing: $\begin{cases} 3x - 2y \geq 12 \\ y \geq \frac{3}{2}x + 1 \end{cases}$

Answer

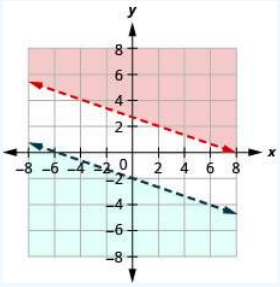


No solution.

Try It! 7.5.10

Solve the system by graphing: $\begin{cases} x + 3y > 8 \\ y < -\frac{1}{3}x - 2 \end{cases}$

Answer



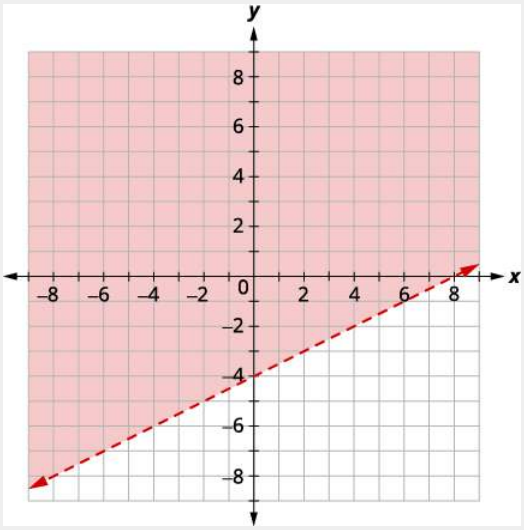
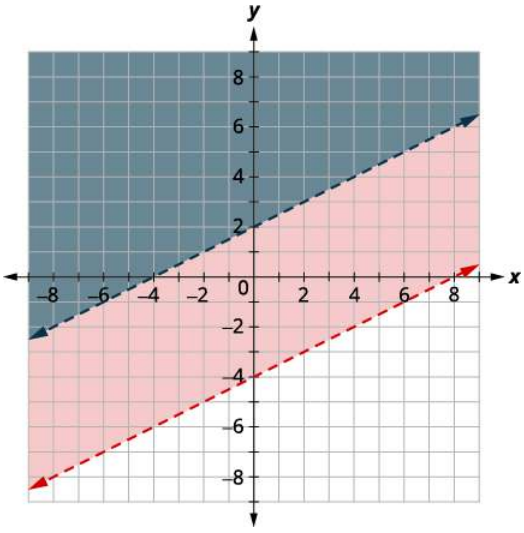
No solution.

Some systems of linear inequalities where the boundary lines are parallel will have a solution. We'll see this in the next example.

Example 7.5.6

Solve the system by graphing: $\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$

Solution:

	$\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$
Graph $y > \frac{1}{2}x - 4$ by graphing $y = \frac{1}{2}x - 4$ using the slope $m = \frac{1}{2}$ and the intercept $b = -4$. The boundary line will be dashed. Test $(0, 0)$ which makes the inequality true, so shade (red) the side that contains $(0, 0)$.	
Graph $x - 2y < -4$ by graphing $x - 2y = -4$ and testing a point. The intercepts are $x = -4$ and $y = 2$ and the boundary line will be dashed. Choose a test point in the solution and verify that it is a solution to both inequalities. Test $(0, 0)$ which makes the inequality false, so shade (blue) the side that does not contain $(0, 0)$.	

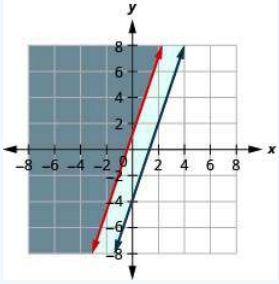
No point on the boundary lines is included in the solution as both lines are dashed.

The solution is the region that is shaded twice which is also the solution to $x - 2y < -4$.

Try It! 7.5.11

Solve the system by graphing: $\begin{cases} y \geq 3x+1 \\ -3x+y \geq -4 \end{cases}$

Answer

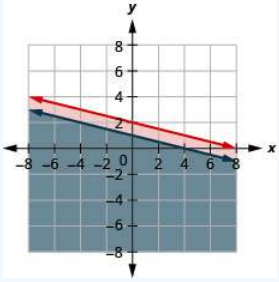


The solution is the grey region.

Try It! 7.5.12

Solve the system by graphing: $\begin{cases} y \leq -\frac{1}{4}x+2 \\ x+4y \leq 4 \end{cases}$

Answer



The solution is the grey region.

Solve Applications of Systems of Inequalities

The first thing we'll need to do to solve applications of systems of inequalities is to translate each condition into an inequality. Then we graph the system, as we did above, to see the region that contains the solutions. Many situations will be realistic only if both variables are positive, so we add inequalities to the system as additional requirements.

Example 7.5.7

Christy sells her photographs at a booth at a street fair. At the start of the day, she wants to have at least 25 photos to display at her booth. Each small photo she displays costs her $4 and each large photo costs her $10. She doesn't want to spend more than $200 on photos to display.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could she display 10 small and 20 large photos?
d. Could she display 20 large and 10 small photos?

Solution:

a.

Let $x =$ the number of small photos.

$y =$ the number of large photos

To find the system of equations translate the information.

She wants to have at least 25 photos.

The number of small plus the number of large should be at least 25.

$$x + y \geq 25$$

\$4 for each small and \$10 for each large must be no more than \$200

$$4x + 10y \leq 200$$

The number of small photos must be greater than or equal to 0.

$$x \geq 0$$

The number of large photos must be greater than or equal to 0.

$$y \geq 0$$

We have our system of equations.

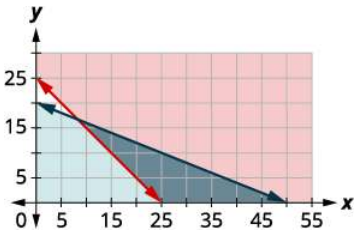
$$\begin{cases} x + y \geq 25 \\ 4x + 10y \leq 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

b.

Since $x \geq 0$ and $y \geq 0$ (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $x + y \geq 25$, graph $x + y = 25$ as a solid line.
Choose $(0, 0)$ as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point $(0, 0)$.

To graph $4x + 10y \leq 200$, graph $4x + 10y = 200$ as a solid line.
Choose $(0, 0)$ as a test point. Since it does make the inequality true, shade (blue) the side that include the point $(0, 0)$.



The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly-shaded section are included in the solution as are the points on the x-axis from $(25, 0)$ to $(55, 0)$.

c. To determine if 10 small and 20 large photos would work, we look at the graph to see if the point $(10, 20)$ is in the solution region. We could also test the point to see if it is a solution of both equations.

It is not, Christy would not display 10 small and 20 large photos.

d. To determine if 20 small and 10 large photos would work, we look at the graph to see if the point $(20, 10)$ is in the solution region. We could also test the point to see if it is a solution of both equations.

It is, so Christy could choose to display 20 small and 10 large photos.

Notice that we could also test the possible solutions by substituting the values into each inequality.

Try It! 7.5.13

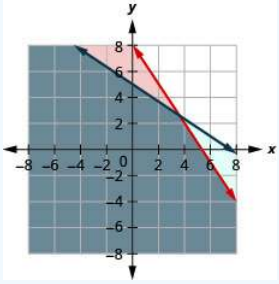
A trailer can carry a maximum weight of 160 pounds and a maximum volume of 15 cubic feet. A microwave oven weighs 30 pounds and has 2 cubic feet of volume, while a printer weighs 20 pounds and has 3 cubic feet of space.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could 4 microwaves and 2 printers be carried on this trailer?
d. Could 7 microwaves and 3 printers be carried on this trailer?

Answer

a. $\begin{cases} 30m+20p \le 160 \\ 2m+3p \le 15 \end{cases}$

b.



c. yes
d. no

Try It! 7.5.14

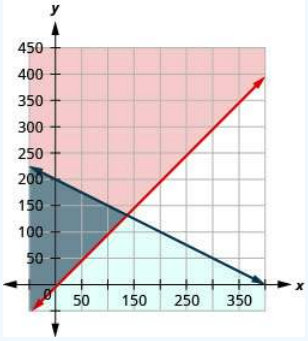
Mary needs to purchase supplies of answer sheets and pencils for a standardized test to be given to the juniors at her high school. The number of the answer sheets needed is at least 5 more than the number of pencils. The pencils cost $2 and the answer sheets cost $1. Mary's budget for these supplies allows for a maximum cost of $400.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could Mary purchase 100 pencils and 100 answer sheets?
d. Could Mary purchase 150 pencils and 150 answer sheets?

Answer

a. $\begin{cases} a \ge p+5 \\ a+2p \le 400 \end{cases}$

b.



c. no
d. no

When we use variables other than x and y to define an unknown quantity, we must change the names of the axes of the graph as well.

Example 7.5.8

Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers and cookies, and he doesn't want to spend more than \$5. At the hamburger restaurant near his college, each hamburger has 240 calories and costs \$1.40. Each cookie has 160 calories and costs \$0.50.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could he eat 3 hamburgers and 1 cookie?
d. Could he eat 2 hamburgers and 4 cookies?

Solution:

a.

Let $h =$ the number of hamburgers.
$c =$ the number of cookies

To find the system of equations translate the information.

The calories from hamburgers at 240 calories each, plus the calories from cookies at 160 calories each must be more that 800.

$$240h + 160c \geq 800$$

The amount spent on hamburgers at \$1.40 each, plus the amount spent on cookies at \$0.50 each must be no more than \$5.00.

$$1.40h + 0.50c \leq 5$$

The number of hamburgers must be greater than or equal to 0.

$$h \geq 0$$

The number of cookies must be greater than or equal to 0.

$$c \geq 0$$

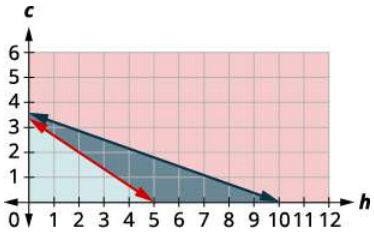
We have our system of equations.

$$\begin{cases} 240h + 160c \geq 800 \\ 1.40h + 0.50c \leq 5 \\ h \geq 0 \\ c \geq 0 \end{cases}$$

b.
Since $h \geq 0$ and $c \geq 0$ (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $240h + 160c \geq 800$, graph $240h + 160c = 800$ as a solid line.

Choose $(0, 0)$ as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point $(0, 0)$.



Graph $1.40h + 0.50c \leq 5$. The boundary line is $1.40h + 0.50c = 5$. We test $(0, 0)$ and it makes the inequality true. We shade the side of the line that includes $(0, 0)$.

The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly shaded section are included in the solution as are the points on the x-axis from $(5, 0)$ to $(10, 0)$.

c. To determine if 3 hamburgers and 2 cookies would meet Omar's criteria, we see if the point $(3, 2)$ is in the solution region. It is, so Omar might choose to eat 3 hamburgers and 2 cookies.

d. To determine if 2 hamburgers and 4 cookies would meet Omar's criteria, we see if the point $(2, 4)$ is in the solution region. It is, Omar might choose to eat 2 hamburgers and 4 cookies.

We could also test the possible solutions by substituting the values into each inequality.

Try It! 7.5.15

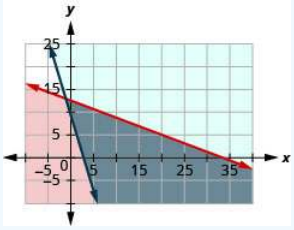
Tension needs to eat at least an extra 1,000 calories a day to prepare for running a marathon. He has only $25 to spend on the extra food he needs and will spend it on $0.75 donuts which have 360 calories each and $2 energy drinks which have 110 calories.

a. Write a system of inequalities that models this situation.
b. Graph the system.
c. Can he buy 8 donuts and 4 energy drinks and satisfy his caloric needs?
d. Can he buy 1 donut and 3 energy drinks and satisfy his caloric needs?

Answer

a. $\begin{cases} 0.75d + 2e \leq 25 \\ 360d + 110e \geq 1000 \end{cases}$

b.



c. yes
d. no

? Try It! 7.5.16

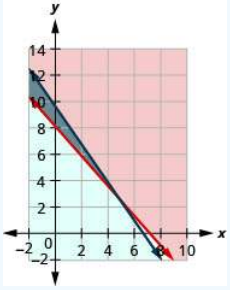
Philip's doctor tells him he should add at least 1,000 more calories per day to his usual diet. Philip wants to buy protein bars that cost $1.80 each and have 140 calories and juice that costs $1.25 per bottle and have 125 calories. He doesn't want to spend more than $12.

a. Write a system of inequalities that models this situation.
b. Graph the system.
c. Can he buy 3 protein bars and 5 bottles of juice?
d. Can he buy 5 protein bars and 3 bottles of juice?

Answer

a. $\begin{cases} 140p + 125j \geq 1000 \\ 1.80p + 1.25j \leq 12 \end{cases}$

b.



c. yes
d. no

Access these online resources for additional instruction and practice with solving systems of linear inequalities by graphing.

- Solving Systems of Linear Inequalities by Graphing
- Systems of Linear Inequalities

Key Concepts

- **Solutions of a System of Linear Inequalities:** Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true. The solution of a system of linear inequalities is shown as a shaded region in the xy-coordinate system that includes all the points whose ordered pairs make the inequalities true.
- **How to solve a system of linear inequalities by graphing.**
  1. Graph the first inequality.
     Graph the boundary line.
     Shade in the side of the boundary line where the inequality is true.
  2. On the same grid, graph the second inequality.
     Graph the boundary line.
     Shade in the side of that boundary line where the inequality is true.
  3. The solution is the region where the shading overlaps.
  4. Check by choosing a test point.

Glossary

system of linear inequalities

Two or more linear inequalities grouped together form a system of linear inequalities.

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CHAPTER OVERVIEW

8: Sequences, Series, and the Binomial Theorem

8.1: Introduction to Sequences and Series
8.2: Arithmetic Sequences and Series
8.3: Geometric Sequences and Series
8.4: Binomial Theorem

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8.1: Introduction to Sequences and Series

Learning Objectives

- Find any element of a sequence given a formula for its general term.
- Use sigma notation and expand corresponding series.
- Distinguish between a sequence and a series.
- Calculate the nth partial sum of sequence.

Sequences

A **sequence**[1] is a function whose domain is a set of consecutive natural numbers beginning with 1. For example, the following equation with domain $\{1, 2, 3, \ldots\}$defines an **infinite sequence**[2]:

$$a(n) = 5n - 3 \text{ or } a_n = 5n - 3$$

The elements in the range of this function are called terms of the sequence. It is common to define the nth term, or the **general term of a sequence**[3], using the subscripted notation a_n, which reads "a sub n." Terms can be found using substitution as follows:

$$\begin{aligned} General\ term: \quad & a_n = 5n - 3 \\ First\ term (n = 1): \quad & a_1 = 5(1) - 3 = 2 \\ Second\ term (n = 2): \quad & a_2 = 5(2) - 3 = 7 \\ Third\ term (n = 3): \quad & a_3 = 5(3) - 3 = 12 \\ Fourth\ term (n = 4): \quad & a_4 = 5(4) - 3 = 17 \\ Fifth\ term (n = 5): \quad & a_5 = 5(5) - 3 = 22 \\ & \vdots \end{aligned}$$

This produces an ordered list,

$$2, 7, 12, 17, 22, \ldots$$

The ellipsis $(\ldots)$ indicates that this sequence continues forever. Unlike a set, order matters. If the domain of a sequence consists of natural numbers that end, such as $\{1, 2, 3, \ldots, k\}$, then it is called a **finite sequence**[4].

Example 8.1.1:

Given the general term of a sequence, find the first 5 terms as well as the 100th term: $a_n = \frac{n(n-1)}{2}$.

Solution:

To find the first 5 terms, substitute $1, 2, 3, 4$, and 5 for n and then simplify.

$$\begin{aligned} a_1 &= \frac{1(1-1)}{2} = \frac{1(0)}{2} = \frac{0}{2} = 0 \\ a_2 &= \frac{2(2-1)}{2} = \frac{2(1)}{2} = \frac{2}{2} = 1 \\ a_3 &= \frac{3(3-1)}{2} = \frac{3(2)}{2} = \frac{6}{2} = 3 \\ a_4 &= \frac{4(4-1)}{2} = \frac{4(3)}{2} = \frac{12}{2} = 6 \\ a_5 &= \frac{5(5-1)}{2} = \frac{5(4)}{2} = \frac{20}{2} = 10 \end{aligned}$$

Use $n = 100$ to determine the 100th term in the sequence.

$$a_{100} = \frac{100(100-1)}{2} = \frac{100(99)}{2} = \frac{9{,}900}{2} = 4,950$$

Answer:

First five terms: $0, 1, 3, 6, 10$, $a_{100} = 4,950$

Sometimes the general term of a sequence will alternate in sign and have a variable other than n.

Example 8.1.2:

Find the first 5 terms of the sequence: $a_n = (-1)^n x^{n+1}$.

Solution:

Here we take care to replace n with the first 5 natural numbers and not x.

$$a_1 = (-1)^1 x^{1+1} = -x^2$$
$$a_2 = (-1)^2 x^{2+1} = x^3$$
$$a_3 = (-1)^3 x^{3+1} = -x^4$$
$$a_4 = (-1)^4 x^{4+1} = x^5$$
$$a_5 = (-1)^5 x^{5+1} = -x^6$$

Answer:

$-x^2, x^3, -x^4, x^5, -x^6$

Exercise 8.1.1

Find the first 5 terms of the sequence: $a_n = (-1)^{n+1} 2^n$.

Answer

$2, -4, 8, -16, 32.$

http://www.youtube.com/v/uuQ3jYL-g_I

One interesting example is the Fibonacci sequence. The first two numbers in the Fibonacci sequence are 1, and each successive term is the sum of the previous two. Therefore, the general term is expressed in terms of the previous two as follows:

$$F_n = F_{n-2} + F_{n-1}$$

Here $F_1 = 1, F_2 = 1$, and $n > 2$. A formula that describes a sequence in terms of its previous terms is called a **recurrence relation**[5].

Example 8.1.3:

Find the first 7 Fibonacci numbers.

Solution:

Given that $F_1 = 1$ and $F_2 = 1$, use the recurrence relation $F_n = F_{n-2} + F_{n-1}$ where n is an integer starting with $n = 3$ to find the next 5 terms:

$$F_3 = F_{3-2} + F_{3-1} = F_1 + F_2 = 1 + 1 = 2$$
$$F_4 = F_{4-2} + F_{4-1} = F_2 + F_3 = 1 + 2 = 3$$
$$F_5 = F_{5-2} + F_{5-1} = F_3 + F_4 = 2 + 3 = 5$$
$$F_6 = F_{6-2} + F_{6-1} = F_4 + F_5 = 3 + 5 = 8$$
$$F_7 = F_{7-2} + F_{7-1} = F_5 + F_6 = 5 + 8 = 13$$

Answer:

$1, 1, 2, 3, 5, 8, 13$



***Figure 9.1.1**: Leonardo Fibonacci (1170–1250)*

Fibonacci numbers appear in applications ranging from art to computer science and biology. The beauty of this sequence can be visualized by constructing a Fibonacci spiral. Consider a tiling of squares where each side has a length that matches each Fibonacci number:

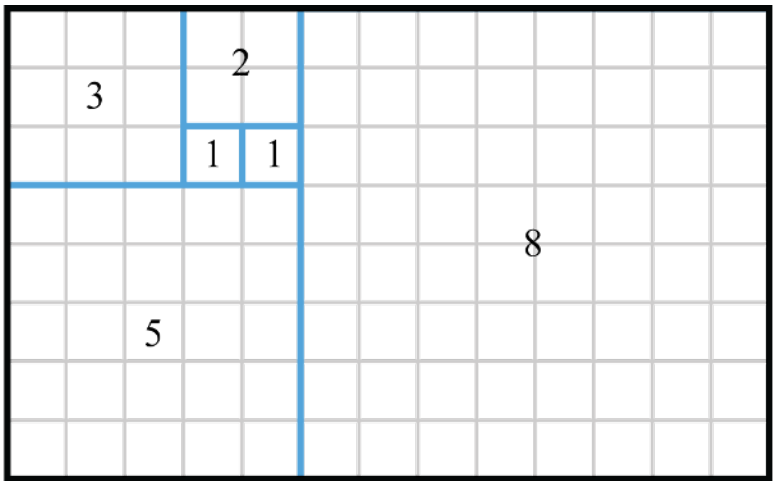


Figure 9.1.2

Connecting the opposite corners of the squares with an arc produces a special spiral shape.

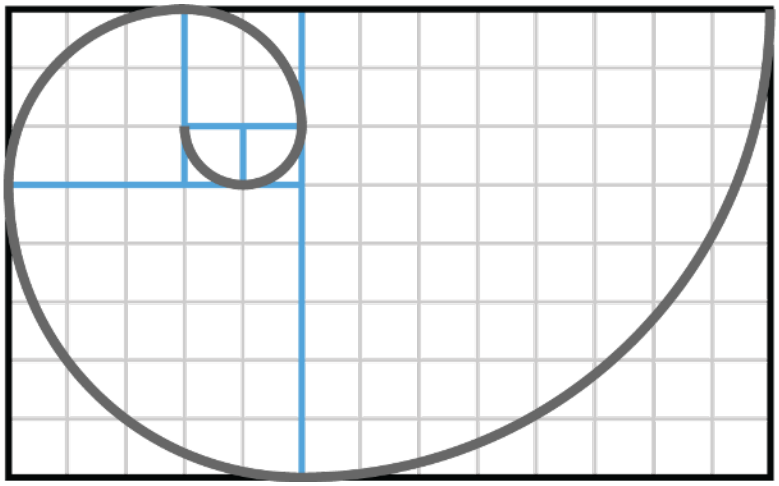


Figure 9.1.3

This shape is called the Fibonacci spiral and approximates many spiral shapes found in nature.

Series

A **series**[6] is the sum of the terms of a sequence. The sum of the terms of an infinite sequence results in an **infinite series**[7], denoted S_∞. The sum of the first n terms in a sequence is called a **partial sum**[8], denoted S_n. For example, given the sequence of positive odd integers $1, 3, 5, \ldots$ we can write:

$$S_\infty = 1+3+5+7+9+\ldots \quad \textit{Infinite series}$$
$$S_5 = 1+3+5+7+9 = 25 \quad \textit{5th partial sum}$$

Example 8.1.4:

Determine the 3rd and 5th partial sums of the sequence: $3, -6, 12, -24, 48, \ldots$

Solution:

$$S_3 = 3+(-6)+12 = 9$$
$$S_5 = 3+(-6)+12+(-24)+48 = 33$$

Answer:

$S_3 = 9; S_5 = 33$

If the general term is known, then we can express a series using **sigma**[9] (or **summation**[10]) **notation**:

$$S_\infty = \sum_{n=1}^{\infty} n^2 = 1^2+2^2+3^2+\ldots \quad \textit{Infinite series}$$
$$S_3 = \sum_{n=1}^{3} n^2 = 1^2+2^2+3^2 \quad \textit{3rd partial sum}$$

The symbol Σ (upper case Greek letter sigma) is used to indicate a series. The expressions above and below indicate the range of the **index of summation**[11], in this case represented by n. The lower number indicates the starting integer and the upper value indicates the ending integer. The nth partial sum S_n can be expressed using sigma notation as follows:

$$S_n = \textstyle\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

This is read, "the sum of a_k as k goes from 1 to n." Replace n with ∞ to indicate an infinite sum.

Example 8.1.5:

Evaluate: $\sum_{k=1}^{5}(-3)^{n-1}$.

$$\begin{aligned}\sum_{k=1}^{5}(-3)^{k-1} &= (-3)^{1-1}+(-3)^{2-1}+(-3)^{3-1}+(-3)^{4-1}+(-3)^{5-1}\\ &= (-3)^0+(-3)^1+(-3)^2+(-3)^3+(-3)^4\\ &= 1-3+9-27+81\\ &= 61\end{aligned}$$

Answer:

61

When working with sigma notation, the index does not always start at 1.

Example 8.1.1:

Evaluate: $\sum_{k=2}^{5}(-1)^k(3k)$

Solution:

Here the index is expressed using the variable k, which ranges from 2 to 5.

Exercise 8.1.2

Evaluate: $\sum_{n=1}^{5}(15-9n)$.

Answer

-60

http://www.youtube.com/v/aZ3sPd8N1TU

Infinity is used as the upper bound of a sum to indicate an infinite series.

Example 8.1.7:

Write in expanded form: $\sum_{i=0}^{\infty}\frac{n}{n+1}$.

Solution:

In this case we begin with $n=0$ and add three dots to indicate that this series continues forever.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{n}{n+1} &= \frac{0}{0+1}+\frac{1}{1+1}+\frac{2}{2+1}+\frac{3}{3+1}+\cdots\\ &= \frac{0}{1}+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots\\ &= 0+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots\end{aligned}$$

Answer:

$0+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$

When expanding a series, take care to replace only the variable indicated by the index.

Example 8.1.8:

Write in expanded form: $\sum_{i=1}^{\infty}(-1)^{i-1}x^{2i}$.

Solution:

$$\begin{aligned}\sum_{i=1}^{\infty}(-1)^{i-1}x^{2i} &= (-1)^{1-1}x^{2(1)}+(-1)^{2-1}x^{2(2)}+(-1)^{3-1}x^{2(3)}+\cdots\\ &= (-1)^{0}x^{2(1)}+(-1)^{1}x^{2(2)}+(-1)^{2}x^{2(3)}+\cdots\\ &= x^2-x^4+x^6-\cdots\end{aligned}$$

Answer:

$x^2-x^4+x^6-\cdots$

Key Takeaways

- A sequence is a function whose domain consists of a set of natural numbers beginning with 1. In addition, a sequence can be thought of as an ordered list.
- Formulas are often used to describe the nth term, or general term, of a sequence using the subscripted notation a_n.

- A series is the sum of the terms in a sequence. The sum of the first n terms is called the nth partial sum and is denoted S_n.
- Use sigma notation to denote summations in a compact manner. The nth partial sum, using sigma notation, can be written $S_n = \sum_{k=1}^{n} a_k$. The symbol Σ denotes a summation where the expression below indicates that the index k starts at 1 and iterates through the natural numbers ending with the value n above.

Footnotes

[1]A function whose domain is a set of consecutive natural numbers starting with 1.

[2]A sequence whose domain is the set of natural numbers $\{1, 2, 3, \ldots\}$.

[3]An equation that defines the nth term of a sequence commonly denoted using subscripts a_n.

[4]A sequence whose domain is $\{1, 2, 3, \ldots, k\}$where k is a natural number.

[5]A formula that uses previous terms of a sequence to describe subsequent terms.

[6]The sum of the terms of a sequence.

[7]The sum of the terms of an infinite sequence denoted S_∞.

[8]The sum of the first n terms in a sequence denoted S_n.

[9]A sum denoted using the symbol Σ (upper case Greek letter sigma).

[10]Used when referring to sigma notation.

[11]The variable used in sigma notation to indicate the lower and upper bounds of the summation.

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8.2: Arithmetic Sequences and Series

Learning Objectives

- Identify the common difference of an arithmetic sequence.
- Find a formula for the general term of an arithmetic sequence.
- Calculate the nth partial sum of an arithmetic sequence.

Arithmetic Sequences

An **arithmetic sequence**[12], or **arithmetic progression**[13], is a sequence of numbers where each successive number is the sum of the previous number and some constant d.

$$a_n = a_{n-1} + d \quad \textit{Arithmetic sequence}$$

And because $a_n - a_{n-1} = d$, the constant d is called the **common difference**[14]. For example, the sequence of positive odd integers is an arithmetic sequence,

$$1, 3, 5, 7, 9, \ldots$$

Here $a_1 = 1$ and the difference between any two successive terms is 2. We can construct the general term $a_n = a_{n-1} + 2$ where,

$$\begin{aligned} a_1 &= 1 \\ a_2 &= a_1 + 2 = 1 + 2 = 3 \\ a_3 &= a_2 + 2 = 3 + 2 = 5 \\ a_4 &= a_3 + 2 = 5 + 2 = 7 \\ a_5 &= a_4 + 2 = 7 + 2 = 9 \\ &\vdots \end{aligned}$$

In general, given the first term a_1 of an arithmetic sequence and its common difference d, we can write the following:

$$\begin{aligned} a_2 &= a_1 + d \\ a_3 &= a_2 + d = (a_1 + d) + d = a_1 + 2d \\ a_4 &= a_3 + d = (a_1 + 2d) + d = a_1 + 3d \\ a_5 &= a_4 + d = (a_1 + 3d) + d = a_1 + 4d \\ &\vdots \end{aligned}$$

From this we see that any arithmetic sequence can be written in terms of its first element, common difference, and index as follows:

$$a_n = a_1 + (n-1)d \quad \textit{Arithmetic Sequence}$$

In fact, any general term that is linear in n defines an arithmetic sequence.

Example 8.2.1:

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 100th term: $7, 10, 13, 16, 19, \ldots$

Solution

Begin by finding the common difference,

$$d = 10 - 7 = 3$$

Note that the difference between any two successive terms is 3. The sequence is indeed an arithmetic progression where $a_1 = 7$ and $d = 3$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 7 + (n-1) \cdot 3 \\ &= 7 + 3n - 3 \\ &= 3n + 4 \end{aligned}$$

Therefore, we can write the general term $a_n = 3n + 4$. Take a minute to verify that this equation describes the given sequence. Use this equation to find the 100th term:

$$a_{100} = 3(100) + 4 = 304$$

Answer:

$a_n = 3n + 4; a_{100} = 304$

The common difference of an arithmetic sequence may be negative.

Example 8.2.2:

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 75th term: $6, 4, 2, 0, -2, \ldots$

Solution

Begin by finding the common difference,

$$d = 4 - 6 = -2$$

Next find the formula for the general term, here $a_1 = 6$ and $d = -2$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 6 + (n-1) \cdot (-2) \\ &= 6 - 2n + 2 \\ &= 8 - 2n \end{aligned}$$

Therefore, $a_n = 8 - 2n$ and the 75th term can be calculated as follows:

$$\begin{aligned} a_{75} &= 8 - 2(75) \\ &= 8 - 150 \\ &= -142 \end{aligned}$$

Answer:

$a_n = 8 - 2n; a_{100} = -142$

The terms between given terms of an arithmetic sequence are called **arithmetic means**[15].

Example 8.2.3:

Find all terms in between $a_1 = -8$ and $a_7 = 10$ of an arithmetic sequence. In other words, find all arithmetic means between the 1st and 7th terms.

Solution

Begin by finding the common difference d. In this case, we are given the first and seventh term:

$$\begin{aligned} a_n &= a_1 + (n-1)d \quad \textit{Use } n = 7. \\ a_7 &= a_1 + (7-1)d \\ a_7 &= a_1 + 6d \end{aligned}$$

Substitute $a_1 = -8$ and $a_7 = 10$ into the above equation and then solve for the common difference d.

$$\begin{aligned} 10 &= -8 + 6d \\ 18 &= 6d \\ 3 &= d \end{aligned}$$

Next, use the first term $a_1 = -8$ and the common difference $d = 3$ to find an equation for the nth term of the sequence.

$$\begin{aligned} a_n &= -8 + (n-1) \cdot 3 \\ &= -8 + 3n - 3 \\ &= -11 + 3n \end{aligned}$$

With $a_n = 3n - 11$, where n is a positive integer, find the missing terms.

$$\left.\begin{aligned} a_1 &= 3(1)-11 = 3-11 = -8 \\ a_2 &= 3(2)-11 = 6-11 = -5 \\ a_3 &= 3(3)-11 = 9-11 = -2 \\ a_4 &= 3(4)-11 = 12-11 = 1 \\ a_5 &= 3(5)-11 = 15-11 = 4 \\ a_6 &= 3(6)-11 = 18-11 = 7 \\ a_7 &= 3(7)-11 = 21-111 = 10 \end{aligned}\right\} \textit{arithmetic means}$$

Answer:

$-5, -2, 1, 4, 7$

In some cases, the first term of an arithmetic sequence may not be given.

Example 8.2.4:

Find the general term of an arithmetic sequence where $a_3 = -1$ and $a_{10} = 48$.

Solution

To determine a formula for the general term we need a_1 and d. A linear system with these as variables can be formed using the given information and $a_n = a_1 + (n-1)d$:

$$\begin{cases} a_3 = a_1 + (3-1)d \\ a_{10} = a_1 + (10-1)d \end{cases} \Rightarrow \begin{cases} -1 = a_1 + 2d & Use\ a_3 = -1. \\ 48 = a_1 + 9d & Use\ a_{10} = 48. \end{cases}$$

Eliminate a_1 by multiplying the first equation by -1 and add the result to the second equation.

$$\begin{cases} -1 = a_1 + 2d \\ 48 = a_1 + 9d \end{cases} \overset{x(-1)}{\Longrightarrow} + \begin{cases} 1 = -a_1 - 2d \\ 48 = a_1 + 9d \end{cases}$$

$$\begin{aligned} 49 &= 7d \\ 7 &= d \end{aligned}$$

Substitute $d = 7$ into $-1 = a_1 + 2d$ to find a_1.

$$\begin{aligned} -1 &= a_1 + 2(7) \\ -1 &= a_1 + 14 \\ -15 &= a_1 \end{aligned}$$

Next, use the first term $a_1 = -15$ and the common difference $d = 7$ to find a formula for the general term.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= -15 + (n-1)\cdot 7 \\ &= -15 + 7n - 7 \\ &= -22 + 7n \end{aligned}$$

Answer:

$a_n = 7n - 22$

Exercise 8.2.1

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 100th term: $\frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$

Answer

$a_n = \frac{1}{2}n + 1; a_{100} = 51$

www.youtube.com/v/_ovjvVKtKpQ

Arithmetic Series

An **arithmetic series**[16] is the sum of the terms of an arithmetic sequence. For example, the sum of the first 5 terms of the sequence defined by $a_n = 2n-1$ follows:

$$\begin{aligned} S_5 &= \sum_{n=1}^{5}(2n-1) \\ &= [2(1)-1]+[2(2)-1]+[2(3)-1]+[2(4)-1]+[2(5)-1] \\ &= 1+3+5+7+9 \\ &= 25 \end{aligned}$$

Adding 5 positive odd integers, as we have done above, is managable. However, consider adding the first 100 positive odd integers. This would be very tedious. Therefore, we next develop a formula that can be used to calculate the sum of the first n terms, denoted S_n, of any arithmetic sequence. In general,

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + a_n$$

Writing this series in reverse we have,

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \ldots + a_1$$

And adding these two equations together, the terms involving d add to zero and we obtain n factors of $a_1 + a_n$:

$$\begin{aligned} 2S_n &= (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_n + a_1) \\ 2S_n &= n\,(a_1 + a_n) \end{aligned}$$

Dividing both sides by 2 leads us the formula for the nth partial sum of an **arithmetic sequence**[17]:

$$S_n = \tfrac{n(a_1+a_n)}{2}$$

Use this formula to calculate the sum of the first 100 terms of the sequence defined by $a_n = 2n-1$. Here $a_1 = 1$ and $a_{100} = 199$.

$$\begin{aligned} S_{100} &= \frac{100\,(a_1 + a_{100})}{2} \\ &= \frac{100(1+199)}{2} \\ &= 10,000 \end{aligned}$$

Example 8.2.5:

Find the sum of the first 50 terms of the given sequence: $4, 9, 14, 19, 24, \ldots$

Solution

Determine whether or not there is a common difference between the given terms.

$$d = 9 - 4 = 5$$

Note that the difference between any two successive terms is 5. The sequence is indeed an arithmetic progression and we can write

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 4 + (n-1)\cdot 5 \\ &= 4 + 5n - 5 \\ &= 5n - 1 \end{aligned}$$

Therefore, the general term is $a_n = 5n-1$. To calculate the 50th partial sum of this sequence we need the 1st and the 50th terms:

$$\begin{aligned} a_1 &= 4 \\ a_{50} &= 5(50) - 1 = 249 \end{aligned}$$

Next use the formula to determine the 50th partial sum of the given arithmetic sequence.

$$
\begin{aligned}
S_n &= \frac{n\left(a_1 + a_n\right)}{2} \\
S_{50} &= \frac{50 \cdot \left(a_1 + a_{50}\right)}{2} \\
&= \frac{50(4+249)}{2} \\
&= 25(253) \\
&= 6,325
\end{aligned}
$$

Answer:

$S_{50} = 6,325$

Example 8.2.6:

Evaluate: $\sum_{n=1}^{35}(10-4n)$.

Solution

In this case, we are asked to find the sum of the first 35 terms of an arithmetic sequence with general term $a_n = 10-4n$. Use this to determine the 1st and the 35th term.

$$
\begin{aligned}
a_1 &= 10-4(1) = 6 \\
a_{35} &= 10-4(35) = -130
\end{aligned}
$$

Next use the formula to determine the 35th partial sum.

$$
\begin{aligned}
S_n &= \frac{n\left(a_1 + a_n\right)}{2} \\
S_{35} &= \frac{35 \cdot \left(a_1 + a_{35}\right)}{2} \\
&= \frac{35[6+(-130)]}{2} \\
&= \frac{35(-124)}{2} \\
&= -2,170
\end{aligned}
$$

Answer:

$-2,170$

Example 8.2.7:

The first row of seating in an outdoor amphitheater contains 26 seats, the second row contains 28 seats, the third row contains 30 seats, and so on. If there are 18 rows, what is the total seating capacity of the theater?



Figure 8.2.1: Roman Theater

Solution

Begin by finding a formula that gives the number of seats in any row. Here the number of seats in each row forms a sequence:

$$26, 28, 30, \ldots$$

Note that the difference between any two successive terms is 2. The sequence is an arithmetic progression where $a_1 = 26$ and $d = 2$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 26 + (n-1)\cdot 2 \\ &= 26 + 2n - 2 \\ &= 2n + 24 \end{aligned}$$

Therefore, the number of seats in each row is given by $a_n = 2n + 24$. To calculate the total seating capacity of the 18 rows we need to calculate the 18th partial sum. To do this we need the 1st and the 18th terms:

$$\begin{aligned} a_1 &= 26 \\ a_{18} &= 2(18) + 24 = 60 \end{aligned}$$

Use this to calculate the 18th partial sum as follows:

$$\begin{aligned} S_n &= \frac{n\,(a_1 + a_n)}{2} \\ S_{18} &= \frac{18\cdot(a_1 + a_{18})}{2} \\ &= \frac{18(26+60)}{2} \\ &= 9(86) \\ &= 774 \end{aligned}$$

Answer:

There are 774 seats total.

Exercise 8.2.2

Find the sum of the first 60 terms of the given sequence: $5, 0, -5, -10, -15, \ldots$

Answer

$S_{60} = -8,550$

www.youtube.com/v/baYq2a_kBKo

Key Takeaways

- An arithmetic sequence is a sequence where the difference d between successive terms is constant.
- The general term of an arithmetic sequence can be written in terms of its first term a_1, common difference d, and index n as follows: $a_n = a_1 + (n-1)d$.
- An arithmetic series is the sum of the terms of an arithmetic sequence.
- The nth partial sum of an arithmetic sequence can be calculated using the first and last terms as follows: $S_n = \frac{n(a_1+a_n)}{2}$.

Footnotes

[12]A sequence of numbers where each successive number is the sum of the previous number and some constant d.

[13]Used when referring to an arithmetic sequence.

[14]The constant d that is obtained from subtracting any two successive terms of an arithmetic sequence; $a_n - a_{n-1} = d$.

[15]The terms between given terms of an arithmetic sequence.

[16]The sum of the terms of an arithmetic sequence.

[17]The sum of the first n terms of an arithmetic sequence given by the formula: $S_n = \frac{n(a_1+a_n)}{2}$.

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8.3: Geometric Sequences and Series

Learning Objectives

- Identify the common ratio of a geometric sequence.
- Find a formula for the general term of a geometric sequence.
- Calculate the nth partial sum of a geometric sequence.
- Calculate the sum of an infinite geometric series when it exists.

Geometric Sequences

A **geometric sequence**[18], or **geometric progression**[19], is a sequence of numbers where each successive number is the product of the previous number and some constant r.

$$a_n = ra_{n-1} \quad \textit{Geometric Sequence}$$

And because $\frac{a_n}{a_{n-1}} = r$, the constant factor r is called the **common ratio**[20]. For example, the following is a geometric sequence,

$$9, 27, 81, 243, 729 \ldots$$

Here $a_1 = 9$ and the ratio between any two successive terms is 3. We can construct the general term $a_n = 3a_{n-1}$ where,

$$\begin{aligned} a_1 &= 9 \\ a_2 &= 3a_1 = 3(9) = 27 \\ a_3 &= 3a_2 = 3(27) = 81 \\ a_4 &= 3a_3 = 3(81) = 243 \\ a_5 &= 3a_4 = 3(243) = 729 \\ &\vdots \end{aligned}$$

In general, given the first term a_1 and the common ratio r of a geometric sequence we can write the following:

$$\begin{aligned} a_2 &= ra_1 \\ a_3 &= ra_2 = r\left(a_1 r\right) = a_1 r^2 \\ a_4 &= ra_3 = r\left(a_1 r^2\right) = a_1 r^3 \\ a_5 &= ra_3 = r\left(a_1 r^3\right) = a_1 r^4 \\ &\vdots \end{aligned}$$

From this we see that any geometric sequence can be written in terms of its first element, its common ratio, and the index as follows:

$$a_n = a_1 r^{n-1} \quad \textit{Geometric Sequence}$$

In fact, any general term that is exponential in n is a geometric sequence.

Example 8.3.1:

Find an equation for the general term of the given geometric sequence and use it to calculate its 10th term: $3, 6, 12, 24, 48 \ldots$

Solution

Begin by finding the common ratio,

$$r = \frac{6}{3} = 2$$

Note that the ratio between any two successive terms is 2. The sequence is indeed a geometric progression where $a_1 = 3$ and $r = 2$.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ &= 3(2)^{n-1} \end{aligned}$$

Therefore, we can write the general term $a_n = 3(2)^{n-1}$ and the 10th term can be calculated as follows:

$$\begin{aligned} a_{10} &= 3(2)^{10-1} \\ &= 3(2)^9 \\ &= 1,536 \end{aligned}$$

Answer:

$a_n = 3(2)^{n-1}; a_{10} = 1,536$

The terms between given terms of a geometric sequence are called **geometric means**[21].

Example 8.3.2:

Find all terms between $a_1 = -5$ and $a_4 = -135$ of a geometric sequence. In other words, find all geometric means between the 1st and 4th terms.

Solution

Begin by finding the common ratio r. In this case, we are given the first and fourth terms:

$$\begin{aligned} a_n &= a_1 r^{n-1} \quad Use\ n = 4 \\ a_4 &= a_1 r^{4-1} \\ a_4 &= a_1 r^3 \end{aligned}$$

Substitute $a_1 = -5$ and $a_4 = -135$ into the above equation and then solve for r.

$$\begin{aligned} -135 &= -5r^3 \\ 27 &= r^3 \\ 3 &= r \end{aligned}$$

Next use the first term $a_1 = -5$ and the common ratio $r = 3$ to find an equation for the nth term of the sequence.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_n &= -5(3)^{n-1} \end{aligned}$$

Now we can use $a_n = -5(3)^{n-1}$ where n is a positive integer to determine the missing terms.

$$\left.\begin{aligned} a_1 &= -5(3)^{1-1} = -5 \cdot 3^0 = -5 \\ a_2 &= -5(3)^{2-1} = -5 \cdot 3^1 = -15 \\ a_3 &= -5(3)^{3-1} = -5 \cdot 3^2 = -45 \\ a_4 &= -5(3)^{4-1} = -5 \cdot 3^3 = -135 \end{aligned}\right\} geometric\ means$$

Answer:

$-15, -45$

The first term of a geometric sequence may not be given.

Example 8.3.3:

Find the general term of a geometric sequence where $a_2 = -2$ and $a_5 = \frac{2}{125}$.

Solution

To determine a formula for the general term we need a_1 and r. A nonlinear system with these as variables can be formed using the given information and $a_n = a_1 r^{n-1}$: :

$$\begin{cases} a_2 = a_1 r^{2-1} \\ a_5 = a_1 r^{5-1} \end{cases} \Longrightarrow \begin{cases} -2 = a_1 r & Use\ a_2 = -2. \\ \frac{2}{125} = a_1 r^4 & Use\ a_5 = \frac{2}{125}. \end{cases}$$

Solve for a_1 in the first equation,

$$\begin{aligned} -2 = a_1 r \quad &\Rightarrow \quad \frac{-2}{r} = a_1 \\ \frac{2}{125} &= a_1 r^4 \end{aligned}$$

Substitute $a_1 = \frac{-2}{r}$ into the second equation and solve for r.

$$\begin{aligned}\frac{2}{125} &= a_1 r^4 \\ \frac{2}{125} &= \left(\frac{-2}{r}\right) r^4 \\ \frac{2}{125} &= -2r^3 \\ -\frac{1}{125} &= r^3 \\ -\frac{1}{5} &= r\end{aligned}$$

Back substitute to find a_1:

$$\begin{aligned}a_1 &= \frac{-2}{r} \\ &= \frac{-2}{\left(-\frac{1}{5}\right)} \\ &= 10\end{aligned}$$

Therefore, $a_1 = 10$ and $r = -\frac{1}{5}$.

Answer:

$a_n = 10\left(-\frac{1}{5}\right)^{n-1}$

? Exercise 8.3.1

Find an equation for the general term of the given geometric sequence and use it to calculate its 6th term: $2, \frac{4}{3}, \frac{8}{9}, \ldots$

Answer

$a_n = 2\left(\frac{2}{3}\right)^{n-1}; a_6 = \frac{64}{243}$

www.youtube.com/v/IGPEl9vloLY

Geometric Series

A **geometric series**[22] is the sum of the terms of a geometric sequence. For example, the sum of the first 5 terms of the geometric sequence defined by $a_n = 3^{n+1}$ follows:

$$\begin{aligned}S_5 &= \sum_{n=1}^{5} 3^{n+1} \\ &= 3^{1+1} + 3^{2+1} + 3^{3+1} + 3^{4+1} + 3^{5+1} \\ &= 3^2 + 3^3 + 3^4 + 3^5 + 3^6 \\ &= 9 + 27 + 81 + 3^5 + 3^6 \\ &= 1,089\end{aligned}$$

Adding 5 positive integers is manageable. However, the task of adding a large number of terms is not. Therefore, we next develop a formula that can be used to calculate the sum of the first n terms of any geometric sequence. In general,

$$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1}$$

Multiplying both sides by r we can write,

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^n$$

Subtracting these two equations we then obtain,

$$\begin{aligned}S_n - rS_n &= a_1 - a_1 r^n \\ S_n(1-r) &= a_1\left(1 - r^n\right)\end{aligned}$$

Assuming $r \neq 1$ dividing both sides by $(1-r)$ leads us to the formula for the ***n*th partial sum of a geometric sequence**[23]:

$$S_n = \frac{a_1(1-r^n)}{1-r} (r \neq 1)$$

In other words, the nth partial sum of any geometric sequence can be calculated using the first term and the common ratio. For example, to calculate the sum of the first 15 terms of the geometric sequence defined by $a_n = 3^{n+1}$, use the formula with $a_1 = 9$ and $r = 3$.

$$\begin{aligned} S_{15} &= \frac{a_1\left(1-r^{15}\right)}{1-r} \\ &= \frac{9\cdot\left(1-3^{15}\right)}{1-3} \\ &= \frac{9(-14,348,906)}{-2} \\ &= 64,570,077 \end{aligned}$$

Example 8.3.4:

Find the sum of the first 10 terms of the given sequence: $4, -8, 16, -32, 64, \ldots$

Solution

Determine whether or not there is a common ratio between the given terms.

$$r = \frac{-8}{4} = -2$$

Note that the ratio between any two successive terms is -2; hence, the given sequence is a geometric sequence. Use $r = -2$ and the fact that $a_1 = 4$ to calculate the sum of the first 10 terms,

$$\begin{aligned} S_n &= \frac{a_1\left(1-r^n\right)}{1-r} \\ S_{10} &= \frac{4\left[1-(-2)^{10}\right]}{1-(-2)}] \\ &= \frac{4(1-1,024)}{1+2} \\ &= \frac{4(-1,023)}{3} \\ &= -1,364 \end{aligned}$$

Answer:

$S_{10} = -1,364$

Example 8.3.5:

Evaluate: $\sum_{n=1}^{6} 2(-5)^n$.

Solution

In this case, we are asked to find the sum of the first 6 terms of a geometric sequence with general term $a_n = 2(-5)^n$. Use this to determine the 1st term and the common ratio r:

$$a_1 = 2(-5)^1 = -10$$

To show that there is a common ratio we can use successive terms in general as follows:

$$\begin{aligned} r &= \frac{a_n}{a_{n-1}} \\ &= \frac{2(-5)^n}{2(-5)^{n-1}} \\ &= (-5)^{n-(n-1)} \\ &= (-5)^1 \\ &= -5 \end{aligned}$$

Use $a_1 = -10$ and $r = -5$ to calculate the 6th partial sum.

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}$$
$$S_6 = \frac{-10\left[1-(-5)^6\right]}{1-(-5)}$$
$$= \frac{-10(1-15,625)}{1+5}$$
$$= \frac{-10(-15,624)}{6}$$
$$= 26,040$$

Answer:

$26,040$

? Exercise 8.3.2

Find the sum of the first 9 terms of the given sequence: $-2, 1, -1/2, \ldots$

Answer

$S_9 = -\frac{171}{128}$

www.youtube.com/v/v-t3P95rWe8

If the common ratio r of an infinite geometric sequence is a fraction where $|r| < 1$ (that is $-1 < r < 1$), then the factor $\left(1-r^n\right)$ found in the formula for the nth partial sum tends toward 1 as n increases. For example, if $r = \frac{1}{10}$ and $n = 2, 4, 6$ we have,

$$1-\left(\tfrac{1}{10}\right)^2 = 1-0.01 = 0.99$$
$$1-\left(\tfrac{1}{10}\right)^4 = 1-0.0001 = 0.9999$$
$$1-\left(\tfrac{1}{10}\right)^6 = 1-0.00001 = 0.999999$$

Here we can see that this factor gets closer and closer to 1 for increasingly larger values of n. This illustrates the idea of a limit, an important concept used extensively in higher-level mathematics, which is expressed using the following notation:

$$\lim_{n\to\infty}\left(1-r^n\right) = 1 \text{ where } |r| < 1$$

This is read, "the limit of $\left(1-r^n\right)$ as n approaches infinity equals 1." While this gives a preview of what is to come in your continuing study of mathematics, at this point we are concerned with developing a formula for special infinite geometric series. Consider the nth partial sum of any geometric sequence,

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{a_1}{1-r}(1-r^n)$$

If $|r| < 1$ then the limit of the partial sums as n approaches infinity exists and we can write,

$$S_n = \frac{a_1}{1-r}(1-r^n) \quad n \xrightarrow{} \infty \quad S_\infty = \frac{a_1}{1-4}\cdot 1$$

Therefore, a **convergent geometric series**[24] is an infinite geometric series where $|r| < 1$; its sum can be calculated using the formula:

$$S_\infty = \frac{a_1}{1-r}$$

✓ Example 8.3.6:

Find the sum of the infinite geometric series: $\frac{3}{2} + \frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \ldots$

Solution

Determine the common ratio, Since the common ratio $r = \frac{1}{3}$ is a fraction between -1 and 1, this is a convergent geometric series. Use the first term $a_1 = \frac{3}{2}$ and the common ratio to calculate its sum

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{\frac{3}{2}}{1-\left(\frac{1}{3}\right)} \\ &= \frac{\frac{3}{3}}{\frac{2}{3}} \\ &= \frac{3}{2}\cdot\frac{3}{2} \\ &= \frac{9}{4} \end{aligned}$$

Answer:

$S_\infty = \frac{9}{4}$

Note

In the case of an infinite geometric series where $|r| \geq 1$, the series diverges and we say that there is no sum. For example, if $a_n = (5)^{n-1}$ then $r = 5$ and we have

$$S_\infty = \sum_{n=1}^{\infty}(5)^{n-1} = 1+5+25+\cdots$$

We can see that this sum grows without bound and has no sum.

Exercise 8.3.3

Find the sum of the infinite geometric series: $\sum_{n=1}^{\infty} -2\left(\frac{5}{9}\right)^{n-1}$

Answer

$-\frac{9}{2}$

www.youtube.com/v/KxsPVUyle_A

A repeating decimal can be written as an infinite geometric series whose common ratio is a power of $1/10$. Therefore, the formula for a convergent geometric series can be used to convert a repeating decimal into a fraction.

Example 8.3.7:

Write as a fraction: $1.181818\ldots$

Solution

Begin by identifying the repeating digits to the right of the decimal and rewrite it as a geometric progression.

$$\begin{aligned} 0.181818\ldots &= 0.18+0.0018+0.000018+\ldots \\ &= \frac{18}{100}+\frac{18}{10,000}+\frac{18}{1,000,000}+\ldots \end{aligned}$$

In this form we can determine the common ratio,

$$\begin{aligned} r &= \frac{\frac{18}{10,000}}{\frac{18}{100}} \\ &= \frac{18}{10,000}\times\frac{100}{18} \\ &= \frac{1}{100} \end{aligned}$$

Note that the ratio between any two successive terms is $\frac{1}{100}$. Use this and the fact that $a_1 = \frac{18}{100}$ to calculate the infinite sum:

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{\frac{18}{100}}{1-\left(\frac{1}{100}\right)} \\ &= \frac{\frac{18}{100}}{\frac{90}{100}} \\ &= \frac{18}{100} \cdot \frac{100}{99} \\ &= \frac{2}{11} \end{aligned}$$

Therefore, $0.181818\ldots = \frac{2}{11}$ and we have,

$$1.181818\ldots = 1 + \tfrac{2}{11} = 1\tfrac{2}{11}$$

Answer:

$1\frac{2}{11}$

Example 8.3.8:

A certain ball bounces back to two-thirds of the height it fell from. If this ball is initially dropped from 27 feet, approximate the total distance the ball travels.

Solution

We can calculate the height of each successive bounce:

$27 \cdot \frac{2}{3} = 18$ feet *Height of the first bounce*

$18 \cdot \frac{2}{3} = 12$ feet *Height of the second bounce*

$12 \cdot \frac{2}{3} = 8$ feet *Height of the third bounce*

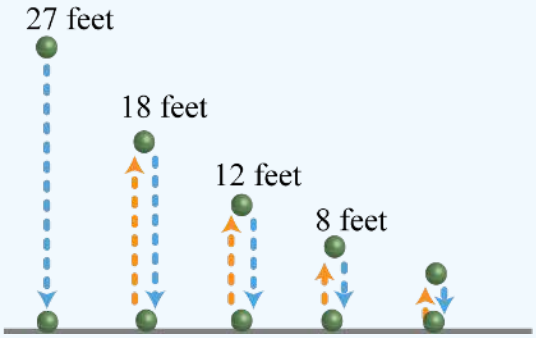


Figure 8.3.1

The total distance that the ball travels is the sum of the distances the ball is falling and the distances the ball is rising. The distances the ball falls forms a geometric series,

$27 + 18 + 12 + \ldots$ *Distance the ball is falling*

where $a_1 = 27$ and $r = \frac{2}{3}$. Because r is a fraction between -1 and 1, this sum can be calculated as follows:

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{27}{1-\frac{2}{3}} \\ &= \frac{27}{\frac{1}{3}} \\ &= 81 \end{aligned}$$

Therefore, the ball is falling a total distance of 81 feet. The distances the ball rises forms a geometric series,

$$18+12+8+\cdots \quad \textit{Distance the ball is rising}$$

where $a_1=18$ and $r=\frac{2}{3}$. Calculate this sum in a similar manner:

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{18}{1-\frac{2}{3}} \\ &= \frac{18}{\frac{1}{3}} \\ &= 54 \end{aligned}$$

Therefore, the ball is rising a total distance of 54 feet. Approximate the total distance traveled by adding the total rising and falling distances:

$$81+54=135 \text{ feet}$$

Answer:

135 feet

Key Takeaways

- A geometric sequence is a sequence where the ratio r between successive terms is constant.
- The general term of a geometric sequence can be written in terms of its first term a_1, common ratio r, and index n as follows: $a_n=a_1r^{n-1}$.
- A geometric series is the sum of the terms of a geometric sequence.
- The nth partial sum of a geometric sequence can be calculated using the first term a_1 and common ratio r as follows: $S_n=\frac{a_1(1-r^n)}{1-r}$.
- The infinite sum of a geometric sequence can be calculated if the common ratio is a fraction between -1 and 1 (that is $|r|<1$) as follows: $S_\infty=\frac{a_1}{1-r}$. If $|r|\geq 1$, then no sum exists.

Footnotes

[18]A sequence of numbers where each successive number is the product of the previous number and some constant r.

[19]Used when referring to a geometric sequence.

[20]The constant r that is obtained from dividing any two successive terms of a geometric sequence; $\frac{a_n}{a_{n-1}}=r$.

[21]The terms between given terms of a geometric sequence.

[22]The sum of the terms of a geometric sequence.

[23]The sum of the first n terms of a geometric sequence, given by the formula: $S_n=\frac{a_1(1-r^n)}{1-r}, r\neq 1$.

[24]An infinite geometric series where $|r|<1$ whose sum is given by the formula:$S_\infty=\frac{a_1}{1-r}$.

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8.4: Binomial Theorem

Learning Objectives

- Evaluate expressions involving factorials.
- Calculate binomial coefficients.
- Expand powers of binomials using the binomial theorem.

Factorials and the Binomial Coefficient

We begin by defining the *factorial*[25] of a natural number n, denoted $n!$, as the product of all natural numbers less than or equal to n.

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

For example,

$$\begin{aligned} 7! &= 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5,040 && \textit{Seven factorial} \\ 5! &= 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120 && \textit{Five factorial} \\ 3! &= 3\cdot 2\cdot 1 = 6 && \textit{Three factorial} \\ 1! &= 1 = 1 && \textit{One factorial} \end{aligned}$$

We define *zero factorial*[26] to be equal to 1,

$$0! = 1 \quad \textit{Zero factorial}$$

The factorial of a negative number is not defined.

On most modern calculators you will find a factorial function. Some calculators do not provide a button dedicated to it. However, it usually can be found in the menu system if one is provided.

The factorial can also be expressed using the following recurrence relation,

$$n! = n(n-1)!$$

For example, the factorial of 8 can be expressed as the product of 8 and 7!:

$$\begin{aligned} 8! &= 8\cdot 7! \\ &= 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 \\ &= 40,320 \end{aligned}$$

When working with ratios involving factorials, it is often the case that many of the factors cancel.

Example 8.4.1

Evaluate: $\frac{12!}{6!}$.

Solution

$$\begin{aligned} \frac{12!}{6!} &= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} \\ &= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot \cancel{6!}}{\cancel{6!}} \\ &= 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7 \\ &= 665,280 \end{aligned}$$

Answer

$665,280$

The *binomial coefficient*[27], denoted ${}_nC_k = \binom{n}{k}$, is read "n choose k" and is given by the following formula:

$$ {}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!} $$

This formula is very important in a branch of mathematics called combinatorics. It gives the number of ways k elements can be chosen from a set of n elements where order does not matter. In this section, we are concerned with the ability to calculate this quantity.

Example 8.4.2

Calculate $\binom{7}{3}$.

Solution

Use the formula for the binomial coefficient,

$$ \binom{n}{k} = \frac{n!}{k!(n-k)!} $$

where $n = 7$ and $k = 3$. After substituting, look for factors to cancel.

$$ \begin{aligned} \binom{7}{3} &= \frac{7!}{3!(7-3)!} \\ &= \frac{7!}{3!4!} \\ &= \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{3!\ \cancel{4!}} \\ &= \frac{210}{6} \\ &= 35 \end{aligned} $$

Answer:

35

Check the menu system of your calculator for a function that calculates this quantity. Look for the notation ${}_nC_k$ in the probability subsection.

Exercise 8.4.1

Calculate $\binom{8}{5}$.

Answer

56

www.youtube.com/v/Rpb8KD1HQGc

Consider the following binomial raised to the 3rd power in its expanded form:

$$ (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 $$

Compare it to the following calculations,

$$\binom{3}{0} = \frac{3!}{0!(3-0)!} = \frac{3!}{1 \cdot 3!} = 1$$
$$\binom{3}{1} = d\frac{3!}{1!(3-1)!} = \frac{3 \cdot 2!}{1 \cdot 2!} = 3$$
$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3 \cdot 2!}{2!} = 3$$
$$\binom{3}{3} = \frac{3!}{3!(3-3)!} = \frac{3!}{3!0!} = 1$$

Notice that there appears to be a connection between these calculations and the coefficients of the expanded binomial. This observation is generalized in the next section.

Binomial Theorem

Consider expanding $(x+2)^5$:

$$(x+2)^5 = (x+2)(x+2)(x+2)(x+2)(x+2)$$

One quickly realizes that this is a very tedious calculation involving multiple applications of the distributive property. The **binomial theorem**[28] provides a method of expanding binomials raised to powers without directly multiplying each factor:

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x^1 y^{n-1}$$

More compactly we can write,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad \textit{Binomial theorem}$$

Example 8.4.3

Expand using the binomial theorem: $(x+2)^5$.

Solution

Use the binomial theorem where $n = 5$ and $y = 2$.

$$(x+2)^5 = \binom{5}{0} x^5 2^0 + \binom{5}{1} x^4 2^1 + \binom{5}{2} x^3 2^2 + \binom{5}{3} x^2 2^3 + \binom{5}{4} x^1 2^4$$

Sometimes it is helpful to identify the pattern that results from applying the binomial theorem. Notice that powers of the variable x start at 5 and decrease to zero. The powers of the constant term start at 0 and increase to 5. The binomial coefficients can be calculated off to the side and are left to the reader as an exercise.

$$\begin{aligned}(x+2)^5 &= \binom{5}{0} x^5 2^0 + \binom{5}{1} x^4 2^1 + \binom{5}{2} x^3 2^2 + \binom{5}{3} x^2 2^3 + \binom{5}{4} x^1 2^4 \\ &= 1x^5 \times 1 + 5x^4 \times 2 + 10x^3 \times 4 + 10x^2 \times 8 + 5x^1 \times 16 + 1 \times 1 \\ &= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32\end{aligned}$$

Answer

$x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

The binomial may have negative terms, in which case we will obtain an alternating series.

Example 8.4.4

Expand using the binomial theorem: $(u-2v)^4$.

Solution

Use the binomial theorem where $n=4, x=u$, and $y=-2v$ and then simplify each term.

$$\begin{aligned}(u-2v)^4 &= \binom{4}{0}u^4(-2v)^0+\binom{4}{1}u^3(-2v)^1+\binom{4}{2}u^2(-2v)^2+\binom{4}{3}u^1(-2v)^3+\binom{4}{4}u^0(-2v)^4\\ &= 1\times u^4\times 1+4u^3(-2v)+6u^2\left(4v^2\right)+4u\left(-8v^3\right)+16v^4\\ &= u^4-8u^3v+24u^2v^2-32uv^3+16v^4\end{aligned}$$

Answer

$u^4-8u^3v+24u^2v^2-32uv^3+16v^4$

Exercise 8.4.2

Expand using the binomial theorem: $\left(a^2-3\right)^4$

Answer

$a^8-12a^6+54a^4-108a^2+81$

www.youtube.com/v/wICbqmoa4T4

Next we study the coefficients of the expansions of $(x+y)^n$ starting with $n=0$:

$$\begin{aligned}(x+y)^0 &= 1\\ (x+y)^1 &= x+y\\ (x+y)^2 &= x+y\\ (x+y)^3 &= x^2+2xy+y^2\\ (x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\ (x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4\end{aligned}$$

Write the coefficients in a triangular array and note that each number below is the sum of the two numbers above it, always leaving a 1 on either end.

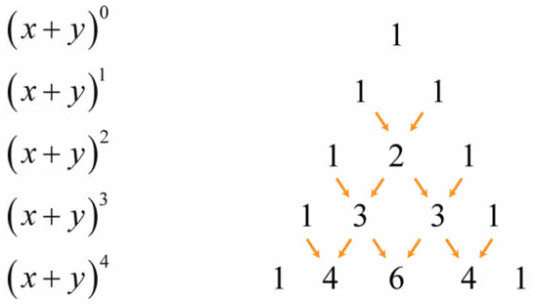


Figure 8.4.1

This is **Pascal's triangle**[29]; it provides a quick method for calculating the binomial coefficients. Use this in conjunction with the binomial theorem to streamline the process of expanding binomials raised to powers. For example, to expand $(x-1)^6$ we would need two more rows of Pascal's triangle,

$(x+y)^0$ 1

$(x+y)^1$ 1 1

$(x+y)^2$ 1 2 1

$(x+y)^3$ 1 3 3 1

$(x+y)^4$ 1 4 6 4 1

$(x+y)^5$ 1 5 10 10 5 1

$(x+y)^6$ 1 6 15 20 15 6 1

Figure 8.4.2

The binomial coefficients that we need are in blue. Use these numbers and the binomial theorem to quickly expand $(x-1)^6$ as follows:

$$\begin{aligned}(x-1)^6 &= 1x^6(-1)^0+6x^5(-1)^1+15x^4(-1)^2+20x^3(-1)^3+15x^2(-1)^4+6x(-1)^5\\ &= x^6-6x^5+15x^4-20x^3+15x^2-6x+1\end{aligned}$$

Example 8.4.5

Expand using the binomial theorem and Pascal's triangle: $(2x-5)^4$.

Solution

From Pascal's triangle we can see that when $n=4$ the binomial coefficients are $1, 4, 6, 4$, and 1.Use these numbers and the binomial theorem as follows:

$$\begin{aligned}(2x-5)^4 &= 1(2x)^4(-5)^0+4(2x)^3(-5)^1+6(2x)^2(-5)^2+4(2x)^1(-5)^3+(2x)^0(-5)^4\\ &= 16x^4\cdot 1+4\cdot 8x^3(-5)+6\cdot 4x^2\cdot 25+4\cdot 2x(-125)+1\cdot 625\\ &= 16x^4-160x^3+600x^2-1,000x+625\end{aligned}$$

Answer:

$16x^4-160x^3+600x^2-1,000x+625$

Key Takeaways

- To calculate the factorial of a natural number, multiply that number by all natural numbers less than it: $5!=5\cdot 4\cdot 3\cdot 2\cdot 1=120$. Remember that we have defined $0!=1$.
- The binomial coefficients are the integers calculated using the formula:

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}.$$

- The binomial theorem provides a method for expanding binomials raised to powers without directly multiplying each factor:

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

- Use Pascal's triangle to quickly determine the binomial coefficients.

Footnotes

[25]The product of all natural numbers less than or equal to a given natural number, denoted $n!$.

[26]The factorial of zero is defined to be equal to $1; 0!=1$.

[27] An integer that is calculated using the formula: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

[28] Describes the algebraic expansion of binomials raised to powers: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

[29] A triangular array of numbers that correspond to the binomial coefficients.

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CHAPTER OVERVIEW

9: The Unit Circle - Sine and Cosine Functions

The trigonometric functions are functions of an angle. and relate the angles of a triangle to the lengths of its sides. They are important in the study of triangles and modeling periodic phenomena, among many other applications.

9.1: Angles
9.2: Right Triangle Trigonometry
9.3: Unit Circle
9.4: The Other Trigonometric Functions

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9.1: Angles

Learning Objectives

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

Drawing Angles in Standard Position

Properly defining an angle first requires that we define a *ray*. A ray consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its *endpoint* and any other point on it. The ray in Figure 9.1.1 can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

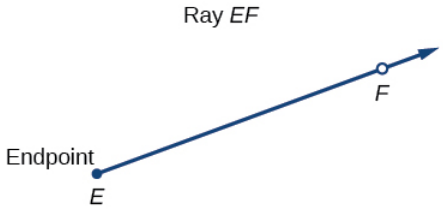


Figure 9.1.1

An *angle* is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle in Figure 9.1.2 is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbol form $\angle DEF$.

