ELEMENTARY ALGEBRA



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Elementary Algebra

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1: REAL NUMBERS AND THEIR OPERATIONS

1.1: Real numbers and the Number Line

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1.1: Real numbers and the Number Line

Learning Objectives

- Construct a number line and graph points on it.
- Use a number line to determine the order of real numbers.
- Determine the opposite of a real number.
- Determine the absolute value of a real number.

Definitions

A **set** is a collection of objects, typically grouped within braces { }, where each object is called an **element**. For example, $\{\text{red, green, blue}\}$ is a set of colors. A **subset** is a set consisting of elements that belong to a given set. For example, $\{\text{green, blue}\}$ is a subset of the color set above. A set with no elements is called the **empty set** and has its own special notation, { } or $\varnothing$.

When studying mathematics, we focus on special sets of numbers. The set of **natural (or counting) numbers**, denoted $\mathbf{N}$, is

$$\{1, 2, 3, 4, 5, ...\} \quad \textit{Natural Numbers}$$

The three periods $(...)$ is called an ellipsis and indicates that the numbers continue without bound. The set of **whole numbers**, denoted $\mathbf{W}$, is the set of natural numbers combined with zero.

$$\{0, 1, 2, 3, 4, 5, ...\} \quad \textit{Whole Numbers}$$

The set of **integers**, denoted $\mathbf{Z}$, consists of both positive and negative whole numbers, as well as zero.

$$\{..., -3, -2, -1, 0, 1, 2, 3, ...\} \quad \textit{Integers}$$

Notice that the sets of natural and whole numbers are both subsets of the set of integers.

Rational numbers, denoted $\mathbf{Q}$, are defined as any number of the form $\frac{a}{b}$, where a and b are integers and b is nonzero. Decimals that repeat or terminate are rational. For example,

$$0.7 = \tfrac{7}{10} \quad \text{and} \quad 0.\overline{3} = 0.3333\cdots = \tfrac{1}{3}$$

The set of integers is a subset of the set of rational numbers because every integer can be expressed as a ratio of the integer and 1. In other words, any integer can be written over 1 and can be considered a rational number. For example,

$$5 = \tfrac{5}{1}$$

Irrational numbers are defined as any number that cannot be written as a ratio of two integers. Nonterminating decimals that do not repeat are irrational. For example,

$$\pi = 3.14159\ldots \quad \text{and} \quad \sqrt{2} = 1.41421\ldots$$

The set of **real numbers**, denoted $\mathbf{R}$, is defined as the set of all rational numbers combined with the set of all irrational numbers. Therefore, all the numbers defined so far are subsets of the set of real numbers. In summary,

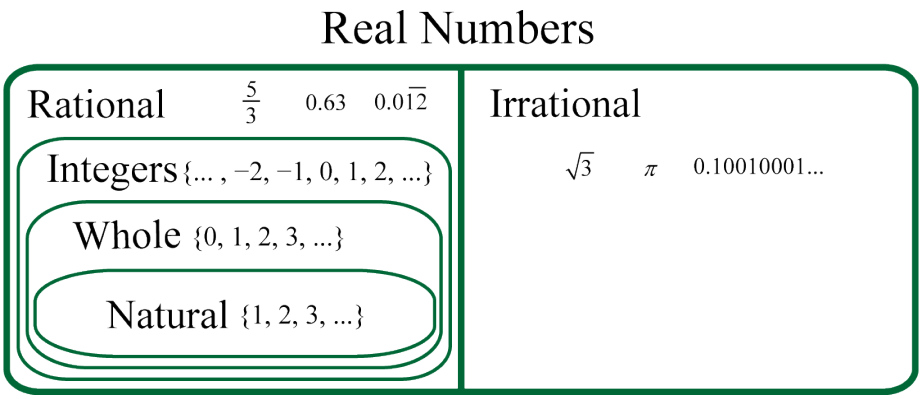


Figure 1.1.1: Real Numbers

Number Line

A **real number line**, or simply number line, allows us to visually display real numbers by associating them with unique points on a line. The real number associated with a point is called a **coordinate**. A point on the real number line that is associated with a coordinate is called its **graph**.

To construct a number line, draw a horizontal line with arrows on both ends to indicate that it continues without bound. Next, choose any point to represent the number zero; this point is called the **origin**.

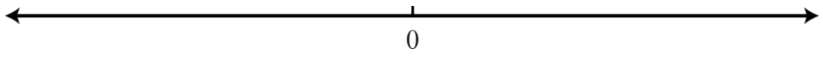


Figure 1.1.2

Mark off consistent lengths on both sides of the origin and label each tick mark to define the scale. Positive real numbers lie to the right of the origin and negative real numbers lie to the left. The number zero (0) is neither positive nor negative. Typically, each tick represents one unit.

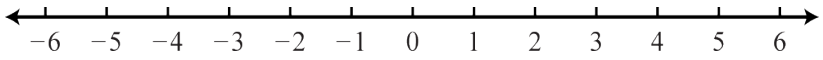


Figure 1.1.3

As illustrated below, the scale need not always be one unit. In the first number line, each tick mark represents two units. In the second, each tick mark represents $\frac{1}{7}$ of a unit.

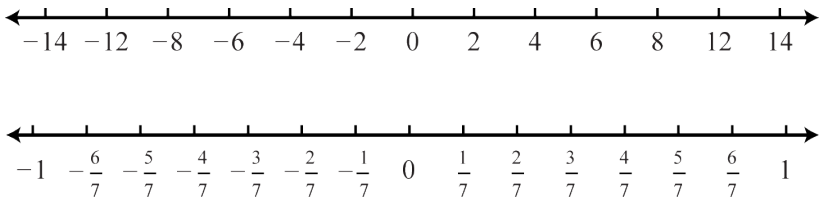


Figure 1.1.4

The graph of each real number is shown as a dot at the appropriate point on the number line. A partial graph of the set of integers $\mathbf{Z}$ follows:

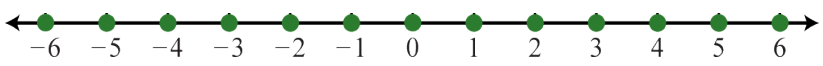


Figure 1.1.5

Example 1.1.1

Graph the following set of real numbers:

Solution

Graph the numbers on a number line with a scale where each tick mark represents $\frac{1}{2}$ of a unit.

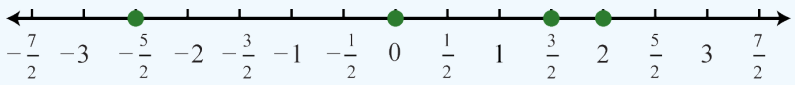


Figure 1.1.6

Ordering Real Numbers

When comparing real numbers on a number line, the larger number will always lie to the right of the smaller one. It is clear that 15 is greater than 5, but it may not be so clear to see that -1 is greater than -5 until we graph each number on a number line.

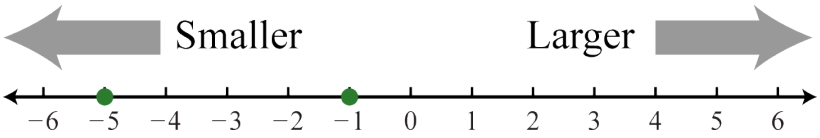


Figure 1.1.7

We use symbols to help us efficiently communicate relationships between numbers on the number line. The symbols used to describe an **equality relationship** between numbers follow:

$=$ *is equal to*
$\neq$ *is not equal to*
$\approx$ *is approximately equal to*

These symbols are used and interpreted in the following manner:

$5 = 5$ *5 is equal to 5*
$0 \neq 5$ *0 is not equal to 5*
$\pi \approx 3.14$ *pi is approximately equal to 3.14*

We next define symbols that denote an **order relationship** between real numbers.

$<$ *Less than*
$>$ *Greater than*
$\leq$ *Less than or equal to*
$\geq$ *Greater than or equal to*

These symbols allow us to compare two numbers. For example,

Since the graph of -120 is to the left of the graph of -10 on the number line, that number is less than -10. We could write an equivalent statement as follows:

Similarly, since the graph of zero is to the right of the graph of any negative number on the number line, zero is greater than any negative number.

The symbols $<$ and $>$ are used to denote **strict inequalities**, and the symbols and are used to denote **inclusive inequalities**. In some situations, more than one symbol can be correctly applied. For example, the following two statements are both true:

In addition, the “or equal to” component of an inclusive inequality allows us to correctly write the following:

The logical use of the word “or” requires that only one of the conditions need be true: the “less than” or the “equal to.”

Fill in the blank with $<$, $=$, or $>$: -2 ___ -12.

Solution

Use > because the graph of -2 is to the right of the graph of -12 on a number line. Therefore, $-2 > -12$, which reads “negative two is greater than negative twelve.”

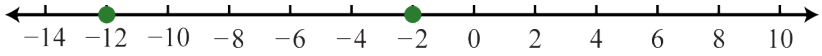


Figure 1.1.8

Answer:

$-2 > -12$

In this text, we will often point out the equivalent notation used to express mathematical quantities electronically using the standard symbols available on a keyboard. We begin with the equivalent textual notation for inequalities:

$$\begin{aligned} \geq &\quad ''>=''\\ \leq &\quad ''<=''\\ \neq &\quad ''!=''\end{aligned}$$

Many calculators, computer algebra systems, and programming languages use this notation.

Opposites

The **opposite** of any real number a is $-a$. Opposite real numbers are the same distance from the origin on a number line, but their graphs lie on opposite sides of the origin and the numbers have opposite signs.

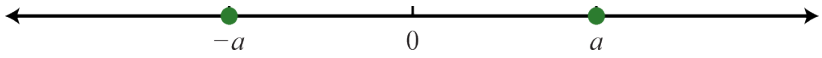


Figure 1.1.9

For example, we say that the opposite of 10 is -10.

Next, consider the opposite of a negative number. Given the integer -7, the integer the same distance from the origin and with the opposite sign is $+7$, or just 7.

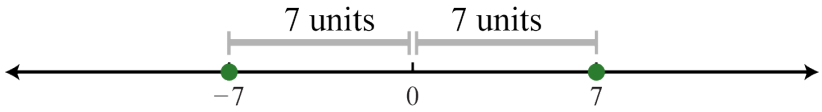


Figure 1.1.10

Therefore, we say that the opposite of -7 is $-(-7) = 7$. This idea leads to what is often referred to as the **double-negative property**. For any real number a,

$$-(-a) = a$$

Example 1.1.3

What is the opposite of $-\frac{3}{4}$?

Solution

Here we apply the double-negative property.

$$-(-\tfrac{3}{4}) = \tfrac{3}{4}$$

Example 1.1.4

Simplify $-(-(4))$

Solution

Start with the innermost parentheses by finding the opposite of $+4$.

$$\begin{aligned} -(-(4)) &= -(\color{cyan}{-(4)}) \\ &= -(\color{cyan}{-4}) \\ &= 4 \end{aligned}$$

Answer

4

Example 1.1.5

Simplify: $-(-(-2))$.

Solution

Apply the double-negative property starting with the innermost parentheses.

$$\begin{aligned} -(-(-2)) &= -(\color{cyan}{-(-2)}) \\ &= -(\color{cyan}{2}) \\ &= -2 \end{aligned}$$

Answer

-2

tip

If there is an even number of consecutive negative signs, then the result is positive. If there is an odd number of consecutive negative signs, then the result is negative.

Try this!

Exercise 1.1.1

Simplify: $-(-(-(5)))$.

Answer available at the end of this section.

Absolute Value

The **absolute value** of a real number a, denoted $|a|$, is defined as the distance between zero (the origin) and the graph of that real number on the number line. Since it is a distance, it is always positive. For example,

$$|-4| = 4 \quad \text{and} \quad |4| = 4$$

Both 4 and -4 are four units from the origin, as illustrated below:

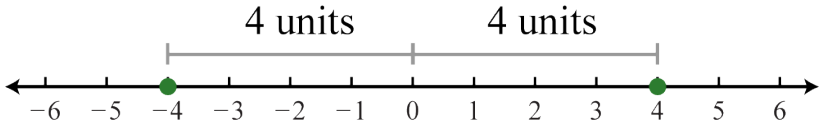


Figure 1.1.11

Example 1.1.6

Simplify:

a. $|-12|$

b.$|12|$

Solution

Both -12 and 12 are twelve units from the origin on a number line. Therefore,

$$|-12| = 12 \quad \text{and} \quad |12| = 12$$

Answer

a.12 b.12

Also, it is worth noting that

$$|0| = 0$$

The absolute value can be expressed textually using the notation abs(a). We often encounter negative absolute values, such as $-|3|$ or $-$ abs(3). Notice that the negative sign is in front of the absolute value symbol. In this case, work the absolute value first and then find the opposite of the result.

Try not to confuse this with the double-negative property, which states that $-(-7) = +7$.

Simplfy: $-|-(-7)|$.

Solution

First, find the opposite of -7 inside the absolute value. Then find the opposite of the result.

$$\begin{aligned} -|-(-7)| &= -|7| \\ &= -7 \end{aligned}$$

Answer

-7

At this point, we can determine what real numbers have a particular absolute value. For example,

$$|?| = 5$$

Think of a real number whose distance to the origin is 5 units. There are two solutions: the distance to the right of the origin and the distance to the left of the origin, namely, $\{\pm 5\}$. The symbol $(\pm)$ is read “plus or minus” and indicates that there are two answers, one positive and one negative.

$$|-5| = 5 \quad \text{and} \quad |5| = 5$$

Now consider the following:

$$|?| = -5$$

Here we wish to find a value for which the distance to the origin is negative. Since negative distance is not defined, this equation has no solution. If an equation has no solution, we say the solution is the empty set: $\varnothing$.

Key Takeaways

- Any real number can be associated with a point on a line.
- Create a number line by first identifying the origin and marking off a scale appropriate for the given problem.
- Negative numbers lie to the left of the origin and positive numbers lie to the right.
- Smaller numbers always lie to the left of larger numbers on the number line.
- The opposite of a positive number is negative and the opposite of a negative number is positive.
- The absolute value of any real number is always positive because it is defined to be the distance from zero (the origin) on a number line.
- The absolute value of zero is zero.

Exercise 1.1.2

Use set notation to list the described elements.

1. The hours on a clock.
2. The days of the week.
3. The first ten whole numbers.
4. The first ten natural numbers.
5. The first five positive even integers.
6. The first five positive odd integers.

Answer available at the end of this section.

Exercise 1.1.3

Determine whether the following real numbers are integers, rational, or irrational.

1. 12
2. -3
3. 4.5
4. -5
5. $0.3\overline{6}$
6. $0.\overline{3}$
7. $1.001000100001\ldots$
8. $1.00\overline{1}$
9. $e = 2.71828\ldots$
10. $\sqrt{7} = 2.645751\ldots$
11. -7
12. 3.14
13. 227
14. 1.33
15. 0
16. $8,675,309$

Answer available at the end of this section.

Exercise 1.1.4

True or false.

1. All integers are rational numbers.
2. All integers are whole numbers.
3. All rational numbers are whole numbers.
4. Some irrational numbers are rational.
5. All terminating decimal numbers are rational.
6. All irrational numbers are real.

Answer available at the end of this section.

Exercise 1.1.5

Choose an appropriate scale and graph the following sets of real numbers on a number line.

1. $\{-3, 0, 3\}$
2. $\{-2, 2, 4, 6, 8, 10\}$
3. $\{-2,\ -1/3,\ 2/3,\ 5/3\}$
4. $\{-5/2,\ -1/2,\ 0,\ 1/2,\ 2\}$
5. $\{-5/7,\ 0,\ 2/7,\ 1\}$
6. $\{-5,\ -2,\ -1,\ 0\}$
7. $\{-3,\ -2,\ 0,\ 2,\ 5\}$
8. $\{-2.5, -1.5, 0, 1, 2.5\}$
9. $\{0, 0.3, 0.6, 0.9, 1.2\}$
10. $\{-10, 30, 50\}$
11. $\{-6, 0, 3, 9, 12\}$
12. $\{-15, -9, 0, 9, 15\}$

Answer available at the end of this section.

Exercise 1.1.6

Fill in the blank with $<$, $=$, or $>$.

1. -7 ___ 0
2. 30 ___ 2
3. 10 ___ -10
4. -150 ___ -75
5. -0.5 ___ -1.5
6. 0 ___ 0
7. -500 ___ 200
8. -1 ___ -200
9. -10 ___ -10
10. -40 ___ -41

Answer available at the end of this section.

Exercise 1.1.7

True or false.

1. $5 \neq 7$
2. $4 = 5$
3. $1 \neq 1$
4. $-5 > -10$
5. $4 \leq 4$
6. $-12 \geq 0$
7. $-10 = -10$
8. $3 > 3$
9. $-1000 < -20$
10. $0 = 0$

Answer available at the end of this section.

Exercise 1.1.8

List the numbers.

1. List three integers less than -5.
2. List three integers greater than -10.
3. List three rational numbers less than zero.
4. List three rational numbers greater than zero.
5. List three integers between -20 and -5.
6. List three rational numbers between 0 and 1.

Answer available at the end of this section.

Exercise 1.1.9

Translate each statement into an English sentence.

1. $10 < 20$
2. $-50 \leq -10$
3. $-4 \neq 0$
4. $30 \geq -1$
5. $0 = 0$
6. $e \approx 2.718$

Answer available at the end of this section.

Exercise 1.1.10

Translate the following into a mathematical statement.

1. Negative seven is less than zero.
2. Twenty-four is not equal to ten.
3. Zero is greater than or equal to negative one.
4. Four is greater than or equal to negative twenty-one.
5. Negative two is equal to negative two.
6. Negative two thousand is less than negative one thousand.

Answer available at the end of this section.

Exercise 1.1.11

Simplify.

1. $-(-9)$
2. $-(-35)$
3. $-(10)$
4. $-(3)$
5. $-(5)$
6. $-(34)$
7. $-(-1)$
8. $-(-(-1))$
9. $-(-(1))$
10. $-(-(-3))$
11. $-(-(-(-11)))$

Answer available at the end of this section.

Exercise 1.1.12

Answer the following questions.

1. What is the opposite of -12
2. What is the opposite of π ?
3. What is the opposite -0.01?
4. Is the opposite of -12 smaller or larger than -11?
5. Is the opposite of 7 smaller or larger than -6?

Answer available at the end of this section.

Exercise 1.1.13

Fill in the blank with $<$, $=$, or $>$.

1. -7 ___ $-(-8)$
2. 6 ___ $-(6)$
3. 13 ___ $-(-12)$
4. $-(-5)$ ___ $-(-2)$
5. -100 ___ $-(-(-50))$
6. 44 ___ $-(-44)$

Answer available at the end of this section.

Exercise 1.1.14

Simplify.

1. $|20|$
2. $|-20|$
3. $|-33|$
4. $|-0.75|$
5. $|-\frac{3}{5}|$
6. $|38|$

7. $|0|$
8. $|1|$
9. $-|12|$
10. $-|-20|$
11. $-|20|$
12. $-|-8|$
13. $-|7|$
14. $-|-316|$
15. $-(-|\frac{8}{9}|)$
16. $|-(-2)|$
17. $-|-(-3)|$
18. $-(-|5|)$
19. $-(-|-45|)$
20. $-|-(-21)|$
21. $\text{abs}(6)$
22. $\text{abs}(-7)$
23. $-\text{abs}(5)$
24. $-\text{abs}(-19)$
25. $-(\ -\text{abs}(9))$
26. $-\text{abs}(-(-12))$

Answer available at the end of this section.

Exercise 1.1.15

Determine the unknown.

1. $|\ ?\ | = 9$
2. $|\ ?\ | = 15$
3. $|\ ?\ | = 0$
4. $|\ ?\ | = 1$
5. $|\ ?\ | = -8$
6. $|\ ?\ | = -20$
7. $|?| - 10 = -2$
8. $|?| + 5 = 14$

Answer available at the end of this section.

Exercise 1.1.16

Fill in the blank with $<$, $=$, or $>$.

1. $|-2|$ ____ 0
2. $|-7|$ ____ $|-10|$
3. -10 ____ $-|-2|$
4. $|-6|$ ____ $|-(-6)|$
5. $-|3|$ ____ $|-(-5)|$
6. 0 ____ $-|-(-4)|$

Answer available at the end of this section.

Exercise 1.1.17

Discussion Board Topics.

1. Research and discuss the history of the number zero.
2. Research and discuss the various numbering systems throughout history.
3. Research and discuss the definition and history of π.
4. Research the history of irrational numbers. Who is credited with proving that the square root of 2 is irrational and what happened to him?
5. Research and discuss the history of absolute value.
6. Discuss the “just make it positive” definition of absolute value

Answers

Exercise 1.1.1

-5

Procedure:

$$\begin{aligned} -(-(-(5))) &= -(-(-(5))) \\ &= -(-(-5)) \\ &= -(5) \\ &= -5 \end{aligned}$$

Video Solution:

(click to see video)

Exercise 1.1.2

1. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

3. $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

5. $\{2, 4, 6, 8, 10\}$

Exercise 1.1.3

1: Integer, Rational

3: Rational

5: Rational

7: Irrational

9: Irrational

11: Integer, Rational

13: Rational

15: Integer, Rational

Exercise 1.1.4

1: True

3: False

5: True

Exercise 1.1.5

1. $\{-3, 0, 3\}$

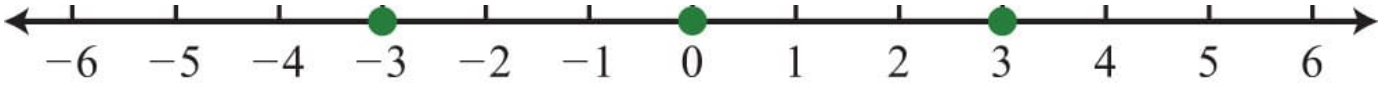


Figure 1.1.12

3. $\{-2, -1/3, 2/3, 5/3\}$

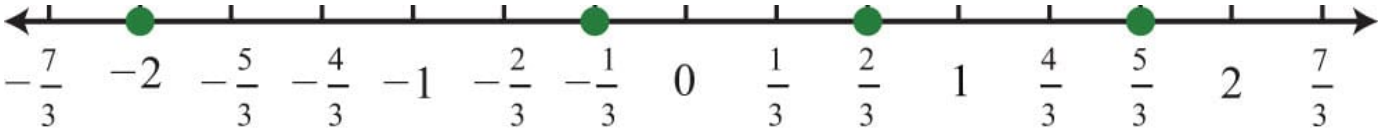


Figure 1.1.13

5. $\{-5/7, 0, 2/7, 1\}$

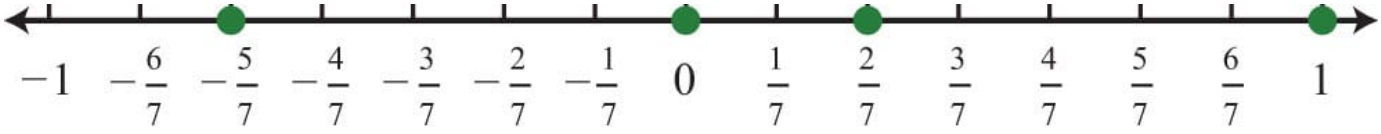


Figure 1.1.14

7. $\{-3, -2, 0, 2, 5\}$

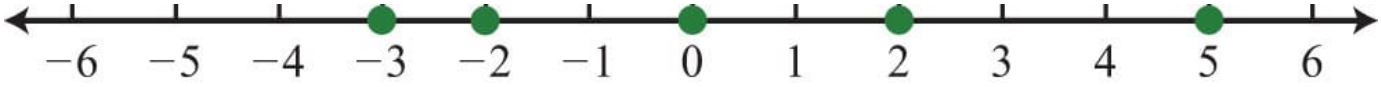


Figure 1.1.15

9. $\{0, 0.3, 0.6, 0.9, 1.2\}$

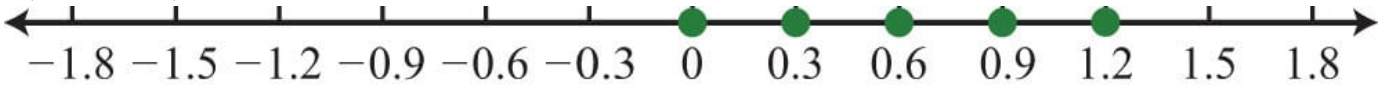


Figure 1.1.16

11. $\{-6, 0, 3, 9, 12\}$

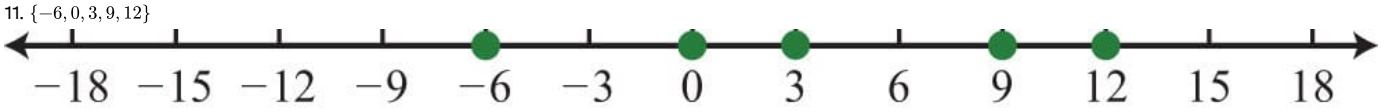


Figure 1.1.17

Exercise 1.1.6

1. $<$
3. $>$
5. $>$
7. $<$
9. $=$

Exercise 1.1.7

1. True
3. False
5. True
7. True
9. True

Exercise 1.1.8

1. $-10, -7, -6$ (answers may vary)
3. $-1, -2/3, -1/3$ (answers may vary)
5. $-15, -10, -7$ (answers may vary)

Exercise 1.1.9

1. Ten is less than twenty.
3. Negative four is not equal to zero.
5. Zero is equal to zero.

Exercise 1.1.10

1. $-7 < 0$
3. $0 \geq -1$
5. $-2 = -2$

Exercise 1.1.11

1. 9
3. -10
5. -5
7. 1
9. 1
11. 11

Exercise 1.1.12

2. $-\pi$
4. Larger

Exercise 1.1.13

1. $<$
3. $>$
5. $<$

Exercise 1.1.14

1. 20
3. 33
5. $\frac{3}{5}$
7. 0
9. -12
11. -20
13. -7

15. $\frac{8}{9}$
17. -3
19. 45
21. 6
23. -5
25. 9

Exercise $1.1.15$

1. ± 9
3. 0
5. $\varnothing$, No solution
7. ± 8

Exercise $1.1.16$

1. $>$
3. $<$
5. $<$

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1.2: Adding and Subtracting Integers

Learning Objectives

- Add and subtract signed integers.
- Translate English sentences involving addition and subtraction into mathematical statements.
- Calculate the distance between two numbers on a number line.

Addition and Subtraction $(+, -)$

Visualize adding $3 + 2$ on the number line by moving from zero three units to the right then another two units to the right, as illustrated below:

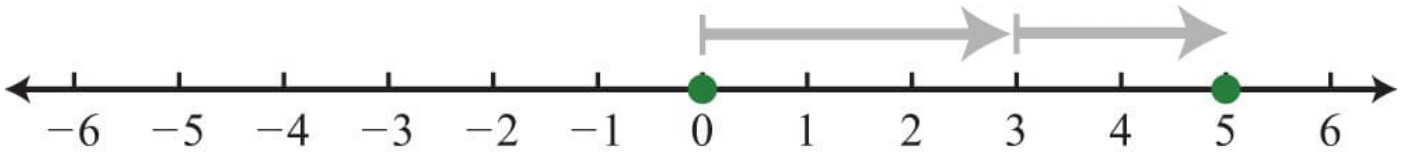


Figure 1.2.1

The illustration shows that $3 + 2 = 5$. Similarly, visualize adding two negative numbers $(-3) + (-2)$ by first moving from the origin three units to the left and then moving another two units to the left.

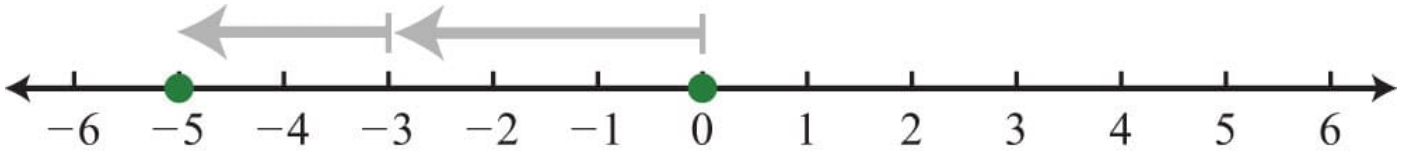


Figure 1.2.2

In this example, the illustration shows $(-3) + (-2) = -5$, which leads to the following two properties of real numbers.

$$\textit{positive number} + \textit{positive number} = \textit{positive number}$$
$$\textit{negative number} + \textit{negative number} = \textit{negative number}$$

Next, we will explore addition of numbers with unlike signs. To add $3 + (-7)$, first move from the origin three units to the right, then move seven units to the left as shown:

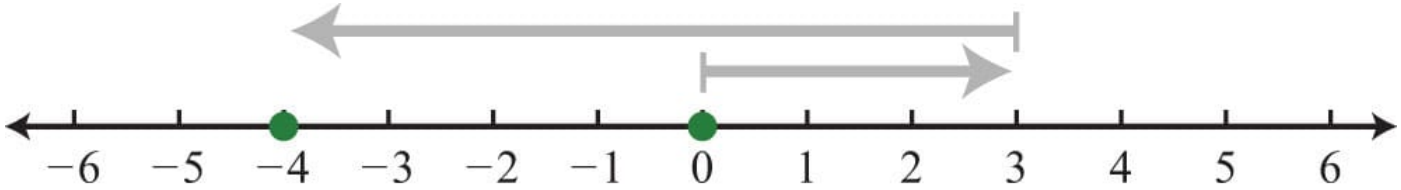


Figure 1.2.3

In this case, we can see that adding a negative number is equivalent to subtraction:

$$3 + (-7) = 3 - 7 = -4$$

It is tempting to say that a positive number plus a negative number is negative, but that is not always true: $7 + (-3) = 7 - 3 = 4$. The result of adding numbers with unlike signs may be positive or negative. The sign of the result is the same as the sign of the number with the greatest distance from the origin. For example, the following results depend on the sign of the number 12 because it is farther from zero than 5:

$$12 + (-5) = 7$$
$$-12 + 5 = -7$$

Example 1.2.1

Simplify: $14 + (-25)$.

Solution

Here -25 is the greater distance from the origin. Therefore, the result is negative.

$$\begin{aligned} 14 + (-25) &= 14 - 25 \\ &= -11 \end{aligned}$$

Answer

-11

Properties of Addition

Given any real numbers a, b, and c, we have the following properties of addition:

- **Additive Identity Property**:

$$a + 0 = 0 + a = a \tag{1.2.1}$$

- **Additive Inverse Property**:

$$a + (-a) = (-a) + a = 0 \tag{1.2.2}$$

- **Associative Property**:

$$(a + b) + c = a + (b + c) \tag{1.2.3}$$

- **Commutative Property**:

$$a + b = b + a \tag{1.2.4}$$

Below are some examples of these properties in action.

Example 1.2.2

Simplify:

a. $5 + 0$

b. $10 + (-10)$

Solution

a. Adding zero to any real number results in the same real number.

$$5 + 0 = 5$$

b. Adding opposites results in zero.

Answer

a. 5; b. 0

Simplify:

a. $(3+7)+4$

b. $3+(7+4)$

Solution

Parentheses group the operations that are to be performed first.

a.

$$\begin{aligned}(3+7)+4 &= 10+4\\ &= 14\end{aligned}$$

b.

$$\begin{aligned}3+(7+4) &= 3+10\\ &= 14\end{aligned}$$

These two examples both result in 14: changing the grouping of the numbers does not change the result.

$$(3+7)+4 = 3+(7+4) = 14$$

Answer

a. 14; b. 14

At this point, we highlight the fact that addition is commutative: the order in which we add does not matter and yields the same result.

$$\begin{aligned}2+9 &= 9+2\\ 11 &= 11\end{aligned}$$

On the other hand, subtraction is not commutative.

$$\begin{aligned}2-9 &\neq 9-2\\ -7 &\neq 7\end{aligned}$$

We will use these properties, along with the double-negative property for real numbers, to perform more involved sequential operations. To simplify things, we will make it a general rule to first replace all sequential operations with either addition or subtraction and then perform each operation in order from left to right.

Example 1.2.4

Simplify: $4-(-10)+(-5)$.

Solution

Replace the sequential operations and then perform them from left to right.

$$\begin{aligned}4-(-10)+(-5) &= 4+10-5 \qquad \begin{array}{l}\textit{Replace } -(-) \textit{ with addition } (+).\\ \textit{Replace } +(-) \textit{ with addition } (-).\end{array}\\ &= 14-5\\ &= 9\end{aligned}$$

Answer

9

Example 1.2.5

Simplify: $-3+(-8)-(-7)$.

Solution

$$\begin{aligned} -3+(-8)-(-7) &= -3-8+7 \qquad \begin{array}{l} \textit{Replace } +(-) \textit{ with } (-). \\ \textit{Replace } -(-) \textit{ with } (+). \end{array} \\ &= -11+7 \\ &= -4 \end{aligned}$$

Answer

-4

Example 1.2.6

Try this!

Simplify: $12-(-9)+(-6)$.

Solution

(click to see video)

Often we find the need to translate English sentences involving addition and subtraction to mathematical statements. Listed below are some key words that translate to the given operation.

Table 1.2.2

Key Words	Operation
Sum, increased by, more than, plus, added to, total	$+$
Difference, decreased by, subtracted from, less, minus	$-$

Example 1.2.7

What is the difference of 7 and -3?

Solution

The key word "difference" implies that we should subtract the numbers.

$$\begin{aligned} 7-(-3) &= 7+3 \\ &= 10 \end{aligned}$$

Answer

The difference of 7 and -3 is 10.

Example 1.2.8

What is the sum of the first five positive integers?

Solution

The initial key word to focus on is “sum”; this means that we will be adding the five numbers. The first five positive integers are $\{1, 2, 3, 4, 5\}$. Recall that 0 is neither positive nor negative.

$$1 + 2 + 3 + 4 + 5 = 15$$

Answer

The sum of the first five positive integers is 15.

Example 1.2.9

What is 10 subtracted from the sum of 8 and 6?

Solution

We know that subtraction is not commutative; therefore, we must take care to subtract in the correct order. First, add 8 and 6 and then subtract 10 as follows:

$$\overbrace{(8 \quad + \quad 6)}^{\text{the sum of 8 and 6}} \qquad \overbrace{- \qquad 10}^{\text{subtract 10 from the sum}}$$

It is important to notice that the phrase “10 subtracted from” does not translate to a mathematical statement in the order it appears. In other words, $10 - (8 + 6)$ would be an incorrect translation and leads to an incorrect answer. After translating the sentence, perform the operations.

$$\begin{aligned}(8 + 6) - 10 &= 14 - 10 \\ &= 4\end{aligned}$$

Answer

Ten subtracted from the sum of 8 and 6 is 4.

Distance on a Number Line

One application of the absolute value is to find the distance between any two points on a number line. For real numbers a and b, the **distance formula for a number line** is given as,

$$d = |b - a|$$

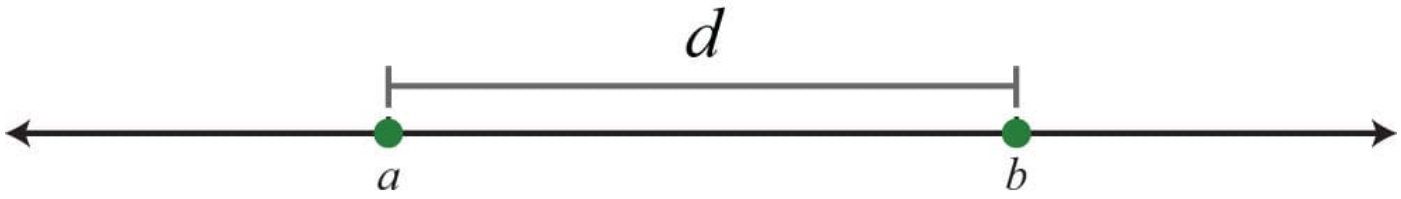


Figure 1.2.4

Example 1.2.10

Determine the distance between 2 and 7 on a number line.

Solution

On the graph we see that the distance between the two given integers is 5 units.

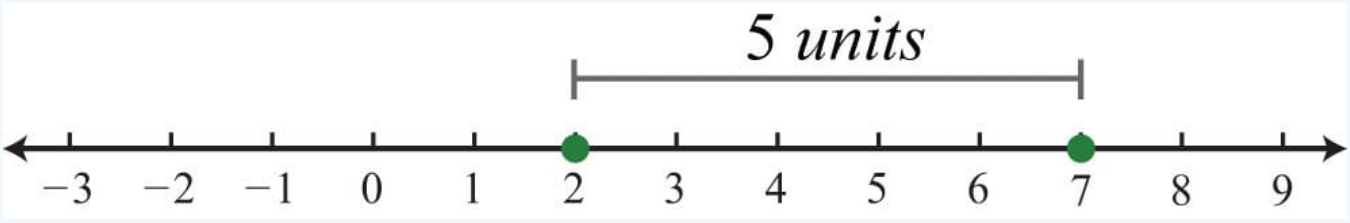


Figure 1.2.5

Using the distance formula we obtain the same result.

$$\begin{aligned} d &= |7-2| \\ &= |5| \\ &= 5 \end{aligned}$$

Answer

5 units

Example 1.2.11

Determine the distance between -4 and 7 on a number line.

Solution

Use the distance formula for a number line $d = |b-a|$, where $a = -4$ and $b = 7$.

$$\begin{aligned} d &= |7-(-4)| \\ &= |7+4| \\ &= |11| \\ &= 11 \end{aligned}$$

Answer

11 units

It turns out that it does not matter which points are used for a and$b$; the absolute value always ensures a positive result.

Table 1.2.3

Using $a = -4$ and $b = 7$	**Using $a = 7$ and $b = -4$**

$d = \|7 - (-4)\| = \|7 + 4\| = \|11\| = 11$	$d = \|-4 - 7\| = \|-11\| = 11$

Exercise 1.2.1

Determine the distance between -12 and -9 on the number line.

Answer available at the end of this section.

Key Takeaways

- A positive number added to a positive number is positive. A negative number added to a negative number is negative.
- The sign of a positive number added to a negative number is the same as the sign of the number with the greatest distance from the origin.
- Addition is commutative and subtraction is not.
- When simplifying, it is a best practice to first replace sequential operations and then work the operations of addition and subtraction from left to right.
- The distance between any two numbers on a number line is the absolute value of their difference. In other words, given any real numbers *a* and *b*, use the formula $d = |b - a|$ to calculate the distance *d* between them.

Exercise 1.2.2

Add and subtract.

1. $24 + (-18)$
2. $9 + (-11)$
3. $-31 + 5$
4. $-12 + 15$
5. $-30 + (-8)$
6. $-50 + (-25)$
7. $-7 + (-7)$
8. $-13 - (-13)$
9. $8 - 12 + 5$
10. $-3 - 7 + 4$
11. $-1 - 2 - 3 - 4$
12. $6 - (-5) + (-10) - 14$
13. $-5 + (-3) - (-7)$
14. $2 - 7 + (-9)$
15. $-30 + 20 - 8 - (-18)$
16. $10 - (-12) + (-8) - 20$
17. $5 - (-2) + (-6)$
18. $-3 + (-17) - (-13)$
19. $-10 + (-12) - (-20)$
20. $-13 + (-5) - (-25)$
21. $20 - (-4) - (-5)$
22. $17 + (-12) - (-2)$

Answer available at the end of this section.

Exercise 1.2.3

Translate each sentence to a mathematical statement and then simplify.

1. Find the sum of 3, 7, and -8.

2. Find the sum of -12, -5, and 7.
3. Determine the sum of the first ten positive integers.
4. Determine the sum of the integers in the set $\{-2, -1, 0, 1, 2\}$.
5. Find the difference of 10 and 6.
6. Find the difference of 10 and -6.
7. Find the difference of -16 and -5.
8. Find the difference of -19 and 7.
9. Subtract 12 from 10.
10. Subtract -10 from -20.
11. Subtract 5 from -31.
12. Subtract -3 from 27.
13. Two less than 8.
14. Five less than -10.
15. Subtract 8 from the sum of 4 and 7.
16. Subtract -5 from the sum of 10 and -3.
17. Subtract 2 from the difference of 8 and 5.
18. Subtract 6 from the difference of -1 and 7.
19. Mandy made a $\$200$ deposit into her checking account on Tuesday. She then wrote 4 checks for $\$50.00$, $\$125.00$, $\$60.00$, and $\$45.00$. How much more than her deposit did she spend?
20. The quarterback ran the ball three times in last Sunday's football game. He gained 7 yards on one run but lost 3 yards and 8 yards on the other two. What was his total yardage running for the game?
21. The revenue for a local photographer for the month is $\$1,200$. His costs include a studio rental of $\$600$, props costing $\$105$, materials fees of $\$135$, and a make-up artist who charges $\$120$. What is his total profit for the month?
22. An airplane flying at $30,000$ feet lost $2,500$ feet in altitude and then rose $1,200$ feet. What is the new altitude of the plane?
23. The temperature was $22\,^\circ$ at $6:00$ p.m. and dropped $26\,^\circ$ by midnight. What was the temperature at midnight?
24. A nurse has 30 milliliters of saline solution but needs 75 milliliters of the solution. How much more does she need?
25. The width of a rectangle is 2 inches less than its length. If the length measures 16 inches, determine the width.
26. The base of a triangle is 3 feet shorter than its height. If the height measures 5 feet, find the length of the base.

Answer available at the end of this section.

Exercise 1.2.4

Find the distance between the given numbers on a number line.

1. -3 and 12
2. 8 and -13
3. -25 and -10
4. -100 and -130
5. -7 and -20
6. 0 and -33
7. -10 and 10
8. -36 and 36
9. The coldest temperature on earth, $-129\,^\circ$F, was recorded in 1983 at Vostok Station, Antarctica. The hottest temperature on earth, $136\,^\circ$F, was recorded in 1922 at Al 'Aziziyah, Libya. Calculate earth's temperature range.
10. The daily high temperature was recorded as $91\,^\circ$F and the low was recorded as $63\,^\circ$F. What was the temperature range for the day?
11. A student earned 67 points on his lowest test and 87 points on his best. Calculate his test score range.
12. On a busy day, a certain website may have $12,500$ hits. On a slow day, it may have as few as 750 hits. Calculate the range of the number of hits.

Answer available at the end of this section.

Exercise 1.2.5

Discussion Board Topics

1. Share an example of adding signed numbers in a real-world application.
2. Demonstrate the associative property of addition with any three real numbers.
3. Show that subtraction is not commutative

Answers

Exercise 1.2.1

3

Video Solution

(click to see video)

Exercise 1.2.2

1: 6
3: −26
5: −38
7: −14
9: 1
11: −10
13: −1
15: 0
17: 1
19: −2
21: 29

Exercise 1.2.3

1: 2
3: 55
5: 4
7: −11
9: −2
11: −36
13: 6
15: 3
17: 1
19: $80
21: $240
23: −4 °
25: 14 inches

Exercise 1.2.4

1: 15 units
3: 15 units
5: 13 units
7: 20 units
9: 265 °F
11: 20 points

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1.3: Multiplying and Dividing Integers

Learning Objectives

- Multiply and divide signed integers.
- Translate English sentences involving multiplication and division into mathematical statements.
- Determine the prime factorization of composite numbers.
- Interpret the results of quotients involving zero.

Multiplication and Division

We begin with a review of what it means to multiply and divide signed numbers. The result of multiplying real numbers is called the **product** and the result of dividing is called the **quotient**. Recall that multiplication is equivalent to adding:

$$3 \cdot 4 = 4 + 4 + 4 = 12$$

Clearly, the product of two positive numbers is positive. Similarly, the product of a positive number and negative number can be written as shown:

We see that the product of a positive number and a negative number is negative. Next, explore the results of multiplying two negative numbers. Consider the products in the following illustration and try to identify the pattern:

$$\left.\begin{aligned} 3(-3) &= -9 \\ 2(-3) &= -6 \\ 1(-3) &= -3 \end{aligned}\right\} \quad \textit{The products increase by 3.}$$

$$0(-3) = 0 \quad \textit{Zero times any real number is zero.}$$

$$\left.\begin{aligned} (-1)(-3) &= 3 \\ (-2)(-3) &= 6 \\ (-3)(-3) &= 9 \end{aligned}\right\} \quad \textit{The pattern continues by increasing the products by 3.}$$

This shows that the product of two negative numbers is positive. To summarize,

$$\begin{aligned} \textit{positive} \times \textit{positive} &= \textit{positive} \\ \textit{positive} \times \textit{negative} &= \textit{negative} \\ \textit{negative} \times \textit{negative} &= \textit{negative} \end{aligned}$$

The rules for division are the same because division can always be rewritten as multiplication:

The rules for multiplication and division should not be confused with the fact that the sum of two negative numbers is negative.

Example 1.3.1

Simplify:

a. $(-3) + (-5)$

b. $(-3)(-5)$

Solution

Here we add and multiply the same two negative numbers.

a. The result of adding two negative numbers is negative.

$$\begin{aligned}(-3)+(-5) &= -3-5\\ &= -8\end{aligned}$$

b. The result of multiplying two negative numbers is positive.

$$(-3)(-5) = 15$$

Answer

a, -8 b.15

Given any real numbers a, b, and c, we have the following properties of multiplication:

Zero factor property:	$a \cdot 0 = 0 \cdot a = 0$
Multiplicative identity property:	$a \cdot 1 = 1 \cdot a = a$
Associative property:	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Commutative property:	$a \cdot b = b \cdot a$

Example 1.3.2

Simplify:

a. $5 \cdot 0$

b. $10 \cdot 1$

Solution

a. Multiplying by zero results in zero.

$$5 \cdot 0 = 0$$

b. Multiplying any real number by one results in the same real number.

$$10 \cdot 1 = 10$$

Answers:

a. 0 b. 10

Example 1.3.3

Simplify:

a. $(3 \cdot 7) \cdot 2$

b. $3 \cdot (7 \cdot 2)$

Solution

a.

$$(3 \cdot 7) \cdot 2 = 21 \cdot 2$$
$$= 42$$

b.

$$3 \cdot (7 \cdot 2) = 3 \cdot 14$$
$$= 42$$

The value of each expression is 42. Changing the grouping of the numbers does not change the result.

$$(3 \cdot 7) \cdot 2 = 3 \cdot (7 \cdot 2) = 42$$

Answer

a. 42 b. 42

At this point, we highlight that multiplication is commutative: the order in which we multiply does not matter and yields the same result.

$$2 \cdot 9 = 9 \cdot 2$$
$$18 = 18$$

On the other hand, division is not commutative.

$$10 \div 5 \neq 5 \div 10$$
$$2 \neq \frac{1}{2}$$

Use these properties to perform sequential operations involving multiplication and division. When doing so, it is important to perform these operations in order from left to right.

Example 1.3.4

Simplify: $3(-2)(-5)(-1)$

Solution

Multiply two numbers at a time as follows:

$$\begin{aligned} 5(-3)(-2)(-4) &= \underbrace{5(-3)}(-2)(-4) \\ &= \underbrace{-15(-2)}(-4) \\ &= \underbrace{30(-4)} \\ &= -120 \end{aligned}$$

Figure 1.3.1

Answer

-120

Because multiplication is commutative, the order in which we multiply does not affect the final answer. When sequential operations involve multiplication and division, order does matter; hence we must work the operations from left to right to obtain a correct result.

Example 1.3.5

Simplify: $10 \div (-2)(-5)$

Solution

Perform the division first; otherwise the result will be incorrect.

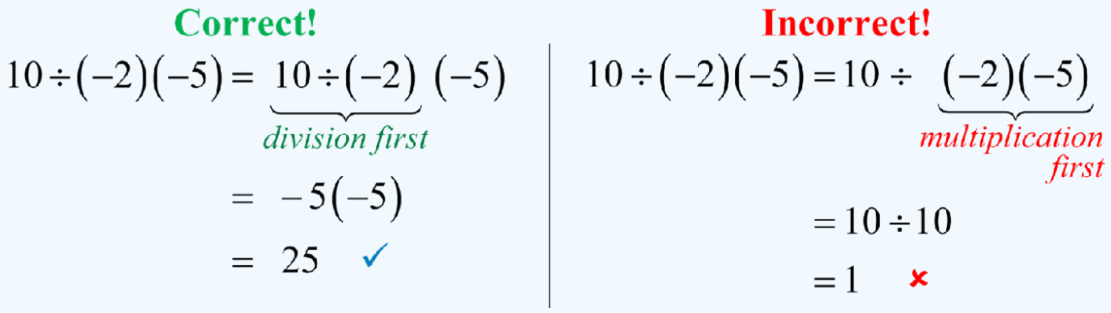


Figure 1.3.2

Notice that the order in which we multiply and divide does affect the result. Therefore, it is important to perform the operations of multiplication and division as they appear from left to right.

Answer

25

Notice that the order in which we multiply and divide does affect the final result. Therefore, it is important to perform the operations of multiplication and division as they appear from left to right.

Example 1.3.6

Simplify: $-6(3) \div (-2)(-3)$

Solution

Work the operations one at a time from left to right.

$$\begin{aligned} & -6(2) \div (-2)(-3) \\ =& -18 \div (-2)(-3) \\ =& 9(-3) \\ =& -27 \end{aligned}$$

Example 1.3.7

Try this! Simplify: $-5 \div 5 \cdot 2(-3)$

Video Solution:

(click to see video)

Within text-based applications, the symbol used for multiplication is the **asterisk** (*) and the symbol used for division is the forward slash (/).

$5 * 3$ and $14/2 = 7$

The set of **even integers** is the set of all integers that are evenly divisible by 2. We can also obtain the set of even integers by multiplying each integer by 2.

$$\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} \quad \textit{Even integers}$$

The set of **odd integers** is the set of all integers that are not evenly divisible by 2.

$$\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\} \quad \textit{Odd integers}$$

A **prime number** is an integer greater than 1 that is divisible only by 1 and itself. The smallest prime number is 2 and the rest are necessarily odd.

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\} \quad \textit{Prime numbers}$$

Any integer greater than 1 that is not prime is called a **composite number** and can be written as a product of primes. When a composite number, such as 30, is written as a product, $30 = 2 \cdot 15$, we say that $2 \cdot 15$ is a **factorization** of 30 and that 2 and 15 are **factors**. Note that factors divide the number evenly. We can continue to write composite factors as products until only a product of primes remains.

$$\begin{aligned} &30 \\ &= 2 \cdot 15 \\ &= 2 \cdot 3 \cdot 5 \end{aligned}$$

Figure $1.3.3$

The **prime factorization** of 30 is $2 \cdot 3 \cdot 5$.

Example $1.3.8$

Determine the prime factorization of 70.

Solution

Begin by writing 70 as a product with 2 as a factor. Then express any composite factor as a product of prime numbers.

$$\begin{aligned} 70 &= 2 \cdot 35 \\ &= 2 \cdot 5 \cdot 7 \end{aligned}$$

Since the prime factorization is unique, it does not matter how we choose to initially factor the number because the end result is the same.

Answer available at the end of this section.

$$\begin{aligned} 70 &= 7 \cdot 10 \\ &= 7 \cdot 2 \cdot 5 \\ &= 2 \cdot 5 \cdot 7 \end{aligned}$$

Answer

The prime factorization of 70 is $2 \cdot 5 \cdot 7$.

Some tests (called divisibility tests) useful for finding prime factors of composite numbers follow:

1. If the integer is even, then 2 is a factor.
2. If the sum of the digits is evenly divisible by 3, then 3 is a factor.

3. If the last digit is a 5 or 0, then 5 is a factor.

Often we find the need to translate English sentences that include multiplication and division terms to mathematical statements. Listed below are some key words that translate to the given operation.

Key Words	Operation
Product, multiplied by, of, times	* or $\cdot$
Quotient, divided by, ratio, per	/ or $\div$

Example 1.3.9

Calculate the quotient of 20 and -10.

Solution

The key word "quotient" implies that we should divide.

$$20 \div (-10) = -2$$

Answer:

The quotient of 20 and -10 is -2.

Example 1.3.10

What is the product of the first three positive even integers?

Solution

The first three positive even integers are {2, 4, 6} and the key word "product" implies that we should multiply.

$$\begin{aligned} 2 \cdot 4 \cdot 6 &= 8 \cdot 6 \\ &= 48 \end{aligned}$$

Answer

The product of the first three positive even integers is 48.

Example 1.3.11

Joe is able to drive 342 miles on 18 gallons of gasoline. How many miles per gallon of gas is this?

Solution

The key word "per" indicates that we must divide the number of miles driven by the number of gallons used:

$$\frac{342 \text{ miles}}{18 \text{ gallons}} = 19 \text{ miles per gallon (mpg)}$$

Answer

Joe gets 19 miles per gallon from his vehicle.

In everyday life, we often wish to use a single value that typifies a set of values. One way to do this is to use what is called the **arithmetic mean** or **average**. To calculate an average, divide the sum of the values in the set by the number of values in that set.

Example 1.3.12

A student earns 75, 86, and 94 on his first three exams. What is the student's test average?

Solution

Add the scores and divide the sum by 3.

$$\begin{aligned}\frac{75+86+94}{3} &= \frac{255}{3}\\ &= 85\end{aligned}$$

Answer

The student's test average is 85.

Zero and Division

Recall the relationship between multiplication and division:

$$\underbrace{12}_{\text{dividend}} \div \underbrace{6}_{\text{divisor}} = \underbrace{2}_{\text{quotient}} \quad \text{because} \quad 6 \cdot 2 = 12$$

Figure $1.3.4$

In this case, the **dividend** 12 is evenly divided by the **divisor** 6 to obtain the quotient, 2. It is true in general that if we multiply the divisor by the quotient we obtain the dividend. Now consider the case where the dividend is zero and the divisor is nonzero:

$$0 \div 6 = 0 \text{ Since } 6 \cdot 0 = 0$$

This demonstrates that zero divided by any nonzero real number must be zero. Now consider a nonzero number divided by zero:

$$12 \div 0 = \,? \text{ or } 0 \cdot \,? = \, 0$$

Here any real number seems to work. For example, $0 \cdot 5 = 0$ and $0 \cdot 3 = 0$. Therefore, the quotient is uncertain or **indeterminate**.

$$0 \div 0 = \tfrac{0}{0} \; \textit{Is indeterminate.}$$

In this course, we state that $0 \div 0$ is undefined.

Key Takeaways

- A positive number multiplied by a negative number is negative. A negative number multiplied by a negative number is positive.
- Multiplication is commutative and division is not.
- When simplifying, work the operations of multiplication and division in order from left to right.
- Even integers are numbers that are evenly divisible by 2 or multiples of 2, and all other integers are odd.
- A prime number is an integer greater than 1 that is divisible only by 1 and itself.
- Composite numbers are integers greater than 1 that are not prime. Composite numbers can be written uniquely as a product of primes.
- The prime factorization of a composite number is found by continuing to divide it into factors until only a product of primes remains.
- To calculate an average of a set of numbers, divide the sum of the values in the set by the number of values in the set.
- Zero divided by any nonzero number is zero. Any number divided by zero is undefined.

Exercise 1.3.1

Multiply and divide.

1. $5(-7)$
2. $-3(-8)$
3. $2(-4)(-9)$
4. $-3 \cdot 2 \cdot 5$
5. $-12(3)(0)$
6. $0(-12)(-5)$
7. $(-1)(-1)(-1)(-1)$
8. $(-1)(-1)(-1)$
9. $-100 \div 25$
10. $25 \div 5(-5)$
11. $-15(-2) \div 10(-3)$
12. $-5 \cdot 10 \div 2(-5)$
13. $(-3)(25) \div (-5)$
14. $6 * (-3)/(-9)$
15. $20/(-5) * 2$
16. $-50/2 * 5$
17. Determine the product of 11 and -3.
18. Determine the product of -7 and -22.
19. Find the product of 5 and -12.
20. Find the quotient of negative twenty-five and five.
21. Determine the quotient of -36 and 3.
22. Determine the quotient of 26 and -13.
23. Calculate the product of 3 and -8 divided by -2.
24. Calculate the product of -1 and -3 divided by 3.
25. Determine the product of the first three positive even integers.
26. Determine the product of the first three positive odd integers.

Answer available at the end of this section.

Exercise 1.3.2

Determine the prime factorization of the following integers.

1. 105
2. 78
3. 138
4. 154
5. 165
6. 330

Answer available at the end of this section.

Exercise 1.3.3

Calculate the average of the numbers in each of the following sets.

1. $\{50, 60, 70\}$
2. $\{9, 12, 30\}$
3. $\{3, 9, 12, 30, 36\}$
4. $\{72, 84, 69, 71\}$

5. The first four positive even integers.
6. The first four positive odd integers.

Answer available at the end of this section.

Exercise 1.3.4

The distance traveled D is equal to the average rate r times the time traveled t at that rate: $D = rt$. Determine the distance traveled given the rate and the time.

1. 60 miles per hour for 3 hours
2. 55 miles per hour for 3 hours
3. 15 miles per hour for 5 hours
4. 75 feet per second for 5 seconds
5. 60 kilometers per hour for 10 hours
6. 60 meters per second for 30 seconds
7. A student club ran a fund-raiser in the quad selling hot dogs. The students sold 122 hot dog meals for $\$3.00$ each. Their costs included $\$50.00$ for the hot dogs and buns, $\$25.00$ for individually wrapped packages of chips, and $\$35.00$ for the sodas. What was their profit?
8. A 230-pound man loses 4 pounds each week for 8 weeks. How much does he weigh at the end of 8 weeks?
9. Mary found that she was able to drive 264 miles on 12 gallons of gas. How many miles per gallon does her car get?
10. After filling his car with gasoline, Bill noted that his odometer reading was $45,346$ miles. After using his car for a week, he filled up his tank with 14 gallons of gas and noted that his odometer read $45,724$ miles. In that week, how many miles per gallon did Bill's car get?

Answer available at the end of this section.

Exercise 1.3.5

Perform the operations.

1. $0 \div 9$
2. $15 \div 0$
3. $4(-7) \div 0$
4. $7(0) \div (-15)$
5. $-5(0) \div 9(0)$
6. $5 \cdot 2(-3)(-5)$
7. $-8 - 5 + (-13)$
8. $-4(-8) \div 16(-2)$
9. $50 \div (-5) \div (-10)$
10. $49 \div 7 \div (-1)$
11. $3 \cdot 4 \div 12$
12. $0 - (-8) - 12$
13. $-8 \cdot 4(-3) \div 2$
14. $0/(-3 * 8 * 5)$
15. $(-4 * 3)/(2 * (-3))$
16. $-16/(-2 * 2) * 3$
17. $-44/11 * 2$
18. $-5 * 3/(-15)$
19. $4 * 3 * 2/6$
20. $-6 * 7/(-2)$
21. During 5 consecutive winter days, the daily lows were -7 °, -3 °, 0 °, -5 °, and -10 °. Calculate the average low temperature.
22. On a very cold day the temperature was recorded every 4 hours with the following results: -16 °, -10 °, 2 °, 6 °, -5 °, and -13 °. Determine the average temperature.

23. A student earns 9, 8, 10, 7, and 6 points on the first 5 chemistry quizzes. What is her quiz average?
24. A website tracked hits on its homepage over the Thanksgiving holiday. The number of hits for each day from Thursday to Sunday was $12,250$; $4,400$; $7,750$; and $10,200$, respectively. What was the average number of hits per day over the holiday period?

Answer available at the end of this section.

Exercise 1.3.1

Discussion Board Topics.

1. Demonstrate the associative property of multiplication with any three real numbers.
2. Show that division is not commutative.
3. Discuss the importance of working multiplication and division operations from left to right. Make up an example where order does matter and share the solution.
4. Discuss division involving 0. With examples, explain why the result is sometimes 0 and why it is sometimes undefined.
5. Research and discuss the fundamental theorem of arithmetic.
6. Research and discuss other divisibility tests. Provide an example for each test.
7. The arithmetic mean is one way to typify a set of values. Research other methods used to typify a set of values

Answers

Example 1.3.8

Exercise 1.3.1

1: -35
3: 72
5: 0
7: 1
9: -4
11: -9
13: 15
15: -8
17: -33
19: -60
21: -12
23: 12
25: 48

Exercise 1.3.2

1: $3 \cdot 5 \cdot 7$
3: $2 \cdot 3 \cdot 23$
5: $3 \cdot 5 \cdot 11$

Exercise 1.3.3

1: 60
2: 18
5: 5

Exercise 1.3.4

1: 180 miles
3: 75 miles
5: 600 kilometers
7: $\$256.00$
9: 22 miles per gallon
11: 0

Exercise 1.3.5

1: 0
3: Undefined
5: 0

7: -26
9: 1
11: 1
13: 48
15: 2
17: -8
19: 4
21: $-5\ ^\circ$
23: 8 points

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1.4: Fractions

Learning Objectives

- Reduce a fraction to lowest terms.
- Multiply and divide fractions.
- Add and subtract fractions.

Reducing

A **fraction** is a real number written as a quotient, or ratio, of two integers a and b, where $b \neq 0$.



Figure 1.4.1

The integer above the fraction bar is called the **numerator** and the integer below is called the **denominator**. The numerator is often called the "part" and the denominator is often called the "whole." **Equivalent fractions** are two equal ratios expressed using different numerators and denominators. For example,

$$\frac{50}{100} = \frac{1}{2}$$

Fifty parts out of 100 is the same ratio as 1 part out of 2 and represents the same real number. Consider the following factorizations of 50 and 100:

$$\begin{aligned} 50 &= 2 \cdot 25 \\ 100 &= 4 \cdot 25 \end{aligned}$$

The numbers 50 and 100 share the factor 25. A shared factor is called a common factor. We can rewrite the ratio $\frac{50}{100}$ as follows:

$$\frac{50}{100} = \frac{2 \cdot 25}{4 \cdot 25}$$

Making use of the multiplicative identity property and the fact that $\frac{25}{25} = 1$, we have

$$\frac{50}{100} = \frac{2 \cdot \cancel{25}}{4 \cdot \cancel{25}} = \frac{2}{4} \cdot 1 = \frac{2}{4}$$

Figure 1.4.2

Dividing $\frac{25}{25}$ and replacing this factor with a 1 is called **canceling**. Together, these basic steps for finding equivalent fractions define the process of **reducing**. Since factors divide their product evenly, we achieve the same result by dividing both the numerator and denominator by 25 as follows:

$$\frac{50 \div 25}{100 \div 25} = \frac{2}{4}$$

Figure 1.4.3

Finding equivalent fractions where the numerator and denominator have no common factor other than 1 is called **reducing to lowest terms**. When learning how to reduce to lowest terms, it is helpful to first rewrite the numerator and denominator as a product of primes and then cancel. For example,

$$\frac{50}{100} = \frac{\cancel{2}^{1} \cdot \cancel{5}^{1} \cdot \cancel{5}^{1}}{2 \cdot \cancel{2}^{1} \cdot \cancel{5}^{1} \cdot \cancel{5}^{1}}$$
$$= \frac{1}{2}$$

Figure 1.4.4

We achieve the same result by dividing the numerator and denominator by the **greatest common factor** (GCF). The GCF is the largest number that divides both the numerator and denominator evenly. One way to find the GCF of 50 and 100 is to list all the factors of each and identify the largest number that appears in both lists. Remember, each number is also a factor of itself.

$\{1, 2, 5, 10, 25, 50\}$ $\quad$ *Factors of* 50
$\{1, 2, 4, 5, 10, 20, 25, 50, 100\}$ $\quad$ *Factors of* 100

Common factors are listed in bold, and we see that the greatest common factor is 50. We use the following notation to indicate the GCF of two numbers: $\text{GCF}(50, 100) = 50$. After determining the GCF, reduce by dividing both the numerator and the denominator as follows:

$$\frac{50 \div 50}{100 \div 50} = \frac{1}{2}$$

Figure 1.4.5

Example 1.4.1

Reduce to lowest terms: $\frac{105}{300}$.

Solution

Rewrite the numerator and denominator as a product of primes and then cancel.

$$\frac{105}{300} = \frac{\overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{5}} \cdot 7}{2 \cdot 2 \cdot \underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{5}} \cdot 5}$$
$$= \frac{7}{2 \cdot 2 \cdot 5}$$
$$= \frac{7}{20}$$

Figure 1.4.6

Alternatively, we achieve the same result if we divide both the numerator and denominator by the GCF$(105, 300)$. A quick way to find the GCF of the two numbers requires us to first write each as a product of primes. The GCF is the product of all the common prime factors.

$$\left.\begin{aligned} 105 &= 3 \cdot 5 \cdot 7 \\ 300 &= 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \end{aligned}\right\} \quad \text{GCF}(105, 300) = 3 \cdot 5 = 15$$

Figure 1.4.7

In this case, the common prime factors are 3 and 5 and the greatest common factor of 105 and 300 is 15.

$$\frac{105}{300} = \frac{105 \div 15}{300 \div 15} = \frac{7}{20}$$

Figure 1.4.8

Answer:

$\frac{7}{20}$

Example 1.4.2

Try this! Reduce to lowest terms: $\frac{32}{96}$.

Video Solution:

(click to see video)

An **improper fraction** is one where the numerator is larger than the denominator. A mixed number is a number that represents the sum of a whole number and a fraction. For example, $5\frac{1}{2}$ is a mixed number that represents the sum $5 + \frac{1}{2}$. Use long division to convert an improper fraction to a mixed number; the remainder is the numerator of the fractional part.

Example 1.4.3

Write $\frac{23}{5}$ as a mixed number.

Solution

Notice that 5 divides into 23 four times with a remainder of 3.

$$\begin{array}{r} 4 \\ 5\overline{)\,23} \\ -20 \\ \hline 3 \end{array}$$

Figure 1.4.9

We then can write

$$\begin{aligned} \frac{23}{5} &= 4 + \frac{3}{5} \\ &= 4\frac{3}{5} \end{aligned}$$

Note that the denominator of the fractional part of the mixed number remains the same as the denominator of the original fraction.

Answer

$4\frac{3}{5}$

To convert mixed numbers to improper fractions, multiply the whole number by the denominator and then add the numerator; write this result over the original denominator.

Example 1.4.4

Write $3\frac{5}{7}$ as an improper fraction.

Solution

Obtain the numerator by multiplying 7 times 3 and then add 5.

$$\begin{aligned} 3\frac{5}{7} &= \frac{7 \cdot 3 + 5}{7} \\ &= \frac{21 + 5}{7} \\ &= \frac{26}{7} \end{aligned}$$

Answer

$\frac{26}{7}$

It is important to note that converting to a mixed number is not part of the reducing process. We consider improper fractions, such as 267, to be reduced to lowest terms. In algebra it is often preferable to work with improper fractions, although in some applications, mixed numbers are more appropriate.

Example 1.4.5

Try this! Convert $10\frac{1}{2}$ to an improper fraction.

Solution

(click to see video)

Multiplying and Dividing Fractions

In this section, assume that a, b, c, and d are all nonzero integers. The product of two fractions is the fraction formed by the product of the numerators and the product of the denominators. In other words, to multiply fractions, multiply the numerators and multiply the denominators:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Example 1.4.6

Multiply: $\frac{2}{3} \cdot \frac{5}{7}$

Solution

Multiply the numerators and multiply the denominators.

$$\begin{aligned} \frac{2}{3} \cdot \frac{5}{7} &= \frac{2 \cdot 5}{3 \cdot 7} \\ &= \frac{10}{21} \end{aligned}$$

Answer:

$\frac{10}{21}$

Example 1.4.7

Multiply: $\frac{5}{9}\left(-\frac{1}{4}\right)$

Solution

Recall that the product of a positive number and a negative number is negative.

$$\begin{aligned} \frac{5}{9}\left(-\frac{1}{4}\right) &= -\frac{5 \cdot 1}{9 \cdot 4} \\ &= -\frac{5}{36} \end{aligned}$$

Answer:

$-\frac{5}{36}$

Example 1.4.8

Multiply: $\frac{2}{3} \cdot 5\frac{3}{4}$

Solution

Begin by converting $5\frac{3}{4}$ to an improper fraction.

$$\begin{aligned}\frac{2}{3} \cdot 5\frac{3}{4} &= \frac{2}{3} \cdot \frac{23}{4} \\ &= \frac{\cancel{2}^{1}}{3} \cdot \frac{23}{\cancel{4}_{2}} \\ &= \frac{23}{6} \\ &= 3\frac{5}{6}\end{aligned}$$

Figure 1.4.10

In this example, we noticed that we could reduce before we multiplied the numerators and the denominators. Reducing in this way is called **cross canceling**, and can save time when multiplying fractions.

Answer

$3\frac{5}{6}$

Two real numbers whose product is 1 are called **reciprocals**. Therefore, $\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals because $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = 1$. For example,

$$\frac{2}{3} \cdot \frac{3}{2} = \frac{6}{6} = 1$$

Because their product is 1, $\frac{2}{3}$ and $\frac{3}{2}$ are reciprocals. Some other reciprocals are listed below:

$$\frac{5}{8} \text{ and } \frac{8}{5} \qquad 7 \text{ and } \frac{1}{7} \qquad -\frac{4}{5} \text{ and } -\frac{5}{4}$$

This definition is important because dividing fractions requires that you multiply the dividend by the reciprocal of the divisor.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Example 1.4.9

Divide: $\frac{2}{3} \div \frac{5}{7}$

Solution

Multiply $\frac{2}{3}$ by the reciprocal of $\frac{5}{7}$.

$$\begin{aligned}\frac{2}{3} \div \frac{5}{7} &= \frac{2}{3} \cdot \frac{7}{5} \\ &= \frac{2 \cdot 7}{3 \cdot 5} \\ &= \frac{14}{15}\end{aligned}$$

Answer:

\(\frac{14}{15\)

You also need to be aware of other forms of notation that indicate division: / and —. For example,

$$5/(1/2) = 5 * (2/1) = (5/1) * (2/1) = 10/1 = 10$$

Or

$$\frac{\frac{7}{8}}{\frac{2}{3}} = \frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot \frac{3}{2} = \frac{21}{16}$$

The latter is an example of a **complex fraction**, which is a fraction whose numerator, denominator, or both are fractions.

Note

Students often ask why dividing is equivalent to multiplying by the reciprocal of the divisor. A mathematical explanation comes from the fact that the product of reciprocals is 1. If we apply the multiplicative identity property and multiply numerator and denominator by the reciprocal of the denominator, then we obtain the following:

$$\frac{\frac{7}{8}}{\frac{2}{3}} = \frac{\frac{7}{8}}{\frac{2}{3}} \cdot 1 = \frac{\frac{7}{8}}{\frac{2}{3}} \cdot \frac{\frac{3}{2}}{\frac{3}{2}} = \frac{\frac{7}{8} \cdot \frac{3}{2}}{1} = \frac{7}{8} \cdot \frac{3}{2}$$

Figure 1.4.11

Before multiplying, look for common factors to cancel; this eliminates the need to reduce the end result.

Example 1.4.10

Divide: $\frac{\frac{5}{2}}{\frac{7}{4}}$.

Solution

$$\frac{\frac{5}{2}}{\frac{7}{4}} = \frac{5}{2} \div \frac{7}{4}$$
$$= \frac{5}{2} \cdot \frac{4}{7}$$
$$= \frac{5}{\cancel{2}_{1}} \cdot \frac{\cancel{4}^{2}}{7}$$
$$= \frac{5 \cdot 2}{1 \cdot 7}$$
$$= \frac{10}{7}$$

Figure 1.4.12

Answer

$\frac{10}{7}$

When dividing by an integer, it is helpful to rewrite it as a fraction over 1.

Example 1.4.11

Divide: $\frac{2}{3} \div 6$

Solution

Rewrite 6 as $\frac{6}{1}$ and multiply by its reciprocal.

$$\frac{2}{3} \div 6 = \frac{2}{3} \div \frac{6}{1}$$
$$= \frac{2}{3} \cdot \frac{1}{6}$$
$$= \frac{\cancel{2}^{1}}{3} \cdot \frac{1}{\cancel{6}_{3}}$$
$$= \frac{1 \cdot 1}{3 \cdot 3}$$
$$= \frac{1}{9}$$

Figure 1.4.13

Answer:

$\frac{1}{9}$

Also, note that we only cancel when working with multiplication. Rewrite any division problem as a product *before* canceling.

Example 1.4.12

Try this! Divide: $5 \div 2\frac{3}{5}$

Video Solution:

(click to see video)

Adding and Subtracting Fractions

Negative fractions are indicated with the negative sign in front of the fraction bar, in the numerator, or in the denominator. All such forms are equivalent and interchangeable.

$$\frac{-3}{4} = -\frac{3}{4} = \frac{3}{-4}$$

Adding or subtracting fractions requires a **common denominator**. In this section, assume the common denominator c is a nonzero integer.

It is good practice to use positive common denominators by expressing negative fractions with negative numerators. In short, avoid negative denominators.

Example 1.4.13

Subtract: $\frac{12}{15} - \frac{3}{15}$

Solution

The two fractions have a common denominator 15. Therefore, subtract the numerators and write the result over the common denominator:

$$\begin{aligned} \frac{12}{15} - \frac{3}{15} &= \frac{12-3}{15} && \textit{Subtract the numerators.} \\ &= \frac{9}{15} \\ &= \frac{9 \div 3}{15 \div 3} && \textit{Reduce.} \\ &= \frac{3}{5} \end{aligned}$$

Answer

$\frac{3}{5}$

Most problems that you are likely to encounter will have **unlike denominators**. In this case, first find equivalent fractions with a common denominator before adding or subtracting the numerators. One way to obtain equivalent fractions is to divide the numerator and the denominator by the same number. We now review a technique for finding equivalent fractions by multiplying the numerator and the denominator by the same number. It should be clear that $5/5$ is equal to 1 and that 1 multiplied times any number is that number:

$$\frac{1}{2} = \frac{1}{2} \cdot 1 = \frac{1}{2} \cdot \frac{5}{5} = \frac{5}{10}$$

We have equivalent fractions $\frac{1}{2} = \frac{5}{10}$. Use this idea to find equivalent fractions with a common denominator to add or subtract fractions. The steps are outlined in the following example.

Example 1.4.14

Subtract: $\frac{7}{15} - \frac{3}{10}$

Solution

Step 1: Determine a common denominator. To do this, use the **least common multiple** (LCM) of the given denominators. The LCM of 15 and 10 is indicated by LCM$(15, 10)$. Try to think of the smallest number that both denominators divide into evenly. List the multiples of each number:

$$\{10, 20, \mathbf{30}, 40, 50, \mathbf{60}, \ldots\} \quad \textit{Multiples of 10}$$
$$\{15, \mathbf{30}, 45, \mathbf{60}, 75, 90, \ldots\} \quad \textit{Multiples of 15}$$

Common multiples are listed in bold, and the least common multiple is 30.

$$\text{LCM}(10, 15) = 30$$

Step 2: Multiply the numerator and the denominator of each fraction by values that result in equivalent fractions with the determined common denominator.

$$\begin{aligned} \frac{7}{15} - \frac{3}{10} &= \frac{7 \cdot 2}{15 \cdot 3} - \frac{3 \cdot 3}{10 \cdot 3} \\ &= \frac{14}{30} - \frac{9}{30} \end{aligned}$$

Step 3: Add or subtract the numerators, write the result over the common denominator and then reduce if possible.

$$\begin{aligned} \frac{14}{30} - \frac{9}{30} &= \frac{14 - 9}{30} \\ &= \frac{5}{30} \\ &= \frac{5 \div 5}{30 \div 5} \\ &= \frac{1}{6} \end{aligned}$$

Answer:

$\frac{1}{6}$

The least common multiple of the denominators is called the **least common denominator** (LCD). Finding the LCD is often the difficult step. It is worth finding because if any common multiple other than the least is used, then there will be more steps involved when reducing.

Example 1.4.15

Add: $\frac{5}{10} + \frac{1}{18}$

Solution

First, determine that the LCM$(10, 18)$ is 90 and then find equivalent fractions with 90 as the denominator.

$$\begin{aligned}\frac{5}{10}+\frac{1}{18} &= \frac{5\cdot 9}{10\cdot 9}+\frac{1\cdot 5}{18\cdot 5}\\ &= \frac{45}{90}+\frac{5}{90}\\ &= \frac{45+5}{90}\\ &= \frac{50}{90}\\ &= \frac{50\div 10}{90\div 10}\\ &= \frac{5}{9}\end{aligned}$$

Answer

$\frac{5}{9}$

Example 1.4.16

Try this! Add: $\frac{2}{30}+\frac{5}{21}$

Video Solution:

(click to see video)

Example 1.4.17

Simplify: $2\frac{1}{3}+\frac{3}{5}-\frac{1}{2}$

Solution

Begin by converting $2\frac{1}{3}$ to an improper fraction.

$$\begin{aligned}2\frac{1}{3}+\frac{3}{5}-\frac{1}{2} &= \frac{7}{3}+\frac{3}{5}-\frac{1}{2} && \textit{Convert to improper fractions.}\\ &= \frac{7\cdot 10}{3\cdot 10}+\frac{3\cdot 6}{5\cdot 6}-\frac{1\cdot 15}{2\cdot 15} && \text{LCM}(3,\ 5,\ 2)\ =\ 30\\ &= \frac{70}{30}+\frac{18}{30}-\frac{15}{30} && \textit{Equivalent fractions with a common denominator}\\ &= \frac{70+18-15}{30}\\ &= \frac{73}{30}\\ &= 2\frac{13}{30}\end{aligned}$$

Answer:

$2\frac{13}{30}$

In general, it is preferable to work with improper fractions. However, when the original problem involves mixed numbers, if appropriate, present your answers as mixed numbers. Also, mixed numbers are often preferred when working with numbers on a number line and with real-world applications.

Example 1.4.18

Subtract: $\frac{5}{7} - 2\frac{1}{7}$

Example 1.4.19

How many $\frac{1}{2}$ inch thick paperback books can be stacked to fit on a shelf that is $1\frac{1}{2}$ feet in height?

Solution

First, determine the height of the shelf in inches. To do this, use the fact that there are 12 inches in 1 foot and multiply as follows:

$$\begin{aligned} 1\frac{1}{2}\text{ ft} &= 1\frac{1}{2}\text{ ft} \cdot \left(\frac{12\text{ in}}{1\text{ ft}}\right) \\ &= \frac{3}{\cancel{2}_{1}} \cdot \frac{\cancel{12}^{6}}{1}\text{ in} \\ &= 18\text{ in} \end{aligned}$$

Next, determine how many notebooks will fit by dividing the height of the shelf by the thickness of each book.

$$\begin{aligned} 18\text{ in} \div \frac{1}{2}\text{ in} &= 18 \cdot \frac{2}{1} \\ &= 18 \cdot 2 \\ &= 36 \end{aligned}$$

Answer

36 books can be stacked on the shelf.

Key Takeaways:

- Fractions are not unique; there are many ways to express the same ratio. Find equivalent fractions by multiplying or dividing the numerator and the denominator by the same real number.
- Equivalent fractions in lowest terms are generally preferred. It is a good practice to always reduce.

- In algebra, improper fractions are generally preferred. However, in real-life applications, mixed number equivalents are often preferred. We may present answers as improper fractions unless the original question contains mixed numbers, or it is an answer to a real-world or geometric application.
- Multiplying fractions does not require a common denominator; multiply the numerators and multiply the denominators to obtain the product. It is a best practice to cancel any common factors in the numerator and the denominator before multiplying.
- Reciprocals are rational numbers whose product is equal to 1. Given a fraction $\frac{a}{b}$, its reciprocal is $\frac{b}{a}$.
- Divide fractions by multiplying the dividend by the reciprocal of the divisor. In other words, multiply the numerator by the reciprocal of the denominator.
- Rewrite any division problem as a product *before* canceling.
- Adding or subtracting fractions requires a common denominator. When the denominators of any number of fractions are the same, simply add or subtract the numerators and write the result over the common denominator.
- Before adding or subtracting fractions, ensure that the denominators are the same by finding equivalent fractions with a common denominator. Multiply the numerator and the denominator of each fraction by the appropriate value to find the equivalent fractions.
- Typically, it is best to convert all mixed numbers to improper fractions before beginning the process of adding, subtracting, multiplying, or dividing.

Exercise 1.4.1

Reduce each fraction to lowest terms.

1. $\frac{5}{30}$
2. $\frac{6}{24}$
3. $\frac{30}{70}$
4. $\frac{18}{27}$
5. $\frac{44}{84}$
6. $\frac{54}{90}$
7. $\frac{135}{30}$
8. $\frac{105}{300}$
9. $\frac{18}{6}$
10. $\frac{256}{16}$
11. $\frac{126}{45}$
12. $\frac{52}{234}$
13. $\frac{54}{162}$
14. $\frac{2000}{3000}$
15. $\frac{270}{360}$

Answer available at the end of this section.

Exercise 1.4.2

Rewrite as an improper fraction.

1. $4\frac{3}{4}$
2. $2\frac{1}{2}$
3. $5\frac{7}{15}$
4. $1\frac{1}{2}$
5. $3\frac{5}{8}$
6. $1\frac{3}{4}$
7. $-2\frac{1}{2}$
8. $-1\frac{3}{4}$

Answer available at the end of this section.

Exercise 1.4.3

Rewrite as a mixed number.

1. $\frac{15}{2}$
2. $\frac{9}{2}$
3. $\frac{40}{13}$
4. $\frac{103}{25}$
5. $\frac{73}{10}$
6. $\frac{-52}{7}$
7. $\frac{-59}{6}$

Answer available at the end of this section.

Exercise 1.4.4

Multiply and reduce to lowest terms.

1. $\frac{2}{3} \cdot \frac{5}{7}$
2. $\frac{1}{5} \cdot \frac{4}{8}$
3. $\frac{1}{2} \cdot \frac{1}{3}$
4. $\frac{3}{4} \cdot \frac{20}{9}$
5. $\frac{5}{7} \cdot \frac{49}{10}$
6. $\frac{2}{3} \cdot \frac{9}{12}$
7. $\frac{6}{14} \cdot \frac{21}{12}$
8. $\frac{44}{15} \cdot \frac{15}{11}$
9. $3\frac{3}{4} \cdot 2\frac{1}{3}$
10. $2\frac{7}{10} \cdot 5\frac{5}{6}$
11. $\frac{3}{11}(-\frac{5}{2})$
12. $-\frac{4}{5}(\frac{9}{5})$
13. $(-\frac{9}{5}(-\frac{3}{10})$
14. $\frac{6}{7}(-\frac{14}{3})$
15. $(-\frac{9}{12})(-\frac{4}{8})$
16. $-\frac{3}{8}(-\frac{4}{15})$
17. $\frac{1}{7} \cdot \frac{1}{2} \cdot \frac{1}{3}$
18. $\frac{3}{5} \cdot \frac{15}{21} \cdot \frac{7}{27}$
19. $\frac{2}{5} \cdot 3\frac{1}{8} \cdot \frac{4}{5}$
20. $2\frac{4}{9} \cdot \frac{2}{5} \cdot 2\frac{5}{11}$

Answer available at the end of this section.

Exercise 1.4.5

Determine the reciprocal of the following numbers.

1. $\frac{1}{2}$
2. $\frac{8}{5}$
3. $-\frac{2}{3}$
4. $-\frac{4}{3}$
5. 10
6. -4
7. $2\frac{1}{3}$
8. $1\frac{5}{8}$

Answer available at the end of this section.

Exercise 1.4.6

Divide and reduce to lowest terms.

1. $\frac{1}{2} \div \frac{2}{3}$
2. $\frac{5}{9} \div \frac{1}{3}$
3. $\frac{5}{8} \div (-\frac{4}{5})$
4. $(-\frac{2}{5}) \div \frac{15}{3}$
5. $\frac{-\frac{6}{7}}{-\frac{6}{7}}$
6. $\frac{-\frac{1}{2}}{\frac{1}{4}}$
7. $\frac{-\frac{10}{3}}{-\frac{5}{20}}$
8. $\frac{\frac{2}{3}}{\frac{9}{2}}$
9. $\frac{\frac{30}{50}}{\frac{5}{3}}$
10. $\frac{\frac{1}{2}}{2}$
11. $\frac{3}{\frac{2}{5}}$
12. $\frac{-6}{\frac{5}{4}}$
13. $2\frac{1}{2} \div \frac{5}{3}$
14. $4\frac{2}{3} \div 3\frac{1}{2}$
15. $5 \div 2\frac{3}{5}$
16. $4\frac{3}{5} \div 23$

Answer available at the end of this section.

Exercise 1.4.7

Add or subtract and reduce to lowest terms.

1. $\frac{17}{20} - \frac{5}{20}$
2. $\frac{4}{9} - \frac{13}{9}$
3. $\frac{3}{5} + \frac{1}{5}$
4. $\frac{11}{15} + \frac{9}{15}$
5. $\frac{5}{7} - 2\frac{1}{7}$
6.
7. $\frac{1}{2} + \frac{1}{3}$
8. $\frac{1}{5} - \frac{1}{4}$
9. $\frac{3}{4} - \frac{5}{2}$
10. $\frac{3}{8} + \frac{7}{16}$
11. $\frac{7}{15} - \frac{3}{10}$
12. $\frac{3}{10} + \frac{2}{14}$
13. $\frac{2}{30} + \frac{5}{21}$
14. $\frac{3}{18} - \frac{1}{24}$
15. $5\frac{1}{2} + 2\frac{1}{3}$
16. $1\frac{3}{4} + 2\frac{1}{10}$
17. $\frac{1}{2} + \frac{1}{3} + \frac{1}{6}$
18. $\frac{2}{3} + \frac{3}{5} - \frac{2}{9}$
19. $\frac{7}{3} - \frac{3}{2} + \frac{2}{15}$
20. $\frac{9}{4} - \frac{3}{2} + \frac{3}{8}$
21.
22. $\frac{2}{3} - 4\frac{1}{2} + 3\frac{1}{6}$
23. $1 - \frac{6}{16} + \frac{3}{18}$
24. $3 - \frac{1}{21} - \frac{1}{15}$

Answer available at the end of this section.

Exercise 1.4.8

Perform the operations. Reduce answers to lowest terms.

1. $\frac{3}{14} \cdot \frac{7}{3} \div \frac{1}{8}$
2. $\frac{1}{2} \cdot (-\frac{4}{5}) \div \frac{14}{15}$
3. $\frac{1}{2} \div \frac{3}{4} \cdot \frac{1}{5}$
4. $-\frac{5}{9} \div \frac{5}{3} \cdot \frac{5}{2}$
5.
6.
7. $\frac{4}{5} \div 4 \cdot \frac{1}{2}$
8. $\frac{5}{3} \div 15 \cdot \frac{2}{3}$
9. What is the product of $\frac{3}{16}$ and $\frac{4}{9}$?
10. What is the product of $-\frac{24}{5}$ and $\frac{25}{8}$?
11. What is the quotient of $\frac{5}{9}$ and $\frac{25}{3}$?
12. What is the quotient of $-\frac{16}{5}$ and 32?
13. Subtract $\frac{1}{6}$ from the sum of $\frac{9}{2}$ and $\frac{2}{3}$.
14. Subtract $\frac{1}{4}$ from the sum of $\frac{3}{4}$ and $\frac{6}{5}$.
15. What is the total width when 3 boards, each with a width of $2\frac{5}{8}$ inches, are glued together?
16. The precipitation in inches for a particular 3-day weekend was published as $\frac{3}{10}$ inches on Friday, $1\frac{1}{2}$ inches on Saturday, and $\frac{3}{4}$ inches on Sunday. Calculate the total precipitation over this period.
17. A board that is $5\frac{1}{4}$ feet long is to be cut into 7 pieces of equal length. What is length of each piece?
18. How many $\frac{3}{4}$ inch thick notebooks can be stacked into a box that is 2 feet high?
19. In a mathematics class of 44 students, one-quarter of the students signed up for a special Saturday study session. How many students signed up?
20. Determine the length of fencing needed to enclose a rectangular pen with dimensions $35\frac{1}{2}$ feet by $20\frac{2}{3}$ feet.
21. Each lap around the track measures $\frac{1}{4}$ mile. How many laps are required to complete a $2\frac{1}{2}$ mile run?

22. A retiree earned a pension that consists of three-fourths of his regular monthly salary. If his regular monthly salary was $\$5,200$, then what monthly payment can the retiree expect from the pension plan?

Answer available at the end of this section.

Exercise 1.4.1

Discussion Board Topics

1. Does 0 have a reciprocal? Explain.
2. Explain the difference between the LCM and the GCF. Give an example.
3. Explain the difference between the LCM and LCD.
4. Why is it necessary to find an LCD in order to add or subtract fractions?
5. Explain how to determine which fraction is larger, $\frac{7}{16}$ or $\frac{1}{2}$.

Answers

Exercise 1.4.1

1: $1/6$
3: $3/7$
5: $11/21$
7: $9/2$
9: 3
11: $14/5$
13: $1/3$
15: $3/4$

Exercise 1.4.2

1: $5/2$
3: $3/2$
5: $7/4$
7: $-7/4$

Exercise 1.4.3

2: $4\frac{1}{2}$
4: $4\frac{3}{25}$
6: $-7\frac{3}{7}$

Exercise 1.4.4

1: $10/21$
3: $1/6$
5: $7/2$
7: $3/4$
9: 834
11: $-15/22$
13:$27/50$
15: $3/8$
17: $1/42$
19: 1

Exercise 1.4.5

1: 2
3: $-3/2$
5: $1/10$
7: $3/7$
9: $3/4$

Exercise 1.4.6

1: $3/4$
3: $-25/32$
5: 1
7: $40/3$

9: $9/25$
11: $25/2$
13: $1\frac{1}{2}$
15: $1\frac{12}{13}$

Exercise 1.4.7

1: $3/5$
3: $4/5$
5: $-1\frac{3}{7}$
7: $5/6$
9: $-7/4$
11: $1/6$
13: $32/105$
15: $7\frac{5}{6}$
17: 1
19: $29/30$
21: $2\frac{2}{3}$
23: $19/24$

Exercise 1.4.8

1: 4
3: $2/15$
5: $9/28$
7: $1/10$
9: $1/12$
11: $1/15$
13: 5
15: $7\frac{7}{8}$ inches
17: $\frac{3}{4}$ feet
19: 11 students
21: 10 laps

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1.5: Review of Decimals and Percents

Skills to Develop

- Convert fractions to decimals and back.
- Perform operations with decimals.
- Round off decimals to a given place.
- Define a percent.
- Convert percents to decimals and back.
- Convert fractions to percents and back.

Decimals

In this section, we provide a brief review of the decimal system. A real number in decimal form, a **decimal** consists of a decimal point, digits (0 through 9) to the left of the decimal point representing the whole number part, and digits to the right of the decimal point representing the fractional part. The digits represent powers of 10 as shown in the set $\{\ldots, 1,000, 100, 10, 1, 1/10, 1/100, 1/1,000, \ldots\}$according to the following diagram:

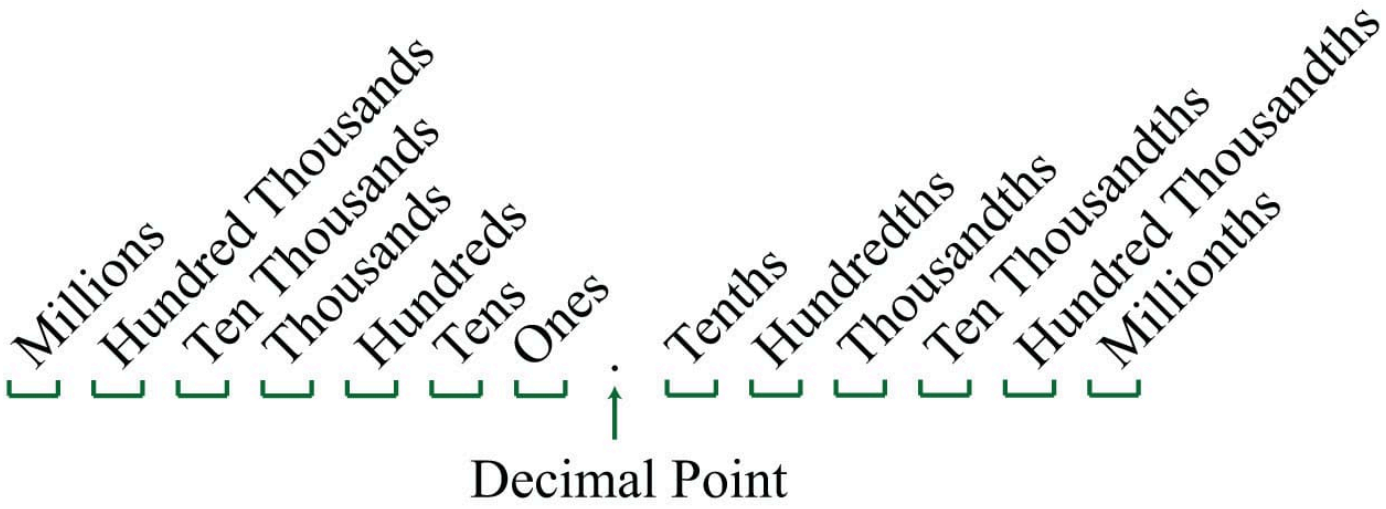


Figure 1.5.1

For example, the decimal 538.3 can be written in the following expanded form:

$$5 \cdot 100 + 3 \cdot 10 + 8 \cdot 1 + 3 \cdot \tfrac{1}{10}$$

After simplifying, we obtain the mixed number $528\frac{3}{10}$. Use this process to convert decimals to mixed numbers.

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1.6: Exponents and Square Roots

Learning Objectives

- Interpret exponential notation with positive integer exponents.
- Calculate the nth power of a real number.
- Calculate the exact and approximate value of the square root of a real number.

Exponential Notation and Positive Integer Exponents

If a number is repeated as a factor numerous times, then we can write the product in a more compact form using exponential notation. For example,

$$5 \cdot 5 \cdot 5 \cdot 5 = 5^4$$

The base is the factor, and the positive integer exponent indicates the number of times the base is repeated as a factor. In the above example, the base is 5 and the exponent is 4. In general, if a is the base that is repeated as a factor n times, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a}$$

Figure $1.6.1$

When the exponent is 2, we call the result a square. For example,

$$3^2 = 3 \cdot 3 = 9$$

The number 3 is the base and the integer 2 is the exponent. The notation 3^2 can be read two ways: “three squared” or “3 raised to the second power.” The base can be any real number.

It is important to study the difference between the ways the last two examples are calculated. In the example $(-7)^2$, the base is -7 as indicated by the parentheses. In the example -5^2, the base is 5, not -5, so only the 5 is squared and the result remains negative. To illustrate this, write

$-5^2 = -1 \cdot 5^2 = -1 \cdot 5 \cdot 5 = -25$

This subtle distinction is very important because it determines the sign of the result.

The textual notation for exponents is usually denoted using the caret $()$ symbol as follows:

$$8^2 = 8 \wedge 2 = 8 * 8 = 64$$
$$-5.1^2 = -5.1 \wedge 2 = -5.1 * 5.1 = -26.01$$

The square of an integer is called a perfect square. The ability to recognize perfect squares is useful in our study of algebra. The squares of the integers from 1 to 15 should be memorized. A partial list of perfect squares follows:

$$\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, \ldots\}$$

Exercise 1.6.1

Simplify

$$(-12)^2.$$

Answer available at the end of this section.

When the exponent is 3 we call the result a cube. For example,

$$3^3 = 3 \cdot 3 \cdot 3 = 27$$

The notation 3^3 can be read two ways: "three cubed" or "3 raised to the third power." As before, the base can be any real number.

Note that the result of cubing a negative number is negative. The cube of an integer is called a perfect cube. The ability to recognize perfect cubes is useful in our study of algebra. The cubes of the integers from 1 to 10 should be memorized. A partial list of perfect cubes follows:

$\{0, 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...\}$

Exercise 1.6.2

Simplify $(-2)^3$.

Answer available at the end of this section.

If the exponent is greater than 3, then the notation an is read "a raised to the nth power."

$$\begin{aligned} 10^6 &= 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1,000,000 \\ (-1)^4 &= (-1)(-1)(-1)(-1) = 1 \\ \left(\frac{1}{3}\right)^5 &= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{243} \end{aligned}$$

Notice that the result of a negative base with an even exponent is positive. The result of a negative base with an odd exponent is negative. These facts are often confused when negative numbers are involved. Study the following four examples carefully:

Table 1.6.1

The base is (-2)	The base is 2
$(-2)^4 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) = +16$ $(-2)^3 = (-2) \cdot (-2) \cdot (-2) = -8$	$-2^4 = -2 \cdot 2 \cdot 2 \cdot 2 = -16$ $-2^3 = -2 \cdot 2 \cdot 2 = -8$

The parentheses indicate that the negative number is to be used as the base.

Example 1.6.1

Calculate:

a. $\left(-\frac{1}{3}\right)^3$
b. $\left(-\frac{1}{3}\right)^4$

Solution:

The base is $-\frac{1}{3}$ for both problems.

a. Use the base as a factor three times.

$$\begin{aligned}\left(-\frac{1}{3}\right)^3 &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\\ &= -\frac{1}{27}\end{aligned}$$

b. Use the base as a factor four times.

$$\begin{aligned}\left(-\frac{1}{3}\right)^4 &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\\ &= +\frac{1}{81}\end{aligned}$$

Answer:

a. $-\frac{1}{27}$; b. $\frac{1}{81}$

Exercise 1.6.3

Simplify:

-10^4 and $(-10)^4$.

Answer available at the end of this section.

Square Root of a Real Number

Think of finding the square root of a number as the inverse of squaring a number. In other words, to determine the square root of 25 the question is, "What number squared equals 25?" Actually, there are two answers to this question, 5 and -5.

$$5^2 = 25 \quad \text{and} (-5)^2 = 25$$

When asked for the square root of a number, we implicitly mean the principal (nonnegative) square root. Therefore we have,

$\sqrt{a^2} = a$, if $a \geq 0$ or more generally $\sqrt{a^2} = |a|$

As an example, $\sqrt{25} = 5$, which is read "square root of 25 equals 5." The symbol $\sqrt{}$ is called the radical sign and 25 is called the radicand. The alternative textual notation for square roots follows:

$$\sqrt{16} = textsqrt(16) = 4$$

It is also worthwhile to note that

$$\sqrt{1} = 1 \quad \text{and} \quad \sqrt{0} = 0$$

This is the case because $1^2 = 1$ and $0^2 = 0$.

Example 1.6.2

Simplify:

$\sqrt{10,000}$.

Solution:

$10,000$ is a perfect square because $100 \cdot 100 = 10,000$.

$$\begin{aligned}\sqrt{10,000} &= \sqrt{(100)^2} \\ &= 100\end{aligned}$$

Answer:

$$100$$

Example 1.6.3

Simplify:

$$\sqrt{\tfrac{1}{9}}.$$

Solution:

Here we notice that $\frac{1}{9}$ is a square because $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

$$\begin{aligned}\sqrt{\frac{1}{9}} &= \sqrt{\left(\frac{1}{3}\right)^2} \\ &= \frac{1}{3}\end{aligned}$$

Answer:

$$\tfrac{1}{3}$$

Given a and b as positive real numbers, use the following property to simplify square roots whose radicands are not squares:

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

The idea is to identify the largest square factor of the radicand and then apply the property shown above. As an example, to simplify $\sqrt{8}$ notice that 8 is not a perfect square. However, $8 = 4 \cdot 2$ and thus has a perfect square factor other than 1. Apply the property as follows:

$$\begin{aligned}\sqrt{8} &= \sqrt{4 \cdot 2} \\ &= \sqrt{4} \cdot \sqrt{2} \\ &= 2 \cdot \sqrt{2} \\ &= 2\sqrt{2}\end{aligned}$$

Here $2\sqrt{2}$ is a simplified irrational number. You are often asked to find an approximate answer rounded off to a certain decimal place. In that case, use a calculator to find the decimal approximation using either the original problem or the simplified equivalent.

$$\sqrt{8} = 2\sqrt{2} \approx 2.83$$

On a calculator, try $2.83 \wedge 2$. What do you expect? Why is the answer not what you would expect?

It is important to mention that ***the radicand must be positive***. For example, $\sqrt{-9}$ is undefined since there is no real number that when squared is negative. Try taking the square root of a negative number on your calculator. What does it say?

Note

Taking the square root of a negative number is defined later in the course.

Example 1.6.4

Simplify and give an approximate answer rounded to the nearest hundredth:

$$\sqrt{75}.$$

Solution:

The radicand 75 can be factored as $25 \cdot 3$ where the factor 25 is a perfect square.

$$\begin{aligned} \sqrt{75} &= \sqrt{25 \cdot 3} && \textit{The largest perfect square} \\ &= \sqrt{25} \cdot \sqrt{3} && \textit{factor of } 75 \textit{ is } 25. \\ &= 5 \cdot \sqrt{3} \\ &= 5\sqrt{3} && \textit{Exact answer} \\ &\approx 8.66 && \textit{Approximate answer} \end{aligned}$$

Answer:

$$\sqrt{75} \approx 8.66$$

As a check, calculate (\sqrt{75}\) and $5\sqrt{3}$ on a calculator and verify that the both results are approximately 8.66.

Example 1.6.5

Simplify:

$$\sqrt{180}.$$

Solution:

$$\begin{aligned} \sqrt{180} &= \sqrt{36 \cdot 5} \\ &= \sqrt{36} \cdot \sqrt{5} \\ &= 6 \cdot \sqrt{5} \\ &= 6\sqrt{5} \end{aligned}$$

Since the question did not ask for an approximate answer, we present the exact answer.

Answer:

$$6\sqrt{5}$$

Example 1.6.6

Simplify:

$$-5\sqrt{162}.$$

Solution:

$$\begin{aligned} -5\sqrt{162} &= -5 \cdot \sqrt{81 \cdot 2} \\ &= -5 \cdot \sqrt{81} \cdot \sqrt{2} \\ &= -5 \cdot 9 \cdot \sqrt{2} \\ &= -45 \cdot \sqrt{2} \\ &= -45\sqrt{2} \end{aligned}$$

Answer:

$$-45\sqrt{2}$$

Exercise 1.6.4

Simplify and give an approximate answer rounded to the nearest hundredth:

$$\sqrt{128}.$$

Answer available at the end of this section.

A right triangle is a triangle where one of the angles measures $90\,°$. The side opposite the right angle is the longest side, called the hypotenuse, and the other two sides are called legs. Numerous real-world applications involve this geometric figure. The Pythagorean theorem states that given any right triangle with legs measuring a and b units, the square of the measure of the hypotenuse c is equal to the sum of the squares of the measures of the legs: $a^2 + b^2 = c^2$. In other words, the hypotenuse of any right triangle is equal to the square root of the sum of the squares of its legs.

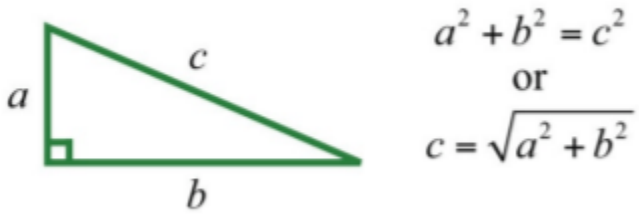


Figure 1.6.1

Example 1.6.7

If the two legs of a right triangle measure 3 units and 4 units, then find the length of the hypotenuse.

Solution:

Given the lengths of the legs of a right triangle, use the formula $c = \sqrt{a^2 + b^2}$ to find the length of the hypotenuse.

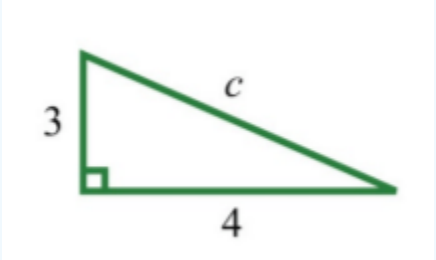


Figure $1.6.2$

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ c &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Answer:

$c = 5$ units

When finding the hypotenuse of a right triangle using the Pythagorean theorem, the radicand is not always a perfect square.

Example $1.6.8$

If the two legs of a right triangle measure 2 units and 6 units, find the length of the hypotenuse.

Solution:

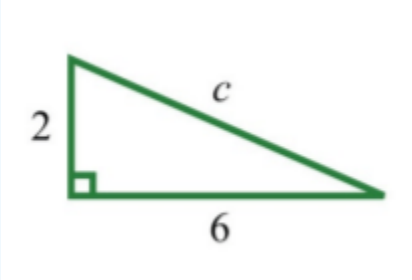


Figure $1.6.3$

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{2^2 + 6^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \\ &= \sqrt{4 \cdot 10} \\ &= \sqrt{4} \cdot \sqrt{10} \\ &= 2 \cdot \sqrt{10} \end{aligned}$$

Answer:

$c = 2\sqrt{10}$ units

Key Takeaways

- When using exponential notation a^n, the base a is used as a factor n times.
- When the exponent is 2, the result is called a square. When the exponent is 3, the result is called a cube.
- Memorize the squares of the integers up to 15 and the cubes of the integers up to 10. They will be used often as you progress in your study of algebra.
- When negative numbers are involved, take care to associate the exponent with the correct base. Parentheses group a negative number raised to some power.
- A negative base raised to an even power is positive.

- A negative base raised to an odd power is negative.
- The square root of a number is a number that when squared results in the original number. The principal square root is the positive square root.
- Simplify a square root by looking for the largest perfect square factor of the radicand. Once a perfect square is found, apply the property $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$, where a and b are nonnegative, and simplify.
- Check simplified square roots by calculating approximations of the answer using both the original problem and the simplified answer on a calculator to verify that the results are the same.
- Find the length of the hypotenuse of any right triangle given the lengths of the legs using the Pythagorean theorem.

Exercise 1.6.5 Square of a Number

Simplify.

1. 10^2
2. 12^2
3. $(-9)^2$
4. -12^2
5. 11^2
6. $(-20)^2$
7. 0^2
8. 1^2
9. $-(-8)^2$
10. $-(13)^2$
11. $(\frac{1}{2})^2$
12. $(-\frac{2}{3})^2$
13. 0.5^2
14. 1.25^2
15. $(-2.6)^2$
16. $-(-5.1)^2$
17. $(2\frac{1}{3})^2$
18. $(5\frac{3}{4})^2$

Answer available at the end of this section.

Exercise 1.6.6 Square of a Number

If s is the length of the side of a square, then the area is given by $A = s^2$.

1. Determine the area of a square given that a side measures 5 inches.
2. Determine the area of a square given that a side measures 2.3 feet.
3. List all the squares of the integers 0 through 15.
4. List all the squares of the integers from -15 to 0.
5. List the squares of all the rational numbers in the set $\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\}$.
6. List the squares of all the rational numbers in the set $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}\}$.

Answer available at the end of this section.

Exercise 1.6.7 Integer Exponents

Simplify.

1. 5^3
2. 2^6
3. $(-1)^4$

4. $(-3)^3$
5. -1^4
6. $(-2)^4$
7. -7^3
8. $(-7)^3$
9. $-(-3)^3$
10. $-(-10)^4$
11. $(-1)^{20}$
12. $(-1)^{21}$
13. $(-6)^3$
14. -3^4
15. 1^{100}
16. 0^{100}
17. $-(\frac{1}{2})^3$
18. $(\frac{1}{2})^6$
19. $(\frac{5}{2})^3$
20. $(-\frac{3}{4})^4$
21. List all the cubes of the integers -5 through 5.
22. List all the cubes of the integers from -10 to 0.
23. List all the cubes of the rational numbers in the set $\{-\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}\}$.
24. List all the cubes of the rational numbers in the set $\{-\frac{3}{7}, -\frac{1}{7}, 0, \frac{1}{7}, \frac{3}{7}\}$.

Answer available at the end of this section.

Exercise 1.6.8 Square Root of a Number

Determine the exact answer in simplified form.

1. $\sqrt{121}$
2. $\sqrt{81}$
3. $\sqrt{100}$
4. $\sqrt{169}$
5. $-\sqrt{25}$
6. $-\sqrt{144}$
7. $\sqrt{12}$
8. $\sqrt{27}$
9. $\sqrt{45}$
10. $\sqrt{50}$
11. $\sqrt{98}$
12. $\sqrt{2000}$
13. $\sqrt{\frac{1}{4}}$
14. $\sqrt{\frac{9}{16}}$
15. $\sqrt{\frac{5}{9}}$
16. $\sqrt{\frac{8}{36}}$
17. $\sqrt{0.64}$
18. $\sqrt{0.81}$
19. $\sqrt{30^2}$
20. $\sqrt{15^2}$
21. $\sqrt{(-2)^2}$
22. $\sqrt{(-5)^2}$
23. $\sqrt{-9}$
24. $\sqrt{-16}$
25. $3\sqrt{16}$
26. $5\sqrt{18}$

27. $-2\sqrt{36}$
28. $-3\sqrt{32}$
29. $6\sqrt{200}$
30. $10\sqrt{27}$

Answer available at the end of this section.

Exercise 1.6.9 Square Root of a Number

Approximate the following to the nearest hundredth.

1. $\sqrt{2}$
2. $\sqrt{3}$
3. $\sqrt{10}$
4. $\sqrt{15}$
5. $2\sqrt{3}$
6. $5\sqrt{2}$
7. $-6\sqrt{5}$
8. $-4\sqrt{6}$
9. $\sqrt{79}$
10. $\sqrt{54}$
11. $-\sqrt{162}$
12. $-\sqrt{86}$
13. If the two legs of a right triangle measure 6 units and 8 units, then find the length of the hypotenuse.
14. If the two legs of a right triangle measure 5 units and 12 units, then find the length of the hypotenuse.
15. If the two legs of a right triangle measure 9 units and 12 units, then find the length of the hypotenuse.
16. If the two legs of a right triangle measure $\frac{3}{2}$ units and 2 units, then find the length of the hypotenuse.
17. If the two legs of a right triangle both measure 1 unit, then find the length of the hypotenuse.
18. If the two legs of a right triangle measure 1 unit and 5 units, then find the length of the hypotenuse.
19. If the two legs of a right triangle measure 2 units and 4 units, then find the length of the hypotenuse.
20. If the two legs of a right triangle measure 3 units and 9 units, then find the length of the hypotenuse.

Answer available at the end of this section.

Exercise 1.6.10 Discussion Board Topics

1. Why is the result of an exponent of 2 called a square? Why is the result of an exponent of 3 called a cube?
2. Research and discuss the history of the Pythagorean theorem.
3. Research and discuss the history of the square root.
4. Discuss the importance of the principal square root.

Answer available at the end of this section.

Answers

Exercise 1.6.1

144

Exercise 1.6.2

-8

Exercise 1.6.3

$-10,000$ and $10,000$

Exercise 1.6.4

$8\sqrt{2} \approx 11.31$

Exercise 1.6.5 **Square of a Number**

1. 100

3. 81
5. 121
7. 0
9. -64
11. $\frac{1}{4}$
13. $.25$
15. 6.76
17. $5\frac{4}{9}$

Exercise $1.6.6$ **Square of a Number**

1. 25 square inches
3. $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225\}$
5. $\{0, \frac{1}{9}, \frac{4}{9}, 1, \frac{16}{9}, \frac{25}{9}, 4\}$

Exercise $1.6.7$ **Integer Exponents**

1. 125
3. 1
5. -1
7. -343
9. 27
11. 1
13. -216
15. 1
17. $-\frac{1}{8}$
19. $\frac{12}{58}$
21. $\{-125, -64, -27, -8, -1, 0, 1, 8, 27, 64, 125\}$
23. $\{-\frac{8}{27}, -\frac{1}{27}, 0, \frac{1}{27}, \frac{8}{27}\}$

Exercise $1.6.8$ **Square Root of a Number**

1. 11
3. 10
5. -5
7. $2\sqrt{3}$
9. $3\sqrt{5}$
11. $7\sqrt{2}$
13. $\frac{1}{2}$
15. $5\sqrt{3}$
17. 0.8
19. 30
21. 2
23. Not real
25. 12
27. -12
29. $60\sqrt{2}$

Exercise $1.6.9$ **Square Root of a Number**

1. 1.41
3. 3.16
5. 3.46
7. -13.42
9. 8.89
11. -12.73
13. 10 units
15. 15 units
17. $\sqrt{2}$ units
19. $2\sqrt{5}$ units

Exercise $1.6.10$ **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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1.7: Order of Operations

Learning Objectives

- Identify and work with grouping symbols.
- Understand the order of operations.
- Simplify using the order of operations.

Grouping Symbols

In a computation where more than one operation is involved, grouping symbols help tell us which operations to perform first. The grouping symbols commonly used in algebra are

$$\begin{array}{ll} (\ ) & \textit{Parentheses} \\ [\ ] & \textit{Brackets} \\ \{\ \} & \textit{Braces} \\ \text{—} & \textit{Fraction bar} \end{array}$$

All of the above grouping symbols, as well as absolute value, have the same order of precedence. Perform operations inside the innermost grouping symbol or absolute value first.

Example 1.7.1

Simplify:

$$5-(4-12).$$

Solution:

Perform the operations within the parentheses first. In this case, first subtract 12 from 4.

$$\begin{aligned} 5-(4-12) &= 5-(-8) \\ &= 5+8 \\ &= 13 \end{aligned}$$

Answer:

$$13$$

Example 1.7.2

Simplify:

$$3\{-2[-(-3-1)]\}.$$

Solution:

Answer:

-24

Example 1.7.3

Simplify:

$$\frac{5-|4-(-3)|}{|-3|-(5-7)}.$$

Solution:

$$\begin{aligned}\frac{5-|4-(-3)|}{|-3|-(5-7)} &= \frac{5-|4+3|}{|-3|-(-2)} \\ &= \frac{5-|7|}{|-3|+2} \\ &= \frac{5-7}{3+2} \\ &= \frac{-2}{5} \\ &= -\frac{2}{5}\end{aligned}$$

Answer:

$$-\frac{2}{5}$$

Exercise 1.7.1

Simplify:

$$-[-3(2+3)].$$

Answer available at the end of this section.

Order of Operations

When several operations are to be applied within a calculation, we must follow a specific order to ensure a single correct result.

1. Perform all calculations within the innermost **parentheses** or grouping symbols.
2. Evaluate all **exponents**.
3. Perform **multiplication and division** operations from left to right.
4. Finally, perform all remaining **addition and subtraction** operations from left to right.

Note

Caution: Note that multiplication and division operations ***must*** be worked from ***left to right***.

Example 1.7.4

Simplify:

$$5^2 - 4 \cdot 3 \div 12.$$

Solution:

First, evaluate 5^2 and then perform multiplication and division as they appear from left to right.

$$\begin{aligned} 5^2 - 4 \cdot 3 \div 12 &= 25 - \underbrace{4 \cdot 3} \div 12 \\ &= 25 - \underbrace{12 \div 12} \\ &= \underbrace{25} - 1 \\ &= 24 \end{aligned}$$

Because multiplication and division operations should be worked from left to right, it is sometimes correct to perform division before multiplication.

Example 1.7.5

Simplify:

$$2^4 - 12 \div 3 \cdot 2 + 11.$$

Solution:

Begin by evaluating the exponent, $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

$$\begin{aligned} 2^4 - 12 \div 3 \cdot 2 + 11 &= 16 - \underbrace{12 \div 3} \cdot 2 + 11 \\ &= 16 - \underbrace{4 \cdot 2} + 11 \\ &= \underbrace{16 - 8} + 11 \\ &= \underbrace{8 + 11} \\ &= 19 \end{aligned}$$

Multiplying first leads to an incorrect result.

$$\begin{aligned} 2^4 - 12 \div 3 \cdot 2 + 11 &= 16 - 12 \div \underbrace{3 \cdot 2}_{Incorrect} + 11 \\ &= 16 - \underbrace{12 \div 6} + 11 \\ &= \underbrace{16 - 2} + 11 \\ &= \underbrace{14 + 11} \\ &= 25 \quad x \end{aligned}$$

Answer:

$$19$$

Example 1.7.6

Simplify:

$$-3 - 5^2 + (-7)^2.$$

Solution:

Take care to correctly identify the base when squaring.

$$\begin{aligned} -3 - 5^2 + (-7)^2 &= -3 - 25 + 49 \\ &= -28 + 49 \\ &= 21 \end{aligned}$$

Answer:

$$21$$

Example 1.7.7

Simplify:

$$5-3[2^3-5+7(-3)].$$

Solution:

It is tempting to first subtract $5-3$, but this will lead to an incorrect result. The order of operations requires us to simplify within the brackets first.

$$\begin{aligned} 5-3[2^3-5+7(-3)] &= 5-3[8-5-21] \\ &= 5-3[-18] \\ &= 5+54 \\ &= 59 \end{aligned}$$

Subtracting $5-3$ first leads to an incorrect result.

$$\begin{aligned} 5-3[2^3-5+7(-3)] &= \underbrace{5-3}_{Incorrect}\,[8-5-21] \\ &= 2[-18] \\ &= -36 \quad x \end{aligned}$$

Answer:

$$59$$

Example 1.7.8

Simplify:

$$-3^2-[5-(4^2-10)].$$

Solution:

Perform the operations within the innermost parentheses first.

Answer:

-8

Example 1.7.9

Simplify:

$$\left(-\tfrac{2}{3}\right)^2 \div \left[\tfrac{5}{3}-\left(-\tfrac{1}{2}\right)^3\right].$$

Solution:

$$\begin{aligned}\left(-\frac{2}{3}\right)^2 \div \left[\frac{5}{3}-\left(-\frac{1}{2}\right)^3\right] &= \left(-\frac{2}{3}\right)^2 \div \left[\frac{5}{3}-\left(-\frac{1}{8}\right)\right] \\ &= \left(-\frac{2}{3}\right)^2 \div \left[\frac{5}{3}+\frac{1}{8}\right] \\ &= \left(-\frac{2}{3}\right)^2 \div \left[\frac{40}{24}+\frac{3}{24}\right] \\ &= \left(-\frac{2}{3}\right)^2 \div \frac{43}{24} \\ &= \frac{4}{\cancel{9}_{3}} \cdot \frac{\cancel{24}^{8}}{43} \\ &= \frac{32}{129}\end{aligned}$$

Answer:

$\frac{32}{129}$

We are less likely to make a mistake if we work one operation at a time. Some problems may involve an absolute value, in which case we assign it the same order of precedence as parentheses.

Example 1.7.10

Simplify:

$$2-4|-4-3|+(-2)^4.$$

Solution:

We begin by evaluating the absolute value and then the exponent $(-2)^4=(-2)(-2)(-2)(-2)=+16$.

Answer:

-10

Exercise 1.7.2

Simplify:

$$10 \div 5 \cdot 2|(-4)+|-3|| + (-3)^2.$$

Answer available at the end of this section.

Key Takeaways

- Grouping symbols indicate which operations to perform first. We usually group mathematical operations with parentheses, brackets, braces, and the fraction bar. We also group operations within absolute values. All groupings have the same order of precedence: the operations within the innermost grouping are performed first.
- When applying operations within a calculation, follow the order of operations to ensure a single correct result.
  - Address innermost parentheses or groupings first.
  - Simplify all exponents.
  - Perform multiplication and division operations from left to right.
  - Finally, perform addition and subtraction operations from left to right.
- It is important to highlight the fact that multiplication and division operations should be applied as they appear from left

to right. It is a common mistake to always perform multiplication before division, which, as we have seen, in some cases produces incorrect results.

Exercise 1.7.3 Order of Operations

Simplify.

1. $-7-3\cdot 5$
2. $3+2\cdot 3$
3. $-3(2)-6^2$
4. $2(-3)^2+5(-4)$
5. $\frac{6}{3} * 2$
6. $6/(3*2)$
7. $-\frac{1}{2}-\frac{3}{5}\cdot\frac{2}{3}$
8. $\frac{5}{8}\div\frac{1}{2}-\frac{5}{6}$
9. $3.2^2-6.9\div 2.3$
10. $8.2-3\div 1.2\cdot 2.1$
11. $2+3(-2)-7$
12. $8\div 2-3\cdot 2$
13. $3+6^2\div 12$
14. $5-4^2\div(-8)$
15. $-9-3\cdot 2\div 3(-2)$
16. $-2-3^2+(-2)^2$
17. $12\div 6\cdot 2-2^2$
18. $4\cdot 3\div 12\cdot 2-(-2)^2$
19. $(-5)^2-2(5)^2\div 10$
20. $-3(4-7)+2$
21. $(-2+7)^2-10^2$
22. $10-7(3+2)+7^2$
23. $-7-3(4-2\cdot 8)$
24. $5-3[6-(2+7)]$
25. $1+2[(-2)^3-(-3)^2]$
26. $-3[2(7-5)\div 4\cdot(-2)+(-3)^3]$
27. $-7^2-[-20-(-3)^2]-(-10)$
28. $4.7-3.2(4-1.2^3)$
29. $-5.4(6.1-3.1\div 0.1)-8.2^2$
30. $-7.3^2+(-9.3)^2-37.8\div 1.8$
31. $2-7(3^2-3+4\cdot 3)$
32. $(\frac{1}{2})^2-(-\frac{2}{3})^2$
33. $(\frac{1}{2})^3+(-2)^3$
34. $(-\frac{1}{3})^2-(-\frac{2}{3})^3$
35. $\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{5}$
36. $\frac{5}{8}\div\frac{3}{2}\cdot\frac{14}{15}$
37. $5\cdot\frac{2}{15}-(\frac{1}{2})^3$
38. $\frac{5}{17}(\frac{3}{5}-\frac{4}{35})$
39. $\frac{3}{16}\div(\frac{5}{12}-\frac{1}{2}+\frac{2}{3})\cdot 4$
40. $(\frac{2}{3})^2-(\frac{1}{2})^2$
41. $\frac{1}{2}[\frac{3}{4}\cdot(-4)^2-2]^2$
42. $6\cdot[(\frac{2}{3})^2-(\frac{1}{2})^2]\div(-2)^2$
43. $(-5)^2+\frac{3}{2}-\frac{4}{2}+2\cdot 7$
44. $(-3.2-3.3)(8.7-4.7)(-4.7+3.9+2.1)$
45. $2-[3-(5-7)^2]3(6-32)$
46. $\frac{2+3\cdot 6-4\cdot 32}{2-32}$
47. $(2+7)\cdot 2-\frac{23}{10}+\frac{9}{2}+3^3$
48. $\frac{(-1-3)^2-1}{5-3\cdot(-7+22)-5}$
49. $(7+4*(-2))/(-3+(-2)^2)$

50. $4+3*((-3)^3+5^2)/6-2^2$

Answer available at the end of this section.

Exercise 1.7.4 Order of Operations

1. Mary purchased 14 bottles of water at $\$0.75$ per bottle, 4 pounds of assorted candy at $\$3.50$ per pound, and 16 packages of microwave popcorn costing $\$0.50$ each for her party. What was her total bill?
2. Joe bought four 8-foot 2-by-4 boards for $\$24.00$. How much did he spend per linear foot?
3. Margaret bought two cases of soda at the local discount store for $\$23.52$. If each case contained 24 bottles, how much did she spend per bottle?
4. Billy earns $\$12.00$ per hour and "time and a half" for every hour he works over 40 hours a week. What is his pay for 47 hours of work this week?
5. Audry bought 4 bags of marbles each containing 15 assorted marbles. If she wishes to divide them up evenly between her 3 children, how many will each child receive?
6. Mark and Janet carpooled home from college for the Thanksgiving holiday. They shared the driving, but Mark drove twice as far as Janet. If Janet drove 135 miles, then how many miles was the entire trip?

Answer available at the end of this section.

Exercise 1.7.5 Order of Operations with Absolute Values

Simplify.

1. $3+2|-5|$
2. $9-4|-3|$
3. $-(-|2|+|-10|)$
4. $-(|-6|-|-8|)$
5. $|-(40-|-22|)|$
6. $||-5|-|10||$
7. $-(|-8|-5)^2$
8. $(|-1|-|-2|)^2$
9. $-4+2|2^2-3^2|$
10. $-10-|4-5^2|$
11. $-|(-5)^2+4^2\div 8|$
12. $-(-3-[6-|-7|])$
13. $-2[7-(4+|-7|)]$
14. $3-7|-2-3|+4^3$
15. $7-5|6^2-5^2|+(-7)^2$
16. $(-4)^2-|-5+(-2)^3|-3^2$
17. $\frac{2}{3}-|\frac{1}{2}-(-\frac{4}{3})^2|$
18. $-30|\frac{10}{3}-\frac{1}{2}\div\frac{1}{5}|$
19. $(-4)^3-(2-|-4|)\div|-3^2+7|$
20. $[10-3(6-|-8|)]\div 4-5^2$

Answer available at the end of this section.

Exercise 1.7.6 Order of Operations with Absolute Values

Find the distance between the given numbers on a number line.

1. $\frac{1}{2}$ and $-\frac{1}{4}$
2. $-\frac{3}{4}$ and $-\frac{2}{3}$
3. $-\frac{5}{8}$ and $-\frac{3}{4}$

4. $-\frac{7}{5}$ and $\frac{3}{7}$
5. -0.5 and 8.3
6. 10.7 and -2.8
7. $3\frac{1}{5}$ and $-2\frac{1}{3}$
8. $5\frac{3}{4}$ and 0

Answer available at the end of this section.

Exercise 1.7.7 Discussion Board Topics

1. Convert various examples in this section to equivalent expressions using text-based symbols.
2. What is PEMDAS and what is it missing?
3. Discuss the importance of proper grouping and give some examples.
4. Experiment with the order of operations on a calculator and share your results.

Answer available at the end of this section.

Answers

Exercise 1.7.1

15

Exercise 1.7.2

13

Exercise 1.7.3 **Order of Operations**

1. -22
3. -42
5. 4
7. $-\frac{9}{10}$
9. 7.24
11. -11
13. 6
15. -5
17. 0
19. 20
21. -75
23. 29
25. -33
27. -10
29. 67.22
31. -124
33. $-\frac{63}{8}$
35. $\frac{7}{30}$
37. $\frac{13}{24}$
39. $\frac{9}{7}$
41. 50
43. -17
45. $-\frac{1}{3}$
47. $\frac{5}{59}$
49. -1

Exercise 1.7.4 **Order of Operations**

1. $\$32.50$
3. $\$0.49$
5. 20 marbles

Exercise 1.7.5 **Order of Operations with Absolute Values**

1. 13
3. -8
5. 18

7. -9
11. 6
13. -27
15. 8
17. 1
19. $-\frac{11}{18}$
21. -63

Exercise $1.7.6$ **Order of Operations with Absolute Values**

1. $\frac{3}{4}$ unit
3. $\frac{1}{8}$ unit
5. 8.8 units
7. $5\frac{8}{15}$ units

Exercise $1.7.7$ **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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1.E: Review Exercises and Sample Exam

Review Exercises

Exercise $1.E.1$ Real Numbers and the Number Line

Choose an appropriate scale and graph the following sets of real numbers on a number line.

1. $\{-4, 0, 4\}$
2. $\{-30, 10, 40\}$
3. $\{-12, -3, 9\}$
4. $\{-10, 8, 10\}$

Answer available at the end of this section.

Exercise $1.E.2$ Real Numbers and the Number Line

Fill in the blank with $<, =, \text{ or } >$.

1. $0\_\_ - 9$
2. $-75\_\_ - 5$
3. $-12\_\_ - (-3)$
4. $-(-23)\_\_23$
5. $|-20|\_\_ - |-30|$
6. $-|6|\_\_ - |-(-8)|$

Answer available at the end of this section.

Exercise $1.E.3$ Real Numbers and the Number Line

Determine the unknown.

1. $|?| = 2$
2. $|?| = 1$
3. $|?| = -7$
4. $|?| = 0$

Answer available at the end of this section.

Exercise $1.E.4$ Real Numbers and the Number Line

Translate the following into a mathematical statement.

1. Negative eight is less than or equal to zero.
2. Seventy-eight is not equal to twelve.
3. Negative nine is greater than negative ten.
4. Zero is equal to zero.

Answer available at the end of this section.

Exercise $1.E.5$ Adding and Subtracting Integers

Simplify.

1. $12 + (-7)$
2. $20 + (-32)$
3. $-23 - (-7)$
4. $-8 - (-8)$
5. $-3 - (-13) + (-1)$
6. $9 + (-15) - (-8)$
7. $(7 - 10) - 3$
8. $(-19 + 6) - 2$

Answer available at the end of this section.

Exercise $1.E.6$ Adding and Subtracting Integers

Find the distance between the given numbers on a number line.

1. -8 and 14
2. -35 and -6
3. What is 2 less than 17?
4. What is 3 less than -20?
5. Subtract 30 from the sum of 8 and 12.
6. Subtract 7 from the difference of -5 and 7.
7. An airplane flying at $22,000$ feet descended $8,500$ feet and then ascended $5,000$ feet. What is the new altitude of the airplane?
8. The width of a rectangle is 5 inches less than its length. If the length measures 22 inches, then determine the width.

Answer available at the end of this section.

Exercise $1.E.7$ Multiplying and Dividing Integers

Simplify.

1. $10 \div 5 \cdot 2$
2. $36 \div 6 \cdot 2$
3. $-6(4) \div 2(-3)$
4. $120 \div (-5)(-3)(-2)$
5. $-8(-5) \div 0$
6. $-24(0) \div 8$
7. Find the product of -6 and 9.
8. Find the quotient of -54 and -3.
9. James was able to drive 234 miles on 9 gallons of gasoline. How many miles per gallon did he get?
10. If a bus travels at an average speed of 54 miles per hour for 3 hours, then how far does the bus travel?

Answer available at the end of this section.

Exercise $1.E.8$ Fractions

Reduce each fraction to lowest terms.

1. $\frac{180}{300}$

2. $\frac{252}{324}$
3. Convert to a mixed number: $\frac{23}{8}$.
4. Convert to an improper fraction: $3\frac{5}{9}$.

Answer available at the end of this section.

Exercise $1.E.9$ Fractions

Simplify.

1. $\frac{3}{5}(-\frac{2}{7})$
2. $-\frac{5}{8}(-\frac{1}{3})$
3. $-\frac{3}{4} \div \frac{6}{7}$
4. $\frac{4}{15} \div \frac{28}{3}$
5. $4\frac{4}{5} \div 6$
6. $5 \div 8\frac{1}{3}$
7. $\frac{5}{4} \div \frac{15}{2} \cdot 6$
8. $\frac{5}{24} \div \frac{3}{2} \div \frac{5}{12}$
9. $\frac{1}{12} - \frac{1}{4}$
10. $\frac{5}{6} - \frac{3}{14}$
11. $\frac{3}{4} + \frac{2}{3} - \frac{1}{12}$
12. $\frac{3}{10} + \frac{5}{12} - \frac{1}{6}$
13. Subtract $\frac{2}{3}$ from the sum of $-\frac{1}{2}$ and $\frac{2}{9}$.
14. Subtract $\frac{5}{6}$ from the difference of $\frac{1}{3}$ and $\frac{7}{2}$.
15. If a bus travels at an average speed of 54 miles per hour for $2\frac{1}{3}$ hours, then how far does the bus travel?
16. Determine the length of fencing needed to enclose a rectangular pen with dimensions $12\frac{1}{2}$ feet by $8\frac{3}{4}$ feet.

Answer available at the end of this section.

Exercise $1.E.10$ Decimals and Percents

1. Write as a mixed number: 5.32.
2. Write as a decimal: $7\frac{3}{25}$

Answer available at the end of this section.

Exercise $1.E.11$ Decimals and Percents

Perform the operations.

1. $6.032 + 2.19$
2. $12.106 - 9.21$
3. 4.23×5.13
4. $9.246 \div 4.02$

Answer available at the end of this section.

Exercise $1.E.12$ Decimals and Percents

Convert to a decimal.

1. 7.2%

2. $5\frac{3}{8}\%$
3. 147%
4. $27\frac{1}{2}\%$

Answer available at the end of this section.

Exercise $1.E.13$ Decimals and Percents

Convert to a percent.

1. 0.055
2. 1.75
3. $\frac{9}{10}$
4. $\frac{5}{6}$
5. Mary purchased 3 boxes of t-shirts for a total of $\$126$. If each box contains 24 t-shirts, then what is the cost of each t-shirt?
6. A retail outlet is offering 12% off the original $\$39.99$ price of tennis shoes. What is the price after the discount?
7. If an item costs $\$129.99$, then what is the total after adding $7\frac{1}{4}\%$ sales tax?
8. It is estimated that 8.3% of the total student population carpools to campus each day. If there are $13,000$ students, then estimate the number of students that carpool to campus.

Answer available at the end of this section.

Exercise $1.E.14$ Exponents and Square Roots

Simplify.

1. 8^2
2. $(-5)^2$
3. -4^2
4. $-(-3)^2$
5. $(\frac{2}{9})^2$
6. $(1\frac{2}{3})^2$
7. 3^3
8. $(-4)^3$
9. $(\frac{2}{5})^3$
10. $(-\frac{1}{6})^3$
11. $-(-2)^4$
12. $-(-1)^5$
13. $\sqrt{49}$
14. $\sqrt{225}$
15. $2\sqrt{25}$
16. $-\sqrt{121}$
17. $3\sqrt{50}$
18. $-4\sqrt{12}$
19. $4\sqrt{9}$
20. $8\sqrt{25}$
21. Calculate the area of a square with sides measuring 3 centimeters. $(A = s^2)$
22. Calculate the volume of a cube with sides measuring 3 centimeters. $(V = s^3)$
23. Determine the length of the diagonal of a square with sides measuring 3 centimeters.
24. Determine the length of the diagonal of a rectangle with dimensions 2 inches by 4 inches.

Answer available at the end of this section.

Exercise $1.E.15$ Order of Operations

Simplify.

1. $-5(2)-7^2$
2. $1-4^2+2(-3)^2$
3. $2+3(6-2\cdot 4)^3$
4. $5-3(8-3\cdot 4)^2$
5. $-2^3+6(3^2-4)+(-3)^2$
6. $5^2-40\div 5(-2)^2-(-4)$
7. $\frac{3}{4}[\frac{2}{9}(-3)^2-4]^2$
8. $(\frac{1}{2})^2-\frac{3}{4}\div\frac{9}{16}-\frac{1}{3}$
9. $\frac{2-3(6-3^2)^2}{4\cdot 5-5^2}$
10. $\frac{(2\cdot 8-6^2)^2-10}{273-(2(-5)^3-7)}$
11. $8-5|3\cdot 4-(-2)^4|$
12. $|14|-|-3-52|$

Answer available at the end of this section.

Exercise $1.E.16$ Order of Operations

Find the distance between the given numbers on a number line.

1. -14 and 22
2. -42 and -2
3. $\frac{7}{8}$ and $-\frac{1}{5}$
4. $-5\frac{1}{2}$ and $-1\frac{1}{4}$

Answer available at the end of this section.

Sample Exam

Exercise $1.E.17$

1. List three integers greater than -10.
2. Determine the unknown(s): $|?|=13$.
3. Fill in the blank with $<, =, \text{ or } :-|-100|\_9^2$.
4. Convert to a fraction: $33\frac{1}{3}\%$
5. Convert to a percent: $2\frac{3}{4}$.
6. Reduce: $\frac{75}{225}$.

Answer available at the end of this section.

Exercise $1.E.18$

Calculate the following.

1. a. $(-7)^2$; b. $-(-7)^2$; c. -7^2
2. a. $(-3)^3$; b. $-(-3)^3$; c. -3^3
3. a. $|10|$; b. $-|-10|$; c. $-|10|$

Answer available at the end of this section.

Exercise $1.E.19$

Simplify.

1. $-(-(-1))$
2. $\frac{2}{3}+\frac{1}{5}-\frac{3}{10}$
3. $10-(-12)+(-8)-20$
4. $-8(4)(-3)\div 2$
5. $\frac{1}{2}\cdot(-\frac{4}{5})\div\frac{14}{15}$
6. $\frac{3}{5}\cdot\frac{1}{2}-\frac{2}{3}$
7. $4\cdot 5-20\div 5\cdot 2$
8. $10-7(3-8)-5^2$
9. $3+2|-2^2-(-1)|+(-2)^2$
10. $\frac{1}{3}[5^2-(7-|-2|)+15\cdot 2\div 3]$
11. $\sqrt{116}$
12. $3\sqrt{72}$

Answer available at the end of this section.

Exercise $1.E.20$

1. Subtract 2 from the sum of 8 and -10.
2. Subtract 10 from the product of 8 and -10.
3. A student earns $9, 8, 10, 7,$ and 8 points on the first 5 chemistry quizzes. What is her quiz average?
4. An $8\frac{3}{4}$ foot plank is cut into 5 pieces of equal length. What is the length of each piece?

Answer available at the end of this section.

Answers

Exercise $1.E.1$ Real Numbers and the Number Line

1.

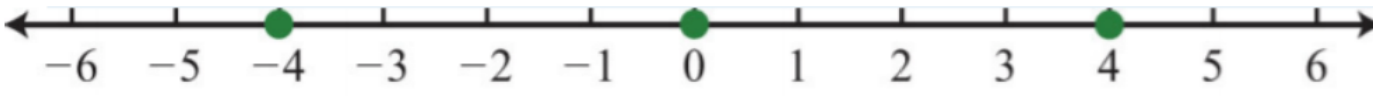


Figure 1.E.1

3.

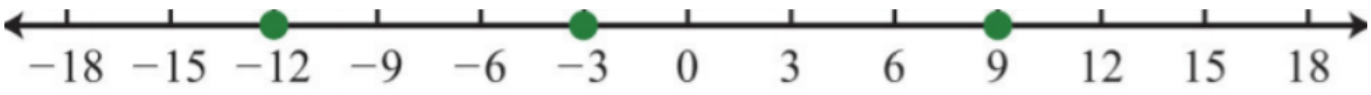


Figure 1.E.2

Exercise $1.E.2$ Real Numbers and the Number Line

1. $>$
3. $<$
5. $>$

Exercise $1.E.3$ Real Numbers and the Number Line

1. ± 2

3. $\emptyset$, No solution

Exercise $1.E.4$ **Real Numbers and the Number Line**

1. $-8 \leq 0$
3. $-9 > -10$

Exercise $1.E.5$ **Adding and Subtracting Integers**

1. 5
3. -16
5. 9
7. -6

Exercise $1.E.6$ **Adding and Subtracting Integers**

1. 22 units
3. 15
5. -10
7. $18,500$ feet

Exercise $1.E.7$ **Multiplying and Dividing Integers**

1. 4
3. 36
5. Undefined
7. -54
9. 26 miles per gallon

Exercise $1.E.8$ **Fractions**

1. $\frac{3}{5}$
3. $2\frac{7}{8}$

Exercise $1.E.9$ **Fractions**

1. $-\frac{6}{35}$
3. $-\frac{7}{8}$
5. $\frac{4}{5}$
7. 1
9. $-\frac{1}{6}$
11. $\frac{4}{3}$
13. $-\frac{17}{18}$
15. 126 miles

Exercise $1.E.10$ **Decimals and Percents**

1. $5\frac{8}{25}$

Exercise $1.E.11$ **Decimals and Percents**

1. 8.222
3. 21.6999

Exercise $1.E.12$ **Decimals and Percents**

1. 0.072
3. 1.47

Exercise $1.E.13$ **Decimals and Percents**

1. 5.5%
3. 90%
5. $\$1.75$
7. $\$139.41$

Exercise $1.E.14$ **Exponents and Square Roots**

1. 64
3. -16
5. $\frac{4}{81}$
7. 27
9. $\frac{81}{25}$
11. -16
13. 7
15. 10
17. $15\sqrt{2}$
19. $\frac{2}{3}$
21. 9 square centimeters
23. $3\sqrt{2}$ centimeters

Exercise $1.E.15$ **Order of Operations**

1. -59

3. -22
5. 31
7. 3
9. 5
11. -12

Exercise $1.E.16$ **Order of Operations**

1. 36 units
3. $\frac{43}{40}$ units

Exercise $1.E.17$

1. $\{-5, 0, 5\}$ (answers may vary)
3. $<$
5. 275%

Exercise $1.E.18$

1. a. 49; b. -49; c. -49
3. a. 10; b. -10; c. -10

Exercise $1.E.19$

2. $\frac{17}{30}$
4. 48
6. $-\frac{11}{30}$
8. 20
10. 10
12. $18\sqrt{2}$

Exercise $1.E.20$

2. -90
4. $1\frac{3}{4}$ feet

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2: LINEAR EQUATIONS AND INEQUALITIES

2.1: Introduction to Algebra

2.2: Simplifying Algebraic Expressions

2.3: Solving Linear Equations- Part 1

2.4: Solving Linear Equations- Part II

2.5: Applications of Linear Equations

2.6: Ratio and Proportion Applications

2.7: Introduction to Inequalities and Interval Notation

2.8: Linear Inequalities (One Variable)

2.E: Review Exercises and Sample Exam

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2.1: Introduction to Algebra

Learning Objectives

- Identify an algebraic expression and its parts.
- Evaluate algebraic expressions.
- Use formulas to solve problems in common applications.

Preliminary Definitions

In algebra, letters are used to represent numbers. The letters used to represent these numbers are called variables. Combinations of variables and numbers along with mathematical operations form algebraic expressions, or just **expressions**.

The following are some examples of expressions with one variable, x:

$$2x+3 \qquad x^2-9 \qquad 3x^2+2x-1 \qquad \frac{x-5}{x^2-25}$$

Terms in an algebraic expression are separated by addition operators, and factors are separated by multiplication operators. The numerical factor of a term is called the coefficient. For example, the algebraic expression $3x^2+2x-1$ can be thought of as $3x^2+2x+(-1)$ and has three terms. The first term, $3x^2$, represents the quantity $3 \cdot x \cdot x$, where 3 is the coefficient and x is the variable. All of the variable factors, with their exponents, form the variable part of a term. If a term is written without a variable factor, then it is called a constant term. Consider the components of $3x^2+2x-1$,

Table 2.1.1

Terms	Coefficient	Variable Part
$3x^2$	3	x^2
$2x$	2	x
-1	-1	

The third term in this expression, -1, is a constant term because it is written without a variable factor. While a variable represents an unknown quantity and may change, the constant term does not change.

Example 2.1.1

List all coefficients and variable parts of each term:

$$5x^2-4xy-y^2.$$

Solution:

Think of the third term in this example, $-y^2$, as $-1y^2$.

Table 2.1.2

Terms	Coefficient	Variable Part
$5x^2$	5	x^2
$-4xy$	-4	xy
$-y^2$	-1	y^2

Answer:

Coefficients: $\{-4, -1, 5\}$; variable parts: $\{x^2, xy, y^2\}$

Some terms, such as y^2 and $-y^2$, appear not to have a coefficient. The multiplicative identity property states that 1 times anything is itself and occurs so often that it is customary to omit this factor and write

$$1y^2 = y^2$$
$$-1y^2 = -y^2$$

Therefore, the coefficient of y^2 is actually 1 and the coefficient of $-y^2$ is -1. In addition, you will encounter terms that have variable parts composed of algebraic expressions as factors.

Example 2.1.2

List all coefficients and variable parts of each term:

$$-3(x+y)^3 + (x+y)^2$$

Solution:

This is an expression with two terms:

Table 2.1.3

Terms	Coefficient	Variable Part
$-3(x+y)^3$	-3	$(x+y)^3$
$(x+y)^2$	1	$(x+y)^2$

Answer:

Coefficients: $\{-3, 1\}$; variable parts: $\{(x+y)^3, (x+y)^2\}$

In our study of algebra, we will encounter a wide variety of algebraic expressions. Typically, expressions use the two most common variables, x and y. However, expressions may use any letter (or symbol) for a variable, even Greek letters, such as alpha (α) and beta (β). Some letters and symbols are reserved for constants, such as $\pi \approx 3.14159$ and $e \approx 2.71828$. Since there is only a limited number of letters, you will also use subscripts, $x^1, x^2, x^3, x^4, \ldots$, to indicate different variables.

Exercise 2.1.1

List all coefficients and variable parts of the expression:

$$-5a^2 + ab - 2b^2 - 3.$$

Answer available at the end of this section.

Evaluating Algebraic Expressions

Think of an algebraic expression as a generalization of particular arithmetic operations. Performing these operations after substituting given values for variables is called evaluating. In algebra, a variable represents an unknown value. However, if the problem specifically assigns a value to a variable, then you can replace that letter with the given number and evaluate using the order of operations.

Example 2.1.3

Evaluate:

a. $2x + 3$, where $x = -4$
b. $\frac{2}{3}y$, where $y = 9$

Solution:

To avoid common errors, it is a best practice to first replace all variables with parentheses and then replace, or substitute, the given value.

a.

b.

$$\begin{aligned}\frac{2}{3}y &= \frac{2}{3}(\ ) \\ &= \frac{2}{3}(9) \\ &= 2 \cdot 3 \\ &= 6\end{aligned}$$

Answer:

a. -5
b. 6

If parentheses are not used in part (a) of the previous example, the result is quite different: $2x + 3 = 2 - 4 + 4$. Without parentheses, the first operation is subtraction, which leads to an incorrect result.

Example 2.1.4

Evaluate:

$$-2x - y, \text{ where } x = -5 \text{ and } y = -3.$$

Solution:

After substituting the given values for the variables, simplify using the order of operations.

$$\begin{aligned}-2x - y &= -2(\ ) - (\ ) && \textit{Replace variables with parentheses.} \\ &= -2(-5) - (-3) && \textit{Substitute values for } x \textit{ and } y. \\ &= 10 + 3 && \textit{Simplify.} \\ &= 13\end{aligned}$$

Answer:

Example 2.1.5

Evaluate:

$$9a^2 - b^2, \text{ where } a = 2 \text{ and } b = -5.$$

Solution:

Answer:

11

Example 2.1.6

Evaluate:

$$-x^2-4x+1, \text{ where } x=-\tfrac{1}{2}.$$

Solution:

$$\begin{aligned} -x^2-4x+1 &= -(\ )^2-4(\ )+1 && \textit{Apply the exponent first.}\\ &= -\left(-\frac{1}{2}\right)^2-4\left(-\frac{1}{2}\right)+1 && (-\tfrac{1}{2})^2=(-\tfrac{1}{2})(-\tfrac{1}{2})=\tfrac{1}{4}\\ &= -\left(\frac{1}{4}\right)+2+1\\ &= -\frac{1}{4}+3\\ &= -\frac{1}{4}+\frac{12}{4}\\ &= \frac{11}{4} \end{aligned}$$

Answer:

$\frac{11}{4}$

The answer to the previous example is $\frac{11}{4}$, which can be written as a mixed number $2\frac{3}{4}$. In algebra, improper fractions are generally preferred. Unless the original problem has mixed numbers in it, or it is an answer to a real-world application, solutions will be expressed as reduced improper fractions.

Example 2.1.7

Evaluate:

$$(3x-2)(x-7), \text{ where } x=\tfrac{2}{3}$$

Solution:

The order of operations requires us to perform the operations within the parentheses first.

$$\begin{aligned} (3x-2)(x-7) &= (3(\ )-2)((\ )-7)\\ &= \left(3\left(\frac{2}{3}\right)-2\right)\left(\left(\frac{2}{3}\right)-7\right)\\ &= (2-2)\left(\frac{2}{3}-\frac{21}{3}\right)\\ &= (0)\left(-\frac{19}{3}\right)\\ &= 0 \end{aligned}$$

Answer:

0

Example 2.1.8

Evaluate:

$$b^2 - 4ac, \text{ where } a = -1, b = -3, \text{ and } c = 2.$$

Solution:

The expression $b^2 - 4ac$ is called the discriminant; it is an essential quantity seen later in our study of algebra.

$$\begin{aligned} b^2 - 4ac &= (\ )^2 - 4(\ )(\ ) \qquad \textit{Exponents first, then multiplication} \\ &= (-3)^2 - 4(-1)(2) \qquad (-3)^2 = (-3)(-3) = +9 \\ &= 9 + 4(2) \\ &= 9 + 8 \\ &= 17 \end{aligned}$$

Answer:

17

Exercise 2.1.2

Evaluate $a^3 - b^3$, where $a = 2$ and $b = -3$.

Answer available at the end of this section.

Using Formulas

The main difference between algebra and arithmetic is the organized use of variables. This idea leads to reusable formulas, which are mathematical models using algebraic expressions to describe common applications. For example, the area of a rectangle is modeled by the formula:

$$A = l \cdot w \tag{2.1.1}$$

In this equation, variables are used to describe the relationship between the area of a rectangle and the length of its sides. The area is the product of the length and width of the rectangle. If the length of a rectangle measures 3 meters and the width measures 2 meters, then the area can be calculated using the formula as follows:

$$\begin{aligned} A &= l \cdot w \\ &= 3\text{m} \cdot (2\text{m}) \\ &= 6 \text{ square meters}(\text{m}^2) \end{aligned}$$

Example 2.1.9

The cost of a daily truck rental is $\$48.00$ plus an additional $\$0.45$ for every mile driven. This cost in dollars can be modeled by the formula $cost = 0.45x + 48$, where x represents the number of miles driven in one day. Use this formula to calculate the cost to rent the truck for a day and drive it 120 miles.

Solution:

Use the formula to find the cost when the number of miles $x = 120$.

$$cost = 0.45x + 48$$

Substitute 120 into the given formula for x and then simplify.

$$\begin{aligned} cost &= 0.45(\ ) + 48 \\ &= 0.45(120) + 48 \\ &= 54 + 48 \\ &= 102 \end{aligned}$$

Answer:

The rental costs \$$102$.

Uniform motion is modeled by the formula $D = rt$, which expresses distance D in terms of the average rate r, or speed, and the time t traveled at that rate. This formula, $D = rt$, is used often and is read "distance equals rate times time."

Example 2.1.10

Jim's road trip takes $2\frac{1}{2}$ hours at an average speed of 66 miles per hour. How far does he travel?

Solution:

Substitute the appropriate values into the formula and then simplify.

$$\begin{aligned} D &= r \cdot t \\ &= \left(66\ \frac{mi}{hr}\right) \cdot \left(2\frac{1}{2}\ hr\right) \\ &= \frac{66}{1} \cdot \frac{5}{2} mi \\ &= 33 \cdot 5mi \\ &= 165mi \end{aligned}$$

Answer:

Jim travels 165 miles.