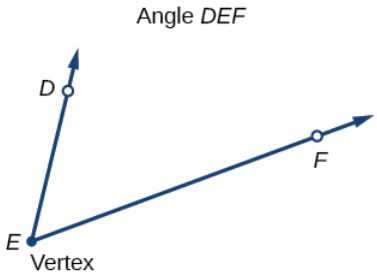


Figure 9.1.2

Greek letters are often used as variables for the measure of an angle. Table 9.1.1 is a list of Greek letters commonly used to represent angles, and a sample angle is shown in Figure 9.1.3.

Table 9.1.1

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

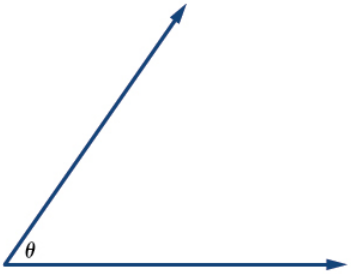


***Figure* 9.1.3**: *Angle theta, shown as* $\angle\theta$

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in Figure 9.1.4.

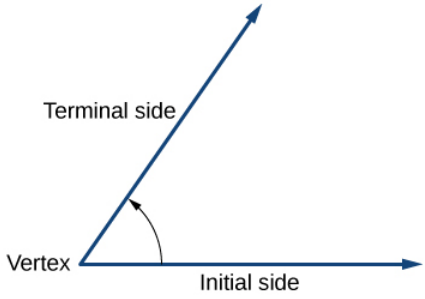


Figure 9.1.4

As we discussed at the beginning of the section, there are many applications for angles, but in order to use them correctly, we must be able to measure them. The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol °. For example, 90 degrees = 90°.

To formalize our work, we will begin by drawing angles on an *x*-*y* coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive *x*-axis. See Figure 9.1.5.

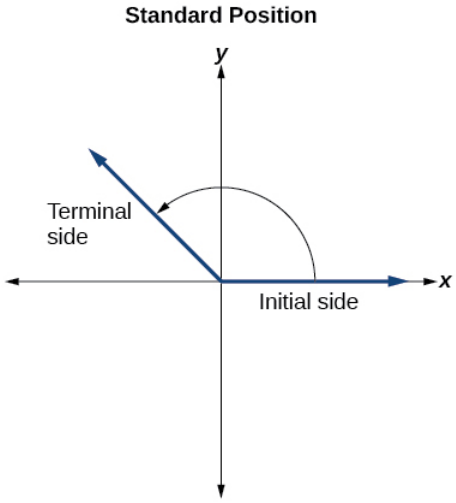


Figure 9.1.5

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

Drawing an angle in standard position always starts the same way—draw the initial side along the positive *x*-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90^\circ}{360^\circ} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive *x*-axis. To draw a 360° angle, we calculate that $\frac{360^\circ}{360^\circ} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive *x*-axis. In this case, the initial side and the terminal side overlap. See Figure 9.1.6.

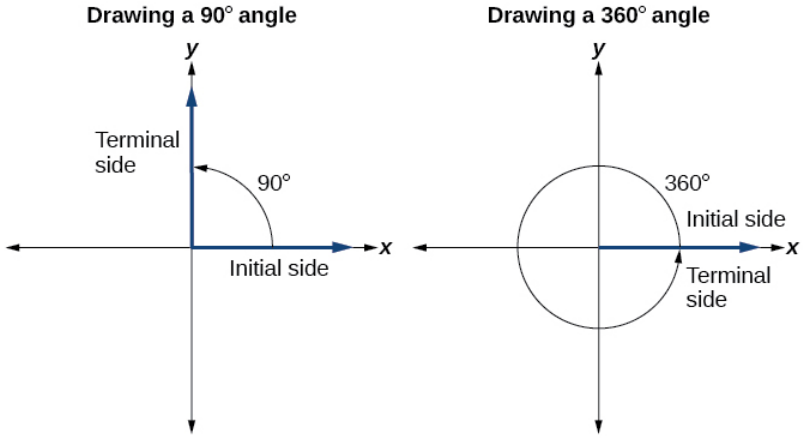


Figure 9.1.6

Since we define an angle in **standard position** by its initial side, we have a special type of angle whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of 0°, 90°, 180°, 270° or 360°. See Figure 9.1.7.

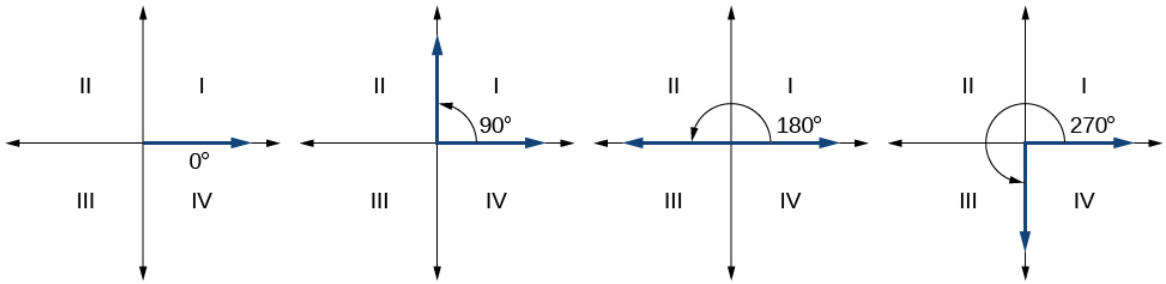


Figure 9.1.7: Quadrantal angles are angles in standard position whose terminal side lies along an axis. Examples are shown.

QUADRANTAL ANGLES

Quadrantal angles are angles in standard position whose terminal side lies on an axis, including 0°, 90°, 180°, 270°, or 360°.

Given an angle measure in degrees, draw the angle in standard position

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive *x*-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example 9.1.1: Drawing an Angle in Standard Position Measured in Degrees

1. Sketch an angle of 30° in standard position.
2. Sketch an angle of −135° in standard position.

Solution

1. Divide the angle measure by 360°.

$$\frac{30^\circ}{360^\circ} = \frac{1}{12} \tag{9.1.1}$$

To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}\left(\frac{1}{4}\right) \tag{9.1.2}$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as in Figure 9.1.8.

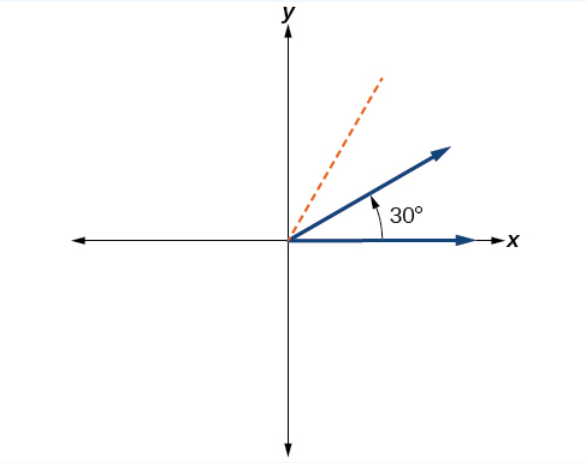


***Figure* 9.1.8**

2. Divide the angle measure by 360°.

$$\frac{-135^\circ}{360^\circ} = -\frac{3}{8} \tag{9.1.3}$$

In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}\left(\frac{1}{4}\right) \tag{9.1.4}$$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in Figure 9.1.9.

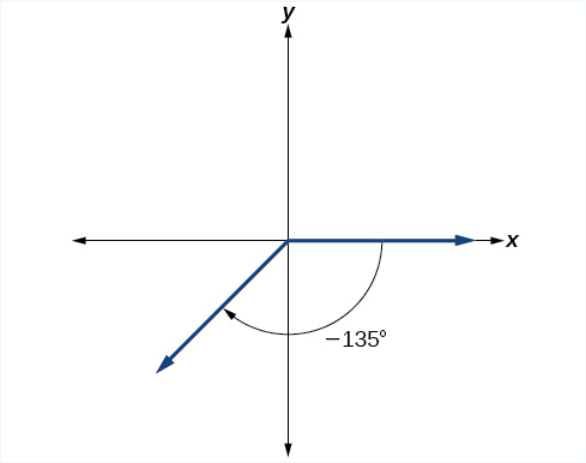


***Figure* 9.1.9**

9.1.1:

Show an angle of 240° on a circle in standard position.

Solution

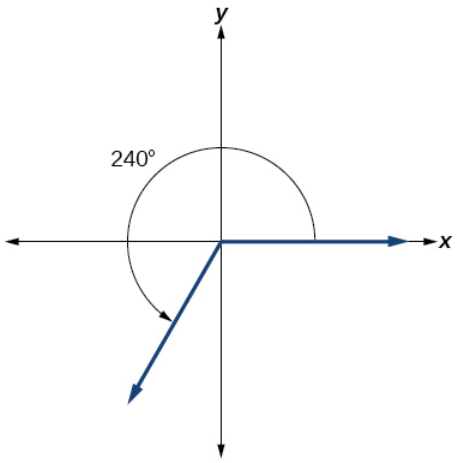


Figure 9.1.10

Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An **arc** may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the **circumference** of that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in Figure 9.1.11.

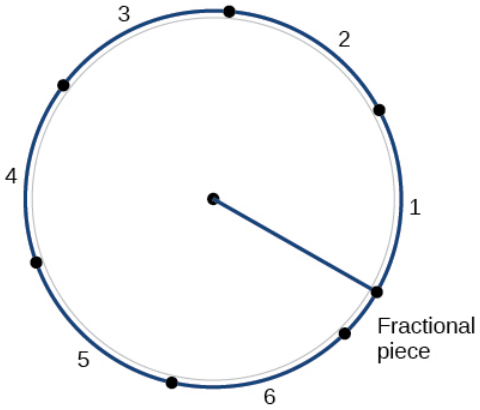


Figure 9.1.11

This brings us to our new angle measure. One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians } = 360^\circ \tag{9.1.5}$$

$$\pi \text{ radians } = \frac{360^\circ}{2} = 180^\circ \tag{9.1.6}$$

$$1 \text{ radian } = \frac{180^\circ}{\pi} \approx 57.3^\circ \tag{9.1.7}$$

See Figure 9.1.12. Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.

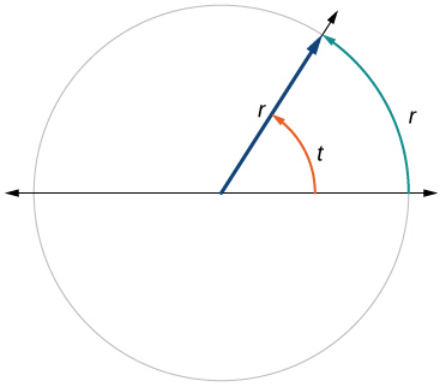


Figure 9.1.12: The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

Relating Arc Lengths to Radius

An **arc length** s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

This ratio, called the **radian measure**, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s s to the radius r. See Figure 9.1.13.

$$s = r\theta \tag{9.1.8}$$

$$\theta = \frac{s}{r} \tag{9.1.9}$$

If $s = r$, then $\theta = \frac{r}{r} = 1$ radian.

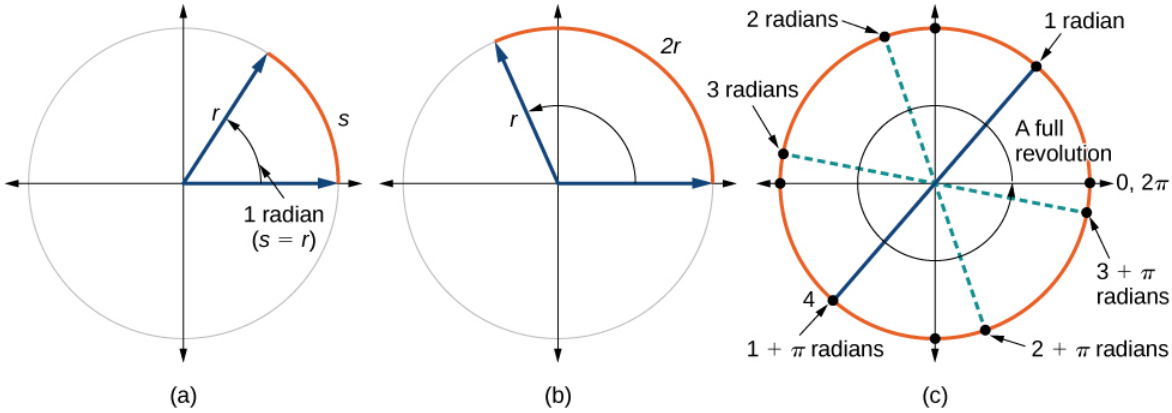


Figure 9.1.13: (a) In an angle of 1 radian, the arc length s equals the radius r. (b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$,where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45° angle on the two circles, as in Figure 9.1.14.

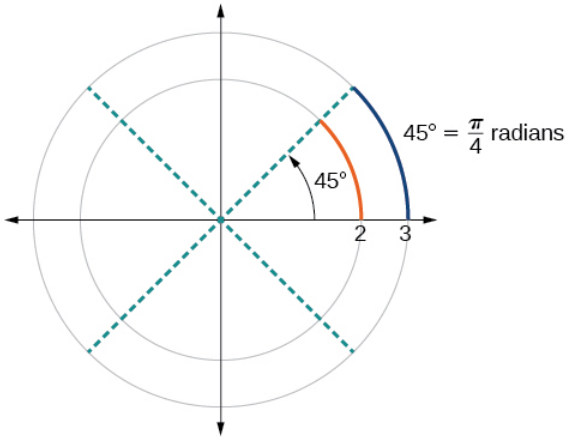


Figure 9.1.14: A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius.

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle: } \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi \tag{9.1.10}$$

$$\text{Larger circle: } \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi \tag{9.1.11}$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

RADIANS

One **radian** is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.

The **radian measure** of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

A measure of 1 radian looks to be about 60°. Is that correct?

Yes. It is approximately 57.3°. Because 2π radians equals 360°, 1 radian equals $\frac{360^\circ}{2\pi} \approx 57.3^\circ$.

Using Radians

Because **radian** measure is the ratio of two lengths, it is a unitless measure. For example, in Figure 9.1.14, suppose the radius were 2 inches and the distance along the arc were also 2 inches. When we calculate the radian measure of the angle, the "inches" cancel, and we have a result without units. Therefore, it is not necessary to write the label "radians" after a radian measure, and if we see an angle that is not labeled with "degrees" or the degree symbol, we can assume that it is a radian measure.

Considering the most basic case, the **unit circle** (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$\begin{aligned} 1\text{rotation} &= 360^\circ &&= 2\pi \quad \text{radians} \\ \frac{1}{2}\text{rotation} &= 180^\circ &&= \pi \quad \text{radians} \\ \frac{1}{4}\text{rotation} &= 90^\circ &&= \frac{\pi}{2} \quad \text{radians} \end{aligned} \tag{9.1.12}$$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in Figure 9.1.15. Memorizing these angles will be very useful as we study the properties associated with angles.

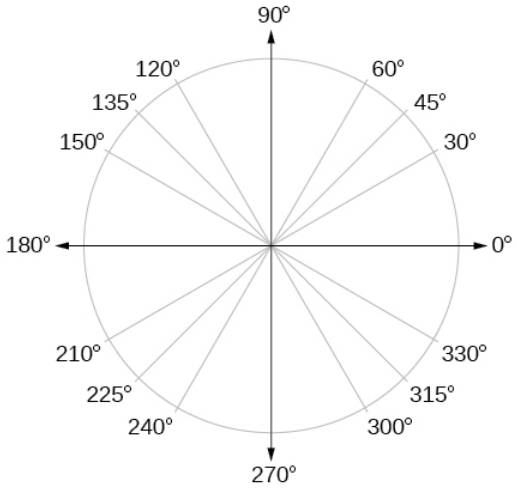


Figure 9.1.15: Commonly encountered angles measured in degrees

Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in Figure 9.1.15, which are shown in Figure 9.1.16. Be sure you can verify each of these measures.

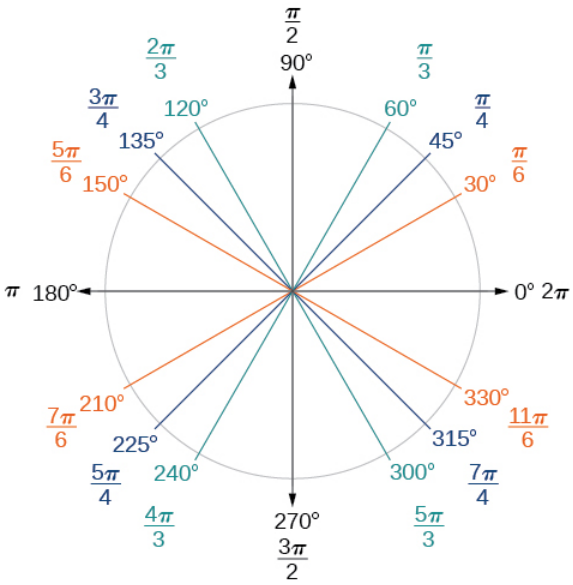


Figure 9.1.16: Commonly encountered angles measured in radians

Example 9.1.2: Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

Solution

For any circle, the arc length along such a rotation would be one-third of the circumference. We know that

$$1 \text{ rotation} = 2\pi r \tag{9.1.13}$$

So,

$$s = \frac{1}{3}(2\pi r) \tag{9.1.14}$$

$$= \frac{2\pi r}{3} \tag{9.1.15}$$

The radian measure would be the arc length divided by the radius.

$$\text{radian measure} = \frac{\frac{2\pi r}{3}}{r} \tag{9.1.16}$$

$$= \frac{2\pi r}{3r} \tag{9.1.17}$$

$$= \frac{2\pi}{3} \tag{9.1.18}$$

9.1.2

Find the radian measure of three-fourths of a full rotation.

Solution

$$\frac{3\pi}{2} \tag{9.1.19}$$

Converting between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{9.1.20}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

$$\frac{\text{Degrees}}{180} = \frac{Radians}{\pi} \tag{9.1.21}$$

CONVERTING BETWEEN RADIANS AND DEGREES

To convert between degrees and radians, use the proportion

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{9.1.22}$$

Example 9.1.3: Converting Radians to Degrees

Convert each radian measure to degrees.

1. $\frac{\pi}{6}$
2. 3

Solution

Because we are given radians and we want degrees, we should set up a proportion and solve it.

1. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{9.1.23}$$

$$\frac{\theta}{180} = \frac{\frac{\pi}{6}}{\pi} \tag{9.1.24}$$

$$\theta = \frac{180}{6} \tag{9.1.25}$$

$$\theta = 30^\circ \tag{9.1.26}$$

2. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{9.1.27}$$

$$\frac{\theta}{180} = \frac{3}{\pi} \tag{9.1.28}$$

$$\theta = \frac{3(180)}{\pi} \tag{9.1.29}$$

$$\theta \approx 172^\circ \tag{9.1.30}$$

9.1.3

Convert $-\frac{3\pi}{4}$ radians to degrees.

Solution

$-135°$

Example 9.1.4: Converting Degrees to Radians

Convert 15 degrees to radians.

Solution

In this example, we start with degrees and want radians, so we again set up a proportion and solve it, but we substitute the given information into a different part of the proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{9.1.31}$$

$$\frac{15}{180} = \frac{\theta^R}{\pi} \tag{9.1.32}$$

$$\frac{15\pi}{180} = \theta^R \tag{9.1.33}$$

$$\frac{\pi}{12} = \theta^R \tag{9.1.34}$$

Analysis

Another way to think about this problem is by remembering that $30^\circ = \frac{\pi}{6}$. Because $15^\circ = \frac{1}{2}(30^\circ)$, ,we can find that $\frac{1}{2}(\frac{\pi}{6})$ is $\frac{\pi}{12}$.

9.1.4

Convert 126° to radians.

Solution

$\frac{7\pi}{10}$

Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at Figure 9.1.17. The angle of 140° is a **positive angle**, measured counterclockwise. The angle of –220° is a **negative angle**, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are **coterminal** angles. Every angle greater than 360° or less than 0° is coterminal with an angle between 0° and 360°, and it is often more convenient to find the coterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.

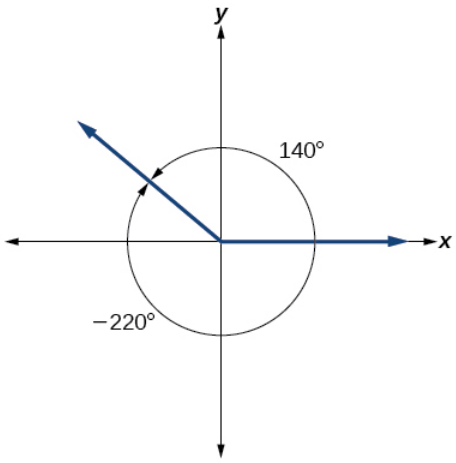


Figure 9.1.17: An angle of 140° and an angle of –220° are coterminal angles.

Any angle has infinitely many **coterminal angles** because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is −260°. Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions.

An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See Figure 9.1.18 for examples of reference angles for angles in different quadrants.

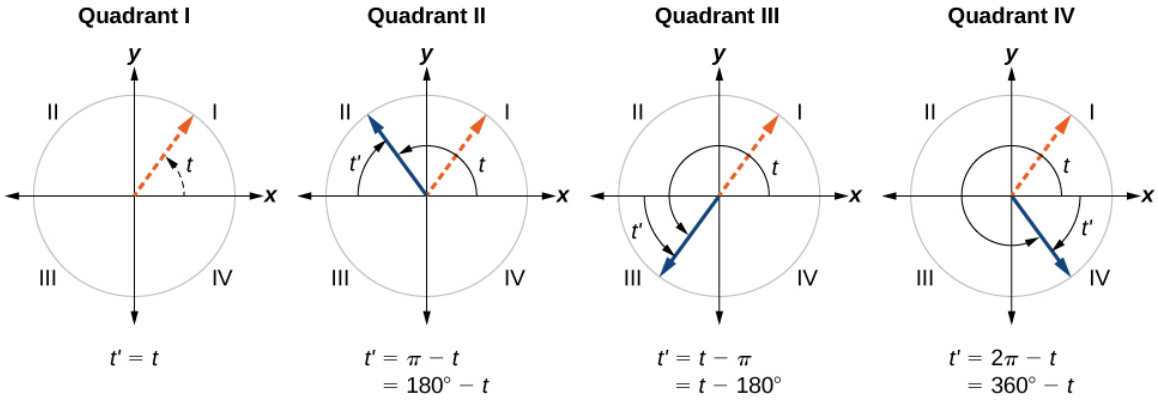


Figure 9.1.18

COTERMINAL AND REFERENCE ANGLES

- *Coterminal angles* are two angles in standard position that have the same terminal side.
- An angle's *reference angle* is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

Given an angle greater than 360°, find a coterminal angle between 0° and 360°

1. Subtract 360° from the given angle.
2. If the result is still greater than 360°, subtract 360° again till the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 9.1.5: Finding an Angle Coterminal with an Angle of Measure Greater Than 360°

Find the least positive angle θ that is coterminal with an angle measuring 800°, where $0^\circ \le \theta < 360^\circ$.

Solution

An angle with measure 800° is coterminal with an angle with measure 800 − 360 = 440°, but 440° is still greater than 360°, so we subtract 360° again to find another coterminal angle: 440 − 360 = 80°.

The angle $\theta = 80^\circ$ is coterminal with 800°. To put it another way, 800° equals 80° plus two full rotations, as shown in Figure 9.1.19.

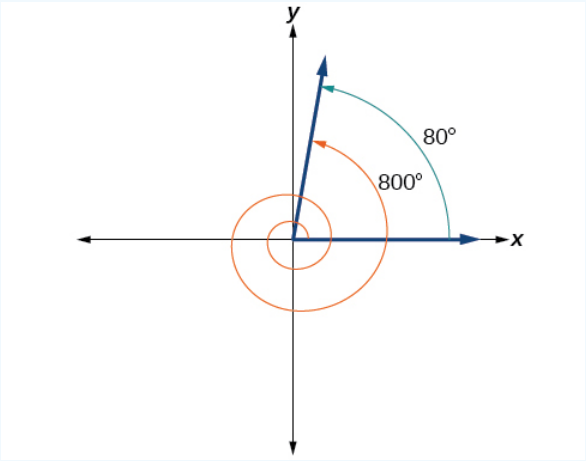


Figure 9.1.19

9.1.5

Find an angle α that is coterminal with an angle measuring 870°, where $0^\circ \le \alpha < 360^\circ$.

Solution

$\alpha = 150^\circ$

Given an angle with measure less than 0°, find a coterminal angle having a measure between 0° and 360°.

1. Add 360° to the given angle.
2. If the result is still less than 0°, add 360° again until the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 9.1.6: Finding an Angle Coterminal with an Angle Measuring Less Than 0°

Show the angle with measure −45° on a circle and find a positive coterminal angle α such that 0° ≤ α < 360°.

Solution

Since 45° is half of 90°, we can start at the positive horizontal axis and measure clockwise half of a 90° angle.

Because we can find coterminal angles by adding or subtracting a full rotation of 360°, we can find a positive coterminal angle here by adding 360°:

$$-45^\circ + 360^\circ = 315^\circ \tag{9.1.35}$$

We can then show the angle on a circle, as in Figure 9.1.20.

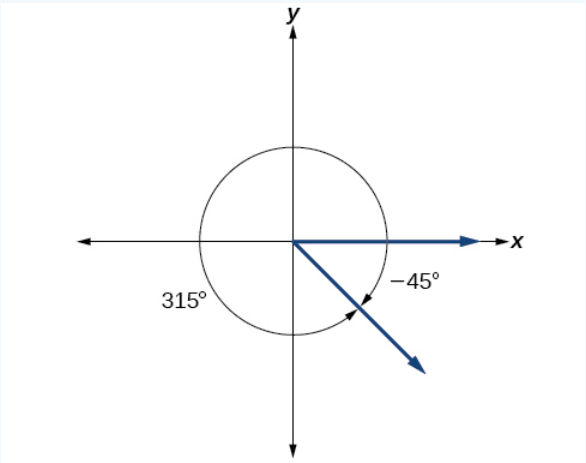


Figure 9.1.20

9.1.6

Find an angle β that is coterminal with an angle measuring −300° such that $0^\circ \le \beta < 360^\circ$.

Solution

$\beta = 60^\circ$

Finding Coterminal Angles Measured in Radians

We can find **coterminal** angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Example 9.1.7: Finding Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \le \beta < 2\pi$.

Solution

When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{8\pi}{4} \tag{9.1.36}$$

$$= \frac{11\pi}{4} \tag{9.1.37}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation:

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} \tag{9.1.38}$$

$$= \frac{3\pi}{4} \tag{9.1.39}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown in Figure 9.1.21.

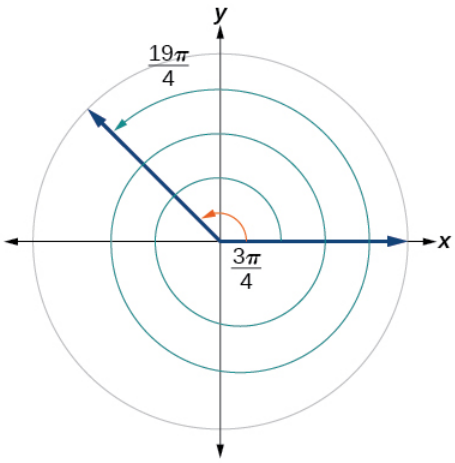


Figure 9.1.21

9.1.7

Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \leq \theta < 2\pi$.

Solution

$\frac{7\pi}{6}$

Determining the Length of an Arc

Recall that the **radian measure** θ of an angle was defined as the ratio of the **arc length** s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

ARC LENGTH ON A CIRCLE

In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians, shown in Figure 9.1.22, is

$$s = r\theta \tag{9.1.40}$$

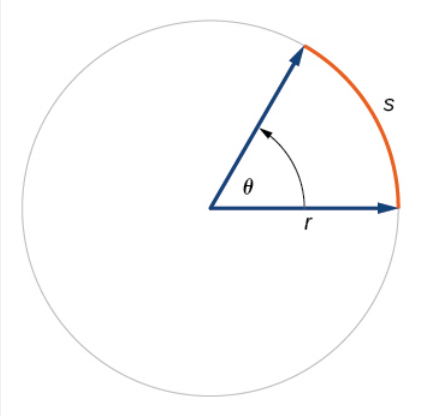


Figure 9.1.22

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of θ : $s = r\theta$.

Example 9.1.8: Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

1. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
2. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

Solution

1. Let's begin by finding the circumference of Mercury's orbit.

$$C = 2\pi r \tag{9.1.41}$$
$$= 2\pi(36 \text{ million miles}) \tag{9.1.42}$$
$$\approx 226 \text{ million miles} \tag{9.1.43}$$

Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the distance traveled:

$$(0.0114)226 \text{ million miles} = 2.58 \text{ million miles} \tag{9.1.44}$$

2. Now, we convert to radians:

$$\text{radian} = \frac{\text{arc length}}{\text{radius}} \tag{9.1.45}$$
$$= \frac{2.58 \text{ million miles}}{36 \text{ million miles}} \tag{9.1.46}$$
$$= 0.0717 \tag{9.1.47}$$

9.1.8

Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

Solution

$$\frac{215\pi}{18} = 37.525 \text{ units} \tag{9.1.48}$$

Finding the Area of a Sector of a Circle

In addition to arc length, we can also use angles to find the area of a **sector of a circle**. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found using the formula $A = \pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\text{Area of sector} = (\frac{\theta}{2\pi})\pi r^2 \tag{9.1.49}$$
$$= \frac{\theta \pi r^2}{2\pi} \tag{9.1.50}$$
$$= \frac{1}{2}\theta r^2 \tag{9.1.51}$$

AREA OF A SECTOR

The area of a **sector of a circle** with radius r subtended by an angle θ, measured in radians, is

$$A = \frac{1}{2}\theta r^2 \tag{9.1.52}$$

See Figure 9.1.23.

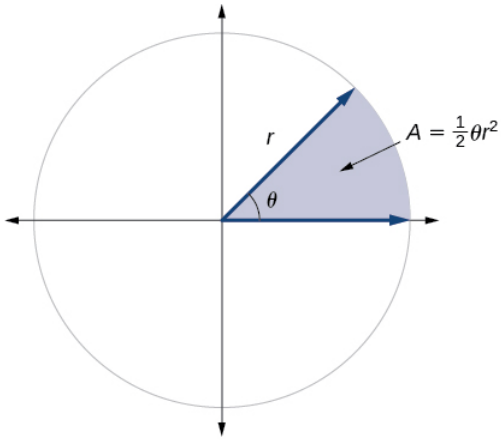


Figure 9.1.23: The area of the sector equals half the square of the radius times the central angle measured in radians.

Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r : $A = \frac{1}{2}\theta r^2$.

Example 9.1.9: Finding the Area of a Sector

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in Figure 9.1.24. What is the area of the sector of grass the sprinkler waters?

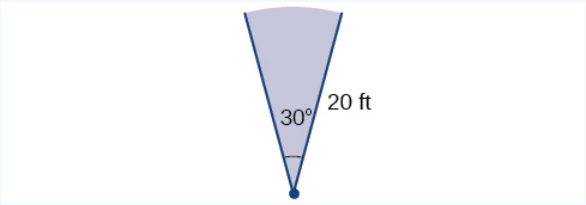


Figure 9.1.24: The sprinkler sprays 20 ft within an arc of 30°.

Solution

First, we need to convert the angle measure into radians. Because 30 degrees is one of our special angles, we already know the equivalent radian measure, but we can also convert:

$$30 \text{ degrees} = 30 \cdot \frac{\pi}{180} \tag{9.1.53}$$

$$= \frac{\pi}{6} \text{ radians} \tag{9.1.54}$$

The area of the sector is then

$$\text{Area} = \frac{1}{2}(\frac{\pi}{6})(20)^2 \tag{9.1.55}$$

$$\approx 104.72 \tag{9.1.56}$$

So the area is about 104.72 ft^2.

9.1.9

In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Write the answer in radian measure to two decimal places.

Solution

1.88

Use Linear and Angular Speed to Describe Motion on a Circular Path

In addition to finding the area of a sector, we can use angles to describe the speed of a moving object. An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it moves along (its **displacement**) in a given time interval. For instance, if a wheel with radius 5 inches rotates once a second, a point on the edge of the wheel moves a distance equal to the circumference, or 10π inches, every second. So the linear speed of the point is 10π in./s. The equation for linear speed is as follows where v is linear speed, s is displacement, and t is time.

$$v = \frac{s}{t} \tag{9.1.57}$$

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval. In other words, angular speed is angular rotation per unit time. So, for instance, if a gear makes a full rotation every 4 seconds, we can calculate its angular speed as $\frac{360 \text{ degrees}}{4 \text{ seconds}} = 90$ degrees per second. Angular speed can be given in radians per second, rotations per minute, or degrees per hour for example. The equation for angular speed is as follows, where ω (read as omega) is angular speed, θ is the angle traversed, and t is time.

$$\omega = \frac{\theta}{t} \tag{9.1.58}$$

Combining the definition of angular speed with the arc length equation, $s = r\theta$, we can find a relationship between angular and linear speeds. The angular speed equation can be solved for θ, giving $\theta = \omega t$. Substituting this into the arc length equation gives:

$$s = r\theta \tag{9.1.59}$$
$$= r\omega t \tag{9.1.60}$$

Substituting this into the linear speed equation gives:

$$v = \frac{s}{t} \qquad = \frac{r\omega t}{t} = r\omega \tag{9.1.61}$$

ANGULAR AND LINEAR SPEED

As a point moves along a circle of radius r, its **angular speed**, ω, is the angular rotation θ per unit time, t.

$$\omega = \frac{\theta}{t} \tag{9.1.62}$$

The **linear speed.** v, of the point can be found as the distance traveled, arc length s, per unit time, t.

$$v = \frac{s}{t} \tag{9.1.63}$$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$$v = r\omega \tag{9.1.64}$$

This equation states that the angular speed in radians, ω, representing the amount of rotation occurring in a unit of time, can be multiplied by the radius r to calculate the total arc length traveled in a unit of time, which is the definition of linear speed.

Given the amount of angle rotation and the time elapsed, calculate the angular speed

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega = \frac{\theta}{t}$.
3. The resulting speed will be in radians per time unit.

Example 9.1.10: Finding Angular Speed

A water wheel, shown in Figure 9.1.25, completes 1 rotation every 5 seconds. Find the angular speed in radians per second.

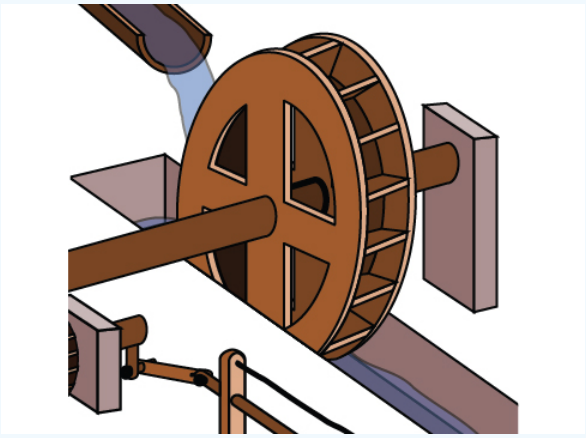


Figure 9.1.25

Solution

The wheel completes 1 rotation, or passes through an angle of 2π radians in 5 seconds, so the angular speed would be $\omega = \frac{2\pi}{5} \approx 1.257$ radians per second.

9.1.10

An old vinyl record is played on a turntable rotating clockwise at a rate of 45 rotations per minute. Find the angular speed in radians per second.

Solution

$-\frac{3\pi}{2}$ rad/s

Given the radius of a circle, an angle of rotation, and a length of elapsed time, determine the linear speed

1. Convert the total rotation to radians if necessary.
2. Divide the total rotation in radians by the elapsed time to find the angular speed: apply $\omega = \frac{\theta}{t}$.
3. Multiply the angular speed by the length of the radius to find the linear speed, expressed in terms of the length unit used for the radius and the time unit used for the elapsed time: apply $v = r\omega$.

Example 9.1.11: Finding a Linear Speed

A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Solution

Here, we have an angular speed and need to find the corresponding linear speed, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road.

We begin by converting from rotations per minute to radians per minute. It can be helpful to utilize the units to make this conversion:

$$180 \frac{\text{rotations}}{\text{minute}} \cdot \frac{2\pi \text{ radians}}{\text{rotation}} = 360\pi \frac{\text{radians}}{\text{minute}} \tag{9.1.65}$$

Using the formula from above along with the radius of the wheels, we can find the linear speed:

$$v = (14 \text{ inches})(360\pi \frac{\text{radians}}{\text{minute}}) \tag{9.1.66}$$

$$= 5040\pi \frac{\text{inches}}{\text{minute}} \tag{9.1.67}$$

Remember that radians are a unitless measure, so it is not necessary to include them.unitless measure, so it is not necessary to include them.

Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour.

$$5040\pi \frac{\cancel{\text{inches}}}{\cancel{\text{minute}}} \cdot \frac{1 \ \cancel{\text{feet}}}{12 \ \cancel{\text{inches}}} \cdot \frac{1 \ \cancel{\text{mile}}}{5280 \ \cancel{\text{feet}}} \cdot \frac{60 \ \cancel{\text{minutes}}}{1 \text{ hour}} \approx 14.99 \text{ miles per hour (mph)} \tag{9.1.68}$$

Try It 9.1.11

A satellite is rotating around Earth at 0.25 radians per hour at an altitude of 242 km above Earth. If the radius of Earth is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

Solution

1655 kilometers per hour

Media

Access these online resources for additional instruction and practice with angles, arc length, and areas of sectors.

- Angles in Standard Position
- Angle of Rotation
- Coterminal Angles
- Determining Coterminal Angles
- Positive and Negative Coterminal Angles
- Radian Measure
- Coterminal Angles in Radians
- Arc Length and Area of a Sector

Key Equations

arc length	$s = r\theta$
area of a sector	$A = \frac{1}{2}\theta r^2$
angular speed	$\omega = \frac{\theta}{t}$
linear speed	$v = \frac{s}{t}$
linear speed related to angular speed	$v = r\omega$

Key Concepts

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive *x*-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive *x*-axis and then place the terminal side according to the fraction of a full rotation the angle represents. See **Example**.
- In addition to degrees, the measure of an angle can be described in radians. See **Example**.
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta^R}{\pi}$. See Example and Example.
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting 360° or 2π. See Example and Example.

- Coterminal angles can be found using radians just as they are for degrees. See **Example**.
- The length of a circular arc is a fraction of the circumference of the entire circle. See **Example**.
- The area of sector is a fraction of the area of the entire circle. See **Example**.
- An object moving in a circular path has both linear and angular speed.
- The angular speed of an object traveling in a circular path is the measure of the angle through which it turns in a unit of time. See **Example**.
- The linear speed of an object traveling along a circular path is the distance it travels in a unit of time. See **Example**.

Glossary

angle
the union of two rays having a common endpoint

angular speed
the angle through which a rotating object travels in a unit of time

arc length
the length of the curve formed by an arc

area of a sector
area of a portion of a circle bordered by two radii and the intercepted arc; the fraction $\frac{\theta}{2\pi}$ multiplied by the area of the entire circle

coterminal angles
description of positive and negative angles in standard position sharing the same terminal side

degree
a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

initial side
the side of an angle from which rotation begins

linear speed
the distance along a straight path a rotating object travels in a unit of time; determined by the arc length

measure of an angle
the amount of rotation from the initial side to the terminal side

negative angle
description of an angle measured clockwise from the positive x-axis

positive angle
description of an angle measured counterclockwise from the positive x-axis

quadrantal angle
an angle whose terminal side lies on an axis

radian measure
the ratio of the arc length formed by an angle divided by the radius of the circle

radian
the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

ray
one point on a line and all points extending in one direction from that point; one side of an angle

reference angle

the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

standard position

the position of an angle having the vertex at the origin and the initial side along the positive *x*-axis

terminal side

the side of an angle at which rotation ends

vertex

the common endpoint of two rays that form an angle

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9.2: Right Triangle Trigonometry

Learning Objectives

- Use right triangles to evaluate trigonometric functions.
- Find function values for $30°(\frac{\pi}{6})$,$45°(\frac{\pi}{4})$,and $60°(\frac{\pi}{3})$.
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

We have previously defined the sine and cosine of an angle in terms of the coordinates of a point on the unit circle intersected by the terminal side of the angle:

$$\cos t = x$$
$$\sin t = y$$

In this section, we will see another way to define trigonometric functions using properties of *right triangles*.

Using Right Triangles to Evaluate Trigonometric Functions

In earlier sections, we used a unit circle to define the *trigonometric functions*. In this section, we will extend those definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. Figure $9.2.1$ shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x, y) to the *x*-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

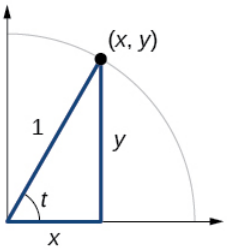


Figure $9.2.1$

We know

$$\cos t = \frac{x}{1} = x \tag{9.2.1}$$

Likewise, we know

$$\sin t = \frac{y}{1} = y \tag{9.2.2}$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x, y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x,we will call the side between the given angle and the right angle the **adjacent side** to angle t. (Adjacent means "next to.") Instead of y,we will call the side most distant from the given angle the **opposite side** from angle t. And instead of 1,we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure $9.2.2$.

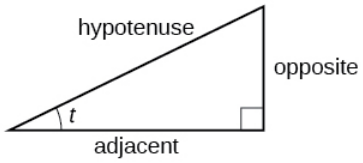


Figure 9.2.2: The sides of a right triangle in relation to angle t.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}} \tag{9.2.3}$$

$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{9.2.4}$$

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} \tag{9.2.5}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

how to: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent is the ratio of the opposite side to the adjacent side.

Example 9.2.1: Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in Figure 9.2.3, find the value of $\cos\alpha$.

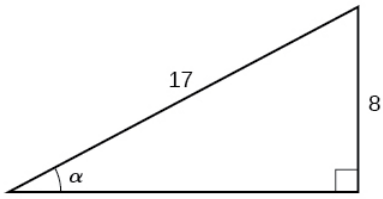


Figure 9.2.3

Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so via Equation 9.2.4:

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{15}{17}$$

Exercise 9.2.1

Given the triangle shown in Figure 9.2.4, find the value of $\sin t$.

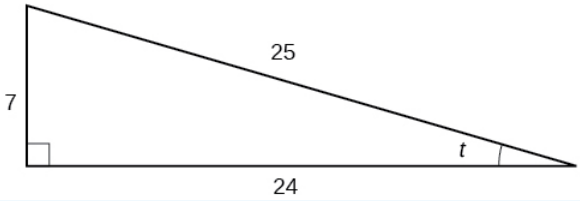


Figure 9.2.4

Answer

$\frac{7}{25}$

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in Figure 9.2.5. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

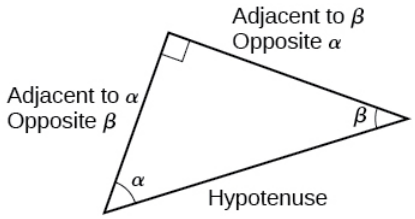


Figure 9.2.5: The side adjacent to one angle is opposite the other.

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

how to: Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example 9.2.2: Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in Figure 9.2.6, evaluate $\sin\alpha, \cos\alpha, \tan\alpha, \sec\alpha, \csc\alpha,$ and $\cot\alpha$.

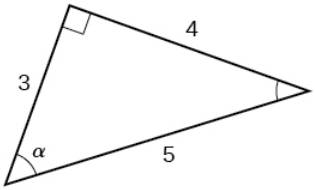


Figure 9.2.6

Solution

$$\sin\alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos\alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan\alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3}$$
$$\sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3}$$
$$\csc\alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4}$$
$$\cot\alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

Exercise 9.2.2

Using the triangle shown in Figure 9.2.7, evaluate $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

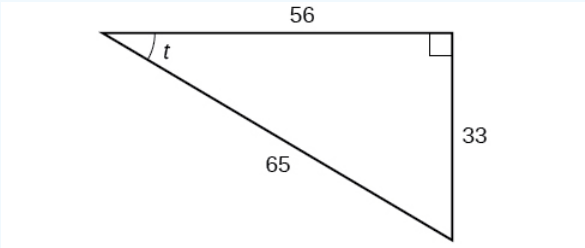


Figure 9.2.7

Answer

$$\sin t = \frac{33}{65}, \cos t = \frac{56}{65}, \tan t = \frac{33}{56},$$

$$\sec t = \frac{65}{56}, \csc t = \frac{65}{33}, \cot t = \frac{56}{33}$$

Finding Trigonometric Functions of Special Angles Using Side Lengths

We have already discussed the trigonometric functions as they relate to the **special angles** on the unit circle. Now, we can use those relationships to evaluate triangles that contain those special angles. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of $30^\circ, 60^\circ,$ and 45°, however, remember that when dealing with right triangles, we are limited to angles between 0° and 90°.

Suppose we have a $30^\circ, 60^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45^\circ, 45^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in Figure 9.2.8.

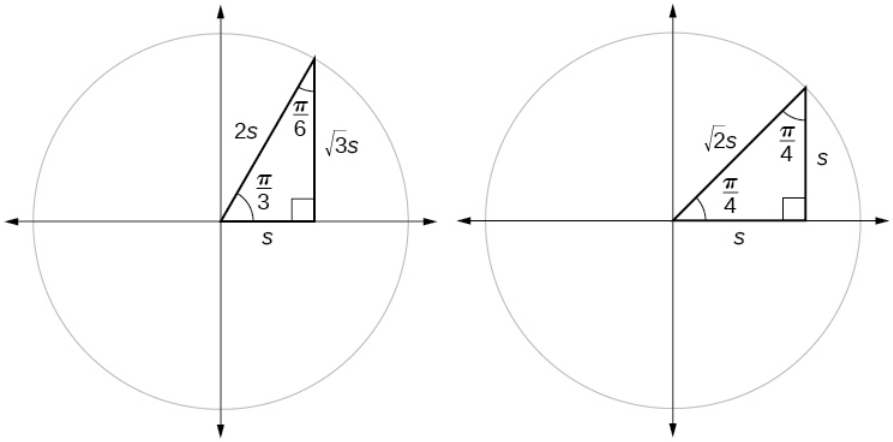


Figure 9.2.8: Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure 9.2.8 for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 9.2.3: Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin(\frac{\pi}{3}) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\cos(\frac{\pi}{3}) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$
$$\tan(\frac{\pi}{3}) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$
$$\sec(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$
$$\csc(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cot(\frac{\pi}{3}) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 9.2.3

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$ using side lengths.

Answer

$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{\pi}{4}) = 1,$
$\sec(\frac{\pi}{4}) = \sqrt{2}, \csc(\frac{\pi}{4}) = \sqrt{2}, \cot(\frac{\pi}{4}) = 1$

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the

sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$ (Figure 9.2.9).

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

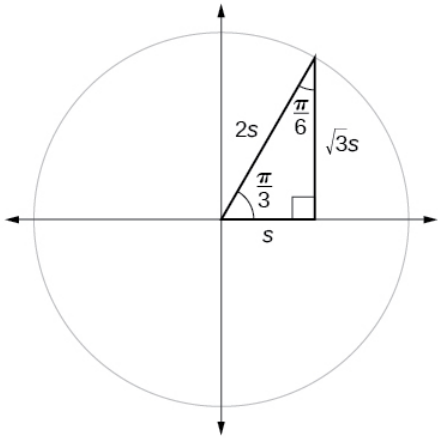
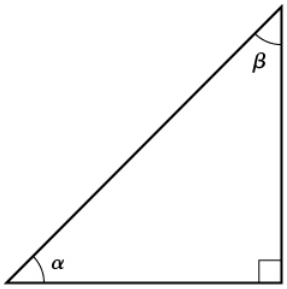


Figure 9.2.9: The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from Figure 9.2.9, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{6})$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos(\frac{\pi}{3})$ and $\sin(\frac{\pi}{6})$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure 9.2.10.



$\sin\alpha = \cos\beta$
$\sin\beta = \cos\alpha$

Figure 9.2.10: Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well.

COFUNCTION IDENTITIES

The **cofunction identities** in radians are listed in Table 9.2.1.

Table 9.2.1

$\cos t = \sin(\frac{\pi}{2} - t)$	$\sin t = \cos(\frac{\pi}{2} - t)$
$\tan t = \cot(\frac{\pi}{2} - t)$	$\cot t = \tan(\frac{\pi}{2} - t)$

$\sec t = \csc(\frac{\pi}{2} - t)$	$\csc t = \sec(\frac{\pi}{2} - t)$

how to: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 9.2.4: Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $(\cos\frac{\pi}{2} - t)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin t = \cos(\frac{\pi}{2} - t).$$

So

$$\cos(\frac{\pi}{2} - t) = \frac{5}{12}.$$

Exercise 9.2.4

If $\csc(\frac{\pi}{6}) = 2$, find $\sec(\frac{\pi}{3})$.

Solution

2

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

how to: Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 9.2.5: Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in Figure 9.2.11.

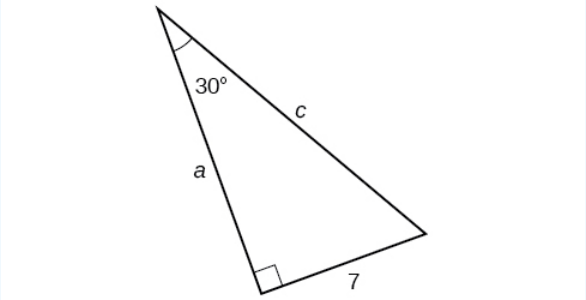


Figure 9.2.11

Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30^\circ) = \frac{7}{a}$$

We rearrange to solve for a.

$$a = \frac{7}{\tan(30^\circ)} \tag{9.2.6}$$

$$= 12.1 \tag{9.2.7}$$

We can use the sine to find the hypotenuse.

$$\sin(30^\circ) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$c = \frac{7}{\sin(30^\circ)} = 14$$

Exercise 9.2.5:

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Answer

$\text{adjacent} = 10; \text{opposite} = 10\sqrt{3}$; missing angle is $\frac{\pi}{6}$

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See Figure 9.2.12

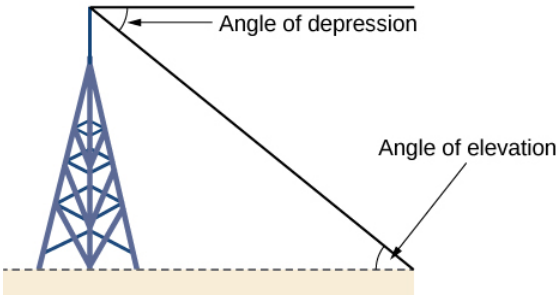


Figure 9.2.12

how to: Given a tall object, measure its height indirectly

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.

4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 9.2.6: Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° 57° between a line of sight to the top of the tree and the ground, as shown in Figure 9.2.13. Find the height of the tree.

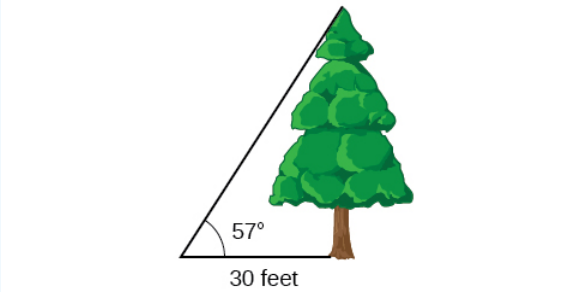


Figure 9.2.13

Solution

We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57°, letting h be the unknown height.

$$\begin{aligned} \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan(57^\circ) &= \frac{h}{30} && \text{Solve for } h. \\ h &= 30\tan(57^\circ) && \text{Multiply.} \\ h &\approx 46.2 && \text{Use a calculator.} \end{aligned} \tag{9.2.8}$$

The tree is approximately 46 feet tall.

Exercise 9.2.6:

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

Answer

About 52 ft

media:

Access these online resources for additional instruction and practice with right triangle trigonometry.

- Finding Trig Functions on Calculator
- Finding Trig Functions Using a Right Triangle
- Relate Trig Functions to Sides of a Right Triangle
- Determine Six Trig Functions from a Triangle
- Determine Length of Right Triangle Side

Visit this website for additional practice questions from Learningpod.

Key Equations

Cofunction Identities

$$\cos t = \sin\left(\frac{\pi}{2} - t\right)$$
$$\sin t = \cos\left(\frac{\pi}{2} - t\right)$$
$$\tan t = \cot\left(\frac{\pi}{2} - t\right)$$
$$\cot t = \tan\left(\frac{\pi}{2} - t\right)$$
$$\sec t = \csc\left(\frac{\pi}{2} - t\right)$$
$$\csc t = \sec\left(\frac{\pi}{2} - t\right)$$

Key Concepts

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See Example.
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See Example.
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See Example.
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See Example.
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See Example.
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See Example.

Glossary

adjacent side
in a right triangle, the side between a given angle and the right angle

angle of depression
the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation
the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side
in a right triangle, the side most distant from a given angle

hypotenuse
the side of a right triangle opposite the right angle

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9.3: Unit Circle

Learning Objectives

- Find function values for the sine and cosine of $30°$ or $(\frac{\pi}{6})$,$45°$ or $(\frac{\pi}{4})$,and $60°$ or $(\frac{\pi}{3})$.
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.



Figure 9.3.1: The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

Finding Function Values for the Sine and Cosine

To define our trigonometric functions, we begin by drawing a unit circle, a circle centered at the origin with radius 1, as shown in Figure 9.3.2. The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a **unit circle**, $s = t$.

Recall that the *x*- and *y*-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y). The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.

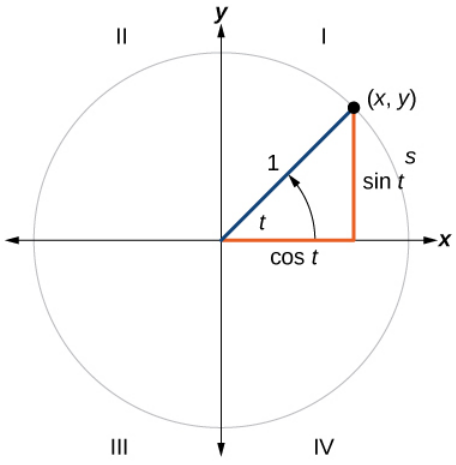


Figure 9.3.2: Unit circle where the central angle is t radians

UNIT CIRCLE

A unit circle has a center at $(0, 0)$ and radius 1. The length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions

Now that we have our unit circle labeled, we can learn how the (x, y) coordinates relate to the **arc length** and **angle**. The **sine function** relates a real number t to the y-coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t. In Figure $9.3.3$, the sine is equal to y. Like all functions, the sine function has an input and an output. Its input is the measure of the angle; its output is the y-coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the x-value of the endpoint on the unit circle of an arc of length t. In Figure $9.3.1$, the cosine is equal to x.

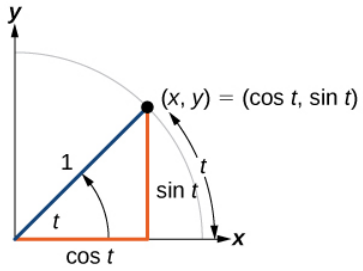


Figure 9.3.3

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

SINE AND COSINE FUNCTIONS

If t is a real number and a point (x, y) on the unit circle corresponds to an angle of t, then

$$\cos t = x \tag{9.3.1}$$
$$\sin t = y \tag{9.3.2}$$

How To: Given a point $P(x, y)$ on the unit circle corresponding to an angle of t, find the sine and cosine

1. The sine of t is equal to the y-coordinate of point P: $\sin t = y$.
2. The cosine of t is equal to the x-coordinate of point P: $\cos t = x$.

Example 9.3.1: Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t, as shown in Figure $9.3.4$. Find $\cos(t)$ and $\sin(t)$.

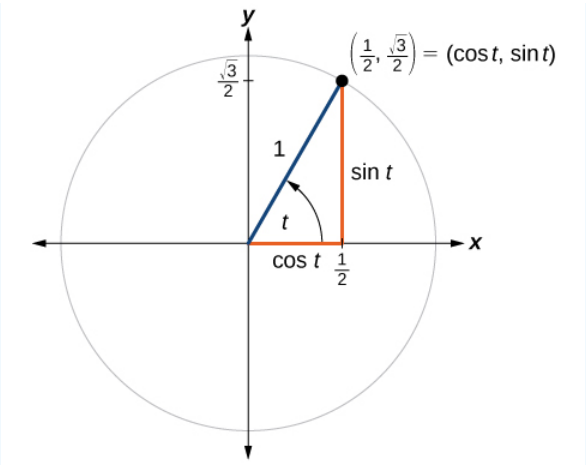


Figure 9.3.4

Solution

We know that $\cos\, t$ is the x-coordinate of the corresponding point on the unit circle and $\sin\, t$ is the y-coordinate of the corresponding point on the unit circle. So:

$$x = \cos\, t = \frac{1}{2}$$
$$y = \sin\, t = \frac{\sqrt{3}}{2}$$

Try It 9.3.1

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in Figure 9.3.5. Find $\cos\, t$ and $\sin\, t$.

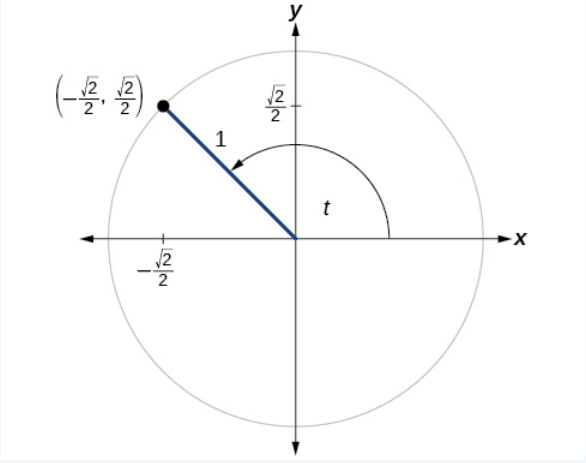


Figure 9.3.5

Answer

$$\cos(t) = -\frac{\sqrt{2}}{2}, \sin(t) = \frac{\sqrt{2}}{2}$$

Finding Sines and Cosines of Angles on an Axis

For quadrantral angles, the corresponding point on the unit circle falls on the x- or y-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Example 9.3.2: Calculating Sines and Cosines along an Axis

Find $\cos(90^\circ)$ and $\sin(90^\circ)$.

Solution

Moving 90° counterclockwise around the unit circle from the positive x-axis brings us to the top of the circle, where the (x, y) coordinates are (0, 1), as shown in Figure 9.3.6.

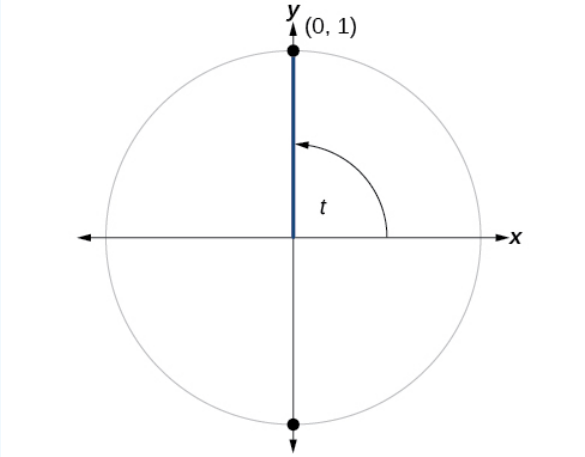


Figure 9.3.6

Using our definitions of cosine and sine,

$$x = \cos t = \cos(90^\circ) = 0$$
$$y = \sin t = \sin(90^\circ) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

Try It 9.3.2

Find cosine and sine of the angle π.

Answer

$\cos(\pi) = -1,\ \sin(\pi) = 0$

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See Figure 9.3.7.

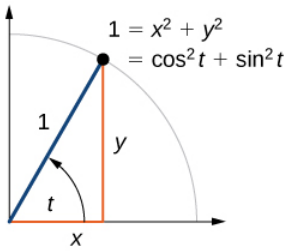


Figure 9.3.7

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

PYTHAGOREAN IDENTITY

The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1 \tag{9.3.3}$$

How To: Given the sine of some angle t and its quadrant location, find the cosine of t

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

Example 9.3.3: Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Solution

If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See Figure 9.3.8.

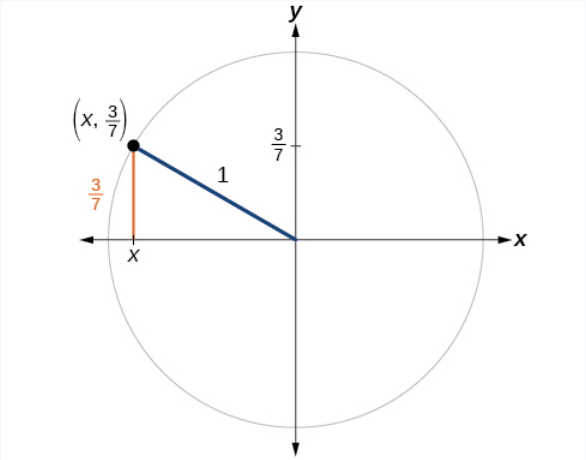


Figure 9.3.8

Substituting the known value for sine into the Pythagorean Identity,

$$\begin{aligned} \cos^2(t) + \sin^2(t) &= 1 \\ \cos^2(t) + \frac{9}{49} &= 1 \\ \cos^2(t) &= \frac{40}{49} \\ \cos(t) &= \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7} \end{aligned}$$

Because the angle is in the second quadrant, we know the x-value is a negative real number, so the cosine is also negative. So

$$\cos(t) = -\frac{2\sqrt{10}}{7}$$

Try It 9.3.3

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Answer

$$\sin(t) = -\frac{7}{25}$$

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees. We can also calculate sines and cosines of the special angles using the **Pythagorean Identity** and our knowledge of triangles.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of 45° or $\frac{\pi}{4}$, as shown in Figure 9.3.9. A $45^\circ - 45^\circ - 90^\circ$ triangle is an isosceles triangle, so the x- and y-coordinates of the corresponding point on the circle are the same. Because the x- and y-values are the same, the sine and cosine values will also be equal.

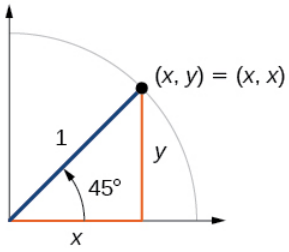


Figure 9.3.9

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first **quadrantal angle**. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y (with $y = x$), and a radius = 1. See Figure 9.3.10.

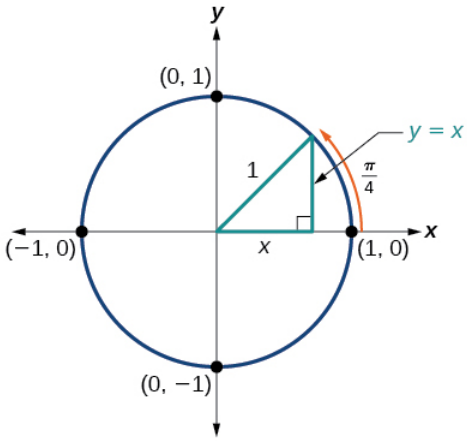


Figure 9.3.10

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

Substituting $y = x$, we get

$$x^2 + x^2 = 1$$

Combining like terms we get

$$2x^2 = 1$$

And solving for x, we get

$$x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$(x, y) = (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}}$$
$$\cos t = \frac{1}{\sqrt{2}},\ \sin t = \frac{1}{\sqrt{2}}$$

If we then rationalize the denominators, we get

$$\cos t = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\sin t = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of 30°, or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of 30°, and another at an angle of -30°, as shown in Figure 9.3.11. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60°, as shown in Figure 9.3.12

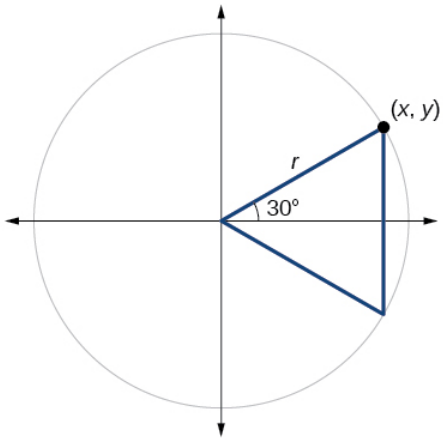


Figure 9.3.11

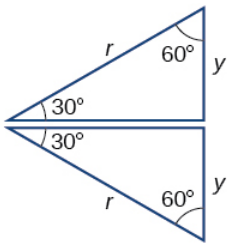


Figure 9.3.12

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

And since $r = 1$ in our **unit circle**,

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1)\\ &= \frac{1}{2}\end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\begin{aligned}\cos^2\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{6}\right) &= 1\\ \cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 &= 1\\ \cos^2\left(\frac{\pi}{6}\right) &= \frac{3}{4} && \text{Use the square root property.}\\ \cos\left(\frac{\pi}{6}\right) &= \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} && \text{Since } y \text{ is positive, choose the positive root.}\end{aligned}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of 30° are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. At $t = \frac{\pi}{3}$ (60°), the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in Figure 9.3.13. Angle A has measure 60°. 60°. At point B, we draw an angle ABC with measure of 60°. We know the angles in a triangle sum to 180°, so the measure of angle C is also 60°. Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

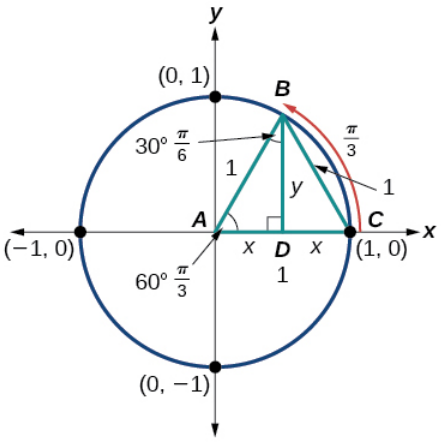


Figure 9.3.13

The measure of angle ABD is 30°. So, if double, angle ABC is 60°. BD is the perpendicular bisector of AC, so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the x-coordinate of point B, which is at the intersection of the 60°

angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2+y^2=1$$

Substituting $x=\frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2+y^2=1$$

Solving for y, we get

$$\begin{aligned}\frac{1}{4}+y^2&=1\\ y^2&=1-\frac{1}{4}\\ y^2&=\frac{3}{4}\\ y&=\pm\frac{\sqrt{3}}{2}\end{aligned}$$

Since $t=\frac{\pi}{3}$ has the terminal side in quadrant I where the y-coordinate is positive, we choose $y=\frac{\sqrt{3}}{2}$, the positive value.

At $t=\frac{\pi}{3}$ (60°), the (x,y) coordinates for the point on a circle of radius 1 at an angle of 60° are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$(x,y)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$$

$$x=\frac{1}{2},\ y=\frac{\sqrt{3}}{2}$$

$$\cos t=\frac{1}{2},\ \sin t=\frac{\sqrt{3}}{2}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. Table 9.3.1 summarizes these values.

Table 9.3.1

Angle	0	$\frac{\pi}{6}$, or 30	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Figure 9.3.14 shows the common angles in the first quadrant of the unit circle.

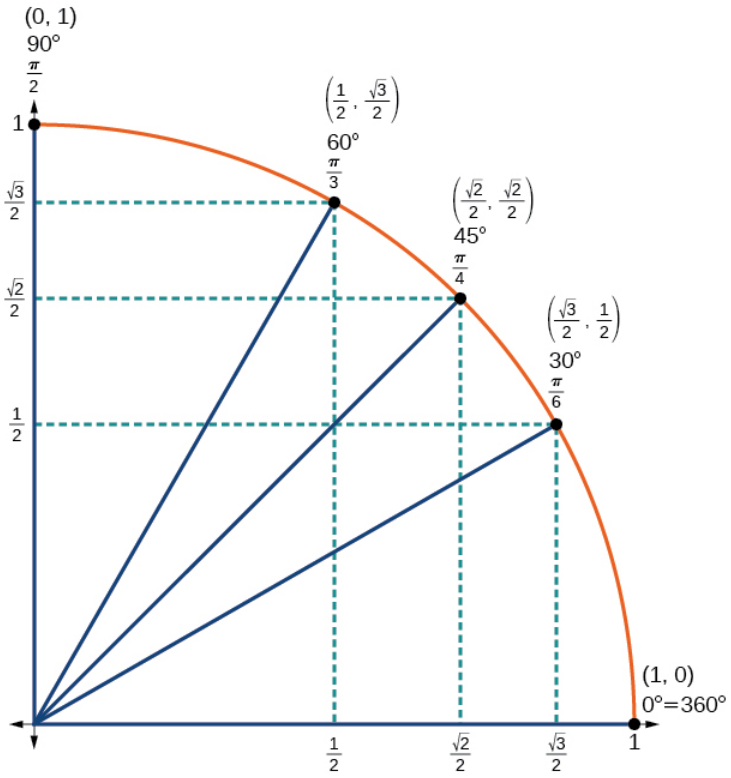


Figure 9.3.14

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the **special angles**, we turn to a computer or calculator. **Be aware**: Most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To: Given an angle in radians, use a graphing calculator to find the cosine

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

Example 9.3.4: Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\dfrac{5\pi}{3}\right)$ using a graphing calculator or computer.

Solution

Enter the following keystrokes:

$$\text{COS}(5 \times \pi \div 3) \text{ ENTER}$$

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

Analysis

We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of 20°, for example, by including the conversion factor to radians as part of the input:

$$\text{SIN}(20 \times \pi \div 180) \text{ ENTER}$$

Try It 9.3.4

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Answer

approximately 0.866025403

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x, y, and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the **unit circle**, as shown in Figure 9.3.15. The bounds of the x-coordinate are $[-1, 1]$. The bounds of the y-coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

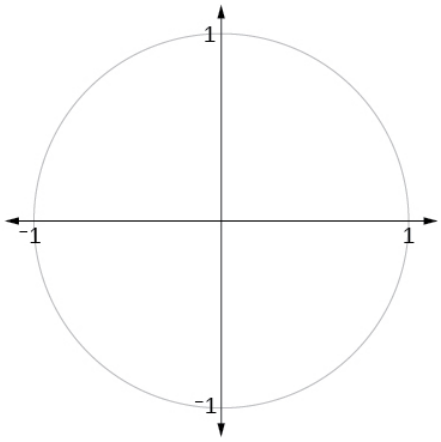


Figure 9.3.15

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the y-coordinate on the unit circle, the other angle with the same sine will share the same y-value, but have the opposite x-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same x-value but will have the opposite y-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in Figure 9.3.16, angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$$\sin(t) = \sin(\alpha) \quad \text{and} \quad \cos(t) = -\cos(\alpha)$$
$$\sin(t) = -\sin(\beta) \quad \text{and} \quad \cos(t) = \cos(\beta)$$

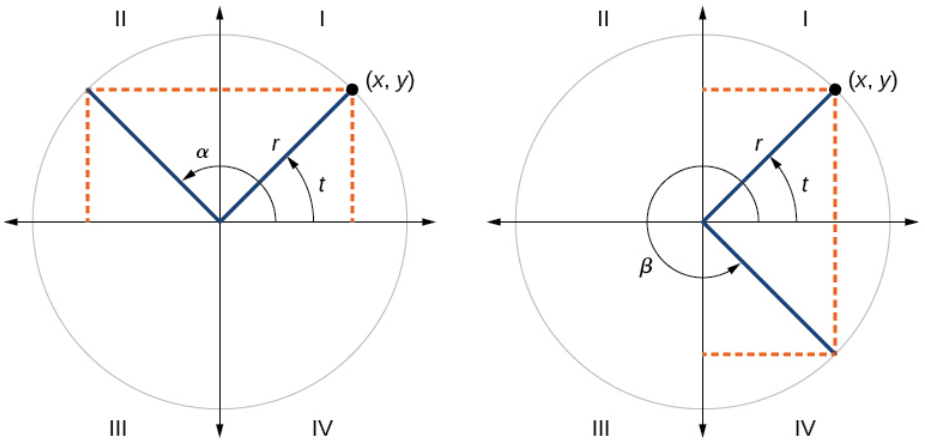


Figure 9.3.16

Recall that an angle's reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and 90°, or 0 and $\frac{\pi}{2}$ radians. As we can see from Figure $9.3.17$, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

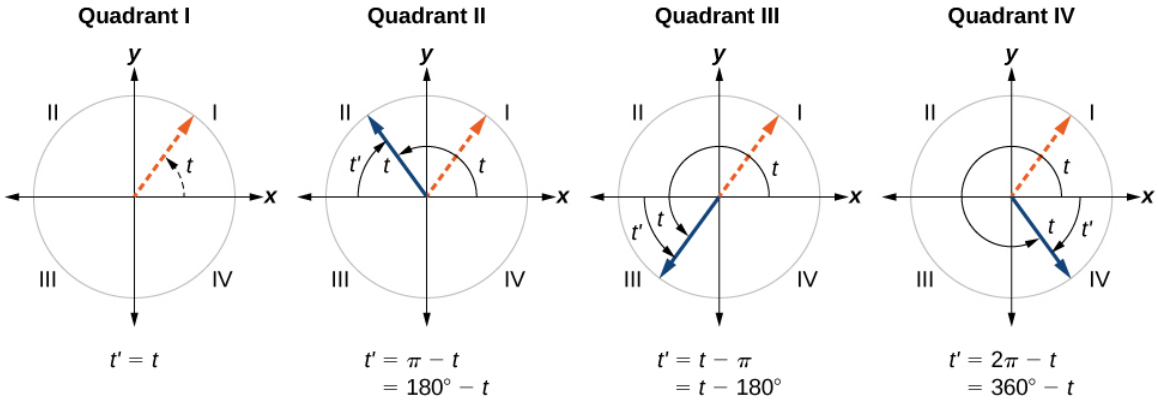


Figure 9.3.17

How To: Given an angle between 0 and 2π, find its reference angle

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180^\circ - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360^\circ - t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Example $9.3.5$: Finding a Reference Angle

Find the reference angle of 225° as shown in Figure $9.3.18$

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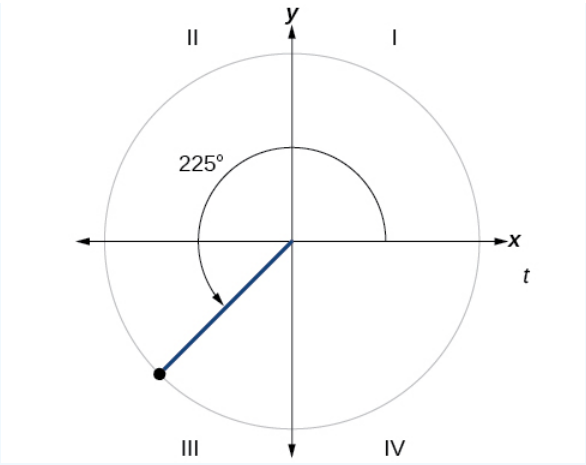


Figure 9.3.18

Solution

Because 225° is in the third quadrant, the reference angle is

$$|(180^\circ - 225^\circ)| = |-45^\circ| = 45^\circ$$

Try It 9.3.5

Find the reference angle of $\frac{5\pi}{3}$.

Answer

(\dfrac{π}{3}\)

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a **negative angle**. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the **reference angle** of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x-values in that quadrant. The sine will be positive or negative depending on the sign of the y-values in that quadrant.

USING REFERENCE ANGLES TO FIND COSINE AND SINE

Angles have cosines and sines with the same absolute value as cosines and sines of their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To: Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

Example 9.3.6: Using Reference Angles to Find Sine and Cosine

1. Using a reference angle, find the exact value of $\cos(150^\circ)$ and $\sin(150^\circ)$.
2. Using the reference angle, find $\cos\ \frac{5\pi}{4}$ and $\sin\ \frac{5\pi}{4}$.

Solution

This tells us that 150° has the same sine and cosine values as 30°, except for the sign. We know that

Since 150° is in the second quadrant, the x-coordinate of the point on the circle is negative, so the cosine value is negative. The y-coordinate is positive, so the sine value is positive.

$\frac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\left|\frac{5\pi}{4}-\pi\right|=\frac{\pi}{4}$. The cosine and sine of $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$. In the third quadrant, both x and y are negative, so:

$$\cos\ \frac{5\pi}{4}=-\frac{\sqrt{2}}{2}\ \text{and}\ \sin\ \frac{5\pi}{4}=-\frac{\sqrt{2}}{2}$$

Try It 9.3.6

1. Use the reference angle of 315° to find $\cos(315^\circ)$ and $\sin(315^\circ)$.
2. Use the reference angle of $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$

Answer

1. $\cos(315^\circ)=\frac{\sqrt{2}}{2}, \sin(315^\circ)=\frac{-\sqrt{2}}{2}$
2. $\cos\left(-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}, \sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}$

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in Figure 9.3.19. Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

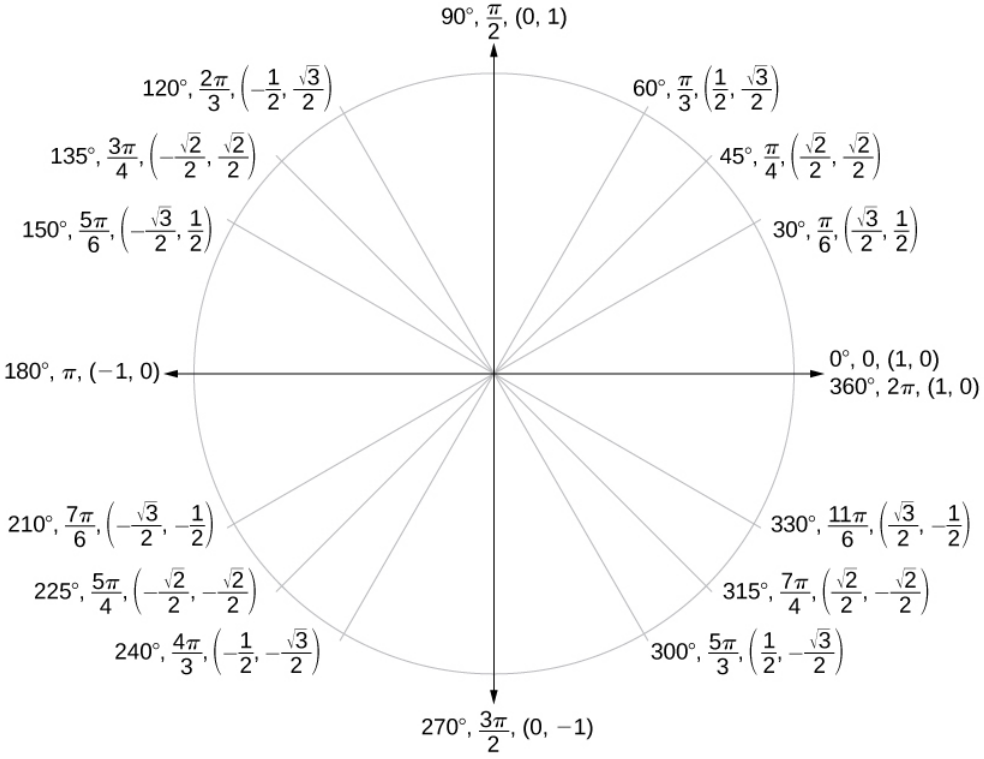


Figure 9.3.19: Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the **identities**

$$x = \cos t$$
$$y = \sin t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the **reference angle**, and give them the signs corresponding to the y- and x-values of the quadrant.

How To: Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point

1. Find the reference angle by measuring the smallest angle to the x-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Example 9.3.7: Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution

We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the x-axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π.

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$$

Next, we will find the cosine and sine of the reference angle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{3}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$
$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos\,\theta$ and $y = \sin\,\theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Try It 9.3.7

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Answer

$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

media

Access these online resources for additional instruction and practice with sine and cosine functions.

- Trigonometric Functions Using the Unit Circle
- Sine and Cosine from the Unit Circle
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four
- Trigonometric Functions Using Reference Angles

Key Equations

Cosine	$\cos\,t = x$
Sine	$\sin\,t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$

Key Concepts

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the x-value of the endpoint. See Example.
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See Example.
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See Example.
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See Example.
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the *x*- and y-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, t, formed by the terminal side of the angle t and the horizontal axis. See Example.

- Reference angles can be used to find the sine and cosine of the original angle. See Example.
- Reference angles can also be used to find the coordinates of a point on a circle. See Example.

Glossary

cosine function

the x-value of the point on a unit circle corresponding to a given angle

Pythagorean Identity

a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

sine function

the y-value of the point on a unit circle corresponding to a given angle

unit circle

a circle with a center at $(0, 0)$ and radius 1.

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9.4: The Other Trigonometric Functions

Learning Objectives

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.
- Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $\frac{1}{12}$ or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

To define the remaining functions, we will once again draw a unit circle with a point (x, y) corresponding to an angle of t,as shown in Figure 9.4.1. As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.

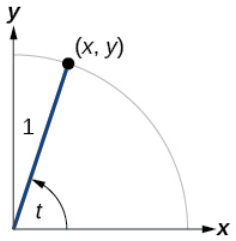


Figure 9.4.1

The first function we will define is the tangent. The **tangent** of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle. In Figure 9.4.1, the tangent of angle t is equal to $\frac{y}{x}, x \neq 0$. Because the *y*-value is equal to the sine of t, and the *x*-value is equal to the cosine of t, the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}, \cos t \neq 0$. The tangent function is abbreviated as tan. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In Figure 9.4.1, the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}, x \neq 0$. The secant function is abbreviated as sec.
- The **cotangent** function is the reciprocal of the tangent function. In Figure 9.4.1, the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}, y \neq 0$. The cotangent function is abbreviated as cot.
- The **cosecant** function is the reciprocal of the sine function. In Figure 9.4.1, the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}, y \neq 0$. The cosecant function is abbreviated as csc.

TANGENT, SECANT, COSECANT, AND COTANGENT FUNCTIONS

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x}, x \neq 0 \tag{9.4.1}$$

$$\sec t = \frac{1}{x}, x \neq 0 \tag{9.4.2}$$

$$\csc t = \frac{1}{y}, y \neq 0 \tag{9.4.3}$$

$$\cot t = \frac{x}{y}, y \neq 0 \tag{9.4.4}$$

Example 9.4.1: Finding Trigonometric Functions from a Point on the Unit Circle

The point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ is on the unit circle, as shown in Figure 9.4.2. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

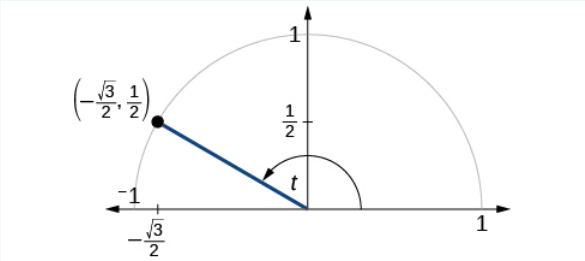


Figure 9.4.2

Solution

Because we know the (x, y) coordinates of the point on the unit circle indicated by angle t, we can use those coordinates to find the six functions:

$$\sin t = y = \frac{1}{2}$$

$$\cos t = x = -\frac{\sqrt{3}}{2}$$

$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\sec t = \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\csc t = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot t = \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$

Exercise 9.4.1:

The point $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ is on the unit circle, as shown in Figure 9.4.3. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

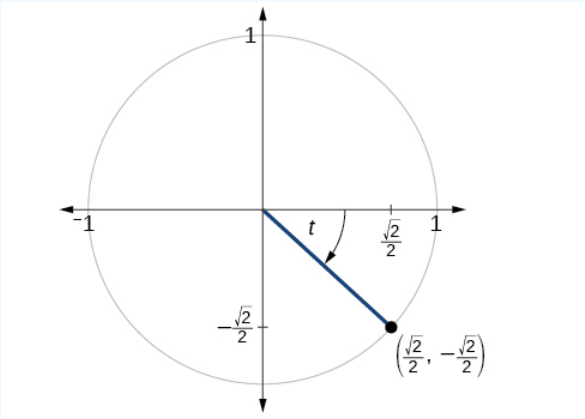


Figure 9.4.3

Solution

$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = -1, \sec t = \sqrt{2}, \csc t = -\sqrt{2}, \cot t = -1$

Example 9.4.2: Finding the Trigonometric Functions of an Angle

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t = \frac{\pi}{6}$.

Solution

We have previously used the properties of equilateral triangles to demonstrate that $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\begin{aligned}
\tan\frac{\pi}{6} &= \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} \\
&= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\
\sec\frac{\pi}{6} &= \frac{1}{\cos\frac{\pi}{6}} \\
&= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\
\csc\frac{\pi}{6} &= \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2 \\
\cot\frac{\pi}{6} &= \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} \\
&= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}
\end{aligned}$$

Exercise 9.4.2:

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t = \frac{\pi}{3}$.

Solution

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \tag{9.4.5}$$

$$\cos\frac{\pi}{3} = \frac{1}{2} \tag{9.4.6}$$

$$\tan\frac{\pi}{3} = \sqrt{3} \tag{9.4.7}$$

$$\sec\frac{\pi}{3} = 2 \tag{9.4.8}$$

$$\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3} \tag{9.4.9}$$

$$\cot\frac{\pi}{3} = \frac{\sqrt{3}}{3} \tag{9.4.10}$$

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x x equal to the cosine and y y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in Table 9.4.1.

Table 9.4.1

Angle	**0**	$\frac{\pi}{6}$, or **30**$^\circ$	$\frac{\pi}{4}$, or **45**$^\circ$	$\frac{\pi}{3}$, or **60**$^\circ$	$\frac{\pi}{2}$, or **90**$^\circ$
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the **reference angle** formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by *x*- and *y*-values in the original quadrant. Figure 9.4.4 shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "A Smart Trig Class." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive. In quadrant II, "**S**mart," only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, "**T**rig," only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lass," only **c**osine and its reciprocal function, secant, are positive.

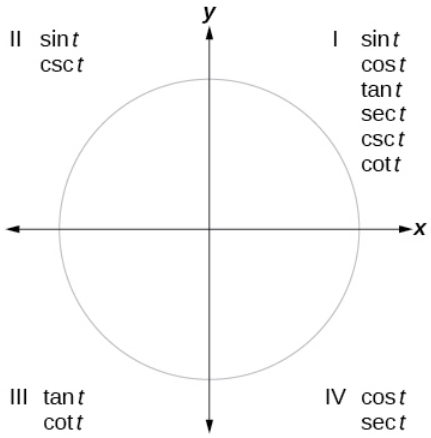


Figure 9.4.4

HOWTO: Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Example 9.4.3: Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Solution

The angle between this angle's terminal side and the *x*-axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both x and y are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$$\cos(-\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}, \sin(-\frac{5\pi}{6}) = -\frac{1}{2}, \tan(-\frac{5\pi}{6}) = \frac{\sqrt{3}}{3} \quad (9.4.11)$$

$$\sec(-\frac{5\pi}{6}) = -\frac{2\sqrt{3}}{3}, \csc(-\frac{5\pi}{6}) = -2, \cot(-\frac{5\pi}{6}) = \sqrt{3} \quad (9.4.12)$$

Exercise 9.4.3

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Solution

$\sin(-\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{-7\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{-7\pi}{4}) = 1,$

$\sec(\frac{-7\pi}{4}) = \sqrt{2}, \csc(\frac{-7\pi}{4}) = \sqrt{2}, \cot(\frac{-7\pi}{4}) = 1$

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard.

Consider the function $f(x) = x^2$, shown in Figure 9.4.5. The graph of the function is symmetrical about the *y*-axis. All along the curve, any two points with opposite *x*-values have the same function value. This matches the result of calculation: $(4)^2 = (-4)^2, (-5)^2 = (5)^2$, and so on. So $f(x) = x^2$ is an **even function**, a function such that two inputs that are opposites have the same output. That means $f(-x) = f(x)$.

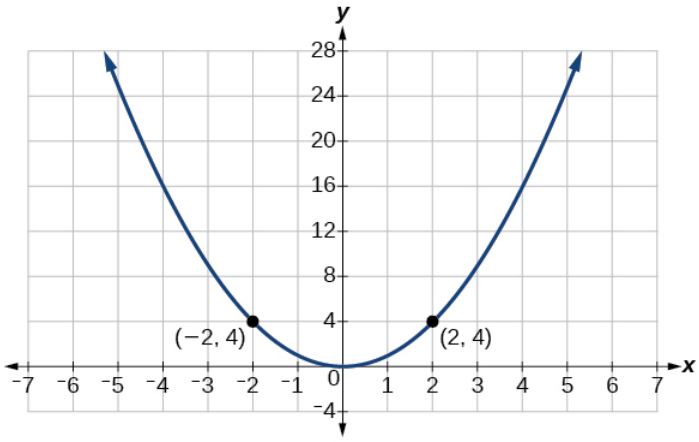


Figure 9.4.5: The function $f(x) = x^2$ is an even function.

Now consider the function $f(x) = x^3$, shown in Figure 9.4.6. The graph is not symmetrical about the *y*-axis. All along the graph, any two points with opposite *x*-values also have opposite *y*-values. So $f(x) = x^3$ is an **odd function**, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.

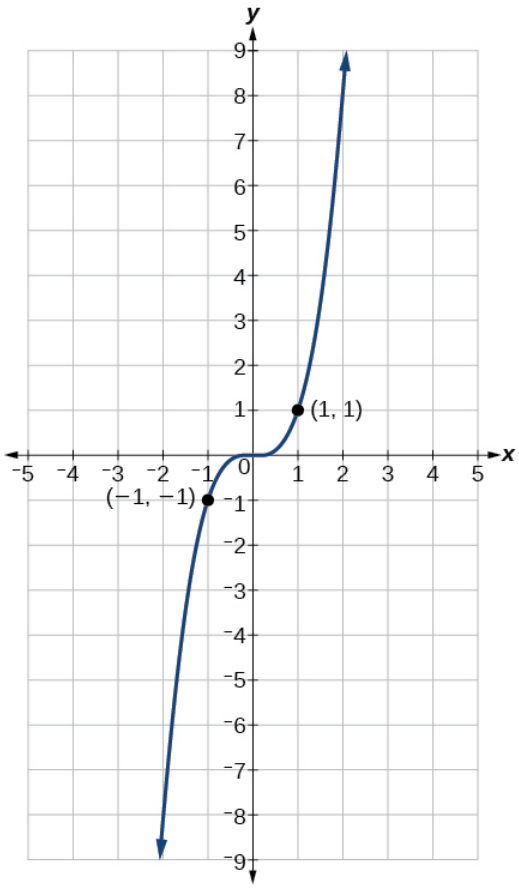


Figure 9.4.6: The function $f(x) = x^3$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a **unit circle** with a positive and a negative angle, as in Figure 9.4.7. The sine of the positive angle is y. The sine of the negative angle is $-y$. The **sine function**, then, is an odd function. We can test each of the six trigonometric functions in this fashion. The results are shown in Table 9.4.2.

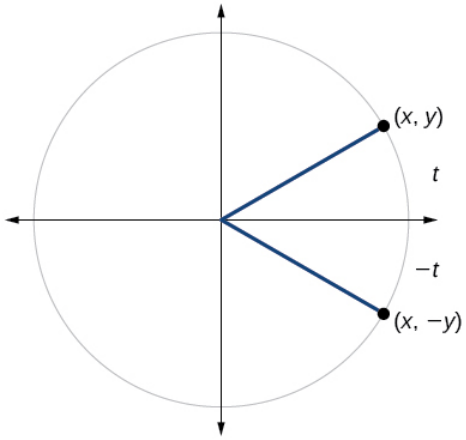


Figure 9.4.7

Table 9.4.2

$$\sin t = y \quad (9.4.13)$$
$$\sin(-t) = -y \quad (9.4.14)$$
$$\sin t \neq sin(-t) \quad (9.4.15)$$

$$\cos t = x \quad (9.4.16)$$
$$\cos(-t) = x \quad (9.4.17)$$
$$\cos t = \cos(-t) \quad (9.4.18)$$

$$\tan(t) = \frac{y}{x} \quad (9.4.19)$$
$$\tan(-t) = -\frac{y}{x} \quad (9.4.20)$$
$$\tan t \neq \tan(-t) \quad (9.4.21)$$

$$\sec t = \frac{1}{x} \quad (9.4.22)$$
$$\sec(-t) = \frac{1}{x} \quad (9.4.23)$$
$$\sec t = \sec(-t) \quad (9.4.24)$$
$$\csc t = \frac{1}{y} \quad (9.4.25)$$
$$\csc(-t) = \frac{1}{-y} \quad (9.4.26)$$
$$\csc t \neq \csc(-t) \quad (9.4.27)$$
$$\cot t = \frac{x}{y} \quad (9.4.28)$$
$$\cot(-t) = \frac{x}{-y} \quad (9.4.29)$$
$$\cot t \neq \cot(-t) \quad (9.4.30)$$

EVEN AND ODD TRIGONOMETRIC FUNCTIONS

- An **even function** is one in which $f(-x) = f(x)$.
- An **odd function** is one in which $f(-x) = -f(x)$.

Cosine and secant are even:

$$\cos(-t) = \cos t \quad (9.4.31)$$
$$\sec(-t) = \sec t \quad (9.4.32)$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t \quad (9.4.33)$$
$$\tan(-t) = -\tan t \quad (9.4.34)$$
$$\csc(-t) = -\csc t \quad (9.4.35)$$
$$\cot(-t) = -\cot t \quad (9.4.36)$$

Example 9.4.4: Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t is 2, what is the secant of $-t$?

Solution

Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Exercise 9.4.4:

If the cotangent of angle t is $\sqrt{3}$, what is the cotangent of $-t$?

Solution

$-\sqrt{3}$

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

FUNDAMENTAL IDENTITIES

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t} \quad (9.4.37)$$
$$\sec t = \frac{1}{\cos t} \quad (9.4.38)$$
$$\csc t = \frac{1}{\sin t} \quad (9.4.39)$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t} \tag{9.4.40}$$

Example 9.4.5: Using Identities to Evaluate Trigonometric Functions

a. Given $\sin(45^\circ) = \frac{\sqrt{2}}{2}, \cos(45^\circ) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45^\circ)$.

b. Given $\sin(\frac{5\pi}{6}) = \frac{1}{2}, \cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$, evaluate $\sec(\frac{5\pi}{6})$.

Solution

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a.
$$\begin{aligned} \tan(45^\circ) &= \frac{\sin(45^\circ)}{\cos(45^\circ)} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1 \end{aligned}$$

b.
$$\begin{aligned} \sec(\frac{5\pi}{6}) &= \frac{1}{\cos(\frac{5\pi}{6})} \\ &= \frac{1}{-\frac{\sqrt{3}}{2}} \\ &= \frac{-2\sqrt{3}}{1} \\ &= \frac{-2}{\sqrt{3}} \\ &= -\frac{2\sqrt{3}}{3} \end{aligned}$$

Exercise 9.4.5

Evaluate $\csc(\frac{7\pi}{6})$.

Solution

-2

Example 9.4.6: Using Identities to Simplify Trigonometric Expressions

Simplify $\frac{\sec t}{\tan t}$.

Solution

We can simplify this by rewriting both functions in terms of sine and cosine.

$$\begin{aligned} \frac{\sec t}{\tan t} &= \frac{1/\cos t}{\sin t/\cos t} && \text{To divide the functions, we multiply by the reciprocal.} \\ &= \frac{1}{\cos t}\frac{\cos t}{\sin t} && \text{Divide out the cosines.} \\ &= \frac{1}{\sin t} && \text{Simplify and use the identity.} \\ &= \csc t \end{aligned} \tag{9.4.41}$$

By showing that $\frac{\sec t}{\tan t}$ can be simplified to $\csc t$,we have, in fact, established a new identity.

$$\frac{\sec t}{\tan t} = \csc t$$

Exercise 9.4.6

Simplify $(\tan t)(\cos t)$.

Solution

$\sin t$

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternative forms of the **Pythagorean Identity**, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \tag{9.4.42}$$

$$1 + \tan^2 t = \sec^2 t \tag{9.4.43}$$

The other form is obtained by dividing both sides by $\sin^2 t$:

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \tag{9.4.44}$$

$$\cot^2 t + 1 = \csc^2 t \tag{9.4.45}$$

ALTERNATE FORMS OF THE PYTHAGOREAN IDENTITY

$$1 + \tan^2 t = \sec^2 t \tag{9.4.46}$$

$$\cot^2 t + 1 = \csc^2 t \tag{9.4.47}$$

Example 9.4.7: Using Identities to Relate Trigonometric Functions

If cos(t)=1213 cos(t)=1213 and t t is in quadrant IV, as shown in Figure 9.4.8, find the values of the other five trigonometric functions.

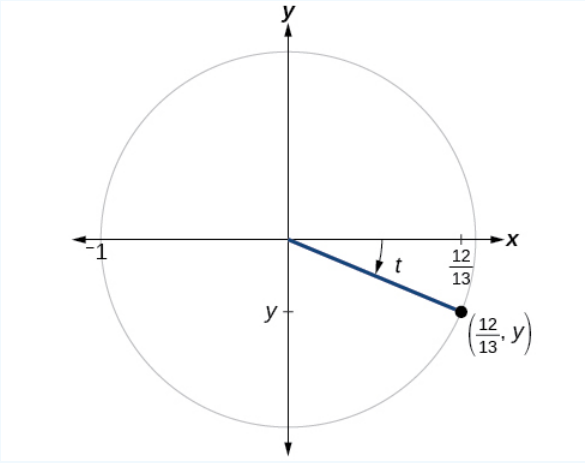


Figure 9.4.8

Solution

We can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$, and the remaining functions by relating them to sine and cosine.

$$(\frac{12}{13})^2+\sin^2 t=1 \tag{9.4.48}$$

$$\sin^2 t=1-(\frac{12}{13})^2 \tag{9.4.49}$$

$$\sin^2 t=1-\frac{144}{169} \tag{9.4.50}$$

$$\sin^2 t=\frac{25}{169} \tag{9.4.51}$$

$$\sin t=\pm\sqrt{\frac{25}{169}} \tag{9.4.52}$$

$$\sin t=\pm\frac{\sqrt{25}}{\sqrt{169}} \tag{9.4.53}$$

$$\sin t=\pm\frac{5}{13} \tag{9.4.54}$$

The sign of the sine depends on the *y*-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the *y*-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t=\frac{\sin t}{\cos t}=\frac{-\frac{5}{13}}{\frac{12}{13}}=-\frac{5}{12} \tag{9.4.55}$$

$$\sec t=\frac{1}{\cos t}=\frac{1}{\frac{12}{13}}=\frac{13}{12} \tag{9.4.56}$$

$$\csc t=\frac{1}{\sin t}=\frac{1}{-\frac{5}{13}}=-\frac{13}{5} \tag{9.4.57}$$

$$\cot t=\frac{1}{\tan t}=\frac{1}{-\frac{5}{12}}=-\frac{12}{5} \tag{9.4.58}$$

Exercise 9.4.7:

If $\sec(t)=-\frac{17}{8}$ and $0<t<\pi,$ find the values of the other five functions.

Solution

$\cos t=-\frac{8}{17}, \sin t=\frac{15}{17}, \tan t=-\frac{15}{8}$

$\csc t=\frac{17}{15}, \cot t=-\frac{8}{15}$

As we discussed in the chapter opening, a function that repeats its values in regular intervals is known as a **periodic function**. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π,will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x+4)=f(x)$. This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

PERIOD OF A FUNCTION

The **period** P of a repeating function f f is the number representing the interval such that $f(x+P)=f(x)$ for any value of x.

The period of the cosine, sine, secant, and cosecant functions is 2π.

The period of the tangent and cotangent functions is π.

Example 9.4.8: Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t based on Figure 9.4.9.

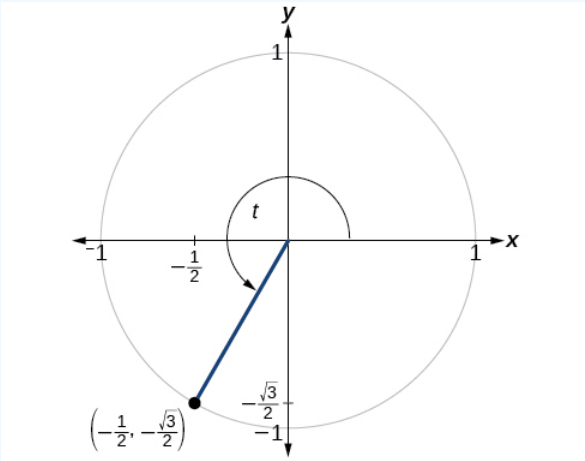


Figure 9.4.9

Solution

$$\sin t = y = -\frac{\sqrt{3}}{2}$$

$$\cos t = x = -\frac{1}{2}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -2$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 9.4.8

Find the values of the six trigonometric functions of angle t based on Figure 9.4.10.

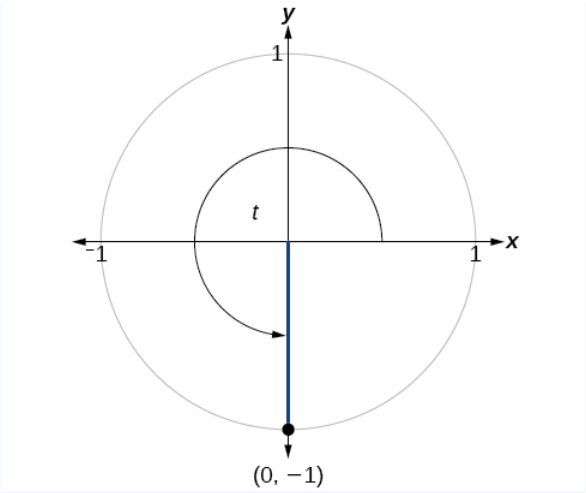


Figure 9.4.10

Solution

$$\sin t = -1, \cos t = 0, \tan t = \text{Undefined} \tag{9.4.59}$$

$$\tag{9.4.60}$$

$$sect = \text{Undefined}, \csc t = -1, \cot t = 0 \tag{9.4.61}$$

Example 9.4.9: Finding the Value of Trigonometric Functions

If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$,find $\sec(t), \csc(t), \tan(t), \cot(t)$.

Solution

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2 \tag{9.4.62}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} - \frac{2\sqrt{3}}{3} \tag{9.4.63}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \tag{9.4.64}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3} \tag{9.4.65}$$

Exercise 9.4.9:

If $\sin(t) = \frac{\sqrt{2}}{2}$ and $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sec(t), \csc(t), \tan(t)$, and $\cot(t)$.

Solution

$\sec t = \sqrt{2}, \csc t = \sqrt{2}, \tan t = 1, \cot t = 1$

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the TAN key. For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of 30°, we could press

$$(\text{for a scientific calculator}): \frac{1}{30 \times \frac{\pi}{180}} \text{COS} \tag{9.4.66}$$

or

$$(\text{for a graphing calculator}): \frac{1}{\cos(\frac{30\pi}{180})} \tag{9.4.67}$$

how to: Given an angle measure in radians, use a scientific calculator to find the cosecant

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Enter the value of the angle inside parentheses.
4. Press the SIN key.
5. Press the = key.

how to: Given an angle measure in radians, use a graphing utility/calculator to find the cosecant

1. If the graphing utility has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Press the SIN key.
4. Enter the value of the angle inside parentheses.
5. Press the ENTER key.

Example 9.4.10: Evaluating the Cosecant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

Solution

For a scientific calculator, enter information as follows:

$$1/(5 \times \pi/7)\text{SIN} = \tag{9.4.68}$$

$$\csc(\frac{5\pi}{7}) \approx 1.279 \tag{9.4.69}$$

Exercise 9.4.10:

Evaluate the cotangent of $-\frac{\pi}{8}$.

≈ -2.414

media

Access these online resources for additional instruction and practice with other trigonometric functions.

- Determining Trig Function Values
- More Examples of Determining Trig Functions
- Pythagorean Identities
- Trig Functions on a Calculator

Key Equations

Tangent function	$\tan t = \frac{\sin t}{\cos t}$
Secant function	$\sec t = \frac{1}{\cos t}$

Cosecant function	$\csc t = \frac{1}{\sin t}$
Cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Key Concepts

- The tangent of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle. See Example.
- Trigonometric functions can also be found from an angle. See Example.
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See Example.
- A function is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See Example.
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See Example and Example.
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See Example.
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f f is the smallest interval such that $f(x + P) = f(x)$ for any value of x.
- The values of trigonometric functions of special angles can be found by mathematical analysis.
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software. See Example.

Glossary

cosecant

the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y}, y \neq 0$

cotangent

the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y}, y \neq 0$

identities

statements that are true for all values of the input on which they are defined

period

the smallest interval P of a repeating function f such that $f(x + P) = f(x)$

secant

the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x}, x \neq 0$

tangent

the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x}, x \neq 0$

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CHAPTER OVERVIEW

10: Periodic Functions

In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

10.1: Graphs of the Sine and Cosine Functions
10.2: Graphs of the Other Trigonometric Functions
10.3: Inverse Trigonometric Functions

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10.1: Graphs of the Sine and Cosine Functions

Learning Objectives

- Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
- Use phase shifts of sine and cosine curves.

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.



Figure 10.1.1: Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/ Flickr)

Light waves can be represented graphically by the sine function. In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the x- and y-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. Table 10.1.1lists some of the values for the sine function on a unit circle.

Table 10.1.1

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Plotting the points from the table and continuing along the *x*-axis gives the shape of the sine function. See Figure 10.1.2

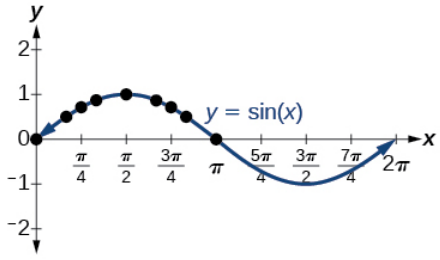


Figure 10.1.2: The sine function

Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in

quadrants III and IV on the unit circle. See Figure 10.1.3

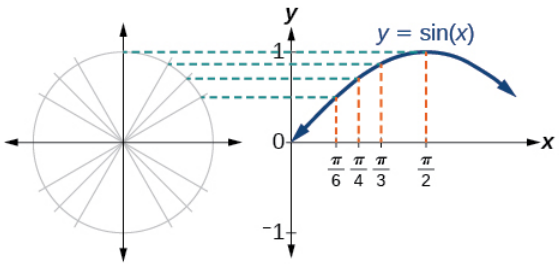


Figure 10.1.3: Plotting values of the sine function

Now let's take a similar look at the **cosine function**. Again, we can create a table of values and use them to sketch a graph. Table 10.1.2 lists some of the values for the cosine function on a unit circle.

Table 10.1.2

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

As with the sine function, we can plots points to create a graph of the cosine function as in Figure 10.1.4

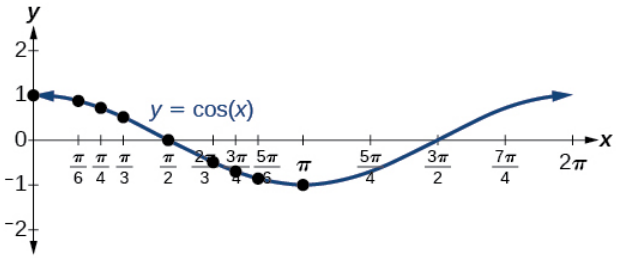


Figure 10.1.4: The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after 2π, which means the functions are periodic with a period of 2π. A **periodic function** is a function for which a specific **horizontal shift**, P, results in a function equal to the original function: $f(x+P) = f(x)$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift with $P > 0$ the **period** of the function. Figure 10.1.5 shows several periods of the sine and cosine functions.

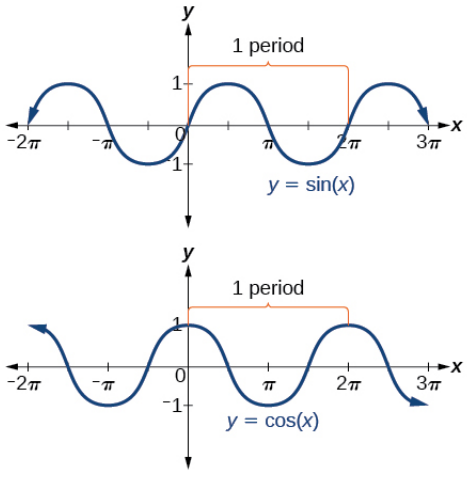


Figure 10.1.5

https://math.libretexts.org/@go/page/186174

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. As we can see in Figure 10.1.6, the sine function is symmetric about the origin. Recall from The Other Trigonometric Functions that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin\ x$. Now we can clearly see this property from the graph.

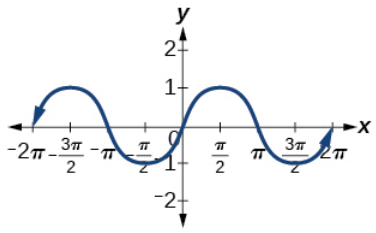


Figure 10.1.6: Odd symmetry of the sine function

Figure 10.1.7 shows that the cosine function is symmetric about the y-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x) = \cos\ x$.

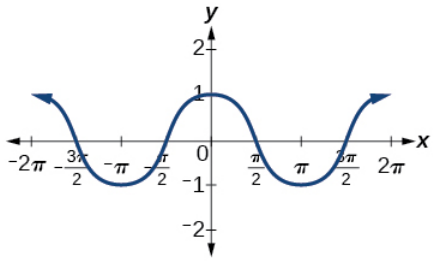


Figure 10.1.7: Even symmetry of the cosine function

CHARACTERISTICS OF SINE AND COSINE FUNCTIONS

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin\ x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos\ x$ is symmetric about they- y-axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or **cosine function** is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D \tag{10.1.1}$$

and

$$y = A\cos(Bx - C) + D \tag{10.1.2}$$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \dfrac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π,which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π

and the graph is compressed. If $f(x)=\sin\left(\frac{x}{2}\right)$, then $B=\frac{1}{2}$, so the period is 4π and the graph is stretched. Notice in Figure 10.1.8how the period is indirectly related to $|B|$.

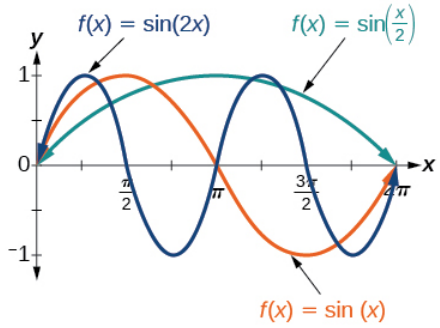


Figure 10.1.8

PERIOD OF SINUSOIDAL FUNCTIONS

If we let $C=0$ and $D=0$ in the general form equations of the sine and cosine functions, we obtain the forms

- $y=A\sin(Bx)$
- $y=A\cos(Bx)$

The period is $\frac{2\pi}{|B|}$.

Example 10.1.1: Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x)=\sin\left(\frac{\pi}{6}x\right)$.

Solution

Let's begin by comparing the equation to the general form $y=A\sin(Bx)$.

In the given equation, $B=\frac{\pi}{6}$, so the period will be

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{6}} \\ &= 2\pi\cdot\frac{6}{\pi} \\ &= 12 \end{aligned}$$

Exercise 10.1.1

Determine the period of the function $g(x)=\cos(\frac{x}{3})$.

Answer

6π

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical **midline** of the

graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the x-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4sinx$ is twice the amplitude of

$$f(x) = 2\sin x$$

If $|A| < 1$, the function is compressed. Figure $10.1.9$ compares several sine functions with different amplitudes.

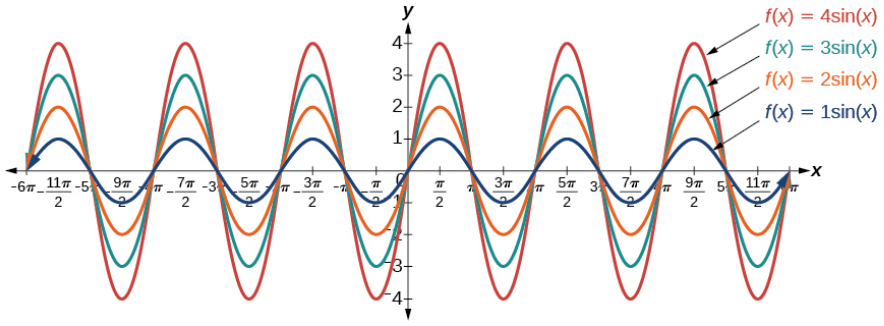


Figure 10.1.9

AMPLITUDE OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A\sin(Bx) \text{ and } y = A\cos(Bx) \tag{10.1.3}$$

The amplitude is A, and the vertical height from the midline is $|A|$. In addition, notice in the example that

$$|A| = amplitude = \frac{1}{2}\,|\,maximum - minimum| \tag{10.1.4}$$

Example 10.1.2: Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Solution

Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

Analysis

The negative value of A results in a reflection across the x-axis of the sine function, as shown in Figure $10.1.10$

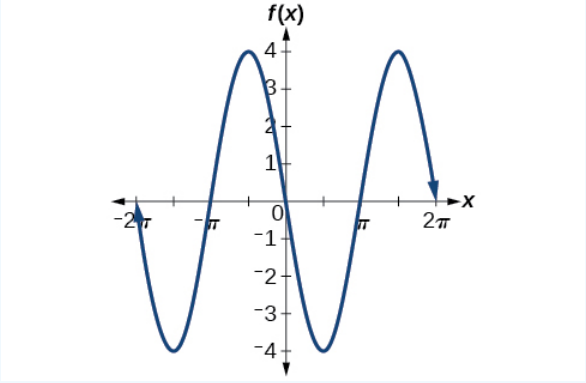


Figure 10.1.10

Exercise 10.1.2

What is the amplitude of the sinusoidal function $f(x) = \frac{1}{2}\sin(x)$? Is the function stretched or compressed vertically?

Answer

$\frac{1}{2}$ compressed

Analyzing Graphs of Variations of $y = \sin\ x$ and $y = \cos\ x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$$y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D \tag{10.1.5}$$

or

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \quad \text{and} \quad y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D \tag{10.1.6}$$

The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. Figure 10.1.11shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by π units, which is more than we see in the graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

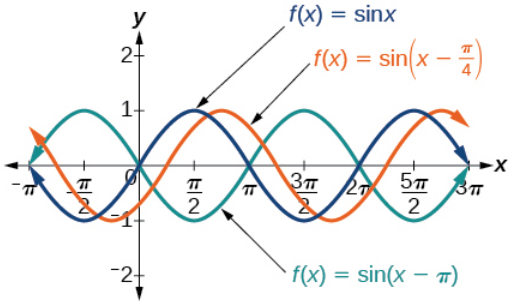


Figure 10.1.11

While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See Figure 10.1.12 The function $y = \cos(x) + D$ has its midline at $y = D$.

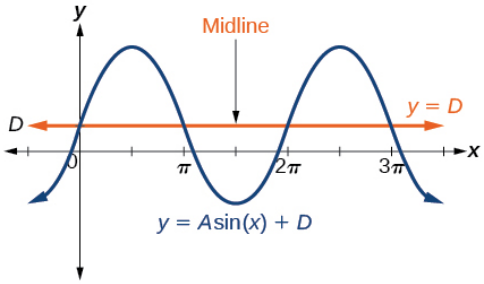


Figure 10.1.12

Any value of D other than zero shifts the graph up or down. Figure 10.1.13compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

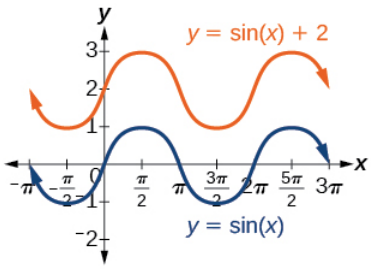


Figure 10.1.13

VARIATIONS OF SINE AND COSINE FUNCTIONS

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{B}$ is the **phase shift** and D is the **vertical shift.**

Example 10.1.3: Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

In the given equation, notice that $B = 1$ and $C = -\frac{\pi}{6}$. So the phase shift is

$$\begin{aligned} \frac{C}{B} &= -\frac{\frac{\pi}{6}}{1} \\ &= -\frac{\pi}{6} \end{aligned}$$

or $\frac{\pi}{6}$ units to the left.

Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore $f(x) = \sin(x + \frac{\pi}{6}) - 2$ can be rewritten as $f(x) = \sin\left(x - \left(-\frac{\pi}{6}\right)\right) - 2$. If the value of C is negative, the shift is to the left.

Exercise 10.1.3

Determine the direction and magnitude of the phase shift for $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$.

Answer

$\frac{\pi}{2}$; right

Example 10.1.4: Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for $f(x) = \cos(x) - 3$.

Solution

Let's begin by comparing the equation to the general form $y = A\cos(Bx - C) + D$.

In the given equation, $D = -3$ so the shift is 3 units downward.

Exercise 10.1.4

Determine the direction and magnitude of the vertical shift for $f(x) = 3\sin(x) + 2$.

Answer

2 units up

How to: Given a sinusoidal function in the form $f(x) = A\sin(Bx - C) + D$,identify the midline, amplitude, period, and phase shift

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$.
4. Determine the midline as $y = D$.

Example 10.1.5: Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

$A = 3$, so the amplitude is $|A| = 3$.

Next, $B = 2$, so the period is $P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{2} = \pi$.

There is no added constant inside the parentheses, so $C = 0$ and the phase shift is $\dfrac{C}{B} = \dfrac{0}{2} = 0$.

Finally, $D = 1$, so the midline is $y = 1$.

Analysis

Inspecting the graph, we can determine that the period is π, the midline is $y = 1$, and the amplitude is 3. See Figure 10.1.14

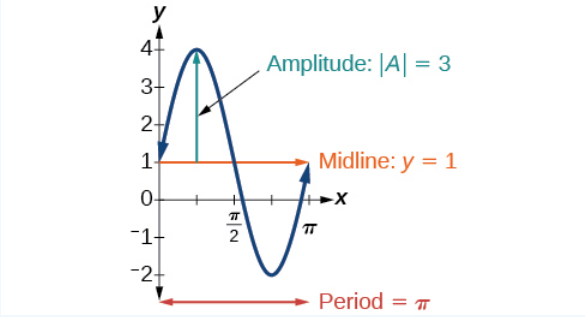


Figure 10.1.14

Exercise 10.1.5

Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$.

Answer

midline: $y = 0$; amplitude: $|A| = \frac{1}{2}$; period: $P = \frac{2\pi}{|B|} = 6\pi$; phase shift: $\frac{C}{B} = \pi$

Example 10.1.6: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function in Figure 10.1.15

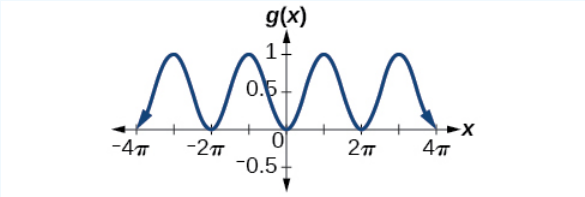


Figure 10.1.15

Solution

To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$$y = A\sin(Bx - C) + D$$

$$y = A\cos(Bx - C) + D$$

The graph could represent either a sine or a **cosine function** that is shifted and/or reflected. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the cosine function has an extreme point for $x = 0$, let us write our equation in terms of a cosine function.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the midline, is D in the equation, so $D = 0.5$.

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5$. Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2} = 0.5$. Also, the graph is reflected about the x-axis so that $A = -0.5$.

The graph is not horizontally stretched or compressed, so $B = 1$; and the graph is not shifted horizontally, so $C = 0$.

Putting this all together,

$$g(x) = -0.5\cos(x) + 0.5$$

Exercise 10.1.6

Determine the formula for the sine function in Figure 10.1.16

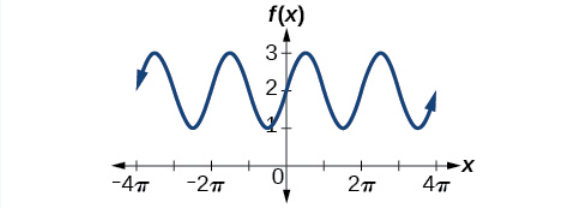


Figure 10.1.16

Answer

$f(x) = \sin(x) + 2$

Example 10.1.7: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in Figure 10.1.17

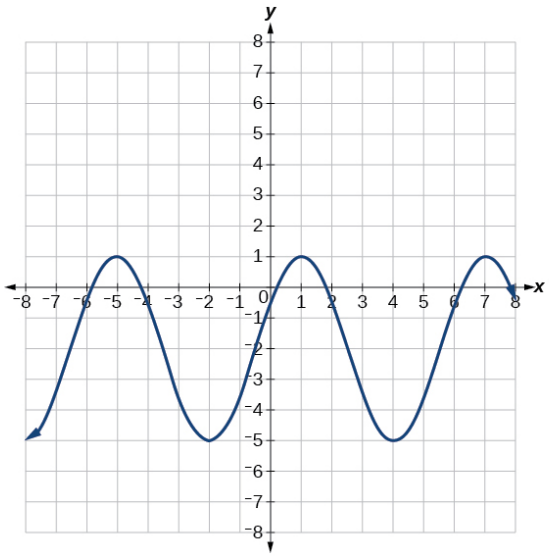


Figure 10.1.17

Solution

With the highest value at 1 and the lowest value at -5, the midline will be halfway between at -2. So $D=-2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A|=3$.

The period of the graph is 6, which can be measured from the peak at $x=1$ to the next peak at $x=7$,or from the distance between the lowest points. Therefore, $P=\dfrac{2\pi}{|B|}=6$. Using the positive value for B,we find that

$$\begin{aligned} B &= \frac{2\pi}{P} \\ &= \frac{2\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

So far, our equation is either $y=3\sin\left(\dfrac{\pi}{3}x-C\right)-2$ or $y=3\cos\left(\dfrac{\pi}{3}x-C\right)-2$.For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes

$$\begin{aligned} y &= 3\cos\left(\frac{\pi}{3}x-\frac{\pi}{3}\right)-2 \qquad \text{or} \\ y &= -3\cos\left(\frac{\pi}{3}x+\frac{2\pi}{3}\right)-2 \end{aligned}$$

Again, these functions are equivalent, so both yield the same graph.

? Exercise 10.1.7

Write a formula for the function graphed in Figure 10.1.18

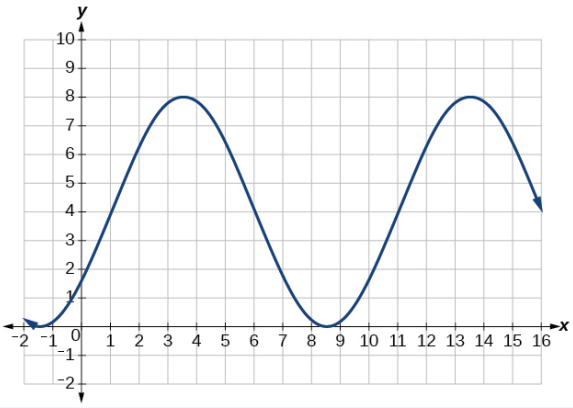


Figure 10.1.18

Answer

two possibilities: $y = 4\sin\left(\frac{\pi}{5}x - \frac{\pi}{5}\right) + 4$ or $y = -4\sin\left(\frac{\pi}{5}x + \frac{4\pi}{5}\right) + 4$

Graphing Variations of $y = \sin\ x$ and $y = \cos\ x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

Given the function $y = A\sin(Bx)$, sketch its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \frac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \frac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the *x*-axis at $x = \frac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \frac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the *x*-axis at $x = \frac{\pi}{2|B|}$.

Example 10.1.8: Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin\left(\frac{\pi x}{2}\right)$.

Solution

Let's begin by comparing the equation to the form $y = A\sin(Bx)$.

- *Step 1.* We can see from the equation that $A = -2$, so the amplitude is 2.

  $|A| = 2$

- *Step 2.* The equation shows that $B = \frac{\pi}{2}$, so the period is

$$P = \frac{2\pi}{\frac{\pi}{2}}$$
$$= 2\pi \cdot \frac{2}{\pi}$$
$$= 4$$

- *Step 3.* Because A is negative, the graph descends as we move to the right of the origin.
- *Step 4.* The x-intercepts are at the beginning of one period, $x = 0$, the horizontal midpoints are at $x = 2$ and at the end of one period at $x = 4$.

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3$. A local minimum will occur 2 units below the midline, at $x = 1$, and a local maximum will occur at 2 units above the midline, at $x = 3$. Figure 10.1.19 shows the graph of the function.

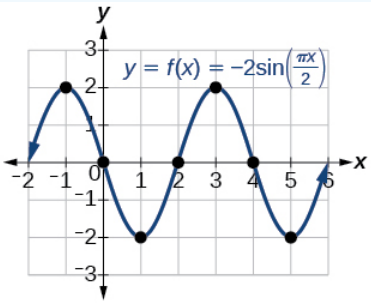


Figure 10.1.19

Exercise 10.1.8

Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, period, and phase shift.

Answer

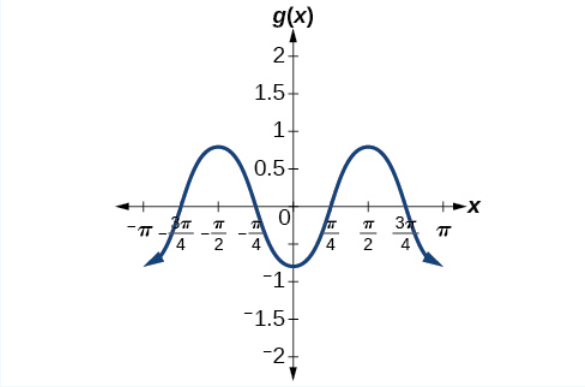


Figure 10.1.20

midline: $y = 0$; amplitude: $|A| = 0.8$; period: $P = \frac{2\pi}{|B|} = \pi$; phase shift: $\frac{C}{B} = 0$ or none

How to: Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph

1. Express the function in the general form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \frac{2\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.

5. Draw the graph of $f(x) = A\sin(Bx)$ shifted to the right or left by $\frac{C}{B}$ and up or down by D.

Example 10.1.9: Graphing a Transformed Sinusoid

Sketch a graph of $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.

Solution

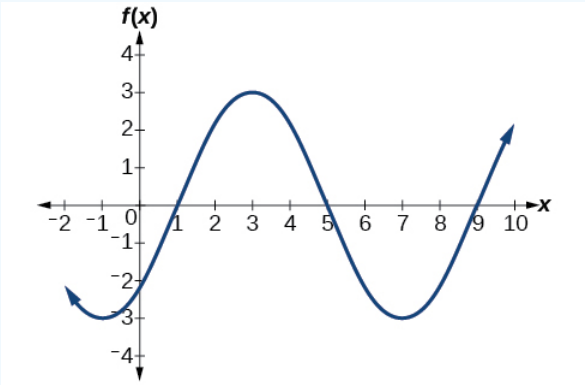


Figure 10.1.21: A horizontally compressed, vertically stretched, and horizontally shifted sinusoid

- *Step 1.* The function is already written in general form: $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.This graph will have the shape of a sine function, starting at the midline and increasing to the right.
- *Step 2.* $|A| = |3| = 3$. The amplitude is 3.
- *Step 3.* Since $|B| = |\frac{\pi}{4}| = \frac{\pi}{4}$, we determine the period as follows.

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{4}} \\ &= 2\pi \cdot \frac{4}{\pi} \\ &= 8 \end{aligned}$$

The period is 8.

- *Step 4.* Since $C = \frac{\pi}{4}$, the phase shift is

$$\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1 \tag{10.1.7}$$

.

The phase shift is 1 unit.

- *Step 5.* Figure 10.1.21shows the graph of the function.

Exercise 10.1.9

Draw a graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Answer

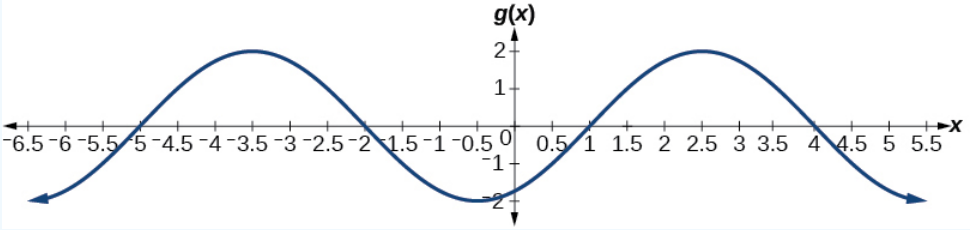


Figure 10.1.22

midline: $y = 0$; amplitude: $|A| = 2$; period: $P = \dfrac{2\pi}{|B|} = 6$; phase shift: $\dfrac{C}{B} = -\dfrac{1}{2}$

Example 10.1.10: Identifying the Properties of a Sinusoidal Function

Given $y = -2cos\left(\dfrac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Solution

Begin by comparing the equation to the general form.

$$y = A\cos(Bx - C) + D$$

- *Step 1.* The function is already written in general form.
- *Step 2.* Since $A = -2$, the amplitude is $|A| = 2$.
- *Step 3.* $|B| = \dfrac{\pi}{2}$, so the period is $P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{\frac{pi}{2}} = 2\pi \cdot \dfrac{2}{\pi} = 4$. The period is 4.
- *Step 4.* $C = -\pi$,so we calculate the phase shift as $\dfrac{C}{B} = -\dfrac{\pi}{\frac{\pi}{2}} = -\pi \cdot \dfrac{2}{\pi} = -2$. The phase shift is -2.
- *Step 5.* $D = 3$,so the midline is $y = 3$, and the vertical shift is up 3.

Since A is negative, the graph of the cosine function has been reflected about the x-axis.

Figure 10.1.23shows one cycle of the graph of the function.

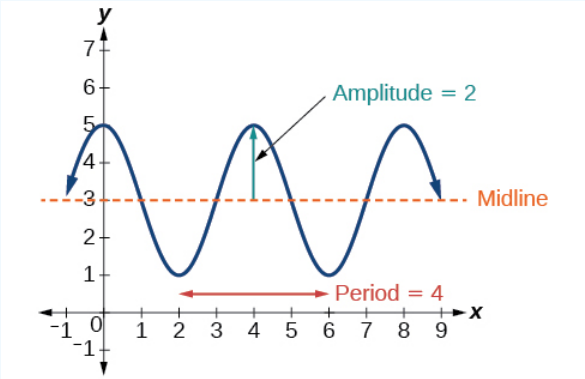


Figure 10.1.23

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter,**circular motion** can be modeled using either the sine or **cosine function**.

Example 10.1.11: Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution

Recall that, for a point on a circle of radius r, the y-coordinate of the point is $y = r\sin(x)$, so in this case, we get the equation $y(x) = 3\sin(x)$. The constant 3 causes a vertical stretch of the y-values of the function by a factor of 3, which we can see in the graph in Figure 10.1.24

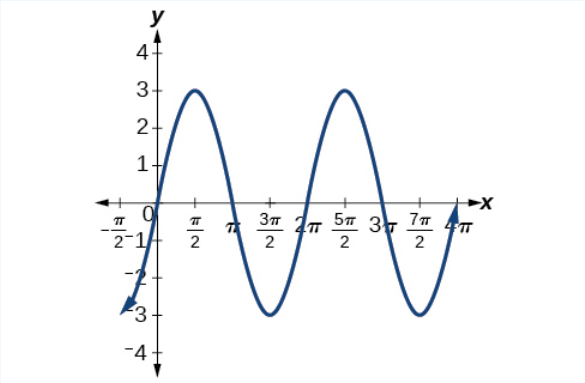


Figure 10.1.24

Analysis

Notice that the period of the function is still 2π; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi$,...Because the outputs of the graph will now oscillate between -3 and 3, the amplitude of the sine wave is 3.

? Exercise 10.1.10

What is the amplitude of the function $f(x) = 7\cos(x)$? Sketch a graph of this function.

Answer

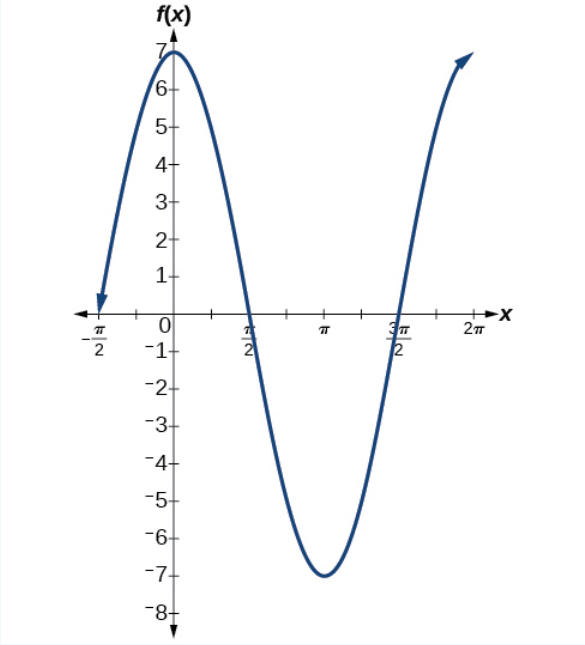


Figure 10.1.25

✔ Example 10.1.12: Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P, as shown in Figure 10.1.26 Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

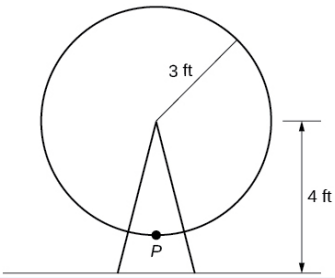


Figure 10.1.26

Solution

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in Figure 10.1.27.

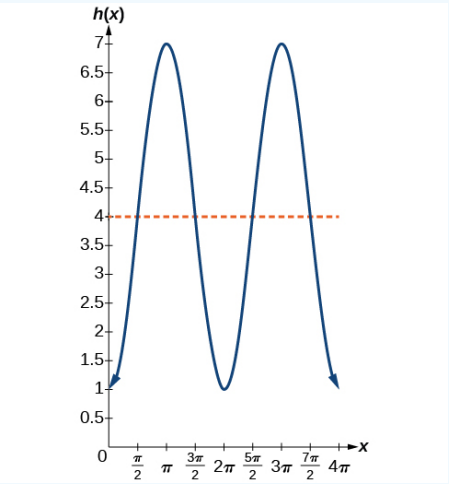


Figure 10.1.27

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a **sine function** starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

? Exercise 10.1.11

A weight is attached to a spring that is then hung from a board, as shown in Figure 10.1.28. As the spring oscillates up and down, the position y of the weight relative to the board ranges from -1 in. (at time $x = 0$) to -7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x. Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x.

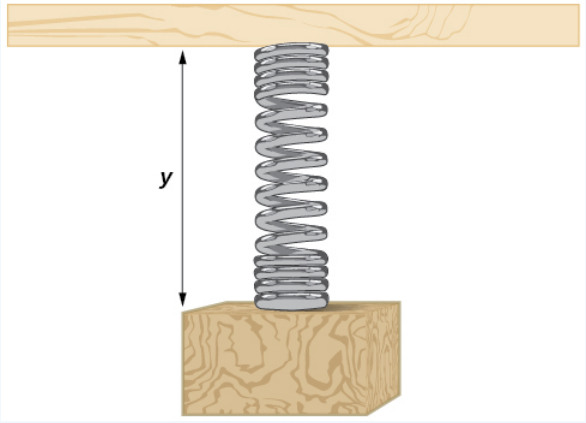


Figure 10.1.28

Answer

$y = 3\cos(x) - 4$

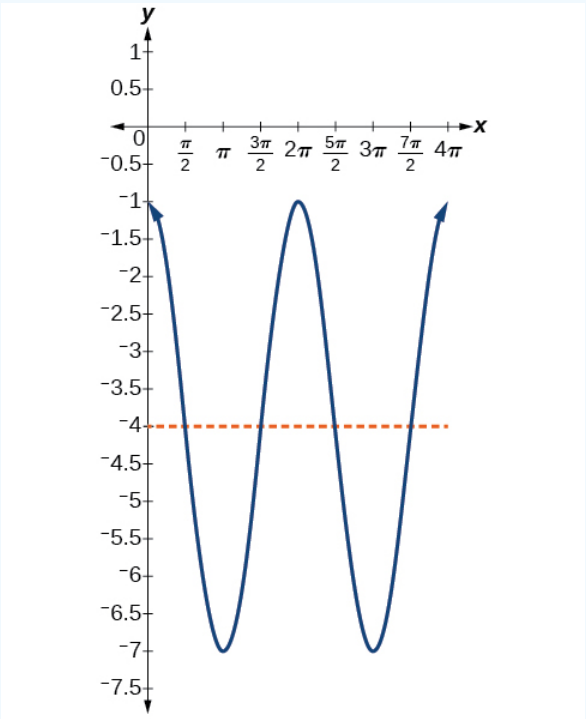


Figure 10.1.29

Example 10.1.13: Determining a Rider's Height on a Ferris Wheel

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period:30, so $B = \dfrac{2\pi}{30} = \dfrac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$

where t is in minutes and y is measured in meters.