The volume in cubic units of a rectangular box is given by the formula $V = lwh$, where l represents the length, w represents the width, and h represents the height.

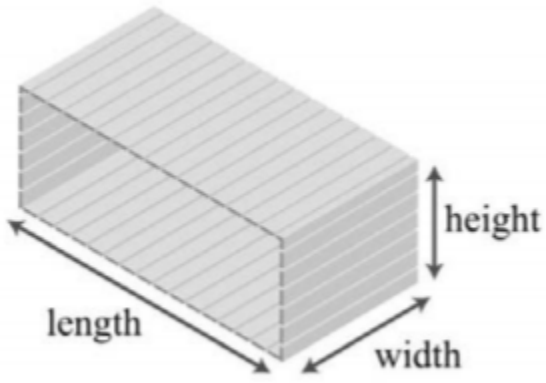


Figure $2.1.1$

Example 2.1.11

A wooden box is 1 foot in length, 5 inches wide, and 6 inches high. Find the volume of the box in cubic inches.

Solution:

Take care to ensure that all the units are consistent and use 12 inches for the length instead of 1 foot.

$$\begin{aligned} V &= lwh \\ V &= (\ )(\ )(\ ) \\ &= (12in)(5in)(6in) \\ &= 360 \text{ cubic inches } (\text{in}^3) \end{aligned}$$

Answer:

The volume of the box is 360 cubic inches.

Simple interest I is given by the formula $I = prt$, where p represents the principal amount invested at an annual interest rate r for t years.

Example 2.1.12

Calculate the simple interest earned on a 2-year investment of $\$1,250$ at an annual interest rate of $3\frac{3}{4}\%$.

Solution:

Convert $3\frac{3}{4}\%$ to a decimal number before using it in the formula.

$$r = 3\tfrac{3}{4}\% = 3.75\% = 0.0375$$

Use this value for r and the fact that $p = \$1,250$ and $t = 2$ years to calculate the simple interest.

$$\begin{aligned} I &= prt \\ &= (1,250)(0.0375)(2) \\ &= 93.75 \end{aligned}$$

Answer:

The simple interest earned is $\$93.75$.

Exercise 2.1.3

The perimeter of a rectangle is given by the formula $P = 2l + 2w$, where l represents the length and w represents the width. Use the formula to calculate the perimeter of a rectangle with a length of 5 feet and a width of $2\frac{1}{2}$ feet.

Answer available at the end of this section.

Key Takeaways

- Think of algebraic expressions as generalizations of common arithmetic operations that are formed by combining numbers, variables, and mathematical operations.
- It is customary to omit the coefficient if it is 1, as in $x^2 = 1x^2$.
- To avoid common errors when evaluating, it is a best practice to replace all variables with parentheses and then substitute the appropriate values.

- The use of algebraic expressions allows us to create useful and reusable formulas that model common applications.

Exercise 2.1.4 Definitions

List all of the coefficients and variable parts of the following expressions.

1. $4x - 1$
2. $-7x^2 - 2x + 1$
3. $-x^2 + 5x - 3$
4. $3x^2y^2 - \frac{2}{3}xy + 7$
5. $\frac{1}{3}y^2 - \frac{1}{2}y + \frac{5}{7}$
6. $-4a^2b + 5ab^2 - ab + 1$
7. $2(a + b)^3 - 3(a + b)^5$
8. $5(x + 2)^2 - 2(x + 2) - 7$
9. $m^2n - mn^2 + 10mn - 27$
10. $x^4 - 2x^3 - 3x^2 - 4x - 1$

Answer available at the end of this section.

Exercise 2.1.5 Evaluating Algebraic Expressions

Evaluate.

1. $x + 3$, where $x = -4$
2. $2x - 3$, where $x = -3$
3. $-5x + 20$, where $x = 4$
4. $-5y$, where $y = -1$
5. $\frac{3}{4}a$, where $a = 32$
6. $2(a - 4)$, where $a = -1$
7. $-10(5 - z)$, where $z = 14$
8. $5y - 1$, where $y = -\frac{1}{5}$
9. $-2a + 1$, where $a = -\frac{1}{3}$
10. $4x + 3$, where $x = \frac{3}{16}$
11. $-x + \frac{1}{2}$, where $x = -2$
12. $\frac{2}{3}x - \frac{1}{2}$, where $x = -\frac{1}{4}$

Answer available at the end of this section.

Exercise 2.1.6 Evaluating Algebraic Expressions

For each problem below, evaluate $b^2 - 4ac$, given the following values for a, b, and c.

1. $a = 1, b = 2, c = 3$
2. $a = 3, b = -4, c = -1$
3. $a = -6, b = 0, c = -2$
4. $a = \frac{1}{2}, b = 1, c = \frac{2}{3}$
5. $a = -3, b = -\frac{1}{2}, c = \frac{1}{9}$
6. $a = -\frac{1}{3}, b = -\frac{2}{3}, c = 0$

Answer available at the end of this section.

Exercise 2.1.7 Evaluating Algebraic Expressions

Evaluate.

1. $-4xy^2$, where $x = -3$ and $y = 2$
2. $\frac{5}{8}x^2y$, where $x = -1$ and $y = 16$
3. $a^2 - b^2$, where $a = 2$ and $b = 3$
4. $a^2 - b^2$, where $a = -1$ and $b = -2$
5. $x^2 - y^2$, where $x = \frac{1}{2}$ and $y = -\frac{1}{2}$
6. $3x^2 - 5x + 1$, where $x = -3$
7. $y^2 - y - 6$, where $y = 0$
8. $1 - y^2$, where $y = -\frac{1}{2}$
9. $(x + 3)(x - 2)$, where $x = -4$
10. $(y - 5)(y + 6)$, where $y = 5$
11. $3(\alpha - \beta) + 4$, where $\alpha = -1$ and $\beta = 6$
12. $3\alpha^2 - \beta^2$, where $\alpha = 2$ and $\beta = -3$
13. Evaluate $4(x + h)$, given $x = 5$ and $h = 0.01$.
14. Evaluate $-2(x + h) + 3$, given $x = 3$ and $h = 0.1$.
15. Evaluate $2(x + h)^2 - 5(x + h) + 3$, given $x = 2$ and $h = 0.1$.
16. Evaluate $3(x + h)^2 + 2(x + h) - 1$, given $x = 1$ and $h = 0.01$.

Answer available at the end of this section.

Exercise 2.1.8 Using Formulas

Convert the following temperatures to degrees Celsius given $C = \frac{5}{9}(F - 32)$, where F represents degrees Fahrenheit.

1. $86\,^\circ F$
2. $95\,^\circ F$
3. $-13\,^\circ F$
4. $14\,^\circ F$
5. $32\,^\circ F$
6. $0\,^\circ F$

Answer available at the end of this section.

Exercise 2.1.9 Using Formulas

Given the base and height of a triangle, calculate the area. $(A = \frac{1}{2}bh)$.

1. $b = 25$ centimeters and $h = 10$ centimeters
2. $b = 40$ inches and $h = 6$ inches
3. $b = \frac{1}{2}$ foot and $h = 2$ feet
4. $b = \frac{3}{4}$ inches and $h = \frac{5}{8}$ inches

Answer available at the end of this section.

Exercise 2.1.10 Using Formulas

1. A certain cellular phone plan charges \$23.00 per month plus \$0.09 for each minute of usage. The monthly charge is given by the formula monthly $charge = 0.09x + 23$, where x represents the number of minutes of usage per month. What is the charge for a month with 5 hours of usage?

2. A taxi service charges $\$3.75$ plus $\$1.15$ per mile given by the formula $charge = 1.15x + 3.75$, where x represents the number of miles driven. What is the charge for a 17-mile ride?
3. If a calculator is sold for $\$14.95$, then the revenue in dollars, R, generated by this item is given by the formula $R = 14.95q$, where q represents the number of calculators sold. Use the formula to determine the revenue generated by this item if 35 calculators are sold.
4. Yearly subscriptions to a tutoring website can be sold for $\$49.95$. The revenue in dollars, R, generated by subscription sales is given by the formula $R = 49.95q$, where q represents the number of yearly subscriptions sold. Use the formula to calculate the revenue generated by selling 250 subscriptions.
5. The cost of producing pens with the company logo printed on them consists of a onetime setup fee of $\$175$ plus $\$0.85$ for each pen produced. This cost can be calculated using the formula $C = 175 + 0.85q$, where q represents the number of pens produced. Use the formula to calculate the cost of producing $2,000$ pens.
6. The cost of producing a subscription website consists of an initial programming and setup fee of $\$4,500$ plus a monthly Web hosting fee of $\$29.95$. The cost of creating and hosting the website can be calculated using the formula $C = 4500 + 29.95n$, where n represents the number of months the website is hosted. How much will it cost to set up and host the website for 1 year?
7. The perimeter of a rectangle is given by the formula $P = 2l + 2w$, where l represents the length and w represents the width. What is the perimeter of a fenced-in rectangular yard measuring 70 feet by 100 feet?
8. Calculate the perimeter of an 8-by-10-inch picture.
9. Calculate the perimeter of a room that measures 12 feet by 18 feet.
10. A computer monitor measures 57.3 centimeters in length and 40.9 centimeters high. Calculate the perimeter.
11. The formula for the area of a rectangle in square units is given by $A = l \cdot w$, where l represents the length and w represents the width. Use this formula to calculate the area of a rectangle with length 12 centimeters and width 3 centimeters.
12. Calculate the area of an 8-by-12-inch picture.
13. Calculate the area of a room that measures 12 feet by 18 feet.
14. A computer monitor measures 57.3 centimeters in length and 40.9 centimeters in height. Calculate the total area of the screen.
15. A concrete slab is poured in the shape of a rectangle for a shed measuring 8 feet by 10 feet. Determine the area and perimeter of the slab.
16. Each side of a square deck measures 8 feet. Determine the area and perimeter of the deck.
17. The volume of a rectangular solid is given by $V = lwh$, where l represents the length, w represents the width, and h is the height of the solid. Find the volume of a rectangular solid if the length is 2 inches, the width is 3 inches, and the height is 4 inches.
18. If a trunk measures 3 feet by 2 feet and is $2\frac{1}{2}$ feet tall, then what is the volume of the trunk?
19. The interior of an industrial freezer measures 3 feet wide by 3 feet deep and 4 feet high. What is the volume of the freezer?
20. A laptop case measures 1 feet 2 inches by 10 inches by 2 inches. What is the volume of the case?
21. If the trip from Fresno to Sacramento can be made by car in $2\frac{1}{2}$ hours at an average speed of 67 miles per hour, then how far is Sacramento from Fresno?
22. A high-speed train averages 170 miles per hour. How far can it travel in $1\frac{1}{2}$ hours?
23. A jumbo jet can cruises at an average speed of 550 miles per hour. How far can it travel in 4 hours?
24. A fighter jet reaches a top speed of $1,316$ miles per hour. How far will the jet travel if it can sustain this speed for 15 minutes?
25. The Hubble Space Telescope is in low earth orbit traveling at an average speed of $16,950$ miles per hour. What distance does it travel in $1\frac{1}{2}$ hours?
26. Earth orbits the sun a speed of about $66,600$ miles per hour. How far does earth travel around the sun in 1 day?
27. Calculate the simple interest earned on a $\$2,500$ investment at 3% annual interest rate for 4 years.
28. Calculate the simple interest earned on a $\$1,000$ investment at 5% annual interest rate for 20 years.
29. How much simple interest is earned on a $\$3,200$ investment at a 2.4% annual interest for 1 year?
30. How much simple interest is earned on a $\$500$ investment at a 5.9% annual interest rate for 3 years?
31. Calculate the simple interest earned on a $\$10,500$ investment at a $4\frac{1}{4}\%$ annual interest rate for 4 years.
32. Calculate the simple interest earned on a $\$6,250$ investment at a $6\frac{3}{4}\%$ annual interest rate for 1 year.

Answer available at the end of this section.

Exercise 2.1.11 Discussion Board Topics

1. Research and discuss the history of the symbols for addition $(+)$ and subtraction $(-)$.
2. What are mathematical models and why are they useful in everyday life?
3. Find and post a useful formula. Demonstrate its use with some values.
4. Discuss the history and importance of the variable. How can you denote a variable when you run out of letters?
5. Find and post a useful resource describing the Greek alphabet.

Answer available at the end of this section.

Answers

Exercise 2.1.1

Coefficients: $\{-5, -3, -2, 1\}$; variable parts: $\{a^2, ab, b^2\}$

Exercise 2.1.2

35

Exercise 2.1.3

15 feet

Exercise 2.1.4 **Definitions**

1. Coefficients: $\{-1, 4\}$; variable parts: $\{x\}$
3. Coefficients: $\{-3, -1, 5\}$; variable parts: $\{x^2, x\}$
5. Coefficients: $\{-\frac{1}{2}, \frac{1}{3}, \frac{5}{7}\}$; variable parts: $\{y^2, y\}$
7. Coefficients: $\{-3, 2\}$; variable parts: $\{(a+b)^3, (a+b)^5\}$
9. Coefficients: $\{-27, -1, 1, 10\}$; variable parts: $\{m^2n, mn^2, mn\}$

Exercise 2.1.5 **Evaluating Algebraic Expressions**

1. -1
3. 0
5. 24
7. 90
9. $\frac{5}{3}$
11. $\frac{5}{2}$

Exercise 2.1.6 **Evaluating Algebraic Expressions**

1. -8
3. -48
5. $\frac{19}{12}$

Exercise 2.1.7 **Evaluating Algebraic Expressions**

1. 48
3. -5
5. 0
7. -6
9. 6
11. -17
13. 20.04
15. 1.32

Exercise 2.1.8 **Using Formulas**

1. $30\ ^\circ C$
3. $-25\ ^\circ C$
5. $0\ ^\circ C$

Exercise 2.1.9 **Using Formulas**

1. 125 square centimeters
3. $\frac{1}{2}$ square feet

Exercise 2.1.10 **Using Formulas**

1. $\$50$
3. $\$523.25$
5. $\$1,875.00$
7. 340 feet
9. 60 feet
11. 36 square centimeters

13. 216 square feet
15. Area: 80 square feet; Perimeter: 36 feet
17. 24 cubic inches
19. 36 cubic feet
21. 167.5 miles
23. $2,200$ miles
25. $25,425$ miles
27. $\$300$
29. $\$76.80$
31. $\$1,785$

Exercise $2.1.11$ **Discussion Board Topics**

1. Answers may vary
3. Answers may vary
5. Answers may vary

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2.2: Simplifying Algebraic Expressions

Learning Objectives

- Apply the distributive property to simplify an algebraic expression.
- Identify and combine like terms.

Distributive Property

The properties of real numbers are important in our study of algebra because a variable is simply a letter that represents a real number. In particular, the distributive property states that given any real numbers $a, b,$ and c,

$$a(b+c) = ab + ac \tag{2.2.1}$$

This property is applied when simplifying algebraic expressions. To demonstrate how it is used, we simplify $2(5-3)$ in two ways, and observe the same correct result.

Certainly, if the contents of the parentheses can be simplified, do that first. On the other hand, when the contents of parentheses cannot be simplified, multiply every term within the parentheses by the factor outside of the parentheses using the distributive property. Applying the distributive property allows you to multiply and remove the parentheses.

Simplify:

$$5(7y+2).$$

Solution:

Multiply 5 times each term inside the parentheses.

$$\begin{aligned} 5(7y+2) &= 5 \cdot 7y + 5 \cdot 2 \\ &= 35y + 10 \end{aligned}$$

Answer:

$$35y + 10$$

Example 2.2.2

Simplify:

$$-3(2x^2+5x+1).$$

Solution:

Multiply -3 times each of the coefficients of the terms inside the parentheses.

Answer:

$-6x^2 - 15x - 3$

Example 2.2.3

Simplify:

$$5(-2a+5b)-2c.$$

Solution:

Apply the distributive property by multiplying only the terms grouped within the parentheses by 5.

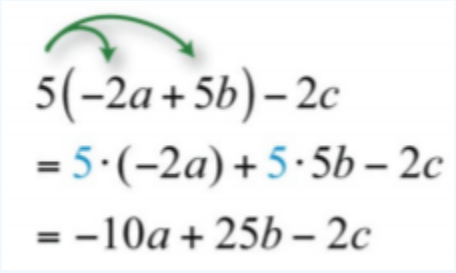


Figure 2.2.1

Answer:

$$-10a+25b-2c$$

Because multiplication is commutative, we can also write the distributive property in the following manner:

$$(b+c)a = ba+ca \tag{2.2.2}$$

Example 2.2.4

Simplify:

$$(3x-4y+1)\cdot 3.$$

Solution:

Multiply each term within the parentheses by 3.

$$\begin{aligned}(3x-4y+1)\cdot 3 &= 3x\cdot 3 - 4y\cdot 3 + 1\cdot 3\\ &= 9x-12y+3\end{aligned}$$

Answer:

$$9x-12y+3$$

Division in algebra is often indicated using the fraction bar rather than with the symbol ($\div$). And sometimes it is useful to rewrite expressions involving division as products:

$$\begin{aligned}\tfrac{x}{5} &= \tfrac{1x}{5} = \tfrac{1}{5}\cdot x\\ \tfrac{3ab}{7} &= \tfrac{3}{7}\cdot\tfrac{ab}{1} = \tfrac{3}{7}\cdot ab\\ \tfrac{x+y}{3} &= \tfrac{1}{3}\cdot\tfrac{(x+y)}{1} = \tfrac{1}{3}\cdot(x+y)\end{aligned}$$

Rewriting algebraic expressions as products allows us to apply the distributive property.

Example 2.2.5

Divide:

\(\frac{25x^{2}-5x+10}{5}.

Solution:

First, treat this as $\frac{1}{5}$ times the expression in the numerator and then distribute.

$$\begin{aligned}\frac{25x^2-5x+10}{5} &= \frac{1}{5}\cdot\frac{(25x^2-5x+10)}{1}\\ &= \frac{1}{5}\cdot(25x^2-5x+10) && \textit{Multiply each term by } \frac{1}{5}.\\ &= \frac{1}{5}\cdot 25x^2-\frac{1}{5}\cdot 5x+\frac{1}{5}\cdot 10 && \textit{Simplify.}\\ &= 5x^2-x+2\end{aligned}$$

Alternate Solution:

Think of 5 as a common denominator and divide each of the terms in the numerator by 5:

$$\begin{aligned}\frac{25x^2-5x+10}{5} &= \frac{25x^2}{5}-\frac{5x}{5}+\frac{10}{5}\\ &= 5x^2-x+2\end{aligned}$$

Answer:

$$5x^2-x+2$$

We will discuss the division of algebraic expressions in more detail as we progress through the course.

Exercise 2.2.1

Simplify:

$$\tfrac{1}{3}(-9x+27y-3).$$

Answer available at the end of this section.

Combining Like Terms

Terms with the same variable parts are called like terms, or similar terms. Furthermore, constant terms are considered to be like terms. If an algebraic expression contains like terms, apply the distributive property as follows:

$$\begin{aligned}2a+3a &= (2+3)a = 5a\\ 7xy-5xy &= (7-5)xy = 2xy\\ 10x^2+4x^2-6x^2 &= (10+4-6)x^2 = 8x^2\end{aligned}$$

In other words, if the variable parts of terms are ***exactly the same***, then we may add or subtract the coefficients to obtain the coefficient of a single term with the same variable part. This process is called combining like terms. For example,

$$3a^2b+2a^2b=5a^2b$$

Notice that the variable factors and their exponents do not change. Combining like terms in this manner, so that the expression contains no other similar terms, is called simplifying the expression. Use this idea to simplify algebraic expressions with multiple

like terms.

Example 2.2.6

Simplify:

$$3a + 2b - 4a + 9b.$$

Solution:

Identify the like terms and combine them.

$$\begin{aligned} 3a + 2b - 4a + 9b &= 3a - 4a + 2b + 9b && \textit{Commutative property of addition} \\ &= -1a + 11b && \textit{Combine like terms.} \\ &= -a + 11b \end{aligned}$$

Answer:

$$-a + 11b$$

In the previous example, rearranging the terms is typically performed mentally and is not shown in the presentation of the solution.

Example 2.2.7

Simplify:

$$x^2 + 3x + 2 + 4x^2 - 5x - 7.$$

Solution:

Identify the like terms and add the corresponding coefficients.

$$\begin{aligned} & 1x^2 + 3x + 2 + 4x^2 - 5x - 7 && \textit{Identify like terms.} \\ &= 5x^2 - 2x - 5 && \textit{Combine like terms.} \end{aligned}$$

Answer:

$$5x^2 - 2x - 5$$

Example 2.2.8

Simplify:

$$5x^2y - 3xy^2 + 4x^2y - 2xy^2.$$

Solution:

Remember to leave the variable factors and their exponents unchanged in the resulting combined term.

$$\begin{aligned} & 5x^2y - 3xy^2 + 4x^2y - 2xy^2 \\ &= 9x^2y - 5xy^2 \end{aligned}$$

Answer:

$$9x^2y - 5xy^2$$

Example 2.2.9

Simplify:

$$\tfrac{1}{2}a - \tfrac{1}{3}b + \tfrac{3}{4}a + b.$$

To add the fractional coefficients, use equivalent coefficients with common denominators for each like term.

$$\begin{aligned}\frac{1}{2}a - \frac{1}{3}b + \frac{3}{4}a + 1b &= \frac{1}{2}a + \frac{3}{4}a - \frac{1}{3}b + 1b \\ &= \frac{2}{4}a + \frac{3}{4}a - \frac{1}{3}b + \frac{3}{3}b \\ &= \frac{5}{4}a + \frac{2}{3}b\end{aligned}$$

Answer:

$$\tfrac{5}{4}a + \tfrac{2}{3}b$$

Example 2.2.10

Simplify:

$$-12x(x+y)^3 + 26x(x+y)^3.$$

Solution:

Consider the variable part to be $x(x+y)^3$. Then this expression has two like terms with coefficients -12 and 26.

$$\begin{aligned}&-12x(x+y)^3 + 26x(x+y)^3 \qquad \textit{Add the coefficients.} \\ &= 14x(x+y)^3\end{aligned}$$

Answer:

$$14x(x+y)^3$$

Exercise 2.2.2

Simplify:

$$-7x + 8y - 2x - 3y.$$

Answer available at the end of this section.

Distributive Property and Like Terms

When simplifying, we will often have to combine like terms after we apply the distributive property. This step is consistent with the order of operations: multiplication before addition.

Example 2.2.11

Simplify:

$$2(3a-b)-7(-2a+3b).$$

Solution:

Distribute 2 and -7 and then combine like terms.

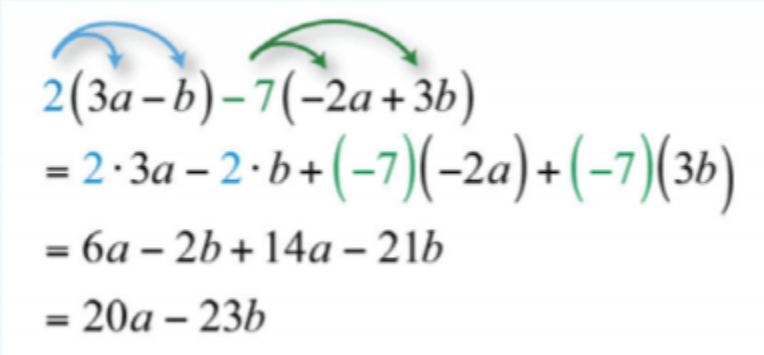


Figure 2.2.2

Answer:

$$20a-23b$$

In the example above, it is important to point out that you can remove the parentheses and collect like terms because you multiply the second quantity by -7, not just by 7. To correctly apply the distributive property, think of this as adding -7 times the given quantity, $2(3a-b)+(-7)(-2a+3b)$.

Exercise 2.2.3

Simplify:

$$-5(2x-3)+7x.$$

Answer available at the end of this section.

Often we will encounter algebraic expressions like $+(a+b)$ or $-(a+b)$. As we have seen, the coefficients are actually implied to be $+1$ and -1, respectively, and therefore, the distributive property applies using $+1$ or -1 as the factor. Multiply each term within the parentheses by these factors:

$$+(a+b)=+1(a+b)=(+1)a+(+1)b=a+b \tag{2.2.3}$$

$$-(a+b)=-1(a+b)=(-1)a+(-1)b=-a-b \tag{2.2.4}$$

This leads to two useful properties,

$$+(a+b)=a+b \tag{2.2.5}$$

$$-(a+b)=-a-b \tag{2.2.6}$$

Example 2.2.12

Simplify:

$$5x - (-2x^2 + 3x - 1).$$

Solution:

Multiply each term within the parentheses by -1 and then combine like terms.

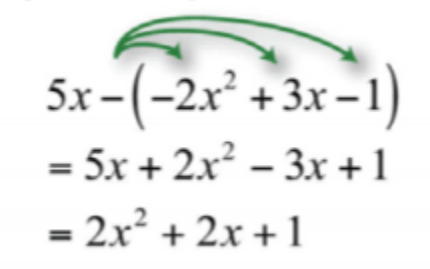


Figure 2.2.3

Answer:

$$2x^2 + 2x + 1$$

When distributing a negative number, all of the signs within the parentheses will change. Note that $5x$ in the example above is a separate term; hence the distributive property does not apply to it.

Example 2.2.13

Simplify:

$$5 - 2(x^2 - 4x - 3).$$

Solution:

The order of operations requires that we multiply before subtracting. Therefore, distribute -2 and then combine the constant terms. Subtracting $5 - 2$ first leads to an incorrect result, as illustrated below:

Incorrect!		*Correct!*	
$5 - 2(x^2 - 4x - 3)$		$5-2(x^2 - 4x - 3)$	
$= 3(x^2 - 4x - 3)$		$= 5-2x^2+8x+6$	
$= 3x^2 - 12x - 9$	x	$= -2x^2 + 8x + 11$	✓

Answer:

$$-2x^2 + 8x + 11$$

Note

It is worth repeating that you must ***follow the order of operations***: multiply and divide before adding and subtracting!

Exercise 2.2.4

Simplify:

$$8 - 3(-x^2 + 2x - 7).$$

Answer available at the end of this section.

Example 2.2.14

Subtract $3x - 2$ from twice the quantity $-4x^2 + 2x - 8$.

Solution:

First, group each expression and treat each as a quantity:

$$(3x - 2) \qquad \text{and} \qquad (-4x^2 + 2x - 8)$$

Next, identify the key words and translate them into a mathematical expression.

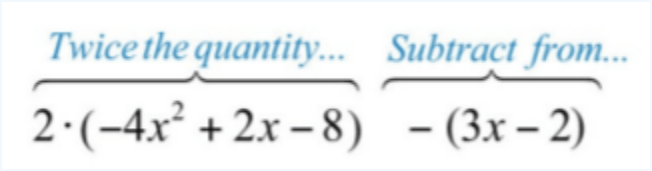


Figure 2.2.4

Finally, simplify the resulting expression.

Answer:

$-8x^2 + x - 14$

Key Takeaways

- The properties of real numbers apply to algebraic expressions, because variables are simply representations of unknown real numbers.
- Combine like terms, or terms with the same variable part, to simplify expressions.
- Use the distributive property when multiplying grouped algebraic expressions, $a(b + c) = ab + ac$.
- It is a best practice to apply the distributive property only when the expression within the grouping is completely simplified.
- After applying the distributive property, eliminate the parentheses and then combine any like terms.
- Always use the order of operations when simplifying.

Exercise 2.2.5 Distributive Property

Multiply.

1. $3(3x - 2)$
2. $12(-5y + 1)$
3. $-2(x + 1)$
4. $5(a - b)$
5. $\frac{5}{8}(8x - 16)$

6. $-\frac{3}{5}(10x-5)$
7. $(2x+3)\cdot 2$
8. $(5x-1)\cdot 5$
9. $(-x+7)(-3)$
10. $(-8x+1)(-2)$
11. $-(2a-3b)$
12. $-(x-1)$
13. $\frac{1}{3}(2x+5)$
14. $-\frac{3}{4}(y-2)$
15. $-3(2a+5b-c)$
16. $-(2y^2-5y+7)$
17. $5(y^2-6y-9)$
18. $-6(5x^2+2x-1)$
19. $7x^2-(3x-11)$
20. $-(2a-3b)+c$
21. $3(7x^2-2x)-3$
22. $\frac{1}{2}(4a^2-6a+4)$
23. $-\frac{1}{3}(9y^2-3y+27)$
24. $(5x^2-7x+9)(-5)$
25. $6(\frac{1}{3}x^2-\frac{1}{6}x+\frac{1}{2})$
26. $-2(3x^3-2x^2+x-3)$
27. $\frac{20x+30y-10z}{10}$
28. $\frac{-4a+20b-8c}{4}$
29. $\frac{3x^2-9x+81}{-3}$
30. $\frac{15y^2+20y-5}{5}$

Answer available at the end of this section.

Exercise 2.2.6 Distributive Property

Translate the following sentences into algebraic expressions and then simplify.

1. Simplify two times the expression $25x^2-9$.
2. Simplify the opposite of the expression $6x^2+5x-1$.
3. Simplify the product of 5 and x^2-8.
4. Simplify the product of -3 and $-2x^2+x-8$.

Answer available at the end of this section.

Exercise 2.2.7 Combining Like Terms

Simplify.

1. $2x-3x$
2. $-2a+5a-12a$
3. $10y-30-15y$
4. $\frac{1}{3}x+\frac{5}{12}x$
5. $-\frac{1}{4}x+\frac{4}{5}+\frac{3}{8}x$
6. $2x-4x+7x-x$
7. $-3y-2y+10y-4y$
8. $5x-7x+8y+2y$
9. $-8\alpha+2\beta-5\alpha-6\beta$
10. $-6\alpha+7\beta-2\alpha+\beta$
11. $3x+5-2y+7-5x+3y$

12. $-y+8x-3+14x+1-y$
13. $4xy-6+2xy+8$
14. $-12ab-3+4ab-20$
15. $\frac{1}{3}x-\frac{2}{5}y+\frac{2}{3}x-\frac{3}{5}y$
16. $\frac{3}{8}a-\frac{2}{7}b-\frac{1}{4}a+\frac{3}{14}b$
17. $-4x^2-3xy+7+4x^2-5xy-3$
18. $x^2+y^2-2xy-x^2+5xy-y^2$
19. $x^2-y^2+2x^2-3y$
20. $\frac{1}{2}x^2-\frac{2}{3}y^2-\frac{1}{8}x^2+\frac{1}{5}y^2$
21. $\frac{3}{16}a^2-\frac{4}{5}+\frac{1}{4}a^2-\frac{1}{4}$
22. $\frac{1}{5}y^2-\frac{3}{4}+\frac{7}{10}y^2-\frac{1}{2}$
23. $6x^2y-3xy^2+2x^2y-5xy^2$
24. $12x^2y^2+3xy-13x^2y^2+10xy$
25. $-ab^2+a^2b-2ab^2+5a^2b$
26. $m^2n^2-mn+mn-3m^2n+4m^2n^2$
27. $2(x+y)^2+3(x+y)^2$
28. $\frac{1}{5}(x+2)^3-\frac{2}{3}(x+2)^3$
29. $-3x(x^2-1)+5x(x^2-1)$
30. $5(x-3)-8(x-3)$
31. $-14(2x+7)+6(2x+7)$
32. $4xy(x+2)^2-9xy(x+2)^2+xy(x+2)^2$

Answer available at the end of this section.

Exercise 2.2.8 Mixed Practice

Simplify.

1. $5(2x-3)+7$
2. $-2(4y+2)-3y$
3. $5x-2(4x-5)$
4. $3-(2x+7)$
5. $2x-(3x-4y-1)$
6. $(10y-8)-(40x+20y-7)$
7. $\frac{1}{2}y-\frac{3}{4}x-(\frac{2}{3}y-\frac{1}{5}x)$
8. $\frac{1}{5}a-\frac{3}{4}b+\frac{3}{15}a-\frac{1}{2}b$
9. $\frac{2}{3}(x-y)+x-2y$
10. $-\frac{1}{3}(6x-1)+\frac{1}{2}(4y-1)-(-2x+2y-\frac{1}{6})$
11. $(2x^2-7x+1)+(x^2+7x-5)$
12. $6(-2x^2+3x-1)+10x^2-5x$
13. $-(x^2-3x+8)+x^2-12$
14. $2(3a-4b)+4(-2a+3b)$
15. $-7(10x-7y)-6(8x+4y)$
16. $10(6x-9)-(80x-35)$
17. $10-5(x^2-3x-1)$
18. $4+6(y^2-9)$
19. $\frac{3}{4}x-(\frac{1}{2}x^2+\frac{2}{3}x-\frac{7}{5})$
20. $-\frac{7}{3}x^2+(-\frac{1}{6}x^2+7x-1)$
21. $(2y^2-3y+1)-(5y^2+10y-7)$
22. $(-10a^2-b^2+c)+(12a^2+b^2-4c)$
23. $-4(2x^2+3x-2)+5(x^2-4x-1)$
24. $2(3x^2-7x+1)-3(x^2+5x-1)$
25. $x^2y+3xy^2-(2x^2y-xy^2)$
26. $3(x^2y^2-12xy)-(7x^2y^2-20xy+18)$
27. $3-5(ab-3)+2(ba-4)$
28. $-9-2(xy+7)-(yx-1)$

29. $-5(4\alpha - 2\beta + 1) + 10(\alpha - 3\beta + 2)$
30. $\frac{1}{2}(100\alpha^2 - 50\alpha\beta + 2\beta^2) - \frac{1}{5}(50\alpha^2 + 10\alpha\beta - 5\beta^2)$

Answer available at the end of this section.

Exercise 2.2.9 Mixed Practice

Translate the following sentences into algebraic expressions and then simplify.

1. What is the difference of $3x - 4$ and $-2x + 5$?
2. Subtract $2x - 3$ from $5x + 7$.
3. Subtract $4x + 3$ from twice the quantity $x - 2$.
4. Subtract three times the quantity $-x + 8$ from $10x - 9$.

Answer available at the end of this section.

Exercise 2.2.10 Discussion Board Topics

1. Do we need a distributive property for division, $(a + b) \div c$? Explain.
2. Do we need a separate distributive property for three terms, $a(b + c + d)$? Explain.
3. Explain how to subtract one expression from another. Give some examples and demonstrate the importance of the order in which subtraction is performed.
4. Given the algebraic expression $8 - 5(3x + 4)$, explain why subtracting $8 - 5$ is not the first step.
5. Can you apply the distributive property to the expression $5(abc)$? Explain why or why not and give some examples.
6. How can you check to see if you have simplified an expression correctly? Give some examples.

Answer available at the end of this section.

Answers

Exercise 2.2.1

$-3x + 9y - 1$

Exercise 2.2.2

$-9x + 5y$

Exercise 2.2.3

$-3x + 15$

Exercise 2.2.4

$3x^2 - 6x + 29$

Exercise 2.2.5 **Distributive Property**

1. $9x - 6$
3. $-2x - 2$
5. $5x - 10$
7. $4x + 6$
9. $3x - 21$
11. $-2a + 3b$
13. $\frac{2}{3}x + \frac{5}{3}$
15. $-6a - 15b + 3c$
17. $5y^2 - 30y - 45$
19. $7x^2 - 3x + 11$
21. $21x^2 - 6x - 3$
23. $-3y^2 + y - 9$
25. $2x^2 - x + 3$
27. $2x + 3y - z$
29. $-x^2 + 3x - 27$

Exercise 2.2.6 **Distributive Property**

1. $50x^2 - 18$
3. $5x^2 - 40$

Exercise 2.2.7 **Combining Like Terms**

1. $-x$
3. $-5y - 30$
5. $\frac{1}{8}x + \frac{4}{5}$
7. y
9. $-13\alpha - 4\beta$
11. $-2x + y + 12$
13. $6xy + 2$
15. $x - y$
17. $-8xy + 4$
19. $3x^2 - y^2 - 3y$
21. $\frac{7}{16}a^2 - \frac{21}{20}$
23. $8x^2y - 8xy^2$
25. $6a^2b - 3ab^2$
27. $5(x + y)^2$
29. $2x(x^2 - 1)$
31. $-8(2x + 7)$

Exercise 2.2.8 **Mixed Practice**

1. $10x - 8$
3. $-3x + 10$
5. $-x + 4y + 1$
7. $-\frac{11}{20}x - \frac{1}{6}y$
9. $\frac{5}{3}x - \frac{8}{3}y$
11. $3x^2 - 4$
13. $3x - 20$
15. $-118x + 25y$
17. $-5x^2 + 15x + 15$
19. $-\frac{1}{2}x^2 + \frac{1}{12}x + \frac{7}{5}$
21. $-3y^2 - 13y + 8$
23. $-3x^2 - 32x + 3$
25. $-x^2y + 4xy^2$
27. $-3ab + 10$
29. $-10\alpha - 20\beta + 15$

Exercise 2.2.9 **Mixed Practice**

1. $5x - 9$
3. $-2x - 7$

Exercise 2.2.10 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary
5. Answers may vary

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2.3: Solving Linear Equations- Part 1

Learning Objectives

- Identify linear equations with one variable and verify their solutions.
- Use the properties of equality to solve basic linear equations.
- Use multiple steps to solve linear equations by isolating the variable.
- Solve linear equations where the coefficients are fractions or decimals.

Linear Equations with One Variable and Their Solutions

Learning how to solve various algebraic equations is one of the main goals in algebra. This section introduces the basic techniques used for solving linear equations with one variable.

An equation is a statement indicating that two algebraic expressions are equal. A linear equation with one variable, x, is an equation that can be written in the general form $ax + b = 0$, where a and b are real numbers and $a \neq 0$. Here are some examples of linear equations, all of which are solved in this section:

$$x + 3 = -5 \qquad 2x - 5 = 15 \qquad \tfrac{5}{3}x + 2 = -8$$

A solution to a linear equation is any value that can replace the variable to produce a true statement. The variable in the linear equation $2x + 3 = 13$ is x, and the solution is $x = 5$. To verify this, substitute the value 5 for x and check that you obtain a true statement.

$$\begin{aligned} 2x + 13 &= 13 \\ 2(5) + 3 &= 13 \\ 10 + 3 &= 13 \\ 13 &= 13 \quad \checkmark \end{aligned}$$

Alternatively, when an equation is equal to a constant, we can verify a solution by substituting the value for the variable and show that the result is equal to that constant. In this sense, we say that solutions satisfy the equation.

Example 2.3.1

Is $x = 3$ a solution to $-2x - 3 = -9$?

Solution:

$$-2x - 3 = -2(3) - 3 = -6 - 3 = -9 \quad \checkmark$$

Answer:

Yes, it is a solution, because $x = 3$ satisfies the equation.

Example 2.3.2

Is $a = -\frac{1}{2}$ a solution to $-10a + 5 = 25$?

Solution:

$$-10a + 5 = -10\left(-\tfrac{1}{2}\right) + 5 = 5 + 5 = 10 \neq 25 \quad x$$

Answer:

No, it is not a solution, because $a = -\frac{1}{2}$ does not satisfy the equation.

Recall that when evaluating expressions, it is a good practice to first replace all variables with parentheses, then substitute the appropriate values. By making use of parentheses we avoid some common errors when working the order of operations.

Example 2.3.3

Is $y = -3$ a solution to $2y - 5 = -y - 14$?

Solution:

Answer:

Yes, it is a solution, because $y = -3$ produces a true statement.

Exercise 2.3.1

Is $x = -3$ a solution to $-2x + 5 = -1$?

Answer available at the end of this section.

Solving Basic Linear Equations

We begin by defining equivalent equations as equations with the same solution set. Consider the following two linear equations and check to see if the solution is $x = 7$.

$$\begin{aligned} 3x - 5 &= 16 \\ 3(7) - 5 &= 16 \\ 21 - 5 &= 16 \\ 16 &= 16 \quad \checkmark \end{aligned} \qquad \begin{aligned} 3x &= 21 \\ 3(7) &= 21 \\ 21 &= 21 \quad \checkmark \end{aligned}$$

Here we can see that the two linear equations $3x - 5 = 16$ and $3x = 21$ are equivalent because they share the same solution set, namely, $\{7\}$. The goal is to develop a systematic process to find equivalent equations until the variable is isolated:

$$\left.\begin{aligned} 3x - 5 &= 16 \\ 3x &= 21 \\ x &= 7 \end{aligned}\right\} \quad \textit{Equivalent equations}$$

To do this, use the properties of equality. Given algebraic expressions A and B, where c is a real number, we have the following:

$$\begin{aligned} &\textbf{Addition Property of Equality}: && \textit{If } A = B, \textit{ then } A + c = B + c \\ &\textbf{Subtraction Property of Equality:} && \textit{If } A = B, \textit{ then } A - c = B - c \\ &\textbf{Multiplication Property of Equality:} && \textit{If } A = B, \textit{ then } cA = cB \\ &\textbf{Division Property of Equality:} && \textit{If } A = B, \textit{ then } \tfrac{A}{c} = \tfrac{B}{c}\ c \neq 0 \end{aligned}$$

Note

Multiplying or dividing both sides of an equation by 0 is carefully avoided. Dividing by 0 is undefined and multiplying both sides by 0 results in the equation $0 = 0$.

To summarize, equality is retained and you obtain an equivalent equation if you add, subtract, multiply, or divide ***both sides*** of

an equation by any nonzero real number. The technique for solving linear equations involves applying these properties in order to ***isolate the variable*** on one side of the equation. If the linear equation has a constant term, then we add to or subtract it from both sides of the equation to obtain an equivalent equation where the variable term is isolated.

Example 2.3.4

Solve:

$$x + 3 = -5.$$

Solution:

To isolate the variable x on the left side, subtract 3 from both sides.

Answer:

The solution is $x = -8$.

To check that this is true, substitute -8 into the original equation and simplify to see that it is satisfied: $x + 3 = -8 + 3 = -5$ ✓.

In the previous example, after subtracting 3 from both sides, you get $x + 0 = -8$. By the additive identity property of real numbers, this is equivalent to $x = -8$. This step is often left out in the presentation of the solution.

If the ***variable term*** of the equation (including the coefficient) is isolated, then apply the multiplication or division property of equality to obtain an equivalent equation with the ***variable*** isolated. In other words, our goal is to obtain an equivalent equation with x or $1x$ isolated on one side of the equal sign.

Example 2.3.5

Solve:

$$-5x = -35.$$

Solution:

The coefficient of x is -5, so divide both sides by -5.

Answer:

The solution is $x = 7$. Perform the check mentally by substituting 7 for x in the original equation.

In the previous example, after dividing both sides by $-5, x$ is left with a coefficient of 1, because $\frac{-5}{-5} = 1$. In fact, when we say "isolate the variable," we mean to change the coefficient of the variable to 1, because $1x = 7$ is equivalent to $x = 7$. This step is often left out of the instructional examples even though its omission is sometimes a source of confusion.

Another important property is the symmetric property: for any algebraic expressions A and B,

$$\text{If } A = B, \text{ then } B = A \tag{2.3.1}$$

The equation $2 = x$ is equivalent to $x = 2$. It does not matter on which side we choose to isolate the variable.

Example 2.3.6

Solve:

$$2 = 5 + x.$$

Solution:

Isolate the variable x by subtracting 5 from both sides of the equation.

Answer:

The solution is -3, and checking the solution shows that $2 = 5 - 3$.

Exercise 2.3.2

Solve:

$$6 = x - 4$$

Answer available at the end of this section.

Isolating the Variable in Two Steps

A linear equation of the form $ax + b = c$ takes two steps to solve. First, use the appropriate equality property of addition or subtraction to isolate the variable term. Next, isolate the variable using the equality property of multiplication or division. Checking solutions in the following examples is left to the reader.

Example 2.3.7

Solve:

$$2x - 5 = 15.$$

Solution:

$$\begin{aligned} 2x - 5 &= 15 \\ 2x - 5{+5} &= 15{+5} && \textit{Add } 5 \textit{ to both sides.} \\ 2x &= 20 \\ \frac{2x}{2} &= \frac{20}{2} && \textit{Divide both sides by } 2. \\ x &= 10 \end{aligned}$$

Answer:

The solution is 10.

Example 2.3.8

Solve:

$$-3x - 2 = 9.$$

Solution:

$$\begin{aligned} -3x-2 &= 9 \\ -3x-2{\color{cyan}+2} &= 9{\color{cyan}+2} && \textit{Add 2 to both sides.} \\ -3x &= 11 \\ \frac{-3x}{{\color{cyan}-3}} &= \frac{11}{{\color{cyan}-3}} && \textit{Divide both sides by } -3. \\ x &= -\frac{11}{3} \end{aligned}$$

Answer:

The solution is $-\frac{11}{3}$.

Example 2.3.9

Solve:

$$6-5y=-14.$$

Solution:

When no sign precedes the term, it is understood to be positive. In other words, think of this as $+6-5y=-14$. Begin by subtracting 6 from both sides of the equal sign.

Answer:

The solution is 4.

Example 2.3.10

Solve:

$$3x+\tfrac{1}{2}=\tfrac{2}{3}.$$

Solution:

$$\begin{aligned} 3x+\frac{1}{2} &= \frac{2}{3} \\ 3x+\frac{1}{2}{\color{cyan}-\frac{1}{2}} &= \frac{2}{3}{\color{cyan}-\frac{1}{2}} && \textit{Subtract } \tfrac{1}{2} \textit{ from both sides.} \\ 3x &= \frac{2\cdot 2}{3\cdot 2}-\frac{1\cdot 3}{2\cdot 3} && \textit{Obtain equivalent fractions with a common denominator.} \\ 3x &= \frac{4}{6}-\frac{3}{6} \\ 3x &= \frac{1}{6} \\ \frac{3x}{{\color{cyan}3}} &= \frac{6}{{\color{cyan}3}} && \textit{Divide both sides by } 3. \\ x &= \frac{1}{6}\div 3 = \frac{1}{6}\cdot{\color{cyan}\frac{1}{3}} = \frac{1}{18} \end{aligned}$$

Answer:

The solution is $\frac{1}{18}$.

Example 2.3.11

Solve:

$$3 - y = 1.$$

Solution:

Recall that $-y$ is equivalent to $-1y$; divide both sides of the equation by -1.

$$\begin{aligned} -y &= -2 \\ \frac{-1y}{-1} &= \frac{-2}{-1} \\ y &= 2 \end{aligned}$$

Alternatively, multiply both sides of $-y = -2$ by -1 and achieve the same result:

Answer:

The solution is 2.

In summary, to retain equivalent equations, we must perform the same operation on both sides of the equation. First, apply the addition or subtraction property of equality to isolate the variable term and then apply the multiplication or division property of equality to isolate the variable on one side of the equation.

Exercise 2.3.3

Solve:

$$-7x + 6 = 27.$$

Answer available at the end of this section.

Multiplying by the Reciprocal

To solve an equation like $\frac{3}{4}x = 1$, we can isolate the variable by dividing both sides by the coefficient. For example

$$\begin{aligned} \frac{3}{4}x &= 1 \\ \frac{\frac{3}{4}x}{\frac{3}{4}} &= \frac{1}{\frac{3}{4}} \\ x &= 1 \cdot \frac{4}{3} \\ x &= \frac{4}{3} \end{aligned}$$

On the left side of the equal sign, the fraction cancels. On the right side, we have a complex fraction and multiply by the reciprocal of the coefficient. You can save a step by recognizing this and start by multiplying both sides of the equation by the reciprocal of the coefficient.

$$\begin{aligned} \frac{3}{4}x &= 1 \\ \frac{4}{3} \cdot \frac{3}{4}x &= \frac{4}{3} \cdot 1 \\ x &= \frac{4}{3} \end{aligned}$$

Recall that the product of reciprocals is 1, in this case $\frac{4}{3} \cdot \frac{3}{4} = 1$, leaving the variable isolated.

Example 2.3.12

Solve:

$$\frac{5}{3}x + 2 = -8.$$

Solution:

Isolate the variable term using the addition property of equality and then multiply both sides of the equation by the reciprocal of the coefficient $\frac{5}{3}$.

Answer:

The solution is -6.

Example 2.3.13

Solve:

$$-\frac{4}{5}x - 5 = 15.$$

Solution:

$$\begin{aligned} -\frac{4}{5}x - 5 &= 15 \\ -\frac{4}{5}x - 5 \color{#1a9cd8}{+5} &= 15 \color{#1a9cd8}{+5} \\ -\frac{4}{5}x &= 20 \end{aligned}$$

The reciprocal of $-\frac{4}{5}$ is $-\frac{5}{4}$ because $(-\frac{5}{4})(-\frac{4}{5}) = +\frac{20}{20} = 1$. Therefore, to isolate the variable x, multiply both sides by $-\frac{5}{4}$.

Answer:

The solution is -25.

Exercise 2.3.4

Solve:

$$\frac{2}{3}x - 9 = -4.$$

Answer available at the end of this section.

Key Takeaways

- Linear equations with one variable can be written in the form $ax + b = 0$, where a and b are real numbers and $a \neq 0$.
- To “solve a linear equation” means to find a numerical value that can replace the variable and produce a true statement.
- The properties of equality provide tools for isolating the variable and solving equations.
- To solve a linear equation, first isolate the *variable term* by adding the opposite of the constant term to both sides of the equation. Then isolate the *variable* by dividing both sides of the equation by its coefficient.
- After isolating a variable term with a fraction coefficient, solve by multiplying both sides by the reciprocal of the coefficient.

Exercise 2.3.5 Solutions to Linear Equations

Is the given value a solution to the linear equation?

1. $x - 6 = 20; x = 26$
2. $y + 7 = -6; y = -13$
3. $-x + 5 = 17; x = 12$
4. $-2y = 44; y = 11$
5. $4x = -24; x = -6$
6. $5x - 1 = 34; x = -7$
7. $-2a - 7 = -7; a = 0$
8. $-\frac{1}{3}x - 4 = -5; x = -3$
9. $-\frac{1}{2}x + \frac{2}{3} = -\frac{1}{4}; x = \frac{11}{6}$
10. $-8x - 33 = 3x; x = 3$
11. $3y - 5 = -2y - 15; y = -2$
12. $3(2x + 1) = -4x - 3; x = -\frac{1}{2}$
13. $\frac{1}{2}y - \frac{1}{3} = \frac{1}{3}y + \frac{1}{6}; y = 3$
14. $-\frac{4}{3}y + \frac{1}{9} = -\frac{2}{3}y - \frac{1}{9}; y = \frac{1}{3}$

Answer available at the end of this section.

Exercise 2.3.6 Solving Basic Linear Equations

Solve.

1. $x + 3 = 13$
2. $y - 4 = 22$
3. $-6 + x = 12$
4. $9 + y = -4$
5. $x - \frac{1}{2} = \frac{1}{3}$
6. $x + \frac{2}{3} = -\frac{1}{5}$
7. $x + 2\frac{1}{2} = 3\frac{1}{3}$
8. $-37 + x = -37$
9. $4x = -44$
10. $-9x = 63$
11. $-y = 13$
12. $-x = -10$
13. $-9x = 0$
14. $-3a = -33$
15. $27 = 18y$
16. $14 = -7x$
17. $31.5.6a = -39.2$
18. $-1.2y = 3.72$
19. $\frac{1}{3}x = -\frac{1}{2}$
20. $-\frac{t}{12} = \frac{1}{4}$
21. $-\frac{7}{3}x = \frac{1}{2}$
22. $\frac{x}{5} = -3$
23. $\frac{4}{9}y = -\frac{2}{3}$
24. $-\frac{5}{8}y = -\frac{5}{2}$

Answer available at the end of this section.

Exercise 2.3.7 Solving Linear Equations

Solve.

1. $5x + 7 = 32$
2. $4x - 3 = 21$
3. $3a - 7 = 23$
4. $12y + 1 = 1$
5. $21x - 7 = 0$
6. $-3y + 2 = -13$
7. $-5x + 9 = 8$
8. $22x - 55 = -22$
9. $4.5x - 2.3 = 6.7$
10. $1.4 - 3.2x = 3$
11. $9.6 - 1.4y = -10.28$
12. $4.2y - 3.71 = 8.89$
13. $3 - 2y = -11$
14. $-4 - 7a = 24$
15. $-10 = 2x - 5$
16. $24 = 6 - 12y$
17. $\frac{5}{6}x - \frac{1}{2} = \frac{2}{3}$
18. $\frac{1}{2}x + \frac{1}{3} = \frac{2}{5}$
19. $4a - \frac{2}{3} = -\frac{1}{6}$
20. $\frac{3}{5}x - \frac{1}{2} = \frac{1}{10}$
21. $-\frac{4}{5}y + \frac{1}{3} = \frac{1}{15}$
22. $-\frac{9}{16}x + \frac{4}{3} = \frac{4}{3}$
23. $-x + 5 = 14$
24. $-y - 7 = -12$
25. $75 - a = 200$
26. $15 = 5 - x$
27. $-8 = 4 - 2x$
28. $33 - x = 33$
29. $18 = 6 - y$
30. $-12 = -2x + 3$
31. $-3 = 3.36 - 1.2a$
32. $0 = -3.1a + 32.55$
33. $\frac{1}{4} = -\frac{3}{8} + 10x$
34. $70 = 50 - \frac{1}{2}y$

Answer available at the end of this section.

Exercise 2.3.8 Solving Linear Equations

Translate the following sentences into linear equations and then solve.

1. The sum of $2x$ and 5 is equal to 15.
2. The sum of $-3x$ and 7 is equal to 14.
3. The difference of $5x$ and 6 is equal to 4.
4. Twelve times x is equal to 36.
5. A number n divided by 8 is 5.
6. Six subtracted from two times a number x is 12.
7. Four added to three times a number n is 25.
8. Three-fourths of a number x is 9.
9. Negative two-thirds times a number x is equal to 20.
10. One-half of a number x plus 3 is equal to 10.

Answer available at the end of this section.

Exercise 2.3.9 Solving Linear Equations

Find a linear equation of the form $ax + b = 0$ with the given solution, where a and b are integers. (Answers may vary.)

1. $x = 2$
2. $x = -3$
3. $x = -\frac{1}{2}$
4. $x = \frac{2}{3}$

Answer available at the end of this section.

Exercise 2.3.10 Discussion Board Topics

1. How many steps are needed to solve any equation of the form $ax + b = c$? Explain.
2. Instead of dividing by 6 when $6x = 12$, could you multiply by the reciprocal of 6? Does this always work?

Answer available at the end of this section.

Answers

Exercise 2.3.1

No

Exercise 2.3.2

$x = 10$

Exercise 2.3.3

$x = -3$

Exercise 2.3.4

$x = \frac{15}{2}$

Exercise 2.3.5 **Solutions to Linear Equations**

1. Yes
3. No
5. Yes
7. Yes
9. Yes
11. Yes
13. Yes

Exercise 2.3.6 **Solving Basic Linear Equations**

1. 10
3. 18
5. $\frac{5}{6}$
7. $\frac{5}{6}$
9. -11
11. -13
13. 0
15. $\frac{3}{2}$
17. -7
19. $-\frac{3}{2}$
21. $-\frac{3}{14}$
23. $-\frac{3}{2}$

Exercise 2.3.7 **Solving Linear Equations**

1. 5
3. 10
5. $\frac{1}{3}$

7. $\frac{1}{5}$
9. 2
11. 14.2
13. 7
15. $-\frac{5}{2}$
17. $\frac{7}{5}$
19. $\frac{1}{8}$
21. $\frac{1}{3}$
23. -9
25. -125
27. 6
29. -12
31. 5.3
33. $\frac{1}{16}$

Exercise $2.3.8$ **Solving Linear Equations**

1. $2x + 5 = 15; x = 5$
3. $5x - 6 = 4; x = 2$
5. $\frac{n}{8} = 5; n = 40$
7. $3n + 4 = 25; n = 7$
9. $-\frac{2}{3}x = 20; x = -30$

Exercise $2.3.9$ **Solving Linear Equations**

1. $x - 2 = 0$
3. $2x + 1 = 0$

Exercise $2.3.10$ **Discussion Board Topics**

1. Answers may vary

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2.4: Solving Linear Equations- Part II

Learning Objectives

- Solve general linear equations.
- Identify and solve conditional equations, identities, and contradictions.
- Clear decimals and fractions from equations.
- Solve literal equations or formulas for a given variable.

Combining Like Terms and Simplifying

Linear equations typically are not given in standard form, so solving them requires additional steps. These additional steps include simplifying expressions on each side of the equal sign using the order of operations.

Same-Side Like Terms

We will often encounter linear equations where the expressions on each side of the equal sign can be simplified. Typically, this involves combining same-side like terms. If this is the case, then it is best to simplify each side first before solving.

Example 2.4.1

Solve:

$$-4a + 2 - a = 3 - 2.$$

Solution:

First, combine the like terms on each side of the equal sign.

Answer:

The solution is $\frac{1}{5}$.

Opposite-Side Like Terms

Given a linear equation in the form $ax + b = cx + d$, we begin by combining like terms on opposite sides of the equal sign. To combine opposite-side like terms, use the addition or subtraction property of equality to effectively "move terms" from one side to the other so that they can be combined.

Example 2.4.2

Solve:

$$-2y - 3 = 5y + 11.$$

Solution:

To "move" the term $5y$ to the left side, subtract it on both sides.

$$\begin{aligned} -2y-3 &= 5y+11 \\ -2y-3\color{#1a9cd8}{-5y} &= 5y+11\color{#1a9cd8}{-5y} \qquad \textit{Subtract } 5y \textit{ from both sides.} \\ -7y-3 &= 11 \end{aligned}$$

From here, solve using the techniques developed previously.

Always check to see that the solution is correct by substituting the solution back into the original equation and simplifying to see if you obtain a true statement.

$$\begin{aligned} -2y-3 &= 5y+11 \\ -2(\color{#1a9cd8}{-2}\color{black})-3 &= 5(\color{#1a9cd8}{-2}\color{black})+1 \\ 4-3 &= -10+11 \\ 1 &= 1 \quad \checkmark \end{aligned}$$

Answer:

The solution is -2.

General Guidelines for Solving Linear Equations

When solving linear equations, the goal is to determine what value, if any, will produce a true statement when substituted in the original equation. Do this by isolating the variable using the following steps:

Step 1: Simplify both sides of the equation using the order of operations and combine all same-side like terms.

Step 2: Use the appropriate properties of equality to combine opposite-side like terms with the variable term on one side of the equation and the constant term on the other.

Step 3: Divide or multiply as needed to isolate the variable.

Step 4: Check to see if the answer solves the original equation.

Example 2.4.3

Solve:

$$-\tfrac{1}{2}(10y-2)+3=14$$

Solution:

Simplify the linear expression on the left side before solving.

To check,

$$\begin{aligned} -\frac{1}{2}(10(\color{#1a9cd8}{-2}\color{black})-2)+3 &= 14 \\ -\frac{1}{2}(-20-2)+3 &= 14 \\ -\frac{1}{2}(-22)+3 &= 14 \\ 11+3 &= 14 \\ 14 &= 14 \quad \checkmark \end{aligned}$$

Answer:

The solution is -2.

Example 2.4.4

Solve:

$$5(3x+2)-2=-2(1-7x).$$

Solution:

First, simplify the expressions on both sides of the equal sign.

Answer:

The solution is -10. The check is left as an exercise.

Exercise 2.4.1

Solve:

$$6-3(4x-1)=4x-7.$$

Answer available at the end of this section.

Conditional Equations, Identities, and Contradictions

There are three different types of equations. Up to this point, we have been solving conditional equations. These are equations that are true for particular values. An identity is an equation that is true for all possible values of the variable. For example,

$$x=x \quad \textit{Identity}$$

has a solution set consisting of all real numbers, R. A contradiction is an equation that is never true and thus has no solutions. For example,

$$x+1=x \quad \textit{Contradiction}$$

has no solution. We use the empty set, $\emptyset$, to indicate that there are no solutions.

If the end result of solving an equation is a true statement, like $0=0$, then the equation is an identity and any real number is a solution. If solving results in a false statement, like $0=1$, then the equation is a contradiction and there is no solution.

Example 2.4.5

Solve:

$$4(x+5)+6=2(2x+3).$$

Solution:

$$\begin{aligned} 4(x+5)+6 &= 2(2x+3) && \textit{Distribute.}\\ 4x+20+6 &= 4x+6 && \textit{Combine same - side like terms.}\\ 4x+26 &= 4x+6 && \textit{Combine opposite - side like terms.}\\ 4x+26-4x &= 4x+6-4x\\ 26 &= 6 \quad x && \textit{False} \end{aligned}$$

Answer:

$\emptyset$. Solving leads to a false statement; therefore, the equation is a contradiction and there is no solution.

Example 2.4.6

Solve:

$$3(3y+5)+5=10(y+2)-y.$$

Solution:

$$\begin{aligned} 3(3y+5)+5 &= 10(y+2)-y && \textit{Distribute.}\\ 9y+15+5 &= 10y+20-y && \textit{Combine same - side like terms.}\\ 9y+20 &= 9y+20 && \textit{Combine opposite - side like terms.}\\ 9y+20-9y &= 9y+20-9y\\ 20 &= 20 \quad \checkmark && \textit{True} \end{aligned}$$

Answer:

R. Solving leads to a true statement; therefore, the equation is an identity and any real number is a solution.

If it is hard to believe that any real number is a solution to the equation in the previous example, then choose your favorite real number, and substitute it in the equation to see that it leads to a true statement. Choose $x=7$ and check:

Solve:

$$-2(3x+1)-(x-3)=-7x+1.$$

Answer

R

Clearing Decimals and Fractions

The coefficients of linear equations may be any real number, even decimals and fractions. When decimals and fractions are used, it is possible to use the multiplication property of equality to clear the coefficients in a single step. If given decimal coefficients, then multiply by an appropriate power of 10 to clear the decimals. If given fractional coefficients, then multiply both sides of the equation by the least common multiple of the denominators (LCD).

Example 2.4.7

Solve:

$$2.3x+2.8=-1.2x+9.8.$$

Solution:

Notice that all decimal coefficients are expressed with digits in the tenths place; this suggests that we can clear the decimals by multiplying both sides by 10. Take care to distribute 10 to each term on both sides of the equation.

$$\begin{aligned} 10\cdot(2.3x+2.8) &= 10\cdot -1.2x+9.8 && \textit{Multiply both sides by } 10.\\ 10\cdot 2.3x+10\cdot 2.8 &= 10\cdot(-1.2x)+10\cdot 9.8 && \\ 23x+28 &= -12x+98 && \textit{Integer coefficients}\\ 23x+28+12x &= -12x+98+12x && \textit{Solve.}\\ 35x+28 &= 98 && \\ 35x+28-28 &= 98-28 && \\ 35x &= 70 && \\ \frac{35x}{35} &= \frac{70}{35} && \\ x &= 2 && \end{aligned}$$

Answer:

The solution is 2.

Example 2.4.8

Solve:

$$\tfrac{1}{3}x+\tfrac{1}{5}=\tfrac{1}{5}x-1.$$

Solution:

Clear the fractions by multiplying both sides by the least common multiple of the given denominators. In this case, the LCM $(3,5)=15$.

Answer:

The solution is -9.

It is important to know that these techniques only work for equations. ***Do not try to clear fractions when simplifying expressions***. As a reminder

Expression	*Equation*
$\frac{1}{2}x+\frac{5}{3}$	$\frac{1}{2}x+\frac{5}{3}=0$

Solve equations and simplify expressions. If you multiply an expression by 6, you will change the problem. However, if you multiply both sides of an equation by 6, you obtain an equivalent equation.

Incorrect	*Correct*
$\frac{1}{2}x+\frac{5}{3}$	$\frac{1}{2}x+\frac{5}{3}=0$
$\neq 6\cdot\left(\frac{1}{2}x+\frac{5}{3}\right)$	$6\cdot\left(\frac{1}{2}x+\frac{5}{3}\right)=6\cdot 0$
$=3x+10$ $\quad x$	$3x+10=10$ $\quad \checkmark$

Literal Equations (Linear Formulas)

Algebra lets us solve whole classes of applications using literal equations, or formulas. Formulas often have more than one variable and describe, or model, a particular real-world problem. For example, the familiar formula $D=rt$ describes the distance traveled in terms of the average rate and time; given any two of these quantities, we can determine the third. Using algebra, we can solve the equation for any one of the variables and derive two more formulas.

$$\begin{aligned} D &= rt \\ \frac{D}{r} &= \frac{rt}{r} && \textit{Divide both sides by } r.\\ \frac{D}{r} &= t \end{aligned}$$

If we divide both sides by r, we obtain the formula $t = Dr$. Use this formula to find the time, given the distance and the rate

$$\begin{aligned} D &= rt \\ \frac{D}{t} &= \frac{rt}{t} \qquad \textit{Divide both sides by } t. \\ \frac{D}{t} &= r \end{aligned}$$

If we divide both sides by t, we obtain the formula $r = Dt$. Use this formula to find the rate, given the distance traveled and the time it takes to travel that distance. Using the techniques learned up to this point, we now have three equivalent formulas relating distance, average rate, and time:

$$D = rt \qquad t = \tfrac{D}{r} \qquad r = \tfrac{D}{t}$$

When given a literal equation, it is often necessary to solve for one of the variables in terms of the others. Use the properties of equality to isolate the indicated variable.

Example 2.4.9

Solve for a:

$$P = 2a + b.$$

Solution:

The goal is to isolate the variable a.

$$\begin{aligned} P &= 2a + b \\ P - b &= 2a + b - b \qquad \textit{Subtract } b \textit{ from both sides.} \\ P - b &= 2a \\ \frac{P-b}{2} &= \frac{2a}{2} \qquad \textit{Divide both sides by } 2. \\ \frac{P-b}{2} &= a \end{aligned}$$

Answer:

$$a = \tfrac{P-b}{2}$$

Example 2.4.10

Solve for y:

$$z = \tfrac{x+y}{2}.$$

Solution:

The goal is to isolate the variable y.

$$\begin{aligned} z &= \frac{x+y}{2} \\ 2 \cdot z &= 2 \cdot \frac{x+y}{2} \qquad \textit{Multiply both sides by } 2. \\ 2z &= x + y \\ 2z - x &= x + y - x \qquad \textit{Subtract } x \textit{ from both sides.} \\ 2z - x &= y \end{aligned}$$

Answer:

$$y = 2z - x$$

Exercise 2.4.3

Solve for b:

$$2a - 3b = c.$$

Answer available at the end of this section.

Key Takeaways

- Solving general linear equations involves isolating the variable, with coefficient 1, on one side of the equal sign.
- The steps for solving linear equations are:
  - Simplify both sides of the equation and combine all same-side like terms.
  - Combine opposite-side like terms to obtain the variable term on one side of the equal sign and the constant term on the other.
  - Divide or multiply as needed to isolate the variable.
  - Check the answer.
- Most linear equations that you will encounter are conditional and have one solution.
- If solving a linear equation leads to a true statement like $0 = 0$, then the equation is an identity and the solution set consists of all real numbers, R.
- If solving a linear equation leads to a false statement like $0 = 5$, then the equation is a contradiction and there is no solution, $\emptyset$.
- Clear fractions by multiplying both sides of a linear equation by the least common multiple of all the denominators. Distribute and multiply all terms by the LCD to obtain an equivalent equation with integer coefficients.
- Given a formula, solve for any variable using the same techniques for solving linear equations. This works because variables are simply representations of real numbers.

Exercise 2.4.4 Checking for Solutions

Is the given value a solution to the linear equation?

1. $2(3x + 5) - 6 = 3x - 8; x = -4$
2. $-x + 17 - 8x = 9 - x; x = -1$
3. $4(3x - 7) - 3(x + 2) = -1; x = \frac{1}{3}$
4. $-5 - 2(x - 5) = -(x + 3); x = -8$
5. $7 - 2(\frac{1}{2}x - 6) = x - 1; x = 10$
6. $3x - \frac{2}{3}(9x - 2) = 0; x = \frac{4}{9}$

Answer available at the end of this section.

Exercise 2.4.5 Solving Linear Equations

Solve.

1. $4x - 7 = 7x + 5$
2. $-5x + 3 = -8x - 9$
3. $3x - 5 = 2x - 17$
4. $-2y - 52 = 3y + 13$
5. $-4x + 2 = 7x - 20$
6. $4x - 3 = 6x - 15$
7. $9x - 25 = 12x - 25$

8. $12y + 15 = -6y + 23$
9. $1.2x - 0.7 = 3x + 4.7$
10. $2.1x + 6.1 = -1.3x + 4.4$
11. $2.02x + 4.8 = 14.782 - 1.2x$
12. $-3.6x + 5.5 + 8.2x = 6.5 + 4.6x$
13. $\frac{1}{2}x - \frac{2}{3} = x + \frac{1}{5}$
14. $\frac{1}{3}x - \frac{1}{2} = -\frac{1}{4}x - \frac{1}{3}$
15. $-\frac{1}{10}y + \frac{2}{5} = \frac{1}{5}y + \frac{3}{10}$
16. $x - \frac{20}{3} = \frac{5}{2}x + \frac{5}{6}$
17. $\frac{2}{3}y + \frac{1}{2} = \frac{5}{8}y + \frac{37}{24}$
18. $\frac{1}{3} + \frac{4}{3}x = \frac{10}{7}x + \frac{1}{3} - \frac{2}{21}x$
19. $\frac{8}{9} - \frac{11}{18}x = \frac{7}{6} - 12x$
20. $\frac{1}{3} - 9x = \frac{4}{9} + \frac{1}{2}x$
21. $12x - 5 + 9x = 44$
22. $10 - 6x - 13 = 12$
23. $-2 + 4x + 9 = 7x + 8 - 2x$
24. $20x - 5 + 12x = 6 - x + 7$
25. $3a + 5 - a = 2a + 7$
26. $-7b + 3 = 2 - 5b + 1 - 2b$
27. $7x - 2 + 3x = 4 + 2x - 2$
28. $-3x + 8 - 4x + 2 = 10$
29. $6x + 2 - 3x = -2x - 13$
30. $3x - 0.75 + 0.21x = 1.24x + 7.13$
31. $-x - 2 + 4x = 5 + 3x - 7$
32. $-2y - 5 = 8y - 6 - 10y$
33. $\frac{1}{10}x - \frac{1}{3} = \frac{1}{30} - \frac{1}{15}x - \frac{7}{15}$
34. $\frac{5}{8} - \frac{4}{3}x + \frac{1}{3} = -\frac{3}{9}x - \frac{1}{4} + \frac{1}{3}x$

Answer available at the end of this section.

Exercise 2.4.6 Solving Linear Equations Involving Parentheses

Solve.

1. $-5(2y - 3) + 2 = 12$
2. $3(5x + 4) + 5x = -8$
3. $4 - 2(x - 5) = -2$
4. $10 - 5(3x + 1) = 5(x - 4)$
5. $9 - (x + 7) = 2(x - 1)$
6. $-5(2x - 1) + 3 = -12$
7. $3x - 2(x + 1) = x + 5$
8. $5x - 3(2x - 1) = 2(x - 3)$
9. $-6(x - 1) - 3x = 3(x + 8)$
10. $-\frac{3}{5}(5x + 10) = \frac{1}{2}(4x - 12)$
11. $3.1(2x - 3) + 0.5 = 22.2$
12. $4.22 - 3.13(x - 1) = 5.2(2x + 1) - 11.38$
13. $6(x - 2) - (7x - 12) = 14$
14. $-9(x - 3) - 3x = -3(4x + 9)$
15. $3 - 2(x + 4) = -3(4x - 5)$
16. $12 - 2(2x + 1) = 4(x - 1)$
17. $3(x + 5) - 2(2x + 3) = 7x + 9$
18. $3(2x - 1) - 4(3x - 2) = -5x + 10$
19. $-3(2a - 3) + 2 = 3(a + 7)$
20. $-2(5x - 3) - 1 = 5(-2x + 1)$
21. $\frac{1}{2}(2x + 1) - \frac{1}{4}(8x + 2) = 3(x - 4)$
22. $-\frac{2}{3}(6x - 3) - \frac{1}{2} = \frac{3}{2}(4x + 1)$

23. $\frac{1}{2}(3x-1)+\frac{1}{3}(2x-5)=0$
24. $\frac{1}{3}(x-2)+\frac{1}{5}=\frac{1}{9}(3x+3)$
25. $-2(2x-7)-(x+3)=6(x-1)$
26. $10(3x+5)-5(4x+2)=2(5x+20)$
27. $2(x-3)-6(2x+1)=-5(2x-4)$
28. $5(x-2)-(4x-1)=-2(3-x)$
29. $6(3x-2)-(12x-1)+4=0$
30. $-3(4x-2)-(9x+3)-6x=0$

Answer available at the end of this section.

Exercise 2.4.7 Literal Equations

Solve for the indicated variable.

1. Solve for w: $A=l\cdot w$.
2. Solve for a: $F=ma$.
3. Solve for w: $P=2l+2w$.
4. Solve for r: $C=2\pi r$.
5. Solve for b: $P=a+b+c$.
6. Solve for C: $F=\frac{9}{5}C+32$.
7. Solve for h: $A=\frac{1}{2}bh$.
8. Solve for t: $I=Prt$.
9. Solve for y: $ax+by=c$.
10. Solve for h: $S=2\pi r^2+2\pi rh$.
11. Solve for x: $z=\frac{2x+y}{5}$.
12. Solve for c: $a=3b-\frac{2c}{3}$.
13. Solve for b: $y=mx+b$.
14. Solve for m: $y=mx+b$.
15. Solve for y: $3x-2y=6$.
16. Solve for y: $-5x+2y=12$.
17. Solve for y: $\frac{x}{3}-\frac{y}{5}=1$.
18. Solve for y: $\frac{3}{4}x-\frac{1}{5}y=\frac{1}{2}$.

Answer available at the end of this section.

Exercise 2.4.8 Literal Equations

Translate the following sentences into linear equations and then solve.

1. The sum of $3x$ and 5 is equal to the sum of $2x$ and 7.
2. The sum of $-5x$ and 6 is equal to the difference of $4x$ and 2.
3. The difference of $5x$ and 25 is equal to the difference of $3x$ and 51.
4. The sum of $\frac{1}{2}x$ and $\frac{3}{4}$ is equal to $\frac{2}{3}x$.
5. A number n divided by 5 is equal to the sum of twice the number and 3.
6. Negative ten times a number n is equal to the sum of three times the number and 13.

Answer available at the end of this section.

Exercise 2.4.9 Discussion Board Topics

1. What is the origin of the word algebra?
2. What is regarded as the main business of algebra?

3. Why is solving equations such an important algebra topic?
4. Post some real-world linear formulas not presented in this section.
5. Research and discuss the contributions of Diophantus of Alexandria.
6. Create an identity or contradiction of your own and share on the discussion board. Provide a solution and explain how you found it.

Answer available at the end of this section.

Answers

Exercise 2.4.1

$x = 1$

Exercise 2.4.3

$b = \frac{2a-c}{3}$

Exercise 2.4.4 **Checking for Solutions**

1. Yes
3. No
5. Yes

Exercise 2.4.5 **Solving Linear Equations**

1. -4
3. -12
5. 2
7. 0
9. -3
11. 3.1
13. $-\frac{26}{15}$
15. $\frac{1}{3}$
17. 25
19. $-\frac{5}{2}$
21. $\frac{7}{3}$
23. -1
25. $\emptyset$
27. $\frac{1}{2}$
29. -3
31. R
33. $-\frac{3}{5}$

Exercise 2.4.6 **Solving Linear Equations Involving Parentheses**

1. $\frac{1}{2}$
3. 8
5. $\frac{4}{3}$
7. $\emptyset$
9. $-\frac{3}{2}$
11. 5
13. -14
15. 2
17. 0
19. $-\frac{10}{9}$
21. 3
23. 1
25. $\frac{17}{11}$
27. $\emptyset$
29. $\frac{7}{6}$

Exercise 2.4.7 **Literal Equations**

1. $w = \frac{A}{l}$
3. $w = \frac{P-2l}{2}$
5. $b = P - a - c$
7. $h = \frac{2A}{b}$
9. $y = \frac{-ax+c}{b}$
11. $x = \frac{5z-y}{2}$

13. $b = y - mx$
15. $y = \frac{3x-6}{2}$
17. $y = \frac{5x-15}{3}$

Exercise $2.4.8$ **Literal Equations**

1. $3x + 5 = 2x + 7$; $x = 2$
3. $5x - 25 = 3x - 51$; $x = -13$
5. $\frac{n}{5} = 2n + 3$; $n = -\frac{5}{3}$

Exercise $2.4.9$ **Discussion Board Topics**

1. Answers may vary
3. Answers may vary
5. Answers may vary

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2.5: Applications of Linear Equations

Learning Objectives

- Identify key words and phrases, translate sentences to mathematical equations, and develop strategies to solve problems.
- Solve word problems involving relationships between numbers.
- Solve geometry problems involving perimeter.
- Solve percent and money problems including simple interest.
- Set up and solve uniform motion problems.

Key Words, Translation, and Strategy

Algebra simplifies the process of solving real-world problems. This is done by using letters to represent unknowns, restating problems in the form of equations, and offering systematic techniques for solving those equations. To solve problems using algebra, first translate the wording of the problem into mathematical statements that describe the relationships between the given information and the unknowns. Usually, this translation to mathematical statements is the difficult step in the process. The key to the translation is to carefully read the problem and identify certain key words and phrases.

Table 2.5.1

Key Words	Translation
Sum, increased by, more than, plus, added to, total	$+$
Difference, decreased by, subtracted from, less, minus	$-$
Product, multiplied by, of, times, twice	$*$
Quotient, divided by, ratio, per	$/$
Is, total, result	$=$

Here are some examples of translated key phrases.

Table 2.5.2

Key Phrases	Translation
The **sum** of a number and 7.	$x+7$
Seven **more than** a number.	
The **difference of** a number and 7.	$x-7$
Seven **less than** a number.	
Seven **subtracted from** a number.	
The **product of** 2 and a number.	$2x$
Twice a number.	
One-half **of a** number.	$\frac{1}{2}x$
The **quotient** of a number and 7.	$\frac{x}{7}$

When translating sentences into mathematical statements, be sure to read the sentence several times and identify the key words and phrases.

Example 2.5.1

Translate:

Four less than twice some number is 16.

Solution:

First, choose a variable for the unknown number and identify the key words and phrases. Let x represent the unknown indicated by "some number."

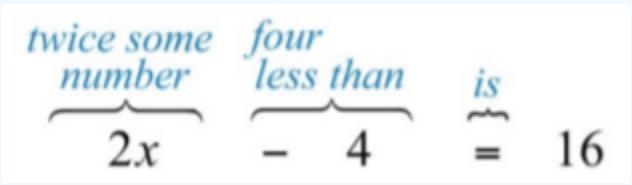


Figure 2.5.1

Remember that subtraction is not commutative. For this reason, take care when setting up differences. In this example, $4 - 2x = 16$ is an incorrect translation.

Answer:

$$2x - 4 = 16$$

It is important to first identify the variable—***let x represent...***—and state in words what the unknown quantity is. This step not only makes your work more readable but also forces you to think about what you are looking for. Usually, if you know what you are asked to find, then the task of finding it is achievable.

Example 2.5.2

Translate:

When 7 is subtracted from 3 times the sum of a number and 12, the result is 20.

Solution:

Let n represent the unknown number.

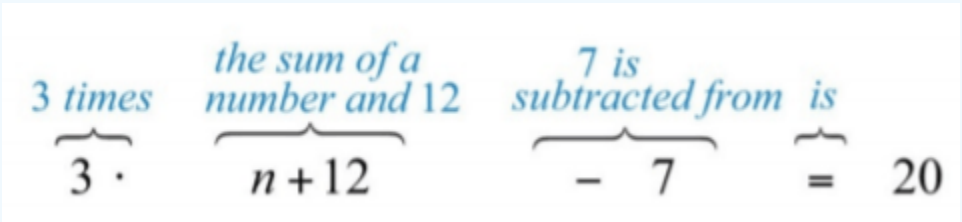


Figure 2.5.2

Answer:

$$3(n + 12) - 7 = 20$$

To understand why parentheses are needed, study the structures of the following two sentences and their translations:

Table 2.5.3

"**3 times the sum** of a number and 12"	$3(n+12)$
"the sum of **3 times a number** and 12"	$3n+12$

The key is to focus on the phrase "**3 times the sum**." This prompts us to group the sum within parentheses and then multiply by 3. Once an application is translated into an algebraic equation, solve it using the techniques you have learned.

Guidelines for Setting Up and Solving Word Problems

- **Step 1**: Read the problem several times, identify the key words and phrases, and organize the given information.
- **Step 2**: Identify the variables by assigning a letter or expression to the unknown quantities.
- **Step 3**: Translate and set up an algebraic equation that models the problem.
- **Step 4**: Solve the resulting algebraic equation.
- **Step 5**: Finally, answer the question in sentence form and make sure it makes sense (check it).

For now, set up all of your equations using only one variable. Avoid two variables by looking for a relationship between the unknowns.

Problems Involving Relationships between Real Numbers

We classify applications involving relationships between real numbers broadly as number problems. These problems can sometimes be solved using some creative arithmetic, guessing, and checking. Solving in this manner is not a good practice and should be avoided. Begin by working through the basic steps outlined in the general guidelines for solving word problems.

Example 2.5.3

A larger integer is 2 less than 3 times a smaller integer. The sum of the two integers is 18. Find the integers.

Solution:

Identify variables: Begin by assigning a variable to the smaller integer.

$$\text{Let } x \text{ represent the smaller integer.}$$

Use the first sentence to identify the larger integer in terms of the variable x: "A larger integer is **2 less** than **3 times** a smaller."

$$\text{Let } 3x-2 \text{ represent the larger integer.}$$

Set up an equation: Add the expressions that represent the two integers, and set the resulting expression equal to 18 as indicated in the second sentence: "The **sum** of the two integers **is** 18."

$$x+(3x-2)=18$$

Solve: Solve the equation to obtain the smaller integer x.

$$\begin{aligned} x+(3x-2) &= 18 \\ x+3x-2 &= 18 \\ 4x-2 &= 18 \\ 4x-2{\color{#1E90FF}+2} &= 18{\color{#1E90FF}+2} \\ 4x &= 20 \\ \frac{4x}{{\color{#1E90FF}4}} &= \frac{20}{{\color{#1E90FF}4}} \\ x &= 5 \end{aligned}$$

Back substitute: Use the expression $3x-2$ to find the larger integer—this is called back substituting.

$$3x-2=3(5)-2=15-2=13$$

Answer the question: The two integers are 5 and 13.

Check: $5+13=18$. The answer makes sense.

Example 2.5.4

The difference between two integers is 2. The larger integer is 6 less than twice the smaller. Find the integers.

Solution:

Use the relationship between the two integers in the second sentence, "The larger integer is **6 less** than **twice the smaller**," to identify the unknowns in terms of one variable.

Let x represent the smaller integer.

Let $2x-6$ represent the larger integer.

Since the difference is positive, subtract the smaller integer from the larger.

$$(2x-6)-x=2$$

Solve.

$$\begin{aligned} 2x-6-x &= 2 \\ x-6 &= 2 \\ x-6+6 &= 2+6 \\ x &= 8 \end{aligned}$$

Use $2x-6$ to find the larger integer.

$$2x-6=2(8)-6=16-6=10$$

Answer:

The two integers are 8 and 10. These integers clearly solve the problem.

It is worth mentioning again that you can often find solutions to simple problems by guessing and checking. This is so because the numbers are chosen to simplify the process of solving, so that the algebraic steps are not too tedious. You learn how to set up algebraic equations with easier problems, so that you can use these ideas to solve more difficult problems later.

Example 2.5.5

The sum of two consecutive even integers is 46. Find the integers.

Solution:

The key phrase to focus on is "consecutive even integers."

Let x represent the first even integer.

Let $x+2$ represent the next even integer.

Add the even integers and set them equal to 46.

$$x + (x + 2) = 46$$

Solve.

$$\begin{aligned} x + x + 2 &= 46 \\ 2x + 2 &= 46 \\ 2x + 2 - 2 &= 46 - 2 \\ 2x &= 44 \\ x &= 22 \end{aligned}$$

Use $x + 2$ to find the next even integer.

$$x + 2 = 22 + 2 = 24$$

Answer:

The consecutive even integers are 22 and 24.

It should be clear that consecutive even integers are separated by two units. However, it may not be so clear that odd integers are as well.

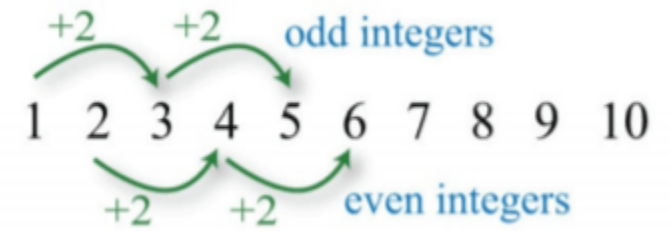


Figure $2.5.3$

Example $2.5.6$

The sum of two consecutive odd integers is 36. Find the integers.

Solution:

The key phrase to focus on is “consecutive odd integers.”

Let x represent the first odd integer.

Let $x + 2$ represent the next odd integer.

Add the two odd integers and set the expression equal to 36.

$$x + (x + 2) = 36$$

Solve.

$$\begin{aligned} x + x + 2 &= 36 \\ 2x + 2 &= 36 \\ 2x + 2 - 2 &= 36 - 2 \\ 2x &= 34 \\ \frac{2x}{2} &= \frac{34}{2} \\ x &= 17 \end{aligned}$$

Use $x+2$ to find the next odd integer.

$$x+2=17+2=19$$

Answer:

The consecutive odd integers are 17 and 19.

The algebraic setup for even and odd integer problems is the same. A common mistake is to use x and $x+3$ when identifying the variables for consecutive odd integers. This is incorrect because adding 3 to an odd number yields an even number: for example, $5+3=8$. An incorrect setup is very likely to lead to a decimal answer, which may be an indication that the problem was set up incorrectly.

Example 2.5.7

The sum of three consecutive integers is 24. Find the integers.

Solution:

Consecutive integers are separated by one unit.

Let x represent the first integer.

Let $x+1$ represent the next integer.

Let $x+2$ represent the third integer.

Add the integers and set the sum equal to 24.

$$x+(x+1)+(x+2)=24$$

Solve.

$$\begin{aligned} x+x+1+x+2 &= 24 \\ 3x+3 &= 24 \\ 3x+3-3 &= 24-3 \\ 3x &= 21 \\ x &= 7 \end{aligned}$$

Back substitute to find the other two integers.

$$x+1=7+1=8$$

$$x+2=7+2=9$$

Answer:

The three consecutive integers are $7, 8$ and 9, where $7+8+9=24$.

Exercise 2.5.1

The sum of three consecutive odd integers is 87. Find the integers.

Answer available at the end of this section.

Geometry Problems (Perimeter)

Recall that the perimeter of a polygon is the sum of the lengths of all the outside edges. In addition, it is helpful to review the following perimeter formulas $(\pi \approx 3.14159)$.

Table 2.5.4

Perimeter of a rectangle:	$P = 2l + 2w$
Perimeter of a square:	$P = 4s$
Perimeter of a triangle:	$P = a + b + c$
Perimeter of a circle (circumference):	$C = 2\pi r$

Keep in mind that you are looking for a relationship between the unknowns so that you can set up algebraic equations using only one variable. When working with geometry problems, it is often helpful to draw a picture.

Example 2.5.8

A rectangle has a perimeter measuring 64 feet. The length is 4 feet more than 3 times the width. Find the dimensions of the rectangle.

Solution:

The sentence "The length is **4 feet more** than **3 times the width**" gives the relationship between the two variables.

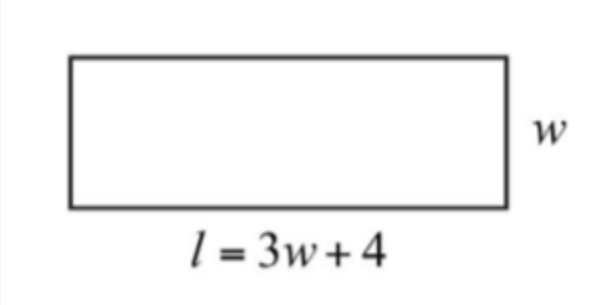


Figure 2.5.4

Let w represent the width of the rectangle.

Let $3w + 4$ represent the length.

The sentence "A rectangle has a **perimeter** measuring 64 feet" suggests an algebraic setup. Substitute 64 for the perimeter and the expression for the length into the appropriate formula as follows:

$$\begin{array}{ccc} P = & & 2l + 2w \\ \downarrow & & \downarrow \\ 64 = & 2(3w + 4) & + 2w \end{array}$$

Once you have set up an algebraic equation with one variable, solve for the width, w.

$$\begin{aligned} 64 &= 6w+8+2w \\ 64 &= 8w+8 \\ 64-8 &= 8w+8-8 \\ 56 &= 8w \\ \frac{56}{8} &= \frac{8w}{8} \\ 7 &= w \end{aligned}$$

Use $3w+4$ to find the length.

$$l = 3w+4 = 3(7)+4 = 21+4 = 25$$

Answer:

The rectangle measures 7 feet by 25 feet. To check, add all of the sides:

$$P = 7 \text{ ft}+7 \text{ ft}+25 \text{ ft}+25 \text{ ft} = 64 \text{ ft}$$

Example 2.5.9

Two sides of a triangle are 5 and 7 inches longer than the third side. If the perimeter measures 21 inches, find the length of each side.

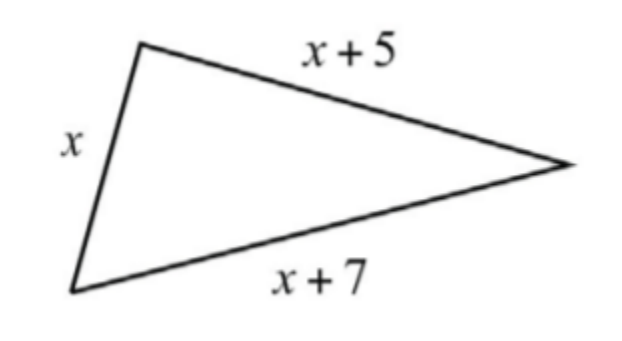


Figure 2.5.5

Solution:

The first sentence describes the relationships between the unknowns.

Let x represent the length of the third side.

Let $x+5$ and $x+7$ represent the lengths of the other two sides.

Substitute these expressions into the appropriate formula and use 21 for the perimeter P.

$$\begin{aligned} P &= a+b+c \\ 21 &= x+(x+5)+(x+7) \end{aligned}$$

You now have an equation with one variable to solve.

$$\begin{aligned} 21 &= x + x + 5 + x + 7 \\ 21 &= 3x + 12 \\ 21 - 12 &= 3x + 12 - 12 \\ 9 &= 3x \\ \frac{9}{3} &= \frac{3x}{3} \\ 3 &= x \end{aligned}$$

Back substitute.

$$x + 5 = 3 + 5 = 8$$

$$x + 5 = 3 + 7 = 10$$

Answer:

The three sides of the triangle measure 3 inches, 8 inches, and 10 inches. The check is left to the reader.

Exercise 2.5.2

The length of a rectangle is 1 foot less than twice its width. If the perimeter is 46 feet, find the dimensions.

Answer available at the end of this section.

Problems Involving Money and Percents

Whenever setting up an equation involving a percentage, we usually need to convert the percentage to a decimal or fraction. If the question asks for a percentage, then do not forget to convert your answer to a percent at the end. Also, when money is involved, be sure to round off to two decimal places.

Example 2.5.10

If a pair of shoes costs $\$52.50$ including a $7\frac{1}{4}\%$ tax, what is the original cost of the item before taxes are added?

Solution:

Begin by converting $7\frac{1}{4}\%$ to a decimal.

$$7\tfrac{1}{4}$$

The amount of tax is this rate times the original cost of the item. The original cost of the item is what you are asked to find.

Let c represent the cost of the item $\underline{\text{before taxes}}$ are added.

$$amount\ of\ tax\ =\ tax\ rate\ \cdot\ cost\ of\ item$$

$$= 0.0725 \cdot c$$

$$total\ cost\ =\ cost\ of\ item\ +\ amount\ of\ tax$$

$$52.50 = c + 0.0725c$$

Use this equation to solve for c, the original cost of the item.

$$\begin{aligned} 52.50 &= 1c + 0.0725c \\ 52.50 &= 1.0725c \\ \frac{52.50}{1.0725} &= \frac{1.0725c}{1.0725} \\ 48.95 &\approx c \end{aligned}$$

Answer:

The cost of the item before taxes is $\$48.95$. Check this by multiplying $\$48.95$ by 0.0725 to obtain the tax and add it to this cost.

Example 2.5.11

Given a $5\frac{1}{8}\%$ annual interest rate, how long will it take $\$1,200$ to yield $\$307.50$ in simple interest?

Solution:

Let t represent the time needed to earn $\$307.50$ at 5.125%.

Organize the data needed to use the simple interest formula $I = prt$.

Table 2.5.5

Given interest for the time period:	$I = \$307.50$
Given principal:	$p = \$1200$
Given rate:	$r = 5\frac{1}{8}\% = 5.125\% = 0.05125$

Next, substitute all of the known quantities into the formula and then solve for the only unknown, t.

$$\begin{aligned} I &= prt \\ 307.50 &= 1200(0.05125)t \\ 307.50 &= 61.5t \\ \frac{307.50}{61.5} &= \frac{61.5t}{61.5} \\ 5 &= t \end{aligned}$$

Answer:

It takes 5 years for $\$1,200$ invested at $5\frac{1}{8}\%$ to earn $\$307.50$ in simple interest.

Example 2.5.12

Mary invested her total savings of $\$3,400$ in two accounts. Her mutual fund account earned 8% last year and her CD earned 5%. If her total interest for the year was $\$245$, how much was in each account?

Solution:

The relationship between the two unknowns is that they total $\$3{,}400$. When a total is involved, a common technique used to avoid two variables is to represent the second unknown as the difference of the total and the first unknown.

Let x represent the amount invested in the mutual fund at $8\% = 0.08$.

Let $3,400 - x$ represent the remaining amount invested in the CD at $5\% = 0.05$.

The total interest is the sum of the interest earned from each account.

Table 2.5.6

Interest earned in the mutual fund:	$I = Prt = x \cdot 0.08 \cdot 1 = 0.08x$
Interest earned in the CD:	$I = Prt = (3,400 - x) \cdot 0.05 \cdot 1 = 0.05(3,400 - x)$

$$mutual\ fund\ interest\ +\ CD\ interest\ =\ total\ interest$$

$$0.08x + 0.05(3,400 - x) = 245$$

This equation models the problem with one variable. Solve for x.

$$\begin{aligned} 0.08x + 0.05(3,400 - x) &= 245 \\ 0.08x + 170 - 0.05x &= 245 \\ 0.03x + 170 - 170 &= 245 - 170 \\ 0.03x &= 75 \\ \frac{0.03x}{0.03} &= \frac{75}{0.03} \\ x &= 2,500 \end{aligned}$$

Back substitute.

$$3,400 - x = 3,400 - 2,500 = 900$$

Answer:

Mary invested $\$2,500$ at 8% in a mutual fund and $\$900$ at 5% in a CD.

Example 2.5.13

Joe has a handful of dimes and quarters that values $\$5.30$. He has one fewer than twice as many dimes than quarters. How many of each coin does he have?

Solution:

Begin by identifying the variables.

Let q represent the number of quarters Joe is holding.

Let $2q - 1$ represent the number of dimes.

To determine the total value of a number of coins, multiply the number of coins by the value of each coin. For example, 5 quarters have a value $\$0.25 \cdot 5 = \$1.25$.

$$value\ in\ quarters\ +\ value\ in\ dimes\ =\ total\ value\ of\ coins$$

$$0.25q + 0.10(2q - 1) = 5.30$$

Solve for the number of quarters, q.

$$\begin{aligned} 0.25q + 0.10(2q - 1) &= 5.30 \\ 0.25q + 0.20q - 0.10 &= 5.30 \\ 0.45q - 0.10 &= 5.30 \\ 0.45q - 0.10 + 0.10 &= 5.30 + 0.10 \\ 0.45q &= 5.40 \\ \frac{0.45q}{0.45} &= \frac{5.40}{0.45} \\ q &= 12 \end{aligned}$$

Back substitute into $2q-1$ to find the number of dimes.

$$2q-1=2(12)-1=24-1=23$$

Answer:

Joe has 12 quarters and 23 dimes. Check by multiplying $\$0.25 \cdot 12 = \$3.00$ and $\$0.10 \cdot 23 = \$2.30$. Then add to obtain the correct amount: $\$3.00 + \$2.30 = \$5.30$.

Exercise 2.5.3

A total amount of $\$5,900$ is invested in two accounts. One account earns 3.5% interest and another earns 4.5%. If the interest for 1 year is $\$229.50$, then how much is invested in each account?

Answer available at the end of this section.

Uniform Motion Problems (Distance Problems)

Uniform motion refers to movement at a speed, or rate that does not change. We can determine the distance traveled by multiplying the average rate by the time traveled at that rate with the formula $D=r \cdot t$. Applications involving uniform motion usually have a lot of data, so it helps to first organize the data in a chart and then set up an algebraic equation that models the problem.

Example 2.5.14

Two trains leave the station at the same time traveling in opposite directions. One travels at 70 miles per hour and the other at 60 miles per hour. How long does it take for the distance between them to reach 390 miles?

Solution:

First, identify the unknown quantity and organize the data.

Let t represent the time it takes to separate 390 miles.

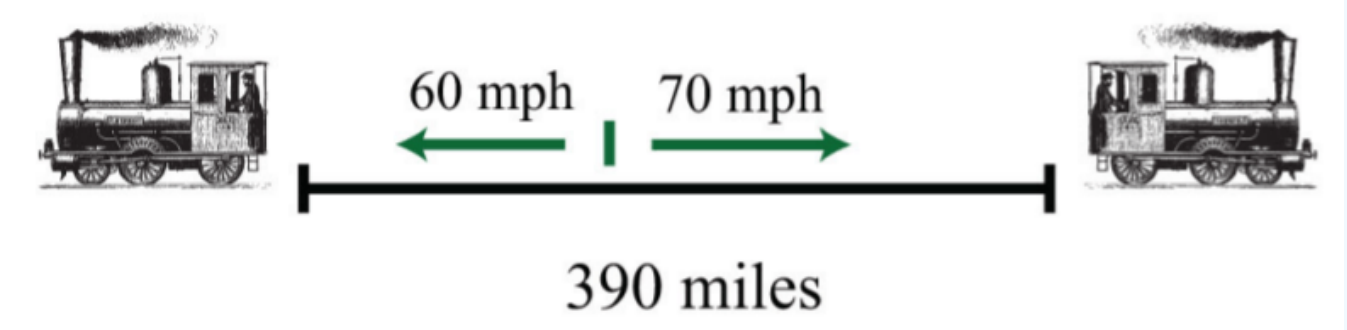


Figure $2.5.6$

The given information is filled in on the following chart. The time for each train is equal.

	Distance =	Rate ×	Time
Train 1		70 mph	t
Train 2		60 mph	t
Total	390 mi		

Figure $2.5.7$

To avoid introducing two more variables, use the formula $D = r \cdot t$ to fill in the unknown distances traveled by each train.

Distance traveled by train 1: $D = r \cdot t = 70 \cdot t$

Distance traveled by tain 2: $D = r \cdot t = 60 \cdot t$

We can now completely fill in the chart.

	Distance =	Rate ×	Time
Train 1	$70t$	70 mph	t
Train 2	$60t$	60 mph	t
Total	390 mi		

Figure $2.5.8$

The algebraic setup is defined by the distance column. The problem asks for the time it takes for the total distance to reach 390 miles.

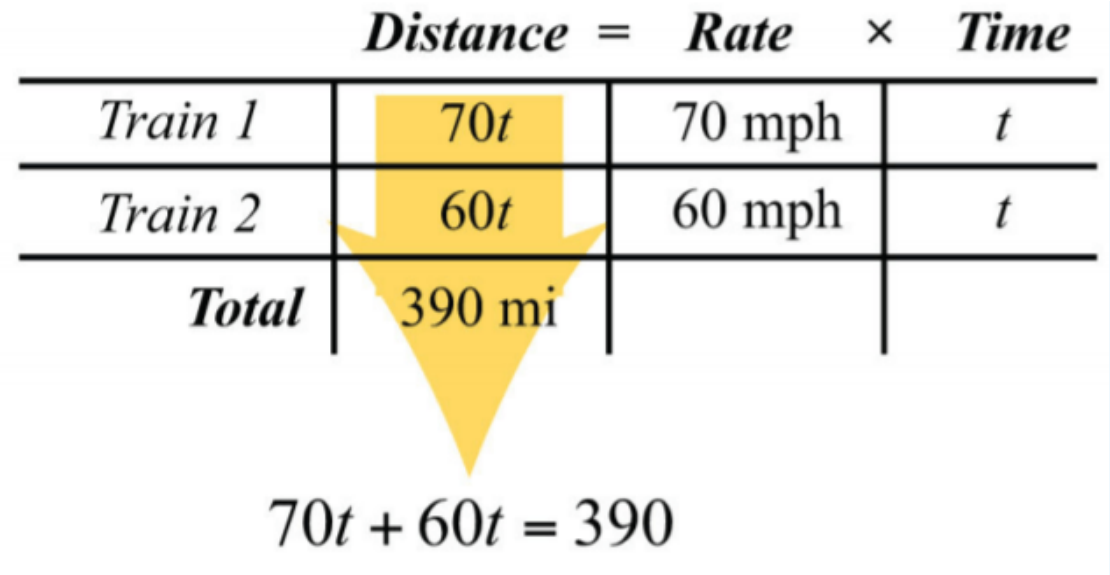


Figure $2.5.9$

Solve for t.

$$\begin{aligned} 70t + 60t &= 390 \\ 130t &= 390 \\ \frac{130t}{130} &= \frac{390}{130} \\ t &= 3 \end{aligned}$$

Answer:

It takes 3 hours for the distance between the trains to reach 390 miles.

Example $2.5.15$

A train traveling nonstop to its destination is able to make the trip at an average speed of 72 miles per hour. On the return trip, the train makes several stops and is only able to average 48 miles per hour. If the return trip takes 2 hours longer than the initial trip to the destination, then what is the travel time each way?

Solution:

First, identify the unknown quantity and organize the data.

Let t represent the time it takes to arrive at the destination.

Let $t + 2$ represent the time it takes for the return trip.

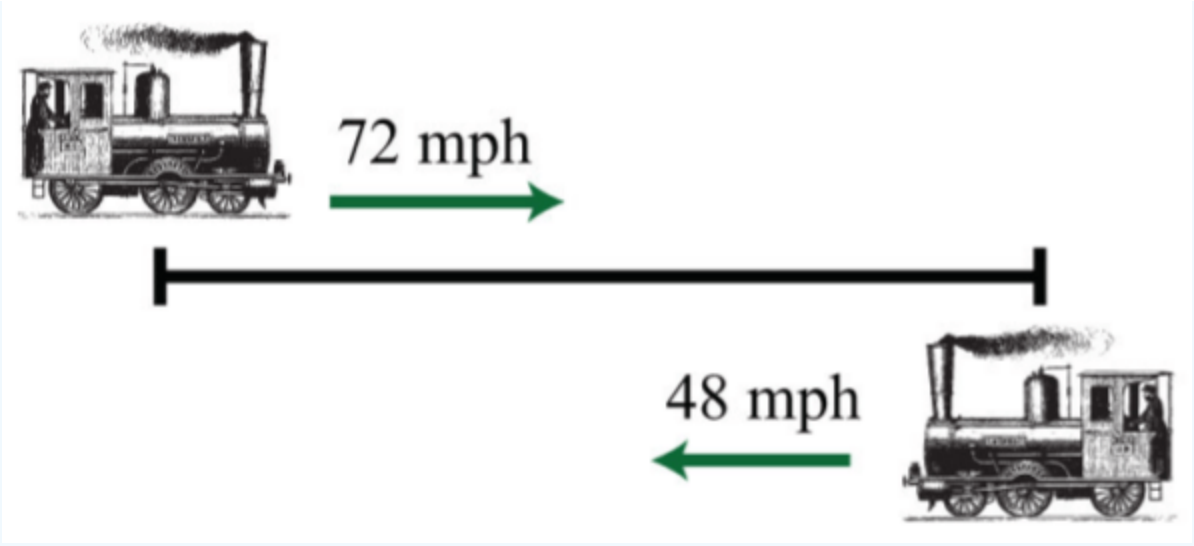


Figure 2.5.10

The given information is filled in the following chart:

	Distance =	***Rate*** ×	***Time***
To Destination		72 mph	t
Return Trip		48 mph	$t+2$
Total			

Figure 2.5.11

Use the formula $D = r \cdot t$ to fill in the unknown distances.

Distance traveled on the destination: $D = r \cdot t = 72 \cdot t$

Distance traveled on the return trip: $D = r \cdot t = 48 \cdot (t + 2)$

Use these expressions to complete the chart.

	Distance =	Rate ×	Time
To Destination	$72t$	72 mph	t
Return Trip	$48(t+2)$	48 mph	$t+2$
Total			

Figure 2.5.12

The algebraic setup is again defined by the distance column. In this case, the distance to the destination and back is the same, and the equation is

$$72t = 48(t+2)$$

Solve for t.

$$\begin{aligned} 72t &= 48(t+2) \\ 722 &= 48t + 96 \\ 72t - 48t &= 48t + 96 - 48t \\ 24t &= 96 \\ \frac{24t}{24} &= \frac{96}{24} \\ t &= 4 \end{aligned}$$

The return trip takes $t + 2 = 4 + 2 = 6$ hours.

Answer:

It takes 4 hours to arrive at the destination and 6 hours to return.

Exercise 2.5.4

Mary departs for school on a bicycle at an average rate of 6 miles per hour. Her sister Kate, running late, leaves 15 minutes later and cycles at twice that speed. How long will it take Kate to catch up to Mary? Be careful! Pay attention to the units given in the problem.

Answer available at the end of this section.

Key Takeaways

- Simplify the process of solving real-world problems by creating mathematical models that describe the relationship between unknowns. Use algebra to solve the resulting equations.
- Guessing and checking for solutions is a poor practice. This technique might sometimes produce correct answers, but is unreliable, especially when problems become more complex.
- Read the problem several times and search for the key words and phrases. Identify the unknowns and assign variables or expressions to the unknown quantities. Look for relationships that allow you to use only one variable. Set up a mathematical model for the situation and use algebra to solve the equation. Check to see if the solution makes sense and present the solution in sentence form.
- Do not avoid word problems: solving them can be fun and rewarding. With lots of practice you will find that they really are not

so bad after all. Modeling and solving applications is one of the major reasons to study algebra.
- Do not feel discouraged when the first attempt to solve a word problem does not work. This is part of the process. Try something different and learn from incorrect attempts.

Exercise 2.5.5 Translate

Translate the following into algebraic equations.

1. The sum of a number and 6 is 37.
2. When 12 is subtracted from twice some number the result is 6.
3. Fourteen less than 5 times a number is 1.
4. Twice some number is subtracted from 30 and the result is 50.
5. Five times the sum of 6 and some number is 20.
6. The sum of 5 times some number and 6 is 20.
7. When the sum of a number and 3 is subtracted from 10 the result is 5.
8. The sum of three times a number and five times that same number is 24.
9. Ten is subtracted from twice some number and the result is the sum of the number and 2.
10. Six less than some number is ten times the sum of that number and 5.

Answer available at the end of this section.

Exercise 2.5.6 Number Problems

Set up an algebraic equation and then solve.

1. A larger integer is 1 more than twice another integer. If the sum of the integers is 25, find the integers.
2. If a larger integer is 2 more than 4 times another integer and their difference is 32, find the integers.
3. One integer is 30 more than another integer. If the difference between the larger and twice the smaller is 8, find the integers.
4. The quotient of some number and 4 is 22. Find the number.
5. Eight times a number is decreased by three times the same number, giving a difference of 20. What is the number?
6. One integer is two units less than another. If their sum is -22, find the two integers.
7. The sum of two consecutive integers is 139. Find the integers.
8. The sum of three consecutive integers is 63. Find the integers.
9. The sum of three consecutive integers is 279. Find the integers.
10. The difference of twice the smaller of two consecutive integers and the larger is 39. Find the integers.
11. If the smaller of two consecutive integers is subtracted from two times the larger, then the result is 17. Find the integers.
12. The sum of two consecutive even integers is 46. Find the integers.
13. The sum of two consecutive even integers is 238. Find the integers.
14. The sum of three consecutive even integers is 96. Find the integers.
15. If the smaller of two consecutive even integers is subtracted from 3 times the larger the result is 42. Find the integers.
16. The sum of three consecutive even integers is 90. Find the integers.
17. The sum of two consecutive odd integers is 68. Find the integers.
18. The sum of two consecutive odd integers is 180. Find the integers.
19. The sum of three consecutive odd integers is 57. Find the integers.
20. If the smaller of two consecutive odd integers is subtracted from twice the larger the result is 23. Find the integers.
21. Twice the sum of two consecutive odd integers is 32. Find the integers.
22. The difference between twice the larger of two consecutive odd integers and the smaller is 59. Find the integers.

Answer available at the end of this section.

Exercise 2.5.7 Geometry Problems

Set up an algebraic equation and then solve.

1. If the perimeter of a square is 48 inches, then find the length of each side.
2. The length of a rectangle is 2 inches longer than its width. If the perimeter is 36 inches, find the length and width.
3. The length of a rectangle is 2 feet less than twice its width. If the perimeter is 26 feet, find the length and width.
4. The width of a rectangle is 2 centimeters less than one-half its length. If the perimeter is 56 centimeters, find the length and width.
5. The length of a rectangle is 3 feet less than twice its width. If the perimeter is 54 feet, find the dimensions of the rectangle.
6. If the length of a rectangle is twice as long as the width and its perimeter measures 72 inches, find the dimensions of the rectangle.
7. The perimeter of an equilateral triangle measures 63 centimeters. Find the length of each side.
8. An isosceles triangle whose base is one-half as long as the other two equal sides has a perimeter of 25 centimeters. Find the length of each side.
9. Each of the two equal legs of an isosceles triangle are twice the length of the base. If the perimeter is 105 centimeters, then how long is each leg?
10. A triangle has sides whose measures are consecutive even integers. If the perimeter is 42 inches, find the measure of each side.
11. A triangle has sides whose measures are consecutive odd integers. If the perimeter is 21 inches, find the measure of each side.
12. A triangle has sides whose measures are consecutive integers. If the perimeter is 102 inches, then find the measure of each side.
13. The circumference of a circle measures 50π units. Find the radius.
14. The circumference of a circle measures 10π units. Find the radius.
15. The circumference of a circle measures 100 centimeters. Determine the radius to the nearest tenth.
16. The circumference of a circle measures 20 centimeters. Find the diameter rounded off to the nearest hundredth.
17. The diameter of a circle measures 5 inches. Determine the circumference to the nearest tenth.
18. The diameter of a circle is 13 feet. Calculate the exact value of the circumference.

Answer available at the end of this section.

Exercise $2.5.8$ Percent and Money Problems

Set up an algebraic equation and then solve.

1. Calculate the simple interest earned on a 2-year investment of $\$1,550$ at a $8\frac{3}{4}$% annual interest rate.
2. Calculate the simple interest earned on a 1-year investment of $\$500$ at a 6% annual interest rate.
3. For how many years must $\$10,000$ be invested at an $8\frac{1}{2}$% annual interest rate to yield $\$4,250$ in simple interest?
4. For how many years must $\$1,000$ be invested at a 7.75% annual interest rate to yield $\$503.75$ in simple interest?
5. At what annual interest rate must $\$2,500$ be invested for 3 years in order to yield $\$412.50$ in simple interest?
6. At what annual interest rate must $\$500$ be invested for 2 years in order to yield $\$93.50$ in simple interest?
7. If the simple interest earned for 1 year was $\$47.25$ and the annual rate was 6.3%, what was the principal?
8. If the simple interest earned for 2 years was $\$369.60$ and the annual rate was $5\frac{1}{4}$%, what was the principal?
9. Joe invested last year's $\$2,500$ tax return in two different accounts. He put most of the money in a money market account earning 5% simple interest. He invested the rest in a CD earning 8% simple interest. How much did he put in each account if the total interest for the year was $\$138.50$?
10. James invested $\$1,600$ in two accounts. One account earns 4.25% simple interest and the other earns 8.5%. If the interest after 1 year was $\$85$, how much did he invest in each account?
11. Jane has her $\$5,400$ savings invested in two accounts. She has part of it in a CD at 3% annual interest and the rest in a savings account that earns 2% annual interest. If the simple interest earned from both accounts is $\$140$ for the year, then how much does she have in each account?
12. Marty put last year's bonus of $\$2,400$ into two accounts. He invested part in a CD with 2.5% annual interest and the rest in a money market fund with 1.3% annual interest. His total interest for the year was $\$42.00$. How much did he invest in each account?
13. Alice puts money into two accounts, one with 2% annual interest and another with 3% annual interest. She invests 3 times as much in the higher yielding account as she does in the lower yielding account. If her total interest for the year is $\$27.50$, how much did she invest in each account?

14. Jim invested an inheritance in two separate banks. One bank offered 5.5% annual interest rate and the other $6\frac{1}{4}$%. He invested twice as much in the higher yielding bank account than he did in the other. If his total simple interest for 1 year was $\$4,860$, then what was the amount of his inheritance?
15. If an item is advertised to cost $\$29.99$ plus 9.25% tax, what is the total cost?
16. If an item is advertised to cost $\$32.98$ plus $8\frac{3}{4}$% tax, what is the total cost?
17. An item, including an 8.75% tax, cost $\$46.49$. What is the original pretax cost of the item?
18. An item, including a 5.48% tax, cost $\$17.82$. What is the original pretax cost of the item?
19. If a meal costs $\$32.75$, what is the total after adding a 15% tip?
20. How much is a 15% tip on a restaurant bill that totals $\$33.33$?
21. Ray has a handful of dimes and nickels valuing $\$3.05$. He has 5 more dimes than he does nickels. How many of each coin does he have?
22. Jill has 3 fewer half-dollars than she has quarters. The value of all 27 of her coins adds to $\$9.75$. How many of each coin does Jill have?
23. Cathy has to deposit $\$410$ worth of five- and ten-dollar bills. She has 1 fewer than three times as many tens as she does five-dollar bills. How many of each bill does she have to deposit?
24. Billy has a pile of quarters, dimes, and nickels that values $\$3.75$. He has 3 more dimes than quarters and 5 more nickels than quarters. How many of each coin does Billy have?
25. Mary has a jar with one-dollar bills, half-dollar coins, and quarters valuing $\$14.00$. She has twice as many quarters than she does half-dollar coins and the same amount of half-dollar coins as one-dollar bills. How many of each does she have?
26. Chad has a bill-fold of one-, five-, and ten-dollar bills totaling $\$118$. He has 2 more than 3 times as many ones as he does five-dollar bills and 1 fewer ten- than five-dollar bills. How many of each bill does Chad have?

Answer available at the end of this section.

Exercise 2.5.9 Uniform Motion (Distance Problems)

Set up an algebraic equation then solve.

1. Two cars leave a location traveling in opposite directions. If one car averages 55 miles per hour and the other averages 65 miles per hour, then how long will it take for them to separate a distance of 300 miles?
2. Two planes leave the airport at the same time traveling in opposite directions. The average speeds for the planes are 450 miles per hour and 395 miles per hour. How long will it take the planes to be a distance of $1,478.75$ miles apart?
3. Bill and Ted are racing across the country. Bill leaves 1 hour earlier than Ted and travels at an average rate of 60 miles per hour. If Ted intends to catch up at a rate of 70 miles per hour, then how long will it take?
4. Two brothers leave from the same location, one in a car and the other on a bicycle, to meet up at their grandmother's house for dinner. If one brother averages 30 miles per hour in the car and the other averages 12 miles per hour on the bicycle, then it takes the brother on the bicycle 1 hour less than 3 times as long as the other in the car. How long does it take each of them to make the trip?
5. A commercial airline pilot flew at an average speed of 350 miles per hour before being informed that his destination airfield may be closed due to poor weather conditions. In an attempt to arrive before the storm, he increased his speed 400 miles per hour and flew for another 3 hours. If the total distance flown was $2,950$ miles, then how long did the trip take?
6. Two brothers drove the $2,793$ miles from Los Angeles to New York. One of the brothers, driving during the day, was able to average 70 miles per hour, and the other, driving at night, was able to average 53 miles per hour. If the brother driving at night drove 3 hours less than the brother driving in the day, then how many hours did they each drive?
7. Joe and Ellen live 21 miles apart. Departing at the same time, they cycle toward each other. If Joe averages 8 miles per hour and Ellen averages 6 miles per hour, how long will it take them to meet?
8. If it takes 6 minutes to drive to the automobile repair shop at an average speed of 30 miles per hour, then how long will it take to walk back at an average rate of 4 miles per hour?
9. Jaime and Alex leave the same location and travel in opposite directions. Traffic conditions enabled Alex to average 14 miles per hour faster than Jaime. After $1\frac{1}{2}$ hours they are 159 miles apart. Find the speed at which each was able to travel.
10. Jane and Holly live 51 miles apart and leave at the same time traveling toward each other to meet for lunch. Jane traveled on the freeway at twice the average speed as Holly. They were able to meet in a half hour. At what rate did each travel?

Answer available at the end of this section.

Exercise 2.5.10 Discussion Board Topics

1. Discuss ideas for calculating taxes and tips mentally.
2. Research historical methods for representing unknowns.
3. Research and compare simple interest and compound interest. What is the difference?
4. Discuss why algebra is a required subject.
5. Research ways to show that a repeating decimal is rational. Share your findings on the discussion board.

Answer available at the end of this section.

Answers

Exercise 2.5.1

The integers are $27, 29$, and 31.

Exercise 2.5.2

Width: 8 feet; length: 15 feet

Exercise 2.5.3

$\$3,600$ is invested at 3.5% and $\$2,300$ at 4.5%.

Exercise 2.5.4

It takes 15 minutes for Kate to catch up.

Exercise 2.5.5 **Translate**

1. $x + 6 = 37$
3. $5x - 14 = 1$
5. $5(x + 6) = 20$
7. $10 - (x + 3) = 5$
9. $2x - 10 = x + 2$

Exercise 2.5.6 **Number Problems**

1. $8, 17$
3. $22, 52$
5. 4
7. $69, 70$
9. $92, 93, 94$
11. $15, 16$
13. $118, 120$
15. $18, 20$
17. $33, 35$
19. $17, 19, 21$
21. $7, 9$

Exercise 2.5.7 **Geometry Problems**

1. 12 inches
3. Width: 5 feet; length: 8 feet
5. Width: 10 feet; length: 17 feet
7. 21 centimeters
9. 21 centimeters, 42 centimeters, 42 centimeters
11. 5 inches, 7 inches, 9 inches
13. 25 units
15. 15.9 centimeters
17. 15.7 inches

Exercise 2.5.8 **Percent and Money Problems**

1. $\$271.25$
3. 5 years
5. 5.5%
7. $\$750.00$
9. Joe invested $\$2,050$ in the money market account and $\$450$ in the CD.
11. Jane has $\$3,200$ in the CD and $\$2,200$ in savings.

13. Alice invested $\$250$ at 2% and $\$750$ at a 3%.
15. $\$32.76$
17. $\$42.75$
19. $\$37.66$
21. He has 17 nickels and 22 dimes.
23. Cathy has 12 fives and 35 ten-dollar bills.
25. Mary has 7 one-dollar bills, 7 half-dollar coins, and 14 quarters.

Exercise $2.5.9$ **Uniform Motion (Distance Problems)**

1. 2.5 hours
3. 6 hours
5. 8 hours
7. $1\frac{1}{2}$ hours
9. Jaime: 46 miles per hour; Alex: 60 miles per hour

Exercise $2.5.10$ **Discussion Board Topics**

1. Answers may vary
3. Answers may vary
5. Answers may vary

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2.6: Ratio and Proportion Applications

Learning Objectives

- Understand the difference between a ratio and a proportion.
- Solve proportions using cross multiplication.
- Solve applications involving proportions, including similar triangles.

Definitions

A ratio is a relationship between two numbers or quantities usually expressed as a quotient. Ratios are typically expressed using the following notation:

$$\frac{a}{b} \qquad a \text{ to } b \qquad a:b$$

All of the above are equivalent forms used to express a ratio. However, the most familiar way to express a ratio is in the form of a fraction. When writing ratios, it is important to pay attention to the units. If the units are the same, then the ratio can be written without them.

Example 2.6.1

Express the ratio 12 feet to 48 feet in reduced form.

Solution:

$$\begin{aligned} 12 \text{ feet to } 48 \text{ feet} &= \frac{12 \text{ feet}}{48 \text{ feet}} \\ &= \frac{12 \div 12}{48 \div 12} \qquad \textit{Reduce.} \\ &= \frac{1}{4} \end{aligned}$$

Answer:

$$1 \text{ to } 4$$

If the units are different, then we must be sure to include them because the ratio represents a rate.

Example 2.6.2

Express the ratio 220 miles to 4 hours in reduced form.

Solution:

$$\begin{aligned} 220 \text{ miles to } 4 \text{ hours} &= \frac{220 \text{ miles}}{4 \text{ hours}} \\ &= \frac{55 \text{miles}}{1 \text{ hour}} \\ &= 55 \text{ miles} \backslash \text{hour} \end{aligned}$$

Answer:

55 miles to 1 hours (or 55 miles per hour)

Rates are useful when determining unit cost, or the price of each unit. We use the unit cost to compare values when the quantities are not the same. To determine the unit cost, divide the cost by the number of units.

Example 2.6.3

A local supermarket offers a pack of 12 sodas for $\$3.48$ on sale, and the local discount warehouse offers the soda in a 36-can case for $\$11.52$. Which is the better value?

Solution:

Divide the cost by the number of cans to obtain the unit price.

Table 2.6.1

Supermarket	Discount warehouse
$\$3.4812$ cans=$\$0.29$/can	$\$11.5236$ cans=$\$0.32$/can

Answer:

The supermarket sale price of $\$3.48$ for a 12-pack is a better value at $\$0.29$ per can.

A proportion is a statement of equality of two ratios.

$$\frac{a}{b}=\frac{c}{d} \tag{2.6.1}$$

This proportion is often read "a is to b as c is to d." Here is an example of a simple proportion,

$$\tfrac{1}{2}=\tfrac{2}{4}$$

If we clear the fractions by multiplying both sides of the proportion by the product of the denominators, 8, then we obtain the following true statement:

$$\begin{aligned} 8\cdot\frac{1}{2}&=8\cdot\frac{2}{4}\\ 4\cdot 1&=2\cdot 2\\ 4&=4 \end{aligned}$$

Given any nonzero real numbers $a, b, c,$ and d that satisfy a proportion, multiply both sides by the product of the denominators to obtain the following:

$$\begin{aligned} \frac{a}{b}&=\frac{c}{d}\\ bd\cdot\frac{a}{b}&=bd\cdot\frac{c}{d}\\ ad&=bc \end{aligned}$$

This shows that cross products are equal, and is commonly referred to as cross multiplication.

$$\text{If } \frac{a}{b}=\frac{c}{d} \text{ then } ad=bc \tag{2.6.2}$$

Solving Proportions

Cross multiply to solve proportions where terms are unknown.

Example 2.6.4

Solve:

$$\frac{5}{8} = \frac{n}{4}.$$

Solution:

Cross multiply and then solve for n.



Figure 2.6.1

$$\begin{aligned} 5 \cdot 4 &= 8 \cdot n && \textit{Cross multiply.} \\ 20 &= 8n \\ \frac{20}{8} &= \frac{8n}{8} && \textit{Divide by } 8. \\ \frac{20 \div 4}{8 \div 4} &= n && \textit{Reduce.} \\ \frac{5}{2} &= n \end{aligned}$$

Answer:

$$n = \frac{5}{2}$$

Example 2.6.5

Solve:

$$\frac{15}{x} = \frac{5}{6}.$$

Solution:

Cross multiply then solve for x.

$$\begin{aligned} \frac{15}{x} &= \frac{5}{6} \\ 15 \cdot 6 &= x \cdot 5 \\ 15 \cdot 6 &= 5x \\ \frac{15 \cdot 6}{5} &= \frac{5x}{5} \\ 3 \cdot 6 &= 1x \\ 18 &= x \end{aligned}$$

Answer:

$$x = 18$$

Example 2.6.6

Solve:

$$\tfrac{n+3}{5} = \tfrac{7}{2}.$$

Solution:

When cross multiplying, be sure to group $n+3$. Apply the distributive property in the next step.

$$\begin{aligned} \frac{n+3}{5} &= \frac{7}{2} && \textit{Cross multiply.} \\ (n+3)\cdot 2 &= 5\cdot 7 && \textit{Distribute.} \\ 2n+6 &= 35 && \textit{Solve.} \\ 2n+6-6 &= 35-6 \\ 2n &= 29 \\ \frac{2n}{2} &= \frac{29}{2} \\ n &= \frac{29}{2} \end{aligned}$$

Answer:

$$n = \tfrac{29}{2}$$

Exercise 2.6.1

Solve:

$$\tfrac{5}{3} = \tfrac{3n-1}{2}.$$

Answer available at the end of this section.

Applications

When setting up proportions, consistency with the units of each ratio is critical. Units for the numerators should be the same and units for the denominators should also be the same.

Example 2.6.7

It is claimed that 2 out of 3 dentists prefer a certain brand of toothpaste. If 600 dentists are surveyed, then how many will say they prefer that brand?

Solution:

First, identify the unknown and assign it a variable.

Let n represent the number of dentists surveyed who prefer the brand name.

Since you are looking for the number of dentists who prefer the brand name out of a total of 600 surveyed, construct the ratios with the number of dentists who prefer the brand in the numerator and the total number surveyed in the denominator.

$$\begin{array}{ll} \underline{2} = \underline{n} & \leftarrow \textit{number of dentists who prefer the brand} \\ 3 \;\; 600 & \leftarrow \textit{total number of dentists surveyed} \end{array}$$

Cross multiply and solve for n,

$$\begin{aligned} 2 \cdot 600 &= 3 \cdot n \\ \frac{2 \cdot 600}{3} &= \frac{3 \cdot n}{3} \\ 2 \cdot 200 &= 1n \\ 400 &= n \end{aligned}$$

Answer:

The claim suggests that 400 out of 600 dentists surveyed prefer the brand name.

Example 2.6.8

In Tulare County, 3 out of every 7 voters said yes to Proposition 40. If $42,000$ people voted, how many said no to Proposition 40?

Solution:

The problem gives the ratio of voters who said yes, but it asks for the number who said no.

Let n represent the number of voters who said no.

If 3 out of 7 said yes, then we can assume 4 out of 7 said no. Set up the ratios with the number of voters who said no in the numerator and the total number of voters in the denominator.

$$\begin{aligned} \frac{4}{7} &= \frac{n}{42,000} \quad \begin{matrix} \leftarrow \textit{voters who voted no} \\ \leftarrow \textit{total number of voters} \end{matrix} \end{aligned}$$

Cross multiply and solve for n.

$$\begin{aligned} 4 \cdot 42,000 &= 7 \cdot n \\ \frac{4 \cdot 42,000}{7} &= \frac{7 \cdot n}{7} \\ 24,000 &= n \end{aligned}$$

Answer:

$24,000$ voters out of $42,000$ said no.

Example 2.6.9

The sum of two integers in the ratio of 4 to 5 is 27. Find the integers.

Solution:

The sum of two integers is 27; use this relationship to avoid two variables.

Let n represent one of the integers.

Let $27 - n$ represent the other integer.

The integers are given to be in the ratio of 4 to 5. Set up the following proportion:

$$\begin{aligned}\frac{4}{5} &= \frac{n}{27-n}\\ 4\cdot(27-n) &= 5\cdot n\\ 108-4n &= 5n\\ 108-4n+4n &= 5n+4n\\ 108 &= 9n\\ \frac{108}{9} &= \frac{9n}{9}\\ 12 &= n\end{aligned}$$

Use $27-n$ to determine the other integer.

$$27-n = 27-12 = 15$$

Answer:

The integers are 12 and 15.

Exercise 2.6.2

A recipe calls for 5 tablespoons of sugar for every 8 cups of flour. How many tablespoons of sugar are required for 32 cups of flour?

Answer available at the end of this section.

Similar Triangles

We will often encounter proportion problems in geometry and trigonometry. One application involves similar triangles, which have the same shape, but not necessarily the same size. The measures of the corresponding angles of similar triangles are equal, and the measures of the corresponding sides are proportional. Given similar triangles ABC and RST,

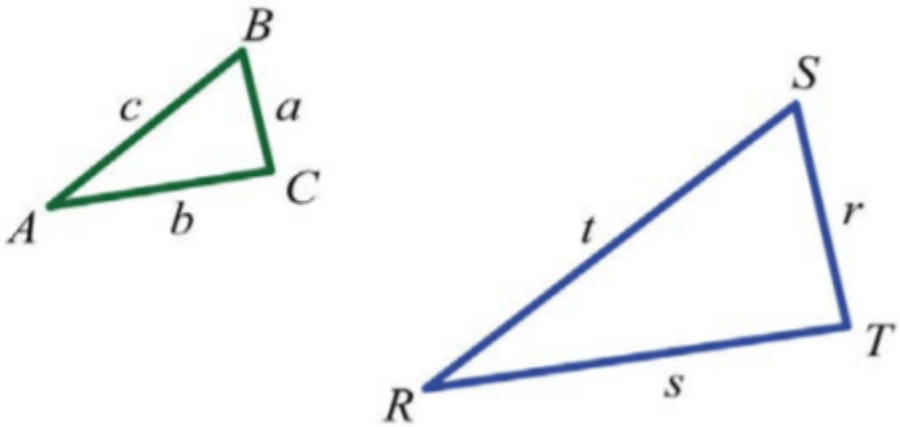


Figure 2.6.2

We may write $ABC\ RST$ and conclude that all of the corresponding angles are equal. The notation indicates that angle A corresponds to angle R and that the measures of these angles are equal: $A = R$.

$$ABC \sim RST$$

Figure 2.6.3

In addition, the measures of other pairs of corresponding angles are equal: $B = S$ and $C = T$.



Figure 2.6.4

Use uppercase letters for angles and a lowercase letter to denote the side opposite of the given angle. Denote the proportionality of the sides as follows:

$$\frac{a}{r} = \frac{b}{s} = \frac{c}{t}$$

Example 2.6.10

If triangle ABC is similar to RST, where $a = 3, b = 4, c = 5,$ and $r = 9$, then find the remaining two sides.

Solution:

Draw a picture and identify the variables pictorially. Represent the remaining unknown sides by s and t. Set up proportions and solve for the missing sides.

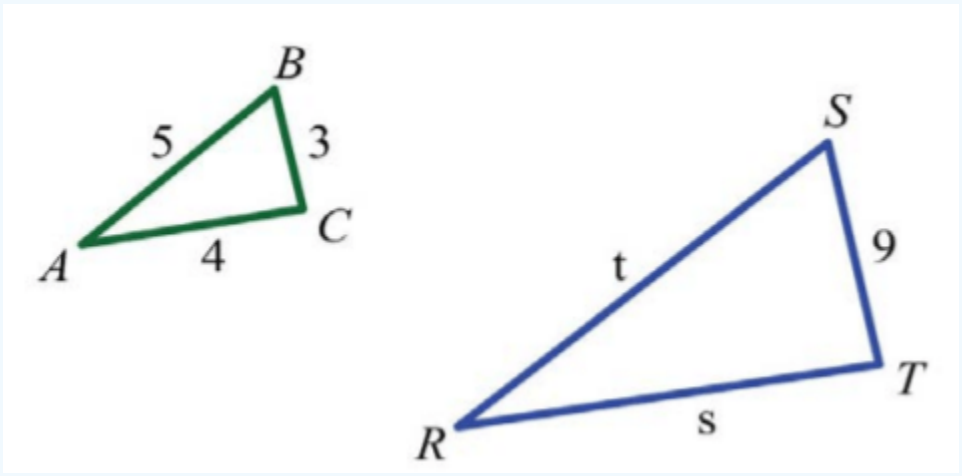


Figure 2.6.5

$$\begin{array}{ll} \underline{Find\ t} & \underline{Find\ s} \\ \frac{5}{t} = \frac{3}{9} & \frac{4}{s} = \frac{3}{9} \\ 5 \cdot 9 = t \cdot 3 & 4 \cdot 9 = s \cdot 3 \\ \frac{5 \cdot 9}{3} = \frac{t \cdot 3}{3} & \frac{4 \cdot 9}{3} = \frac{s \cdot 3}{3} \\ 5 \cdot 3 = t & 4 \cdot 3 = s \\ 15 = t & 12 = s \end{array}$$

Answer:

The two remaining sides measure 12 units and 15 units.

The reduced ratio of any two corresponding sides of similar triangles is called the scale factor. In the previous example, the ratio

of the two given sides a and r is

$$\frac{3}{9} = \frac{1}{3}$$

Therefore, triangle ABC is similar to triangle RST with a scale factor of $\frac{1}{3}$. This means that each leg of triangle ABC is $\frac{1}{3}$ of the measure of the corresponding legs of triangle RST. Also, another interesting fact is that the perimeters of similar triangles are in the same proportion as their sides and share the same scale factor.

Example 2.6.11

If a triangle ABC has a perimeter of 12 units and is similar to RST with a scale factor of $\frac{1}{3}$, then find the perimeter of triangle RST.

Solution:

Let x represent the perimeter of triangle RST.

Scale factor $\frac{1}{3}$ implies that the perimeters are in proportion to this ratio. Set up a proportion as follows:

$$\begin{array}{ll} \frac{1}{3} = \frac{12}{x} & \begin{array}{l} \leftarrow perimeter\ of\ ABC \\ \leftarrow perimeter\ of\ RST \end{array} \end{array}$$

Cross multiply and solve for x.

$$\begin{aligned} 1 \cdot x &= 3 \cdot 12 \\ x &= 36 \end{aligned}$$

Answer:

The perimeter of triangle RST is 36 units.

Key Takeaways

- Solve proportions by multiplying both sides of the equation by the product of the denominators, or cross multiply.
- When setting up a proportion, it is important to ensure consistent units in the numerators and denominators.
- The corresponding angles of similar triangles are equal and their corresponding sides are proportional. The ratio of any two corresponding sides determines the scale factor, which can be used to solve many applications involving similar triangles.

Exercise 2.6.3 Ratios and Rates

Express each ratio in reduced form.

1. 100 inches : 250 inches
2. 480 pixels : 320 pixels
3. 96 feet : 72 feet
4. 240 miles 4 hours
5. 96 feet 3 seconds
6. $6,000$ revolutions 4 minutes
7. Google's average 2008 stock price and earnings per share were $\$465.66$ and $\$14.89$, respectively. What was Google's average price-to-earnings ratio in 2008? (Source: Wolfram Alpha)
8. The F-22 Raptor has two engines that each produce $35,000$ pounds of thrust. If the takeoff weight of this fighter jet is $50,000$ pounds, calculate the plane's thrust-to-weight ratio. (Source: USAF)
9. A discount warehouse offers a box of 55 individual instant oatmeal servings for $\$11.10$. The supermarket offers smaller boxes of the same product containing 12 individual servings for $\$3.60$. Which store offers the better value?

10. Joe and Mary wish to take a road trip together and need to decide whose car they will take. Joe calculated that his car is able to travel 210 miles on 12 gallons of gasoline. Mary calculates that her car travels 300 miles on 19 gallons. Which of their cars gets more miles to the gallon?

Answer available at the end of this section.

Exercise $2.6.4$ Solving Proportions

Solve.

1. $\frac{2}{3}=\frac{n}{150}$
2. $\frac{7}{n}=\frac{2}{15}$
3. $\frac{1}{3}=\frac{5}{n}$
4. $\frac{1}{25}=\frac{6}{n}$
5. $\frac{n}{8}=-\frac{3}{2}$
6. $\frac{n}{3}=-\frac{5}{7}$
7. $8=\frac{2n}{3}$
8. $\frac{5}{n}=-30$
9. $1=\frac{1}{n-1}$
10. $-1=-\frac{1}{n+1}$
11. $-\frac{40}{n}=-\frac{5}{3}$
12. $\frac{2n+1}{3}=-\frac{3}{5}$
13. $\frac{5}{3n+3}=\frac{2}{3}$
14. $\frac{n+1}{2n-1}=\frac{1}{3}$
15. $\frac{5n+7}{5}=\frac{n-1}{2}$
16. $-\frac{2}{n+3}=\frac{n+7}{6}$
17. Find two numbers in the ratio of 3 to 5 whose sum is 160. (Hint: Use n and $160-n$ to represent the two numbers.)
18. Find two numbers in the ratio of 2 to 7 whose sum is 90.
19. Find two numbers in the ratio of -3 to 7 whose sum is 80.
20. Find two numbers in the ratio of -1 to 3 whose sum is 90.
21. A larger integer is 5 more than a smaller integer. If the two integers have a ratio of 6 to 5 find the integers.
22. A larger integer is 7 less than twice a smaller integer. If the two integers have a ratio of 2 to 3 find the integers.

Answer available at the end of this section.

Exercise $2.6.5$ Solving Proportions

Given the following proportions, determine each ratio, $x:y$.

1. $\frac{x}{3}=\frac{y}{4}$
2. $\frac{x-2y}{3}=-\frac{3y}{5}$
3. $\frac{2x+4y}{2x-4y}=\frac{3}{2}$
4. $\frac{x+y}{x-y}=\frac{3}{5}$

Answer available at the end of this section.

Exercise $2.6.6$ Applications

Set up a proportion and then solve.

1. If 4 out of every 5 voters support the governor, then how many of the $1,200$ people surveyed support the governor?
2. If 1 out of every 3 voters surveyed said they voted yes on Proposition 23, then how many of the 600 people surveyed voted yes?

3. Out of 460 students surveyed, the ratio to support the student union remodel project was 3 to 5. How many students were in favor of the remodel?
4. An estimated 5 out of 7 students carry credit card debt. Estimate the number of students that carry credit card debt out of a total of $14,000$ students.
5. If the ratio of female to male students at the college is 6 to 5, then determine the number of male students out of $11,000$ total students.
6. In the year 2009 it was estimated that there would be 838 deaths in the United States for every $100,000$ people. If the total US population was estimated to be $307,212,123$ people, then how many deaths in the United States were expected in 2009? (Source: CIA World Factbook)
7. In the year 2009 it was estimated that there would be $1,382$ births in the United States for every $100,000$ people. If the total US population was estimated to be $307,212,123$ people, then how many births in the United States were expected in 2009? (Source: CIA World Factbook)
8. If 2 out of every 7 voters approve of a sales tax increase then determine the number of voters out of the 588 surveyed who do not support the increase.
9. A recipe calls for 1 cup of lemon juice to make 4 cups of lemonade. How much lemon juice is needed to make 2 gallons of lemonade?
10. The classic "Shirley Temple" cocktail requires 1 part cherry syrup to 4 parts lemon-lime soda. How much cherry syrup is needed to mix the cocktail given a 12-ounce can of lemon-lime soda?
11. A printer prints 30 pages in 1 minute. How long will it take to print a 720-page booklet?
12. A typist can type 75 words per minute. How long will it take to type 72 pages if there are approximately 300 words per page?
13. On a particular map, every $\frac{1}{16}$ inch represents 1 mile. How many miles does $3\frac{1}{2}$ inches represent?
14. On a graph every 1 centimeter represents 100 feet. What measurement on the map represents one mile?
15. A candy store offers mixed candy at $\$3.75$ for every half-pound. How much will 2.6 pounds of candy cost?
16. Mixed nuts are priced at $\$6.45$ per pound. How many pounds of mixed nuts can be purchased with $\$20.00$?
17. Corn at the farmers market is bundled and priced at $\$1.33$ for 6 ears. How many ears can be purchased with $\$15.00$?
18. If 4 pizzas cost $\$21.00$, then how much will 16 pizzas cost?
19. A sweetened breakfast cereal contains 110 calories in one $\frac{3}{4}$-cup serving. How many calories are in a $1\frac{7}{8}$-cup serving?
20. Chicken-flavored rice contains 300 calories in each 2.5-ounce serving. How many calories are in a 4-ounce scoop of chicken-flavored rice?
21. A 200-pound man would weigh about 33.2 pounds on the moon. How much will a 150-pound man weigh on the moon?
22. A 200-pound man would weigh about 75.4 pounds on Mars. How much will a 150-pound man weigh on Mars?
23. There is a 1 out of 6 chance of rolling a 1 on a six-sided die. How many times can we expect a 1 to come up in 360 rolls of the die?
24. There is a 1 out of 6 chance of rolling a 7 with two six-sided dice. How many times can we expect a 7 to come up in 300 rolls?
25. The ratio of peanuts to all nuts in a certain brand of packaged mixed nuts is 3 to 5. If the package contains 475 nuts, then how many peanuts can we expect?
26. A mixed bag of marbles is packaged with a ratio of 6 orange marbles for every 5 red marbles. If the package contains 216 orange marbles, then how many red marbles can we expect?
27. A graphic designer wishes to create a 720-pixel-wide screen capture. If the width to height ratio is to be $3:2$, then to how many pixels should he set the height?
28. If a video monitor is produced in the width to height ratio of $16:9$ and the width of the monitor is 40 inches, then what is the height?

Answer available at the end of this section.

Exercise 2.6.7 Similar Triangles

If triangle ABC is similar to triangle RST, find the remaining two sides given the information.

1. $a=6, b=8, c=10$, and $s=16$
2. $b=36, c=48, r=20$, and $t=32$
3. $b=2, c=4, r=6$, and $s=4$
4. $b=3, c=2, r=10$, and $t=12$
5. $a=40, c=50, s=3$, and $t=10$

6. $c = 2, r = 7, s = 9$, and $t = 4$
7. At the same time of day, a tree casts a 12-foot shadow while a 6-foot man casts a 3-foot shadow. Estimate the height of the tree.
8. At the same time of day, a father and son, standing side by side, cast a 4-foot and 2-foot shadow, respectively. If the father is 6 feet tall, then how tall is his son?
9. If the $6 - 8 - 10$ right triangle ABC is similar to RST with a scale factor of $\frac{2}{3}$, then find the perimeter of triangle RST.
10. If the $3 - 4 - 5$ right triangle ABC is similar to RST with a scale factor of 5, then find the perimeter of triangle RST.
11. An equilateral triangle with sides measuring 6 units is similar to another with scale factor $3 : 1$. Find the length of each side of the unknown triangle.
12. The perimeter of an equilateral triangle ABC measures 45 units. If triangle ABC RST and $r = 20$, then what is the scale factor?
13. The perimeter of an isosceles triangle ABC, where the two equal sides each measure twice that of the base, is 60 units. If the base of a similar triangle measures 6 units, then find its perimeter.
14. The perimeter of an isosceles triangle ABC measures 11 units and its two equal sides measure 4 units. If triangle ABC is similar to triangle RST and triangle RST has a perimeter of 22 units, then find all the sides of triangle RST.
15. A $6 - 8 - 10$ right triangle ABC is similar to a triangle RST with perimeter 72 units. Find the length of each leg of triangle RST.
16. The perimeter of triangle ABC is 60 units and $b = 20$ units. If ABC RST and $s = 10$ units, then find the perimeter of triangle RST.

Answer available at the end of this section.

Exercise $2.6.8$ Discussion Board Topics

1. What is the golden ratio and where does it appear?
2. Research and discuss the properties of similar triangles.
3. Discuss the mathematics of perspective.
4. Research and discuss the various aspect ratios that are available in modern media devices.

Answer available at the end of this section.

Answers

Exercise $2.6.1$

$n = \frac{13}{9}$

Exercise $2.6.2$

20 tablespoons of sugar

Exercise $2.6.3$ **Ratios and Rates**

1. $2 : 5$
3. $4 : 3$
5. 32 feet per second
7. 31.27
9. The discount warehouse

Exercise $2.6.4$ **Solving Proportions**

1. $n = 100$
3. $n = 15$
5. $n = -12$
7. $n = 12$
9. $n = 2$
11. $n = 24$
13. $n = \frac{3}{2}$
15. $n = -\frac{19}{5}$
17. $60, 100$
19. $-60, 140$
21. $25, 30$

Exercise $2.6.5$ **Solving Proportions**

1. $\frac{3}{4}$

3. 10

Exercise $2.6.6$ **Applications**

1. 960 people
3. 276 students
5. $5,000$ male students
7. $4,245,672$ births
9. 8 cups of lemon juice
11. 24 minutes
13. 56 miles
15. $\$19.50$
17. 66 ears
19. 275 calories
21. 24.9 pounds
23. 60 times
25. 285 peanuts
27. 480 pixels

Exercise $2.6.7$ **Similar Triangles**

1. $t = 20, r = 12$
3. $a = 3, t = 8$
5. $r = 8, b = 15$
7. 24 feet
9. 36 units
11. 2 units
13. 30 units
15. $r = 18$ units, $s = 24$ units, $t = 30$ units

Exercise $2.6.8$ **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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2.7: Introduction to Inequalities and Interval Notation

Learning Objectives

- Graph the solutions of a single inequality on a number line and express the solutions using interval notation.
- Graph the solutions of a compound inequality on a number line, and express the solutions using interval notation.

Unbounded Intervals

An algebraic inequality, such as $x \geq 2$, is read "x is greater than or equal to 2." This inequality has infinitely many solutions for x. Some of the solutions are $2, 3, 3.5, 5, 20,$ and 20.001. Since it is impossible to list all of the solutions, a system is needed that allows a clear communication of this infinite set. Two common ways of expressing solutions to an inequality are by graphing them on a number line and using interval notation.

To express the solution graphically, draw a number line and shade in all the values that are solutions to the inequality. Interval notation is textual and uses specific notation as follows:

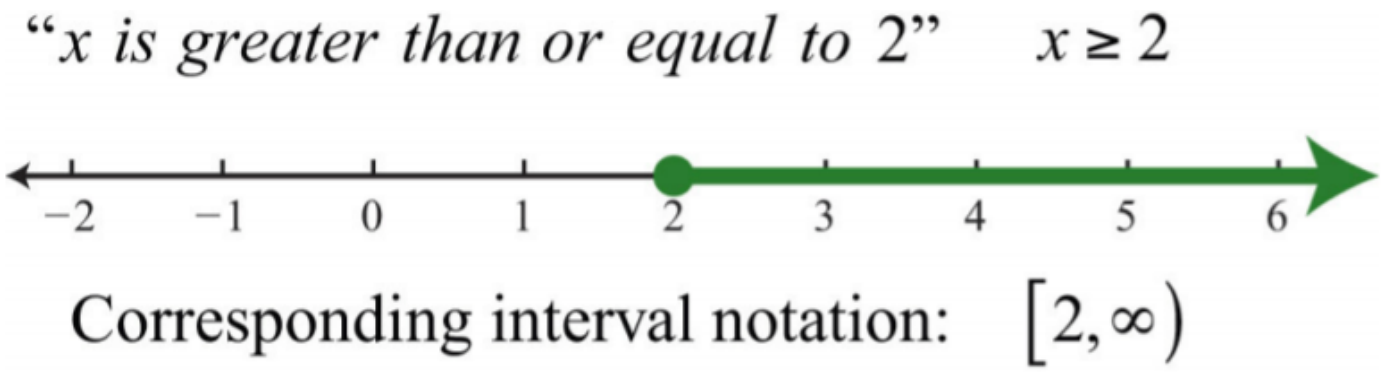


Figure $2.7.1$

Determine the interval notation after graphing the solution set on a number line. The numbers in interval notation should be written in the same order as they appear on the number line, with smaller numbers in the set appearing first. In this example, there is an inclusive inequality, which means that the lower-bound 2 is included in the solution. Denote this with a closed dot on the number line and a square bracket in interval notation. The symbol (∞) is read as infinity and indicates that the set is unbounded to the right on a number line. Interval notation requires a parenthesis to enclose infinity. The square bracket indicates the boundary is included in the solution. The parentheses indicate the boundary is not included. ***Infinity*** is an upper bound to the real numbers, but is not itself a real number: it cannot be included in the solution set.

Now compare the interval notation in the previous example to that of the strict, or noninclusive, inequality that follows:

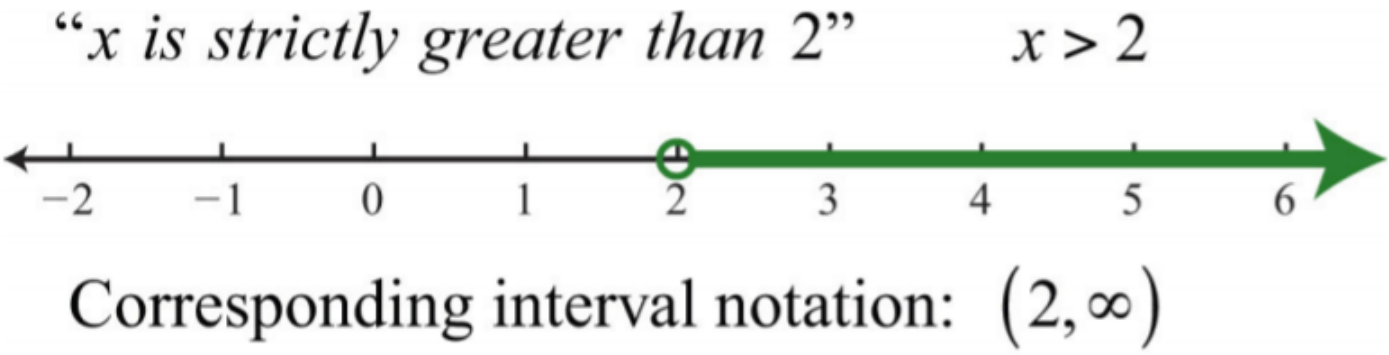


Figure 2.7.2

Strict inequalities imply that solutions may get very close to the boundary point, in this case 2, but not actually include it. Denote this idea with an open dot on the number line and a round parenthesis in interval notation.