Media

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine
- Translations of Sine and Cosine
- Graphing Sine and Cosine Transformations
- Graphing the Sine Function

Key Equations

Sinusoidal functions	$f(x) = A\sin(Bx - C) + D$ $f(x) = A\cos(Bx - C) + D$

Key Concepts

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the *y*-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \dfrac{2\pi}{|B|}$. See Example 10.1.1.
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See Example 10.1.2
- The value $\dfrac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See Example 10.1.3
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See Example 10.1.4
- Combinations of variations of sinusoidal functions can be detected from an equation. See Example 10.1.5.
- The equation for a sinusoidal function can be determined from a graph. See Example 10.1.6 and Example 10.1.7.
- A function can be graphed by identifying its amplitude and period. See Example 10.1.8 and Example 10.1.9.
- A function can also be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See Example 10.1.10
- Sinusoidal functions can be used to solve real-world problems. See Example 10.1.11, Example 10.1.12, and Example 10.1.13

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10.2: Graphs of the Other Trigonometric Functions

Learning Objectives

- Analyze the graph of $y = \tan x$.
- Graph variations of $y = \tan x$.
- Analyze the graphs of $y = \sec x$ and $y = \csc x$.
- Graph variations of $y = \sec x$ and $y = \csc x$.
- Analyze the graph of $y = \cot x$.
- Graph variations of $y = \cot x$.

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

Analyzing the Graph of $y = \tan x$

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

$$\tan x = \frac{\sin x}{\cos x} \tag{10.2.1}$$

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is a constant. If we graph the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

$$\begin{aligned} \tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{Definition of tangent} \\ &= \frac{-\sin x}{\cos x} && \text{Sine is an odd function, cosine is even} \\ &= -\frac{\sin x}{\cos x} && \text{The quotient of an odd and an even function is odd} \\ &= -\tan x && \text{Definition of tangent} \end{aligned}$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in Table 10.2.1.

Table 10.2.1

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when $\frac{\pi}{3} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{3} \approx 1.05$ and $\frac{\pi}{2} \approx 1.57$, we will evaluate x at radian measures $1.05 < x < 1.57$ as shown in Table 10.2.2

Table 10.2.2

x	1.3	1.5	1.55	1.56
$\tan x$	3.6	14.1	48.1	92.6

As x approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan x$ is an odd function, we see the corresponding table of negative values in Table 10.2.3.

Table 10.2.3

x	−1.3	−1.5	−1.55	−1.56
$\tan x$	−3.6	−14.1	−48.1	−92.6

We can see that, as x approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of x for which $\cos x = 0$. For example, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{3\pi}{2}\right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. Figure 10.2.1 represents the graph of $y = \tan x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from π to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

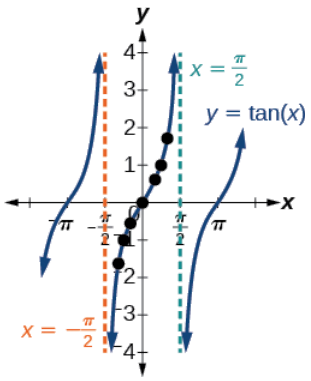


Figure 10.2.1: Graph of the tangent function

Graphing Variations of $y = \tan x$

As with the sine and cosine functions, the *tangent function* can be described by a general equation.

$$y = A\tan(Bx)$$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

FEATURES OF THE GRAPH OF $Y = A\tan(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.
- The domain is all real numbers x,where $x \neq \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$ such that k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$ where k is an integer.
- $y = A\tan(Bx)$ is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the **tangent function** to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f(x) = A\tan(Bx)$. We focus on a single **period** of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is

then the interval $\left(-\frac{P}{2}, \frac{P}{2}\right)$ and the graph has vertical asymptotes at $\pm\frac{P}{2}$ where $P = \frac{\pi}{B}$. On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the graph will come up from the left asymptote at $x = -\frac{\pi}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at $x = \frac{\pi}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

$$f\left(\frac{P}{4}\right) = A\tan\left(B\frac{P}{4}\right) = A\tan\left(B\frac{\pi}{4B}\right) = A$$

because $\tan\left(\frac{\pi}{4}\right) = 1$.

Howto: Given the function $f(x) = A\tan(Bx)$, graph one period.

1. Identify the stretching factor, $|A|$.
2. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
3. Draw vertical asymptotes at $x = -\frac{P}{2}$ and $x = \frac{P}{2}$.
4. For $A > 0$, the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for $A < 0$).
5. Plot reference points at $\left(\frac{P}{4}, A\right)$, $(0, 0)$, and $\left(-\frac{P}{4}, -A\right)$, and draw the graph through these points.

Example 10.2.1: Sketching a Compressed Tangent

Sketch a graph of one period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$.

Solution

First, we identify A and B.

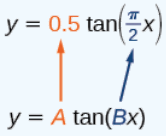


Figure 10.2.2

Because $A = 0.5$ and $B = \frac{\pi}{2}$, we can find the stretching/compressing factor and period. The period is $\frac{\pi}{\frac{\pi}{2}} = 2$, so the asymptotes are at $x = \pm 1$. At a quarter period from the origin, we have

$$\begin{aligned} f(0.5) &= 0.5\tan\left(\frac{0.5\pi}{2}\right) \\ &= 0.5\tan\left(\frac{\pi}{4}\right) \\ &= 0.5 \end{aligned}$$

This means the curve must pass through the points $(0.5, 0.5)$, $(0, 0)$,and $(-0.5, -0.5)$. The only inflection point is at the origin. Figure $10.2.3$shows the graph of one period of the function.

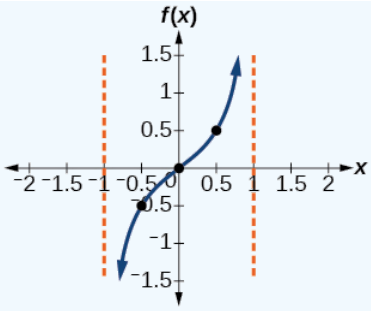


Figure 10.2.3

? Exercise 10.2.1

Sketch a graph of $f(x) = 3\tan\left(\frac{\pi}{6}x\right)$.

Answer

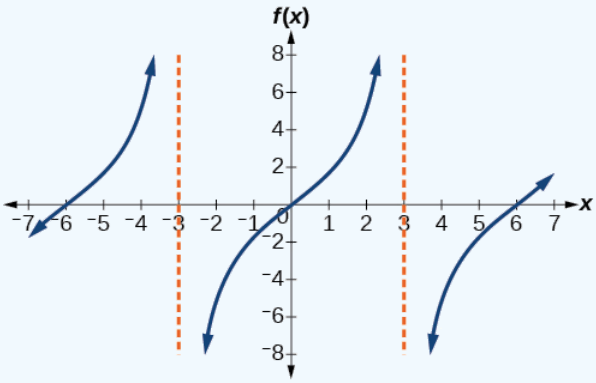


Figure 10.2.4

Graphing One Period of a Shifted Tangent Function

Now that we can graph a **tangent function** that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add C and D to the general form of the tangent function.

$$f(x) = A\tan(Bx - C) + D$$

The graph of a transformed tangent function is different from the basic tangent function $\tan x$ in several ways:

FEATURES OF THE GRAPH OF $Y = A\tan(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an odd integer.
- There is no amplitude.
- $y = A\tan(Bx)$ is and odd function because it is the qoutient of odd and even functions(sin and cosine perspectively).

Howto: Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

1. Express the function given in the form $y = A\tan(Bx - C) + D$.
2. Identify the **stretching/compressing** factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{\pi}{|B|}$.
4. Identify C and determine the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
7. Plot any three reference points and draw the graph through these points.

Example 10.2.2: Graphing One Period of a Shifted Tangent Function

Graph one period of the function $y = -2\tan(\pi x + \pi) - 1$.

Solution

- *Step 1.* The function is already written in the form $y = A\tan(Bx - C) + D$.
- *Step 2.*\(A=−2\), so the stretching factor is $|A| = 2$.
- *Step 3.* \(B=\pi\), so the period is $P = \dfrac{\pi}{|B|} = \dfrac{\pi}{pi} = 1$.
- *Step 4.* \(C=−\pi\), so the phase shift is $CB = \dfrac{-\pi}{\pi} = -1$.
- *Step 5-7.* The asymptotes are at $x = -\dfrac{3}{2}$ and $x = -\dfrac{1}{2}$ and the three recommended reference points are $(-1.25, 1)$, $(-1, -1)$, and $(-0.75, -3)$. The graph is shown in Figure 10.2.5

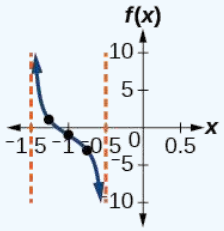


Figure 10.2.5

Analysis

Note that this is a decreasing function because $A < 0$.

Exercise 10.2.2

How would the graph in Example 10.2.2look different if we made $A = 2$ instead of -2?

Answer

It would be reflected across the line $y = -1$, becoming an increasing function.

Howto: Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period P from the spacing between successive vertical asymptotes or *x*-intercepts.
2. Write $f(x) = A\tan\left(\dfrac{\pi}{P}x\right)$.
3. Determine a convenient point $(x, f(x))$ on the given graph and use it to determine A.

Example 10.2.3: Identifying the Graph of a Stretched Tangent

Find a formula for the function graphed in Figure 10.2.6

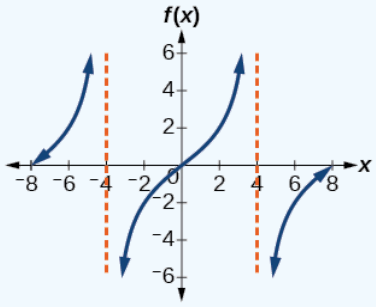


Figure 10.2.6: A stretched tangent function

Solution

The graph has the shape of a tangent function.

- *Step 1.* One cycle extends from -4 to 4, so the period is $P = 8$. Since $P = \frac{\pi}{|B|}$, we have $B = \frac{\pi}{P} = \frac{\pi}{8}$.
- *Step 2.* The equation must have the form $f(x) = A\tan\left(\frac{\pi}{8}x\right)$.
- *Step 3.* To find the vertical stretch A,we can use the point $(2, 2)$.

$$\begin{aligned} 2 &= A\tan\left(\frac{\pi}{8}\cdot 2\right) \\ &= A\tan\left(\frac{\pi}{4}\right) \end{aligned}$$

Because $\tan\left(\frac{\pi}{4}\right) = 1$, $A = 2$.

This function would have a formula $f(x) = 2\tan\left(\frac{\pi}{8}x\right)$.

Exercise 10.2.3

Find a formula for the function in Figure 10.2.7.

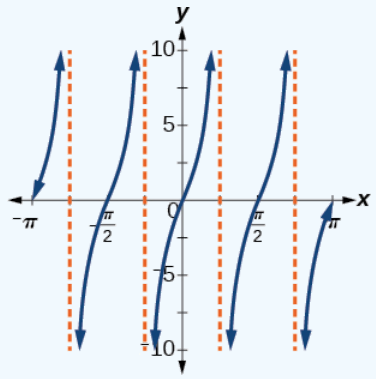


Figure 10.2.7

Answer

$g(x) = 4\tan(2x)$

Analyzing the Graphs of $y = \sec x$ and $y = \csc x$

The secant was defined by the reciprocal identity $sec\, x = \dfrac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\dfrac{\pi}{2}$, $\dfrac{3\pi}{2}$ etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See Figure 10.2.8 The graph of the cosine is shown as a dashed orange wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the x-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

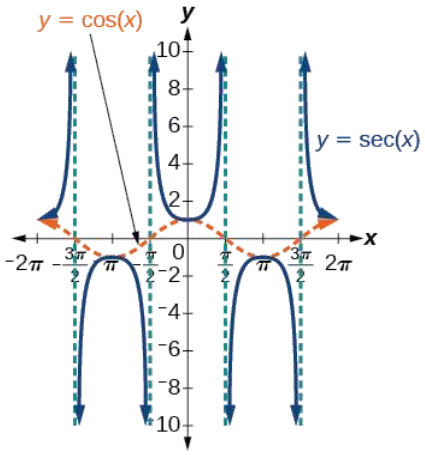


Figure 10.2.8: Graph of the secant function, $f(x) = \sec x = \dfrac{1}{\cos x}$

As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

FEATURES OF THE GRAPH OF $Y = A\sec(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

Similar to the secant, the **cosecant** is defined by the reciprocal identity $\csc x = \dfrac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See Figure 10.2.7. The graph of sine is shown as a dashed orange wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the **cosecant function** decreases.

The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the x-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc(-x) = -\csc x$.

The graph of cosecant, which is shown in Figure 10.2.9, is similar to the graph of secant.

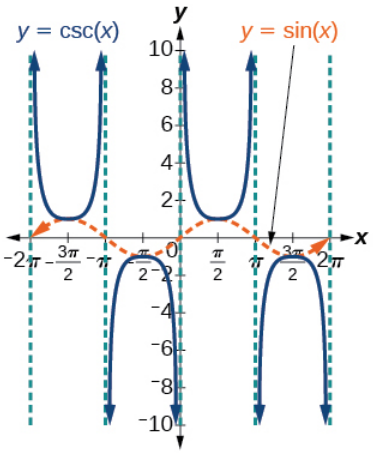


Figure 10.2.9: The graph of the cosecant function, $f(x) = \csc x = \frac{1}{\sin x}$

FEATURES OF THE GRAPH OF $Y = A\csc(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

Graphing Variations of $y = \sec x$ and $y = \csc x$

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions.The equations become the following.

$$y = A\sec(Bx - C) + D \tag{10.2.2}$$

$$y = A\csc(Bx - C) + D \tag{10.2.3}$$

FEATURES OF THE GRAPH OF $Y = A\sec(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
- There is no amplitude.

- $y = A\sec(Bx)$ is an even function because cosine is an even function.

FEATURES OF THE GRAPH OF $Y = A\csc(Bx - C) + D$

1. The stretching factor is $|A|$.
2. The period is $\dfrac{2\pi}{|B|}$.
3. The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an integer.
4. The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
5. The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
6. There is no amplitude.
7. $y = A\csc(Bx)$ is an odd function because sine is an odd function.

HOWTO: Given a function of the form $y = A\sec(Bx)$, graph one period

1. Express the function given in the form $y = A\sec(Bx)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{2\pi}{|B|}$.
4. Sketch the graph of $y = A\cos(Bx)$.
5. Use the reciprocal relationship between $y = \cos\ x$ and $y = \sec\ x$ to draw the graph of $y = A\sec(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 10.2.4: Graphing a Variation of the Secant Function

Graph one period of $f(x) = 2.5\sec(0.4x)$

Solution

- *Step 1.* The given function is already written in the general form, $y = A\sec(Bx)$.
- *Step 2.* $A = 2.5$ so the stretching factor is 2.5.
- *Step 3.* $B = 0.4$ so $P = \dfrac{2\pi}{0.4} = 5\pi$. The period is 5π units.
- *Step 4.* Sketch the graph of the function $g(x) = 2.5\cos(0.4x)$.
- *Step 5.* Use the reciprocal relationship of the cosine and secant functions to draw the cosecant function.
- *Steps 6–7.* Sketch two asymptotes at $x = 1.25\pi$ and $x = 3.75\pi$. We can use two reference points, the local minimum at $(0, 2.5)$ and the local maximum at $(2.5\pi, -2.5)$ Figure $10.2.10$shows the graph.

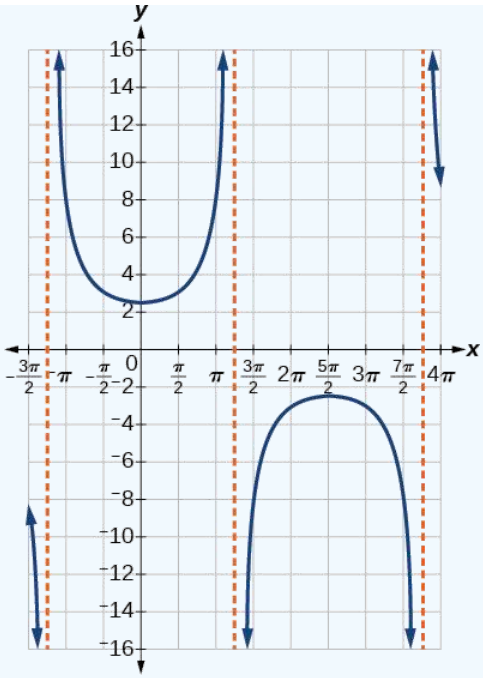


Figure 10.2.10

Exercise 10.2.4

Graph one period of $f(x) = -2.5\sec(0.4x)$.

Answer

This is a vertical reflection of the preceding graph because A is negative.

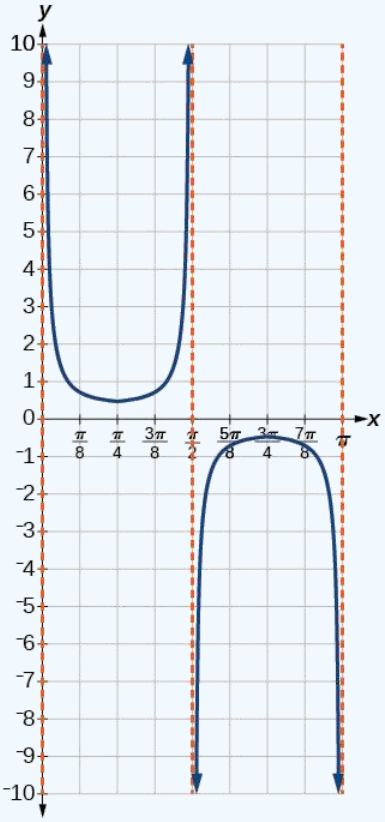


Figure 10.2.11

Q&A: Do the vertical shift and stretch/compression affect the secant's range?

Yes. The range of $f(x) = A\sec(Bx - C) + D$ is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.

Howto: Given a function of the form $f(x) = A\sec(Bx - C) + D$, graph one period.

1. Express the function given in the form $y = A\sec(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\dfrac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\sec(Bx)$. but shift it to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$, where k is an odd integer.

Example 10.2.5: Graphing a Variation of the Secant Function

Graph one period of $y = 4\sec\left(\dfrac{\pi}{3x} - \dfrac{\pi}{2}\right) + 1$.

Solution

- *Step 1.* Express the function given in the form $y = 4\sec\left(\dfrac{\pi}{3x} - \dfrac{\pi}{2}\right) + 1$.
- *Step 2.* The stretching/compressing factor is $|A| = 4$.
- *Step 3.* The period is

$$\begin{aligned}\frac{2\pi}{|B|} &= \frac{2\pi}{\frac{\pi}{3}} \\ &= 2\pi \cdot \frac{3}{\pi} \\ &= 6\end{aligned}$$

- *Step 4.* The phase shift is

$$\begin{aligned}\frac{C}{B} &= \frac{\frac{\pi}{2}}{\frac{\pi}{3}} \\ &= \frac{\pi}{2} \cdot \frac{3}{\pi} \\ &= 1.5\end{aligned}$$

- *Step 5.* Draw the graph of $y = A\sec(Bx)$, but shift it to the right by $\frac{C}{B} = 1.5$ and up by $D = 6$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0$, $x = 3$, and $x = 6$. There is a local minimum at $(1.5, 5)$ and a local maximum at $(4.5, -3)$. Figure $10.2.12$shows the graph.

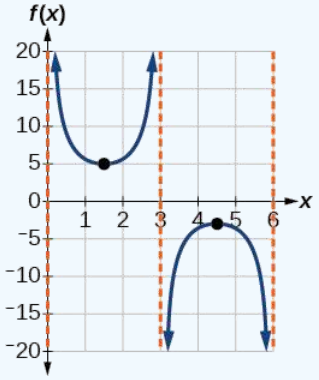


Figure 10.2.12

? Exercise 10.2.5

Graph one period of $f(x) = -6\sec(4x + 2) - 8$.

Answer

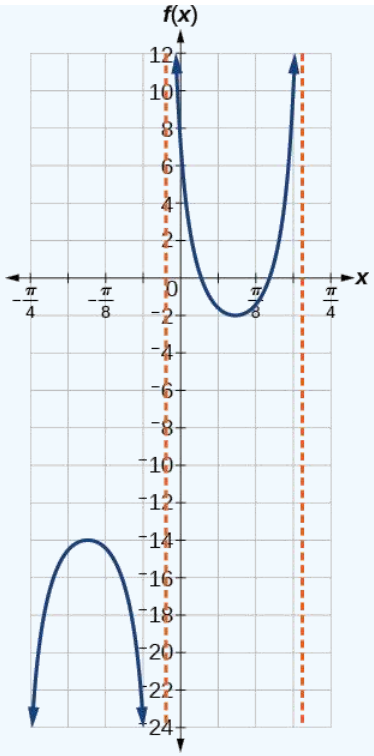


Figure 10.2.13

Q&A: The domain of $\csc\ x$ was given to be all x such that $x \neq k\pi$ for any integer k. Would the domain of $y = A\csc(Bx - C) + D$ be $x \neq \dfrac{C + k\pi}{B}$?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function's input.

Howto: Given a function of the form $y = A\csc(Bx)$, graph one period.

1. Express the function given in the form $y = A\csc(Bx)$.
2. $|A|$.
3. Identify B and determine the period, $P = \dfrac{2\pi}{|B|}$.
4. Draw the graph of $y = A\sin(Bx)$.
5. Use the reciprocal relationship between $y = sin\ x$ and $y = \csc\ x$ to draw the graph of $y = A\csc(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 10.2.6: Graphing a Variation of the Cosecant Function

Graph one period of $f(x) = -3\csc(4x)$.

Solution

- *Step 1.* The given function is already written in the general form, $y = A\csc(Bx)$.
- *Step 2.* $|A| = |-3| = 3$,so the stretching factor is 3.
- *Step 3.* $B = 4$,so $P = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$. The period is $\dfrac{\pi}{2}$ units.
- *Step 4.* Sketch the graph of the function $g(x) = -3\sin(4x)$.
- *Step 5.* Use the reciprocal relationship of the sine and cosecant functions to draw the cosecant function.

- *Steps 6–7.* Sketch three asymptotes at $x = 0$, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$. We can use two reference points, the local maximum at $\left(\frac{\pi}{8}, -3\right)$ and the local minimum at $\left(\frac{3\pi}{8}, 3\right)$. Figure 10.2.14shows the graph.

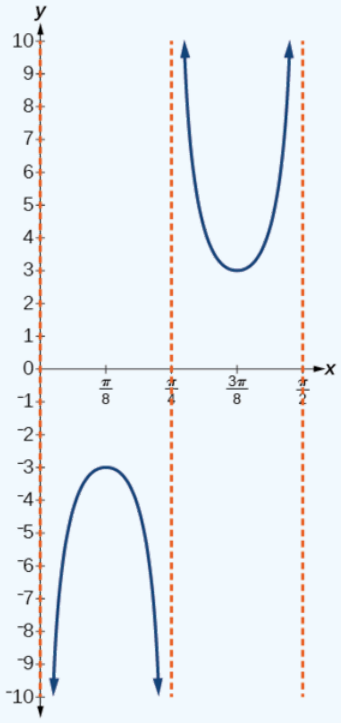


Figure 10.2.14

Exercise 10.2.6

Graph one period of $f(x) = 0.5\csc(2x)$.

Answer

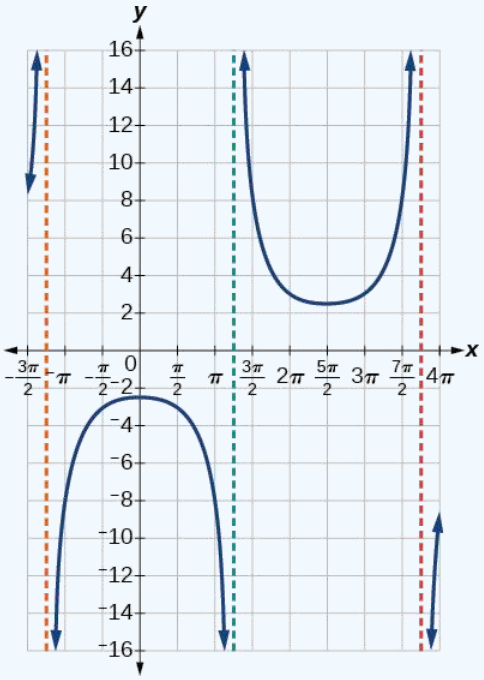


Figure 10.2.15

Howto: Given a function of the form $f(x) = A\csc(Bx - C) + D$, graph one period

1. Express the function given in the form $y = A\csc(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\csc(Bx)$ but shift it to the right by and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.

Example 10.2.7: Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant

Sketch a graph of $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$. What are the domain and range of this function?

Solution

- *Step 1.* Express the function given in the form $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$.
- *Step 2.* Identify the stretching/compressing factor, $|A| = 2$.
- *Step 3.* The period is $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$.
- *Step 4.* The phase shift is $\frac{0}{\frac{\pi}{2}} = 0$.
- *Step 5.* Draw the graph of $y = A\csc(Bx)$ but shift it up $D = 1$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0$, $x = 2$, $x = 4$.

The graph for this function is shown in Figure 10.2.16

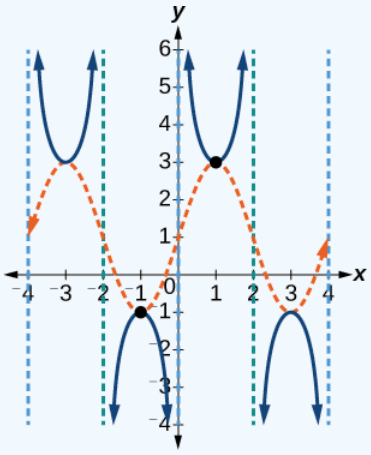


Figure 10.2.16: A transformed cosecant function

Analysis

The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$,shown as the orange dashed wave.

Exercise 10.2.7

Given the graph of $f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 1$ shown in Figure 10.2.17, sketch the graph of $g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1$ on the same axes.

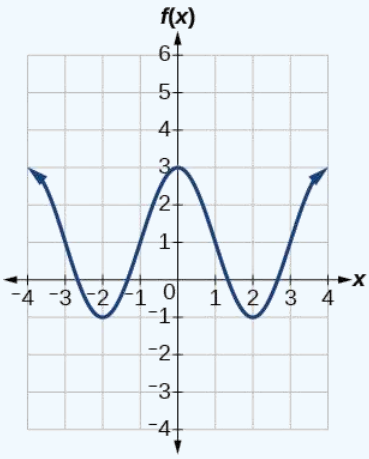


Figure 10.2.17

Answer

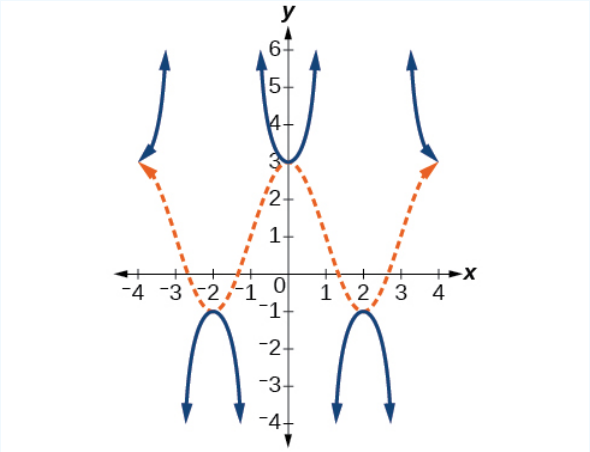


Figure 10.2.18

Analyzing the Graph of $y = \cot x$

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $cot\, x = \dfrac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See Figure 10.2.19. Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.

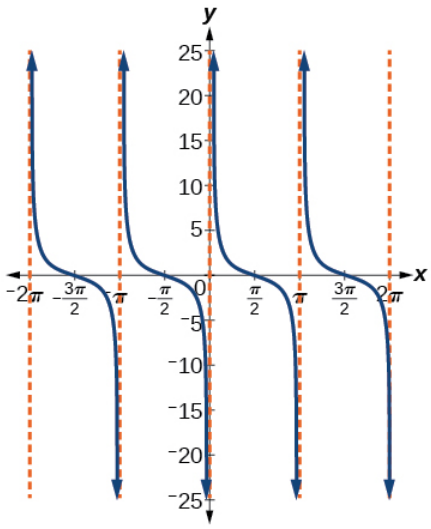


Figure 10.2.19: The cotangent function

FEATURES OF THE GRAPH OF $Y = A\cot(BX)$

- The stretching factor is $|A|$.
- The period is $P = \dfrac{\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\cot(Bx)$ is an odd function.

Graphing Variations of $y = \cot x$

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

$$y = A\cot(Bx - C) + D \tag{10.2.4}$$

PROPERTIES OF THE GRAPH OF $Y = A\cot(Bx - c) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{\pi}{|B|}$
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
- There is no amplitude.
- $y = A\cot(Bx)$ is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively)

Howto: Given a modified cotangent function of the form $f(x) = A\cot(Bx)$,graph one period.

1. Express the function in the form $f(x) = A\cot(Bx)$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \dfrac{\pi}{|B|}$.
4. Draw the graph of $y = A\tan(Bx)$.
5. Plot any two reference points.

6. Use the reciprocal relationship between tangent and cotangent to draw the graph of $y = Acot(Bx)$.
7. Sketch the asymptotes.

Example 10.2.8: Graphing Variations of the Cotangent Function

Determine the stretching factor, period, and phase shift of $y = 3\cot(4x)$, and then sketch a graph.

Solution

- *Step 1.* Expressing the function in the form $f(x) = A\cot(Bx)$ gives $f(x) = 3\cot(4x)$.
- *Step 2.* The stretching factor is $|A| = 3$.
- *Step 3.* The period is $P = \frac{\pi}{4}$.
- *Step 4.* Sketch the graph of $y = 3\tan(4x)$.
- *Step 5.* Plot two reference points. Two such points are $\left(\frac{\pi}{16}, 3\right)$ and $\left(\frac{3\pi}{16}, -3\right)$.
- *Step 6.* Use the reciprocal relationship to draw $y = 3\cot(4x)$.
- *Step 7.* Sketch the asymptotes, $x = 0$, $x = \frac{\pi}{4}$.

The orange graph in Figure 10.2.20shows $y = 3\tan(4x)$ and the blue graph shows $y = 3\cot(4x)$.

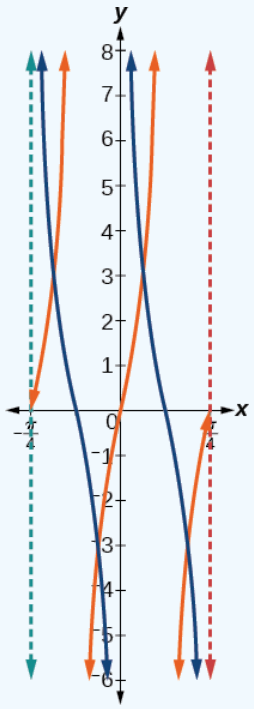


Figure 10.2.20

Howto: Given a modified cotangent function of the form $f(x) = A\cot(Bx - C) + D$, graph one period.

1. Express the function in the form $f(x) = A\cot(Bx - C) + D$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the asymptotes $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
7. Plot any three reference points and draw the graph through these points.

Example 10.2.9: Graphing a Modified Cotangent

Sketch a graph of one period of the function $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.

Solution

- *Step 1.* The function is already written in the general form $f(x) = A\cot(Bx - C) + D$.
- *Step 2.* $A = 4$,so the stretching factor is 4.
- *Step 3.* $B = \frac{\pi}{8}$, so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8$.
- *Step 4.* $C = \frac{\pi}{2}$,so the phase shift is $CB = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$.
- *Step 5.* We draw $f(x) = 4\tan\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 6-7.* Three points we can use to guide the graph are $(6, 2)$, $(8, -2)$, and $(10, -6)$. We use the reciprocal relationship of tangent and cotangent to draw $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 8.* The vertical asymptotes are $x = 4$ and $x = 12$.

The graph is shown in Figure 10.2.21

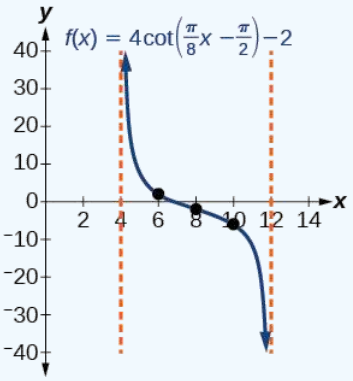


Figure 10.2.21: One period of a modified cotangent function

Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let's return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the **tangent function**.

Example 10.2.10: Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y = 5\tan(\frac{\pi}{4}t)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.
b. Graph on the interval $[0, 5]$.
c. Evaluate $f(1)$ and discuss the function's value at that input.

Solution

a. We know from the general form of $y = A\tan(Bt)$ that $|A|$ is the stretching factor and $\frac{\pi}{B}$ is the period.

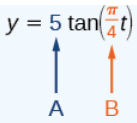


Figure 10.2.22

We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is $\frac{\pi}{\frac{\pi}{4}} = \frac{\pi}{1} \cdot \frac{4}{\pi} = 4$. This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at $t = 2$ and use the stretching factor and period. See Figure 10.2.23

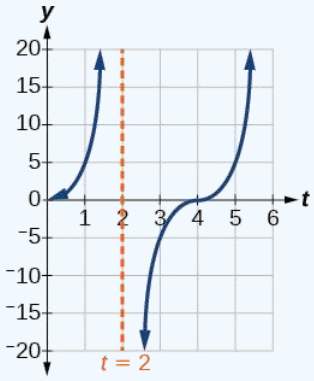


Figure 10.2.23

c. period: $f(1) = 5\tan(\frac{\pi}{4}(1)) = 5(1) = 5$; after 1 second, the beam of has moved 5 ft from the spot across from the police car.

Media

Access these online resources for additional instruction and practice with graphs of other trigonometric functions.

- Graphing the Tangent
- Graphing Cosecant and Secant
- Graphing the Cotangent

Key Equations

Shifted, compressed, and/or stretched tangent function	$y = A\tan(Bx - C) + D$
Shifted, compressed, and/or stretched secant function	$y = A\sec(Bx - C) + D$
Shifted, compressed, and/or stretched cosecant function	$y = A\csc(Bx - C) + D$
Shifted, compressed, and/or stretched cotangent function	$y = A\cot(Bx - C) + D$

Key Concepts

- The tangent function has period π.
- $f(x) = A\tan(Bx - C) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift. See Example 10.2.1, Example 10.2.2, and Example 10.2.3.
- The secant and cosecant are both periodic functions with a period of 2π. $f(x) = A\sec(Bx - C) + D$ gives a shifted, compressed, and/or stretched secant function graph. See Example 10.2.4 and Example 10.2.5.
- $f(x) = A\csc(Bx - C) + D$ gives a shifted, compressed, and/or stretched cosecant function graph. See Example 10.2.6 and Example 10.2.7.
- The cotangent function has period π and vertical asymptotes at $0, \pm\pi, \pm 2\pi, ...$.
- The range of cotangent is $(-\infty, \infty)$, and the function is decreasing at each point in its range.

- The cotangent is zero at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$,....
- $f(x) = A\cot(Bx - C) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift. See Example $10.2.8$ and Example $10.2.9$.
- Real-world scenarios can be solved using graphs of trigonometric functions. See Example $10.2.10$

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10.3: Inverse Trigonometric Functions

Learning Objectives

- Understand and use the inverse sine, cosine, and tangent functions.
- Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Use a calculator to evaluate inverse trigonometric functions.
- Find exact values of composite functions with inverse trigonometric functions.

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.

Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in Figure 10.3.1.

Trig Functions	Inverse Trig Functions
Domain: Measure of an angle	Domain: Ratio
Range: Ratio	Range: Measure of an angle

Figure 10.3.1

For example, if $f(x) = \sin\ x$, then we would write $f^{-1}(x) = \sin^{-1} x$. Be aware that $\sin^{-1} x$ does not mean $\dfrac{1}{\sin\ x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$, then $\dfrac{\pi}{6} = \sin^{-1}\left(\dfrac{1}{2}\right)$.
- Since $\cos(\pi) = -1$, then $\pi = \cos^{-1}(-1)$.
- Since $\tan\left(\dfrac{\pi}{4}\right) = 1$, then $\dfrac{\pi}{4} = \tan^{-1}(1)$.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the **domain** of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. Figure 10.3.2 shows the graph of the sine function limited to $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ and the graph of the cosine function limited to $[0, \pi]$.

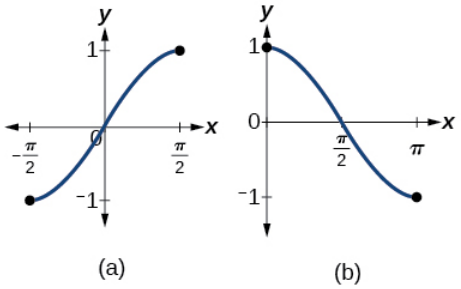


Figure 10.3.2: (a) Sine function on a restricted domain of $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$; (b) Cosine function on a restricted domain of $[0, \pi]$

Figure 10.3.3 shows the graph of the tangent function limited to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

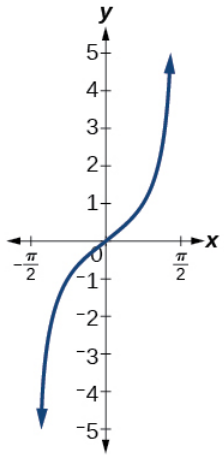


Figure 10.3.3: Tangent function on a restricted domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one **vertical asymptote** to the next instead of being divided into two parts by an asymptote.

On these restricted domains, we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1} x$ means $x = \sin\ y$. The inverse sine function is sometimes called the **arcsine** function, and notated $\arcsin\ x$.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

- The inverse cosine function $y = \cos^{-1} x$ means $x = \cos\ y$. The inverse cosine function is sometimes called the **arccosine** function, and notated $\arccos\ x$.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

- The inverse tangent function $y = \tan^{-1} x$ means $x = \tan\ y$. The inverse tangent function is sometimes called the **arctangent** function, and notated $\arctan\ x$.

$$y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

The graphs of the inverse functions are shown in Figures 10.3.4 - 10.3.6 Notice that the output of each of these inverse functions is a *number,* an angle in radian measure. We see that $\sin^{-1} x$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$, and $\tan^{-1} x$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

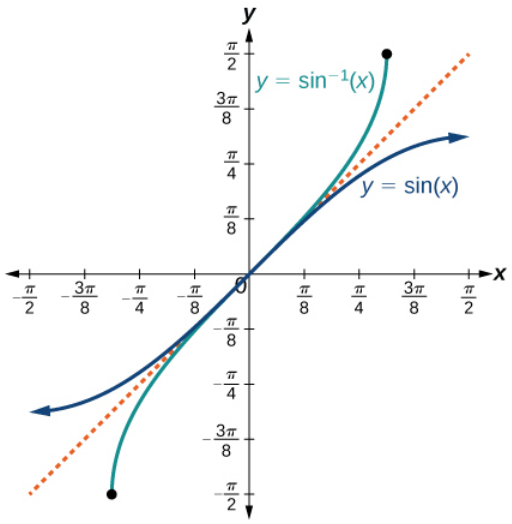


Figure 10.3.4: The sine function and inverse sine (or arcsine) function

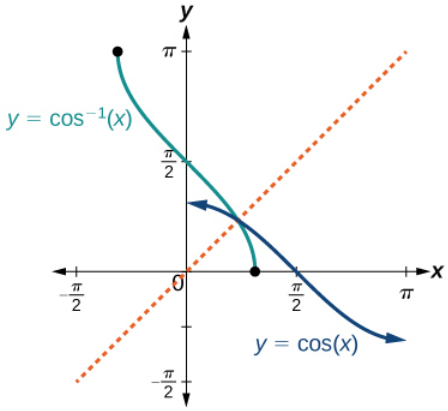


Figure 10.3.5: The cosine function and inverse cosine (or arccosine) function

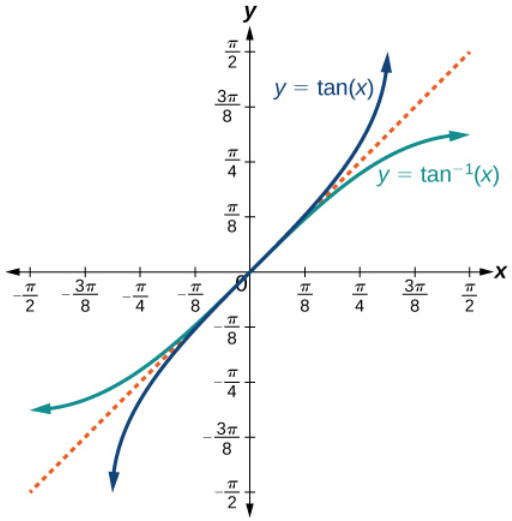


Figure 10.3.6: The tangent function and inverse tangent (or arctangent) function

RELATIONS FOR INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$,then $\tan^{-1} x = y$.

Example 10.3.1: Writing a Relation for an Inverse Function

Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, write a relation involving the inverse sine.

Solution

Use the relation for the inverse sine. If $\sin y = x$, then $\sin^{-1} x = y$.

In this problem, $x = 0.96593$, and $y = \frac{5\pi}{12}$.

$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}$

Exercise 10.3.1

Given $\cos(0.5) \approx 0.8776$,write a relation involving the inverse cosine.

Answer

$\arccos(0.8776) \approx 0.5$

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}(30°)$, $\frac{\pi}{4}(45°)$, and $\frac{\pi}{3}(60°)$, and their reflections into other quadrants.

Given a “special” input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x,depending on which corresponds to the given inverse function.

Example 10.3.2: Evaluating Inverse Trigonometric Functions for Special Input Values

Evaluate each of the following.

a. $\sin^{-1}\left(\frac{1}{2}\right)$

b. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

c. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

d. $\tan^{-1}(1)$

Solution

a. Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as determining the angle that would have a sine value of $\frac{1}{2}$. In other words, what angle x would satisfy $\sin(x) = \frac{1}{2}$? There are multiple values that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Remember that the inverse is a

function, so for each input, we will get exactly one output.

b. To evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

c. To evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0, \pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

d. Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1) = \frac{\pi}{4}$.

? Exercise 10.3.2

Evaluate each of the following.

a. $\sin^{-1}(-1)$
b. $\tan^{-1}(-1)$
c. $\cos^{-1}(-1)$
d. $\cos^{-1}\left(\frac{1}{2}\right)$

Answer a

$-\frac{\pi}{2}$

Answer b

$-\frac{\pi}{4}$

Answer c

π

Answer d

$\frac{\pi}{3}$

Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate **inverse trigonometric functions** that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, SIN-1, ARCSIN, or ASIN.

In the previous chapter, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In these examples and exercises, the answers will be interpreted as angles and we will use θ as the independent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

✓ Example 10.3.3: Evaluating the Inverse Sine on a Calculator

Evaluate $\sin^{-1}(0.97)$ using a calculator.

Solution

Because the output of the inverse function is an angle, the calculator will give us a degree value if in degree mode and a radian value if in radian mode. Calculators also use the same domain restrictions on the angles as we are using.

In radian mode, $\sin^{-1}(0.97) \approx 1.3252$. In degree mode, $\sin^{-1}(0.97) \approx 75.93^\circ$. Note that in calculus and beyond we will use radians in almost all cases.

Exercise 10.3.3

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

Answer

$1.9823 \text{or } 113.578^\circ$

Given two sides of a right triangle like the one shown in Figure 8.4.7, find an angle.

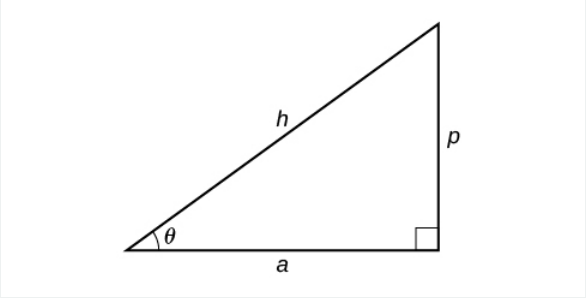


Figure 10.3.7

1. If one given side is the hypotenuse of length h and the side of length a adjacent to the desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.
2. If one given side is the hypotenuse of length h and the side of length p opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$.
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

Example 10.3.4: Applying the Inverse Cosine to a Right Triangle

Solve the triangle in Figure $10.3.8$ for the angle θ.

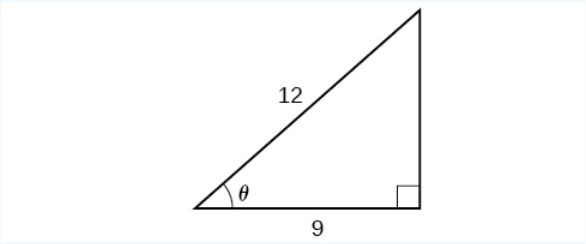


Figure 10.3.8

Solution

Because we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\cos\theta = \frac{9}{12}$$
$$\theta = \cos^{-1}\left(\frac{9}{12}\right) \quad \text{Apply definition of the inverse}$$
$$\theta \approx 0.7227 \quad \text{or about } 41.4096^\circ \text{ Evaluate}$$

Exercise 10.3.4

Solve the triangle in Figure 10.3.9 for the angle θ.

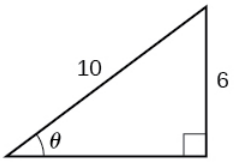


Figure 10.3.9

Answer

$\sin^{-1}(0.6) = 36.87^\circ = 0.6435$ radians

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let $f(x)$ and $g(x)$ be two different trigonometric functions belonging to the set$\{\ \sin(x), \cos(x), \tan(x)\ \}$ and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function, $f(f^{-1}(y)) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

COMPOSITIONS OF A TRIGONOMETRIC FUNCTION AND ITS INVERSE

$$\begin{aligned}
\sin(\sin^{-1}x) &= x \quad && \text{for } -1 \le x \le 1 \\
\cos(\cos^{-1}x) &= x \quad && \text{for } -1 \le x \le 1 \\
\tan(\tan^{-1}x) &= x \quad && \text{for } -\infty < x < \infty \\
\sin^{-1}(\sin x) &= x \quad && \text{only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\
\cos^{-1}(\cos x) &= x \quad && \text{only for } 0 \le x \le \pi \\
\tan^{-1}(\tan x) &= x \quad && \text{only for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{aligned}$$

Q&A

Is it correct that $\sin^{-1}(\sin x) = x$?

No. This equation is correct ifx x belongs to the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but sine is defined for all real input values, and for x outside the restricted interval, the equation is not correct because its inverse always returns a value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The situation is similar for cosine and tangent and their inverses. For example, $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain off f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

Example 10.3.5: Using Inverse Trigonometric Functions

Evaluate the following:

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

b. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

c. $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$

d. $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Solution

a. $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

b. $\frac{2\pi}{3}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $sin\left(\frac{2\pi}{3}\right) = sin\left(\frac{\pi}{3}\right)$, so $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$.

c. $\frac{2\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$.

d. $-\frac{\pi}{3}$ is not in $[0, \pi]$, but $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ because cosine is an even function. $\frac{\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

Exercise 10.3.5

Evaluate $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$ and $\tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right)$.

Answer

$\frac{\pi}{8}; \frac{2\pi}{9}$

Evaluating Compositions of the Form $f^{-1}(g(x))$

Now that we can compose a trigonometric function with its inverse, we can explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will begin with compositions of the form $f^{-1}(g(x))$. For special values of x,we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by considering the relation between the two acute angles of a right triangle where one is θ, making the other $\frac{\pi}{2} - \theta$.Consider the sine and cosine of each angle of the right triangle in Figure 10.3.10

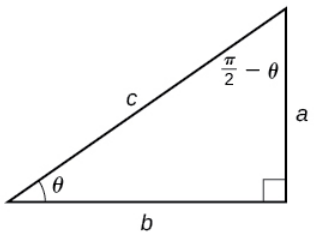


Figure 10.3.10: Right triangle illustrating the cofunction relationships

Because $\cos\theta = \frac{b}{c} = sin\left(\frac{\pi}{2} - \theta\right)$, we have $\sin^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$ if $0 \le \theta \le \pi$. If θ is not in this domain, then we need to find another angle that has the same cosine as θ and does belong to the restricted domain; we then subtract this angle from $\frac{\pi}{2}$.Similarly, $\sin\theta = \frac{a}{c} = \cos\left(\frac{\pi}{2} - \theta\right)$, so $\cos^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$ if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. These are just the function-cofunction relationships presented in another way.

Given functions of the form $\sin^{-1}(\cos x)$ and $\cos^{-1}(\sin x)$, evaluate them.

1. If x is in $[0, \pi]$, then $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$.
2. If x is not in $[0, \pi]$, then find another angle y in $[0, \pi]$ such that $\cos y = \cos x$.

$$\sin^{-1}(\cos x) = \frac{\pi}{2} - y \tag{10.3.1}$$

3. If x is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$.
4. If x is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then find another angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \sin x$.

$$\cos^{-1}(\sin x) = \frac{\pi}{2} - y \tag{10.3.2}$$

Example 10.3.6: Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

a. by direct evaluation.
b. by the method described previously.

Solution

a. Here, we can directly evaluate the inside of the composition.

$$\begin{aligned} \cos\left(\frac{13\pi}{6}\right) &= \cos\left(\frac{\pi}{6} + 2\pi\right) \\ &= \cos\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

Now, we can evaluate the inverse function as we did earlier.

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \tag{10.3.3}$$

b. We have $x = \frac{13\pi}{6}$, $y = \frac{\pi}{6}$, and

$$\begin{aligned} \sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) &= \frac{\pi}{2} - \frac{\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

Exercise 10.3.6

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

Answer

$\frac{3\pi}{4}$

Evaluating Compositions of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f(g^{-1}(x))$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas, such as $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of

Pythagoras's relation between the lengths of the sides. We can use the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, to solve for one when given the other. We can also use the inverse trigonometric functions to find compositions involving algebraic expressions.

Example 10.3.7: Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Solution

Beginning with the inside, we can say there is some angle such that $\theta = \cos^{-1}\left(\frac{4}{5}\right)$, which means $\cos\theta = \frac{4}{5}$, and we are looking for $\sin\theta$. We can use the Pythagorean identity to do this.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 && \text{Use our known value for cosine} \\ \sin^2\theta + \left(\frac{4}{5}\right)^2 &= 1 && \text{Solve for sine} \\ \sin^2\theta &= 1 - \frac{16}{25} \\ \sin\theta &= \pm\frac{9}{25} \\ &= \pm\frac{3}{5} \end{aligned}$$

Since $\theta = \cos^{-1}\left(\frac{4}{5}\right)$ is in quadrant I, $\sin\theta$ must be positive, so the solution is 35. See Figure 10.3.11

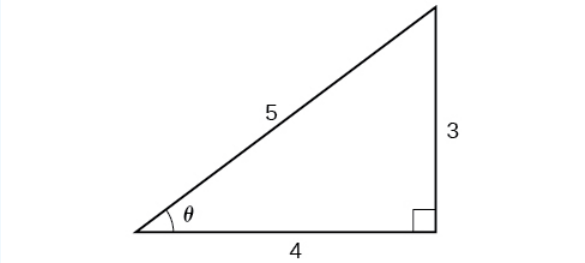


Figure 10.3.11: Right triangle illustrating that if $\cos\theta = \frac{4}{5}$, then $\sin\theta = \frac{3}{5}$

We know that the inverse cosine always gives an angle on the interval $[0, \pi]$, so we know that the sine of that angle must be positive; therefore $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin\theta = \frac{3}{5}$.

Exercise 10.3.7

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Answer

$\frac{12}{13}$

Example 10.3.8: Evaluating the Composition of a Sine with an Inverse Tangent

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Solution

While we could use a similar technique as in Example 10.3.6, we will demonstrate a different technique here. From the inside, we know there is an angle such that $\tan\theta = \frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle, as shown in Figure 10.3.12

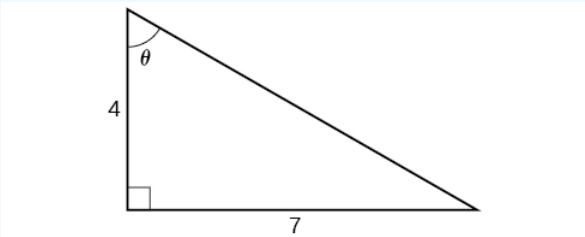


Figure 10.3.12: A right triangle with two sides known

Using the Pythagorean Theorem, we can find the hypotenuse of this triangle.

$$\begin{aligned} 4^2 + 7^2 &= hypotenuse^2 \\ hypotenuse &= \sqrt{65} \end{aligned}$$

Now, we can evaluate the sine of the angle as the opposite side divided by the hypotenuse.

$$\sin\theta = \frac{7}{\sqrt{65}}$$

This gives us our desired composition.

$$\begin{aligned} \sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) &= \sin\theta \\ &= \frac{7}{\sqrt{65}} \\ &= \frac{7\sqrt{65}}{65} \end{aligned}$$

? Exercise 10.3.8

Evaluate $\cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right)$.

Answer

$\frac{4\sqrt{2}}{9}$

✓ Example 10.3.9: Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ for $-3 \le x \le 3$.

Solution

We know there is an angle θ such that $\sin\theta = \frac{x}{3}$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \qquad \text{Use the Pythagorean Theorem} \\ \left(\frac{x}{3}\right)^2 + \cos^2\theta &= 1 \qquad \text{Solve for cosine} \\ \cos^2\theta &= 1 - \frac{x^2}{9} \\ \cos\theta &= \pm\sqrt{\frac{9-x^2}{9}} \\ &= \pm\sqrt{\frac{9-x^2}{3}} \end{aligned}$$

Because we know that the inverse sine must give an angle on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we can deduce that the cosine of that angle must be positive.

$cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \sqrt{\frac{9-x^2}{3}}$

Exercise 10.3.9

Find a simplified expression for $\sin(\tan^{-1}(4x))$ for $-\frac{1}{4} \leq x \leq \frac{1}{4}$.

Answer

$$\frac{4x}{\sqrt{16x^2+1}}$$

Media

Access this online resource for additional instruction and practice with inverse trigonometric functions.

- Evaluate Expressions Involving Inverse Trigonometric Functions

Visit this website for additional practice questions from Learningpod.

Key Concepts

- An inverse function is one that "undoes" another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, inverse trigonometric functions are defined for restricted domains.
- For any trigonometric function $f(x)$, if $x = f^{-1}(y)$, then $f(x) = y$. However, $f(x) = y$ only implies $x = f^{-1}(y)$ if x is in the restricted domain of f. See Example 10.3.1.
- Special angles are the outputs of inverse trigonometric functions for special input values; for example, $\frac{\pi}{4} = \tan^{-1}(1)$ and $\frac{\pi}{6} = \sin^{-1}(\frac{1}{2})$ .See Example 10.3.2
- A calculator will return an angle within the restricted domain of the original trigonometric function. See Example 10.3.3
- Inverse functions allow us to find an angle when given two sides of a right triangle. See Example 10.3.4
- In function composition, if the inside function is an inverse trigonometric function, then there are exact expressions; for example,$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$. See Example 10.3.5
- If the inside function is a trigonometric function, then the only possible combinations are $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ if $0 \leq x \leq \pi$ and $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$ if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. See Example 10.3.6 and Example 10.3.7.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function. See Example 10.3.8
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig identities to assist in determining the ratio of sides. See Example 10.3.9

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CHAPTER OVERVIEW

11: Trigonometric Identities and Equations

In this chapter, we discuss how to manipulate trigonometric equations algebraically by applying various formulas and trigonometric identities. We will also investigate some of the ways that trigonometric equations are used to model real-life phenomena.

11.1: Trigonometric Identities
11.2: Sum and Difference Identities
11.3: Double-Angle, Half-Angle, and Reduction Formulas
11.4: Sum-to-Product and Product-to-Sum Formulas
11.5: Solving Trigonometric Equations

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11.1: Trigonometric Identities

Learning Objectives

- Verify the fundamental trigonometric identities.
- Simplify trigonometric expressions using algebra and the identities.

In espionage movies, we see international spies with multiple passports, each claiming a different identity. However, we know that each of those passports represents the same person. The trigonometric identities act in a similar manner to multiple passports—there are many ways to represent the same trigonometric expression. Just as a spy will choose an Italian passport when traveling to Italy, we choose the identity that applies to the given scenario when solving a trigonometric equation.



Figure 11.1.1: International passports and travel documents

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Verifying the Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations, just as factoring, finding common denominators, and using special formulas are the basic tools of solving algebraic equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result. In this first section, we will work with the fundamental identities: the **Pythagorean identities**, the even-odd identities, the reciprocal identities, and the quotient identities.

We will begin with the **Pythagorean identities** (Table 11.1.1), which are equations involving trigonometric functions based on the properties of a right triangle. We have already seen and used the first of these identifies, but now we will also use additional identities.

Table 11.1.1: Pythagorean Identities

$\sin^2\theta+\cos^2\theta=1$	$1+\cot^2\theta=\csc^2\theta$	$1+\tan^2\theta=\sec^2\theta$

The second and third identities can be obtained by manipulating the first. The identity $1+\cot^2\theta=\csc^2\theta$ is found by rewriting the left side of the equation in terms of sine and cosine.

Prove: $1+\cot^2\theta=\csc^2\theta$

$$\begin{aligned} 1+\cot^2\theta &= (1+\frac{\cos^2}{\sin^2}) && \text{Rewrite the left side} \\ &= \left(\frac{\sin^2}{\sin^2}\right)+\left(\frac{\cos^2}{\sin^2}\right) && \text{Write both terms with the common denominator} \\ &= \frac{\sin^2+\cos^2}{\sin^2} \\ &= \frac{1}{\sin^2} \\ &= \csc^2 \end{aligned}$$

Similarly,$1+\tan^2\theta=\sec^2\theta$ can be obtained by rewriting the left side of this identity in terms of sine and cosine. This gives

$$\begin{aligned} 1+\tan^2\theta &= 1+\left(\frac{\sin\theta}{\cos\theta}\right)^2 && \text{Rewrite left side} \\ &= \left(\frac{\cos\theta}{\cos\theta}\right)^2+\left(\frac{\sin\theta}{\cos\theta}\right)^2 && \text{Write both terms with the common denominator} \\ &= \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} \\ &= \frac{1}{\cos^2\theta} \\ &= \sec^2\theta \end{aligned}$$

Recall that we determined which trigonometric functions are odd and which are even. The next set of fundamental identities is the set of **even-odd identities.** The **even-odd identities** relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle (Table 11.1.2).

Table 11.1.2: Even-Odd Identities

$\tan(-\theta)=-\tan\theta$	$\sin(-\theta)=-\sin\theta$	$\cos(-\theta)=\cos\theta$
$\cot(-\theta)=-\cot\theta$	$\csc(-\theta)=-\csc\theta$	$\sec(-\theta)=\sec\theta$

Recall that an **odd function** is one in which $f(-x)=-f(x)$ for all x in the domain off. f. The sine function is an odd function because $\sin(-\theta)=-\sin\theta$. The graph of an odd function is symmetric about the origin. For example, consider corresponding inputs of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. The output of $\sin\left(\frac{\pi}{2}\right)$ is opposite the output of $\sin\left(-\frac{\pi}{2}\right)$. Thus,

$$\begin{aligned} \sin\left(\frac{\pi}{2}\right) &= 1 \\ \sin\left(-\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

This is shown in Figure 11.1.2

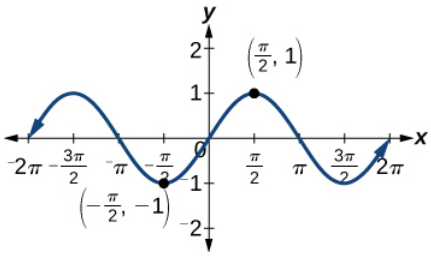


Figure 11.1.2: Graph of $y=\sin\theta$

Recall that an **even function** is one in which

$f(-x)=f(x)$ for all x in the domain of f

The graph of an even function is symmetric about the *y*-axis. The cosine function is an even function because $\cos(-\theta) = \cos\theta$. For example, consider corresponding inputs $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. The output of $\cos\left(\frac{\pi}{4}\right)$ is the same as the output of $\cos\left(-\frac{\pi}{4}\right)$. Thus,

$$\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$
$$\approx 0.707$$

See Figure 11.1.3

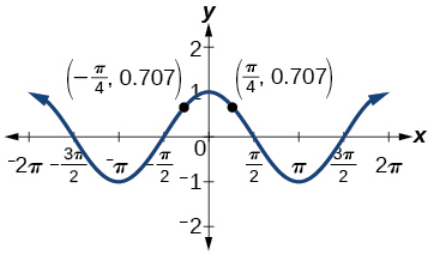


Figure 11.1.3: Graph of $y = \cos\theta$

For all θ in the domain of the sine and cosine functions, respectively, we can state the following:

- Since $\sin(-\theta) = -\sin\theta$,sine is an odd function.
- Since $\cos(-\theta) = \cos\theta$,cosine is an even function.

The other even-odd identities follow from the even and odd nature of the sine and cosine functions. For example, consider the tangent identity,$\tan(-\theta) = -\tan\theta$. We can interpret the tangent of a negative angle as

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta.$$

Tangent is therefore an odd function, which means that $\tan(-\theta) = -\tan(\theta)$ for all θ in the domain of the **tangent function.**

The cotangent identity, $\cot(-\theta) = -\cot\theta$,also follows from the sine and cosine identities. We can interpret the cotangent of a negative angle as

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta.$$

Cotangent is therefore an odd function, which means that $\cot(-\theta) = -\cot(\theta)$ for all θ in the domain of the **cotangent function.**

The **cosecant function** is the reciprocal of the sine function, which means that the cosecant of a negative angle will be interpreted as

$$\csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta.$$

The cosecant function is therefore odd.

Finally, the secant function is the reciprocal of the cosine function, and the secant of a negative angle is interpreted as

$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta.$$

The secant function is therefore even.

To sum up, only two of the trigonometric functions, cosine and secant, are even. The other four functions are odd, verifying the even-odd identities.

The next set of fundamental identities is the set of reciprocal identities, which, as their name implies, relate trigonometric functions that are reciprocals of each other. (Table 11.1.3). Recall that we first encountered these identities when defining trigonometric functions from right angles in Right Angle Trigonometry.

Table 11.1.3: Reciprocal Identities

$\sin\theta = \dfrac{1}{\csc\theta}$	$\csc\theta = \dfrac{1}{\sin\theta}$
$\cos\theta = \dfrac{1}{\sec\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$
$\tan\theta = \dfrac{1}{\cot\theta}$	$\cot\theta = \dfrac{1}{\tan\theta}$

The final set of identities is the set of **quotient identities,** which define relationships among certain trigonometric functions and can be very helpful in verifying other identities (Table 11.1.4).

Table 11.1.4: Quotient Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

The reciprocal and quotient identities are derived from the definitions of the basic trigonometric functions.

SUMMARIZING TRIGONOMETRIC IDENTITIES

The Pythagorean identities are based on the properties of a right triangle.

$$\cos^2\theta + \sin^2\theta = 1 \tag{11.1.1}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{11.1.2}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{11.1.3}$$

The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle.

$$\tan(-\theta) = -\tan\theta \tag{11.1.4}$$

$$\cot(-\theta) = -\cot\theta \tag{11.1.5}$$

$$\sin(-\theta) = -\sin\theta \tag{11.1.6}$$

$$\csc(-\theta) = -\csc\theta \tag{11.1.7}$$

$$\cos(-\theta) = \cos\theta \tag{11.1.8}$$

$$\sec(-\theta) = \sec\theta \tag{11.1.9}$$

The reciprocal identities define reciprocals of the trigonometric functions.

$$\sin\theta = \frac{1}{\csc\theta} \tag{11.1.10}$$

$$\cos\theta = \frac{1}{\sec\theta} \tag{11.1.11}$$

$$\tan\theta = \frac{1}{\cot\theta} \tag{11.1.12}$$

$$\csc\theta = \frac{1}{\sin\theta} \tag{11.1.13}$$

$$\sec\theta = \frac{1}{\cos\theta} \tag{11.1.14}$$

$$\cot\theta = \frac{1}{\tan\theta} \tag{11.1.15}$$

The **quotient identities** define the relationship among the trigonometric functions.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{11.1.16}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{11.1.17}$$

Example 11.1.1: Graphing the Equations of an Identity

Graph both sides of the identity $\cot\theta = \dfrac{1}{\tan\theta}$. In other words, on the graphing calculator, graph $y = \cot\theta$ and $y = \dfrac{1}{\tan\theta}$.

Solution

See Figure 11.1.4

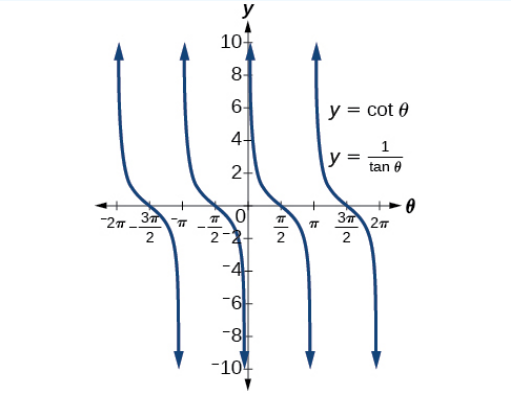


Figure 11.1.4

Analysis

We see only one graph because both expressions generate the same image. One is on top of the other. This is a good way to prove any identity. If both expressions give the same graph, then they must be identities.

How to: Given a trigonometric identity, verify that it is true.

1. Work on one side of the equation. It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. Look for opportunities to factor expressions, square a binomial, or add fractions.
3. Noting which functions are in the final expression, look for opportunities to use the identities and make the proper substitutions.
4. If these steps do not yield the desired result, try converting all terms to sines and cosines.

Example 11.1.2: Verifying a Trigonometric Identity

Verify $\tan\theta\cos\theta = \sin\theta$.

Solution

We will start on the left side, as it is the more complicated side:

$$\begin{aligned}\tan\theta\cos\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta \\ &= \sin\theta\end{aligned}$$

Analysis

This identity was fairly simple to verify, as it only required writing $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Exercise 11.1.1

Verify the identity $\csc\theta\cos\theta\tan\theta = 1$.

Answer

$$\begin{aligned}\csc\theta\cos\theta\tan\theta &= \left(\frac{1}{\sin\theta}\right)\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{\cos\theta}{\sin\theta}\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}\\ &= 1\end{aligned}$$

Example 11.1.3A: Verifying the Equivalency Using the Even-Odd Identities

Verify the following equivalency using the even-odd identities:

$$(1+\sin x)[1+\sin(-x)] = \cos^2 x$$

Solution

Working on the left side of the equation, we have

$$(1+\sin x)[1+\sin(-x)] = (1+\sin x)(1-\sin x)$$

Since

$$\begin{aligned}\sin(-x) &= -\sin x\\ [5pt] &= 1-\sin^2 x \qquad \text{Difference of squares}\\ [5pt] &= \cos^2 x\\ \cos^2 x &= 1-\sin^2 x\end{aligned}$$

Example 11.1.3B: Verifying a Trigonometric Identity Involving $\sec^2\theta$

Verify the identity $\dfrac{\sec^2\theta-1}{\sec^2\theta} = \sin^2\theta$

Solution

As the left side is more complicated, let's begin there.

$$\begin{aligned}\frac{\sec^2\theta-1}{\sec^2\theta} &= \frac{(\tan^2\theta+1)-1}{\sec^2\theta}\\ \sec^2\theta &= \tan^2\theta+1\\ &= \frac{\tan^2\theta}{\sec^2\theta}\\ &= \tan^2\theta\left(\frac{1}{\sec^2\theta}\right)\\ &= \tan^2\theta\left(\cos^2\theta\right)\\ \cos^2\theta &= \frac{1}{\sec^2\theta}\\ &= \left(\frac{\sin^2\theta}{\cos^2\theta}\right)\\ \tan^2\theta &= \frac{\sin^2\theta}{\cos^2\theta}\\ &= \sin^2\theta\end{aligned}$$

There is more than one way to verify an identity. Here is another possibility. Again, we can start with the left side.

$$\begin{aligned}\frac{\sec^2\theta-1}{\sec^2\theta} &= \frac{\sec^2\theta}{\sec^2\theta}-\frac{1}{\sec^2\theta}\\ &= 1-\cos^2\theta\\ &= \sin^2\theta\end{aligned}$$

Analysis

In the first method, we used the identity $\sec^2\theta=\tan^2\theta+1$ and continued to simplify. In the second method, we split the fraction, putting both terms in the numerator over the common denominator. This problem illustrates that there are multiple ways we can verify an identity. Employing some creativity can sometimes simplify a procedure. As long as the substitutions are correct, the answer will be the same.

Exercise 11.1.2

Show that $\dfrac{\cot\theta}{\csc\theta}=\cos\theta$.

Answer

$$\begin{aligned}\frac{\cot\theta}{\csc\theta} &= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}}\\ &= \frac{\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{1}\\ &= \cos\theta\end{aligned}$$

Example 11.1.4: Creating and Verifying an Identity

Create an identity for the expression $2\tan\theta\sec\theta$ by rewriting strictly in terms of sine.

Solution

There are a number of ways to begin, but here we will use the quotient and reciprocal identities to rewrite the expression:

$$\begin{aligned}2\tan\theta\sec\theta &= 2\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right)\\ &= \frac{2\sin\theta}{\cos^2\theta}\\ &= \frac{2\sin\theta}{1-\sin^2\theta} \qquad \text{Substitute } 1-\sin^2\theta \text{ for } \cos^2\theta\end{aligned}$$

Thus,

$$2\tan\theta\sec\theta=\frac{2\sin\theta}{1-\sin^2\theta}$$

Example 11.1.5: Verifying an Identity Using Algebra and Even/Odd Identities

Verify the identity:

$$\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)}=\cos\theta-\sin\theta$$

Solution

Let's start with the left side and simplify:

$$\begin{aligned}\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} &= \frac{[\sin(-\theta)]^2-[\cos(-\theta)]^2}{\sin(-\theta)-\cos(-\theta)} \\ &= \frac{(-\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta} \quad ,\sin(-x)=-\sin\ x \text{ and } \cos(-x)=\cos\ x \\ &= \frac{(\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta} \quad \text{Difference of squares} \\ &= \frac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{-(\sin\theta+\cos\theta)} \\ &= \cos\theta-\sin\theta\end{aligned}$$

? Exercise 11.1.3

Verify the identity $\dfrac{\sin^2\theta-1}{\tan\theta\sin\theta-\tan\theta}=\dfrac{\sin\theta+1}{\tan\theta}$.

Answer

$$\begin{aligned}\frac{\sin^2\theta-1}{\tan\theta\sin\theta-\tan\theta} &= \frac{(\sin\theta+1)(\sin\theta-1)}{\tan\theta(\sin\theta-1)} \\ &= \frac{\sin\theta+1}{\tan\theta}\end{aligned}$$

✓ Example 11.1.6: Verifying an Identity Involving Cosines and Cotangents

Verify the identity: $(1-\cos^2x)(1+\cot^2x)=1$.

Solution

$$\begin{aligned}(1-\cos^2x)(1+\cot^2x) &= (1-\cos^2x)\left(1+\frac{\cos^2x}{\sin^2x}\right) \\ &= (1-\cos^2x)\left(\frac{\sin^2x}{\sin^2x}+\frac{\cos^2x}{\sin^2x}\right) \quad \text{Find the common denominator} \\ &= (1-\cos^2x)\left(\frac{\sin^2x+\cos^2x}{\sin^2x}\right) \\ &= (\sin^2x)\left(\frac{1}{\sin^2x}\right) \\ &= 1\end{aligned}$$

Using Algebra to Simplify Trigonometric Expressions

We have seen that algebra is very important in verifying trigonometric identities, but it is just as critical in simplifying trigonometric expressions before solving. Being familiar with the basic properties and formulas of algebra, such as the difference of squares formula, the perfect square formula, or substitution, will simplify the work involved with trigonometric expressions and equations.

For example, the equation $(\sin x+1)(\sin x-1)=0$ resembles the equation $(x+1)(x-1)=0$, which uses the factored form of the difference of squares. Using algebra makes finding a solution straightforward and familiar. We can set each factor equal to zero and solve. This is one example of recognizing algebraic patterns in trigonometric expressions or equations.

Another example is the difference of squares formula, $a^2-b^2=(a-b)(a+b)$, which is widely used in many areas other than mathematics, such as engineering, architecture, and physics. We can also create our own identities by continually expanding an expression and making the appropriate substitutions. Using algebraic properties and formulas makes many trigonometric equations easier to understand and solve.

Example 11.1.7A: Writing the Trigonometric Expression as an Algebraic Expression

Write the following trigonometric expression as an algebraic expression: $2\cos^2\theta+\cos\theta-1$.

Solution

Notice that the pattern displayed has the same form as a standard quadratic expression,ax^2+bx+c . Letting $\cos\theta=x$,we can rewrite the expression as follows:

$$2x^2+x-1$$

This expression can be factored as $(2x+1)(x-1)$. If it were set equal to zero and we wanted to solve the equation, we would use the zero factor property and solve each factor for x. At this point, we would replace x with $\cos\theta$ and solve for θ.

Example 11.1.7B: Rewriting a Trigonometric Expression Using the Difference of Squares

Rewrite the trigonometric expression using the difference of squares: $4cos^2\theta-1$.

Solution

Notice that both the coefficient and the trigonometric expression in the first term are squared, and the square of the number 1 is 1. This is the difference of squares.

$$\begin{aligned}4\cos^2\theta-1 &= (2\cos\theta)^2-1\\ &= (2\cos\theta-1)(2\cos\theta+1)\end{aligned}$$

Analysis

If this expression were written in the form of an equation set equal to zero, we could solve each factor using the zero factor property. We could also use substitution like we did in the previous problem and let $\cos\theta=x$, rewrite the expression as $4x^2-1$, and factor $(2x-1)(2x+1)$. Then replace x with $\cos\theta$ and solve for the angle.

Exercise 11.1.4

Rewrite the trigonometric expression using the difference of squares: $25-9\sin^2\theta$.

Answer

This is a difference of squares formula: $25-9\sin^2\theta=(5-3\sin\theta)(5+3\sin\theta)$.

Example 11.1.8: Simplify by Rewriting and Using Substitution

Simplify the expression by rewriting and using identities:

$$\csc^2\theta-\cot^2\theta$$

Solution

We can start with the Pythagorean identity.

$$1+\cot^2\theta=\csc^2\theta$$

Now we can simplify by substituting $1+\cot^2\theta$ for $\csc^2\theta$

$$\begin{aligned}\csc^2\theta-\cot^2\theta &= 1+\cot^2\theta-\cot^2\theta\\ &= 1\end{aligned}$$

Exercise 11.1.5

Use algebraic techniques to verify the identity: $\dfrac{\cos\theta}{1+\sin\theta}=\dfrac{1-\sin\theta}{\cos\theta}$.

(Hint: Multiply the numerator and denominator on the left side by $1-\sin\theta$.)

Answer

$$\frac{\cos\theta}{1+\sin\theta}\left(\frac{1-\sin\theta}{1-\sin\theta}\right)=\frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta}$$
$$=\frac{\cos\theta(1-\sin\theta)}{\cos^2\theta}$$
$$=\frac{1-\sin\theta}{\cos\theta}$$

Media

Access these online resources for additional instruction and practice with the fundamental trigonometric identities.

- Fundamental Trigonometric Identities
- Verifying Trigonometric Identities

Key Equations

Pythagorean identities	$\cos^2\theta+\sin^2\theta=1$ $1+\cot^2\theta=\csc^2\theta$ $1+\tan^2\theta=\sec^2\theta$
Even-odd identities	$\tan(-\theta)=-\tan\theta$ $\cot(-\theta)=-\cot\theta$ $\sin(-\theta)=-\sin\theta$ $\csc(-\theta)=-\csc\theta$ $\cos(-\theta)=\cos\theta$ $\sec(-\theta)=\sec\theta$
Reciprocal identities	$\sin\theta=\frac{1}{\csc\theta}$ $\cos\theta=\frac{1}{\sec\theta}$ $\tan\theta=\frac{1}{\cot\theta}$ $\csc\theta=\frac{1}{\sin\theta}$ $\sec\theta=\frac{1}{\cos\theta}$ $\cot\theta=\frac{1}{\tan\theta}$
Quotient identities	$\tan\theta=\frac{\sin\theta}{\cos\theta}$ $\cot\theta=\frac{\cos\theta}{\sin\theta}$

Key Concepts

- There are multiple ways to represent a trigonometric expression. Verifying the identities illustrates how expressions can be rewritten to simplify a problem.
- Graphing both sides of an identity will verify it. See Example 11.1.1.
- Simplifying one side of the equation to equal the other side is another method for verifying an identity. See Example 11.1.2 and Example 11.1.3.
- The approach to verifying an identity depends on the nature of the identity. It is often useful to begin on the more complex side of the equation. See Example 11.1.4.
- We can create an identity and then verify it. See Example 11.1.5.
- Verifying an identity may involve algebra with the fundamental identities. See Example 11.1.6 and Example 11.1.7.

- Algebraic techniques can be used to simplify trigonometric expressions. We use algebraic techniques throughout this text, as they consist of the fundamental rules of mathematics. See Example 11.1.8, Example 11.1.9, and Example 11.1.10

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11.2: Sum and Difference Identities

Learning Objectives

- Use sum and difference formulas for cosine.
- Use sum and difference formulas for sine.
- Use sum and difference formulas for tangent.
- Use sum and difference formulas for cofunctions.
- Use sum and difference formulas to verify identities.

How can the height of a mountain be measured? What about the distance from Earth to the sun? Like many seemingly impossible problems, we rely on mathematical formulas to find the answers. The trigonometric identities, commonly used in mathematical proofs, have had real-world applications for centuries, including their use in calculating long distances.

The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.



Figure 11.2.1: Mount McKinley, in Denali National Park, Alaska, rises 20,237 feet (6,168 m) above sea level. It is the highest peak in North America. (credit: Daniel A. Leifheit, Flickr)

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word *identity*.

Using the Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special angles, which we can review in the unit circle shown in Figure 11.2.2

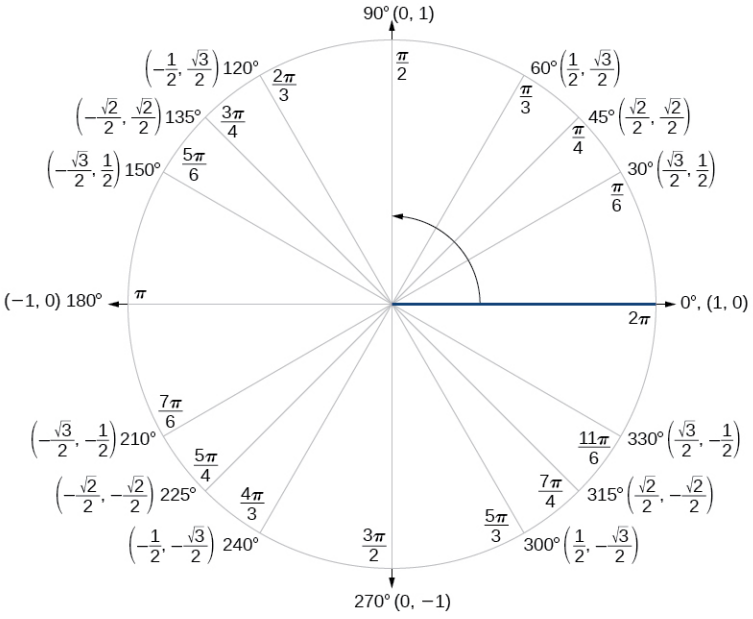


Figure 11.2.2: The Unit Circle

We will begin with the sum and difference formulas for cosine, so that we can find the cosine of a given angle if we can break it up into the sum or difference of two of the special angles (Table 11.2.1).

Table 11.2.1

Sum formula for cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference formula for cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

First, we will prove the difference formula for cosines. Let's consider two points on the unit circle (Figure 11.2.3). Point P is at an angle α from the positive x-axis with coordinates $(\cos\alpha,\sin\alpha)$ and point Q is at an angle of β from the positive x-axis with coordinates $(\cos\beta,\sin\beta)$. Note the measure of angle POQ is $\alpha-\beta$.

Label two more points: A at an angle of $(\alpha-\beta)$ from the positive x-axis with coordinates $(\cos(\alpha-\beta),\sin(\alpha-\beta))$; and point B with coordinates $(1,0)$. Triangle POQ is a rotation of triangle AOB and thus the distance from P to Q is the same as the distance from A to B.

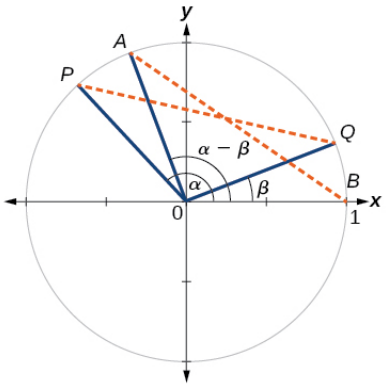


Figure 11.2.3

We can find the distance from P to Q using the distance formula.

$$\begin{aligned} d_{PQ} &= \sqrt{(\cos\alpha-\cos\beta)^2+(sin\alpha-sin\beta)^2} \\ &= \sqrt{\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta} && \text{Apply Pythagorean identity and simplify.} \\ &= \sqrt{(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \\ &= \sqrt{1+1-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \\ &= \sqrt{2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \end{aligned}$$

Similarly, using the distance formula we can find the distance from A to B.

$$\begin{aligned} d_{AB} &= \sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2} \\ &= \sqrt{\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)} && \text{Apply Pythagorean identity and simplify} \\ &= \sqrt{(\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta))-2\cos(\alpha-\beta)+1} \\ &= \sqrt{1-2\cos(\alpha-\beta)+1} \\ &= \sqrt{2-2\cos(\alpha-\beta)} && \text{Subtract 2 from both sides and divide both sides by } -2. \end{aligned}$$

$$\cos\alpha\cos\beta+\sin\alpha\sin\beta = \cos(\alpha-\beta)$$

Thus, we have the difference formula for cosine. We can use similar methods to derive the cosine of the sum of two angles.

SUM AND DIFFERENCE FORMULAS FOR COSINE

These formulas can be used to calculate the cosine of sums and differences of angles.

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \tag{11.2.1}$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \tag{11.2.2}$$

How to: Given two angles, find the cosine of the difference between the angles

1. Write the difference formula for cosine.
2. Substitute the values of the given angles into the formula.
3. Simplify.

Example 11.2.1: Finding the Exact Value Using the Formula for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4}-\frac{\pi}{6}\right)$.

Solution

Begin by writing the formula for the cosine of the difference of two angles. Then substitute the given values.

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\ \cos\left(\frac{5\pi}{4}-\frac{\pi}{6}\right) &= \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\ &= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= -\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in radian mode.

Exercise 11.2.1:

Find the exact value of $\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$.

Answer

$\frac{\sqrt{2}+\sqrt{6}}{4}$

Example 11.2.2: Finding the Exact Value Using the Formula for the Sum of Two Angles for Cosine

Find the exact value of $\cos(75^\circ)$.

Solution

As $75^\circ = 45^\circ + 30^\circ$,we can evaluate $\cos(75^\circ)$ as $\cos(45^\circ + 30^\circ)$.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\ \cos(45^\circ+30^\circ) &= \cos(45^\circ)\cos(30^\circ)-\sin(45^\circ)\sin(30^\circ)\\ &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in degree mode.

Analysis

Note that we could have also solved this problem using the fact that $75^\circ = 135^\circ - 60^\circ$.

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\ \cos(135^\circ-60^\circ) &= \cos(135^\circ)\cos(60^\circ)+\sin(135^\circ)\sin(60^\circ)\\ &= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= -\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

? Exercise 11.2.2

Find the exact value of $\cos(105^\circ)$.

Answer

$\frac{\sqrt{2}-\sqrt{6}}{4}$

Using the Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they resemble the cosine formulas.

SUM AND DIFFERENCE FORMULAS FOR SINE

These formulas can be used to calculate the sines of sums and differences of angles.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta \tag{11.2.3}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta-\cos\alpha\sin\beta \tag{11.2.4}$$

How to: Given two angles, find the sine of the difference between the angles

1. Write the difference formula for sine.
2. Substitute the given angles into the formula.
3. Simplify.

Example 11.2.3: Using Sum and Difference Identities to Evaluate the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

a. $\sin(45^\circ-30^\circ)$
b. $\sin(135^\circ-120^\circ)$

Solution

a. Let's begin by writing the formula and substitute the given angles.

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \sin(45^\circ-30^\circ) &= \sin(45^\circ)\cos(30^\circ)-\cos(45^\circ)\sin(30^\circ)\end{aligned}$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45^\circ)=\tfrac{\sqrt{2}}{2},\qquad \cos(30^\circ)=\tfrac{\sqrt{3}}{2},\qquad \cos(45^\circ)=\tfrac{\sqrt{2}}{2},\qquad \sin(30^\circ)=\tfrac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(45^\circ-30^\circ) &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

b. Again, we write the formula and substitute the given angles.

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \sin(135^\circ-120^\circ) &= \sin(135^\circ)\cos(120^\circ)-\cos(135^\circ)\sin(120^\circ)\end{aligned}$$

Next, we find the values of the trigonometric expressions.

$$\sin(135^\circ)=\tfrac{\sqrt{2}}{2},\qquad \cos(120^\circ)=-\tfrac{1}{2},\qquad \cos(135^\circ)=\tfrac{\sqrt{2}}{2},\qquad \sin(120^\circ)=\tfrac{\sqrt{3}}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(135^\circ-120^\circ) &= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)-\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example 11.2.4: Finding the Exact Value of an Expression Involving an Inverse Trigonometric Function

Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{3}{5}\right)$. Then check the answer with a graphing calculator.

Solution

The pattern displayed in this problem is $\sin(\alpha+\beta)$. Let $\alpha=\cos^{-1}\frac{1}{2}$ and $\beta=\sin^{-1}\frac{3}{5}$. Then we can write

$$\begin{aligned}\cos\alpha &= \frac{1}{2}, \quad 0\le\alpha\le\pi\\ \sin\beta &= \frac{3}{5}, \quad -\frac{\pi}{2}\le\beta\le\frac{\pi}{2}\end{aligned}$$

We will use the Pythagorean identities to find $\sin\alpha$ and $\cos\beta$

$$\begin{aligned}\sin\alpha &= \sqrt{1-\cos^2\alpha}\\ &= \sqrt{1-\frac{1}{4}}\\ &= \sqrt{\frac{3}{4}}\\ &= \frac{\sqrt{3}}{2}\\ \cos\beta &= \sqrt{1-\sin^2\beta}\\ &= \sqrt{1-\frac{9}{25}}\\ &= \sqrt{\frac{16}{25}}\\ &= \frac{4}{5}\end{aligned}$$

Using the sum formula for sine,

$$\begin{aligned}\sin\left(\cos^{-1}\tfrac{1}{2}+\sin^{-1}\tfrac{3}{5}\right) &= \sin(\alpha+\beta)\\ &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \frac{\sqrt{3}}{2}\cdot\frac{4}{5}+\frac{1}{2}\cdot\frac{3}{5}\\ &= \frac{4\sqrt{3}+3}{10}\end{aligned}$$

Using the Sum and Difference Formulas for Tangent

Finding exact values for the tangent of the sum or difference of two angles is a little more complicated, but again, it is a matter of recognizing the pattern.

Finding the sum of two angles formula for tangent involves taking quotient of the sum formulas for sine and cosine and simplifying. Recall, $\tan x=\dfrac{\sin x}{\cos x}$, when $\cos x\neq 0$.

Let's derive the sum formula for tangent.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\ &= \frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\end{aligned}$$

We can derive the difference formula for tangent in a similar way.

SUM AND DIFFERENCE FORMULAS FOR TANGENT

The sum and difference formulas for tangent are:

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \tag{11.2.5}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \tag{11.2.6}$$

How to: Given two angles, find the tangent of the sum of the angles

1. Write the sum formula for tangent.
2. Substitute the given angles into the formula.
3. Simplify.

Example 11.2.5: Finding the Exact Value of an Expression Involving Tangent

Find the exact value of $\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$.

Solution

Let's first write the sum formula for tangent and then substitute the given angles into the formula.

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\tan\left(\frac{\pi}{6}\right)+\tan\left(\frac{\pi}{4}\right)}{1-\left(\tan\left(\frac{\pi}{6}\right)\right)\left(\tan\left(\frac{\pi}{4}\right)\right)}$$

Next, we determine the individual function values within the formula:

$$\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}},\quad \text{and}\quad \tan\left(\frac{\pi}{4}\right)=1 \tag{11.2.7}$$

So we have,

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\frac{1}{\sqrt{3}}+1}{1-\left(\frac{1}{\sqrt{3}}\right)(1)}$$

$$=\frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$

$$=\frac{1+\sqrt{3}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}-1}$$

$$=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

? Exercise 11.2.5:

Find the exact value of $\tan\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)$.

Answer

$\frac{1-\sqrt{3}}{1+\sqrt{3}}$

✓ Example 11.2.6: Finding Multiple Sums and Differences of Angles

Given $\sin\alpha=\frac{3}{5}, \quad 0<\alpha<\frac{\pi}{2}$, and $\cos\beta=-\frac{5}{13}, \quad \pi<\beta<\frac{3\pi}{2}$,

find

a. $\sin(\alpha+\beta)$
b. $\cos(\alpha+\beta)$
c. $\tan(\alpha+\beta)$
d. $\tan(\alpha-\beta)$

Solution

We can use the sum and difference formulas to identify the sum or difference of angles when the ratio of sine, cosine, or tangent is provided for each of the individual angles. To do so, we construct what is called a reference triangle to help find each component of the sum and difference formulas.

a. To find $\sin(\alpha+\beta)$, we begin with $\sin\alpha=\frac{3}{5}$ and $0<\alpha<\frac{\pi}{2}$. The side opposite α has length 3, the hypotenuse has length 5, and α is in the first quadrant. See Figure 11.2.4 Using the Pythagorean Theorem, we can find the length of side a:

$$a^2+3^2=5^2$$
$$a^2=16$$
$$a=4$$

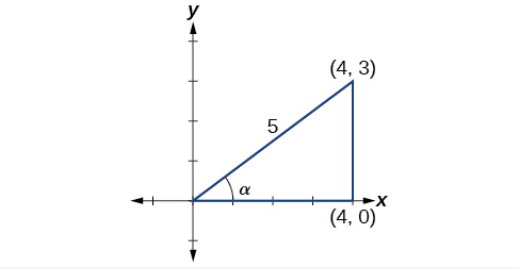


Figure 11.2.4

Since $\cos\beta=-\frac{5}{13}$ and $\pi<\beta<\frac{3\pi}{2}$,the side adjacent to β is -5,the hypotenuse is 13, and β is in the third quadrant. See Figure 11.2.5 Again, using the Pythagorean Theorem, we have

$$\begin{aligned}(-5)^2+a^2 &= 13^2\\ 25+a^2 &= 169\\ a^2 &= 144\\ a &= \pm 12\end{aligned}$$

Since β is in the third quadrant, $a = -12$.

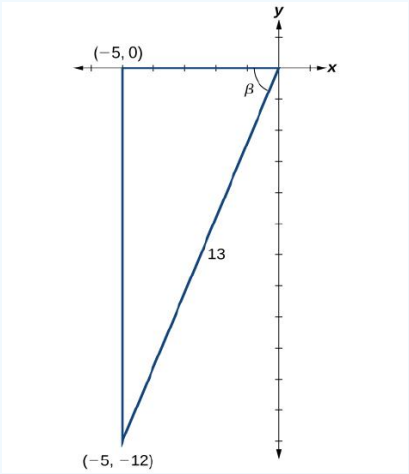


Figure 11.2.5

The next step is finding the cosine of α and the sine of β. The cosine ofα α is the adjacent side over the hypotenuse. We can find it from the triangle in Figure 11.2.5: $\cos\alpha = \frac{4}{5}$. We can also find the sine of β from the triangle in Figure 11.2.5, as opposite side over the hypotenuse: $\sin\beta = -\frac{12}{13}$. Now we are ready to evaluate $\sin(\alpha+\beta)$.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)+\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)\\ &= -\frac{15}{65}-\frac{48}{65}\\ &= -\frac{63}{65}\end{aligned}$$

b. We can find $\cos(\alpha+\beta)$ in a similar manner. We substitute the values according to the formula.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right)-\left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)\\ &= -\frac{20}{65}+\frac{36}{65}\\ &= \frac{16}{65}\end{aligned}$$

c. For $\tan(\alpha+\beta)$,if $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, then

$$\tan\alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

If $\sin\beta = -\frac{12}{13}$ and $\cos\beta = -\frac{5}{13}$, then

$$\tan\beta = \frac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{5}$$

Then,

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\ &= \frac{\frac{3}{4}+\frac{12}{5}}{1-\frac{3}{4}\left(\frac{12}{5}\right)}\\ &= \frac{\frac{63}{20}}{-\frac{16}{20}}\\ &= -\frac{63}{16}\end{aligned}$$

d. To find $\tan(\alpha-\beta)$, we have the values we need. We can substitute them in and evaluate.

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\\ &= \frac{\frac{3}{4}-\frac{12}{5}}{1+\frac{3}{4}\left(\frac{12}{5}\right)}\\ &= \frac{-\frac{33}{20}}{\frac{56}{20}}\\ &= -\frac{33}{56}\end{aligned}$$

Analysis

A common mistake when addressing problems such as this one is that we may be tempted to think that α and β are angles in the same triangle, which of course, they are not. Also note that

$$\tan(\alpha+\beta) = \sin(\alpha+\beta)\cos(\alpha+\beta)$$

Using Sum and Difference Formulas for Cofunctions

Now that we can find the sine, cosine, and tangent functions for the sums and differences of angles, we can use them to do the same for their cofunctions. You may recall from Right Triangle Trigonometry that, if the sum of two positive angles is $\frac{\pi}{2}$, those two angles are complements, and the sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In Figure 11.2.6, notice that if one of the acute angles is labeled as θ, then the other acute angle must be labeled $\frac{\pi}{2}-\theta$.

Notice also that $\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$,which is opposite over hypotenuse. Thus, when two angles are complimentary, we can say that the sine of θ equals the cofunction of the complement of θ. Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.

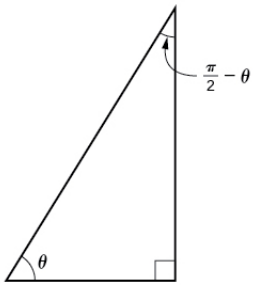


Figure 11.2.6

From these relationships, the cofunction identities are formed. Recall that you first encountered these identities in The Unit Circle: Sine and Cosine Functions.

COFUNCTION IDENTITIES

The cofunction identities are summarized in Table 11.2.2

Table 11.2.2

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$
$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$
$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Notice that the formulas in the table may also justified algebraically using the sum and difference formulas. For example, using

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

We can write

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \\ &= (0)\cos\theta + (1)\sin\theta \\ &= \sin\theta\end{aligned}$$

Example 11.2.7: Finding a Cofunction with the Same Value as the Given Expression

Write $\tan\frac{\pi}{9}$ in terms of its cofunction.

Solution

The cofunction of $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$. Thus,

$$\begin{aligned}\tan\left(\frac{\pi}{9}\right) &= \cot\left(\frac{\pi}{2} - \frac{\pi}{9}\right) \\ &= \cot\left(\frac{9\pi}{18} - \frac{2\pi}{18}\right) \\ &= \cot\left(\frac{7\pi}{18}\right)\end{aligned}$$

Exercise 11.2.7

Write $\sin\frac{\pi}{7}$ in terms of its cofunction.

Answer

$\cos\left(\frac{5\pi}{14}\right)$

Using the Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems. Reviewing the general rules presented earlier may help simplify the process of verifying an identity.

How to: Given an identity, verify using sum and difference formulas

1. Begin with the expression on the side of the equal sign that appears most complex. Rewrite that expression until it matches the other side of the equal sign. Occasionally, we might have to alter both sides, but working on only one side is the most efficient.
2. Look for opportunities to use the sum and difference formulas.
3. Rewrite sums or differences of quotients as single quotients.
4. If the process becomes cumbersome, rewrite the expression in terms of sines and cosines.

Example 11.2.8A: Verifying an Identity Involving Sine

Verify the identity $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$.

Solution

We see that the left side of the equation includes the sines of the sum and the difference of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

We can rewrite each using the sum and difference formulas.

$$\begin{aligned}\sin(\alpha+\beta)+\sin(\alpha-\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta\\ &= 2\sin\alpha\cos\beta\end{aligned}$$

We see that the identity is verified.

Example 11.2.8B: Verifying an Identity Involving Tangent

Verify the following identity.

$$\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\tan\alpha-\tan\beta$$

Solution

We can begin by rewriting the numerator on the left side of the equation.

$$\begin{aligned}\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} &= \frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\\ &= \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta} && \text{Rewrite using a common denominator}\\ &= \frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta} && \text{Cancel}\\ &= \tan\alpha-\tan\beta && \text{Rewrite in terms of tangent}\end{aligned}$$

Exercise 11.2.8:

Verify the identity: $\tan(\pi-\theta)=-\tan\theta$.

Answer

$$\begin{aligned}\tan(\pi-\theta) &= \frac{\tan(\pi)-\tan\theta}{1+\tan(\pi)\tan\theta}\\ &= \frac{0-\tan\theta}{1+0\cdot\tan\theta}\\ &= -\tan\theta\end{aligned}$$

Example 11.2.9A: Using Sum and Difference Formulas to Solve an Application Problem

Let L_1 and L_2 denote two non-vertical intersecting lines, and let θ denote the acute angle between L_1 and L_2. See Figure 11.2.7. Show that

$$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 respectively. (**Hint:** Use the fact that $\tan\theta_1=m_1$ and $\tan\theta_2=m_2$.)

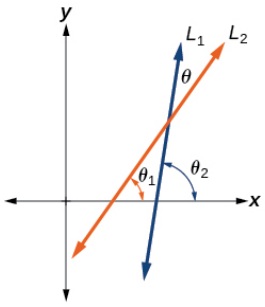


Figure 11.2.7

Solution

Using the difference formula for tangent, this problem does not seem as daunting as it might.

$$\begin{aligned}\tan\theta &= \tan(\theta_2 - \theta_1)\\ &= \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_1\tan\theta_2}\\ &= \frac{m_2 - m_1}{1+m_1 m_2}\end{aligned}$$

Example 11.2.9B: Investigating a Guy-wire Problem

For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole. Added support is provided by another guy-wire S attached 40 feet above ground on the same pole. If the wires are attached to the ground 50 feet from the pole, find the angle α between the wires. See Figure 11.2.8

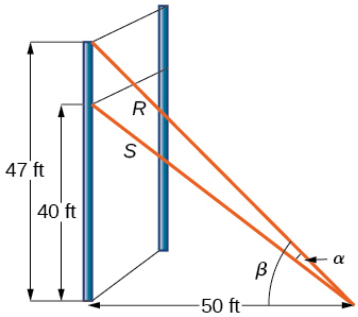


Figure 11.2.8

Solution

Let's first summarize the information we can gather from the diagram. As only the sides adjacent to the right angle are known, we can use the tangent function. Notice that $\tan\beta = \frac{47}{50}$, and $\tan(\beta-\alpha) = \frac{40}{50} = \frac{4}{5}$. We can then use difference formula for tangent.

$$\tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha} \tag{11.2.8}$$

Now, substituting the values we know into the formula, we have,

$$\frac{4}{5} = \frac{\frac{47}{50} - \tan\alpha}{1+\frac{47}{50}\tan\alpha}$$

$$4\left(1+\tfrac{47}{50}\tan\alpha\right) = 5\left(\tfrac{47}{50} - \tan\alpha\right)$$

Use the distributive property, and then simplify the functions.

$$\begin{aligned} 4(1)+4\left(\tfrac{47}{50}\right)\tan\alpha &= 5\left(\tfrac{47}{50}\right)-5\tan\alpha \\ 4+3.76\tan\alpha &= 4.7-5\tan\alpha \\ 5\tan\alpha+3.76\tan\alpha &= 0.7 \\ 8.76\tan\alpha &= 0.7 \\ \tan\alpha &\approx 0.07991 \\ \tan^{-1}(0.07991) &\approx .079741 \end{aligned}$$

Now we can calculate the angle in degrees.

$$\begin{aligned} \alpha &\approx 0.079741\left(\frac{180}{\pi}\right) \\ &\approx 4.57^\circ \end{aligned}$$

Analysis

Occasionally, when an application appears that includes a right triangle, we may think that solving is a matter of applying the Pythagorean Theorem. That may be partially true, but it depends on what the problem is asking and what information is given.

Media

Access these online resources for additional instruction and practice with sum and difference identities.

- Sum and Difference Identities for Cosine
- Sum and Difference Identities for Sine
- Sum and Difference Identities for Tangent

Key Equations

Sum Formula for Cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference Formula for Cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
Sum Formula for Sine	$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
Difference Formula for Sine	$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
Sum Formula for Tangent	$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
Difference Formula for Tangent	$\tan(\alpha-\beta)=\dfrac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$
Cofunction identities	$\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$ $\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$ $\tan\theta=\cot\left(\frac{\pi}{2}-\theta\right)$ $\cot\theta=\tan\left(\frac{\pi}{2}-\theta\right)$ $\sec\theta=\csc\left(\frac{\pi}{2}-\theta\right)$ $\csc\theta=\sec\left(\frac{\pi}{2}-\theta\right)$

Key Concepts

- The sum formula for cosines states that the cosine of the sum of two angles equals the product of the cosines of the angles minus the product of the sines of the angles. The difference formula for cosines states that the cosine of the difference of two angles equals the product of the cosines of the angles plus the product of the sines of the angles.
- The sum and difference formulas can be used to find the exact values of the sine, cosine, or tangent of an angle. See Example 11.2.1 and Example 11.2.2
- The sum formula for sines states that the sine of the sum of two angles equals the product of the sine of the first angle and cosine of the second angle plus the product of the cosine of the first angle and the sine of the second angle. The difference formula for sines states that the sine of the difference of two angles equals the product of the sine of the first angle and cosine of the second angle minus the product of the cosine of the first angle and the sine of the second angle. See Example 11.2.3
- The sum and difference formulas for sine and cosine can also be used for inverse trigonometric functions. See Example 11.2.4

- The sum formula for tangent states that the tangent of the sum of two angles equals the sum of the tangents of the angles divided by 1 minus the product of the tangents of the angles. The difference formula for tangent states that the tangent of the difference of two angles equals the difference of the tangents of the angles divided by 1 plus the product of the tangents of the angles. See Example 11.2.5
- The Pythagorean Theorem along with the sum and difference formulas can be used to find multiple sums and differences of angles. See Example 11.2.6
- The cofunction identities apply to complementary angles and pairs of reciprocal functions. See Example 11.2.7.
- Sum and difference formulas are useful in verifying identities. See Example 11.2.8and Example 11.2.9
- Application problems are often easier to solve by using sum and difference formulas. See Example 11.2.10and Example 11.2.11

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11.3: Double-Angle, Half-Angle, and Reduction Formulas

Learning Objectives

- Use double-angle formulas to find exact values
- Use double-angle formulas to verify identities
- Use reduction formulas to simplify an expression
- Use half-angle formulas to find exact values

Bicycle ramps made for competition (see Figure 11.3.1) must vary in height depending on the skill level of the competitors. For advanced competitors, the angle formed by the ramp and the ground should be θ such that $\tan\theta = \frac{5}{3}$. The angle is divided in half for novices. What is the steepness of the ramp for novices? In this section, we will investigate three additional categories of identities that we can use to answer questions such as this one.

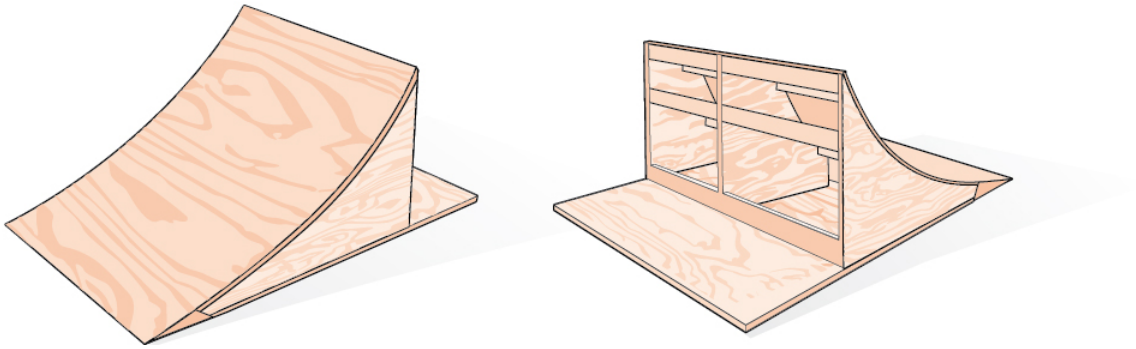


Figure 11.3.1: Bicycle ramps for advanced riders have a steeper incline than those designed for novices.

Using Double-Angle Formulas to Find Exact Values

In the previous section, we used addition and subtraction formulas for trigonometric functions. Now, we take another look at those same formulas. The **double-angle formulas** are a special case of the sum formulas, where $\alpha = \beta$. Deriving the double-angle formula for sine begins with the sum formula,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{11.3.1}$$

If we let $\alpha = \beta = \theta$, then we have

$$\begin{aligned}\sin(\theta+\theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ \sin(2\theta) &= 2\sin\theta\cos\theta\end{aligned}$$

Deriving the double-angle for cosine gives us three options. First, starting from the sum formula, $\cos(\alpha+\beta) = \cos\,\alpha\,\cos\,\beta - \sin\,\alpha\,\sin\,\beta$,and letting $\alpha = \beta = \theta$, we have

$$\begin{aligned}\cos(\theta+\theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Using the Pythagorean properties, we can expand this double-angle formula for cosine and get two more variations. The first variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ &= (1-\sin^2\theta) - \sin^2\theta\end{aligned}$$

The second variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1-\cos^2\theta) \\ &= 2\cos^2\theta - 1\end{aligned}$$

Similarly, to derive the double-angle formula for tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\theta+\theta) = \frac{\tan\theta + \tan\theta}{1-\tan\theta\tan\theta}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

DOUBLE-ANGLE FORMULAS

The *double-angle formulas* are summarized as follows:

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{11.3.2}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{11.3.3}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} \tag{11.3.4}$$

How to: Given the tangent of an angle and the quadrant in which it is located, use the double-angle formulas to find the exact value

1. Draw a triangle to reflect the given information.
2. Determine the correct double-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 11.3.1: Using a Double-Angle Formula to Find the Exact Value Involving Tangent

Given that $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$
b. $\cos(2\theta)$
c. $\tan(2\theta)$

Solution

If we draw a triangle to reflect the information given, we can find the values needed to solve the problems on the image. We are given $\tan\theta = -\frac{3}{4}$,such that θ is in quadrant II. The tangent of an angle is equal to the opposite side over the adjacent side, and because θ is in the second quadrant, the adjacent side is on the *x*-axis and is negative. Use the **Pythagorean Theorem** to find the length of the hypotenuse:

$$\begin{aligned}(-4)^2 + (3)^2 &= c^2 \\ 16+9 &= c^2 \\ 25 &= c^2 \\ c &= 5\end{aligned}$$

Now we can draw a triangle similar to the one shown in Figure 11.3.2

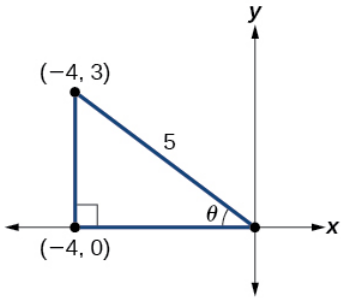


Figure 11.3.2

a. Let's begin by writing the double-angle formula for sine.

$$\sin(2\theta) = 2\ \sin\ \theta\ \cos\ \theta$$

We see that we to need to find $\sin\ \theta$ and $\cos\ \theta$. Based on Figure 11.3.2, we see that the hypotenuse equals 5, so $\sin\theta = 35$, $\sin\ \theta = 35$, and $\cos\theta = -45$. Substitute these values into the equation, and simplify.

Thus,

$$\begin{aligned}\sin(2\theta) &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

b. Write the double-angle formula for cosine.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Again, substitute the values of the sine and cosine into the equation, and simplify.

$$\begin{aligned}\cos(2\theta) &= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2\\ &= \frac{16}{25} - \frac{9}{25}\\ &= \frac{7}{25}\end{aligned}$$

c. Write the double-angle formula for tangent.

$$\tan(2\theta) = \frac{2\ \tan\ \theta}{1 - \tan^2\theta}$$

In this formula, we need the tangent, which we were given as $\tan\ \theta = -\frac{3}{4}$. Substitute this value into the equation, and simplify.

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(-\frac{3}{4}\right)}{1 - \left(-\frac{3}{4}\right)^2}\\ &= \frac{-\frac{3}{2}}{1 - \frac{9}{16}}\\ &= -\frac{3}{2}\left(\frac{16}{7}\right)\\ &= -\frac{24}{7}\end{aligned}$$

Exercise 11.3.1

Given $\sin\alpha = \frac{5}{8}$,with θ in quadrant I, find $\cos(2\alpha)$.

Answer

$$\cos(2\alpha) = \frac{7}{32}$$

Example 11.3.2: Using the Double-Angle Formula for Cosine without Exact Values

Use the double-angle formula for cosine to write $\cos(6x)$ in terms of $cos(3x)$.

Solution

$$\begin{aligned}\cos(6x) &= \cos(3x+3x)\\ &= \cos 3x\cos 3x - \sin 3x\sin 3x\\ &= \cos^2 3x - \sin^2 3x\end{aligned}$$

Analysis

This example illustrates that we can use the double-angle formula without having exact values. It emphasizes that the pattern is what we need to remember and that identities are true for all values in the domain of the trigonometric function.

Using Double-Angle Formulas to Verify Identities

Establishing identities using the double-angle formulas is performed using the same steps we used to derive the sum and difference formulas. Choose the more complicated side of the equation and rewrite it until it matches the other side.

Example 11.3.3: Using the Double-Angle Formulas to Verify an Identity

Verify the following identity using double-angle formulas:

$$1+\sin(2\theta) = (\sin\theta+\cos\theta)^2$$

Solution

We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= (\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta\\ &= 1+2\sin\theta\cos\theta\\ &= 1+\sin(2\theta)\end{aligned}$$

Analysis

This process is not complicated, as long as we recall the perfect square formula from algebra:

$$(a\pm b)^2 = a^2 \pm 2ab + b^2$$

where $a = \sin\theta$ and $b = \cos\theta$. Part of being successful in mathematics is the ability to recognize patterns. While the terms or symbols may change, the algebra remains consistent.

Exercise 11.3.2

Verify the identity: $\cos^4\theta - \sin^4\theta = \cos(2\theta)$.

Answer

$$\cos^4\theta - \sin^4\theta = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) = \cos(2\theta)$$

Example 11.3.4: Verifying a Double-Angle Identity for Tangent

Verify the identity: $\tan(2\theta) = 2\cot\theta - \tan\theta$

Solution

In this case, we will work with the left side of the equation and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned}
\tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} && \text{Double-angle formula} \\
&= \frac{2\tan\theta\left(\frac{1}{\tan\theta}\right)}{(1-\tan^2\theta)\left(\frac{1}{\tan\theta}\right)} && \text{Multiply by a term that results in desired numerator} \\
&= \frac{2}{\frac{1}{\tan\theta} - \frac{\tan^2\theta}{\tan\theta}} \\
&= \frac{2}{\cot\theta - \tan\theta} && \text{Use reciprocal identity for } \frac{1}{\tan\theta}
\end{aligned}$$

Analysis

Here is a case where the more complicated side of the initial equation appeared on the right, but we chose to work the left side. However, if we had chosen the left side to rewrite, we would have been working backwards to arrive at the equivalency. For example, suppose that we wanted to show

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{2}{\cot\theta - \tan\theta}$$

Lets work on the right side

$$\begin{aligned}
\frac{2}{\cot\theta - \tan\theta} &= \frac{2}{\frac{1}{\tan\theta} - \tan\theta}\left(\frac{\tan\theta}{\tan\theta}\right) \\
&= \frac{2\tan\theta}{\frac{1}{\tan\theta}(\tan\theta) - \tan\theta(\tan\theta)} \\
&= \frac{2\tan\theta}{1-\tan^2\theta}
\end{aligned}$$

When using the identities to simplify a trigonometric expression or solve a trigonometric equation, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated. However, we should begin with the guidelines set forth earlier.

Exercise 11.3.3

Verify the identity: $\cos(2\theta)\cos\theta = \cos^3\theta - \cos\theta\sin^2\theta$.

Answer

$$\cos(2\theta)\cos\theta = (\cos^2\theta - \sin^2\theta)\cos\theta = \cos^3\theta - \cos\theta\sin^2\theta$$

Use Reduction Formulas to Simplify an Expression

The double-angle formulas can be used to derive the **reduction formulas,** which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of

the first power of cosine. These formulas are especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2\sin^2\theta$. Solve for $\sin^2\theta$:

$$\begin{aligned} \cos(2\theta) &= 1 - 2\sin^2\theta \\ 2\sin^2\theta &= 1 - \cos(2\theta) \\ \sin^2\theta &= \frac{1-\cos(2\theta)}{2} \end{aligned}$$

Next, we use the formula $\cos(2\theta) = 2\cos^2\theta - 1$. Solve for $\cos^2\theta$:

$$\begin{aligned} \cos(2\theta) &= 2\cos^2\theta - 1 \\ 1 + \cos(2\theta) &= 2\cos^2\theta \\ \frac{1+\cos(2\theta)}{2} &= \cos^2\theta \end{aligned}$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned} \tan^2\theta &= \frac{\sin^2\theta}{\cos^2\theta} \\ &= \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} && \text{(Substitute the reduction formulas)} \\ &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{2}{1+\cos(2\theta)}\right) \\ &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \end{aligned}$$

REDUCTION FORMULAS

The reduction formulas are summarized as follows:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \tag{11.3.5}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \tag{11.3.6}$$

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \tag{11.3.7}$$

Example 11.3.5: Writing an Equivalent Expression Not Containing Powers Greater Than 1

Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Solution

We will apply the reduction formula for cosine twice.

$$\begin{aligned}\cos^4 x &= \left(\cos^2 x\right)^2 \\ &= \left(\frac{1+\cos(2x)}{2}\right)^2 \qquad \text{Substitute reduction formula} \\ &= \frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) \\ &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\left(\frac{1+\cos^2(2x)}{2}\right) \qquad \text{Substitute reduction formula for } \cos^2 x \\ &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{8}+\frac{1}{8}\cos(4x) \\ &= \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\end{aligned}$$

Analysis

The solution is found by using the reduction formula twice, as noted, and the perfect square formula from algebra.

Example 11.3.6: Using the Power-Reducing Formulas to Prove an Identity

Use the power-reducing formulas to prove $\sin^3(2x) = \left[\frac{1}{2}\ \sin(2x)\right]\ [1-\cos(4x)$

Solution

We will work on simplifying the left side of the equation:

$$\begin{aligned}\sin^3(2x) &= [\sin(2x)][\sin^2(2x)] \\ &= \sin(2x)\left[\frac{1-\cos(4x)}{2}\right] \qquad \text{Substitute the power-reduction formula.} \\ &= \sin(2x)\left(\frac{1}{2}\right)[1-\cos(4x)] \\ &= \frac{1}{2}[\sin(2x)][1-\cos(4x)]\end{aligned}$$

Analysis

Note that in this example, we substituted $\frac{1-\cos(4x)}{2}$ for $\sin^2(2x)$. The formula states $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

We let $\theta = 2x$, so $2\theta = 4x$.

Exercise 11.3.4

Use the power-reducing formulas to prove that $10\cos^4 x = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)$.

Answer

$$
\begin{aligned}
10\cos^4 x &= 10(\cos^2 x)^2 \\
&= 10\left[\frac{1+\cos(2x)}{2}\right]^2 \qquad \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{10}{4}[1+2\cos(2x)+\cos^2(2x)] \\
&= \frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{4}\left(\frac{1+\cos^2(2x)}{2}\right) \qquad \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{8}+\frac{10}{8}\cos(4x) \\
&= \frac{30}{8}+5\cos(2x)+\frac{10}{8}\cos(4x) \\
&= \frac{15}{4}+5\cos(2x)+\frac{5}{4}\cos(4x)
\end{aligned}
$$

Using Half-Angle Formulas to Find Exact Values

The next set of identities is the set of **half-angle formulas**, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with $\frac{\alpha}{2}$,the half-angle formula for sine is found by simplifying the equation and solving for $\sin\left(\frac{\alpha}{2}\right)$. Note that the half-angle formulas are preceded by a $\pm$ sign. This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which $\frac{\alpha}{2}$ terminates.

The half-angle formula for sine is derived as follows:

$$
\begin{aligned}
\sin^2\theta &= \frac{1-\cos(2\theta)}{2} \\
\sin^2\left(\frac{\alpha}{2}\right) &= \frac{1-\left(\cos 2\cdot\frac{\alpha}{2}\right)}{2} \\
&= \frac{1-\cos\alpha}{2} \\
\sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos\alpha}{2}}
\end{aligned}
$$

To derive the half-angle formula for cosine, we have

$$
\begin{aligned}
\cos^2\theta &= \frac{1+\cos(2\theta)}{2} \\
\cos^2\left(\frac{\alpha}{2}\right) &= \frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{2} \\
&= \frac{1+\cos\alpha}{2} \\
\cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos\alpha}{2}}
\end{aligned}
$$

For the tangent identity, we have

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$$\tan^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

HALF-ANGLE FORMULAS

The **half-angle formulas** are as follows:

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}} \tag{11.3.8}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \tag{11.3.9}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha} \tag{11.3.10}$$

Example 11.3.7

Using a Half-Angle Formula to Find the Exact Value of a Sine Function. Find $\sin(15^\circ)$ using a half-angle formula.

Solution

Since $15^\circ = \frac{30^\circ}{2}$, we use the half-angle formula for sine (Equation 11.3.8):

$$\begin{aligned}\sin\frac{30^\circ}{2} &= \sqrt{\frac{1-\cos 30^\circ}{2}}\\ &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

Remember that we can check the answer with a graphing calculator.

Analysis

Notice that we used only the positive root because $\sin(15^\circ)$ is positive.

Howto: Given the tangent of an angle and the quadrant in which the angle lies, find the exact values of trigonometric functions of half of the angle.

1. Draw a triangle to represent the given information.
2. Determine the correct half-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 11.3.8: Finding Exact Values Using Half-Angle Identities

Given that $\tan\alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

a. $\sin\left(\frac{\alpha}{2}\right)$
b. $\cos\left(\frac{\alpha}{2}\right)$
c. $\tan\left(\frac{\alpha}{2}\right)$

Solution

Using the given information, we can draw the triangle shown in Figure 11.3.3 Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate $\sin\alpha = -\frac{8}{17}$ and $\cos\alpha = -\frac{15}{17}$.

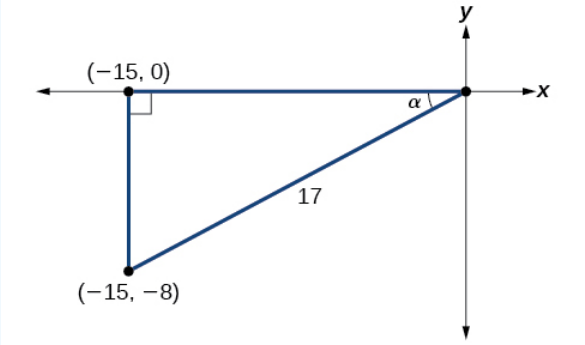


Figure 11.3.3

a. Before we start, we must remember that if α is in quadrant III, then $180^\circ < \alpha < 270^\circ$,so $\frac{180^\circ}{2} < \frac{\alpha}{2} < \frac{270^\circ}{2}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II, since $90^\circ < \frac{\alpha}{2} < 135^\circ$. To find $\sin\frac{\alpha}{2}$,we begin by writing the half-angle formula for sine. Then we substitute the value of the cosine we found from the triangle in Figure 11.3.3and simplify.

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
&= \pm\sqrt{\frac{1-(-\frac{15}{17})}{2}} \\
&= \pm\sqrt{\frac{\frac{32}{17}}{2}} \\
&= \pm\sqrt{\frac{32}{17}\cdot\frac{1}{2}} \\
&= \pm\sqrt{\frac{16}{17}} \\
&= \pm\frac{4}{\sqrt{17}} \\
&= \frac{4\sqrt{17}}{17}
\end{aligned}$$

We choose the positive value of $\sin\frac{\alpha}{2}$ because the angle terminates in quadrant II and sine is positive in quadrant II.

b. To find $\cos\frac{\alpha}{2}$, we will write the half-angle formula for cosine, substitute the value of the cosine we found from the triangle in Figure 11.3.3, and simplify.

$$\begin{aligned}\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}}\\ &= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}}\\ &= \pm\sqrt{\frac{\frac{2}{17}}{2}}\\ &= \pm\sqrt{\frac{2}{17}\cdot\frac{1}{2}}\\ &= \pm\sqrt{\frac{1}{17}}\\ &= -\frac{\sqrt{17}}{17}\end{aligned}$$

We choose the negative value of $\cos\frac{\alpha}{2}$ because the angle is in quadrant II because cosine is negative in quadrant II.

c. To find $\tan\frac{\alpha}{2}$, we write the half-angle formula for tangent. Again, we substitute the value of the cosine we found from the triangle in Figure 11.3.3 and simplify.

$$\begin{aligned}\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}\\ &= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{1+\left(-\frac{15}{17}\right)}}\\ &= \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}}\\ &= \pm\sqrt{\frac{32}{2}}\\ &= -\sqrt{16}\\ &= -4\end{aligned}$$

We choose the negative value of $\tan\frac{\alpha}{2}$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

? Exercise 11.3.5

Given that $\sin\alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\left(\frac{\alpha}{2}\right)$.

Answer

$-\frac{2}{\sqrt{5}}$

Example 11.3.9: Finding the Measurement of a Half Angle

Now, we will return to the problem posed at the beginning of the section. A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $tan\ \theta = 53$ for higher-level competition, what is the measurement of the angle for novice competition?

Solution

Since the angle for novice competition measures half the steepness of the angle for the high level competition, and $\tan\theta = \frac{5}{3}$ for high competition, we can find $\cos\theta$ from the right triangle and the Pythagorean theorem so that we can use the half-angle identities. See Figure 11.3.4

$$3^2 + 5^2 = 34$$
$$c = \sqrt{34}$$

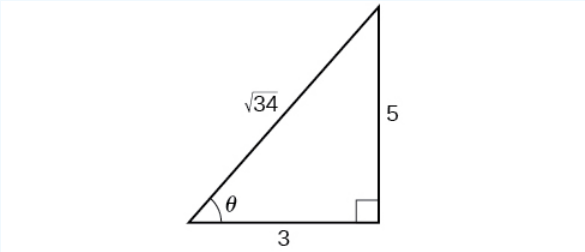


Figure 11.3.4

We see that $\cos\theta = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34}$. We can use the half-angle formula for tangent: $\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$. Since $\tan\theta$ is in the first quadrant, so is $\tan\frac{\theta}{2}$.

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\frac{3\sqrt{34}}{34}}{1+\frac{3\sqrt{34}}{34}}}$$
$$= \sqrt{\frac{\frac{34-3\sqrt{34}}{34}}{\frac{34+3\sqrt{34}}{34}}}$$
$$= \sqrt{\frac{34-3\sqrt{34}}{34+3\sqrt{34}}}$$
$$\approx 0.57$$

We can take the inverse tangent to find the angle: $\tan^{-1}(0.57) \approx 29.7^\circ$. So the angle of the ramp for novice competition is $\approx 29.7^\circ$.

Media

Access these online resources for additional instruction and practice with double-angle, half-angle, and reduction formulas.

- Double-Angle Identities
- Half-Angle Identities

Key Equations

Double-angle formulas	$\sin(2\theta) = 2\sin\,\theta\,\cos\,\theta$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $= 1 - 2\sin^2\theta$ $= 2\cos^2\theta - 1$ $\tan(2\theta) = \dfrac{2\tan\,\theta}{1-\tan^2\theta}$
Reduction formulas	$\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$ $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$ $\tan^2\theta = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Half-angle formulas	$\sin\,\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\,\alpha}{2}}$ $\cos\,\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\,\alpha}{2}}$ $\tan\,\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\,\alpha}{1+\cos\,\alpha}}$ $= \dfrac{\sin\,\alpha}{1+\cos\,\alpha}$ $= \dfrac{1-\cos\,\alpha}{\sin\,\alpha}$

Key Concepts

- Double-angle identities are derived from the sum formulas of the fundamental trigonometric functions: sine, cosine, and tangent. See Example 11.3.1, Example 11.3.2, Example 11.3.3, and Example 11.3.4.
- Reduction formulas are especially useful in calculus, as they allow us to reduce the power of the trigonometric term. See Example 11.3.5 and Example 11.3.6.
- Half-angle formulas allow us to find the value of trigonometric functions involving half-angles, whether the original angle is known or not. See Example 11.3.7, Example 11.3.8, and Example 11.3.9.

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11.4: Sum-to-Product and Product-to-Sum Formulas

Learning Objectives

- Express products as sums.
- Express sums as products.

A band marches down the field creating an amazing sound that bolsters the crowd. That sound travels as a wave that can be interpreted using trigonometric functions.



Figure 11.4.1: The UCLA marching band (credit: Eric Chan, Flickr).

For example, Figure 11.4.2 represents a sound wave for the musical note A. In this section, we will investigate trigonometric identities that are the foundation of everyday phenomena such as sound waves.

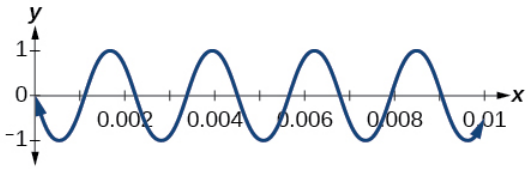


Figure 11.4.2

Expressing Products as Sums

We have already learned a number of formulas useful for expanding or simplifying trigonometric expressions, but sometimes we may need to express the product of cosine and sine as a sum. We can use the **product-to-sum formulas,** which express products of trigonometric functions as sums. Let's investigate the cosine identity first and then the sine identity.

Expressing Products as Sums for Cosine

We can derive the product-to-sum formula from the sum and difference identities for **cosine**. If we add the two equations, we get:

$$\begin{aligned} \cos\alpha\cos\beta+\sin\alpha\sin\beta &= \cos(\alpha-\beta) \\ +\cos\alpha\cos\beta-\sin\alpha\sin\beta &= \cos(\alpha+\beta) \\ \hline 2\cos\alpha\cos\beta &= \cos(\alpha-\beta)+\cos(\alpha+\beta) \end{aligned}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{11.4.1}$$

How to: Given a product of cosines, express as a sum

1. Write the formula for the product of cosines.
2. Substitute the given angles into the formula.
3. Simplify.

Example 11.4.1: Writing the Product as a Sum Using the Product-to-Sum Formula for Cosine

Write the following product of cosines as a sum: $2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right)$.

Solution

We begin by writing the formula for the product of cosines (Equation 11.4.1):

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

We can then substitute the given angles into the formula and simplify.

$$\begin{aligned}2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right) &= 2\left(\frac{1}{2}\right)\left[\cos\left(\frac{7x}{2}-\frac{3x}{2}\right)+\cos\left(\frac{7x}{2}+\frac{3x}{2}\right)\right]\\ &=\cos\left(\frac{4x}{2}\right)+\cos\left(\frac{10x}{2}\right)\\ &=\cos 2x+\cos 5x\end{aligned}$$

Exercise 11.4.1

Use the product-to-sum formula (Equation 11.4.1) to write the product as a sum or difference: $\cos(2\theta)\cos(4\theta)$.

Answer

$\frac{1}{2}(\cos 6\theta+\cos 2\theta)$

Expressing the Product of Sine and Cosine as a Sum

Next, we will derive the product-to-sum formula for sine and cosine from the sum and difference formulas for **sine**. If we add the sum and difference identities, we get:

$$\begin{aligned}\cos\alpha\cos\beta+\sin\alpha\sin\beta &= \cos(\alpha-\beta)\\ +\cos\alpha\cos\beta-\sin\alpha\sin\beta &= \cos(\alpha+\beta)\\ \hline 2\cos\alpha\cos\beta &= \cos(\alpha-\beta)+\cos(\alpha+\beta)\end{aligned}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

Example 11.4.2: Writing the Product as a Sum Containing only Sine or Cosine

Express the following product as a sum containing only sine or cosine and no products: $\sin(4\theta)\cos(2\theta)$.

Solution

Write the formula for the product of sine and cosine. Then substitute the given values into the formula and simplify.

$$\begin{aligned}\sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]\\ \sin(4\theta)\cos(2\theta) &= \frac{1}{2}[\sin(4\theta+2\theta)+\sin(4\theta-2\theta)]\\ &= \frac{1}{2}[\sin(6\theta)+\sin(2\theta)]\end{aligned}$$

Exercise 11.4.2

Use the product-to-sum formula to write the product as a sum: $\sin(x+y)\cos(x-y)$.

Answer

$\frac{1}{2}(\sin 2x+\sin 2y)$

Expressing Products of Sines in Terms of Cosine

Expressing the product of sines in terms of **cosine** is also derived from the sum and difference identities for cosine. In this case, we will first subtract the two cosine formulas:

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\ -\cos(\alpha+\beta) &= -(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\\ \cos(\alpha-\beta)-\cos(\alpha+\beta) &= 2\sin\alpha\sin\beta\end{aligned}$$

Then, we divide by 2 to isolate the product of sines:

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

Similarly we could express the product of cosines in terms of sine or derive other product-to-sum formulas.

THE PRODUCT-TO-SUM FORMULAS

The **product-to-sum** formulas are as follows:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{11.4.2}$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \tag{11.4.3}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \tag{11.4.4}$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)] \tag{11.4.5}$$

Example 11.4.3: Express the Product as a Sum or Difference

Write $\cos(3\theta)\cos(5\theta)$ as a sum or difference.

Solution

We have the product of cosines, so we begin by writing the related formula. Then we substitute the given angles and simplify.

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\cos(3\theta)\cos(5\theta) = \frac{1}{2}[\cos(3\theta-5\theta)+\cos(3\theta+5\theta)]$$

$$= \frac{1}{2}[\cos(2\theta)+\cos(8\theta)] \qquad \text{Use even-odd identity}$$

? Exercise 11.4.3

Use the product-to-sum formula to evaluate $\cos\frac{11\pi}{12}\cos\frac{\pi}{12}$.

Answer

$$\frac{-2-\sqrt{3}}{4}$$

Expressing Sums as Products

Some problems require the reverse of the process we just used. The **sum-to-product formulas** allow us to express sums of sine or cosine as products. These formulas can be derived from the product-to-sum identities. For example, with a few substitutions, we can derive the sum-to-product identity for **sine**. Let $\frac{u+v}{2}=\alpha$ and $\frac{u-v}{2}=\beta$.

Then,

$$\alpha+\beta = \frac{u+v}{2}+\frac{u-v}{2}$$

$$= \frac{2u}{2}$$

$$= u$$

$$\alpha-\beta = \frac{u+v}{2}-\frac{u-v}{2}$$

$$= \frac{2v}{2}$$

$$= v$$

Thus, replacing α and β in the product-to-sum formula with the substitute expressions, we have

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \frac{1}{2}[\sin u+\sin v] \qquad \text{Substitute for } (\alpha+\beta) \text{ and } (\alpha\beta)$$

$$2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \sin u+\sin v$$

The other sum-to-product identities are derived similarly.

SUM-TO-PRODUCT FORMULAS

The **sum-to-product formulas** are as follows:

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{11.4.6}$$

$$\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \tag{11.4.7}$$

$$\cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{11.4.8}$$

$$\cos\alpha+\cos\beta=2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{11.4.9}$$

Example 11.4.4: Writing the Difference of Sines as a Product

Write the following difference of sines expression as a product: $\sin(4\theta)-\sin(2\theta)$.

Solution

We begin by writing the formula for the difference of sines.

$$\sin\alpha-\sin\beta=2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

Substitute the values into the formula, and simplify.

$$\begin{aligned}\sin(4\theta)-\sin(2\theta)&=2\sin\left(\frac{4\theta-2\theta}{2}\right)\cos\left(\frac{4\theta+2\theta}{2}\right)\\&=2\sin\left(\frac{2\theta}{2}\right)\cos\left(\frac{6\theta}{2}\right)\\&=2\sin\theta\cos(3\theta)\end{aligned}$$

Exercise 11.4.4

Use the sum-to-product formula to write the sum as a product: $\sin(3\theta)+\sin(\theta)$.

Answer

$2\sin(2\theta)\cos(\theta)$

Example 11.4.5: Evaluating Using the Sum-to-Product Formula

Evaluate $\cos(15^\circ)-\cos(75^\circ)$. Check the answer with a graphing calculator.

Solution

We begin by writing the formula for the difference of cosines.

$$\cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Then we substitute the given angles and simplify.

$$\begin{aligned}\cos(15^\circ)-\cos(75^\circ)&=-2\sin\left(\frac{15^\circ+75^\circ}{2}\right)\sin\left(\frac{15^\circ-75^\circ}{2}\right)\\&=-2\sin(45^\circ)\sin(-30^\circ)\\&=-2\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)\\&=\frac{\sqrt{2}}{2}\end{aligned}$$

Example 11.4.6: Proving an Identity

Prove the identity:

$$\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)}=-\tan t \tag{11.4.10}$$

Solution

We will start with the left side, the more complicated side of the equation, and rewrite the expression until it matches the right side.

$$\begin{aligned}\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} &= \frac{-2\sin\left(\frac{4t+2t}{2}\right)\sin\left(\frac{4t-2t}{2}\right)}{2\sin\left(\frac{4t+2t}{2}\right)\cos\left(\frac{4t-2t}{2}\right)}\\ &= \frac{-2\sin(3t)\sin t}{2\sin(3t)\cos t}\\ &= -\frac{\sin t}{\cos t}\\ &= -\tan t\end{aligned}$$

Analysis

Recall that verifying trigonometric identities has its own set of rules. The procedures for solving an equation are not the same as the procedures for verifying an identity. When we prove an identity, we pick one side to work on and make substitutions until that side is transformed into the other side.

Example 11.4.7: Verifying the Identity Using Double-Angle Formulas and Reciprocal Identities

Verify the identity $\csc^2\theta-2=\cos(2\theta)\sin 2\theta$.

Solution

For verifying this equation, we are bringing together several of the identities. We will use the double-angle formula and the reciprocal identities. We will work with the right side of the equation and rewrite it until it matches the left side.

$$\begin{aligned}\cos(2\theta)\sin 2\theta &= \frac{1-2\sin^2\theta}{\sin^2\theta}\\ &= \frac{1}{\sin^2\theta}-\frac{2\sin^2\theta}{\sin^2\theta}\\ &= \csc^2\theta-2\end{aligned}$$

Exercise 11.4.5

Verify the identity $\tan\theta\cot\theta-\cos^2\theta=\sin^2\theta$.