Example 2.7.1

Graph and give the interval notation equivalent:

$$x < 3.$$

Solution:

Use an open dot at 3 and shade all real numbers strictly less than 3. Use negative infinity $(-\infty)$ to indicate that the solution set is unbounded to the left on a number line.

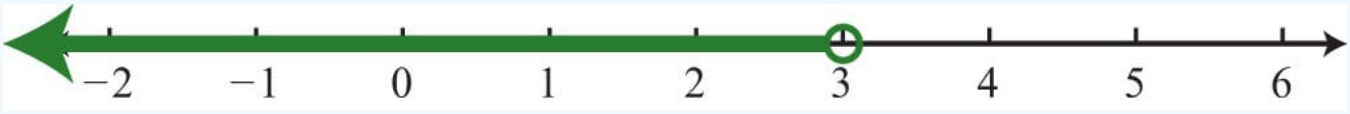


Figure 2.7.3

Answer:

Interval notation: $(-\infty, 3)$

Example 2.7.2

Graph and give the interval notation equivalent:

$$x \leq 5.$$

Solution:

Use a closed dot and shade all numbers less than and including 5.

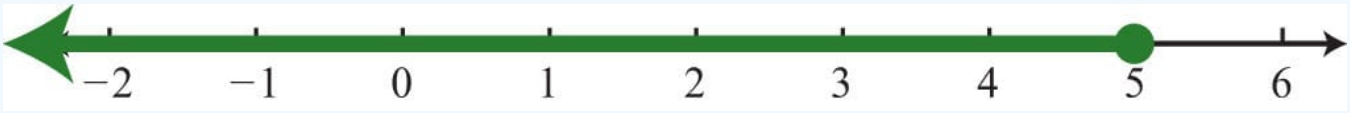


Figure 2.7.4

Answer:

Interval notation: $(-\infty, 5]$

It is important to see that $5 \geq x$ is the same as $x \leq 5$. Both require values of x to be smaller than or equal to 5. To avoid confusion, it is good practice to rewrite all inequalities with the variable on the left. Also, when using text, use "inf" as a shortened form of infinity. For example, $(-\infty, 5]$ can be expressed textually as $(\ -\text{inf}, 5]$.

A compound inequality is actually two or more inequalities in one statement joined by the word "and" or by the word "or." Compound inequalities with the logical "or" require that either condition must be satisfied. Therefore, the solution set of this type of compound inequality consists of all the elements of the solution sets of each inequality. When we join these individual solution sets it is called the union, denoted $\cup$. For example, the solutions to the compound inequality $x < 3$ or $x \geq 6$ can be graphed as follows:

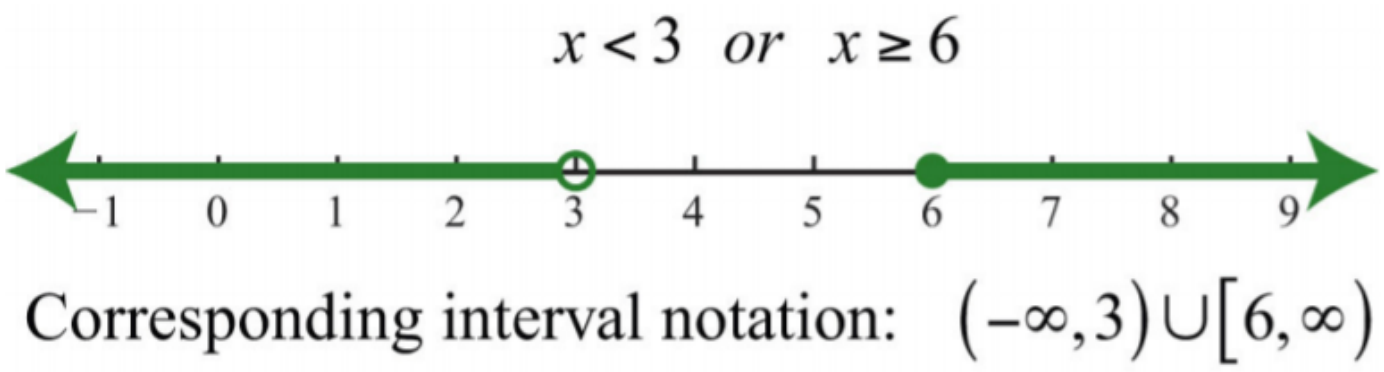


Figure $2.7.5$

Sometimes we encounter compound inequalities where the separate solution sets overlap. In the case where the compound inequality contains the word "or," we combine all the elements of both sets to create one set containing all the elements of each.

Example $2.7.3$

Graph and give the interval notation equivalent:

$$x \leq -1 \text{ or } x < 3.$$

Solution:

Combine all solutions of both inequalities. The solutions to each inequality are sketched above the number line as a means to determine the union, which is graphed on the number line below.

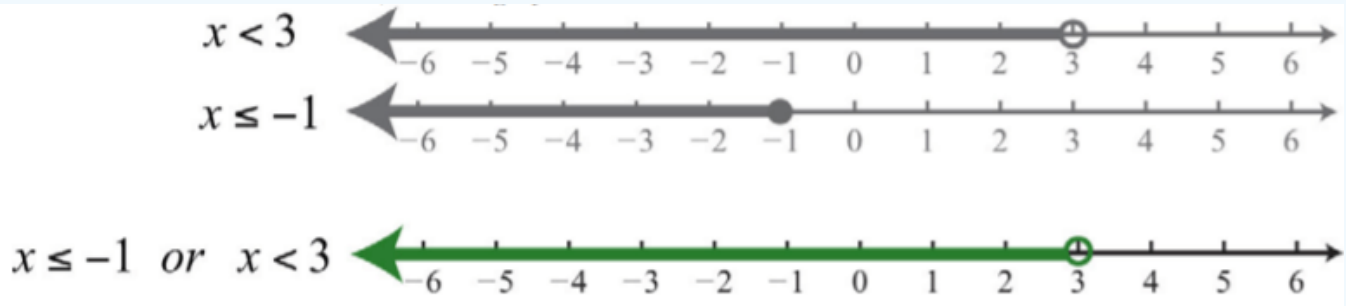


Figure $2.7.6$

Answer:

Interval notation: $(-\infty, 3)$

Any real number less than 3 in the shaded region on the number line will satisfy at least one of the two given inequalities.

Example 2.7.4

Graph and give the interval notation equivalent:

$$x < 3 \text{ or } x \geq -1.$$

Solution:

Both solution sets are graphed above the union, which is graphed below.

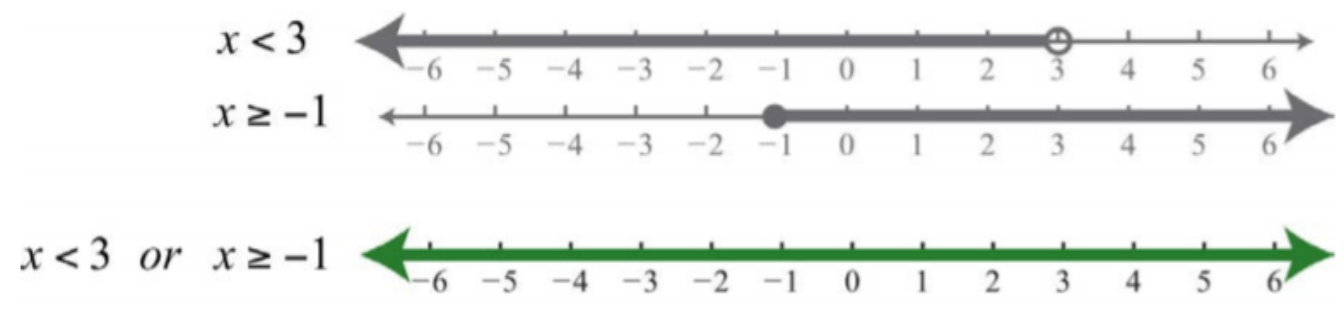


Figure 2.7.7

Answer:

Interval notation: $R = (-\infty, \infty)$

When you combine both solution sets and form the union, you can see that all real numbers satisfy the original compound inequality.

In summary,

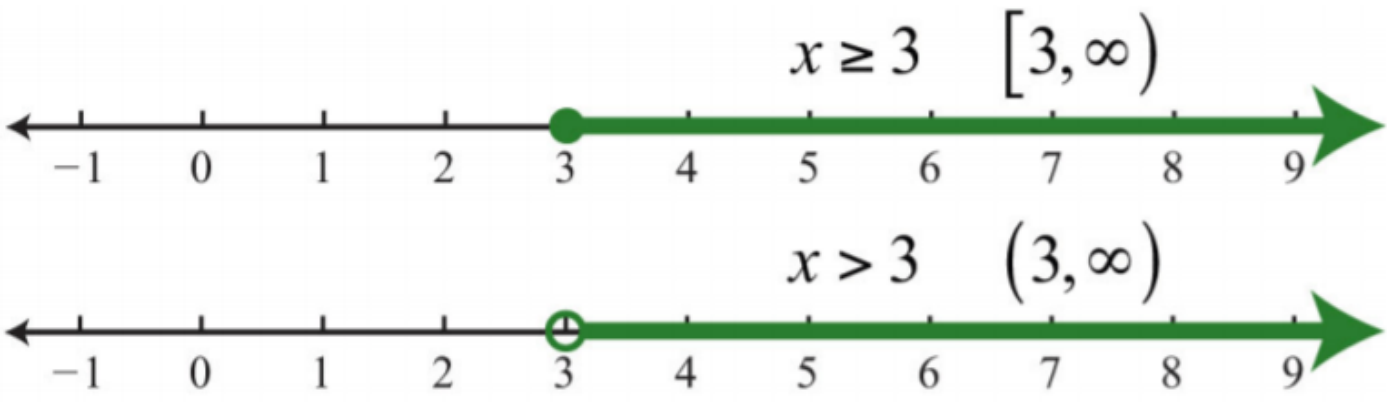


Figure 2.7.8

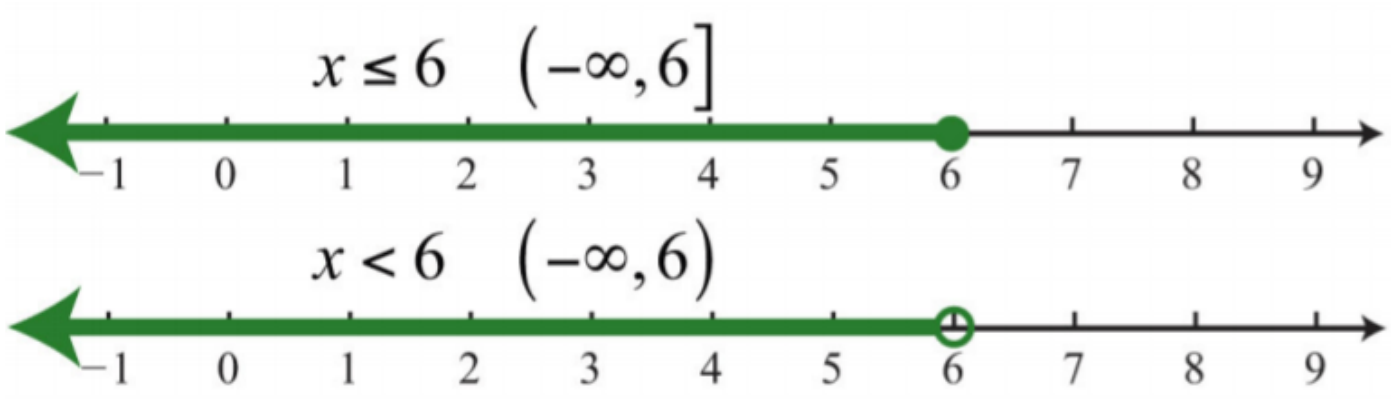


Figure 2.7.9

and

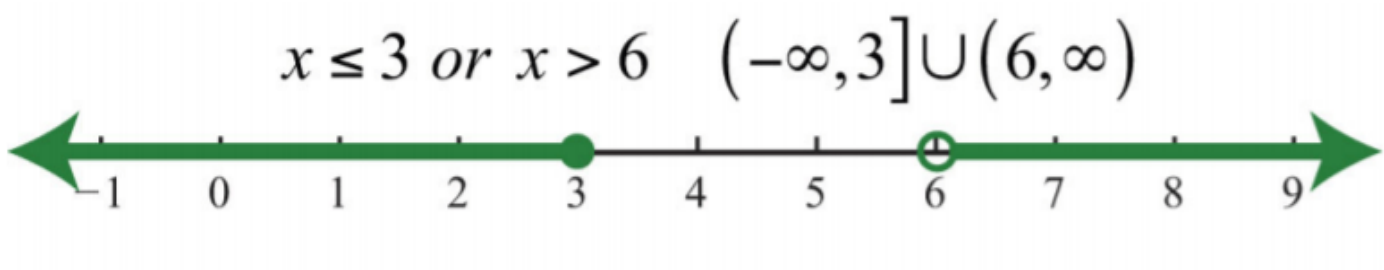


Figure 2.7.10

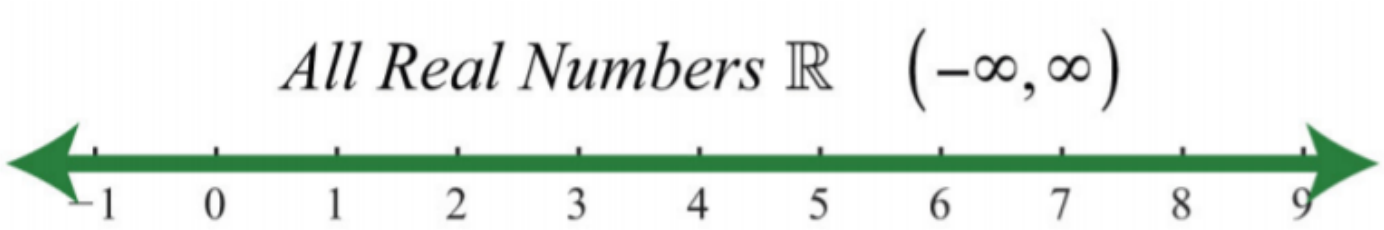


Figure 2.7.11

Bounded Intervals

An inequality such as

$$-1 \leq x < 3$$

reads "-1 one is less than or equal to x and x is less than three." This is a compound inequality because it can be decomposed as follows:

$$-1 \leq x \text{ and } x < 3$$

The logical "and" requires that both conditions must be true. Both inequalities are satisfied by all the elements in the intersection, denoted $\cap$, of the solution sets of each.

Example 2.7.5

Graph and give the interval notation equivalent:

$$x < 3 \text{ and } x \geq -1.$$

Solution:

Determine the intersection, or overlap, of the two solution sets. The solutions to each inequality are sketched above the number line as a means to determine the intersection, which is graphed on the number line below.

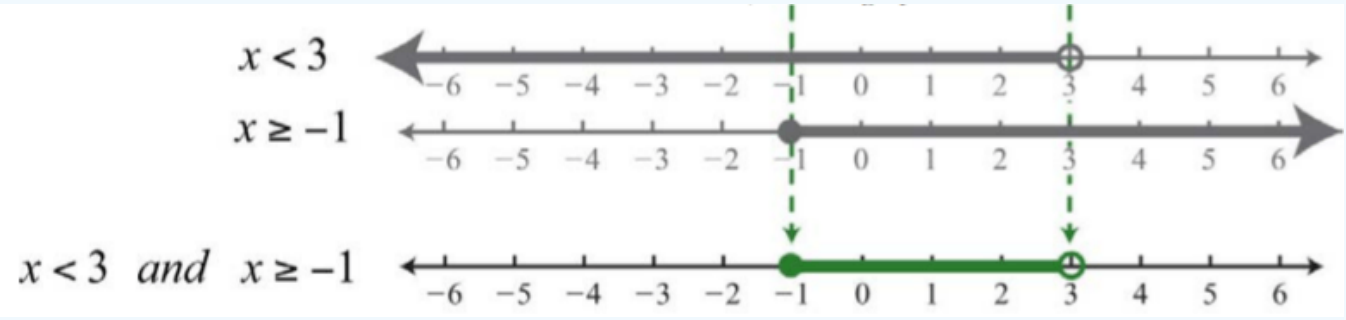


Figure 2.7.12

Here $x = 3$ is not a solution because it solves only one of the inequalities.

Answer:

Interval notation: $[-1, 3)$

Alternatively, we may interpret $-1 \leq x < 3$ as all possible values for x between or bounded by -1 and 3 on a number line. For example, one such solution is $x = 1$. Notice that 1 is between -1 and 3 on a number line, or that $-1 < 1 < 3$. Similarly, we can see that other possible solutions are $-1, -0.99, 0, 0.0056, 1.8$, and 2.99. Since there are infinitely many real numbers between -1 and 3, we must express the solution graphically and/or with interval notation, in this case $[-1, 3)$.

Example 2.7.6

Graph and give the interval notation equivalent:

$$-\tfrac{3}{2} < x < 2.$$

Solution:

Shade all real numbers bounded by, or strictly between, $-\frac{3}{2} = -1\frac{1}{2}$ and 2.



Figure 2.7.13

Answer:

Interval notation: $(-\frac{3}{2}, 2)$

Example 2.7.7

Graph and give the interval notation equivalent:

$$-5 < x \leq 15$$

Solution:

Shade all real numbers between -5 and 15, and indicate that the upper bound, 15, is included in the solution set by using a closed dot.



Figure 2.7.14

Answer:

Interval notation: $(-5, 15]$

In the previous two examples, we did not decompose the inequalities; instead we chose to think of all real numbers between the two given bounds.

In summary,

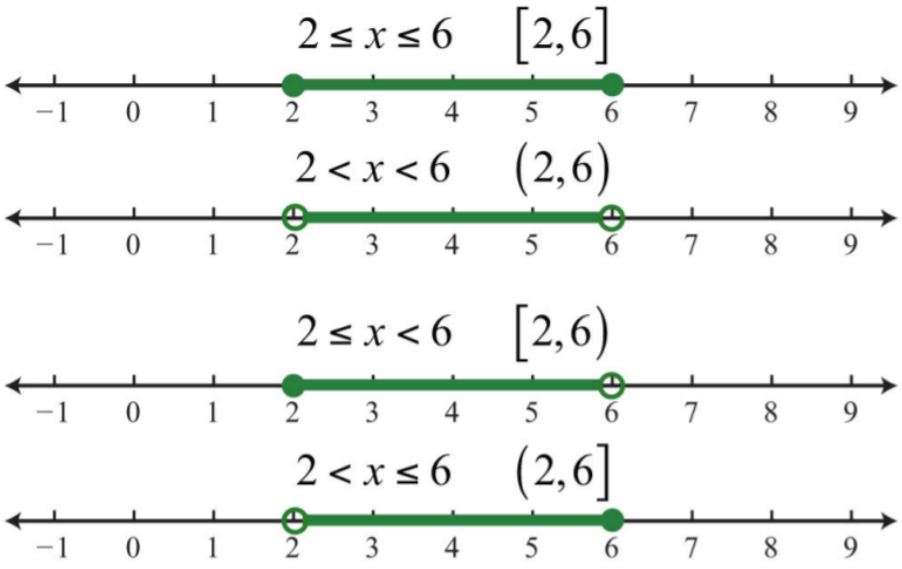


Figure 2.7.15

Set-Builder Notation

In this text, we use interval notation. However, other resources that you are likely to encounter use an alternate method for describing sets called set-builder notation. We have used set notation to list the elements such as the integers

$$\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

The braces group the elements of the set and the ellipsis marks indicate that the integers continue forever. In this section, we wish to describe intervals of real numbers—for example, the real numbers greater than or equal to 2.

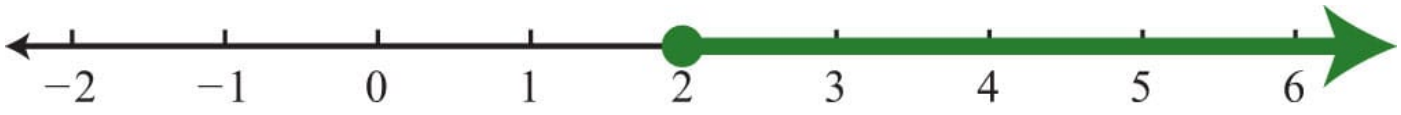


Figure $2.7.16$

Since the set is too large to list, set-builder notation allows us to describe it using familiar mathematical notation. An example of set-builder notation follows:

$$\{x \in \mathbf{R} | x \geq 2\}$$

Here $x \in R$ describes the type of number, where the symbol $(\in)$ is read "element of." This implies that the variable x represents a real number. The vertical bar $(|)$ is read "such that." Finally, the statement $x \geq 2$ is the condition that describes the set using mathematical notation. At this point in our study of algebra, it is assumed that all variables represent real numbers. For this reason, you can omit the "$\in R$" and write $\{x | x \geq 2\}$, which is read "the set of all real numbers x such that x is greater than or equal to 2."

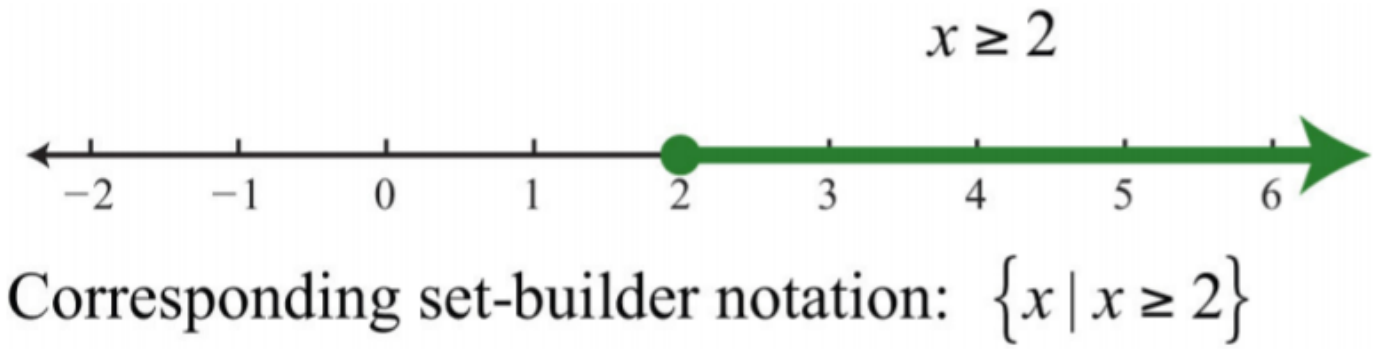


Figure $2.7.17$

To describe compound inequalities such as $x < 3$ or $x \geq 6$, write $\{x | x < 3 \text{ or } x \geq 6\}$, which is read "the set of all real numbers x such that x is less than 3 or x is greater than or equal to 6."

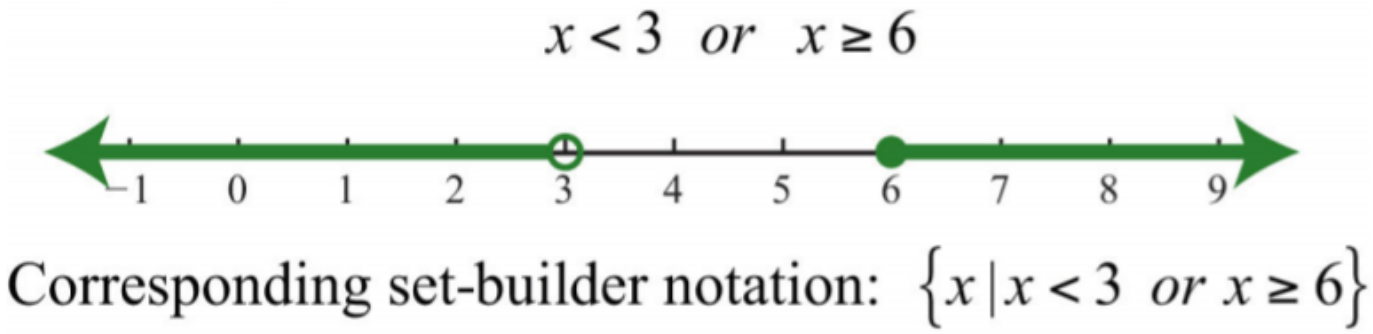


Figure $2.7.18$

Write bounded intervals, such as $-1 \leq x < 3$, as $\{x| -1 \leq x < 3\}$, which is read "the set of all real numbers x such that x is greater than or equal to -1 and less than 3."

Key Takeaways

- Inequalities usually have infinitely many solutions, so rather than presenting an impossibly large list, we present such solutions sets either graphically on a number line or textually using interval notation.
- Inclusive inequalities with the "or equal to" component are indicated with a closed dot on the number line and with a square bracket using interval notation.
- Strict inequalities without the "or equal to" component are indicated with an open dot on the number line and a parenthesis using interval notation.
- Compound inequalities that make use of the logical "or" are solved by solutions of either inequality. The solution set is the union of each individual solution set.
- Compound inequalities that make use of the logical "and" require that all inequalities are solved by a single solution. The solution set is the intersection of each individual solution set.
- Compound inequalities of the form $n < A < m$ can be decomposed into two inequalities using the logical "and." However, it is just as valid to consider the argument A to be bounded between the values n and m.

Exercise 2.7.1 Simple Inequalities

Graph all solutions on a number line and provide the corresponding interval notation.

1. $x \leq 10$
2. $x > -5$
3. $x > 0$
4. $x \leq 0$
5. $x \leq -3$
6. $x \geq -1$
7. $-4 < x$
8. $1 \geq x$
9. $x < -\frac{1}{2}$
10. $x \geq -\frac{3}{2}$
11. $x \geq -1\frac{3}{4}$
12. $x < \frac{3}{4}$

Answer available at the end of this section.

Exercise 2.7.2 Compound Inequalities

Graph all solutions on a number line and give the corresponding interval notation.

1. $-2 < x < 5$
2. $-5 \leq x \leq -1$
3. $-5 < x \leq 20$
4. $0 \leq x < 15$
5. $10 < x \leq 40$
6. $-40 \leq x < -10$
7. $0 < x \leq 50$
8. $-30 < x < 0$
9. $-\frac{5}{8} < x < \frac{1}{8}$
10. $-\frac{3}{4} \leq x \leq \frac{1}{2}$
11. $-1 \leq x < 1\frac{1}{2}$
12. $-1\frac{1}{2} < x < -\frac{1}{2}$
13. $x < -3$ or $x > 3$
14. $x < -2$ or $x \geq 4$

15. $x \leq 0$ or $x > 10$
16. $x \leq -20$ or $x \geq -10$
17. $x < -\frac{2}{3}$ or $x > \frac{1}{3}$
18. $x \leq -\frac{4}{3}$ or $x > -\frac{1}{3}$
19. $x > -5$ or $x < 5$
20. $x < 12$ or $x > -6$
21. $x < 3$ or $x \geq 3$
22. $x \leq 0$ or $x > 0$
23. $x < -7$ or $x < 2$
24. $x \geq -3$ or $x > 0$
25. $x \geq 5$ or $x > 0$
26. $x < 15$ or $x \leq 10$
27. $x > -2$ and $x < 3$
28. $x \geq 0$ and $x < 5$
29. $x \geq -5$ and $x \leq -1$
30. $x < -4$ and $x > 2$
31. $x \leq 3$ and$x > 3$
32. $x \leq 5$ and $x \geq 5$
33. $x \leq 0$ and $x \geq 0$
34. $x < 2$ and $x \leq -1$
35. $x > 0$ and $x \geq -1$
36. $x < 5$ and $x < 2$

Answer available at the end of this section.

Exercise 2.7.3 Interval Notation

Determine the inequality given the answers expressed in interval notation.

1. $(-\infty, 7]$
2. $(-4, \infty)$
3. $[-\frac{1}{2}, \infty)$
4. $(-\infty, -3)$
5. $(-8, 10]$
6. $(-20, 0]$
7. $(-14, -2)$
8. $[\frac{2}{3}, \frac{4}{3}]$
9. $(-\frac{3}{4}, \frac{1}{2})$
10. $(-\infty, -8)$
11. $(8, \infty)$
12. $(-\infty, 4) \cup [8, \infty)$
13. $(-\infty, -2] \cup [0, \infty)$
14. $(-\infty, -5] \cup (5, \infty)$
15. $(-\infty, 0) \cup (2, \infty)$
16. $(-\infty, -15) \cup (-5, \infty)$

Answer available at the end of this section.

Exercise 2.7.4 Interval Notation

Write an equivalent inequality.

1. All real numbers less than 27.
2. All real numbers less than or equal to zero.
3. All real numbers greater than 5.

4. All real numbers greater than or equal to -8.
5. All real numbers strictly between -6 and 6.
6. All real numbers strictly between -80 and 0.

Answer available at the end of this section.

Exercise 2.7.5 Discussion Board Topics

1. Compare interval notation with set-builder notation. Share an example of a set described using both systems.
2. Explain why we do not use a bracket in interval notation when infinity is an endpoint.
3. Research and discuss the different compound inequalities, particularly unions and intersections.
4. Research and discuss the history of infinity.
5. Research and discuss the contributions of Georg Cantor.
6. What is a Venn diagram? Explain and post an example.

Answer available at the end of this section.

Answers

Exercise 2.7.1 Simple Inequalities

1. $(-\infty, 10]$



Figure 2.7.19

3. $(0, \infty)$



Figure 2.7.20

5. $(-\infty, -3]$

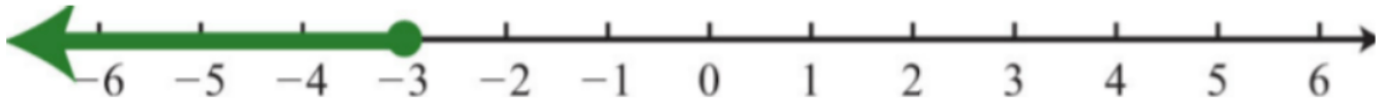


Figure 2.7.21

7. $(-4, \infty)$



Figure 2.7.22

9. $(-\infty, -\frac{1}{2})$

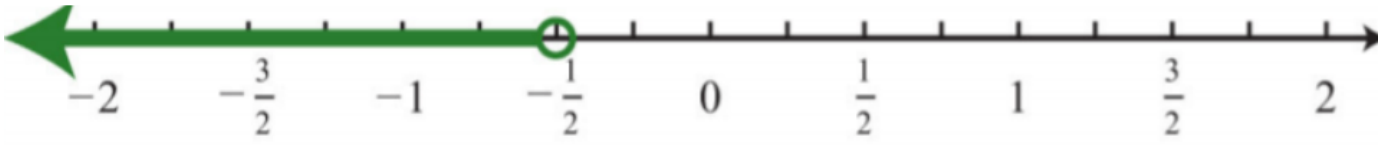


Figure 2.7.23

11. $[-1\frac{3}{4}, \infty)$

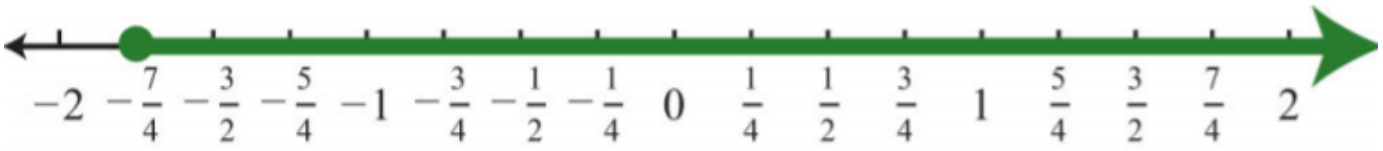


Figure 2.7.24

Exercise 2.7.2 **Compound Inequalities**

1. $(-2, 5)$



Figure 2.7.25

3. $(-5, 20]$



Figure 2.7.26

5. $(10, 40]$

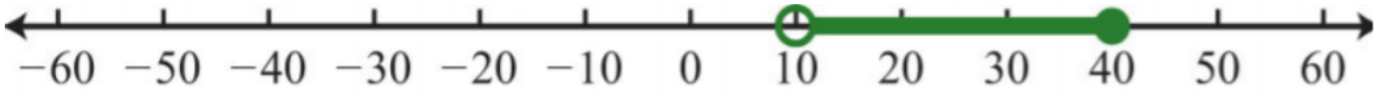


Figure 2.7.27

7. $(0, 50]$



Figure 2.7.28

9. $(-\frac{5}{8}, \frac{1}{8})$



Figure 2.7.29

11. $[-1, 1\frac{1}{2})$



Figure 2.7.30

13. $(-\infty, -3) \cup (3, \infty)$

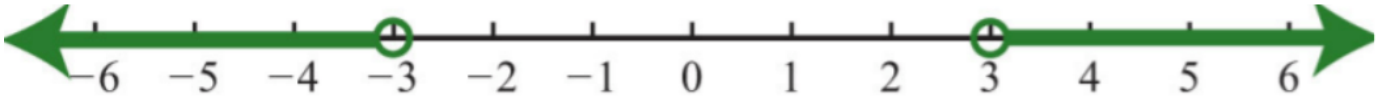


Figure 2.7.31

15. $(-\infty, 0] \cup (10, \infty)$



Figure 2.7.32

17. $(-\infty, -\frac{2}{3}) \cup (\frac{1}{3}, \infty)$



Figure 2.7.33

19. R



Figure 2.7.34

21. R



Figure 2.7.35

23. $(-\infty, 2)$

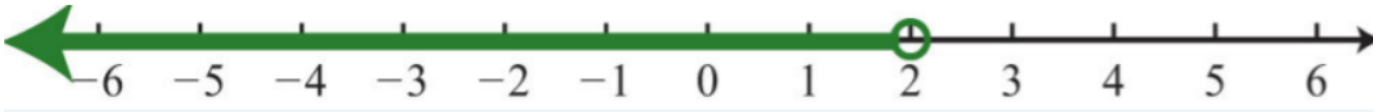


Figure 2.7.36

25. $(0, \infty)$



Figure 2.7.37

27. $(-2, 3)$



Figure 2.7.38

29. $[-5, -1]$



Figure 2.7.39

31. $\emptyset$

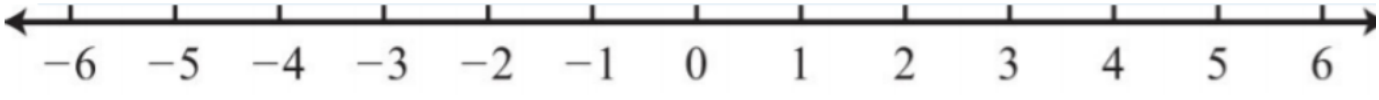


Figure 2.7.40

33. $\{0\}$

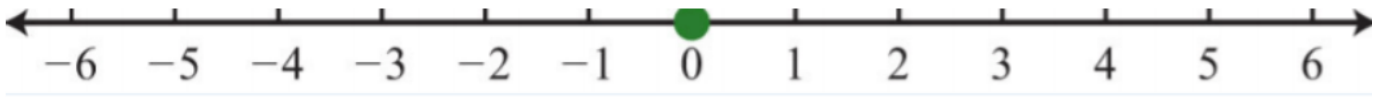


Figure 2.7.41

35. $(0, \infty)$



Figure 2.7.42

Exercise 2.7.3 **Interval Notation**

1. $x \leq 7$
3. $x \geq -\frac{1}{2}$
5. $-8 < x \leq 10$
7. $-14 < x < -2$
9. $-\frac{3}{4} < x < \frac{1}{2}$
11. $x > 8$
13. $x \leq -2$ or $x \geq 0$
15. $x < 0$ or $x > 2$

Exercise 2.7.4 **Interval Notation**

1. $x < 27$
3. $x > 5$
5. $-6 < x < 6$

Exercise 2.7.5 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary
5. Answers may vary

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2.8: Linear Inequalities (One Variable)

Learning Objectives

- Identify linear inequalities and check solutions.
- Solve linear inequalities and express the solutions graphically on a number line and in interval notation.
- Solve compound linear inequalities and express the solutions graphically on a number line and in interval notation.
- Solve applications involving linear inequalities and interpret the results.

Definition of a Linear Inequality

A linear inequality is a mathematical statement that relates a linear expression as either less than or greater than another. The following are some examples of linear inequalities, all of which are solved in this section:

$$3x+7<16 \qquad -2x+1\geq 21 \qquad -7(2x+1)<1$$

A solution to a linear inequality is a real number that will produce a true statement when substituted for the variable. Linear inequalities have either infinitely many solutions or no solution. If there are infinitely many solutions, graph the solution set on a number line and/or express the solution using interval notation.

Example 2.8.1

Are $x=-2$ and $x=4$ solutions to $3x+7<16$?

Solution:

Substitute the values for x, simplify, and check to see if we obtain a true statement.

$$\begin{array}{rcl} \underline{Check\ x=-2} & & \underline{Check\ x=4} \\ 3(-2)+7<16 & & 3(4)+7<16 \\ -6+7<16 & & 12+7<16 \\ 1<16 & \checkmark & 19<16 \quad x \end{array}$$

Answer:

$x=-2$ is a solution and $x=4$ is not.

Algebra of Linear Inequalities

All but one of the techniques learned for solving linear equations apply to solving linear inequalities. You may add or subtract any real number to both sides of an inequality, and you may multiply or divide both sides by any ***positive*** real number to create equivalent inequalities. For example,

-5\color{Cerulean}{-7} &\color{Cerulean}{Subtract\:7\:from\:both\:sides.} \ 3&>-12\quad\color{Cerulean}{\checkmark} &\color{Cerulean}{True} \ 10&>-5 \ \frac{10}{\color{Cerulean}{5}}&>\frac{-5}{\color{Cerulean}{5}} &\color{Cerulean}{Divide\:both\:sides\:by\:5.} \ 2&>-1\quad\color{Cerulean}{\checkmark} &\color{Cerulean}{True} \end{aligned}\)

Both subtracting 7 from each side and dividing each side by $+5$ results in an equivalent inequality that is true.

Solve and graph the solution set:

$$3x + 7 < 16.$$

Solution:

$$\begin{aligned} 3x + 7 &< 16 \\ 3x + 7 \color{cerulean}{-7} &< 16 \color{cerulean}{-7} \\ 3x &< 9 \\ \frac{3x}{\color{cerulean}{3}} &< \frac{9}{\color{cerulean}{3}} \\ x &< 3 \end{aligned}$$



Figure 2.8.1

It is helpful to take a minute and choose a few values in and out of the solution set, substitute them into the original inequality, and then verify the results. As indicated, you should expect $x = 0$ to solve the original inequality, but $x = 5$ should not.

$$\begin{array}{rl} \underline{Check\ x = 0} & \\ 3(0) + 7 < 16 & \\ 7 < 16 & \checkmark \end{array} \qquad \begin{array}{rl} \underline{Check\ x = 5} & \\ 3(5) + 7 < 16 & \\ 15 + 7 < 16 & \\ 22 < 16 & x \end{array}$$

Checking in this manner gives a good indication that the inequality is solved correctly. This can be done mentally.

Answer:

Interval notation: $(-\infty, 3)$

When working with linear inequalities, a different rule applies when multiplying or dividing by a negative number. To illustrate the problem, consider the true statement $10 > -5$ and divide both sides by -5.

\frac{-5}{\color{Cerulean}{-5}} &\color{Cerulean}{Divide\:both\:sides\:by\:-5.} \ -2&\color{red}{>}\color{black}{1}\quad\ color{red}{x} &\color{Cerulean}{False} \end{aligned}\)

Dividing by -5 results in a false statement. To retain a true statement, the inequality must be reversed.

$$\begin{array}{rll} 10 > -5 & & \\ \frac{10}{-5} < \frac{-5}{-5} & Reverse\ the\ inequality. & \\ -2 < 1 \quad \checkmark & & True \end{array}$$

The same problem occurs when multiplying by a negative number. This leads to the following new rule: ***when multiplying or dividing by a negative number, reverse the inequality***. It is easy to forget to do this so take special care to watch for negative coefficients.

In general, given algebraic expressions A and B, where c is a positive nonzero real number, we have the following properties of inequalities:

Addition property of inequalities:	If $A < B$ then, $A+c< B+c$
Subtraction property of inequalities:	If $A < B$, then $A-c< B-c$
Multiplication property of inequalities:	If $A < B$, then $cA <cB$
	If $A < B$, then $-cA>-cB$
Division property of inequalities:	If $A < B$, then $\frac{A}{c}< \frac{B}{c}$
	If $A < B$, then $\frac{A}{-c}>\frac{B}{-c}$

We use these properties to obtain an equivalent inequality, one with the same solution set, where the variable is isolated. The process is similar to solving linear equations.

Example 2.8.3

Solve:

$$-2x + 1 \geq 21.$$

Solution:



Figure 2.8.2

Answer:

Interval notation: $(-\infty, -10]$

Example 2.8.4

Solve:

$$-7(2x + 1) < 1.$$

Solution:

$$\begin{aligned} -7(2x+1) &< 1 && \textit{Distribute.} \\ -14x - 7 &< 1 \\ -14x - 7+7 &< 1+7 \\ 14x &< 8 \\ \frac{-14x}{-14} &> \frac{8}{-14} && \textit{Reverse the inequality.} \\ x &> -\frac{8\div 2}{14\div 2} && \textit{Reduce.} \\ x &> -\frac{4}{7} \end{aligned}$$

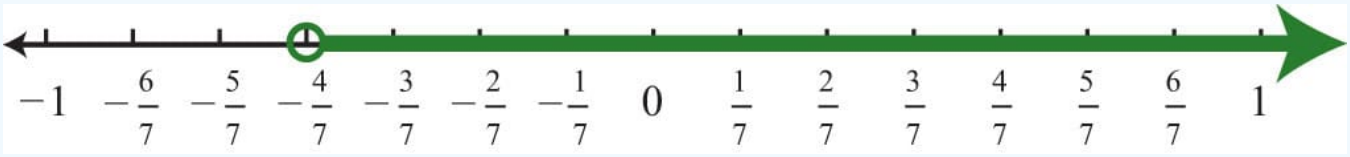


Figure 2.8.3

Answer:

Interval notation: $(-\frac{4}{7}, \infty)$

Example 2.8.5

Solve:

$$5x - 3(2x - 1) \geq 2(x - 3).$$

Solution:



Figure 2.8.4

Answer:

Interval notation: $(-\infty, 3]$

Exercise 2.8.1

Solve:

$$3 - 5(x - 1) \leq 28.$$

Answer available at the end of this section.

Compound Inequalities

Following are some examples of compound linear inequalities:

$$-3 < 2x + 5 < 17$$

$$-1 \leq \tfrac{1}{2}x - 3 < 1$$

$$3x + 1 < 10 \quad \textit{or} \quad 2x - 1 \geq 11$$

These compound inequalities are actually two inequalities in one statement joined by the word “and” or by the word “or.” For example,

$$-3 < 2x + 5 < 17$$

is a compound inequality because it can be decomposed as follows:

$$-3 < 2x + 5 \quad \text{and} \quad 2x + 5 < 17$$

Solve each inequality individually, and the intersection of the two solution sets solves the original compound inequality. While this method works, there is another method that usually requires fewer steps. Apply the properties of this section to all three parts of the compound inequality with the goal of ***isolating the variable in the middle*** of the statement to determine the bounds of the solution set.

Example 2.8.6

Solve:

$$-3 < 2x + 5 < 17.$$

Solution:

$$\begin{gathered} -3 < 2x + 5 < 17 \\ -3 \color{blue}{-5} < 2x + 5 \color{blue}{-5} < 17 \color{blue}{-5} \\ -8 < 2x < 12 \\ \tfrac{-8}{\color{blue}{2}} < \tfrac{2x}{\color{blue}{2}} < \tfrac{12}{\color{blue}{2}} \\ -4 < x < 6 \end{gathered}$$

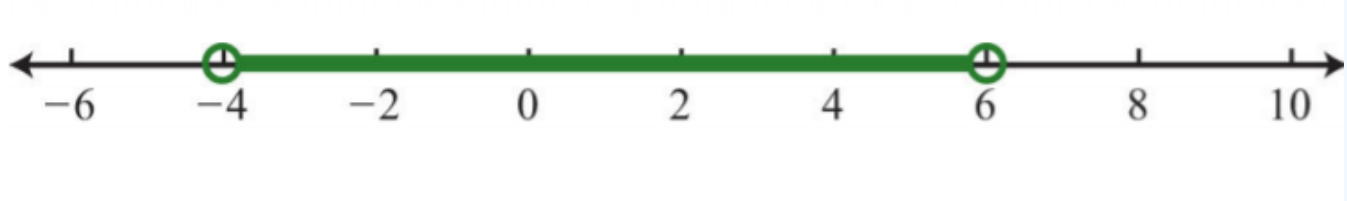


Figure 2.8.5

Answer:

Interval notation: $(-4, 6)$

Example 2.8.7

Solve:

$$-1 \leq \tfrac{1}{2}x - 3 < 1.$$

Solution:

$$\begin{gathered} -1 \leq \tfrac{1}{2}x - 3 < 1 \\ -1 \color{blue}{+3} \leq \tfrac{1}{2}x - 3 \color{blue}{+3} < 1 \color{blue}{+3} \\ 2 \leq \tfrac{1}{2}x < 4 \\ \color{blue}{2 \cdot} 2 \leq \color{blue}{2 \cdot} \tfrac{1}{2}x < \color{blue}{2 \cdot} 4 \\ 4 \leq x < 8 \end{gathered}$$

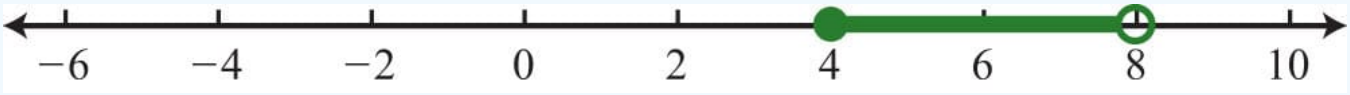


Figure 2.8.6

Answer:

Interval notation: $[4, 8)$

It is important to note that when multiplying or dividing all three parts of a compound inequality by a negative number, you must reverse all of the inequalities in the statement. For example,

$$-10 < -2x < 20$$
$$\frac{-10}{-2} > \frac{-2x}{-2} > \frac{20}{-2}$$
$$5 > x > -10$$

The answer above can be written in an equivalent form, where smaller numbers lie to the left and the larger numbers lie to the right, as they appear on a number line.

$$-10 < x < 5$$

Using interval notation, write $(-10, 5)$.

Exercise 2.8.2

Solve:

$$-8 \leq 2(-3x + 5) < 34.$$

Answer available at the end of this section.

For compound inequalities with the word "or" you ***must*** work both inequalities separately and then consider the union of the solution sets. Values in this union solve either inequality.

Example 2.8.8

Solve:

$$3x + 1 < 10 \text{ or } 2x - 1 \geq 11.$$

Solution:

Solve each inequality and form the union by combining the solution sets.

$$\begin{array}{rcl} 3x + 1 < 10 & & 2x - 1 \geq 11 \\ 3x + 1 - 1 < 10 - 1 & & 2x - 1 + 1 \geq 11 + 1 \\ 3x < 9 & \text{or} & 2x \geq 12 \\ \frac{3x}{3} < \frac{9}{3} & & \frac{2x}{2} \geq \frac{12}{2} \\ x < 3 & & x \geq 6 \end{array}$$

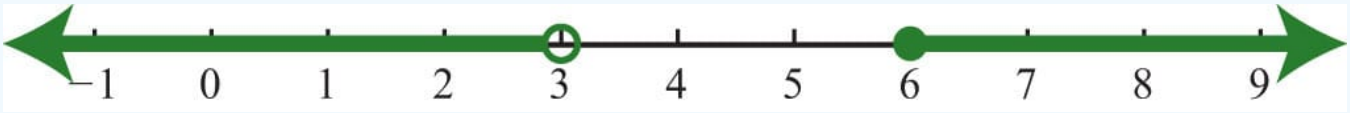


Figure 2.8.7

Answer:

Interval notation: $(-\infty, 3) \cup [6, \infty)$

Exercise 2.8.3

Solve:

$$4x - 1 < -5 \text{ or } 4x - 1 > 5.$$

Answer available at the end of this section.

Applications of Linear Inequalities

Some of the key words and phrases that indicate inequalities are summarized below:

Table 2.8.1

Key Phrases	Translation
A number is **at least** 5.	$x \geq 5$
A number is 5 **or more inclusive**.	
A number is **at most** 3.	$x \leq 3$
A number is 3 **or less inclusive**.	
A number is **strictly less than** 4.	$x < 4$
A number is **less than** 4, **noninclusive**.	
A number is **greater than** 7.	$x > 7$
A number is **more than** 7, **noninclusive**.	
A number is **in between** 2 and 10.	$2 < x < 10$
A number is at least 5 **and** at most 15.	$5 \leq x \leq 15$
A number may **range** from 5 to 15.	

As with all applications, carefully read the problem several times and look for key words and phrases. Identify the unknowns and assign variables. Next, translate the wording into a mathematical inequality. Finally, use the properties you have learned to solve the inequality and express the solution graphically or in interval notation.

Example 2.8.9

Translate:

Five less than twice a number is at most 25.

Solution:

First, choose a variable for the unknown number and identify the key words and phrases.

$$\overbrace{2n}^{\textit{twice a number}} \quad \overbrace{-5}^{\textit{five less than}} \quad \overbrace{\leq}^{\textit{is at most}} \quad 25$$

Answer:

$$2n - 5 \leq 25.$$

The key phrase “is at most” indicates that the quantity has a maximum value of 25 or smaller.

Example 2.8.10

The temperature in the desert can range from $10\ °C$ to $45\ °C$ in one 24-hour period. Find the equivalent range in degrees Fahrenheit, F, given that $C = \frac{5}{9}(F - 32)$.

Solution:

Set up a compound inequality where the temperature in Celsius is inclusively between $10\ °C$ and $45\ °C$. Then substitute the expression equivalent to the Celsius temperature in the inequality and solve for F.

$$\begin{aligned} 10°C \leq\ & temperature\ in\ Celsius \leq 45°C \\ 10 \leq\ & \tfrac{5}{9}(F - 32) \leq 45 \\ \tfrac{9}{5}\cdot 10 \leq\ & \tfrac{9}{5}\cdot\tfrac{5}{9}(F - 32) \leq \tfrac{9}{5}\cdot 45 \\ 18 \leq\ & F - 32 \leq 81 \\ 18+32 \leq\ & F - 32 + 32 \leq 81 + 32 \\ 50 \leq\ & F \leq 113 \end{aligned}$$

Answer:

The equivalent Fahrenheit range is from $50\ °F$ to $113\ °F$.

Example 2.8.11

In the first four events of a meet, a gymnast scores $7.5, 8.2, 8.5$, and 9.0. What must she score on the fifth event to average at least 8.5?

Solution:

The average must be at least 8.5; this means that the average must be greater than or equal to 8.5.

$$\begin{aligned} average &\geq 5 \\ \frac{7.5 + 8.2 + 8.5 + 9.0 + x}{5} &\geq 8.5 \\ \frac{33.2 + x}{5} &\geq 8.5 \\ 5\cdot\frac{33.2 + x}{5} &\geq 5\cdot 8.5 \qquad \textit{Multiply both sides by } 5. \\ 33.2 + x &\geq 42.5 \\ 33.2 + x - 33.2 &\geq 42.5 - 33.2 \\ x &\geq 9.3 \end{aligned}$$

Answer:

She must score at least 9.3 on the fifth event.

Key Takeaways

- Inequalities typically have infinitely many solutions. The solutions are presented graphically on a number line or using interval notation or both.
- All but one of the rules for solving linear inequalities are the same as for solving linear equations. If you divide or multiply an inequality by a negative number, reverse the inequality to obtain an equivalent inequality.
- Compound inequalities involving the word “or” require us to solve each inequality and form the union of each solution set. These are the values that solve at least one of the given inequalities.
- Compound inequalities involving the word “and” require the intersection of the solution sets for each inequality. These are the values that solve both or all of the given inequalities.

- The general guidelines for solving word problems apply to applications involving inequalities. Be aware of a new list of key words and phrases that indicate a mathematical setup involving inequalities.

Exercise 2.8.4 Checking for Solutions

Determine whether the given number is a solution to the given inequality.

1. $2x - 3 < 6; x = -1$
2. $-3x + 1 \leq 0; x = -2$
3. $5x - 20 > 0; x = 3$
4. $\frac{1}{2}x + 1 > -\frac{3}{4}; x = -\frac{1}{4}$
5. $-5 < 7x + 1 < 9; x = 0$
6. $-20 \leq -3x - 5 \leq -10; x = 5$
7. $x < -3 \text{ or } x > 3; x = -10$
8. $x < 0 \text{ or } x \geq 1; x = \frac{1}{2}$
9. $2x + 1 < -3 \text{ or } 2x + 1 \geq 5; x = 2$
10. $4x - 1 < -17 \text{ or } 3x + 2 \geq 6; x = 1$

Answer available at the end of this section.

Exercise 2.8.5 Solving Linear Inequalities

Solve and graph the solution set. In addition, present the solution set in interval notation.

1. $x + 5 > 1$
2. $x - 3 < -4$
3. $6x \leq 24$
4. $4x > -8$
5. $-7x \leq 14$
6. $-2x + 5 > 9$
7. $7x - 3 \leq 25$
8. $12x + 7 > -53$
9. $-2x + 5 < -7$
10. $-2x + 4 \leq 4$
11. $-15x + 10 > 20$
12. $-8x + 1 \leq 29$
13. $\frac{1}{7}x - 3 < 1$
14. $\frac{1}{2}x - \frac{1}{3} > \frac{2}{3}$
15. $\frac{5}{3}x + \frac{1}{2} \leq \frac{1}{3}$
16. $-\frac{3}{4}x - \frac{1}{2} \geq \frac{5}{2}$
17. $-\frac{1}{5}x + \frac{3}{4} < -\frac{1}{5}$
18. $-\frac{2}{3}x + 1 < -3$
19. $2(-3x + 1) < 14$
20. $-7(x - 2) + 1 < 15$
21. $9x - 3(3x + 4) > -12$
22. $12x - 4(3x + 5) \leq -2$
23. $5 - 3(2x - 6) \geq -1$
24. $9x - (10x - 12) < 22$
25. $2(x - 7) - 3(x + 3) \leq -3$
26. $5x - 3 > 3x + 7$
27. $4(3x - 2) \leq -2(x + 3) + 12$
28. $5(x - 3) \geq 15x - (10x + 4)$
29. $12x + 1 > 2(6x - 3) - 5$
30. $3(x - 2) + 5 > 2(3x + 5) + 2$
31. $-4(3x - 1) + 2x \leq 2(4x - 1) - 3$

32. $-2(x-2)+14x<7(2x+1)$

Answer available at the end of this section.

Exercise 2.8.6 Solving Linear Inequalities

Set up an algebraic inequality and then solve it.

1. The sum of three times a number and 4 is greater than negative 8.
2. The sum of 7 and three times a number is less than or equal to 1.
3. When a number is subtracted from 10, the result is at most 12.
4. When 5 times a number is subtracted from 6, the result is at least 26.
5. If five is added to three times a number, then the result is less than twenty.
6. If three is subtracted from two times a number, then the result is greater than or equal to nine.
7. Bill earns $\$12.00$ for the day plus $\$0.25$ for every person he gets to register to vote. How many people must he register to earn at least $\$50.00$ for the day?
8. With a golf club membership costing $\$100$ per month, each round of golf costs only $\$25.00$. How many rounds of golf can a member play if he wishes to keep his costs to $\$250$ per month at most?
9. Joe earned scores of $72, 85$, and 75 on his first three algebra exams. What must he score on the fourth exam to average at least 80?
10. Maurice earned $4, 7$, and 9 points out of 10 on the first three quizzes. What must he score on the fourth quiz to average at least 7?
11. A computer is set to shut down if the temperature exceeds $40\ ^\circ C$. Give an equivalent statement using degrees Fahrenheit. (Hint: $C=\frac{5}{9}(F-32)$.)
12. A certain brand of makeup is guaranteed not to run if the temperature is less than $35\ ^\circ C$. Give an equivalent statement using degrees Fahrenheit.

Answer available at the end of this section.

Exercise 2.8.7 Compound Inequalities

Solve and graph the solution set. In addition, present the solution set in interval notation.

1. $-1<x+3<5$
2. $-10\le 5x<20$
3. $-2\le 4x+6<10$
4. $-10\le 3x-1\le -4$
5. $-15<3x-6\le 6$
6. $-22<5x+3\le 3$
7. $-1\le \frac{1}{2}x-5\le 1$
8. $1<8x+5<5$
9. $-\frac{1}{5}\le \frac{2}{3}x-\frac{1}{5}<\frac{4}{5}$
10. $-\frac{1}{2}<\frac{3}{4}x-\frac{2}{3}\le \frac{1}{2}$
11. $-3\le 3(x-1)\le 3$
12. $-12<6(x-3)\le 0$
13. $4<-2(x+3)<6$
14. $-5\le 5(-x+1)<15$
15. $-\frac{3}{2}\le \frac{1}{4}(\frac{1}{2}x-1)+\frac{3}{4}<\frac{3}{2}$
16. $-4\le -\frac{1}{3}(3x+12)<4$
17. $-2\le 12-2(x-3)\le 20$
18. $-5<2(x-1)-3(x+2)<5$
19. $3x\le -15$ or $2x>6$
20. $4x-1<-17$ or $3x+2\ge 8$
21. $-2x+1<-1$ or $-2x+1>1$
22. $7x+4\le 4$ or $6x-5\ge 1$

23. $3x - 7 < 14$ or $2x + 3 > 7$
24. $-3x + 1 < -5$ or $-4x - 3 > -23$
25. $\frac{1}{2}x - 2 < -1$ or $\frac{1}{2}x - 2 > 1$
26. $\frac{1}{3}x + 3 \geq -2$ or $\frac{1}{3}x + 3 \leq 2$
27. $3x + 7 \leq 7$ or $-5x + 6 > 6$
28. $-10x - 3 \leq 17$ or $20x - 6 > -26$
29. $2x - 10 < -2$ or $-3x + 4 > -5$
30. $5x + 3 < 4$ or $5 - 10x > 4$
31. $3x < 18$ and $5x > -20$
32. $x + 7 \leq 5$ and $x - 3 \geq -10$
33. $2x - 1 < 5$ and $3x - 1 < 10$
34. $5x + 2 < -13$ and $3x + 4 > 13$

Answer available at the end of this section.

Exercise 2.8.8 Compound Inequalities

Set up a compound inequality for the following and then solve.

1. Five more than two times some number is between 15 and 25.
2. Four subtracted from three times some number is between -4 and 14.
3. Clint wishes to earn a B, which is at least 80 but less than 90. What range must he score on the fourth exam if the first three were $65, 75$, and 90?
4. A certain antifreeze is effective for a temperature range of $-35\ ^\circ C$ to $120\ ^\circ C$. Find the equivalent range in degrees Fahrenheit.
5. The average temperature in London ranges from $23\ ^\circ C$ in the summer to $14\ ^\circ C$ in the winter. Find the equivalent range in degrees Fahrenheit.
6. If the base of a triangle measures 5 inches, then in what range must the height be for the area to be between 10 square inches and 20 square inches?
7. A rectangle has a length of 7 inches. Find all possible widths if the area is to be at least 14 square inches and at most 28 square inches.
8. A rectangle has a width of 3 centimeters. Find all possible lengths, if the perimeter must be at least 12 centimeters and at most 26 centimeters.
9. The perimeter of a square must be between 40 feet and 200 feet. Find the length of all possible sides that satisfy this condition.
10. If two times an angle is between 180 degrees and 270 degrees, then what are the bounds of the original angle?
11. If three times an angle is between 270 degrees and 360 degrees then what are the bounds of the original angle?

Answer available at the end of this section.

Exercise 2.8.9 Discussion Board Topics

1. Research and discuss the use of set-builder notation with intersections and unions.
2. Can we combine logical “or” into one statement like we do for logical “and”?

Answer available at the end of this section.

Answers

Exercise 2.8.1

$[-4, \infty)$

Exercise 2.8.2

$(-4, 3]$

Exercise 2.8.3

$(-\infty, -1) \cup (\frac{3}{2}, \infty)$

Exercise 2.8.4 **Checking for Solutions**

1. Yes
3. No
5. Yes
7. Yes
9. Yes

Exercise 2.8.5 **Solving Linear Inequalities**

1. $x > -4; (-4, \infty)$

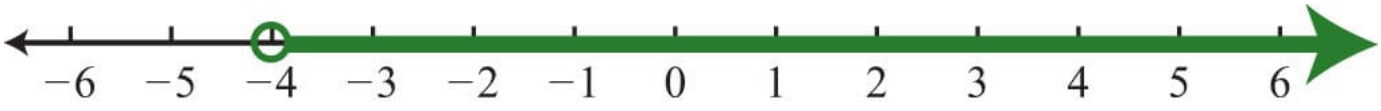


Figure 2.8.8

3. $x \leq 4; (-\infty, 4]$



Figure 2.8.9

5. $x \geq -2; [-2, \infty)$

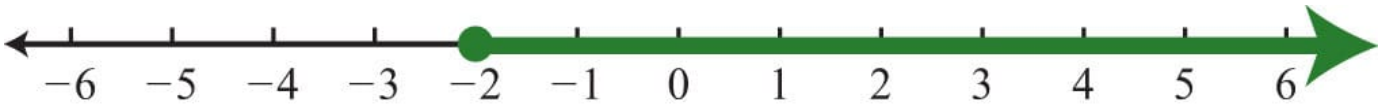


Figure 2.8.10

7. $x \leq 4; (-\infty, 4]$



Figure 2.8.11

9. $x > 6; (6, \infty)$

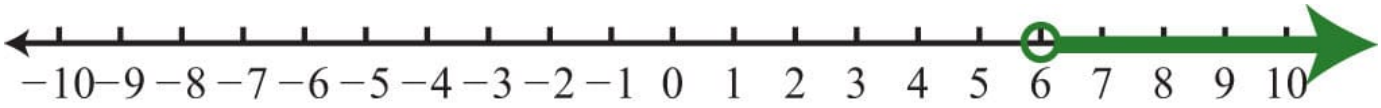


Figure 2.8.12

11. $x < -\frac{2}{3}; (-\infty, -\frac{2}{3})$

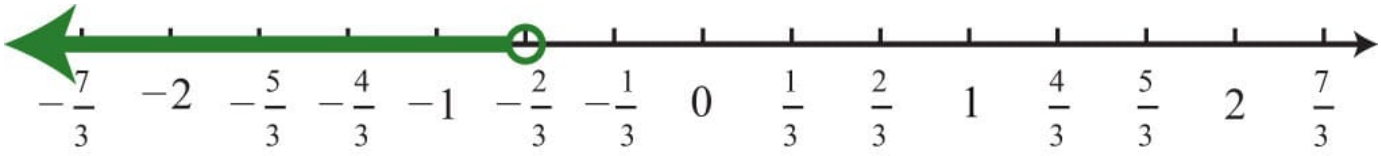


Figure 2.8.13

13. $x < 28; (-\infty, 28)$



Figure 2.8.14

15. $x \leq -\frac{1}{10}; (-\infty, -\frac{1}{10}]$

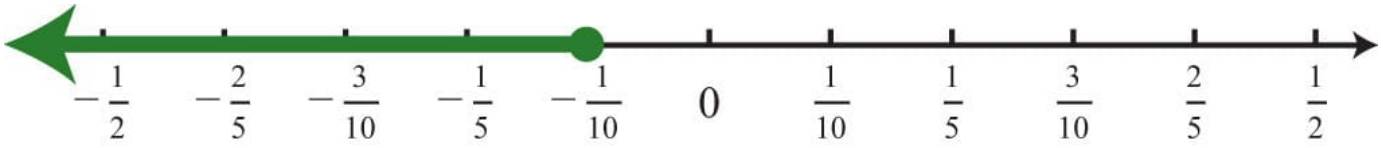


Figure 2.8.15

17. $x > \frac{19}{4}; (\frac{19}{4}, \infty)$

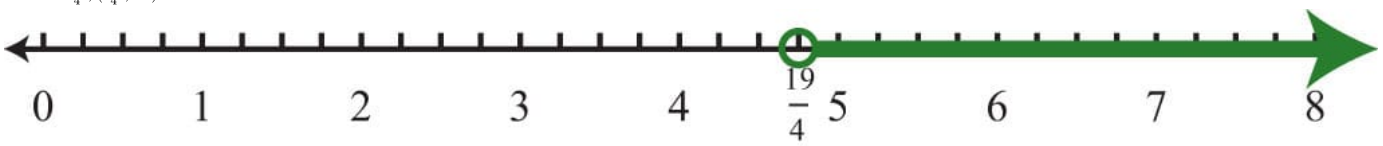


Figure 2.8.16

19. $x > -2; (-2, \infty)$

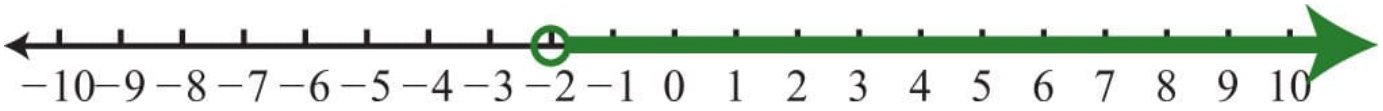


Figure 2.8.17

21. $\emptyset$

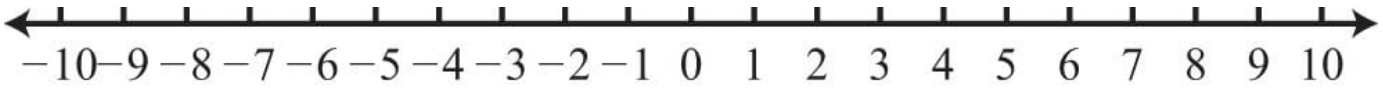


Figure 2.8.18

23. $x \leq 4; (-\infty, 4]$



Figure 2.8.19

25. $x \geq -20; [-20, \infty)$

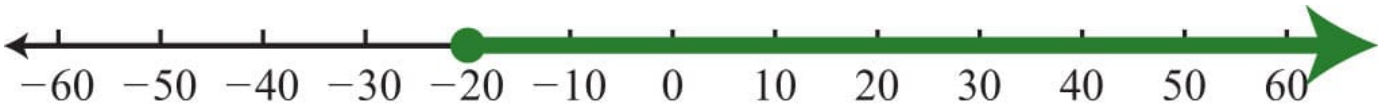


Figure 2.8.20

27. $x \leq 1; (-\infty, 1]$



Figure 2.8.21

29. R



Figure 2.8.22

31. $x \geq \frac{1}{2}; [\frac{1}{2}, \infty)$

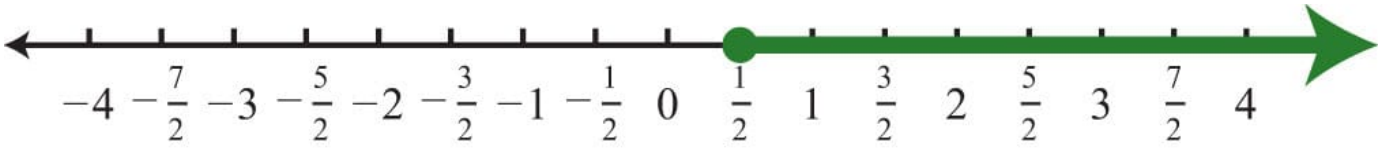


Figure 2.8.23

Exercise 2.8.6 **Solving Linear Inequalities**

1. $n > -4$
3. $n \geq -2$
5. $n < 5$
7. Bill must register at least 152 people.
9. Joe must earn at least an 88 on the fourth exam.
11. The computer will shut down when the temperature exceeds 104°F.

Exercise 2.8.7 **Compound Inequalities**

1. $-4 < x < 2; (-4, 2)$



Figure 2.8.24

3. $-2 \leq x < 1; [-2, 1)$

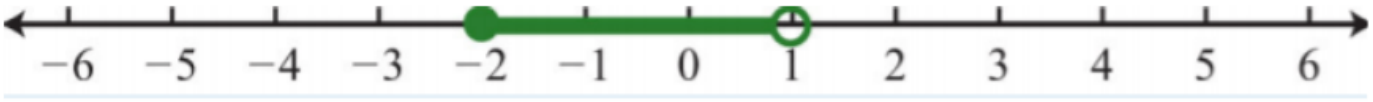


Figure 2.8.25

5. $-3 < x \leq 4; (-3, 4]$



Figure 2.8.26

7. $8 \leq x \leq 12; [8, 12]$



Figure 2.8.27

9. $0 \leq x < \frac{3}{2}; [0, \frac{3}{2})$

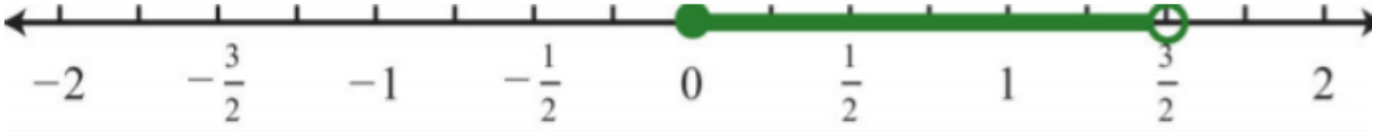


Figure 2.8.28

11. $0 \leq x \leq 2; [0, 2]$

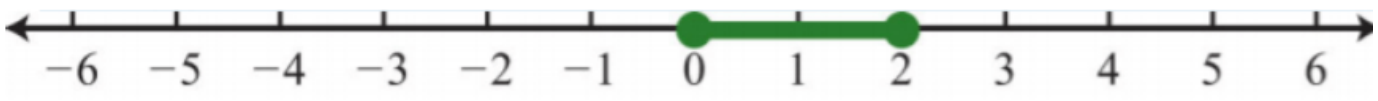


Figure 2.8.29

13. $-6 < x < -5; (-6, -5)$



Figure 2.8.30

15. $-16 \leq x < 8; [-16, 8)$

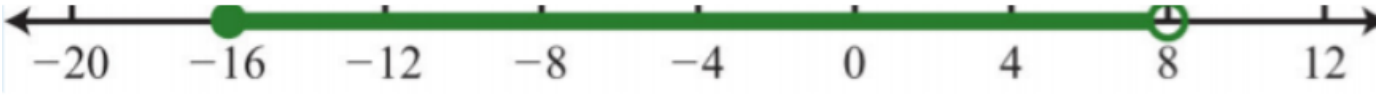


Figure 2.8.31

17. $-1 \leq x \leq 10; [-1, 10]$



Figure 2.8.32

19. $x \leq -5$ or $x > 3; (-\infty, -5] \cup (3, \infty)$



Figure 2.8.33

21. $x > 1$ or $x < 0; (-\infty, 0) \cup (1, \infty)$



Figure 2.8.34

23. R



Figure 2.8.35

25. $x < 2$ or $x > 6; (-\infty, 2) \cup (6, \infty)$



Figure 2.8.36

27. $x \leq 0; (-\infty, 0]$



Figure 2.8.37

29. $x < 4; (-\infty, 4)$



Figure 2.8.38

31. $-4 < x < 6; (-4, 6)$



Figure 2.8.39

33. $x < 3; (-\infty, 3)$



Figure 2.8.40

Exercise 2.8.8 **Compound Inequalities**

1. $5 < n < 20$
3. Clint must earn a score in the range from 90 to 100.
5. The average temperature in London ranges from $57.2\,^{\circ}F$ to $73.4\,^{\circ}F$.
7. The width must be at least 2 inches and at most 4 inches.

9. Sides must be between 10 feet and 50 feet.

11. The angle is between 90 degrees and 120 degrees.

Exercise $2.8.9$ **Discussion Board Topics**

2. Answers may vary

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2.E: Review Exercises and Sample Exam

Review Exercises

Exercise $2.E.1$ Introduction to Algebra

Evaluate.

1. $2x + 7$, where $x = -4$
2. $-4x + 1$, where $x = -2$
3. $\frac{2}{3}y - \frac{1}{2}$, where $y = \frac{3}{5}$
4. $-\frac{3}{4}y + \frac{5}{3}$, where $y = \frac{2}{3}$
5. $b^2 - 4ac$, where $a = 5, b = -2$, and $c = \frac{1}{2}$
6. $b^2 - 4ac$, where $a = -\frac{1}{4}, b = -1$, and $c = -3$
7. $2x^2 - x + 3$, where $x = -3$
8. $5x^2 - 2x + 4$, where $x = -1$
9. Calculate the simple interest earned on a 3-year investment of $\$750$ at an annual interest rate of 8%.
10. A bus traveled for $1\frac{2}{3}$ hours at an average speed of 48 miles per hour. What distance did the bus travel?
11. Calculate the area of a rectangle with dimensions $4\frac{1}{2}$ feet by 6 feet.
12. Calculate the volume of a rectangular box with dimensions $4\frac{1}{2}$ feet by 6 feet by 1 foot.

Answer available at the end of this section.

Exercise $2.E.2$ Simplifying Algebraic Expressions

Multiply.

1. $-5(3x - 2)$
2. $(6x - 9) \cdot 3$
3. $\frac{3}{4}(4x^2 - 8x + 32)$
4. $-20(\frac{1}{10}x^2 - \frac{2}{5}x - \frac{5}{4})$
5. $-(3a - 2b + 5c - 1)$
6. $-6(y^3 + 3y^2 - 7y + 5)$

Answer available at the end of this section.

Exercise $2.E.3$ Simplifying Algebraic Expressions

Simplify.

1. $5a - 7b - 3a + 5b$
2. $6x^2 - 4x + 7x^2 - 3x$
3. $\frac{3}{5}xy + \frac{1}{2} - \frac{1}{10}xy - \frac{1}{4}$
4. $-\frac{3}{4}a - \frac{4}{21}b + \frac{1}{3}a - \frac{1}{7}b$
5. $a^2b + 2ab^2 - 7a^2b + 9ab^2$
6. $y^2 - 3y + 5 - y^2 + 9$
7. $-8(8x - 3) - 7$
8. $7 - (6x - 9)$
9. $2(3x^2 - 2x + 1) - (5x - 7)$
10. $(2y^2 + 6y - 8) - (5y^2 - 12y + 1)$
11. $6 - 3(a - 2b) + 7(5a - 3b)$

12. $10-5(x^2-x+1)-(3x^2+5x-1)$
13. Subtract $5x-1$ from $2x-3$.
14. Subtract $x-3$ from twice the quantity $x-1$.

Answer available at the end of this section.

Exercise $2.E.4$ Solving Linear Equations: Part 1

Is the given value a solution to the linear equation?

1. $-x+3=-18; x=-15$
2. $4x-3=-3x; x=-2$
3. $8x+2=5x+1; x=-\frac{1}{3}$
4. $2x+4=3x-2; x=-1$

Answer available at the end of this section.

Exercise $2.E.5$ Solving Linear Equations: Part 1

Solve.

1. $y+23=25$
2. $-3x=54$
3. $\frac{x}{4}=8$
4. $\frac{5}{2}x=\frac{2}{3}$
5. $7x-5=-54$
6. $-2x+7=43$
7. $7x+3=0$
8. $4x+5=5$
9. $1=10-3x$
10. $10-5y=15$
11. $7-y=28$
12. $33-x=16$
13. $\frac{5}{6}x+\frac{1}{3}=\frac{3}{2}$
14. $-\frac{2}{3}y+\frac{1}{5}=-\frac{1}{3}$
15. The sum of $9x$ and 6 is 51.
16. The difference of $3x$ and 8 is 25.

Answer available at the end of this section.

Exercise $2.E.6$ Solving Linear Equations: Part II

Solve.

1. $5x-2=3x+6$
2. $7x+1=2x-29$
3. $14x+1=15x-11$
4. $6y-13=3+7y$
5. $8y+6-3y=22-3y$
6. $12-5y+6=y-6$
7. $5-2(7x-1)=2x+1$
8. $10-5(x-1)=5-x$
9. $2x-(3x-4)=7-x$
10. $9x-3(2x+1)=3x-3$

11. $2(5x-2)-3(2x+1)=5(x-3)$
12. $3(5x-1)-4(x-4)=-5(2x+10)$
13. $\frac{3}{2}(4x-3)+\frac{1}{4}=1$
14. $\frac{3}{4}-\frac{1}{6}(4x-9)=2$
15. $\frac{2}{3}(9x-3)+\frac{1}{2}=3(2x-\frac{1}{2})$
16. $1-\frac{5}{4}(4x-1)=5(\frac{1}{2}-x)$
17. The sum of $4x$ and 3 is equal to the difference of $7x$ and 8.
18. The difference of $5x$ and 1 is equal to the sum of $12x$ and 1.
19. Solve for x: $y=9x+1$
20. Solve for y: $5x+2y=3$
21. Solve for l: $P=2l+2w$
22. Solve for b: $A=\frac{1}{2}bh$

Answer available at the end of this section.

Exercise $2.E.7$ Applications of Linear Equations

1. A larger integer is 3 more than twice a smaller integer. If their sum is 39, then find the integers.
2. A larger integer is 5 more than 3 times a smaller integer. If their sum is 49, then find the integers.
3. The sum of three consecutive odd integers is 45. Find the integers.
4. The sum of three consecutive even integers is 72. Find the integers.
5. The sum of three consecutive integers is 60. Find the integers.
6. The length of a rectangle is 7 centimeters less than twice its width. If the perimeter measures 46 centimeters, then find the dimensions of the rectangle.
7. A triangle has sides whose measures are consecutive even integers. If the perimeter is 24 meters, then find the measure of each side.
8. The circumference of a circle measures 24π inches. Find the radius of the circle.
9. Mary invested $\$1,800$ in two different accounts. One account earned 3.5% simple interest and the other earned 4.8%. If the total interest after 1 year was $\$79.25$, then how much did she invest in each account?
10. James has $\$6$ in dimes and quarters. If he has 4 fewer quarters than he does dimes, then how many of each coin does he have?
11. Two brothers leave the house at the same time traveling in opposite directions. One averages 40 miles per hour and the other 36 miles per hour. How long does it take for the distance between them to reach 114 miles?
12. Driving to her grandmother's house, Jill made several stops and was only able to average 40 miles per hour. The return trip took 2 hours less time because she drove nonstop and was able to average 60 miles per hour. How long did it take Jill to drive home from her grandmother's house?

Answer available at the end of this section.

Exercise $2.E.8$ Ratio and Proportion Applications

Solve.

1. $\frac{3}{4}=\frac{n}{8}$
2. $\frac{7}{3}=\frac{2}{8n}$
3. $\frac{6}{n}=\frac{30}{11}$
4. $\frac{n}{5}=\frac{2}{3}$
5. $\frac{3n-1}{3}=\frac{1}{2}$
6. $\frac{4}{2n+5}=-\frac{1}{3}$
7. $-3=\frac{1}{n-1}$
8. $\frac{2}{n-6}=\frac{1}{2n+1}$
9. Find two numbers in the proportion 4 to 5 whose sum is 27.
10. A larger number is 2 less than twice a smaller number. If the two numbers are in the proportion 5 to 9, then find the numbers.

11. A recipe calls for $1\frac{1}{2}$ teaspoons of vanilla extract for every 3 cups of batter. How many teaspoons of vanilla extract should be used with 7 cups of batter?
12. The ratio of female to male employees at a certain bank is 4 to 5. If there are 80 female employees at the bank, then determine the total number of employees.

Answer available at the end of this section.

Exercise $2.E.9$ Ratio and Proportion Applications

If triangle ABC is similar to triangle RST, then find the remaining two sides given the following.

1. $a = 4, b = 9, c = 12$, and $s = 3$
2. $b = 7, c = 10, t = 15$, and $r = 6$
3. At the same time of day, a pole casts a 27-foot shadow and 4-foot boy casts a 6-foot shadow. Calculate the height of the pole.
4. An equilateral triangle with sides measuring 10 units is similar to another equilateral triangle with scale factor of $2 : 3$. Find the perimeter of the unknown triangle.

Answer available at the end of this section.

Exercise $2.E.10$ Introduction to Inequalities and Interval Notation

Graph all solutions on a number line and provide the corresponding interval notation.

1. $x < -1$
2. $x \leq 10$
3. $x \geq 0$
4. $x > -2$
5. $-\frac{1}{2} \leq x < \frac{3}{2}$
6. $-20 < x < 30$
7. $x < 5$ or $x \geq 15$
8. $x < 2$ or $x > 0$

Answer available at the end of this section.

Exercise $2.E.11$ Introduction to Inequalities and Interval Notation

Determine the inequality given the answers expressed in interval notation.

1. $(-\infty, 3)$
2. $[-4, \infty)$
3. $(-2, 2)$
4. $(-3, 8]$
5. $(-\infty, 1) \cup [3, \infty)$
6. $(-\infty, -8] \cup [8, \infty)$

Answer available at the end of this section.

Exercise $2.E.12$ Linear Inequalities (One Variable)

Solve and graph. In addition, present the solution set in interval notation.

1. $x + 2 > -1$

2. $-4x \geq 16$
3. $9x + 4 \leq -5$
4. $5x - 7 < 13$
5. $7x + 5 - 8x \geq 15$
6. $5x - 6 + 3x < 2 + 9x - 5$
7. $3x - (x - 4) > x + 4$
8. $3(2x - 1) - 3(x - 2) \leq 2(x + 4)$
9. $2 - 5(x - 4) > 12$
10. $3x - 5(x - 2) \geq 11 - 5x$
11. $-1 < 2x + 5 \leq 11$
12. $-2 \leq \frac{1}{4}x - \frac{7}{2} \leq 2$
13. $5x + 3 < -2$ or $6x - 5 \geq 7$
14. $20 - 3x \leq 5$ or $5 - 2x \geq 25$

Answer available at the end of this section.

Sample Exam

Exercise $2.E.13$

1. Evaluate $b^2 - 4ac$, where $a = -1, b = -2$ $and\backslash(c = \frac{1}{2}$.
2. Determine the area of a triangle given that the base measures 10 centimeters and the height measures 5 centimeters. $(A = \frac{1}{2}bh)$

Answer available at the end of this section.

Exercise $2.E.14$

Simplify.

1. $5 - 2(4x - 1)$
2. $\frac{1}{4}x - \frac{2}{3}y + \frac{1}{2}x - \frac{3}{5}y$
3. $(5a + 4ab - 2b) - (3a + 2ab - 3b)$
4. $3x - (x^2 + 5x - 1) + (x^2 - x + 4)$

Answer available at the end of this section.

Exercise $2.E.15$

Solve.

1. $2 - 5x = 27$
2. $\frac{1}{2}x - \frac{3}{4} = -\frac{1}{8}$
3. $5x - 7 = 3x - 5$
4. $3(y - 3) - (4y + 2) = 1$
5. $5(x - 2) - 3(x + 2) = 2x - 3$
6. $\frac{5}{8} = \frac{n}{32}$
7. $\frac{3}{n+1} = -\frac{6}{4}$
8. Solve for b: $A = a + 2b$.

Answer available at the end of this section.

Exercise $2.E.16$

Solve and graph the solution set. In addition, present the solution set in interval notation.

1. $2x + 3 > 23$
2. $5(-2x + 1) \leq 35$
3. $4(3x - 2) < 3(2x + 1) + 1$
4. $-9 \leq 3(x + 4) \leq 21$
5. $6(x - \frac{1}{3}) < -2$ or $\frac{1}{5}(x + 10) \geq 3$

Answer available at the end of this section.

Exercise $2.E.17$

1. An algebra student earns $75, 79$, and 89 points on the first three quizzes. What must she score on the fourth quiz to earn an average of at least 80?
2. The sum of three consecutive odd integers is 117. Find the integers.
3. The length of a rectangle is 6 inches less than twice the width. If the perimeter measures 39 inches, then find the dimensions of the rectangle.
4. Millie invested her $\$5,350$ savings in two accounts. One account earns 5% annual interest and the other earns 6.2% in annual interest. If she earned $\$317.30$ simple interest in 1 year, then how much was in each account?
5. Because of traffic, Joe was only able to drive an average of 42 miles per hour on the trip to a conference. He was able to average 63 miles per hour on the return trip and it took 1 hour less time. How long did it take Joe to drive home from the conference?
6. A graphic designer wishes to crop an image in the width-to-height ratio of $3 : 2$. If the height is required to be 400 pixels, then to how many pixels should the width be set?

Answer available at the end of this section.

Answers

Exercise $2.E.1$ Introduction to Algebra

1. -1
3. $-\frac{1}{10}$
5. -6
7. 24
9. $\$180$
11. 27 square feet

Exercise $2.E.2$ Simplifying Algebraic Expressions

1. $-15x + 10$
3. $3x^2 - 6x + 24$
5. $-3a + 2b - 5c + 1$

Exercise $2.E.3$ Simplifying Algebraic Expressions

1. $2a - 2b$
3. $\frac{1}{2}xy + \frac{1}{4}$
5. $-6a^2b + 11ab^2$
7. $-64x + 17$
9. $6x^2 - 9x + 9$
11. $32a - 15b + 6$
13. $-3x - 2$

Exercise $2.E.4$ Solving Linear Equations: Part 1

1. No
3. Yes

Exercise $2.E.5$ Solving Linear Equations: Part 1

1. 2
3. 32

5. -7
7. $-\frac{3}{7}$
9. 3
11. -21
13. $\frac{7}{5}$
15. 5

Exercise $2.E.6$ **Solving Linear Equations: Part II**

1. 4
3. 12
5. 2
7. $\frac{3}{8}$
9. $\emptyset$
11. 8
13. $\frac{7}{8}$
15. R
17. $\frac{11}{3}$
19. $x = \frac{y-1}{9}$
21. $l = \frac{P-2w}{2}$

Exercise $2.E.7$ **Applications of Linear Equations**

1. $12, 27$
3. $13, 15, 17$
5. $19, 20, 21$
7. 6 meters, 8 meters, 10 meters
9. Mary invested \$$550$ at 3.5% and \$$1,250$ at 4.8%.
11. They will be 114 miles apart in $1\frac{1}{2}$ hours.

Exercise $2.E.8$ **Ratio and Proportion Applications**

1. 6
3. $\frac{11}{5}$
5. $\frac{5}{6}$
7. $\frac{2}{3}$
9. $12, 15$
11. $3\frac{1}{2}$ teaspoons

Exercise $2.E.9$ **Ratio and Proportion Applications**

1. $t = 4, r = \frac{4}{3}$
3. 18 feet

Exercise $2.E.10$ **Introduction to Inequalities and Interval Notation**

1. $(-\infty, -1)$



Figure 2.E.1

3. $[0, \infty)$



Figure 2.E.2

5. $[-\frac{1}{2}, \frac{3}{2})$

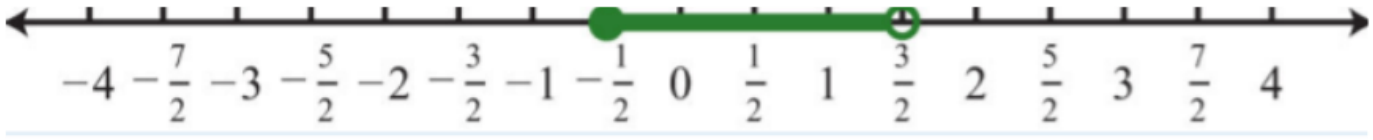


Figure 2.E.3

7. $(-\infty, 5) \cup [15, \infty)$



Figure 2.E.4

Exercise $2.E.11$ **Introduction to Inequalities and Interval Notation**

1. $x < 3$
3. $-2 < x < 2$
5. $x < 1$ or $x \geq 3$

Exercise $2.E.12$ **Linear Inequalities (One Variable)**

1. $x > -3; (-3, \infty)$

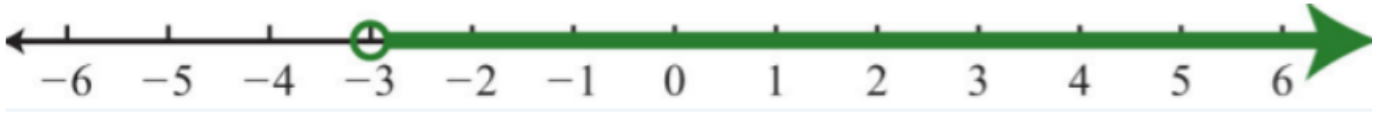


Figure 2.E.5

3. $x \leq -1; (-\infty, -1]$

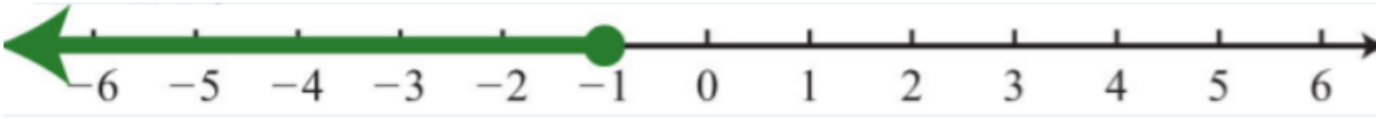


Figure 2.E.6

5. $x \leq -10; (-\infty, -10]$

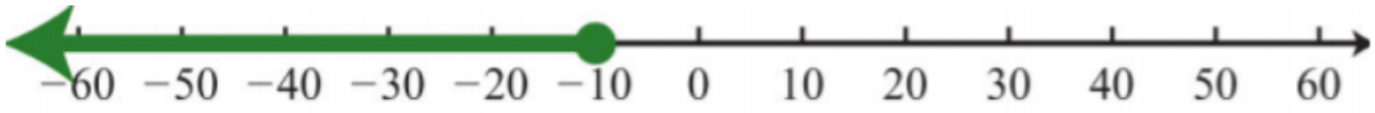


Figure 2.E.7

7. $x > 0; (0, \infty)$

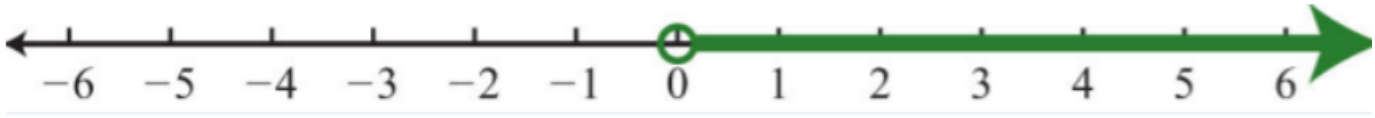


Figure 2.E.8

9. $x < 2; (-\infty, 2)$



Figure 2.E.9

11. $-3 < x \leq 3; (-3, 3]$

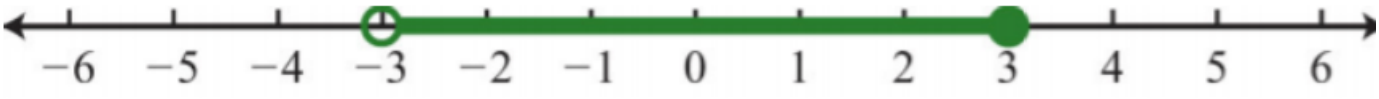


Figure 2.E.10

13. $x < -1 \text{ or } x \geq 2; (-\infty, -1) \cup [2, \infty)$

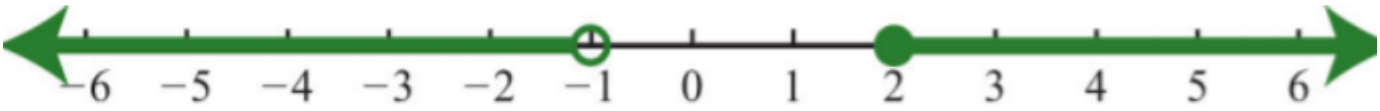


Figure 2.E.11

Exercise $2.E.13$

1. 6

Exercise $2.E.14$

1. $-8x + 7$
3. $2ab + 2a + b$

Exercise $2.E.15$

1. -5
3. 1
5. $\emptyset$
7. -3

Exercise $2.E.16$

1. $x > 10; (10, \infty)$



Figure 2.E.12

3. $x < 2; (-\infty, 2)$



Figure 2.E.13

5. $x < 0 \text{ or } x \geq 5; (-\infty, 0) \cup [5, \infty)\backslash)$



Figure 2.E.14

Exercise $2.E.17$

2. The three odd integers are $37, 39$, and 41.
4. Millie invested \$$1,200$ in the account earning 5% annual interest and \$$4,150$ in the account earning 6.2%.
6. The width should be set to 600 pixels.

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3: GRAPHING LINES

3.1: Rectangular Coordinate System

3.2: Graph by Plotting Points

3.3: Graph Using Intercepts

3.4: Graph Using the y-Intercept and Slope

3.5: Finding Linear Equations

3.6: Parallel and Perpendicular Lines

3.7: Introduction to Functions

3.8: Linear Inequalities (Two Variables)

3.E: Review Exercises and Sample Exam

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3.1: Rectangular Coordinate System

Learning Objectives

- Plot points using the rectangular coordinate system.
- Calculate the distance between any two points in the rectangular coordinate plane.
- Determine the midpoint between any two points.

Rectangular Coordinate System

The rectangular coordinate system consists of two real number lines that intersect at a right angle. The horizontal number line is called the x-axis, and the vertical number line is called the y-axis. These two number lines define a flat surface called a plane, and each point on this plane is associated with an ordered pair of real numbers (x, y). The first number is called the **x-coordinate**, and the second number is called the **y-coordinate**. The intersection of the two axes is known as the origin, which corresponds to the point $(0, 0)$.

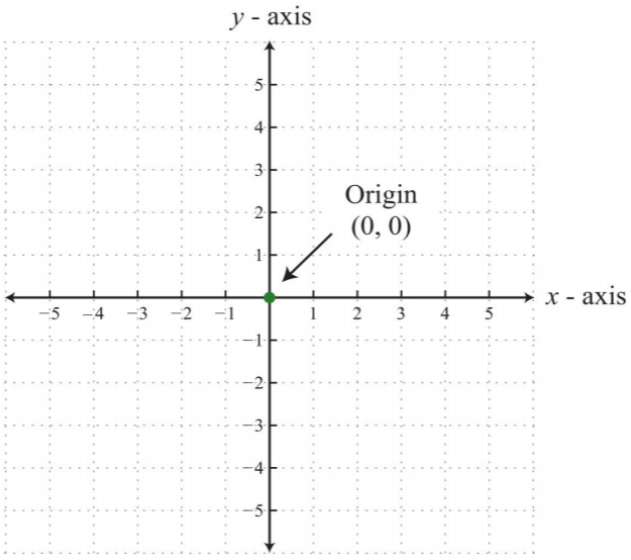


Figure 3.1.1

An ordered pair (x, y) represents the position of a point relative to the origin. The x-coordinate represents a position to the right of the origin if it is positive and to the left of the origin if it is negative. The y-coordinate represents a position above the origin if it is positive and below the origin if it is negative. Using this system, every position (point) in the plane is uniquely identified. For example, the pair $(2, 3)$ denotes the position relative to the origin as shown:

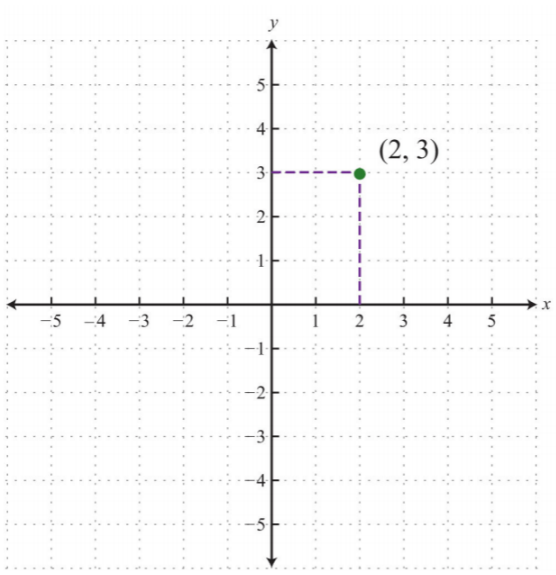


Figure 3.1.2

This system is often called the Cartesian coordinate system, named after the French mathematician René Descartes (1596–1650).

The x- and y-axes break the plane into four regions called quadrants, named using roman numerals I, II, III, and IV, as pictured. In quadrant I, both coordinates are positive. In quadrant II, the x-coordinate is negative and the y-coordinate is positive. In quadrant III, both coordinates are negative. In quadrant IV, the x-coordinate is positive and the y-coordinate is negative.

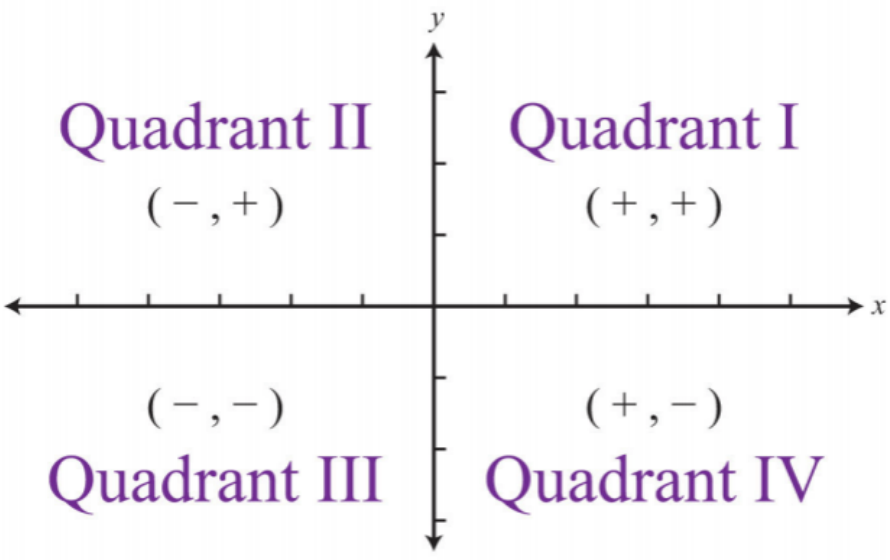


Figure 3.1.3

Example 3.1.1

Plot the ordered pair $(-3, 5)$ and determine the quadrant in which it lies.

Solution:

The coordinates $x = -3$ and $y = 5$ indicate a point 3 units to the left of and 5 units above the origin.

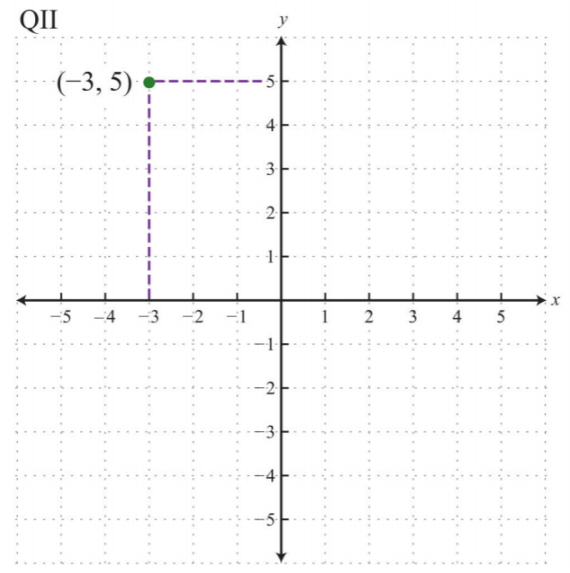


Figure 3.1.4

Answer:

The point is plotted in quadrant II (QII) because the x-coordinate is negative and the y-coordinate is positive.

Ordered pairs with 0 as one of the coordinates do not lie in a quadrant; these points are on one axis or the other (or the point is the origin if both coordinates are 0). Also, the scale indicated on the x-axis may be different from the scale indicated on the y-axis. Choose a scale that is convenient for the given situation.

Example 3.1.2

Plot this set of ordered pairs:

$$\{(4,0),(-6,0),(0,3),(-2,6),(-4,-6)\}.$$

Solution:

Each tick mark on the x-axis represents 2 units and each tick mark on the y-axis represents 3 units.

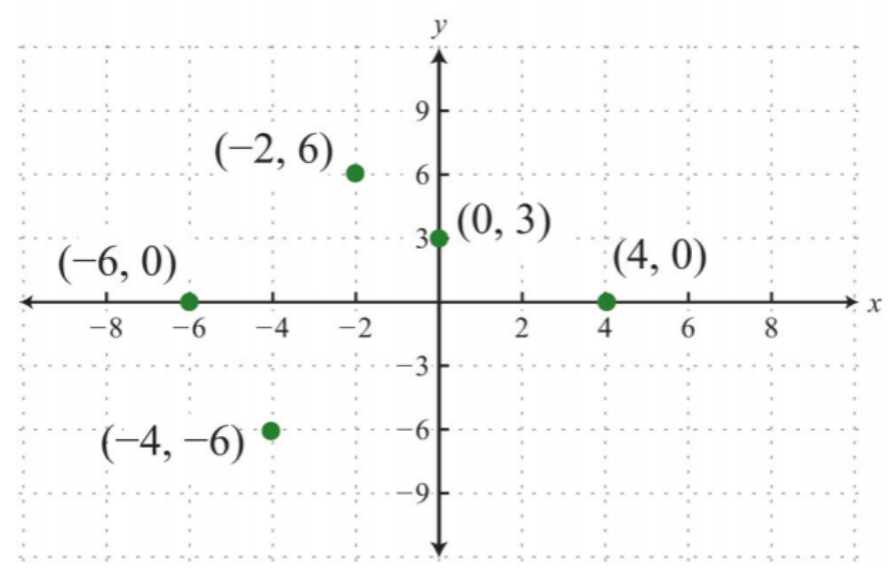


Figure 3.1.5

Example 3.1.3

Plot this set of ordered pairs:

$$\{(-6,-5),(-3,-3),(0,-1),(3,1),(6,3)\}.$$

Solution:

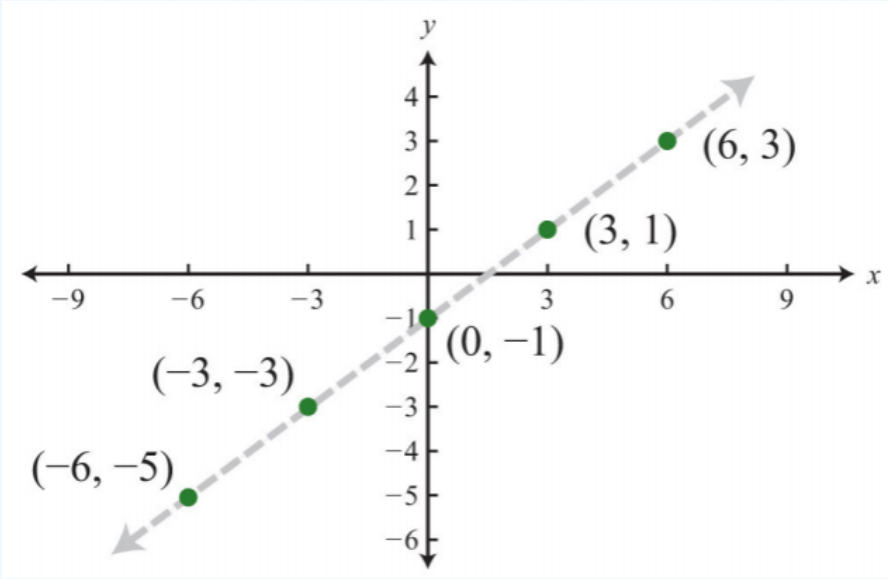


Figure 3.1.6

In this example, the points appear to be collinear, or to lie on the same line. The entire chapter focuses on finding and expressing points with this property.

Exercise 3.1.1

Plot the set of points $\{(5,3),(-3,2),(-2,-4),(4,-3)\}$ and indicate in which quadrant they lie.

Answer available at the end of this section.

Graphs are used in everyday life to display data visually. A line graph consists of a set of related data values graphed on a coordinate plane and connected by line segments. Typically, the independent quantity, such as time, is displayed on the x-axis and the dependent quantity, such as distance traveled, on the y-axis.

Example 3.1.4

The following line graph shows the number of mathematics and statistics bachelor's degrees awarded in the United States each year since 1970.

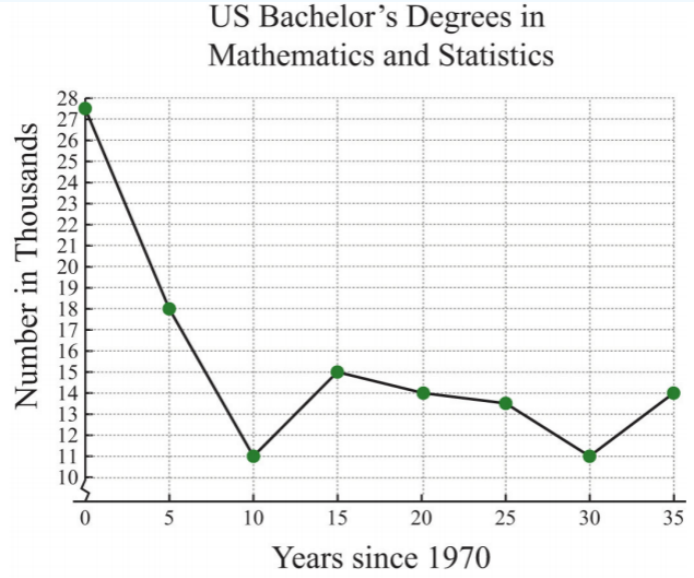


Figure 3.1.8: *Source: Digest of Education Statistics.*

a. How many mathematics and statistics bachelor's degrees were awarded in 1975?
b. In which years were the number of mathematics and statistics degrees awarded at the low of $11,000$?

Solution:

a. The scale on the x-axis represents time since 1970, so to determine the number of degrees awarded in 1975, read the y -value of the graph at $x = 5$.



Figure 3.1.9: *Source: Digest of Education Statistics.*

The y-value corresponding to $x = 5$ is 18. The graph indicates that this is in thousands; there were $18,000$ mathematics and statistics degrees awarded in 1975.

b. To find the year a particular number of degrees was awarded, first look at the y-axis. In this case, $11,000$ degrees is represented by 11 on the y-axis; look to the right to see in which years this occurred.

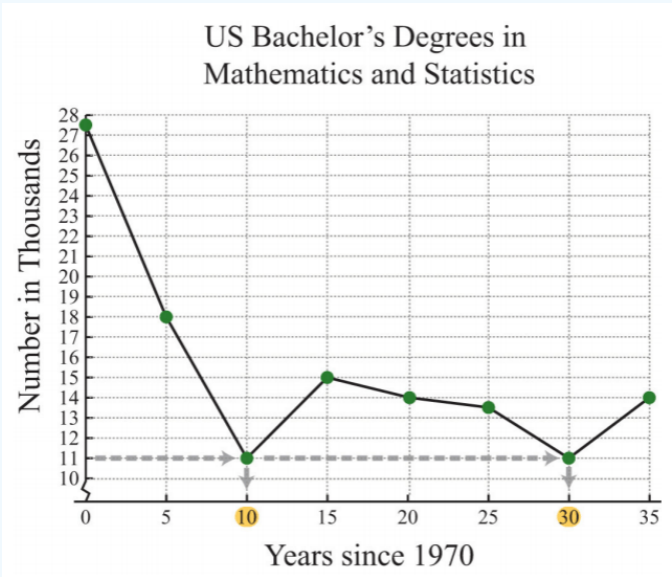


Figure 3.1.10: *Source: Digest of Education Statistics.*

The y-value of 11 occurs at two data points, one where $x = 10$ and the other where $x = 30$. These values correspond to the years 1980 and 2000, respectively.

Answers:

a. In the year 1975, $18,000$ mathematics and statistics degrees were awarded.
b. In the years 1980 and 2000, the lows of $11,000$ mathematics and statistics degrees were awarded.

Distance Formula

Frequently you need to calculate the distance between two points in a plane. To do this, form a right triangle using the two points as vertices of the triangle and then apply the Pythagorean theorem. Recall that the **Pythagorean theorem** states that if given any right triangle with legs measuring a and b units, then the square of the measure of the hypotenuse c is equal to the sum of the squares of the legs: $a^2 + b^2 = c^2$. In other words, the hypotenuse of any right triangle is equal to the square root of the sum of the squares of its legs.

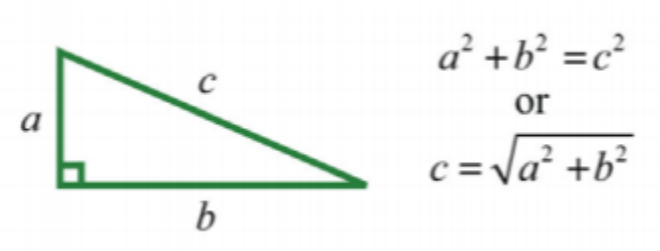


Figure 3.1.11

Example 3.1.5

Find the distance between $(-1, 2)$ and $(3, 5)$.

Solution:

Form a right triangle by drawing horizontal and vertical lines through the two points. This creates a right triangle as shown below:

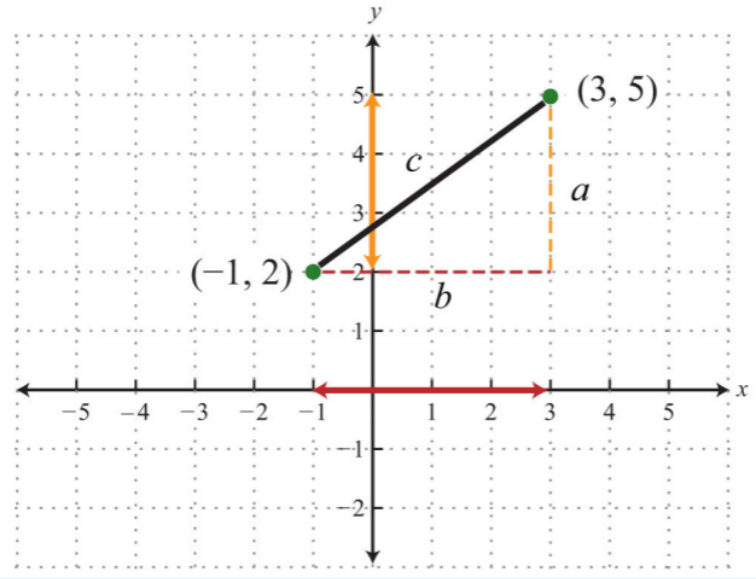


Figure 3.1.12

The length of leg b is calculated by finding the distance between the x-values of the given points, and the length of leg a is calculated by finding the distance between the given y-values.

$$a = 5 - 2 = 3\text{units}$$
$$b = 3 - (-1) = 3 + 1 = 4\text{units}$$

Next, use the Pythagorean theorem to find the length of the hypotenuse.

$$\begin{aligned} c &= \sqrt{3^2+4^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \text{ units} \end{aligned}$$

Answer:

The distance between the two points is 5 units.

Generalize this process to produce a formula that can be used to algebraically calculate the distance between any two given points.

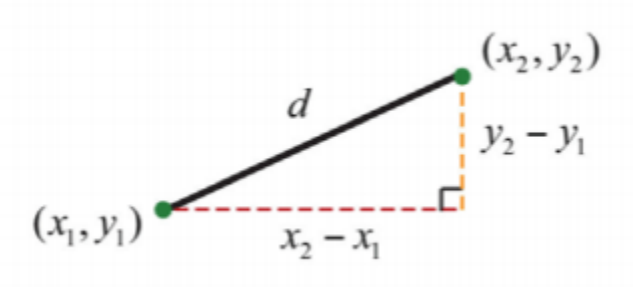


Figure 3.1.13

Given two points, (x_1, y_1) and (x_2, y_2), then the distance, d, between them is given by the distance formula:

$$\sqrt{(x_2+x_1)^2+(y_2+y_1)^2} \tag{3.1.1}$$

Example 3.1.6

Calculate the distance between $(-3,-1)$ and $(-2,4)$.

Solution:

Use the distance formula.

x_1, y_1	x_2, y_2
$(-3,-1)$	$(-2,4)$

It is a good practice to include the formula in its general form as a part of the written solution before substituting values for the variables. This improves readability and reduces the chance for errors.

$$\begin{aligned} d &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{(-2-(-3))^2+(4-(-1))^2} \\ &= \sqrt{(-2+3)^2+(4+1)^2} \\ &= \sqrt{(1)^2+(5)^2} \\ &= \sqrt{1+25} \\ &= \sqrt{26} \end{aligned}$$

Answer:

$\sqrt{26}$ units

Exercise 3.1.2

Calculate the distance between $(-7, 5)$ and $(-1, 13)$.

Answer available at the end of this section.

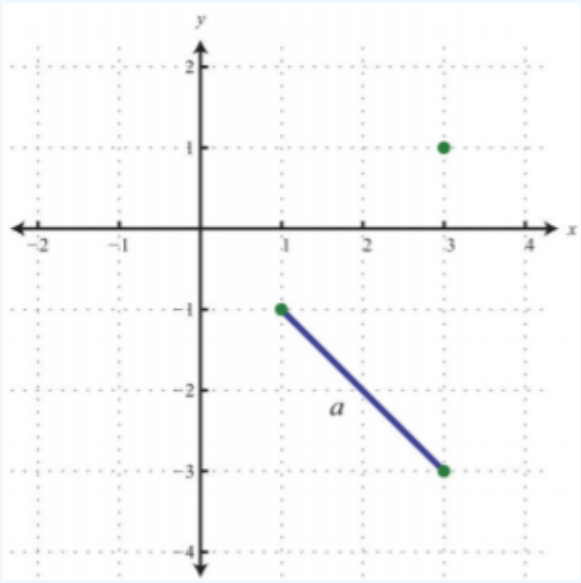
Example 3.1.7

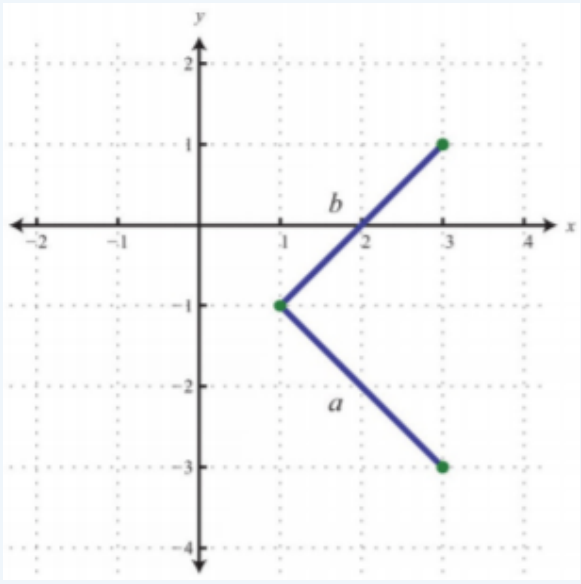
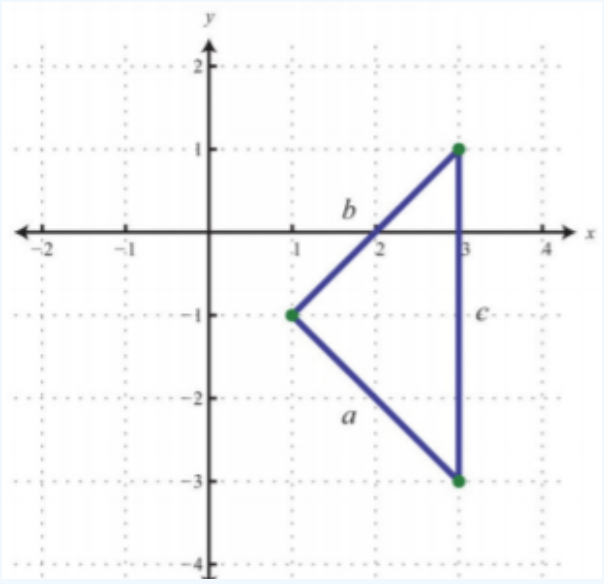
Do the three points $(1, -1), (3, -3)$, and $(3, 1)$ form a right triangle?

Solution:

The Pythagorean theorem states that having side lengths that satisfy the property $a^2 + b^2 = c^2$ is a necessary and sufficient condition of right triangles. In other words, if you can show that the sum of the squares of the leg lengths of the triangle is equal to the square of the length of the hypotenuse, then the figure must be a right triangle. First, calculate the length of each side using the distance formula.

Table 3.1.1

Geometry	Calculation
 Figure 3.1.14	Points: $(1, -1)$ and $(3, -3)$ $a = \sqrt{(3-1)^2 + (-3-(-1))^2}$ $= \sqrt{(a)^2 + (-3+1)^2}$ $= \sqrt{4 + (-2)^2}$ $= \sqrt{4+4}$ $= \sqrt{8}$

Geometry	Calculation
 Figure 3.1.15	Points: $(1,-1)$ and $(3,1)$ $b = \sqrt{(3-1)^2+(1-(-1))^2}$ $= \sqrt{2^2+(1+1)^2}$ $= \sqrt{4+(2)^2}$ $= \sqrt{4+4}$ $= \sqrt{8}$
 Figure 3.1.16	Points: $(3,-3)$ and $(3,1)$ $c = \sqrt{(3-3)^2+(1-(-3))^2}$ $= \sqrt{(0)^2+(1+3)^2}$ $= \sqrt{0+(4)^2}$ $= \sqrt{16}$ $= 4$

Now we check to see if $a^2+b^2=c^2$.

$$a^2+b^2=c^2$$

$$(\sqrt{8})^2+(\sqrt{8})^2=(4)^2$$
$$\sqrt{64}+\sqrt{64}=16$$
$$8+8=16$$
$$16=16 \quad \checkmark$$

Answer:

Yes, the three points form a right triangle. In fact, since two of the legs are equal in length, the points form an isosceles right triangle.

Midpoint Formula

The point that bisects the line segment formed by two points, (x_1, y_1) and (x_2, y_2), is called the midpoint and is given by the following formula:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \tag{3.1.2}$$

The midpoint is an ordered pair formed by finding the average of the x-values and the average of the y-values of the given points.

Example 3.1.8

Calculate the midpoint between $(-1, -2)$ and $(7, 4)$.

Solution:

First, calculate the average of the x- and y-values of the given points.

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-1, -2) & (7, 4) \end{array}$$

$$\begin{aligned} \frac{x_1 + x_2}{2} &= \frac{-1 + 7}{2} &&= \frac{6}{2} = 3 \\ \frac{y_1 + y_2}{2} &= \frac{-2 + 4}{2} &&= \frac{2}{2} = 1 \end{aligned}$$

Next, form the midpoint as an ordered pair using the averaged coordinates.

$$\left(\tfrac{x_1+x_2}{2}, \tfrac{y_1+y_2}{2}\right)$$

$$(3, 1)$$

Answer:

$$(3, 1)$$

To verify that this is indeed the midpoint, calculate the distance between the two given points and verify that the result is equal to the sum of the two equal distances from the endpoints to this midpoint. This verification is left to the reader as an exercise.

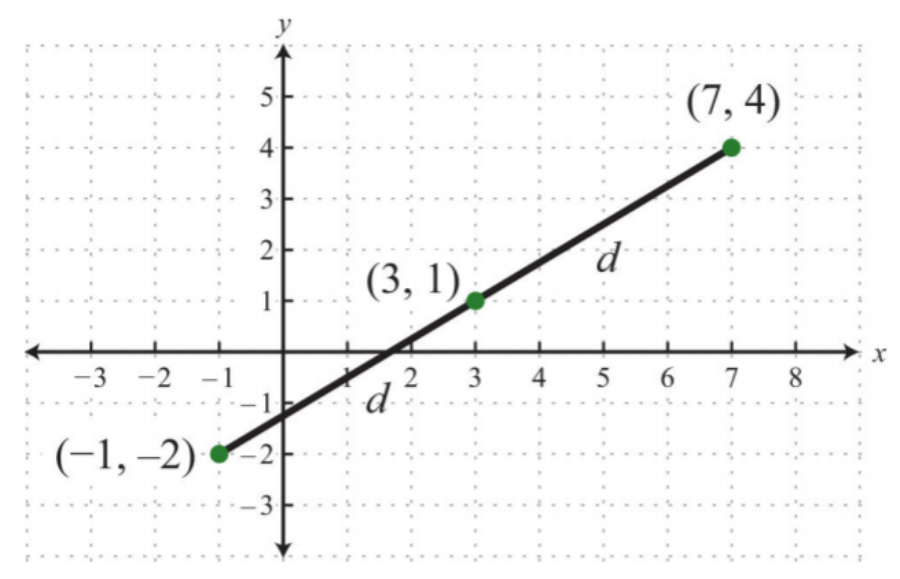


Figure 3.1.17

Exercise 3.1.3

Find the midpoint between $(-6, 5)$ and $(6, -11)$.

Answer available at the end of this section.

Key Takeaways

- Use the rectangular coordinate system to uniquely identify points in a plane using ordered pairs (x, y). Ordered pairs indicate position relative to the origin. The x-coordinate indicates position to the left and right of the origin. The y-coordinate indicates position above or below the origin.
- The scales on the x-axis and y-axis may be different. Choose a scale for each axis that is appropriate for the given problem.
- Graphs are used to visualize real-world data. Typically, independent data is associated with the x-axis and dependent data is associated with the y-axis.
- The Pythagorean theorem gives us a necessary and sufficient condition of right triangles. Given a right triangle, then the measures of the sides satisfy $a^2 + b^2 = c^2$. Conversely, if the sides satisfy $a^2 + b^2 = c^2$, then the triangle must be a right triangle.
- The distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, is derived from the Pythagorean theorem and gives us the distance between any two points, (x_1, y_1) and (x_2, y_2), in a rectangular coordinate plane.
- The midpoint formula, $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, is derived by taking the average of each coordinate and forming an ordered pair.

Exercise 3.1.4 Ordered Pairs

Give the coordinates of points A, B, C, D, and E.

1.

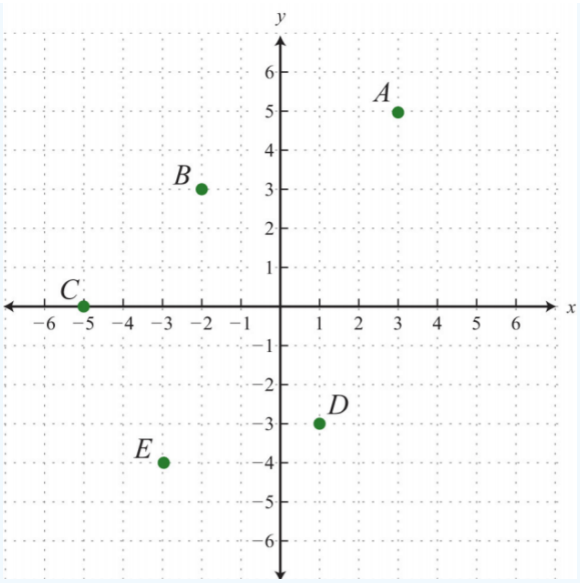


Figure 3.1.18

2.

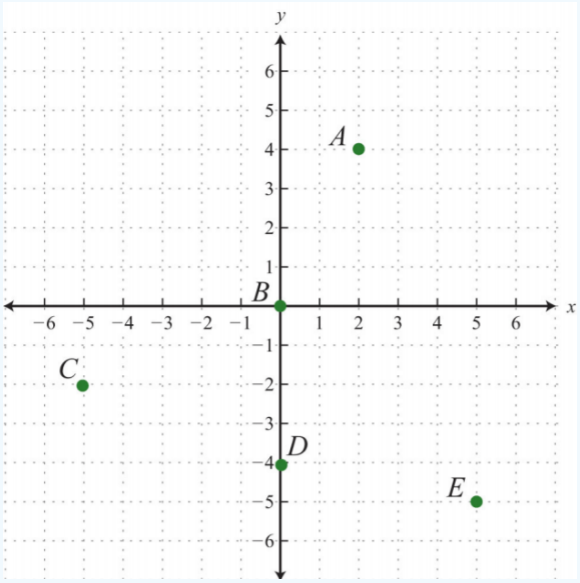


Figure 3.1.19

3.

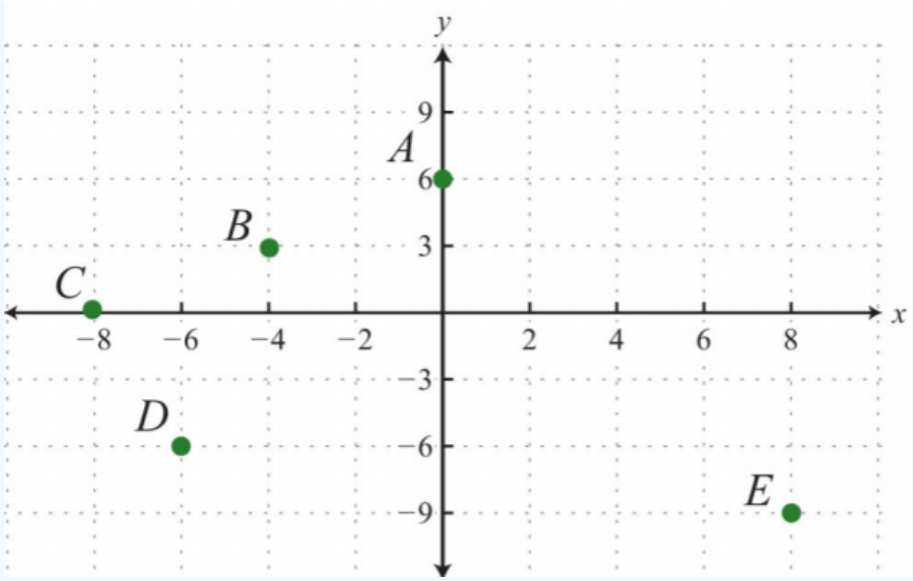


Figure 3.1.20

4.

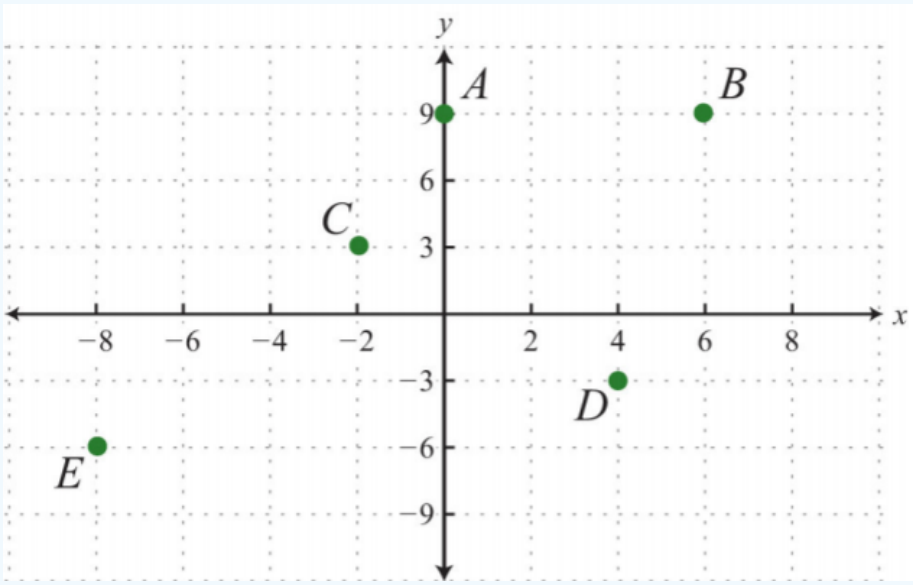


Figure 3.1.21

5.

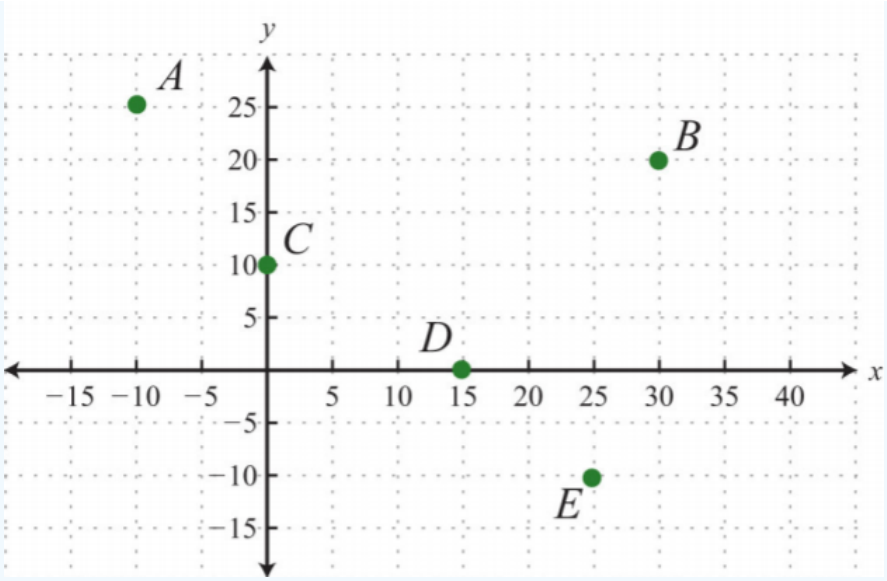


Figure 3.1.22

6.

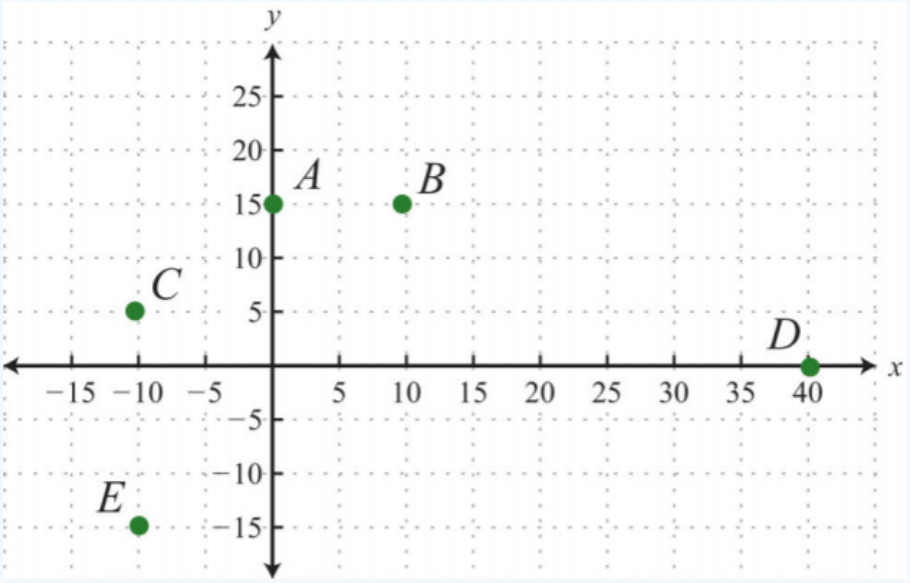


Figure 3.1.23

Answer available at the end of this section.

Exercise 3.1.5 Ordered Pairs

Graph the given set of ordered pairs.

1. $\{(-4,5),(-1,1),(-3,-2),(5,-1)\}$
2. $\{(-15,-10),(-5,10),(15,10),(5,-10)\}$
3. $\{(-2,5),(10,0),(2,-5),(6,-10)\}$
4. $\{(-8,3),(-4,6),(0,-6),(6,9)\}$
5. $\{(-10,5),(20,-10),(30,15),(50,0)\}$
6. $\{(-\frac{5}{3},-\frac{1}{2}),(-\frac{1}{3},\frac{1}{2}),(\frac{2}{3},-1),(\frac{5}{3},1)\}$

7. $\{(-\frac{3}{5}, -\frac{4}{3}), (\frac{2}{5}, \frac{4}{3}), (1, -\frac{2}{3}), (0, 1)\}$
8. $\{(-3.5, 0), (-1.5, 2), (0, 1.5), (2.5, -1.5)\}$
9. $\{(-0.8, 0.2), (-0.2, -0.4), (0, -1), (0.6, -0.4)\}$
10. $\{(-1.2, -1.2), (-0.3, -0.3), (0, 0), (0.6, 0.6), (1.2, 1.2)\}$

Answer available at the end of this section.

Exercise 3.1.6 Ordered Pairs

State the quadrant in which the given point lies.

1. $(-3, 2)$
2. $(5, 7)$
3. $(-12, -15)$
4. $(7, -8)$
5. $(-3.8, 4.6)$
6. $(17.3, 1.9)$
7. $(-\frac{1}{8}, -\frac{5}{8})$
8. $(\frac{3}{4}, -\frac{1}{4})$
9. $x > 0$ and $y < 0$
10. $x < 0$ and $y < 0$
11. $x < 0$ and $y > 0$
12. $x > 0$ and $y > 0$

Answer available at the end of this section.

Exercise 3.1.7 Ordered Pairs

The average price of a gallon of regular unleaded gasoline in US cities is given in the following line graph. Use the graph to answer the following questions.

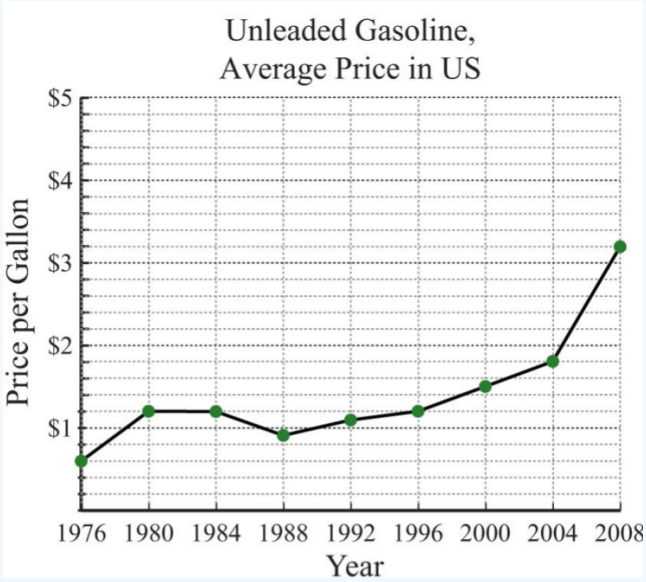


Figure 3.1.29: *Source: Bureau of Labor Statistics*

1. What was the average price of a gallon of unleaded gasoline in 2004?
2. What was the average price of a gallon of unleaded gasoline in 1976?
3. In which years were the average price of a gallon of unleaded gasoline $\$1.20$?

4. What is the price increase of a gallon of gasoline from 1980 to 2008?
5. What was the percentage increase in the price of a gallon of unleaded gasoline from 1976 to 1980?
6. What was the percentage increase in the price of a gallon of unleaded gasoline from 2000 to 2008?

Answer available at the end of this section.

Exercise 3.1.8 Ordered Pairs

The average price of all-purpose white flour in US cities from 1980 to 2008 is given in the following line graph. Use the graph to answer the questions that follow.

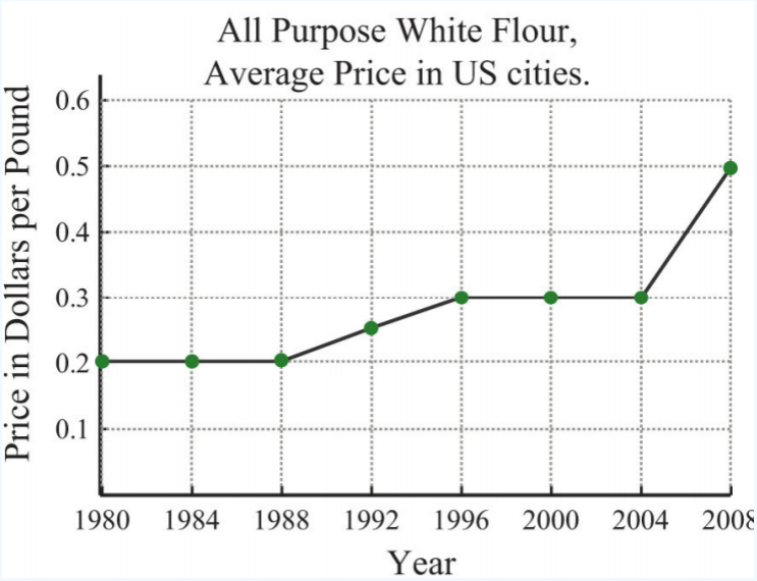


Figure 3.1.30: *Source: Bureau of Labor Statistics*

1. What was the average price per pound of all-purpose white flour in 2000?
2. What was the average price per pound of all-purpose white flour in 2008?
3. In which year did the price of flour average $0.25 per pound?
4. In which years did the price of flour average $0.20 per pound?
5. What was the percentage increase in flour from the year 2000 to 2008?
6. What was the percentage increase in flour from the year 1992 to 2000?

Answer available at the end of this section.

Exercise 3.1.9 Ordered Pairs

Given the following data, create a line graph.

1. The percentage of total high school graduates who enrolled in college.

Table 3.1.2: Source: Digest of Education Statistics

Year	Percentage
1969	36%
1979	40%
1989	47%

Year	Percentage
1999	42%

2. The average daily temperature given in degrees Fahrenheit in May.

Table 3.1.3

Exam	Temperature
8:00 am	60
12:00 pm	72
4:00 pm	75
8:00 pm	67
12:00 am	60
4:00 am	55

Answer available at the end of this section.

Exercise 3.1.10 Ordered Pairs

Calculate the area of the shape formed by connecting the following set of vertices.

1. $\{(0,0),(0,3),(5,0),(5,3)\}$
2. $\{(-1,-1),(-1,1),(1,-1),(1,1)\}$
3. $\{(-2,-1),(-2,3),(5,3),(5,-1)\}$
4. $\{(-5,-4),(-5,5),(3,5),(3,-4)\}$
5. $\{(0,0),(4,0),(2,2)\}$
6. $\{(-2,-2),(2,-2),(0,2)\}$
7. $\{(0,0),(0,6),(3,4)\}$
8. $\{(-2,0),(5,0),(3,-3)\}$

Answer available at the end of this section.

Exercise 3.1.11 Distance Formula

Calculate the distance between the given two points.

1. $(-5,3)$ and $(-1,6)$
2. $(6,-2)$ and $(-2,4)$
3. $(0,0)$ and $(5,12)$
4. $(-6,-8)$ and $(0,0)$
5. $(-7,8)$ and $(5,-1)$
6. $(-1,-2)$ and $(9,22)$
7. $(-1,2)$ and $(-\frac{7}{2},-4)$
8. $(-\frac{1}{2},\frac{1}{3})$ and $(\frac{5}{2},-\frac{11}{3})$
9. $(-\frac{1}{3},\frac{2}{3})$ and $(1,-\frac{1}{3})$
10. $(\frac{1}{2},-\frac{3}{4})$ and $(\frac{3}{2},\frac{1}{4})$
11. $(1,2)$ and $(4,3)$
12. $(2,-4)$ and $(-3,-2)$
13. $(-1,5)$ and $(1,-3)$
14. $(1,-7)$ and $(5,-1)$
15. $(-7,-3)$ and $(-1,6)$

16. $(0,1)$ and $(1,0)$
17. $(-0.2,-0.2)$ and $(1.8,1.8)$
18. $(1.2,-3.3)$ and $(2.2,-1.7)$

Answer available at the end of this section.

Exercise 3.1.12 Distance Formula

For each problem, show that the three points form a right triangle.

1. $(-3,-2),(0,-2)$, and $(0,4)$
2. $(7,12),(7,-13)$, and $(-5,-4)$
3. $(-1.4,0.2),(1,2)$, and $(1,-3)$
4. $(2,-1),(-1,2)$, and $(6,3)$
5. $(-5,2),(-1,-2)$, and $(-2,5)$
6. $(1,-2),(2,3)$, and $(-3,4)$

Answer available at the end of this section.

Exercise 3.1.13 Distance Formula

Isosceles triangles have two legs of equal length. For each problem, show that the following points form an isosceles triangle.

1. $(1,6),(-1,1)$, and $(3,1)$
2. $(-6,-2),(-3,-5)$, and $(-9,-5)$
3. $(-3,0),(0,3)$, and $(3,0)$
4. $(0,-1),(0,1)$, and $(1,0)$

Answer available at the end of this section.

Exercise 3.1.14 Distance Formula

Calculate the area and the perimeter of the triangles formed by the following set of vertices.

1. $\{(-4,-5),(-4,3),(2,3)\}$
2. $\{(-1,1),(3,1),(3,-2)\}$
3. $\{(-3,1),(-3,5),(1,5)\}$
4. $\{(-3,-1),(-3,7),(1,-1)\}$

Answer available at the end of this section.

Exercise 3.1.15 Midpoint Formula

Find the midpoint between the given two points.

1. $(-1,6)$ and $(-7,-2)$
2. $(8,0)$ and $(4,-3)$
3. $(-10,0)$ and $(10,0)$
4. $(-3,-6)$ and $(-3,6)$
5. $(-10,5)$ and $(14,-5)$
6. $(0,1)$ and $(2,2)$
7. $(5,-3)$ and $(4,-5)$
8. $(0,0)$ and $(1,1)$

9. $(-1,-1)$ and $(4,4)$
10. $(3,-5)$ and $(3,5)$
11. $(-\frac{1}{2},-\frac{1}{3})$ and $(\frac{3}{2},\frac{7}{3})$
12. $(\frac{3}{4},-\frac{2}{3})$ and $(\frac{1}{8},-\frac{1}{2})$
13. $(\frac{5}{3},\frac{1}{4})$ and $(-\frac{1}{6},-\frac{3}{2})$
14. $(-\frac{1}{5},-\frac{5}{2})$ and $(\frac{7}{10},-\frac{1}{4})$
15. Given the right triangle formed by the vertices $(0,0),(6,0)$, and $(6,8)$, show that the midpoints of the sides form a right triangle.
16. Given the isosceles triangle formed by the vertices $(-10,-12),(0,12)$, and $(10,-12)$, show that the midpoints of the sides also form an isosceles triangle.
17. Calculate the area of the triangle formed by the vertices $(-4,-3),(-1,1)$, and $(2,-3)$. (Hint: The vertices form an isosceles triangle.)
18. Calculate the area of the triangle formed by the vertices $(-2,1),(4,1)$, and $(1,-5)$.

Answer available at the end of this section.

Exercise 3.1.16 Discussion Board Topics

1. Research and discuss the life and contributions to mathematics of René Descartes.
2. Research and discuss the history of the right triangle and the Pythagorean theorem.
3. What is a Pythagorean triple? Provide some examples.
4. Explain why you cannot use a ruler to calculate distance on a graph.
5. How do you bisect a line segment with only a compass and a straightedge?

Answer available at the end of this section.

Answers

Exercise 3.1.1

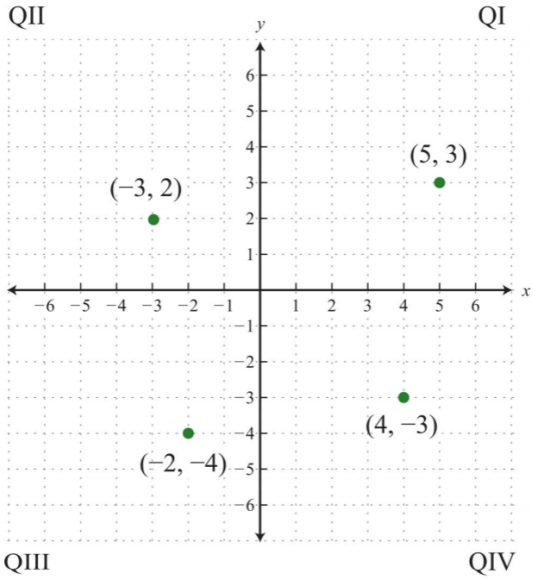


Figure 3.1.7

Exercise 3.1.2

10 units

Exercise 3.1.3

$(0,-3)$

Exercise 3.1.4 **Ordered Pairs**

1. $A:(3,5); B:(-2,3); C:(-5,0); D:(1,-3); E:(-3,-4)$
3. $A:(0,6); B:(-4,3); C:(-8,0); D:(-6,-6); E:(8,-9)$

5. $A:(-10,25); B:(30,20); C:(0,10); D:(15,0); E:(25,-10)$

Exercise 3.1.5 **Ordered Pairs**

1.

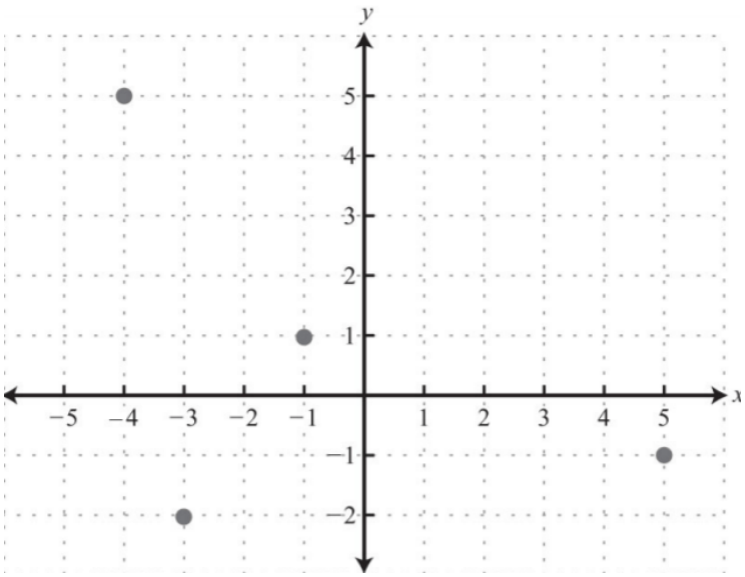


Figure 3.1.24

3.

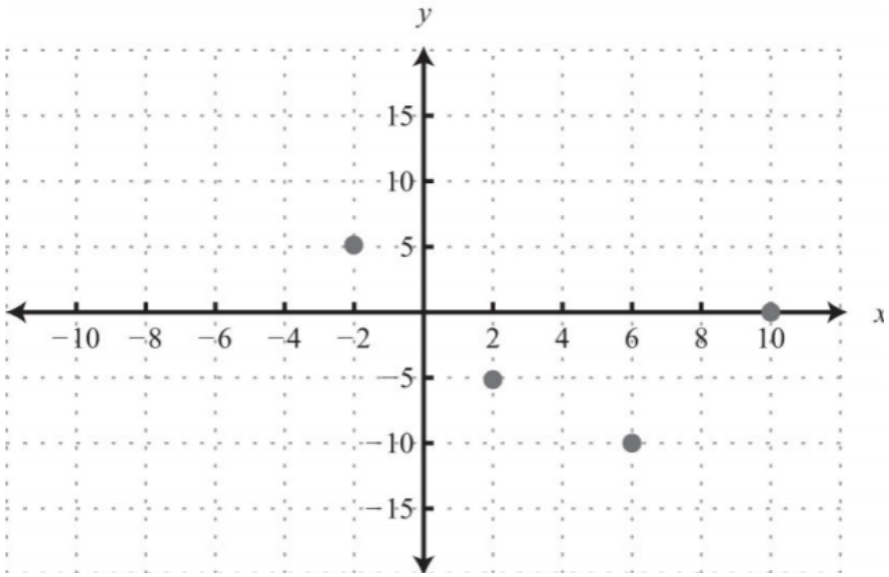


Figure 3.1.25

5.

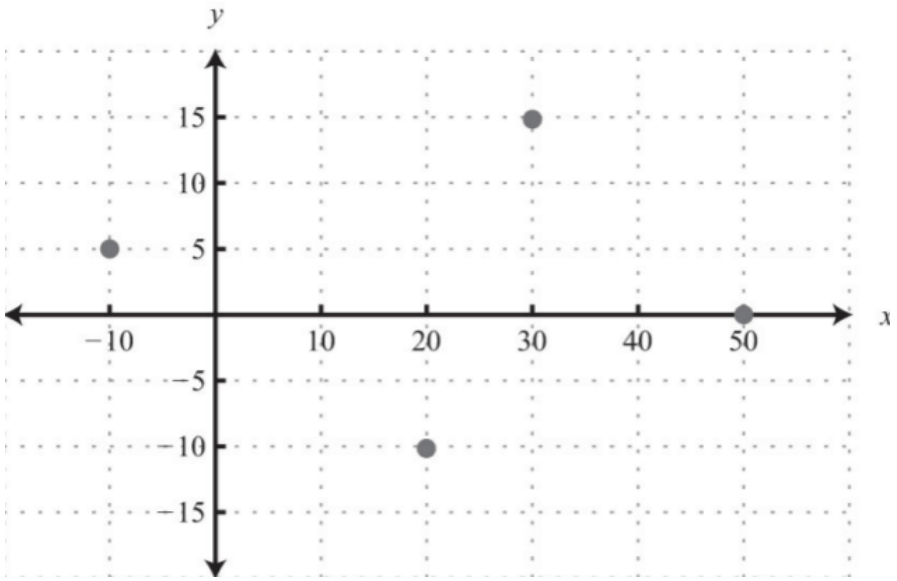


Figure 3.1.26

7.

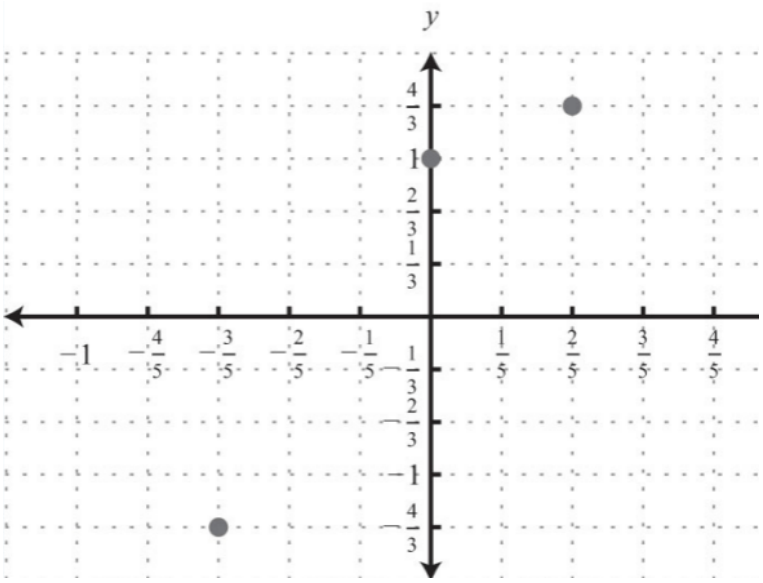


Figure 3.1.27

9.