Answer

$$\begin{aligned}\tan\theta\cot\theta-\cos^2\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)-\cos^2\theta\\ &= 1-\cos^2\theta\\ &= \sin^2\theta\end{aligned}$$

Media

Access these online resources for additional instruction and practice with the product-to-sum and sum-to-product identities.

- Sum to Product Identities
- Sum to Product and Product to Sum Identities

Key Equations

Product-to-sum Formulas

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

Sum-to-product Formulas

$$\sin\alpha+\sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$$

$$\sin\alpha-\sin\beta = 2\sin(\frac{\alpha-\beta}{2})\cos(\frac{\alpha+\beta}{2})$$

$$\cos\alpha-\cos\beta = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$$

$$\cos\alpha+\cos\beta = 2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$$

Key Concepts

- From the sum and difference identities, we can derive the product-to-sum formulas and the sum-to-product formulas for sine and cosine.
- We can use the product-to-sum formulas to rewrite products of sines, products of cosines, and products of sine and cosine as sums or differences of sines and cosines. See Example 11.4.1, Example 11.4.2, and Example 11.4.3
- We can also derive the sum-to-product identities from the product-to-sum identities using substitution.
- We can use the sum-to-product formulas to rewrite sum or difference of sines, cosines, or products sine and cosine as products of sines and cosines. See Example 11.4.4
- Trigonometric expressions are often simpler to evaluate using the formulas. See Example 11.4.5
- The identities can be verified using other formulas or by converting the expressions to sines and cosines. To verify an identity, we choose the more complicated side of the equals sign and rewrite it until it is transformed into the other side. See Example 11.4.6 and Example 11.4.7.

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11.5: Solving Trigonometric Equations

Learning Objectives

- Solve linear trigonometric equations in sine and cosine.
- Solve equations involving a single trigonometric function.
- Solve trigonometric equations using a calculator.
- Solve trigonometric equations that are quadratic in form.
- Solve trigonometric equations using fundamental identities.
- Solve trigonometric equations with multiple angles.
- Solve right triangle problems.

Thales of Miletus (circa 625–547 BC) is known as the founder of geometry. The legend is that he calculated the height of the Great Pyramid of Giza in Egypt using the theory of *similar triangles*, which he developed by measuring the shadow of his staff. Based on proportions, this theory has applications in a number of areas, including fractal geometry, engineering, and architecture. Often, the angle of elevation and the angle of depression are found using similar triangles.



Figure 11.5.1: Egyptian pyramids standing near a modern city. (credit: Oisin Mulvihill)

In earlier sections of this chapter, we looked at trigonometric identities. Identities are true for all values in the domain of the variable. In this section, we begin our study of trigonometric equations to study real-world scenarios such as the finding the dimensions of the pyramids.

Solving Linear Trigonometric Equations in Sine and Cosine

Trigonometric equations are, as the name implies, equations that involve trigonometric functions. Similar in many ways to solving polynomial equations or rational equations, only specific values of the variable will be solutions, if there are solutions at all. Often we will solve a trigonometric equation over a specified interval. However, just as often, we will be asked to find all possible solutions, and as trigonometric functions are periodic, solutions are repeated within each period. In other words, trigonometric equations may have an infinite number of solutions. Additionally, like rational equations, the domain of the function must be considered before we assume that any solution is valid. The **period** of both the sine function and the cosine function is 2π. In other words, every 2π units, the *y*-values repeat. If we need to find all possible solutions, then we must add $2\pi k$,where k is an integer, to the initial solution. Recall the rule that gives the format for stating all possible solutions for a function where the period is 2π:

$$\sin\theta = \sin(\theta \pm 2k\pi) \tag{11.5.1}$$

There are similar rules for indicating all possible solutions for the other trigonometric functions. Solving trigonometric equations requires the same techniques as solving algebraic equations. We read the equation from left to right, horizontally, like a sentence. We look for known patterns, factor, find common denominators, and substitute certain expressions with a variable to make solving a more straightforward process. However, with trigonometric equations, we also have the advantage of using the identities we developed in the previous sections.

Example 11.5.1A: Solving a Linear Trigonometric Equation Involving the Cosine Function

Find all possible exact solutions for the equation $\cos\theta = \frac{1}{2}$.

Solution

From the unit circle, we know that

$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3},\ \frac{5\pi}{3}$$

These are the solutions in the interval $[0, 2\pi]$. All possible solutions are given by

$$\theta = \frac{\pi}{3} \pm 2k\pi \quad \text{and} \quad \theta = \frac{5\pi}{3} \pm 2k\pi$$

where k is an integer.

Example 11.5.1B: Solving a Linear Equation Involving the Sine Function

Find all possible exact solutions for the equation $\sin t = \frac{1}{2}$.

Solution

Solving for all possible values of t means that solutions include angles beyond the period of 2π. From the section on Sum and Difference Identities, we can see that the solutions are $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. But the problem is asking for all possible values that solve the equation. Therefore, the answer is

$$t = \frac{\pi}{6} \pm 2\pi k \quad \text{and} \quad t = \frac{5\pi}{6} \pm 2\pi k$$

where k is an integer.

Howto: Given a trigonometric equation, solve using algebra

1. Look for a pattern that suggests an algebraic property, such as the difference of squares or a factoring opportunity.
2. Substitute the trigonometric expression with a single variable, such as x or u.
3. Solve the equation the same way an algebraic equation would be solved.
4. Substitute the trigonometric expression back in for the variable in the resulting expressions.
5. Solve for the angle.

Example 11.5.2: Solve the Linear Trigonometric Equation

Solve the equation exactly: $2\cos\theta - 3 = -5$, $0 \le \theta < 2\pi$.

Solution

Use algebraic techniques to solve the equation.

$$\begin{aligned} 2\cos\theta - 3 &= -5 \\ 2\cos\theta &= -2 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Exercise 11.5.2

Solve exactly the following linear equation on the interval $[0, 2\pi)$: $2\sin x + 1 = 0$.

Answer

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Solving Equations Involving a Single Trigonometric Function

When we are given equations that involve only one of the six trigonometric functions, their solutions involve using algebraic techniques and the unit circle (see [link]). We need to make several considerations when the equation involves trigonometric functions other than sine and cosine. Problems involving the reciprocals of the primary trigonometric functions need to be viewed from an algebraic perspective. In other words, we will write the reciprocal function, and solve for the angles using the function. Also, an equation involving the tangent function is slightly different from one containing a sine or cosine function. First, as we know, the period of tangent is π,not 2π. Further, the domain of tangent is all real numbers with the exception of odd integer multiples of $\frac{\pi}{2}$,unless, of course, a problem places its own restrictions on the domain.

Solving a Problem Involving a Single Trigonometric Function

Solve the problem exactly: $2\sin^2\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Solution

As this problem is not easily factored, we will solve using the square root property. First, we use algebra to isolate $\sin\theta$. Then we will find the angles.

$$\begin{aligned} 2\sin^2\theta - 1 &= 0 \\ 2\sin^2\theta &= 1 \\ \sin^2\theta &= \frac{1}{2} \\ \sqrt{\sin^2\theta} &= \pm\sqrt{\frac{1}{2}} \\ \sin\theta &= \pm\frac{1}{\sqrt{2}} \\ &= \pm\frac{\sqrt{2}}{2} \\ \theta &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

Example 11.5.3B: Solving a Trigonometric Equation Involving Cosecant

Solve the following equation exactly: $\csc\theta = -2$, $0 \le \theta < 4\pi$.

Solution

We want all values of θ for which $\csc\theta = -2$ over the interval $0 \le \theta < 4\pi$.

$$\begin{aligned} \csc\theta &= -2 \\ \frac{1}{\sin\theta} &= -2 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \end{aligned}$$

Analysis

As $\sin\theta = -\frac{1}{2}$, notice that all four solutions are in the third and fourth quadrants.

Example 11.5.3C: Solving an Equation Involving Tangent

Solve the equation exactly: $\tan\left(\theta - \frac{\pi}{2}\right) = 1$, $0 \le \theta < 2\pi$.

Solution

Recall that the tangent function has a period of π. On the interval $[0, \pi)$,and at the angle of $\frac{\pi}{4}$,the tangent has a value of 1. However, the angle we want is $\left(\theta - \frac{\pi}{2}\right)$. Thus, if $\tan\left(\frac{\pi}{4}\right) = 1$,then

$$\theta - \frac{\pi}{2} = \frac{\pi}{4}$$
$$\theta = \frac{3\pi}{4} \pm k\pi$$

Over the interval $[0, 2\pi)$,we have two solutions:

$\theta = \frac{3\pi}{4}$ and $\theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$

Exercise 11.5.3

Find all solutions for $\tan x = \sqrt{3}$.

Answer

$\frac{\pi}{3} \pm \pi k$

Example 11.5.4: Identify all Solutions to the Equation Involving Tangent

Identify all exact solutions to the equation $2(\tan x + 3) = 5 + \tan x$, $0 \le x < 2\pi$.

Solution

We can solve this equation using only algebra. Isolate the expression $\tan x$ on the left side of the equals sign.

$$\begin{aligned} 2(\tan x) + 2(3) &= 5 + \tan x \\ 2\tan x + 6 &= 5 + \tan x \\ 2\tan x - \tan x &= 5 - 6 \\ \tan x &= -1 \end{aligned}$$

There are two angles on the unit circle that have a tangent value of -1: $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$.

Solve Trigonometric Equations Using a Calculator

Not all functions can be solved exactly using only the unit circle. When we must solve an equation involving an angle other than one of the special angles, we will need to use a calculator. Make sure it is set to the proper mode, either degrees or radians, depending on the criteria of the given problem.

Example 11.5.5A: Using a Calculator to Solve a Trigonometric Equation Involving Sine

Use a calculator to solve the equation $\sin\theta = 0.8$,where θ is in radians.

Solution

Make sure mode is set to radians. To find θ, use the inverse sine function. On most calculators, you will need to push the 2ND button and then the SIN button to bring up the $\sin^{-1}$ function. What is shown on the screen is $\sin^{-1}$.The calculator is ready for

the input within the parentheses. For this problem, we enter $\sin^{-1}(0.8)$, and press ENTER. Thus, to four decimals places,

$\sin^{-1}(0.8) \approx 0.9273$

The solution is

$\theta \approx 0.9273 \pm 2\pi k$

The angle measurement in degrees is

$$\begin{aligned} \theta &\approx 53.1^\circ \\ \theta &\approx 180^\circ - 53.1^\circ \\ &\approx 126.9^\circ \end{aligned}$$

Analysis

Note that a calculator will only return an angle in quadrants I or IV for the sine function, since that is the range of the inverse sine. The other angle is obtained by using $\pi - \theta$.

Example 11.5.5B: Using a Calculator to Solve a Trigonometric Equation Involving Secant

Use a calculator to solve the equation $\sec\theta = -4$, giving your answer in radians.

Solution

We can begin with some algebra.

$$\begin{aligned} \sec\theta &= -4 \\ \frac{1}{\cos\theta} &= -4 \\ \cos\theta &= -\frac{1}{4} \end{aligned}$$

Check that the MODE is in radians. Now use the inverse cosine function

$$\begin{aligned} \cos^{-1}\left(-\frac{1}{4}\right) &\approx 1.8235 \\ \theta &\approx 1.8235 + 2\pi k \end{aligned}$$

Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$,1.8235 is between these two numbers, thus $\theta \approx 1.8235$ is in quadrant II. Cosine is also negative in quadrant III. Note that a calculator will only return an angle in quadrants I or II for the cosine function, since that is the range of the inverse cosine. See Figure 11.5.2

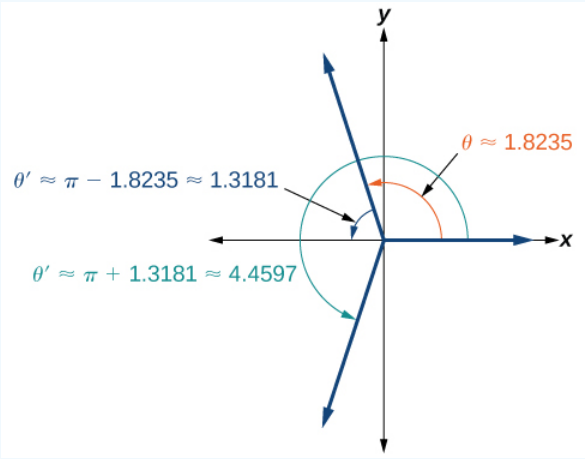


Figure 11.5.2

So, we also need to find the measure of the angle in quadrant III. In quadrant III, the reference angle is $\theta' \approx \pi - 1.8235 \approx 1.3181$. The other solution in quadrant III is $\theta' \approx \pi + 1.3181 \approx 4.4597$.

The solutions are $\theta \approx 1.8235 \pm 2\pi k$ and $\theta \approx 4.4597 \pm 2\pi k$.

? Exercise 11.5.5

Solve $\cos\theta = -0.2$.

Answer

$\theta \approx 1.7722 \pm 2\pi k$ and $\theta \approx 4.5110 \pm 2\pi k$

Solving Trigonometric Equations in Quadratic Form

Solving a *quadratic equation* may be more complicated, but once again, we can use algebra as we would for any quadratic equation. Look at the pattern of the equation. Is there more than one trigonometric function in the equation, or is there only one? Which trigonometric function is squared? If there is only one function represented and one of the terms is squared, think about the standard form of a quadratic. Replace the trigonometric function with a variable such as x or u. If substitution makes the equation look like a quadratic equation, then we can use the same methods for solving quadratics to solve the trigonometric equations.

✓ Example 11.5.6A: Solving a Trigonometric Equation in Quadratic Form

Solve the equation exactly: $\cos^2\theta + 3\cos\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Solution

We begin by using substitution and replacing $\cos\theta$ with x. It is not necessary to use substitution, but it may make the problem easier to solve visually. Let $\cos\theta = x$. We have

$$x^2 + 3x - 1 = 0$$

The equation cannot be factored, so we will use the **quadratic formula:** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2} \\ &= \frac{-3 \pm \sqrt{13}}{2} \end{aligned}$$

Replace x with $\cos\theta$ and solve.

$$\begin{aligned} \cos\theta &= \frac{-3 \pm \sqrt{13}}{2} \\ \theta &= \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \end{aligned}$$

Note that only the + sign is used. This is because we get an error when we solve $\theta = \cos^{-1}\left(\dfrac{-3 - \sqrt{13}}{2}\right)$ on a calculator, since the domain of the inverse cosine function is $[-1, 1]$. However, there is a second solution:

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \\ &\approx 1.26 \end{aligned}$$

This terminal side of the angle lies in quadrant I. Since cosine is also positive in quadrant IV, the second solution is

$$\begin{aligned} \theta &= 2\pi - \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \\ &\approx 5.02 \end{aligned}$$

Example 11.5.6B: Solving a Trigonometric Equation in Quadratic Form by Factoring

Solve the equation exactly: $2\sin^2\theta - 5\sin\theta + 3 = 0$, $0 \le \theta \le 2\pi$.

Solution

Using grouping, this quadratic can be factored. Either make the real substitution, $\sin\theta = u$,or imagine it, as we factor:

$$\begin{aligned} 2\sin^2\theta - 5\sin\theta + 3 &= 0 \\ (2\sin\theta - 3)(\sin\theta - 1) &= 0 \qquad \text{Now set each factor equal to zero.} \\ 2\sin\theta - 3 &= 0 \\ 2\sin\theta &= 3 \\ \sin\theta &= \frac{3}{2} \\ \sin\theta - 1 &= 0 \\ \sin\theta &= 1 \end{aligned}$$

Next solve for θ: $\sin\theta \neq \frac{3}{2}$, as the range of the sine function is $[-1, 1]$. However, $\sin\theta = 1$, giving the solution $\theta = \frac{\pi}{2}$.

Analysis

Make sure to check all solutions on the given domain as some factors have no solution.

Exercise 11.5.6

Solve $\sin^2\theta = 2\cos\theta + 2$, $0 \le \theta \le 2\pi$. [Hint: Make a substitution to express the equation only in terms of cosine.]

Answer

$\cos\theta = -1$, $\theta = \pi$

Example 11.5.7A: Solving a Trigonometric Equation Using Algebra

Solve exactly: $2\sin^2\theta + \sin\theta = 0;\ 0 \le \theta < 2\pi$

Solution

This problem should appear familiar as it is similar to a quadratic. Let $\sin\theta = x$. The equation becomes $2x^2 + x = 0$. We begin by factoring:

$$\begin{aligned} 2x^2 + x &= 0 \\ x(2x + 1) &= 0 \qquad \text{Set each factor equal to zero.} \\ x &= 0 \\ 2x + 1 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

Then, substitute back into the equation the original expression $\sin\theta$ for x. Thus,

$$\begin{aligned} \sin\theta &= 0 \\ \theta &= 0, \pi \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

The solutions within the domain $0 \le \theta < 2\pi$ are $\theta = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$.

If we prefer not to substitute, we can solve the equation by following the same pattern of factoring and setting each factor equal to zero.

$$\begin{aligned} \sin^2\theta+\sin\theta &= 0 \\ \sin\theta(2\sin\theta+1) &= 0 \\ \sin\theta &= 0 \\ \theta &= 0,\pi \\ 2\sin\theta+1 &= 0 \\ 2\sin\theta &= -1 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6},\frac{11\pi}{6} \end{aligned}$$

Analysis

We can see the solutions on the graph in Figure 11.5.3 On the interval $0 \le \theta < 2\pi$,the graph crosses the x-axis four times, at the solutions noted. Notice that trigonometric equations that are in quadratic form can yield up to four solutions instead of the expected two that are found with quadratic equations. In this example, each solution (angle) corresponding to a positive sine value will yield two angles that would result in that value.

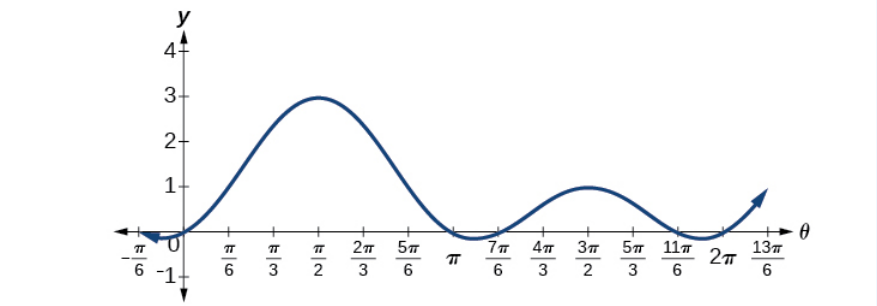


Figure 11.5.3

We can verify the solutions on the unit circle in via the result in the section on Sum and Difference Identities as well.

Example 11.5.7B: Solving a Trigonometric Equation Quadratic in Form

Solve the equation quadratic in form exactly: $2\sin^2\theta-3\sin\theta+1=0$, $0 \le \theta < 2\pi$.

Solution

We can factor using grouping. Solution values of θ can be found on the unit circle.

$$\begin{aligned} (2\sin\theta-1)(\sin\theta-1) &= 0 \\ 2\sin\theta-1 &= 0 \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6},\frac{5\pi}{6} \\ \sin\theta &= 1 \\ \theta &= \frac{\pi}{2} \end{aligned}$$

Exercise 11.5.7

Solve the quadratic equation $2\cos^2\theta+\cos\theta=0$.

Answer

$\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$

Solving Trigonometric Equations Using Fundamental Identities

While algebra can be used to solve a number of trigonometric equations, we can also use the fundamental identities because they make solving equations simpler. Remember that the techniques we use for solving are not the same as those for verifying identities. The basic rules of algebra apply here, as opposed to rewriting one side of the identity to match the other side. In the next example, we use two identities to simplify the equation.

Example 11.5.8A: Use Identities to Solve an Equation

Use identities to solve exactly the trigonometric equation over the interval $0 \le x < 2\pi$.

$$\cos x \cos(2x) + \sin x \sin(2x) = \frac{\sqrt{3}}{2}$$

Solution

Notice that the left side of the equation is the difference formula for cosine.

$$\begin{aligned} \cos x \cos(2x) + \sin x \sin(2x) &= \frac{\sqrt{3}}{2} \\ \cos(x - 2x) &= \frac{\sqrt{3}}{2} && \text{Difference formula for cosine} \\ \cos(-x) &= \frac{\sqrt{3}}{2} && \text{Use the negative angle identity.} \\ \cos x &= \frac{\sqrt{3}}{2} \end{aligned}$$

From the unit circle in the section on Sum and Difference Identities, we see that $\cos x = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{6}, \frac{11\pi}{6}$.

Example 11.5.8B: Solving the Equation Using a Double-Angle Formula

Solve the equation exactly using a double-angle formula: $\cos(2\theta) = \cos\theta$.

Solution

We have three choices of expressions to substitute for the double-angle of cosine. As it is simpler to solve for one trigonometric function at a time, we will choose the double-angle identity involving only cosine:

$$\begin{aligned} \cos(2\theta) &= \cos\theta \\ 2\cos^2\theta - 1 &= \cos\theta \\ 2\cos^2\theta - \cos\theta - 1 &= 0 \\ (2\cos\theta + 1)(\cos\theta - 1) &= 0 \\ 2\cos\theta + 1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \cos\theta - 1 &= 0 \\ \cos\theta &= 1 \end{aligned}$$

So, if $\cos\theta = -\frac{1}{2}$,then $\theta = \frac{2\pi}{3} \pm 2\pi k$ and $\theta = \frac{4\pi}{3} \pm 2\pi k$; if $\cos\theta = 1$,then $\theta = 0 \pm 2\pi k$.

Example 11.5.8C: Solving an Equation Using an Identity

Solve the equation exactly using an identity: $3\cos\theta + 3 = 2\sin^2\theta$, $0 \le \theta < 2\pi$.

Solution

If we rewrite the right side, we can write the equation in terms of cosine:

$$\begin{aligned} 3\cos\theta+3 &= 2\sin^2\theta \\ 3\cos\theta+3 &= 2(1-\cos^2\theta) \\ 3\cos\theta+3 &= 2-2\cos^2\theta \\ 2\cos^2\theta+3\cos\theta+1 &= 0 \\ (2\cos\theta+1)(\cos\theta+1) &= 0 \\ 2\cos\theta+1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3},\ \frac{4\pi}{3} \\ \cos\theta+1 &= 0 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Our solutions are $\theta = \frac{2\pi}{3},\ \frac{4\pi}{3},\ \pi$.

Solving Trigonometric Equations with Multiple Angles

Sometimes it is not possible to solve a trigonometric equation with identities that have a multiple angle, such as $\sin(2x)$ or $\cos(3x)$. When confronted with these equations, recall that $y = \sin(2x)$ is a horizontal compression by a factor of 2 of the function $y = \sin x$. On an interval of 2π,we can graph two periods of $y = \sin(2x)$,as opposed to one cycle of $y = \sin x$. This compression of the graph leads us to believe there may be twice as many x-intercepts or solutions to $\sin(2x) = 0$ compared to $\sin x = 0$. This information will help us solve the equation.

Example 11.5.9: Solving a Multiple Angle Trigonometric Equation

Solve exactly: $\cos(2x) = \frac{1}{2}$ on $[0, 2\pi)$.

Solution

We can see that this equation is the standard equation with a multiple of an angle. If $\cos(\alpha) = \frac{1}{2}$,we know α is in quadrants I and IV. While $\theta = \cos^{-1}\frac{1}{2}$ will only yield solutions in quadrants I and II, we recognize that the solutions to the equation $\cos\theta = \frac{1}{2}$ will be in quadrants I and IV.

Therefore, the possible angles are $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. So, $2x = \frac{\pi}{3}$ or $2x = \frac{5\pi}{3}$, which means that $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Does this make sense? Yes, because $\cos\left(2\left(\frac{\pi}{6}\right)\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Are there any other possible answers? Let us return to our first step.

In quadrant I, $2x = \frac{\pi}{3}$, so $x = \frac{\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{\pi}{3} + 2\pi \\ &= \frac{\pi}{3} + \frac{6\pi}{3} \\ &= \frac{7\pi}{3} \\ x &= \frac{7\pi}{6} \end{aligned}$$

One more rotation yields

$$\begin{aligned} 2x &= \frac{\pi}{3} + 4\pi \\ &= \frac{\pi}{3} + \frac{12\pi}{3} \\ &= \frac{13\pi}{3} \end{aligned}$$

$x = \frac{13\pi}{6} > 2\pi$, so this value for x is larger than 2π, so it is not a solution on $[0, 2\pi)$.

In quadrant IV, $2x = \frac{5\pi}{3}$, so $x = \frac{5\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 2\pi \\ &= \frac{5\pi}{3} + \frac{6\pi}{3} \\ &= \frac{11\pi}{3} \end{aligned}$$

so $x = \frac{11\pi}{6}$.

One more rotation yields

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 4\pi \\ &= \frac{5\pi}{3} + \frac{12\pi}{3} \\ &= \frac{17\pi}{3} \end{aligned}$$

$x = \frac{17\pi}{6} > 2\pi$,so this value for x is larger than 2π,so it is not a solution on $[0, 2\pi)$.

Our solutions are $x = \frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Note that whenever we solve a problem in the form of $sin(nx) = c$, we must go around the unit circle n times.

Solving Right Triangle Problems

We can now use all of the methods we have learned to solve problems that involve applying the properties of right triangles and the Pythagorean Theorem. We begin with the familiar Pythagorean Theorem,

$$a^2 + b^2 = c^2 \tag{11.5.2}$$

and model an equation to fit a situation.

Example 11.5.10A: Using the Pythagorean Theorem to Model an Equation

One of the cables that anchors the center of the London Eye Ferris wheel to the ground must be replaced. The center of the Ferris wheel is 69.5 meters above the ground, and the second anchor on the ground is 23 meters from the base of the Ferris wheel. Approximately how long is the cable, and what is the angle of elevation (from ground up to the center of the Ferris wheel)? See Figure 11.5.4

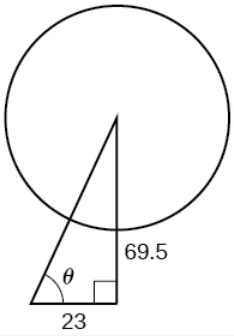


Figure 11.5.4

Solution

Use the Pythagorean Theorem (Equation 11.5.2) and the properties of right triangles to model an equation that fits the problem. Using the information given, we can draw a right triangle. We can find the length of the cable with the Pythagorean Theorem.

$$\begin{aligned} a^2+b^2 &= c^2 \\ (23)^2+(69.5)^2 &\approx 5359 \\ \sqrt{5359} &\approx 73.2\ m \end{aligned}$$

The angle of elevation is θ,formed by the second anchor on the ground and the cable reaching to the center of the wheel. We can use the tangent function to find its measure. Round to two decimal places.

$$\begin{aligned} \tan\theta &= 69.523 \\ \tan^{-1}(69.523) &\approx 1.2522 \\ &\approx 71.69^\circ \end{aligned}$$

The angle of elevation is approximately 71.7°, and the length of the cable is 73.2 meters.

Example 11.5.10B: Using the Pythagorean Theorem to Model an Abstract Problem

OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length. Find the angle that a ladder of any length forms with the ground and the height at which the ladder touches the wall.

Solution

For any length of ladder, the base needs to be a distance from the wall equal to one fourth of the ladder's length. Equivalently, if the base of the ladder is "a" feet from the wall, the length of the ladder will be $4a$ feet. See Figure 11.5.5

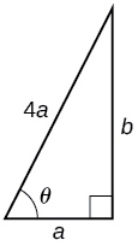


Figure 11.5.5

The side adjacent to θ is a and the hypotenuse is $4a$. Thus,

$$\begin{aligned} \cos\theta &= \frac{a}{4a} \\ &= \frac{1}{4} \\ \cos^{-1}\left(\frac{1}{4}\right) &\approx 75.5^\circ \end{aligned}$$

The elevation of the ladder forms an angle of 75.5° with the ground. The height at which the ladder touches the wall can be found using the Pythagorean Theorem:

$$\begin{aligned} a^2+b^2 &= (4a)^2 \\ b^2 &= (4a)^2-a^2 \\ b^2 &= 16a^2-a^2 \\ b^2 &= 15a^2 \\ b &= a\sqrt{15} \end{aligned}$$

Thus, the ladder touches the wall at $a\sqrt{15}$ feet from the ground.

Media

Access these online resources for additional instruction and practice with solving trigonometric equations.

- Solving Trigonometric Equations I
- Solving Trigonometric Equations II
- Solving Trigonometric Equations III
- Solving Trigonometric Equations IV
- Solving Trigonometric Equations V
- Solving Trigonometric Equations VI

Key Concepts

- When solving linear trigonometric equations, we can use algebraic techniques just as we do solving algebraic equations. Look for patterns, like the difference of squares, quadratic form, or an expression that lends itself well to substitution. See Example 11.5.1, Example 11.5.2, and Example 11.5.3.
- Equations involving a single trigonometric function can be solved or verified using the unit circle. See Example 11.5.4, Example 11.5.5, and Example 11.5.6, and Example 11.5.7.
- We can also solve trigonometric equations using a graphing calculator. See Example 11.5.8 and Example 11.5.9.
- Many equations appear quadratic in form. We can use substitution to make the equation appear simpler, and then use the same techniques we use solving an algebraic quadratic: factoring, the quadratic formula, etc. See Example 11.5.10, Example 11.5.11, Example 11.5.12, and Example 11.5.13.
- We can also use the identities to solve trigonometric equation. See Example 11.5.14, Example 11.5.15, and Example 11.5.16.
- We can use substitution to solve a multiple-angle trigonometric equation, which is a compression of a standard trigonometric function. We will need to take the compression into account and verify that we have found all solutions on the given interval. See Example 11.5.17.
- Real-world scenarios can be modeled and solved using the Pythagorean Theorem and trigonometric functions. See Example 11.5.18.

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CHAPTER OVERVIEW

12: Further Applications of Trigonometry

In this chapter, we will explore applications of trigonometry that will enable us to solve many different kinds of problems, including finding the height of a tree. We extend topics we introduced in Trigonometric Functions and investigate applications more deeply and meaningfully.

12.1: Non-right Triangles - Law of Sines
12.2: Non-right Triangles - Law of Cosines
12.3: Vectors
12.4: Polar Coordinates
12.5: Polar Coordinates - Graphs
12.6: Polar Form of Complex Numbers

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12.1: Non-right Triangles - Law of Sines

Learning Objectives

- Use the Law of Sines to solve oblique triangles.
- Find the area of an oblique triangle using the sine function.
- Solve applied problems using the Law of Sines.

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the **angle of elevation** measured by the second station is 15 degrees. How can we determine the altitude of the aircraft? We see in Figure 12.1.1 that the triangle formed by the aircraft and the two stations is not a right triangle, so we cannot use what we know about right triangles. In this section, we will find out how to solve problems involving **non-right triangles**.

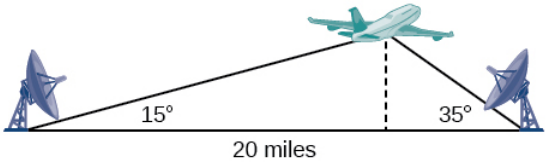


Figure 12.1.1

Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

Any triangle that is not a right triangle is an **oblique triangle**. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

ASA (angle-side-angle) We know the measurements of two angles and the included side. See Figure 12.1.2

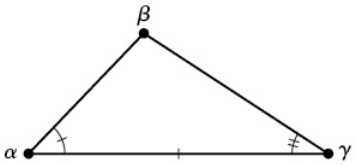


Figure 12.1.2

AAS (angle-angle-side) We know the measurements of two angles and a side that is not between the known angles. See Figure 12.1.3

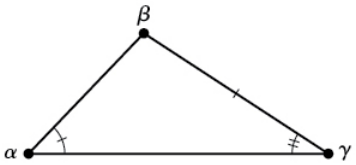


Figure 12.1.3

SSA (side-side-angle) We know the measurements of two sides and an angle that is not between the known sides. See Figure 12.1.4

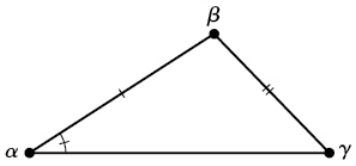


Figure 12.1.4

Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let's see how this statement is derived by considering the triangle shown in Figure 12.1.5.

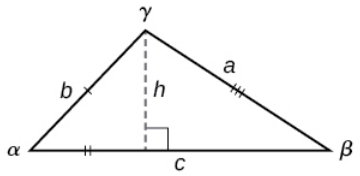


Figure 12.1.5

Using the right triangle relationships, we know that $\sin\alpha = \frac{h}{b}$ and $\sin\beta = \frac{h}{a}$. Solving both equations for h gives two different expressions for h.

$$h = b\sin\alpha \text{ and } h = a\sin\beta$$

We then set the expressions equal to each other.

$$\begin{aligned} b\sin\alpha &= a\sin\beta \\ \left(\frac{1}{ab}\right)(b\sin\alpha) &= (a\sin\beta)\left(\frac{1}{ab}\right) \qquad \text{Multiply both sides by } \frac{1}{ab} \\ \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \end{aligned}$$

Similarly, we can compare the other ratios.

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \text{ and } \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Collectively, these relationships are called the **Law of Sines**.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Note the standard way of labeling triangles: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. See Figure 12.1.6.

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.

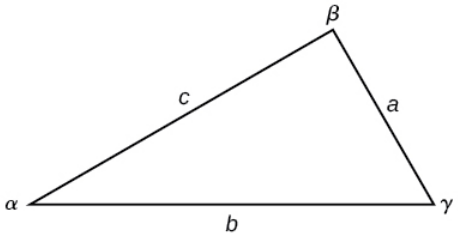


Figure 12.1.6

Note: LAW OF SINES

Given a triangle with angles and opposite sides labeled as in Figure 12.1.6, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The **Law of Sines** is based on proportions and is presented symbolically two ways.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \tag{12.1.1}$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \tag{12.1.2}$$

To solve an oblique triangle, use any pair of applicable ratios.

Example 12.1.1: Solving for Two Unknown Sides and Angle of an AAS Triangle

Solve the triangle shown in Figure 12.1.7 to the nearest tenth.

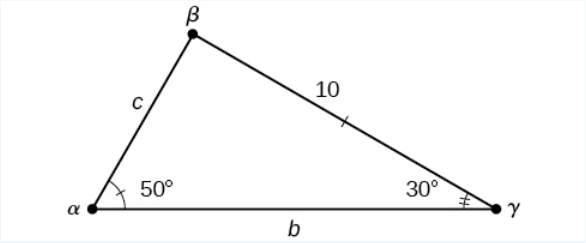


Figure 12.1.7

Solution

The three angles must add up to 180 degrees. From this, we can determine that

$$\begin{aligned}\beta &= 180^\circ - 50^\circ - 30^\circ \\ &= 100^\circ\end{aligned}$$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know that angle $\alpha = 50^\circ$ and its corresponding side $a = 10$. We can use the following proportion from the Law of Sines to find the length of c.

$$\begin{aligned}\frac{\sin(50^\circ)}{10} &= \frac{\sin(30^\circ)}{c} \\ c\frac{\sin(50^\circ)}{10} &= \sin(30^\circ) \qquad \text{Multiply both sides by } c \\ c &= \sin(30^\circ)\frac{10}{\sin(50^\circ)} \qquad \text{Multiply by the reciprocal to isolate } c \\ c &\approx 6.5\end{aligned}$$

Similarly, to solve for b, we set up another proportion.

$$\begin{aligned}\frac{\sin(50^\circ)}{10} &= \frac{\sin(100^\circ)}{b} \\ b\sin(50^\circ) &= 10\sin(100^\circ) \qquad \text{Multiply both sides by } b \\ b &= \frac{10\sin(100^\circ)}{\sin(50^\circ)} \qquad \text{Multiply by the reciprocal to isolate } b \\ b &\approx 12.9\end{aligned}$$

Therefore, the complete set of angles and sides is

$$\begin{aligned}\alpha &= 50^\circ \quad & a &= 10 \\ \beta &= 100^\circ \quad & b &\approx 12.9 \\ \gamma &= 30^\circ \quad & c &\approx 6.5\end{aligned}$$

Exercise 12.1.1

Solve the triangle shown in Figure 12.1.8 to the nearest tenth.

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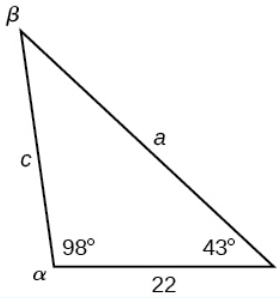


Figure 12.1.8

Answer

$\alpha = 98^\circ \quad a = 34.6$
$\beta = 39^\circ \quad b = 22$
$\gamma = 43^\circ \quad c = 23.8$

Using The Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an ambiguous case. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

Note: POSSIBLE OUTCOMES FOR SSA TRIANGLES

Oblique triangles in the category SSA may have four different outcomes. Figure 12.1.9 illustrates the solutions with the known sides a and b and known angle α.

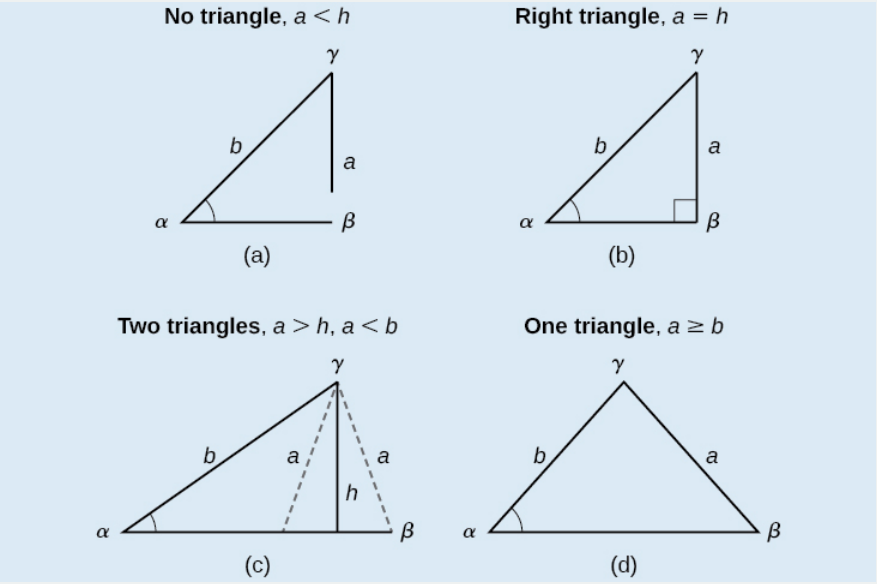


Figure 12.1.9

Example 12.1.2: Solving an Oblique SSA Triangle

Solve the triangle in Figure 12.1.10 for the missing side and find the missing angle measures to the nearest tenth.

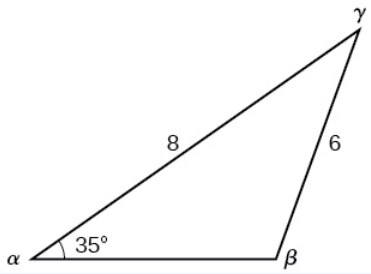


Figure 12.1.10

Solution

Use the Law of Sines to find angle β and angle γ, and then side c. Solving for β, we have the proportion

$$\begin{aligned} \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \\ \frac{\sin(35^\circ)}{6} &= \frac{\sin\beta}{8} \\ \frac{8\sin(35^\circ)}{6} &= \sin\beta \\ 0.7648 &\approx \sin\beta \\ \sin^{-1}(0.7648) &\approx 49.9^\circ \\ \beta &\approx 49.9^\circ \end{aligned}$$

However, in the diagram, angle β appears to be an obtuse angle and may be greater than 90°. How did we get an acute angle, and how do we find the measurement of β? Let's investigate further. Dropping a perpendicular from γ and viewing the triangle from a right angle perspective, we have Figure 12.1.11. It appears that there may be a second triangle that will fit the given criteria.

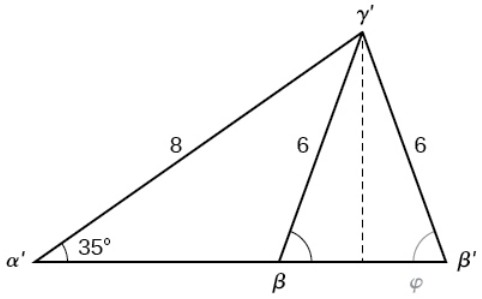


Figure 12.1.11

The angle supplementary to β is approximately equal to 49.9°, which means that $\beta = 180^\circ - 49.9^\circ = 130.1^\circ$. (Remember that the sine function is positive in both the first and second quadrants.) Solving for γ, we have

$$\begin{aligned} \gamma &= 180^\circ - 35^\circ - 130.1^\circ \\ &\approx 14.9^\circ \end{aligned}$$

We can then use these measurements to solve the other triangle. Since β' is supplementary to β, we have

$$\begin{aligned} \gamma' &= 180^\circ - 35^\circ - 49.5^\circ \\ &\approx 95.1^\circ \end{aligned}$$

Now we need to find c and c'.

We have

$$\begin{aligned} \frac{c}{\sin(14.9^\circ)} &= \frac{6}{\sin(35^\circ)} \\ c &= \frac{6\sin(14.9^\circ)}{\sin(35^\circ)} \\ &\approx 2.7 \end{aligned}$$

Finally,

$$\begin{aligned}\frac{c'}{\sin(95.1^\circ)} &= \frac{6}{\sin(35^\circ)} \\ c' &= \frac{6\sin(95.1^\circ)}{\sin(35^\circ)} \\ &\approx 10.4\end{aligned}$$

To summarize, there are two triangles with an angle of 35°, an adjacent side of 8, and an opposite side of 6, as shown in Figure 12.1.12

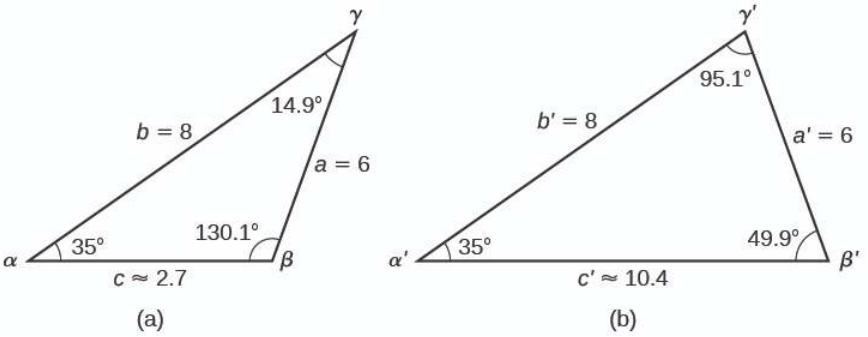


Figure 12.1.12

However, we were looking for the values for the triangle with an obtuse angle β. We can see them in the first triangle (a) in Figure 12.1.12

? Exercise 12.1.2

Given $\alpha = 80^\circ$, $a = 120$, and $b = 121$, find the missing side and angles. If there is more than one possible solution, show both.

Answer

Solution 1

$\alpha = 80^\circ$	$a = 120$
$\beta \approx 83.2^\circ$	$b = 121$
$\gamma \approx 16.8^\circ$	$c \approx 35.2$

Solution 2

$\alpha' = 80^\circ$	$a' = 120$
$\beta' \approx 96.8^\circ$	$b' = 121$
$\gamma' \approx 3.2^\circ$	$c' \approx 6.8$

✓ Example 12.1.3: Solving for the Unknown Sides and Angles of a SSA Triangle

In the triangle shown in Figure 12.1.13, solve for the unknown side and angles. Round your answers to the nearest tenth.

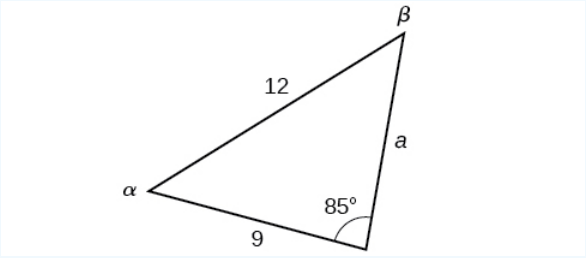


Figure 12.1.13

Solution

In choosing the pair of ratios from the Law of Sines to use, look at the information given. In this case, we know the angle, $\gamma = 85^\circ$, and its corresponding side $c = 12$, and we know side $b = 9$. We will use this proportion to solve for β.

$$\frac{\sin(85^\circ)}{12} = \frac{\sin\beta}{9} \qquad \text{Isolate the unknown.}$$
$$\frac{9\sin(85^\circ)}{12} = \sin\beta$$

To find β, apply the inverse sine function. The inverse sine will produce a single result, but keep in mind that there may be two values for β. It is important to verify the result, as there may be two viable solutions, only one solution (the usual case), or no solutions.

$$\beta = \sin^{-1}\left(\frac{9\sin(85^\circ)}{12}\right)$$
$$\beta \approx \sin^{-1}(0.7471)$$
$$\beta \approx 48.3^\circ$$

In this case, if we subtract β from 180°, we find that there may be a second possible solution. Thus, $\beta = 180^\circ - 48.3^\circ \approx 131.7^\circ$. To check the solution, subtract both angles, 131.7° and 85°, from 180°. This gives

$$\begin{aligned}\alpha &= 180^\circ - 85^\circ - 131.7^\circ \\ &\approx -36.7^\circ\end{aligned}$$

which is impossible, and so $\beta \approx 48.3^\circ$.

To find the remaining missing values, we calculate $\alpha = 180^\circ - 85^\circ - 48.3^\circ \approx 46.7^\circ$. Now, only side a is needed. Use the Law of Sines to solve for a by one of the proportions.

$$\frac{\sin(85^\circ)}{12} = \frac{\sin(46.7^\circ)}{a}$$
$$a\frac{\sin(85^\circ)}{12} = \sin(46.7^\circ)$$
$$a = \frac{12\sin(46.7^\circ)}{\sin(85^\circ)}$$
$$\approx 8.8$$

The complete set of solutions for the given triangle is

$$\begin{aligned}\alpha &\approx 46.7^\circ & a &\approx 8.8 \\ \beta &\approx 48.3^\circ & b &= 9 \\ \gamma &= 85^\circ & c &= 12\end{aligned}$$

? Exercise 12.1.3

Given $\alpha = 80^\circ$, $a = 100$, $b = 10$, find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

Answer

$\beta \approx 5.7^\circ$, $\gamma \approx 94.3^\circ$, $c \approx 101.3$

✓ Example 12.1.4: Finding the Triangles That Meet the Given Criteria

Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Solution

Using the given information, we can solve for the angle opposite the side of length 10. See Figure 12.1.14

$$\frac{\sin\alpha}{10} = \frac{\sin(50^\circ)}{4}$$
$$\sin\alpha = \frac{10\sin(50^\circ)}{4}$$
$$\sin\alpha \approx 1.915$$

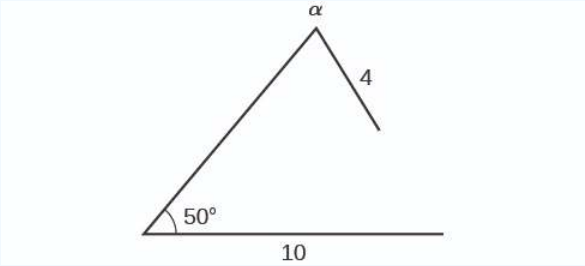


Figure 12.1.14

We can stop here without finding the value of α. Because the range of the sine function is $[-1, 1]$, it is impossible for the sine value to be 1.915. In fact, inputting $\sin^{-1}(1.915)$in a graphing calculator generates an ERROR DOMAIN. Therefore, no triangles can be drawn with the provided dimensions.

? Exercise 12.1.4

Determine the number of triangles possible given $a = 31$, $b = 26$, $\beta = 48^\circ$.

Answer

two

Finding the Area of an Oblique Triangle Using the Sine Function

Now that we can solve a triangle for missing values, we can use some of those values and the sine function to find the area of an oblique triangle. Recall that the area formula for a triangle is given as $Area = \frac{1}{2}bh$, where b is base and h is height. For oblique triangles, we must find h before we can use the area formula. Observing the two triangles in Figure 12.1.15, one acute and one obtuse, we can drop a perpendicular to represent the height and then apply the trigonometric property $\sin\alpha = \frac{opposite}{hypotenuse}$ to write an equation for area in oblique triangles. In the acute triangle, we have $\sin\alpha = \frac{h}{c}$ or $c\sin\alpha = h$. However, in the obtuse triangle, we drop the perpendicular outside the triangle and extend the base b to form a right triangle. The angle used in calculation is α', or $180 - \alpha$.

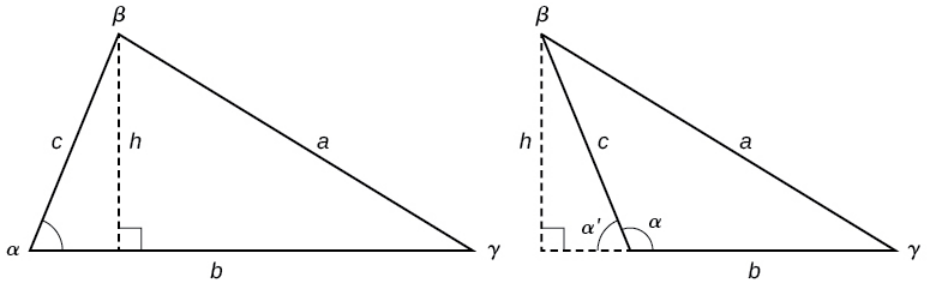


Figure 12.1.15

Thus,

$$Area = \frac{1}{2}(base)(height) = \frac{1}{2}b(c\sin\alpha)$$

Similarly,

$$Area = \frac{1}{2}a(b\sin\gamma) = \frac{1}{2}a(c\sin\beta)$$

Note: AREA OF AN OBLIQUE TRIANGLE

The formula for the area of an oblique triangle is given by

$$Area = \frac{1}{2}bc\sin\alpha \tag{12.1.3}$$

$$Area = \frac{1}{2}ac\sin\beta \tag{12.1.4}$$

$$Area = \frac{1}{2}ab\sin\gamma \tag{12.1.5}$$

This is equivalent to one-half of the product of two sides and the sine of their included angle.

Example 12.1.5: Finding the Area of an Oblique Triangle

Find the area of a triangle with sides $a = 90$, $b = 52$, and angle $\gamma = 102^\circ$. Round the area to the nearest integer.

Solution

Using the formula, we have

$$Area = \frac{1}{2}ab\sin\gamma$$
$$Area = \frac{1}{2}(90)(52)\sin(102^\circ)$$
$$Area \approx 2289 \text{ square units}$$

Exercise 12.1.5

Find the area of the triangle given $\beta = 42^\circ$, $a = 7.2\ ft$, $c = 3.4\ ft$. Round the area to the nearest tenth.

Answer

about 8.2 square feet

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

Example 12.1.6: Finding an Altitude

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in Figure 12.1.16 Round the altitude to the nearest tenth of a mile.

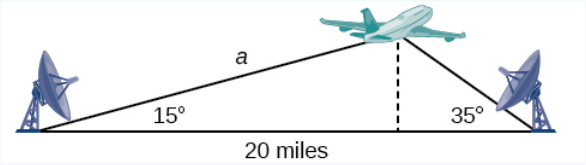


Figure 12.1.16

Solution

To find the elevation of the aircraft, we first find the distance from one station to the aircraft, such as the side a, and then use right triangle relationships to find the height of the aircraft, h.

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180^\circ - 15^\circ - 35^\circ = 130^\circ$. This angle is opposite the side of length 20, allowing us to set up a Law of Sines relationship.

$$\begin{aligned} \frac{\sin(130^\circ)}{20} &= \frac{\sin(35^\circ)}{a} \\ a\sin(130^\circ) &= 20\sin(35^\circ) \\ a &= \frac{20\sin(35^\circ)}{\sin(130^\circ)} \\ a &\approx 14.98 \end{aligned}$$

The distance from one station to the aircraft is about 14.98 miles.

Now that we know a, we can use right triangle relationships to solve for h.

$$\begin{aligned} \sin(15^\circ) &= \frac{opposite}{hypotenuse} \\ \sin(15^\circ) &= \frac{h}{a} \\ \sin(15^\circ) &= \frac{h}{14.98} \\ h &= 14.98\sin(15^\circ) \\ h &\approx 3.88 \end{aligned}$$

The aircraft is at an altitude of approximately 3.9 miles.

Exercise 12.1.6

The diagram shown in Figure $12.1.17$ represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point A, is 70°, the angle of elevation from the northern end zone, point B, is 62°, and the distance between the viewing points of the two end zones is 145 yards.

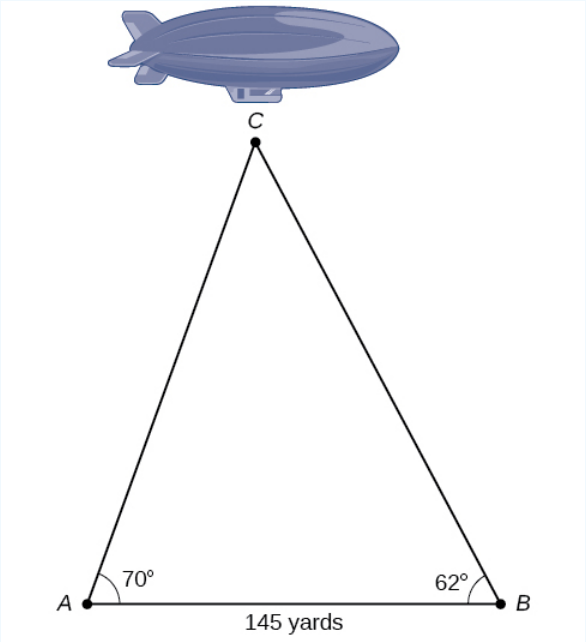


Figure 12.1.17

Answer

161.9yd.

Media

Access these online resources for additional instruction and practice with trigonometric applications.

- Law of Sines: The Basics
- Law of Sines: The Ambiguous Case

Key Equations

Law of Sines	$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$ $\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$
Area for oblique triangles	$Area=\frac{1}{2}bc\sin\alpha$ $=\frac{1}{2}ac\sin\beta$ $=\frac{1}{2}ab\sin\gamma$

Key Concepts

- The Law of Sines can be used to solve oblique triangles, which are non-right triangles.
- According to the Law of Sines, the ratio of the measurement of one of the angles to the length of its opposite side equals the other two ratios of angle measure to opposite side.
- There are three possible cases: ASA, AAS, SSA. Depending on the information given, we can choose the appropriate equation to find the requested solution. See Example $12.1.1$.
- The ambiguous case arises when an oblique triangle can have different outcomes.
- There are three possible cases that arise from SSA arrangement—a single solution, two possible solutions, and no solution. See Example $12.1.2$and Example $12.1.3$
- The Law of Sines can be used to solve triangles with given criteria. See Example $12.1.4$
- The general area formula for triangles translates to oblique triangles by first finding the appropriate height value. See Example $12.1.5$
- There are many trigonometric applications. They can often be solved by first drawing a diagram of the given information and then using the appropriate equation. See Example $12.1.6$

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12.2: Non-right Triangles - Law of Cosines

Learning Objectives

In this section, you will:

- Use the Law of Cosines to solve oblique triangles.
- Solve applied problems using the Law of Cosines.
- Use Heron's formula to find the area of a triangle.

Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles as shown in Figure 12.2.1 How far from port is the boat?

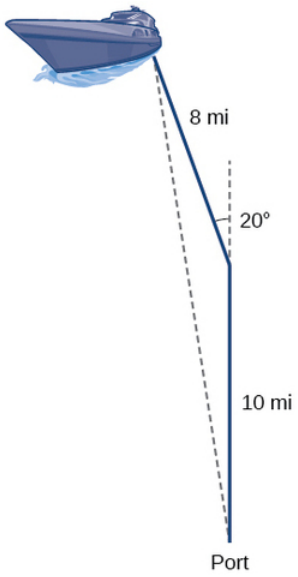


Figure 12.2.1

Unfortunately, while the Law of Sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool for solving oblique triangles described by these last two cases.

Using the Law of Cosines to Solve Oblique Triangles

The tool we need to solve the problem of the boat's distance from the port is the *Law of Cosines*, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the Generalized Pythagorean Theorem, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle ABC is placed in the coordinate plane with vertex A at the origin, side c drawn along the x-axis, and vertex C located at some point (x, y) in the plane, as illustrated in Figure 12.2.2 Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.

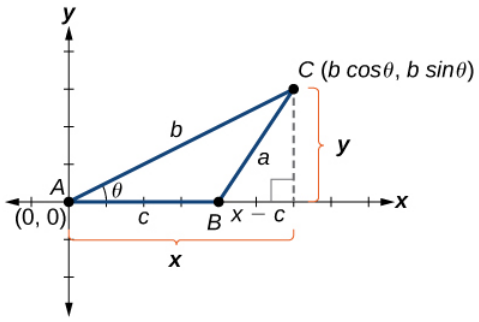


Figure 12.2.2

We can drop a perpendicular from C to the x-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$$\cos\theta = \frac{x(adjacent)}{b(hypotenuse)} \text{ and } \sin\theta = \frac{y(opposite)}{b(hypotenuse)}$$

In terms of θ, $x = b\cos\theta$ and $y = b\sin\theta$. The (x, y) point located at C has coordinates $(b\cos\theta, b\sin\theta)$. Using the side $(x - c)$ as one leg of a right triangle and y as the second leg, we can find the length of hypotenuse a using the Pythagorean Theorem. Thus,

$$\begin{aligned} a^2 &= (x-c)^2 + y^2 \\ &= (b\cos\theta - c)^2 + (b\sin\theta)^2 && \text{Substitute } (b\cos\theta) \text{ for } x \text{ and } (b\sin\theta) \text{ for } y \\ &= (b^2\cos^2\theta - 2bc\cos\theta + c^2) + b^2\sin^2\theta && \text{Expand the perfect square.} \\ &= b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta && \text{Group terms noting that } \cos^2\theta + \sin^2\theta = 1 \\ &= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta && \text{Factor out } b^2 \\ a^2 &= b^2 + c^2 - 2bc\cos\theta \end{aligned}$$

The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

The LAW OF COSINES

The Law of Cosines states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle.

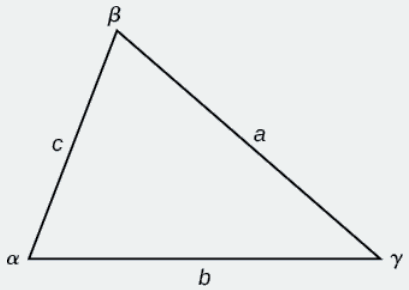


Figure 12.2.3

For triangles labeled as in Figure 12.2.3, with angles α, β and γ, and opposite corresponding sides a, b, and c, respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \tag{12.2.1}$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \tag{12.2.2}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{12.2.3}$$

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} \tag{12.2.4}$$

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac} \tag{12.2.5}$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} \tag{12.2.6}$$

How to: Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the length of the unknown side or angle.
3. Apply the Law of Sines or Cosines to find the measure of a second angle.
4. Compute the measure of the remaining angle.

Example 12.2.1: Finding the Unknown Side and Angles of a SAS Triangle

Find the unknown side and angles of the triangle in Figure 12.2.4

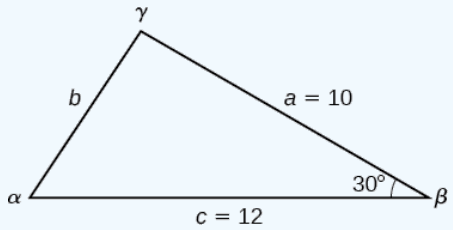


Figure 12.2.4

Solution

First, make note of what is given: two sides and the angle between them. This arrangement is classified as SAS and supplies the data needed to apply the Law of Cosines.

Each one of the three laws of cosines begins with the square of an unknown side opposite a known angle. For this example, the first side to solve for is side b, as we know the measurement of the opposite angle β.

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$b^2 = 10^2 + 12^2 - 2(10)(12)\cos(30^\circ) \quad \text{Substitute the measurements for the known quantities.}$$

$$b^2 = 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right) \quad \text{Evaluate the cosine and begin to simplify.}$$

$$b^2 = 244 - 120\sqrt{3}$$

$$b = \sqrt{244 - 120\sqrt{3}} \quad \text{Use the square root property.}$$

$$b \approx 6.013$$

Because we are solving for a length, we use only the positive square root. Now that we know the length b, we can use the Law of Sines to fill in the remaining angles of the triangle. Solving for angle α, we have

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

$$\frac{\sin\alpha}{10} = \frac{\sin(30^\circ)}{6.013}$$

$$\sin\alpha = \frac{10\sin(30^\circ)}{6.013} \qquad \text{Multiply both sides of the equation by } 10.$$

$$\alpha = \sin^{-1}\left(\frac{10\sin(30^\circ)}{6.013}\right) \qquad \text{Find the inverse sine of } \frac{10\sin(30^\circ)}{6.013}.$$

$$\alpha \approx 56.3^\circ$$

The other possibility for α would be $\alpha = 180^\circ - 56.3^\circ \approx 123.7^\circ$. In the original diagram,α is adjacent to the longest side, so α is an acute angle and, therefore, 123.7° does not make sense. Notice that if we choose to apply the Law of Cosines, we arrive at a unique answer. We do not have to consider the other possibilities, as cosine is unique for angles between 0° and 180°. Proceeding with $\alpha \approx 56.3^\circ$, we can then find the third angle of the triangle.

$$\begin{aligned}\gamma &= 180^\circ - 30^\circ - 56.3^\circ \\ &\approx 93.7^\circ\end{aligned}$$

The complete set of angles and sides is

$$\alpha \approx 56.3^\circ \; a = 10$$

$$\beta = 30^\circ \; b \approx 6.013$$

$$\gamma \approx 93.7^\circ \; c = 12$$

? Exercise 12.2.1

Find the missing side and angles of the given triangle: $\alpha = 30^\circ$, $b = 12$, $c = 24$.

Answer

$a \approx 14.9$, $\beta \approx 23.8^\circ$, $\gamma \approx 126.2^\circ$.

✓ Example 12.2.2: Solving for an Angle of a SSS Triangle

Find the angle α for the given triangle if side $a = 20$, side $b = 25$, and side $c = 18$.

Solution

For this example, we have no angles. We can solve for any angle using the Law of Cosines. To solve for angle α, we have

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$20^2 = 25^2 + 18^2 - 2(25)(18)\cos\alpha \qquad \text{Substitute the appropriate measurements.}$$

$$400 = 625 + 324 - 900\cos\alpha \qquad \text{Simplify in each step.}$$

$$400 = 949 - 900\cos\alpha$$

$$-549 = -900\cos\alpha \qquad \text{Isolate } \cos\alpha.$$

$$0.61 \approx \cos\alpha \qquad \text{Find the inverse cosine.}$$

$$\alpha \approx 52.4^\circ$$

See Figure 12.2.5

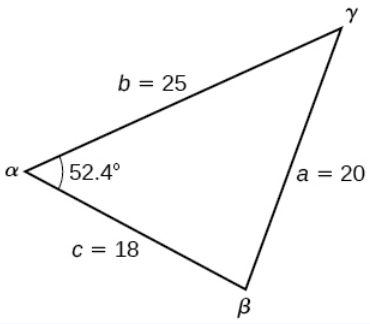


Figure 12.2.5

Analysis

Because the inverse cosine can return any angle between 0 and 180 degrees, there will not be any ambiguous cases using this method.

? Exercise 12.2.2

Given $a = 5$, $b = 7$, and $c = 10$, find the missing angles.

Answer

$\alpha \approx 27.7^\circ, \beta \approx 40.5^\circ, \gamma \approx 111.8^\circ$

Solving Applied Problems Using the Law of Cosines

Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

✓ Example 12.2.3A: Using the Law of Cosines to Solve a Communication Problem

On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points. Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5050 feet from the first tower and 2420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

Solution

For simplicity, we start by drawing a diagram similar to Figure $12.2.6$ and labeling our given information.

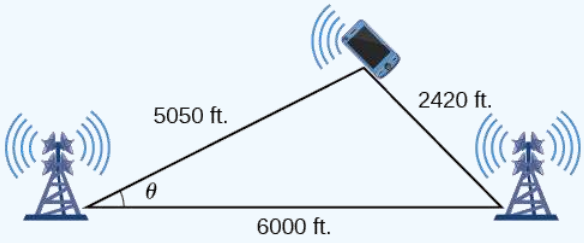


Figure 12.2.6

Using the Law of Cosines, we can solve for the angle θ. Remember that the Law of Cosines uses the square of one side to find the cosine of the opposite angle. For this example, let $a = 2420$, $b = 5050$, and $c = 6000$. Thus, θ corresponds to the opposite side $a = 2420$.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\theta \\ (2420)^2 &= (5050)^2 + (6000)^2 - 2(5050)(6000)\cos\theta \\ \cos\theta &\approx 0.9183 \\ \theta &\approx \cos^{-1}(0.9183) \\ \theta &\approx 23.3^\circ \end{aligned}$$

To answer the questions about the phone's position north and east of the tower, and the distance to the highway, drop a perpendicular from the position of the cell phone, as in Figure 12.2.7. This forms two right triangles, although we only need the right triangle that includes the first tower for this problem.

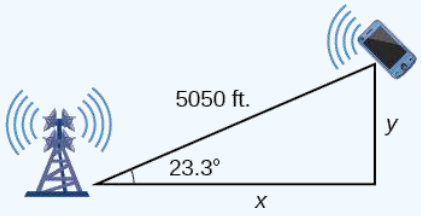


Figure 12.2.7

Using the angle $\theta = 23.3°$ and the basic trigonometric identities, we can find the solutions. Thus

$$\begin{aligned} \cos(23.3^\circ) &= \frac{x}{5050} \\ x &= 5050\cos(23.3^\circ) \\ x &\approx 4638.15\ feet \\ \sin(23.3^\circ) &= \frac{y}{5050} \\ y &= 5050\sin(23.3^\circ) \\ y &\approx 1997.5\ feet \end{aligned}$$

The cell phone is approximately 4638 feet east and 1998 feet north of the first tower, and 1998 feet from the highway.

Example 12.2.3B: Calculating Distance Traveled Using a SAS Triangle

Returning to our problem at the beginning of this section, suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat? The diagram is repeated here in Figure 12.2.8

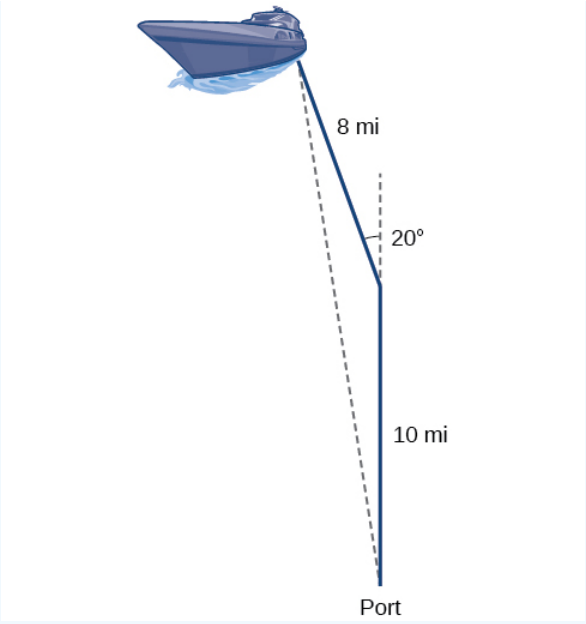


Figure 12.2.8

Solution

The boat turned 20 degrees, so the obtuse angle of the non-right triangle is the supplemental angle, $180^\circ - 20^\circ = 160^\circ$. With this, we can utilize the Law of Cosines to find the missing side of the obtuse triangle—the distance of the boat to the port.

$$\begin{aligned} x^2 &= 8^2 + 10^2 - 2(8)(10)\cos(160^\circ) \\ x^2 &= 314.35 \\ x &= \sqrt{314.35} \\ x &\approx 17.7\, miles \end{aligned}$$

The boat is about 17.7 miles from port.

Using Heron's Formula to Find the Area of a Triangle

We already learned how to find the area of an oblique triangle when we know two sides and an angle. We also know the formula to find the area of a triangle using the base and the height. When we know the three sides, however, we can use **Heron's formula** instead of finding the height. **Heron of Alexandria** was a geometer who lived during the first century A.D. He discovered a formula for finding the area of oblique triangles when three sides are known.

HERON'S FORMULA

Heron's formula finds the area of oblique triangles in which sides a, b,and c are known.

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \tag{12.2.7}$$

where $s = \dfrac{(a+b+c)}{2}$ is one half of the perimeter of the triangle, sometimes called the semi-perimeter.

Example 12.2.4: Using Heron's Formula to Find the Area of a Given Triangle

Find the area of the triangle in Figure $12.2.9$using Heron's formula.

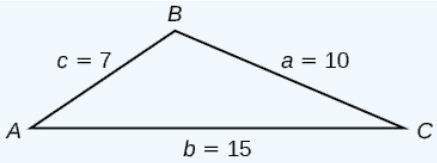


Figure 12.2.9

Solution

First, we calculate s.

$$s = \frac{(a+b+c)}{2}$$
$$s = \frac{(10+15+7)}{2}$$
$$= 16$$

Then we apply the formula.

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$
$$Area = \sqrt{16(16-10)(16-15)(16-7)}$$
$$Area \approx 29.4$$

The area is approximately 29.4 square units.

Exercise 12.2.3

Use Heron's formula to find the area of a triangle with sides of lengths $a = 29.7$ ft, $b = 42.3$ ft, and $c = 38.4$ ft.

Answer

Area = 552 square feet

Example 12.2.5: Applying Heron's Formula to a Real-World Problem

A Chicago city developer wants to construct a building consisting of artist's lofts on a triangular lot bordered by Rush Street, Wabash Avenue, and Pearson Street. The frontage along Rush Street is approximately 62.4 meters, along Wabash Avenue it is approximately 43.5 meters, and along Pearson Street it is approximately 34.1 meters. How many square meters are available to the developer? See Figure $12.2.10$ for a view of the city property.



Figure 12.2.10

Solution

Find the measurement for s, which is one-half of the perimeter.

$$s = \frac{(62.4 + 43.5 + 34.1)}{2}$$
$$s = 70\ m$$

Apply Heron's formula.

$$Area = \sqrt{70(70 - 62.4)(70 - 43.5)(70 - 34.1)}$$
$$Area = \sqrt{506,118.2}$$
$$Area \approx 711.4$$

The developer has about 711.4 square meters.

? Exercise 12.2.4

Find the area of a triangle given $a = 4.38$ ft , $b = 3.79$ ft, and $c = 5.22$ ft.

Answer

about 8.15 square feet

Media

Access these online resources for additional instruction and practice with the Law of Cosines.

- Law of Cosines
- Law of Cosines: Applications
- Law of Cosines: Applications 2

Key Equations

Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos\alpha$ $b^2 = a^2 + c^2 - 2ac\cos\beta$ $c^2 = a^2 + b^2 - 2ab\cos\gamma$
Heron's formula	$Area = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{(a+b+c)}{2}$

Key Concepts

- The Law of Cosines defines the relationship among angle measurements and lengths of sides in oblique triangles.
- The Generalized Pythagorean Theorem is the Law of Cosines for two cases of oblique triangles: SAS and SSS. Dropping an imaginary perpendicular splits the oblique triangle into two right triangles or forms one right triangle, which allows sides to be related and measurements to be calculated. See Example $12.2.1$ and Example $12.2.2$
- The Law of Cosines is useful for many types of applied problems. The first step in solving such problems is generally to draw a sketch of the problem presented. If the information given fits one of the three models (the three equations), then apply the Law of Cosines to find a solution. See Example $12.2.3$ and Example $12.2.4$
- Heron's formula allows the calculation of area in oblique triangles. All three sides must be known to apply Heron's formula. See Example $12.2.5$ and See Example $12.2.6$

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12.3: Vectors

Learning Objectives

- View vectors geometrically.
- Find magnitude and direction.
- Perform vector addition and scalar multiplication.
- Find the component form of a vector.
- Find the unit vector in the direction of v.
- Perform operations with vectors in terms of i and j.
- Find the dot product of two vectors.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour, as shown in Figure $12.3.1$. What are the ground speed and actual bearing of the plane?

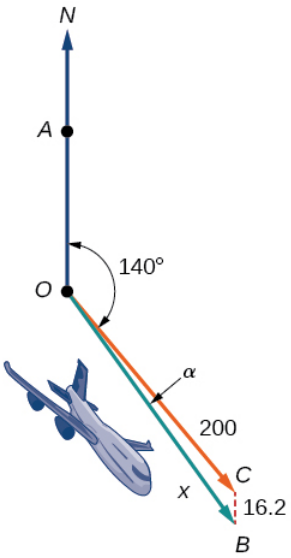


Figure 12.3.1

Ground speed refers to the speed of a plane relative to the ground. Airspeed refers to the speed a plane can travel relative to its surrounding air mass. These two quantities are not the same because of the effect of wind. In an earlier section, we used triangles to solve a similar problem involving the movement of boats. Later in this section, we will find the airplane's ground speed and bearing, while investigating another approach to problems of this type. First, however, let's examine the basics of vectors.

A Geometric View of Vectors

A vector is a specific quantity drawn as a line segment with an arrowhead at one end. It has an **initial point**, where it begins, and a **terminal point**, where it ends. A vector is defined by its magnitude, or the length of the line, and its direction, indicated by an arrowhead at the terminal point. Thus, a vector is a directed line segment. There are various symbols that distinguish vectors from other quantities:

- Lower case, boldfaced type, with or without an arrow on top such as $\boldsymbol{u}$, $\boldsymbol{w}$, $\vec{\boldsymbol{v}}$, $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{w}}$.
- Given initial point P and terminal point Q, a vector can be represented as $\overrightarrow{PQ}$. The arrowhead on top is what indicates that it is not just a line, but a directed line segment.
- Given an initial point of $(0, 0)$ and terminal point (a, b), a vector may be represented as $\langle a, b\rangle$.

This last symbol $\langle a, b\rangle$ has special significance. It is called the standard position. The position vector has an initial point $(0, 0)$ and a terminal point $\langle a, b\rangle$. To change any vector into the position vector, we think about the change in the *x*-coordinates and the change in the *y*-coordinates. Thus, if the initial point of a vector $\overrightarrow{CD}$ is $C(x_1, y_1)$ and the terminal point is $D(x_2, y_2)$, then the position vector is found by calculating

$$\begin{aligned}\overrightarrow{AB} &= \langle x_2 - x_1, y_2 - y_1 \rangle \\ &= \langle a, b \rangle\end{aligned}$$

In Figure 12.3.2, we see the original vector $\overrightarrow{CD}$ and the position vector $\overrightarrow{AB}$.

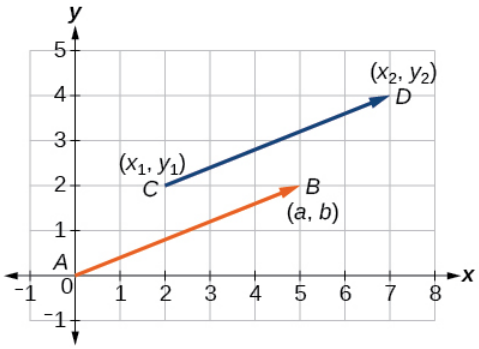


Figure 12.3.2

PROPERTIES OF VECTORS

A vector is a directed line segment with an initial point and a terminal point. Vectors are identified by magnitude, or the length of the line, and direction, represented by the arrowhead pointing toward the terminal point. The position vector has an initial point at $(0, 0)$ and is identified by its terminal point $\langle a, b \rangle$.

Example 12.3.1A: Find the Position Vector

Consider the vector whose initial point is $P(2, 3)$ and terminal point is $Q(6, 4)$. Find the position vector.

Solution

The position vector is found by subtracting one x-coordinate from the other x-coordinate, and one y-coordinate from the other y-coordinate. Thus

$$\begin{aligned}v &= \langle 6 - 2, 4 - 3 \rangle \\ &= \langle 4, 1 \rangle\end{aligned}$$

The position vector begins at $(0, 0)$ and terminates at $(4, 1)$. The graphs of both vectors are shown in Figure 12.3.3

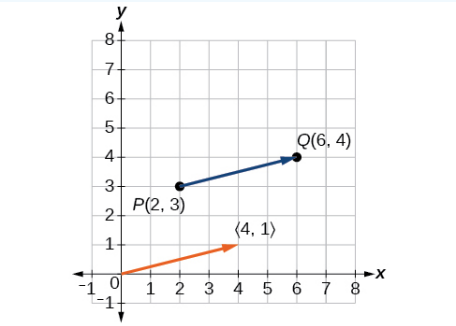


Figure 12.3.3

We see that the position vector is $\langle 4, 1 \rangle$.

Example 12.3.1B: Drawing a Vector with the Given Criteria and Its Equivalent Position Vector

Find the position vector given that vector v has an initial point at $(-3, 2)$ and a terminal point at $(4, 5)$, then graph both vectors in the same plane.

Solution

The position vector is found using the following calculation:

$$v = \langle 4-(-3), 5-2\rangle$$
$$= \langle 7, 3\rangle$$

Thus, the position vector begins at $(0, 0)$ and terminates at $(7, 3)$. See Figure 12.3.4

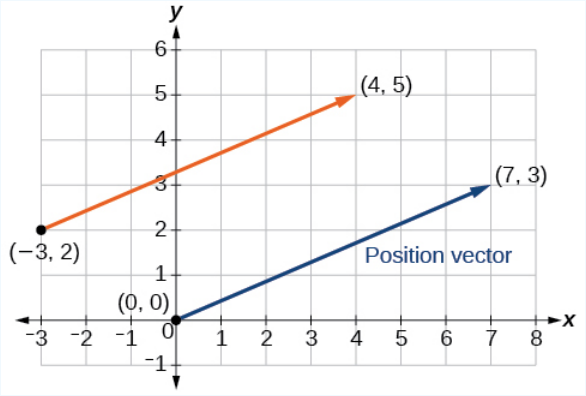


Figure 12.3.4

? Exercise 12.3.1

Draw a vector $\vec{v}$ that connects from the origin to the point $(3, 5)$.

Answer

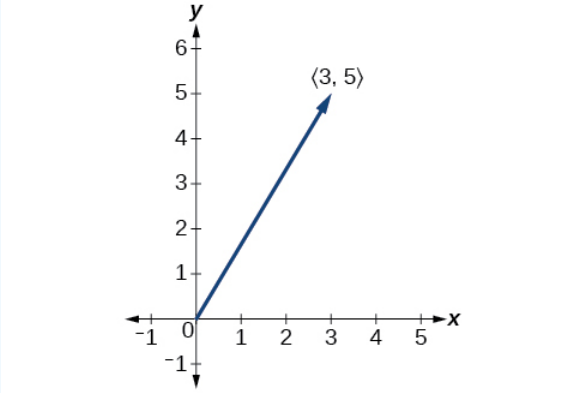


Figure 12.3.5

Finding Magnitude and Direction

To work with a vector, we need to be able to find its magnitude and its direction. We find its magnitude using the Pythagorean Theorem or the distance formula, and we find its direction using the inverse tangent function.

MAGNITUDE AND DIRECTION OF A VECTOR

Given a position vector $\vec{v} = \langle a, b\rangle$,the magnitude is found by $|v| = \sqrt{a^2 + b^2}$ .The direction is equal to the angle formed with the x-axis, or with the y-axis, depending on the application. For a position vector, the direction is found by $\tan\theta = \left(\frac{b}{a}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$, as illustrated in Figure 12.3.6

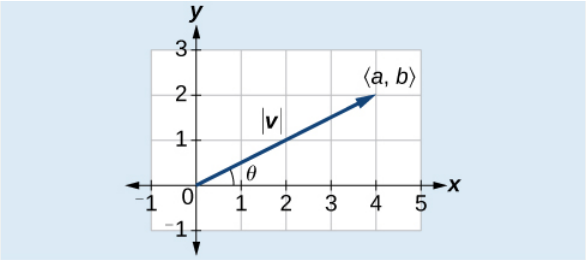


Figure 12.3.6

Two vectors $\vec{v}$ and $\vec{u}$ are considered equal if they have the same magnitude and the same direction. Additionally, if both vectors have the same position vector, they are equal.

Example 12.3.2A: Finding the Magnitude and Direction of a Vector

Find the magnitude and direction of the vector with initial point $P(-8, 1)$ and terminal point $Q(-2, -5)$.Draw the vector.

Solution

First, find the position vector.

$$\begin{aligned} u &= \langle -2, -(-8), -5-1 \rangle \\ &= \langle 6, -6 \rangle \end{aligned}$$

We use the Pythagorean Theorem to find the magnitude.

$$\begin{aligned} |u| &= \sqrt{(6)^2 + (-6)^2} \\ &= \sqrt{72} \\ &= \sqrt{62} \end{aligned}$$

The direction is given as

$$\begin{aligned} \tan\theta &= \frac{-6}{6} = -1 \rightarrow \theta = \tan^{-1}(-1) \\ &= -45^\circ \end{aligned}$$

However, the angle terminates in the fourth quadrant, so we add 360° to obtain a positive angle. Thus, $-45^\circ + 360^\circ = 315^\circ$. See Figure 12.3.7.

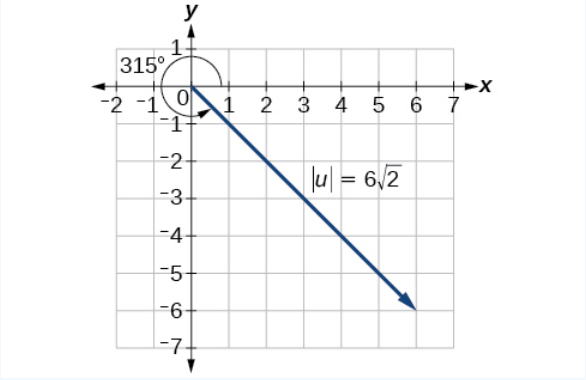


Figure 12.3.7

Example 12.3.2B: Showing That Two Vectors Are Equal

Show that vector $\vec{v}$ with initial point at $(5, -3)$ and terminal point at $(-1, 2)$ is equal to vector $\vec{u}$ with initial point at $(-1, -3)$ and terminal point at $(-7, 2)$. Draw the position vector on the same grid as $\vec{v}$ and $\vec{u}$. Next, find the magnitude and direction of each vector.

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Solution

As shown in Figure 12.3.8, draw the vector $\vec{v}$ starting at initial $(5,-3)$ and terminal point $(-1,2)$. Draw the vector $\vec{u}$ with initial point $(-1,-3)$ and terminal point $(-7,2)$. Find the standard position for each.

Next, find and sketch the position vector for $\vec{v}$ and $\vec{u}$. We have

$$\begin{aligned} v &= \langle -1-5, 2-(-3)\rangle \\ &= \langle -6, 5\rangle u \\ &= \langle -7-(-1), 2-(-3)\rangle \\ &= \langle -6, 5\rangle \end{aligned}$$

Since the position vectors are the same, $\vec{v}$ and $\vec{u}$ are the same.

An alternative way to check for vector equality is to show that the magnitude and direction are the same for both vectors. To show that the magnitudes are equal, use the Pythagorean Theorem.

$$\begin{aligned} |v| &= \sqrt{(-1-5)^2+(2-(-3))^2} \\ &= \sqrt{(-6)^2+(5)^2} \\ &= \sqrt{36+25} \\ &= \sqrt{61} \\ |u| &= \sqrt{(-7-(-1))^2+(2-(-3))^2} \\ &= \sqrt{(-6)^2+(5)^2} \\ &= \sqrt{36+25} \\ &= \sqrt{61} \end{aligned}$$

As the magnitudes are equal, we now need to verify the direction. Using the tangent function with the position vector gives

$$\begin{aligned} \tan\theta = -\frac{5}{6} \Rightarrow \theta &= \tan^{-1}\left(-\frac{5}{6}\right) \\ &= -39.8^\circ \end{aligned}$$

However, we can see that the position vector terminates in the second quadrant, so we add 180°. Thus, the direction is $-39.8^\circ + 180^\circ = 140.2^\circ$.

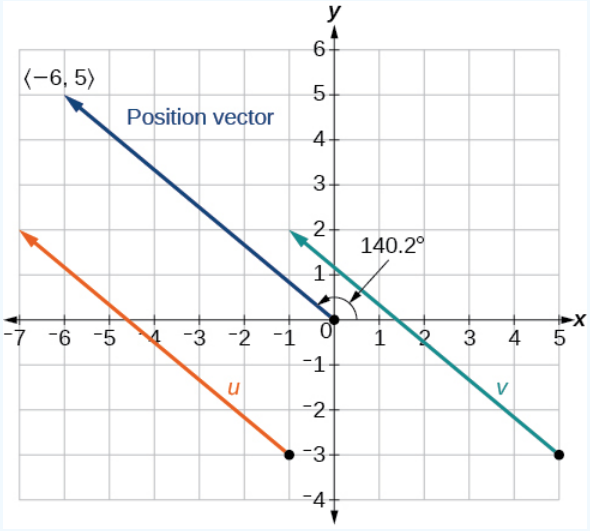


Figure 12.3.8

Performing Vector Addition and Scalar Multiplication

Now that we understand the properties of vectors, we can perform operations involving them. While it is convenient to think of the vector $u = \langle x, y \rangle$ as an arrow or directed line segment from the origin to the point (x, y), vectors can be situated anywhere in the plane. The sum of two vectors $\vec{u}$ and $\vec{v}$, or vector addition, produces a third vector $\overrightarrow{u+v}$, the resultant vector.

To find $\overrightarrow{u+v}$, we first draw the vector $\vec{u}$, and from the terminal end of $\vec{u}$, we drawn the vector $\vec{v}$. In other words, we have the initial point of $\vec{v}$ meet the terminal end of $\vec{u}$. This position corresponds to the notion that we move along the first vector and then, from its terminal point, we move along the second vector. The sum $\overrightarrow{u+v}$ is the resultant vector because it results from addition or subtraction of two vectors. The resultant vector travels directly from the beginning of $\vec{u}$ to the end of $\vec{v}$ in a straight path, as shown in Figure 12.3.9.

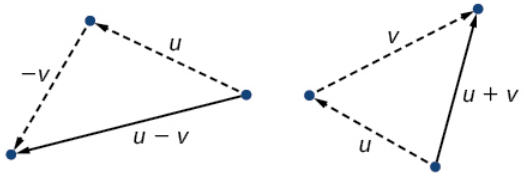


Figure 12.3.9

Vector subtraction is similar to vector addition. To find $\overrightarrow{u-v}$, view it as $\overrightarrow{u+(-v)}$. Adding $\overrightarrow{-v}$ is reversing direction of $\vec{v}$ and adding it to the end of $\vec{u}$. The new vector begins at the start of $\vec{u}$ and stops at the end point of $\overrightarrow{-v}$. See Figure 12.3.10 for a visual that compares vector addition and vector subtraction using parallelograms.

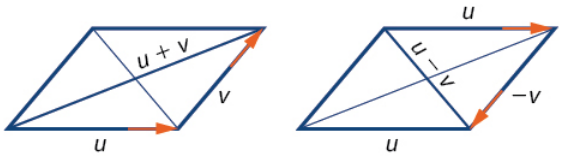


Figure 12.3.10

Example 12.3.3: Adding and Subtracting Vectors

Given $u = \langle 3, -2 \rangle$ and $v = \langle -1, 4 \rangle$, find two new vectors $\overrightarrow{u+v}$, and $\overrightarrow{u-v}$.

Solution

To find the sum of two vectors, we add the components. Thus,

$$\begin{aligned} u+v &= \langle 3, -2 \rangle + \langle -1, 4 \rangle \\ &= \langle 3+(-1), -2+4 \rangle \\ &= \langle 2, 2 \rangle \end{aligned}$$

See Figure 12.3.11a

To find the difference of two vectors, add the negative components of $\vec{v}$ to $\vec{u}$. Thus,

$$\begin{aligned} u+(-v) &= \langle 3, -2 \rangle + \langle 1, -4 \rangle \\ &= \langle 3+1, -2+(-4) \rangle \\ &= \langle 4, -6 \rangle \end{aligned}$$

See Figure 12.3.11b

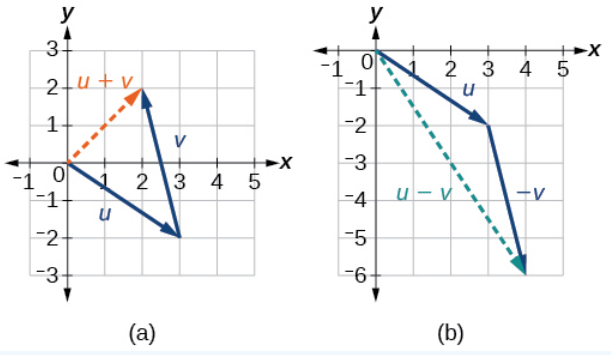


Figure 12.3.11: (a) Sum of two vectors (b) Difference of two vectors

Multiplying By a Scalar

While adding and subtracting vectors gives us a new vector with a different magnitude and direction, the process of multiplying a vector by a **scalar**, a constant, changes only the magnitude of the vector or the length of the line. Scalar multiplication has no effect on the direction unless the scalar is negative, in which case the direction of the resulting vector is opposite the direction of the original vector.

SCALAR MULTIPLICATION

Scalar multiplication involves the product of a vector and a scalar. Each component of the vector is multiplied by the scalar. Thus, to multiply $v = \langle a, b \rangle$ by k, we have

$kv = \langle ka, kb \rangle$

Only the magnitude changes, unless k is negative, and then the vector reverses direction.

Example 12.3.4: Performing Scalar Multiplication

Given vector $\vec{v} = \langle 3, 1 \rangle$, find $3\vec{v}$, $\frac{1}{2}$, and $\overrightarrow{-v}$.

Solution

See Figure 12.3.12for a geometric interpretation. If $\vec{v} = \langle 3, 1 \rangle$, then

$$\begin{aligned} 3v &= \langle 3 \cdot 3, 3 \cdot 1 \rangle \\ &= \langle 9, 3 \rangle \\ \frac{1}{2}v &= \langle \frac{1}{2} \cdot 3, \frac{1}{2} \cdot 1 \rangle \\ &= \langle \frac{3}{2}, \frac{1}{2} \rangle \\ -v &= \langle -3, -1 \rangle \end{aligned}$$

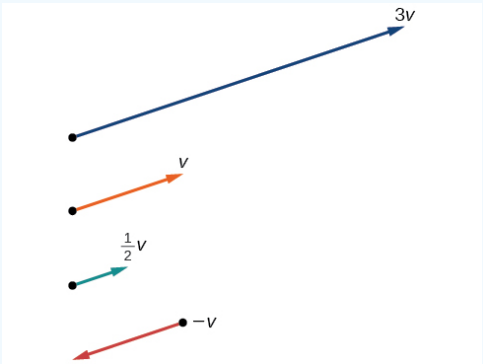


Figure 12.3.12

Analysis

Notice that the vector $3\vec{v}$ is three times the length of $\vec{v}$, $\frac{1}{2}\vec{v}$ is half the length of $\vec{v}$, and $\overrightarrow{-v}$ is the same length of $\vec{v}$, but in the opposite direction.

? Exercise 12.3.2

Find the scalar multiple $3u$ given $\vec{u} = \langle 5, 4 \rangle$.

Answer

$3u = \langle 15, 12 \rangle$

✓ Example 12.3.5

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is 31. Find the two numbers.

Solution

First, we must multiply each vector by the scalar.

$$\begin{aligned} 3u &= 3\langle 3, -2 \rangle \\ &= \langle 9, -6 \rangle \\ 2v &= 2\langle -1, 4 \rangle \\ &= \langle -2, 8 \rangle \end{aligned}$$

Then, add the two together.

$$\begin{aligned} w &= 3u + 2v \\ &= \langle 9, -6 \rangle + \langle -2, 8 \rangle \\ &= \langle 9 - 2, -6 + 8 \rangle \\ &= \langle 7, 2 \rangle \end{aligned}$$

So, $w = \langle 7, 2 \rangle$.

Finding Component Form

In some applications involving vectors, it is helpful for us to be able to break a vector down into its components. Vectors are comprised of two components: the horizontal component is the x direction, and the vertical component is the y direction. For example, we can see in the graph in Figure 12.3.13 that the position vector $\langle 2, 3 \rangle$ comes from adding the vectors v_1 and v_2. We have v_2 with initial point $(0, 0)$ and terminal point $(2, 0)$.

$$\begin{aligned} v_1 &= \langle 2 - 0, 0 - 0 \rangle \\ &= \langle 2, 0 \rangle \end{aligned}$$

We also have v_2 with initial point $(0, 0)$ and terminal point $(0, 3)$.

$$\begin{aligned} v_2 &= \langle 0 - 0, 3 - 0 \rangle \\ &= \langle 0, 3 \rangle \end{aligned}$$

Therefore, the position vector is

$$\begin{aligned} v &= \langle 2 + 0, 3 + 0 \rangle \\ &= \langle 2, 3 \rangle \end{aligned}$$

Using the Pythagorean Theorem, the magnitude of v_1 is 2, and the magnitude of v_2 is 3. To find the magnitude of v, use the formula with the position vector.

$$\begin{aligned} |v| &= \sqrt{|v_1|^2 + |v_2|^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

The magnitude of v is $\sqrt{13}$. To find the direction, we use the tangent function $\tan\theta = \dfrac{y}{x}$.

$$\tan\theta = \frac{v_2}{v_1}$$

$$\tan\theta = \frac{3}{2}$$

$$\theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.3^\circ$$

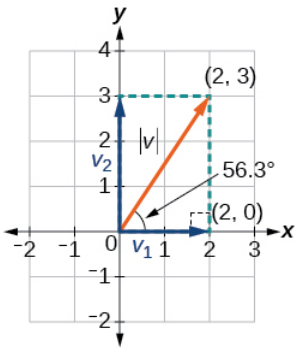


Figure 12.3.13

Thus, the magnitude of $\vec{v}$ is $\sqrt{13}$ and the direction is 56.3° off the horizontal.

Example 12.3.6: Finding the Components of the Vector

Find the components of the vector $\vec{v}$ with initial point $(3, 2)$ and terminal point $(7, 4)$.

Solution

First find the standard position.

$$\begin{aligned} v &= \langle 7-3, 4-2 \rangle \\ &= \langle 4, 2 \rangle \end{aligned}$$

See the illustration in Figure 12.3.14

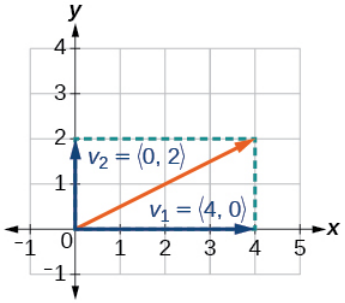


Figure 12.3.14

The horizontal component is $\overrightarrow{v_1} = \langle 4, 0 \rangle$ and the vertical component is $\overrightarrow{v_2} = \langle 0, 2 \rangle$.

Finding the Unit Vector in the Direction of v

In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a unit vector. We can then preserve the direction of the original vector while simplifying calculations.

Unit vectors are defined in terms of components. The horizontal unit vector is written as $\vec{i} = \langle 1, 0 \rangle$ and is directed along the positive horizontal axis. The vertical unit vector is written as $\vec{j} = \langle 0, 1 \rangle$ and is directed along the positive vertical axis. See Figure 12.3.15

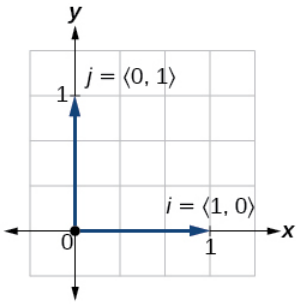


Figure 12.3.15

THE UNIT VECTORS

If $\vec{v}$ is a nonzero vector, then $\dfrac{v}{|v|}$ is a unit vector in the direction of $\boldsymbol{v}$**.** Any vector divided by its magnitude is a unit vector. Notice that magnitude is always a scalar, and dividing by a scalar is the same as multiplying by the reciprocal of the scalar.

Example 12.3.7: Finding the Unit Vector in the Direction of v

Find a unit vector in the same direction as $v = \langle -5, 12 \rangle$.

Solution

First, we will find the magnitude.

$$\begin{aligned} |v| &= \sqrt{(-5)^2 + (12)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

Then we divide each component by $|v|$, which gives a unit vector in the same direction as $\vec{v}$:

$$\frac{v}{|v|} = -\frac{5}{13}i + \frac{12}{13}j$$

or, in component form

$$\frac{v}{|v|} = \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle$$

See Figure 12.3.16

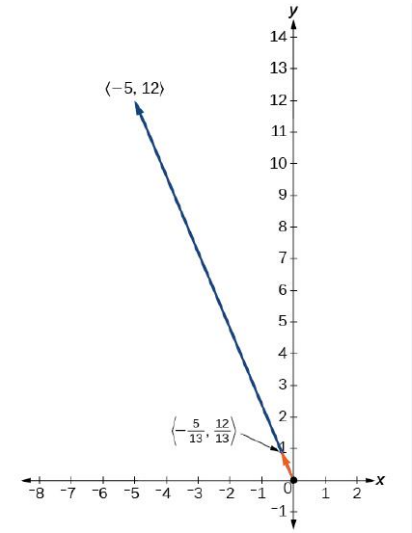


Figure 12.3.16

Verify that the magnitude of the unit vector equals 1. The magnitude of $-\frac{5}{13}i+\frac{12}{13}j$ is given as

$$\begin{aligned}\sqrt{\left(-\frac{5}{13}\right)^2+\left(\frac{12}{13}\right)^2} &= \sqrt{\frac{25}{169}+\frac{144}{169}} \\ &= \sqrt{\frac{169}{169}} \\ &= 1\end{aligned}$$

The vector $u=\frac{5}{13}i+\frac{12}{13}j$ is the unit vector in the same direction as $v=\langle -5,12\rangle$.

Performing Operations with Vectors in Terms of i and j

So far, we have investigated the basics of vectors: magnitude and direction, vector addition and subtraction, scalar multiplication, the components of vectors, and the representation of vectors geometrically. Now that we are familiar with the general strategies used in working with vectors, we will represent vectors in rectangular coordinates in terms of i and j.

VECTORS IN THE RECTANGULAR PLANE

Given a vector $\vec{v}$ with initial point $P=(x_1,y_1)$ and terminal point $Q=(x_2,y_2)$, $\vec{v}$ is written as

$$v=(x_2-x_1)i+(y_1-y_2)j \tag{12.3.1}$$

The position vector from $(0,0)$ to (a,b), where $(x_2-x_1)=a$ and $(y_2-y_1)=b$, is written as $\vec{v}=\overrightarrow{ai}+\overrightarrow{bj}$. This vector sum is called a linear combination of the vectors $\vec{i}$ and $\vec{j}$.

The magnitude of $\vec{v}=\overrightarrow{ai}+\overrightarrow{bj}$ is given as $|v|=\sqrt{a^2+b^2}$. See Figure 12.3.17.

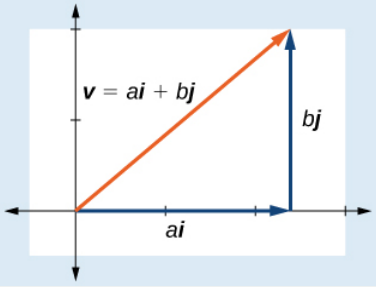


Figure 12.3.17

Example 12.3.8A: Writing a Vector in Terms of i and j

Given a vector $\vec{v}$ with initial point $P=(2,-6)$ and terminal point $Q=(-6,6)$, write the vector in terms of $\vec{i}$ and $\vec{j}$.

Solution

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$\begin{aligned} v &= (x_2-x_1)i+(y_2-y_1)j \\ &= (-6-2)i+(6-(-6))j \\ &= -8i+12j \end{aligned}$$

Example 12.3.8B: Writing a Vector in Terms of i and j Using Initial and Terminal Points

Given initial point $P_1=(-1,3)$ and terminal point $P_2=(2,7)$, write the vector $\vec{v}$ in terms of $\vec{i}$ and $\vec{j}$.

Solution

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$\begin{aligned} v &= (x_2-x_1)i+(y_2-y_1)j \\ v &= (2-(-1))i+(7-3)j \\ &= 3i+4j \end{aligned}$$

Exercise 12.3.3

Write the vector $\vec{u}$ with initial point $P=(-1,6)$ and terminal point $Q=(7,-5)$ in terms of $\vec{i}$ and $\vec{j}$.

Answer

$u=8i-11j$

Performing Operations on Vectors in Terms of i and j

When vectors are written in terms of i and j, we can carry out addition, subtraction, and scalar multiplication by performing operations on corresponding components.

ADDING AND SUBTRACTING VECTORS IN RECTANGULAR COORDINATES

Given $v=ai+bj$ and $u=ci+dj$, then

$$\begin{aligned} v+u &= (a+c)i+(b+d)j \\ v-u &= (a-c)i+(b-d)j \end{aligned}$$

Example 12.3.9: Finding the Sum of the Vectors

Find the sum of $v_1=2i-3j$ and $v_2=4i+5j$.

Solution

$$\begin{aligned} v_1+v_2 &= (2+4)i+(-3+5)j \\ &= 6i+2j \end{aligned}$$

Calculating the Component Form of a Vector: Direction

We have seen how to draw vectors according to their initial and terminal points and how to find the position vector. We have also examined notation for vectors drawn specifically in the Cartesian coordinate plane using i and j. For any of these vectors, we can calculate the magnitude. Now, we want to combine the key points, and look further at the ideas of magnitude and direction.

Calculating direction follows the same straightforward process we used for polar coordinates. We find the direction of the vector by finding the angle to the horizontal. We do this by using the basic trigonometric identities, but with $|v|$ replacing r.

VECTOR COMPONENTS IN TERMS OF MAGNITUDE AND DIRECTION

Given a position vector $v = \langle x, y \rangle$ and a direction angle θ,

$$\cos\theta = \frac{x}{|v|} \text{ and } \sin\theta = y|v|$$

$$x = |v|\cos\theta$$

$$y = |v|\sin\theta$$

Thus, $v = xi + yj = |v|\cos\theta i + |v|\sin\theta j$, and magnitude is expressed as $|v| = \sqrt{x^2 + y^2}$.

Example 12.3.10: Writing a Vector in Terms of Magnitude and Direction

Write a vector with length 7 at an angle of 135° to the positive *x*-axis in terms of magnitude and direction.

Solution

Using the conversion formulas $x = |v|\cos\theta i$ and $y = |v|\sin\theta j$, we find that

$$x = 7\cos(135^\circ)i$$

$$= -\frac{7\sqrt{2}}{2}$$

$$y = 7\sin(135^\circ)j$$

$$= \frac{7\sqrt{2}}{2}$$

This vector can be written as $v = 7\cos(135^\circ)i + 7\sin(135^\circ)j$ or simplified as

$$v = -\frac{7\sqrt{2}}{2}i + \frac{7\sqrt{2}}{2}j$$

Exercise 12.3.4

A vector travels from the origin to the point $(3, 5)$. Write the vector in terms of magnitude and direction.

Answer

$v = \sqrt{34}\cos(59^\circ)i + \sqrt{34}\sin(59^\circ)j$

Magnitude = 34

$\theta = \tan^{-1}\left(\frac{5}{3}\right) = 59.04^\circ$

Finding the Dot Product of Two Vectors

As we discussed earlier in the section, scalar multiplication involves multiplying a vector by a scalar, and the result is a vector. As we have seen, multiplying a vector by a number is called scalar multiplication. If we multiply a vector by a vector, there are two possibilities: the *dot product* and the *cross product*. We will only examine the dot product here; you may encounter the cross product in more advanced mathematics courses.

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar.

DOT PRODUCT

The dot product of two vectors $v = \langle a, b \rangle$ and $u = \langle c, d \rangle$ is the sum of the product of the horizontal components and the product of the vertical components.

$$v \cdot u = ac + bd \tag{12.3.2}$$

To find the angle between the two vectors, use the formula below.

$$\cos\theta = \frac{v}{|v|} \cdot \frac{u}{|u|} \tag{12.3.3}$$

Example 12.3.11A: Finding the Dot Product of Two Vectors

Find the dot product of $v = \langle 5, 12 \rangle$ and $u = \langle -3, 4 \rangle$.

Solution

Using the formula, we have

$$\begin{aligned} v \cdot u &= \langle 5, 12 \rangle \cdot \langle -3, 4 \rangle \\ &= 5 \cdot (-3) + 12 \cdot 4 \\ &= -15 + 48 \\ &= 33 \end{aligned}$$

Example 12.3.11B: Finding the Dot Product of Two Vectors and the Angle between Them

Find the dot product of $v_1 = 5i + 2j$ and $v_2 = 3i + 7j$. Then, find the angle between the two vectors.

Solution

Finding the dot product, we multiply corresponding components.

$$\begin{aligned} v_1 \cdot v_2 &= \langle 5, 2 \rangle \cdot \langle 3, 7 \rangle \\ &= 5 \cdot 3 + 2 \cdot 7 \\ &= 15 + 14 \\ &= 29 \end{aligned}$$

To find the angle between them, we use the formula $\cos\theta = \frac{v}{|v|} \cdot \frac{u}{|u|}$.

$$\begin{aligned} \frac{v}{|v|} \cdot \frac{u}{|u|} &= \left\langle \frac{5}{\sqrt{29}} + \frac{2}{\sqrt{29}} \right\rangle \cdot \left\langle \frac{3}{\sqrt{58}} + \frac{7}{\sqrt{58}} \right\rangle \\ &= \frac{5}{\sqrt{29}} \cdot \frac{3}{\sqrt{58}} + \frac{2}{\sqrt{29}} \cdot \frac{7}{\sqrt{58}} \\ &= \frac{15}{\sqrt{1682}} + \frac{14}{\sqrt{1682}} \\ &= \frac{29}{\sqrt{1682}} \\ &= 0.707107 \end{aligned}$$

$$\cos^{-1}(0.707107) = 45^\circ$$

See Figure 12.3.18

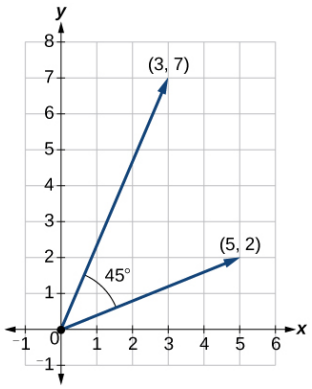


Figure 12.3.18

Example 12.3.11C: Finding the Angle between Two Vectors

Find the angle between $u = \langle -3, 4 \rangle$ and $v = \langle 5, 12 \rangle$.

Solution

Using the formula, we have

$$\theta = \cos^{-1}\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right)$$

$$\begin{aligned}\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right) &= \frac{-3i+4j}{5} \cdot \frac{5i+12j}{13} \\ &= \left(-\frac{3}{5} \cdot \frac{5}{13}\right) + \left(\frac{4}{5} \cdot \frac{12}{13}\right) \\ &= -\frac{15}{65} + \frac{48}{65} \\ &= \frac{33}{65}\end{aligned}$$

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{33}{65}\right) \\ &= 59.5^\circ\end{aligned}$$

See Figure 12.3.19

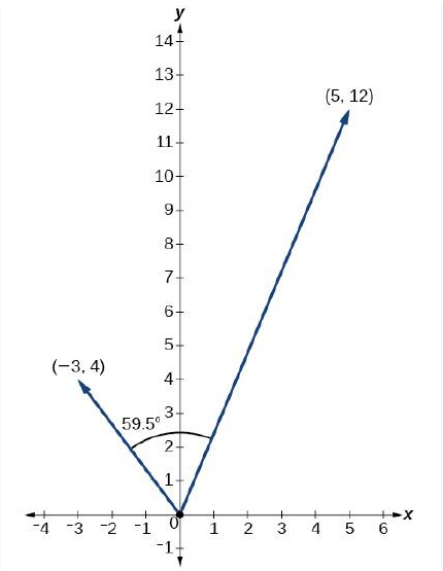


Figure 12.3.19

Example 12.3.11D: Finding Ground Speed and Bearing Using Vectors

We now have the tools to solve the problem we introduced in the opening of the section.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour. What are the ground speed and actual bearing of the plane? See Figure 12.3.20

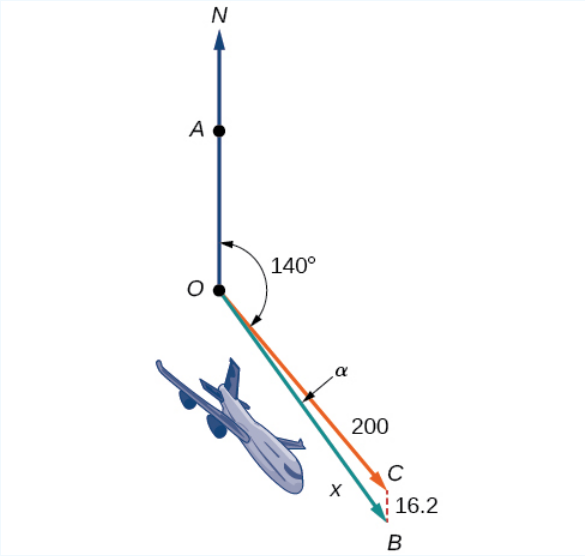


Figure 12.3.20

Solution

The ground speed is represented by x in the diagram, and we need to find the angle α in order to calculate the adjusted bearing, which will be $140^\circ + \alpha$.

Notice in Figure 12.3.20 that angle $\angle BCO$ must be equal to angle $\angle AOC$ by the rule of alternating interior angles, so angle $\angle BCO$ is 140°. We can find x by the Law of Cosines:

$$\begin{aligned} x^2 &= (16.2)^2 + (200)^2 - 2(16.2)(200)\cos(140^\circ) \\ x^2 &= 45,226.41 \\ x &= \sqrt{45,226.41} \\ x &= 212.7 \end{aligned}$$

The ground speed is approximately 213 miles per hour. Now we can calculate the bearing using the Law of Sines.

$$\begin{aligned} \frac{\sin\alpha}{16.2} &= \frac{\sin(140^\circ)}{212.7} \\ \sin\alpha &= \frac{16.2\sin(140^\circ)}{212.7} \\ &= 0.04896 \\ \sin^{-1}(0.04896) &= 2.8^\circ \end{aligned}$$

Therefore, the plane has a SE bearing of $140^\circ + 2.8^\circ = 142.8^\circ$. The ground speed is 212.7 miles per hour.

Media: Access these online resources for additional instruction and practice with vectors.

- Introduction to Vectors
- Vector Operations
- The Unit Vector

Key Concepts

- The position vector has its initial point at the origin. See Example 12.3.1.
- If the position vector is the same for two vectors, they are equal. See Example 12.3.2
- Vectors are defined by their magnitude and direction. See Example 12.3.3
- If two vectors have the same magnitude and direction, they are equal. See Example 12.3.4
- Vector addition and subtraction result in a new vector found by adding or subtracting corresponding elements. See Example 12.3.5.
- Scalar multiplication is multiplying a vector by a constant. Only the magnitude changes; the direction stays the same. See Example 12.3.6 and Example 12.3.7.
- Vectors are comprised of two components: the horizontal component along the positive x-axis, and the vertical component along the positive y-axis. See Example 12.3.8
- The unit vector in the same direction of any nonzero vector is found by dividing the vector by its magnitude.
- The magnitude of a vector in the rectangular coordinate system is $|v| = \sqrt{a^2 + b^2}$. See Example 12.3.9
- In the rectangular coordinate system, unit vectors may be represented in terms of ii and jj where i represents the horizontal component and j represents the vertical component. Then, $v = ai + bj$ is a scalar multiple of v by real numbers a and b. See Example 12.3.10 and Example 12.3.11.
- Adding and subtracting vectors in terms of i and j consists of adding or subtracting corresponding coefficients of i and corresponding coefficients of j. See Example 12.3.12
- A vector $v = ai + bj$ is written in terms of magnitude and direction as $v = |v|\cos\theta i + |v|\sin\theta j$. See Example 12.3.13
- The dot product of two vectors is the product of the i terms plus the product of the j terms. See Example 12.3.14
- We can use the dot product to find the angle between two vectors. Example 12.3.15 and Example 12.3.16
- Dot products are useful for many types of physics applications. See Example 12.3.17.

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12.4: Polar Coordinates

Learning Objectives

- Plot points using polar coordinates.
- Convert from polar coordinates to rectangular coordinates.
- Convert from rectangular coordinates to polar coordinates.
- Transform equations between polar and rectangular forms.
- Identify and graph polar equations by converting to rectangular equations.

Over 12 kilometers from port, a sailboat encounters rough weather and is blown off course by a 16-knot wind (see Figure 12.4.1). How can the sailor indicate his location to the Coast Guard? In this section, we will investigate a method of representing location that is different from a standard coordinate grid.

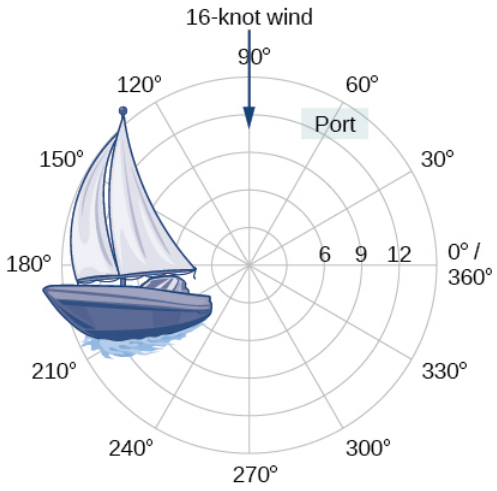
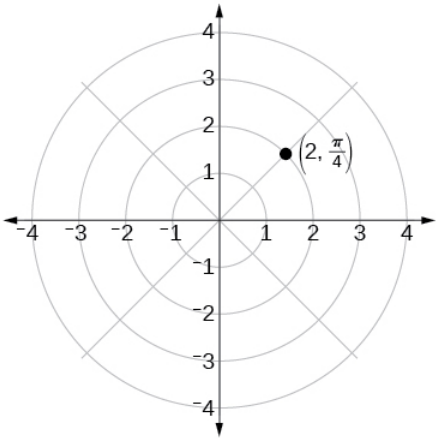


Figure 12.4.1

Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of *rectangular coordinates* (x, y) in the *Cartesian coordinate plane*. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to polar coordinates, which are points labeled (r, θ) and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the pole, or the origin of the coordinate plane.

The polar grid is scaled as the unit circle with the positive x-axis now viewed as the polar axis and the origin as the pole. The first coordinate r is the radius or length of the directed line segment from the pole. The angle θ, measured in radians, indicates the direction of r. We move counterclockwise from the polar axis by an angle of θ,and measure a directed line segment the length of r in the direction of θ. Even though we measure θ first and then r, the polar point is written with the r-coordinate first. For example, to plot the point $\left(2, \frac{\pi}{4}\right)$,we would move $\frac{\pi}{4}$ units in the counterclockwise direction and then a length of 2 from the pole. This point is plotted on the grid in Figure 12.4.2



Polar Grid

Figure 12.4.2

Example 12.4.1: Plotting a Point on the Polar Grid

Plot the point $\left(3, \frac{\pi}{2}\right)$ on the polar grid.

Solution

The angle $\frac{\pi}{2}$ is found by sweeping in a counterclockwise direction 90° from the polar axis. The point is located at a length of 3 units from the pole in the $\frac{\pi}{2}$ direction, as shown in Figure 12.4.3

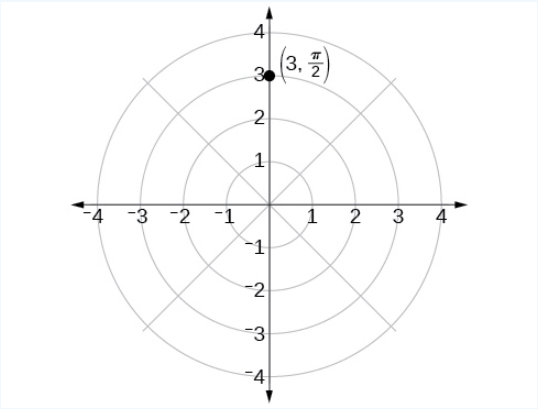


Figure 12.4.3

Exercise 12.4.1

Plot the point $\left(2, \frac{\pi}{3}\right)$ in the polar grid.

Answer

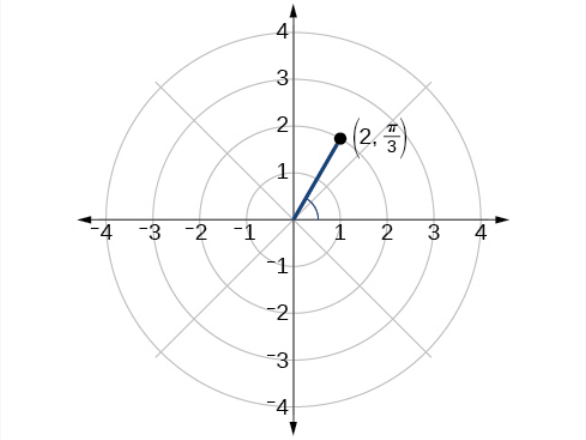


Figure 12.4.4

Example 12.4.2: Plotting a Point in the Polar Coordinate System with a Negative Component

Plot the point $\left(-2, \frac{\pi}{6}\right)$ on the polar grid.

Solution

We know that $\frac{\pi}{6}$ is located in the first quadrant. However, $r = -2$. We can approach plotting a point with a negative r in two ways:

1. Plot the point $\left(2, \frac{\pi}{6}\right)$ by moving $\frac{\pi}{6}$ in the counterclockwise direction and extending a directed line segment 2 units into the first quadrant. Then retrace the directed line segment back through the pole, and continue 2 units into the third quadrant;
2. Move $\frac{\pi}{6}$ in the counterclockwise direction, and draw the directed line segment from the pole 2 units in the negative direction, into the third quadrant.

See Figure $12.4.5a$. Compare this to the graph of the polar coordinate $(2, \pi 6)$ shown in Figure $12.4.5b$

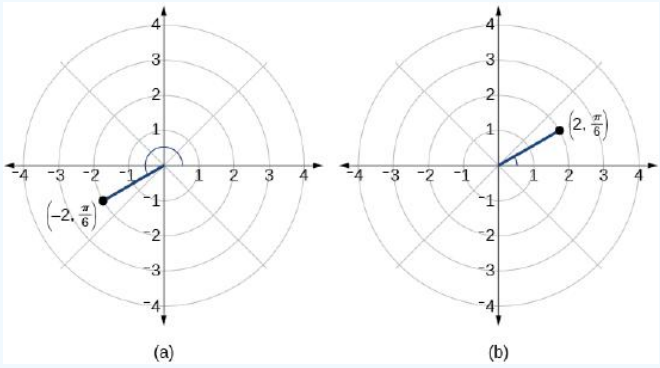


Figure 12.4.5

Exercise 12.4.2

Plot the points $\left(3, -\frac{\pi}{6}\right)$ and $\left(2, \frac{9\pi}{4}\right)$ on the same polar grid.

Answer

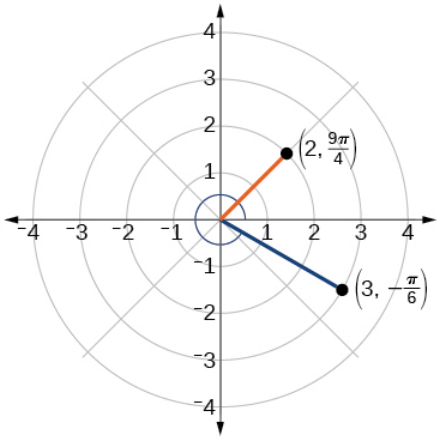


Figure 12.4.6

Converting from Polar Coordinates to Rectangular Coordinates

When given a set of **polar coordinates**, we may need to convert them to rectangular coordinates. To do so, we can recall the relationships that exist among the variables x, y, r, and θ.

$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$$
$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$$

Dropping a perpendicular from the point in the plane to the *x*-axis forms a right triangle, as illustrated in Figure 12.4.7. An easy way to remember the equations above is to think of $\cos\theta$ as the adjacent side over the hypotenuse and $\sin\theta$ as the opposite side over the hypotenuse.

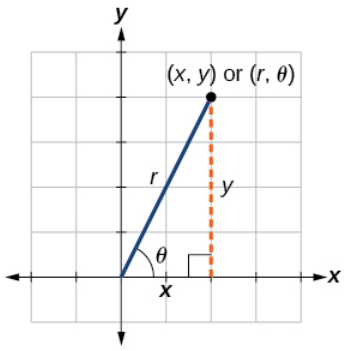


Figure 12.4.7

CONVERTING FROM POLAR COORDINATES TO RECTANGULAR COORDINATES

To convert polar coordinates (r, θ) to rectangular coordinates (x, y), let

$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta \quad (12.4.1)$$

$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta \quad (12.4.2)$$

How to: Given polar coordinates, convert to rectangular coordinates.

1. Given the polar coordinate (r, θ), write $x = r\cos\theta$ and $y = r\sin\theta$.
2. Evaluate $\cos\theta$ and $\sin\theta$.
3. Multiply $\cos\theta$ by r to find the x-coordinate of the rectangular form.
4. Multiply $\sin\theta$ by r to find the y-coordinate of the rectangular form.

Example 12.4.3A: Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $\left(3, \frac{\pi}{2}\right)$ as rectangular coordinates.

Solution

Use the equivalent relationships.

$$\begin{aligned} x &= r\cos\theta \\ x &= 3\cos\frac{\pi}{2} \\ &= 0 \\ y &= r\sin\theta \\ y &= 3\sin\frac{\pi}{2} \\ &= 3 \end{aligned}$$

The rectangular coordinates are $(0, 3)$. See Figure 12.4.8

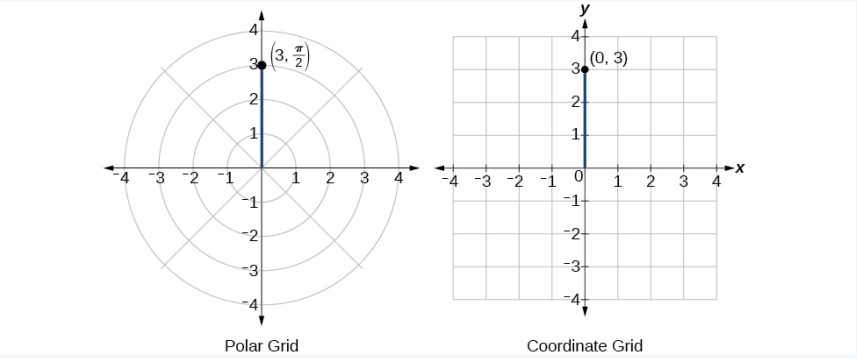


Figure 12.4.8

Example 12.4.3B: Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $(-2, 0)$ as rectangular coordinates.

Solution

See Figure 12.4.9 Writing the polar coordinates as rectangular, we have

$$\begin{aligned} x &= r\cos\theta \\ x &= -2\cos(0) \\ &= -2 \\ y &= r\sin\theta \\ y &= -2\sin(0) \\ &= 0 \end{aligned}$$

The rectangular coordinates are also $(-2, 0)$.

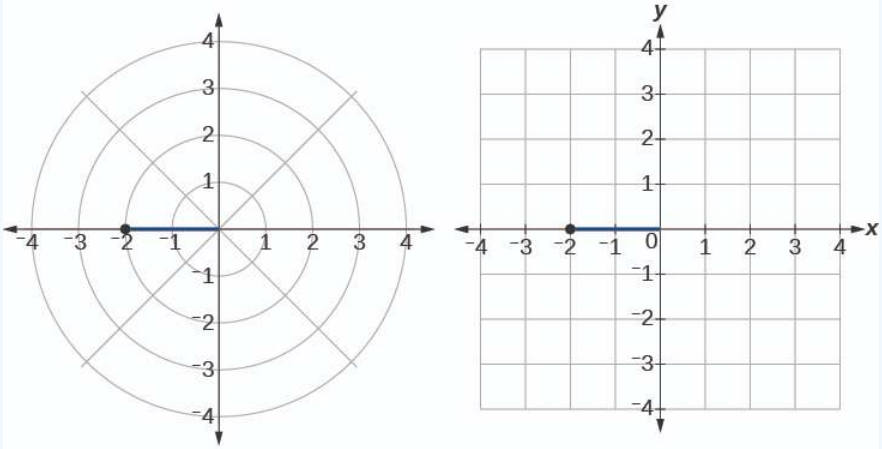


Figure 12.4.9

? Exercise 12.4.3

Write the polar coordinates $\left(-1, \frac{2\pi}{3}\right)$ as rectangular coordinates.

Answer

$$(x, y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships. With this conversion, however, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

CONVERTING FROM RECTANGULAR COORDINATES TO POLAR COORDINATES

Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in Figure 12.4.10

$$\cos\theta = \frac{x}{r} \text{ or } x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \text{ or } y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

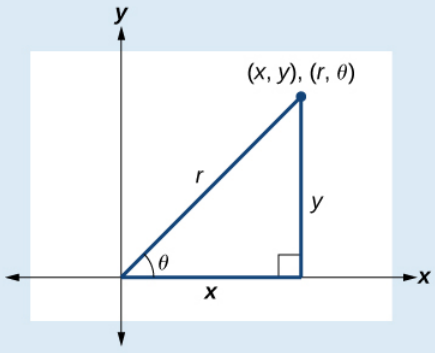


Figure 12.4.10

Example 12.4.4: Writing Rectangular Coordinates as Polar Coordinates

Convert the rectangular coordinates $(3, 3)$ to polar coordinates.

Solution

We see that the original point $(3, 3)$ is in the first quadrant. To find θ, use the formula $\tan\theta = \dfrac{y}{x}$. This gives

$$\begin{aligned} \tan\theta &= \frac{3}{3} \\ \tan\theta &= 1 \\ \tan^{-1}(1) &= \frac{\pi}{4} \end{aligned}$$

To find r, we substitute the values for x and y into the formula $r = \sqrt{x^2 + y^2}$. We know that r must be positive, as $\dfrac{\pi}{4}$ is in the first quadrant. Thus

$$\begin{aligned} r &= \sqrt{3^2 + 3^2} \\ r &= \sqrt{9 + 9} \\ r &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

So, $r = 3\sqrt{2}$ and $\theta = \dfrac{\pi}{4}$, giving us the polar point $(3\sqrt{2}, \dfrac{\pi}{4})$. See Figure 12.4.11

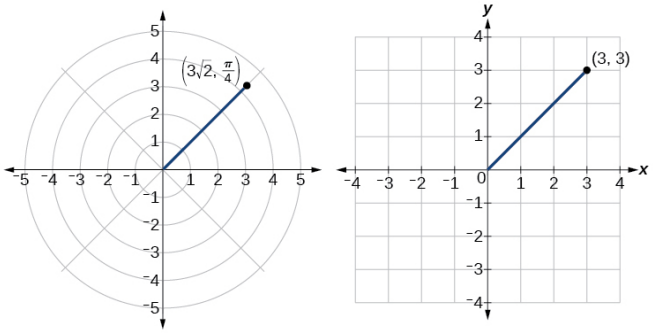


Figure 12.4.11

Analysis

There are other sets of polar coordinates that will be the same as our first solution. For example, the points $\left(-3\sqrt{2}, \dfrac{5\pi}{4}\right)$ and $\left(3\sqrt{2}, -\dfrac{7\pi}{4}\right)$ will coincide with the original solution of $\left(3\sqrt{2}, \dfrac{\pi}{4}\right)$. The point $\left(-3\sqrt{2}, \dfrac{5\pi}{4}\right)$ indicates a move further counterclockwise by π, which is directly opposite $\dfrac{\pi}{4}$. The radius is expressed as $-3\sqrt{2}$. However, the angle $\dfrac{5\pi}{4}$ is located in the third quadrant and, as r is negative, we extend the directed line segment in the opposite direction, into the first quadrant. This is the same point as $\left(3\sqrt{2}, \dfrac{\pi}{4}\right)$. The point $\left(3\sqrt{2}, -\dfrac{7\pi}{4}\right)$ is a move further clockwise by $-\dfrac{7\pi}{4}$, from $\dfrac{\pi}{4}$. The radius, $3\sqrt{2}$, is the same.

Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems. We can then use a graphing calculator to graph either the rectangular form or the polar form of the equation.

How to: Given an equation in polar form, graph it using a graphing calculator

1. Change the **MODE** to **POL**, representing polar form.
2. Press the **Y=** button to bring up a screen allowing the input of six equations: r_1, r_2,..., r_6.
3. Enter the polar equation, set equal to r.
4. Press **GRAPH**.

Example 12.4.5A: Writing a Cartesian Equation in Polar Form

Write the Cartesian equation $x^2+y^2=9$ in polar form.

Solution

The goal is to eliminate x and y from the equation and introduce r and θ. Ideally, we would write the equation r as a function of θ. To obtain the polar form, we will use the relationships between (x,y) and (r,θ). Since $x=r\cos\theta$ and $y=r\sin\theta$, we can substitute and solve for r.

$$\begin{aligned}(r\cos\theta)^2+(r\sin\theta)^2 &= 9\\ r^2\cos^2\theta+r^2\sin^2\theta &= 9\\ r^2(\cos^2\theta+\sin^2\theta) &= 9\\ r^2(1) &= 9 \qquad \text{Substitute } \cos^2\theta+\sin^2\theta=1\\ r &= \pm 3 \qquad \text{Use the square root property.}\end{aligned}$$

Thus, $x^2+y^2=9$,$r=3$,and $r=-3$ should generate the same graph. See Figure 12.4.12

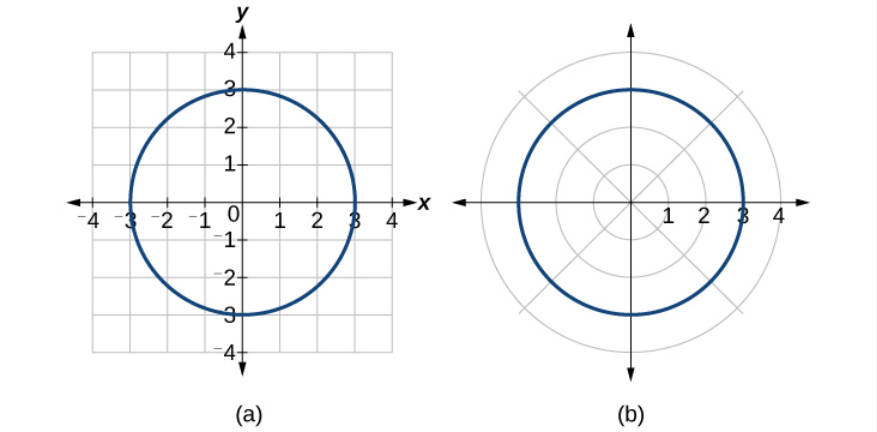


Figure 12.4.12: (a) Cartesian form $x^2+y^2=9$ (b) Polar form $r=3$

To graph a circle in rectangular form, we must first solve for y.

$$\begin{aligned}x^2+y^2 &= 9\\ y^2 &= 9-x^2\\ y &= \pm\sqrt{9-x^2}\end{aligned}$$

Note that this is two separate functions, since a circle fails the vertical line test. Therefore, we need to enter the positive and negative square roots into the calculator separately, as two equations in the form $Y_1=\sqrt{9-x^2}$ and $Y_2=-\sqrt{9-x^2}$. Press **GRAPH.**

Example 12.4.5B: Rewriting a Cartesian Equation as a Polar Equation

Rewrite the **Cartesian equation** $x^2+y^2=6y$ as a polar equation.

Solution

This equation appears similar to the previous example, but it requires different steps to convert the equation.

We can still follow the same procedures we have already learned and make the following substitutions:

$r^2 = 6y$ Use $x^2 + y^2 = r^2$.

$r^2 = 6r\sin\theta$ Substitute $y = r\sin\theta$.

$r^2 - 6r\sin\theta = 0$ Set equal to 0.

$r(r - 6\sin\theta) = 0$ Factor and solve.

$r = 0$ We reject $r = 0$, as it only represents one point, $(0, 0)$.

or $r = 6\sin\theta$

Therefore, the equations $x^2 + y^2 = 6y$ and $r = 6\sin\theta$ should give us the same graph. See Figure 12.4.13

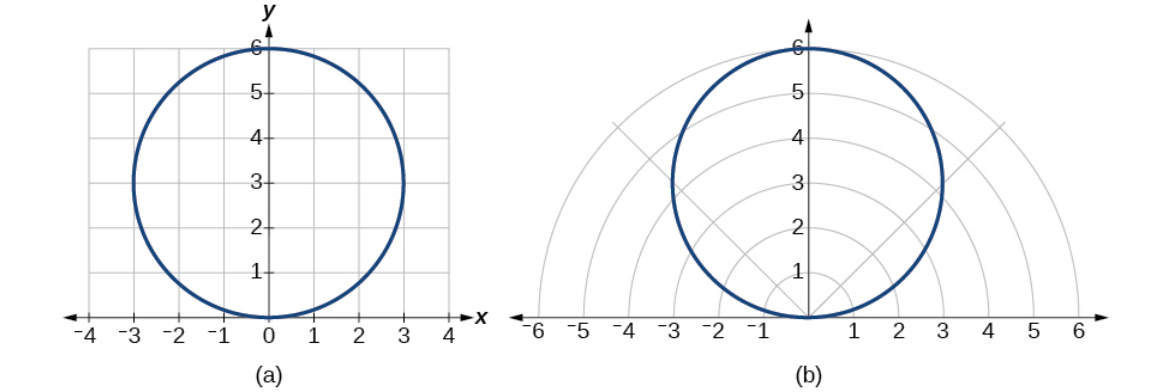


Figure 12.4.13: (a) Cartesian form $x^2 + y^2 = 6y$ (b) polar form $r = 6\sin\theta$

The Cartesian or rectangular equation is plotted on the rectangular grid, and the **polar equation** is plotted on the polar grid. Clearly, the graphs are identical.

? Exercise 12.4.4A:

Rewriting a Cartesian Equation in Polar Form

Rewrite the Cartesian equation $y = 3x + 2$ as a polar equation.

Answer

We will use the relationships $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} y &= 3x + 2 \\ r\sin\theta &= 3r\cos\theta + 2 \\ r\sin\theta - 3r\cos\theta &= 2 \\ r(\sin\theta - 3\cos\theta) &= 2 \quad \text{Isolate } r. \\ r &= \frac{2}{\sin\theta - 3\cos\theta} \quad \text{Solve for } r. \end{aligned}$$

? Exercise 12.4.4B:

Rewrite the Cartesian equation $y^2 = 3 - x^2$ in polar form.

Answer

$r = \sqrt{3}$

Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Example 12.4.6A: Graphing a Polar Equation by Converting to a Rectangular Equation

Covert the polar equation $r = 2\sec\theta$ to a rectangular equation, and draw its corresponding graph.

Solution

The conversion is

$$\begin{aligned} r &= 2\sec\theta \\ r &= \frac{2}{\cos\theta} \\ r\cos\theta &= 2 \\ x &= 2 \end{aligned}$$

Notice that the equation $r = 2\sec\theta$ drawn on the polar grid is clearly the same as the vertical line $x = 2$ drawn on the rectangular grid (see Figure 12.4.14). Just as $x = c$ is the standard form for a vertical line in rectangular form, $r = c\sec\theta$ is the standard form for a vertical line in polar form.