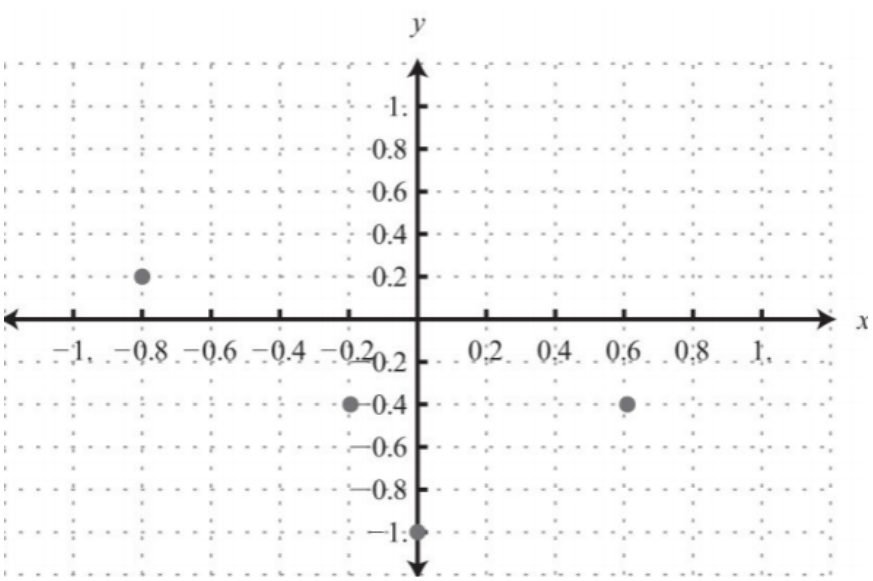


Figure 3.1.28

Exercise 3.1.6 **Ordered Pairs**

1. QII
3. QIII
5. QII
7. QIII
9. QIV
11. QII

Exercise 3.1.7 **Ordered Pairs**

1. $1.80
3. 1980 to 1984, 1996
3. 100%

Exercise 3.1.8 **Ordered Pairs**

1. $0.30
3. 1992
5. 67%

Exercise 3.1.9 **Ordered Pairs**

1.

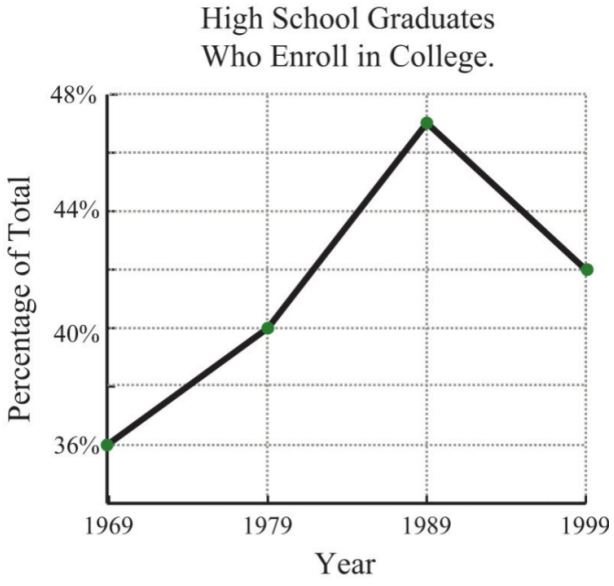


Figure 3.1.31

Exercise 3.1.10 **Ordered Pairs**

1. 15 square units

3. 28 square units
5. 4 square units
7. 9 square units

Exercise $3.1.11$ **Distance Formula**

1. 5 units
3. 13 units
5. 15 units
7. $\frac{13}{2}$ units
9. $\frac{5}{3}$ units
11. $\sqrt{10}$ units
13. $2\sqrt{17}$ units
15. $3\sqrt{13}$ units
17. 2.8 units

Exercise $3.1.12$ **Distance Formula**

1. Proof
3. Proof
5. Proof

Exercise $3.1.13$ **Distance Formula**

1. Proof
3. Proof

Exercise $3.1.14$ **Distance Formula**

1. Perimeter: 24 units; area: 24 square units
3. Perimeter: $8+4\sqrt{2}$ units; area: 8 square units

Exercise $3.1.15$ **Midpoint Formula**

1. $(-4,2)$
3. $(0,0)$
5. $(2,0)$
7. $(\frac{9}{2},-4)$
9. $(\frac{3}{2},\frac{3}{2})$
11. $(\frac{1}{2},1)$
13. $(\frac{3}{4},-\frac{5}{8})$
15. 12 square units

Exercise $3.1.16$ **Discussion Board Topics**

1. Answers may vary
3. Answers may vary
5. Answers may vary

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3.2: Graph by Plotting Points

Learning Objectives

- Verify solutions to linear equations with two variables.
- Graph lines by plotting points.
- Identify and graph horizontal and vertical lines.

Solutions to Equations with Two Variables

A linear equation with two variables has standard form $ax + by = c$, where a, b, and c are real numbers and a and b are not both 0. Solutions to equations of this form are ordered pairs (x, y), where the coordinates, when substituted into the equation, produce a true statement.

Example 3.2.1

Determine whether $(1, -2)$ and $(-4, 1)$ are solutions to $6x - 3y = 12$.

Solution:

Substitute the x- and y-values into the equation to determine whether the ordered pair produces a true statement.

$$\begin{array}{ll} \underline{Check\ (1,-2)} & \underline{Check\ (-4,1)} \\ 6x-3y=12 & 6x-3y=12 \\ 6(1)-3(-2)=12 & 6(-4)-3(1)=12 \\ 6+6=12 & -24-3=12 \\ 12=12 \quad \checkmark & -27=12 \quad x \end{array}$$

Answer:

$(1, -2)$ is a solution, and $(-4, 1)$ is not.

It is often the case that a linear equation is given in a form where one of the variables, usually y, is isolated. If this is the case, then we can check that an ordered pair is a solution by substituting in a value for one of the coordinates and simplifying to see if we obtain the other.

Example 3.2.2

Are $(\frac{1}{2}, -3)$ and $(-5, 14)$ solutions to $y = 2x - 4$?

Solution:

Substitute the x-values and simplify to see if the corresponding y-values are obtained.

$$\begin{array}{ll} \underline{x=\frac{1}{2}} & \underline{x=-5} \\ y=2x-4 & y=2x-4 \\ =2(\frac{1}{2})-4 & =2(-5)-4 \\ =1-4 & =-10-4 \\ =-3 \quad \checkmark & =-14 \quad x \end{array}$$

Answer:

$(\frac{1}{2}, -3)$ is a solution, and $(-5, 14)$ is not.

Exercise 3.2.1

Is $(6, -1)$ a solution to $y = -\frac{2}{3}x + 3$?

Answer available at the end of this section.

When given linear equations with two variables, we can solve for one of the variables, usually y, and obtain an equivalent equation as follows:

Written in this form, we can see that y depends on x. Here x is the independent variable and y is the dependent variable.

$$\left.\begin{aligned} 6x - 3y &= 12 \\ y &= 2x - 4 \end{aligned}\right\} \textit{Equivalent equations}$$

The linear equation $y = 2x - 4$ can be used to find ordered pair solutions. If we substitute any real number for x, then we can simplify to find the corresponding y-value. For example, if $x = 3$, then $y = 2(3) - 4 = 6 - 4 = 2$, and we can form an ordered pair solution, $(3, 2)$. Since there are infinitely many real numbers to choose for x, the linear equation has infinitely many ordered pair solutions (x, y).

Example 3.2.3

Find ordered pair solutions to the equation $5x - y = 14$ with the given x-values $\{-2, -1, 0, 4, 6\}$.

Solution:

First, solve for y.

Next, substitute the x-values in the equation $y = 5x - 14$ to find the corresponding y-values.

$x - value\ y - value\ Solution\ x = -2\ (-2, -24)\ x = -1\ (-1, -19)\ x = 0\ (0, -14)\ x = 4\ (4, 6)\ x = 6\ (6, 16)$ Table 3.2.1

Answer:

$$\{(-2, -24), (-1, -19), (0, -14), (4, 6), (6, 16)\}$$

In the previous example, certain x-values are given, but that is not always going to be the case. When treating x as the independent variable, we can choose any values for x and then substitute them into the equation to find the corresponding y -values. This method produces as many ordered pair solutions as we wish.

Example 3.2.4

Find five ordered pair solutions to $6x + 2y = 10$.

Solution:

First, solve for y.

$$\begin{aligned} 6x+2y &= 10 \\ 6x+2y-6x &= 10-6x \\ 2y &= -6x+10 \\ \frac{2y}{2} &= \frac{-6x+10}{2} \\ y &= \frac{-6x}{2}+\frac{10}{2} \\ y &= -3x+5 \end{aligned}$$

Next, choose any set of x-values. Usually we choose some negative values and some positive values. In this case, we will find the corresponding y-values when x is $\{-2,-1,0,1,2\}$. Make the substitutions required to fill in the following table (often referred to as a t-chart):

x	y	Solutions	
-2	11	$y=-3(-2)+5=6+5=11$	$(-2, 11)$
-1	8	$y=-3(-1)+5=3+5=8$	$(-1, 8)$
0	5	$y=-3(0)+5=0+5=5$	$(0, 5)$
1	2	$y=-3(1)+5=-3+5=2$	$(1, 2)$
2	-1	$y=-3(2)+5=-6+5=-1$	$(2, -1)$

Figure 3.2.1

Answer:

$\{(-2,11),(-1,8),(0,5),(1,2),(2,-1)\}$. Since there are infinitely many ordered pair solutions, answers may vary depending on the choice of values for the independent variable.

Exercise 3.2.2

Find five ordered pair solutions to $10x-2y=2$.

Answer available at the end of this section.

Graph by Plotting Points

Since the solutions to linear equations are ordered pairs, they can be graphed using the rectangular coordinate system. The set of all solutions to a linear equation can be represented on a rectangular coordinate plane using a straight line connecting at least two points; this line is called its graph. To illustrate this, plot five ordered pair solutions, $\{(-2,11),(-1,8),(0,5),(1,2),(2,-1)\}$, to the linear equation $6x+2y=10$.

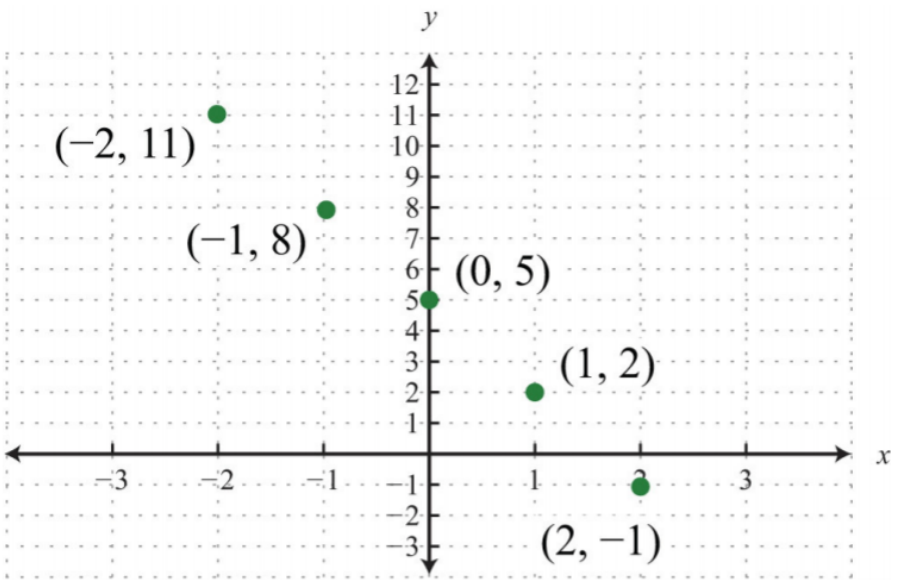


Figure 3.2.2

Notice that the points are collinear; this will be the case for any linear equation. Draw a line through the points with a straightedge, and add arrows on either end to indicate that the graph extends indefinitely.

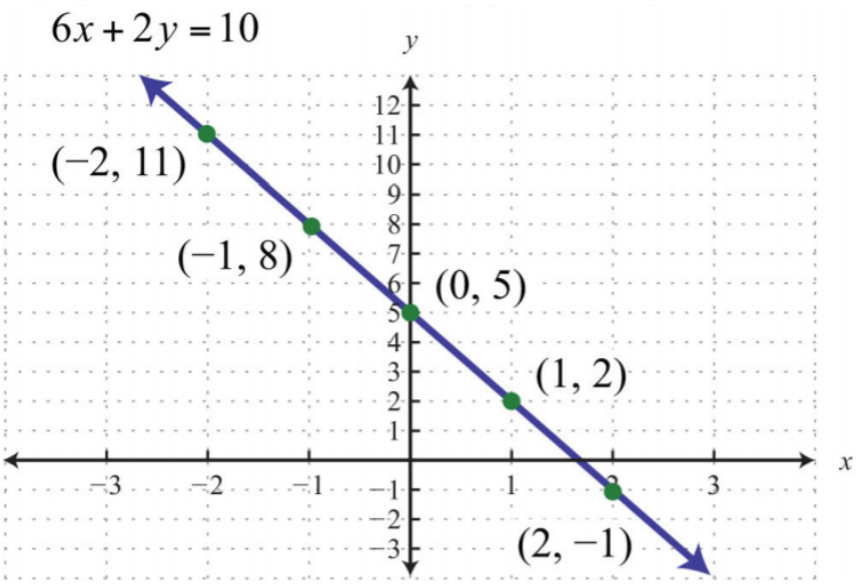


Figure 3.2.3

The resulting line represents all solutions to $6x + 2y = 10$, of which there are infinitely many. The steps for graphing lines by plotting points are outlined in the following example.

Example 3.2.5

Find five ordered pair solutions and graph:

$$10x - 5y = 10.$$

Solution:

Step 1: Solve for y.

Step 2: Choose at least two x-values and find the corresponding y-values. In this section, we will choose five real numbers to use as x-values. It is a good practice to choose 0 and some negative numbers, as well as some positive numbers.

Figure $3.2.4$

Five ordered pair solutions are
$\{(-2,-6),(-1,-4),(0,-2),(1,0),(2,2)\}$

Step 3: Choose an appropriate scale, plot the points, and draw a line through them using a straightedge. In this case, choose a scale where each tick mark on the y-axis represents 2 units because all the y-values are multiples of 2.

Answer:

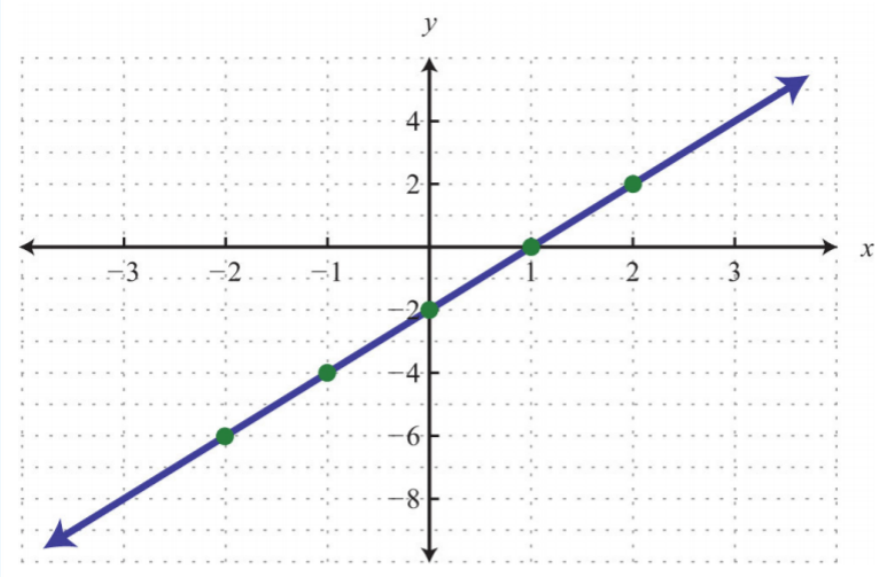


Figure $3.2.5$

It will not always be the case that y can be solved in terms of x with integer coefficients. In fact, the coefficients often turn out to be fractions.

Example $3.2.6$

Find five ordered pair solutions and graph:

$$-5x+2y=10.$$

Solution:

$$\begin{aligned} -5x+2y &= 10 \\ -5x+2y+5x &= 10+5x \\ 2y &= 5x+10 \\ frac2y2 &= \frac{5x+10}{2} \\ y &= \frac{5x}{2}+\frac{10}{2} \\ y &= \frac{5}{2}x+5 \end{aligned}$$

Remember that you can choose any real number for the independent variable x, so choose wisely here. Since the denominator of the coefficient of the variable x is 2, you can avoid fractions by choosing multiples of 2 for the x-values. In this case, choose the set of x-values $\{-6, -4, -2, 0, 2\}$ and find the corresponding y-values.

x	y	
-6	-10	$y = \frac{5}{2}(-6) + 5 = -15 + 5 = -10$
-4	-5	$y = \frac{5}{2}(-4) + 5 = -10 + 5 = -5$
-2	0	$y = \frac{5}{2}(-2) + 5 = -5 + 5 = 0$
0	5	$y = \frac{5}{2}(0) + 5 = 0 + 5 = 5$
2	10	$y = \frac{5}{2}(2) + 5 = 5 + 5 = 10$

Figure 3.2.6

Five solutions are $\{(-6, -10), (-4, -5), (-2, 0), (0, 5), (2, 10)\}$. Here we choose to scale the x-axis with multiples of 2 and the y-axis with multiples of 5.

Answer:

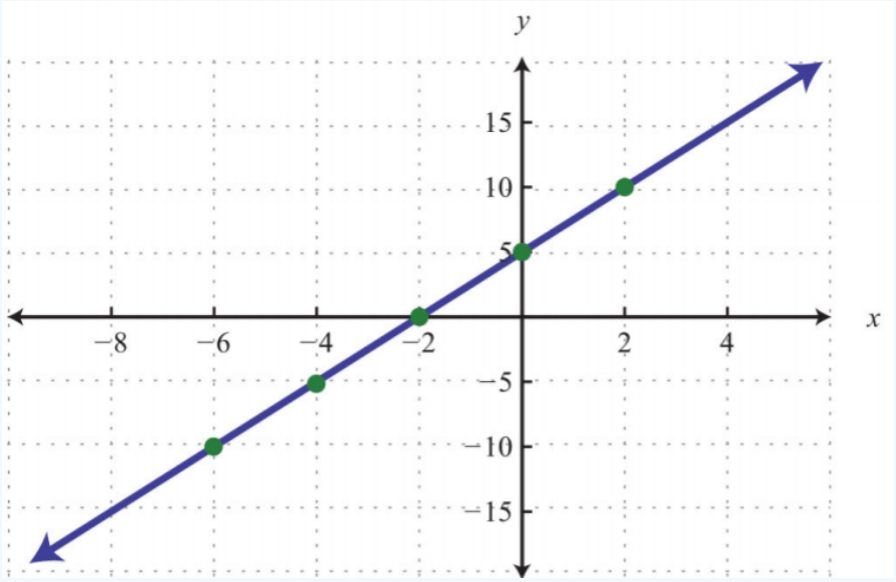


Figure 3.2.7

Exercise 3.2.3

Find five ordered pair solutions and graph: $x + 2y = 6$.

Answer available at the end of this section.

Horizontal and Vertical Lines

We need to recognize by inspection linear equations that represent a vertical or horizontal line.

Example 3.2.7

Graph by plotting five points: $y = -2$.

Solution:

Since the given equation does not have a variable x, we can rewrite it with a 0 coefficient for x.

$$y = 0x - 2$$

Choose any five values for x and see that the corresponding y-value is always -2.

x	y	
-2	-2	$y = 0(-2) - 2 = 0 - 2 = -2$
-1	-2	$y = 0(-1) - 2 = 0 - 2 = -2$
0	-2	$y = 0(0) - 2 = 0 - 2 = -2$
1	-2	$y = 0(1) - 2 = 0 - 2 = -2$
2	-2	$y = 0(2) - 2 = 0 - 2 = -2$

Figure 3.2.9

We now have five ordered pair solutions to plot $\{(-2,-2),(-1,-2),(0,-2),(1,-2),(2,-2)\}$.

Answer:

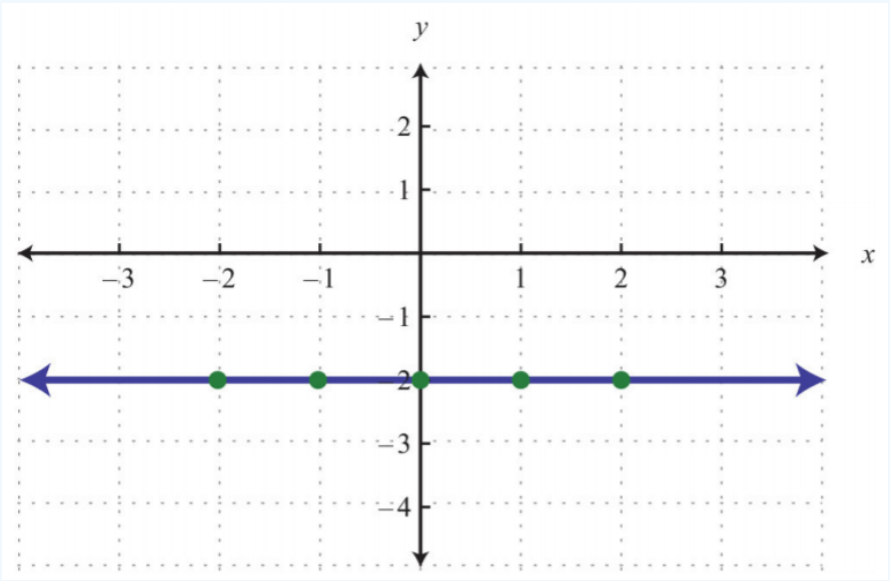


Figure 3.2.10

When the coefficient for the variable x is 0, the graph is a horizontal line. In general, the equation for a horizontal line can be written in the form $y = k$, where k represents any real number.

Example 3.2.8

Graph by plotting five points: $x = 3$.

Solution:

Since the given equation does not have a variable y, rewrite it with a 0 coefficient for y.

$$x = 0y + 3$$

Choose any five values for y and see that the corresponding x-value is always 3.

x	y	
3	−2	$x = 0(-2) + 3 = 0 + 3 = 3$
3	−1	$x = 0(-1) + 3 = 0 + 3 = 3$
3	0	$x = 0(0) + 3 = 0 + 3 = 3$
3	1	$x = 0(1) + 3 = 0 + 3 = 3$
3	2	$x = 0(2) + 3 = 0 + 3 = 3$

Figure 3.2.11

We now have five ordered pair solutions to plot: $\{(3, -2), (3, -1), (3, 0), (3, 1), (3, 2)\}$.

Answer:

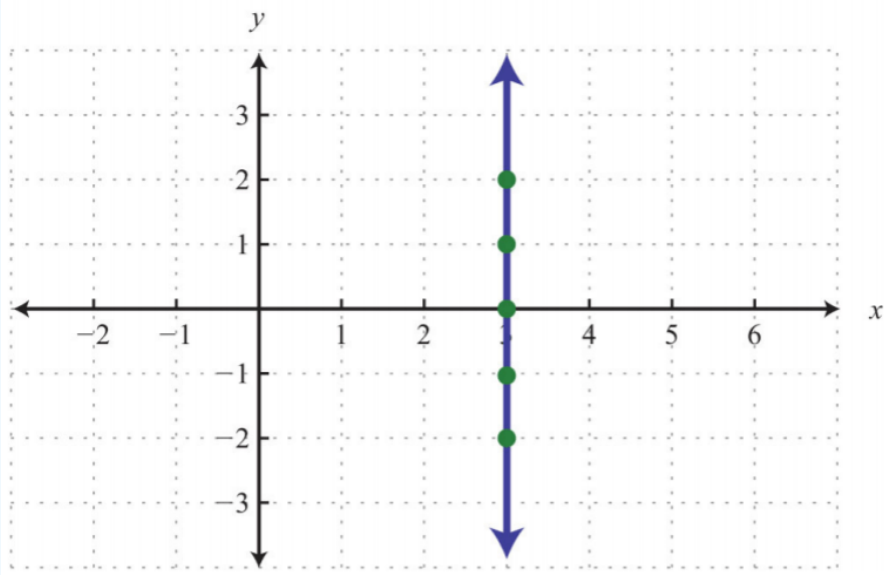


Figure 3.2.12

When the coefficient for the variable y is 0, the graph is a vertical line. In general, the equation for a vertical line can be written as $x = k$, where k represents any real number.

To summarize, if k is a real number.

$$y = k \quad \textit{Horizontal line} \tag{3.2.1}$$

$$x = k \quad \textit{Vertical line} \tag{3.2.2}$$

Exercise 3.2.4

Graph $y = 5$ and $x = -2$ on the same set of axes and determine where they intersect.

Answer available at the end of this section.

Key Takeaways

- Solutions to linear equations with two variables $ax + by = c$ are ordered pairs (x, y), where the coordinates, when substituted into the equation, result in a true statement.
- Linear equations with two variables have infinitely many ordered pair solutions. When the solutions are graphed, they are collinear.
- To find ordered pair solutions, choose values for the independent variable, usually x, and substitute them in the equation to find the corresponding y-values.
- To graph linear equations, determine at least two ordered pair solutions and draw a line through them with a straightedge.
- Horizontal lines are described by $y = k$, where k is any real number.
- Vertical lines are described by $x = k$, where k is any real number.

Exercise 3.2.5 Solutions to Linear Systems

Determine whether the given point is a solution.

1. $5x - 2y = 4; (-1, 1)$
2. $3x - 4y = 10; (2, -1)$
3. $-3x + y = -6; (4, 6)$
4. $-8x - y = 24; (-2, -3)$
5. $-x + y = -7; (5, -2)$
6. $9x - 3y = 6; (0, -2)$
7. $\frac{1}{2}x + \frac{1}{3}y = -\frac{1}{6}; (1, -2)$
8. $\frac{3}{4}x - \frac{1}{2}y = -1; (2, 1)$
9. $4x - 3y = 1; (\frac{1}{2}, \frac{1}{3})$
10. $-10x + 2y = -\frac{9}{5}; (\frac{1}{5}, \frac{1}{10})$
11. $y = \frac{1}{3}x + 3; (6, 3)$
12. $y = -4x + 1; (-2, 9)$
13. $y = \frac{2}{3}x - 3; (0, -3)$
14. $y = -\frac{5}{8}x + 1; (8, -5)$
15. $y = -\frac{1}{2}x + \frac{3}{4}; (-\frac{1}{2}, 1)$
16. $y = -\frac{1}{3}x - \frac{1}{2}; (\frac{1}{2}, -\frac{2}{3})$
17. $y = 2; (-3, 2)$
18. $y = 4; (4, -4)$
19. $x = 3; (3, -3)$
20. $x = 0; (1, 0)$

Answer available at the end of this section.

Exercise 3.2.6 Solutions to Linear Systems

Find the ordered pair solutions given the set of x-values.

1. $y = -2x + 4; \{-2, 0, 2\}$
2. $y = 12x - 3; \{-4, 0, 4\}$
3. $y = -\frac{3}{4}x + \frac{1}{2}; \{-2, 0, 2\}$
4. $y = -3x + 1; \{-\frac{1}{2}, 0, \frac{1}{2}\}$
5. $y = -4; \{-3, 0, 3\}$
6. $y = \frac{1}{2}x + \frac{3}{4}; \{-\frac{1}{4}, 0, \frac{1}{4}\}$
7. $2x - 3y = 1; \{0, 1, 2\}$
8. $3x - 5y = -15; \{-5, 0, 5\}$
9. $-x + y = 3; \{-5, -1, 0\}$
10. $\frac{1}{2}x - \frac{1}{3}y = -4; \{-4, -2, 0\}$
11. $\frac{3}{5}x + \frac{1}{10}y = 2; \{-15, -10, -5\}$
12. $x - y = 0; \{10, 20, 30\}$

Answer available at the end of this section.

Exercise 3.2.7 Solutions to Linear Systems

Find the ordered pair solutions, given the set of y-values.

1. $y = \frac{1}{2}x - 1; \{-5, 0, 5\}$
2. $y = -\frac{3}{4}x + 2; \{0, 2, 4\}$
3. $3x - 2y = 6; \{-3, -1, 0\}$
4. $-x + 3y = 4; \{-4, -2, 0\}$
5. $\frac{1}{3}x - \frac{1}{2}y = -4; \{-1, 0, 1\}$
6. $\frac{3}{5}x + \frac{1}{10}y = 2; \{-20, -10, -5\}$

Answer available at the end of this section.

Exercise 3.2.8 Graphing Lines

Given the set of x-values $\{-2, -1, 0, 1, 2\}$, find the corresponding y-values and graph them.

1. $y = x + 1$
2. $y = -x + 1$
3. $y = 2x - 1$
4. $y = -3x + 2$
5. $y = 5x - 10$
6. $5x + y = 15$
7. $3x - y = 9$
8. $6x - 3y = 9$
9. $y = -5$
10. $y = 3$

Answer available at the end of this section.

Exercise 3.2.9 Graphing Lines

Find at least five ordered pair solutions and graph.

1. $y = 2x - 1$
2. $y = -5x + 3$
3. $y = -4x + 2$
4. $y = 10x - 20$
5. $y = -\frac{1}{2}x + 2$
6. $y = \frac{1}{3}x - 1$
7. $y = \frac{2}{3}x - 6$
8. $y = -\frac{2}{3}x + 2$
9. $y = x$
10. $y = -x$
11. $-2x + 5y = -15$
12. $x + 5y = 5$
13. $6x - y = 2$
14. $4x + y = 12$
15. $-x + 5y = 0$
16. $x + 2y = 0$
17. $\frac{1}{10}x - y = 3$
18. $\frac{3}{2}x + 5y = 30$

Answer available at the end of this section.

Exercise 3.2.10 Horizontal and Vertical Lines

Find at least five ordered pair solutions and graph them.

1. $y = 4$
2. $y = -10$
3. $x = 4$
4. $x = -1$
5. $y = 0$
6. $x = 0$
7. $y = \frac{3}{4}$
8. $x = -\frac{5}{4}$
9. Graph the lines $y = -4$ and $x = 2$ on the same set of axes. Where do they intersect?
10. Graph the lines $y = 5$ and $x = -5$ on the same set of axes. Where do they intersect?
11. What is the equation that describes the x-axis?
12. What is the equation that describes the y-axis?

Answer available at the end of this section.

Exercise 3.2.11 Mixed Practice

Graph by plotting points.

1. $y = -\frac{3}{5}x + 6$
2. $y = \frac{3}{5}x - 3$
3. $y = -3$
4. $x = -5$
5. $3x - 2y = 6$
6. $-2x + 3y = -12$

Answer available at the end of this section.

Exercise 3.2.12 Discussion Board Topics

1. Discuss the significance of the relationship between algebra and geometry in describing lines.
2. Give real-world examples relating two unknowns

Answer available at the end of this section.

Answers

Exercise 3.2.1

Yes

Exercise 3.2.2

$\{(-2,-11),(-1,-6),(0,-1),(1,4),(2,9)\}$ (answers may vary)

Exercise 3.2.3

$\{(-2,4),(0,3),(2,2),(4,1),(6,0)\}$

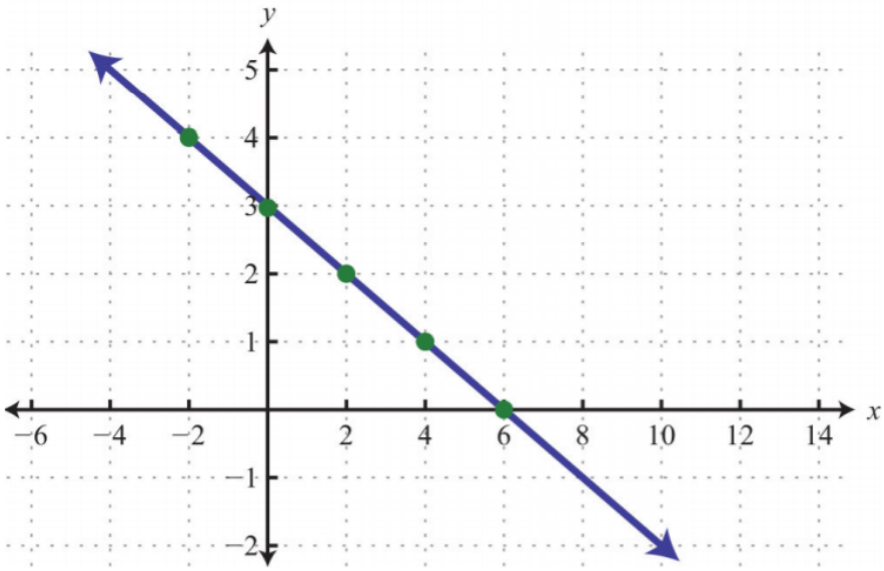


Figure 3.2.8

Exercise 3.2.4

$(-2,5)$

Exercise 3.2.5 **Solutions to Linear Systems**

1. No
3. Yes
5. Yes
7. Yes
9. Yes
11. No
13. Yes
15. Yes
17. Yes
19. Yes

Exercise 3.2.6 **Solutions to Linear Systems**

1. $\{(-2,8),(0,4),(2,0)\}$
3. $\{(-2,2),(0,1/2),(2,-1)\}$
5. $\{(-3,-4),(0,-4),(3,-4)\}$
7. $\{(0,-\frac{1}{3}),(1,\frac{1}{3}),(2,1)\}$
9. $\{(-5,-2),(-1,2),(0,3)\}$
11. $\{(-15,110),(-10,80),(-5,50)\}$

Exercise 3.2.7 **Solutions to Linear Systems**

1. $\{(-8,-5),(2,0),(12,5)\}$

3. $\{(0,-3),(\frac{4}{3},-1),(2,0)\}$
5. $\{(-\frac{27}{2},-1),(-12,0),(-\frac{21}{2},1)\}$

Exercise 3.2.8 **Graphing Lines**

1.

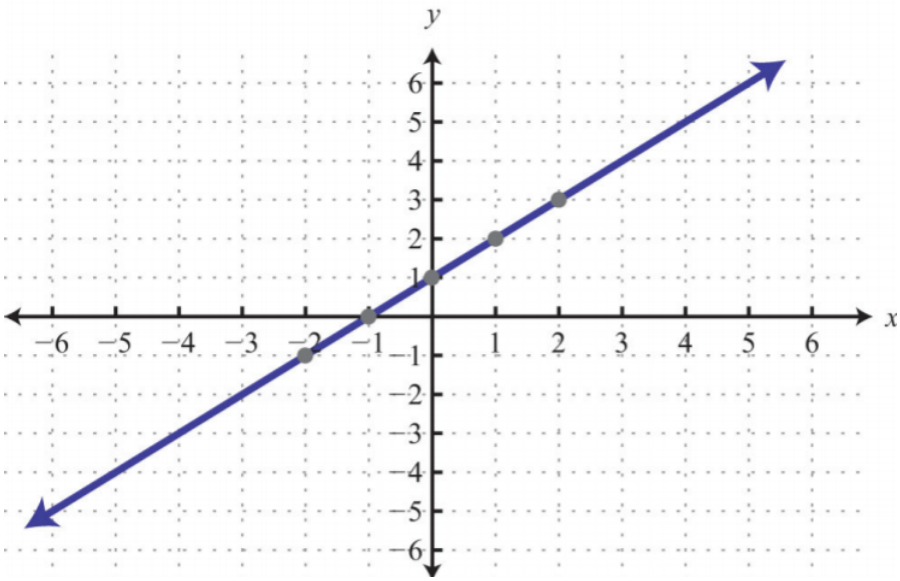


Figure 3.2.13

3.

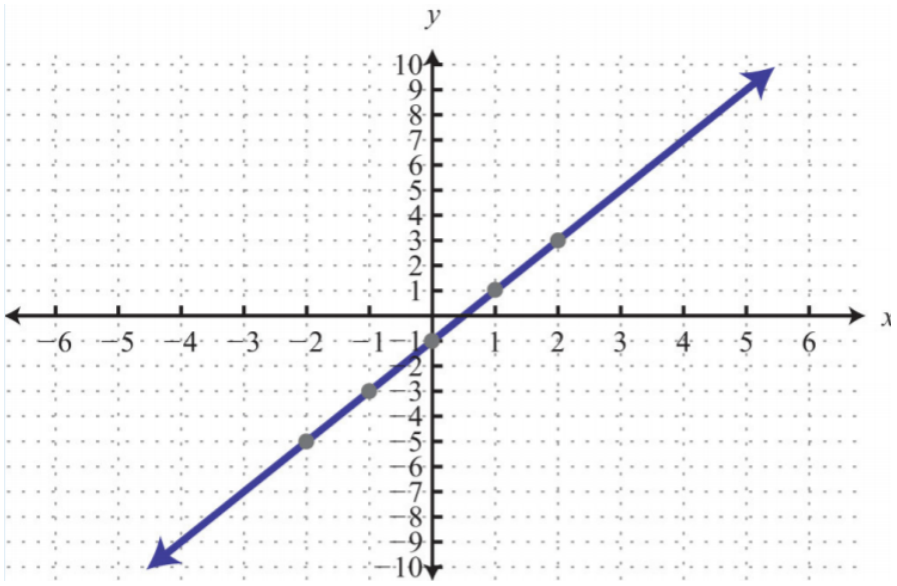


Figure 3.2.14

5.

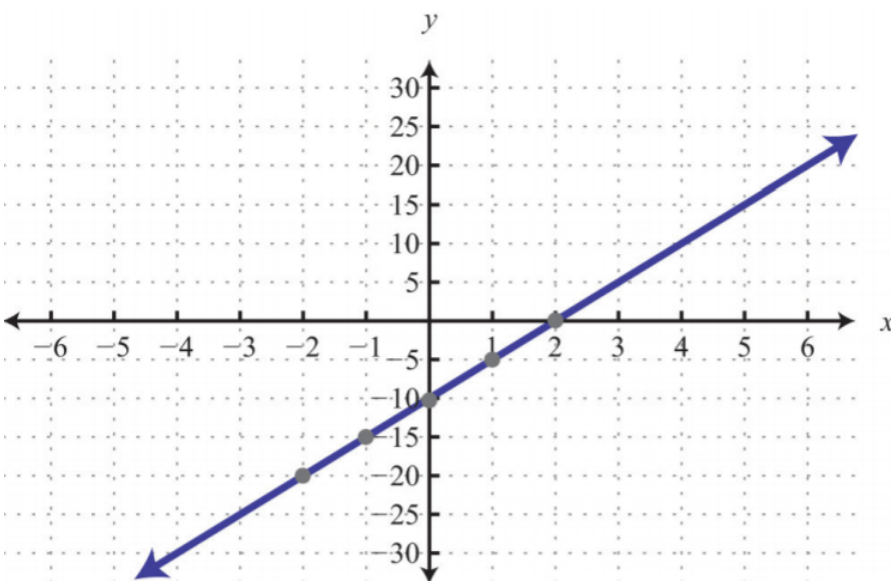


Figure 3.2.15

7.

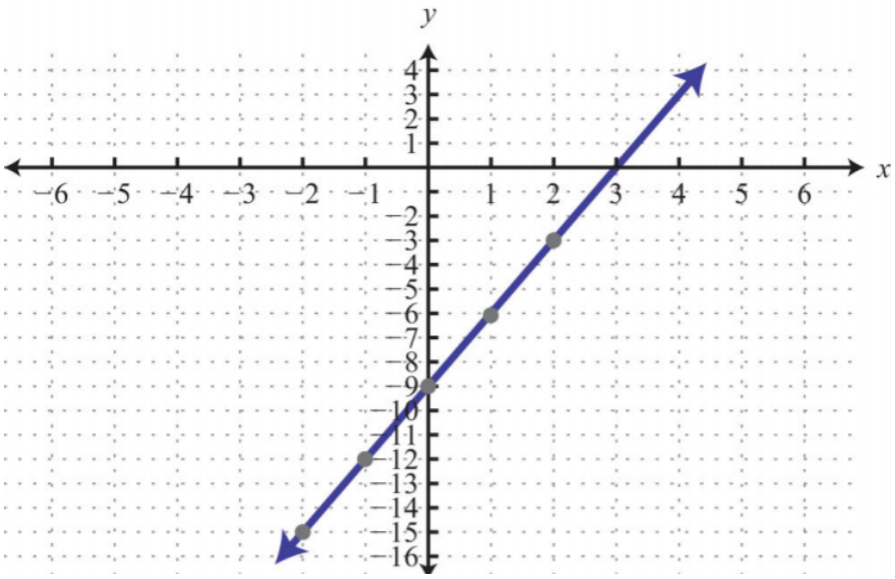


Figure 3.2.16

9.

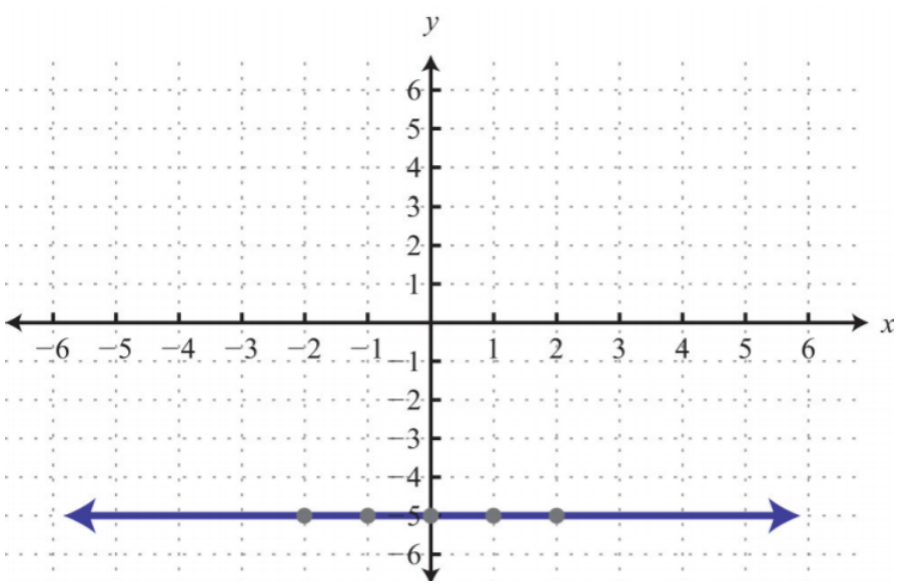


Figure 3.2.17

Exercise 3.2.9 **Graphing Lines**

1.

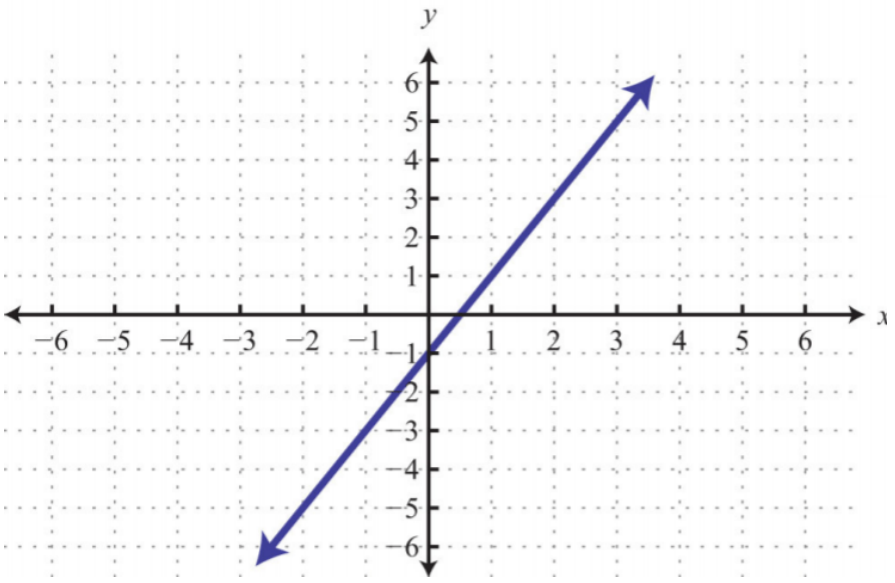


Figure 3.2.18

3.

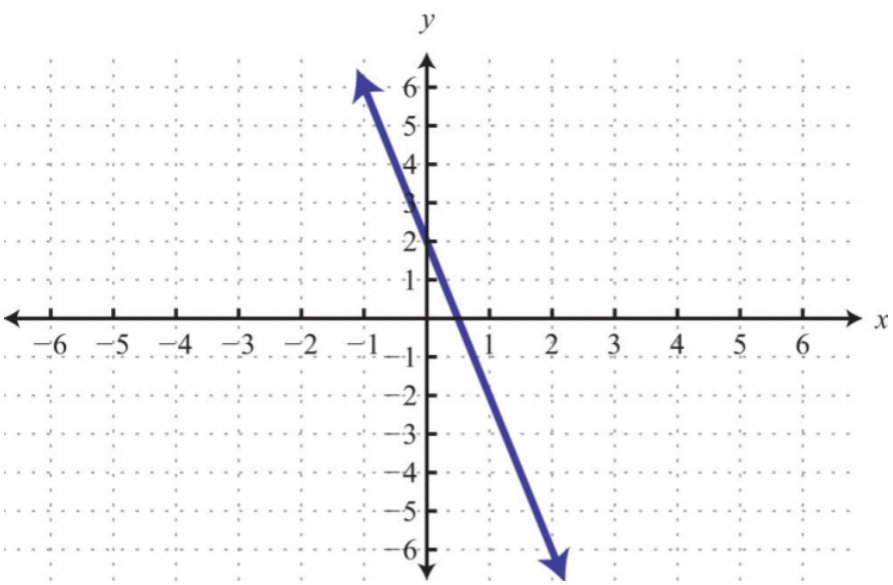


Figure 3.2.19

5.

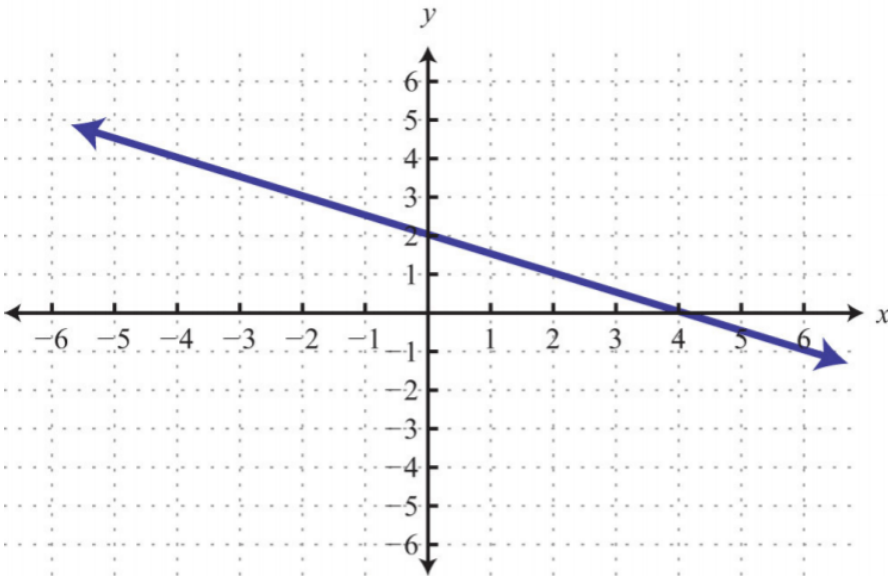


Figure 3.2.20

7.

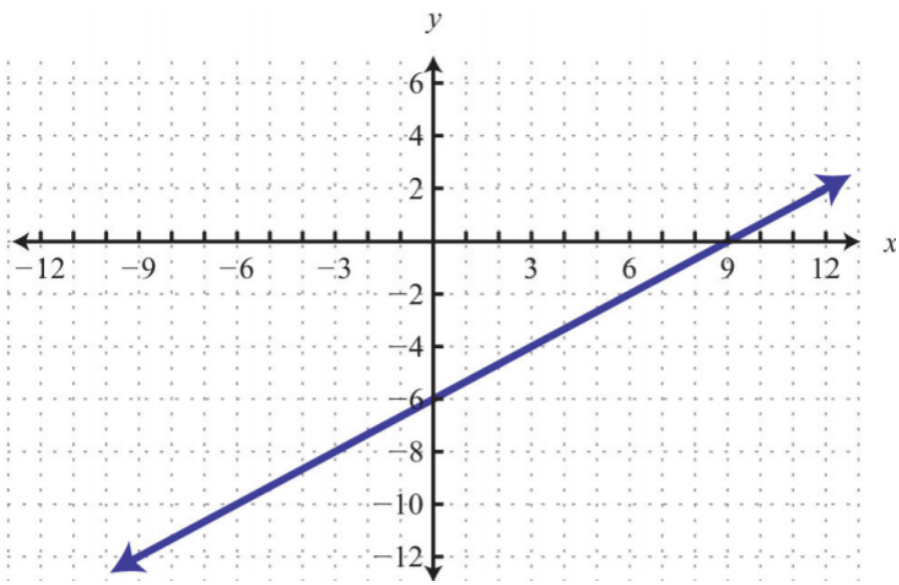


Figure 3.2.21

9.

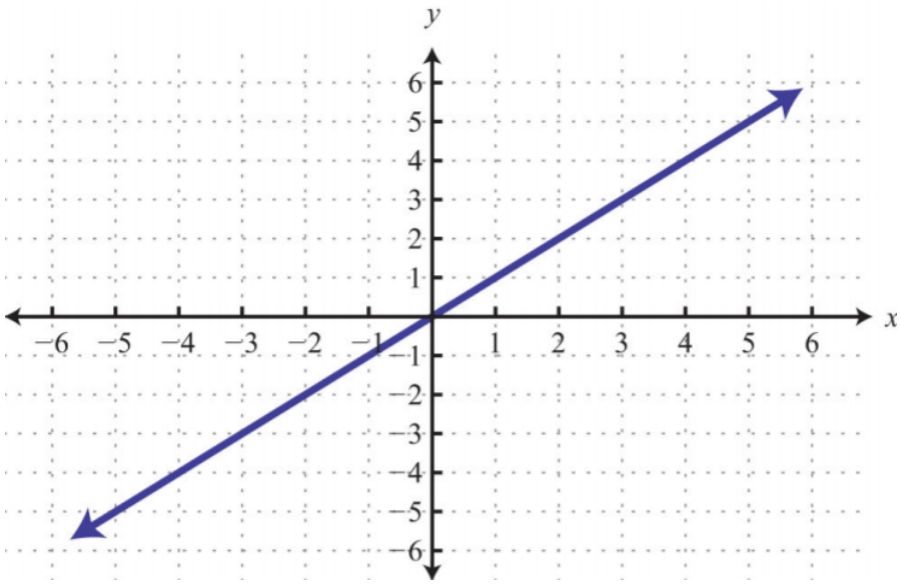


Figure 3.2.22

11.

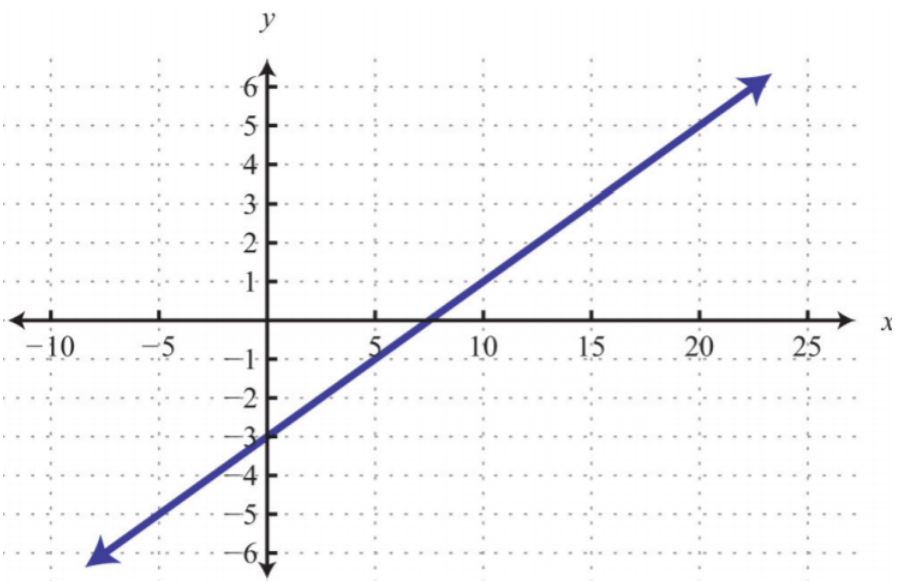


Figure 3.2.23

13.

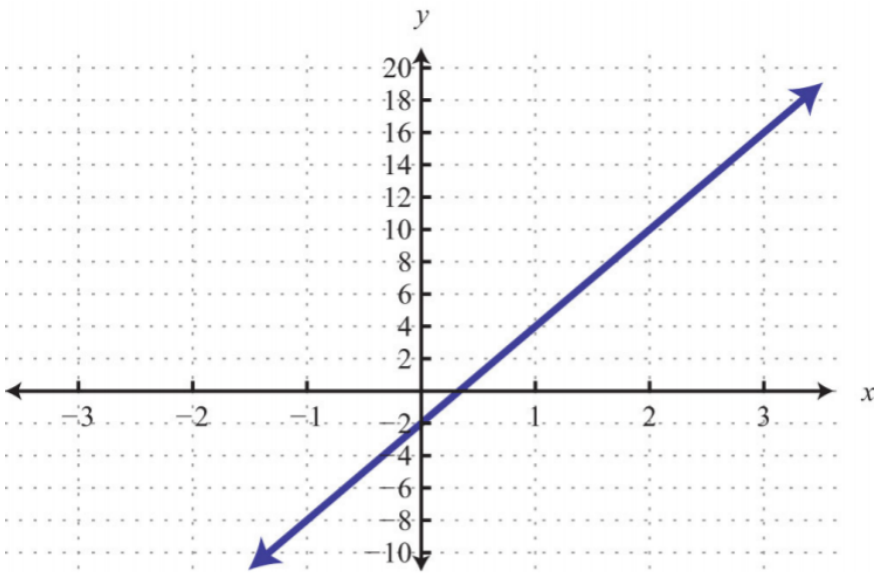


Figure 3.2.24

15.

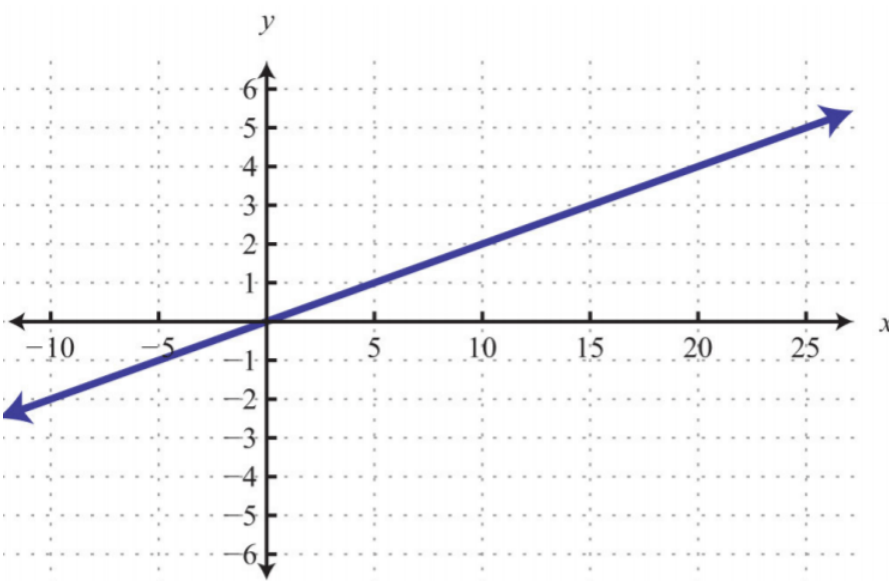


Figure 3.2.25

17.

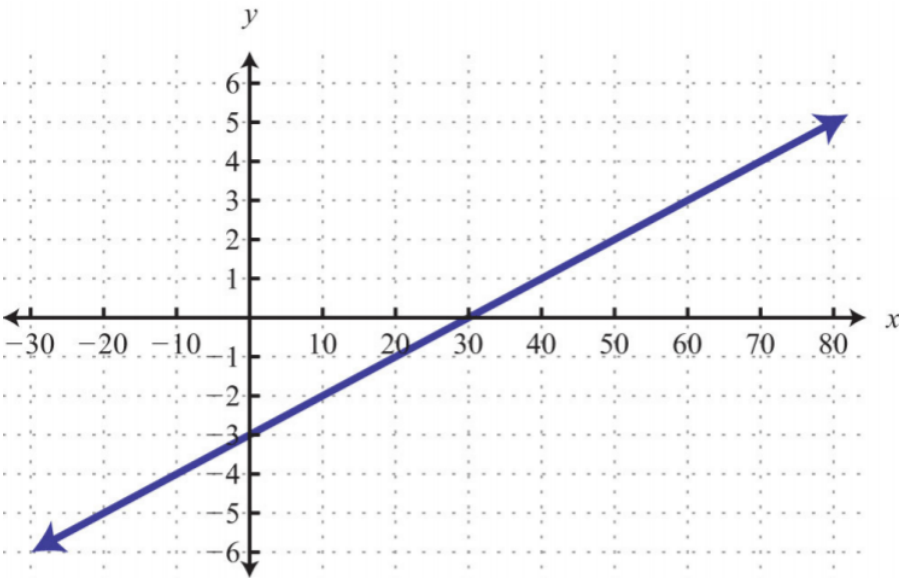


Figure 3.2.26

Exercise 3.2.10 **Horizontal and Vertical Lines**

1.

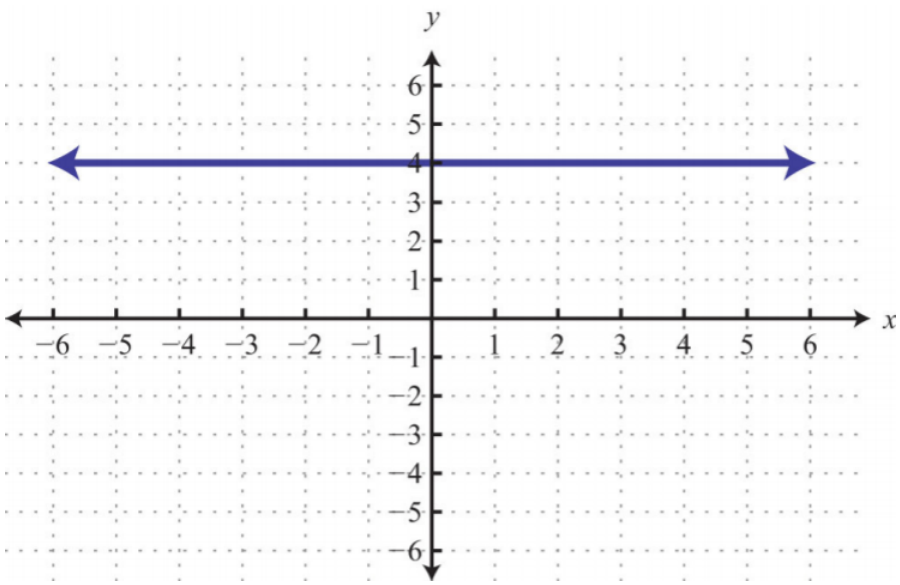


Figure 3.2.27

3.

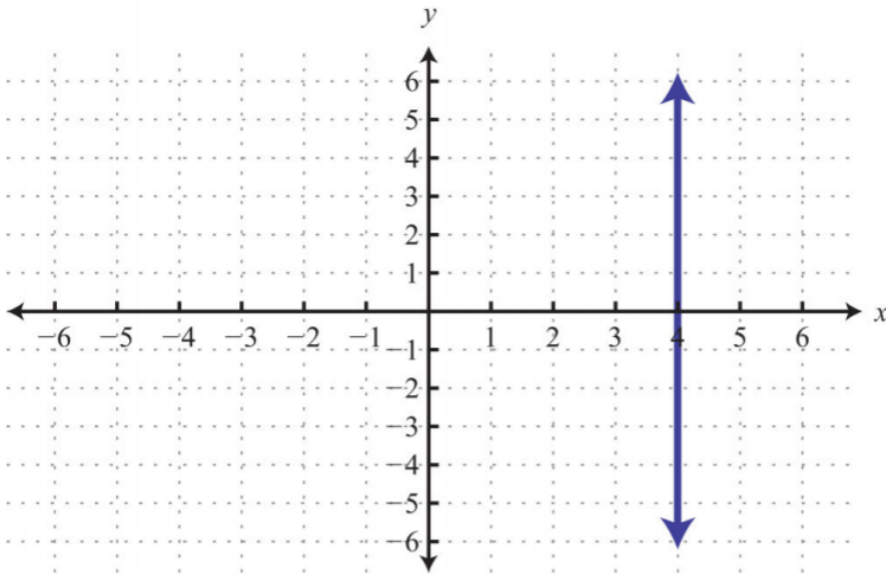


Figure 3.2.28

5.

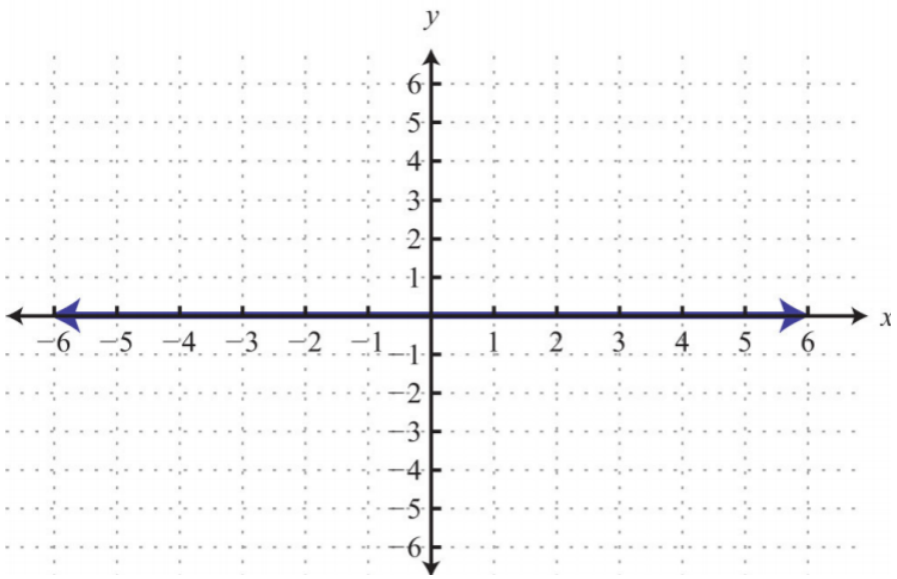


Figure 3.2.29

7.

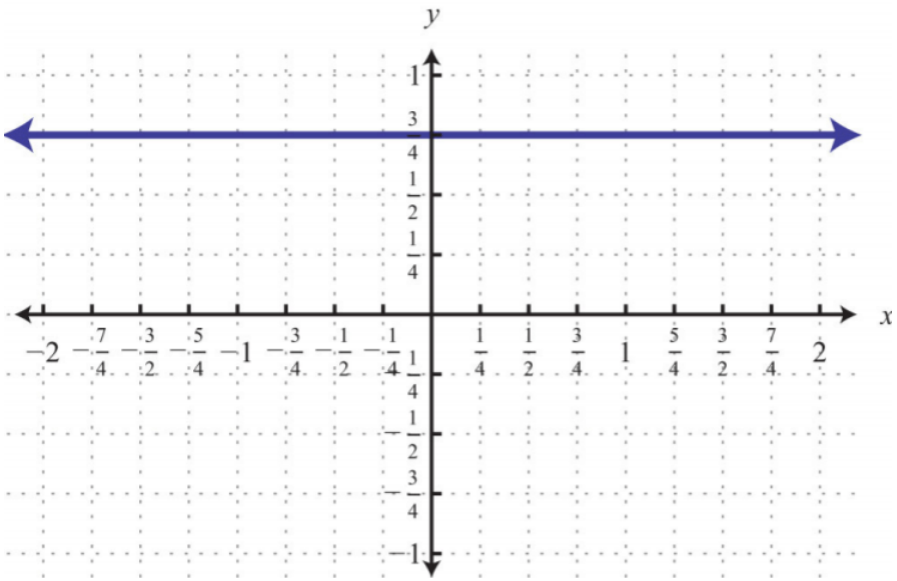


Figure 3.2.30

9.

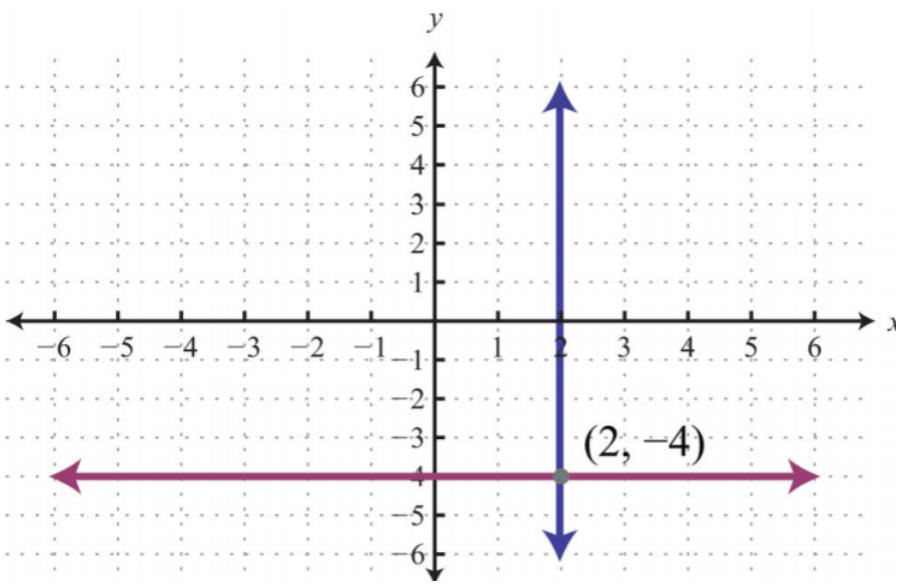


Figure 3.2.31

11. $y = 0$

Exercise 3.2.11 **Mixed Practice**

1.

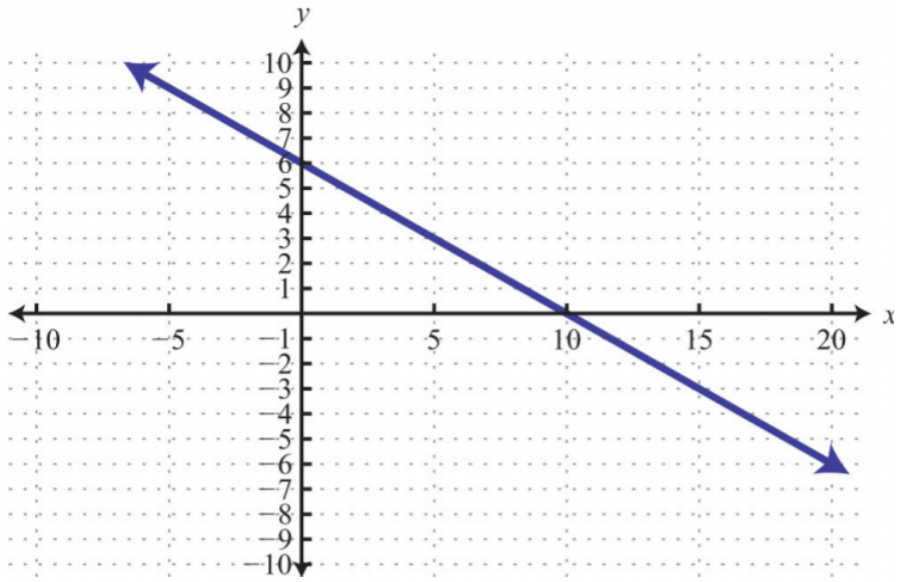


Figure 3.2.32

3.

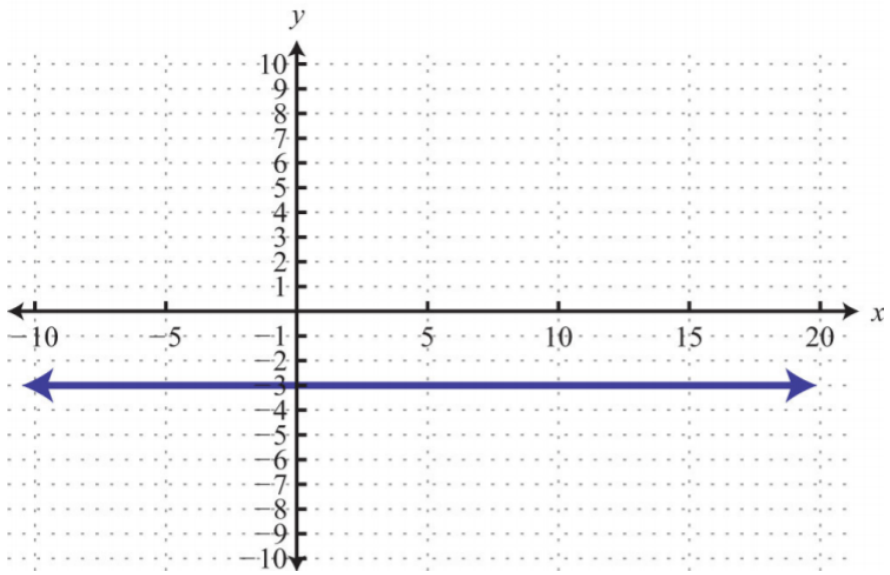


Figure 3.2.33

5.

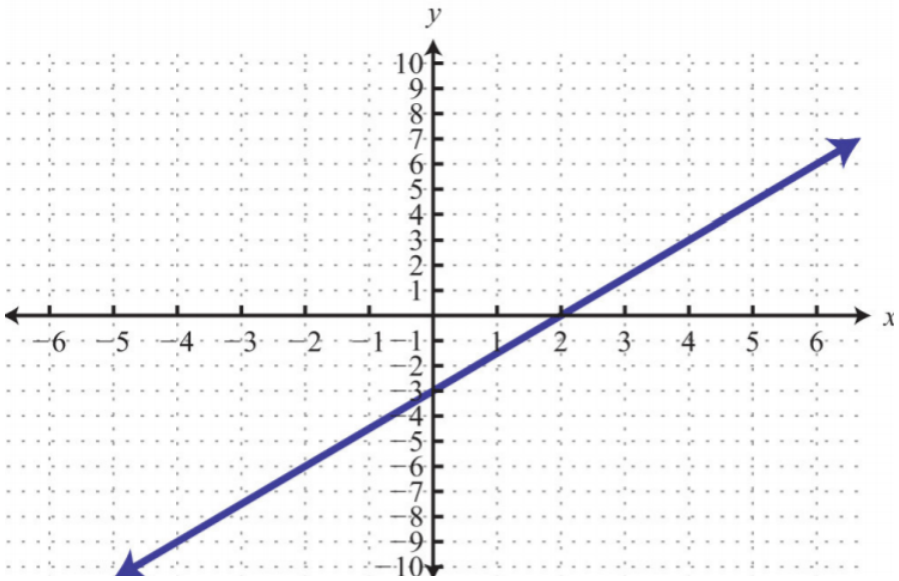


Figure 3.2.34

Exercise 3.2.12 **Discussion Board Topics**

1. Answers may vary

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3.3: Graph Using Intercepts

Learning Objectives

- Identify and find x- and y-intercepts of a graph.
- Graph a line using x- and y-intercepts

Definition of x- and y-intercepts

The **x-intercept** is the point where the graph of a line intersects the x-axis.

The **y-intercept** is the point where the graph of a line intersects the y-axis.

These points have the form $(x, 0)$ and $(0, y)$, respectively.

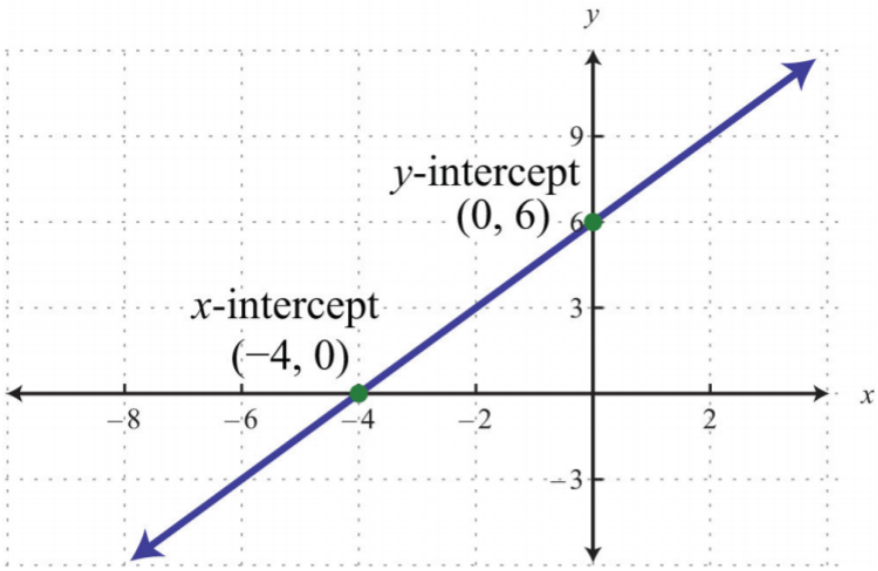


Figure 3.3.1

To find the x- and y-intercepts algebraically, use the fact that all x-intercepts have a y-value of zero and all y-intercepts have an x-value of zero. To find the y-intercept, set $x = 0$ and determine the corresponding y-value. Similarly, to find the x-intercept, set $y = 0$ and determine the corresponding x-value.

Example 3.3.1

Find the x- and y-intercepts:

$$-3x + 2y = 12.$$

Solution:

To find the x-intercept, set $y = 0$.

Therefore, the x-intercept is $(-4, 0)$. To find the y-intercept, set $x = 0$.

$$\begin{aligned} -3x+2y &= 12 \qquad \textit{To find the } y-\textit{intercept,} \\ &\downarrow \qquad\qquad\qquad\quad \textit{set } x=0. \\ -3(0)+2y &= 12 \\ 2y &= 12 \\ y &= 6 \end{aligned}$$

Hence the y-intercept is $(0,6)$. Note that this linear equation is graphed above.

Answer:

x-intercept: $(-4,0)$; y-intercept: $(0,6)$

Example 3.3.2

Find the x- and y-intercepts:

$$y=-3x+9$$

Solution:

Begin by finding the x-intercept.

$$\begin{aligned} y &= -3x+9 \qquad \textit{Set } y=0. \\ &\downarrow \\ 0 &= -3x+9 \qquad \textit{Solve for } x. \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

The x-intercept is $(3,0)$. Next, determine the y-intercept.

$$\begin{aligned} y &= -3x+9 \qquad \textit{Set } x=0. \\ &\downarrow \\ y &= -3(0)+9 \qquad \textit{Solve for } y. \\ y &= 9 \end{aligned}$$

The y-intercept is $(0,9)$.

Answer:

x-intercept: $(3,0)$; y-intercept: $(0,9)$

Keep in mind that the intercepts are ordered pairs and not numbers. In other words, the x-intercept is not $x=2$ but rather $(2,0)$. In addition, not all graphs necessarily have both intercepts: for example,

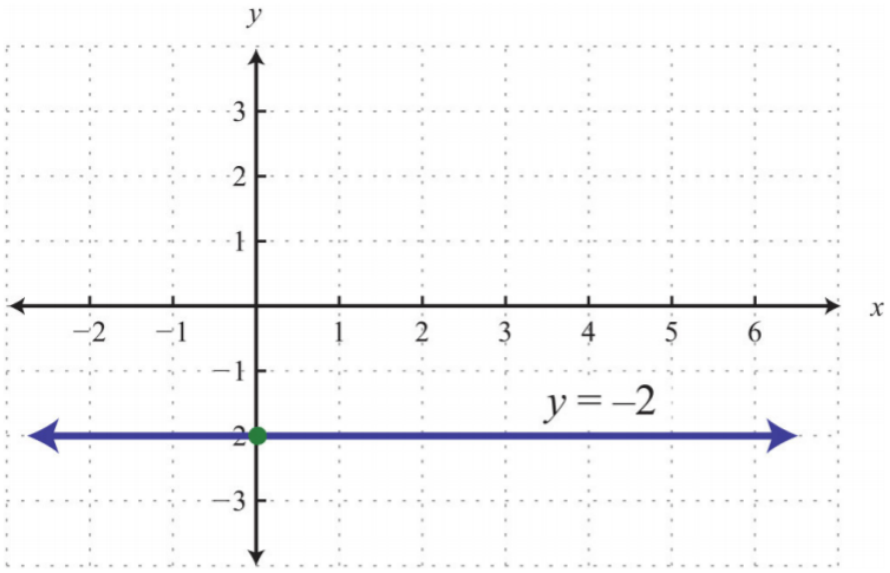


Figure 3.3.2

The horizontal line graphed above has a y-intercept of $(0, -2)$ and no x-intercept.

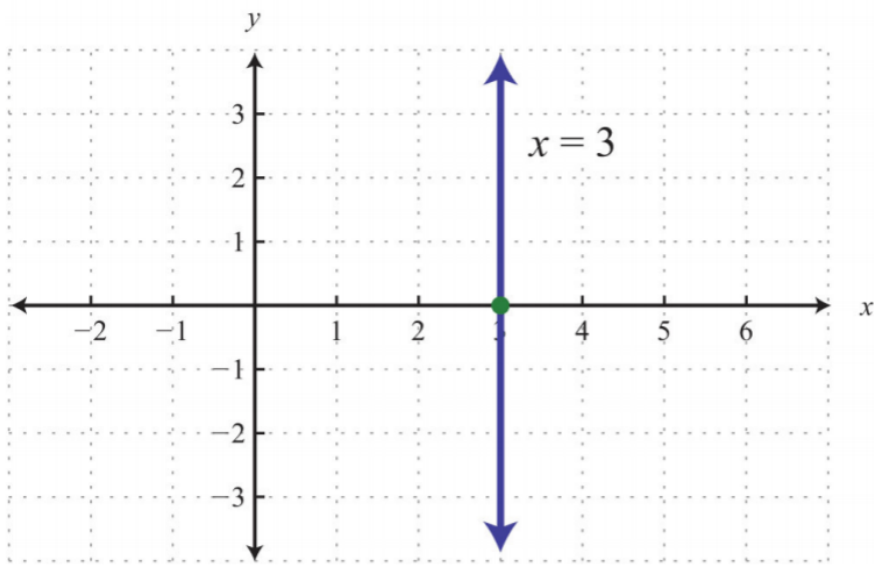


Figure 3.3.3

The vertical line graphed above has an x-intercept $(3, 0)$ and no y-intercept.

Exercise 3.3.1

Find the x- and y-intercepts:

$$4x - y = 2.$$

Answer available at the end of this section.

Graphing Lines Using Intercepts

Since two points determine a line, we can use the x- and y-intercepts to graph linear equations. We have just outlined an easy method for finding intercepts; now we outline the steps for graphing lines using the intercepts.

Example 3.3.3

Graph using intercepts:

$$2x - 3y = 12.$$

Solution:

Step 1: Find the x- and y-intercepts.

$To\ find\ the\ x-intercept,\ set\ y = 0.$	$To\ find\ the\ y-intercept,\ set\ x = 0.$
$2x - 3y = 12$	$2x - 3y = 12$
$2x - 3(0) = 12$	$2(0) - 3y = 12$
$2x = 12$	$-3y = 12$
$x = 6$	$y = -4$
$x-intercept:\ (6, 0)$	$y-intercept:\ (0, -4)$

Step 2: Plot the intercepts and draw the line through them. Use a straightedge to create a nice straight line. Add an arrow on either end to indicate that the line continues indefinitely in either direction.

Answer:

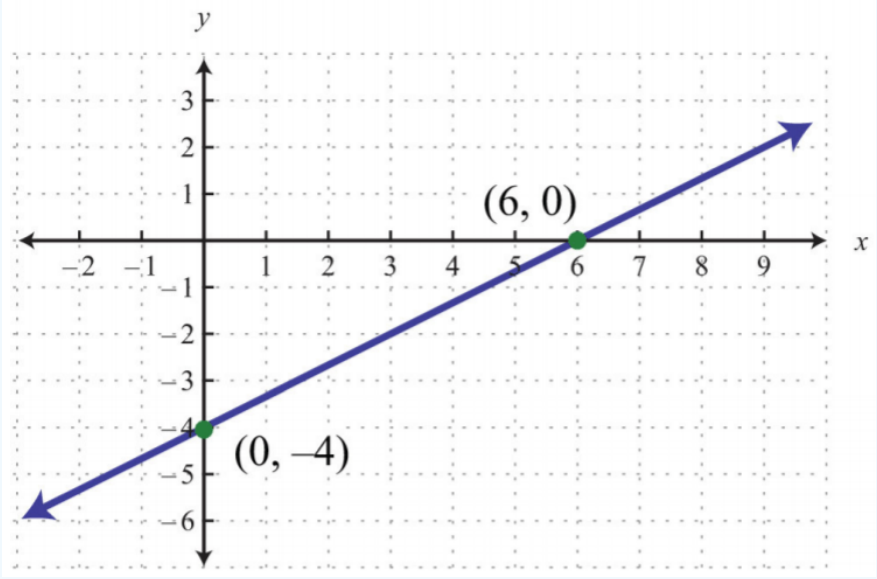


Figure 3.3.4

Example 3.3.4

Graph using intercepts:

$$y = -\tfrac{1}{5}x + 3.$$

Solution:

Begin by determining the x- and \(y\-intercepts.

$$\begin{array}{cc} \underline{x-intercept} & \underline{y-intercept} \\ y=-\frac{1}{5}x+3 & y=-\frac{1}{5}x+3 \\ 0=-\frac{1}{5}x+3 & y=-\frac{1}{5}(0)+3 \\ \frac{1}{5}x=3 & y=3 \\ 5\cdot\frac{1}{5}x=5\cdot 3 & y-intercept:\ (0,3) \\ x=15 & \\ x-intercept:\ (15,0) & \end{array}$$

Next, graph the two points and draw a line through them with a straighedge.

Answer:

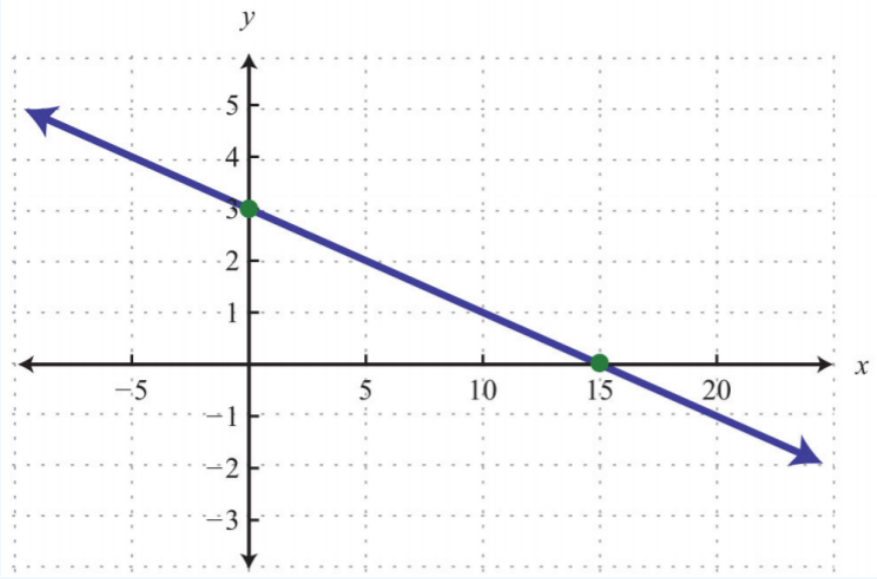


Figure 3.3.5

Example 3.3.5

Graph using intercepts:

$$y=-2x.$$

Solution:

$$\begin{array}{cc} \underline{x-intercept} & \underline{y-intercept} \\ y=-2x & y=-2x \\ 0=-2x & y=-2(0) \\ \frac{0}{-2}=\frac{-2x}{-2} & y=0 \\ 0=x & y-intercept:\ (0,0) \\ x-intercept:\ (0,0) & \end{array}$$

Here the x- and y-intercepts are actually the same point, the origin. We will need at least one more point so that we can graph the line. Choose any value for x and determine the corresponding value for y.

x	y
-1	2
1	-2

$y = -2(-1) = 2 = 2$

$y = -2(1) = -2$

Figure 3.3.6

Use the ordered pair solutions $(0, 0), (-1, 2)$, and $(1, -2)$ to graph the line.

Answer:

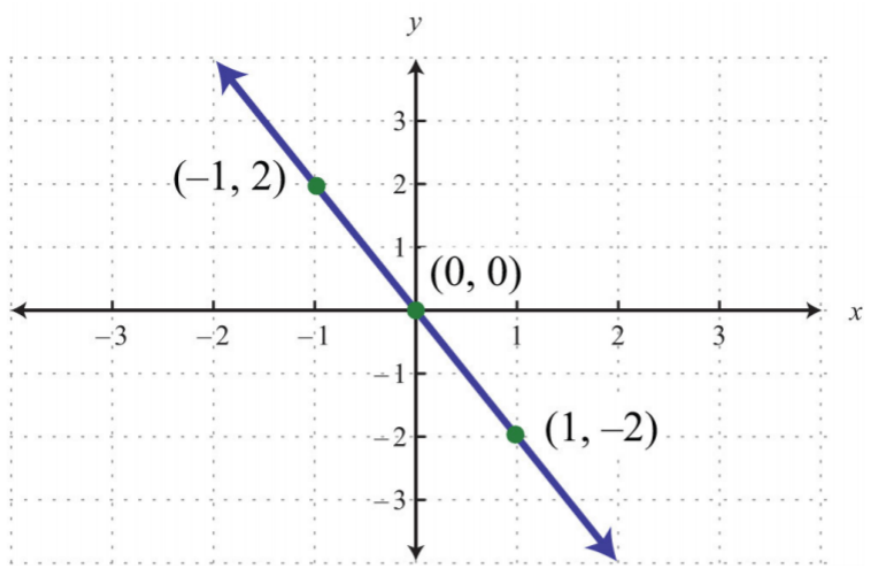


Figure 3.3.7

To summarize, any linear equation can be graphed by finding two points and connecting them with a line drawn with a straightedge. Two important and useful points are the x- and y-intercepts; find these points by substituting $y = 0$ and $x = 0$, respectively. This method for finding intercepts will be used throughout our study of algebra.

Exercise 3.3.2

Graph using intercepts:

$$3x - 5y = 15.$$

Answer available at the end of this section.

Finding Intercepts Given the Graph

The x- and y-intercepts are important points on any graph. This chapter will focus on the graphs of linear equations. However, at this point, we can use these ideas to determine intercepts of nonlinear graphs. Remember that intercepts are ordered pairs that indicate where the graph intersects the axes.

Example 3.3.6

Find the x- and y-intercepts given the following graph:

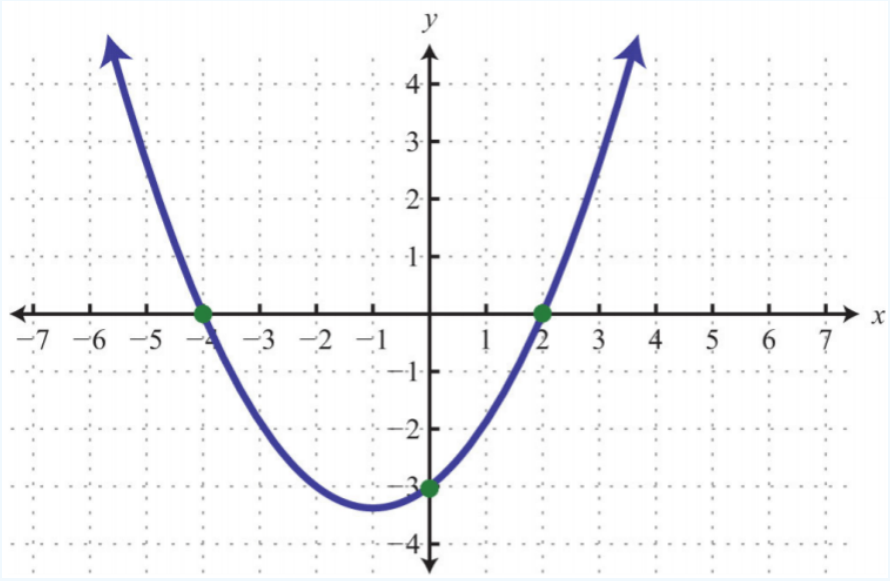


Figure 3.3.8

Solution:

We see that the graph intersects the x-axis in two places. This graph has two x-intercepts, namely, $(-4, 0)$ and $(2, 0)$. Furthermore, the graph intersects the y-axis in one place. The only y-intercept is $(0, -3)$.

Answer:

x-intercepts: $(-4, 0), (2, 0)$; y-intercept: $(0, -3)$

In our study of algebra, we will see that some graphs have many intercepts. Also, we will see that some graphs do not have any.

Example 3.3.7

Given the following graph, find the x- and y-intercepts:

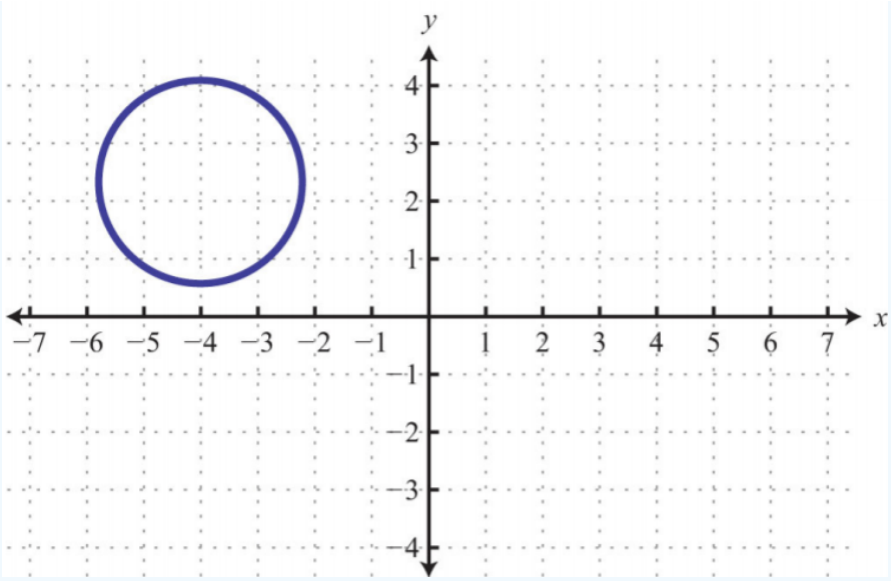


Figure 3.3.9

Solution:

This is a graph of a circle; we can see that it does not intersect either axis. Therefore, this graph does not have any intercepts.

Answer:

None

Key Takeaways

- Since two points determine any line, we can graph lines using the x- and y-intercepts.
- To find the x-intercept, set $y = 0$ and solve for x.
- To find the y-intercept, set $x = 0$ and solve for y.
- This method of finding x- and y-intercepts will be used throughout our study of algebra because it works for any equation.
- To graph a line, find the intercepts, if they exist, and draw a straight line through them. Use a straightedge to create the line and include arrows on either end to indicate that the line extends infinitely in either direction.
- Horizontal and vertical lines do not always have both x- and y-intercepts.

Exercise 3.3.3 Intercepts

Given the graph, find the x- and y-intercepts.

1.

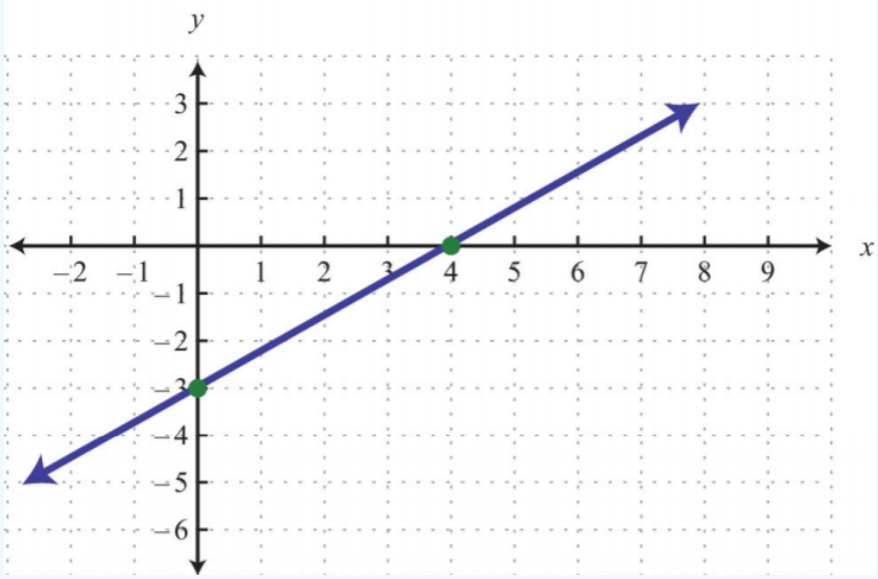


Figure 3.3.10

2.

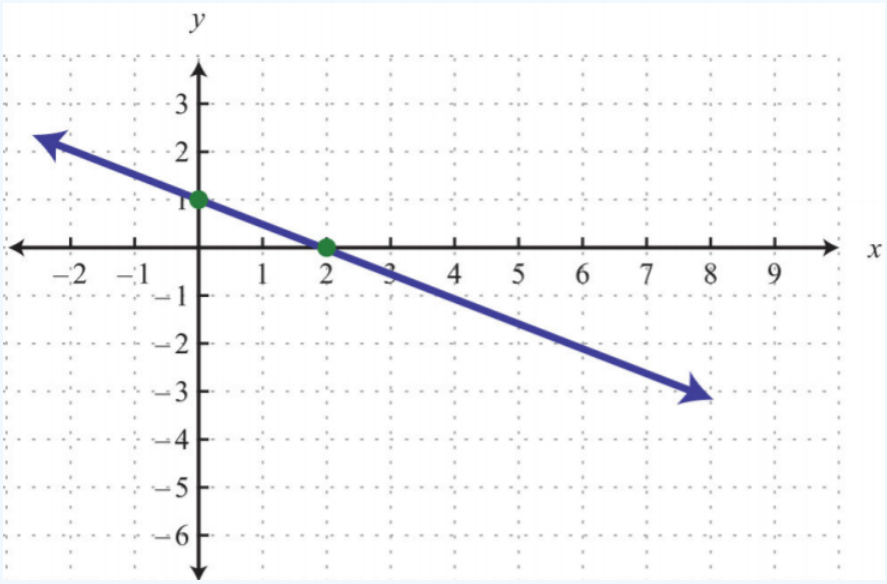


Figure 3.3.11

3.

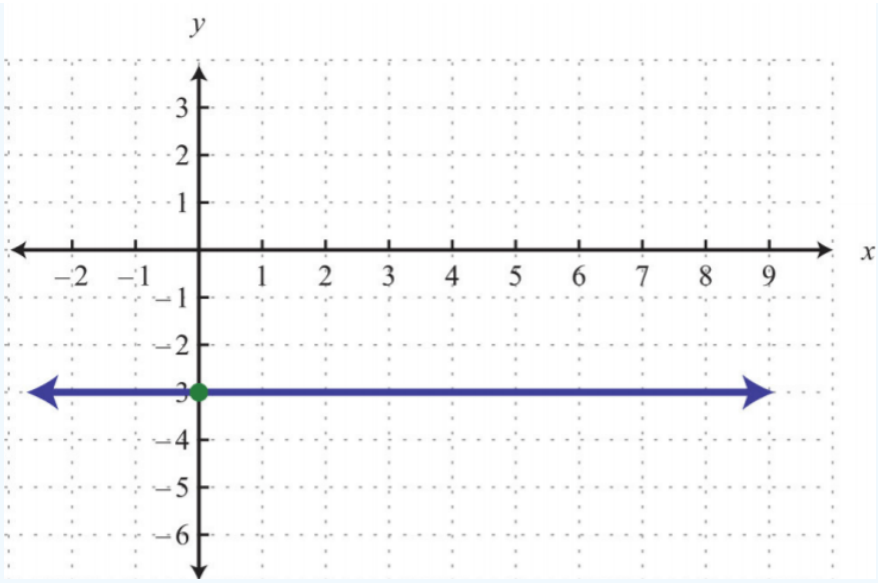


Figure 3.3.12

4.

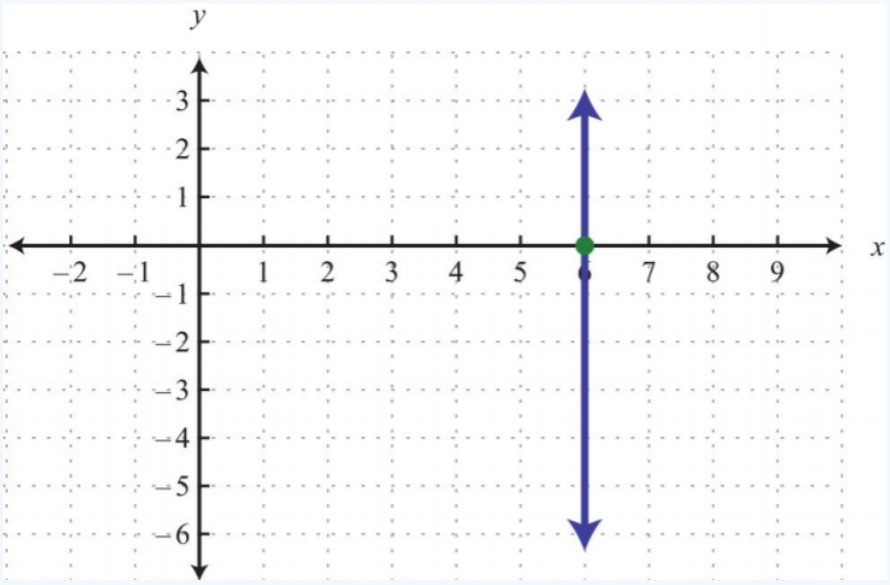


Figure 3.3.13

5.

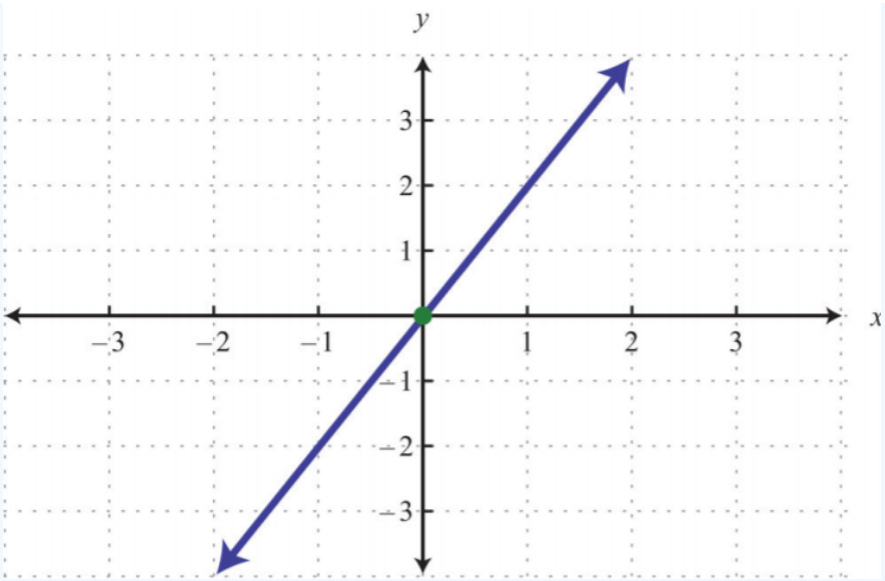


Figure 3.3.14

6.

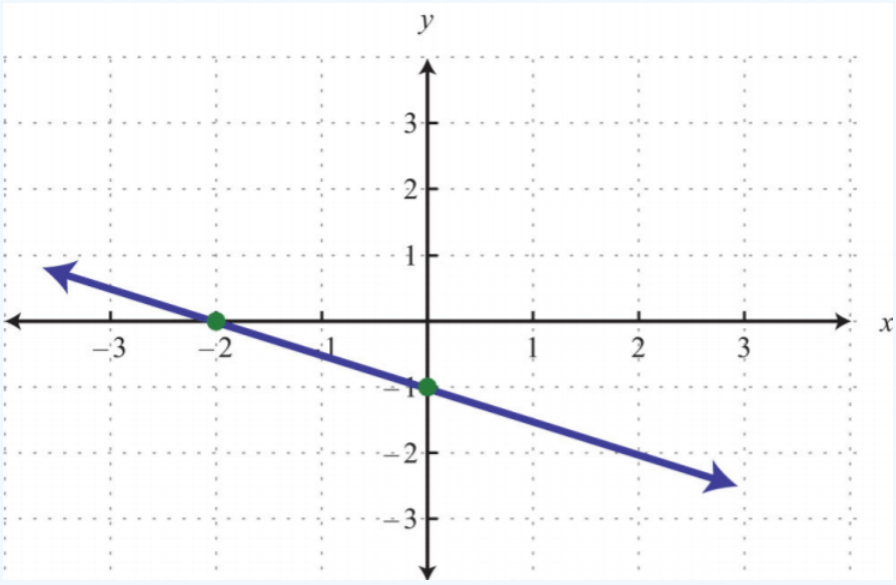


Figure 3.3.15

Answer available at the end of this section.

Exercise 3.3.4 Intercepts

Find the x- and y-intercepts.

1. $5x - 4y = 20$
2. $-2x + 7y = -28$
3. $x - y = 3$
4. $-x + y = 0$

5. $3x - 4y = 1$
6. $-2x + 5y = 3$
7. $\frac{1}{4}x - \frac{1}{3}y = 1$
8. $-\frac{2}{5}x + \frac{3}{4}y = 2$
9. $y = 6$
10. $y = -3$
11. $x = 2$
12. $x = -1$
13. $y = mx + b$
14. $ax + by = c$

Answer available at the end of this section.

Exercise 3.3.5 Graph Using Intercepts

Find the intercepts and graph them.

1. $3x + 4y = 12$
2. $-2x + 3y = 6$
3. $5x - 2y = 10$
4. $-4x - 8y = 16$
5. $-\frac{1}{2}x + \frac{1}{3}y = 1$
6. $\frac{3}{4}x - \frac{1}{2}y = -3$
7. $2x - \frac{5}{2}y = 10$
8. $2x - \frac{7}{3}y = -14$
9. $4x - y = -8$
10. $6x - y = 6$
11. $-x + 2y = 1$
12. $3x + 4y = 6$
13. $2x + y = -1$
14. $-2x + 6y = 3$
15. $15x + 4y = -60$
16. $-25x + 3y = 75$
17. $4x + 2y = 0$
18. $3x - y = 0$
19. $-12x + 6y = -4$
20. $3x + 12y = -4$
21. $y = 2x + 4$
22. $y = -x + 3$
23. $y = \frac{1}{2}x + 1$
24. $y = \frac{2}{3}x - 3$
25. $y = -\frac{2}{5}x + 1$
26. $y = -\frac{5}{8}x - \frac{5}{4}$
27. $y = -\frac{7}{8}x - \frac{7}{2}$
28. $y = -x + \frac{3}{2}$
29. $y = 3$
30. $y = \frac{3}{2}$
31. $x = 5$
32. $x = -2$
33. $y = 5x$
34. $y = -x$

Answer available at the end of this section.

Exercise 3.3.6 Intercepts of Nonlinear Graphs

Given the graph find the x- and y-intercepts.

1.

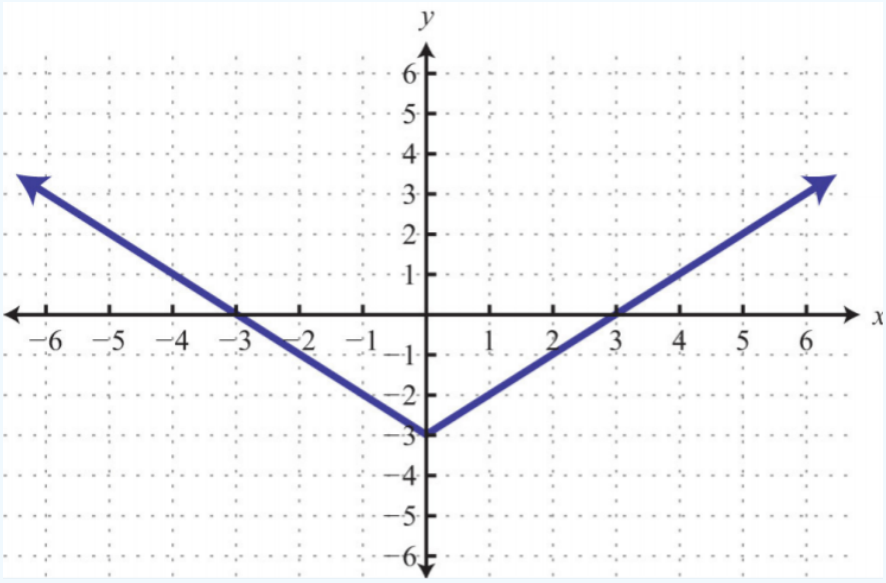


Figure 3.3.33

2.

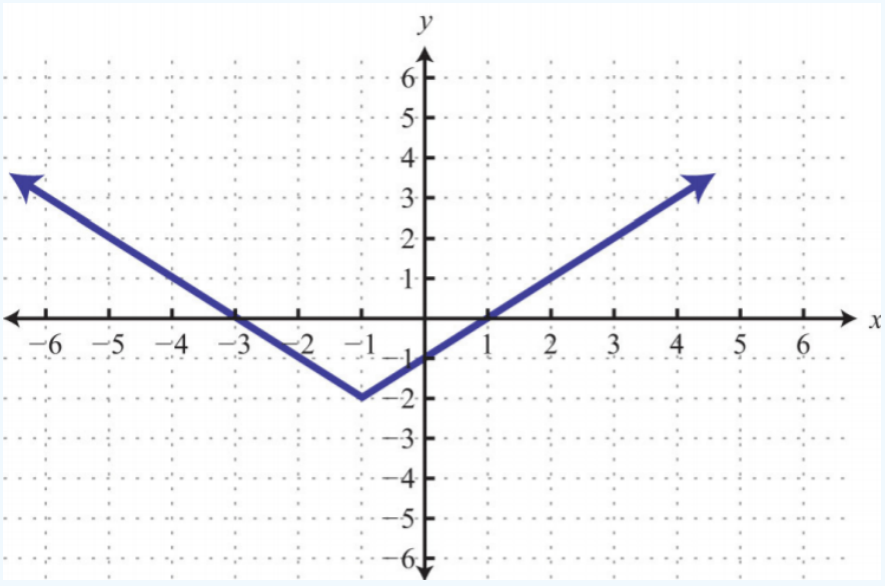


Figure 3.3.34

3.

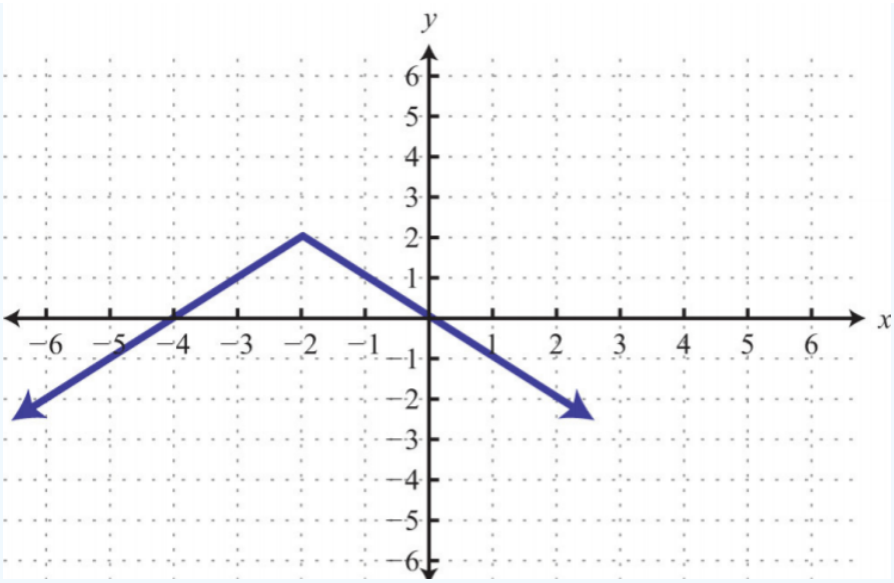


Figure 3.3.35

4.

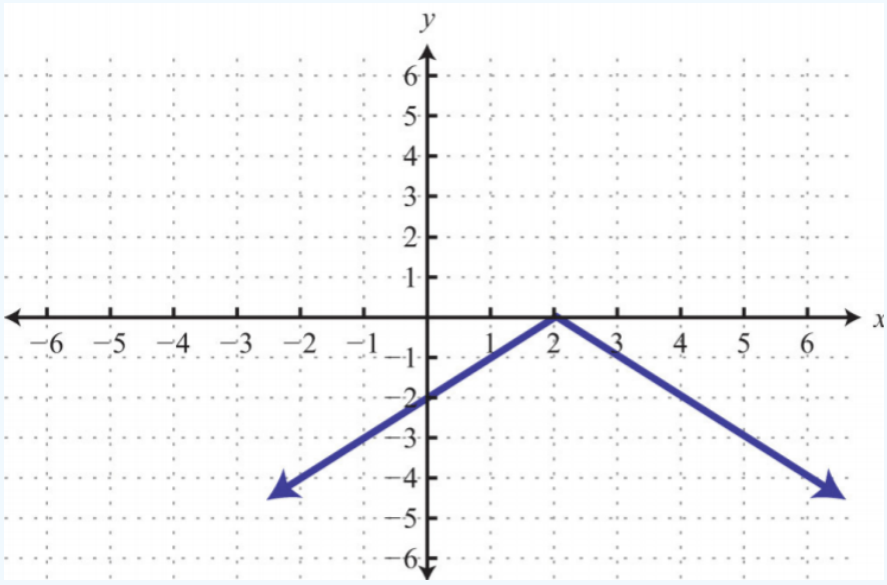


Figure 3.3.36

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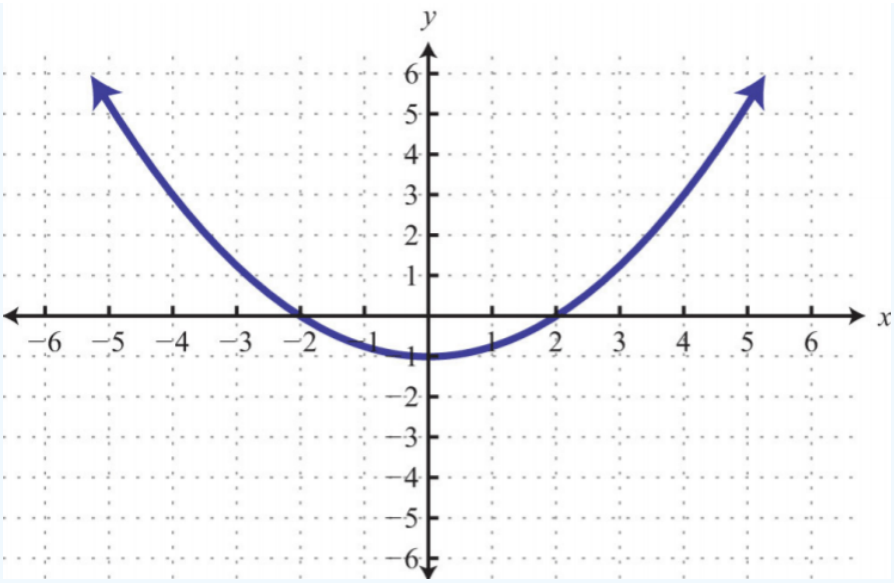


Figure 3.3.37

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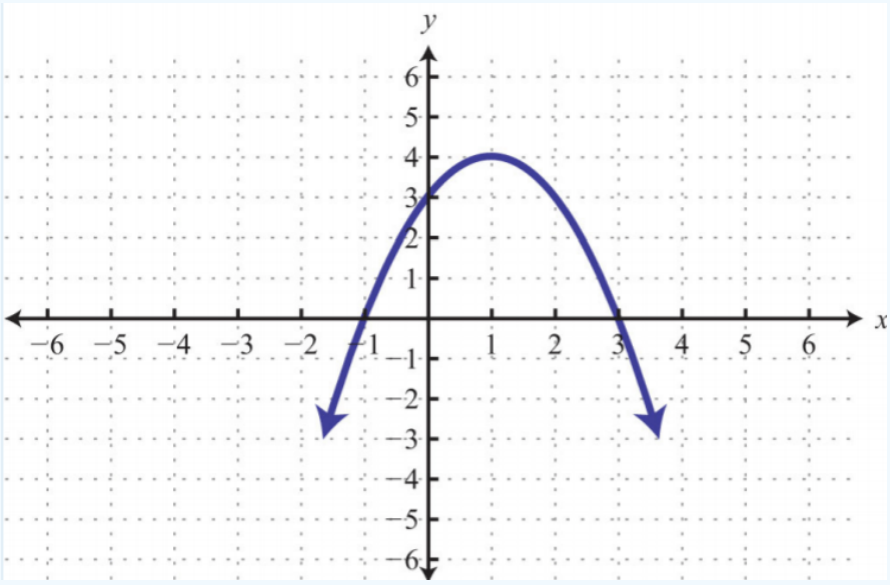


Figure 3.3.38

7.

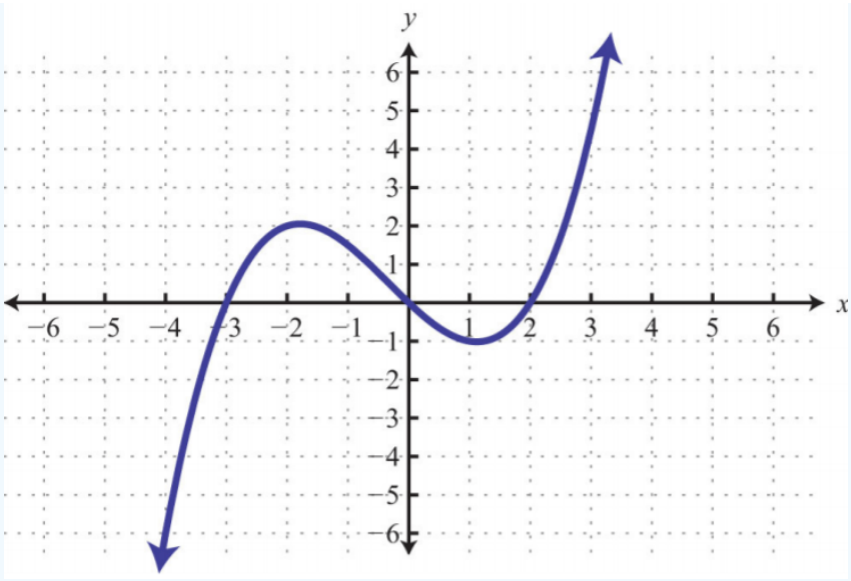


Figure 3.3.39

8.

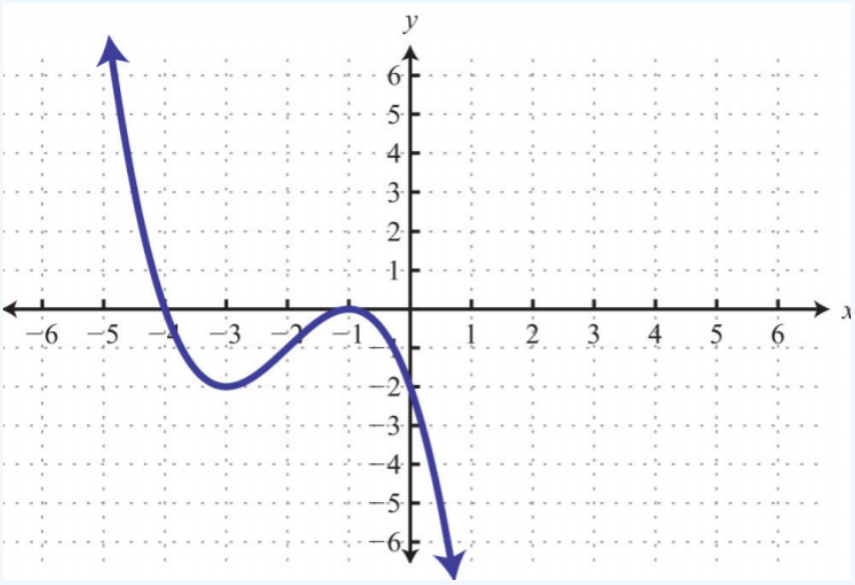


Figure 3.3.40

9.

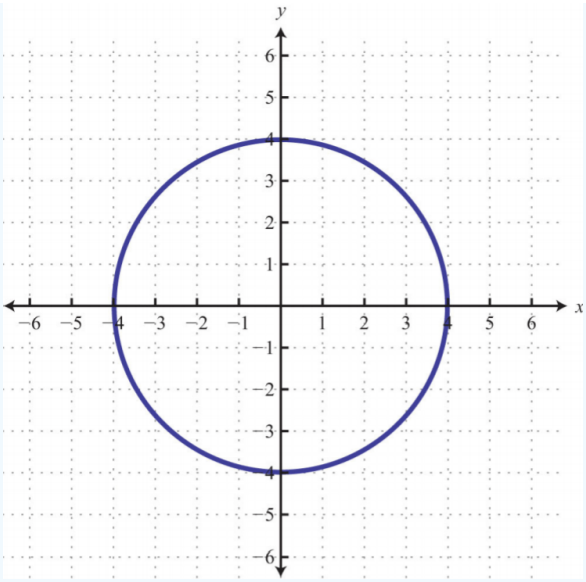


Figure 3.3.41

10.

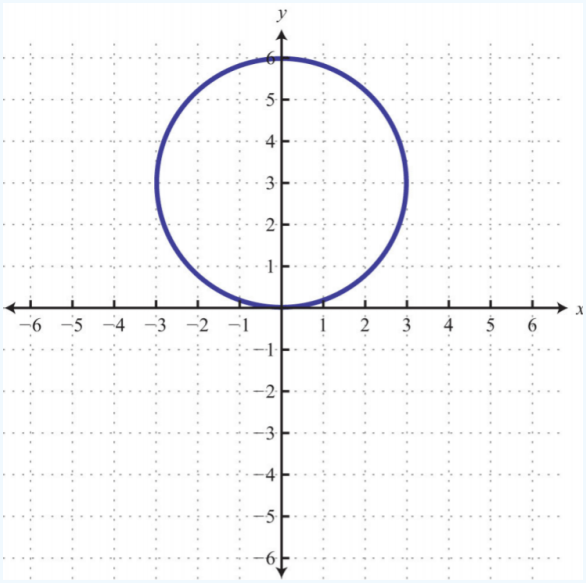


Figure 3.3.42

Answer available at the end of this section.

Exercise 3.3.7 Discussion Board Topics

1. What are the x-intercepts of the line $y = 0$?
2. What are the y-intercepts of the line $x = 0$?
3. Do all lines have intercepts?
4. How many intercepts can a circle have? Draw circles showing all possible numbers of intercepts.
5. Research and post the definitions of line segment, ray, and line. Why are the arrows important?

Answer available at the end of this section.

Answers

Exercise 3.3.1

x-intercept: $(\frac{1}{2}, 0)$; y-intercept: $(0, -2)$

Exercise 3.3.2

x-intercept: $(5, 0)$; y-intercept: $(0, -3)$

Exercise 3.3.3 **Intercepts**

1. y-intercept: $(0, -3)$; x-intercept: $(4, 0)$
3. y-intercept: $(0, -3)$; x-intercept: none
5. y-intercept: $(0, 0)$; x-intercept: $(0, 0)$

Exercise 3.3.4 **Intercepts**

1. x-intercept: $(4, 0)$; y-intercept: $(0, -5)$
3. x-intercept: $(3, 0)$; y-intercept: $(0, -3)$
5. x-intercept: $(\frac{1}{3}, 0)$; y-intercept: $(0, -\frac{1}{4})$
7. x-intercept: $(4, 0)$; y-intercept: $(0, -3)$
9. x-intercept: none; y-intercept: $(0, 6)$
11. x-intercept: $(2, 0)$; y-intercept: none
13. x-intercept: $(-\frac{b}{m}, 0)$; y-intercept: $(0, b)$

Exercise 3.3.5 **Graph Using Intercepts**

1.

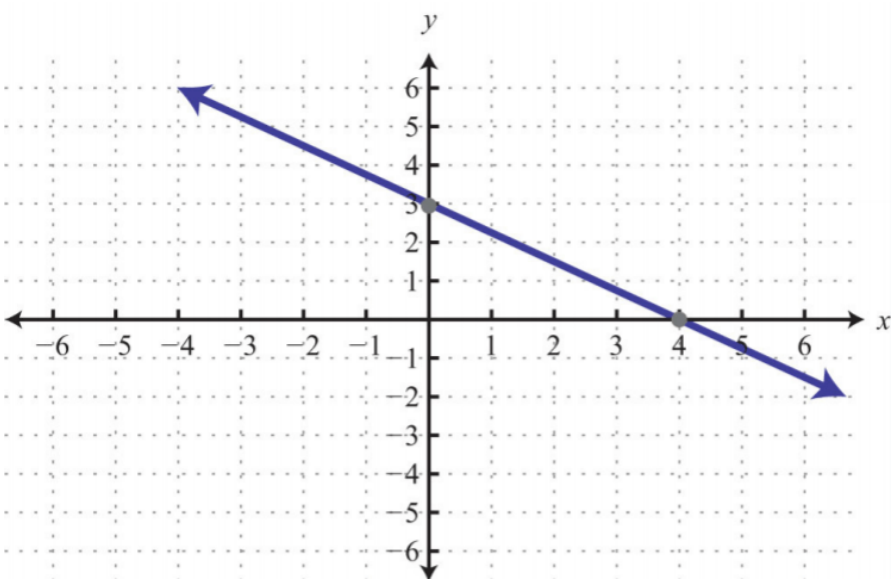


Figure 3.3.16

3.

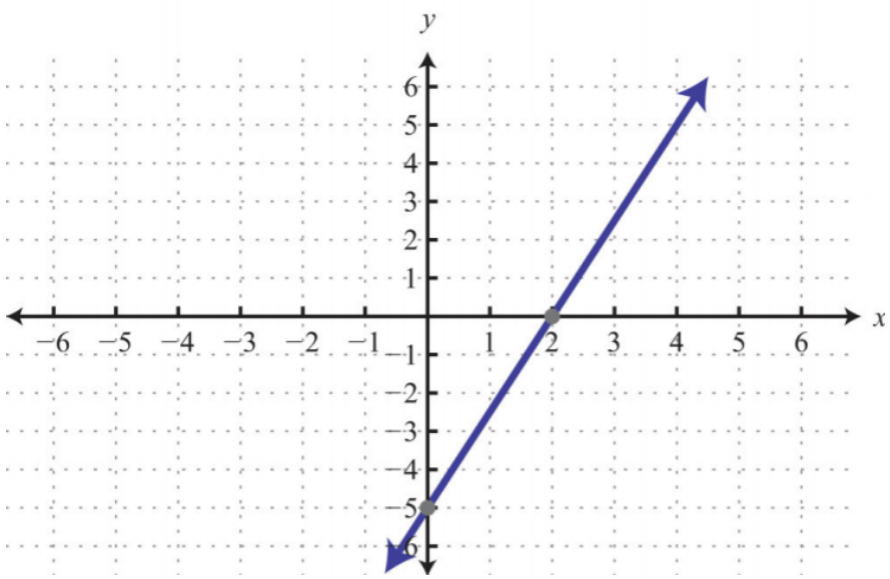


Figure 3.3.17

5.

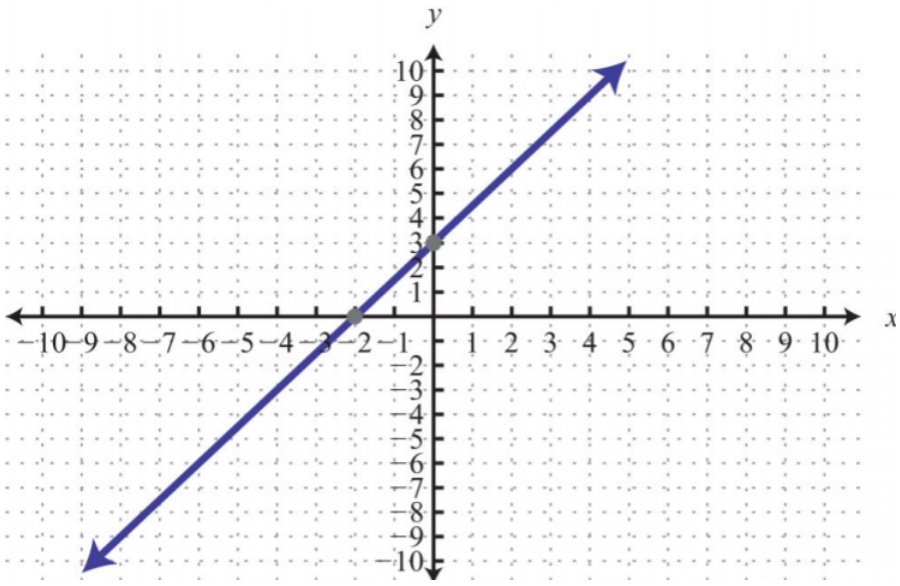


Figure 3.3.18

7.

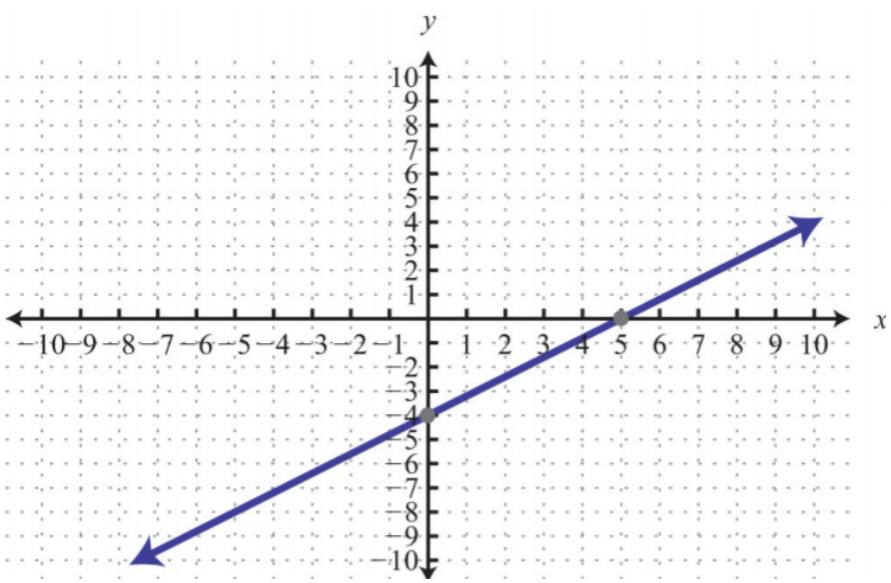


Figure 3.3.19

9.

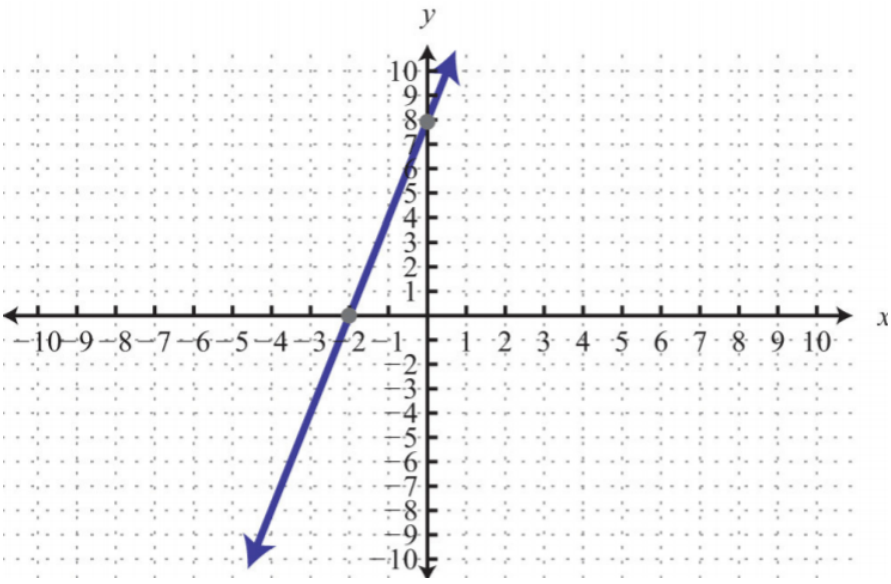


Figure 3.3.20

11.

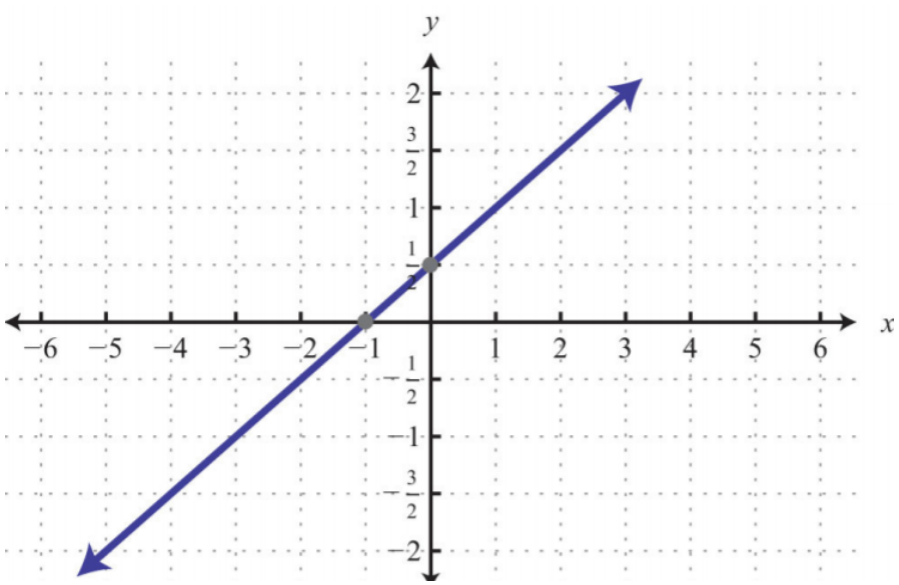


Figure 3.3.21

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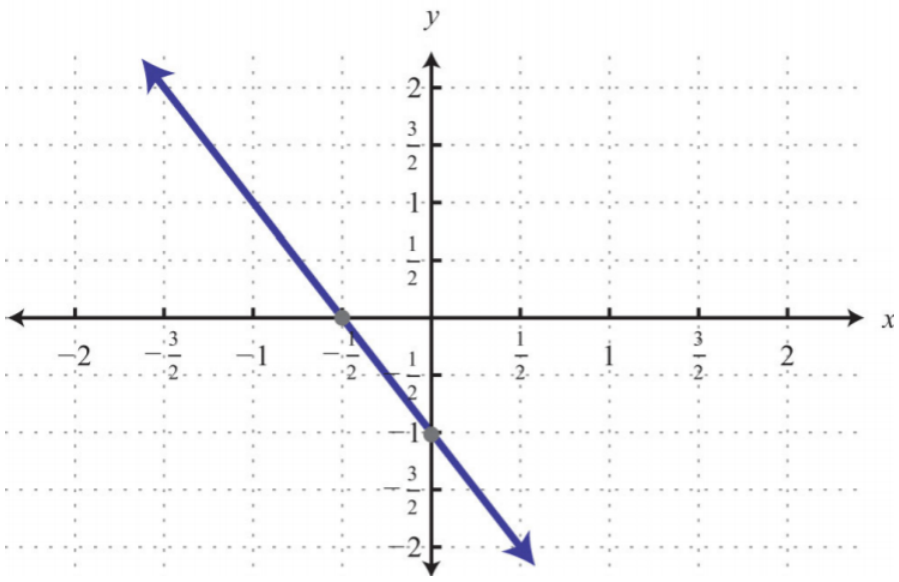


Figure 3.3.22

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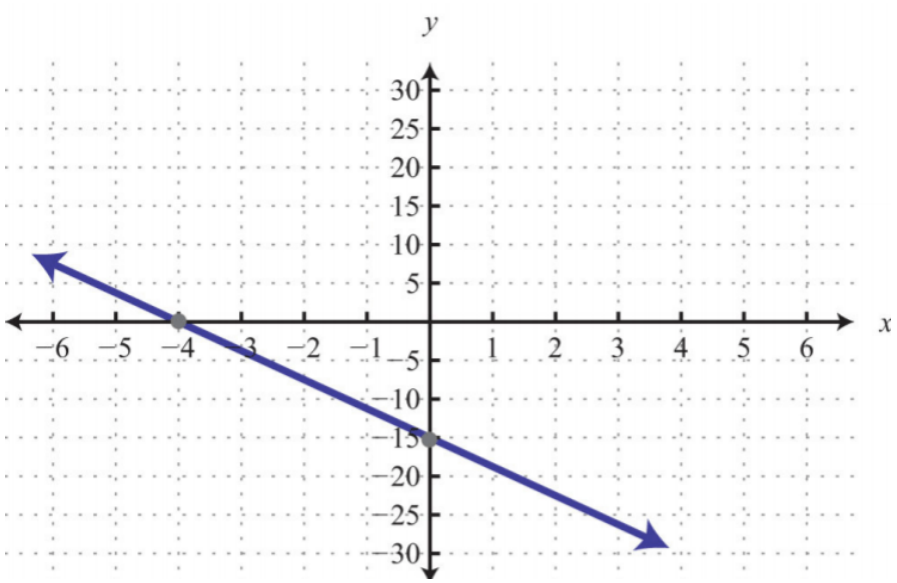


Figure 3.3.23

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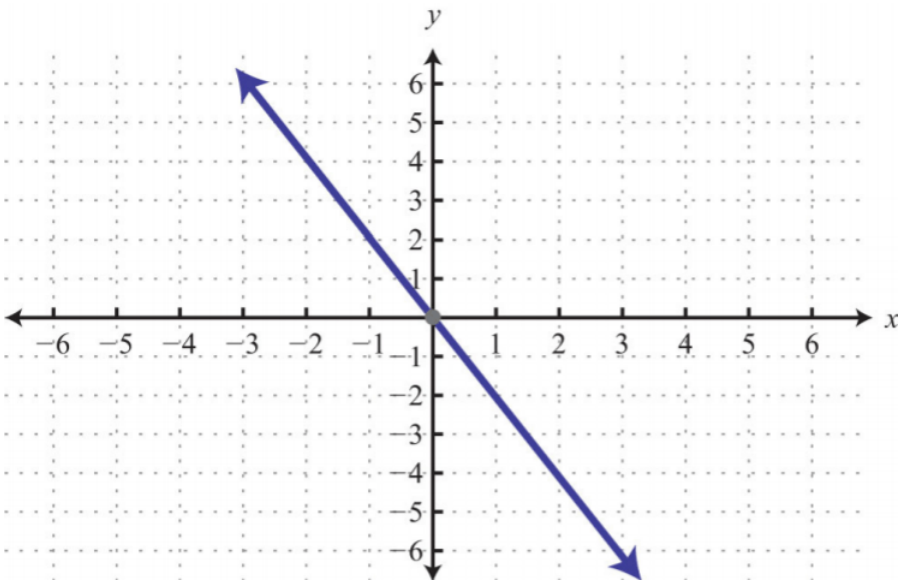


Figure 3.3.24

19.

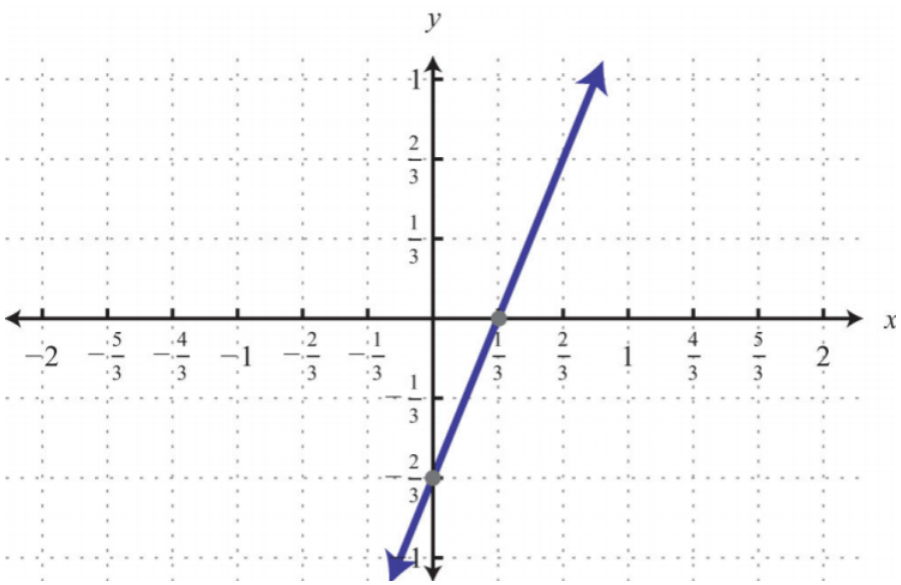


Figure 3.3.25

21.

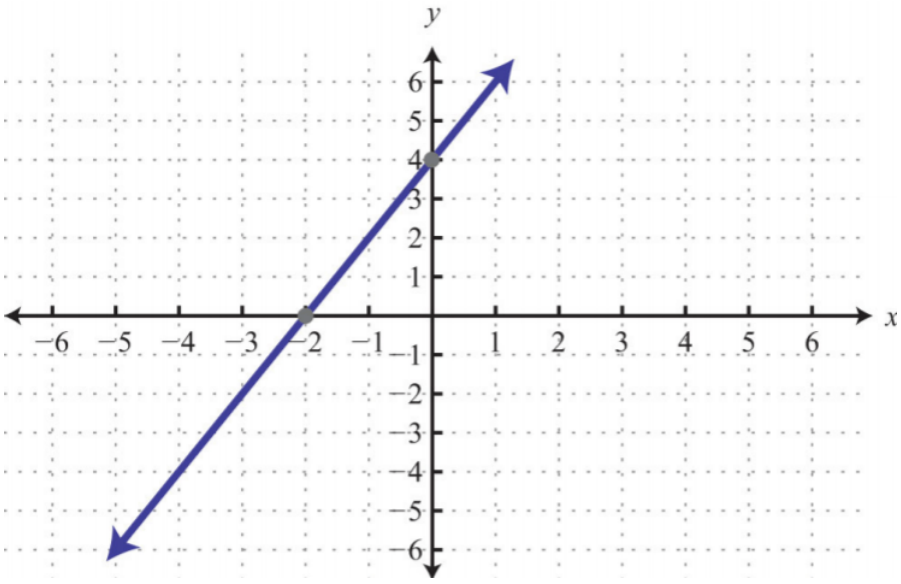


Figure 3.3.26

23.

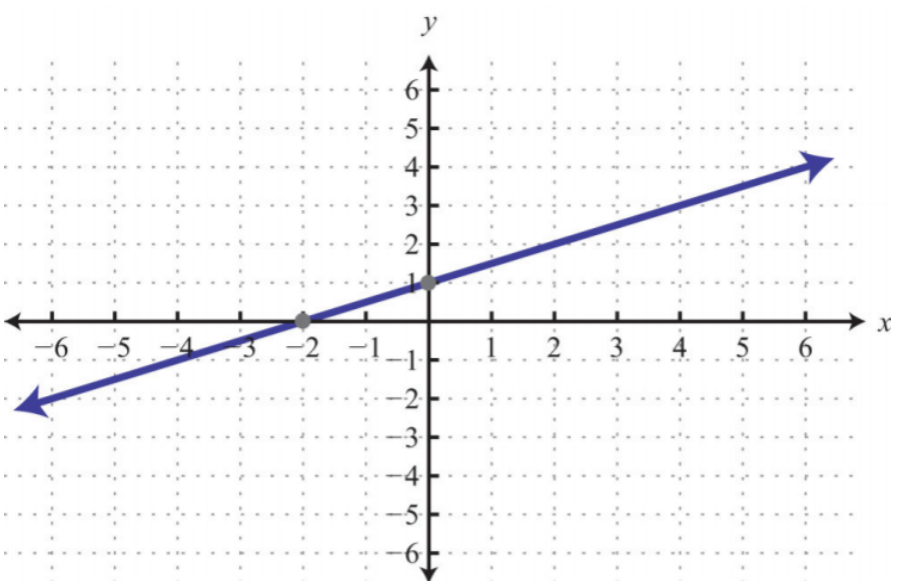


Figure 3.3.27

25.

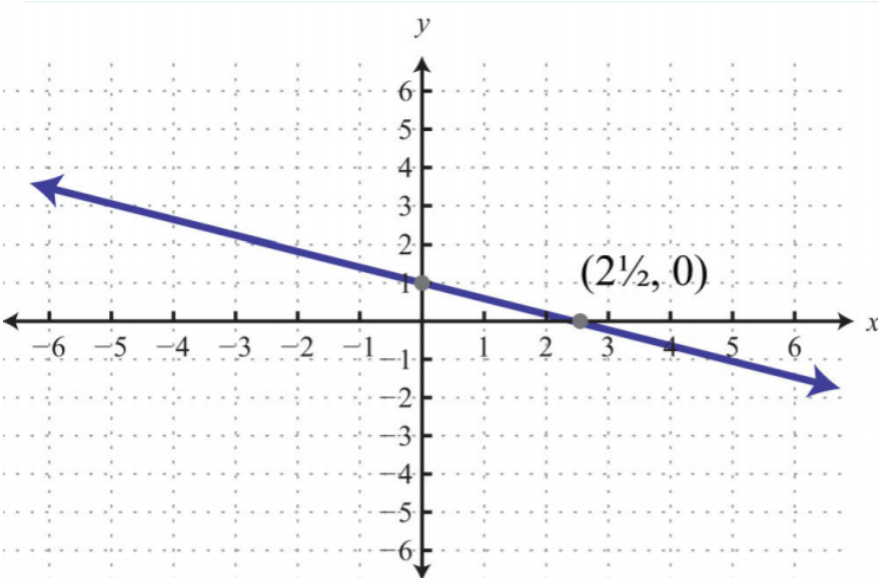


Figure 3.3.28

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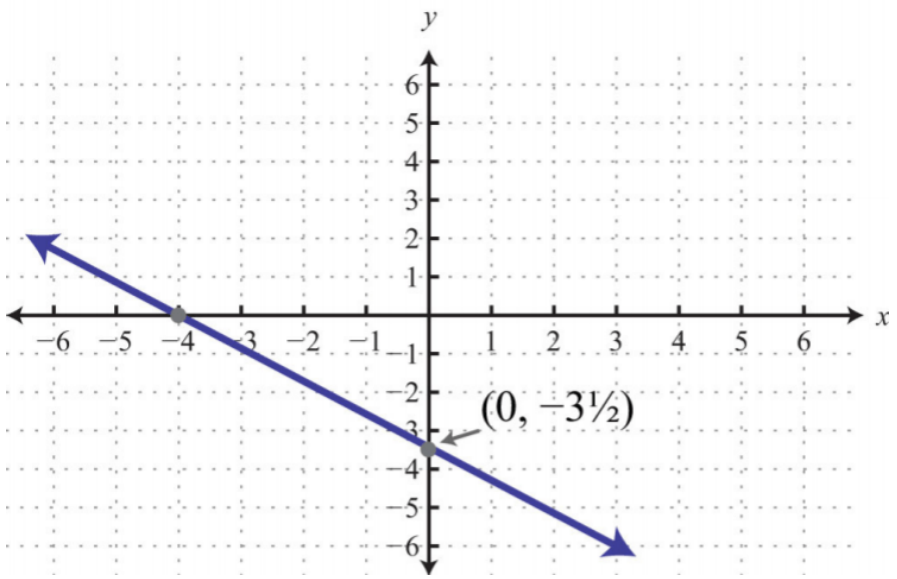


Figure 3.3.29

29.

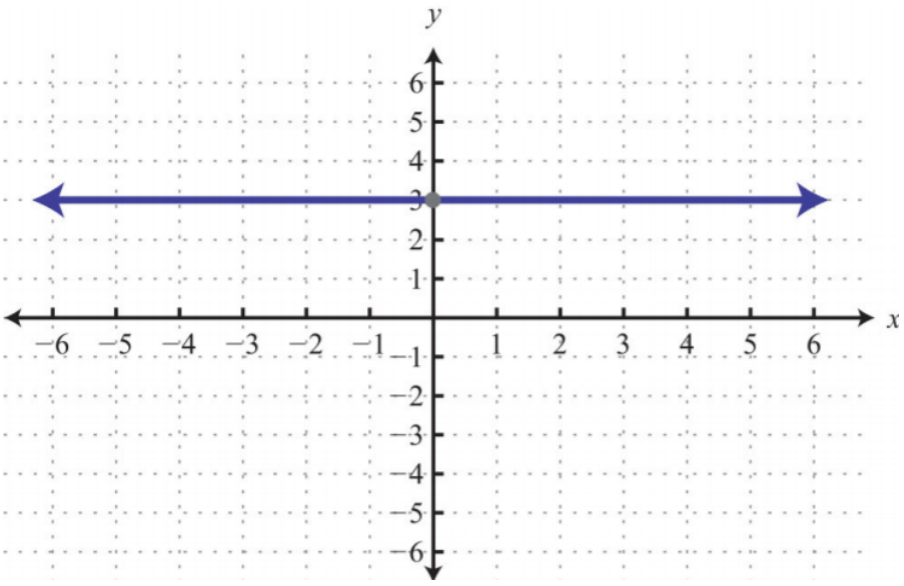


Figure 3.3.30

31.

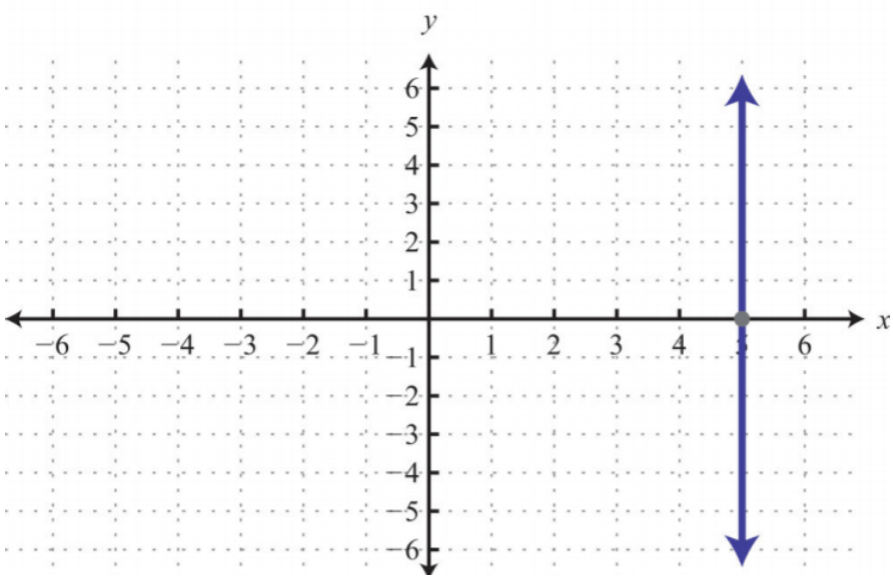


Figure 3.3.31

33.

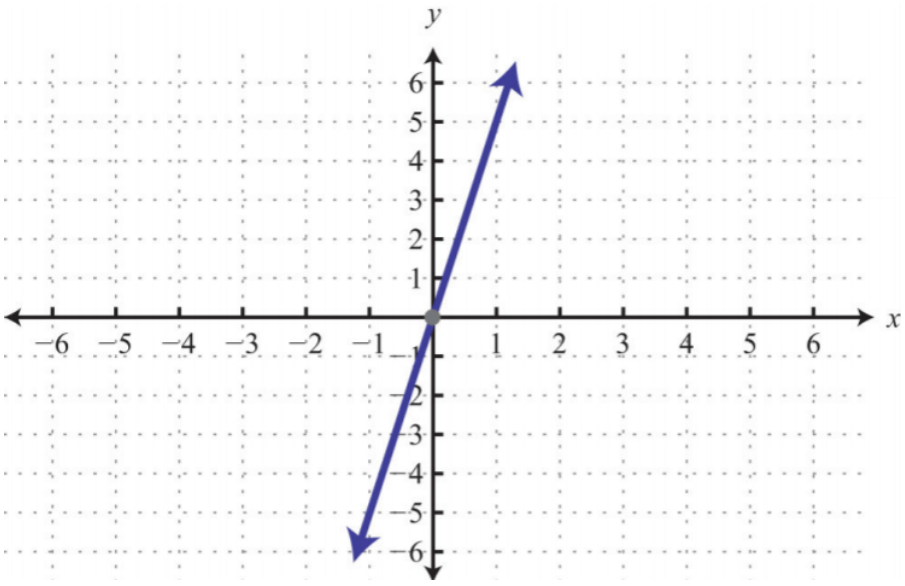


Figure 3.3.32

Exercise 3.3.6 **Intercepts of Nonlinear Graphs**

1. x-intercepts: $(-3,0),(3,0)$; y-intercept: $(0,-3)$
3. x-intercepts: $(-4,0),(0,0)$; y-intercept: $(0,0)$
5. x-intercepts: $(-2,0),(2,0)$; y-intercept: $(0,-1)$
7. x-intercepts: $(-3,0),(0,0),(2,0)$; y-intercept: $(0,0)$
9. x-intercepts: $(-4,0),(4,0)$; y-intercepts: $(0,-4),(0,4)$

Exercise 3.3.7 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary
5. Answers may vary

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3.4: Graph Using the y-Intercept and Slope

Learning Objectives

- Identify and find the slope of a line.
- Graph a line using the slope and y-intercept.