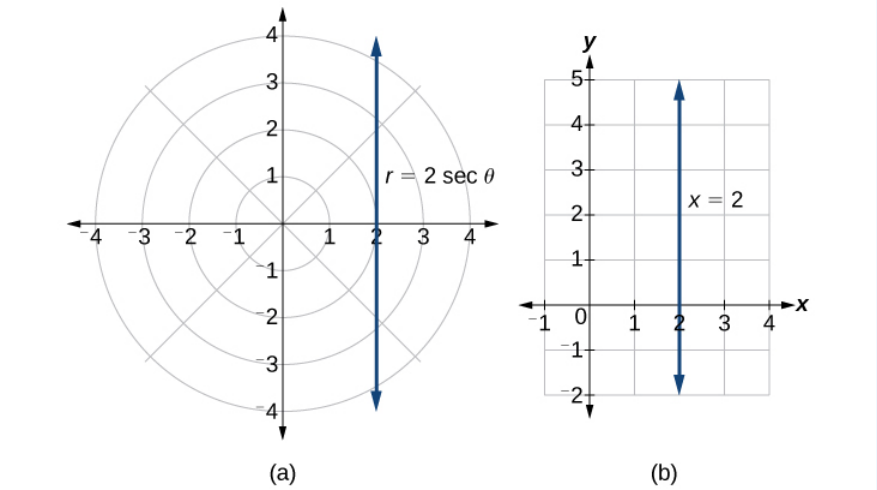


Figure 12.4.14: (a) Polar grid (b) Rectangular coordinate system

A similar discussion would demonstrate that the graph of the function $r = 2\csc\theta$ will be the horizontal line $y = 2$. In fact, $r = c\csc\theta$ is the standard form for a horizontal line in polar form, corresponding to the rectangular form $y = c$.

Example 12.4.6B: Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \dfrac{3}{1-2\cos\theta}$ as a Cartesian equation.

Solution

The goal is to eliminate θ and r,and introduce x and y. We clear the fraction, and then use substitution. In order to replace r with x and y, we must use the expression $x^2 + y^2 = r^2$.

$$\begin{aligned} &= \frac{3}{1-2\cos\theta} \\ r(1-2\cos\theta) &= 3 \\ r\left(1-2\left(\frac{x}{r}\right)\right) &= 3 \quad \text{Use } \cos\theta = \frac{x}{r} \text{ to eliminate } \theta. \\ r - 2x &= 3 \\ r &= 3+2x \quad \text{Isolate } r. \\ r^2 &= (3+2x)^2 \quad \text{Square both sides.} \\ x^2 + y^2 &= (3+2x)^2 \quad \text{Use } x^2+y^2 = r^2. \end{aligned}$$

The Cartesian equation is $x^2+y^2=(3+2x)^2$. However, to graph it, especially using a graphing calculator or computer program, we want to isolate y.

$$\begin{aligned} x^2+y^2 &= (3+2x)^2 \\ y^2 &= (3+2x)^2-x^2 \\ y &= \pm(3+2x)^2-x^2 \end{aligned}$$

When our entire equation has been changed from r and θ to x and y, we can stop, unless asked to solve for y or simplify. See Figure 12.4.15

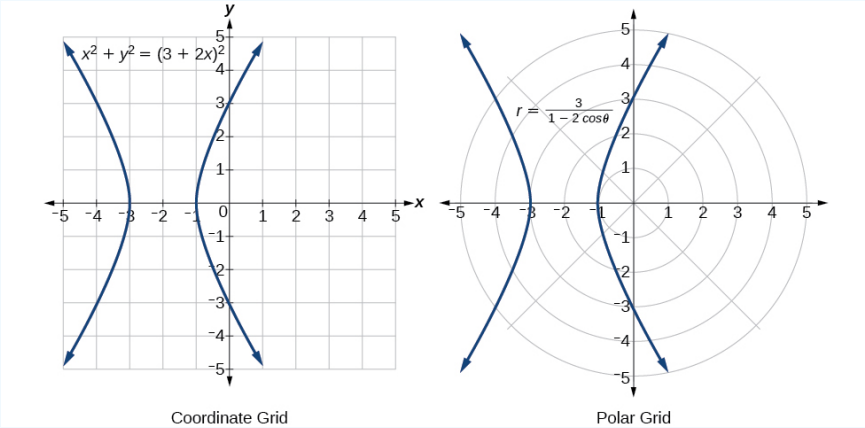


Figure 12.4.15

The "hour-glass" shape of the graph is called a *hyperbola*. Hyperbolas have many interesting geometric features and applications, which we will investigate further in Analytic Geometry.

Analysis

In this example, the right side of the equation can be expanded and the equation simplified further, as shown above. However, the equation cannot be written as a single function in Cartesian form. We may wish to write the rectangular equation in the hyperbola's standard form. To do this, we can start with the initial equation.

$$\begin{aligned} &x^2+y^2=(3+2x)^2 \\ &x^2+y^2-(3+2x)^2=0 \\ &x^2+y^2-(9+12x+4x^2)=0 \\ &x^2+y^2-9-12x-4x^2=0 \\ &-3x^2-12x+y^2=9 && \text{Multiply through by } -1. \\ &3x^2+12x-y^2=-9 \\ &3(x^2+4x)-y2=-9 && \text{Organize terms to complete the square for } x. \\ &3(x^2+4x+4)-y^2=-9+12 \\ &3(x+2)^2-y^2=3 \\ &(x+2)^2-\frac{y^2}{3}=1 \end{aligned}$$

? Exercise 12.4.5

Rewrite the polar equation $r=2\sin\theta$ in Cartesian form.

Answer

$x^2+y^2=2y$ or, in the standard form for a circle, $x^2+(y-1)^2=1$

Example 12.4.7: Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian form.

Solution

$$
\begin{aligned}
r &= \sin(2\theta) && \text{Use the double angle identity for sine.}\\
r &= 2\sin\theta\cos\theta && \text{Use } \cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}.\\
r &= 2\frac{x}{r})(\frac{y}{r}) && \text{Simplify.}\\
r &= \frac{2xy}{r^2} && \text{Multiply both sides by } r^2.\\
r^3 &= 2xy &&\\
(x^2+y^2)^{\,3} &= 2xy && \text{As } x^2+y^2 = r^2, r = \sqrt{x^2+y^2}.
\end{aligned}
$$

This equation can also be written as

$$(x^2+y^2)^{\frac{3}{2}} = 2xy \text{ or } x^2+y^2 = (2xy)^{\frac{2}{3}}$$

Media

Access these online resources for additional instruction and practice with polar coordinates.

- Introduction to Polar Coordinates
- Comparing Polar and Rectangular Coordinates

Key Equations

Conversion formulas	$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$ $\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$ $r^2 = x^2 + y^2$ $\tan\theta = \frac{y}{x}$

Key Concepts

- The polar grid is represented as a series of concentric circles radiating out from the pole, or origin.
- To plot a point in the form (r, θ), $\theta > 0$, move in a counterclockwise direction from the polar axis by an angle of θ, and then extend a directed line segment from the pole the length of r in the direction of θ. If θ is negative, move in a clockwise direction, and extend a directed line segment the length of r in the direction of θ. See Example 12.4.1.
- If r is negative, extend the directed line segment in the opposite direction of θ. See Example 12.4.2
- To convert from polar coordinates to rectangular coordinates, use the formulas $x = r\cos\theta$ and $y = r\sin\theta$. See Example 12.4.3 and Example 12.4.4
- To convert from rectangular coordinates to polar coordinates, use one or more of the formulas: $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, $\tan\theta = \frac{y}{x}$, and $r = \sqrt{x^2+y^2}$. See Example 12.4.5.
- Transforming equations between polar and rectangular forms means making the appropriate substitutions based on the available formulas, together with algebraic manipulations. See Example 12.4.6, Example 12.4.7, and Example 12.4.8.
- Using the appropriate substitutions makes it possible to rewrite a polar equation as a rectangular equation, and then graph it in the rectangular plane. See Example 12.4.9, Example 12.4.10, and Example 12.4.11.

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12.5: Polar Coordinates - Graphs

Learning Objectives

- Test polar equations for symmetry.
- Graph polar equations by plotting points.

Kepler's First Law of Planetary Motion argues that the planets move through space in elliptical, periodic orbits about the sun, as shown in Figure 12.5.1. They are in constant motion, so fixing an exact position of any planet is valid only for a moment. In other words, we can fix only a planet's instantaneous position. This is one application of polar coordinates, represented as (r, θ). We interpret r as the distance from the sun and θ as the planet's angular bearing, or its direction from a fixed point on the sun. In this section, we will focus on the polar system and the graphs that are generated directly from polar coordinates.

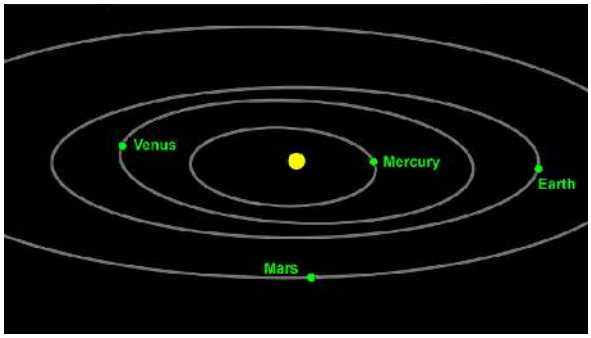


Figure 12.5.1: Planets follow elliptical paths as they orbit around the Sun. (credit: modification of work by NASA/JPL-Caltech)

Testing Polar Equations for Symmetry

Just as a rectangular equation such as $y = x^2$ describes the relationship between x and y on a Cartesian grid, a **polar equation** describes a relationship between r and θ on a polar grid. Recall that the coordinate pair (r, θ) indicates that we move counterclockwise from the polar axis (positive x-axis) by an angle of θ, and extend a ray from the pole (origin) r units in the direction of θ. All points that satisfy the polar equation are on the graph.

Symmetry is a property that helps us recognize and plot the graph of any equation. If an equation has a graph that is symmetric with respect to an axis, it means that if we folded the graph in half over that axis, the portion of the graph on one side would coincide with the portion on the other side. By performing three tests, we will see how to apply the properties of symmetry to polar equations. Further, we will use symmetry (in addition to plotting key points, zeros, and maximums of r) to determine the graph of a polar equation.

In the first test, we consider symmetry with respect to the line $\theta = \frac{\pi}{2}$ (y-axis). We replace (r, θ) with $(-r, -\theta)$ to determine if the new equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin\theta$;

$$\begin{aligned} r &= 2\sin\theta \\ -r &= 2\sin -\theta \quad \text{Replace } (r, \theta) \text{ with } (-r, -\theta). \\ -r &= -2\sin\theta \quad \text{Identity: } \sin(-\theta) = -\sin\theta. \\ r &= 2\sin\theta \quad \text{Multiply both sides by } -1 \end{aligned}$$

This equation exhibits symmetry with respect to the line $\theta = \frac{\pi}{2}$.

In the second test, we consider symmetry with respect to the polar axis (x-axis). We replace (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ to determine equivalency between the tested equation and the original. For example, suppose we are given the equation $r = 1 - 2\cos\theta$.

$$\begin{aligned} r &= 1 - 2\cos\theta \\ r &= 1 - 2\cos(-\theta) \quad \text{Replace } (r, \theta) \text{ with } (r, -\theta). \\ r &= 1 - 2\cos\theta \quad \text{Even/Odd identity} \end{aligned}$$

The graph of this equation exhibits symmetry with respect to the polar axis.

In the third test, we consider symmetry with respect to the pole (origin). We replace (r, θ) with $(-r, \theta)$ to determine if the tested equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin(3\theta)$.

$$\begin{aligned} r &= 2\sin(3\theta) \\ -r &= 2\sin(3\theta) \end{aligned}$$

The equation has failed the symmetry test, but that does not mean that it is not symmetric with respect to the pole. Passing one or more of the symmetry tests verifies that symmetry will be exhibited in a graph. However, failing the symmetry tests does not necessarily indicate that a graph will not be symmetric about the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole. In these instances, we can confirm that symmetry exists by plotting reflecting points across the apparent axis of symmetry or the pole. Testing for symmetry is a technique that simplifies the graphing of polar equations, but its application is not perfect.

Note: SYMMETRY TESTS

A polar equation describes a curve on the polar grid. The graph of a polar equation can be evaluated for three types of symmetry, as shown in Figure 12.5.2

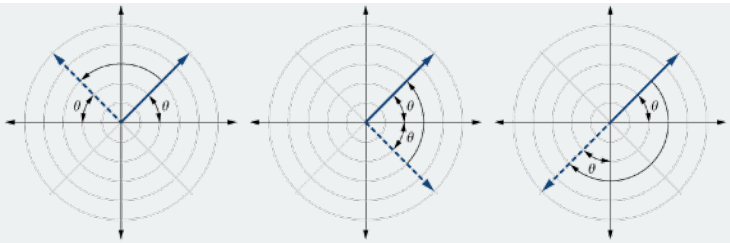


Figure 12.5.2: (a) A graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$ (y-axis) if replacing (r, θ) with $(-r, -\theta)$ yields an equivalent equation. (b) A graph is symmetric with respect to the polar axis (x-axis) if replacing (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ yields an equivalent equation. (c) A graph is symmetric with respect to the pole (origin) if replacing (r, θ) with $(-r, \theta)$ yields an equivalent equation.

How to: Given a polar equation, test for symmetry.

1. Substitute the appropriate combination of components for (r, θ): $(-r, -\theta)$ for $\theta = \frac{\pi}{2}$ symmetry; $(r, -\theta)$ for polar axis symmetry; and $(-r, \theta)$ for symmetry with respect to the pole.
2. If the resulting equations are equivalent in one or more of the tests, the graph produces the expected symmetry.

Example 12.5.1: Testing a Polar Equation for Symmetry

Test the equation $r = 2\sin\theta$ for symmetry.

Solution

Test for each of the three types of symmetry.

1) Replacing (r, θ) with $(-r, -\theta)$ yields the same result. Thus, the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.	$-r = 2\sin(-\theta)$ $-r = -2\sin\theta$ Even-odd identity $r = 2\sin\theta$ Multiply by -1 Passed
2) Replacing θ with $-\theta$ does not yield the same equation. Therefore, the graph fails the test and may or may not be symmetric with respect to the polar axis.	$r = 2\sin(-\theta)$ $r = -2\sin\theta$ Even-odd identity $r = -2\sin\theta \neq 2\sin\theta$ Failed
3) Replacing r with $-r$ changes the equation and fails the test. The graph may or may not be symmetric with respect to the pole.	$-r = 2\sin\theta$ $r = -2\sin\theta \neq 2\sin\theta$ Failed

Analysis

Using a graphing calculator, we can see that the equation $r = 2\sin\theta$ is a circle centered at $(0, 1)$ with radius $r = 1$ and is indeed symmetric to the line $\theta = \frac{\pi}{2}$. We can also see that the graph is not symmetric with the polar axis or the pole. See Figure 12.5.3

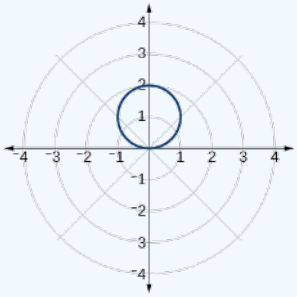


Figure 12.5.3

Exercise 12.5.1

Test the equation for symmetry: $r = -2\cos\theta$.

Answer

The equation fails the symmetry test with respect to the line $\theta = \frac{\pi}{2}$ and with respect to the pole. It passes the polar axis symmetry test.

Graphing Polar Equations by Plotting Points

To graph in the rectangular coordinate system we construct a table of x and y values. To graph in the polar coordinate system we construct a table of θ and r values. We enter values of θ into a polar equation and calculate r. However, using the properties of symmetry and finding key values of θ and r means fewer calculations will be needed.

Finding Zeros and Maxima

To find the zeros of a polar equation, we solve for the values of θ that result in $r = 0$. Recall that, to find the zeros of polynomial functions, we set the equation equal to zero and then solve for x. We use the same process for polar equations. Set $r = 0$, and solve for θ.

For many of the forms we will encounter, the maximum value of a polar equation is found by substituting those values of θ into the equation that result in the maximum value of the trigonometric functions. Consider $r = 5\cos\theta$; the maximum distance between the curve and the pole is 5 units. The maximum value of the cosine function is 1 when $\theta = 0$, so our polar equation is $5\cos\theta$, and the value $\theta = 0$ will yield the maximum $|r|$.

Similarly, the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, and if our polar equation is $r = 5\sin\theta$, the value $\theta = \frac{\pi}{2}$ will yield the maximum $|r|$. We may find additional information by calculating values of r when $\theta = 0$. These points would be polar axis intercepts, which may be helpful in drawing the graph and identifying the curve of a polar equation.

Example 12.5.2: Finding Zeros and Maximum Values for a Polar Equation

Using the equation in Example 12.5.1, find the zeros and maximum $|r|$ and, if necessary, the polar axis intercepts of $r = 2\sin\theta$.

Solution

To find the zeros, set r equal to zero and solve for θ.

$$\begin{aligned} 2\sin\theta &= 0 \\ \sin\theta &= 0 \\ \theta &= \sin^{-1} 0 \\ \theta &= n\pi \qquad \text{where n is an integer} \end{aligned}$$

Substitute any one of the θ values into the equation. We will use 0.

$$\begin{aligned} r &= 2\sin(0) \\ r &= 0 \end{aligned}$$

The points $(0, 0)$ and $(0, \pm n\pi)$ are the zeros of the equation. They all coincide, so only one point is visible on the graph. This point is also the only polar axis intercept.

To find the maximum value of the equation, look at the maximum value of the trigonometric function $\sin\theta$, which occurs when $\theta = \frac{\pi}{2} \pm 2k\pi$ resulting in $\sin\left(\frac{\pi}{2}\right) = 1$. Substitute $\frac{\pi}{2}$ for θ.

$$\begin{aligned} r &= 2\sin\left(\frac{\pi}{2}\right) \\ r &= 2(1) \\ r &= 2 \end{aligned}$$

Analysis

The point $\left(2, \frac{\pi}{2}\right)$ will be the maximum value on the graph. Let's plot a few more points to verify the graph of a circle. See Table 12.5.1 and Figure 12.5.4

Table 12.5.1

θ	$r = 2\sin\theta$	r
0	$r = 2\sin(0) = 0$	0
$\frac{\pi}{6}$	$r = 2\sin\left(\frac{\pi}{6}\right) = 1$	1
$\frac{\pi}{3}$	$r = 2\sin\left(\frac{\pi}{3}\right) \approx 1.73$	1.73
$\frac{pi}{2}$	$r = 2\sin\left(\frac{\pi}{2}\right) = 2$	2
$\frac{2\pi}{3}$	$r = 2\sin\left(\frac{2\pi}{3}\right) \approx 1.73$	1.73
$\frac{5\pi}{6}$	$r = 2\sin\left(\frac{5\pi}{6}\right) = 1$	1
π	$r = 2\sin(\pi) = 0$	0

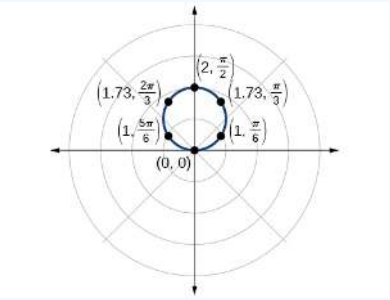


Figure 12.5.4

Exercise 12.5.2

Without converting to Cartesian coordinates, test the given equation for symmetry and find the zeros and maximum values of $|r|$: $r = 3\cos\theta$.

Answer

Tests will reveal symmetry about the polar axis. The zero is $\left(0, \frac{\pi}{2}\right)$, and the maximum value is $(3, 0)$.

Investigating Circles

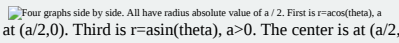
Now we have seen the equation of a circle in the polar coordinate system. In the last two examples, the same equation was used to illustrate the properties of symmetry and demonstrate how to find the zeros, maximum values, and plotted points that produced the graphs. However, the circle is only one of many shapes in the set of polar curves.

There are five classic polar curves**: cardioids**, **limaçons, lemniscates, rose curves**, and **Archimedes' spirals**. We will briefly touch on the polar formulas for the circle before moving on to the classic curves and their variations.

FORMULAS FOR THE EQUATION OF A CIRCLE

Some of the formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$, where a is the diameter of the circle or the distance from the pole to the farthest point on the circumference. The radius is $\frac{|a|}{2}$, or one-half the diameter. For $r = a\cos\theta$, the center is $\left(\frac{a}{2}, 0\right)$. For $r = a\sin\theta$, the center is $\left(\frac{a}{2}, \pi\right)$. Figure 12.5.5 shows the graphs of

these four circles.



0. The center is at (a/2,0). Second is r=acos(theta), a<0. The center is at (a/2,0). Third is r=asin(theta), a>0. The center is at (a/2, pi). Fourth is r=asin(theta), a<0. The center is at (a/2, pi)." src="/api/deki/files/7429/CNX_Precalc_Figure_08_04_005new.jpg">

Figure 12.5.5

Example 12.5.3: Sketching the Graph of a Polar Equation for a Circle

Sketch the graph of $r = 4\cos\theta$.

Solution

First, testing the equation for symmetry, we find that the graph is symmetric about the polar axis. Next, we find the zeros and maximum $|r|$ for $r = 4\cos\theta$. First, set $r = 0$, and solve for θ. Thus, a zero occurs at $\theta = \frac{\pi}{2} \pm k\pi$. A key point to plot is $\left(0, \frac{\pi}{2}\right)$.

To find the maximum value of r, note that the maximum value of the cosine function is 1 when $\theta = 0 \pm 2k\pi$. Substitute $\theta = 0$ into the equation:

$$\begin{aligned} r &= 4\cos\theta \\ r &= 4\cos(0) \\ r &= 4(1) \\ &= 4 \end{aligned}$$

The maximum value of the equation is 4. A key point to plot is $(4, 0)$.

As $r = 4\cos\theta$ is symmetric with respect to the polar axis, we only need to calculate r-values for θ over the interval $[0, \pi]$. Points in the upper quadrant can then be reflected to the lower quadrant. Make a table of values similar to Table 12.5.2 The graph is shown in Figure 12.5.6

Table 12.5.2

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	3.46	2.83	2	0	-2	-2.83	-3.46	4

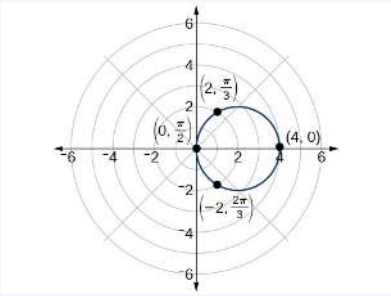


Figure 12.5.6

Investigating Cardioids

While translating from polar coordinates to Cartesian coordinates may seem simpler in some instances, graphing the classic curves is actually less complicated in the polar system. The next curve is called a cardioid, as it resembles a heart. This shape is often included with the family of curves called limaçons, but here we will discuss the cardioid on its own.

FORMULAS FOR A CARDIOID

The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $\frac{a}{b} = 1$. The cardioid graph passes through the pole, as we can see in Figure 12.5.7.

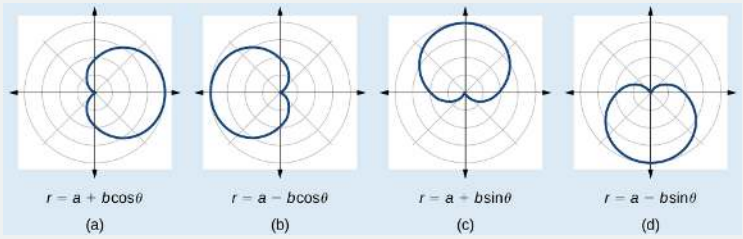


Figure 12.5.7

How to: Given the polar equation of a cardioid, sketch its graph

1. Check equation for the three types of symmetry.
2. Find the zeros. Set $r = 0$.
3. Find the maximum value of the equation according to the maximum value of the trigonometric expression.
4. Make a table of values for r and θ.
5. Plot the points and sketch the graph.

Example 12.5.4: Sketching the Graph of a Cardioid

Sketch the graph of $r = 2 + 2\cos\theta$.

Solution

First, testing the equation for symmetry, we find that the graph of this equation will be symmetric about the polar axis. Next, we find the zeros and maximums. Setting $r = 0$, we have $\theta = \pi + 2k\pi$. The zero of the equation is located at $(0, \pi)$. The graph passes through this point.

The maximum value of $r = 2 + 2\cos\theta$ occurs when $\cos\theta$ is a maximum, which is when $\cos\theta = 1$ or when $\theta = 0$. Substitute $\theta = 0$ into the equation, and solve for r.

$$\begin{aligned} r &= 2 + 2\cos(0) \\ r &= 2 + 2(1) \\ &= 4 \end{aligned}$$

The point $(4, 0)$ is the maximum value on the graph.

We found that the polar equation is symmetric with respect to the polar axis, but as it extends to all four quadrants, we need to plot values over the interval $[0, \pi]$. The upper portion of the graph is then reflected over the polar axis. Next, we make a table of values, as in Table 12.5.3, and then we plot the points and draw the graph. See Figure 12.5.8

Table 12.5.3

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	3.41	2	1	0

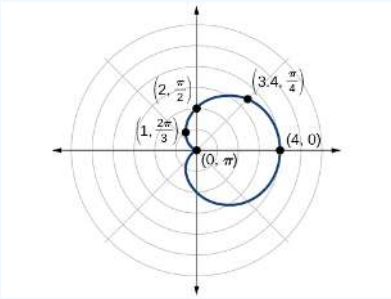


Figure 12.5.8

Investigating Limaçons

The word *limaçon* is Old French for "snail," a name that describes the shape of the graph. As mentioned earlier, the cardioid is a member of the limaçon family, and we can see the similarities in the graphs. The other images in this category include the one-loop limaçon and the two-loop (or inner-loop) limaçon. **One-loop limaçons** are sometimes referred to as dimpled limaçons when $1 < \frac{a}{b} < 2$ and convex limaçons when $\frac{a}{b} \geq 2$.

FORMULAS FOR ONE-LOOP LIMAÇONS

The formulas that produce the graph of a dimpled one-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $1 < ab < 2$. All four graphs are shown in Figure 12.5.9

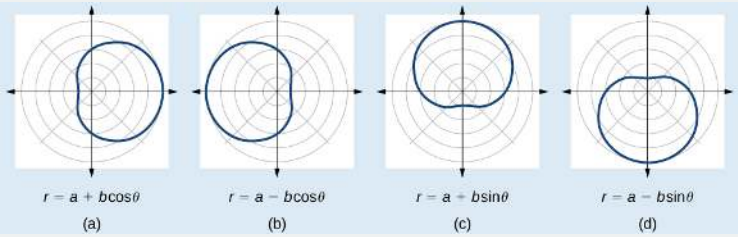


Figure 12.5.9

How to: Given a polar equation for a one-loop limaçon, sketch the graph

1. Test the equation for symmetry. Remember that failing a symmetry test does not mean that the shape will not exhibit symmetry. Often the symmetry may reveal itself when the points are plotted.
2. Find the zeros.
3. Find the maximum values according to the trigonometric expression.
4. Make a table.
5. Plot the points and sketch the graph.

Example 12.5.5: Sketching the Graph of a One-Loop Limaçon

Graph the equation $r = 4 - 3\sin\theta$.

Solution

First, testing the equation for symmetry, we find that it fails all three symmetry tests, meaning that the graph may or may not exhibit symmetry, so we cannot use the symmetry to help us graph it. However, this equation has a graph that clearly displays symmetry with respect to the line $\theta = \frac{\pi}{2}$, yet it fails all the three symmetry tests. A graphing calculator will immediately illustrate the graph's reflective quality.

Next, we find the zeros and maximum, and plot the reflecting points to verify any symmetry. Setting $r = 0$ results in θ being undefined. What does this mean? How could θ be undefined? The angle θ is undefined for any value of $\sin\theta > 1$. Therefore, θ is undefined because there is no value of θ for which $\sin\theta > 1$. Consequently, the graph does not pass through the pole. Perhaps the graph does cross the polar axis, but not at the pole. We can investigate other intercepts by calculating r when $\theta = 0$.

$$\begin{aligned} r(0) &= 4 - 3\sin(0) \\ r &= 4 - 3 \cdot 0 \\ &= 4 \end{aligned}$$

So, there is at least one polar axis intercept at $(4, 0)$.

Next, as the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, we will substitute $\theta = \frac{\pi}{2}$ into the equation and solve for r. Thus, $r = 1$.

Make a table of the coordinates similar to Table 12.5.4

Table 12.5.4

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	4	2.5	1.4	1	1.4	2.5	4	5.5	6.6	7	6.6	5.5	4

The graph is shown in Figure 12.5.10

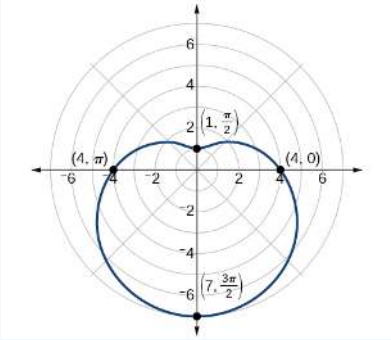


Figure 12.5.10: One-loop limaçon

Analysis

This is an example of a curve for which making a table of values is critical to producing an accurate graph. The symmetry tests fail; the zero is undefined. While it may be apparent that an equation involving $\sin\theta$ is likely symmetric with respect to the line $\theta = \frac{\pi}{2}$, evaluating more points helps to verify that the graph is correct.

Exercise 12.5.3

Sketch the graph of $r = 3 - 2\cos\theta$.

Answer

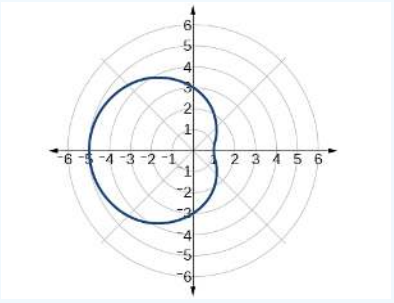


Figure 12.5.11

Another type of limaçon, the **inner-loop limaçon**, is named for the loop formed inside the general limaçon shape. It was discovered by the German artist Albrecht Dürer(1471-1528), who revealed a method for drawing the inner-loop limaçon in his 1525 book *Underweysung der Messing*. A century later, the father of mathematician Blaise Pascal, Étienne Pascal(1588-1651), rediscovered it.

FORMULAS FOR INNER-LOOP LIMAÇONS

The formulas that generate the inner-loop limaçons are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $a < b$. The graph of the inner-loop limaçon passes through the pole twice: once for the outer loop, and once for the inner loop. See Figure 10.5.12 for the graphs.

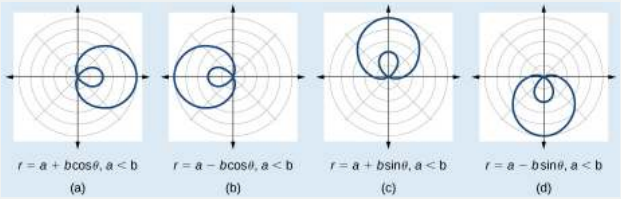


Figure 12.5.12

Example 12.5.6: Sketching the Graph of an Inner-Loop Limaçon

Sketch the graph of $r = 2 + 5\cos\theta$.

Solution

Testing for symmetry, we find that the graph of the equation is symmetric about the polar axis. Next, finding the zeros reveals that when $r = 0$, $\theta = 1.98$. The maximum $|r|$ is found when $\cos\theta = 1$ or when $\theta = 0$. Thus, the maximum is found at the point $(7, 0)$.

Even though we have found symmetry, the zero, and the maximum, plotting more points will help to define the shape, and then a pattern will emerge. See Table 12.5.5

Table 12.5.5

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

r	7	6.3	4.5	2	-0.5	-2.3	-3	-2.3	-0.5	2	4.5	6.3	7

As expected, the values begin to repeat after $\theta = \pi$. The graph is shown in Figure 12.5.13

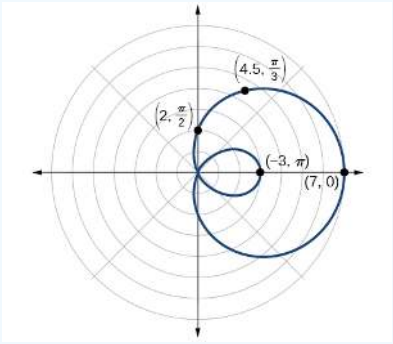


Figure 12.5.13: Inner-loop limaçon

Investigating Lemniscates

The lemniscate is a polar curve resembling the infinity symbol ∞ or a figure 8. Centered at the pole, a lemniscate is symmetrical by definition.

FORMULAS FOR LEMNISCATES

The formulas that generate the graph of a lemniscate are given by $r^2 = a^2 \cos 2\theta$ and $r^2 = a^2 \sin 2\theta$ where $a \neq 0$. The formula $r^2 = a^2 \sin 2\theta$ is symmetric with respect to the pole. The formula $r^2 = a^2 \cos 2\theta$ is symmetric with respect to the pole, the line $\theta = \dfrac{\pi}{2}$, and the polar axis. See Figure 12.5.14for the graphs.

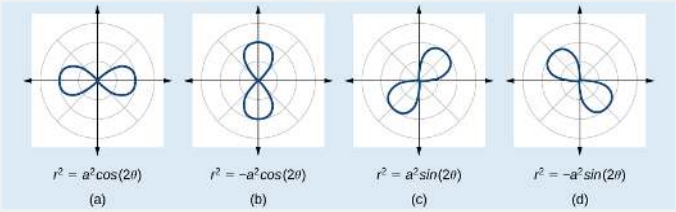


Figure 12.5.14

Example 12.5.7: Sketching the Graph of a Lemniscate

Sketch the graph of $r^2 = 4 \cos 2\theta$.

Solution

The equation exhibits symmetry with respect to the line $\theta = \dfrac{\pi}{2}$, the polar axis, and the pole.

Let's find the zeros. It should be routine by now, but we will approach this equation a little differently by making the substitution $u = 2\theta$.

$$\begin{aligned} 0 &= 4\cos 2\theta \\ 0 &= 4\cos u \\ 0 &= \cos u \\ \cos^{-1} 0 &= \frac{\pi}{2} \\ u &= \frac{\pi}{2} \qquad \text{Substitute } 2\theta \text{ back in for } u. \\ 2\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{4} \end{aligned}$$

So, the point $\left(0, \dfrac{\pi}{4}\right)$ is a zero of the equation.

Now let's find the maximum value. Since the maximum of $\cos u = 1$ when $u = 0$, the maximum $\cos 2\theta = 1$ when $2\theta = 0$. Thus,

$$\begin{aligned} r^2 &= 4\cos(0) \\ r^2 &= 4(1) \\ r^2 &= 4 \\ r &= \pm 4 \\ &= 2 \end{aligned}$$

We have a maximum at $(2, 0)$. Since this graph is symmetric with respect to the pole, the line $\theta = \dfrac{\pi}{2}$, and the polar axis, we only need to plot points in the first quadrant.

Make a table similar to Table 12.5.6

Table 12.5.6

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2	$\sqrt{2}$	0	$\sqrt{2}$	0

Plot the points on the graph, such as the one shown in Figure 12.5.15

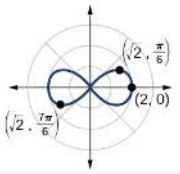


Figure 12.5.15: Lemniscate

Analysis

Making a substitution such as $u = 2\theta$ is a common practice in mathematics because it can make calculations simpler. However, we must not forget to replace the substitution term with the original term at the end, and then solve for the unknown.

Some of the points on this graph may not show up using the Trace function on the TI-84 graphing calculator, and the calculator table may show an error for these same points of r. This is because there are no real square roots for these values of θ. In other words, the corresponding r-values of $\sqrt{4\cos(2\theta)}$ are complex numbers because there is a negative number under the radical.

Investigating Rose Curves

The next type of polar equation produces a petal-like shape called a rose curve. Although the graphs look complex, a simple polar equation generates the pattern.

ROSE CURVES

The formulas that generate the graph of a **rose curve** are given by $r = a\cos n\theta$ and $r = a\sin n\theta$ where $a \neq 0$. If n is even, the curve has $2n$ petals. If n is odd, the curve has n petals. See Figure 12.5.16

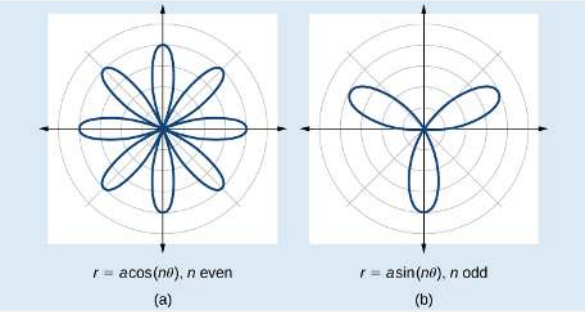


Figure 12.5.16

Example 12.5.8: Sketching the Graph of a Rose Curve (n Even)

Sketch the graph of $r = 2\cos 4\theta$.

Solution

Testing for symmetry, we find again that the symmetry tests do not tell the whole story. The graph is not only symmetric with respect to the polar axis, but also with respect to the line $\theta = \frac{\pi}{2}$ and the pole.

Now we will find the zeros. First make the substitution $u = 4\theta$.

$$\begin{aligned} 0 &= 2\cos 4\theta \\ 0 &= \cos 4\theta \\ 0 &= \cos u \\ \cos^{-1} 0 &= u \\ u &= \frac{\pi}{2} \\ 4\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{8} \end{aligned}$$

The zero is $\theta = \frac{\pi}{8}$. The point $\left(0, \frac{\pi}{8}\right)$ is on the curve.

Next, we find the maximum $|r|$. We know that the maximum value of $\cos u = 1$ when $\theta = 0$. Thus,

$$\begin{aligned} r &= 2\cos(4 \cdot 0) \\ r &= 2\cos(0) \\ r &= 2(1) \\ &= 2 \end{aligned}$$

The point $(2, 0)$ is on the curve.

The graph of the rose curve has unique properties, which are revealed in Table 12.5.7.

Table 12.5.7

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$5\pi 8$	$3\pi 4$
r	2	0	-2	0	2	0	-2

As $r = 0$ when $\theta = \frac{\pi}{8}$, it makes sense to divide values in the table by $\frac{\pi}{8}$ units. A definite pattern emerges. Look at the range of r-values: $2, 0, -2, 0$ and so on. This represents the development of the curve one petal at a time. Starting at $r = 0$, each petal extends out a distance of $r = 2$, and then turns back to zero $2n$ times for a total of eight petals. See the graph in Figure 12.5.17.

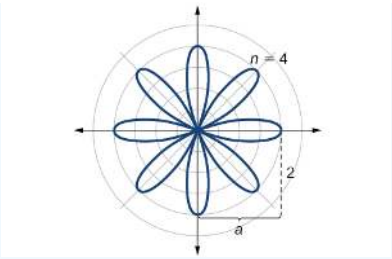


Figure 12.5.17: Rose curve, n even

Analysis

When these curves are drawn, it is best to plot the points in order, as in the Table 12.5.7. This allows us to see how the graph hits a maximum (the tip of a petal), loops back crossing the pole, hits the opposite maximum, and loops back to the pole. The action is continuous until all the petals are drawn.

? Exercise 12.5.4

Sketch the graph of $r = 4\sin(2\theta)$.

Answer

The graph is a rose curve, n even

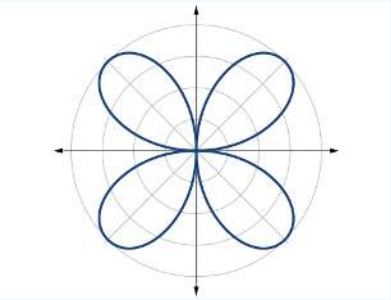


Figure 12.5.18

✓ Example 12.5.9: Sketching the Graph of a Rose Curve (n Odd)

Sketch the graph of $r = 2\sin(5\theta)$.

Solution

The graph of the equation shows symmetry with respect to the line $\theta = \frac{\pi}{2}$. Next, find the zeros and maximum. We will want to make the substitution $u = 5\theta$.

$$\begin{aligned} 0 &= 2\sin(5\theta) \\ 0 &= \sin u \\ \sin^{-1} 0 &= 0 \\ u &= 0 \\ 5\theta &= 0 \\ \theta &= 0 \end{aligned}$$

The maximum value is calculated at the angle where $\sin\theta$ is a maximum. Therefore,

$$\begin{aligned} r &= 2\sin\left(5 \cdot \frac{\pi}{2}\right) \\ r &= 2(1) \\ &= 2 \end{aligned}$$

Thus, the maximum value of the polar equation is 2. This is the length of each petal. As the curve for n odd yields the same number of petals as n, there will be five petals on the graph. See Figure 12.5.19

Create a table of values similar to Table 12.5.8

Table 12.5.8

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	\(0\0	1	**−1.73**	2	**−1.73**	1	0

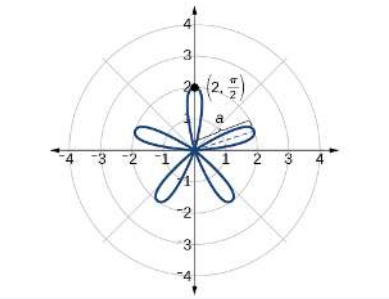


Figure 12.5.19: Rose curve, n odd

Exercise 12.5.5

Sketch the graph of $r = 3\cos(3\theta)$.

Answer

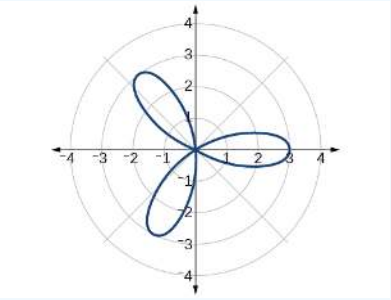


Figure 12.5.20

Rose curve, n odd

Investigating the Archimedes' Spiral

The final polar equation we will discuss is the Archimedes' spiral, named for its discoverer, the Greek mathematician Archimedes (c. 287 BCE - c. 212 BCE), who is credited with numerous discoveries in the fields of geometry and mechanics.

ARCHIMEDES' SPIRAL

The formula that generates the graph of the Archimedes' spiral is given by $r = \theta$ for $\theta \geq 0$. As θ increases, r increases at a constant rate in an ever-widening, never-ending, spiraling path. See Figure 12.5.21

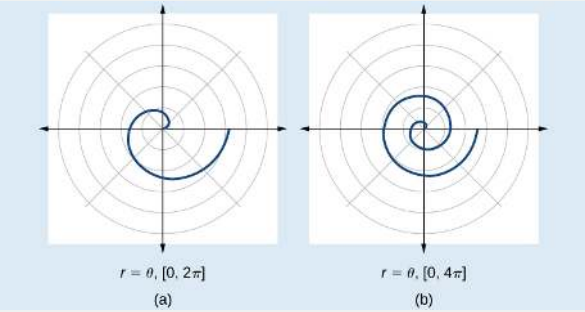


Figure 12.5.21

How to: Given an Archimedes' spiral over $[0, 2\pi]$, sketch the graph

1. Make a table of values for r and θ over the given domain.
2. Plot the points and sketch the graph.

Example 12.5.10: Sketching the Graph of an Archimedes' Spiral

Sketch the graph of $r = \theta$ over $[0, 2\pi]$.

Solution

As r is equal to θ, the plot of the Archimedes' spiral begins at the pole at the point $(0, 0)$. While the graph hints of symmetry, there is no formal symmetry with regard to passing the symmetry tests. Further, there is no maximum value, unless the domain is restricted.

Create a table such as Table 12.5.9

Table 12.5.9

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0.785	1.57	3.14	4.71	5.50	6.28

Notice that the r-values are just the decimal form of the angle measured in radians. We can see them on a graph in Figure 12.5.22

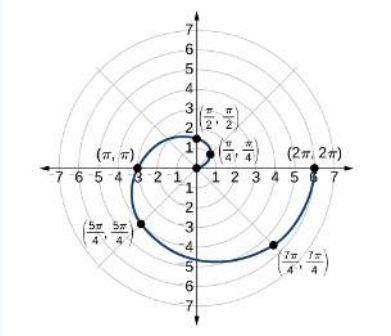


Figure 12.5.22: Archimedes' spiral

Analysis

The domain of this polar curve is $[0, 2\pi]$. In general, however, the domain of this function is $(-\infty, \infty)$. Graphing the equation of the Archimedes' spiral is rather simple, although the image makes it seem like it would be complex.

Exercise 12.5.6

Sketch the graph of $r = -\theta$ over the interval $[0, 4\pi]$.

Answer

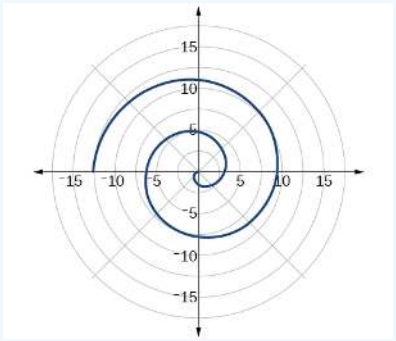


Figure 12.5.23

Summary of Curves

We have explored a number of seemingly complex polar curves in this section. Figure 12.5.24and Figure 12.5.25summarize the graphs and equations for each of these curves.

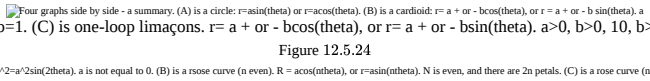


Figure 12.5.24

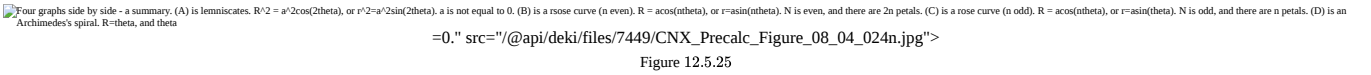


Figure 12.5.25

Media

Access these online resources for additional instruction and practice with graphs of polar coordinates.

- Graphing Polar Equations Part 1
- Graphing Polar Equations Part 2
- Animation: The Graphs of Polar Equations
- Graphing Polar Equations on the TI-84

Key Concepts

- It is easier to graph polar equations if we can test the equations for symmetry with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole.
- There are three symmetry tests that indicate whether the graph of a polar equation will exhibit symmetry. If an equation fails a symmetry test, the graph may or may not exhibit symmetry. See Example 12.5.1.
- Polar equations may be graphed by making a table of values for θ and r.
- The maximum value of a polar equation is found by substituting the value θ that leads to the maximum value of the trigonometric expression.
- The zeros of a polar equation are found by setting $r = 0$ and solving for θ. See Example 12.5.2.
- Some formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$. See Example 12.5.3.
- The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, for $a > 0$, $b > 0$, and $ab = 1$. See Example 12.5.4.
- The formulas that produce the graphs of a one-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $1 < ab < 2$. See Example 12.5.5.
- The formulas that produce the graphs of an inner-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $a > 0$, $b > 0$, and $a < b$. See Example 12.5.6.
- The formulas that produce the graphs of a lemniscates are given by $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$, where $a \neq 0$.See Example 12.5.7.
- The formulas that produce the graphs of rose curves are given by $r = a\cos n\theta$ and $r = a\sin n\theta$, where $a \neq 0$; if n is even, there are $2n$ petals, and if n is odd, there aren n petals. See Example 12.5.8and Example 12.5.9.
- The formula that produces the graph of an Archimedes' spiral is given by $r = \theta$, $\theta \geq 0$. See Example 12.5.10

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12.6: Polar Form of Complex Numbers

Learning Objectives

- Plot complex numbers in the complex plane.
- Find the absolute value of a complex number.
- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find quotients of complex numbers in polar form.
- Find powers of complex numbers in polar form.
- Find roots of complex numbers in polar form.

"God made the integers; all else is the work of man." This rather famous quote by nineteenth-century German mathematician Leopold Kronecker sets the stage for this section on the polar form of a complex number. Complex numbers were invented by people and represent over a thousand years of continuous investigation and struggle by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and others. Complex numbers answered questions that for centuries had puzzled the greatest minds in science.

We first encountered complex numbers in the section on Complex Numbers. In this section, we will focus on the mechanics of working with complex numbers: translation of complex numbers from polar form to rectangular form and vice versa, interpretation of complex numbers in the scheme of applications, and application of De Moivre's Theorem.

Plotting Complex Numbers in the Complex Plane

Plotting a complex number $a+bi$ is similar to plotting a real number, except that the horizontal axis represents the real part of the number, a, and the vertical axis represents the imaginary part of the number, bi.

How to: Given a complex number $a+bi$, plot it in the complex plane.

1. Label the horizontal axis as the *real* axis and the vertical axis as the *imaginary axis.*
2. Plot the point in the complex plane by moving a units in the horizontal direction and b units in the vertical direction.

Example 12.6.1: Plotting a Complex Number in the Complex Plane

Plot the complex number $2-3i$ in the complex plane.

Solution

From the origin, move two units in the positive horizontal direction and three units in the negative vertical direction. See Figure 12.6.1.

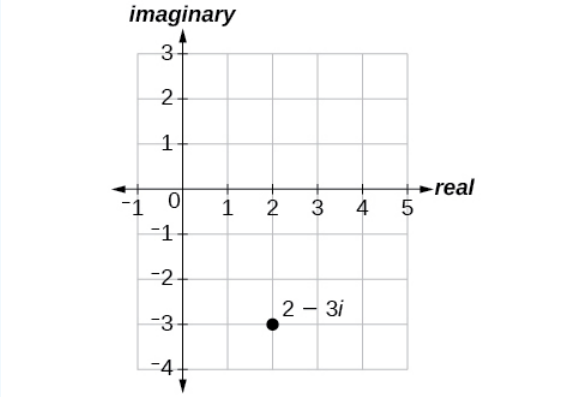


Figure 12.6.1

Exercise 12.6.1

Plot the point $1+5i$ in the complex plane.

Answer

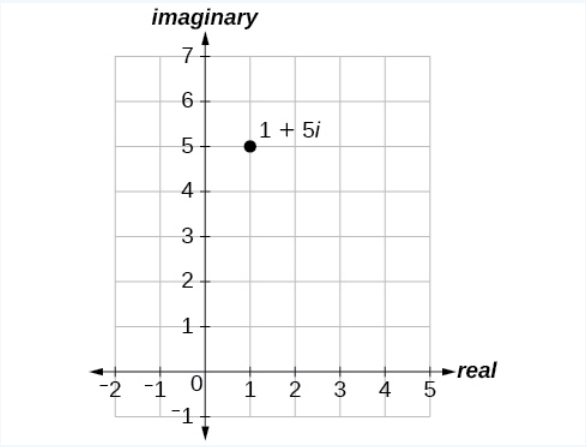


Figure 12.6.2

Finding the Absolute Value of a Complex Number

The first step toward working with a complex number in polar form is to find the absolute value. The absolute value of a complex number is the same as its magnitude, or $|z|$. It measures the distance from the origin to a point in the plane. For example, the graph of $z=2+4i$, in Figure 12.6.3, shows $|z|$.

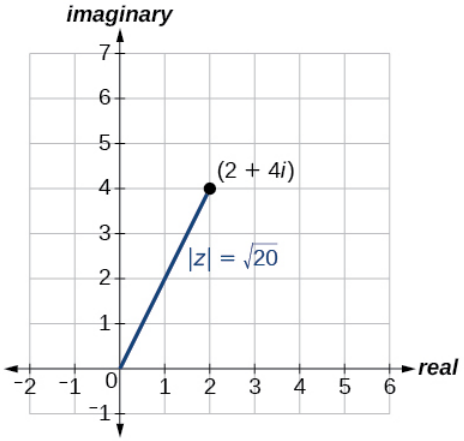


Figure 12.6.3

ABSOLUTE VALUE OF A COMPLEX NUMBER

Given $z=x+yi$, a complex number, the absolute value of z is defined as

$$|z|=\sqrt{x^2+y^2} \tag{12.6.1}$$

It is the distance from the origin to the point (x,y).

Notice that the absolute value of a real number gives the distance of the number from 0, while the absolute value of a complex number gives the distance of the number from the origin, $(0,0)$.

Example 12.6.2: Finding the Absolute Value of a Complex Number with a Radical

Find the absolute value of $z = \sqrt{5} - i$.

Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(\sqrt{5})^2 + (-1)^2}$$
$$|z| = \sqrt{5 + 1}$$
$$|z| = \sqrt{6}$$

See Figure 12.6.4

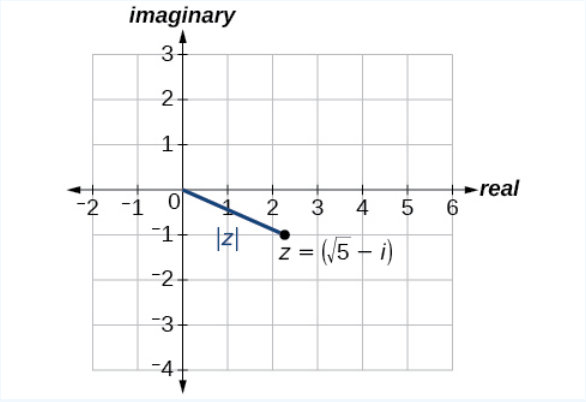


Figure 12.6.4

Exercise 12.6.2

Find the absolute value of the complex number $z = 12 - 5i$.

Answer

13

Example 12.6.3: Finding the Absolute Value of a Complex Number

Given $z = 3 - 4i$, find $|z|$.

Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(3)^2 + (-4)^2}$$
$$|z| = \sqrt{9 + 16}$$
$$|z| = \sqrt{25}$$
$$|z| = 5$$

The absolute value z is 5. See Figure 12.6.5

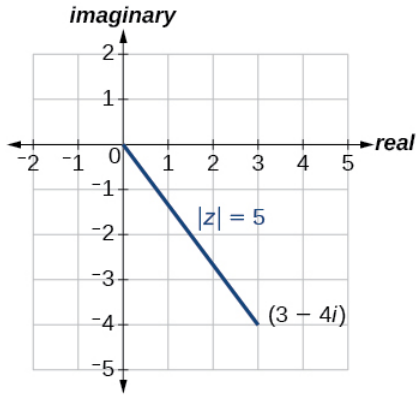


Figure 12.6.5

Exercise 12.6.3

Given $z = 1 - 7i$, find $|z|$.

Answer

$|z| = \sqrt{50} = 5\sqrt{2}$

Writing Complex Numbers in Polar Form

The polar form of a complex number expresses a number in terms of an angle θ and its distance from the origin r. Given a complex number in rectangular form expressed as $z = x + yi$, we use the same conversion formulas as we do to write the number in trigonometric form:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

We review these relationships in Figure 12.6.6

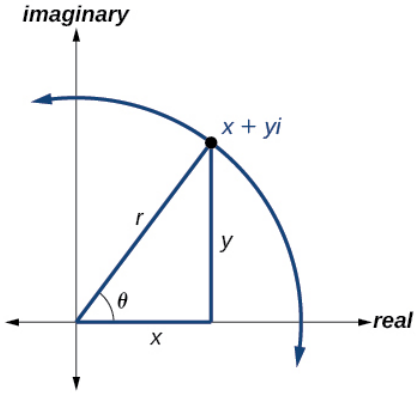


Figure 12.6.6

We use the term **modulus** to represent the absolute value of a complex number, or the distance from the origin to the point (x, y). The modulus, then, is the same as r, the radius in polar form. We use θ to indicate the angle of direction (just as with polar coordinates). Substituting, we have

$$\begin{aligned} z &= x + yi \\ z &= r\cos\theta + (r\sin\theta)i \\ z &= r(\cos\theta + i\sin\theta) \end{aligned}$$

POLAR FORM OF A COMPLEX NUMBER

Writing a complex number in polar form involves the following conversion formulas:

$$x = r\cos\theta \tag{12.6.2}$$
$$y = r\sin\theta \tag{12.6.3}$$
$$r = \sqrt{x^2+y^2} \tag{12.6.4}$$

Making a direct substitution, we have

$$z = x + yi \tag{12.6.5}$$
$$z = (r\cos\theta) + i(r\sin\theta) \tag{12.6.6}$$
$$z = r(\cos\theta + i\sin\theta) \tag{12.6.7}$$

where r is the modulus and θ is the argument. We often use the abbreviation $r\ cis\theta$ to represent $r(\cos\theta + i\sin\theta)$.

Example 12.6.4: Expressing a Complex Number Using Polar Coordinates

Express the complex number $4i$ using polar coordinates.

Solution

On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. Writing it in polar form, we have to calculate r first.

$$r = \sqrt{x^2+y^2}$$
$$r = \sqrt{0^2+4^2}$$
$$r = \sqrt{16}$$
$$r = 4$$

Next, we look at x. If $x = r\cos\theta$, and $x = 0$, then $\theta = \dfrac{\pi}{2}$. In polar coordinates, the complex number $z = 0 + 4i$ can be written as $z = 4\left(\cos\left(\dfrac{\pi}{2}\right) + i\sin\left(\dfrac{\pi}{2}\right)\right)$ or $4\ cis\left(\dfrac{\pi}{2}\right)$. See Figure 12.6.7.

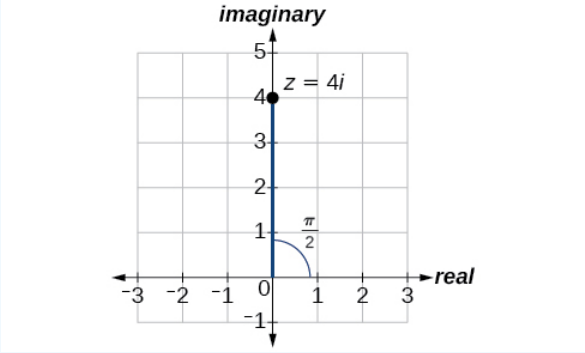


Figure 12.6.7

Exercise 12.6.4

Express $z = 3i$ as $r\ cis\theta$ in polar form.

Answer

$$z = 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

Example 12.6.5: Finding the Polar Form of a Complex Number

Find the polar form of $-4+4i$.

Solution

First, find the value of r.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ r &= \sqrt{(-4)^2+(4^2)} \\ r &= \sqrt{32} \\ r &= 4\sqrt{2} \end{aligned}$$

Find the angle θ using the formula:

$$\begin{aligned} \cos\theta &= \frac{x}{r} \\ \cos\theta &= \frac{-4}{4\sqrt{2}} \\ \cos\theta &= -\frac{1}{\sqrt{2}} \\ \theta &= \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) \\ &= \frac{3\pi}{4} \end{aligned}$$

Thus, the solution is $4\sqrt{2}\,cis\left(\frac{3\pi}{4}\right)$.

Exercise 12.6.5

Write $z=\sqrt{3}+i$ in polar form.

Answer

$$z=2\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$$

Converting a Complex Number from Polar to Rectangular Form

Converting a complex number from polar form to rectangular form is a matter of evaluating what is given and using the distributive property. In other words, given $z=r(\cos\theta+i\sin\theta)$, first evaluate the trigonometric functions $\cos\theta$ and $\sin\theta$. Then, multiply through by r.

Example 12.6.6A: Converting from Polar to Rectangular Form

Convert the polar form of the given complex number to rectangular form:

$$z=12\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$$

Solution

We begin by evaluating the trigonometric expressions.

$$\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2} \text{ and } \sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

After substitution, the complex number is

$$z=12\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$

We apply the distributive property:

$$\begin{aligned} z &= 12\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) \\ &= (12)\frac{\sqrt{3}}{2}+(12)\frac{1}{2}i \\ &= 6\sqrt{3}+6i \end{aligned}$$

The rectangular form of the given point in complex form is $6\sqrt{3}+6i$.

Example 12.6.6B: Finding the Rectangular Form of a Complex Number

Find the rectangular form of the complex number given $r=13$ and $\tan\theta=\frac{5}{12}$.

Solution

If $\tan\theta=\frac{5}{12}$, and $\tan\theta=\frac{y}{x}$, we first determine $r=\sqrt{x^2+y^2}=\sqrt{122+52}=13$. We then find $\cos\theta=\frac{x}{r}$ and $\sin\theta=\frac{y}{r}$.

$$\begin{aligned} z &= 13\left(\cos\theta+i\sin\theta\right) \\ &= 13\left(\frac{12}{13}+\frac{5}{13}i\right) \\ &= 12+5i \end{aligned}$$

The rectangular form of the given number in complex form is $12+5i$.

Exercise 12.6.6

Convert the complex number to rectangular form:

$$z=4\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$$

Answer

$z=2\sqrt{3}-2i$

Finding Products of Complex Numbers in Polar Form

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham **de Moivre** (1667-1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

PRODUCTS OF COMPLEX NUMBERS IN POLAR FORM

If $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$, then the product of these numbers is given as:

$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)] \quad (12.6.8)$$

$$z_1z_2 = r_1r_2\ cis(\theta_1+\theta_2) \quad (12.6.9)$$

Notice that the product calls for multiplying the moduli and adding the angles.

Example 12.6.7: Finding the Product of Two Complex Numbers in Polar Form

Find the product of $z_1 z_2$, given $z_1 = 4(\cos(80^\circ) + i\sin(80^\circ))$ and $z_2 = 2(\cos(145^\circ) + i\sin(145^\circ))$.

Solution

Follow the formula

$$\begin{aligned} z_1 z_2 &= 4 \cdot 2[\cos(80^\circ + 145^\circ) + i\sin(80^\circ + 145^\circ)] \\ z_1 z_2 &= 8[\cos(225^\circ) + i\sin(225^\circ)] \\ z_1 z_2 &= 8\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right] \\ z_1 z_2 &= 8\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right] \\ z_1 z_2 &= -4\sqrt{2} - 4i\sqrt{2} \end{aligned}$$

Finding Quotients of Complex Numbers in Polar Form

The quotient of two complex numbers in polar form is the quotient of the two moduli and the difference of the two arguments.

QUOTIENTS OF COMPLEX NUMBERS IN POLAR FORM

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the quotient of these numbers is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)],\ z_2 \neq 0 \tag{12.6.10}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,cis(\theta_1 - \theta_2),\ z_2 \neq 0 \tag{12.6.11}$$

Notice that the moduli are divided, and the angles are subtracted.

How to: Given two complex numbers in polar form, find the quotient

1. Divide $\frac{r_1}{r_2}$.
2. Find $\theta_1 - \theta_2$.
3. Substitute the results into the formula: $z = r(\cos\theta + i\sin\theta)$. Replace r with $\frac{r_1}{r_2}$, and replace θ with $\theta_1 - \theta_2$.
4. Calculate the new trigonometric expressions and multiply through by r.

Example 12.6.8: Finding the Quotient of Two Complex Numbers

Find the quotient of $z_1 = 2(\cos(213^\circ) + i\sin(213^\circ))$ and $z_2 = 4(\cos(33^\circ) + i\sin(33^\circ))$.

Solution

Using the formula, we have

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2}{4}[\cos(213^\circ - 33^\circ) + i\sin(213^\circ - 33^\circ)] \\ \frac{z_1}{z_2} &= \frac{1}{2}[\cos(180^\circ) + i\sin(180^\circ)] \\ \frac{z_1}{z_2} &= \frac{1}{2}[-1 + 0i] \\ \frac{z_1}{z_2} &= -\frac{1}{2} + 0i \\ \frac{z_1}{z_2} &= -\frac{1}{2} \end{aligned}$$

Exercise 12.6.8

Find the product and the quotient of $z_1 = 2\sqrt{3}(\cos(150^\circ) + i\sin(150^\circ))$ and $z_2 = 2(\cos(30^\circ) + i\sin(30^\circ))$.

Answer

$$z_1 z_2 = -4\sqrt{3}\,;\ \frac{z_1}{z_2} = -\frac{\sqrt{3}}{2} + \frac{3}{2}i$$

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is greatly simplified using **De Moivre's Theorem**. It states that, for a positive integer n, z^n is found by raising the modulus to the nth power and multiplying the argument by n. It is the standard method used in modern mathematics.

DE MOIVRE'S THEOREM

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)] \tag{12.6.12}$$
$$z^n = r^n\, cis(n\theta) \tag{12.6.13}$$

where n is a positive integer.

Example 12.6.9: Evaluating an Expression Using De Moivre's Theorem

Evaluate the expression $(1+i)^5$ using De Moivre's Theorem.

Solution

Since De Moivre's Theorem applies to complex numbers written in polar form, we must first write $(1+i)$ in polar form. Let us find r.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(1)^2 + (1)^2}$$
$$r = \sqrt{2}$$

Then we find θ. Using the formula $\tan\theta = \dfrac{y}{x}$ gives

$$\tan\theta = \frac{1}{1}$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

Use De Moivre's Theorem to evaluate the expression.

$$(a+bi)^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$
$$(1+i)^5 = (\sqrt{2})^5\left[\cos\left(5\cdot\frac{\pi}{4}\right) + i\sin\left(5\cdot\frac{\pi}{4}\right)\right]$$
$$(1+i)^5 = 4\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$(1+i)^5 = 4\sqrt{2}\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$(1+i)^5 = -4 - 4i$$

Finding Roots of Complex Numbers in Polar Form

To find the nth root of a complex number in polar form, we use the nth Root Theorem or De Moivre's Theorem and raise the complex number to a power with a rational exponent. There are several ways to represent a formula for finding n^{th} roots of complex numbers in polar form.

THE N^{TH} ROOT THEOREM

To find the nth root of a complex number in polar form, use the formula given as

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right] \tag{12.6.14}$$

where $k = 0, 1, 2, 3, \ldots, n-1$. We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

th Root of a Complex Number

Evaluate the cube roots of $z = 8\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)$.

Solution

We have

$$z^{\frac{1}{3}} = 8^{\frac{1}{3}}\left[\cos\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)\right]$$
$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)\right]$$

There will be three roots: $k = 0, 1, 2$. When $k = 0$, we have

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)$$

When $k = 1$, we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)\right] \qquad \text{Add } \frac{2(1)\pi}{3} \text{ to each angle.}$$
$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{8\pi}{9}\right)+i\sin\left(\frac{8\pi}{9}\right)\right)$$

When $k = 2$, we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)\right] \qquad \text{Add } \frac{2(2)\pi}{3} \text{ to each angle.}$$
$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{14\pi}{9}\right)+i\sin\left(\frac{14\pi}{9}\right)\right)$$

Remember to find the common denominator to simplify fractions in situations like this one. For $k = 1$, the angle simplification is

$$\frac{\frac{2\pi}{3}}{3}+\frac{2(1)\pi}{3} = \frac{2\pi}{3}\left(\frac{1}{3}\right)+\frac{2(1)\pi}{3}\left(\frac{3}{3}\right)$$
$$= \frac{2\pi}{9}+\frac{6\pi}{9}$$
$$= \frac{8\pi}{9}$$

Exercise 12.6.10

Find the four fourth roots of $16(\cos(120^\circ)+i\sin(120^\circ))$.

Answer

$z_0 = 2(\cos(30^\circ)+i\sin(30^\circ))$

$z_1 = 2(\cos(120^\circ)+i\sin(120^\circ))$

$z_2 = 2(\cos(210^\circ)+i\sin(210^\circ))$

$z_3 = 2(\cos(300^\circ)+i\sin(300^\circ))$

Media

Access these online resources for additional instruction and practice with polar forms of complex numbers.

- The Product and Quotient of Complex Numbers in Trigonometric Form
- De Moivre's Theorem

Key Concepts

- Complex numbers in the form $a+bi$ are plotted in the complex plane similar to the way rectangular coordinates are plotted in the rectangular plane. Label the x-axis as the real axis and the y-axis as the *imaginary* axis. See Example 12.6.1.
- The absolute value of a complex number is the same as its magnitude. It is the distance from the origin to the point: $|z| = \sqrt{a^2+b^2}$. See Example 12.6.2 and Example 12.6.3.
- To write complex numbers in polar form, we use the formulas $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2+y^2}$. Then, $z = r(\cos\theta+i\sin\theta)$. See Example 12.6.4 and Example 12.6.5.
- To convert from polar form to rectangular form, first evaluate the trigonometric functions. Then, multiply through by r. See Example 12.6.6 and Example 12.6.7.
- To find the product of two complex numbers, multiply the two moduli and add the two angles. Evaluate the trigonometric functions, and multiply using the distributive property. See Example 12.6.8.
- To find the quotient of two complex numbers in polar form, find the quotient of the two moduli and the difference of the two angles. See Example 12.6.9.
- To find the power of a complex number z^n, raise r to the power n, and multiply θ by n. See Example 12.6.10.
- Finding the roots of a complex number is the same as raising a complex number to a power, but using a rational exponent. See Example 12.6.11.

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