Slope

The steepness of any incline can be measured as the ratio of the vertical change to the horizontal change. For example, a 5% incline can be written as $\frac{5}{100}$, which means that for every 100 feet forward, the height increases 5 feet.

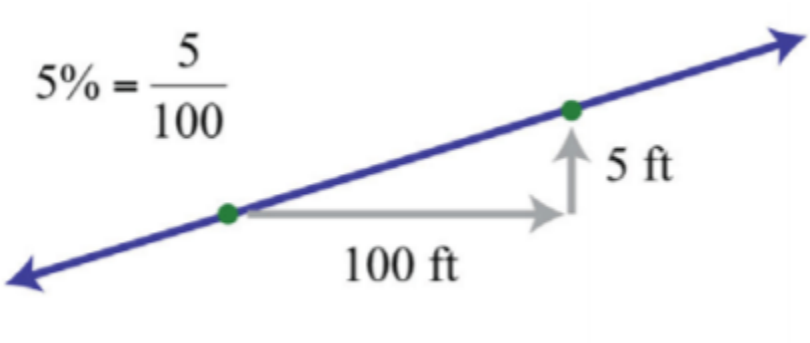


Figure $3.4.1$

In mathematics, we call the incline of a line the slope and use the letter m to denote it. The vertical change is called the rise and the horizontal change is called the run.

$$\textit{Slope} \quad m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{rise}{run} \tag{3.4.1}$$

The rise and the run can be positive or negative. A positive rise corresponds to a vertical change up and a negative rise corresponds to a vertical change down. A positive run denotes a horizontal change to the right and a negative run corresponds to a horizontal change to the left. Given the graph, we can calculate the slope by determining the vertical and horizontal changes between any two points.

Example $3.4.1$

Find the slope of the given line:

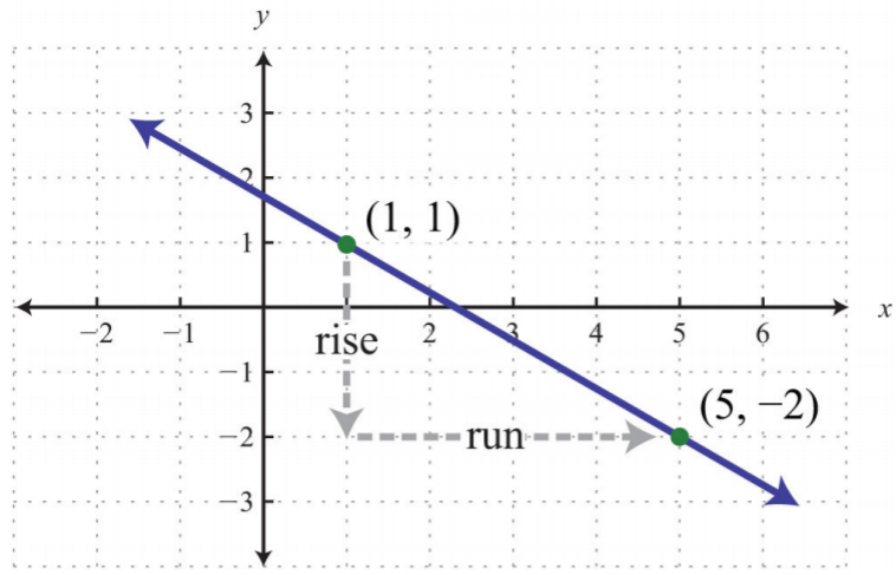


Figure 3.4.2

Solution:

From the given points on the graph, count 3 units down and 4 units right.

$$m = \frac{rise}{run} = \frac{-3 \text{ units}}{4 \text{ units}} = -\frac{3}{4}$$

Answer:

$$m = -\frac{3}{4}$$

Here we have a negative slope, which means that for every 4 units of movement to the right, the vertical change is 3 units downward. There are four geometric cases for the value of the slope.

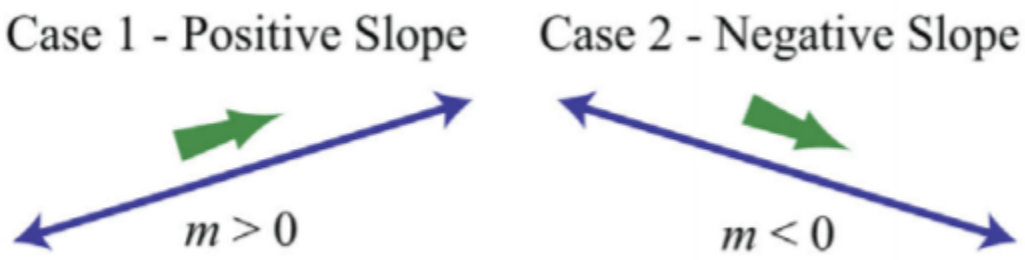


Figure 3.4.3

Reading the graph from left to right, we see that lines with an upward incline have positive slopes and lines with a downward incline have negative slopes.

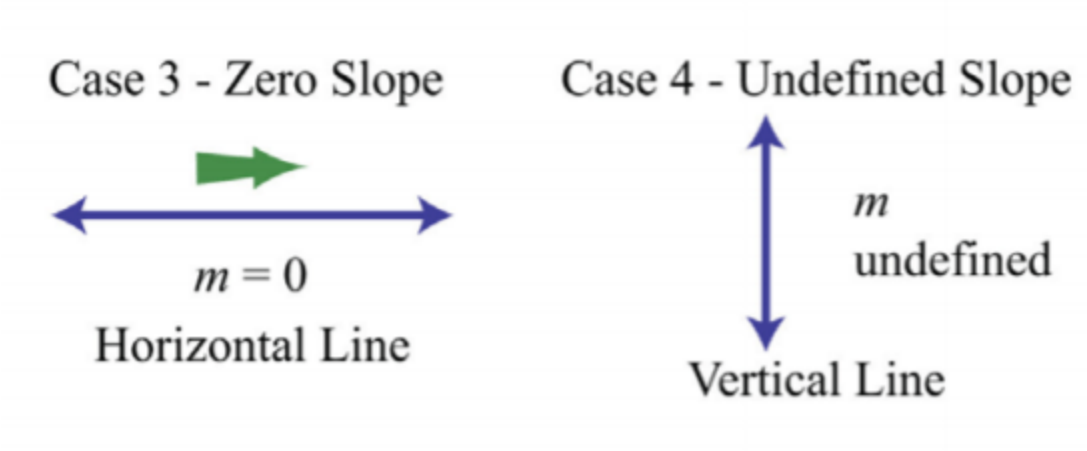


Figure 3.4.4

If the line is horizontal, then the rise is 0:

$$m = \frac{rise}{run} = \frac{0}{run} = 0$$

The slope of a horizontal line is 0. If the line is vertical, then the run is 0:

$$m = \frac{rise}{run} = \frac{rise}{0} \quad Undefined$$

The slope of a vertical line is undefined.

Exercise 3.4.1

Find the slope of the given line:

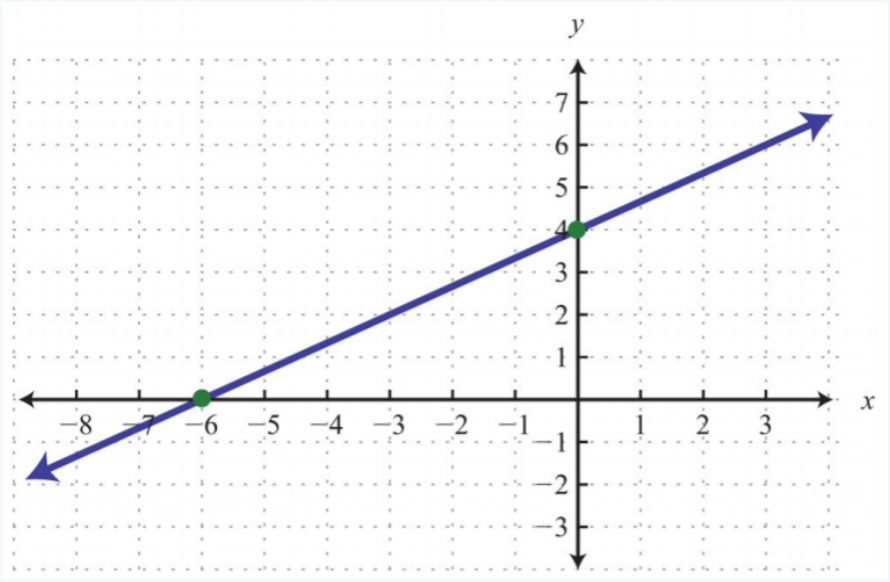


Figure 3.4.5

Answer available at the end of this section.

Calculating the slope can be difficult if the graph does not have points with integer coordinates. Therefore, we next develop a

formula that allows us to calculate the slope algebraically. Given any two points (x_1, y_1) and (x_2, y_2), we can obtain the rise and run by subtracting the corresponding coordinates.

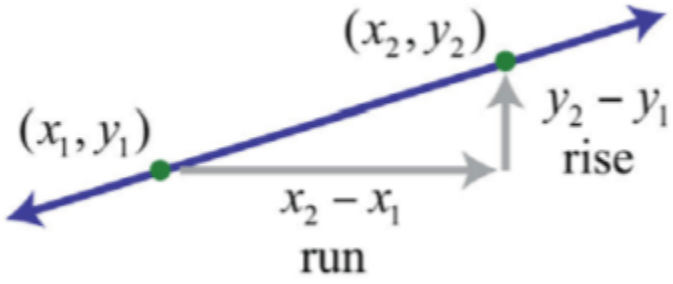


Figure 3.4.6

This leads us to the slope formula. Given any two points (x_1, y_1) and (x_2, y_2), the slope is given by

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 3.4.2

Find the slope of the line passing through $(-3, -5)$ and $(2, 1)$.

Solution:

Given $(-3, -5)$ and $(2, 1)$, calculate the difference of the y-values divided by the difference of the x-values. Since subtraction is not commutative, take care to be consistent when subtracting the coordinates.

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-3, -5) & (2, 1) \end{array}$$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - (-5)}{2 - (-3)} \\ &= \frac{1 + 5}{2 + 3} \\ &= \frac{6}{5} \end{aligned}$$

Answer:

$$m = \frac{6}{5}$$

We can graph the line described in the previous example and verify that the slope is $\frac{6}{5}$.

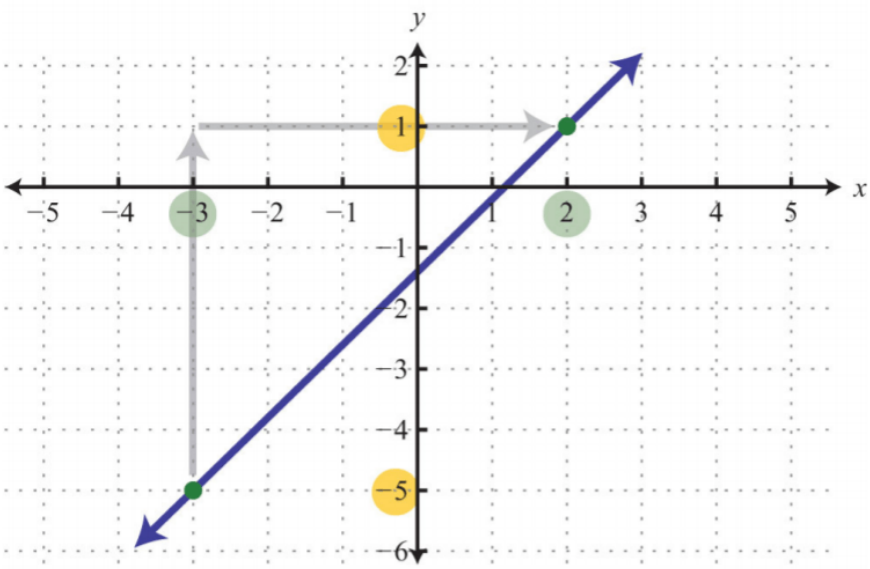


Figure 3.4.7

Certainly the graph is optional; the beauty of the slope formula is that we can obtain the slope, given two points, using only algebra.

Example 3.4.3

Find the slope of the line passing through $(-4, 3)$ and $(-1, -7)$.

Solution:

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-4, 3) & (-1, -7) \end{array}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7-(3)}{-1-(-4)} = \frac{-7-3}{-1+4} = \frac{-10}{3}$$

Answer:

$$m = -\frac{10}{3}$$

When using the slope formula, take care to be consistent since order does matter. You must subtract the coordinates of the first point from the coordinates of the second point for both the numerator and the denominator in the ***same order***.

Example 3.4.4

Find the slope of the line passing through $(7, -2)$ and $(-5, -2)$.

Solution:

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (7, -2) & (-5, -2) \end{array}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-(-2)}{-5-(7)} = \frac{-2+2}{-5-7} = \frac{0}{-12} = 0$$

Answer:

$m = 0$. As an exercise, plot the given two points and verify that they lie on a horizontal line.

Example 3.4.5

Find the slope of the line passing through $(-4, -3)$ and $(-4, 5)$.

Solution:

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-4, -3) & (-4, 5) \end{array}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-3)}{-4-(-4)} = \frac{5+3}{-4+4} = \frac{8}{0} \quad Undefined$$

Answer:

The slope m is undefined. As an exercise, plot the given two points and verify that they lie on a vertical line.

Exercise 3.4.2

Calculate the slope of the line passing through $(-2, 3)$ and $(5, -5)$.

Answer available at the end of this section.

When considering the slope as a rate of change it is important to include the correct units.

Example 3.4.6

A Corvette Coupe was purchased new in 1970 for about $\$5,200$ and depreciated in value over time until it was sold in 1985 for $\$1,300$. At this point, the car was beginning to be considered a classic and started to increase in value. In the year 2000, when the car was 30 years old, it sold at auction for $\$10,450$. The following line graph depicts the value of the car over time.

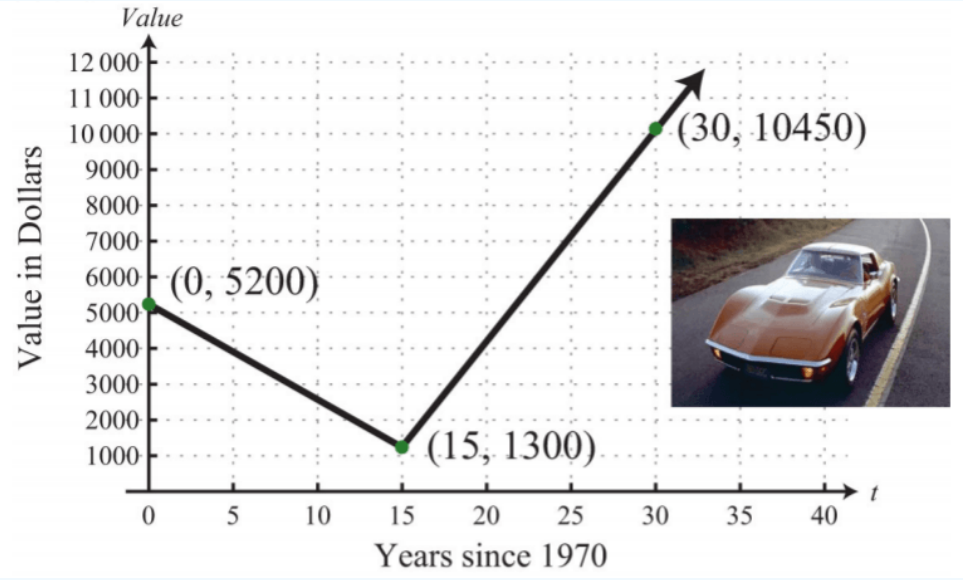


Figure 3.4.8

a. Determine the rate at which the car depreciated in value from 1970 to 1985.

b. Determine the rate at which the car appreciated in value from 1985 to 2000.

Solution:

Notice that the value depends on the age of the car and that the slope measures the rate in dollars per year.

a. The slope of the line segment depicting the value for the first 15 years is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\$1{,}300 - \$5{,}200}{15 \text{ years} - 0 \text{ years}} = \frac{-\$3{,}900}{15 \text{ years}} = -\$260 \text{ per year}$$

b. The slope of the line segment depicting the value for the next 15 years is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\$10{,}450 - \$1{,}300}{30 \text{ years} - 15 \text{ years}} = \frac{\$9{,}150}{15 \text{ years}} = \$610 \text{ per year}$$

Answer:

a. The value of the car depreciated $260 per year from 1970 to 1985.
b. The value of the car appreciated $610 per year from 1985 to 2000.

Slope-Intercept Form of a Line

To this point, we have learned how to graph lines by plotting points and by using the x- and y-intercepts. In addition, we have seen that we need only two points to graph a line. In this section, we outline a process to easily determine two points using the y -intercept and the slope. The equation of any nonvertical line can be written in slope-intercept form $y = mx + b$. In this form, we can identify the slope, m, and the y-intercept, $(0, b)$.

Example 3.4.7

Determine the slope and y-intercept:

$$y = -\tfrac{4}{5}x + 7.$$

Solution:

In this form, the coefficient of x is the slope, and the constant is the y-value of the y-intercept. Therefore, by inspection, we have

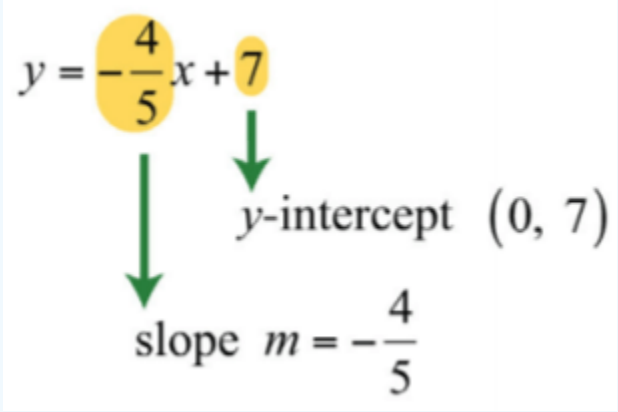


Figure 3.4.9

Answer:

The y-intercept is $(0, 7)$, and the slope is $m = -\frac{4}{5}$.

It is not always the case that the linear equation is given in slope-intercept form. When it is given in standard form, you have to first solve for y to obtain slope-intercept form.

Example 3.4.8

Express $3x + 5y = 30$ in slope-intercept form and then identify the slope and y-intercept.

Solution:

Begin by solving for y. To do this, apply the properties of equality to first isolate $5y$ and then divide both sides by 5.

$$\begin{aligned} 3x + 5y &= 30 \\ 3x + 5y - 3x &= 30 - 3x \\ 5y &= -3x + 30 \\ \frac{5y}{5} &= \frac{-3x + 30}{5} \\ y &= \frac{-3x}{5} + \frac{30}{5} \\ y &= -\frac{3}{5}x + 6 \end{aligned}$$

Answer:

Slope-intercept form: $y = -\frac{3}{5}x + 6$; y-intercept: $(0, 6)$; slope: $m = -\frac{3}{5}$

Once the equation is in slope-intercept form, we immediately have one point to plot, the y-intercept. From the intercept, you can mark off the slope to plot another point on the line. From the previous example we have

$$y = -\tfrac{3}{5}x + 6$$

y-intercept: $(0, 6)$

slope: $m = -\frac{3}{5} = \frac{-3}{5} = \frac{rise}{run}$

Starting from the point $(0, 6)$, use the slope to mark another point 3 units down and 5 units to the right.

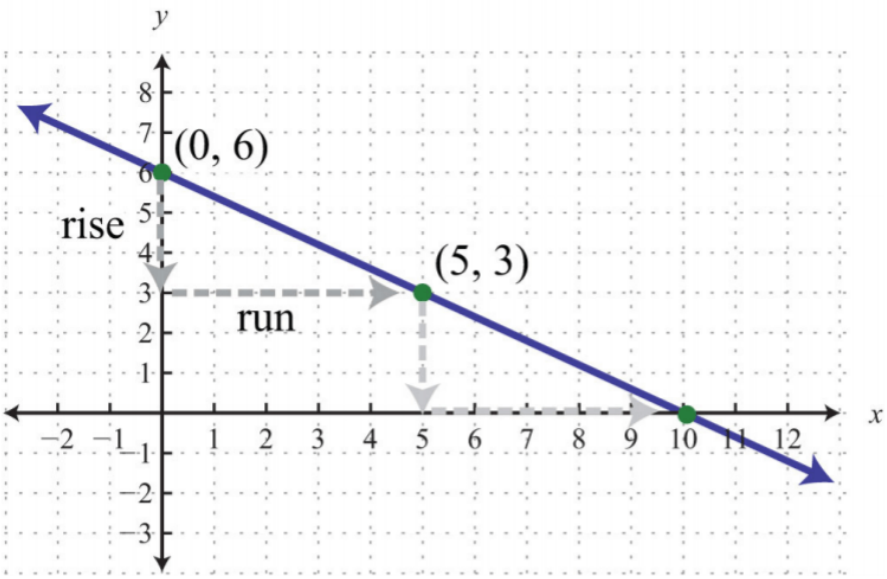


Figure 3.4.10

It is not necessary to check that the second point, (5, 3), solves the original linear equation. However, we do it here for the sake of completeness.

$$\begin{aligned} 3x+5y &= 30 \\ 3(5)+5(3) &= 30 \\ 15+15 &= 30 \\ 30 &= 30 \quad \checkmark \end{aligned}$$

Marking off the slope in this fashion produces as many ordered pair solutions as we desire. Notice that if we mark off the slope again, from the point $(5,3)$, then we obtain the x-intercept, $(10,0)$.

Example 3.4.9

Graph:

$$-x+2y=4.$$

Solution:

In this example, we outline the general steps for graphing a line using slope-intercept form.

Step 1: Solve for y to obtain slope-intercept form.

$$\begin{aligned} -x+2y &= 4 \\ -x+2y+x &= 4+x \\ 2y &= x+4 \\ \frac{2y}{2} &= \frac{x+4}{2} \\ y &= \frac{1x}{2}+\frac{4}{2} \\ y &= \frac{1}{2}x+2 \end{aligned}$$

Step 2: Identify the y-intercept and slope.

y-intercept: $(0,2)$

slope: $m=\frac{1}{2}=\frac{rise}{run}$

Step 3: Plot the y-intercept and use the slope to find another ordered pair solution. Starting from the y-intercept, mark off the slope and identify a second point. In this case, mark a point after a rise of 1 unit and a run of 2 units.

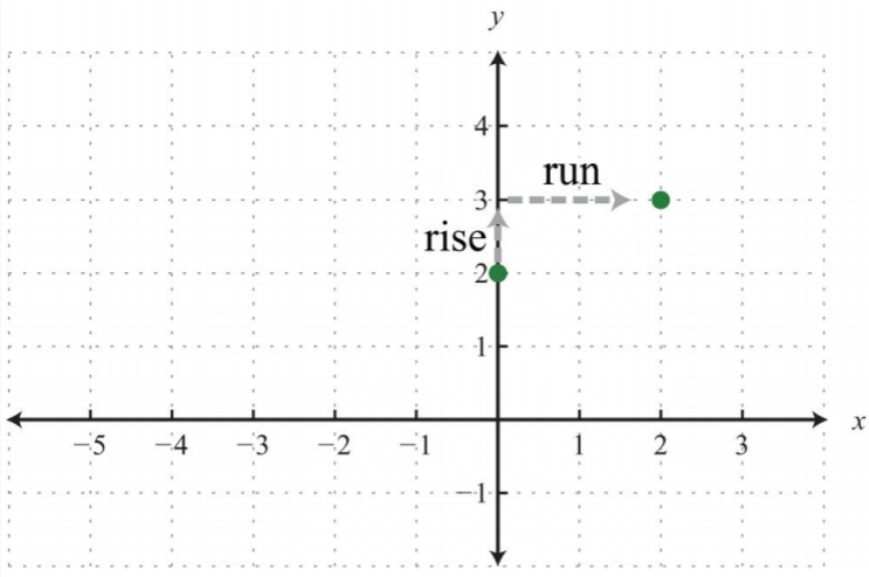


Figure 3.4.11

Step 4: Draw the line through the two points with a straightedge.

Answer:

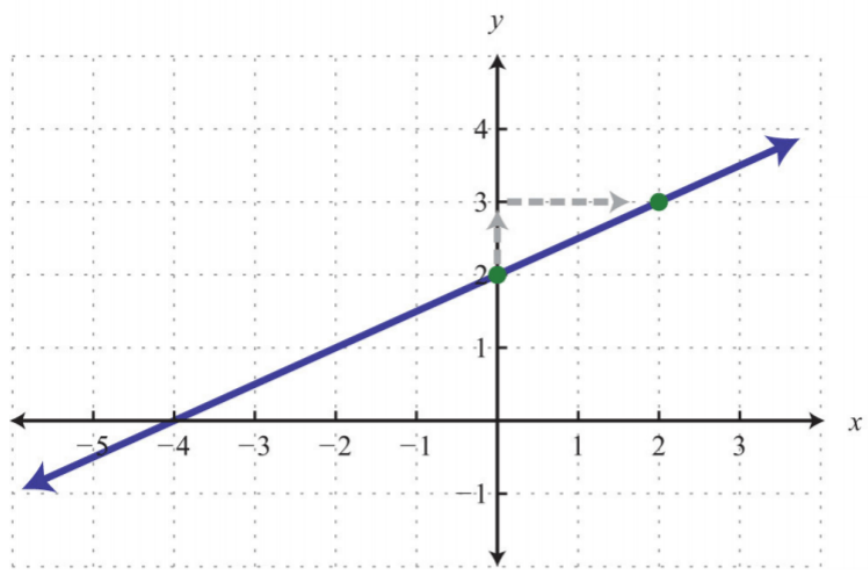


Figure 3.4.12

In this example, we notice that we could get the x-intercept by marking off the slope in a different but equivalent manner. Consider the slope as the ratio of two negative numbers as follows:

$$m = \frac{1}{2} = \frac{-1}{-2} = \frac{rise}{run}$$

We could obtain another point on the line by marking off the equivalent slope down 1 unit and left 2 units. We do this twice to obtain the x-intercept, $(-4, 0)$.

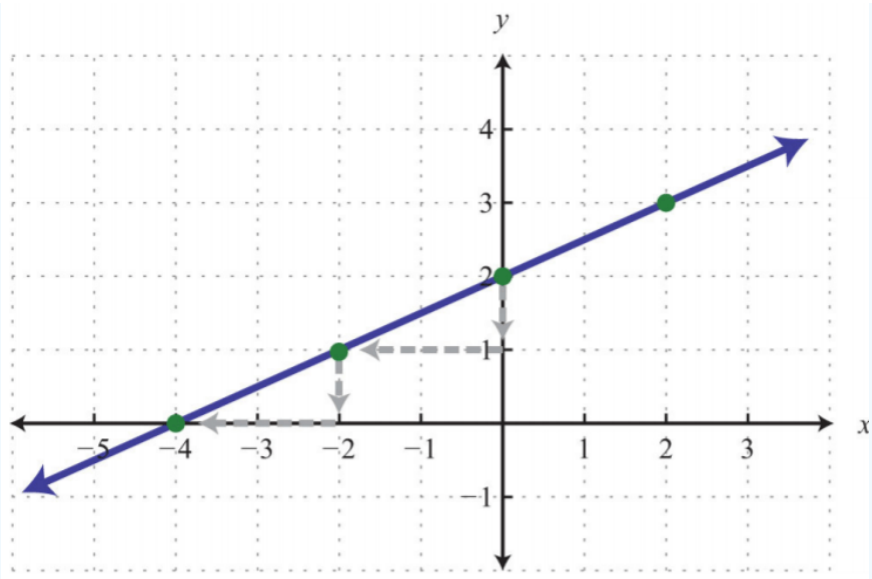


Figure 3.4.13

Marking off the slope multiple times is not necessarily always going to give us the x-intercept, but when it does, we obtain a valuable point with little effort. In fact, it is a good practice to mark off the slope multiple times; doing so allows you to obtain more points on the line and produce a more accurate graph.

Example 3.4.10

Graph and find the x-intercept:

$$y = \frac{3}{4}x - 2.$$

Solution:

The equation is given in slope-intercept form. Therefore, by inspection, we have the y-intercept and slope.

y-intercept: $(0, -2)$

slope: $m = \frac{3}{4} = \frac{rise}{run}$

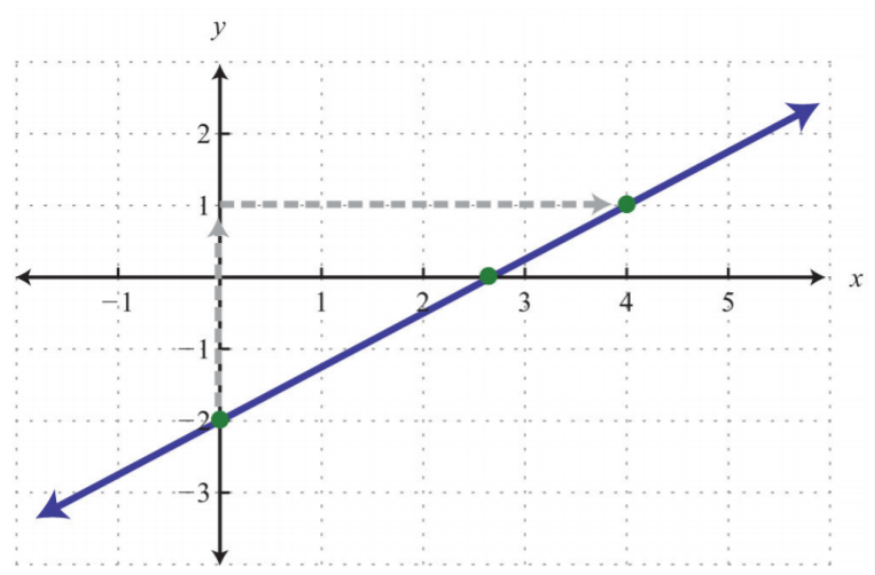


Figure 3.4.14

We can see that the x-value of the x-intercept is a mixed number between 2 and 3. To algebraically find x-intercepts, recall that we must set $y = 0$ and solve for x.

Answer:

The x-intercept is $(2\frac{2}{3}, 0)$.

Example 3.4.11

Graph:

$$x - y = 0.$$

Solution:

Begin by solving for y.

$$\begin{aligned} x - y &= 0 \\ x - y\color{#2ca1e8}{-x} &= 0\color{#2ca1e8}{-x} \\ -y &= -x \\ \color{#2ca1e8}{-1}\cdot(-y) &= \color{#2ca1e8}{-1}\cdot(-x) \\ y &= x \end{aligned}$$

The equation $y = x$ can be written $y = 1x + 0$, and we have

$$y\text{-intercept: } (0, 0)$$

$$\text{slope: } m = 1 = \tfrac{1}{1} = \tfrac{rise}{run}$$

Answer:

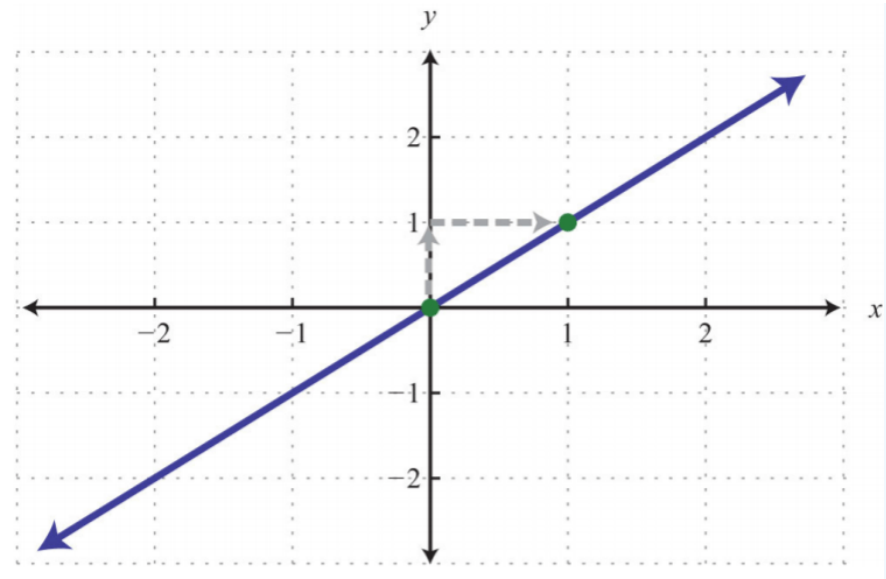


Figure 3.4.15

Exercise 3.4.3

Graph $-2x + 5y = 20$ and label the x-intercept.

Answer available at the end of this section.

Key Takeaways

- Slope measures the steepness of a line as *rise over run*. A positive rise denotes a vertical change up, and a negative rise denotes a vertical change down. A positive run denotes a horizontal change right, and a negative run denotes a horizontal change left.
- Horizontal lines have a slope of zero, and vertical lines have undefined slopes.
- Given any two points on a line, we can algebraically calculate the slope using the slope formula, $m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$.
- Any nonvertical line can be written in slope-intercept form, $y = mx + b$, from which we can determine, by inspection, the slope m and y-intercept $(0, b)$.
- If we know the y-intercept and slope of a line, then we can easily graph it. First, plot the y-intercept, and from this point use the slope as rise over run to mark another point on the line. Finally, draw a line through these two points with a straightedge and add an arrow on either end to indicate that it extends indefinitely.
- We can obtain as many points on the line as we wish by marking off the slope multiple times.

Exercise 3.4.4 Slope

Determine the slope and the y-intercept of the given graph.

1.

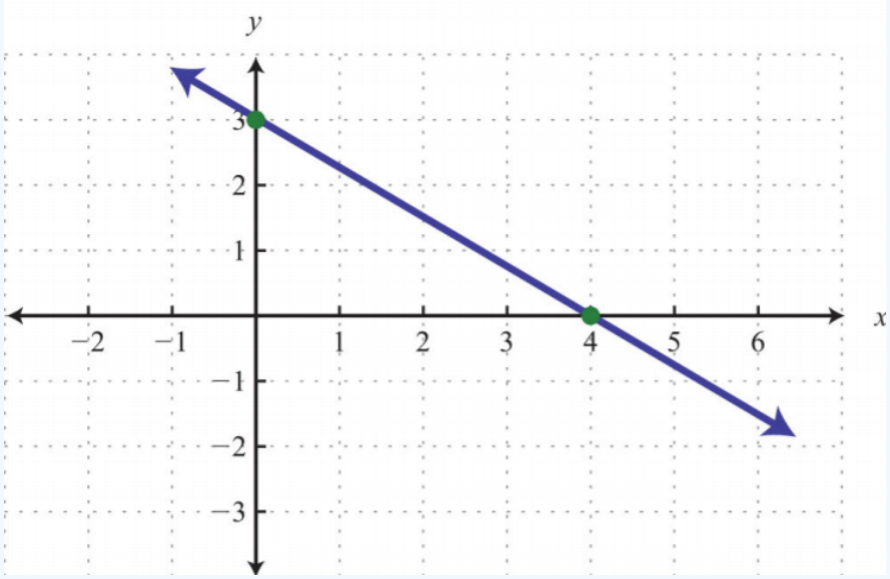


Figure 3.4.17

2.

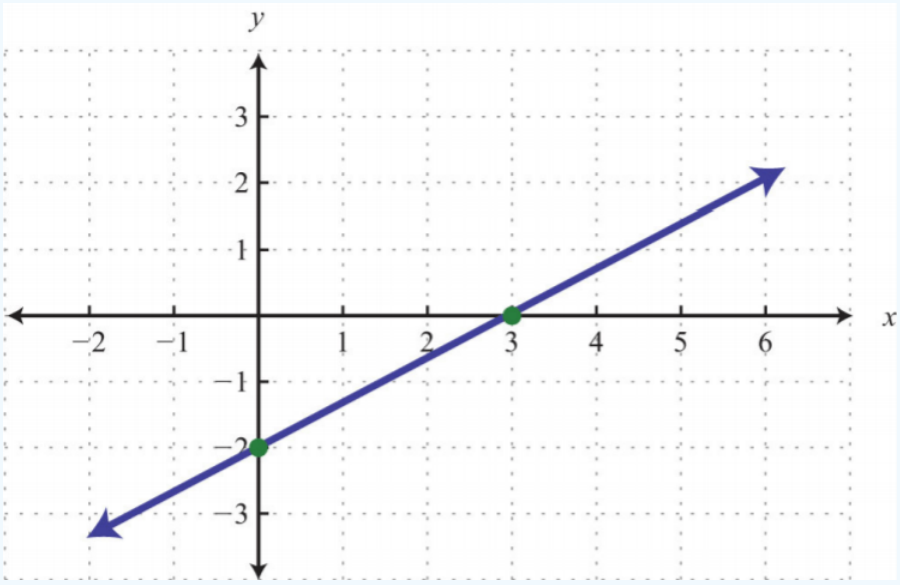


Figure 3.4.18

3.

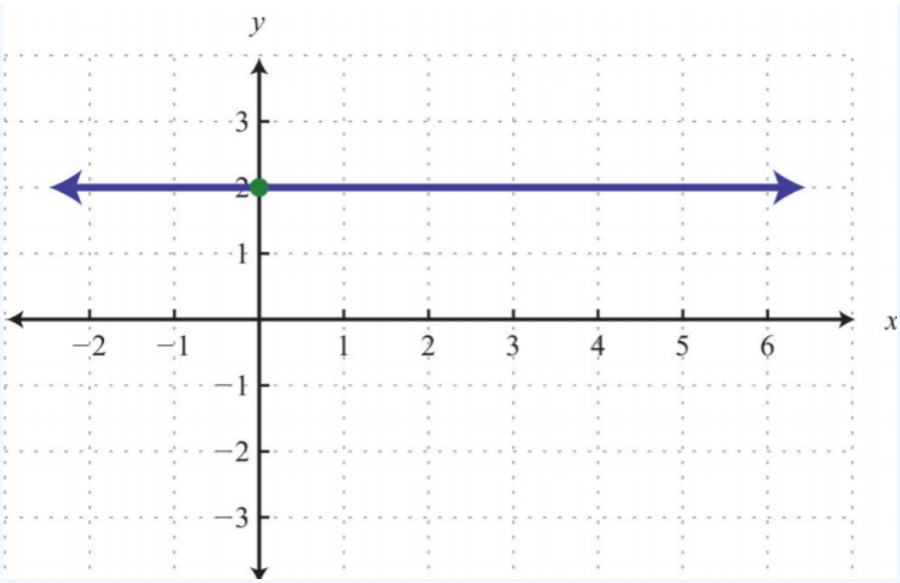


Figure 3.4.19

4.

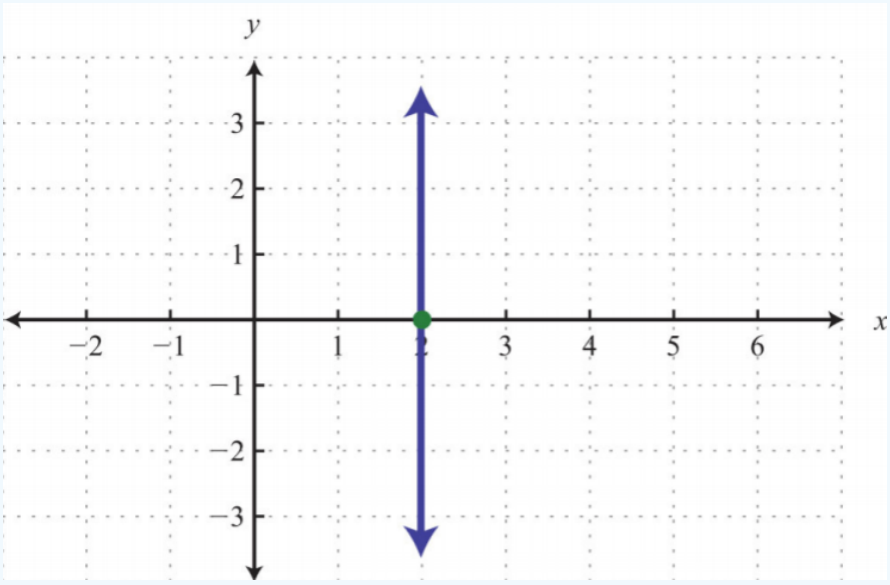


Figure 3.4.20

5.

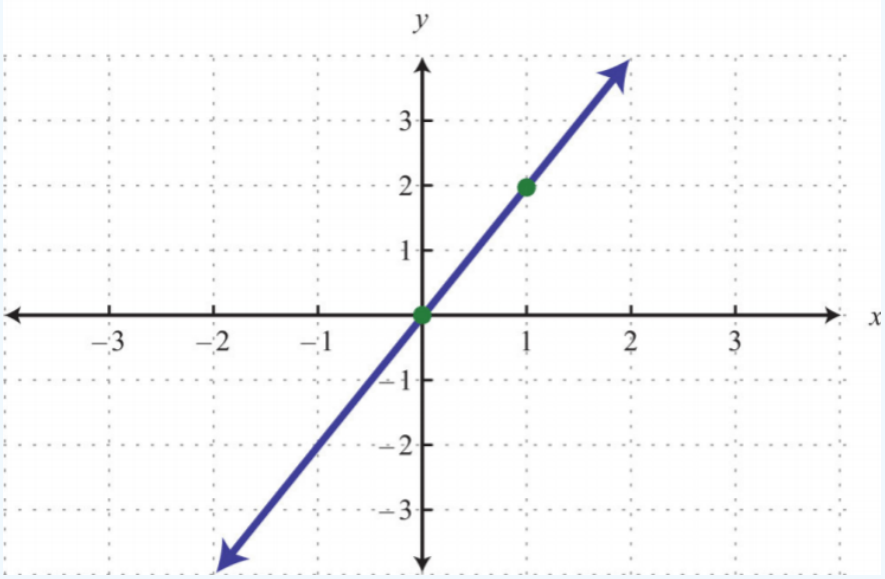


Figure 3.4.21

6.

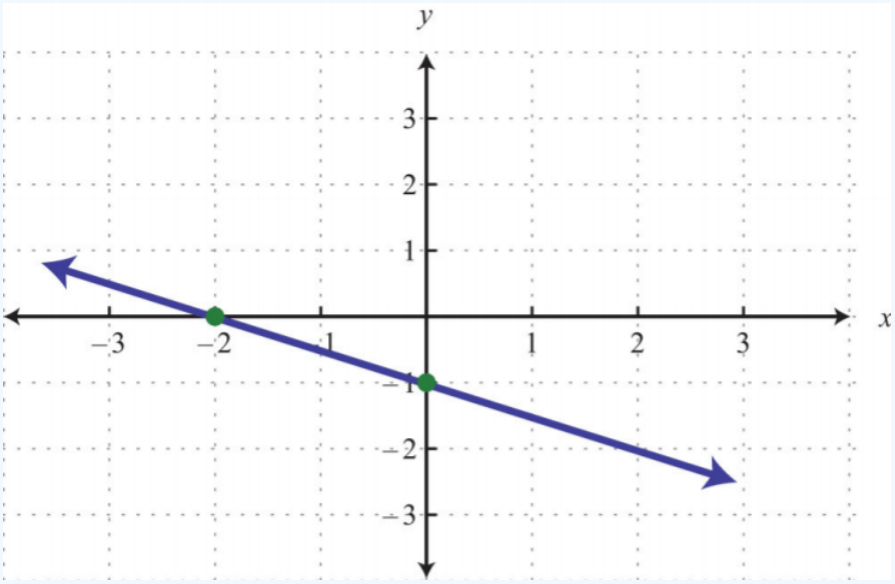


Figure 3.4.22

Answer available at the end of this section.

Exercise 3.4.5 Slope

Determine the slope, given two points.

1. $(3, 2)$ and $(5, 1)$
2. $(7, 8)$ and $(-3, 5)$
3. $(2, -3)$ and $(-3, 2)$
4. $(-3, 5)$ and $(7, -5)$

5. $(-1,-6)$ and $(3,2)$
6. $(5,3)$ and $(4,12)$
7. $(-9,3)$ and $(-6,-5)$
8. $(-22,4)$ and $(-8,-12)$
9. $(\frac{1}{2},-\frac{1}{3})$ and $(-\frac{1}{2},\frac{2}{3})$
10. $(-\frac{3}{4},\frac{3}{2})$ and $(\frac{1}{4},-\frac{1}{2})$
11. $(-\frac{1}{3},\frac{5}{8})$ and $(\frac{1}{2},-\frac{3}{4})$
12. $(-\frac{3}{5},-\frac{3}{2})$ and $(\frac{1}{10},\frac{4}{5})$
13. $(3,-5)$ and $(5,-5)$
14. $(-3,1)$ and $(-14,1)$
15. $(-2,3)$ and $(-2,-4)$
16. $(-4,-4)$ and $(5,5)$
17. A roof drops 4 feet for every 12 feet forward. Determine the slope of the roof.
18. A road drops 300 feet for every $5,280$ feet forward. Determine the slope of the road.
19. The following graph gives the US population of persons 65 years old and over. At what rate did this population increase from 2000 to 2008?

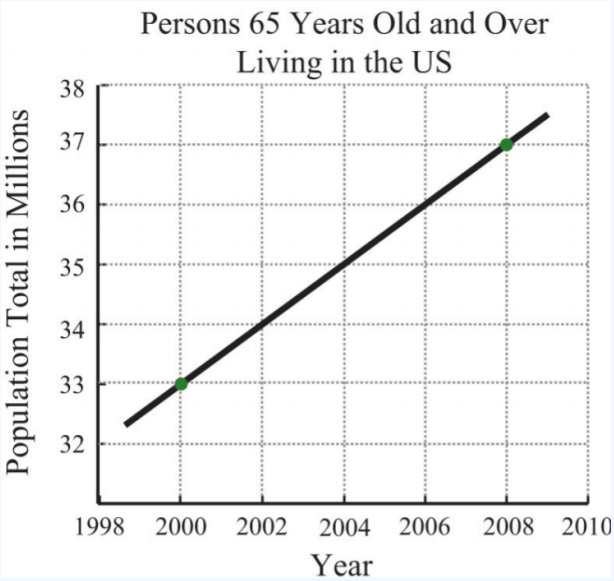


Figure $3.4.23$*: Source: US Census Bureau.*

20. The following graph gives total consumer credit outstanding in the United States. At what rate did consumer credit increase from 2002 to 2008?

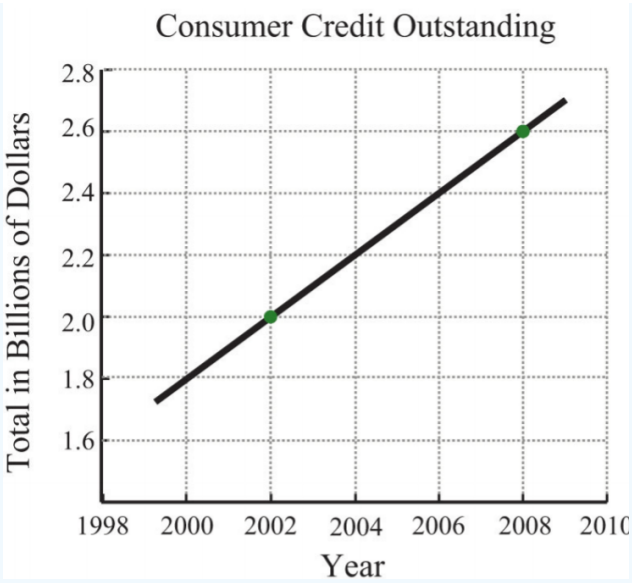


Figure $3.4.24$*: Source: US Census Bureau.*

21. A commercial van was purchased new for $\$20,000$ and is expected to be worth $\$4,000$ in 8 years. Determine the rate at which the van depreciates in value.

22. A commercial-grade copy machine was purchased new for $\$4,800$ and will be considered worthless in 6 years. Determine the rate at which the copy machine depreciates in value.

23. Find y if the slope of the line passing through $(-2,3)$ and $(4,y)$ is 12.

24. Find y if the slope of the line passing through $(5,y)$ and $(6,-1)$ is 10.

25. Find y if the slope of the line passing through $(5,y)$ and $(-4,2)$ is 0.

26. Find x if the slope of the line passing through $(-3,2)$ and $(x,5)$ is undefined.

Answer available at the end of this section.

Exercise $3.4.6$ Slope-Intercept Form

Express the given linear equation in slope-intercept form and identify the slope and y-intercept.

1. $6x-5y=30$
2. $-2x+7y=28$
3. $9x-y=17$
4. $x-3y=18$
5. $2x-3y=0$
6. $-6x+3y=0$
7. $\frac{2}{3}x-\frac{5}{4}y=10$
8. $-\frac{4}{3}x+\frac{1}{5}y=-5$

Answer available at the end of this section.

Exercise 3.4.7 Slope-Intercept Form

Graph the line given the slope and the y-intercept.

1. $m = \frac{1}{3}$ and $(0, -2)$
2. $m = -\frac{2}{3}$ and $(0, 4)$
3. $m = 3$ and $(0, 1)$
4. $m = -2$ and $(0, -1)$
5. $m = 0$ and $(0, 5)$
6. m undefined and $(0, 0)$
7. $m = 1$ and $(0, 0)$
8. $m = -1$ and $(0, 0)$
9. $m = -\frac{15}{3}$ and $(0, 20)$
10. $m = -10$ and $(0, -5)$

Answer available at the end of this section.

Exercise 3.4.8 Slope-Intercept Form

Graph using the slope and y-intercept.

1. $y = \frac{2}{3}x - 2$
2. $y = -\frac{1}{3}x + 1$
3. $y = -3x + 6$
4. $y = 3x + 1$
5. $y = \frac{3}{5}x$
6. $y = -\frac{3}{7}x$
7. $y = -8$
8. $y = 7$
9. $y = -x + 2$
10. $y = x + 1$
11. $y = \frac{1}{2}x + \frac{3}{2}$
12. $y = -\frac{3}{4}x + \frac{5}{2}$
13. $4x + y = 7$
14. $3x - y = 5$
15. $5x - 2y = 10$
16. $-2x + 3y = 18$
17. $x - y = 0$
18. $x + y = 0$
19. $\frac{1}{2}x - \frac{1}{3}y = 1$
20. $-\frac{2}{3}x + \frac{1}{2}y = 2$
21. $3x + 2y = 1$
22. $5x + 3y = 1$
23. On the same set of axes, graph the three lines, where $y = \frac{3}{2}x + b$ and $b = \{-2, 0, 2\}$.
24. On the same set of axes, graph the three lines, where $y = mx + 1$ and $m = \{-\frac{1}{2}, 0, \frac{1}{2}\}$.

Answer available at the end of this section.

Exercise 3.4.9 Discussion Board Topics

1. Name three methods for graphing lines. Discuss the pros and cons of each method.
2. Choose a linear equation and graph it three different ways. Scan the work and share it on the discussion board.
3. Why do we use the letter m for slope?
4. How are equivalent fractions useful when working with slopes?

5. Can we graph a line knowing only its slope?
6. Research and discuss the alternative notation for slope: $m = \frac{\Delta y}{\Delta x}$.
7. What strategies for graphing lines should be brought to an exam? Explain

Answer available at the end of this section.

Answers

Exercise 3.4.1

$m = \frac{2}{3}$

Exercise 3.4.2

$m = -\frac{8}{7}$

Exercise 3.4.3

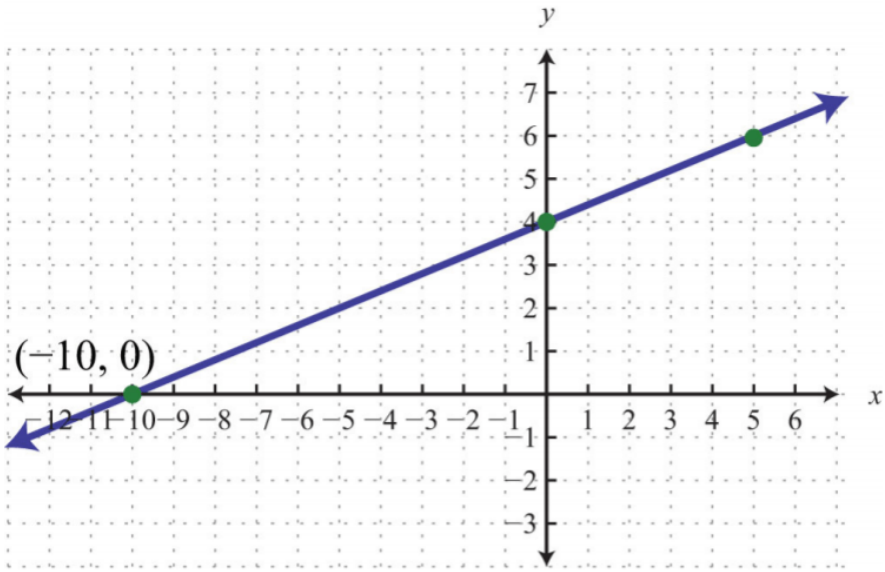


Figure 3.4.16

Exercise 3.4.4 **Slope**

1. y-intercept: $(0, 3)$; slope: $m = -\frac{3}{4}$
3. y-intercept: $(0, 2)$; slope: $m = 0$
5. y-intercept: $(0, 0)$; slope: $m = 2$

Exercise 3.4.5 **Slope**

1. $-\frac{1}{2}$
3. -1
5. 2
7. $-\frac{8}{3}$
9. -1
11. $-\frac{33}{20}$
13. 0
15. Undefined
17. $-\frac{1}{3}$
19. $\frac{1}{2}$ million per year
21. $\$2,000$ per year
23. 75
25. 2

Exercise 3.4.6 **Slope-Intercept Form**

1. $y = \frac{6}{5}x - 6$; slope: $\frac{6}{5}$; y-intercept: $(0, -6)$
3. $y = 9x - 17$; slope: 9; y-intercept: $(0, -17)$
5. $y = \frac{2}{3}x$; slope: $\frac{2}{3}$; y-intercept: $(0, 0)$
7. $y = \frac{8}{15}x - 8$; slope: $\frac{8}{15}$; y-intercept: $(0, -8)$

Exercise 3.4.7 **Slope-Intercept Form**

1.

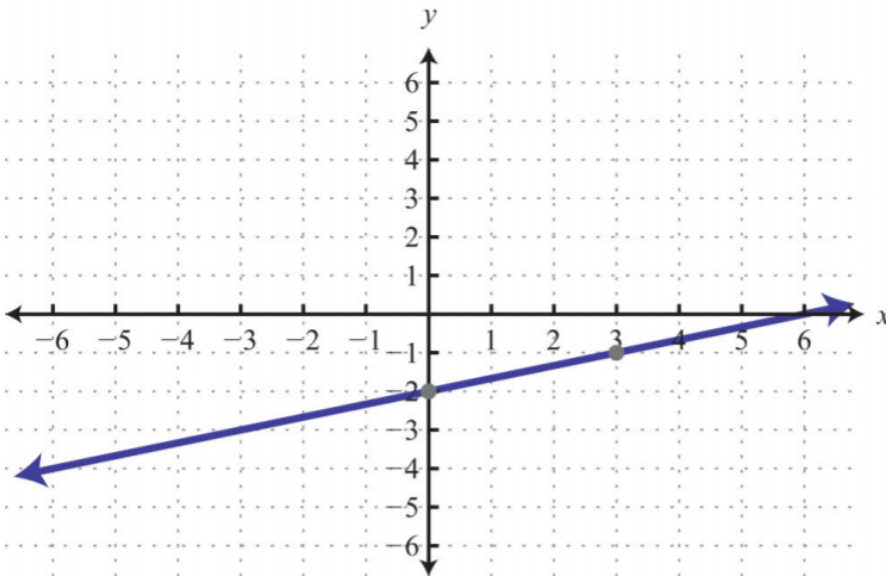


Figure 3.4.25

3.

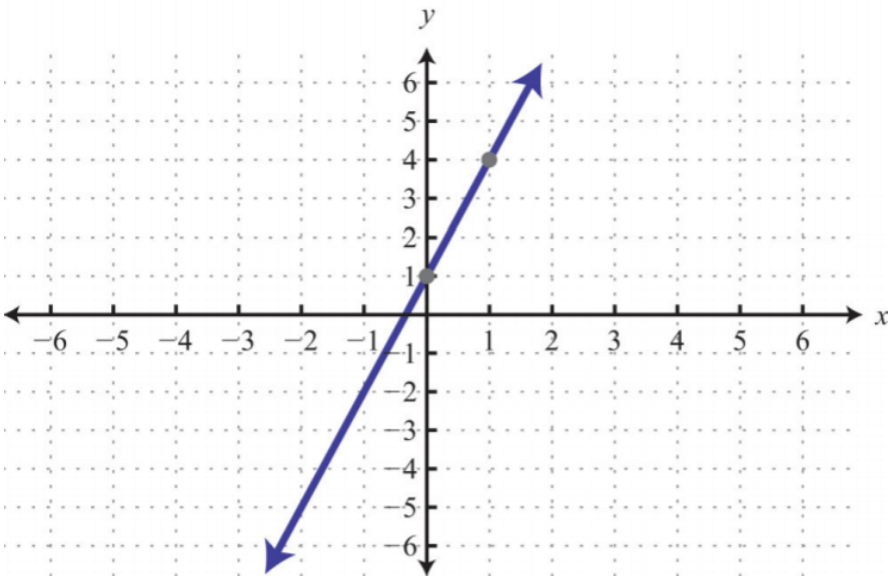


Figure 3.4.26

5.

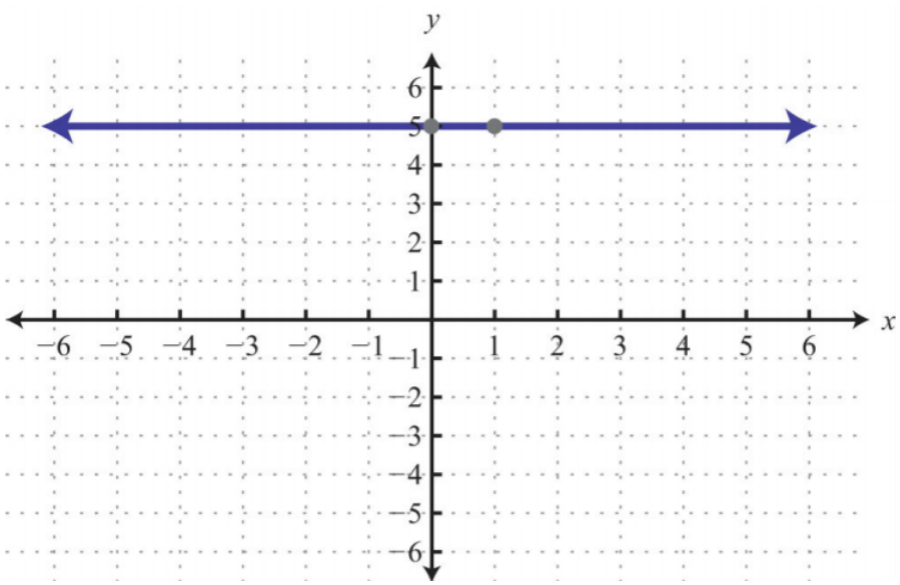


Figure 3.4.27

7.

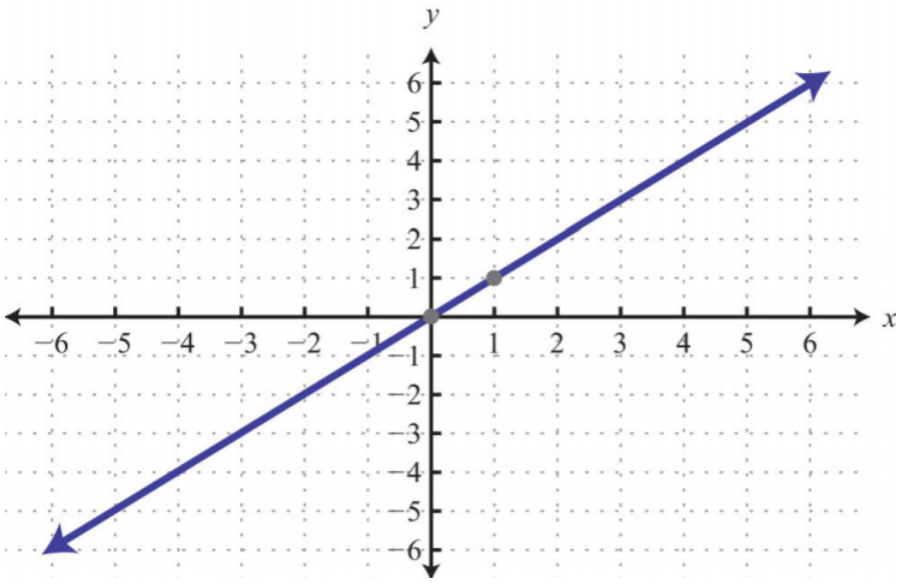


Figure 3.4.28

9.

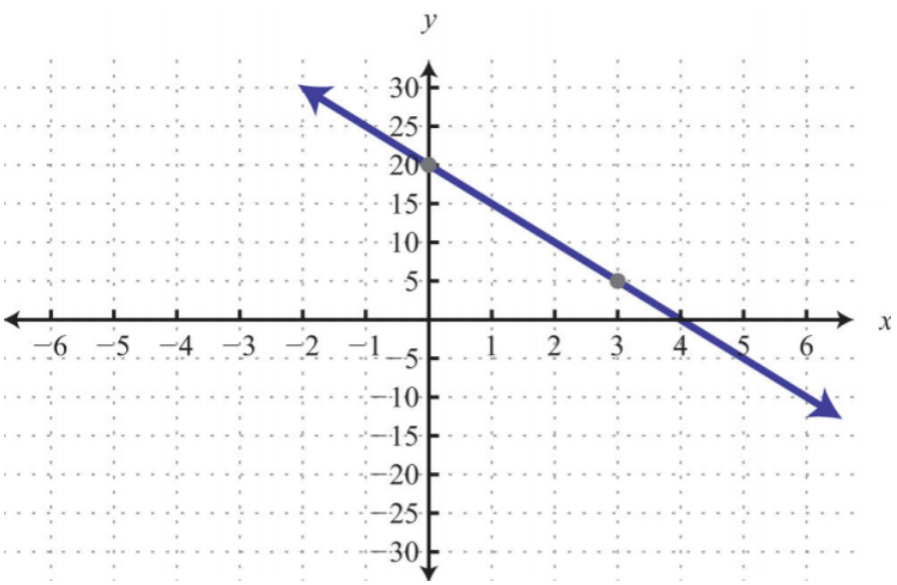


Figure 3.4.29

Exercise 3.4.8 **Slope-Intercept Form**

1.

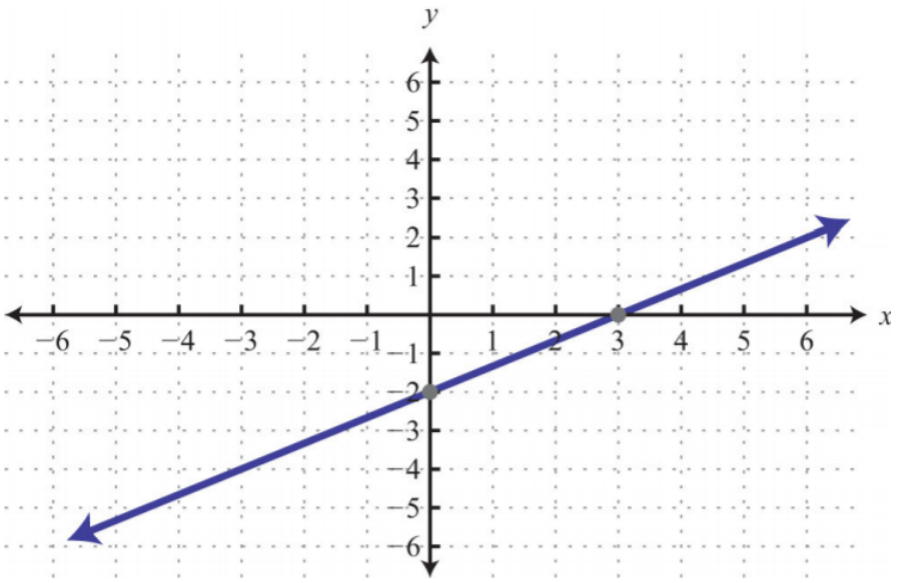


Figure 3.4.30

3.

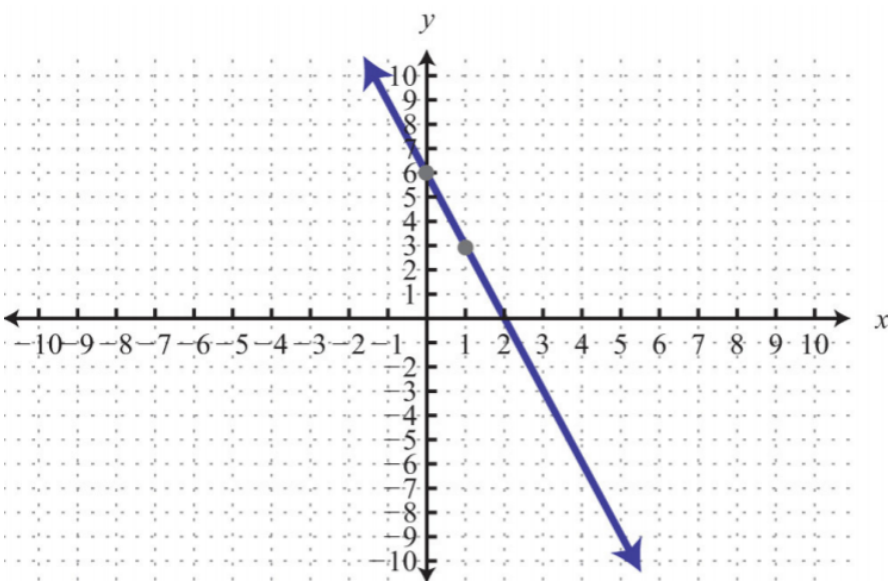


Figure 3.4.31

5.

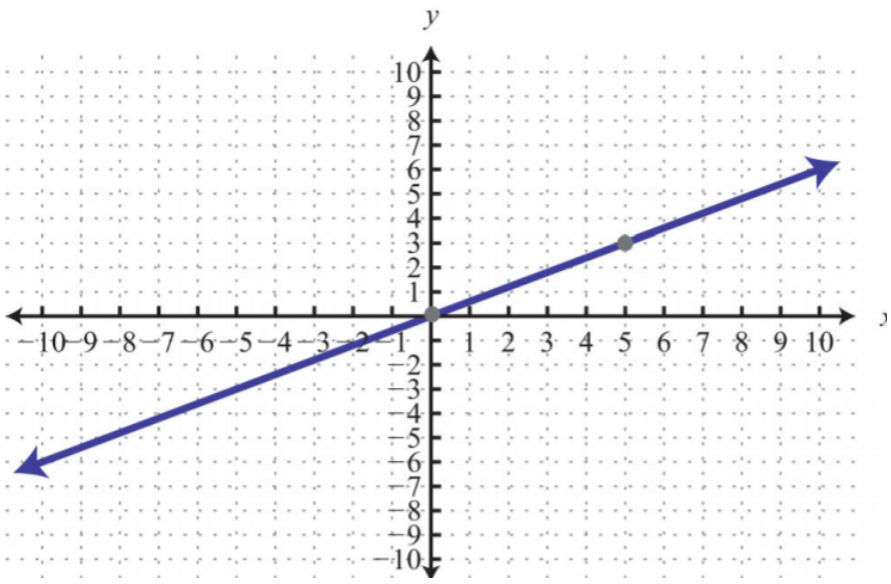


Figure 3.4.32

7.

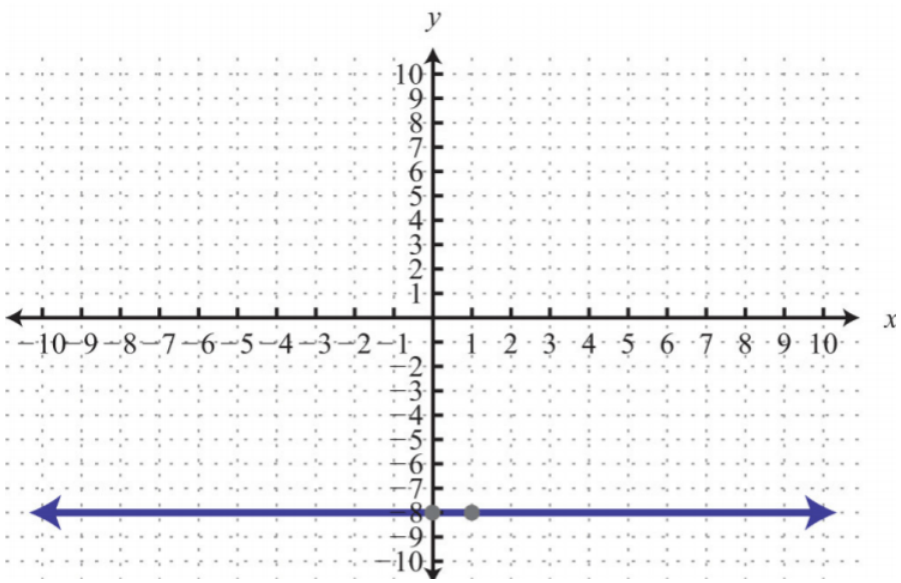


Figure 3.4.33

9.

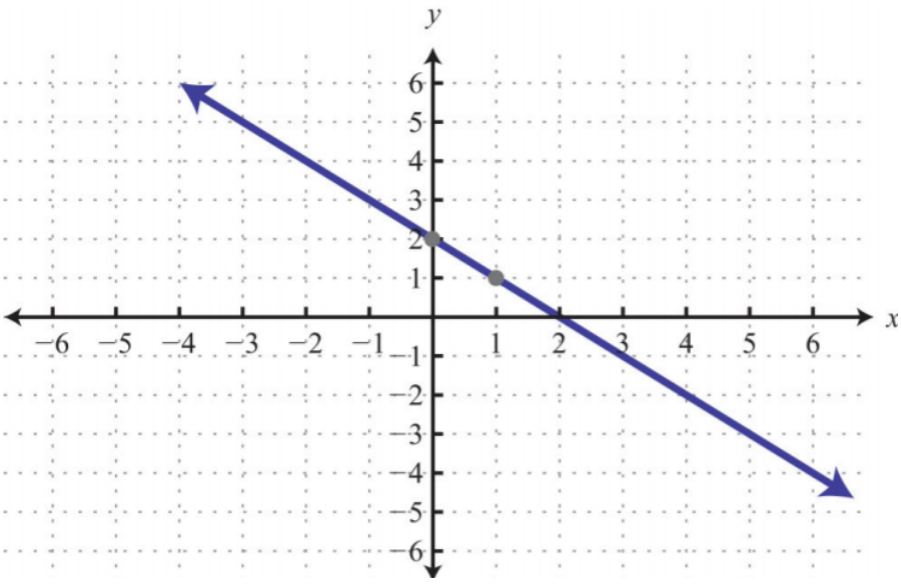


Figure 3.4.34

11.

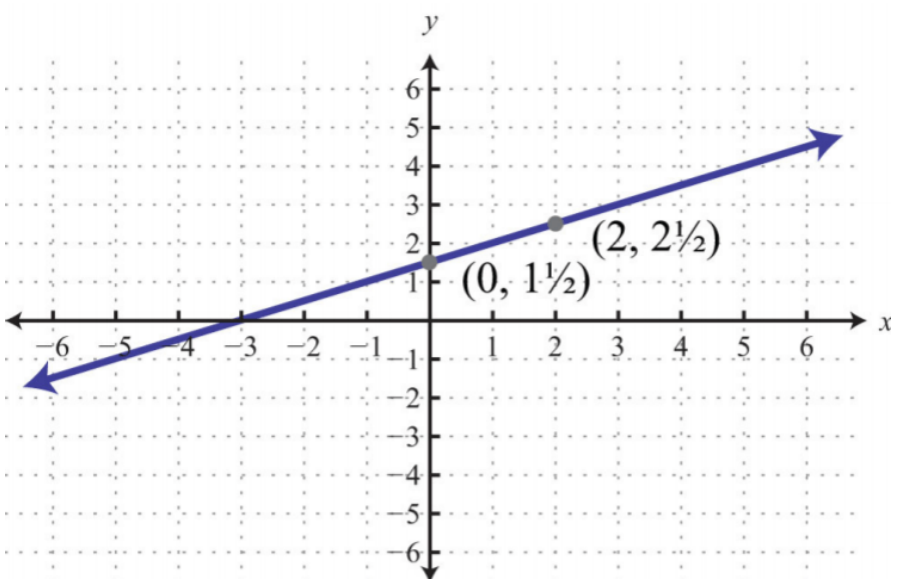


Figure 3.4.35

13.

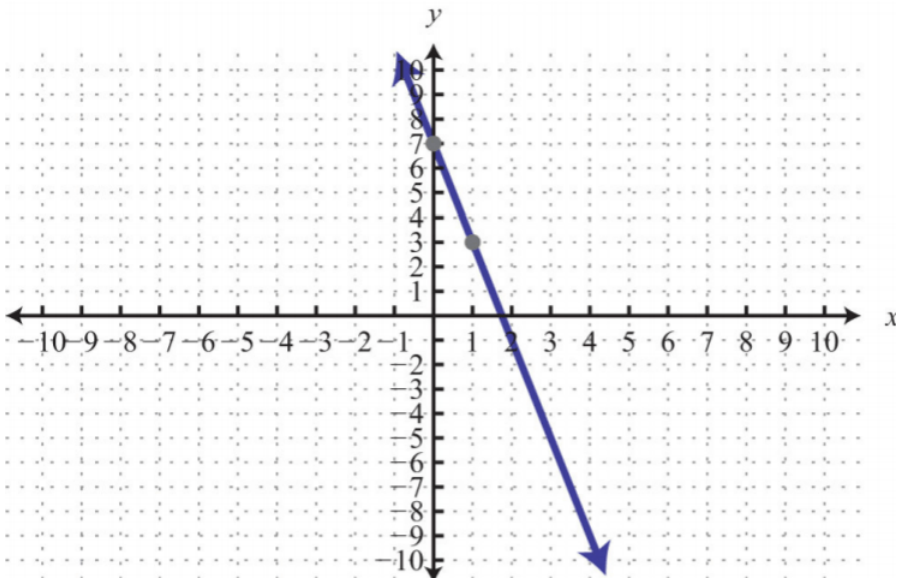


Figure 3.4.36

15.

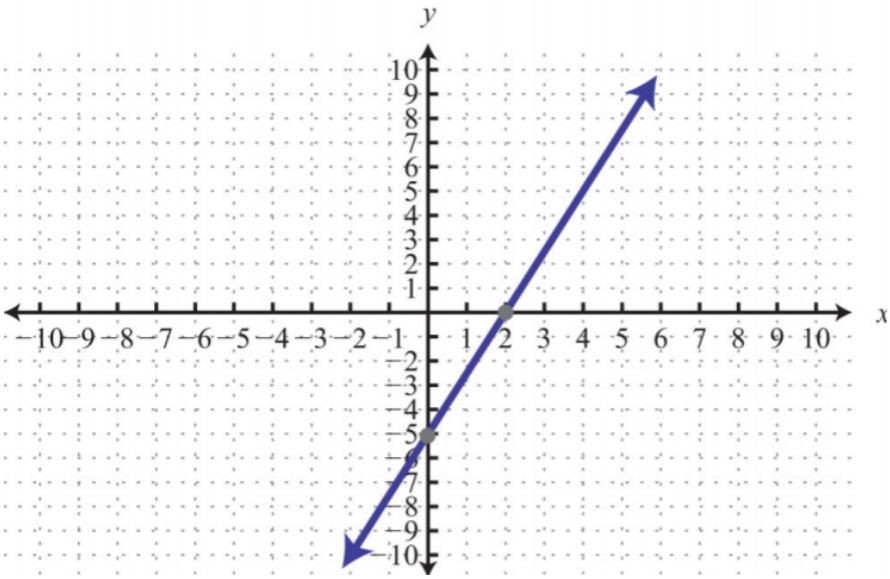


Figure 3.4.37

17.

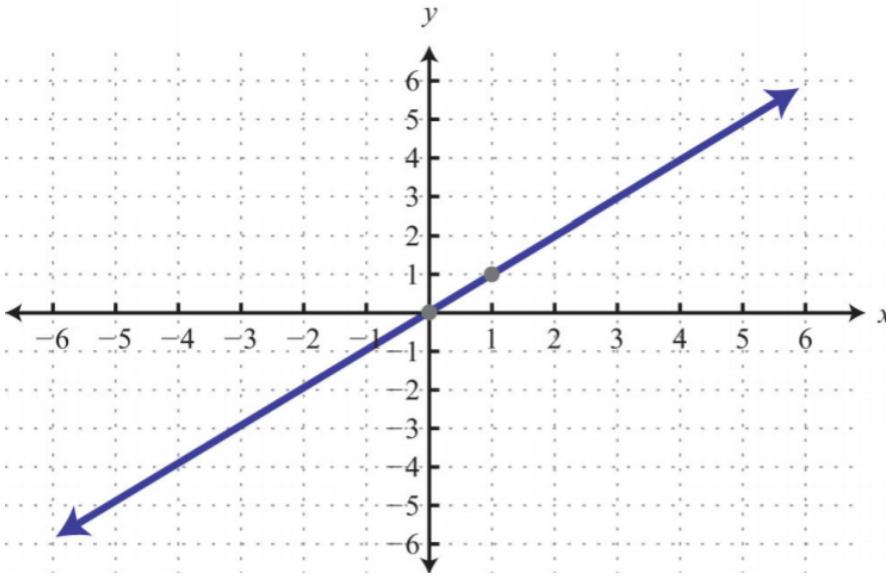


Figure 3.4.38

19.

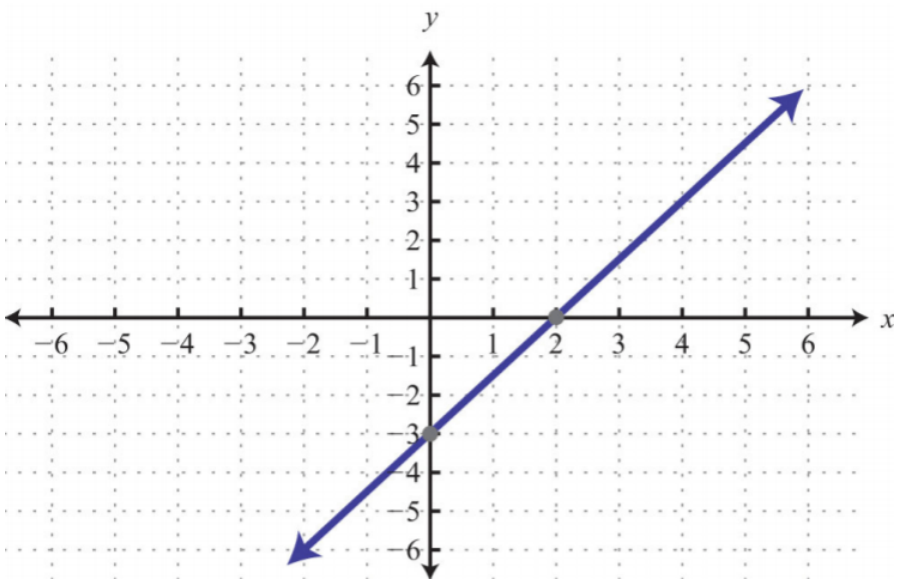


Figure 3.4.39

21.

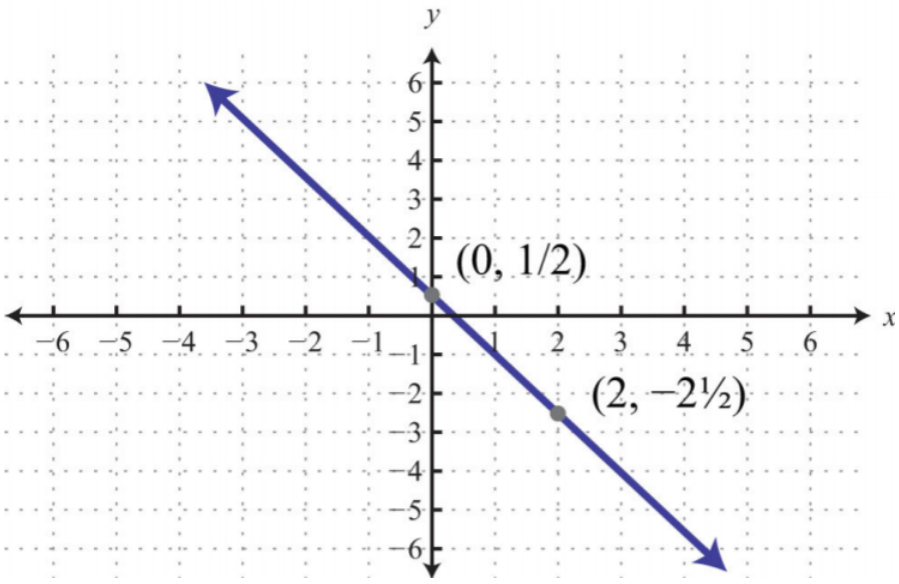


Figure 3.4.40

23.

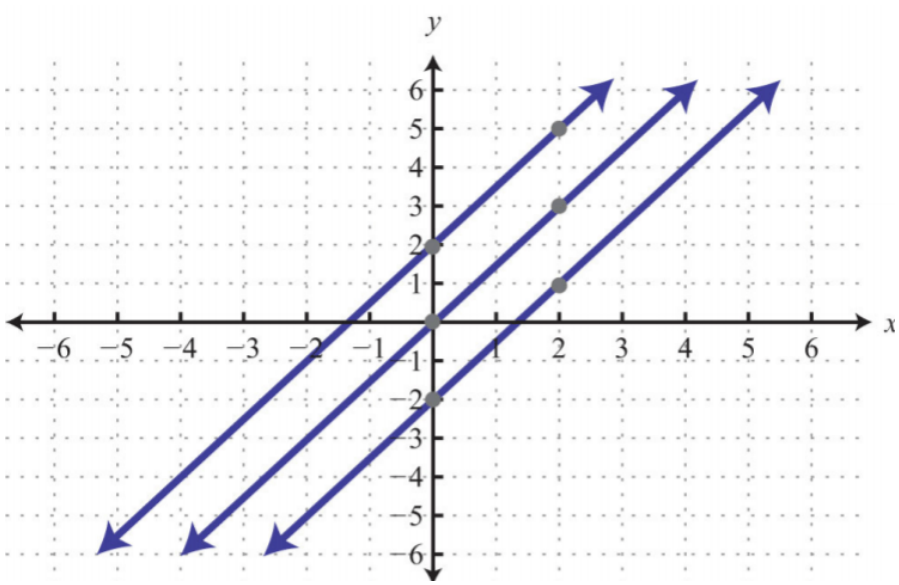


Figure 3.4.41

Exercise 3.4.9 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary
5. Answers may vary
7. Answers may vary

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3.5: Finding Linear Equations

Learning Objectives

- Given a graph, identify the slope and y-intercept.
- Find the equation of the line using the slope and y-intercept.
- Find the equation of the line using point-slope form.

Finding Equations Using Slope-Intercept Form

Given the algebraic equation of a line, we are able to graph it in a number of ways. In this section, we will be given a geometric description of a line and be asked to find the algebraic equation. Finding the equation of a line can be accomplished in a number of ways, the first of which makes use of slope-intercept form, $y = mx + b$. If we know the slope, m, and the y-intercept, $(0, b)$, we can construct the equation.

Example 3.5.1

Find the equation of a line with slope $m = -\frac{5}{8}$ and y-intercept $(0, 1)$.

Solution:

The given y-intercept implies that $b = 1$. Substitute the slope m and the y-value of the y-intercept b into the equation $y = mx + b$.

$$\begin{aligned} y &= m\ x + b \\ &\downarrow \quad\ \downarrow \\ y &= -\frac{5}{8}x + 1 \end{aligned}$$

Answer:

$$y = -\tfrac{5}{8}x + 1$$

Finding a linear equation is very straightforward if the slope and y-intercept are given. This is certainly not always the case; however, the example demonstrates that the algebraic equation of a line depends on these two pieces of information. If the graph is given, then we can often read it to determine the y-intercept and slope.

Example 3.5.2

Find the equation of the line given the graph:

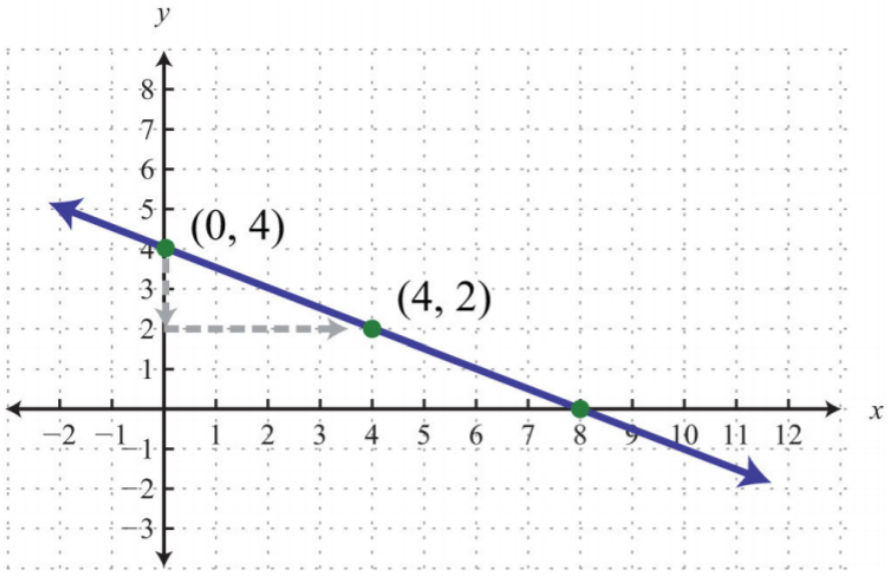


Figure 3.5.1

Solution:

By reading the graph, we can see that the y-intercept is $(0, 4)$, and thus

$$b = 4$$

Furthermore, from the points $(0, 4)$ to $(4, 2)$, we can see that the rise is -2 units and the run is 4 units.

$$m = \frac{rise}{run} = \frac{-2}{4} = -\frac{1}{2}$$

Now substitute m and b into slope-intercept form:

$$y = mx+b$$
$$y = -\frac{1}{2}x+4$$

Answer:

$$y = -\tfrac{1}{2}x + 4$$

Often the y-intercept and slope will not be given or are not easily discernible from the graph. For this reason, we will develop some algebraic techniques that allow us to calculate these quantities.

Example 3.5.3

Find the equation of the line with slope $m = -\frac{2}{3}$ passing through $(-6, 3)$.

Solution:

Begin by substituting the given slope into slope-intercept form.

$$y = mx + b$$
$$y = -\frac{2}{3}x + b$$

For the ordered pair $(-6, 3)$ to be a solution, it must solve the equation. Therefore, we can use it to find b. Substitute the appropriate x- and y-values as follows:

$$\begin{array}{c} y = -\frac{2}{3}x + b \\ \downarrow \quad \downarrow \\ (3) = -\frac{2}{3}(-6) + b \end{array}$$

After substituting the appropriate values, solve for the only remaining variable, b.

$$\begin{aligned} 3 &= -\frac{2}{3}(-6) + b \\ 3 &= -2(-2) + b \\ 3 &= 4 + b \\ -1 &= b \end{aligned}$$

Once we have b, we can then complete the equation:

$$\begin{aligned} y &= mx+b \\ y &= -\frac{2}{3}x-1 \end{aligned}$$

As a check, verify that $(-6, 3)$ solves this linear equation as follows:

Answer:

$y = -\frac{2}{3}x - 1$

Example 3.5.4

Find the equation of the line given the graph:

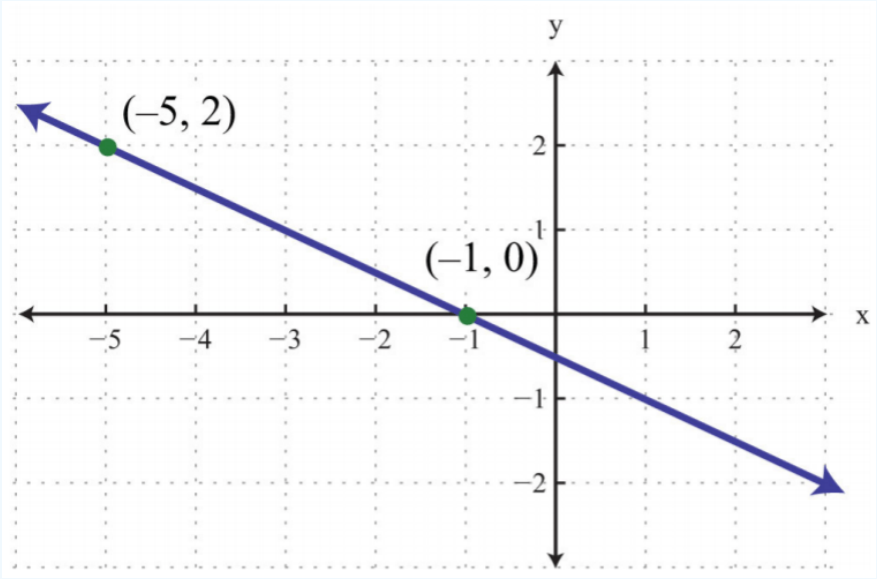


Figure 3.5.2

Solution:

Use the graph to determine the slope. From the points $(-5,2)$ to $(-1,0)$, we can see that the rise between the points is -2 units and the run is 4 units. Therefore, we calculate the slope as follows:

$$m = \frac{rise}{run} = \frac{-2}{4} = -\frac{1}{2}$$

Substitute the slope into slope-intercept form.

$$\begin{aligned} y &= mx + b \\ y &= -\frac{1}{2}x + b \end{aligned}$$

Now substitute the coordinates of one of the given points to find b. It does not matter which one you choose. Here choose $(-1,0)$:

$$\begin{aligned} y &= -\frac{1}{2}x + b \\ 0 &= -\frac{1}{2}(-1) + b \\ 0 &= \frac{1}{2} + b \\ -\frac{1}{2} &= b \end{aligned}$$

Next, put it all together.

$$\begin{aligned} y &= mx + b \\ y &= -\frac{1}{2}x - \frac{1}{2} \end{aligned}$$

Answer:

$$y = -\tfrac{1}{2}x - \tfrac{1}{2}$$

As an exercise, substitute the coordinates of the point $(-5,2)$ to see that b will turn out to be the same value. In fact, you can substitute any ordered pair solution of the line to find b. We next outline an algebraic technique for finding the equation of a nonvertical line passing through two given points.

Example 3.5.5

Find the equation of the line passing through $(-4,-2)$ and $(1,3)$.

Solution:

When finding a linear equation using slope-intercept form $y = mx + b$, the goal is to find m and then b.

Step 1: Find the slope m. In this case, given two points, use the slope formula.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3 - (-2)}{1 - (-4)} \\ &= \frac{3 + 2}{1 + 4} \\ &= \frac{5}{5} \\ &= 1 \end{aligned}$$

Substitute $m = 1$ into slope-intercept form.

$$y = mx + b$$
$$y = 1x + b$$

Step 2: Find b. To do this, substitute the coordinates of any given ordered pair solution. Use $(1, 3)$:

$$y = 1x + b$$
$$3 = 1(1) + b$$
$$3 = 1 + b$$
$$2 = b$$

Step 3: Finish building the equation by substituting in the value for b. In this case, we use $b = 2$.

$$y = 1x + b$$
$$y = 1x + 2$$

Answer:

$$y = x + 2$$

These three steps outline the process for finding the equation of any nonvertical line in slope-intercept form. This is a completely algebraic method, but always keep in mind the geometry behind the technique.

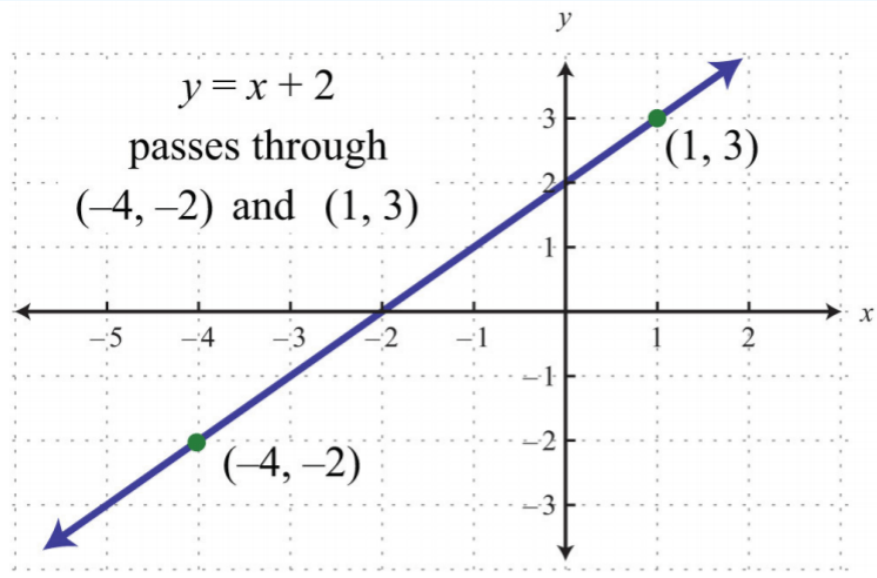


Figure $3.5.3$

Note that the line has a y-intercept at $(0, 2)$, with slope $m = 1$.

Example $3.5.6$

Find the equation of the line passing through $(-1, 3)$ and $(5, 1)$.

Solution:

First, find m, the slope. Given two points, use the slope formula as follows:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1-(3)}{5-(-1)} \\ &= \frac{1-3}{5+1} \\ &= \frac{-2}{6} \\ &= -\frac{1}{3} \end{aligned}$$

Substitute $m = -\frac{1}{3}$ into slope-intercept form.

$$\begin{aligned} y &= mx + b \\ y &= -\frac{1}{3}x + b \end{aligned}$$

Next, find b. Substitute the coordinates of the point $(-1, 3)$.

$$\begin{aligned} y &= -\frac{1}{3}x + b \\ 3 &= -\frac{1}{3}(-1) + b \\ 3 &= \frac{1}{3} + b \\ 3 - \frac{1}{3} &= b \\ \frac{3 \cdot 3}{1 \cdot 3} - \frac{1}{3} &= b \\ \frac{8}{3} &= b \end{aligned}$$

Finally, substitute $b = \frac{8}{3}$ into the equation.

$$\begin{aligned} y &= -\frac{1}{3}x + b \\ y &= -\frac{1}{3}x + \frac{8}{3} \end{aligned}$$

Answer:

$$y = -\tfrac{1}{3}x + \tfrac{8}{3}$$

Exercise 3.5.1

Find the equation of the line passing through $(-3, 4)$ and $(6, -2)$.

Answer available at the end of this section.

Finding Equations Using a Point and the Slope

Given any point on a line and its slope, we can find the equation of that line. Begin by applying the slope formula with a given point (x_1, y_1) and a variable point (x, y).

$$\begin{aligned} m &= \frac{y-y_1}{x-x_1} \\ \frac{m}{1} &= \frac{y-y_1}{x-x_1} && \textit{Cross multiply.} \\ m(x-x_1) &= y-y_1 && \textit{Apply the symmetric property.} \\ y-y_1 &= m(x-x_1) \end{aligned}$$

The equation $y-y_1=m(x-x_1)$ is called the point-slope form of a line. Any nonvertical linear equation can be written in this form. It is useful for finding the equation of a line given the slope and any ordered pair solution.

Example 3.5.7

Find the equation of the line with slope $m=\frac{1}{2}$ passing through $(4,-1)$.

Solution:

Use point-slope form, where $m=\frac{1}{2}$ and $(x_1,y_1)=(4,-1)$.

At this point, we must choose to present the equation of our line in either standard form or slope-intercept form.

$\underline{Standard\ form}$	$\underline{Slope-intercept\ form}$
$y+1=\frac{1}{2}x-2$	$y+1=\frac{1}{2}x=2$
$y+1-1=\frac{1}{2}x-2-1$	$y+1-1=\frac{1}{2}x-2-1$
$y-\frac{1}{2}x=\frac{1}{2}x-3-\frac{1}{2}x$	$y=\frac{1}{2}x-3$
$-\frac{1}{2}x+y=-3$	

In this textbook, we will present our lines in slope-intercept form. This facilitates future graphing.

Answer:

$$y=\frac{1}{2}x-3$$

Example 3.5.8

Find the equation of the line passing through $(-5,3)$ with slope $m=-\frac{2}{5}$.

Solution:

Substitute $(-5,3)$ and $m=-\frac{2}{5}$ into point-slope form.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-(3) &= -\frac{2}{5}(x-(-5)) && \textit{Solve for } y. \\ y-3 &= -\frac{2}{5}(x+5) && \textit{Distribute } -\frac{2}{5}. \\ y-3 &= -\frac{2}{5}x-2 \\ y-3+3 &= -\frac{2}{5}x-2+3 \\ y &= -\frac{2}{5}x+1 \end{aligned}$$

Answer:

$$y=-\frac{2}{5}x+1$$

It is always important to understand what is occurring geometrically. Compare the answer for the last example to the

corresponding graph below.

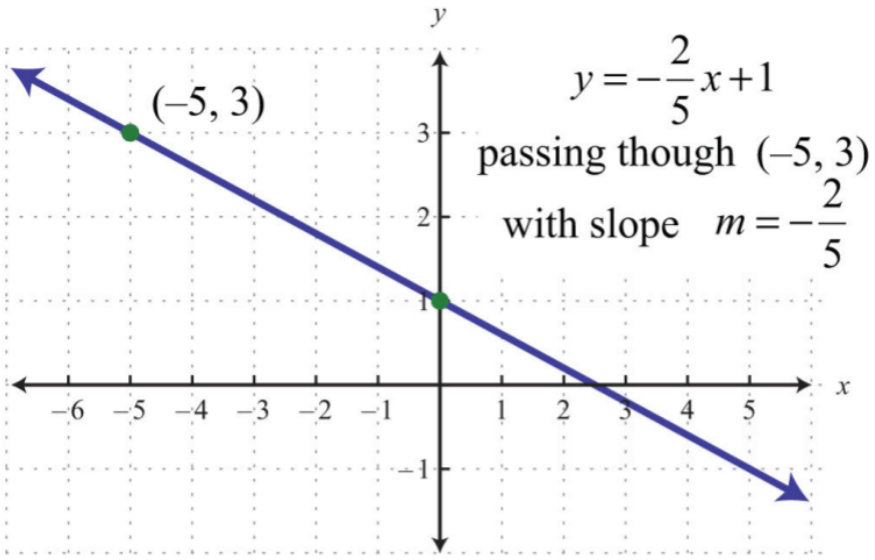


Figure 3.5.4

The geometric understanding is important because you will often be given graphs from which you will need to determine a point on the line and the slope.

Example 3.5.9

Find an equation of the given graph:

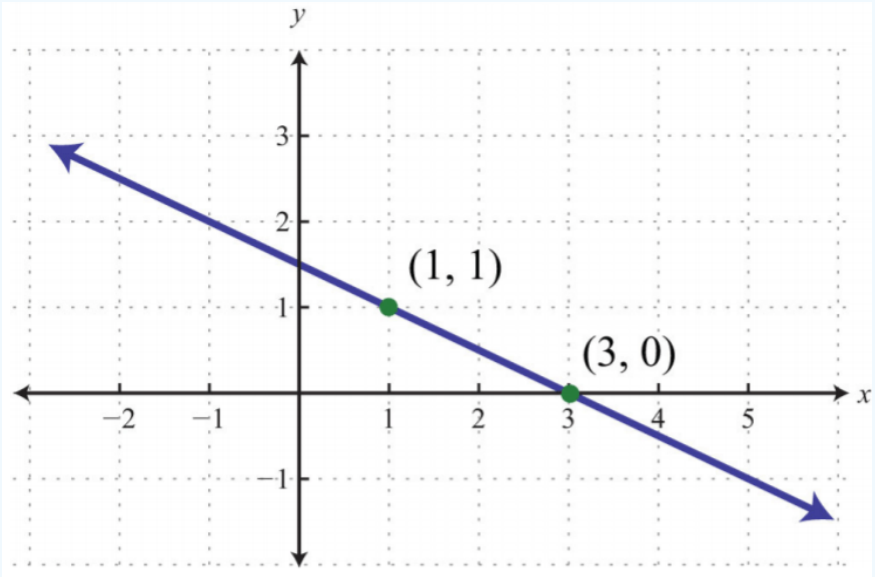


Figure 3.5.5

Solution:

Between the points $(1,1)$ to $(3,0)$, we can see that the rise is -1 unit and the run is 2 units. The slope of the line is $m = \frac{rise}{run} = \frac{-1}{2} = -\frac{1}{2}$. Use this and the point $(3,0)$ to find the equation as follows:

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-0 &= -\frac{1}{2}(x-3) \\ y &= -\frac{1}{2}x+\frac{3}{2} \end{aligned}$$

Answer:

$$y=-\tfrac{1}{2}x+\tfrac{3}{2}$$

Example 3.5.10

Find the equation of the line passing through $(-1,1)$ and $(7,-1)$.

Solution:

Begin by calculating the slope using the slope formula.

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{-1-1}{7-(-1)} \\ &= \frac{-2}{7+1} \\ &= \frac{-2}{8} \\ &= -\frac{1}{4} \end{aligned}$$

Next, substitute into point-slope form using one of the given points; it does not matter which point is used. Use $m=-\frac{1}{4}$ and the point $(-1,1)$.

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-1 &= -\frac{1}{4}(x-(-1)) \\ y-1 &= -\frac{1}{4}(x+1) \\ y-1 &= -\frac{1}{4}x-\frac{1}{4} \\ y &= -\frac{1}{4}x-\frac{1}{4}+1 \\ y &= -\frac{1}{4}x+\frac{3}{4} \end{aligned}$$

Answer:

$$y=-\tfrac{1}{4}x+\tfrac{3}{4}$$

Exercise 3.5.2

Find the equation of the line passing through $(4,-5)$ and $(-4,1)$.

Answer available at the end of this section.

Key Takeaways

- Given the graph of a line, you can determine the equation in two ways, using slope-intercept form, $y=mx+b$, or point-slope

form, $y - y_1 = m(x - x_1)$.

- The slope and one point on the line is all that is needed to write the equation of a line.
- All nonvertical lines are completely determined by their y-intercept and slope.
- If the slope and y-intercept can be determined, then it is best to use slope-intercept form to write the equation.
- If the slope and a point on the line can be determined, then it is best to use point-slope form to write the equation.

Exercise 3.5.3 Slope-Intercept Form

Determine the slope and y-intercept.

1. $5x - 3y = 18$
2. $-6x + 2y = 12$
3. $x - y = 5$
4. $-x + y = 0$
5. $4x - 5y = 15$
6. $-7x + 2y = 3$
7. $y = 3$
8. $y = -\frac{3}{4}$
9. $\frac{1}{5}x - \frac{1}{3}y = -1$
10. $\frac{5}{16}x + \frac{3}{8}y = 9$
11. $-\frac{2}{3}x + \frac{5}{2}y = \frac{5}{4}$
12. $\frac{1}{2}x - \frac{3}{4}y = -\frac{1}{2}$

Answer available at the end of this section.

Exercise 3.5.4 Finding Equations in Slope-Intercept Form

Given the slope and y-intercept, determine the equation of the line.

1. $m = \frac{1}{2}$; $(0, 5)$
2. $m = 4$; $(0, -1)$
3. $m = -\frac{2}{3}$; $(0, -4)$
4. $m = -3$; $(0, 9)$
5. $m = 0$; $(0, -1)$
6. $m = 5$; $(0, 0)$

Answer available at the end of this section.

Exercise 3.5.5 Finding Equations in Slope-Intercept Form

Given the graph, find the equation in slope-intercept form.

1.

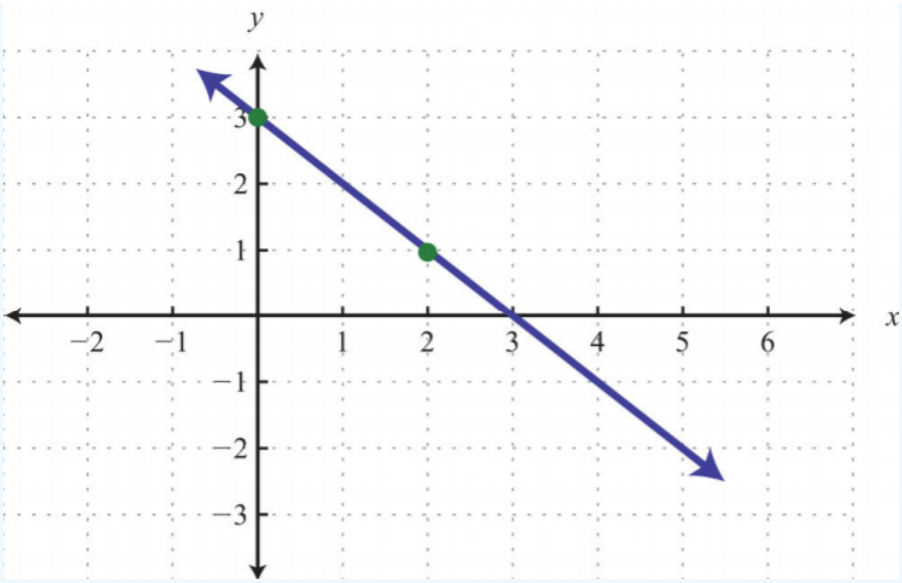


Figure 3.5.6

2.

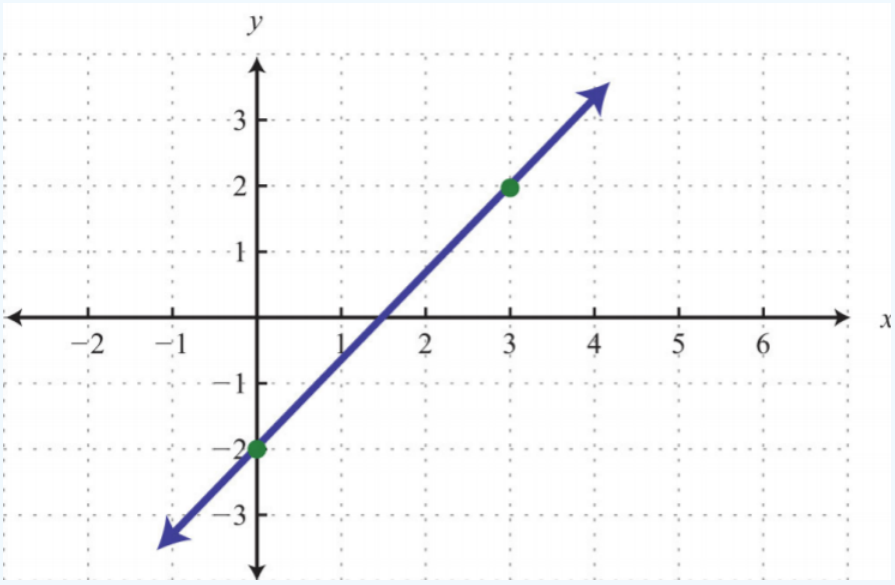


Figure 3.5.7

3.

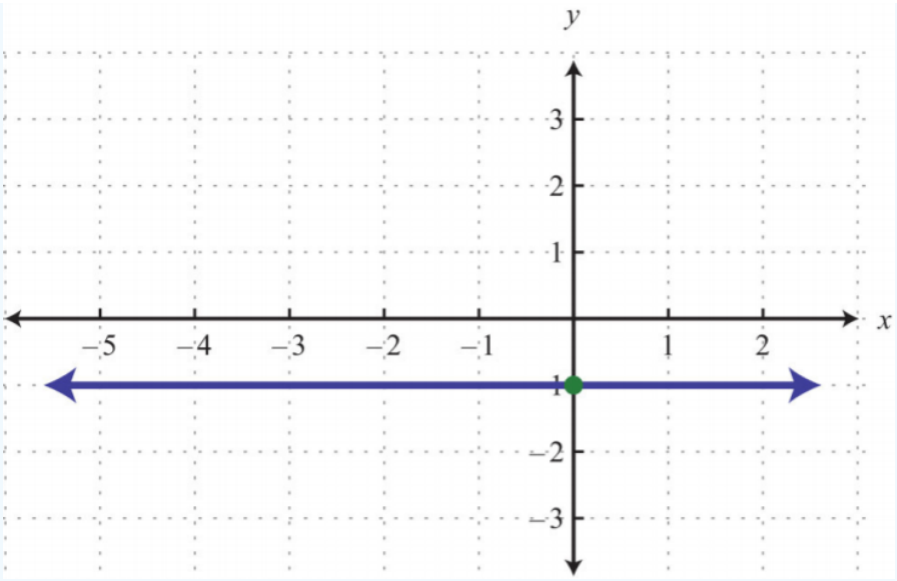


Figure 3.5.8

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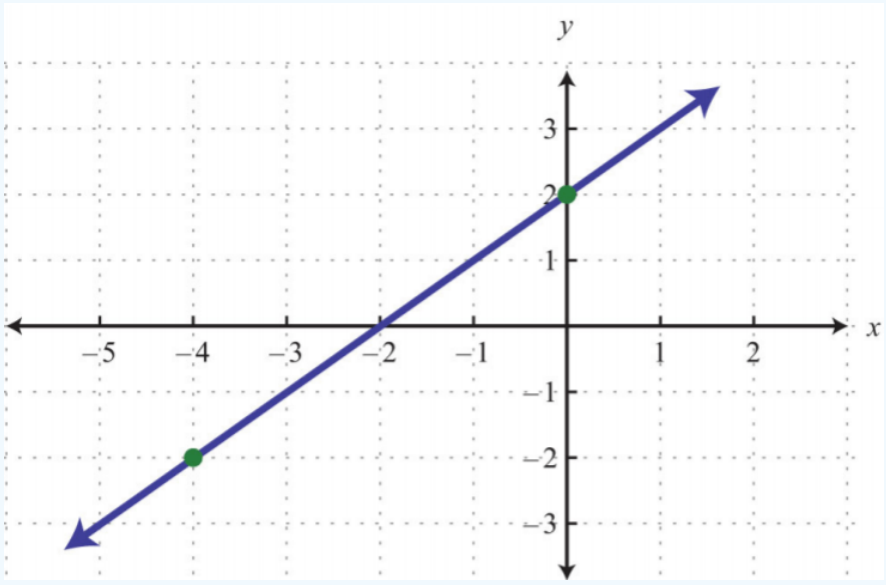


Figure 3.5.9

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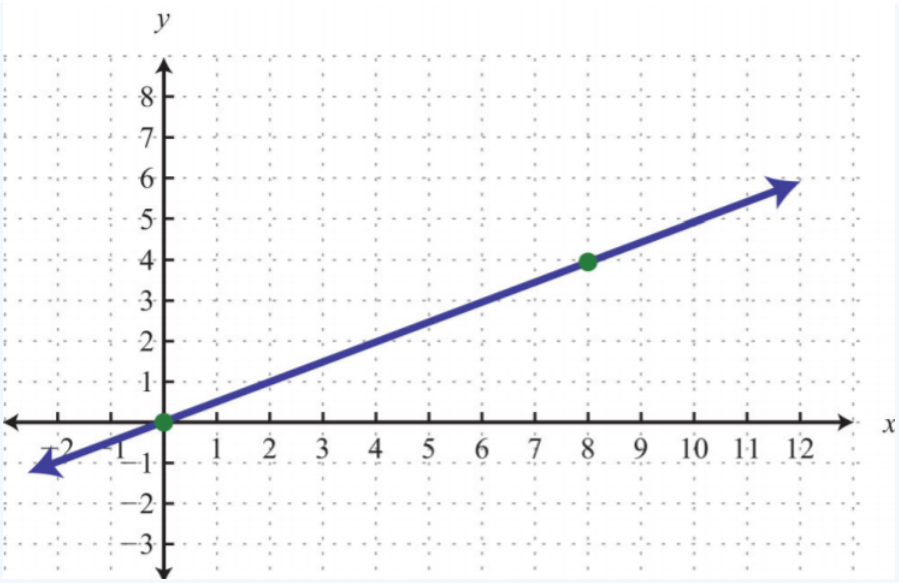


Figure 3.5.10

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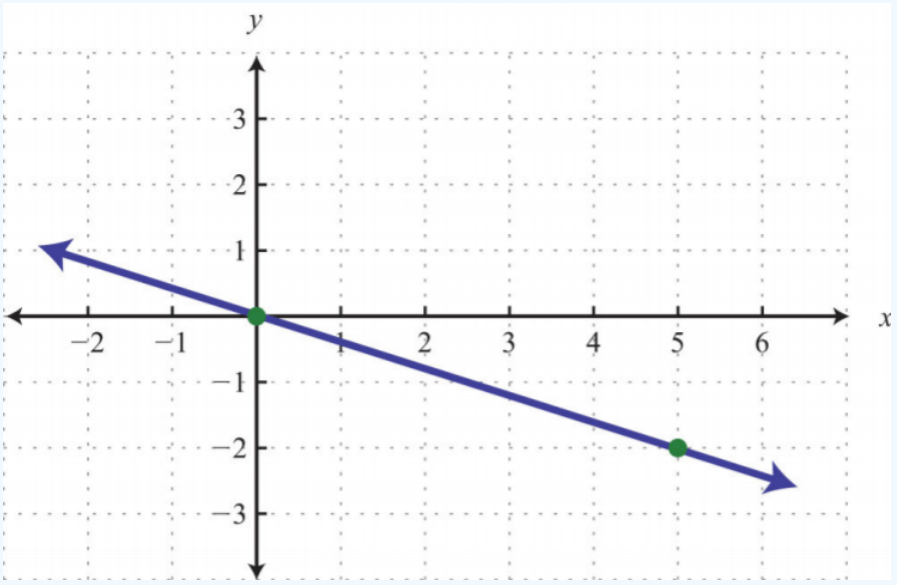


Figure 3.5.11

Answer available at the end of this section.

Exercise 3.5.6 Finding Equations in Slope-Intercept Form

Find the equation, given the slope and a point.

1. $m = \frac{2}{3}$; $(-9, 2)$
2. $m = -\frac{1}{5}$; $(5, -5)$
3. $m = 0$; $(-4, 3)$
4. $m = 3$; $(-2, 1)$

5. $m = -5$; $(-2, 8)$
6. $m = -4$; $(\frac{1}{2}, -\frac{3}{2})$
7. $m = -\frac{1}{2}$; $(3, 2)$
8. $m = \frac{3}{4}$; $(\frac{1}{3}, \frac{5}{4})$
9. $m = 0$; $(3, 0)$
10. m undefined; $(3, 0)$

Answer available at the end of this section.

Exercise 3.5.7 Finding Equations in Slope-Intercept Form

Given two points, find the equation of the line.

1. $(-6, 6), (2, 2)$
2. $(-10, -3), (5, 0)$
3. $(0, \frac{1}{2}), (\frac{1}{2}, -1)$
4. $(\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, 1)$
5. $(3, -4), (-6, -7)$
6. $(-5, 2), (3, 2)$
7. $(-6, 4), (-6, -3)$
8. $(-4, -4), (-1, -1)$
9. $(3, -3), (-5, 5)$
10. $(0, 8), (-4, 0)$

Answer available at the end of this section.

Exercise 3.5.8 Equations Using Point-Slope Form

Find the equation, given the slope and a point.

1. $m = \frac{1}{2}$; $(4, 3)$
2. $m = -\frac{1}{3}$; $(9, -2)$
3. $m = 6$; $(1, -5)$
4. $m = -10$; $(1, -20)$
5. $m = -3$; $(2, 3)$
6. $m = \frac{2}{3}$; $(-3, -5)$
7. $m = -\frac{3}{4}$; $(-8, 3)$
8. $m = 5$; $(\frac{1}{5}, -3)$
9. $m = -3$; $(-\frac{1}{9}, 2)$
10. $m = 0$; $(4, -6)$
11. $m = 0$; $(-5, 10)$
12. $m = \frac{5}{8}$; $(4, 3)$
13. $m = -\frac{3}{5}$; $(-2, -1)$
14. $m = \frac{1}{4}$; $(12, -2)$
15. $m = 1$; $(0, 0)$
16. $m = -\frac{3}{4}$; $(0, 0)$

Answer available at the end of this section.

Exercise 3.5.9 Equations Using Point-Slope Form

Given the graph, use the point-slope formula to find the equation.

1.

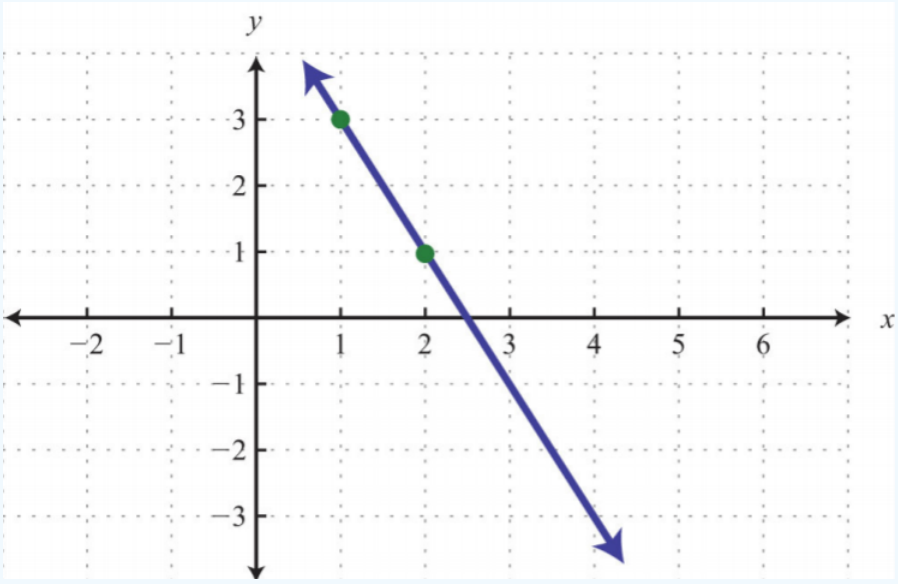


Figure 3.5.12

2.

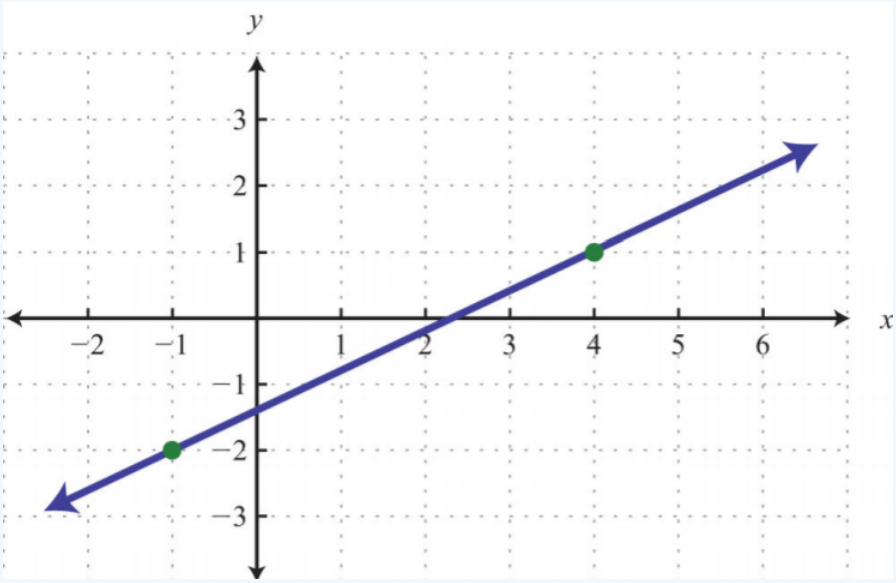


Figure 3.5.13

3.

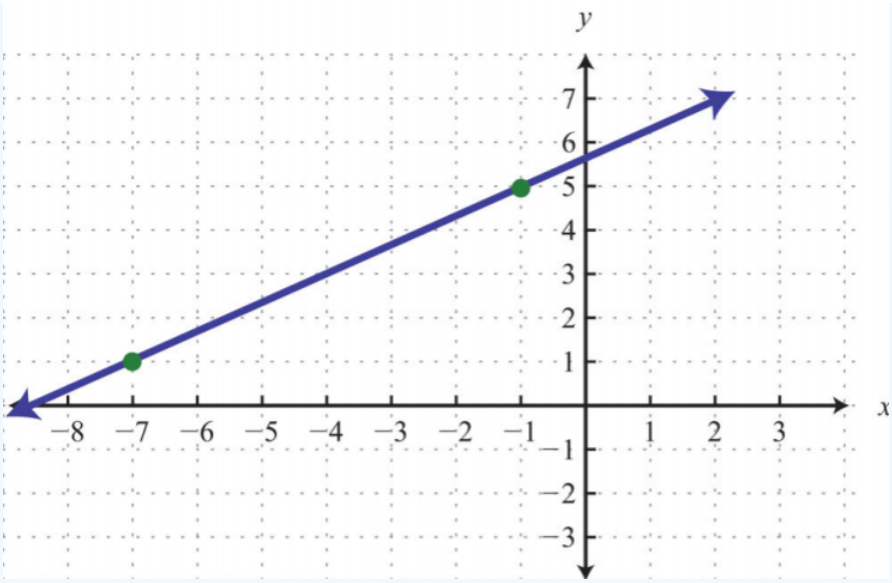


Figure 3.5.14

4.

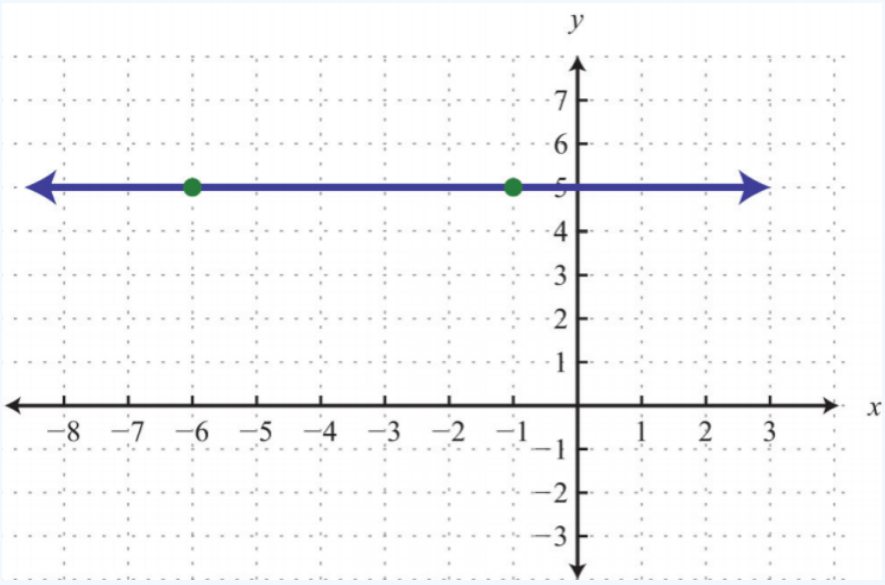


Figure 3.5.15

5.

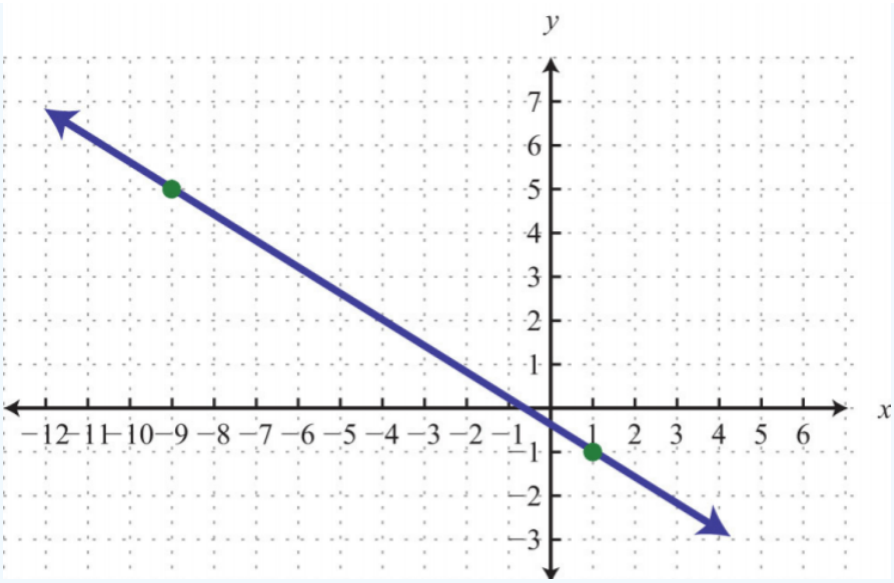


Figure 3.5.16

6.

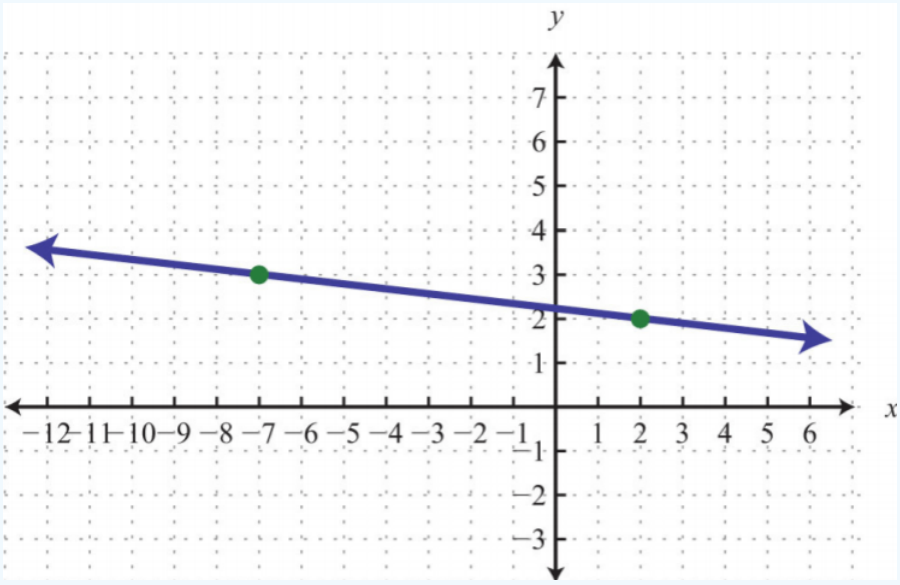


Figure 3.5.17

Answer available at the end of this section.

Exercise 3.5.10 Equations Using Point-Slope Form

Use the point-slope formula to find the equation of the line passing through the two points.

1. $(-4, 0), (0, 5)$
2. $(-1, 2), (0, 3)$
3. $(-3, -2), (3, 2)$
4. $(3, -1), (2, -3)$

5. $(-2,4),(2,-4)$
6. $(-5,-2),(5,2)$
7. $(-3,-1),(3,3)$
8. $(1,5),(0,5)$
9. $(1,2),(2,4)$
10. $(6,3),(2,-3)$
11. $(10,-3),(5,-4)$
12. $(-3,3),(-1,12)$
13. $(\frac{4}{5},-\frac{1}{3}),(-\frac{1}{5},\frac{2}{3})$
14. $(\frac{5}{3},\frac{1}{3}),(-\frac{10}{3},-\frac{5}{3})$
15. $(3,-\frac{1}{4}),(4,-\frac{1}{2})$
16. $(0,0),(-5,1)$
17. $(2,-4),(0,0)$
18. $(3,5),(3,-2)$
19. $(-4,7),(-1,7)$
20. $(-8,0),(6,0)$

Answer available at the end of this section.

Exercise 3.5.11 Applications

1. Joe has been keeping track of his cellular phone bills for the last two months. The bill for the first month was $\$38.00$ for 100 minutes of usage. The bill for the second month was $\$45.50$ for 150 minutes of usage. Find a linear equation that gives the total monthly bill based on the minutes of usage.
2. A company in its first year of business produced 150 training manuals for a total cost of $\$2,350$. The following year, the company produced 50 more manuals at a cost of $\$1,450$. Use this information to find a linear equation that gives the total cost of producing training manuals from the number of manuals produced.
3. A corn farmer in California was able to produce 154 bushels of corn per acre 2 years after starting his operation. Currently, after 7 years of operation, he has increased his yield to 164 bushels per acre. Use this information to write a linear equation that gives the total yield per acre based on the number of years of operation, and use it to predict the yield for next year.
4. A Webmaster has noticed that the number of registered users has been steadily increasing since beginning an advertising campaign. Before starting to advertise, he had $1,200$ registered users, and after 3 months of advertising he now has $1,590$ registered users. Use this data to write a linear equation that gives the total number of registered users, given the number of months after starting to advertise. Use the equation to predict the number of users 7 months into the advertising campaign.
5. A car purchased new cost $\$22,000$ and was sold 10 years later for $\$7,000$. Write a linear equation that gives the value of the car in terms of its age in years.
6. An antique clock was purchased in 1985 for $\$1,500$ and sold at auction in 1997 for $\$5,700$. Determine a linear equation that models the value of the clock in terms of years since 1985.

Answer available at the end of this section.

Exercise 3.5.12 Discussion Board Topics

1. Discuss the merits and drawbacks of point-slope form and y-intercept form.
2. Research and discuss linear depreciation. In a linear depreciation model, what do the slope and y-intercept represent?

Answer available at the end of this section.

Answers

Exercise 3.5.1

$y=-\frac{2}{3}x+2$

Exercise 3.5.2

$y = -\frac{3}{4}x - 2$

Exercise 3.5.3 **Slope-Intercept Form**

1. $m = \frac{5}{3}$; $(0, -6)$
3. $m = 1$; $(0, -5)$
5. $m = \frac{4}{5}$; $(0, -3)$
7. $m = 0$; $(0, 3)$
9. $m = \frac{3}{5}$; $(0, 3)$
11. $m = \frac{4}{15}$; $(0, \frac{1}{2})$

Exercise 3.5.4 **Finding Equations in Slope-Intercept Form**

1. $y = \frac{1}{2}x + 5$
3. $y = -\frac{2}{3}x - 4$
5. $y = -1$

Exercise 3.5.5 **Finding Equations in Slope-Intercept Form**

1. $y = -x + 3$
3. $y = -1$
5. $y = \frac{1}{2}x$

Exercise 3.5.6 **Finding Equations in Slope-Intercept Form**

1. $y = \frac{2}{3}x + 8$
3. $y = 3$
5. $y = -5x - 2$
7. $y = -\frac{1}{2}x + \frac{7}{2}$
9. $y = 0$

Exercise 3.5.7 **Finding Equations in Slope-Intercept Form**

1. $y = -\frac{1}{2}x + 3$
3. $y = -3x + \frac{1}{2}$
5. $y = \frac{1}{3}x - 5$
7. $x = -6$
9. $y = -x$

Exercise 3.5.8 **Equations Using Point-Slope Form**

1. $y = \frac{1}{2}x + 1$
3. $y = 6x - 11$
5. $y = -3x + 9$
7. $y = -\frac{3}{4}x - 3$
9. $y = -3x + \frac{5}{3}$
11. $y = 10$
13. $y = -\frac{3}{5}x - \frac{1}{15}$
15. $y = x$

Exercise 3.5.9 **Equations Using Point-Slope Form**

1. $y = -2x + 5$
3. $y = \frac{2}{3}x + \frac{17}{3}$
5. $y = -\frac{3}{5}x - \frac{2}{5}$

Exercise 3.5.10 **Equations Using Point-Slope Form**

1. $y = \frac{5}{4}x + 5$
3. $y = \frac{2}{3}x$
5. $y = -2x$
7. $y = \frac{2}{3}x + 1$
9. $y = 2x$
11. $y = \frac{1}{5}x - 5$
13. $y = -x + \frac{7}{15}$
15. $y = -\frac{1}{4}x + \frac{1}{2}$
17. $y = -2x$
19. $y = 7$

Exercise 3.5.11 **Applications**

1. cost $= 0.15x + 23$
3. yield $= 2x + 150$; 166 bushels
5. value $= -1,500x + 22,000$

Exercise 3.5.12 **Discussion Board Topics**

1. Answers may vary

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3.6: Parallel and Perpendicular Lines

Learning Objectives

- Determine the slopes of parallel and perpendicular lines.
- Find equations of parallel and perpendicular lines

Definition of Parallel and Perpendicular

Parallel lines are lines in the same plane that never intersect. Two nonvertical lines in the same plane, with slopes m_1 and m_2, are parallel if their slopes are the same, $m_1 = m_2$. Consider the following two lines:

Consider their corresponding graphs:

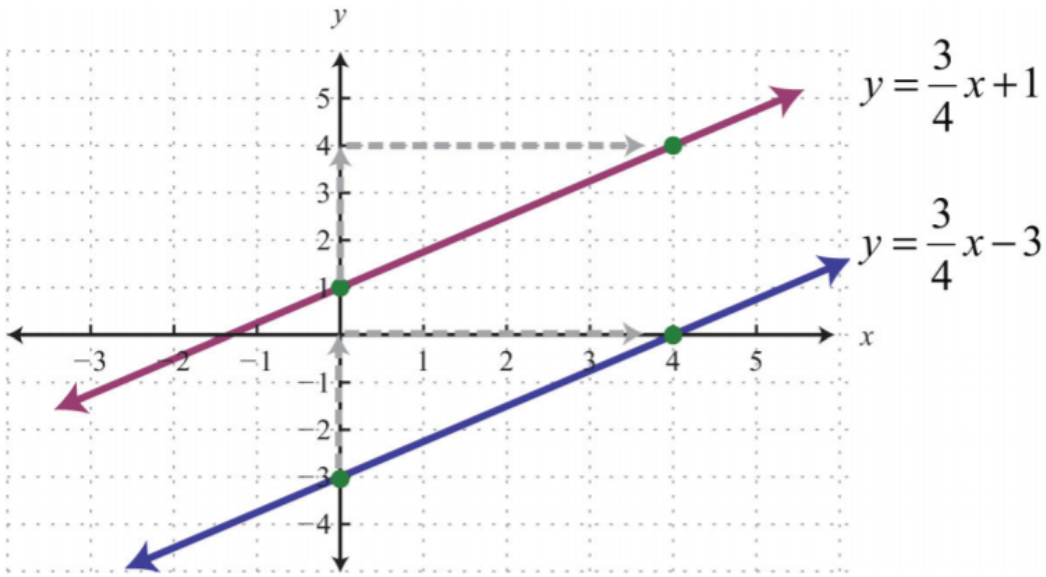


Figure 3.6.1

Both lines have a slope $m = \frac{3}{4}$ and thus are parallel.

Perpendicular lines are lines in the same plane that intersect at right angles (90 degrees). Two nonvertical lines in the same plane, with slopes m_1 and m_2, are perpendicular if the product of their slopes is $-1 : m1 \cdot m2 = -1$. We can solve for m_1 and obtain $m_1 = \frac{-1}{m_2}$. In this form, we see that perpendicular lines have slopes that are **negative reciprocals**, or opposite reciprocals. For example, if given a slope

$$m = -\frac{5}{8}$$

then the slope of a perpendicular line is the opposite reciprocal:

$$m_\perp = \frac{8}{5}$$

The mathematical notation $m_\perp$ reads "m perpendicular." We can verify that two slopes produce perpendicular lines if their product is -1.

$$m \cdot m_\perp = -\frac{5}{8} \cdot \frac{8}{5} = -\frac{40}{40} = -1 \quad \checkmark$$

Geometrically, we note that if a line has a positive slope, then any perpendicular line will have a negative slope. Furthermore, the rise and run between two perpendicular lines are interchanged.

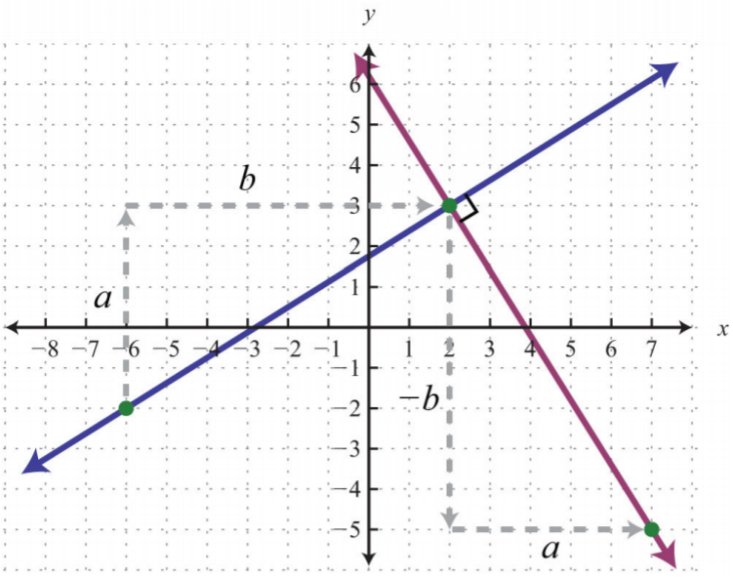


Figure 3.6.2

Perpendicular lines have slopes that are opposite reciprocals, so remember to find the reciprocal and change the sign. In other words,

$$\text{If } m = \tfrac{a}{b}, \text{ then } m_{\perp} = -\tfrac{b}{a}$$

Determining the slope of a perpendicular line can be performed mentally. Some examples follow

Table 3.6.1

Given slope	Slope of perpendicular line
$m = \frac{1}{2}$	$m_{\perp} = -2$
$m = -\frac{3}{4}$	$m_{\perp} = \frac{4}{3}$
$m = 3$	$m_{\perp} = -\frac{1}{3}$
$m = -4$	$m_{\perp} = \frac{1}{4}$

Example 3.6.1

Determine the slope of a line parallel to $y = -5x + 3$.

Solution:

Since the given line is in slope-intercept form, we can see that its slope is $m = -5$. Thus the slope of any line parallel to the given line must be the same, $m_{\parallel} = -5$. The mathematical notation $m_{\parallel}$ reads "m parallel."

Answer:

$$m_{\parallel} = -5$$

Example 3.6.2

Determine the slope of a line perpendicular to $3x - 7y = 21$.

Solution:

First, solve for y and express the line in slope-intercept form.

In this form, we can see that the slope of the given line is $m = \frac{3}{7}$, and thus $m_{\perp} = -\frac{7}{3}$.

Answer:

$m_{\perp} = -\frac{7}{3}$

Exercise 3.6.1

Find the slope of the line perpendicular to $15x + 5y = 20$.

Answer available at the end of this section.

Finding Equations of Parallel and Perpendicular Lines

We have seen that the graph of a line is completely determined by two points or one point and its slope. Often you will be asked to find the equation of a line given some geometric relationship—for instance, whether the line is parallel or perpendicular to another line.

Example 3.6.3

Find the equation of the line passing through $(6, -1)$ and parallel to $y = \frac{1}{2}x + 2$

Solution

Here the given line has slope $m = \frac{1}{2}$, and the slope of a line parallel is $m_{\parallel} = \frac{1}{2}$. Since you are given a point and the slope, use the point-slope form of a line to determine the equation.

Point	*Slope*
$(6, -1)$	$m_{\parallel} = \frac{1}{2}$

Answer:

$y = \frac{1}{2}x - 4$

It is important to have a geometric understanding of this question. We were asked to find the equation of a line parallel to another line passing through a certain point.

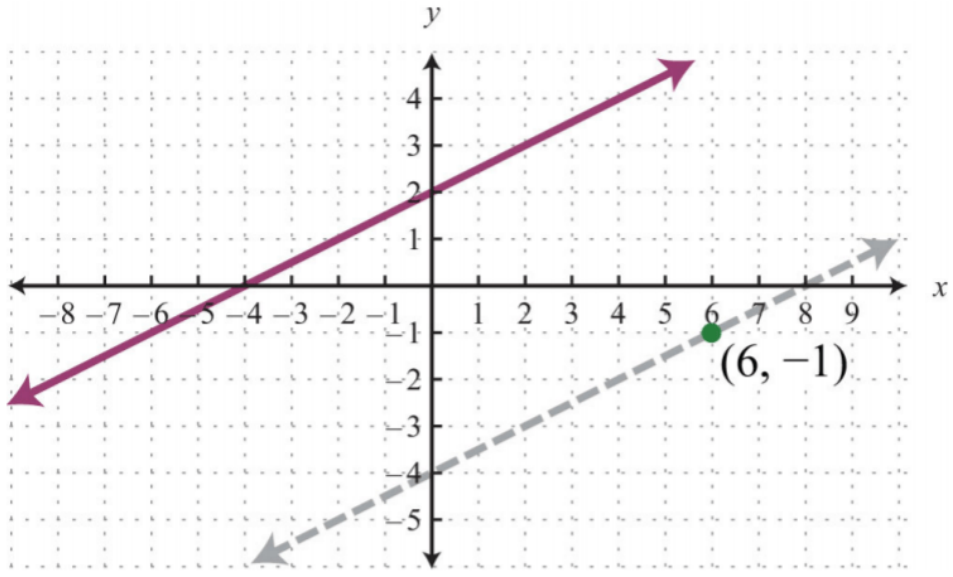


Figure 3.6.3

Through the point $(6,-1)$ we found a parallel line, $y=\frac{1}{2}x-4$, shown dashed. Notice that the slope is the same as the given line, but the y-intercept is different. If we keep in mind the geometric interpretation, then it will be easier to remember the process needed to solve the problem.

Example 3.6.4

Find the equation of the line passing through $(-1,-5)$ and perpendicular to $y=-\frac{1}{4}x+2$.

Solution:

The given line has slope $m=-\frac{1}{4}$, and thus $m_{\perp}=+\frac{4}{1}=4$. Substitute this slope and the given point into point-slope form.

Point	*Slope*
$(-1,-5)$	$m_{\perp}=4$

Answer:

$y=4x-1$

Geometrically, we see that the line $y=4x-1$, shown dashed below, passes through $(-1,-5)$ and is perpendicular to the given line.

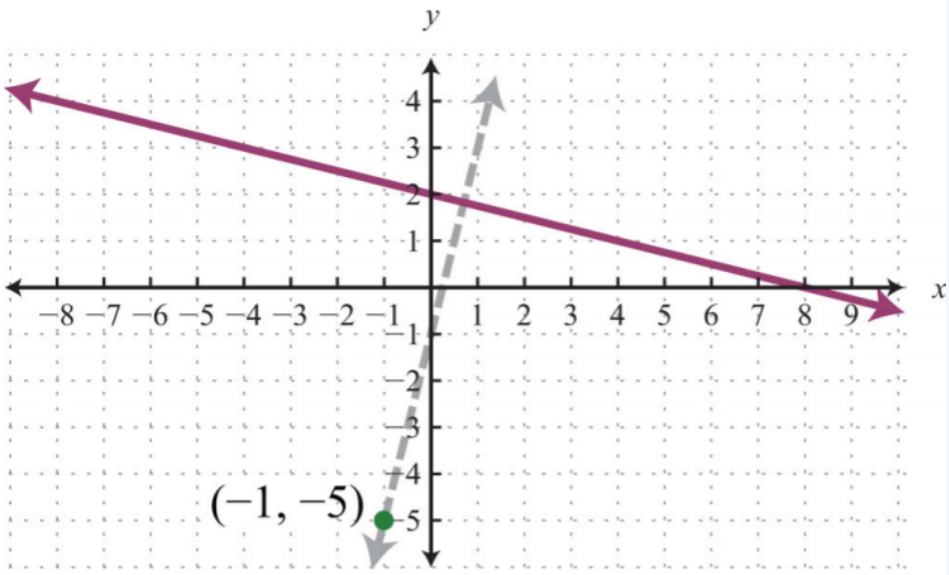


Figure 3.6.4

It is not always the case that the given line is in slope-intercept form. Often you have to perform additional steps to determine the slope. The general steps for finding the equation of a line are outlined in the following example.

Example 3.6.5

Find the equation of the line passing through $(8, -2)$ and perpendicular to $6x + 3y = 1$.

Solution:

Step 1: Find the slope m. First, find the slope of the given line. To do this, solve for y to change standard form to slope-intercept form, $y = mx + b$.

$$\begin{aligned} 6x + 3y &= 1 \\ 6x + 3y - 6x &= 1 - 6x \\ 3y &= -6x + 1 \\ \frac{3y}{3} &= \frac{-6x + 1}{3} \\ y &= \frac{-6x}{3} + \frac{1}{3} \\ y &= -2x + \frac{1}{3} \end{aligned}$$

In this form, you can see that the slope is $m = -2 = -\frac{2}{1}$, and thus $m_\perp = \frac{-1}{-2} = +\frac{1}{2}$.

Step 2: Substitute the slope you found and the given point into the point-slope form of an equation for a line. In this case, the slope is $m_\perp = \frac{1}{2}$ and the given point is $(8, -2)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= \frac{1}{2}(x - 8) \end{aligned}$$

Step 3: Solve for y.

Answer:

$y = \frac{1}{2}x - 6$

Example 3.6.6

Find the equation of the line passing through $(\frac{7}{2}, 1)$ and parallel to $2x + 14y = 7$.

Solution:

Find the slope m by solving for y.

$$\begin{aligned} 2x + 14y &= 7 \\ 2x + 14y - 2x &= 7 - 2x \\ 14y &= -2x + 7 \\ \frac{14y}{14} &= \frac{-2x + 7}{14} \\ y &= \frac{-2x}{14} + \frac{7}{14} \\ y &= -\frac{1}{7}x + \frac{1}{2} \end{aligned}$$

The given line has the slope $m = -\frac{1}{7}$, and so $m_{\parallel} = -\frac{1}{7}$. We use this and the point $(\frac{7}{2}, 1)$ in point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= -\frac{1}{7}\left(x - \frac{7}{2}\right) \\ y - 1 &= -\frac{1}{7}x + \frac{1}{2} \\ y - 1 + 1 &= -\frac{1}{7}x + \frac{1}{2} + 1 \\ y &= -\frac{1}{7}x + \frac{1}{2} + \frac{2}{2} \\ y &= -\frac{1}{7}x + \frac{3}{2} \end{aligned}$$

Answer:

$$y = -\tfrac{1}{7}x + \tfrac{3}{2}$$

Exercise 3.6.2

Find the equation of the line perpendicular to $x - 3y = 9$ and passing through $(-\frac{1}{2}, 2)$.

Answer available at the end of this section.

When finding an equation of a line perpendicular to a horizontal or vertical line, it is best to consider the geometric interpretation.

Example 3.6.7

Find the equation of the line passing through $(-3, -2)$ and perpendicular to $y = 4$.

Solution:

We recognize that $y = 4$ is a horizontal line and we want to find a perpendicular line passing through $(-3, -2)$.

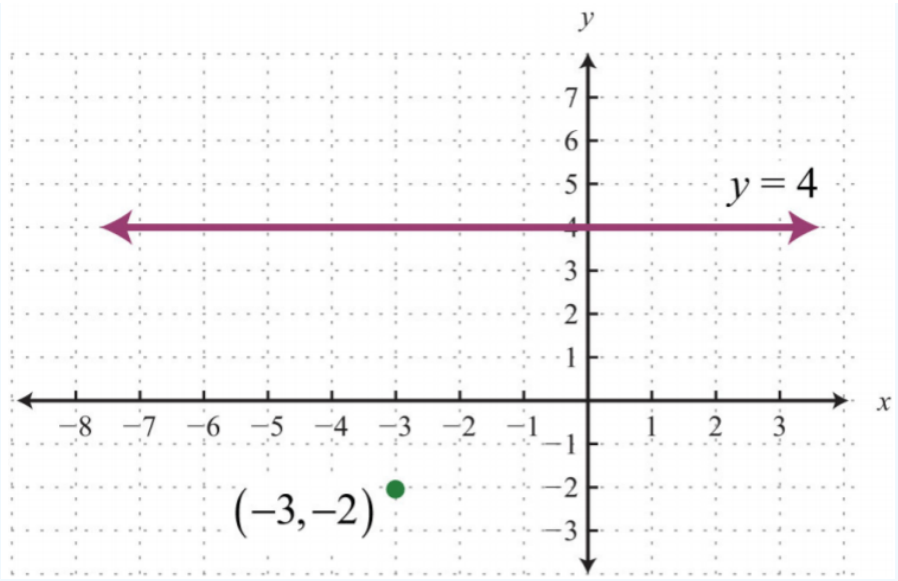


Figure 3.6.5

If we draw the line perpendicular to the given horizontal line, the result is a vertical line.

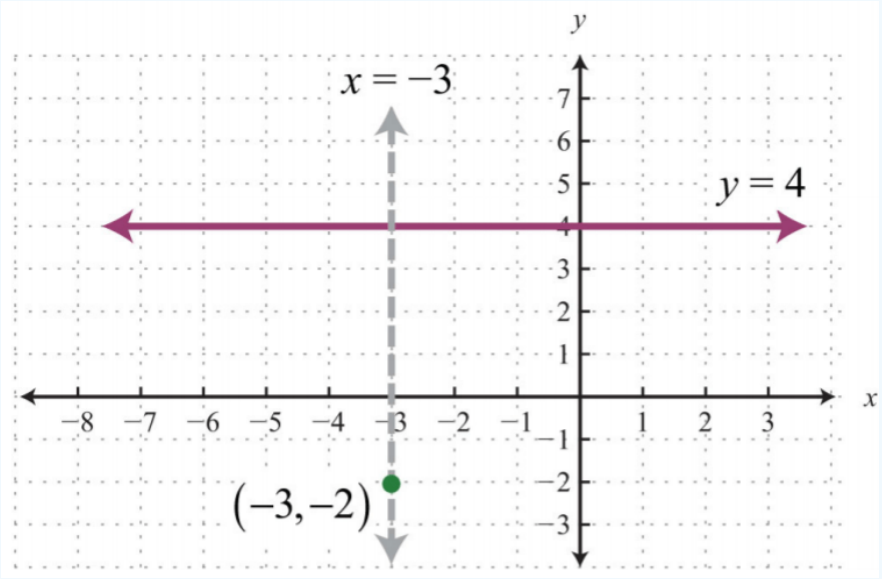


Figure 3.6.6

Equations of vertical lines look like $x = k$. Since it must pass through $(-3, -2)$, we conclude that $x = -3$ is the equation. All ordered pair solutions of a vertical line must share the same x-coordinate.

Answer:

$$x = -3$$

We can rewrite the equation of any horizontal line, $y = k$, in slope-intercept form as follows:

$$y = 0x + k$$

Written in this form, we see that the slope is $m = 0 = \frac{0}{1}$. If we try to find the slope of a perpendicular line by finding the

opposite reciprocal, we run into a problem: $m_\perp = -\frac{1}{0}$, which is undefined. This is why we took care to restrict the definition to two nonvertical lines. Remember that horizontal lines are perpendicular to vertical lines.

Key Takeaways

- Parallel lines have the same slope.
- Perpendicular lines have slopes that are opposite reciprocals. In other words, if $m = \frac{a}{b}$, then $m_\perp = -\frac{b}{a}$.
- To find an equation of a line, first use the given information to determine the slope. Then use the slope and a point on the line to find the equation using point-slope form.
- Horizontal and vertical lines are perpendicular to each other.

Exercise 3.6.3 Parallel and Perpendicular Lines

Determine the slope of parallel lines and perpendicular lines.

1. $y = -\frac{3}{4}x + 8$
2. $y = \frac{1}{2}x - 3$
3. $y = 4x + 4$
4. $y = -3x + 7$
5. $y = -\frac{5}{8}x - 12$
6. $y = \frac{7}{3}x + \frac{3}{2}$
7. $y = 9x - 25$
8. $y = -10x + 15$
9. $y = 5$
10. $x = -12$
11. $x - y = 0$
12. $x + y = 0$
13. $4x + 3y = 0$
14. $3x - 5y = 10$
15. $-2x + 7y = 14$
16. $-x - y = \frac{1}{5}$
17. $\frac{1}{2}x - \frac{1}{3}y = -1$
18. $-\frac{2}{3}x + \frac{4}{5}y = 8$
19. $2x - \frac{1}{5}y = \frac{1}{10}$
20. $-\frac{4}{5}x - 2y = 7$

Answer available at the end of this section.

Exercise 3.6.4 Parallel and Perpendicular Lines

Determine if the lines are parallel, perpendicular, or neither.

1. $\begin{cases} y = \frac{2}{3}x + 3 \\ y = \frac{2}{3}x - 3 \end{cases}$
2. $\begin{cases} y = \frac{3}{4}x - 1 \\ y = \frac{4}{3}x + 3 \end{cases}$
3. $\begin{cases} y = -2x + 1 \\ y = \frac{1}{2}x + 8 \end{cases}$
4. $\begin{cases} y = 3x - \frac{1}{2} \\ y = 3x + 2 \end{cases}$

5. $\begin{cases} y = 5 \\ x = -2 \end{cases}$
6. $\begin{cases} y = 7 \\ y = -\frac{1}{7} \end{cases}$
7. $\begin{cases} 3x - 5y = 15 \\ 5x + 3y = 9 \end{cases}$
8. $\begin{cases} x - y = 7 \\ 3x + 3y = 2 \end{cases}$
9. $\begin{cases} 2x - 6y = 4 \\ -x + 3y = -2 \end{cases}$
10. $\begin{cases} -4x + 2y = 3 \\ 6x - 3y = -3 \end{cases}$
11. $\begin{cases} x + 3y = 9 \\ 2x + 3y = 6 \end{cases}$
12. $\begin{cases} y - 10 = 0 \\ x - 10 = 0 \end{cases}$
13. $\begin{cases} y + 2 = 0 \\ 2y - 10 = 0 \end{cases}$
14. $\begin{cases} 3x + 2y = 6 \\ 2x + 3y = 6 \end{cases}$
15. $\begin{cases} -5x + 4y = 20 \\ 10x - 8y = 16 \end{cases}$
16. $\begin{cases} \frac{1}{2}x - \frac{1}{3}y = 1 \\ \frac{1}{6}x + \frac{1}{4}y = -2 \end{cases}$

Answer available at the end of this section.

Exercise 3.6.5 Equations in Point-Slope Form

Find the equation of the line

1. Parallel to $y = \frac{1}{2}x + 2$ and passing through $(6, -1)$.
2. Parallel to $y = -\frac{3}{4}x - 3$ and passing through $(-8, 2)$.
3. Perpendicular to $y = 3x - 1$ and passing through $(-3, 2)$.
4. Perpendicular to $y = -\frac{1}{3}x + 2$ and passing through $(4, -3)$.
5. Perpendicular to $y = -2$ and passing through $(-1, 5)$.
6. Perpendicular to $x = \frac{1}{5}$ and passing through $(5, -3)$.
7. Parallel to $y = 3$ and passing through $(2, 4)$.
8. Parallel to $x = 2$ and passing through (7, –3)\).
9. Perpendicular to $y = x$ and passing through $(7, -13)$.
10. Perpendicular to $y = 2x + 9$ and passing through $(3, -1)$.
11. Parallel to $y = \frac{1}{4}x - 5$ and passing through $(-2, 1)$.
12. Parallel to $y = -\frac{3}{4}x + 1$ and passing through $(4, \frac{1}{4})$.
13. Parallel to $2x - 3y = 6$ and passing through $(6, -2)$.
14. Parallel to $-x + y = 4$ and passing through $(9, 7)$.
15. Perpendicular to $5x - 3y = 18$ and passing through $(-9, 10)$.
16. Perpendicular to $x - y = 11$ and passing through $(6, -8)$.
17. Parallel to $\frac{1}{5}x - \frac{1}{3}y = 2$ and passing through $(-15, 6)$.
18. Parallel to $-10x - \frac{5}{7}y = 12$ and passing through $(-1, \frac{1}{2})$.
19. Perpendicular to $\frac{1}{2}x - \frac{1}{3}y = 1$ and passing through $(-10, 3)$.
20. Perpendicular to $-5x + y = -1$ and passing through $(-4, 0)$.
21. Parallel to $x + 4y = 8$ and passing through $(-1, -2)$.
22. Parallel to $7x - 5y = 35$ and passing through $(2, -3)$.
23. Perpendicular to $6x + 3y = 1$ and passing through $(8, -2)$.
24. Perpendicular to $-4x - 5y = 1$ and passing through $(-1, -1)$.
25. Parallel to $-5x - 2y = 4$ and passing through $(\frac{1}{5}, -\frac{1}{4})$.

26. Parallel to $6x - \frac{3}{2}y = 9$ and passing through $(\frac{1}{3}, \frac{2}{3})$.
27. Perpendicular to $y - 3 = 0$ and passing through $(-6, 12)$.
28. Perpendicular to $x + 7 = 0$ and passing through $(5, -10)$.

Answer available at the end of this section.

Answers

Exercise 3.6.1

$m_{\perp} = \frac{1}{3}$

Exercise 3.6.2

$y = -3x + \frac{1}{2}$

Exercise 3.6.3 **Parallel and Perpendicular Lines**

1. $m_{\parallel} = -\frac{3}{4}$ and $m_{\perp} = \frac{4}{3}$
3. $m_{\parallel} = 4$ and $m_{\perp} = -\frac{1}{4}$
5. $m_{\parallel} = -\frac{5}{8}$ and $m_{\perp} = \frac{8}{5}$
7. $m_{\parallel} = 9$ and $m_{\perp} = -\frac{1}{9}$
9. $m_{\parallel} = 0$ and $m_{\perp}$ undefined
11. $m_{\parallel} = 1$ and $m_{\perp} = -1$
13. $m_{\parallel} = -\frac{4}{3}$ and $m_{\perp} = \frac{3}{4}$
15. $m_{\parallel} = \frac{2}{7}$ and $m_{\perp} = -\frac{7}{2}$
17. $m_{\parallel} = \frac{3}{2}$ and $m_{\perp} = -\frac{2}{3}$
19. $m_{\parallel} = 10$ and $m_{\perp} = -\frac{1}{10}$

Exercise 3.6.4 **Parallel and Perpendicular Lines**

1. Parallel
3. Perpendicular
5. Perpendicular
7. Perpendicular
9. Parallel
11. Neither
13. Parallel
15. Parallel

Exercise 3.6.5 **Equations in Point-Slope Form**

1. $y = \frac{1}{2}x - 4$
3. $y = -\frac{1}{3}x + 1$
5. $x = -1$
7. $y = 4$
9. $y = -x - 6$
11. $y = \frac{1}{4}x + \frac{3}{2}$
13. $y = \frac{2}{3}x - 6$
15. $y = -\frac{3}{5}x + \frac{23}{5}$
17. $y = \frac{3}{5}x + 15$
19. $y = -\frac{2}{3}x - \frac{11}{3}$
21. $y = -\frac{1}{4}x - \frac{9}{4}$
23. $y = \frac{1}{2}x - 6$
25. $y = -\frac{5}{2}x + \frac{1}{4}$
27. $x = -6$

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3.7: Introduction to Functions

Learning Objectives

- Identify a function.
- State the domain and range of a function.
- Use function notation.

Relations, Functions, Domain, and Range

Relationships between sets occur often in everyday life. For example, for each month in Cape Canaveral, we can associate an average amount of rainfall. In this case, the amount of precipitation depends on the month of the year, and the data can be written in tabular form or as a set of ordered pairs.

Table 3.7.1

Month	Precipitation	Ordered pairs
January	2.4 in	(January, 2.4)
February	3.3 in	(February, 3.3)
March	3.1 in	(March, 3.1)
April	2.0 in	(April, 2.0)
May	3.8 in	(May, 3.8)
June	6.8 in	(June, 6.8)
July	8.1 in	(July, 8.1)
August	7.6 in	(August, 7.6)
September	7.3 in	(September, 7.3)
October	4.1 in	(October, 4.1)
November	3.3 in	(November, 3.3)
December	2.4 in	(December, 2.4)

We define a relation as any set of ordered pairs. Usually we write the independent component of the relation in the first column and the dependent component in the second column. In the opening example, notice that it makes sense to relate the average amount of precipitation as dependent on the month of year. The set of all elements in the first column of a relation is called the domain. The set of all elements that compose the second column is called the range. In this example, the domain consists of the set of all months of the year, and the range consists of the values that represent the average rainfall for each month.

In the context of algebra, the relations of interest are sets of ordered pairs (x, y) in the rectangular coordinate plane. In this case, the x-values define the domain and the y-values define the range. Of special interest are relations where every x-value corresponds to exactly one y-value; these relations are called functions.

Example 3.7.1

Determine the domain and range of the following relation and state whether or not it is a function:

$$\{(-1, 4), (0, 7), (2, 3), (3, 3), (4, -2)\}$$

Solution:

Here we separate the domain and range and depict the correspondence between the values with arrows.

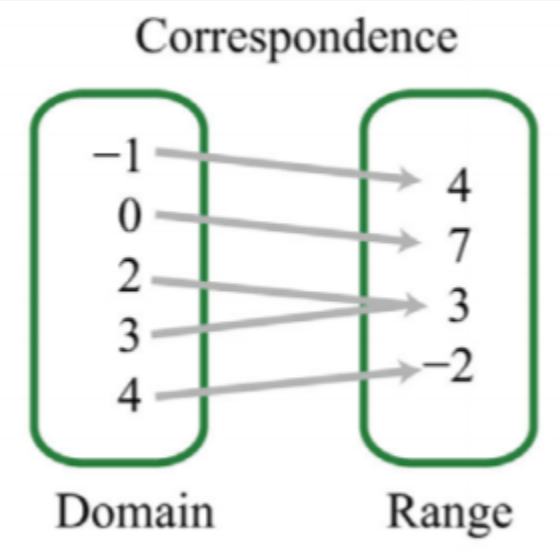


Figure 3.7.1

Answer:

The domain is $\{-1, 0, 2, 3, 4\}$, and the range is $\{-2, 3, 4, 7\}$. The relation is a function because each x-value corresponds to exactly one y-value.

Example 3.7.2

Determine the domain and range of the following relation and state whether or not it is a function:

$$\{(-4, -3), (-2, 6), (0, 3), (3, 5), (3, 7)\}.$$

Solution:

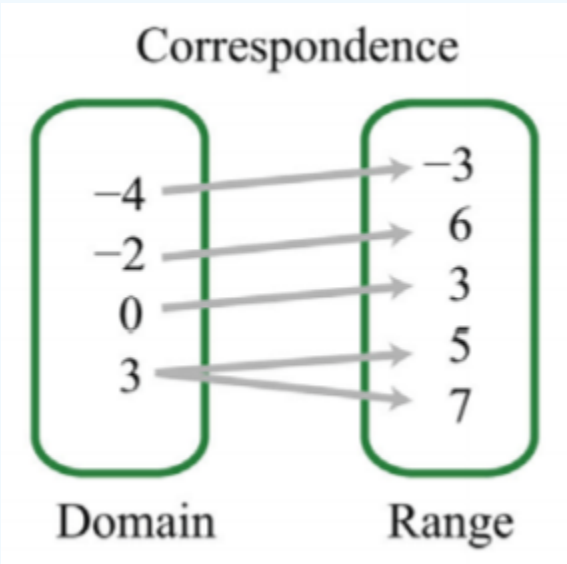


Figure $3.7.2$

Answer:

The domain is $\{-4,-2,0,3\}$, and the range is $\{-3,3,5,6,7\}$. This relation is not a function because the x-value 3 has two corresponding y-values.

In the previous example, the relation is not a function because it contains ordered pairs with the same x-value, $(3,5)$ and $(3,7)$. We can recognize functions as relations where no x-values are repeated.

In algebra, equations such as $y=\frac{3}{4}x-2$ define relations. This linear equation can be graphed as follows:

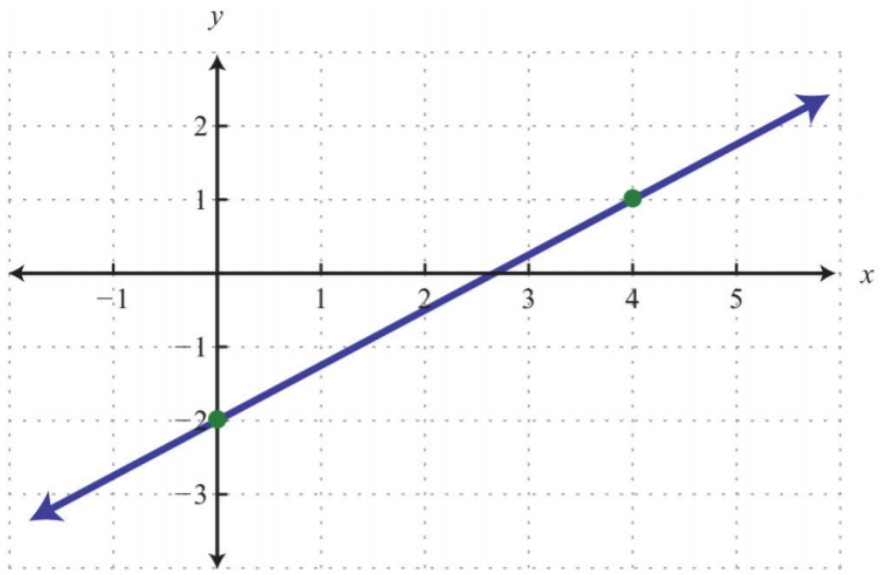


Figure $3.7.3$

The graph is a relation since it represents the infinite set of ordered pair solutions to $y=\frac{3}{4}x-2$. The domain is the set of all x-values, and in this case consists of all real numbers. The range is the set of all possible y-values, and in this case also consists of all real numbers. Furthermore, the graph is a function because for each x-value there is only one corresponding y-value. In fact, any nonvertical or nonhorizontal line is a function with domain and range consisting of all real numbers.

Any graph is a set of ordered pairs and thus defines a relation. Consider the following graph of a circle:

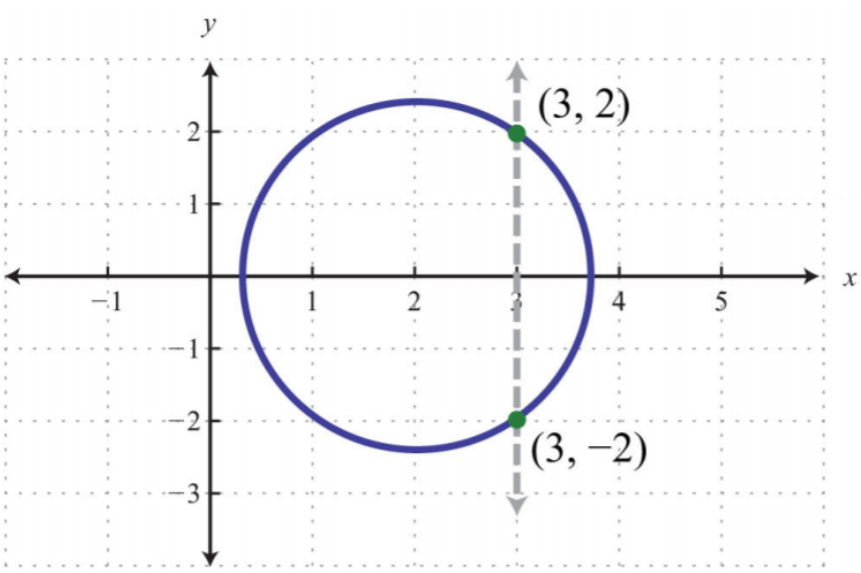


Figure 3.7.4

Here the graph represents a relation where many x-values in the domain correspond to two y-values. If we draw a vertical line, as illustrated, we can see that $(3, 2)$ and $(3, -2)$ are two ordered pairs with the same x-value. Therefore, the x-value 3 corresponds to two y-values; hence the graph does not represent a function. The illustration suggests that if any vertical line intersects a graph more than once, then the graph does not represent a function. This is called the vertical line test.

Example 3.7.3

Given the following graph, determine the domain and range and state whether or not it is a function.

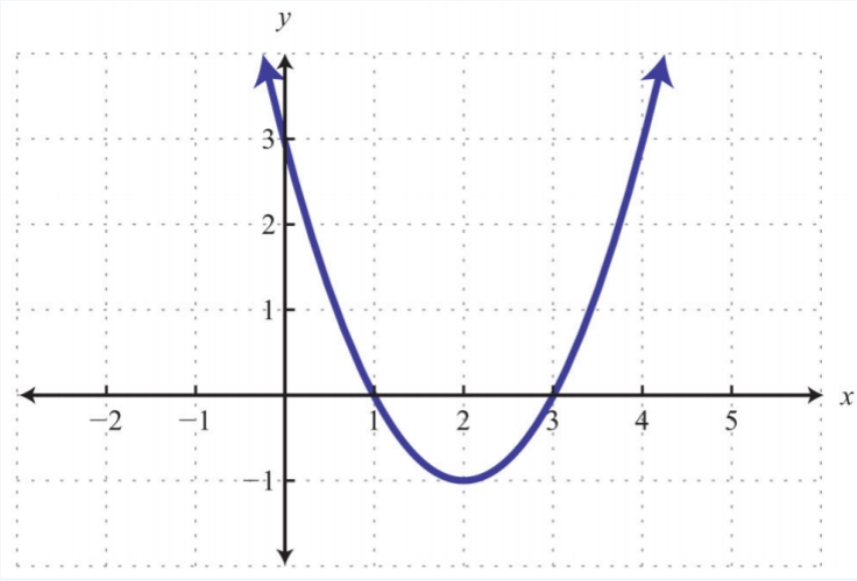


Figure 3.7.5

Solution:

The given shape is called a parabola and extends indefinitely to the left and right as indicated by the arrows. This suggests that if we choose any x-value, then we will be able to find a corresponding point on the graph; therefore, the domain consists of all real numbers. Furthermore, the graph shows that -1 is the minimum y-value, and any y-value greater than that is represented in the relation. Hence the range consists of all y-values greater than or equal to -1, or in interval notation, $[-1, \infty)$.

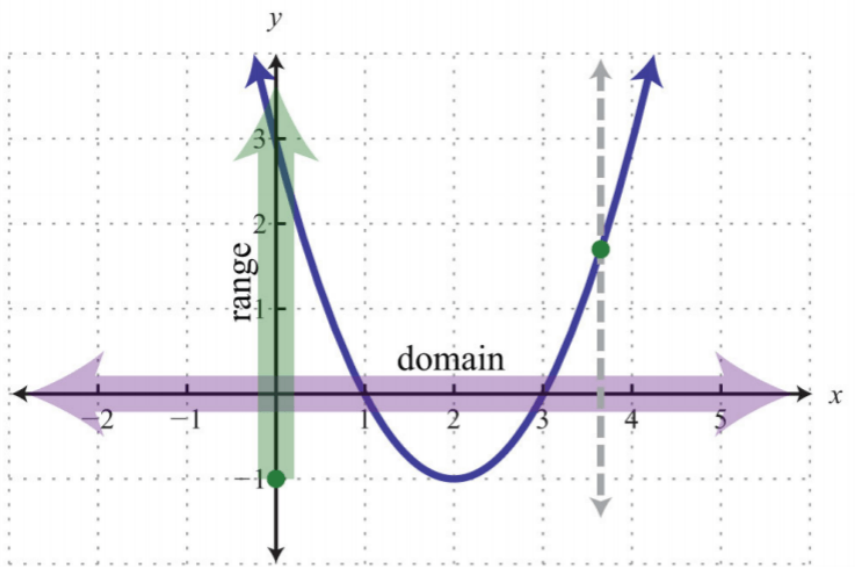


Figure $3.7.6$

Lastly, any vertical line will intersect the graph only once; therefore, it is a function.

Answer:

The domain is all real numbers $R = (-\infty, \infty)$, and the range is $[-1, \infty)$. The graph represents a function because it passes the vertical line test.

Exercise $3.7.1$

Given the graph, determine the domain and range and state whether or not it is a function:

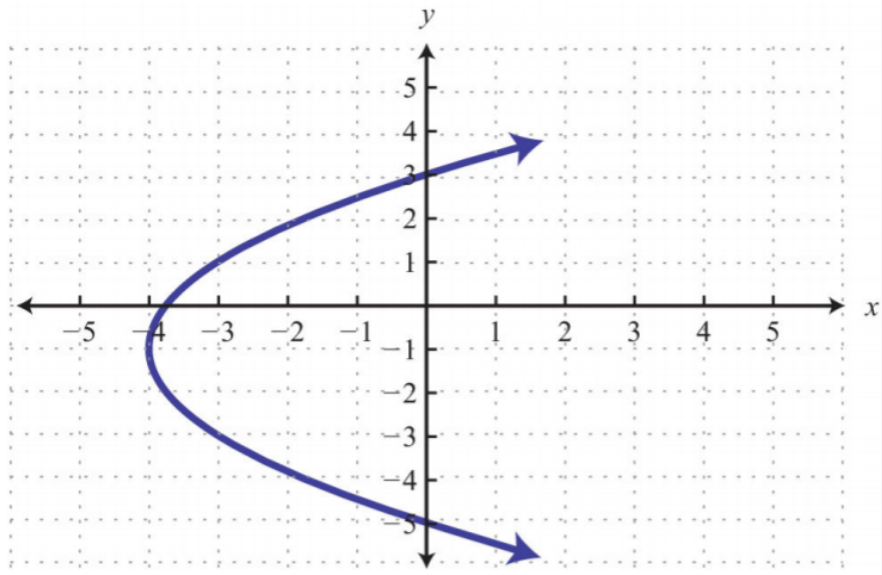


Figure 3.7.7

Answer available at the end of this section.

Function Notation and Linear Functions

With the definition of a function comes special notation. If we consider each x-value to be the input that produces exactly one output, then we can use the notation

$$f(x) = y \tag{3.7.1}$$

The notation $f(x)$ reads "f of x" and should not be confused with multiplication. Most of our study of algebra involves functions, so the notation becomes very useful when performing common tasks. Functions can be named with different letters; some common names for functions are $g(x), h(x), C(x)$, and $R(x)$. First, consider nonvertical lines that we know can be expressed using slope-intercept form, $y = mx + b$. For any real numbers m and b, the equation defines a function, and we can replace y with the new notation $f(x)$ as follows:

$$y = mx + b \tag{3.7.2}$$

$$f(x) = mx + b \tag{3.7.3}$$

Therefore, a linear function is any function that can be written in the form $f(x) = mx + b$. In particular, we can write the following:

The notation also shows values to evaluate in the equation. If the value for x is given as 8, then we know that we can find the corresponding y-value by substituting 8 in for x and simplifying. Using function notation, this is denoted $f(8)$ and can be interpreted as follows:

Finally, simplify:

We have $f(8) = 4$. This notation tells us that when $x = 8$ (***the input***), the function results in 4 (***the output***).

Given the linear function $f(x) = -5x + 7$, find $f(-2)$.

Solution:

In this case, $f(-2)$ indicates that we should evaluate when $x = -2$.

$$\begin{aligned} f(x) &= -5x + 7 \\ f(-2) &= -5(-2) + 7 \qquad \textit{Replace } x \textit{ with } -2. \\ &= 10 + 7 \\ &= 17 \end{aligned}$$

Answer:

$$f(-2) = 17$$

Example 3.7.5

Given the linear function $f(x) = -5x + 7$, find x when $f(x) = 10$.

Solution:

In this case, $f(x) = 10$ indicates that the function should be set equal to 10.

$$\begin{aligned} f(x) &= -5x + 7 \\ 10 &= -5x + 7 \qquad \textit{Replace } f(x) \textit{ with } 10. \\ 10-7 &= -5x + 7-7 \qquad \textit{Solve for } x. \\ 3 &= -5x \\ \frac{3}{-5} &= \frac{-5x}{-5} - \frac{3}{5} \qquad = x \end{aligned}$$

Answer:

Here $x = -\frac{3}{5}$, and we can write $f(-\frac{3}{5}) = 10$.

Example 3.7.6

Given the graph of a linear function $g(x)$,

a. Find $g(2)$.
b. Find x when $g(x) = 3$.

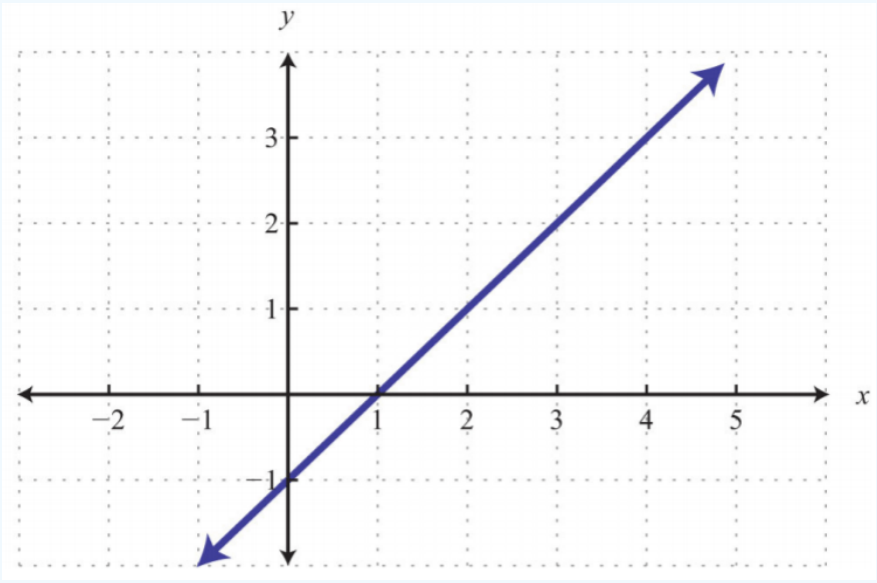


Figure 3.7.8

Solution:

a. The notation $g(2)$ implies that $x = 2$. Use the graph to determine the corresponding y-value.

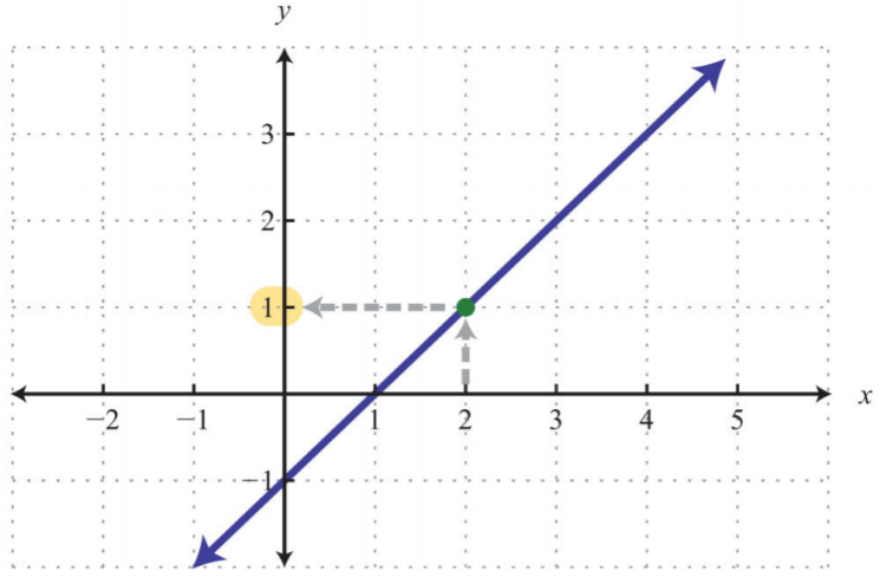


Figure 3.7.9

b. The notation $g(x) = 3$ implies that the y-value is given as 3. Use the graph to determine the corresponding x-value.

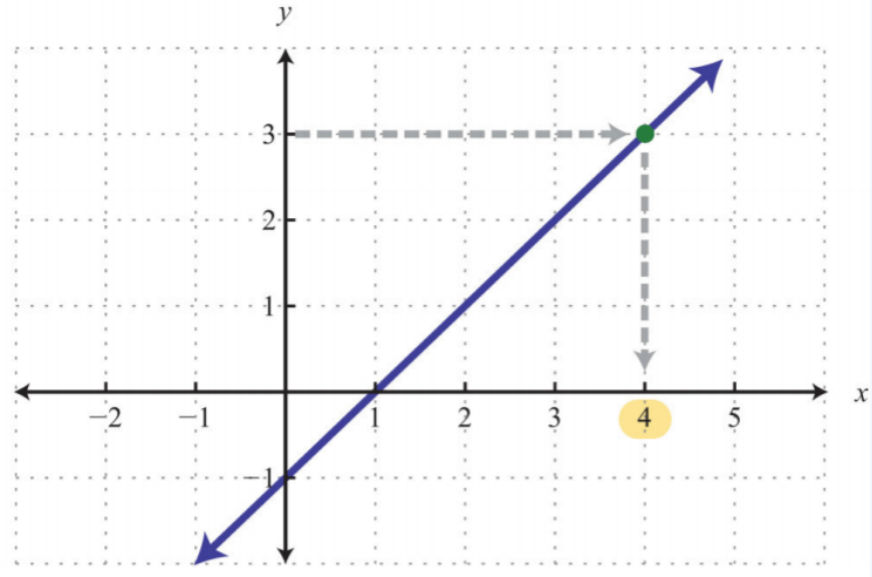


Figure 3.7.10

Answer:

a. $g(2) = 1$
b. $x = 4$

Example 3.7.7

Graph the linear function $f(x) = -\frac{5}{3}x + 6$ and state the domain and range.

Solution:

From the function, we see that $b = 6$ and thus the y-intercept is $(0, 6)$. Also, we can see that the slope is $m = \frac{-5}{3} = -\frac{5}{3} = \frac{rise}{run}$. Starting from the y-intercept, mark a second point down 5 units and right 3 units.

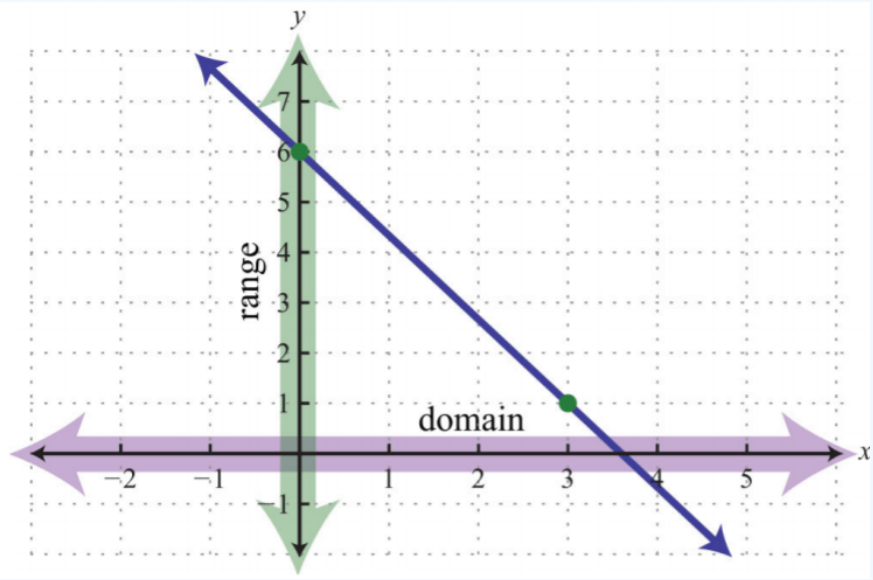


Figure 3.7.11

Given any coordinate on the x-axis, we can find a corresponding point on the graph; the domain consists of all real numbers. Also, for any coordinate on the y-axis, we can find a point on the graph; the range consists of all real numbers.

Answer:

Both the domain and range consist of all real numbers R.

Exercise 3.7.2

Given the linear function $g(x) = -x + 5$,

a. Find $g(-\frac{1}{2})$.
b. Find x when $g(x) = 18$.

Answer available at the end of this section.

Key Takeaways

- A relation is any set of ordered pairs. However, in the context of this course, we will be working with sets of ordered pairs (x, y) in the rectangular coordinate system. The set of x-values defines the domain and the set of y-values defines the range.
- Special relations where every x-value (*input*) corresponds to exactly one y-value (*output*) are called functions.
- We can easily determine whether an equation represents a function by performing the vertical line test on its graph. If any vertical line intersects the graph more than once, then the graph does not represent a function. In this case, there will be more than one point with the same x-value.

- Any nonvertical or nonhorizontal line is a function and can be written using function notation $f(x) = mx + b$. Both the domain and range consist of all real numbers.
  - If asked to find $f(a)$, we substitute a in for the variable and then simplify.
  - If asked to find x when $f(x) = a$, we set the function equal to a and then solve for x.

Exercise 3.7.3 Functions

For each problem below, does the correspondence represent a function?

1. Algebra students to their scores on the first exam.
2. Family members to their ages.
3. Lab computers to their users.
4. Students to the schools they have attended.
5. People to their citizenships.
6. Local businesses to their number of employees.

Answer available at the end of this section.

Exercise 3.7.4 Functions

Determine the domain and range and state whether the relation is a function or not.

1. $\{(3, 2), (5, 3), (7, 4)\}$
2. $\{(-5, -3), (0, 0), (5, 0)\}$
3. $\{(-10, 2), (-8, 1), (-8, 0)\}$
4. $\{(9, 12), (6, 6), (6, 3)\}$

5.

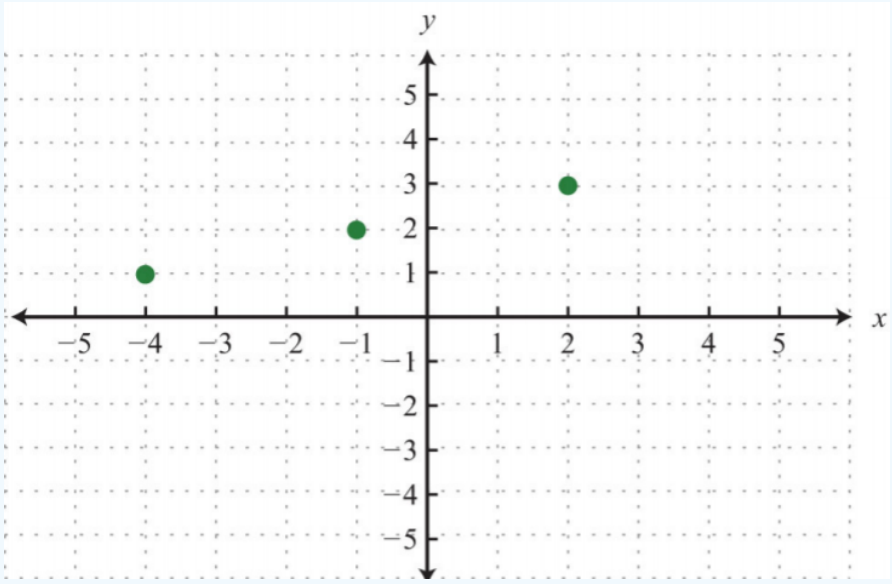


Figure 3.7.12

6.

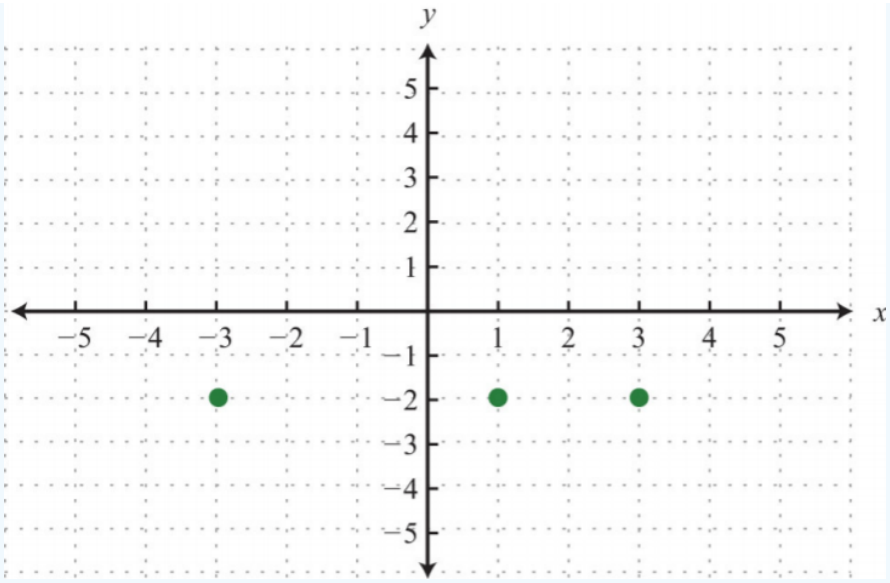


Figure 3.7.13

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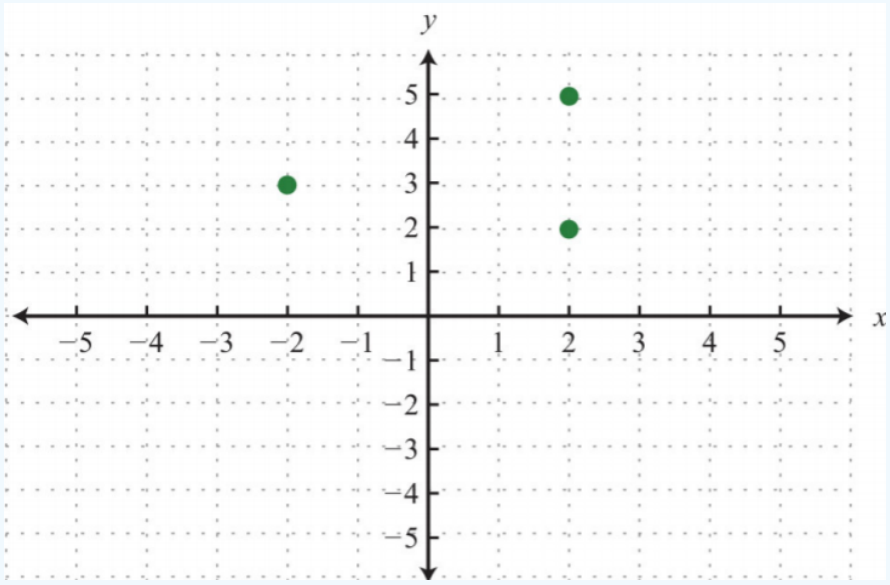


Figure 3.7.1***4***

8.

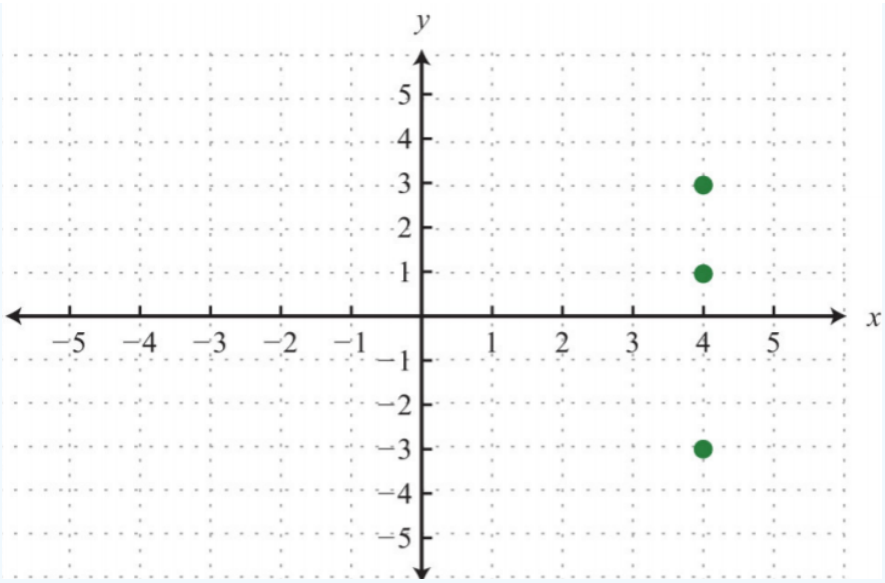


Figure 3.7.15

9.

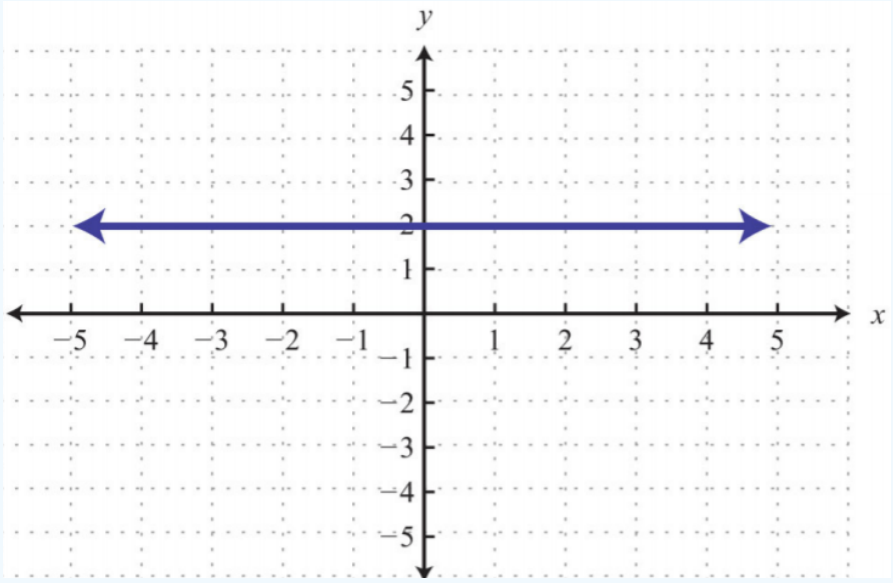


Figure 3.7.16

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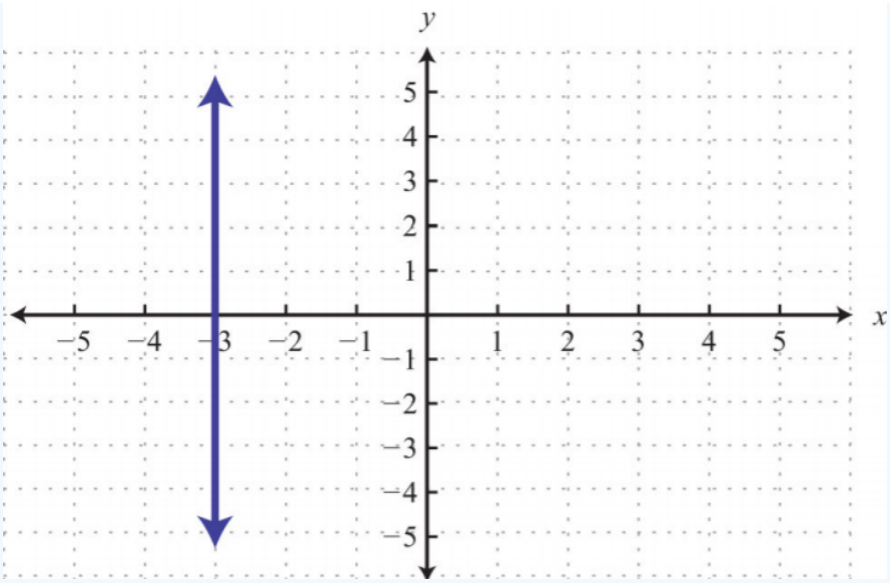


Figure 3.7.17

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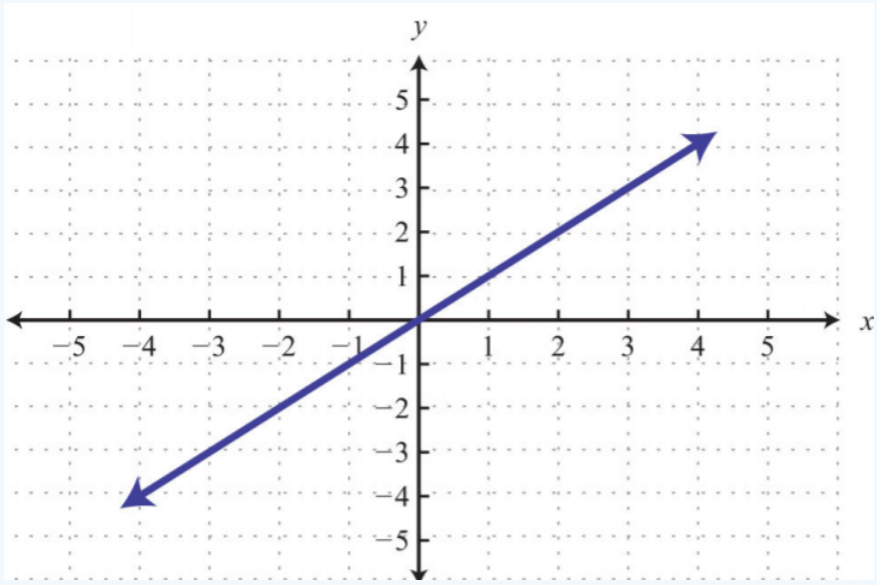


Figure 3.7.18

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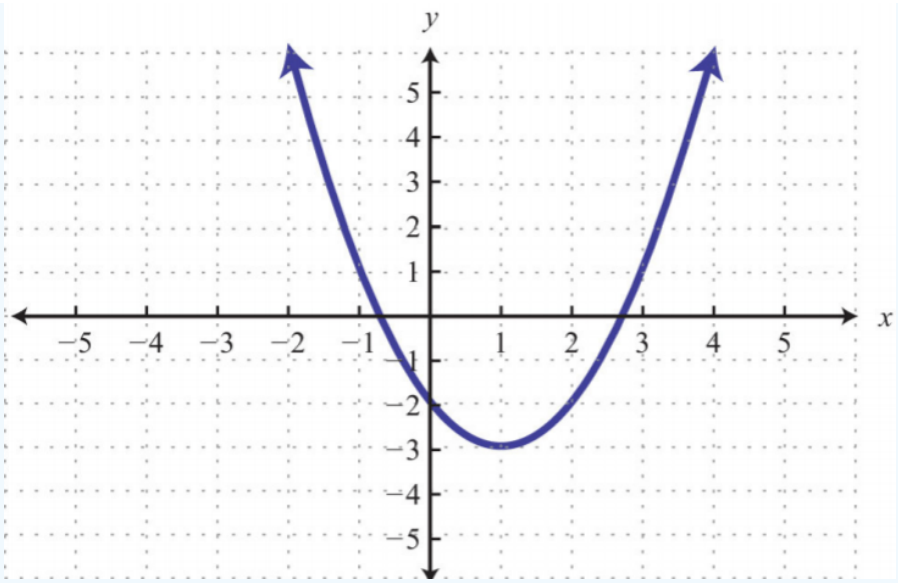


Figure 3.7.19

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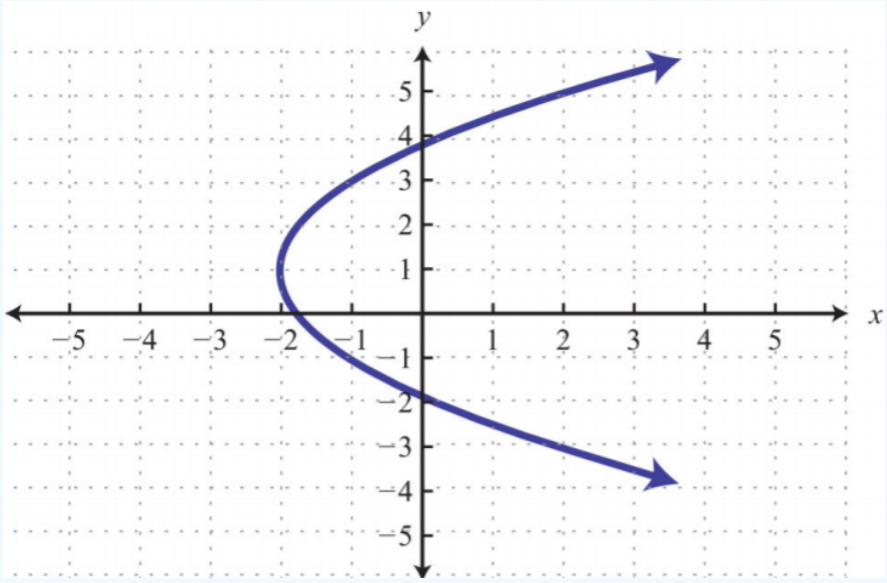


Figure 3.7.20

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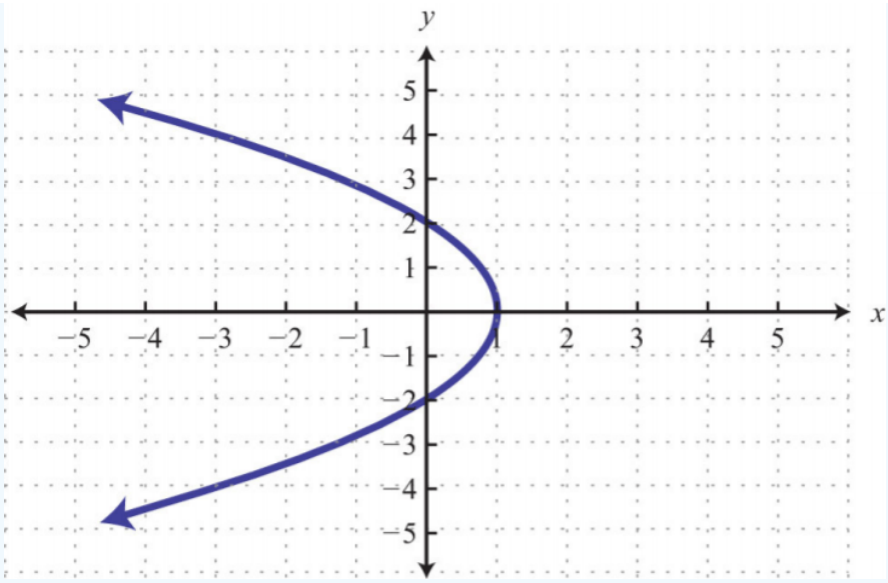


Figure 3.7.21

15.

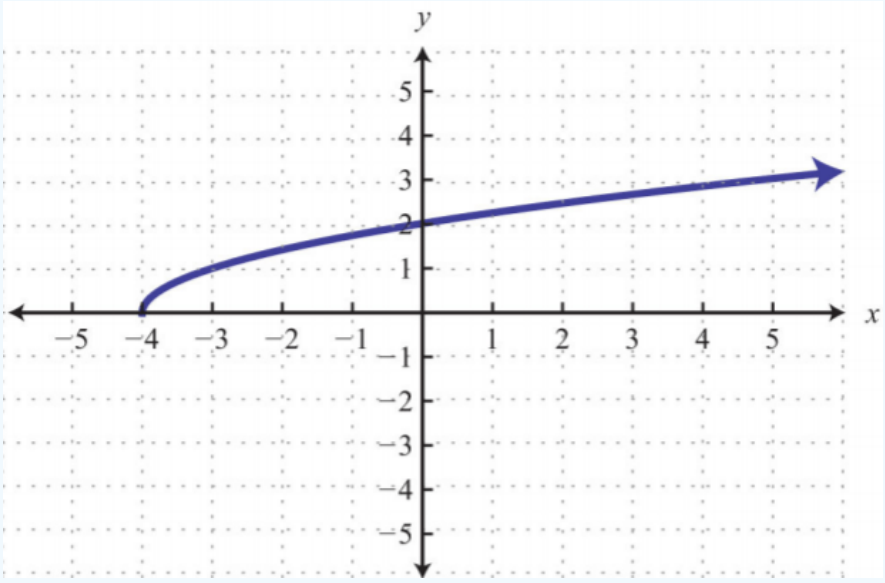


Figure 3.7.22

16.

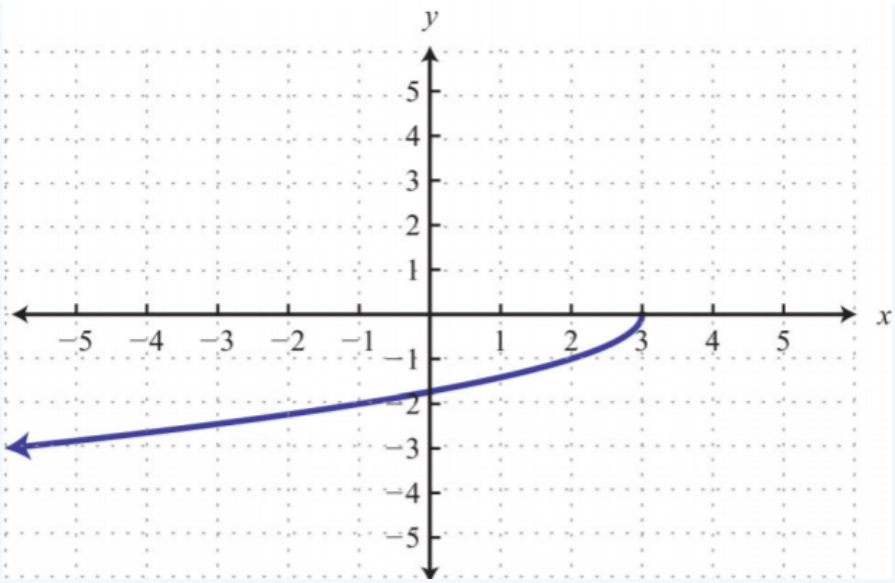


Figure 3.7.23

17.

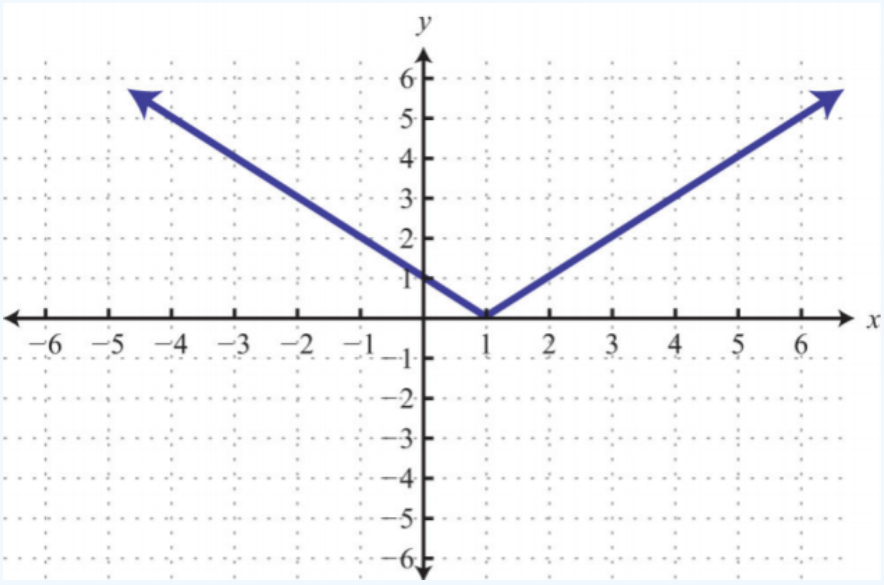


Figure 3.7.24

18.

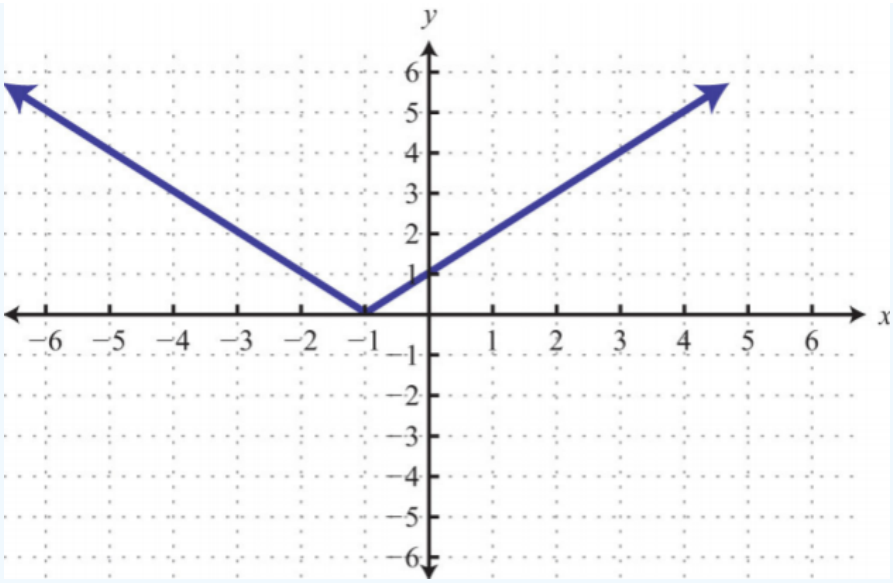


Figure 3.7.25

19.

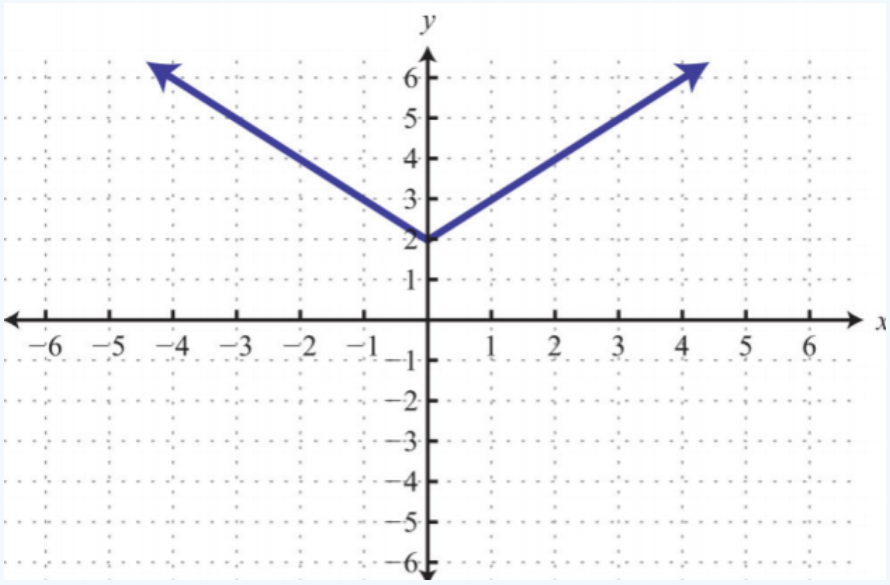


Figure 3.7.26

20.

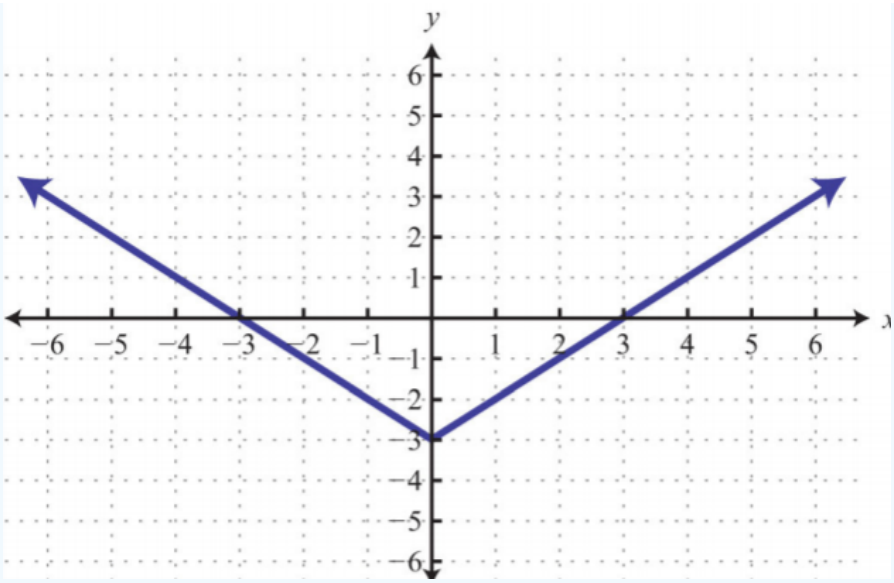


Figure 3.7.27

Answer available at the end of this section.

Exercise 3.7.5 Function Notation

Given the following functions, find the function values.

1. $f(x) = 3x$, find $f(-2)$.
2. $f(x) = -5x + 1$, find $f(-1)$.
3. $f(x) = \frac{3}{5}x - 4$, find $f(15)$.
4. $f(x) = \frac{2}{5}x - \frac{1}{5}$, find $f(3)$.
5. $f(x) = \frac{5}{2}x - \frac{1}{3}$, find $f(-\frac{1}{3})$.
6. $f(x) = -6$, find $f(7)$.
7. $g(x) = 5$, find $g(-4)$.
8. $g(x) = -5x$, find $g(-3)$.
9. $g(x) = -\frac{1}{8}x + \frac{5}{8}$, find $g(\frac{5}{8})$.
10. $g(x) = \frac{5}{3}x - 5$, find $g(3)$.
11. $f(x) = 5x - 9$, find x when $f(x) = 1$.
12. $f(x) = -7x + 2$, find x when $f(x) = 0$.
13. $f(x) = -\frac{7}{5}x - 2$, find x when $f(x) = -9$.
14. $f(x) = -x - 4$, find x when $f(x) = 12$.
15. $g(x) = x$, find x when $g(x) = 12$.
16. $g(x) = -x + 1$, find x when $g(x) = \frac{2}{3}$.
17. $g(x) = -5x + \frac{1}{3}$, find x when $g(x) = -\frac{1}{2}$.
18. $g(x) = -\frac{5}{8}x + 3$, find x when $g(x) = 3$.

Answer available at the end of this section.

Exercise 3.7.6 Function Notation

Given $f(x) = \frac{2}{3}x - 1$ and $g(x) = -3x + 2$ calculate the following.

1. $f(6)$
2. $f(-\frac{1}{2})$

3. $f(0)$
4. $f(1)$
5. $g(\frac{2}{3})$
6. $g(0)$
7. $g(-1)$
8. $g(-\frac{1}{2})$
9. Find x when $f(x) = 0$.
10. Find x when $f(x) = -3$.
11. Find x when $g(x) = -1$.
12. Find x when $g(x) = 0$.

Answer available at the end of this section.

Exercise 3.7.7 Function Notation

Given the graph, find the function values.

1. Given the graph of $f(x)$, find $f(-4), f(-1), f(0)$, and $f(2)$.

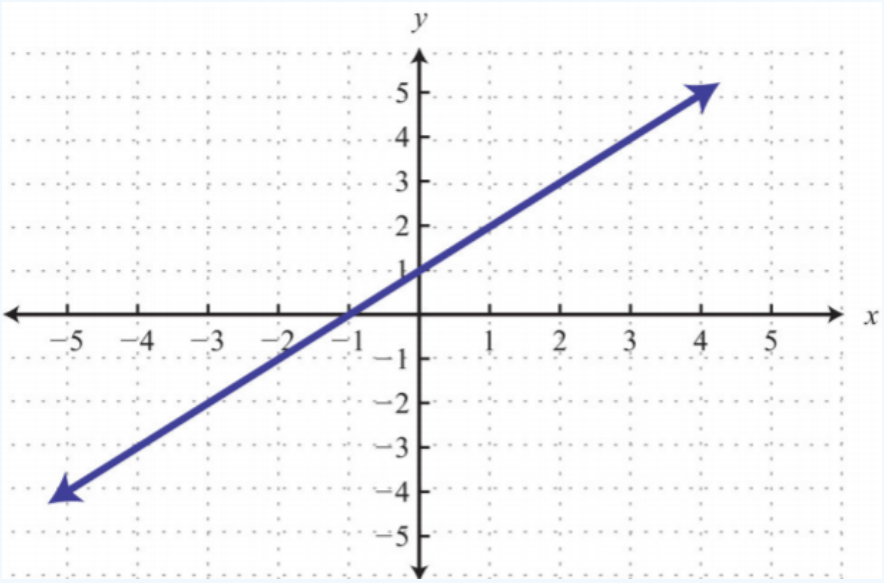


Figure 3.7.28

2. Given the graph of $g(x)$, find $g(-3), g(-1), g(0)$, and $g(1)$.

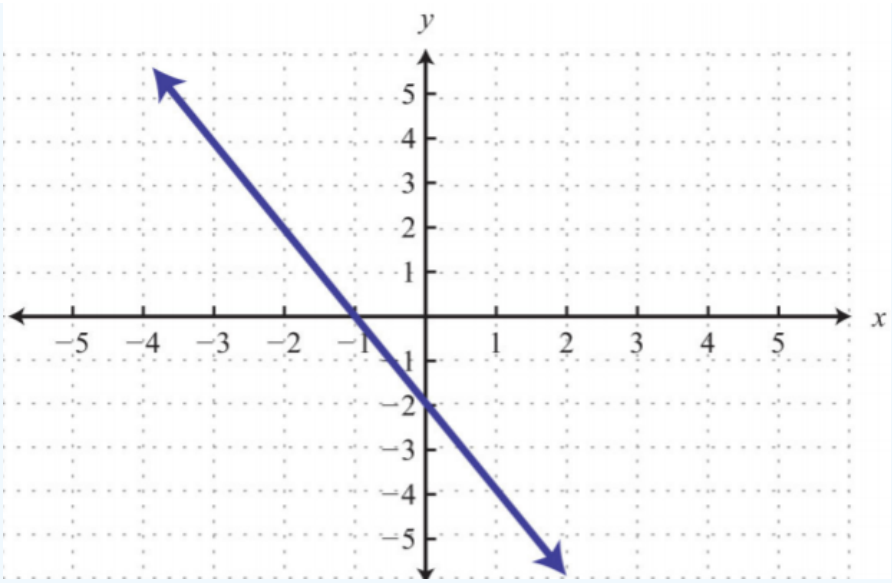


Figure 3.7.29

3. Given the graph of $f(x)$, find $f(-4), f(-1), f(0)$, and $f(2)$.

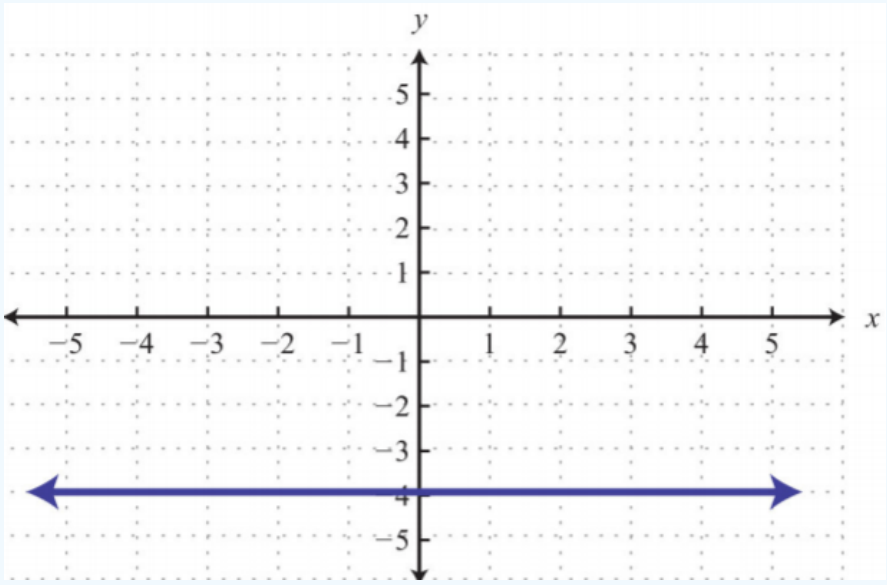


Figure 3.7.30

4. Given the graph of $g(x)$, find $g(-4), g(-1), g(0)$, and $g(2)$.

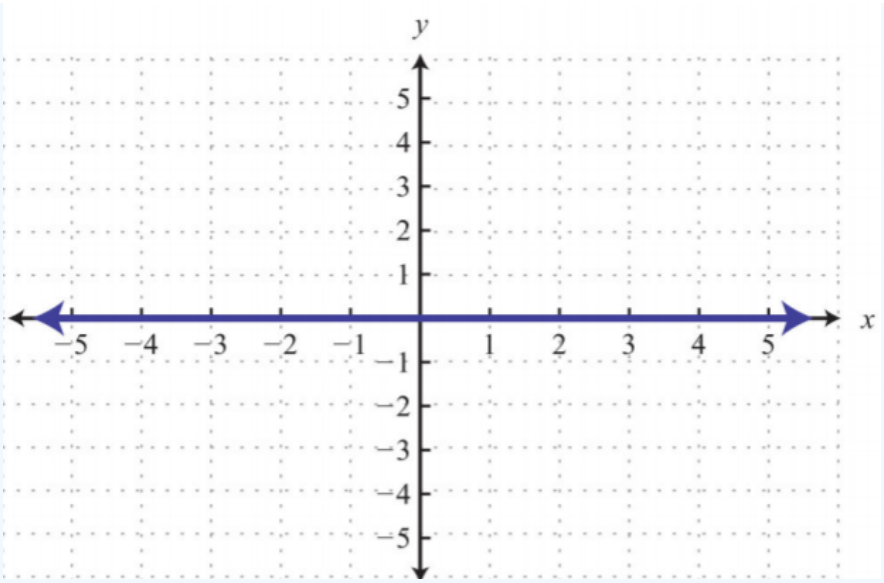


Figure 3.7.31

5. Given the graph of $f(x)$, find $f(-1), f(0), f(1)$, and $f(3)$.

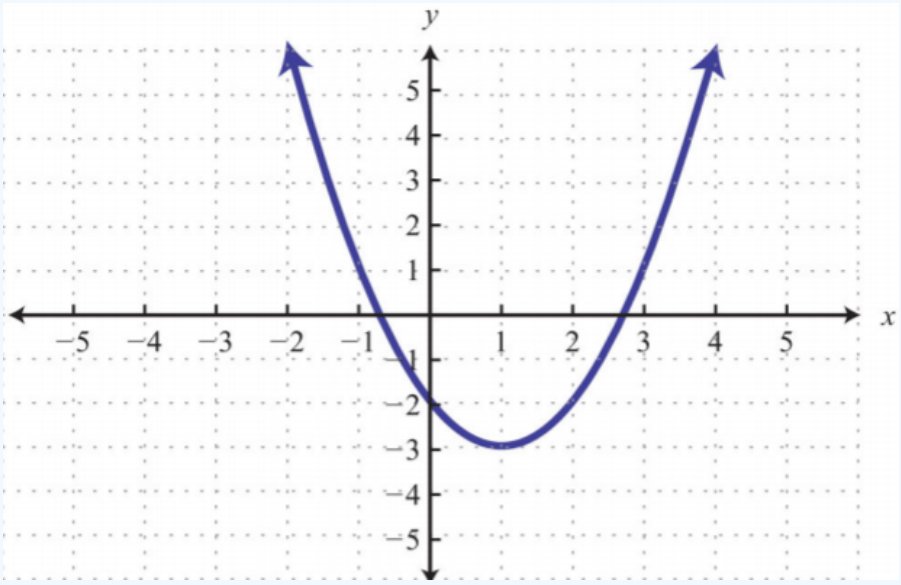


Figure 3.7.32

6. Given the graph of $g(x)$, find $g(-2), g(0), g(2)$, and $g(6)$.

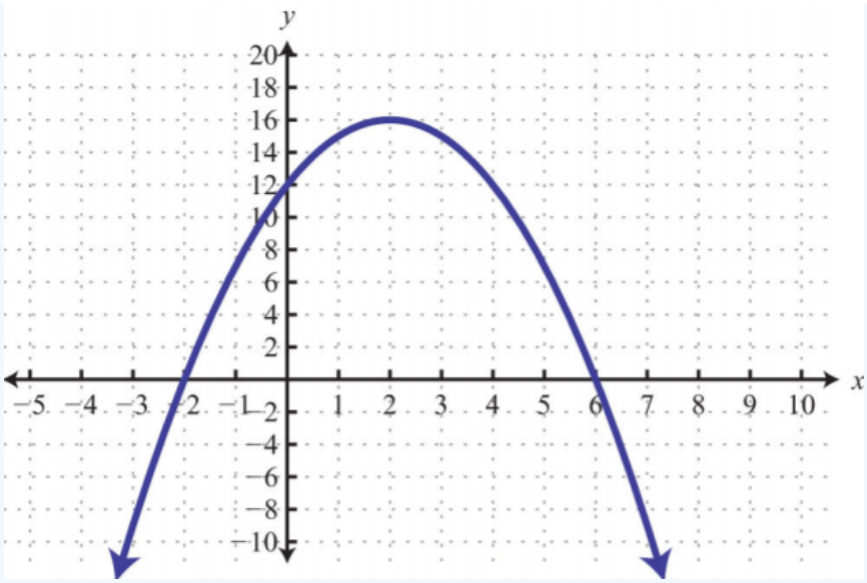


Figure 3.7.33

7. Given the graph of $g(x)$, find $g(-4), g(-3), g(0)$, and $g(4)$.

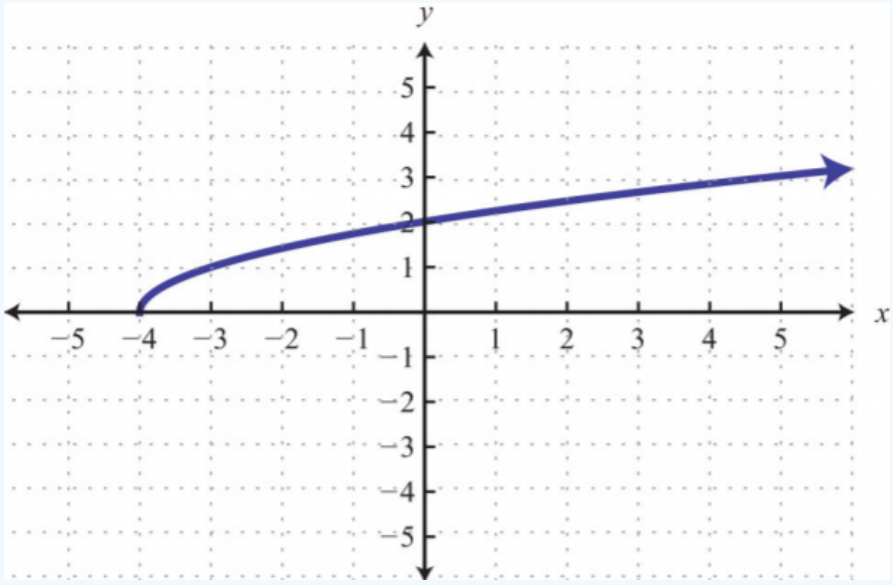


Figure 3.7.34

8. Given the graph of $f(x)$, find $f(-4), f(0), f(1)$, and $f(3)$.

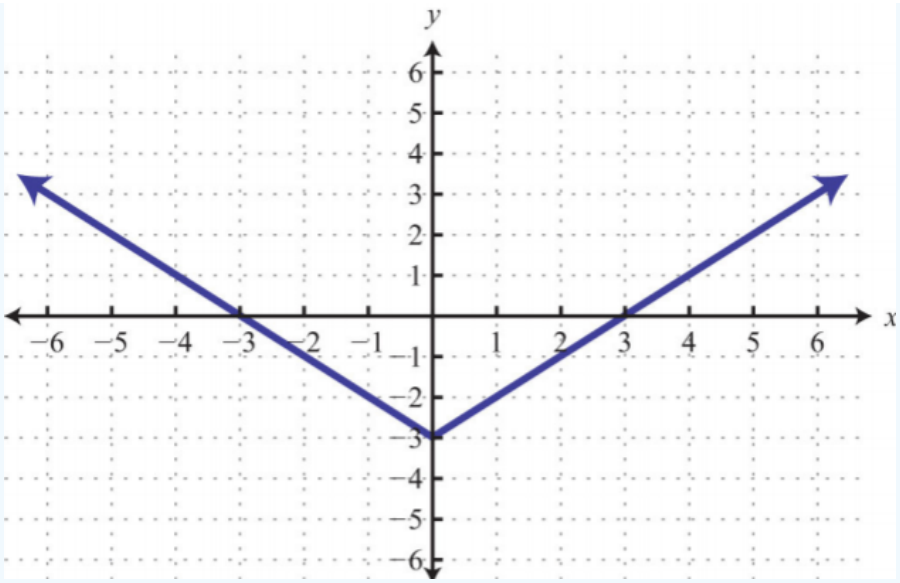


Figure 3.7.35

Answer available at the end of this section.

Exercise 3.7.8 Function Notation

Given the graph, find the x-values.

1. Given the graph of $f(x)$, find x when $f(x) = 3$, $f(x) = 1$, and $f(x) = -3$.

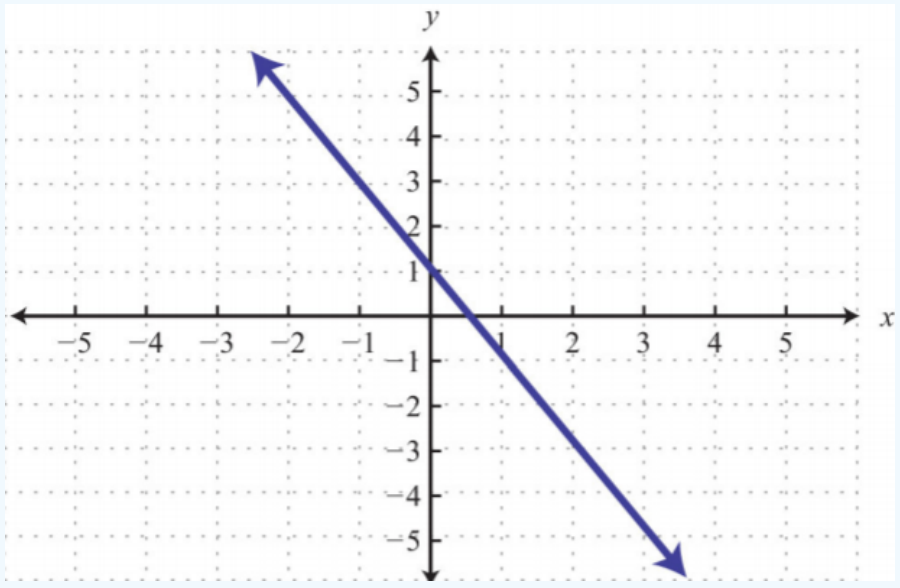


Figure 3.7.36

2. Given the graph of $g(x)$, find x when $g(x) = -1$, $g(x) = 0$, and $g(x) = 1$.

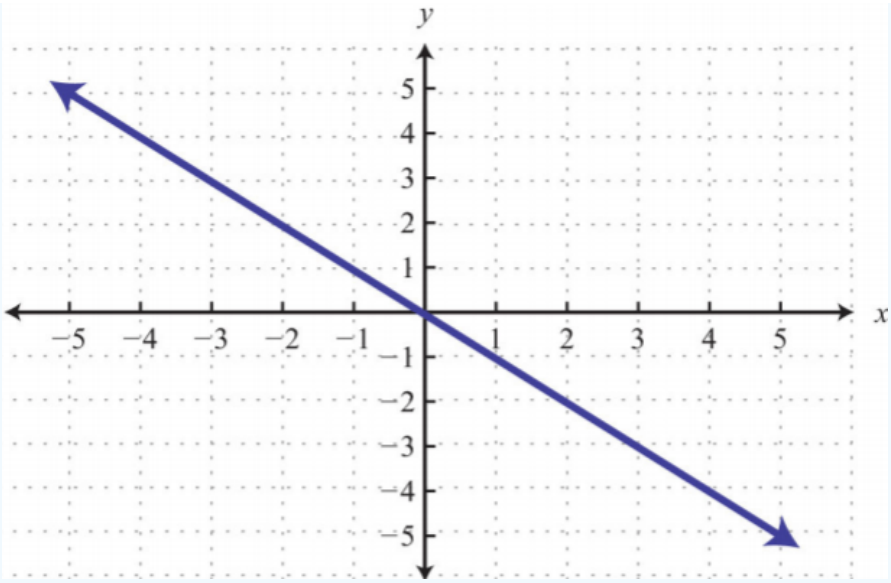


Figure 3.7.37

3. Given the graph of $f(x)$, find x when $f(x) = 3$.

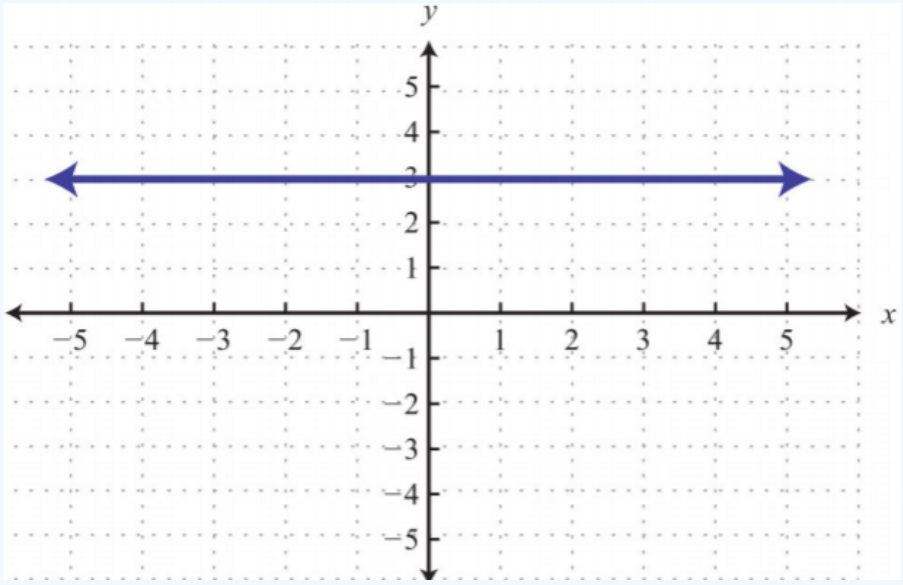


Figure 3.7.38

4. Given the graph of $g(x)$, find x when $g(x) = -2, g(x) = 0$, and $g(x) = 4$.

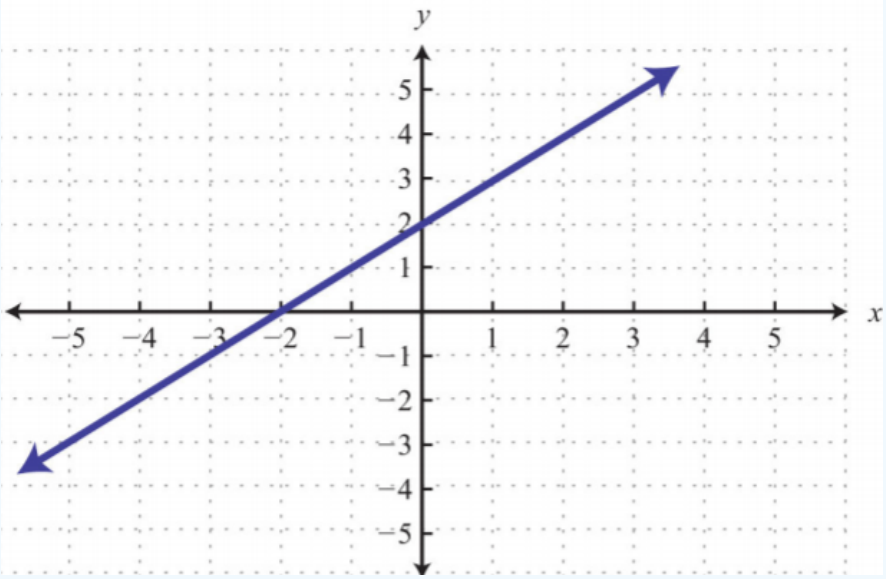


Figure 3.7.39

5. Given the graph of $f(x)$, find x when $f(x) = -16, f(x) = -12$, and $f(x) = 0$.

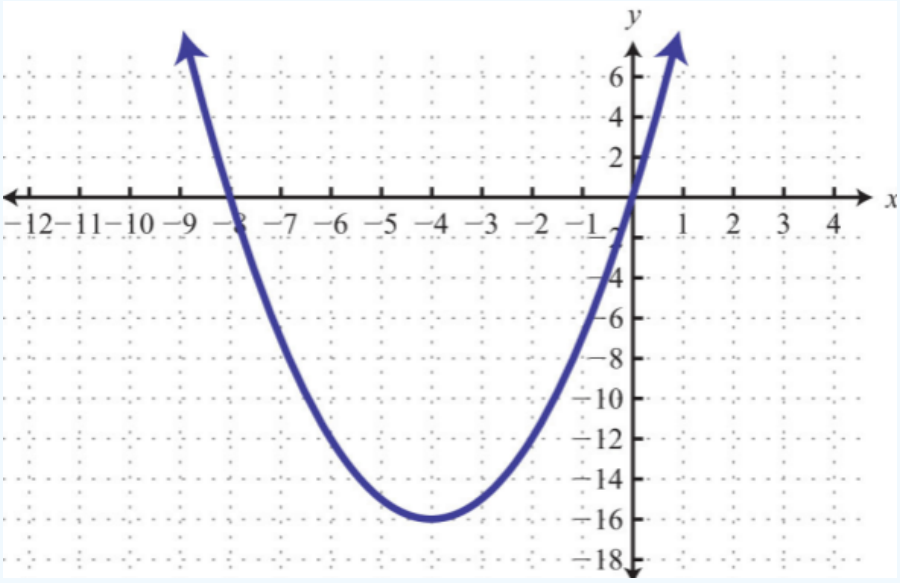


Figure 3.7.40

6. Given the graph of $g(x)$, find x when $g(x) = -3, g(x) = 0$, and $g(x) = 1$.

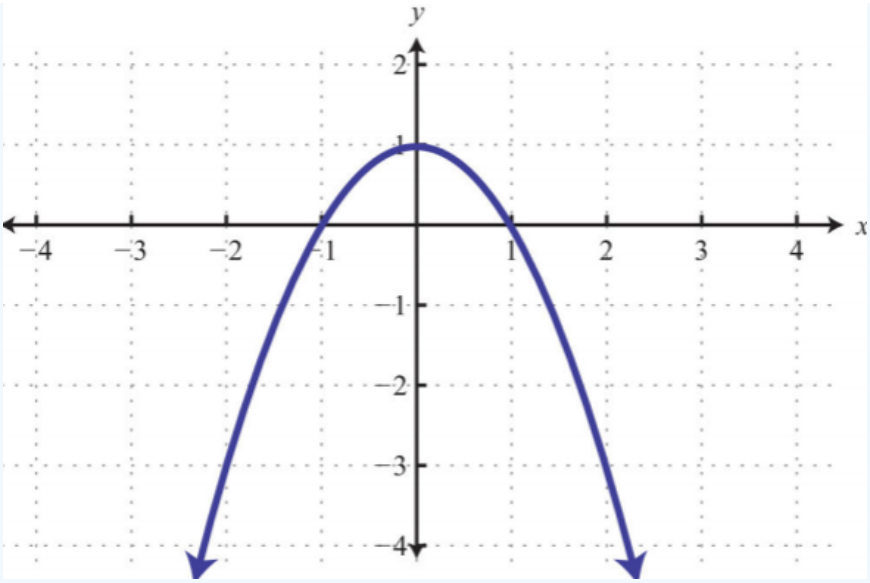


Figure 3.7.41

7. Given the graph of $f(x)$, find x when $f(x) = -4, f(x) = 0$, and $f(x) = -2$.

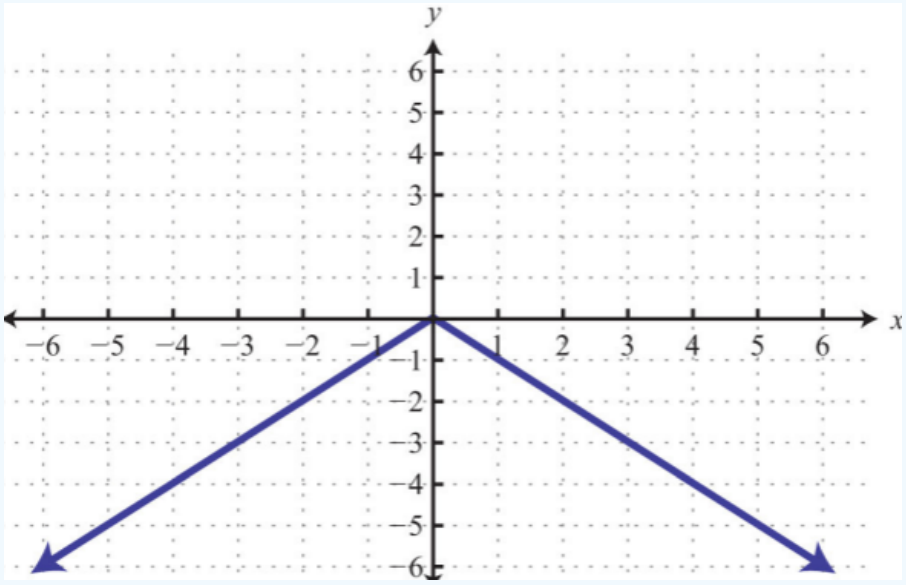


Figure 3.7.42

8. Given the graph of $g(x)$, find x when $g(x) = 5, g(x) = 3$, and $g(x) = 2$.

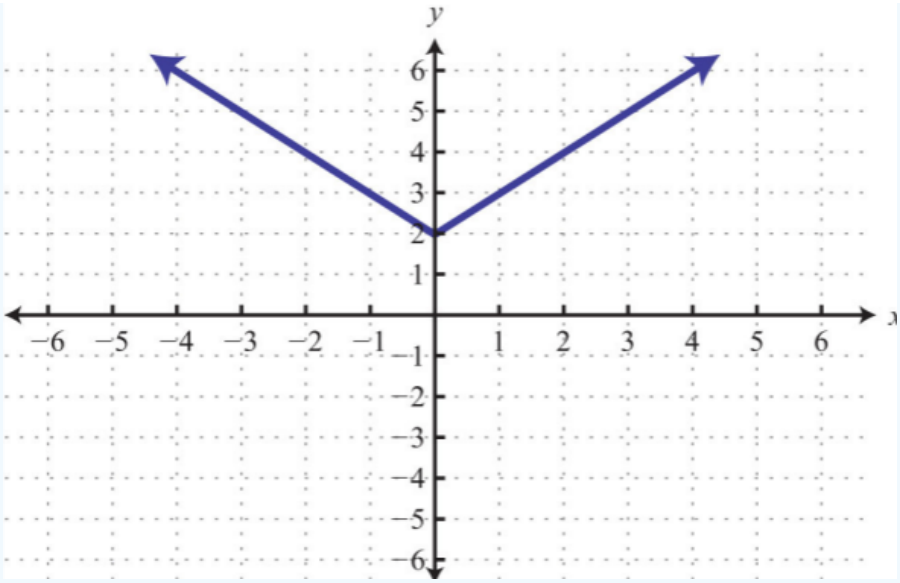


Figure 3.7.43

9. The cost in dollars of producing pens with a company logo is given by the function $C(x) = 1.65x + 120$, where x is the number of pens produced. Use the function to calculate the cost of producing 200 pens.

10. The revenue in dollars from selling sweat shirts is given by the function $R(x) = 29.95x$, where x is the number of sweat shirts sold. Use the function to determine the revenue if 20 sweat shirts are sold.

11. The value of a new car in dollars is given by the function $V(t) = -2,500t + 18,000$, where t represents the age of the car in years. Use the function to determine the value of the car when it is 5 years old. What was the value of the car when new?

12. The monthly income in dollars of a commissioned car salesman is given by the function $I(n) = 550n + 1,250$, where n represents the number of cars sold in the month. Use the function to determine the salesman's monthly income if he sells 3 cars this month. What is his income if he does not sell any cars in a month?

13. The perimeter of an isosceles triangle with a base measuring 10 centimeters is given by the function $P(x) = 2x + 10$, where x represents the length of each of the equal sides. Find the length of each side if the perimeter is 40 centimeters.

14. The perimeter of a square depends on the length of each side s and is modeled by the function $P(s) = 4s$. If the perimeter of a square measures 140 meters, then use the function to calculate the length of each side.

15. A certain cellular phone plan charges $\$18$ per month and $\$0.10$ per minute of usage. The cost of the plan is modeled by the function $C(x) = 0.10x + 18$, where x represents the number of minutes of usage per month. Determine the minutes of usage if the cost for the month was $\$36$.

16. The monthly revenue generated by selling subscriptions to a tutoring website is given by the function $R(x) = 29x$, where x represents the number of subscription sales per month. How many subscriptions were sold if the revenues for the month totaled $\$1,508$?

Answer available at the end of this section.

Exercise 3.7.9 Function Notation

Graph the linear function and state the domain and range.

1. $f(x) = -\frac{5}{2}x + 10$
2. $f(x) = \frac{3}{5}x - 10$

3. $g(x) = 6x + 2$
4. $g(x) = -4x + 6$
5. $h(t) = \frac{1}{2}t - 3$
6. $h(t) = -\frac{3}{4}t + 3$
7. $C(x) = 100 + 50x$
8. $C(x) = 50 + 100x$

Answer available at the end of this section.

Exercise 3.7.10 Discussion Board Topics

1. Is a vertical line a function? What are the domain and range of a vertical line?
2. Is a horizontal line a function? What are the domain and range of a horizontal line?
3. Come up with your own correspondence between real-world sets. Explain why it does or does not represent a function.
4. Can a function have more than one y-intercept? Explain.

Answer available at the end of this section.

Answers

Exercise 3.7.1

Domain: $[-4, \infty)$; range: $(-\infty, \infty)$; function: no

Exercise 3.7.2

a. $g(-\frac{1}{2}) = \frac{1}{12}$

b. $x = -13$

Exercise 3.7.3 **Functions**

1. Yes
3. No
5. No

Exercise 3.7.4 **Functions**

1. Domain: $\{3, 5, 7\}$; range: $\{2, 3, 4\}$; function: yes
3. Domain: $\{-10, -8\}$; range: $\{0, 1, 2\}$; function: no
5. Domain: $\{-4, -1, 2\}$; range: $\{1, 2, 3\}$; function: yes
7. Domain: $\{-2, 2\}$; range: $\{2, 3, 5\}$; function: no
9. Domain: $(-\infty, \infty)$; range: $\{2\}$; function: yes
11. Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$; function: yes
13. Domain: $[-2, \infty)$; range: $(-\infty, \infty)$; function: no
15. Domain: $[-4, \infty)$; range: $[0, \infty)$; function: yes
17. Domain: $(-\infty, \infty)$; range: $[0, \infty)$; function: yes
19. Domain: $(-\infty, \infty)$; range: $[2, \infty)$; function: yes

Exercise 3.7.5 **Function Notation**

1. $f(-2) = -6$
3. $f(15) = 5$
5. $f(-\frac{1}{3}) = -\frac{7}{6}$
7. $g(-4) = 5$
9. $g(\frac{5}{8}) = \frac{35}{64}$
11. $x = 2$
13. $x = 5$
15. $x = 12$
17. $x = \frac{1}{6}$

Exercise 3.7.6 **Function Notation**

1. $f(6) = 3$
3. $f(0) = -1$
5. $g(\frac{2}{3}) = 0$
7. $g(-1) = 5$
9. $x = \frac{3}{2}$
11. $x = 1$

Exercise 3.7.7 **Function Notation**

1. $f(-4) = -3, f(-1) = 0, f(0) = 1$, and $f(2) = 3$
3. $f(-4) = -4, f(-1) = -4, f(0) = -4$, and $f(2) = -4$
5. $f(-1) = 1, f(0) = -2, f(1) = -3$, and $f(3) = 1$
7. $g(-4) = 0, g(-3) = 1, g(0) = 2$, and $g(4) = 3$

Exercise 3.7.8 **Function Notation**

1. $f(-1) = 3, f(0) = 1$, and $f(2) = -3$
3. $f(1) = 3$ (answers may vary)
5. $f(-4) = -16$; $f(-6) = -12$ and $f(-2) = -12$; $f(-8) = 0$ and $f(0) = 0$
7. $f(-4) = -4$ and $f(4) = -4$; $f(0) = 0$; $f(-2) = -2$ and $f(2) = -2$
9. $\$450$
11. New: $\$18,000$; 5 years old: $\$5,500$
13. 15 centimeters
15. 180 minutes

Exercise 3.7.9 **Function Notation**

1. Domain and range: R

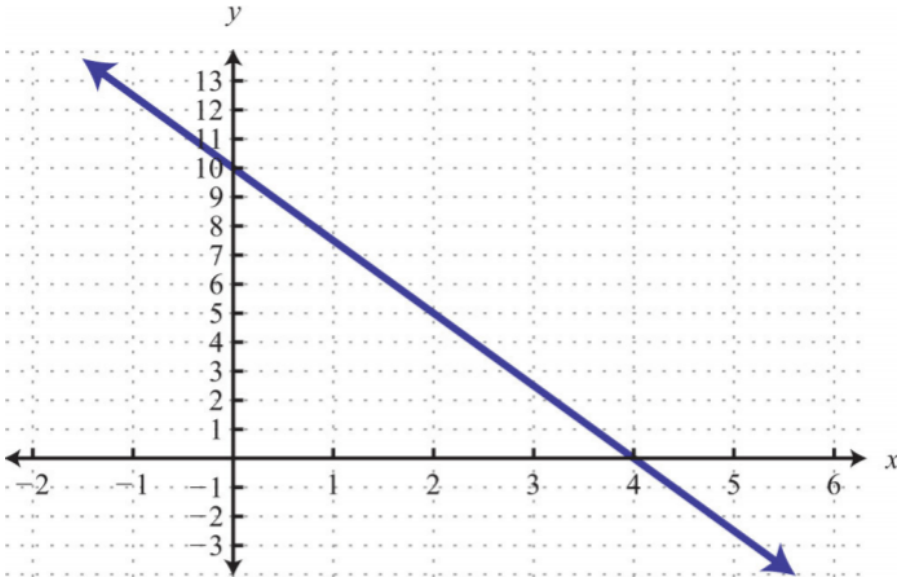


Figure 3.7.44

3. Domain and range: R

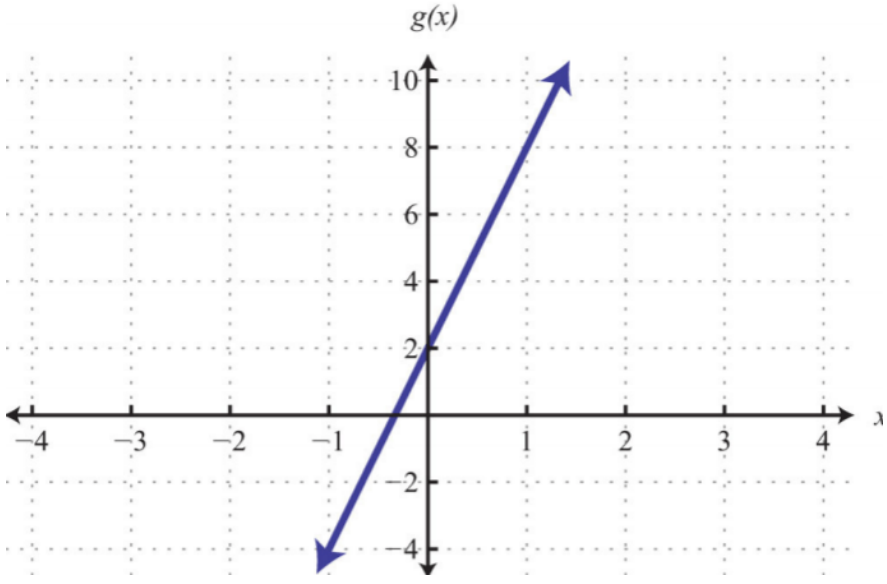


Figure 3.7.45

5. Domain and range: R

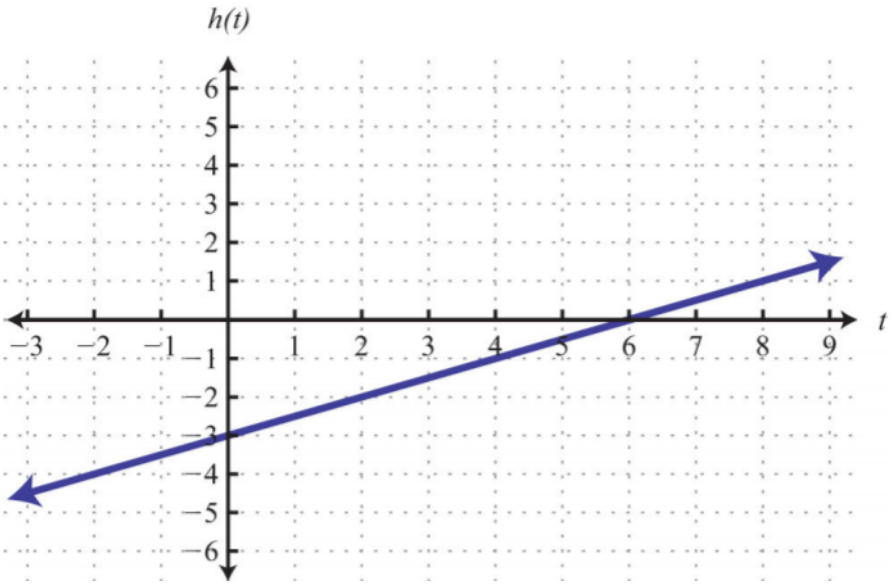


Figure 3.7.46

7. Domain and range: R

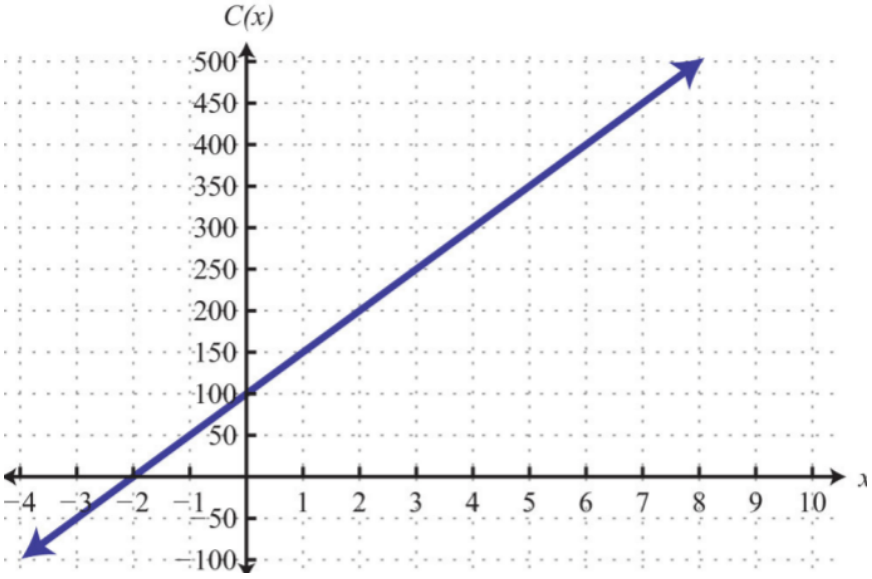


Figure 3.7.47

Exercise 3.7.10 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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3.8: Linear Inequalities (Two Variables)

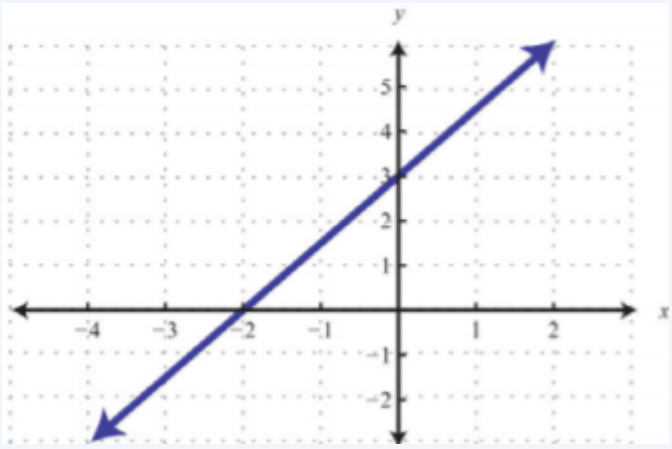
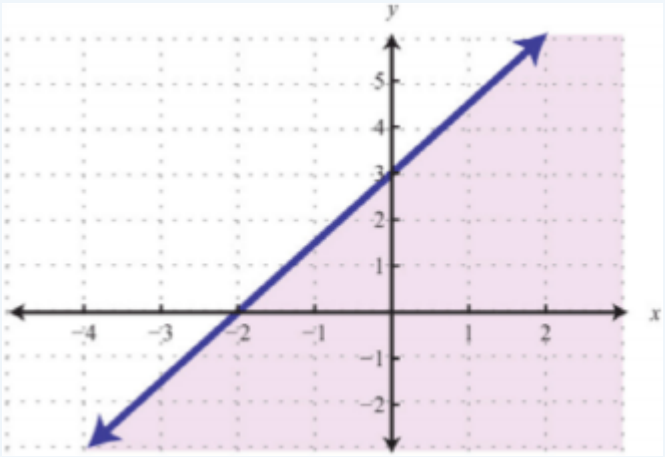
Learning Objectives

- Identify and check solutions to linear inequalities with two variables.
- Graph solution sets of linear inequalities with two variables.

Solutions to Linear Inequalities

We know that a linear equation with two variables has infinitely many ordered pair solutions that form a line when graphed. A linear inequality with two variables, on the other hand, has a solution set consisting of a region that defines half of the plane.

Table 3.8.1

Linear Equation	Linear Inequality
$y = \frac{3}{2}x + 3$	$y \leq \frac{3}{2}x + 3$
	
Figure 3.8.1	***Figure*** 3.8.2

For the inequality, the line defines one boundary of the region that is shaded. This indicates that any ordered pair that is in the shaded region, including the boundary line, will satisfy the inequality. To see that this is the case, choose a few test points and substitute them into the inequality.

Table 3.8.2

Test point	$y \leq \frac{3}{2}x + 3$
$(0, 0)$	$0 \leq \frac{3}{2}(0) + 3$ $0 \leq 3$ ✓
$(2, 1)$	$1 \leq \frac{3}{2}(2) + 3$ $1 \leq 3 + 3$ $1 \leq 6$ ✓

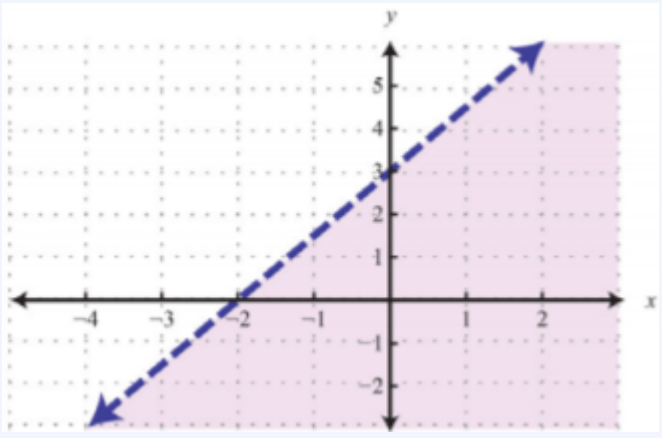
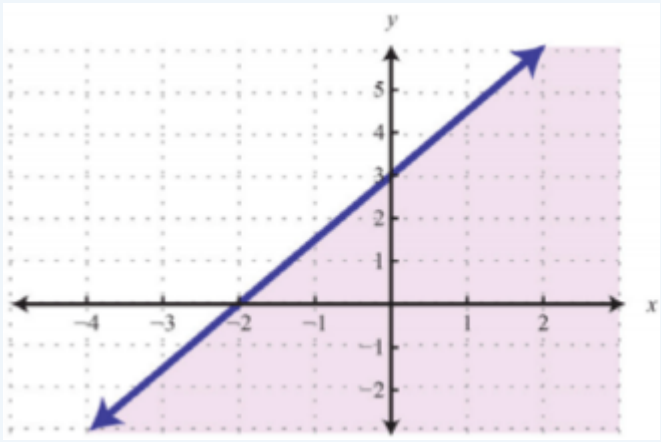
$(-2,-1)$	$-1 \le \frac{3}{2}(-2)+3$ $-1 \le -3+3$ $-1 \le 0 \quad \checkmark$

Also, we can see that ordered pairs outside the shaded region do not solve the linear inequality.

Test point	$y \le \frac{3}{2}x+3$
$(-2,3)$	$3 \le \frac{3}{2}(-2)+3$
	$3 \le -3+3$
	$3 \le 0 \quad x$

The graph of the solution set to a linear inequality is always a region. However, the boundary may not always be included in that set. In the previous example, the line was part of the solution set because of the "or equal to" part of the inclusive inequality $\le$. If we have a strict inequality $<$, we would then use a dashed line to indicate that those points are not included in the solution set.

Table 3.8.3

Non-Inclusive Boundary	Inclusive Boundary
$y < \frac{3}{2}x+3$	$y \le \frac{3}{2}x+3$
	
Figure 3.8.3	***Figure*** 3.8.4

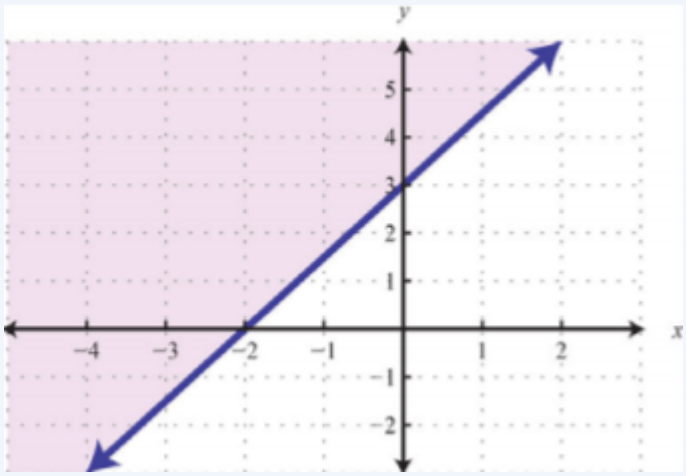
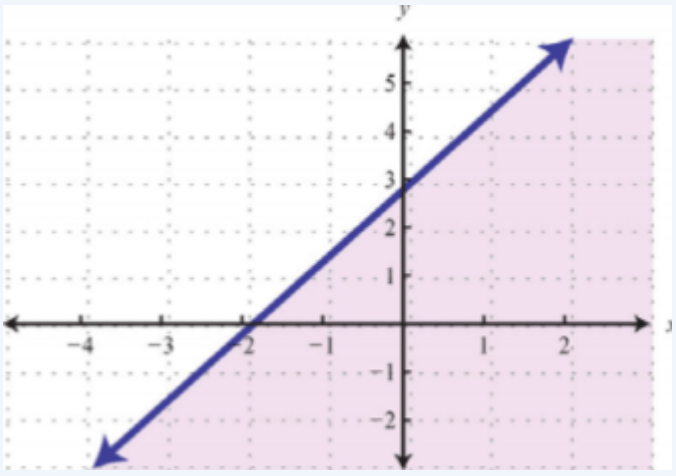
Consider the point $(0,3)$ on the boundary; this ordered pair satisfies the linear equation. It is the "or equal to" part of the inclusive inequality that makes it part of the solution set.

$y < \frac{3}{2}x+3$	$y \le \frac{3}{2}x+3$
$3 < \frac{3}{2}(0)+3$	$3 \le \frac{3}{2}(0)+3$
$3 < 0+3$	$3 \le 0+3$
$3 < 3 \quad x$	$3 \le 3 \quad \checkmark$

So far, we have seen examples of inequalities that were "less than." Now consider the following graphs with the same boundary:

Table 3.8.4

Greater Than (Above)	Less Than (Below)
$y \ge \frac{3}{2}x+3$	$y \le \frac{3}{2}x+3$

Greater Than (Above)	Less Than (Below)
	
Figure 3.8.5	***Figure*** 3.8.6

Given the graphs above, what might we expect if we use the origin $(0,0)$ as a test point?

$y \geq \frac{3}{2}x+3$	$y \leq \frac{3}{2}x+3$
$0 \geq \frac{3}{2}(0)+3$	$0 \leq \frac{3}{2}(0)+3$
$0 \geq 0+3$	$0 \leq 0+3$
$0 \geq 3$ x	$0 \leq 3$ ✓

Exercise 3.8.1

Which of the ordered pairs $(-2,-1),(0,0),(-2,8),(2,1)$, and $(4,2)$ solve the inequality $y > -\frac{1}{2}x+2$?

Answer available at the end of this section.

Graphing Solutions to Linear Inequalities

Solutions to linear inequalities are a shaded half-plane, bounded by a solid line or a dashed line. This boundary is either included in the solution or not, depending on the given inequality. If we are given a strict inequality, we use a dashed line to indicate that the boundary is not included. If we are given an inclusive inequality, we use a solid line to indicate that it is included. The steps for graphing the solution set for an inequality with two variables are outlined in the following example.

Example 3.8.1

Graph the solution set:

$$y > -3x+1.$$

Solution:

Step 1: Graph the boundary line. In this case, graph a dashed line $y = -3x+1$ because of the strict inequality. By inspection, we see that the slope is $m = -3 = -\frac{3}{1} = \frac{rise}{run}$ and the y-intercept is $(0,1)$.

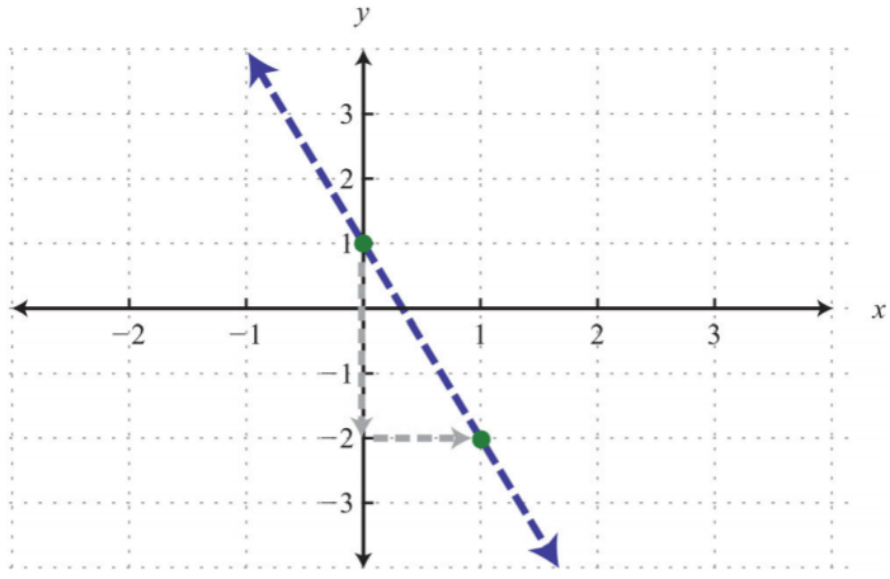


Figure 3.8.7

Step 2: Test a point ***not*** on the boundary. A common test point is the origin $(0, 0)$. The test point helps us determine which half of the plane to shade.

$\underline{Test\ point}$	$\underline{y > -3x + 1}$
$(0, 0)$	$0 > -3(0) + 1$
	$0 > 1 \quad x$

Step 3: Shade the region containing the solutions. Since the test point $(0, 0)$ was not a solution, it does not lie in the region containing all the ordered pair solutions. Therefore, shade the half of the plane that does not contain this test point. In this case, shade above the boundary line.

Answer:

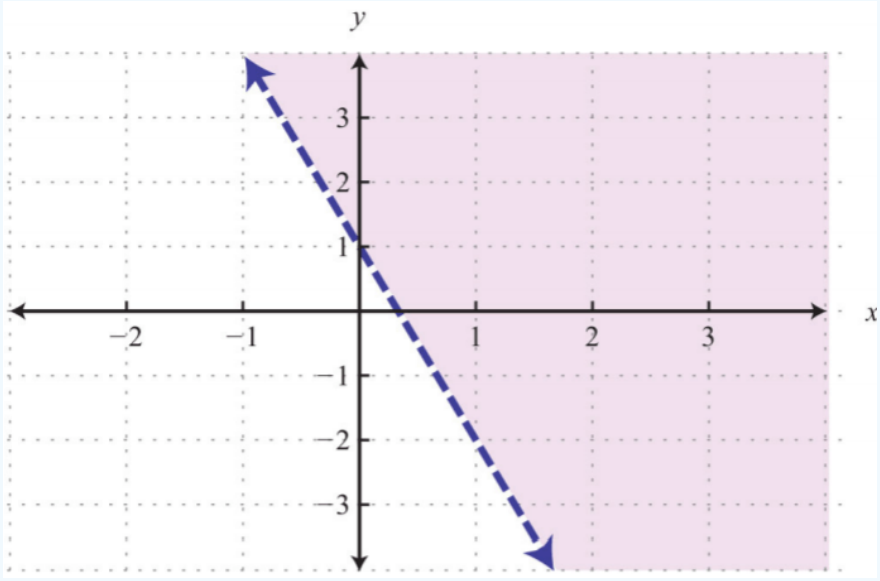


Figure 3.8.8

Consider the problem of shading above or below the boundary line when the inequality is in slope-intercept form. If $y > mx + b$

, then shade above the line. If $y < mx + b$, then shade below the line. Use this with caution; sometimes the boundary is given in standard form, in which case these rules do not apply.

Example 3.8.2

Graph the solution set:

$$2x - 5y \geq -10.$$

Solution:

Here the boundary is defined by the line $2x - 5y = -10$. Since the inequality is inclusive, we graph the boundary using a solid line. In this case, graph the boundary line using intercepts.

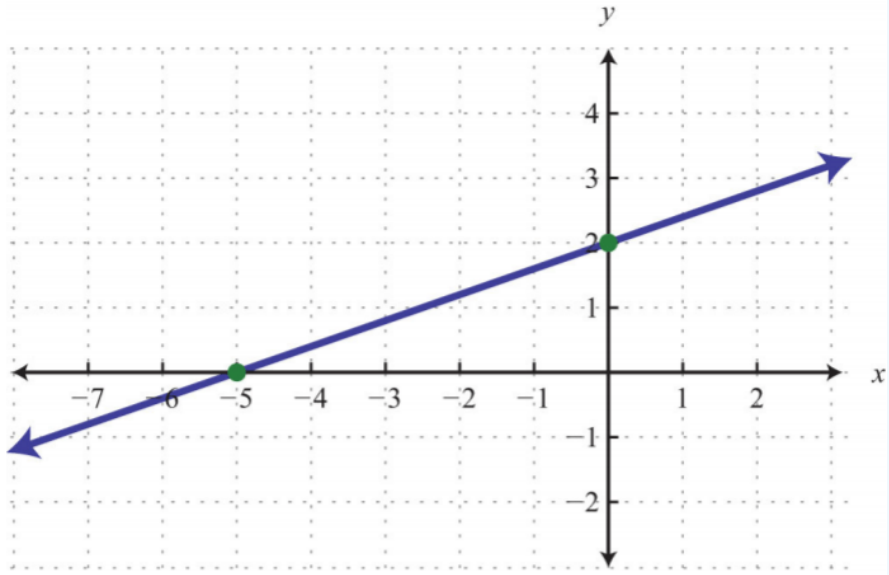


Figure 3.8.9

Next, test a point; this helps decide which region to shade.

Test point	$2x - 5y \geq -10$
$(0, 0)$	$2(0) - 5(0) \geq -10$
	$0 \geq -10$ ✓

Since the test point is in the solution set, shade the half of the plane that contains it.

Answer:

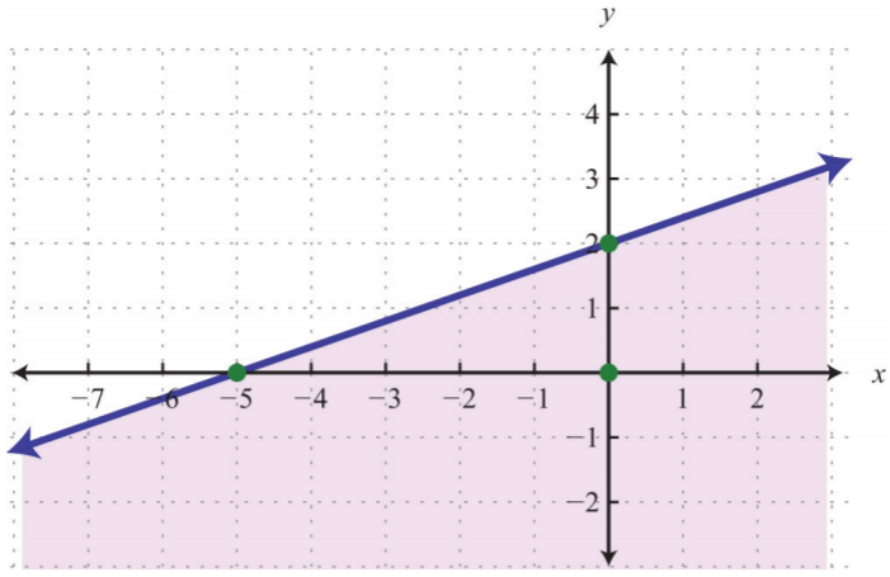


Figure 3.8.10

In this example, notice that the solution set consists of all the ordered pairs below the boundary line. This may be counterintuitive because of the original $\geq$ in the inequality. This illustrates that it is a best practice to actually test a point. Solve for y and you see that the shading is correct.

In slope-intercept form, you can see that the region below the boundary line should be shaded. An alternate approach is to first express the boundary in slope-intercept form, graph it, and then shade the appropriate region.

Graph the solution set:

$$y < 2.$$

Solution:

First, graph the boundary line $y = 2$ with a dashed line because of the strict inequality.

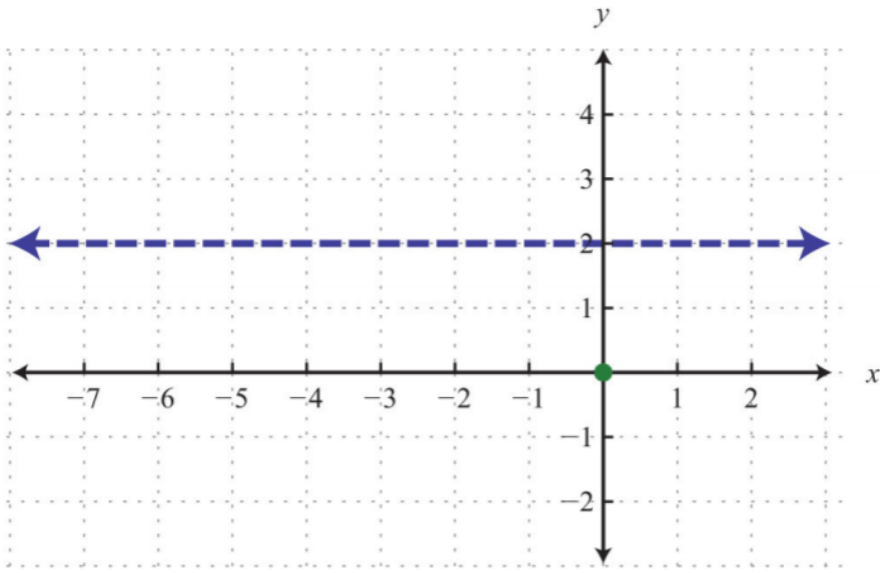


Figure 3.8.11

Now, test a point.

Test point	$y < 2$
$(0, 0)$	$0 < 2$ ✓

In this case, shade the region that contains the test point.

Answer:

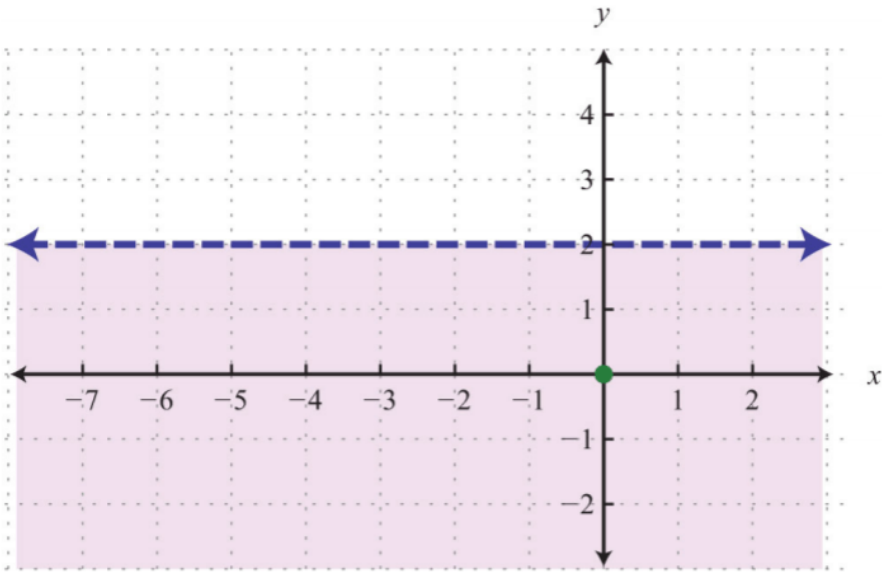


Figure 3.8.12

Exercise 3.8.2

Graph the solution set:

$$5x - y \leq 10.$$

Answer available at the end of this section.

Key Takeaways

- Linear inequalities with two variables have infinitely many ordered pair solutions, which can be graphed by shading in the appropriate half of a rectangular coordinate plane.
- To graph the solution set of a linear inequality with two variables, first graph the boundary with a dashed or solid line depending on the inequality. If given a strict inequality, use a dashed line for the boundary. If given an inclusive inequality, use a solid line. Next, choose a test point not on the boundary. If the test point solves the inequality, then shade the region that contains it; otherwise, shade the opposite side.
- When graphing the solution sets of linear inequalities, it is a good practice to test values in and out of the solution set as a check.

Exercise 3.8.3 Solutions to Linear Inequalities (Two Variables)

Is the ordered pair a solution to the given inequality?

1. $y < 5x + 1$; $(0, 0)$
2. $y > -\frac{1}{2}x - 4$; $(0, -2)$
3. $y \leq \frac{2}{3}x + 1$; $(6, 5)$
4. $y \geq -\frac{1}{3}x - 5$; $(-3, -8)$
5. $y < \frac{1}{5}x - \frac{1}{3}$; $(-\frac{1}{2}, -1)$
6. $4x - 3y \leq 2$; $(-2, -1)$
7. $-x + 4y > 7$; $(0, 0)$
8. $7x - 3y < 21$; $(5, -3)$
9. $y > -5$; $(-3, -1)$
10. $x \leq 0$; $(0, 7)$

Answer available at the end of this section.

Exercise 3.8.4 Graphing Solutions to Linear Inequalities

Graph the solution set.

1. $y < -3x + 3$
2. $y < -\frac{2}{3}x + 4$
3. $y \geq -\frac{1}{2}x$
4. $y \geq \frac{4}{5}x - 8$
5. $y \leq 8x - 7$
6. $y > -5x + 3$
7. $y > -x + 4$
8. $y > x - 2$
9. $y \geq -1$
10. $y < -3$
11. $x < 2$
12. $x \geq 2$
13. $y \leq \frac{3}{4}x - \frac{1}{2}$
14. $y > -\frac{3}{2}x + \frac{5}{2}$
15. $-2x + 3y > 6$
16. $7x - 2y > 14$
17. $5x - y < 10$
18. $x - y < 0$

19. $3x - 2y \geq 0$
20. $x - 5y \leq 0$
21. $-x + 2y \leq -4$
22. $-x + 2y \leq 3$
23. $2x - 3y \geq -1$
24. $5x - 4y < -3$
25. $\frac{1}{2}x - \frac{1}{3}y < 1$
26. $\frac{1}{2}x - \frac{1}{10}y \geq \frac{1}{2}$
27. $x \geq -2y$
28. $x < 2y + 3$
29. $3x - y + 2 > 0$
30. $3 - y - 2x < 0$
31. $-4x \leq 12 - 3y$
32. $5x \leq -4y - 12$
33. Write an inequality that describes all points in the upper half-plane above the x-axis.
34. Write an inequality that describes all points in the lower half-plane below the x-axis.
35. Write an inequality that describes all points in the half-plane left of the y-axis.
36. Write an inequality that describes all points in the half-plane right of the y-axis.
37. Write an inequality that describes all ordered pairs whose y-coordinates are at least 2.
38. Write an inequality that describes all ordered pairs whose x-coordinate is at most 5.

Answer available at the end of this section.

Answers

Exercise 3.8.1

$(-2, 8)$ and $(4, 2)$

Exercise 3.8.2

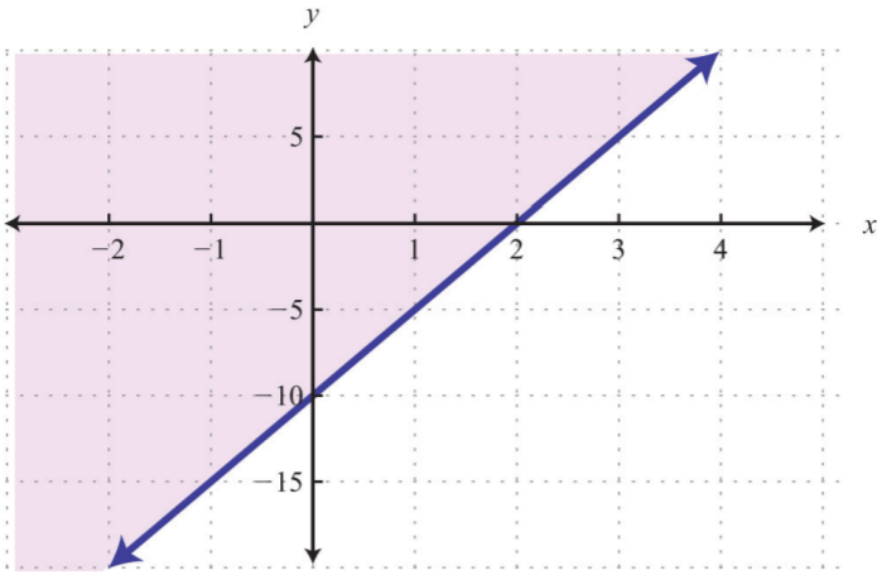


Figure 3.8.13

Exercise 3.8.3 **Solutions to Linear Inequalities (Two Variables)**

1. Yes
3. Yes
5. Yes
7. No
9. Yes

Exercise 3.8.4 **Graphing Solutions to Linear Inequalities**

1.

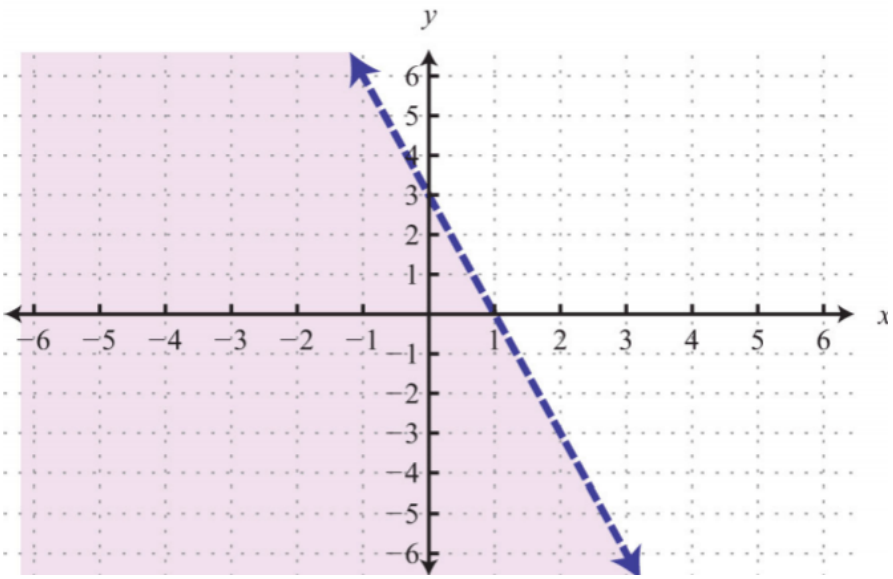


Figure 3.8.14

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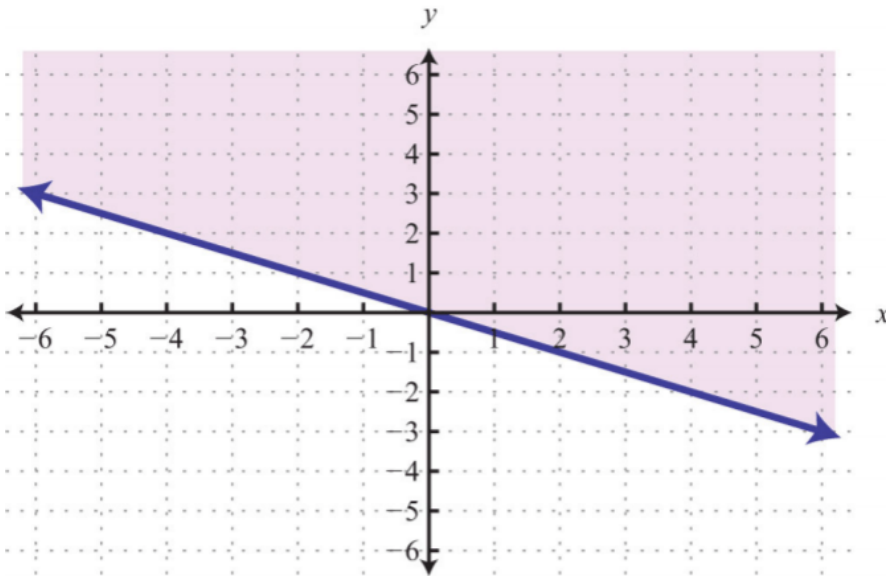


Figure 3.8.15

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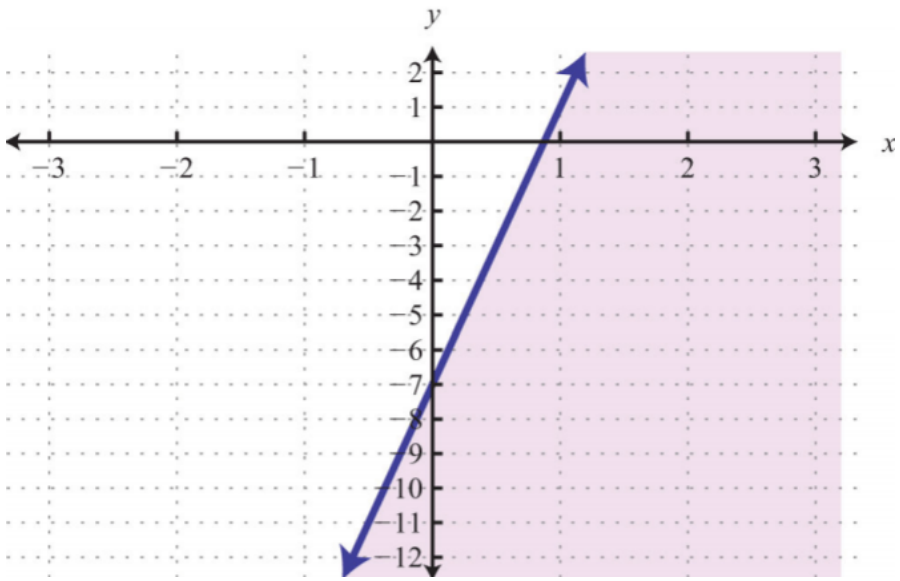


Figure 3.8.16

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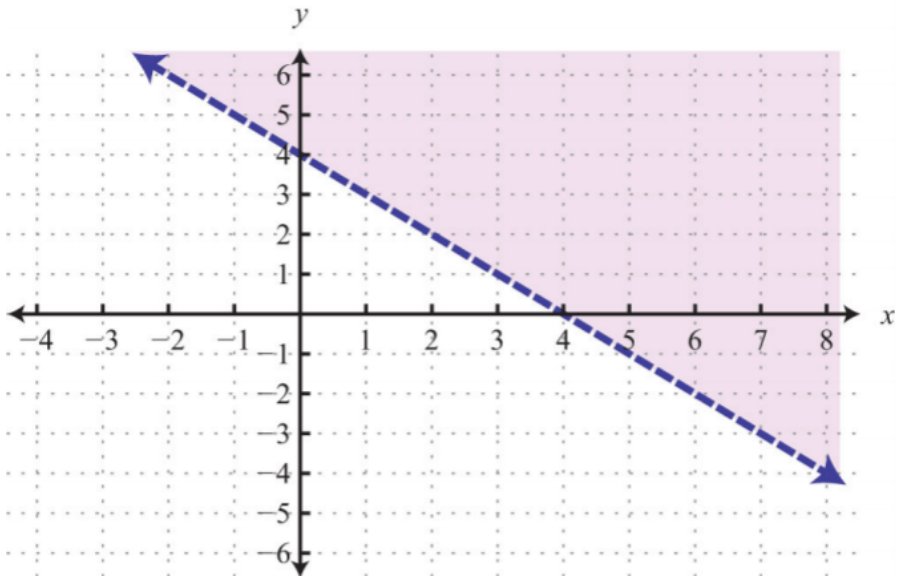


Figure 3.8.17

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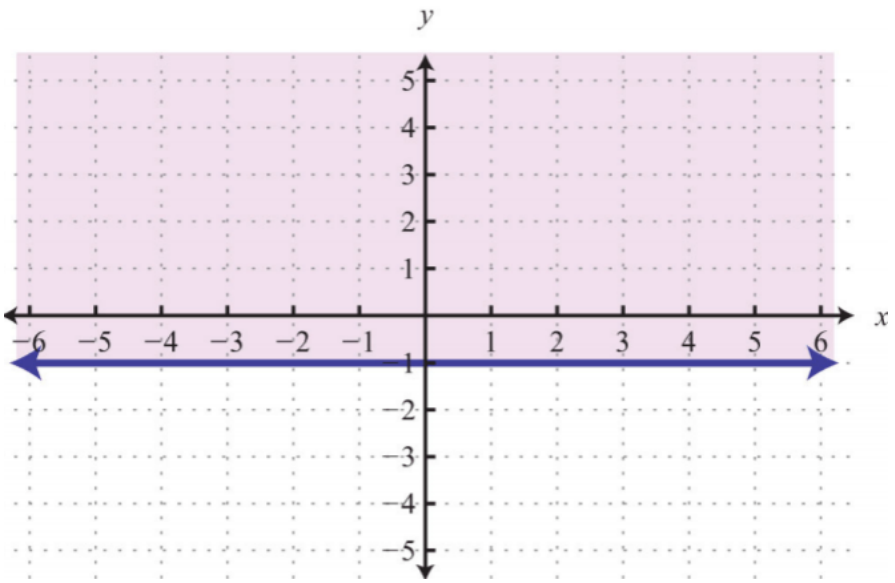


Figure 3.8.18

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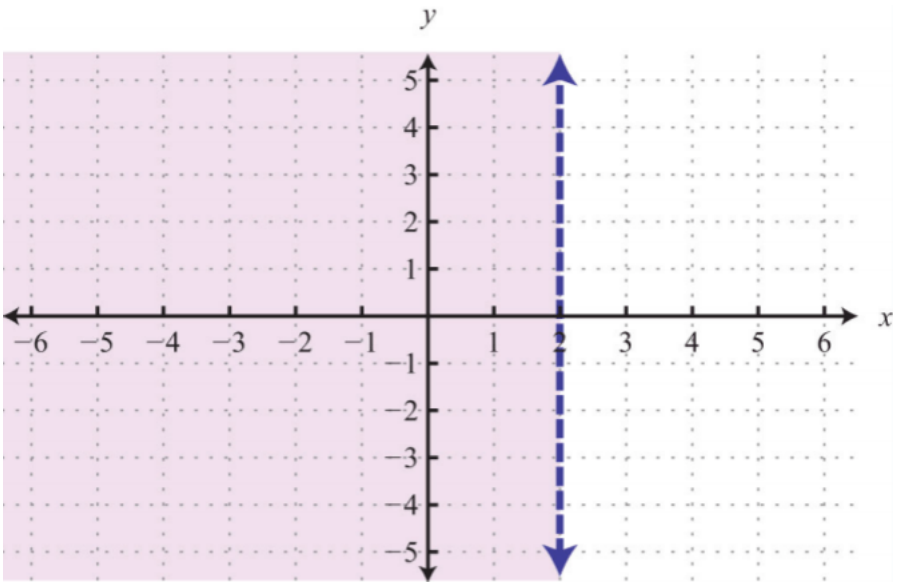


Figure 3.8.19

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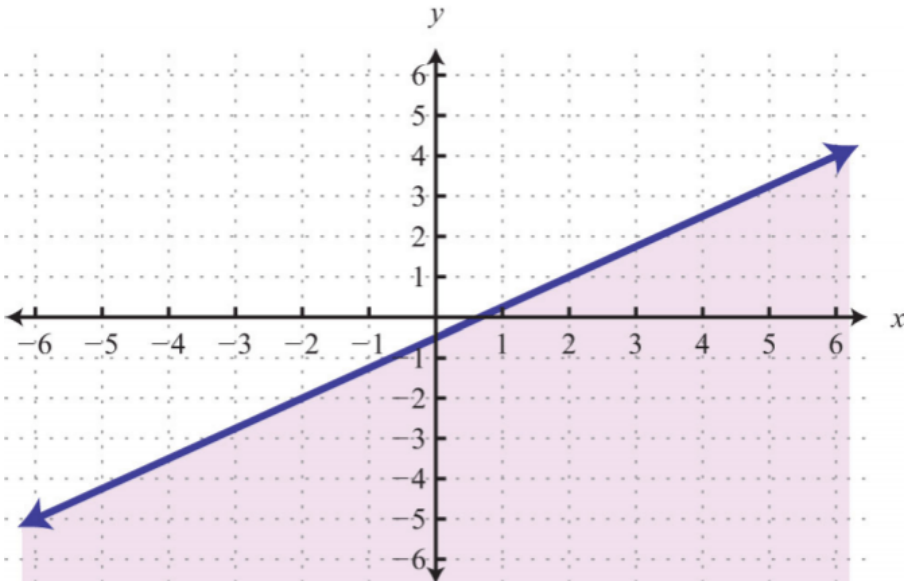


Figure 3.8.20

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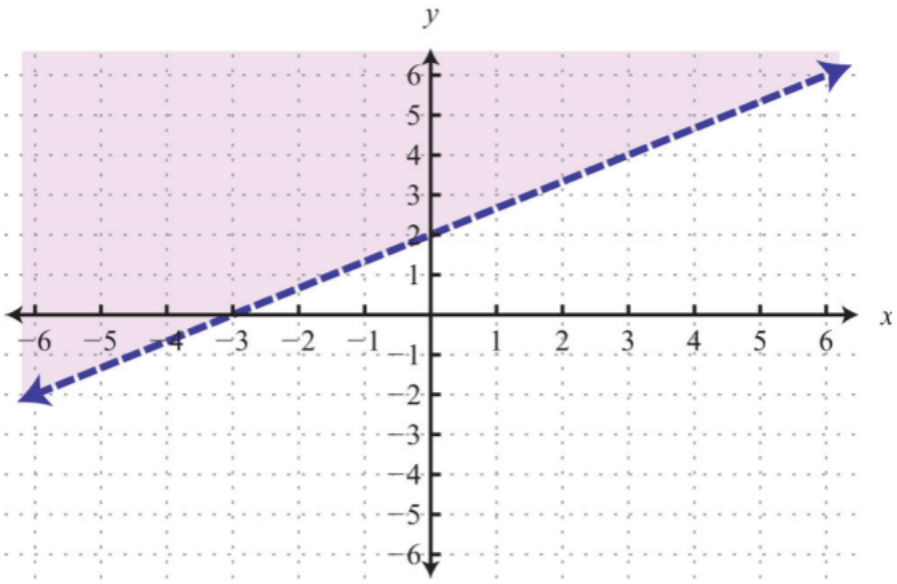


Figure 3.8.21

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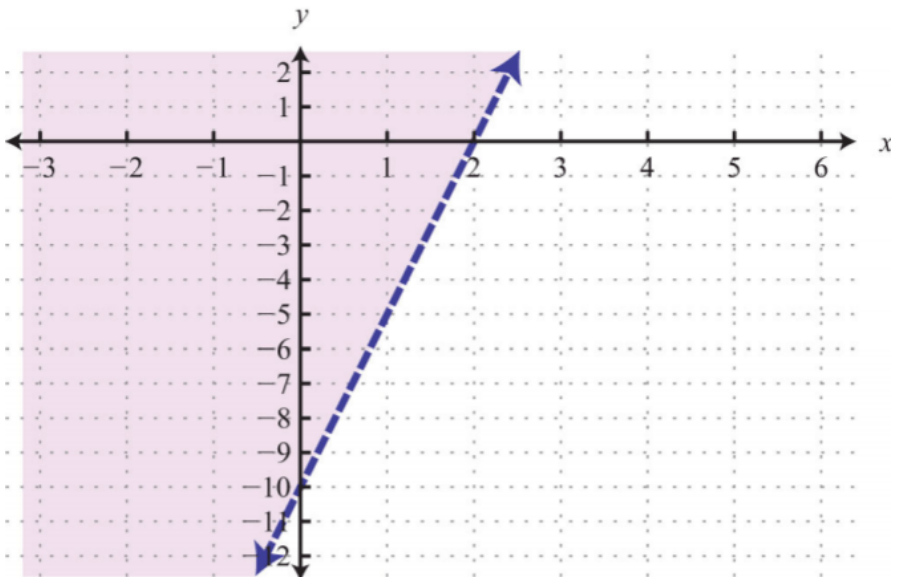


Figure 3.8.22

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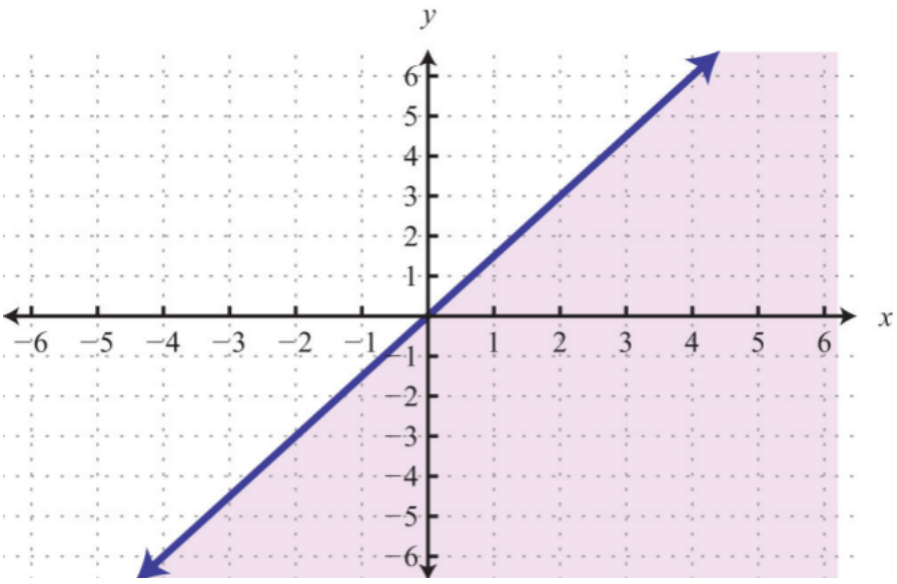


Figure 3.8.23

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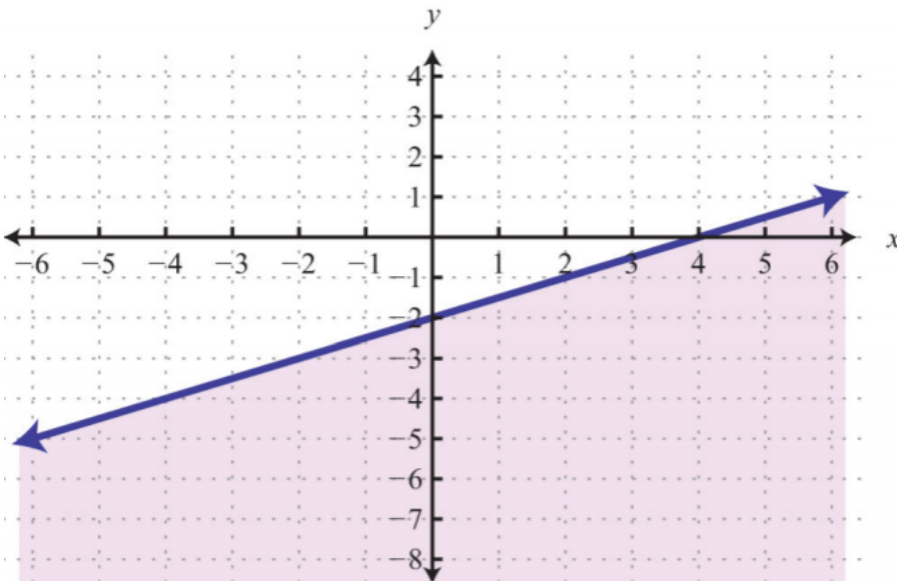


Figure 3.8.24

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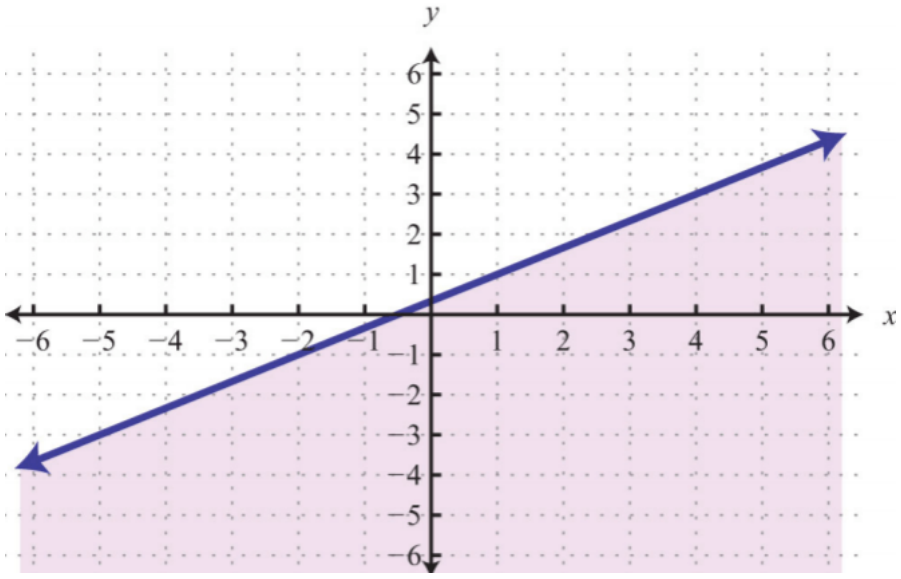


Figure 3.8.25

25.

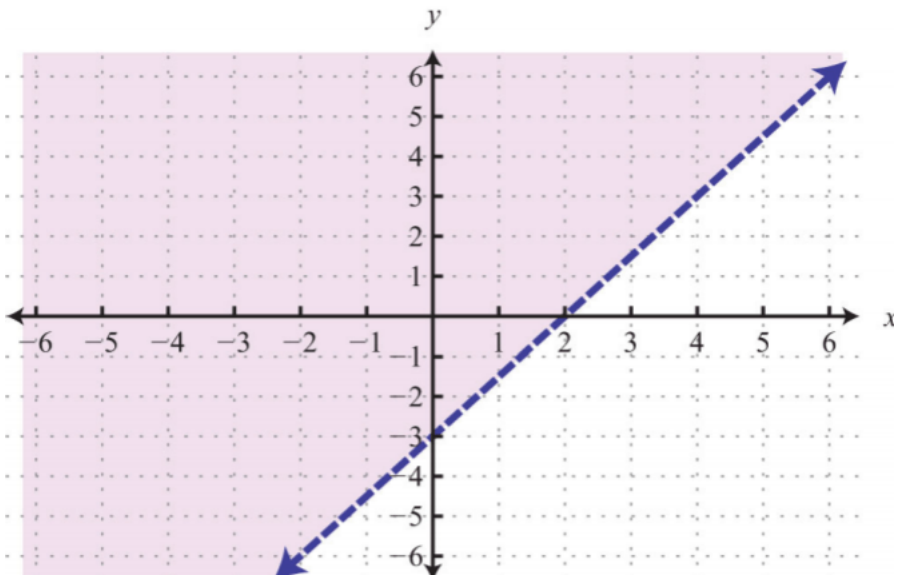


Figure 3.8.26

27.

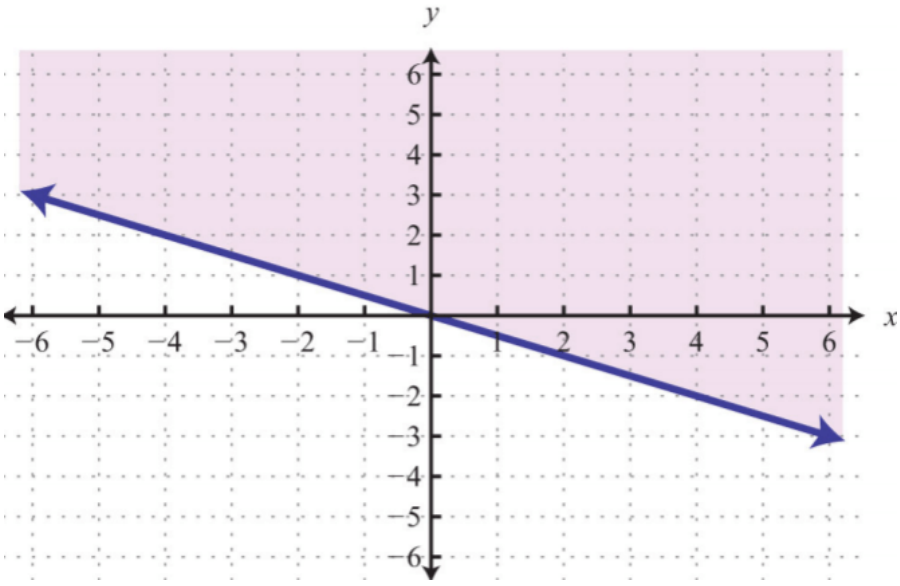


Figure 3.8.27

29.

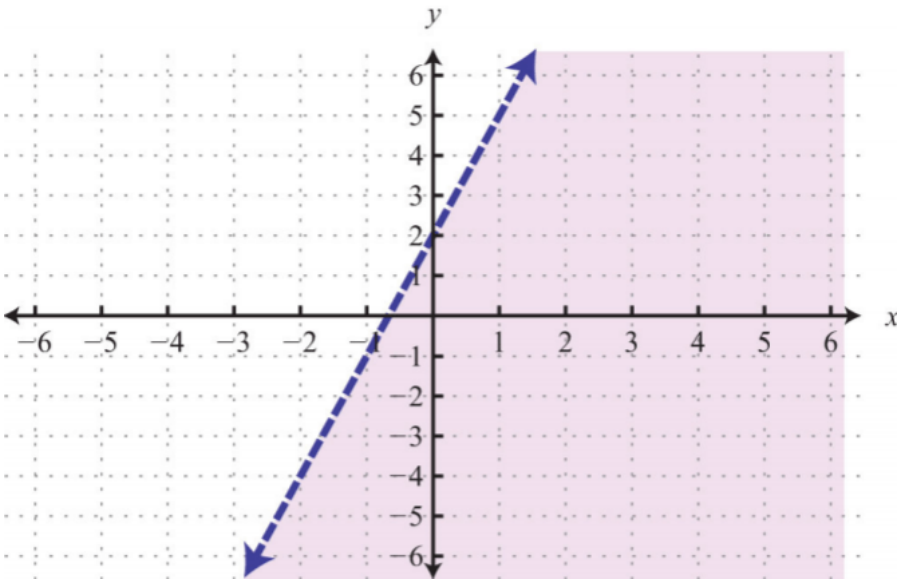


Figure 3.8.28

31.

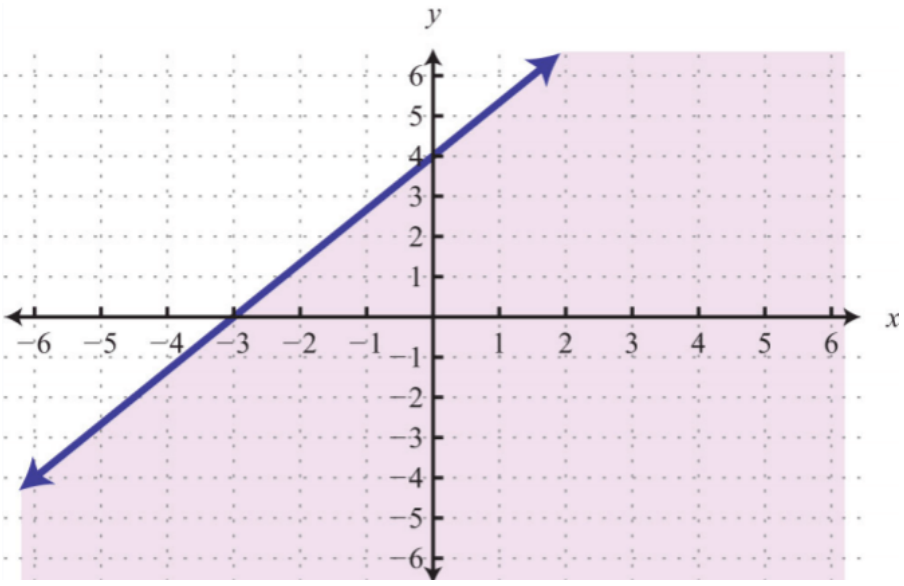


Figure 3.8.29

33. $y > 0$
35. $x < 0$
37. $y \geq 2$

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3.E: Review Exercises and Sample Exam

Review Exercises

Exercise $3.E.1$ Rectangular Coordinate System

Graph the given set of ordered pairs.

1. $\{(-3,4),(-4,0),(0,3),(2,4)\}$
2. $\{(-5,5),(-3,-1),(0,0),(3,2)\}$
3. Graph the points $(-3,5),(-3,-3)$, and $(3,-3)$ on a rectangular coordinate plane. Connect the points and calculate the area of the shape.
4. Graph the points $(-4,1),(0,1),(0,-2)$, and $(-4,-2)$ on a rectangular coordinate plane. Connect the points and calculate the area of the shape.
5. Graph the points $(1,0),(4,0),(1,-5)$, and $(4,-5)$ on a rectangular coordinate plane. Connect the points and calculate the perimeter of the shape.
6. Graph the points $(-5,2),(-5,-3),(1,2)$, and $(1,-3)$ on a rectangular coordinate plane. Connect the points and calculate the perimeter of the shape.

Answer available at the end of this section.

Exercise $3.E.2$ Rectangular Coordinate System

Calculate the distance between the given two points.

1. $(-1,-2)$ and $(5,6)$
2. $(2,-5)$ and $(-2,-2)$
3. $(-9,-3)$ and $(-8,4)$
4. $(-1,3)$ and $(1,-3)$

Answer available at the end of this section.

Exercise $3.E.3$ Rectangular Coordinate System

Calculate the midpoint between the given points.

1. $(-1,3)$ and $(5,-7)$
2. $(6,-3)$ and $(-8,-11)$
3. $(7,-2)$ and $(-6,-1)$
4. $(-6,0)$ and $(0,0)$
5. Show algebraically that the points $(-1,-1),(1,-3)$, and $(2,0)$ form an isosceles triangle.
6. Show algebraically that the points $(2,-1),(6,1)$, and $(5,3)$ form a right triangle.

Answer available at the end of this section.

Exercise $3.E.4$ Graph by Plotting Points

Determine whether the given point is a solution.

1. $-5x+2y=7$; $(1,-1)$

2. $6x - 5y = 4$; $(-1, -2)$
3. $y = \frac{3}{4}x + 1$; $(-\frac{2}{3}, \frac{1}{2})$
4. $y = -\frac{3}{5}x - 2$; $(10, -8)$

Answer available at the end of this section.

Exercise $3.E.5$ Graph by Plotting Points

Find at least five ordered pair solutions and graph.

1. $y = -x + 2$
2. $y = 2x - 3$
3. $y = \frac{1}{2}x - 2$
4. $y = -\frac{2}{3}x$
5. $y = 3$
6. $x = -3$
7. $x - 5y = 15$
8. $2x - 3y = 12$

Answer available at the end of this section.

Exercise $3.E.6$ Graph Using Intercepts

Given the graph, find the x- and y- intercepts.

1.

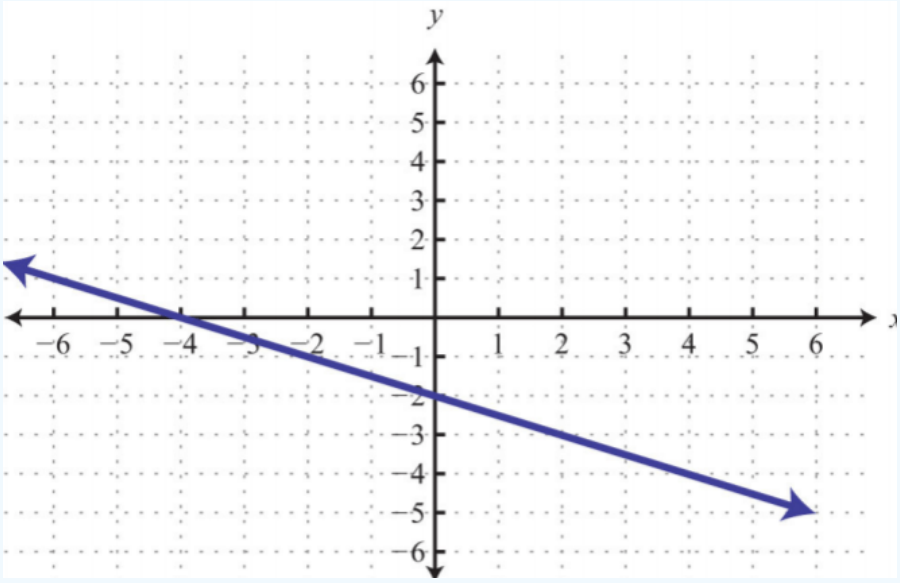


Figure 3.E.8

2.

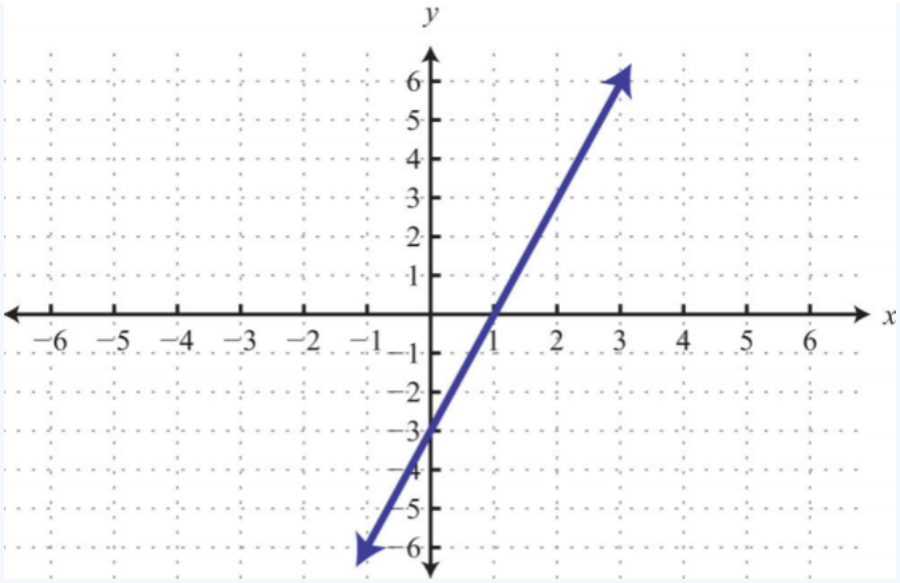


Figure 3.E.9

3.

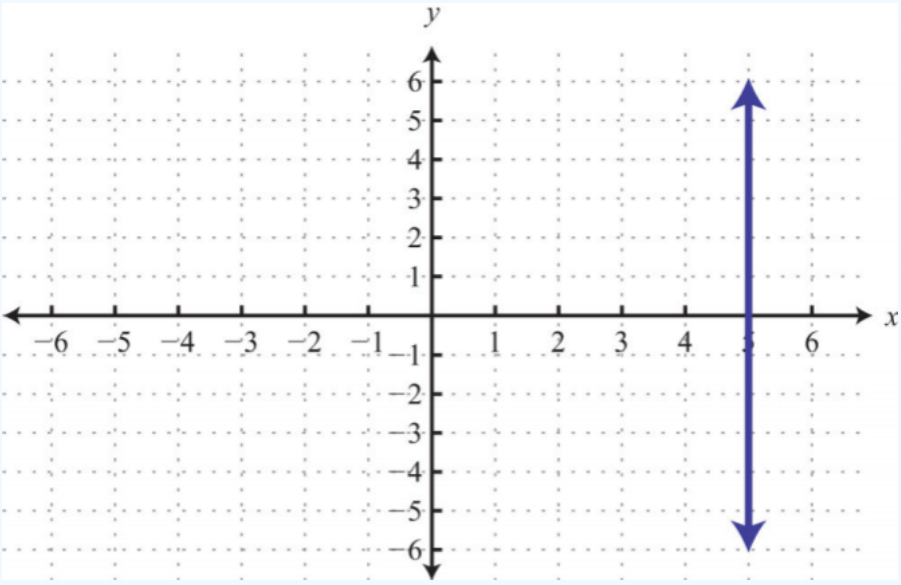


Figure 3.E.10

4.

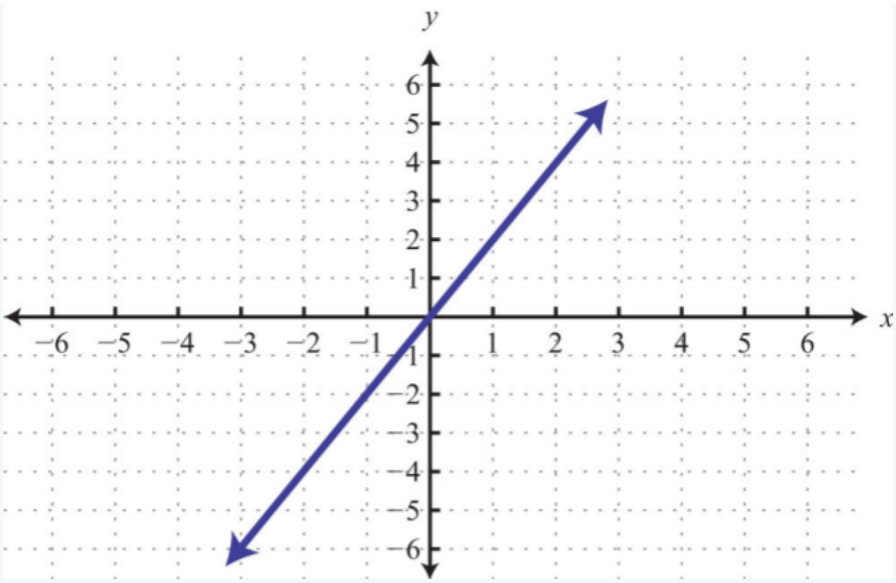


Figure 3.E.11

Answer available at the end of this section.

Exercise $3.E.7$ Graph Using Intercepts

Find the intercepts and graph them.

1. $3x - 4y = 12$
2. $2x - y = -4$
3. $\frac{1}{2}x - \frac{1}{3}y = 1$
4. $-\frac{1}{2}x + \frac{2}{3}y = 2$
5. $y = -\frac{5}{3}x + 5$
6. $y = -3x + 4$

Answer available at the end of this section.

Exercise $3.E.8$ Graph Using the y-Intercept and Slope

Given the graph, determine the slope and y-intercept.

1.

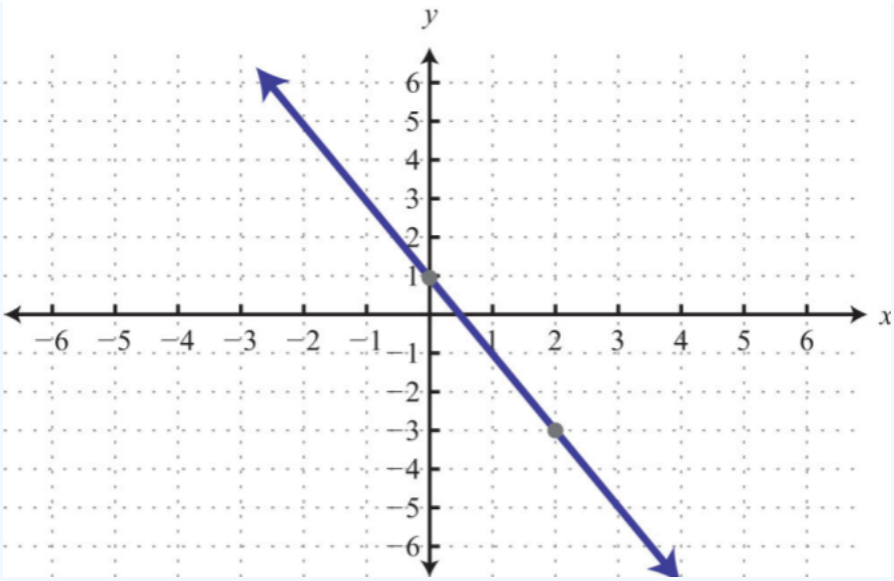


Figure 3.E.15

2.

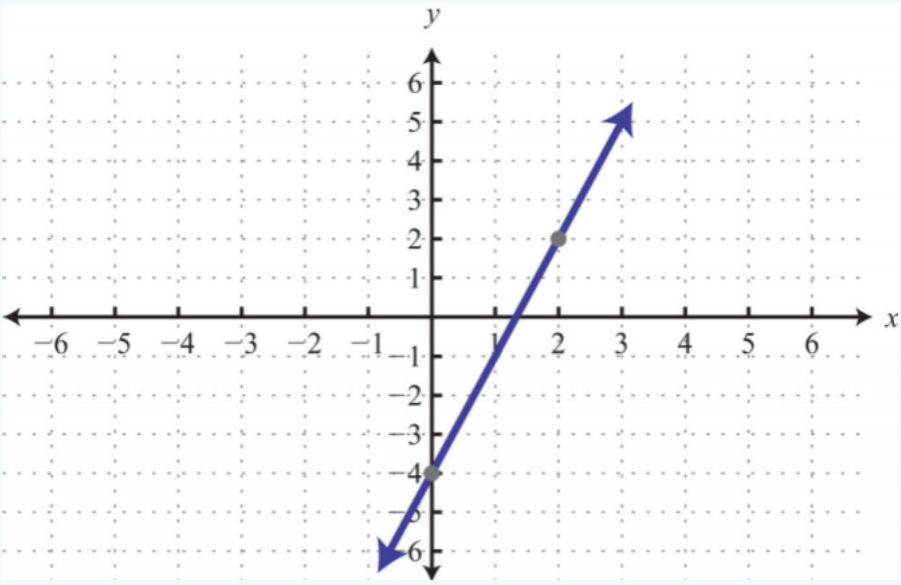


Figure 3.E.16

Answer available at the end of this section.

Exercise $3.E.9$ Graph Using the y-Intercept and Slope

Determine the slope, given two points.

1. $(-3, 8)$ and $(5, -6)$
2. $(0, -5)$ and $(-6, 3)$
3. $(\frac{1}{2}, -\frac{2}{3})$ and $(\frac{1}{4}, -\frac{1}{3})$
4. $(5, -\frac{3}{4})$ and $(2, -\frac{3}{4})$

Answer available at the end of this section.

Exercise $3.E.10$ Graph Using the y-Intercept and Slope

Express in slope-intercept form and identify the slope and y-intercept.

1. $12x - 4y = 8$
2. $3x - 6y = 24$
3. $-\frac{1}{3}x + \frac{3}{4}y = 1$
4. $-5x + 3y = 0$

Answer available at the end of this section.

Exercise $3.E.11$ Graph Using the y-Intercept and Slope

1. $y = -x + 3$
2. $y = 4x - 1$
3. $y = -2x$
4. $y = -\frac{5}{2}x + 3$
5. $2x - 3y = 9$
6. $2x + \frac{3}{2}y = 3$
7. $y = 0$
8. $x - 4y = 0$

Answer available at the end of this section.

Exercise $3.E.12$ Finding Linear Equations

Given the graph, determine the equation of the line.

1.

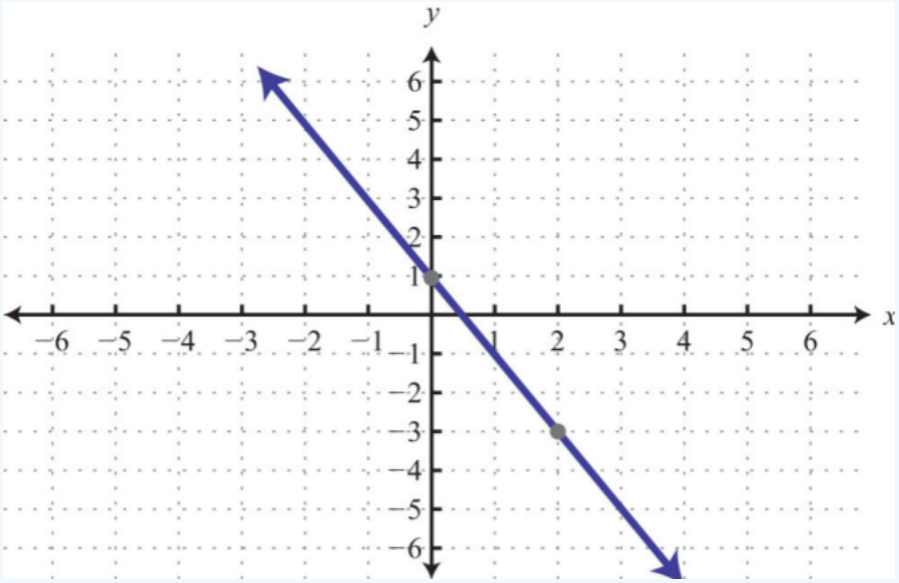


Figure 3.E.21

2.

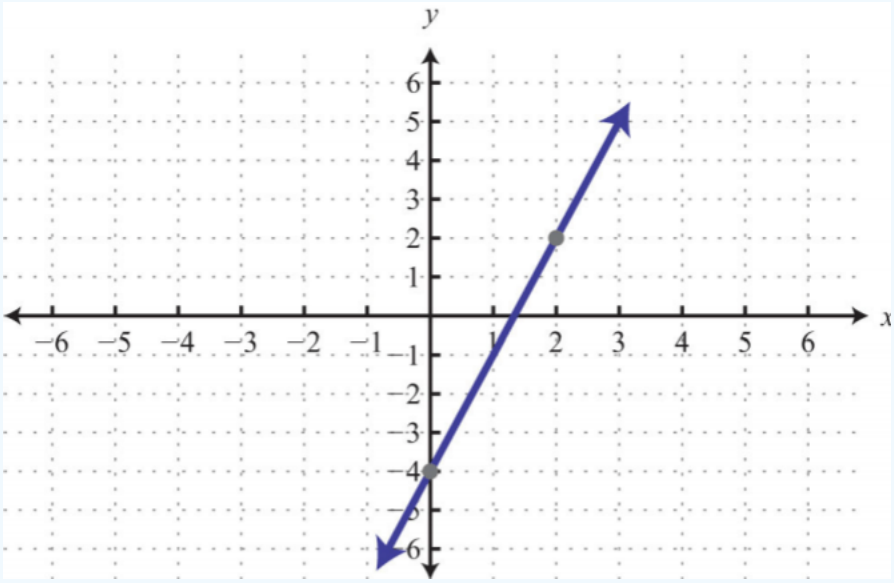


Figure 3.E.22

3.

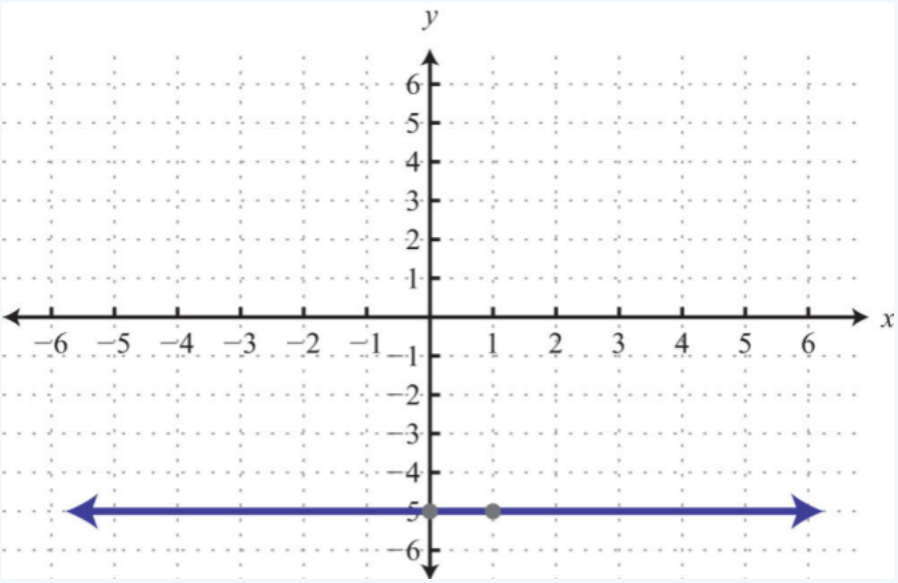


Figure 3.E.23

4.

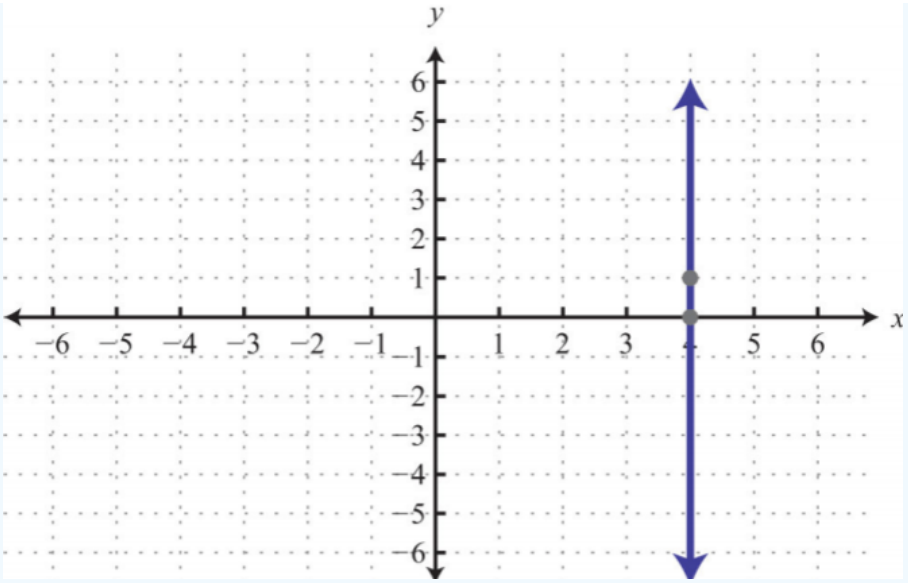


Figure 3.E.24

Answer available at the end of this section.

Exercise $3.E.13$ Finding Linear Equations

Find the equation of a line, given the slope and a point on the line.

1. $m = \frac{1}{2}$; $(-4, 8)$
2. $m = -\frac{1}{5}$; $(-5, -9)$
3. $m = \frac{2}{3}$; $(1, -2)$
4. $m = -\frac{3}{4}$; $(2, -3)$

Answer available at the end of this section.

Exercise $3.E.14$ Finding Linear Equations

Find the equation of the line given two points on the line.

1. $(-5, -5)$ and $(10, 7)$
2. $(-6, 12)$ and $(3, -3)$
3. $(2, -1)$ and $(-2, 2)$
4. $(\frac{5}{2}, -2)$ and $(-5, \frac{5}{2})$
5. $(7, -6)$ and $(3, -6)$
6. $(10, 1)$ and $(10, -3)$

Answer available at the end of this section.

Exercise $3.E.15$ Parallel and Perpendicular Lines

Determine if the lines are parallel, perpendicular, or neither.

1. $\begin{cases} -3x + 7y = 14 \\ 6x - 14y = 42 \end{cases}$

2. $\begin{cases} 2x+3y=18 \\ 2x-3y=36 \end{cases}$
3. $\begin{cases} \quad x+4y=2 \\ 8x-2y=-1 \end{cases}$
4. $\begin{cases} y=2 \\ x=2 \end{cases}$

Answer available at the end of this section.

Exercise $3.E.16$ Parallel and Perpendicular Lines

Find the equation of the line in slope-intercept form.

1. Parallel to $5x-y=15$ and passing through $(-10,-1)$.
2. Parallel to $x-3y=1$ and passing through $(2,-2)$.
3. Perpendicular to $8x-6y=4$ and passing through $(8,-1)$.
4. Perpendicular to $7x+y=14$ and passing through $(5,1)$.
5. Parallel to $y=1$ and passing through $(4,-1)$.
6. Perpendicular to $y=1$ and passing through $(4,-1)$.

Answer available at the end of this section.

Exercise $3.E.17$ Introduction to Functions

Determine the domain and range and state whether it is a function or not.

1. $\{(-10,-1),(-5,2),(5,2)\}$

2. $\{(-12,4),(-1,-3),(-1,-2)\}$

3.

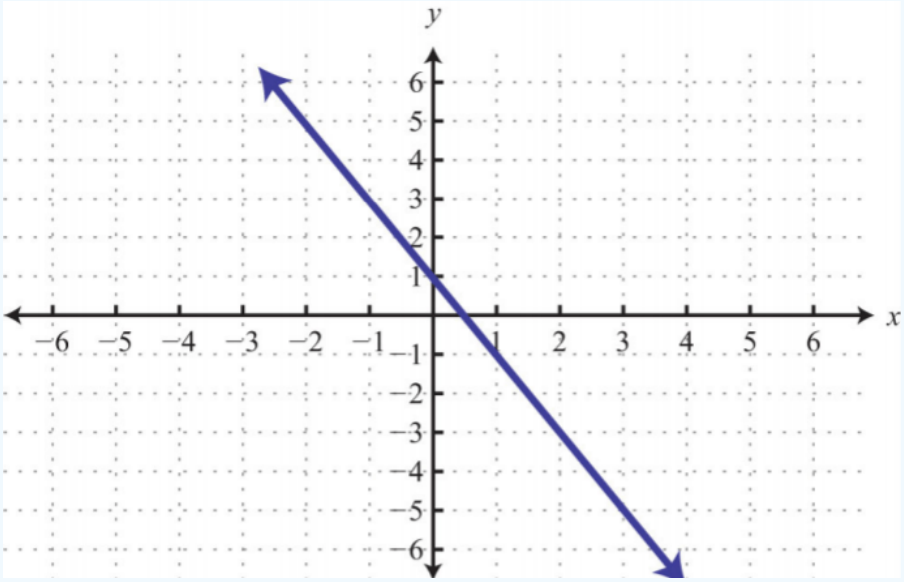


Figure 3.E.25

4.

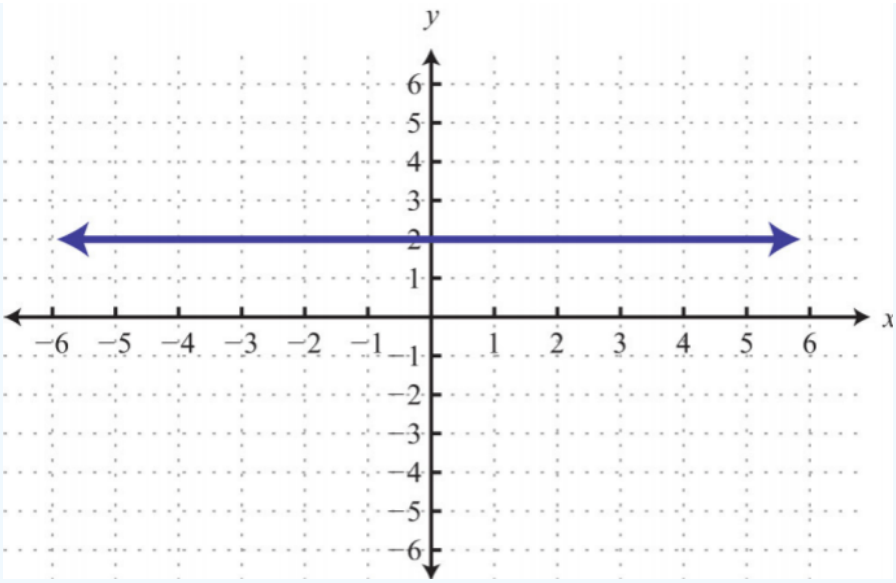


Figure 3.E.26

5.

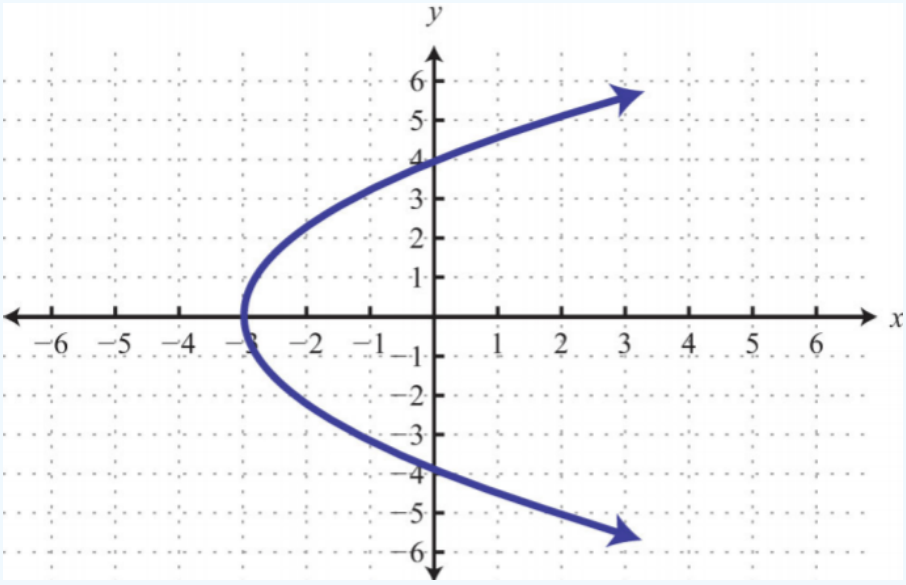


Figure 3.E.27

6.

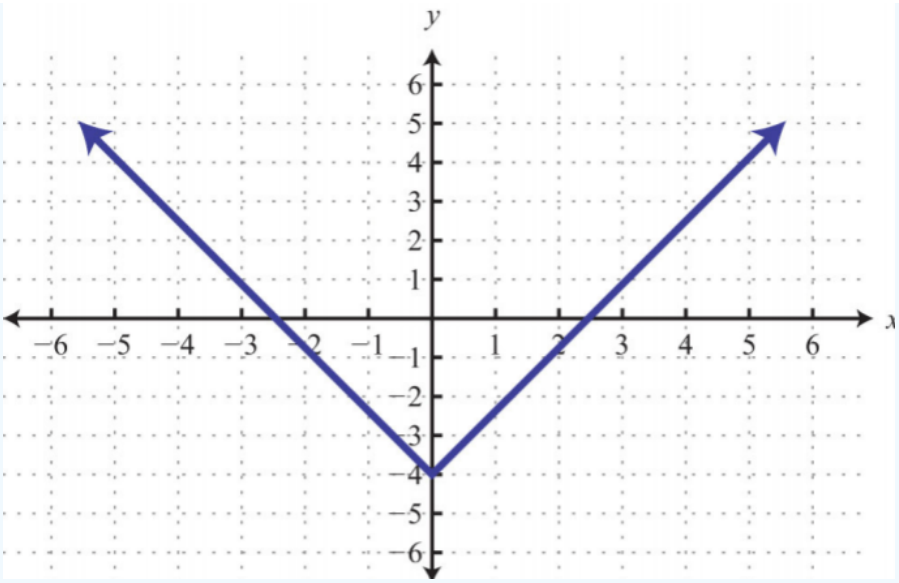


Figure 3.E.28

Answer available at the end of this section.

Exercise $3.E.18$ Introduction to Functions

Given the following,

1. $f(x) = 9x - 4$, find $f(-1)$.
2. $f(x) = -5x + 1$, find $f(-3)$.
3. $g(x) = \frac{1}{2}x - \frac{1}{3}$, find $g(-\frac{1}{3})$.
4. $g(x) = -\frac{3}{4}x + \frac{1}{3}$, find $g(\frac{2}{3})$.
5. $f(x) = 9x - 4$, find x when $f(x) = 0$.
6. $f(x) = -5x + 1$, find x when $f(x) = 2$.
7. $g(x) = \frac{1}{2}x - \frac{1}{3}$, find x when $g(x) = 1$.
8. $g(x) = -\frac{3}{4}x + \frac{1}{3}$, find x when $g(x) = -1$.

Answer available at the end of this section.

Exercise $3.E.19$ Introduction to Functions

Given the graph of a function $f(x)$, determine the following.

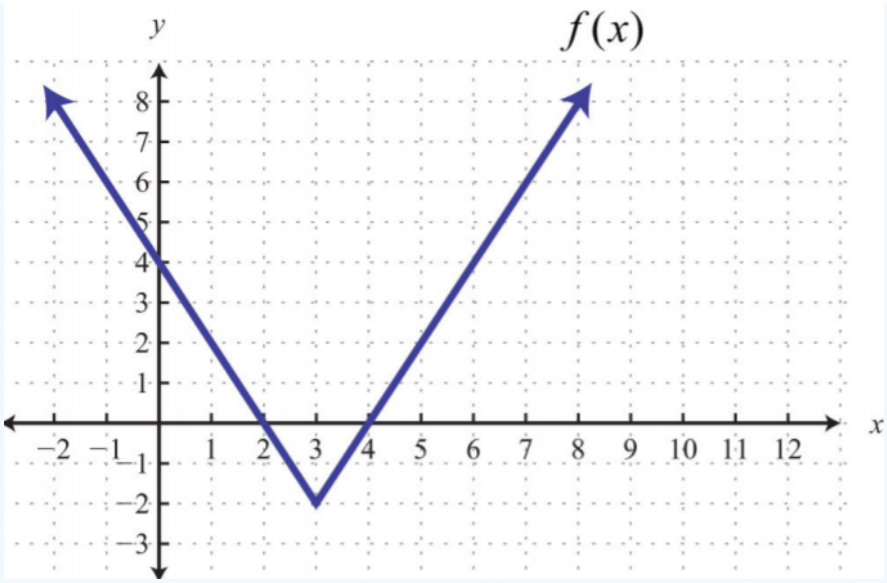


Figure 3.E.29

1. $f(3)$
2. x when $f(x) = 4$

Answer available at the end of this section.

Exercise $3.E.20$ Linear Inequalities (Two Variables)

Is the ordered pair a solution to the given inequality?

1. $6x - 2y \leq 1$; $(-3, -7)$
2. $-3x + y > 2$; $(0, 2)$
3. $6x - 10y < -1$; $(5, -3)$
4. $x - \frac{1}{3}y > 0$; $(1, 4)$
5. $y > 0$; $(-3, -1)$
6. $x \leq -5$; $(-6, 4)$

Answer available at the end of this section.

Exercise $3.E.21$ Linear Inequalities (Two Variables)

Graph the solution set.

1. $y \geq -2x + 1$
2. $y < 3x - 4$
3. $-x + y \leq 3$
4. $\frac{5}{2}x + \frac{1}{2}y \leq 2$
5. $3x - 5y > 0$
6. $y > 0$

Answer available at the end of this section.

Sample Exam

Exercise $3.E.22$

1. Graph the points $(-4,-2),(-4,1)$, and $(0,-2)$ on a rectangular coordinate plane. Connect the points and calculate the area of the shape.
2. Is $(-2,4)$ a solution to $3x-4y=-10$? Justify your answer.

Answer available at the end of this section.

Exercise $3.E.23$

Given the set of x-values $\{-2,-1,0,1,2\}$, find the corresponding y-values and graph the following.

1. $y=x-1$
2. $y=-x+1$
3. On the same set of axes, graph $y=4$ and $x=-3$. Give the point where they intersect.

Answer available at the end of this section.

Exercise $3.E.24$

Find the x- and y-intercepts and use those points to graph the following.

1. $2x-y=8$
2. $12x+5y=15$
3. Calculate the slope of the line passing through $(-4,-5)$ and $(-3,1)$.

Answer available at the end of this section.

Exercise $3.E.25$

Determine the slope and y-intercept. Use them to graph the following.

1. $y=-\frac{3}{2}x+6$
2. $5x-2y=6$
3. Given $m=-3$, determine $m_{\perp}$.
4. Are the given lines parallel, perpendicular, or neither? $\begin{cases}-2x+3y=-12\\4x-6y=30\end{cases}$
5. Determine the slope of the given lines.
   a. $y=-2$
   b. $x=\frac{1}{3}$
   c. Are these lines parallel, perpendicular, or neither?
6. Determine the equation of the line with slope $m=-\frac{3}{4}$ passing through $(8,1)$.
7. Find the equation to the line passing through $(-2,3)$ and $(4,1)$.
8. Find the equation of the line parallel to $5x-y=6$ passing through $(-1,-2)$.
9. Find the equation of the line perpendicular to $-x+2y=4$ passing through $(\frac{1}{2},5)$.

Answer available at the end of this section.

Exercise $3.E.26$

Given a linear function $f(x) = -\frac{4}{5}x + 2$, determine the following.

1. $f(10)$
2. x when $f(x) = 0$
3. Graph the solution set: $3x - 4y > 4$.
4. Graph the solution set: $y - 2x \geq 0$.
5. A rental car company charges $\$32.00$ plus $\$0.52$ per mile driven. Write an equation that gives the cost of renting the car in terms of the number of miles driven. Use the formula to determine the cost of renting the car and driving it 46 miles.
6. A car was purchased new for $\$12,000$ and was sold 5 years later for $\$7,000$. Write a linear equation that gives the value of the car in terms of its age in years.
7. The area of a rectangle is 72 square meters. If the width measures 4 meters, then determine the length of the rectangle.

Answer available at the end of this section.

Answers

Exercise $3.E.1$ Rectangular Coordinate System

1.

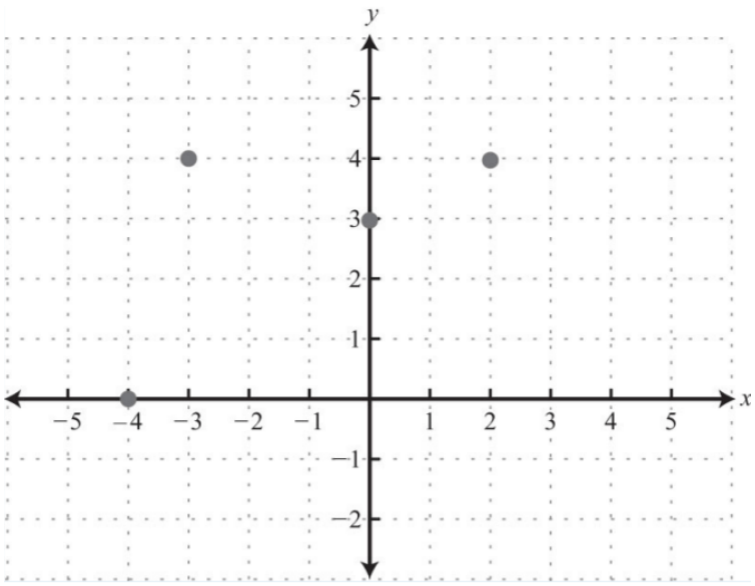


Figure 3.E.1

3. Area: 24 square units

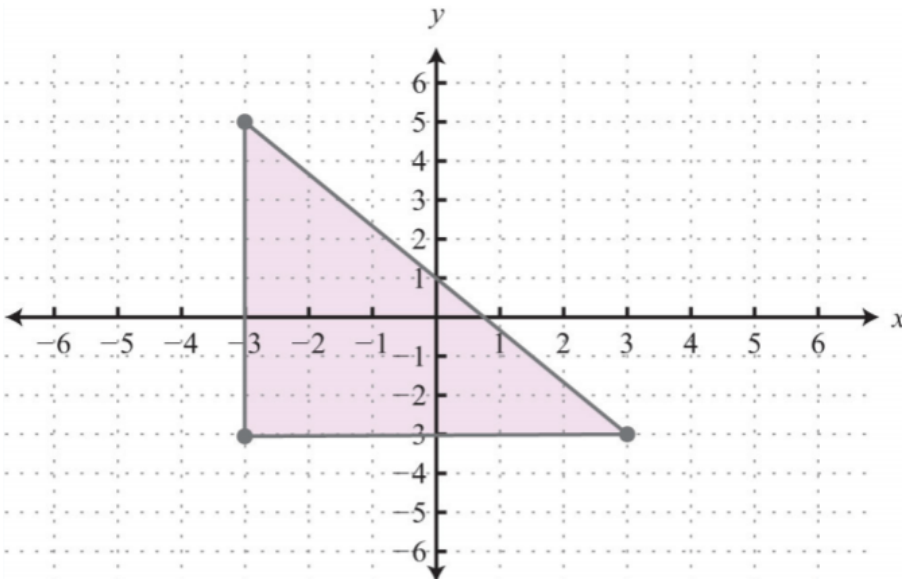


Figure 3.E.2

5. Perimeter: 16 units

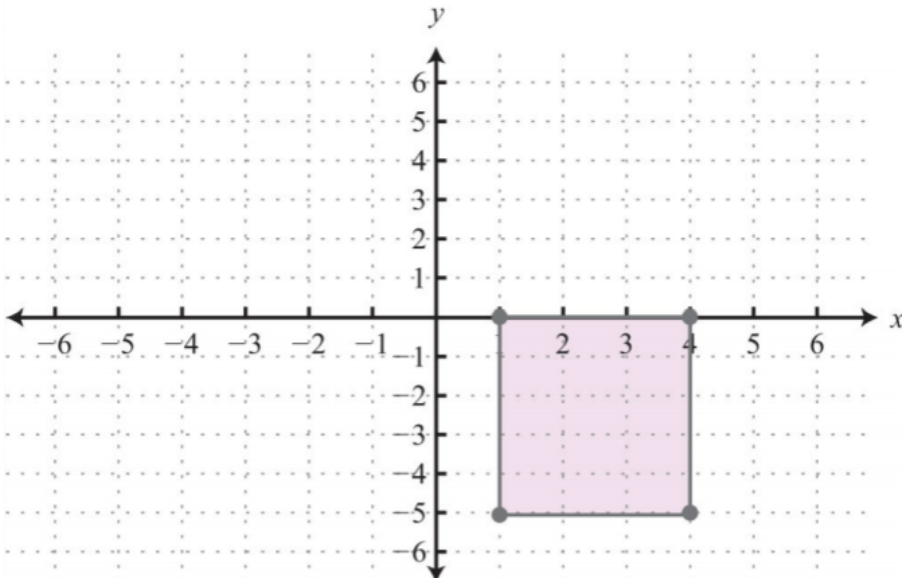


Figure 3.E.3

Exercise $3.E.2$ **Rectangular Coordinate System**

1. 10 units
3. $5\sqrt{2}$ units

Exercise $3.E.3$ **Rectangular Coordinate System**

1. $(2, -2)$
3. $(\frac{1}{2}, -\frac{3}{2})$
5. Answers may vary

Exercise $3.E.4$ **Graph by Plotting Points**

1. No
3. Yes

Exercise $3.E.5$ **Graph by Plotting Points**

1.

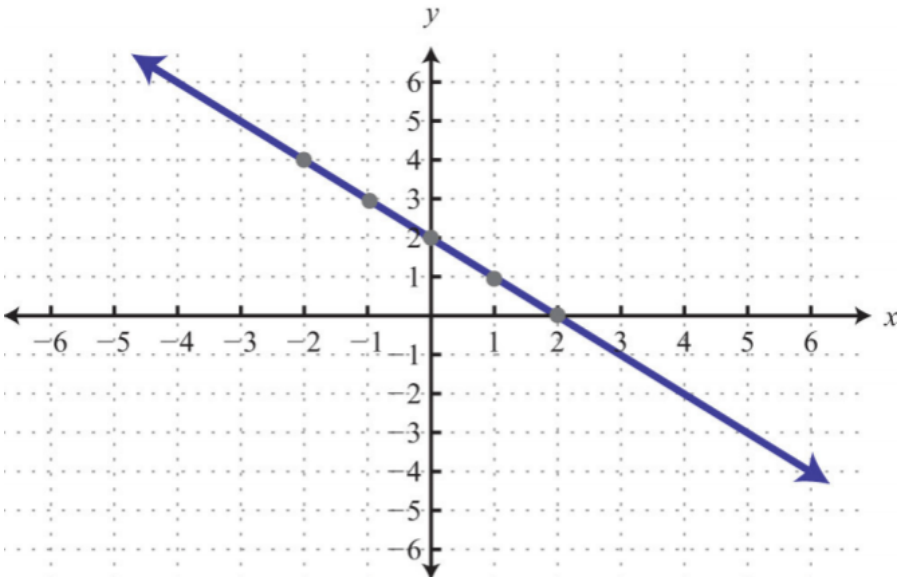


Figure 3.E.4

3.

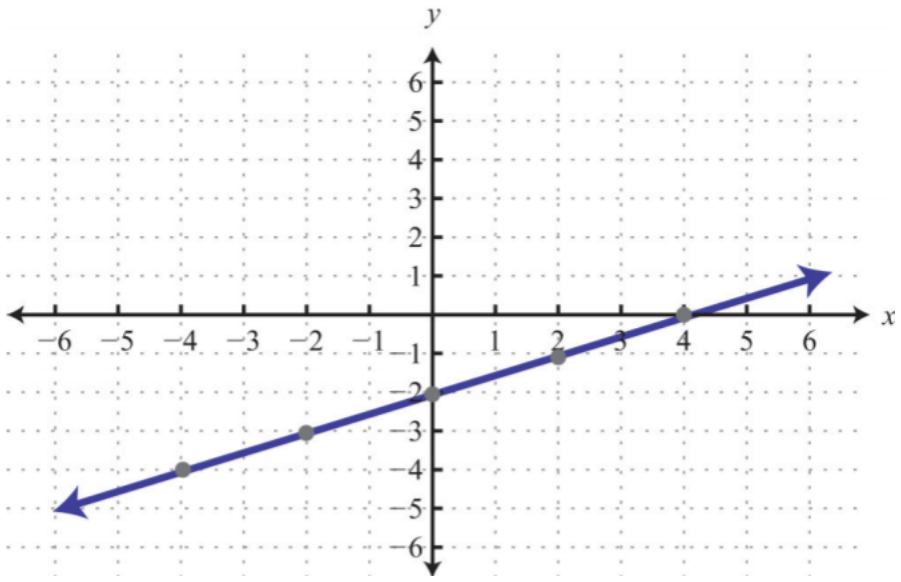


Figure 3.E.5

5.

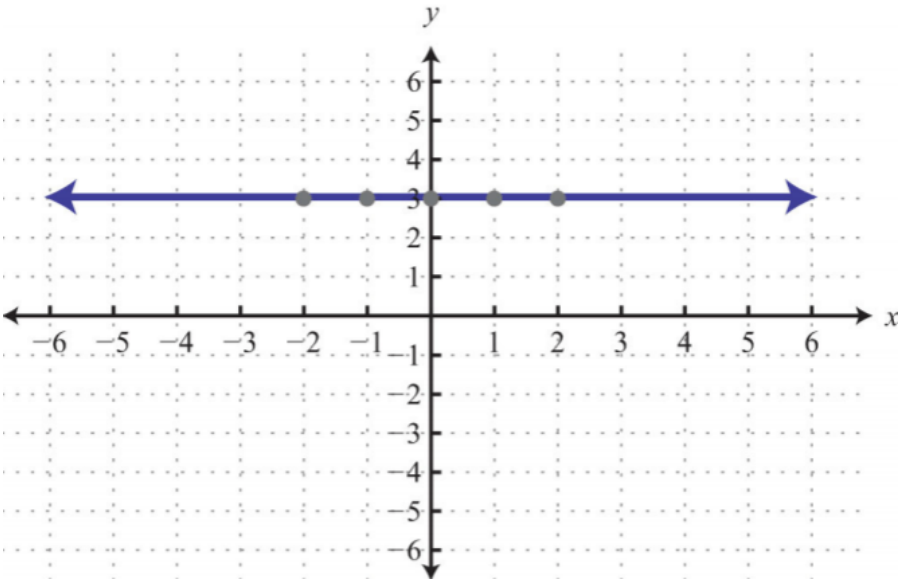


Figure 3.E.6

7.

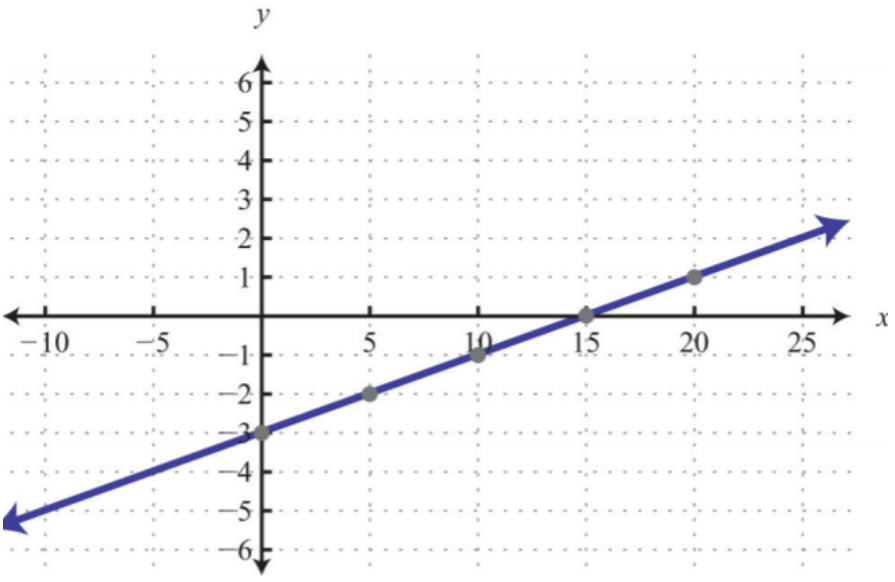


Figure 3.E.7

Exercise $3.E.6$ **Graph Using Intercepts**

1. y-intercept: $(0, -2)$; x-intercept: $(-4, 0)$
3. y-intercept: none; x-intercept: $(5, 0)$

Exercise $3.E.7$ **Graph Using Intercepts**

1.

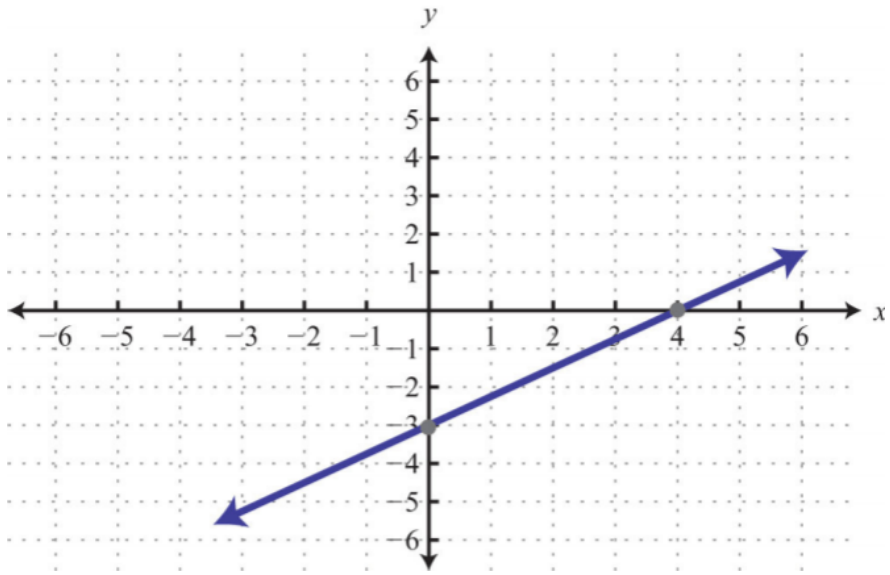


Figure 3.E.12

3.

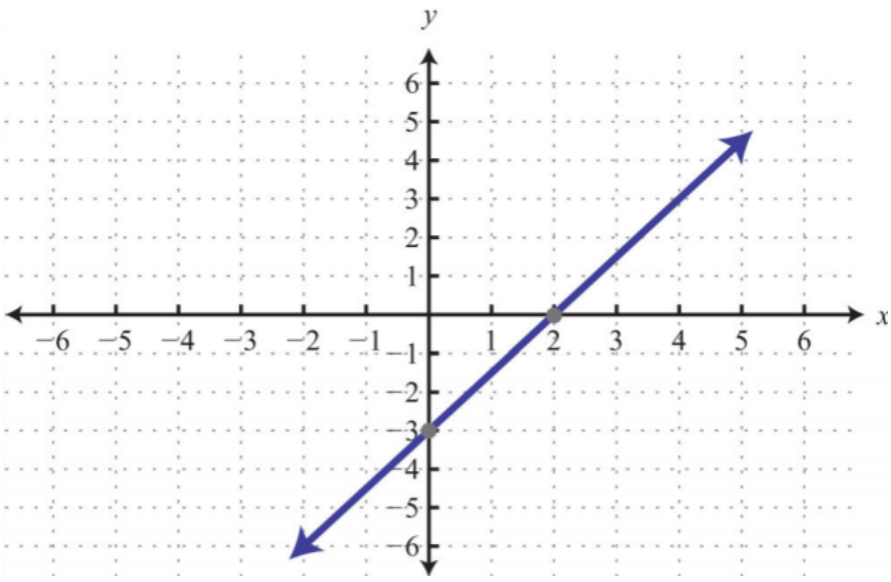


Figure 3.E.13

5.

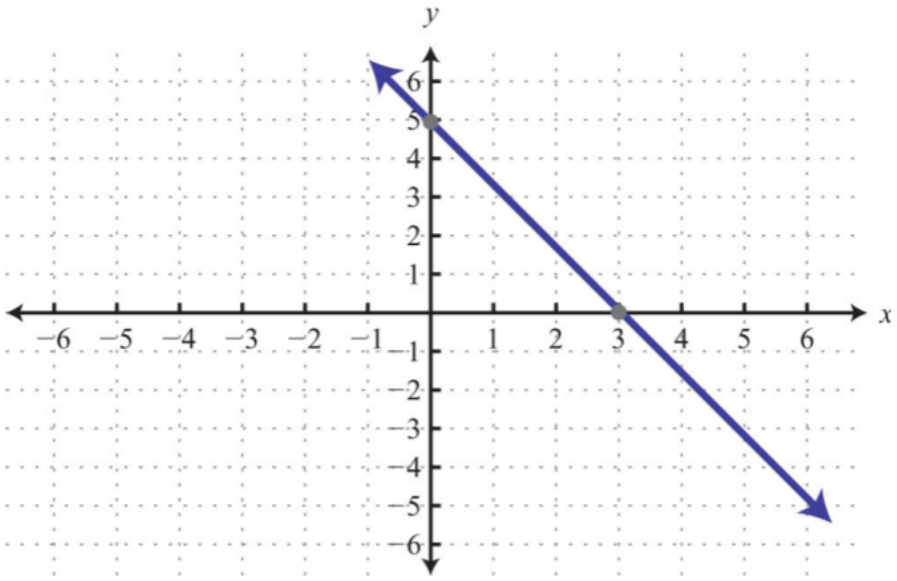


Figure 3.E.14

Exercise $3.E.8$ **Graph Using the** y**-Intercept and Slope**

1. y-intercept: $(0, 1)$; slope: -2

Exercise $3.E.9$ **Graph Using the** y**-Intercept and Slope**

1. $-\frac{7}{4}$
3. $-\frac{4}{3}$

Exercise $3.E.10$ **Graph Using the** y**-Intercept and Slope**

1. $y = 3x - 2$; slope: 3; y-intercept $(0, -2)$
3. $y = \frac{4}{9}x + \frac{4}{3}$; slope: $\frac{4}{9}$; y-intercept $(0, \frac{4}{3})$

Exercise $3.E.11$ **Graph Using the** y**-Intercept and Slope**

1.

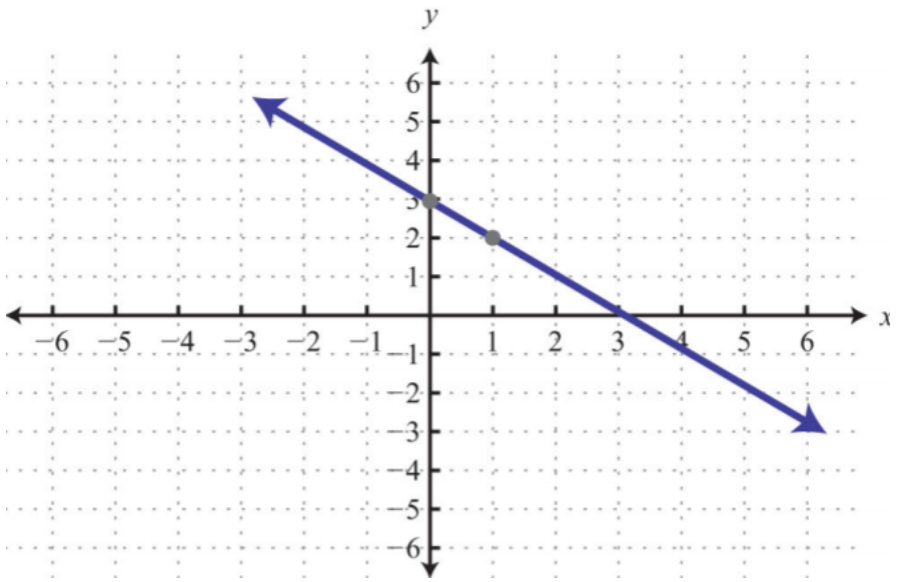


Figure 3.E.17

3.

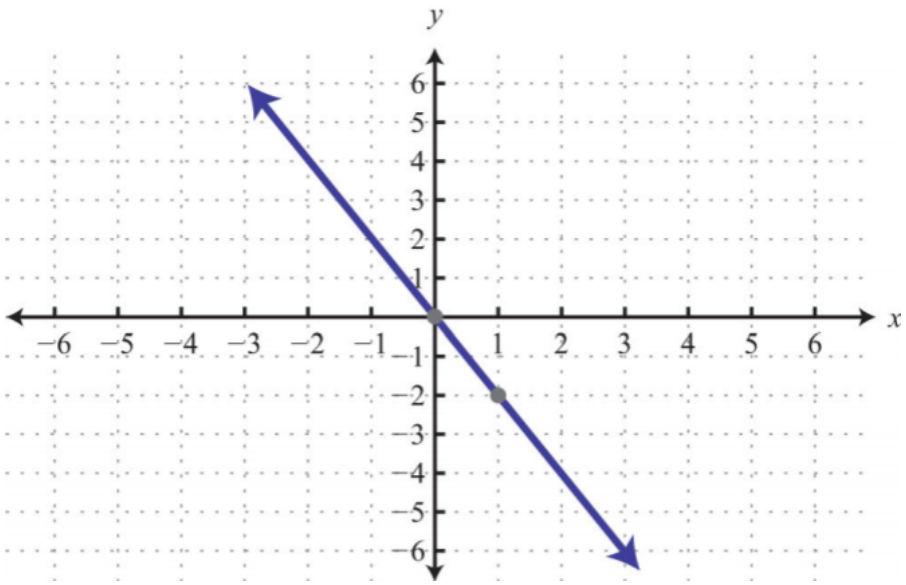


Figure 3.E.18

5.

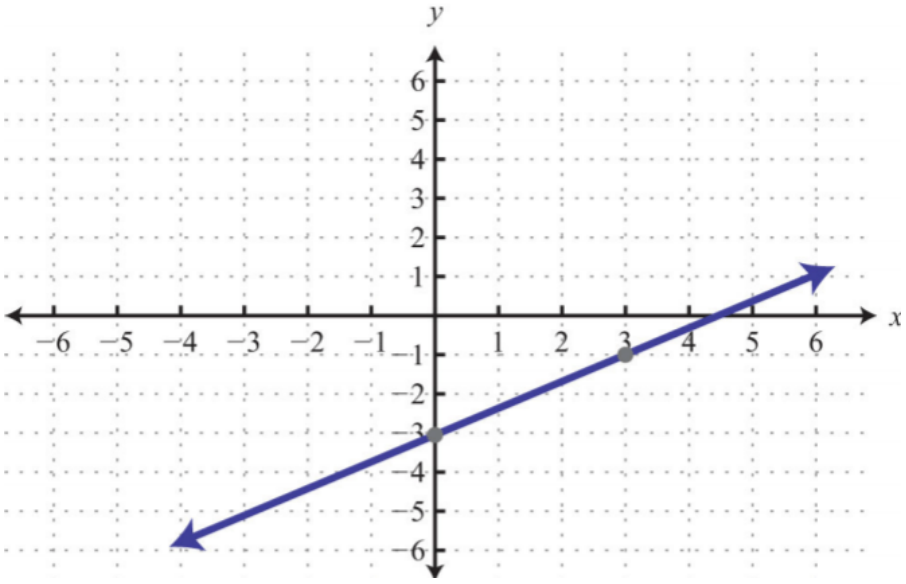


Figure 3.E.19

7.

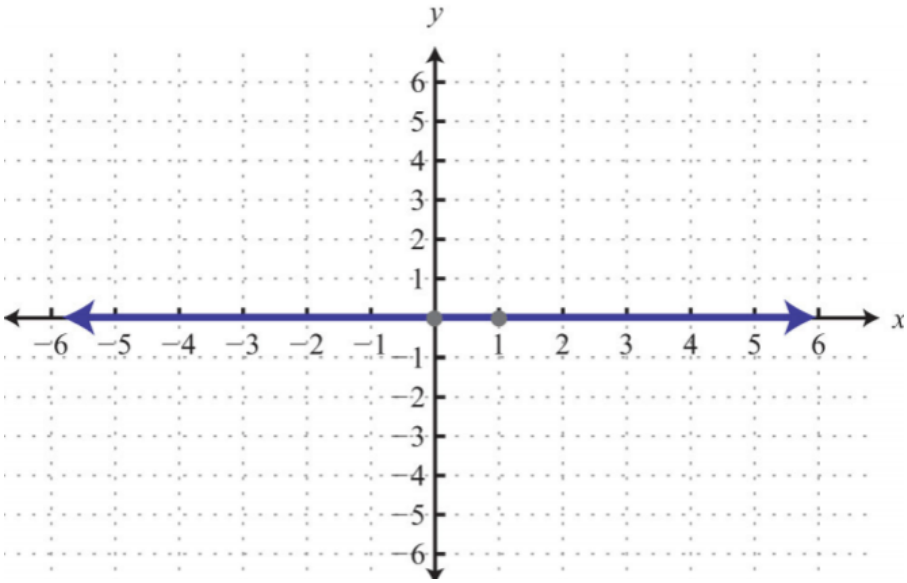


Figure 3.E.20

Exercise $3.E.12$ **Finding Linear Equations**

1. $y = -2x + 1$
3. $y = -5$

Exercise $3.E.13$ **Finding Linear Equations**

1. $y = \frac{1}{2}x + 10$
3. $y = \frac{2}{3}x - \frac{8}{3}$

Exercise $3.E.14$ **Finding Linear Equations**

1. $y = \frac{4}{5}x - 1$
3. $y = -\frac{3}{4}x + \frac{1}{2}$
5. $y = -6$

Exercise $3.E.15$ **Parallel and Perpendicular Lines**

1. Parallel
3. Perpendicular

Exercise $3.E.16$ **Parallel and Perpendicular Lines**

1. $y = 5x + 49$
3. $y = -\frac{3}{4}x + 5$
5. $y = -1$

Exercise $3.E.17$ **Introduction to Functions**

1. Domain: $\{-10, -5, 5\}$; range: $\{-1, 2\}$; function: yes
3. Domain: R; range: R; function: yes
5. Domain: $[-3, \infty)$; range: R; function: no

Exercise $3.E.18$ **Introduction to Functions**

1. $f(-1) = -13$
3. $g(-\frac{1}{3}) = -\frac{1}{2}$
5. $x = \frac{4}{9}$
7. $x = \frac{8}{3}$

Exercise $3.E.19$ **Introduction to Functions**

1. $f(3) = -2$

Exercise $3.E.20$ **Linear Inequalities (Two Variables)**

1. Yes
3. No
5. Yes

Exercise $3.E.21$ **Linear Inequalities (Two Variables)**

1.

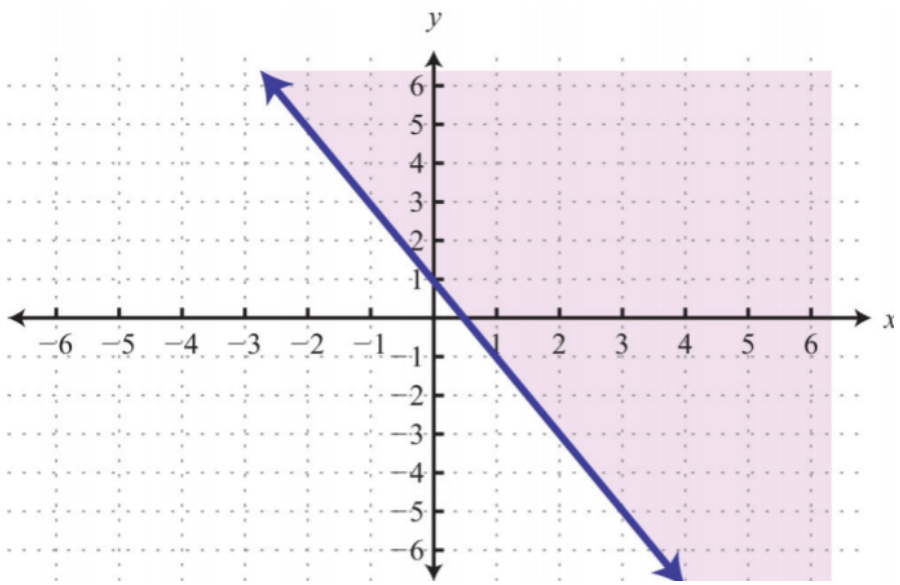


Figure 3.E.30

3.

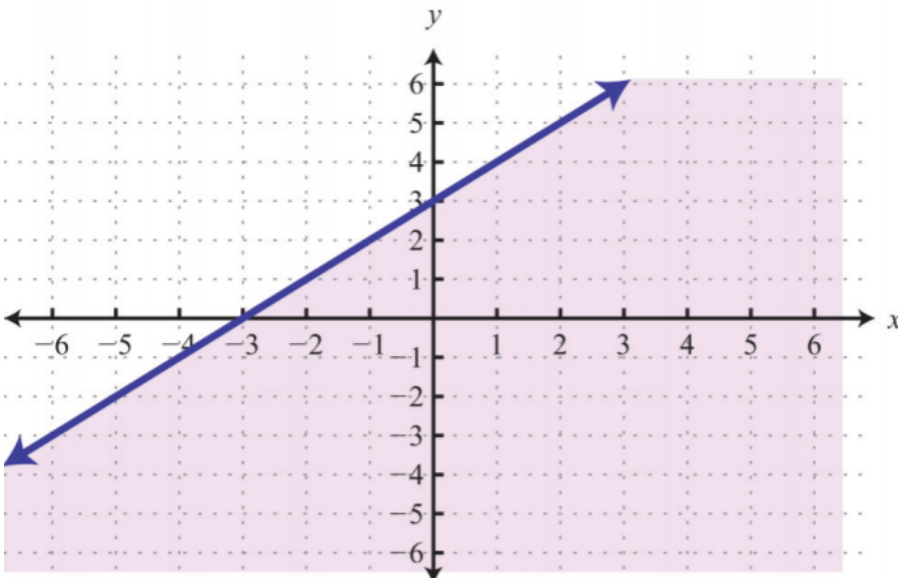


Figure 3.E.31

5.

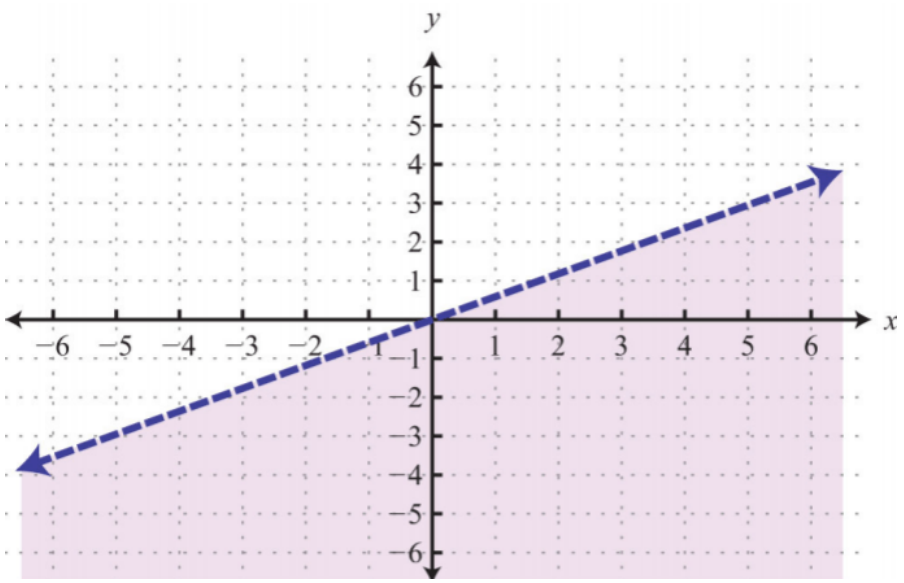


Figure 3.E.32

Exercise $3.E.22$

1. Area: 6 square units

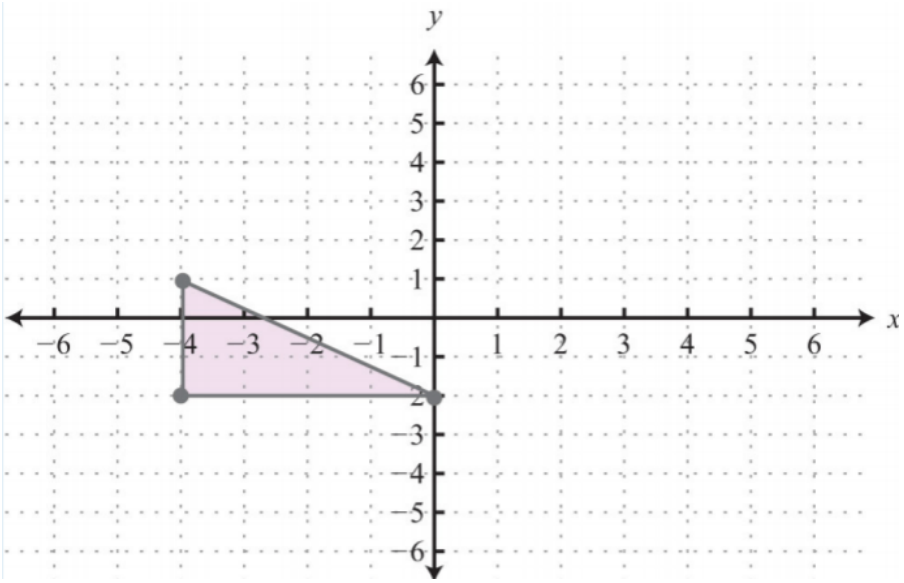


Figure 3.E.33

Exercise $3.E.23$

1.

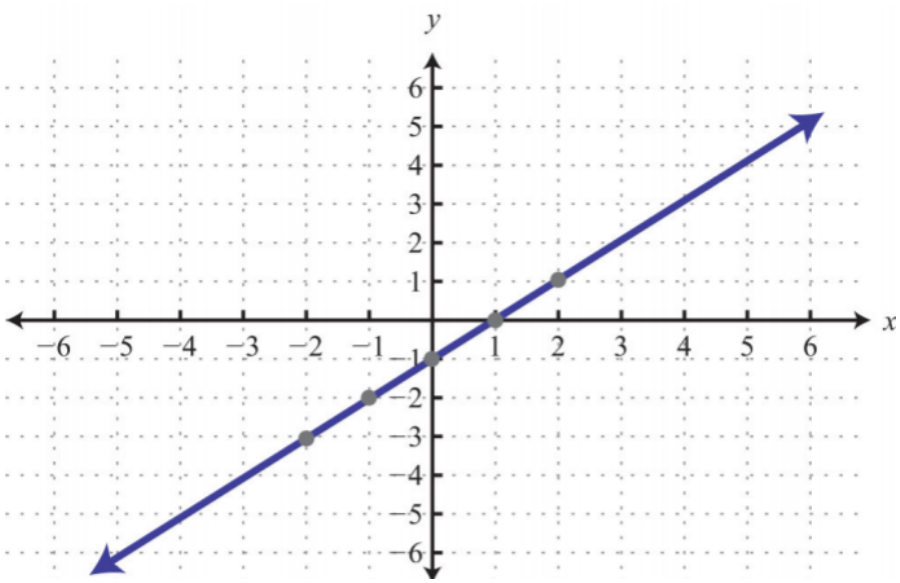


Figure 3.E.34

3. Intersection: $(-3, 4)$

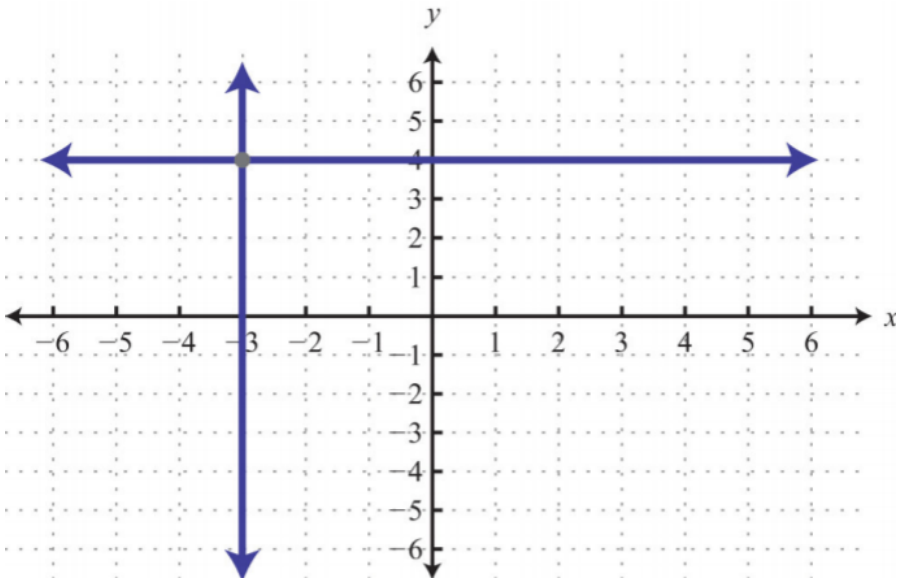


Figure 3.E.35

Exercise $3.E.24$

2.

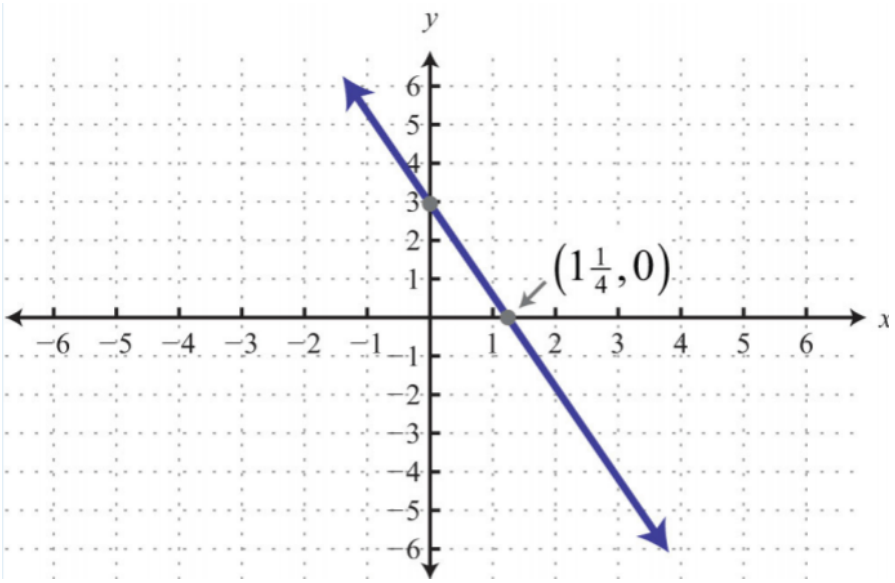


Figure 3.E.36

Exercise $3.E.25$

1. Slope: $-\frac{3}{2}$; y-intercept: $(0, 6)$

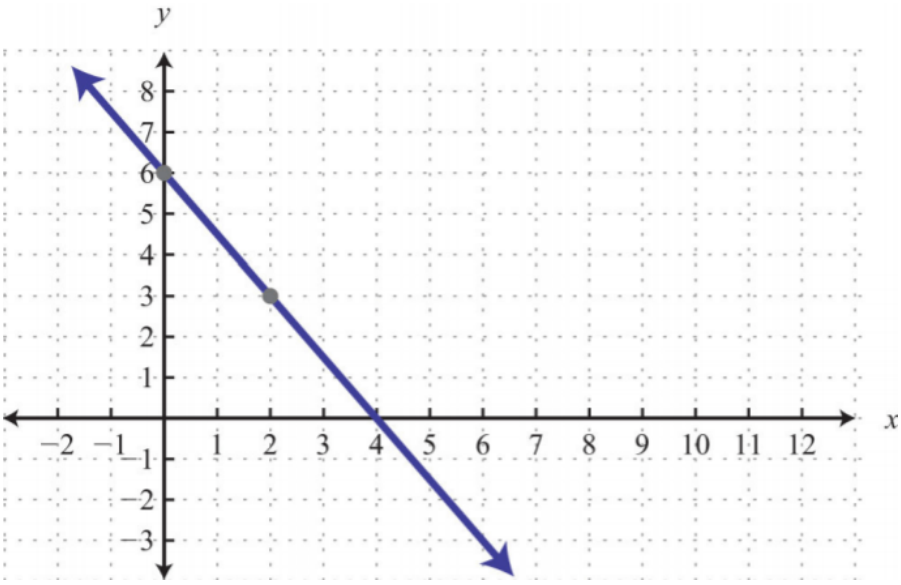


Figure 3.E.37

3. $m_{\perp} = \frac{1}{3}$
5. a. 0; b. Undefined; c. Perpendicular
6. $y = -\frac{3}{4}x + 7$
8. $y = 5x + 3$

Exercise $3.E.26$

1. $f(10) = -6$
3.

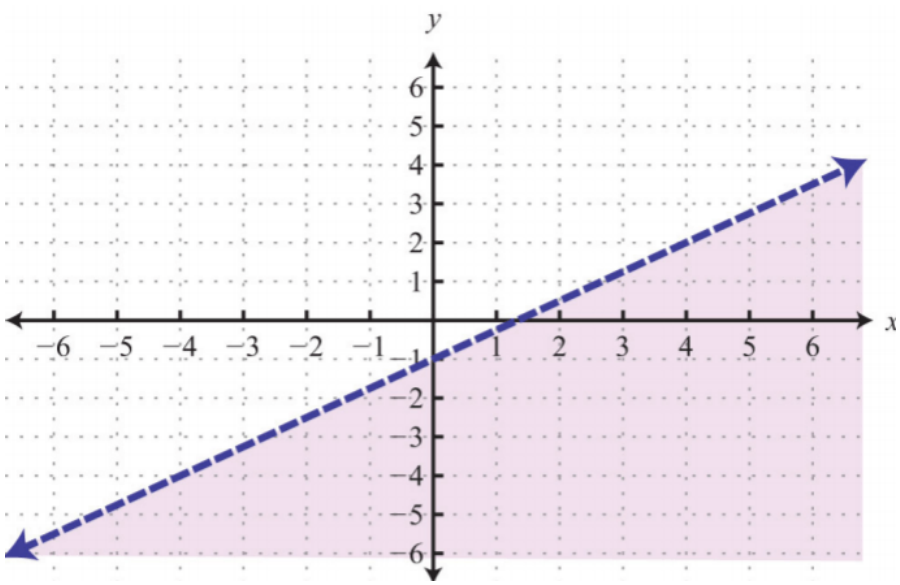


Figure 3.E.38

5. cost $= 0.52x + 32$; \$55.92
7. 18 meters

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4: SOLVING LINEAR SYSTEMS

4.1: Solving Linear Systems by Graphing

4.2: Solving Linear Systems by Substitution

4.3: Solving Linear Systems by Elimination

4.4: Applications of Linear Systems

4.5: Solving Systems of Linear Inequalities (Two Variables)

4.E: Review Exercises and Sample Exam

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4.1: Solving Linear Systems by Graphing

Learning Objectives

- Check solutions to systems of linear equations.
- Solve linear systems using the graphing method.
- Identify dependent and inconsistent systems.

Definition of a Linear System

Real-world applications are often modeled using more than one variable and more than one equation. A system of equations consists of a set of two or more equations with the same variables. In this section, we will study linear systems consisting of two linear equations each with two variables. For example,

A solution to a linear system, or simultaneous solution, to a linear system is an ordered pair (x, y) that solves both of the equations. In this case, $(3, 2)$ is the only solution. To check that an ordered pair is a solution, substitute the corresponding x-and y-values into each equation and then simplify to see if you obtain a true statement for both equations.

$Check:\ (3, 2)$

$$\begin{array}{ll} Equation\ 1: \ 2x - 37 = 0 & Equation\ 2 \ -4x + 2y = -8 \\ 2(3) - 3(2) = 0 & -4(3) + 2(2) = -8 \\ 6 - 6 = 0 & -12 + 4 = -8 \\ 0 = 0 \quad \checkmark & -8 = -8 \quad \checkmark \end{array}$$

Example 4.1.1

Determine whether $(1, 0)$ is a solution to the system

$$\begin{cases} x - y = 1 \\ -2x + 3y = 5 \end{cases}.$$

Solution:

Substitute the appropriate values into both equations.

$$Check:\ (1, 0)$$

$$\begin{array}{ll} Equation\ 1: \ x - y = 1 & Equation\ 2: \ -2x + 3y = 5 \\ (1) - (0) = 1 & -2(1) + 3(0) = 5 \\ 1 - 0 = 1 & -2 + 0 = 5 \\ 1 = 1 \quad \checkmark & -2 = 5 \quad x \end{array}$$

Answer:

Since $(1, 0)$ does not satisfy both equations, it is not a solution.

Exercise 4.1.1

Is $(-2, 4)$ a solution to the system

$$\begin{aligned} x - y &= -6 \\ -2x + 3y &= 16 \end{aligned}$$?

Answer available at the end of this section.

Solve by Graphing

Geometrically, a linear system consists of two lines, where a solution is a point of intersection. To illustrate this, we will graph the following linear system with a solution of $(3, 2)$:

First, rewrite the equations in slope-intercept form so that we may easily graph them.

$$\begin{aligned} 2x - 3y &= 0 \\ 2x - 3y - 2x &= 0 - 2x \\ -3y &= -2x \\ \frac{-3y}{-3} &= \frac{-2x}{-3} \\ y &= \frac{2}{3}x \end{aligned} \qquad \begin{aligned} -4x + 2y &= -8 \\ -4x + 2y + 4x &= -8 + 4x \\ 2y &= 4x - 8 \\ \frac{2y}{2} &= \frac{4x-8}{2} \\ y &= 2x - 4 \end{aligned}$$

Next, replace these forms of the original equations in the system to obtain what is called an equivalent system. Equivalent systems share the same solution set.

$$\begin{aligned} y &= \frac{2}{3}x \\ y &= 2x - 4 \end{aligned} \qquad \begin{aligned} &\textit{Original system} \\ &\textit{Equivalent system} \end{aligned}$$

If we graph both of the lines on the same set of axes, then we can see that the point of intersection is indeed $(3, 2)$, the solution to the system.

Figure 4.1.1

To summarize, linear systems described in this section consist of two linear equations each with two variables. A solution is an ordered pair that corresponds to a point where the two lines in the rectangular coordinate plane intersect. Therefore, we can solve linear systems by graphing both lines on the same set of axes and determining the point where they cross. When graphing the lines, take care to choose a good scale and use a straightedge to draw the line through the points; accuracy is very important here. The steps for solving linear systems using the graphing method are outlined in the following example.

Example 4.1.2

Solve by graphing:

$$\begin{cases} x-y=-4 \\ 2x+y=1 \end{cases}.$$

Solution:

Step 1: Rewrite the linear equations in slope-intercept form.

$$\begin{array}{ll} x-y=-4 & 2x+y=1 \\ x-y\color{cyan}{-x}=-4\color{cyan}{-x} & 2x+y\color{cyan}{-2x}=1\color{cyan}{-2x} \\ -y=-x-4 & y=-2x+1 \\ \frac{-y}{\color{cyan}{-1}}=\frac{-x-4}{\color{cyan}{-1}} & \\ y=x+4 & \end{array}$$

Step 2: Write the equivalent system and graph the lines on the same set of axes.

$$\begin{array}{ll} \textit{Line } 1: y=x+4 & \textit{Line } 2\ y=-2x+1 \\ y-\textit{intercept}:\ (0,4) & y-\textit{intercept}:\ (0,1) \\ \textit{slope}:\ m=1=\frac{1}{1}=\frac{\textit{rise}}{\textit{run}} & \textit{slope}:\ m=-2=\frac{-2}{1}=\frac{\textit{rise}}{\textit{run}} \end{array}$$

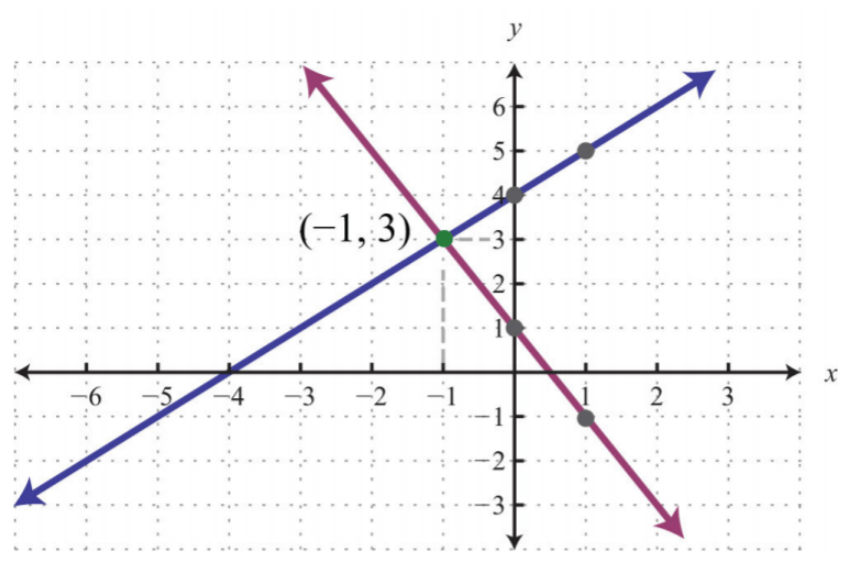


Figure 4.1.2

Step 3: Use the graph to estimate the point where the lines intersect and check to see if it solves the original system. In the above graph, the point of intersection appears to be $(-1,3)$.

Check :
$(-1, 3)$



Answer:

$(-1, 3)$

Example 4.1.3

Solve by graphing:

$$\begin{cases} 2x + y = 2 \\ -2x + 3y = -18 \end{cases}.$$

Solution:

We first solve each equation for y to obtain an equivalent system where the lines are in slope-intercept form.

Graph the lines and determine the point of intersection.

Figure 4.1.3

Answer:

Example 4.1.4

Solve by graphing:

$$\begin{cases} 3x + y = 6 \\ y = -3 \end{cases}$$

Solution:

Figure 4.1.4

Check : $(3, -3)$



Answer:

$(3, -3)$

The graphing method for solving linear systems is not ideal when the solution consists of coordinates that are not integers. There will be more accurate algebraic methods in sections to come, but for now, the goal is to understand the geometry involved when solving systems. It is important to remember that the solutions to a system correspond to the point, or points, where the graphs of the equations intersect.

Exercise 4.1.2

Solve by graphing:

$$\begin{cases} -x + y = 6 \\ 5x + 2y = -2 \end{cases}.$$

Answer available at the end of this section.

Dependent and Inconsistent Systems

Systems with at least one solution are called consistent systems. Up to this point, all of the examples have been of consistent systems with exactly one ordered pair solution. It turns out that this is not always the case. Sometimes systems consist of two linear equations that are equivalent. If this is the case, the two lines are the same and when graphed will coincide. Hence the solution set consists of all the points on the line. This is a dependent system. Given a consistent linear system with two variables, there are two possible results:

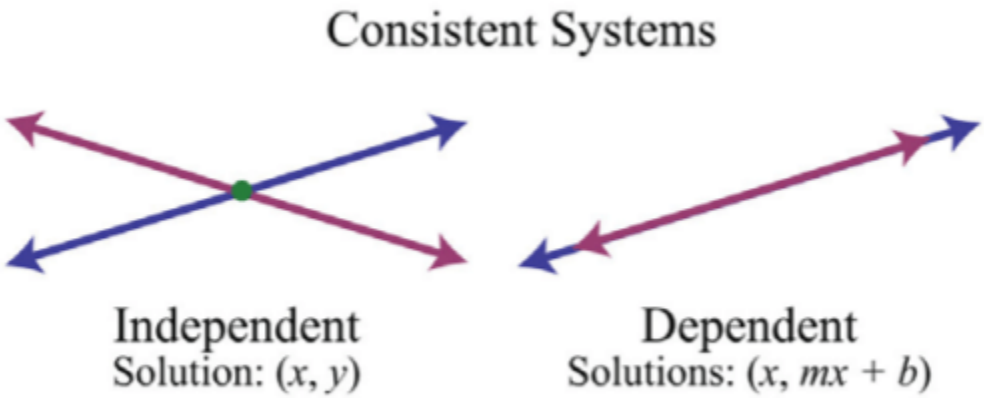


Figure 4.1.5

The solutions to independent systems are ordered pairs (x, y). We need some way to express the solution sets to dependent systems, since these systems have infinitely many solutions, or points of intersection. Recall that any line can be written in slope-intercept form, $y = mx + b$. Here, y depends on x. So we may express all the ordered pair solutions (x, y) in the form $(x, mx + b)$, where x is any real number.

Example 4.1.5

Solve by graphing:

$$\begin{cases} -2x+3y=-9 \\ 4x-6y=18 \end{cases}.$$

Solution:

Determine slope-intercept form for each linear equation in the system.

$$\begin{array}{cc} -2x+3y=-9 & 4x-6y=18 \\ -2x+3y+2x=-9+2x & 4x-6y-4x=18-4x \\ 3y=2x-9 & -6y=-4x+18 \\ \frac{3y}{3}=\frac{2x-9}{3} & \frac{-6y}{-6}=\frac{-4x+18}{-6} \\ y=\frac{2}{3}x-3 & y=\frac{-4}{-6}x+\frac{18}{-6} \\ & y=\frac{2}{3}x-3 \end{array}$$

In slope-intercept form, we can easily see that the system consists of two lines with the same slope and same y-intercept.

They are, in fact, the same line. And the system is dependent.

Figure 4.1.6

Answer:

In this example, it is important to notice that the two lines have the same slope and same y-intercept. This tells us that the two equations are equivalent and that the simultaneous solutions are all the points on the line $y=\frac{2}{3}x-3$. This is a dependent system, and the infinitely many solutions are expressed using the form $(x, mx+b)$. Other resources may express this set using set notation, $\{(x,y)|y=\frac{2}{3}x-3\}$, which reads "the set of all ordered pairs (x,y) such that y equals two-thirds x minus 3." Sometimes the lines do not cross and there is no point of intersection. Such systems have no solution, $\varnothing$, and are called inconsistent systems.

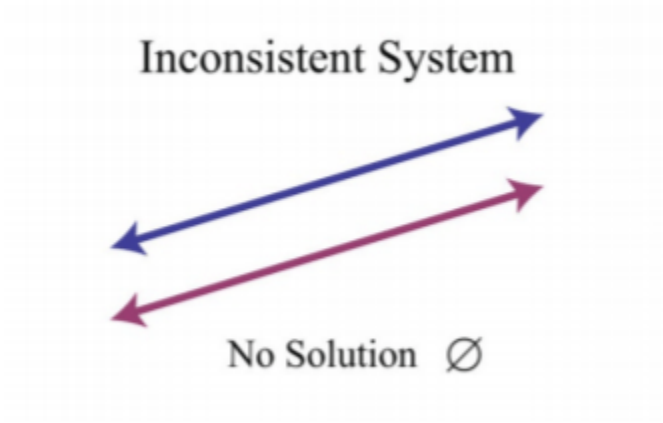


Figure 4.1.7

Example 4.1.6

Solve by graphing:

$$\begin{cases} -2x+5y=-15 \\ -4x+10y=10 \end{cases}.$$

Solution:

Determine slope-intercept form for each linear equation.

$$\begin{array}{cc} -2x+5y=-15 & -4x+10y=10 \\ -2x+5y+2x=-15+2x & -4x+10y+4x=10+4x \\ 5y=2x-15 & 10y=4x+10 \\ \frac{5y}{5}=\frac{2x-15}{5} & \frac{10y}{10}=\frac{4x+10}{10} \\ y=\frac{2}{5}x-3 & y=\frac{2}{5}x+1 \end{array}$$

In slope-intercept form, we can easily see that the system consists of two lines with the same slope and different y-intercepts.

Therefore, they are parallel and will never intersect.

Figure 4.1.8

Answer:

There is no simultaneous solution, $\varnothing$.

Exercise 4.1.3

Solve by graphing:

$$\begin{cases} x+y=-1 \\ -2x-2y=2 \end{cases}.$$

Answer available at the end of this section.

Key Takeaways

- In this section, we limit our study to systems of two linear equations with two variables. Solutions to such systems, if they exist, consist of ordered pairs that satisfy both equations. Geometrically, solutions are the points where the graphs intersect.
- The graphing method for solving linear systems requires us to graph both of the lines on the same set of axes as a means to determine where they intersect.
- The graphing method is not the most accurate method for determining solutions, particularly when the solutions have coordinates that are not integers. It is a good practice to always check your solutions.
- Some linear systems have no simultaneous solution. These systems consist of equations that represent parallel lines with different y-intercepts and do not intersect in the plane. They are called inconsistent systems and the solution set is the empty set, $\varnothing$.
- Some linear systems have infinitely many simultaneous solutions. These systems consist of equations that are equivalent and represent the same line. They are called dependent systems and their solutions are expressed using the notation $(x, mx+b)$, where x is any real number.

Exercise 4.1.4 Solutions to Linear Systems

Determine whether the given ordered pair is a solution to the given system.

1. $(3,-2); \begin{cases} x+y=-1 \\ -2x-2y=2 \end{cases}$
2. $(-5,0); \begin{cases} x+y=-1 \\ -2x-2y=2 \end{cases}$
3. $(-2,-6); \begin{cases} -x+y=-4 \\ 3x-y=-12 \end{cases}$
4. $(2,-7); \begin{cases} 3x+2y=-8 \\ -5x-3y=11 \end{cases}$
5. $(0,-3); \begin{cases} 5x-5y=15 \\ -13x+2y=-6 \end{cases}$
6. $(-12,14); \begin{cases} x+y=-14 \\ -2x-4y=0 \end{cases}$
7. $(\frac{3}{4},\frac{1}{4}); \begin{cases} -x-y=-1 \\ -4x-8y=5 \end{cases}$
8. $(-3,4); \begin{cases} \frac{1}{3}x+\frac{1}{2}y=1 \\ \frac{2}{3}x-\frac{3}{2}y=-8 \end{cases}$
9. $(-5,-3); \begin{cases} y=-35x-10 \\ y=5 \end{cases}$
10. $(4,2); \begin{cases} x=4-7 \\ x+4y=8 \end{cases}$

Answer available at the end of this section.

Exercise 4.1.5 Solutions to Linear Systems

Given the graph, determine the simultaneous solution.

1.

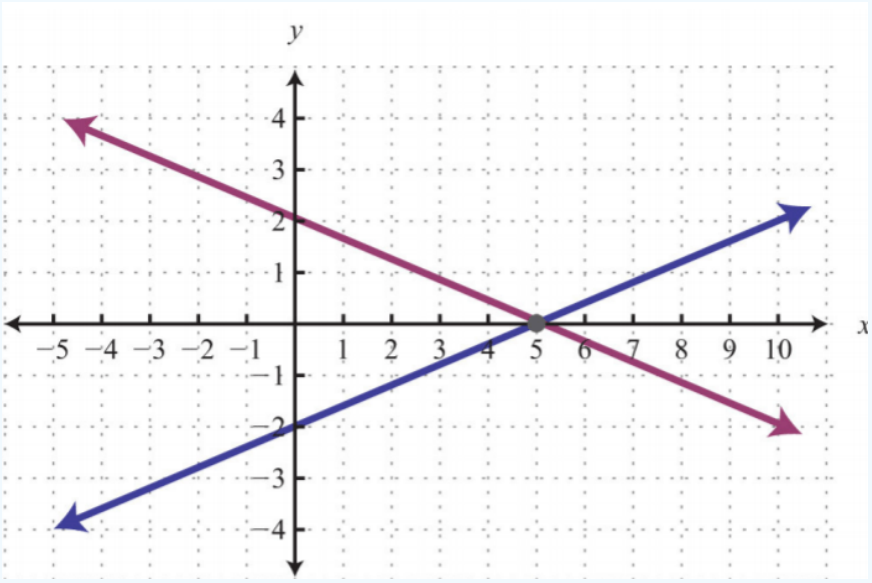


Figure 4.1.9

2.

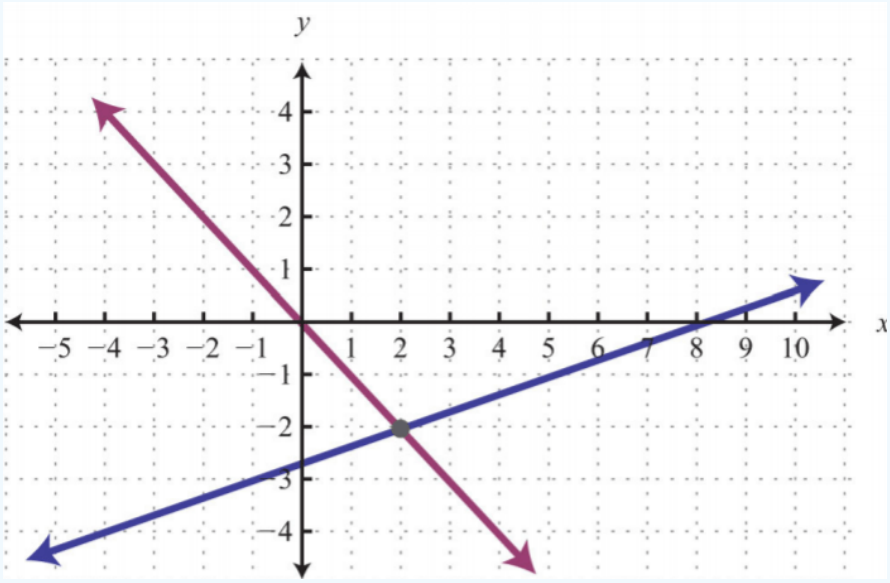


Figure 4.1.10

3.

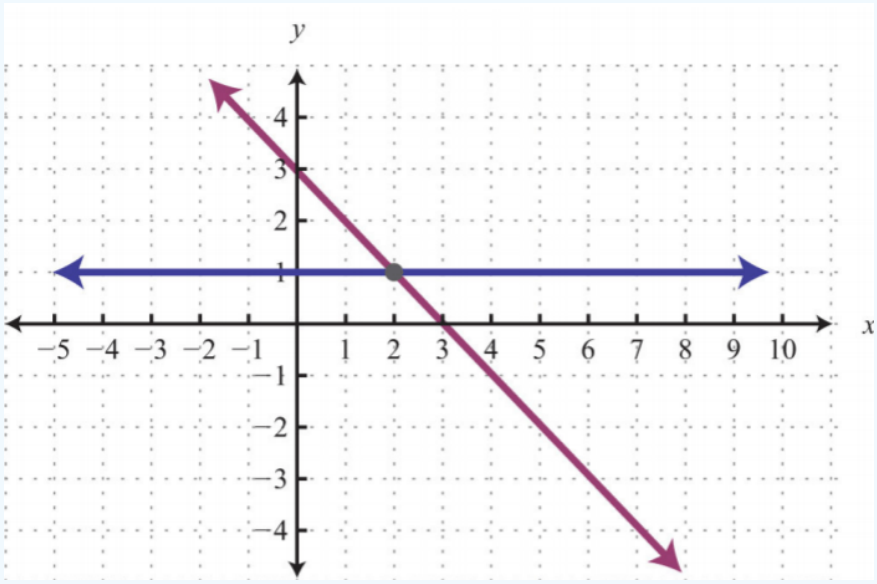


Figure 4.1.11

4.

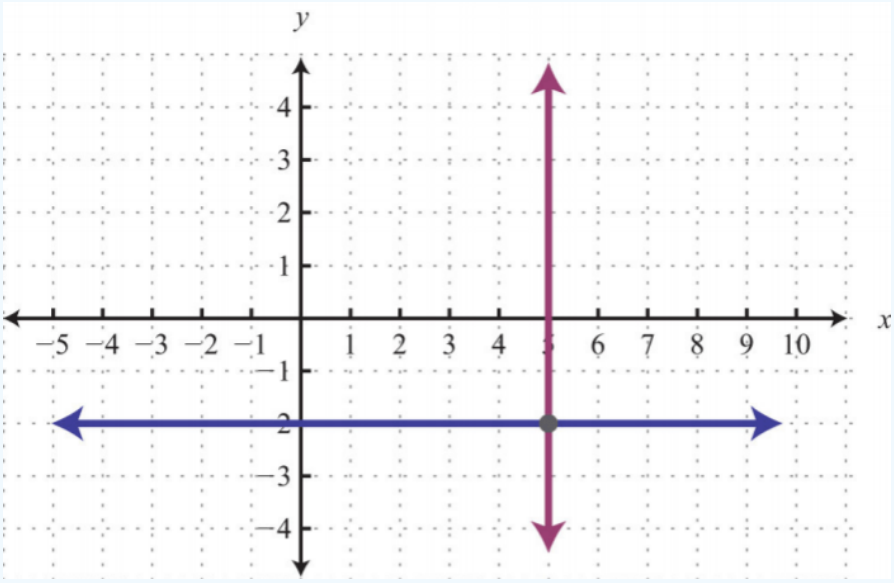


Figure 4.1.12

5.

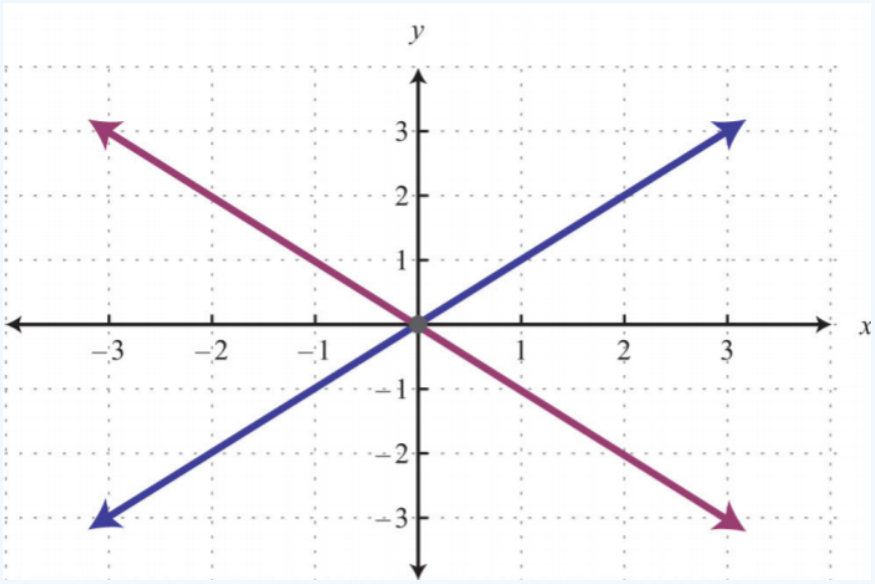


Figure 4.1.13

6.

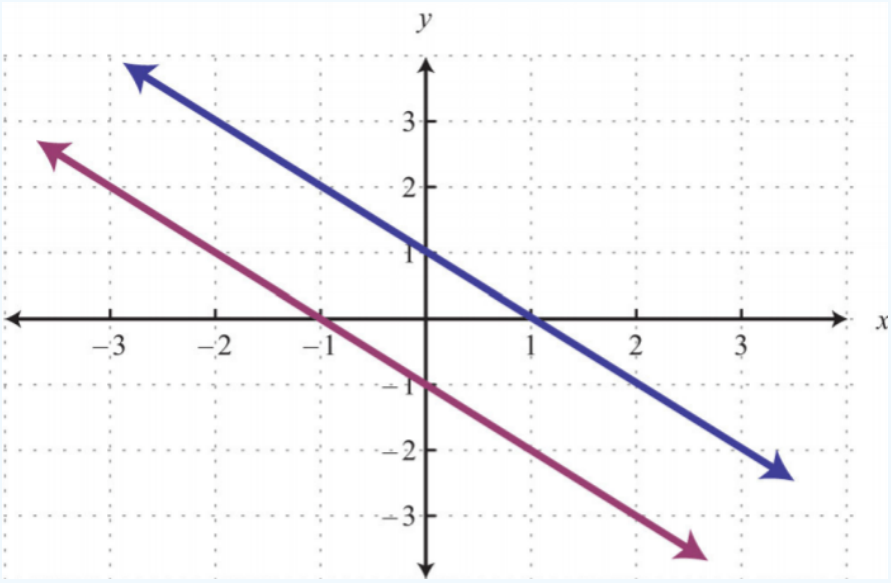


Figure 4.1.14

7.

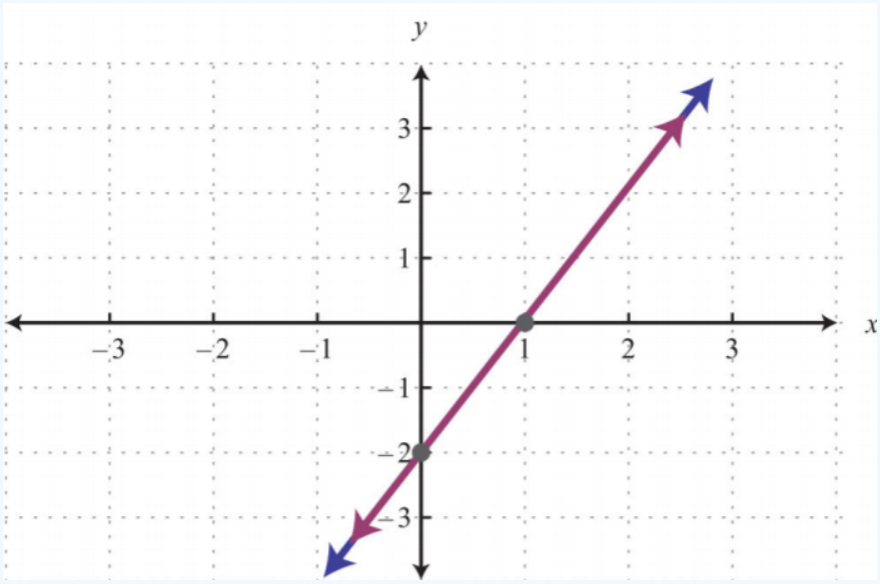


Figure 4.1.15

8.

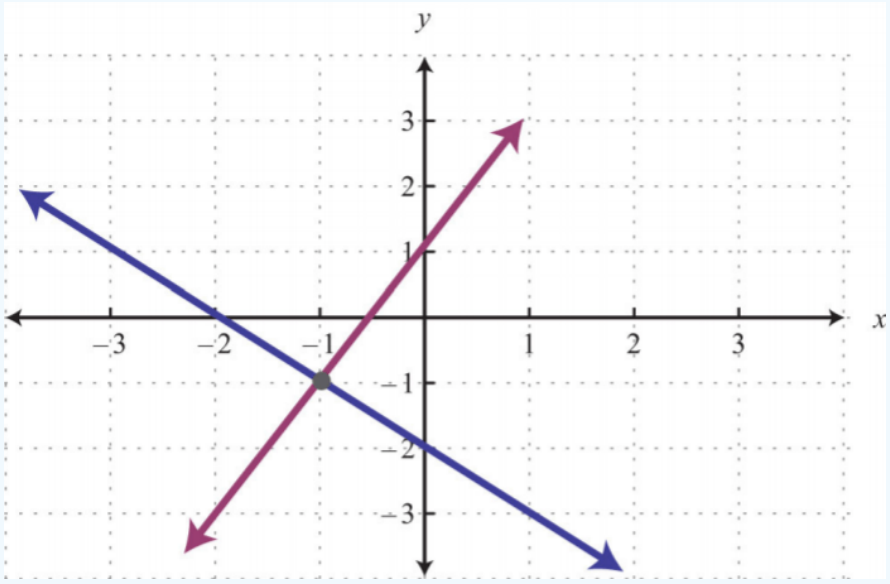


Figure 4.1.16

9.

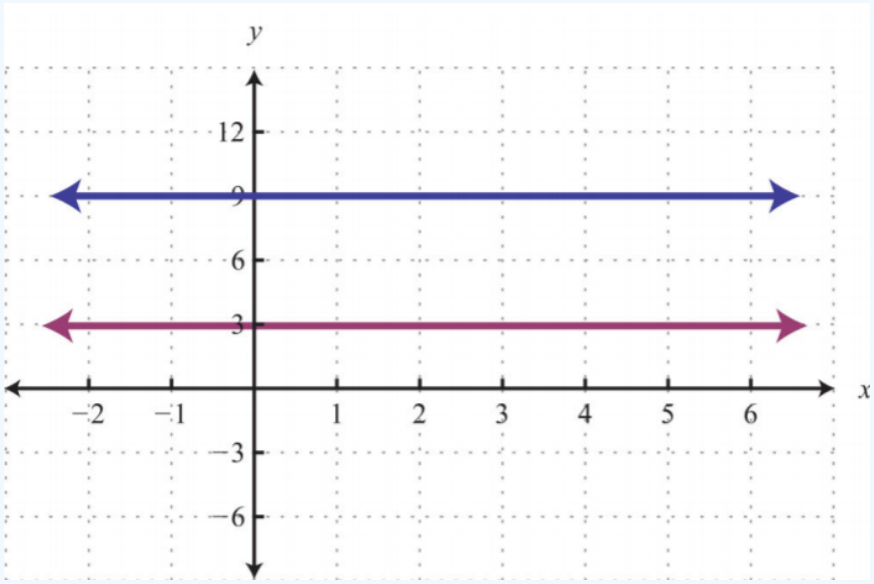


Figure 4.1.17

10.

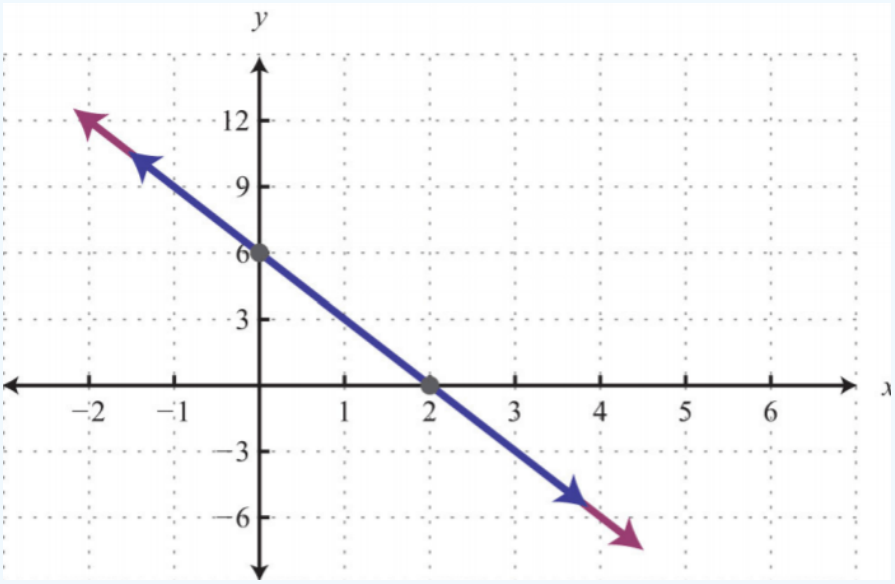


Figure 4.1.18

Answer available at the end of this section.

Exercise 4.1.6 Solving Linear Systems

Solve by graphing.

1. $\begin{cases} y = \frac{3}{2}x + 6 \\ y = -x + 1 \end{cases}$
2. $\begin{cases} y = \frac{3}{4}x + 2 \\ y = -\frac{1}{4}x - 2 \end{cases}$
3. $\begin{cases} y = x - 4 \\ y = -x + 2 \end{cases}$
4. $\begin{cases} y = -5x + 4 \\ y = 4x - 5 \end{cases}$
5. $\begin{cases} y = \frac{2}{5}x + 1 \\ y = \frac{3}{5}x \end{cases}$
6. $\begin{cases} y = -\frac{2}{5}x + 6 \\ y = \frac{2}{5}x + 10 \end{cases}$
7. $\begin{cases} y = -2 \\ y = x + 1 \end{cases}$
8. $\begin{cases} y = 3 \\ x = -3 \end{cases}$
9. $\begin{cases} y = 0 \\ y = \frac{2}{5}x - 4 \end{cases}$
10. $\begin{cases} x = 2 \\ y = 3x \end{cases}$

11. $\begin{cases} y = \frac{3}{5}x - 6 \\ y = \frac{3}{5}x - 3 \end{cases}$
12. $\begin{cases} y = -\frac{1}{2}x + 1 \\ y = -\frac{1}{2}x + 1 \end{cases}$
13. $\begin{cases} 2x + 3y = 18 \\ -6x + 3y = -6 \end{cases}$
14. $\begin{cases} -3x + 4y = 20 \\ 2x + 8y = 8 \end{cases}$
15. $\begin{cases} -2x + y = 1 \\ 2x - 3y = 9 \end{cases}$
16. $\begin{cases} x + 2y = -8 \\ 5x + 4y = -4 \end{cases}$
17. $\begin{cases} 4x + 6y = 36 \\ 2x - 3y = 6 \end{cases}$
18. $\begin{cases} 2x - 3y = 18 \\ 6x - 3y = -6 \end{cases}$
19. $\begin{cases} 3x + 5y = 30 \\ -6x - 10y = -10 \end{cases}$
20. $\begin{cases} -x + 3y = 3 \\ 5x - 15y = -15 \end{cases}$
21. $\begin{cases} x - y = 0 \\ -x + y = 0 \end{cases}$
22. $\begin{cases} y = x \\ y - x = 1 \end{cases}$
23. $\begin{cases} 3x + 2y = 0 \\ x = 2 \end{cases}$
24. $\begin{cases} 2x + \frac{1}{3}y = \frac{2}{3} \\ -3x + 12y = -2 \end{cases}$
25. $\begin{cases} \frac{1}{10}x + \frac{1}{5}y = 2 \\ -\frac{1}{5}x + \frac{1}{5}y = -1 \end{cases}$
26. $\begin{cases} \frac{1}{3}x - \frac{1}{2}y = 1 \\ \frac{1}{3}x + \frac{1}{5}y = 1 \end{cases}$
27. $\begin{cases} \frac{1}{9}x + \frac{1}{6}y = 0 \\ \frac{1}{9}x + \frac{1}{4}y = \frac{1}{2} \end{cases}$
28. $\begin{cases} \frac{5}{16}x - \frac{1}{2}y = 5 \\ -\frac{5}{16}x + \frac{1}{2}y = \frac{5}{2} \end{cases}$
29. $\begin{cases} \frac{1}{6}x - \frac{1}{2}y = \frac{9}{2} \\ -\frac{1}{18}x + \frac{1}{6}y = -\frac{3}{2} \end{cases}$
30. $\begin{cases} \frac{1}{2}x - \frac{1}{4}y = -\frac{1}{2} \\ \frac{1}{3}x - \frac{1}{2}y = 3 \end{cases}$
31. $\begin{cases} y = 4 \\ x = -5 \end{cases}$
32. $\begin{cases} y = -3 \\ x = 2 \end{cases}$
33. $\begin{cases} y = 0 \\ x = 0 \end{cases}$
34. $\begin{cases} y = -2 \\ y = 3 \end{cases}$

35. $\begin{cases} y = 5 \\ y = -5 \end{cases}$
36. $\begin{cases} y = 2 \\ y - 2 = 0 \end{cases}$
37. $\begin{cases} x = -5 \\ x = 1 \end{cases}$
38. $\begin{cases} y = x \\ x = 0 \end{cases}$
39. $\begin{cases} 4x + 6y = 3 \\ -x + y = -2 \end{cases}$
40. $\begin{cases} -2x + 20y = 20 \\ 3x + 10y = -10 \end{cases}$

Answer available at the end of this section.

Exercise 4.1.7 Solving Linear Systems

Set up a linear system of two equations and two variables and solve it using the graphing method.

1. The sum of two numbers is
2. The larger number is 10 less than five times the smaller.
3. The difference between two numbers is 12 and their sum is 4.
4. Where on the graph of $3x - 2y = 6$ does the x-coordinate equal the y-coordinate?
5. Where on the graph of $-5x + 2y = 30$ does the x-coordinate equal the y-coordinate?

Answer available at the end of this section.

Exercise 4.1.8 Solving Linear Systems

A regional bottled water company produces and sells bottled water. The following graph depicts the supply and demand curves of bottled water in the region. The horizontal axis represents the weekly tonnage of product produced, Q. The vertical axis represents the price per bottle in dollars, P.

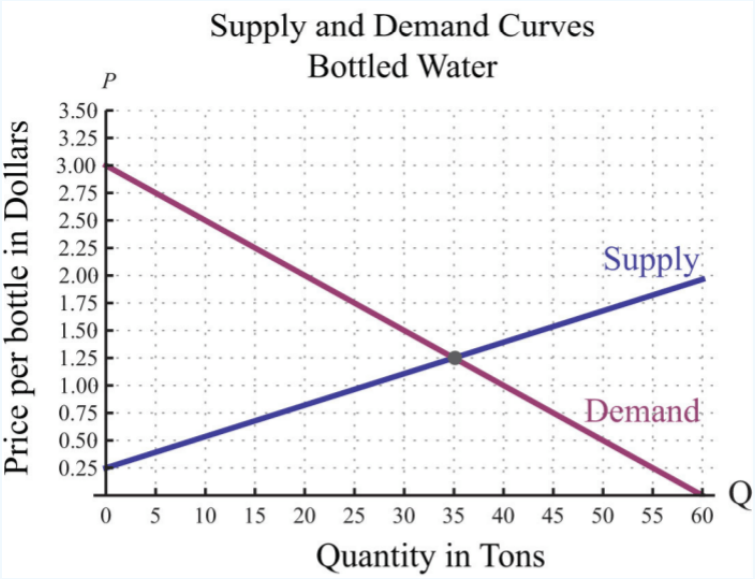


Figure 4.1.19

Use the graph to answer the following questions.

1. Determine the price at which the quantity demanded is equal to the quantity supplied.
2. If production of bottled water slips to 20 tons, then what price does the demand curve predict for a bottle of water?
3. If production of bottled water increases to 40 tons, then what price does the demand curve predict for a bottle of water?
4. If the price of bottled water is set at $\$2.50$ dollars per bottle, what quantity does the demand curve predict?

Answer available at the end of this section.

Exercise $4.1.9$ Discussion Board Topics

1. Discuss the weaknesses of the graphing method for solving systems.
2. Explain why the solution set to a dependent linear system is denoted by $(x, mx + b)$.

Answer available at the end of this section.

Answers

Exercise $4.1.1$

Yes

Exercise $4.1.2$

$(-2, 4)$

Exercise $4.1.3$

$(x, -x - 1)$

Exercise $4.1.4$ **Solutions to Linear Systems**

1. No
3. No
5. Yes
7. No
9. Yes

Exercise $4.1.5$ **Solutions to Linear Systems**

1. $(5, 0)$
3. $(2, 1)$
5. $(0, 0)$
7. $(x, 2x - 2)$
9. $\emptyset$

Exercise $4.1.6$ **Solving Linear Systems**

1. $(-2, 3)$
3. $(3, -1)$
5. $(5, 3)$
7. $(-3, -2)$
9. $(10, 0)$
11. $\emptyset$
13. $(3, 4)$
15. $(-3, -5)$
17. $(6, 2)$
19. $\emptyset$
21. (x, x)
23. $(2, -3)$
25. $(10, 5)$
27. $(-9, 6)$
29. $(x, 13x - 9)$
31. $(-5, 4)$
33. $(0, 0)$
35. $\emptyset$
37. $\emptyset$
39. $(\frac{3}{2}, -\frac{1}{2})$

Exercise $4.1.7$ **Solving Linear Systems**

1. The two numbers are 5 and 15.
3. $(6, 6)$

Exercise $4.1.8$ **Solving Linear Systems**

1. $\$1.25$
3. $\$1.00$

Exercise $4.1.9$ **Discussion Board Topics**

1. Answers may vary

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4.2: Solving Linear Systems by Substitution

Learning Objectives

- Solve linear systems using the substitution method.

The Substitution Method

In this section, we will define a completely algebraic technique for solving systems. The idea is to solve one equation for one of the variables and substitute the result into the other equation. After performing this substitution step, we will be left with a single equation with one variable, which can be solved using algebra. This is called the substitution method, and the steps are outlined in the following example.

Example 4.2.1

Solve by substitution:

Solution:

Step 1: Solve for either variable in either equation. If you choose the first equation, you can isolate y in one step.

$$\begin{aligned} 2x+y &= 7 \\ 2x+y-2x &= 7-2x \\ y &= -2x+7 \end{aligned}$$

Step 2: Substitute the expression $-2x+7$ for the y variable in the ***other*** equation.

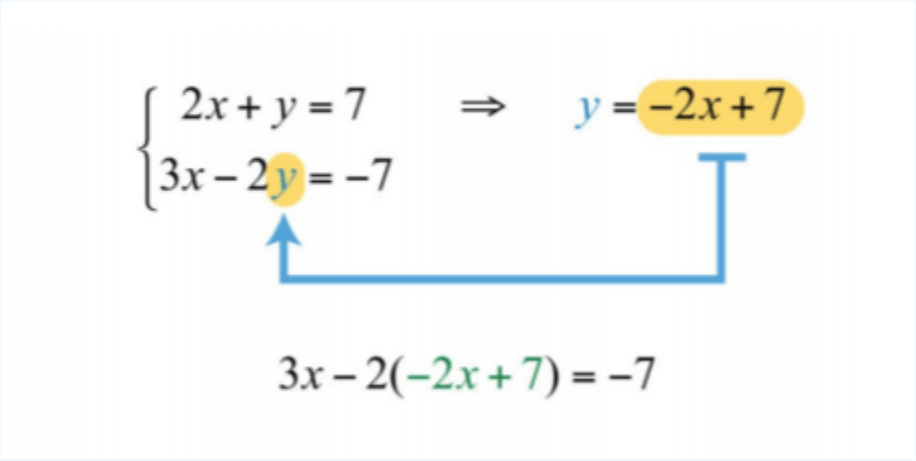


Figure 4.2.1

This leaves you with an equivalent equation with one variable, which can be solved using the techniques learned up to this point.

Step 3: Solve for the remaining variable. To solve for x, first distribute -2:

Step 4: Back substitute to find the value of the other coordinate. Substitute $x=1$ into either of the original equations or their equivalents. Typically, we use the equivalent equation that we found when isolating a variable in step 1.

$$\begin{aligned} y &= -2x + 7 \\ &= -2(1) + 7 \\ &= -2 + 7 \\ &= 5 \end{aligned}$$

The solution to the system is $(1, 5)$. Be sure to present the solution as an ordered pair.

Step 5: Check. Verify that these coordinates solve both equations of the original system:

$$Check: \quad (1, 5)$$

$Equation\ 1:$	$Equation\ 2:$
$2x + y = 7$	$3x - 2y = -7$
$2(1) + (5) = 7$	$3(1) - 2(5) = -7$
$2 + 5 = 7$	$3 - 10 = -7$
$7 = 7$ ✓	$-7 = -7$ ✓

The graph of this linear system follows:

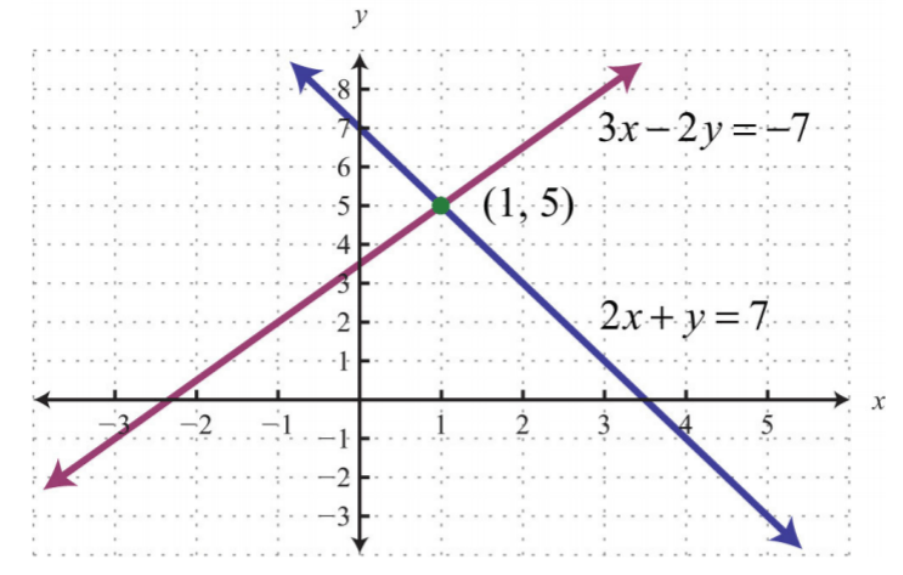


Figure 4.2.2

The substitution method for solving systems is a completely algebraic method. Thus graphing the lines is not required.

Answer:

$$(1, 5)$$

Example 4.2.2

Solve by substitution:

$$\begin{cases} 2x - y = 1 \\ 2x - y = 3 \end{cases}.$$

Solution:

In this example, we can see that x has a coefficient of 1 in the second equation. This indicates that it can be isolated in one step as follows:

$$\begin{aligned} x - y &= 3 \\ x - y{\color{#1E90FF}{+y}} &= 3{\color{#1E90FF}{+y}} \\ x &= 3 + y \end{aligned}$$

$$\begin{cases} 2x - y = 12 \\ x - y = 3 \Rightarrow x = 3 + y \end{cases}$$

Substitute $3 + y$ for x in the first equation. Use parentheses and take care to distribute.

$$\begin{aligned} 2x - y &= 12 \\ 2(3 + y) - y &= 12 \\ 6 + 2y - y &= 12 \\ 6 + y &= 12 \\ 6 + y{\color{#1E90FF}{-6}} &= 12{\color{#1E90FF}{-6}} \\ y &= 6 \end{aligned}$$

Use $x = 3 + y$ to find x.

$$\begin{aligned} x &= 3 + y \\ &= 3 + {\color{#228B22}{+6}} \\ &= 9 \end{aligned}$$

Answer:

$(9, 6)$. The check is left to the reader.

Example 4.2.3

Solve by substitution:

$$\begin{cases} 3x - 5y = 1 \\ 7x = -1 \end{cases}.$$

Solution:

In this example, the variable x is already isolated. Hence we can substitute $x = -1$ into the first equation.

Answer:

$(-1, -4)$. It is a good exercise to graph this particular system to compare the substitution method to the graphing method for solving systems.

Exercise 4.2.1

Solve by substitution:

$$\begin{cases} 3x + y = 4 \\ 8x + 2y = 10 \end{cases}.$$

Answer available at the end of this section.

Solving systems algebraically frequently requires work with fractions.

Example 4.2.4

Solve by substitution:

$$\begin{cases} 2x+8y=5 \\ 24x-4y=-15 \end{cases}.$$

Solution:

Begin by solving for x in the first equation.

$$\begin{aligned} 2x+8y &= 5 \\ 2x+8y-8y &= 5-8y \\ \frac{2x}{2} &= \frac{-8y+5}{2} \\ x &= \frac{-8y}{2}+\frac{5}{2} \\ x &= -4y+\frac{5}{2} \end{aligned}$$

$$\begin{cases} 2x+8y=5 \Rightarrow x=-4y+\frac{5}{2} \\ 24x-4y=-15 \end{cases}$$

Next, substitute into the second equation and solve for y.

Back substitute into the equation used in the substitution step:

$$\begin{aligned} x &= -4y+\frac{5}{2} \\ &= -4\left(\frac{3}{4}\right)+\frac{5}{2} \\ &= -3+\frac{5}{2} \\ &= -\frac{6}{2}+\frac{5}{2} \\ &= -\frac{1}{2} \end{aligned}$$

Answer:

$$(-\tfrac{1}{2}, \tfrac{3}{4})$$

As we know, not all linear systems have only one ordered pair solution. Recall that some systems have infinitely many ordered pair solutions and some do not have any solutions. Next, we explore what happens when using the substitution method to solve a dependent system.

Example 4.2.5

Solve by substitution:

$$\begin{cases} -5x+y=-1 \\ 10x-2y=2 \end{cases}.$$

Solution:

Since the first equation has a term with coefficient 1, we choose to solve for that first.

Next, substitute this expression in for y in the second equation.

$$\begin{aligned} 10x-2y &= 2 \\ 10x-2(5x-1) &= 2 \\ 10x-10x+2 &= 2 \\ 2 &= 2 \quad True \end{aligned}$$

This process led to a true statement; hence the equation is an identity and any real number is a solution. This indicates that the system is dependent. The simultaneous solutions take the form $(x, mx + b)$, or in this case, $(x, 5x - 1)$, where x is any real number.

Answer:

$$(x, 5x - 1)$$

To have a better understanding of the previous example, rewrite both equations in slope-intercept form and graph them on the same set of axes.

Figure 4.2.3

We can see that both equations represent the same line, and thus the system is dependent. Now explore what happens when solving an inconsistent system using the substitution method.

Example 4.2.6

Solve by substitution:

$$\begin{cases} -7x + 3y = 3 \\ 14x - 6y = -16 \end{cases}.$$

Solution:

Solve for y in the first equation.

$$\begin{aligned} -7x + 3y &= 3 \\ -7x + 3y + 7x &= 3 + 7x \\ 3y &= 7x + 3 \\ \frac{3y}{3} &= \frac{7x + 3}{3} \\ y &= \frac{7}{3}x + 1 \end{aligned}$$

$$\begin{cases} -7x + 3y = 3 \Rightarrow y = \frac{7}{3}x + 1 \\ 14x - 6y = -16 \end{cases}$$

Substitute into the second equation and solve.

Solving leads to a false statement. This indicates that the equation is a contradiction. There is no solution for x and hence no solution to the system.

Answer:

No solution, $\varnothing$

A false statement indicates that the system is inconsistent, or in geometric terms, that the lines are parallel and do not intersect. To illustrate this, determine the slope-intercept form of each line and graph them on the same set of axes.

Figure 4.2.4

In slope-intercept form, it is easy to see that the two lines have the same slope but different y-intercepts.

Exercise 4.2.2

Solve by substitution:

$$\begin{cases} 2x-5y=3 \\ 4x-10y=6 \end{cases}.$$

Answer available at the end of this section.

Key Takeaways

- The substitution method is a completely algebraic method for solving a system of equations.
- The substitution method requires that we solve for one of the variables and then substitute the result into the other equation. After performing the substitution step, the resulting equation has one variable and can be solved using the techniques learned up to this point.
- When the value of one of the variables is determined, go back and substitute it into one of the original equations, or their equivalent equations, to determine the corresponding value of the other variable.
- Solutions to systems of two linear equations with two variables, if they exist, are ordered pairs (x, y).
- If the process of solving a system of equations leads to a false statement, then the system is inconsistent and there is no solution, $\varnothing$.
- If the process of solving a system of equations leads to a true statement, then the system is dependent and there are infinitely many solutions that can be expressed using the form $(x, mx+b)$.

Exercise 4.2.3 Substitution Method

Solve by substitution.

1. $\begin{cases} y=4x-1 \\ -3x+y=1 \end{cases}$
2. $\begin{cases} y=3x-8 \\ 4x-y=2 \end{cases}$
3. $\begin{cases} x=2y-3 \\ x+3y=-8 \end{cases}$
4. $\begin{cases} x=-4y+12 \\ x+3y=12 \end{cases}$
5. $\begin{cases} y=3x-5 \\ x+2y=2 \end{cases}$
6. $\begin{cases} y=x \\ 2x+3y=10 \end{cases}$
7. $\begin{cases} y=4x+1 \\ -4x+y=2 \end{cases}$
8. $\begin{cases} y=-3x+5 \\ 3x+y=5 \end{cases}$
9. $\begin{cases} y=2x+3 \\ 2x-y=-3 \end{cases}$
10. $\begin{cases} y=5x-1 \\ x-2y=5 \end{cases}$
11. $\begin{cases} y=-7x+1 \\ 3x-y=4 \end{cases}$
12. $\begin{cases} x=6y+2 \\ 5x-2y=0 \end{cases}$
13. $\begin{cases} y=-2-2x \\ -y=-6 \end{cases}$
14. $\begin{cases} x=-3x-4 \\ y=-3 \end{cases}$
15. $\begin{cases} y=-\frac{1}{5}x+\frac{3}{7} \\ x-5y=9 \end{cases}$

16. $\begin{cases} y = \frac{2}{3}x - \frac{1}{6} \\ x - 9y = 0 \end{cases}$

17. $\begin{cases} y = \frac{1}{2}x + \frac{1}{3} \\ x - 6y = 4 \end{cases}$

18. $\begin{cases} y = -\frac{3}{8}x + \frac{1}{2} \\ 2x + 4y = 1 \end{cases}$

19. $\begin{cases} x + y = 6 \\ 2x + 3y = \frac{1}{6} \end{cases}$

20. $\begin{cases} x - y = 3 \\ -2x + 3y = -2 \end{cases}$

21. $\begin{cases} 2x + y = 2 \\ 3x - 2y = \frac{1}{7} \end{cases}$

22. $\begin{cases} x - 3y = -\frac{1}{13} \\ x + 5y = -5 \end{cases}$

23. $\begin{cases} x + 2y = -3 \\ 3x - 4y = -2 \end{cases}$

24. $\begin{cases} 5x - y = 12 \\ 9x - y = 10 \end{cases}$

25. $\begin{cases} x + 2y = -6 \\ -4x - 8y = 24 \end{cases}$

26. $\begin{cases} x + 3y = -6 \\ -2x - 6y = -12 \end{cases}$

27. $\begin{cases} -3x + y = -4 \\ 6x - 2y = -2 \end{cases}$

28. $\begin{cases} x - 5y = -10 \\ 2x - 10y = -20 \end{cases}$

29. $\begin{cases} 3x - y = 9 \\ 4x + 3y = -1 \end{cases}$

30. $\begin{cases} 2x - y = 5 \\ 4x + 2y = -2 \end{cases}$

31. $\begin{cases} -x + 4y = 0 \\ 2x - 5y = -6 \end{cases}$

32. $\begin{cases} 3y - x = 5 \\ 5x + 2y = -8 \end{cases}$

33. $\begin{cases} 2x - 5y = 1 \\ 4x + 10y = 2 \end{cases}$

34. $\begin{cases} 3x - 7y = -3 \\ 6x + 14y = 0 \end{cases}$

35. $\begin{cases} 10x - y = 3 \\ -5x + 12y = 1 \end{cases}$

36. $\begin{cases} -\frac{1}{3}x + \frac{1}{6}y = \frac{2}{3} \\ \frac{1}{2}x - \frac{1}{3}y = -\frac{3}{2} \end{cases}$

37. $\begin{cases} \frac{1}{3}x + \frac{2}{3}y = 1 \\ \frac{1}{4}x - \frac{1}{3}y = -\frac{1}{12} \end{cases}$

38. $\begin{cases} \frac{1}{7}x - y = \frac{1}{2} \\ \frac{1}{4}x + \frac{1}{2}y = 2 \end{cases}$

39. $\begin{cases} -\frac{3}{5}x + \frac{2}{5}y = \frac{1}{2} \\ \frac{1}{3}x - \frac{1}{12}y = -\frac{1}{3} \end{cases}$

40. $\begin{cases} \frac{1}{2}x = \frac{2}{3}y \\ x - \frac{2}{3}y = 2 \end{cases}$

41. $\begin{cases} -\frac{1}{2}x+\frac{1}{2}y=\frac{5}{8} \\ \frac{1}{4}x+\frac{1}{2}y=\frac{1}{4} \end{cases}$
42. $\begin{cases} x-y=0 \\ -x+2y=3 \end{cases}$
43. $\begin{cases} y=3x \\ 2x-3y=0 \end{cases}$
44. $\begin{cases} 2x+3y=18 \\ -6x+3y=-6 \end{cases}$
45. $\begin{cases} -3x+4y=20 \\ 2x+8y=8 \end{cases}$
46. $\begin{cases} 5x-3y=-1 \\ 3x+2y=7 \end{cases}$
47. $\begin{cases} -3x+7y=2 \\ 2x+7y=1 \end{cases}$
48. $\begin{cases} y=3 \\ y=-3 \end{cases}$
49. $\begin{cases} x=5 \\ x=-2 \end{cases}$
50. $\begin{cases} y=4 \\ y=4 \end{cases}$

Answer available at the end of this section.

Exercise 4.2.4 Substitution Method

Set up a linear system and solve it using the substitution method.

1. The sum of two numbers is 19. The larger number is 1 less than three times the smaller.
2. The sum of two numbers is 15. The larger is 3 more than twice the smaller.
3. The difference of two numbers is 7 and their sum is 1.
4. The difference of two numbers is 3 and their sum is -7.
5. Where on the graph of $-5x+3y=30$ does the x-coordinate equal the y-coordinate?
6. Where on the graph of $\frac{1}{2}x-\frac{1}{3}y=1$ does the x-coordinate equal the y-coordinate?

Answer available at the end of this section.

Exercise 4.2.5 Discussion Board Topics

1. Describe what drives the choice of variable to solve for when beginning the process of solving by substitution.
2. Discuss the merits and drawbacks of the substitution method.

Answer available at the end of this section.

Answers

Exercise 4.2.1

$(1,1)$

Exercise 4.2.2

$(x, \frac{2}{5}x-\frac{3}{5})$

Exercise 4.2.3 **Substitution Method**

1. $(2,7)$
3. $(-5,-1)$
5. $(2,6)$
7. $\emptyset$
9. $(x, 2x+3)$
11. $(\frac{1}{2}, -\frac{5}{2})$

13. $(4, -2)$
15. $(3, \frac{12}{5})$
17. $(-3, -\frac{7}{6})$
19. $(2, 4)$
21. $(3, -4)$
23. $(-\frac{8}{5}, -\frac{7}{10})$
25. $(x, -\frac{1}{2}x - 3)$
27. $\emptyset$
29. $(2, -3)$
31. $(-8, -2)$
33. $(\frac{1}{2}, 0)$
35. $\emptyset$
37. $(1, 1)$
39. $(-\frac{11}{10}, -\frac{2}{5})$
41. $(-\frac{1}{2}, \frac{3}{4})$
43. $(0, 0)$
45. $(-4, 2)$
47. $(-\frac{1}{5}, \frac{1}{5})$
49. $\emptyset$

Exercise $4.2.4$ **Substitution Method**

1. The two numbers are 5 and 14.
3. The two numbers are 4 and -3.
5. $(-\frac{1}{5}, -\frac{1}{5})$

Exercise $4.2.5$ **Discussion Board Topics**

1. Answers may vary

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4.3: Solving Linear Systems by Elimination

Learning Objectives

- Solve linear systems using the elimination method.
- Solve linear systems with fractions and decimals.
- Identify the weaknesses and strengths of each method for solving linear systems.

The Elimination Method

In this section, the goal is to develop another completely algebraic method for solving a system of linear equations. We begin by defining what it means to add equations together. In the following example, notice that if we add the expressions on both sides of the equal sign, we obtain another true statement.

$$\begin{aligned} 2+3 &= 5 \\ +\quad 1+7 &= 8 \\ \hline 3+10 &= 13 \\ 13 &= 13 \quad \checkmark \end{aligned}$$

This is true in general: if A, B, C, and D are algebraic expressions, then we have the following addition property of equations:

$$\text{If } A = B \text{ and } C = D, \text{ then } A + C = B + D. \tag{4.3.1}$$

For the system

$$\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$$

we add the two equations together:

$$\begin{aligned} x+y &= 5 \\ +\quad x-y &= 1 \\ \hline 2x &= 6 \end{aligned}$$

The sum of y and $-y$ is zero and that term is eliminated. This leaves us with a linear equation with one variable that can be easily solved:

$$\begin{aligned} \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

At this point, we have the x coordinate of the simultaneous solution, so all that is left to do back substitute to find the corresponding y-value.

$$\begin{aligned} x + y &= 5 \\ 3+y &= 5 \\ y &= 2 \end{aligned}$$

Hence the solution to the system is $(3, 2)$. This process describes the elimination (or addition) method for solving linear systems. Of course, the variable is not always so easily eliminated. Typically, we have to find an equivalent system by applying the multiplication property of equality to one or both of the equations as a means to line up one of the variables to eliminate. The goal is to arrange that either the x terms or the y terms are opposites, so that when the equations are added, the terms eliminate. The steps for the elimination method are outlined in the following example.

Example 4.3.1

Solve by elimination:

$$\begin{cases} 2x + y = 7 \\ 3x - 2y = -7 \end{cases}.$$

Solution:

Step 1: Multiply one, or both, of the equations to set up the elimination of one of the variables. In this example, we will eliminate the variable y by multiplying both sides of the first equation by 2. Take care to distribute.

$$\begin{aligned} 2(2x + y) &= 2(7) \\ 4x + 2y &= 14 \end{aligned}$$

This leaves us with an equivalent system where the variable y is lined up to eliminate.

Step 2: Add the equations together to eliminate one of the variables.

$$\begin{array}{rr} & 4x+2y = 14 \\ + & 3x-2y = -7 \\ \hline & 7x = 7 \end{array}$$

Step 3: Solve for the remaining variable.

$$\begin{aligned} \frac{7x}{7} &= \frac{7}{7} \\ x &= 1 \end{aligned}$$

Step 3: Back substitute into either equation or its equivalent equation.

$$\begin{aligned} 2x + y &= 7 \\ 2(1) + y &= 7 \\ 2 + y - 2 &= 7 - 2 \\ y &= 5 \end{aligned}$$

Step 4: Check. Remember that the solution must solve both of the original equations.

$$Check: \quad (1, 5)$$

$$\begin{array}{ll} Equation\ 1: & Equation\ 2: \\ 2x + y = 7 & 4x + 2y = 14 \\ 2(1) + (5) = 7 & 4(1) + 2(5) = 14 \\ 2 + 5 = 7 & 4 + 10 = 14 \\ 7 = 7 \quad \checkmark & 14 = 14 \quad \checkmark \end{array}$$

Answer:

$$(1, 5)$$

Occasionally, we will have to multiply both equations to line up one of the variables to eliminate. We want the resulting equivalent equations to have terms with opposite coefficients.

Example 4.3.2

Solve by elimination:

$$\begin{cases} 5x-3y=-1 \\ 3x+2y=7 \end{cases}.$$

Solution:

We choose to eliminate the terms with variable y because the coefficients have different signs. To do this, we first determine the least common multiple of the coefficients; in this case, the LCM$(3,2)$ is 6. Therefore, multiply both sides of both equations by the appropriate values to obtain coefficients of -6 and 6.

$$\begin{aligned} 2(5x-3y) &= 2(-1) \quad \textit{Multiply both sides of the first equation by 2.} \\ 10x-6y &= -2 \end{aligned}$$

$$\begin{aligned} 3(3x+2y) &= 3(7) \quad \textit{Multiply both sides of the second equation by 3.} \\ 9x+6y &= 21 \end{aligned}$$

This results in the following equivalent system:

The y terms are now lined up to eliminate.

Back substitute.

$$\begin{aligned} 3x+2y &= 7 \\ 3(1)+2y &= 7 \\ 3+2y-3 &= 7-3 \\ 2y &= 4 \\ y &= 2 \end{aligned}$$

Answer:

$$(1,2)$$

Sometimes linear systems are not given in standard form. When this is the case, it is best to first rearrange the equations before beginning the steps to solve by elimination.

Example 4.3.3

Solve by elimination:

$$\begin{cases} 5x+12y=11 \\ 3y=4x+1 \end{cases}.$$

Solution:

First, rewrite the second equation in standard form.

$$\begin{aligned} 3y &= 4x+1 \\ 3y-4x &= 4x+1-4x \\ -4x+3y &= 1 \end{aligned}$$

This results in the following equivalent system where like terms are aligned in columns:

$$\begin{cases} 5x+12y=11 \\ 3y=4x+1 \end{cases} \Rightarrow \begin{cases} 5x+12y=11 \\ -4x+3y=1 \end{cases}$$

We can eliminate the term with variable y if we multiply the second equation by -4.

Next, we add the equations together,

$$\begin{aligned} 5x+12y &= 11 \\ +\quad 16x-12y &= -4 \\ \hline 21x &= 7 \\ \frac{21x}{21} &= \frac{7}{21} \\ x &= \frac{1}{3} \end{aligned}$$

Back substitute.

$$\begin{aligned} 3y &= 4x+1 \\ 3y &= 4\left(\frac{1}{3}\right)+1 \\ 3y &= \frac{4}{3}+\frac{3}{3} \\ 3y &= \frac{7}{3} \\ \frac{3y}{3} &= \frac{3}{3} \\ y &= \frac{7}{3}\cdot\frac{1}{3} \\ y &= \frac{7}{9} \end{aligned}$$

Answer:

$$(\tfrac{1}{3}, \tfrac{7}{9})$$

Exercise 4.3.1

Solve by elimination:

$$\begin{cases} 2x+y=-3 \\ -3x-2y=4 \end{cases}.$$

Answer available at the end of this section.

At this point, we explore what happens when solving dependent and inconsistent systems using the elimination method.

Example 4.3.4

Solve by elimination:

$$\begin{cases} 3x-y=7 \\ 6x-2y=14 \end{cases}.$$

Solution:

To eliminate the variable x, we could multiply the first equation by -2.

Now adding the equations we have

A true statement indicates that this is a dependent system. The lines coincide, and we need y in terms of x to present the solution set in the form $(x, mx+b)$. Choose one of the original equations and solve for y. Since the equations are equivalent, it does not matter which one we choose.

Answer:

$(x, 3x-7)$

Example 4.3.5

Solve by elimination:

$$\begin{cases} -x+3y=9 \\ 2x-6y=12 \end{cases}.$$

Solution:

We can eliminate x by multiplying the first equation by 2.

$$\begin{cases} -x+3y=9 \\ 2x-6y=12 \end{cases} \overset{\times 2}{\Rightarrow} \begin{cases} -2x+6y=18 \\ 2x-6y=12 \end{cases}$$

Now adding the equations we have

$$\begin{array}{r} -2x+6y=18 \\ \underline{2x-6y=12} \\ 0=30 \quad False \end{array}$$

A false statement indicates that the system is inconsistent. The lines are parallel and do not intersect.

Answer:

No solution, $\emptyset$

Exercise 4.3.2

Solve by elimination:

$$\begin{cases} 3x+15y=-15 \\ 2x+10y=30 \end{cases}.$$

Answer available at the end of this section.

Clearing Fractions and Decimals

Given a linear system where the equations have fractional coefficients, it is usually best to clear the fractions before beginning the elimination method.

Example 4.3.6

Solve:

$$\begin{cases} -\frac{1}{10}x+\frac{1}{2}y=\frac{4}{5} \\ \frac{1}{7}x+\frac{1}{3}y=-\frac{2}{21} \end{cases}.$$

Solution:

Recall that we can clear fractions by multiplying both sides of an equation by the least common denominator (LCD). Take care to distribute and then simplify.

$$\begin{array}{cc} Equation\ 1: & Equation\ 2: \\ 10\left(-\frac{1}{10}x+\frac{1}{2}y\right)=10\left(\frac{4}{5}\right) & 21\left(\frac{1}{7}x+\frac{1}{3}y\right)=21\left(-\frac{2}{21}\right) \\ 10\cdot\left(-\frac{1}{10}x\right)+10\cdot\frac{1}{2}y=10\cdot\frac{4}{5} & 21\cdot\frac{1}{7}x+21\cdot\frac{1}{3}y=21\left(-\frac{2}{21}\right) \\ -x+5y=8 & 3x+7y=-2 \end{array}$$

This results in an equivalent system where the equations have integer coefficients,

Solve using the elimination method.

$$\begin{aligned} -3x+15y &= 24 \\ +\quad 3x+7y &= -2 \\ \hline 22y &= 22 \\ \frac{22y}{22} &= \frac{22}{22} \\ y &= 1 \end{aligned}$$

Back substitute.

Answer:

$(-3,1)$

We can use a similar technique to clear decimals before solving.

Example 4.3.7

Solve:

$$\begin{cases} 3x-0.6y=-0.9 \\ -0.5x+0.12y=0.16 \end{cases}.$$

Solution:

Multiply each equation by the lowest power of 10 necessary to result in integer coefficients. In this case, multiply the first equation by 10 and the second equation by 100.

$$\begin{array}{cc} Equation\ 1: & Equation\ 2: \\ 10(3x-0.6y)=10(-0.9) & 100(-0.5x+0.12y)=100(0.16) \\ 30x-6y=-9 & -50x+12y=16 \end{array}$$

This results in an equivalent system where the equations have integer coefficients:

Solve using the elimination method.

Back substitute.

$$\begin{aligned} 3x-0.6y &= -0.9 \\ 3(-0.2)-0.6y &= -0.9 \\ -0.6-0.6y+0.6 &= -0.9+0.6 \\ \frac{-0.6y}{-0.6} &= \frac{-0.3}{-0.6} \\ y &= 0.5 \end{aligned}$$

Answer:

$(-0.2, 0.5)$

Exercise 4.3.3

Solve using elimination:

$$\begin{cases} \frac{1}{3}x - \frac{2}{3}y = 3 \\ \frac{1}{3}x - \frac{1}{2}y = \frac{8}{3} \end{cases}$$

Answer available at the end of this section.

Summary of the Methods for Solving Linear Systems

We have developed three methods for solving linear systems of two equations with two variables. In this section, we summarize the strengths and weaknesses of each method.

The graphing method is useful for understanding what a system of equations is and what the solutions must look like. When the equations of a system are graphed on the same set of axes, we can see that the solution is the point where the graphs intersect. The graphing is made easy when the equations are in slope-intercept form. For example

$$\begin{cases} y = 5x + 15 \\ y = -5x + 5 \end{cases}$$

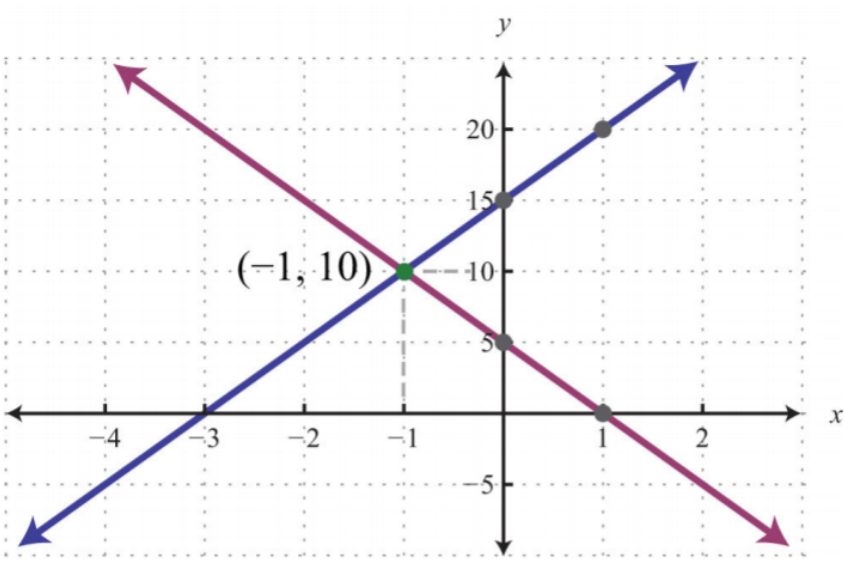


Figure 4.3.1

The simultaneous solution $(-1, 10)$ corresponds to the point of intersection. One drawback of this method is that it is very inaccurate. When the coordinates of the solution are not integers, the method is practically unusable. If we have a choice, we typically avoid this method in favor of the more accurate algebraic techniques.

The substitution method, on the other hand, is a completely algebraic method. It requires you to solve for one of the variables and substitute the result into the other equation. The resulting equation has one variable for which you can solve.

This method is particularly useful when there is a variable within the system with coefficient of 1. For example

$$\begin{cases} 10x + y = 20 \\ 7x + 5y = 14 \end{cases}$$ *Choose the substitution method.*

In this case, it is easy to solve for y in the first equation and then substitute the result into the other equation. One drawback of this method is that it often leads to equivalent equations with fractional coefficients, which are tedious to work with. If there is not a coefficient of 1, then it usually is best to choose the elimination method.

The elimination method is a completely algebraic method that makes use of the addition property of equations. We multiply one or both of the equations to obtain equivalent equations where one of the variables is eliminated if we add them together. For example,

Here we multiply both sides of the first equation by 5 and both sides of the second equation by -2. This results in an equivalent system where the variable x is eliminated when we add the equations together. Of course, there are other combinations of numbers that achieve the same result. We could even choose to eliminate the variable y. No matter which variable is eliminated first, the solution will be the same. Note that the substitution method, in this case, would require tedious calculations with fractional coefficients. One weakness of the elimination method, as we will see later in our study of algebra, is that it does not always work for nonlinear systems.

Key Takeaways

- The elimination method is a completely algebraic method for solving a system of equations.
- Multiply one or both of the equations in a system by certain numbers to obtain an equivalent system consisting of like terms with opposite coefficients. Adding these equivalent equations together eliminates a variable, and the resulting equation has one variable for which you can solve.
- It is a good practice to first rewrite the equations in standard form before beginning the elimination method.
- When the value of one of the variables is determined, back substitute into one of the original equations, or their equivalent equations, and determine the corresponding value of the other variable.

Solve by elimination.

1. $\begin{cases} x+y=3 \\ 2x-y=9 \end{cases}$
2. $\begin{cases} x-y=-6 \\ 5x+y=-18 \end{cases}$
3. $\begin{cases} x+3y=5 \\ -x-2y=0 \end{cases}$
4. $\begin{cases} -x+4y=4 \\ x-y=-7 \end{cases}$
5. $\begin{cases} -x+y=2 \\ x-y=-3 \end{cases}$
6. $\begin{cases} 3x-y=-2 \\ 6x+4y=2 \end{cases}$
7. $\begin{cases} 5x+2y=-3 \\ 10x-y=4 \end{cases}$
8. $\begin{cases} -2x+14y=28 \\ x-7y=21 \end{cases}$
9. $\begin{cases} -2x+y=4 \\ 12x-6y=-24 \end{cases}$
10. $\begin{cases} x+8y=3 \\ 3x+12y=6 \end{cases}$
11. $\begin{cases} 2x-3y=15 \\ 4x+10y=14 \end{cases}$
12. $\begin{cases} 4x+3y=-10 \\ 3x-9y=15 \end{cases}$
13. $\begin{cases} -4x-5y=-3 \\ 8x+3y=-15 \end{cases}$
14. $\begin{cases} -2x+7y=56 \\ 4x-2y=-112 \end{cases}$
15. $\begin{cases} -9x-15y=-15 \\ 3x+5y=-10 \end{cases}$
16. $\begin{cases} 6x-7y=4 \\ 2x+6y=-7 \end{cases}$

17. $\begin{cases} 4x+2y=4 \\ -5x-3y=-7 \end{cases}$
18. $\begin{cases} 5x-3y=-1 \\ 3x+2y=7 \end{cases}$
19. $\begin{cases} 7x+3y=9 \\ 2x+5y=-14 \end{cases}$
20. $\begin{cases} 9x-3y=3 \\ 7x+2y=-15 \end{cases}$
21. $\begin{cases} 5x-3y=-7 \\ -7x+6y=11 \end{cases}$
22. $\begin{cases} 2x+9y=8 \\ 3x+7y=-1 \end{cases}$
23. $\begin{cases} 2x+2y=5 \\ 3x+3y=-5 \end{cases}$
24. $\begin{cases} -3x+6y=-12 \\ 2x-4y=8 \end{cases}$
25. $\begin{cases} 25x+15y=-1 \\ 15x+10y=-1 \end{cases}$
26. $\begin{cases} 2x-3y=2 \\ 18x-12y=5 \end{cases}$
27. $\begin{cases} y=-2x-3 \\ -3x-2y=4 \end{cases}$
28. $\begin{cases} 28x+6y=9 \\ 6y=4x-15 \end{cases}$
29. $\begin{cases} y=5x+15 \\ y=-5x+5 \end{cases}$
30. $\begin{cases} 2x-3y=9 \\ 5x-8y=-16 \end{cases}$
31. $\begin{cases} \frac{1}{2}x-\frac{1}{3}y=\frac{1}{6} \\ \frac{5}{2}x+y=\frac{7}{2} \end{cases}$
32. $\begin{cases} \frac{1}{4}x-\frac{1}{9}y=1 \\ x+y=\frac{3}{4} \end{cases}$
33. $\begin{cases} \frac{1}{2}x-\frac{1}{4}y=\frac{1}{3} \\ \frac{1}{4}x+\frac{1}{2}y=-\frac{19}{6} \end{cases}$
34. $\begin{cases} -\frac{14}{3}x+2y=4 \\ -\frac{1}{3}x+\frac{1}{7}y=\frac{4}{21} \end{cases}$
35. $\begin{cases} 0.025x+0.1y=0.5 \\ 0.11x+0.04y=-0.2 \end{cases}$
36. $\begin{cases} 1.3x+0.1y=0.35 \\ 0.5x+y=-2.75 \end{cases}$
37. $\begin{cases} x+y=5 \\ 0.02x+0.03y=0.125 \end{cases}$
38. $\begin{cases} x+y=30 \\ 0.05x+0.1y=2.4 \end{cases}$

Answer

1. $(4,-1)$
3. $(-10,5)$
5. $\emptyset$
7. $(\frac{1}{5},-2)$
9. $(x,2x+4)$

11. $(6, -1)$

13. $(-3, 3)$

15. $\emptyset$

17. $(-1, 4)$

19. $(3, -4)$

21. $(-1, \frac{2}{3})$

23. $\emptyset$

25. $(\frac{1}{5}, -\frac{2}{5})$

27. $(-2, 1)$

29. $(-1, 10)$

31. $(1, 1)$

33. $(-2, -\frac{16}{3})$

35. $(-4, 6)$

37. $(2.5, 2.5)$

Exercise 4.3.5 Elimination Method

Set up a linear system and solve it using the elimination method.

1. The sum of two numbers is 14. The larger number is 1 less than two times the smaller.
2. The sum of two numbers is 30. The larger is 2 more than three times the smaller.
3. The difference of two numbers is 13 and their sum is 11.
4. The difference of two numbers is 2 and their sum is -12.

Answer available at the end of this section.

Exercise 4.3.6 Mixed Exercises

Solve using any method.

1. $\begin{cases} y = 2x - 3 \\ 3x + y = 12 \end{cases}$
2. $\begin{cases} x + 3y = -5 \\ y = 13x + 5 \end{cases}$
3. $\begin{cases} x = -1 \\ y = 3 \end{cases}$
4. $\begin{cases} y = 1 \\ 2x + 9 = 0 \end{cases}$
5. $\begin{cases} y = x \\ -x + y = 1 \end{cases}$
6. $\begin{cases} y = 5x \\ y = -10 \end{cases}$
7. $\begin{cases} 3y = 2x - 24 \\ 3x + 4y = 2 \end{cases}$

8. $\begin{cases} y = -32x + 1 \\ -2y + 2 = 3x \end{cases}$
9. $\begin{cases} 7y = -2x - 1 \\ 7x = 2y + 23 \end{cases}$
10. $\begin{cases} 5x + 9y - 14 = 0 \\ 3x + 2y - 5 = 0 \end{cases}$
11. $\begin{cases} y = -\frac{5}{16}x + 10 \\ y = \frac{5}{16}x - 10 \end{cases}$
12. $\begin{cases} y = -\frac{6}{5}x + 1 \\ 2x = 6 \end{cases}$
13. $\begin{cases} 2(x-3) + y = 0 \\ 3(2x + y - 1) = 15 \end{cases}$
14. $\begin{cases} 3 - 2(x - y) = -3 \\ 4x - 3(y + 1) = 8 \end{cases}$
15. $\begin{cases} 2(x+1) = 3(2y-1) - 21 \\ 3(x+2) = 1 - (3y - 2) \end{cases}$
16. $\begin{cases} \frac{x}{2} - \frac{y}{3} = -7 \\ \frac{x}{3} - \frac{y}{2} = -8 \end{cases}$
17. $\begin{cases} \frac{x}{4} - \frac{y}{2} = \frac{3}{4} \\ \frac{x}{3} + \frac{y}{6} = \frac{1}{6} \end{cases}$
18. $\begin{cases} \frac{1}{3}x - \frac{2}{3}y = 3 \\ \frac{1}{3}x - \frac{1}{2}y = \frac{8}{3} \end{cases}$
19. $\begin{cases} -\frac{1}{10}x + \frac{1}{2}y = \frac{4}{5} \\ \frac{1}{7}x + \frac{1}{3}y = -\frac{2}{21} \end{cases}$
20. $\begin{cases} y = -\frac{5}{3}x + \frac{1}{2} \\ \frac{1}{3}x + \frac{1}{5}y = \frac{1}{10} \end{cases}$
21. $\begin{cases} -\frac{1}{7}x + y = -\frac{2}{3} \\ -\frac{1}{14}x + \frac{1}{2}y = \frac{1}{3} \end{cases}$
22. $\begin{cases} \frac{1}{15}x - \frac{1}{12}y = \frac{1}{3} \\ -\frac{3}{10}x + \frac{3}{8}y = -\frac{3}{2} \end{cases}$
23. $\begin{cases} x + y = 4,200 \\ 0.03x + 0.0525y = 193.5 \end{cases}$
24. $\begin{cases} x + y = 350 \\ 0.2x + 0.1y = 52.5 \end{cases}$
25. $\begin{cases} 0.2x - 0.05y = 0.43 \\ 0.3x + 0.1y = -0.3 \end{cases}$
26. $\begin{cases} 0.1x + 0.3y = 0.3 \\ 0.05x - 0.5y = -0.63 \end{cases}$
27. $\begin{cases} 0.15x - 0.25y = -0.3 \\ -0.75x + 1.25y = -4 \end{cases}$
28. $\begin{cases} -0.15x + 1.25y = 0.4 \\ -0.03x + 0.25y = 0.08 \end{cases}$

Answer available at the end of this section.

Exercise $4.3.7$ Discussion Board Topics

1. How do we choose the best method for solving a linear system?
2. What does it mean for a system to be dependent? How can we tell if a given system is dependent?

Answer available at the end of this section.

Answers

Exercise $4.3.1$

$(-2, 1)$

Exercise $4.3.2$

No solution, $\emptyset$

Exercise $4.3.3$

$(5, -2)$

Exercise $4.3.5$ **Elimination Method**

1. The two numbers are 5 and 9.

3. The two numbers are 12 and -1.

Exercise $4.3.6$ **Mixed Exercises**

1. $(3, 3)$
3. $(-1, 3)$
5. $\emptyset$
7. $(6, -4)$
9. $(3, -1)$
11. $(32, 0)$
13. $(x, -2x + 6)$
15. $(-4, 3)$
17. $(1, -1)$
19. $(-3, 1)$
21. $\emptyset$
23. $(1,200, 3,000)$
25. $(0.8, -5.4)$
27. $\emptyset$

Exercise $4.3.7$ **Discussion Board Topics**

1. Answers may vary

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4.4: Applications of Linear Systems

Learning Objectives

- Set up and solve applications involving relationships between numbers.
- Set up and solve applications involving interest and money.
- Set up and solve mixture problems.
- Set up and solve uniform motion problems (distance problems).

Problems Involving Relationships between Real Numbers

We now have the techniques needed to solve linear systems. For this reason, we are no longer limited to using one variable when setting up equations that model applications. If we translate an application to a mathematical setup using two variables, then we need to form a linear system with two equations.

Example 4.4.1

The sum of two numbers is 40 and their difference is 8. Find the numbers.

Solution:

Identify variables:

Let x represent one of the unknown numbers.

Let y represent the other unknown number.

Set up equations:

When using two variables, we need to set up two equations. The first key phrase, "the ***sum*** of the two numbers is 40," translates as follows:

$$x + y = 40$$

And the second key phrase, "the ***difference*** is 8," leads us to the second equation:

$$x - y = 8$$

Therefore, our algebraic setup consists of the following system:

$$\begin{cases} x + y = 40 \\ x - y = 8 \end{cases}$$

Solve:

We can solve the resulting system using any method of our choosing. Here we choose to solve by elimination. Adding the equations together eliminates the variable y.

$$\begin{array}{rl} & x+y = 40 \\ + & x-y = 8 \\ \hline & 2x = 48 \\ & x = 24 \end{array}$$

Once we have x, back substitute to find y.

$$\begin{aligned} x+y &= 40 \\ 24+y &= 40 \\ 24+y-24 &= 40-24 \\ y &= 16 \end{aligned}$$

Check:

The sum of the two numbers should be 42 and their difference 8.

$$\begin{aligned} 24+16 &= 40 \\ 24-16 &= 8 \end{aligned}$$

Answer:

The two numbers are 24 and 16.

Example 4.4.2

The sum of 9 times a larger number and twice a smaller is 6. The difference of 3 times the larger and the smaller is 7. Find the numbers.

Solution:

Begin by assigning variables to the larger and smaller number.

Let x represent the larger number.

Let y represent the smaller number.

The first sentence describes a sum and the second sentence describes a difference.

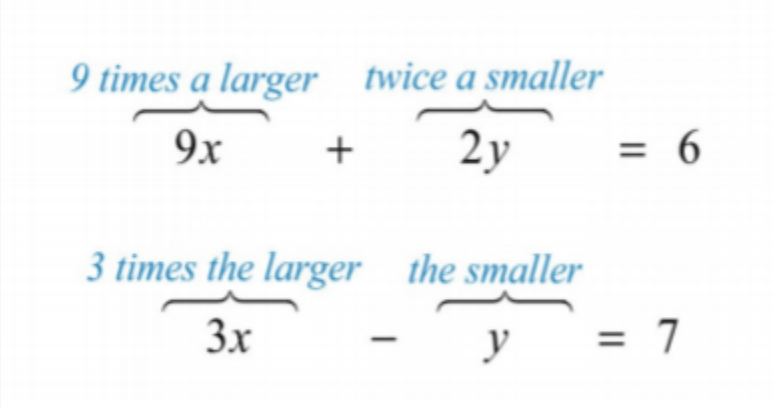


Figure 4.4.1

This leads to the following system:

$$\begin{cases} 9x+2y=6 \\ 3x-y=7 \end{cases}$$

Solve using the elimination method. Multiply the second equation by 2 and add.

$$\begin{cases} 9x+2y=6 \\ 3x-y=7 \end{cases} \overset{\times 2}{\Rightarrow} \begin{cases} 9x+2y=6 \\ 6x-2y=14 \end{cases}$$

$$\begin{aligned} 9x+2y &= 6 \\ +\quad 6x-2y &= 14 \\ \hline 15x &= 20 \\ x &= \frac{20}{15} \\ x &= \frac{4}{3} \end{aligned}$$

Back substitute to find y.

Answer:

The larger number is $\frac{4}{3}$ and the smaller number is -3.

Exercise 4.4.1

The sum of two numbers is 3. When twice the smaller number is subtracted from 6 times the larger the result is 22. Find the numbers.

Answer available at the end of this section.

Interest and Money Problems

In this section, the interest and money problems should seem familiar. The difference is that we will be making use of two variables when setting up the algebraic equations.

Example 4.4.3

A roll of 32 bills contains only \$$5$ bills and \$$10$ bills. If the value of the roll is \$$220$, then how many of each bill are in the roll?

Solution:

Begin by identifying the variables.

Let x represent the number of \$$5$ bills.

Let y represent the number of \$$10$ bills.

When using two variables, we need to set up two equations. The first equation is created from the fact that there are 32 bills.

$$x + y = 32$$

The second equation sums the value of each bill: the total value is \$$220$.

$$\$5 \cdot x + \$10 \cdot y = \$220$$

Present both equations as a system; this is our algebraic setup.

$$\begin{cases} x + y = 32 \\ 5x + 10y = 220 \end{cases}$$

Here we choose to solve by elimination, although substitution would work just as well. Eliminate x by multiplying the first equation by -5.

Now add the equations together:

Once we have y, the number of \$10 bills, back substitute to find x.

$$\begin{aligned} x+y &= 32 \\ x+12 &= 32 \\ x+12-12 &= 32-12 \\ x &= 20 \end{aligned}$$

Answer:

There are twenty \$5 bills and twelve \$10 bills. The check is left to the reader.

Example 4.4.4

A total of $\$6,300$ was invested in two accounts. Part was invested in a CD at a $4\frac{1}{2}\%$ annual interest rate and part was invested in a money market fund at a $3\frac{3}{4}\%$ annual interest rate. If the total simple interest for one year was $\$267.75$, then how much was invested in each account?

Solution:

Let x represent the amount invested at $4\frac{1}{2}\% = 4.5\% = 0.045$

Let y represent the amount invested at $3\frac{3}{4}\% = 3.75\% = 0.0375$

The total amount in both accounts can be expressed as

$$x+y = 6,300$$

To set up a second equation, use the fact that the total interest was $\$267.75$. Recall that the interest for one year is the interest rate times the principal ($I = prt = pr \cdot 1 = pr$). Use this to add the interest in both accounts. Be sure to use the decimal equivalents for the interest rates given as percentages.

$interest\ from\ the\ CD$	$+$	$interest\ from\ the\ fund$	$=$	$total\ interest$
$0.045x$	$+$	$0.375y$	$=$	267.75

These two equations together form the following linear system:

$$\begin{cases} x+y = 6,300 \\ 0.045x+0.0375y = 267.75 \end{cases}$$

Eliminate y by multiplying the first equation by -0.0375.

$$\begin{cases} x+y = 6,300 \\ 0.045x+0.0375y = 267.75 \end{cases} \overset{\times(-0.0375)}{\Rightarrow} \begin{cases} -0.0375x-0.0375y = -236.25 \\ 0.045x+0.0375y = 267.75 \end{cases}$$

Next, add the equations together to eliminate the variable y.

$$\begin{aligned} -0.0375x-0.0375y &= -236.25 \\ +\ 0.045x+0.0375y &= 267.75 \\ \hline 0.0075x &= 31.5 \\ \frac{0.0075x}{0.0075} &= \frac{31.5}{0.0075} \\ x &= 4,200 \end{aligned}$$

Back substitute.

$$\begin{aligned} x+y &= 6,300 \\ 4,200+y &= 6,300 \\ 4,200+y-4,200 &= 6,300-4,200 \\ y &= 2,100 \end{aligned}$$

Answer:

$\$4,200$ was invested at $4\frac{1}{2}\%$ and $\$2,100$ was invested at $3\frac{3}{4}\%$

At this point, we should be able to solve these types of problems in two ways: with one variable and now with two variables. Setting up word problems with two variables often simplifies the entire process, particularly when the relationships between the variables are not so clear.

Exercise 4.4.2

On the first day of a two-day meeting, 10 coffees and 10 doughnuts were purchased for a total of $\$20.00$. Since nobody drank the coffee and all the doughnuts were eaten, the next day only 2 coffees and 14 doughnuts were purchased for a total of $\$13.00$. How much did each coffee and each doughnut cost?

Answer available at the end of this section.

Mixture Problems

Mixture problems often include a percentage and some total amount. It is important to make a distinction between these two types of quantities. For example, if a problem states that a 20-ounce container is filled with a 2% saline (salt) solution, then this means that the container is filled with a mixture of salt and water as follows:

Table 4.4.1

	Percentage	Amount
Salt	$2\% = 0.02$	$0.02(20 \text{ ounces}) = 0.4 \text{ ounces}$
Water	$98\% = 0.98$	$0.98(20 \text{ ounces}) = 19.6 \text{ ounces}$

In other words, we multiply the percentage times the total to get the amount of each part of the mixture.

Example 4.4.5

A 2% saline solution is to be combined and mixed with a 5% saline solution to produce 72 ounces of a 2.5% saline solution. How much of each is needed?

Solution:

Let x represent the amount of 2% saline solution needed.

Let y represent the amount of 5% saline solution needed.

The total amount of saline solution needed is 72 ounces. This leads to one equation,

$$x+y=72$$

The second equation adds up the amount of salt in the correct percentages. The amount of salt is obtained by multiplying the percentage times the amount, where the variables x and y represent the amounts of the solutions.

$$\begin{array}{ccccc} \textit{salt in 2\% solution} & + & \textit{salt in 5\% solution} & = & \textit{salt in the end solution} \\ 0.02x & + & 0.05y & = & 0.025(72) \end{array}$$

Solve.

$$\begin{cases} x+y=72 \\ 0.02x+0.05y=0.025(72) \end{cases} \overset{\times(-0.02)}{\Rightarrow} \begin{cases} -0.02x-0.02y=-1.44 \\ 0.02x+0.05y=1.8 \end{cases}$$

$$\begin{array}{r} -0.02x-0.02y=-1.44 \\ +\quad 0.02x+0.05y=1.8 \\ \hline 0.03y=0.36 \\ \frac{0.03y}{0.03}=\frac{0.36}{0.03} \\ y=12 \end{array}$$

Back substitute.

$$\begin{array}{r} x+y=72 \\ x+12=72 \\ x+12-12=72-12 \\ x=60 \end{array}$$

Answer:

We need 60 ounces of the 2% saline solution and 12 ounces of the 5% saline solution.

Example $4.4.6$

A 50% alcohol solution is to be mixed with a 10% alcohol solution to create an 8-ounce mixture of a 32% alcohol solution. How much of each is needed?

Solution:

Let x represent the amount of 50% alcohol solution needed.

Let y represent the amount of 10% alcohol solution needed.

The total amount of the mixture must be 8 ounces.

$$x+y=8$$

The second equation adds up the amount of alcohol from each solution in the correct percentages. The amount of alcohol in the end result is 32% of 8 ounces, or $0.032(8)$.

$$\begin{array}{ccccc} \textit{alcohol in 50\% solution} & + & \textit{alcohol in 10\% solution} & = & \textit{alcohol in the end solution} \\ 0.50x & + & 0.10y & = & 0.32(8) \end{array}$$

Now we can form a system of two linear equations and two variables as follows:

$$\begin{cases} x+y=8 \\ 0.50x+0.10y=0.32(8) \end{cases}$$

In this example, multiply the second equation by 100 to eliminate the decimals. In addition, multiply the first equation by -10 to line up the variable y to eliminate.

$$\begin{array}{cc} \textit{Equation } 1: & \textit{Equation } 2: \\ -10(x+y)=-10(8) & 100\,0.50x+0.10y=100(0.32)(8) \\ -10x-10y=-80 & 50x+10y=256 \end{array}$$

We obtain the following equivalent system:

Add the equations and then solve for x:

Back substitute.

$$\begin{aligned} x+y &= 8 \\ 4.4+y &= 8 \\ 4.4+y-4.4 &= 8-4.4 \\ x &= 3.6 \end{aligned}$$

Answer:

To obtain 8 ounces of a 32% alcohol mixture we need to mix 4.4 ounces of the 50% alcohol solution and 3.6 ounces of the 10 % solution.

Exercise $4.4.3$

A 70% antifreeze concentrate is to be mixed with water to produce a 5-gallon mixture containing 28% antifreeze. How much water and antifreeze concentrate is needed?

Answer available at the end of this section.

Uniform Motion Problems (Distance Problems)

Recall that the distance traveled is equal to the average rate times the time traveled at that rate, $D = r \cdot t$.

These uniform motion problems usually have a lot of data, so it helps to first organize that data in a chart and then set up a linear system. In this section, you are encouraged to use two variables.

Example $4.4.7$

An executive traveled a total of 8 hours and $1,930$ miles by car and by plane. Driving to the airport by car, she averaged 60 miles per hour. In the air, the plane averaged 350 miles per hour. How long did it take her to drive to the airport?

Solution:

We are asked to find the time it takes her to drive to the airport; this indicates that time is the unknown quantity.

Let x represent the time it took to drive to the airport.

Let y represent the time spent in the air.

	Distance =	Rate ×	Time
Travel by car		60 mph	x
Travel by air		350 mph	y
Total	1,930 mi		8 hours

Figure 4.4.2

Use the formula $D = r \cdot t$ to fill in the unknown distances.

$$\textit{Distance traveled in the car :} \quad D = r \cdot t = 60 \cdot x$$

$$\textit{Distance traveled in the air :} \quad D = r \cdot t = 350 \cdot y$$

	Distance =	Rate ×	Time
Travel by car	$60x$	60 mph	x
Travel by air	$350y$	350 mph	y
Total	1,930 mi		8 hours

Figure 4.4.3

The distance column and the time column of the chart help us to set up the following linear system.

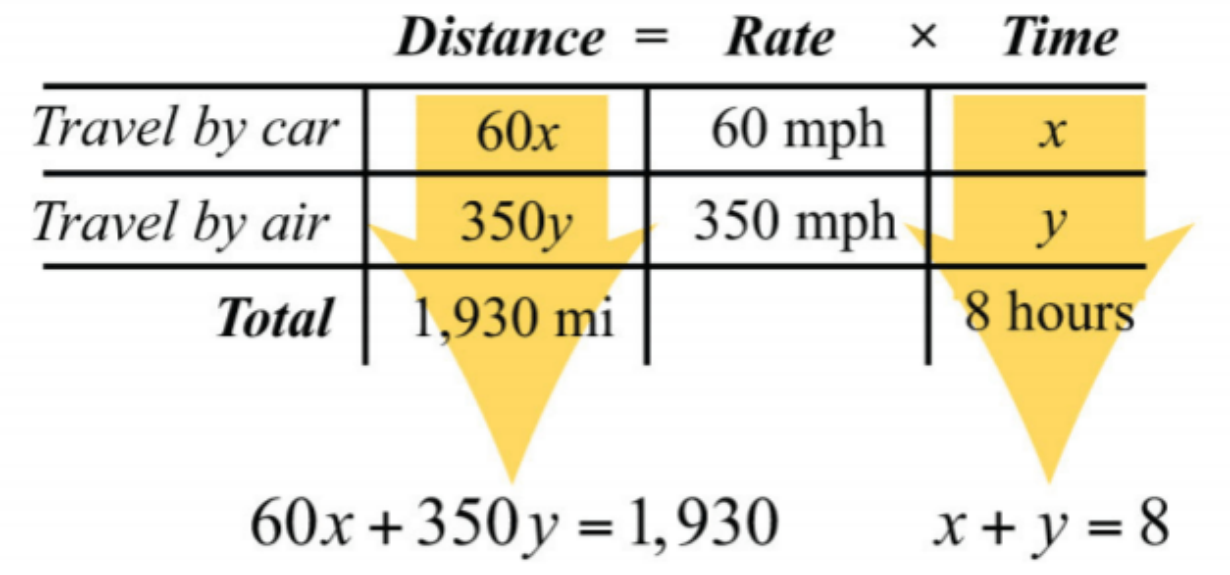


Figure 4.4.4

$$\begin{cases} x+y=8 & \leftarrow \textit{total time traveled} \\ 6-x+350y=1,930 & \leftarrow \textit{total distance traveled} \end{cases}$$

Solve.

Now back substitute to find the time it took to drive to the airport x:

$$\begin{aligned} x+y&=8 \\ x+5&=8 \\ x&=3 \end{aligned}$$

Answer:

It took her 3 hours to drive to the airport.

It is not always the case that time is the unknown quantity. Read the problem carefully and identify what you are asked to find; this defines your variables.

Example 4.4.8

Flying with the wind, an airplane traveled $1,365$ miles in 3 hours. The plane then turned against the wind and traveled another 870 miles in 2 hours. Find the speed of the airplane and the speed of the wind.

Solution:

There is no obvious relationship between the speed of the plane and the speed of the wind. For this reason, use two variables as follows:

Let x represent the speed of the airplane.

Let w represent the speed of the wind.

Use the following chart to organize the data:

	Distance =	Rate ×	Time
Flight with wind	1,365 mi		3 hrs
Flight against wind	870 mi		2 hrs
Total			

Figure 4.4.5

With the wind, the airplane's total speed is $x + w$. Flying against the wind, the total speed is $x - w$.

	Distance =	Rate ×	Time
Flight with wind	1,365 mi	$x + w$	3 hrs
Flight against wind	870 mi	$x - w$	2 hrs
Total	2,235 mi		5 hrs

Figure 4.4.6

Use the rows of the chart along with the formula $D = r \cdot t$ to construct a linear system that models this problem. Take care to group the quantities that represent the rate in parentheses.

	Distance =	Rate ×	Time	
Flight with wind	1,365 mi	$x + w$	3 hrs	$1,365 = (x + w) \cdot 3$
Flight against wind	870 mi	$x - w$	2 hrs	$870 = (x - w) \cdot 2$
Total	2,235 mi		5 hrs	

Figure 4.4.7

$$\begin{cases} 1,365 = (x + w) \cdot 3 & \leftarrow \textit{distance traveled with the wind} \\ 870 = (x - w) \cdot 2 & \leftarrow \textit{distance traveled against the wind} \end{cases}$$

If we divide both sides of the first equation by 3 and both sides of the second equation by 2, then we obtain the following equivalent system:

$$\begin{cases} 1,365 = (x + w) \cdot 3 \\ 870 = (x - w) \cdot 2 \end{cases} \overset{\begin{subarray}{l} \div 3 \\ \Rightarrow \\ \div 2 \\ \Rightarrow \end{subarray}}{} \begin{cases} 455 = x + w \\ 435 = x - w \end{cases}$$

$$\begin{aligned} x+w &= 455 \\ +\quad x-w &= 435 \\ \hline 2x &= 890 \\ \frac{2x}{2} &= \frac{890}{2} \\ x &= 455 \end{aligned}$$

Back substitute.

$$\begin{aligned} x+w &= 455 \\ 455+w &= 455 \\ w &= 10 \end{aligned}$$

Answer:

The speed of the airplane is 445 miles per hour and the speed of the wind is 10 miles per hour.

Exercise 4.4.4

A boat traveled 24 miles downstream in 2 hours. The return trip, which was against the current, took twice as long. What are the speeds of the boat and of the current?

Answer available at the end of this section.

Key Takeaways

- Use two variables as a means to simplify the algebraic setup of applications where the relationship between unknowns is unclear.
- Carefully read the problem several times. If two variables are used, then remember that you need to set up two linear equations in order to solve the problem.
- Be sure to answer the question in sentence form and include the correct units for the answer.

Exercise 4.4.5 Applications Involving Numbers

Set up a linear system and solve.

1. The sum of two integers is 54 and their difference is 10. Find the integers.
2. The sum of two integers is 50 and their difference is 24. Find the integers.
3. The sum of two positive integers is 32. When the smaller integer is subtracted from twice the larger, the result is 40. Find the two integers.
4. The sum of two positive integers is 48. When twice the smaller integer is subtracted from the larger, the result is 12. Find the two integers.
5. The sum of two integers is 74. The larger is 26 more than twice the smaller. Find the two integers.
6. The sum of two integers is 45. The larger is 3 less than three times the smaller. Find the two integers.
7. The sum of two numbers is zero. When 4 times the smaller number is added to 8 times the larger, the result is 1. Find the two numbers.
8. The sum of a larger number and 4 times a smaller number is 5. When 8 times the smaller is subtracted from twice the larger, the result is -2. Find the numbers.
9. The sum of 12 times the larger number and 11 times the smaller is -36. The difference of 12 times the larger and 7 times the smaller is 36. Find the numbers.
10. The sum of 4 times the larger number and 3 times the smaller is 7. The difference of 8 times the larger and 6 times the smaller is 10. Find the numbers.

Answer available at the end of this section.

Exercise 4.4.6 Interest and Money Problems

Set up a linear system and solve.

1. A $\$7,000$ principal is invested in two accounts, one earning 3% interest and another earning 7% interest. If the total interest for the year is $\$262$, then how much is invested in each account?
2. Mary has her total savings of $\$12,500$ in two different CD accounts. One CD earns 4.4% interest and another earns 3.2% interest. If her total interest for the year is $\$463$, then how much does she have in each CD account?
3. Sally's $\$1,800$ savings is in two accounts. One account earns 6% annual interest and the other earns 3%. Her total interest for the year is $\$93$. How much does she have in each account?
4. Joe has two savings accounts totaling $\$4,500$. One account earns $3\frac{3}{4}\%$ annual interest and the other earns $2\frac{5}{8}\%$. If his total interest for the year is $\$141.75$, then how much is in each account?
5. Millicent has $\$10,000$ invested in two accounts. For the year, she earns $\$535$ more in interest from her 7% mutual fund account than she does from her 4% CD. How much does she have in each account?
6. A small business has $\$85,000$ invested in two accounts. If the account earning 3% annual interest earns $\$825$ more in interest than the account earning 4.5% annual interest, then how much is invested in each account?
7. Jerry earned a total of $\$284$ in simple interest from two separate accounts. In an account earning 6% interest, Jerry invested $\$1,000$ more than twice the amount he invested in an account earning 4%. How much did he invest in each account?
8. James earned a total of $\$68.25$ in simple interest from two separate accounts. In an account earning 2.6% interest, James invested one-half as much as he did in the other account that earned 5.2%. How much did he invest in each account?
9. A cash register contains $\$10$ bills and $\$20$ bills with a total value of $\$340$. If there are 23 bills total, then how many of each does the register contain?
10. John was able to purchase a pizza for $\$10.80$ with quarters and dimes. If he uses 60 coins to buy the pizza, then how many of each did he have?
11. Dennis mowed his neighbor's lawn for a jar of dimes and nickels. Upon completing the job, he counted the coins and found that there were 4 less than twice as many dimes as there were nickels. The total value of all the coins is $\$6.60$. How many of each coin did he have?
12. Two families bought tickets for the big football game. One family ordered 2 adult tickets and 3 children's tickets for a total of $\$26.00$. Another family ordered 3 adult tickets and 4 children's tickets for a total of $\$37.00$. How much did each adult ticket cost?
13. Two friends found shirts and shorts on sale at a flea market. One bought 5 shirts and 3 shorts for a total of $\$51.00$. The other bought 3 shirts and 7 shorts for a total of $\$80.00$. How much was each shirt and each pair of shorts?
14. On Monday Joe bought 10 cups of coffee and 5 doughnuts for his office at a cost of $\$16.50$. It turns out that the doughnuts were more popular than the coffee. Therefore, on Tuesday he bought 5 cups of coffee and 10 doughnuts for a total of $\$14.25$. How much was each cup of coffee?

Answer available at the end of this section.

Exercise 4.4.7 Mixture Problems

Set up a linear system and solve.

1. A 15% acid solution is to be mixed with a 25% acid solution to produce 12 gallons of a 20% acid solution. How much of each is needed?
2. One alcohol solution contains 12% alcohol and another contains 26% alcohol. How much of each should be mixed together to obtain 5 gallons of a 14.8% alcohol solution?
3. A nurse wishes to obtain 40 ounces of a 1.2% saline solution. How much of a 1% saline solution must she mix with a 2% saline solution to achieve the desired result?
4. A customer ordered 20 pounds of fertilizer that contains 15% nitrogen. To fill the customer's order, how much of the stock 30% nitrogen fertilizer must be mixed with the 10% nitrogen fertilizer?
5. A customer ordered 2 pounds of a mixed peanut product containing 15% cashews. The inventory consists of only two mixes containing 10% and 30% cashews. How much of each type must be mixed to fill the order?

6. How many pounds of pure peanuts must be combined with a 20% peanut mix to produce 10 pounds of a 32% peanut mix?
7. How much cleaning fluid with 20% alcohol content, must be mixed with water to obtain a 24-ounce mixture with 10% alcohol content?
8. A chemist wishes to create a 32-ounce solution with 12% acid content. He uses two types of stock solutions, one with 30 % acid content and another with 10% acid content. How much of each does he need?
9. A concentrated cleaning solution that contains 50% ammonia is mixed with another solution containing 10% ammonia. How much of each is mixed to obtain 8 ounces of a 32% ammonia cleaning formula?
10. A 50% fruit juice concentrate can be purchased wholesale. Best taste is achieved when water is mixed with the concentrate in such a way as to obtain a 12% fruit juice mixture. How much water and concentrate is needed to make a 50-ounce fruit juice drink?
11. A 75% antifreeze concentrate is to be mixed with water to obtain 6 gallons of a 25% antifreeze solution. How much water is needed?
12. Pure sugar is to be mixed with a fruit salad containing 10% sugar to produce 48 ounces of a salad containing 16% sugar. How much pure sugar is required?

Answer available at the end of this section.

Exercise $4.4.8$ Uniform Motion Problems

Set up a linear system and solve.

1. An airplane averaged 460 miles per hour on a trip with the wind behind it and 345 miles per hour on the return trip against the wind. If the total round trip took 7 hours, then how long did the airplane spend on each leg of the trip?
2. The two legs of a 330-mile trip took 5 hours. The average speed for the first leg of the trip was 70 miles per hour and the average speed for the second leg of the trip was 60 miles per hour. How long did each leg of the trip take?
3. An executive traveled $1,200$ miles, part by helicopter and part by private jet. The jet averaged 320 miles per hour while the helicopter averaged 80 miles per hour. If the total trip took $4\frac{1}{2}$ hours, then how long did she spend in the private jet?
4. Joe took two buses on the 463-mile trip from San Jose to San Diego. The first bus averaged 50 miles per hour and the second bus was able to average 64 miles per hour. If the total trip took 8 hours, then how long was spent in each bus?
5. Billy canoed downstream to the general store at an average rate of 9 miles per hour. His average rate canoeing back upstream was 4 miles per hour. If the total trip took $6\frac{1}{2}$ hours, then how long did it take Billy to get back on the return trip?
6. Two brothers drove the $2,793$ miles from Los Angeles to New York. One of the brothers, driving in the day, was able to average 70 miles per hour, and the other, driving at night, was able to average 53 miles per hour. If the total trip took 45 hours, then how many hours did each brother drive?
7. A boat traveled 24 miles downstream in 2 hours. The return trip took twice as long. What was the speed of the boat and the current?
8. A helicopter flying with the wind can travel 525 miles in 5 hours. On the return trip, against the wind, it will take 7 hours. What are the speeds of the helicopter and of the wind?
9. A boat can travel 42 miles with the current downstream in 3 hours. Returning upstream against the current, the boat can only travel 33 miles in 3 hours. Find the speed of the current.
10. A light aircraft flying with the wind can travel 180 miles in $1\frac{1}{2}$ hours. The aircraft can fly the same distance against the wind in 2 hours. Find the speed of the wind.

Answer available at the end of this section.

Exercise $4.4.9$ Discussion Board

1. Compose a number or money problem that can be solved with a system of equations of your own and share it on the discussion board.
2. Compose a mixture problem that can be solved with a system of equations of your own and share it on the discussion board.
3. Compose a uniform motion problem that can be solved with a system of equations of your own and share it on the discussion board

Answer available at the end of this section.

Answers

Exercise $4.4.1$

The two numbers are $-\frac{1}{2}$ and $\frac{7}{2}$.

Exercise $4.4.2$

Coffee: \$$1.25$; doughnut: \$$0.75$

Exercise $4.4.3$

We need to mix 3 gallons of water with 2 gallons of antifreeze concentrate.

Exercise $4.4.4$

The speed of the boat is 9 miles per hour and the speed of the current is 3 miles per hour.

Exercise $4.4.5$ **Applications Involving Numbers**

1. The integers are 22 and 32.
3. The integers are 8 and 24.
5. The integers are 16 and 58.
7. The two numbers are $-\frac{1}{4}$ and $\frac{1}{4}$.
9. The smaller number is -4 and the larger is $\frac{2}{3}$.

Exercise $4.4.6$ **Interest and Money Problems**

1. \$$5,700$ at 3% and \$$1,300$ at 7%
3. \$$1,300$ at 6% and \$$500$ at 3%
5. \$$8,500$ at 7% and \$$1,500$ at 4%
7. \$$1,400$ at 4% and \$$3,800$ at 6%
9. 12 tens and 11 twenties
11. 52 dimes and 28 nickels
13. Shirts: \$$4.50$; shorts: \$$9.50$

Exercise $4.4.7$ **Mixture Problems**

1. 6 gallons of each
3. 32 ounces of the 1% saline solution and 8 ounces of the 2% saline solution
5. 1.5 pounds of the 10% cashew mix and 0.5 pounds of the 30% cashew mix
7. 12 ounces of cleaning fluid
9. 4.4 ounces of the 50% ammonia solution and 3.6 ounces of the 10% ammonia solution
11. 4 gallons

Exercise $4.4.8$ **Uniform Motion Problems**

1. The airplane flew 3 hours with the wind and 4 hours against the wind.
3. 3.5 hours
5. 4.5 hours
7. Boat: 9 miles per hour; current: 3 miles per hour
9. 1.5 miles per hour

Exercise $4.4.9$ **Discussion Board**

1. Answers may vary
3. Answers may vary

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4.5: Solving Systems of Linear Inequalities (Two Variables)

Learning Objectives

- Check solutions to systems of linear inequalities with two variables.
- Solve systems of linear inequalities.

Solutions to Systems of Linear Inequalities

A system of linear inequalities consists of a set of two or more linear inequalities with the same variables. The inequalities define the conditions that are to be considered simultaneously. For example,

We know that each inequality in the set contains infinitely many ordered pair solutions defined by a region in a rectangular coordinate plane. When considering two of these inequalities together, the intersection of these sets defines the set of simultaneous ordered pair solutions. When we graph each of the above inequalities separately, we have

$y > x - 2$ $y \leq 2x + 2$

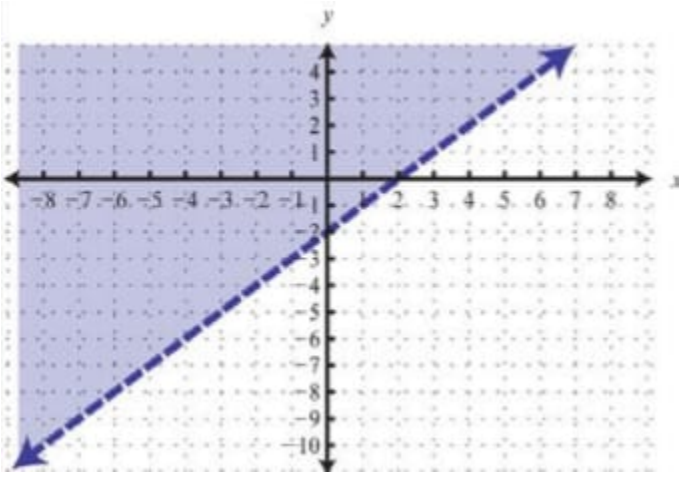


Figure 4.5.1

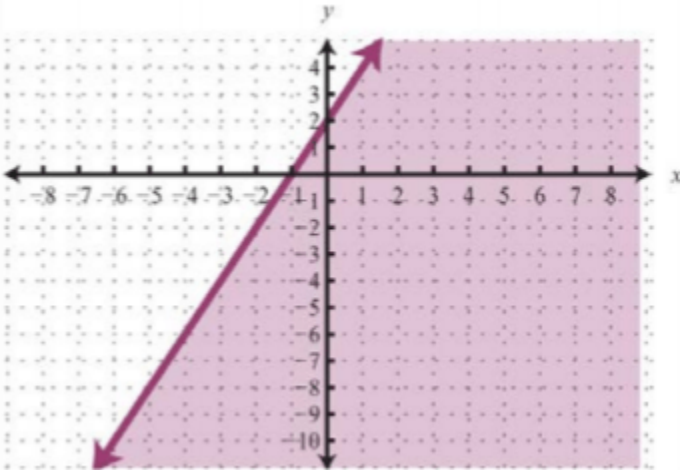


Figure 4.5.2

Table 4.5.1

When graphed on the same set of axes, the intersection can be determined.

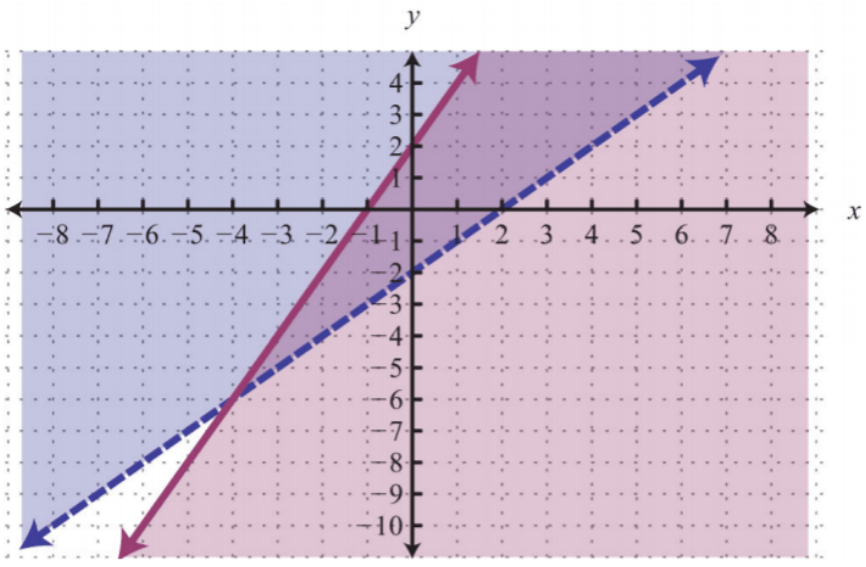


Figure 4.5.3

The intersection is shaded darker and the final graph of the solution set is presented as follows:

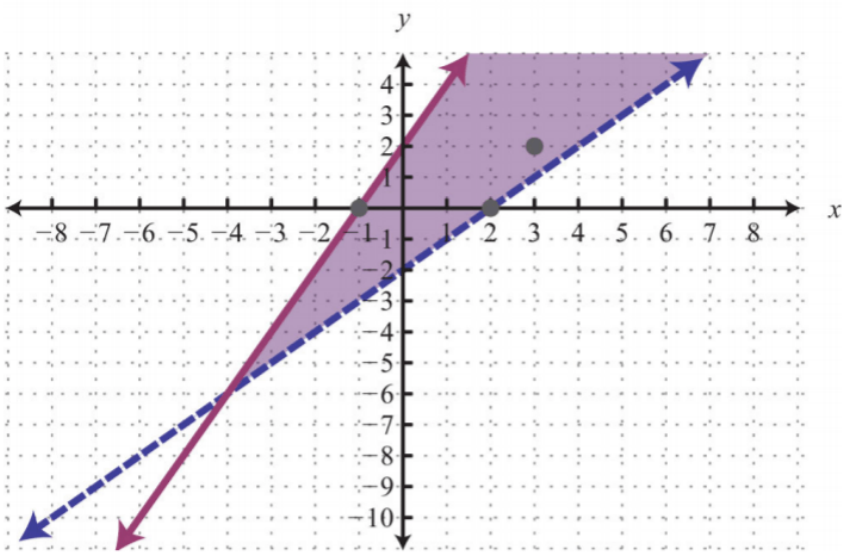


Figure 4.5.4

The graph suggests that $(3, 2)$ is a solution because it is in the intersection. To verify this, show that it solves both of the original inequalities:

$$Check: \quad (3, 2)$$

$$\begin{array}{ll} Inequality\,1: \quad y > x - 2 & Inequality\,2: \quad y/leq2x + 2 \\ 2 > 3 - 2 & 2 \leq 2(3) + 2 \\ 2 > 1 \quad \checkmark & 2 \leq 8 \quad \checkmark \end{array}$$

Points on the solid boundary are included in the set of simultaneous solutions and points on the dashed boundary are not. Consider the point $(-1, 0)$ on the solid boundary defined by $y = 2x + 2$ and verify that it solves the original system:

$$Check: \quad (-1, 0)$$

$$\begin{array}{ll} Inequality\,1: & y > x-2 \\ & 0 > -1-2 \\ & 0 > -3 \quad \checkmark \end{array} \qquad \begin{array}{ll} Inequality\,2: & y \leq 2x+2 \\ & 0 \leq 2(-1)+2 \\ & 0 \leq 0 \quad \checkmark \end{array}$$

Notice that this point satisfies both inequalities and thus is included in the solution set. Now consider the point $(2,0)$ on the dashed boundary defined by $y = x - 2$ and verify that it does not solve the original system:

$$Check: \quad (2,0)$$

$$\begin{array}{ll} Inequality\,1: & y > x-2 \\ & 0 > 2-2 \\ & 0 > 0 \quad x \end{array} \qquad \begin{array}{ll} Inequality\,2: & y \leq 2x+2 \\ & 0 \leq 2(2)+2 \\ & 0 \leq \quad \checkmark \end{array}$$

This point does not satisfy both inequalities and thus is not included in the solution set.

Solving Systems of Linear Inequalities

Solutions to a system of linear inequalities are the ordered pairs that solve all the inequalities in the system. Therefore, to solve these systems, graph the solution sets of the inequalities on the same set of axes and determine where they intersect. This intersection, or overlap, defines the region of common ordered pair solutions.

Example 4.5.1

Graph the solution set:

$$\begin{cases} -2x + y > -4 \\ 3x - 6y \geq 6 \end{cases}.$$

Solution:

To facilitate the graphing process, we first solve for y.

2x-4 \ y&\leq \frac{1}{2}x-1 \end{aligned}\right.\)

For the first inequality, we use a dashed boundary defined by $y = 2x - 4$ and shade all points above the line. For the second inequality, we use a solid boundary defined by $y = \frac{1}{2}x - 1$ and shade all points below. The intersection is darkened.

Figure 4.5.5

Now we present the solution with only the intersection shaded.

Answer:

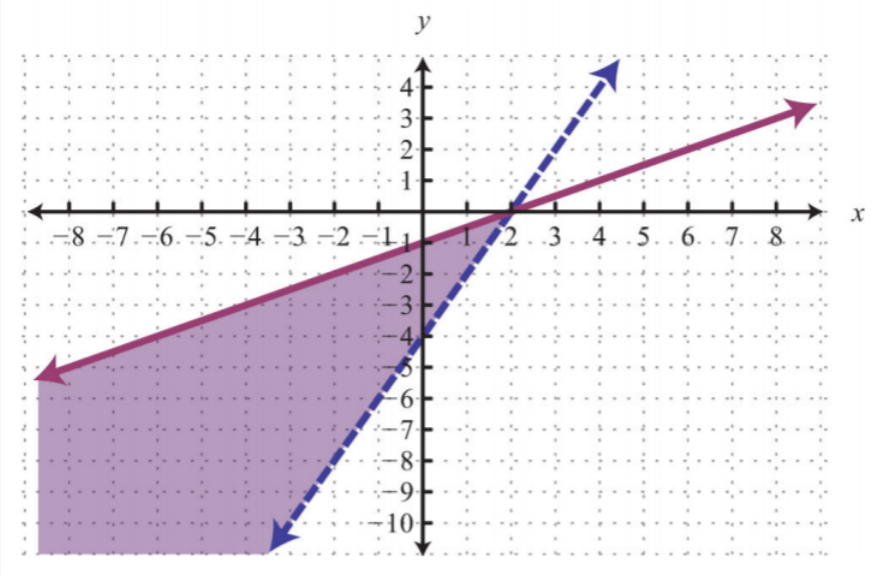


Figure 4.5.6

Example 4.5.2

Graph the solution set:

$$\begin{cases} -2x + 3y > 6 \\ 4x - 6y > 12 \end{cases}.$$

Solution:

Begin by solving both inequalities for y.

Use a dashed line for each boundary. For the first inequality, shade all points above the boundary. For the second inequality, shade all points below the boundary.

Figure 4.5.7

As you can see, there is no intersection of these two shaded regions. Therefore, there are no simultaneous solutions.

Answer:

No solution,
$\emptyset$

Example 4.5.3

Graph the solution set:

Solution:

Figure 4.5.8

After graphing all three inequalities on the same set of axes, we determine that the intersection lies in the triangular region pictured.

Answer:

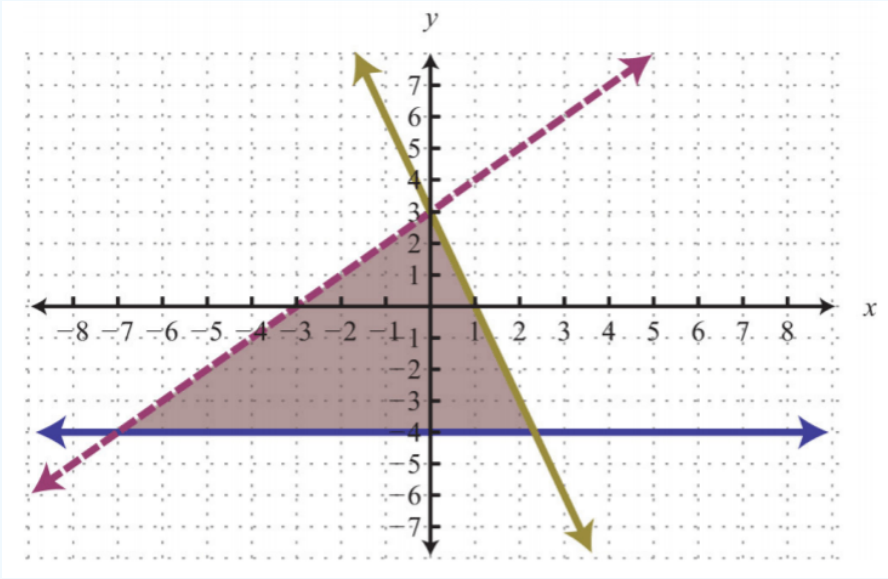


Figure 4.5.9

The graphic suggests that $(-1, 1)$ is a common point. As a check, substitute that point into the inequalities and verify that it solves all three conditions.

Key Takeaways

- To solve systems of linear inequalities, graph the solution sets of each inequality on the same set of axes and determine where they intersect.

Exercise 4.5.1 Solving Systems of Linear Inequalities

Determine whether the given point is a solution to the given system of linear equations.

1. $(3, 2)$; $\begin{cases} y \leq x + 3 \\ y \geq -x + 3 \end{cases}$

2. $(-3,-2)$; $\begin{cases} y < -3x+4 \\ y \geq 2x-1 \end{cases}$
3. $(5,0)$; $\begin{cases} y > -x+5 \\ y \leq \frac{3}{4}x-2 \end{cases}$
4. $(0,1)$; $\begin{cases} y < \frac{2}{3}x+1 \\ y \geq \frac{5}{2}x-2 \end{cases}$
5. $(-1,\frac{8}{3})$; $\begin{cases} -4x+3y \geq -12 \\ 2x+3y < 6 \end{cases}$
6. $(-1,-2)$; $\begin{cases} -x+y < 0 \\ x+y < 0 \\ x+y < -2 \end{cases}$

Answer available at the end of this section.

Exercise 4.5.2 Solving Systems of Linear Inequalities

Graph the solution set.

1. $\begin{cases} y \leq x+3 \\ y \geq -x+3 \end{cases}$
2. $\begin{cases} y < -3x+4 \\ y \geq 2x-1 \end{cases}$
3. $\begin{cases} y > x \\ y < -1 \end{cases}$
4. $\begin{cases} y < \frac{2}{3}x+1 \\ y \geq \frac{5}{2}x-2 \end{cases}$
5. $\begin{cases} y > -x+5 \\ y \leq \frac{3}{4}x-2 \end{cases}$
6. $\begin{cases} y > \frac{3}{5}x+3 \\ y < \frac{3}{5}x-3 \end{cases}$
7. $\begin{cases} x+4y < 12 \\ -3x+12y \geq -12 \end{cases}$
8. $\begin{cases} -x+y \leq 6 \\ 2x+y \geq 1 \end{cases}$
9. $\begin{cases} -2x+3y > 3 \\ 4x-3y < 15 \end{cases}$
10. $\begin{cases} -4x+3y \geq -12 \\ 2x+3y < 6 \end{cases}$
11. $\begin{cases} 5x+y \leq 4 \\ -4x+3y < -6 \end{cases}$
12. $\begin{cases} 3x+5y < 15 \\ -x+2y \leq 0 \end{cases}$
13. $\begin{cases} x \geq 0 \\ 5x+y > 5 \end{cases}$
14. $\begin{cases} x \geq -2 \\ y \geq 1 \end{cases}$
15. $\begin{cases} x-3 < 0 \\ y+2 \geq 0 \end{cases}$
16. $\begin{cases} 5y \geq 2x+5 \\ -2x < -5y-5 \end{cases}$
17. $\begin{cases} x-y \geq 0 \\ -x+y < 1 \end{cases}$
18. $\begin{cases} -x+y \geq 0 \\ y-x < 1 \end{cases}$

19. $\begin{cases} x > -2 \\ x \leq 2 \end{cases}$
20. $\begin{cases} y > -1 \\ y < 2 \end{cases}$
21. $\begin{cases} -x + 2y > 8 \\ 3x - 6y \geq 18 \end{cases}$
22. $\begin{cases} -3x + 4y \leq 4 \\ 6x - 8y > -8 \end{cases}$
23. $\begin{cases} 2x + y < 3 \\ -x \leq 12y \end{cases}$
24. $\begin{cases} 2x + 6y \leq 6 \\ -13x - y \leq 3 \end{cases}$
25. $\begin{cases} y < 3 \\ y > x \\ x > -4 \end{cases}$
26. $\begin{cases} y < 1 \\ y \geq x - 1 \\ y < -3x + 3 \end{cases}$
27. $\begin{cases} -4x + 3y > -12 \\ y \geq 2 \\ 2x + 3y > 6 \end{cases}$
28. $\begin{cases} -x + y < 0 \\ x + y \leq 0 \\ x + y > -2 \end{cases}$
29. $\begin{cases} x + y < 2 \\ x < 3 \\ -x + y \leq 2 \end{cases}$
30. $\begin{cases} y + 4 \geq 0 \\ \frac{1}{2}x + \frac{1}{3}y \leq 1 \\ -\frac{1}{2}x + \frac{1}{3}y \leq 1 \end{cases}$
31. Construct a system of linear inequalities that describes all points in the first quadrant.
32. Construct a system of linear inequalities that describes all points in the second quadrant.
33. Construct a system of linear inequalities that describes all points in the third quadrant.
34. Construct a system of linear inequalities that describes all points in the fourth quadrant.

Answer available at the end of this section.

Answers

Exercise 4.5.1 **Solving Systems of Linear Inequalities**

1. Yes
3. No
5. No

Exercise 4.5.2 **Solving Systems of Linear Inequalities**

1.

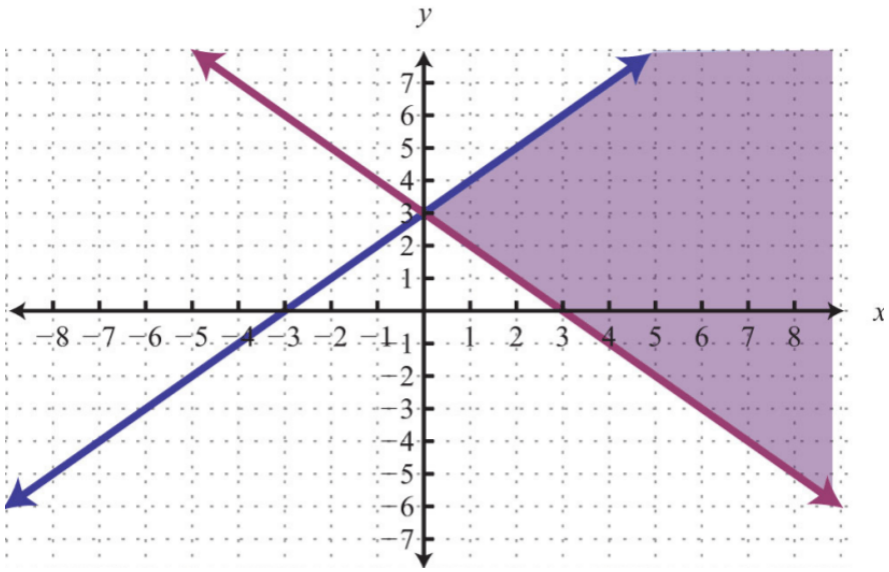


Figure 4.5.10

3.

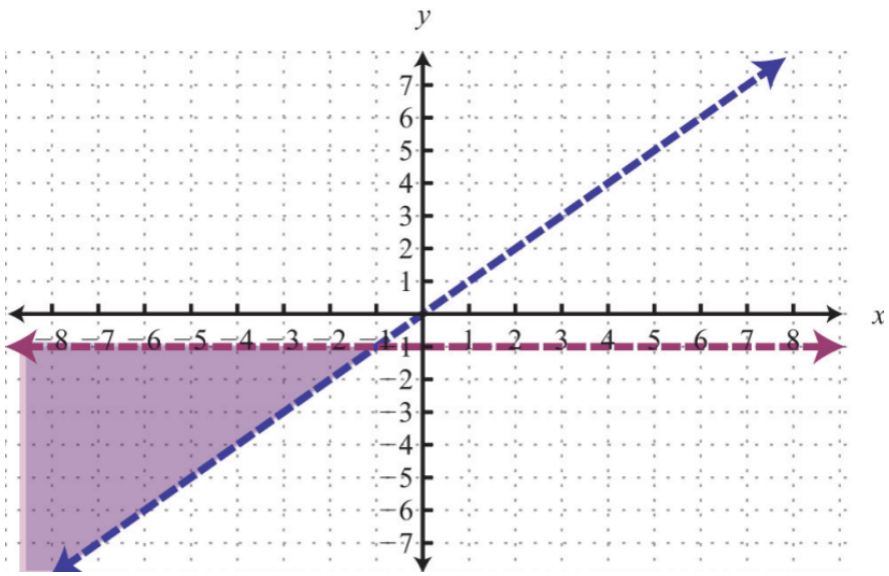


Figure 4.5.11

5.

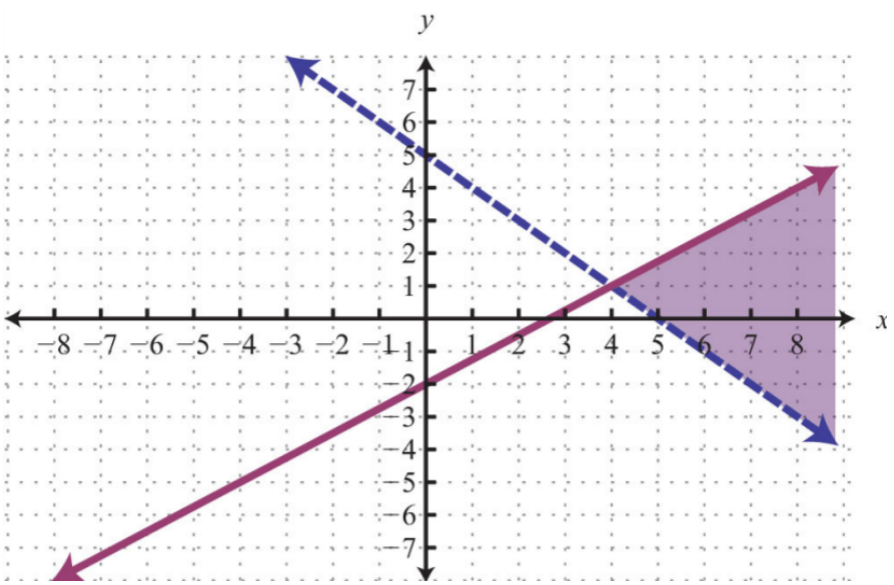


Figure 4.5.12

7.

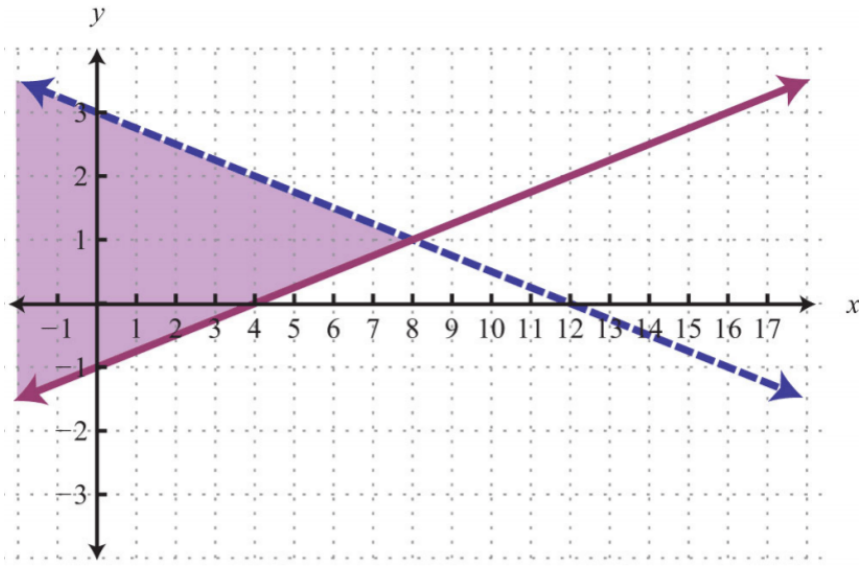


Figure 4.5.13

9.

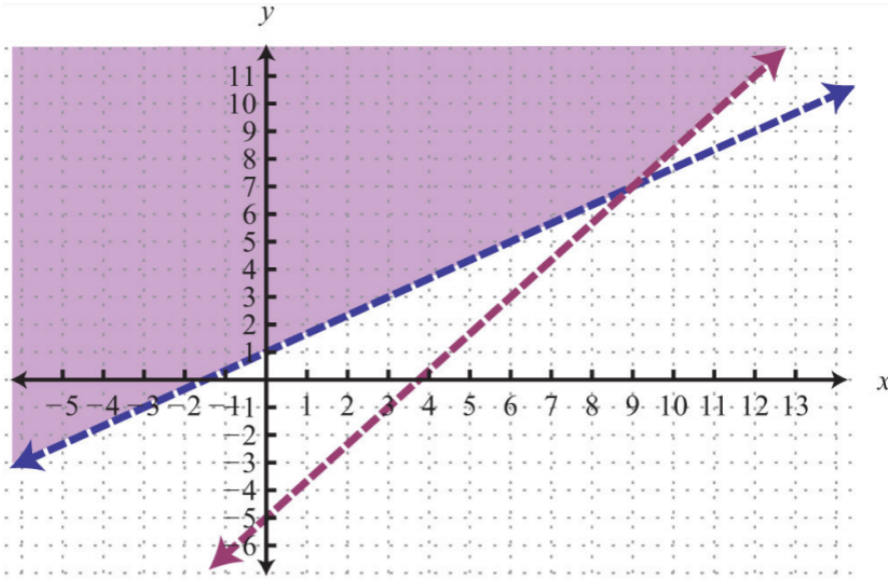


Figure 4.5.14

11.

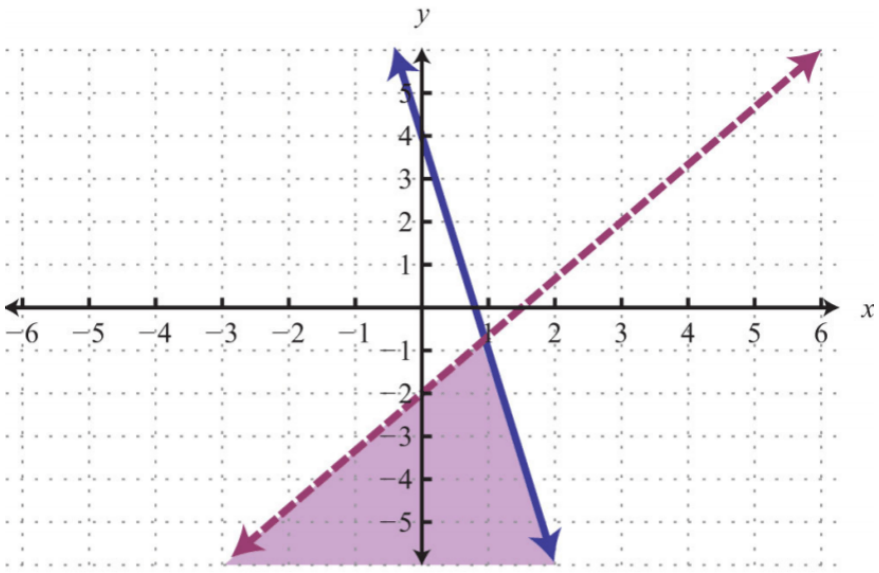


Figure 4.5.15

13.

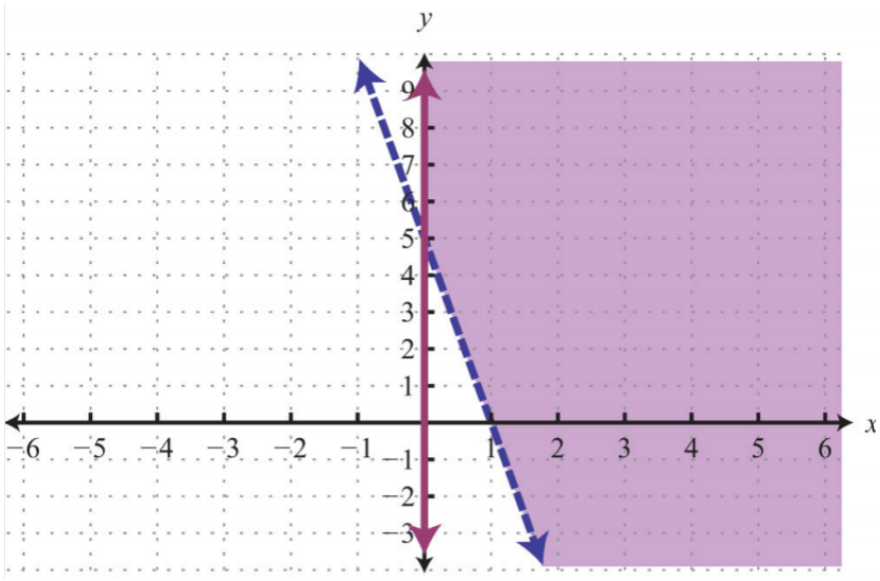


Figure 4.5.16

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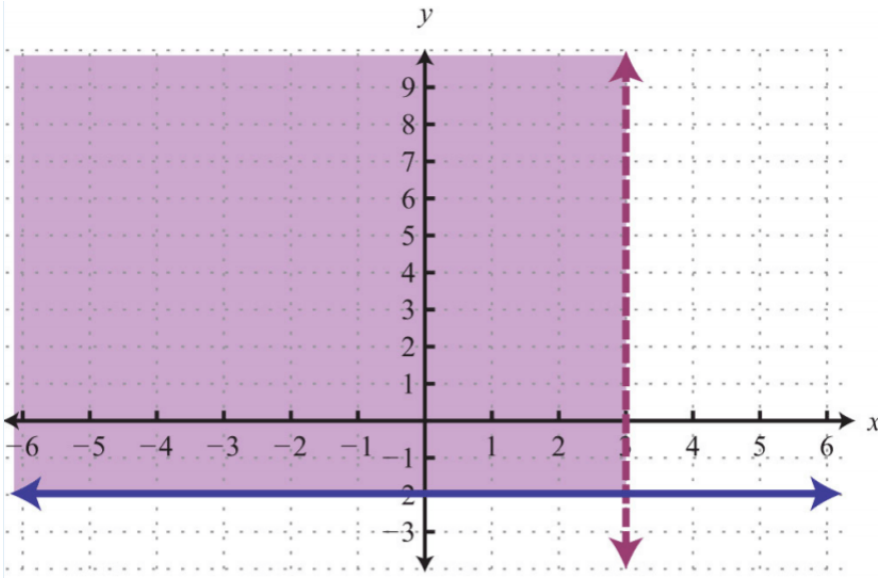


Figure 4.5.17

17.

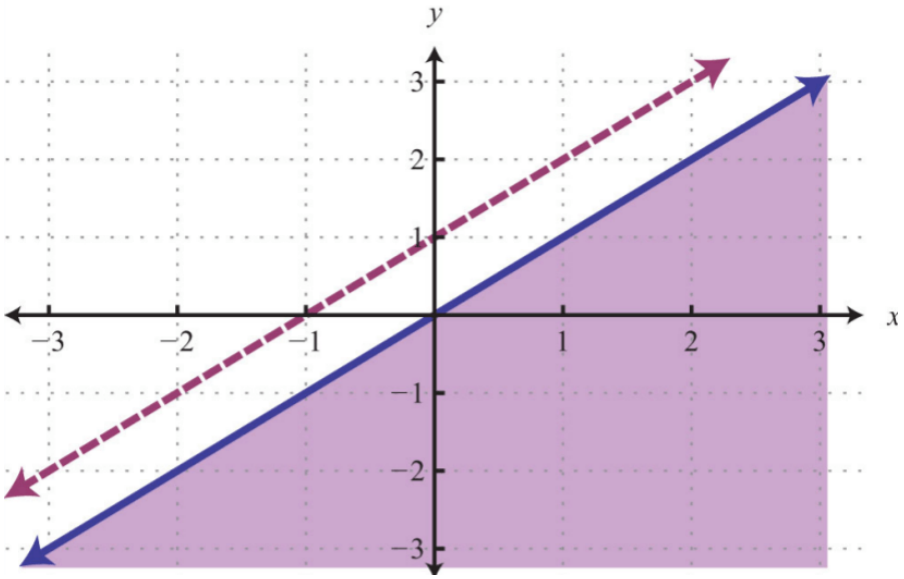


Figure 4.5.18

19.

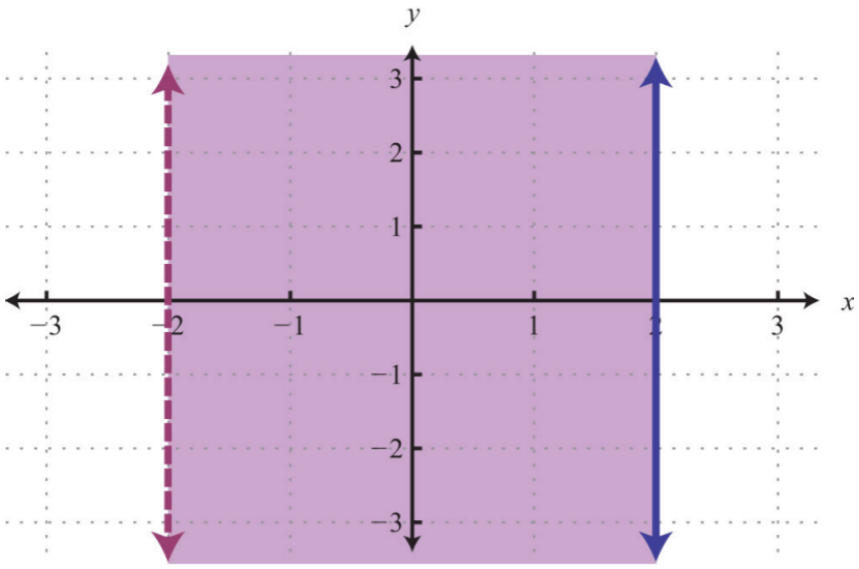


Figure 4.5.19

21. No solution, $\emptyset$

23.

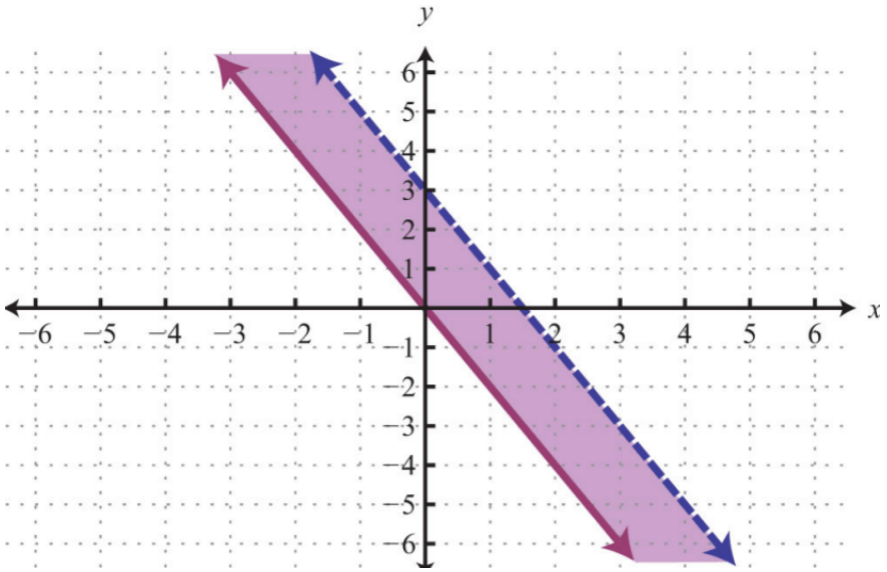


Figure 4.5.20

25.

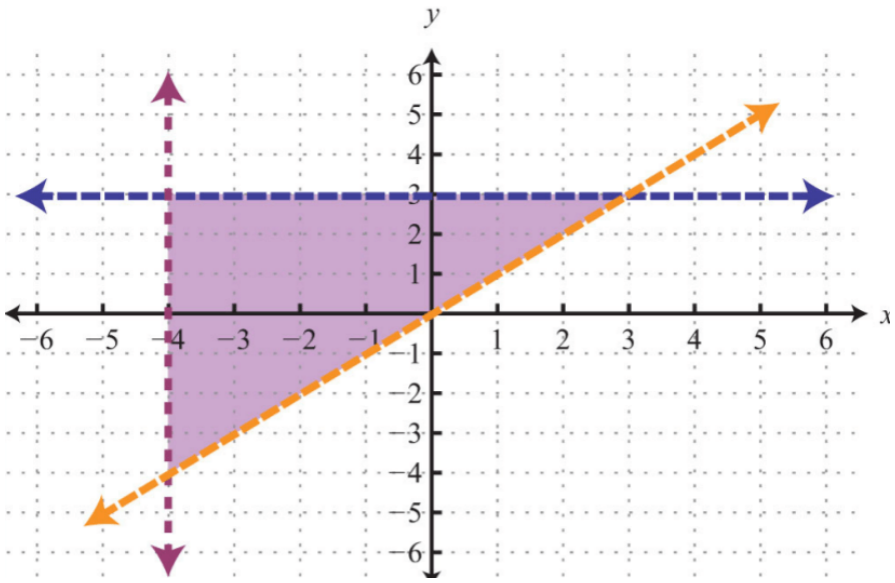


Figure 4.5.21

27.

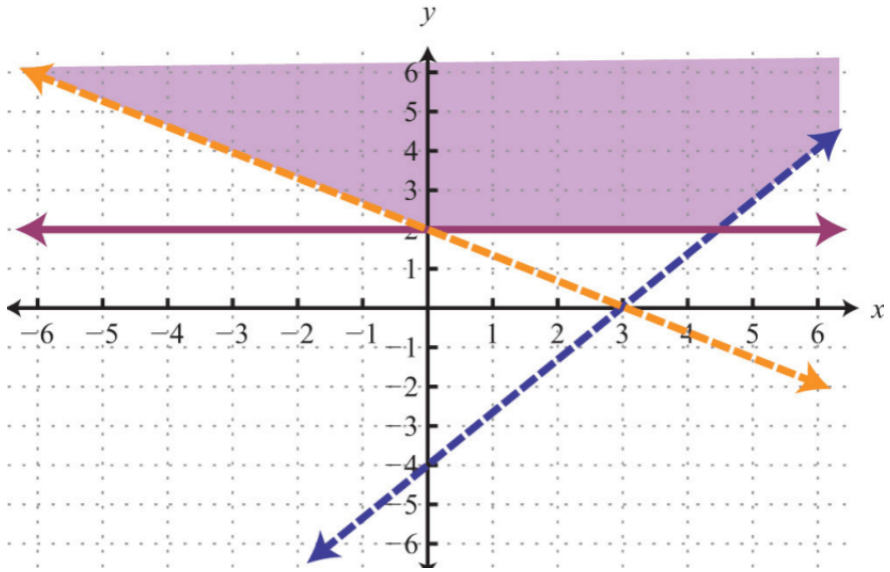


Figure 4.5.22

29.

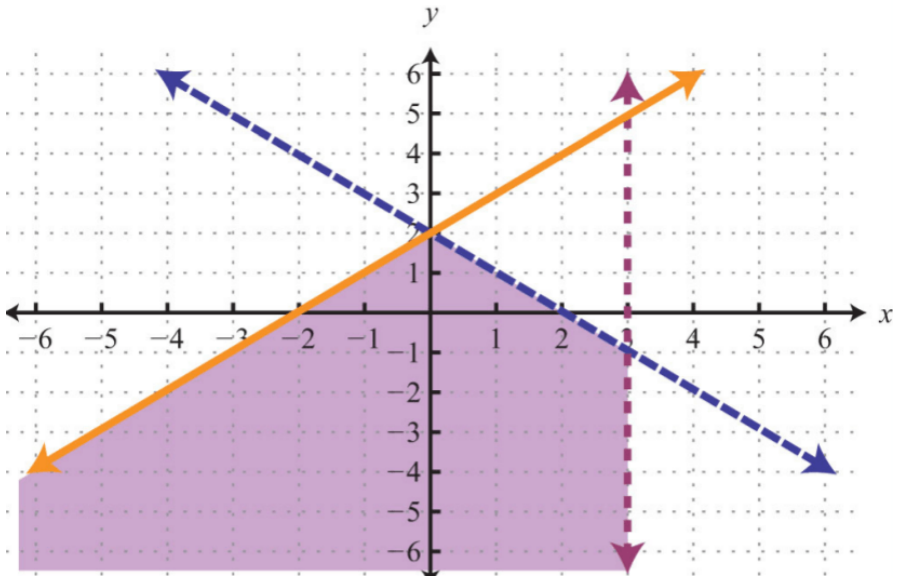


Figure 4.5.23

31. $\begin{cases} x > 0 \\ y > 0 \end{cases}$

33. $\begin{cases} x < 0 \\ y < 0 \end{cases}$

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4.E: Review Exercises and Sample Exam

Review Exercises

Exercise $4.E.1$ Solving Linear Systems by Graphing

Determine whether the given ordered pair is a solution to the given system.

1. $(1,-3)$; $\begin{cases} 5x-y=8 \\ -3x+y=-6 \end{cases}$
2. $(-3,-4)$; $\begin{cases} 4x-12y=-10 \\ 6x-5y=-2 \end{cases}$
3. $(-1,\frac{1}{5})$; $\begin{cases} \frac{3}{5}x-\frac{1}{3}y=-\frac{2}{3} \\ -\frac{1}{5}x-\frac{1}{2}y=\frac{1}{10} \end{cases}$
4. $(\frac{1}{2},-1)$; $\begin{cases} x+\frac{3}{4}y=-\frac{1}{4} \\ \frac{2}{3}x-y=\frac{4}{3} \end{cases}$

Answer available at the end of this section.

Exercise $4.E.2$ Solving Linear Systems by Graphing

Given the graph, determine the simultaneous solution.

1.

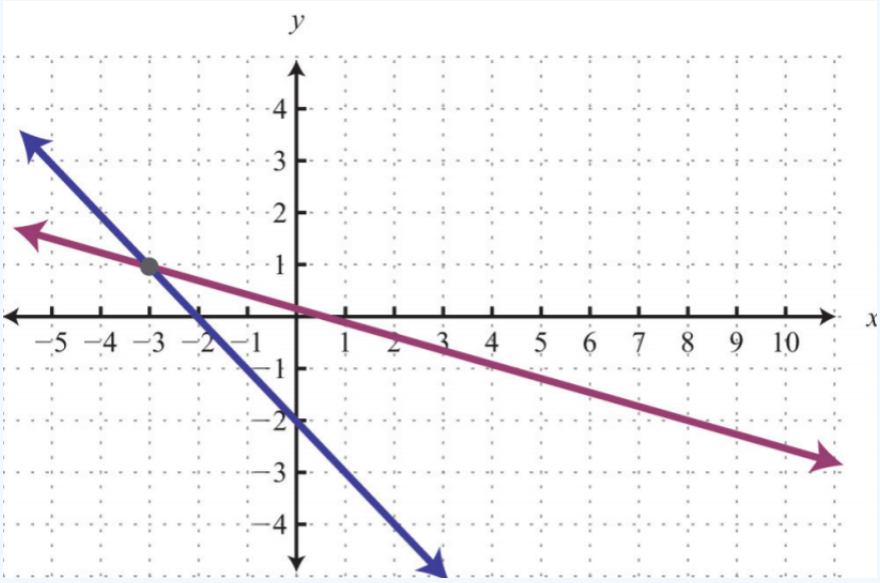


Figure 4.E.1

2.

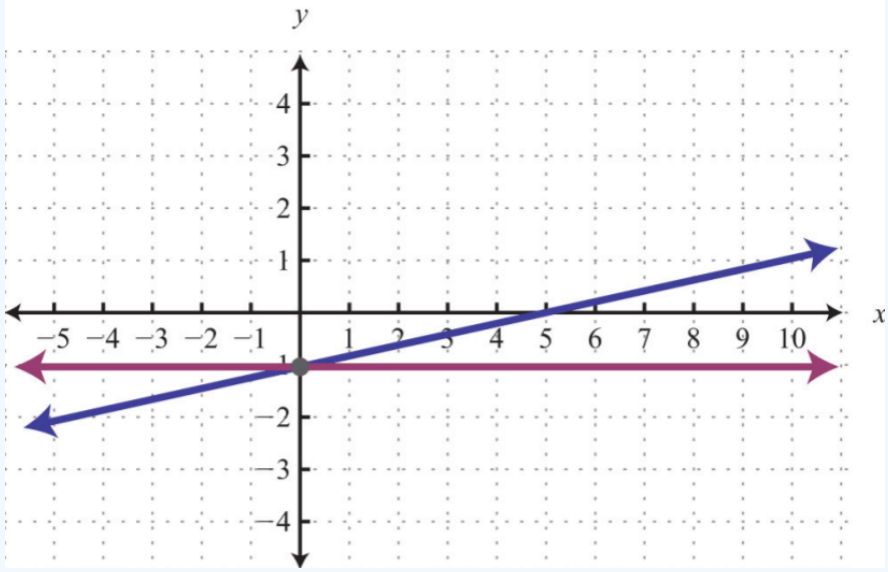


Figure 4.E.2

3.

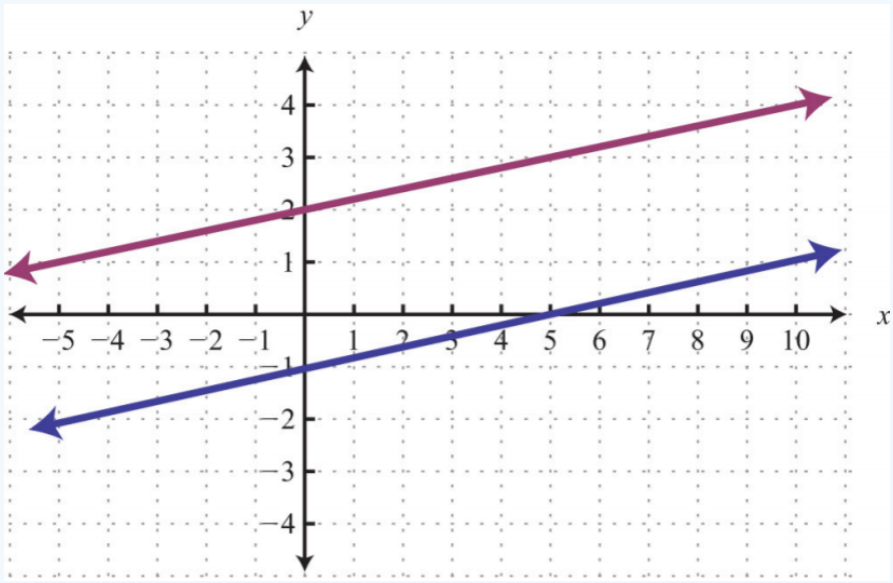


Figure 4.E.3

4.

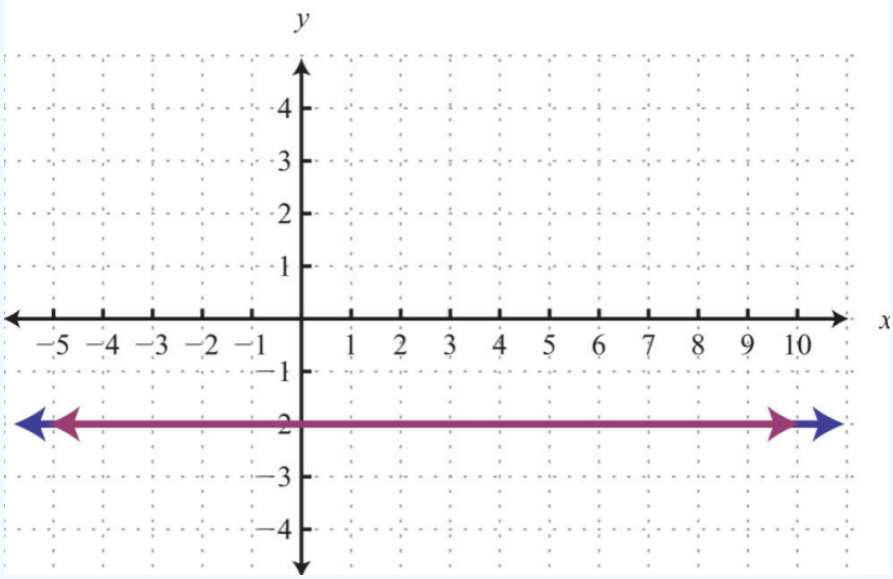


Figure 4.E.4

Answer available at the end of this section.

Exercise $4.E.3$ Solving Linear Systems by Graphing

Solve by graphing.

1. $\begin{cases} y = 12x - 3 \\ y = -\frac{3}{4}x + 2 \end{cases}$
2. $\begin{cases} y = 5 \\ y = -\frac{4}{5}x + 1 \end{cases}$
3. $\begin{cases} x - 2y = 0 \\ 2x - 3y = 3 \end{cases}$
4. $\begin{cases} 5x - y = -11 \\ -4x + 2y = 16 \end{cases}$
5. $\begin{cases} \frac{5}{2}x + 2y = 6 \\ 5x + 4y = 12 \end{cases}$
6. $\begin{cases} 6x - 10y = -2 \\ 3x - 5y = 5 \end{cases}$

Answer available at the end of this section.

Exercise $4.E.4$ Solving Linear Systems by Substitution

Solve by substitution.

1. $\begin{cases} y = 7x - 2 \\ x + y = 6 \end{cases}$
2. $\begin{cases} 2x - 4y = 10 \\ x = -2y - 1 \end{cases}$
3. $\begin{cases} x - y = 0 \\ 5x - 7y = -8 \end{cases}$
4. $\begin{cases} 9x + 2y = -41 \\ -x + y = 7 \end{cases}$

5. $\begin{cases} 6x-3y=4 \\ 2x-9y=4 \end{cases}$
6. $\begin{cases} 8x-y=7 \\ 12x+3y=6 \end{cases}$
7. $\begin{cases} 20x-4y=-3 \\ -5x+y=-12 \end{cases}$
8. $\begin{cases} 3x-y=6 \\ x-13y=2 \end{cases}$
9. $\begin{cases} x=-1 \\ 8x-4y=-10 \end{cases}$
10. $\begin{cases} y=-7 \\ 14x-4y=0 \end{cases}$

Answer available at the end of this section.

Exercise $4.E.5$ Solving Linear Systems by Elimination

Solve by elimination.

1. $\begin{cases} x-y=5 \\ 3x-8y=5 \end{cases}$
2. $\begin{cases} 7x+2y=-10 \\ 9x+4y=-30 \end{cases}$
3. $\begin{cases} 9x-6y=-6 \\ 2x-5y=17 \end{cases}$
4. $\begin{cases} 4x-2y=30 \\ 3x+7y=14 \end{cases}$
5. $\begin{cases} \frac{5}{2}x-2y=-\frac{1}{14} \\ \frac{1}{6}x-\frac{1}{3}y=-\frac{1}{3} \end{cases}$
6. $\begin{cases} 2x-\frac{3}{2}y=20 \\ \frac{3}{32}x-\frac{1}{3}y=\frac{1}{16} \end{cases}$
7. $\begin{cases} 0.1x-0.3y=0.17 \\ 0.6x+0.5y=-0.13 \end{cases}$
8. $\begin{cases} -1.25x-0.45y=-12.23 \\ 0.5x-1.5y=5.9 \end{cases}$
9. $\begin{cases} 6x-52y=-5 \\ -12x+5y=10 \end{cases}$
10. $\begin{cases} 27x+12y=-2 \\ 9x+4y=3 \end{cases}$
11. $\begin{cases} 6x-5y=0 \\ 4x-3y=2 \end{cases}$
12. $\begin{cases} 5x=1 \\ 10x+3y=6 \end{cases}$
13. $\begin{cases} 8y=-2x+6 \\ 3x=6y-18 \end{cases}$
14. $\begin{cases} 6y=3x+1 \\ 9x-27y-3=0 \end{cases}$

Answer available at the end of this section.

Exercise $4.E.6$ Applications of Linear Systems

Set up a linear system and solve.

1. The sum of two numbers is 74 and their difference is 38. Find the numbers.
2. The sum of two numbers is 34. When the larger is subtracted from twice the smaller, the result is 8. Find the numbers.

3. A jar full of 40 coins consisting of dimes and nickels has a total value of $\$2.90$. How many of each coin are in the jar?
4. A total of $\$9,600$ was invested in two separate accounts earning 5.5% and 3.75% annual interest. If the total simple interest earned for the year was $\$491.25$, then how much was invested in each account?
5. A 1% saline solution is to be mixed with a 3% saline solution to produce 6 ounces of a 1.8% saline solution. How much of each is needed?
6. An 80% fruit juice concentrate is to be mixed with water to produce 10 gallons of a 20% fruit juice mixture. How much of each is needed?
7. An executive traveled a total of $4\frac{1}{2}$ hours and 435 miles to a conference by car and by light aircraft. Driving to the airport by car, he averaged 50 miles per hour. In the air, the light aircraft averaged 120 miles per hour. How long did it take him to drive to the airport?
8. Flying with the wind, an airplane traveled $1,065$ miles in 3 hours. On the return trip, against the wind, the airplane traveled 915 miles in 3 hours. What is the speed of the wind?

Answer available at the end of this section.

Exercise $4.E.7$ Systems of Linear Inequalities (Two Variables)

Determine whether the given point is a solution to the system of linear inequalities.

1. $(5,-2)$; $\begin{cases} 5x-y>8 \\ -3x+y\leq -6 \end{cases}$
2. $(2,3)$; $\begin{cases} 2x-3y>-10 \\ -5x+y>1 \end{cases}$
3. $(2,-10)$; $\begin{cases} y<-10x \\ -y\geq 0 \end{cases}$
4. $(0,-2)$; $\begin{cases} y>12x-4 \\ y<-\frac{3}{4}x+2 \end{cases}$

Answer available at the end of this section.

Exercise $4.E.8$ Systems of Linear Inequalities (Two Variables)

Graph the solution set.

1. $\begin{cases} 8x+3y\leq 24 \\ 2x+3y<12 \end{cases}$
2. $\begin{cases} x+y\geq 7 \\ 4x-y\geq 0 \end{cases}$
3. $\begin{cases} x-3y>-12 \\ -2x+6y>-6 \end{cases}$
4. $\begin{cases} y\leq 7 \\ x-y>0 \end{cases}$
5. $\begin{cases} y<4 \\ y\geq \frac{4}{3}x+1 \\ y>-x-1 \end{cases}$
6. $\begin{cases} x-y\geq -3 \\ x-y\leq 3 \\ x+y<1 \end{cases}$

Answer available at the end of this section.

Sample Exam

Exercise $4.E.9$

1. Is $(-3, 2)$ a solution to the system $\begin{cases} 2x - 3y = -12 \\ -4x + y = 14 \end{cases}$?
2. Is $(-2, 9)$ a solution to the system $\begin{cases} x + y \geq 7 \\ 4x - y < 0 \end{cases}$?

Answer available at the end of this section.

Exercise $4.E.10$

Given the graph, determine the simultaneous solution.

1.

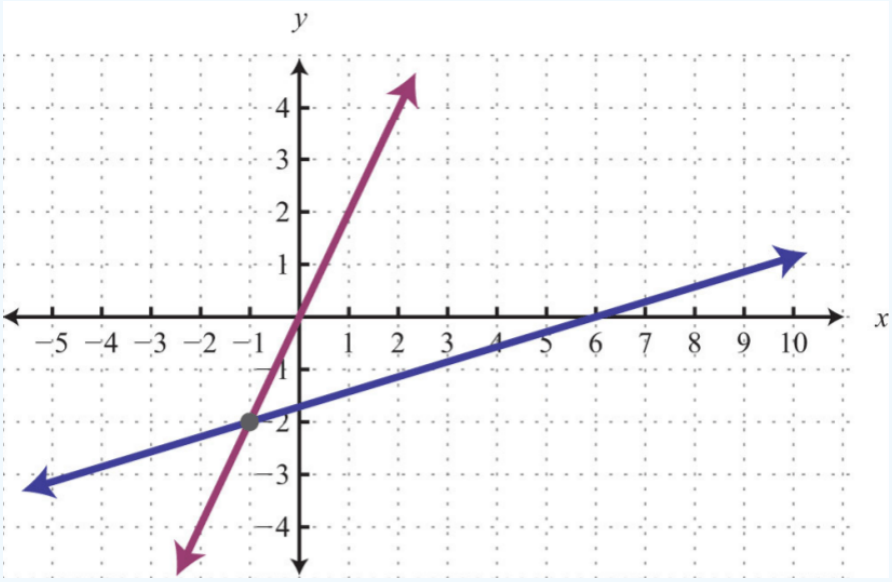


Figure 4.E.8

2.

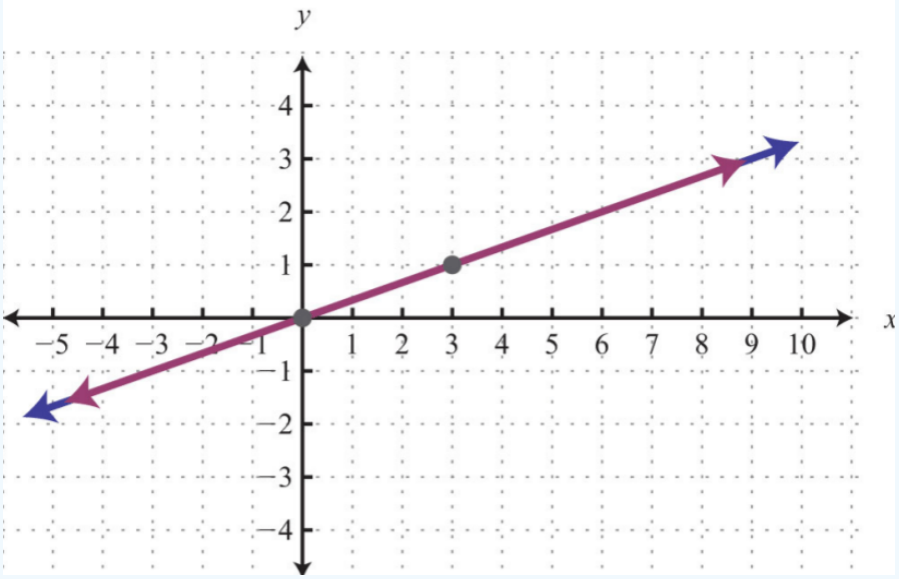


Figure 4.E.9

Answer available at the end of this section.

Exercise $4.E.11$

Solve using the graphing method.

1. $\begin{cases} y = x - 3 \\ y = -12x + 3 \end{cases}$
2. $\begin{cases} 2x + 3y = 6 \\ -x + 6y = -18 \end{cases}$
3. $\begin{cases} y = 2 \\ x + y = 3 \end{cases}$
4. $\begin{cases} y = x \\ x = -5 \end{cases}$

Answer available at the end of this section.

Exercise $4.E.12$

Solve using the substitution method.

1. $\begin{cases} 5x + y = -14 \\ 2x - 3y = -9 \end{cases}$
2. $\begin{cases} 4x - 3y = 1 \\ x - 2y = 2 \end{cases}$
3. $\begin{cases} 5x + y = 1 \\ 10x + 2y = 4 \end{cases}$
4. $\begin{cases} x - 2y = 4 \\ 3x - 6y = 12 \end{cases}$

Answer available at the end of this section.

Exercise $4.E.13$

Solve using the elimination method.

1. $\begin{cases} 4x - y = 13 \\ -5x + 2y = -17 \end{cases}$
2. $\begin{cases} 7x - 3y = -23 \\ 4x + 5y = 7 \end{cases}$
3. $\begin{cases} -3x + 18y = 1 \\ 8x - 6y = 6 \end{cases}$
4. $\begin{cases} -4x + 3y = -3 \\ 8x - 6y = 6 \end{cases}$
5. $\begin{cases} \frac{1}{2}x + \frac{3}{4}y = \frac{7}{4} \\ 4x - \frac{1}{3}y = \frac{4}{3} \end{cases}$
6. $\begin{cases} 0.2x - 0.1y = -0.24 \\ -0.3x + 0.5y = 0.08 \end{cases}$

Answer available at the end of this section.

Exercise $4.E.14$

Graph the solution set.

1. $\begin{cases} 3x + 4y < 2 \\ 4x - 4y < 8 \end{cases}$
2. $\begin{cases} x \leq 8 \\ 3x - 8y \leq 0 \end{cases}$

Answer available at the end of this section.

Exercise $4.E.15$

Set up a linear system of two equations and two variables and solve it using any method.

1. The sum of two integers is 23. If the larger integer is one less than twice the smaller, then find the two integers.
2. James has $\$2,400$ saved in two separate accounts. One account earns 3% annual interest and the other earns 4%. If his interest for the year totals $\$88$, then how much is in each account?
3. Mary drives 110 miles to her grandmother's house in a total of 2 hours. On the freeway, she averages 62 miles per hour. In the city she averages 34 miles per hour. How long does she spend on the freeway?
4. A 15% acid solution is to be mixed with a 35% acid solution to produce 12 ounces of a 22% acid solution. How much of each is needed?
5. Joey has bag full of 52 dimes and quarters with a total value of $\$8.35$. How many of each coin does Joey have?

Answer available at the end of this section.

Answers

Exercise $4.E.1$ Solving Linear Systems by Graphing

1. Yes
3. Yes

Exercise $4.E.2$ Solving Linear Systems by Graphing

1. $(-3, 1)$
3. $\emptyset$

Exercise $4.E.3$ Solving Linear Systems by Graphing

1. $(4, -1)$
3. $(6, 3)$
5. $(x, -\frac{5}{4}x + 3)$

Exercise $4.E.4$ Solving Linear Systems by Substitution

1. $(1, 5)$

3. $(4, 4)$
5. $(\frac{1}{2}, -\frac{1}{3})$
7. $\emptyset$
9. $(-1, \frac{1}{2})$

Exercise $4.E.5$ **Solving Linear Systems by Elimination**

1. $(7, 2)$
3. $(-4, -5)$
5. $(-\frac{1}{2}, \frac{3}{4})$
7. $(0.2, -0.5)$
9. $(x, \frac{12}{5}x + 2)$
11. $(5, 6)$
13. $(-3, \frac{3}{2})$

Exercise $4.E.6$ **Applications of Linear Systems**

1. 18 and 56
3. 18 dimes and 22 nickels
5. 3.6 ounces of the 1% saline solution and 2.4 ounces of the 3% saline solution
7. It took him $1\frac{1}{2}$ hours to drive to the airport

Exercise $4.E.7$ **Systems of Linear Inequalities (Two Variables)**

1. Yes
3. No

Exercise $4.E.8$ **Systems of Linear Inequalities (Two Variables)**

1.

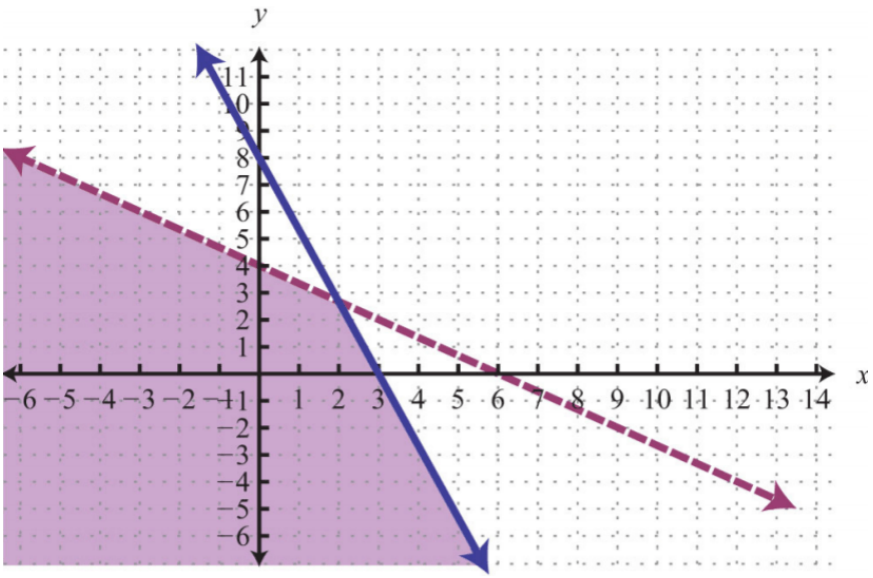


Figure 4.E.5

3.

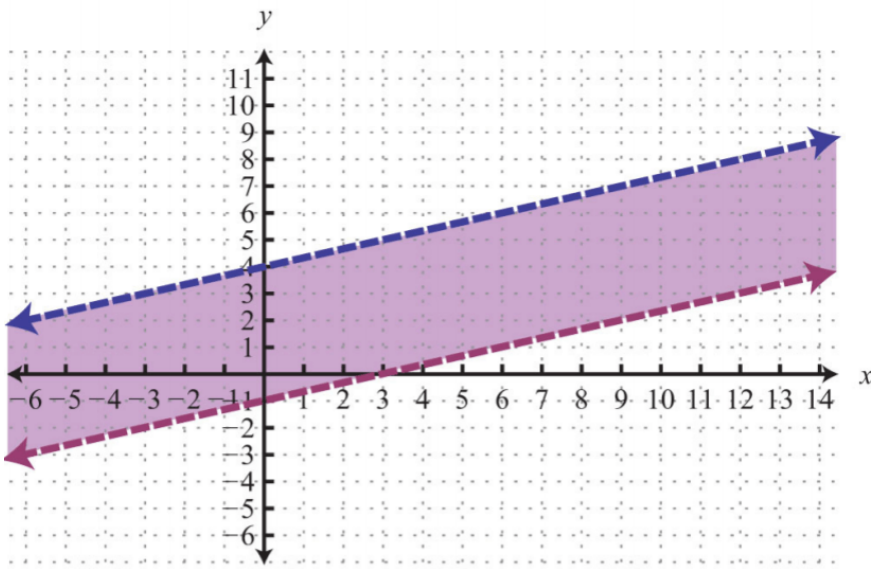


Figure 4.E.6

5.

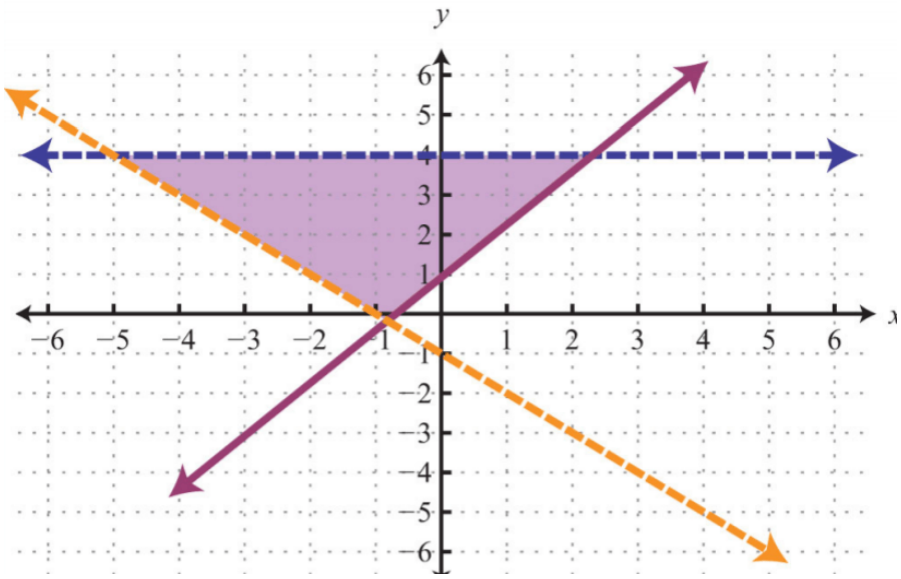


Figure 4.E.7

Exercise $4.E.9$

1. Yes

Exercise $4.E.10$

1. $(-1,-2)$

Exercise $4.E.11$

1. $(4,1)$
3. $(1,2)$

Exercise $4.E.12$

1. $(-3,1)$
3. $\varnothing$

Exercise $4.E.13$

1. $(3,-1)$
3. $\varnothing$
5. $(\frac{1}{2},2)$

Exercise $4.E.14$

1.

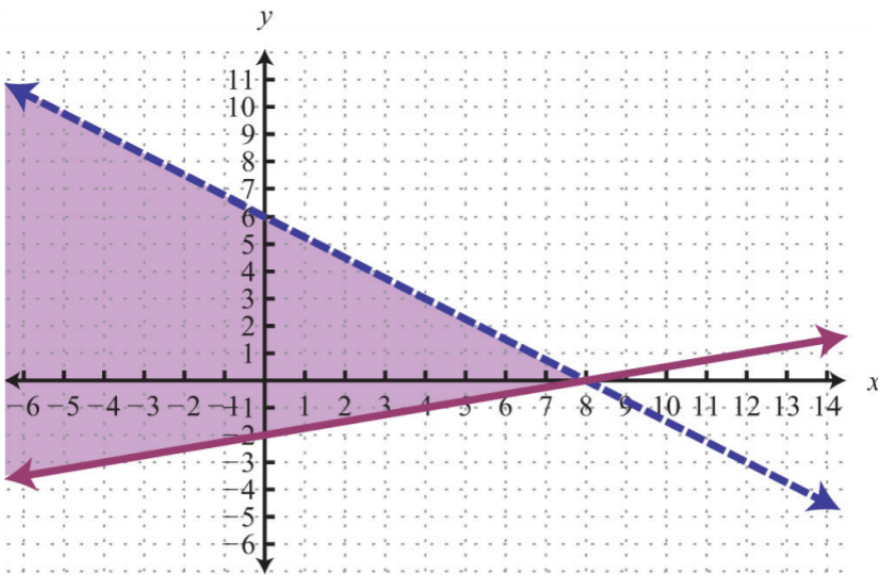


Figure 4.E.10

Exercise $4.E.15$

1. 8 and 15
3. She drives $1\frac{1}{2}$ hours on the freeway.
5. 21 quarters and 31 dimes

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5: POLYNOMIALS AND THEIR OPERATIONS

5.1: Rules of Exponents

5.2: Introduction to Polynomials

5.3: Adding and Subtracting Polynomials

5.4: Multiplying Polynomials

5.5: Dividing Polynomials

5.6: Negative Exponents

5.E: Review Exercises and Sample Exam

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5.1: Rules of Exponents

Learning Objectives

- Simplify expressions using the rules of exponents.
- Simplify expressions involving parentheses and exponents.
- Simplify expressions involving 0 as an exponent.

Product, Quotient, and Power Rule for Exponents

If a factor is repeated multiple times, then the product can be written in exponential form x_n. The positive integer **exponent** n indicates the number of times the **base** x is repeated as a factor

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ times}}$$

For example,

$$5^4 = 5 \cdot 5 \cdot 5 \cdot 5$$

Here the base is 5 and the exponent is 4. Exponents are sometimes indicated with the caret (^) symbol found on the keyboard: 5^ $4 = 5 * 5 * 5 * 5$.

Next consider the product of 2^3 and 2^5,

$$2^3 \cdot 2^5 = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{8 \text{ factors of } 2} = 2^8$$

Expanding the expression using the definition produces multiple factors of the base, which is quite cumbersome, particularly when n is large. For this reason, we will develop some useful rules to help us simplify expressions with exponents. In this example, notice that we could obtain the same result by adding the exponents.

$$2^3 \cdot 2^5 = 2^{3+5} = 2^8$$

In general, this describes the product rule for exponents. If m and n are positive integers, then

$$x^m \cdot x^n = x^{m+n} \tag{5.1.1}$$

In other words, when multiplying two expressions with the same base, add the exponents.

Example 5.1.1

Simplify: $10^5 \cdot 10^{18}$.

Solution:

$$\begin{aligned} 10^5 \cdot 10^{18} &= 10^{5+18} \\ &= 10^{23} \end{aligned}$$

Answer:

$$10^{23}$$

In the previous example, notice that we did not multiply the base 10 times itself. When applying the product rule, add the exponents and leave the base unchanged.

Example 5.1.2

Simplify: $x^6 \cdot x^{12} \cdot x$.

Solution:

Recall that the variable x is assumed to have an exponent of $1 : x = x^1$.

$$\begin{aligned} x^6 \cdot x^{12} \cdot x &= x^6 \cdot x^{12} \cdot x^1 \\ &= x^{6+12+1} \\ &= x^{19} \end{aligned}$$

Answer:

$$x^{19}$$

The base could be any algebraic expression.

Example 5.1.3

Simplify: $(x+y)^9(x+y)^{13}$.

Solution:

Treat the expression $(x+y)$ as the base.

$$\begin{aligned} (x+y)^9(x+y)^{13} &= (x+y)^{9+13} \\ &= (x+y)^{22} \end{aligned}$$

Answer:

$$(x+y)^{22}$$

The commutative property of multiplication allows us to use the product rule for exponents to simplify factors of an algebraic expression.

Example 5.1.4

Simplify: $2x^8y \cdot 3x^4y^7$.

Solution:

Multiply the coefficients and add the exponents of variable factors with the same base.

$$\begin{aligned} 2x^8y \cdot 3x^4y^7 &= 2 \cdot 3 \cdot x^8 \cdot x^4 \cdot y^1 \cdot y^7 && \textit{Commutative property} \\ &= 6 \cdot x^{8+4} \cdot y^{1+7} && \textit{Power rule for exponents} \\ &= 6x^{12}y^8 \end{aligned}$$

Answer:

$$6x^{12}y^8$$

Next, we will develop a rule for division by first looking at the quotient of 2^7 and 2^3.

$$\frac{2^7}{2^3} = \frac{\overbrace{2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2}^{7 \text{ factors of } 2}}{\underbrace{2\cdot2\cdot2}_{3 \text{ factors of } 2}} = \frac{\cancel{2\cdot2\cdot2}\cdot2\cdot2\cdot2\cdot2}{\cancel{2\cdot2\cdot2}} = \frac{2\cdot2\cdot2\cdot2}{1} = 2^4$$

Figure 5.1.3

Here we can cancel factors after applying the definition of exponents. Notice that the same result can be obtained by subtracting the exponents.

$$\frac{2^7}{2^3} = 2^{7-3} = 2^4$$

This describes the quotient rule for exponents. If m and n are positive integers and $x \neq 0$, then

$$\frac{x^m}{x^n} = x^{m-n}$$

In other words, when you divide two expressions with the same base, subtract the exponents.

Example 5.1.5

Simplify: $\frac{12y^{15}}{4y^7}$.

Solution:

Divide the coefficients and subtract the exponents of the variable y.

$$\begin{aligned} \frac{12y^{15}}{4y^7} &= \frac{12}{4} \cdot y^{15-7} \\ &= 3y^8 \end{aligned}$$

Answer:

$$3y^8$$

Example 5.1.6

Simplify: $\frac{20x^{10}(x+5)^6}{10x^9(x+5)^2}$

Solution:

$$\begin{aligned}\frac{20x^{10}(x+5)^6}{10x^9(x+5)^2} &= \frac{20}{10}\cdot x^{10-9}\cdot(x+5)^{6-2}\\ &= 2x^1(x+5)^4\end{aligned}$$

Answer:

$$2x(x+5)^4$$

Now raise 2^3 to the fourth power as follows:

$$\begin{aligned}\left(2^3\right)^4 &= \underbrace{2^3\cdot 2^3\cdot 2^3\cdot 2^3}_{4\ factors\ of\ 2^3}\\ &= \underbrace{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2}_{12\ factors\ of\ 2}\\ &= 2^{12}\end{aligned}$$

After writing the base 2^3 as a factor four times, expand to obtain 12 factors of 2. We can obtain the same result by multiplying the exponents.

$$\left(2^3\right)^4 = 2^{3\cdot 4} = 2^{12}$$

In general, this describes the power rule for exponents. Given positive integers m and n, then

$$\left(x^m\right)^n = x^{m\cdot n} \tag{5.1.2}$$

In other words, when raising a power to a power, multiply the exponents.

Example 5.1.7

Simplify: $\left(y^6\right)^7 = y^{6\cdot 7}$

Solution:

$$\begin{aligned}\left(y^6\right)^7 &= y^{6\cdot 7}\\ &= y^{42}\end{aligned}$$

Answer:

$$y^{42}$$

To summarize, we have developed three very useful rules of exponents that are used extensively in algebra. If given positive integers m and n, then

- Product rule:

$$x^m \cdot x^n = x^{m+n} \tag{5.1.3}$$

- Quotient rule:

$$\frac{x^m}{x^n} = x^{m-n}, x \neq 0 \tag{5.1.4}$$

- Power rule:

$$\left(x^m\right)^n = x^{m \cdot n} \tag{5.1.5}$$

Exercise 5.1.1

Simplify: $y^5 \cdot \left(y^4\right)^6$.

Answer available at the end of this section.

Power Rules for Products and Quotients

Now we consider raising grouped products to a power. For example,

$$\begin{aligned}(xy)^4 &= xy \cdot xy \cdot xy \cdot xy \\ &= x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \quad \textit{Commutative property} \\ &= x^4 \cdot y^4\end{aligned}$$

After expanding, we have four factors of the product xy. This is equivalent to raising each of the original factors to the fourth power. In general, this describes the power rule for a product. If n is a positive integer, then

$$(xy)^n = x^n y^n \tag{5.1.6}$$

Example 5.1.8

Simplify: $(2ab)^7 = 2^7 a^7 b^7$.

Solution:

We must apply the exponent 7 to all the factors, including the coefficient, 2.

$$\begin{aligned}(2ab)^7 &= 2^7 a^7 b^7 \\ &= 128 a^7 b^7\end{aligned}$$

If a coefficient is raised to a relatively small power, then present the real number equivalent, as we did in this example: $2^7 = 128$.

Answer:

$$128 a^7 b^7$$

In many cases, the process of simplifying expressions involving exponents requires the use of several rules of exponents.

Example 5.1.9

Simplify: $(3xy^3)^4$.

Solution:

$$\begin{aligned}(3xy^3)^4 &= 3^4 \cdot x^4 \cdot (y^3)^4 && \textit{Power rule for products}\\ &= 3^4 x^4 y^{3\cdot 4} && \textit{Power rule for exponents}\\ &= 81x^4y^{12}\end{aligned}$$

Answer:

$$81x^4y^{12}$$

Example 5.1.10

Simplify: $(4x^2y^5z)^3$.

Solution:

$$\begin{aligned}(4x^2y^5z)^3 &= 4^3 \cdot (x^2)^3 \cdot (y^5)^3 \cdot z^3\\ &= 64x^6y^{15}z^3\end{aligned}$$

Answer:

$$64x^6y^{15}z^3$$

Example 5.1.11

Simplify: $[5(x+y)^3]^3$

Solution:

$$\begin{aligned}[5(x+y)^3]^3 &= 5^3 \cdot (x+y)^9\\ &= 125(x+y)^9\end{aligned}$$

Answer:

$$125(x+y)^9$$

Next, consider a quotient raised to a power.

$$\begin{aligned}\left(\frac{x}{y}\right)^4 &= \frac{x}{y}\cdot\frac{x}{y}\cdot\frac{x}{y}\cdot\frac{x}{y}\\ &= \frac{x\cdot x\cdot x\cdot x}{y\cdot y\cdot y\cdot y}\\ &= \frac{x^4}{y^4}\end{aligned}$$

Here we obtain four factors of the quotient, which is equivalent to the numerator and the denominator both raised to the fourth power. In general, this describes the power rule for a quotient. If n is a positive integer and $y \neq 0$, then

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \tag{5.1.7}$$

In other words, given a fraction raised to a power, we can apply that exponent to the numerator and the denominator. This rule

requires that the denominator is nonzero. We will make this assumption for the remainder of the section.

Example 5.1.12

Simplify: $\left(\frac{3a}{b}\right)^3$

Solution:

First, apply the power rule for a quotient and then the power rule for a product.

$$\begin{aligned}\left(\frac{3a}{b}\right)^3 &= \frac{(3a)^3}{b^3} && \textit{Power rule for a quotient}\\ &= \frac{3^3 \cdot a^3}{b^3} && \textit{Power rule for a product}\\ &= \frac{27a^3}{b^3}\end{aligned}$$

Answer:

$\frac{27a^3}{b^3}$

In practice, we often combine these two steps by applying the exponent to all factors in the numerator and the denominator.

Example 5.1.13

Simplify: $\left(\frac{ab^2}{2c^3}\right)^5$

Solution:

Apply the exponent 5 to all of the factors in the numerator and the denominator.

$$\begin{aligned}\left(\frac{ab^2}{2c^3}\right)^5 &= \frac{a^5(b^2)^5}{2^5(c^3)^5}\\ &= \frac{a^5b^{10}}{32c^{15}}\end{aligned}$$

Answer:

$\frac{a^5b^{10}}{32c^{15}}$

Example 5.1.14

Simplify: $\left(\frac{5x^5(2x-1)^4}{3y^7}\right)^2$

Solution:

$$\begin{aligned}\left(\frac{5x^5(2x-1)^4}{3y^7}\right)^2 &= \frac{(5x^5(2x-1)^4)^2}{(3y^7)^2} && \textit{Power rule for a quotient}\\ &= \frac{5^2 \cdot (x^5)^2 \cdot [(2x-1)^4]^2}{3^2 \cdot (y^7)^2} && \textit{Power rule for products}\\ &= \frac{25x^{10}(2x-1)^8}{9y^{14}} && \textit{Power rule for exponents}\end{aligned}$$

Answer:

$$\frac{25x^{10}(2x-1)^8}{9y^{14}}$$

It is a good practice to simplify within parentheses before using the power rules; this is consistent with the order of operations.

Example 5.1.15

Simplify: $\left(\frac{-2x^3y^4z}{xy^2}\right)^4$

Solution:

$$\begin{aligned}\left(\frac{-2x^3y^4z}{xy^2}\right)^4 &= (-2\cdot x^{3-1}\cdot y^{4-2}\cdot z)^4 && \textit{Simplify within the parentheses first.}\\ &= (-2\cdot x^2\cdot y^2\cdot z)^4 && \textit{Apply the power rule for a product.}\\ &= (-2)^4\cdot (x^2)^4\cdot (y^2)^4\cdot z^4 && \textit{Apply the power rule for exponents.}\\ &= 16x^8y^8z^4\end{aligned}$$

Answer:

$$16x^8y^8z^4$$

To summarize, we have developed two new rules that are useful when grouping symbols are used in conjunction with exponents. If given a positive integer n, where y is a nonzero number, then

- Power rule for a product:

$$(xy)^n = x^n y^n \tag{5.1.8}$$

- Power rule for a quotient:

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \tag{5.1.9}$$

Exercise 5.1.2

Simplify: $\left(\frac{4x^2(x-y)^3}{3yz^5}\right)^3$

Answer available at the end of this section.

Zero as an Exponent

Using the quotient rule for exponents, we can define what it means to have 0 as an exponent. Consider the following calculation:

\(\color{Cerulean}{1}\color{black}{=\frac{8}{8}=\frac{2^{3}}{2^{3}}=2^{3-3}=}\color{Cerulean}{2^{0}}

Eight divided by 8 is clearly equal to 1, and when the quotient rule for exponents is applied, we see that a 0 exponent results. This leads us to the definition of zero as an exponent, where $x \neq 0$:

$$x^0 = 1 \tag{5.1.10}$$

It is important to note that 0^0 is undefined. If the base is negative, then the result is still $+1$. In other words, any nonzero base raised to the 0 power is defined to be 1. In the following examples, assume all variables are nonzero.

Example 5.1.16

Simplify:

a. $(-5)^0$
b. -5^0

Solution:

a. Any nonzero quantity raised to the 0 power is equal to 1.

$$(-5)^0 = 1$$

b.In the example -5^0, the base is 5, not -5.

Answer:

0. 1
1. -1

Example 5.1.17

Simplify:

$$(5x^3y^0z^2)^2.$$

Solution:

It is good practice to simplify within the parentheses first.

$$\begin{aligned}(5x^3y^0z^2)^2 &= (5x^3 \cdot 1 \cdot z^2)^2 \\ &= (5x^3z^2)^2 \\ &= 5^2x^{3\cdot 2}z^{2\cdot 2} \\ &= 25x^6z^4\end{aligned}$$

Answer:

$$25x^6z^4$$

Example 5.1.18

Simplify:

$$\left(-\frac{8a^{10}b^5}{5c^{12}d^{14}}\right)^0.$$

Solution:

$$\left(-\frac{8a^{10}b^5}{5c^{12}d^{14}}\right)^0 = 1$$

Answer:

$$1$$

Exercise 5.1.3

Simplify:

$$5x^0 \text{ and } (5x)^0$$

Answer available at the end of this section.

Key Takeaways

- The rules of exponents allow you to simplify expressions involving exponents.
- When multiplying two quantities with the same base, add exponents: $x^m \cdot x^n = x^{m+n}$.
- When dividing two quantities with the same base, subtract exponents: $\frac{x^m}{x^n} = x^{m-n}$.
- When raising powers to powers, multiply exponents: $(x^m)^n = x^{m \cdot n}$.
- When a grouped quantity involving multiplication and division is raised to a power, apply that power to all of the factors in the numerator and the denominator: $(xy)^n = x^n y^n$ and $(\frac{x}{y})^n = \frac{x^n}{y^n}$.
- Any nonzero quantity raised to the 0 power is defined to be equal to 1: $x^0 = 1$.

Exercise 5.1.4 Product, Quotient, and Power Rule for Exponents

Write each expression using exponential form.

1. $(2x)(2x)(2x)(2x)(2x)$
2. $(-3y)(-3y)(-3y)$
3. $-10 \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$
4. $12 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y$
5. $-6 \cdot (x-1)(x-1)(x-1)$
6. $(9ab)(9ab)(9ab)(a^2-b)(a^2-b)$

Answer available at the end of this section.

Exercise 5.1.5 Product, Quotient, and Power Rule for Exponents

Simplify.

1. $2^7 \cdot 2^5$
2. $3^9 \cdot 3$
3. -2^4
4. $(-2)^4$
5. -3^3
6. $(-3)^4$
7. $10^{13} \cdot 10^5 \cdot 10^4$
8. $10^8 \cdot 10^7 \cdot 10$
9. $\frac{5^{12}}{5^2}$
10. $\frac{10^7}{10^{10}}$
11. $\frac{10^{12}}{10^9}$
12. $(7^3)^5$
13. $(4^8)^4$
14. $10^6 \cdot (10^5)^4$

Answer available at the end of this section.

Exercise 5.1.6 Product, Quotient, and Power Rule for Exponents

Simplify.

1. $(-x)^6$
2. $a^5 \cdot (-a)^2$
3. $x^3 \cdot x^5 \cdot x$
4. $y^5 \cdot y^4 \cdot y^2$
5. $(a^5)^2 \cdot (a^3)^4 \cdot a$
6. $(x+1)^4 (y^5)^4 \cdot y^2$
7. $(x+1)^5 (x+1)^8$
8. $(2a-b)^{12} (2a-b)^9$
9. $\frac{(3x-1)^5}{(3x-1)^2}$
10. $\frac{(a-5)^{37}}{(a-5)^{13}}$
11. $xy^2 \cdot x^2 y$
12. $3x^2 y^3 \cdot 7xy^5$
13. $-8a^2 b \cdot 2ab$
14. $-3ab^2 c^3 \cdot 9a^4 b^5 c^6$
15. $2a^2 b^4 c(-3abc)$
16. $5a^2 (b^3)^3 c^3 \cdot (-2) 2a^3 (b^2)^4$
17. $2x^2 (x+y)^5 \cdot 3x^5 (x+y)^4$
18. $-5xy^6 (2x-1)^6 \cdot x^5 y (2x-1)^3$
19. $x^2 y \cdot xy^3 \cdot x^5 y^5$
20. $-2x^{10} y \cdot 3x^2 y^{12} \cdot 5xy^3$
21. $3^2 x^4 y^2 z \cdot 3xy^4 z^4$
22. $(-x^2)^3 (x^3)^2 (x^4)^3$
23. $a^{10} \cdot \frac{(a^6)^3}{a^3}$
24. $\frac{10x^9 (x^3)^5}{2x^5}$
25. $\frac{a^6 b^3}{a^2 b^2}$
26. $\frac{m^{10} n^7}{m^3 n^4}$
27. $\frac{20x^5 y^{12} z^3}{10x^2 y^{10} z}$
28. $\frac{-24a^{16} b^{12} c^3}{6a^6 b^{11} c}$
29. $\frac{16x^4 (x+2)^3}{4x(x+2)}$
30. $\frac{50y^2 (x+y)^{20}}{10y(x+y)^{17}}$

Answer available at the end of this section.

Exercise 5.1.7 Power Rules for Products and Quotients

Simplify.

1. $(2x)^5$
2. $(-3y)^4$
3. $(-xy)^3$
4. $(5xy)^3$
5. $(-4abc)^2$
6. $\left(\frac{7}{2x}\right)^2$
7. $-\left(\frac{5}{3y}\right)^3$
8. $(3abc)^3$
9. $\left(\frac{-2xy}{3z}\right)^4$
10. $\left(\frac{5y}{(2x-1)x}\right)^3$
11. $(3x^2)^3$

12. $(-2x^3)^2$
13. $(xy^5)^7$
14. $(x^2y^{10})^2$
15. $\left(\frac{3x^2}{y}\right)^3$
16. $(2x^2y^3z^4)^5$
17. $\left(\frac{-7ab^4}{c^2}\right)^2$
18. $[x^5y^4(x+y)^4]^5$
19. $[2y(x+1)^5]^3$
20. $(ab^3)^3$
21. $\left(\frac{5a^2}{3b}\right)^4$
22. $\left(\frac{-2x^3}{3y^2}\right)^2$
23. $\left(\frac{-x^2}{y^3}\right)^3$
24. $\left(\frac{ab^2}{3c^3d^2}\right)^4$
25. $\left(\frac{2x^7y}{(x-1)^3z^5}\right)^6$
26. $(2x^4)^3 \cdot (x^5)^2$
27. $(x^3y)^2 \cdot (xy^4)^3$
28. $(-2a^2b^3)^2 \cdot (2a^5b)^4$
29. $(-a^2b)^3(3ab^4)^4$
30. $(2x^3(x+y)^4)^5 \cdot (2x^4(x+y)^2)^3$
31. $\left(\frac{-3x^5y^4}{xy^2}\right)^3$
32. $\left(\frac{-3x^5y^4}{xy^2}\right)^2$
33. $\left(\frac{-25x^{10}y^{15}}{5x^5y^{10}}\right)^3$
34. $\left(\frac{10x^3y^5}{5xy^2}\right)^2$
35. $\left(\frac{-24ab^3}{6bc}\right)^5$
36. $\left(\frac{-2x^3y^{16}}{x^2y}\right)^2$
37. $\left(\frac{30ab^3}{3abc}\right)^3$
38. $\left(\frac{3s^3t^2}{2s^2t}\right)^3$
39. $\left(\frac{6xy^5(x+y)^6}{3y^2z(x+y)^2}\right)^5$
40. $\left(\frac{-64a^5b^{12}c^2(2ab-1)^{14}}{32a^2b^{10}c^2(2ab-1)^7}\right)^4$
41. The probability of tossing a fair coin and obtaining n heads in a row is given by the formula $P = (12)^n$. Determine the probability, as a percent, of tossing 5 heads in a row.
42. The probability of rolling a single fair six-sided die and obtaining n of the same faces up in a row is given by the formula $P = (16)^n$. Determine the probability, as a percent, of obtaining the same face up two times in a row.
43. If each side of a square measures $2x^3$ units, then determine the area in terms of the variable x.
44. If each edge of a cube measures $5x^2$ units, then determine the volume in terms of the variable x.

Answer available at the end of this section.

Exercise 5.1.8 Zero Exponents

Simplify. (Assume variables are nonzero.)

1. 7^0
2. $(-7)^0$
3. -10^0
4. $-3^0 \cdot (-7)^0$
5. 8675309^0
6. $5^2 \cdot 3^0 \cdot 2^3$
7. $-3^0 \cdot (-2)^2 \cdot (-3)^0$

8. $\frac{5x^0}{y^2}$
9. $(-3)^2x^2y^0z^5$
10. $-3^2(x^3)^2y^2(z^3)^0$
11. $2x^3y^0z \cdot 3x^0y^3z^5$
12. $-3ab^2c^0 \cdot 3a^2(b^3c^2)^0$
13. $(-8xy^2)^0$
14. $(2x^2y^3)^0$
15. $\frac{9x^0y^4}{3y^3}$

Answer available at the end of this section.

Exercise 5.1.9 Discussion Board Topics

1. René Descartes (1637) established the usage of exponential form: a^2, a^3, and so on. Before this, how were exponents denoted?
2. Discuss the accomplishments accredited to Al-Karismi.
3. Why is 0^0 undefined?
4. Explain to a beginning student why $3^4 \cdot 3^2 \neq 9^6$.

Answer available at the end of this section.

Answers

Exercise 5.1.1

y^{29}

Exercise 5.1.2

$\frac{64x^6(x-y)^9}{27y^3z^{15}}$

Exercise 5.1.3

$5x^0 = 5$ and $(5x)^0 = 1$

Exercise 5.1.4 **Product, Quotient, and Power Rule for Exponents**

1. $(2x)^5$
3. $-10a^7$
5. $-6(x-1)^3$

Exercise 5.1.5 **Product, Quotient, and Power Rule for Exponents**

1. 2^{12}
3. -16
5. -27
7. 10^{22}
9. 5^{10}
11. 10^3
13. 4^{32}

Exercise 5.1.6 **Product, Quotient, and Power Rule for Exponents**

1. x^6
3. x^9
5. a^{23}
7. $(x+1)^{13}$
9. $(3x-1)^3$
11. x^3y^3
13. $-16a^3b^2$
15. $-6a^3b^5c^2$
17. $6x^7(x+y)^9$
19. x^8y^9
21. $27x^5y^6z^5$
23. a^{25}
25. a^4b
27. $2x^3y^2z^2$
29. $4x^3(x+2)^2$

Exercise 5.1.7 **Power Rules for Products and Quotients**

1. $32x^5$
3. $-x^3y^3$
5. $16a^2b^2c^2$
7. $-\frac{125}{27y^3}$
9. $\frac{16x^4y^4}{81z^4}$
11. $27x^6$
13. x^7y^{35}
15. $\frac{27x^6}{y^3}$
17. $\frac{49a^2b^8}{c^4}$
19. $8y^3(x+1)^{15}$
21. $\frac{625a^8}{81b^4}$
23. $-\frac{x^6}{y^9}$
25. $\frac{64x^{42}y^6}{(x-1)^{18}z^{30}}$
27. x^9y^{14}
29. $-81a^{10}b^{19}$
31. $-27x^{12}y^6$
33. $-125x^{15}y^{15}$
35. $\frac{-1024a^5b^{10}}{c^5}$
37. $\frac{1000b^6}{c^3}$
39. $\frac{32x^5y^{15}(x+y)^{20}}{z^5}$
41. $3\frac{1}{8}\%$
43. $A=4x^6$

Exercise 5.1.8 **Zero Exponents**

1. 1
3. -1
5. 1
7. -4
9. $9x^2z^5$
11. $6x^3y^3z^6$
13. 1
15. $3y$

Exercise 5.1.9 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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5.2: Introduction to Polynomials

Learning Objectives

- Identify a polynomial and determine its degree.
- Evaluate a polynomial for given values of the variables.
- Evaluate a polynomial using function notation.

Definitions

A polynomial is a special algebraic expression with terms that consist of real number coefficients and variable factors with whole number exponents.

$$\textit{Examples of polynomials :}$$

$$3x^2 \quad 7xy+5 \quad \tfrac{3}{2}x^3+3x^2-\tfrac{1}{2}x+1 \quad 6x^2y-4xy^3-4xy^3+7$$

Polynomials do not have variables in the denominator of any term.

$$\textit{Examples that are not polynomials :}$$

$$\tfrac{2x^2}{y} \quad 5\sqrt{x}+5 \quad 5x^2+3x^{-2}+7 \quad \tfrac{2}{x}-\tfrac{5}{y}=3$$

The degree of a term in a polynomial is defined to be the exponent of the variable, or if there is more than one variable in the term, the degree is the sum of their exponents. Recall that $x^0=1$; any constant term can be written as a product of x^0 and itself. Hence the degree of a constant term is 0.

Table 5.2.1

Term	Degree
$3x^2$	2
$6x^2y$	$2+1=3$
$7a^2b^3$	$2+3=5$
8	0, since $8=8x^0$
$2x$	1, since $x=x^1$

The degree of a polynomial is the largest degree of all of its terms.

Table 5.2.2

Polynomial	Degree
$4x^5-3x^3+2x-1$	5
$6x^2y-5xy^3+7$	4, because $5xy^3$ has degree 4.
$12x+54$	1, because $x=x^1$

We classify polynomials by the number of terms and the degree as follows:

Table 5.2.3

Expression	Classification	Degree
$5x^7$	Monomial (one term)	7
$8x^6 - 1$	Binomial (two terms)	6
$-3x^2 + x - 1$	Trinomial (three terms)	2
$5x^3 - 2x^2 + 3x - 6$	Polynomial (many terms)	3

In this text, we will call polynomials with four or more terms simply polynomials.

Example 5.2.1

Classify and state the degree:

$$7x^2 - 4x^5 - 1.$$

Solution:

Here there are three terms. The highest variable exponent is 5. Therefore, this is a trinomial of degree 5.

Answer:

Trinomial; degree 5

Example 5.2.2

Classify and state the degree:

$$12a^5bc^3.$$

Solution:

Since the expression consists of only multiplication, it is one term, a monomial. The variable part can be written as $a^5b^1c^3$; hence its degree is $5 + 1 + 3 = 9$.

Answer:

Monomial; degree 9

Example 5.2.3

Classify and state the degree:

$$4x^2y - 6xy^4 + 5x^3y^3 + 4.$$

Solution:

The term $4x^2y$ has degree 3; $-6xy^4$ has degree 5; $5x^3y^3$ has degree 6; and the constant term 4 has degree 0. Therefore, the polynomial has 4 terms with degree 6.

Answer:

Polynomial; degree 6

Of particular interest are polynomials with one variable, where each term is of the form $a_n x^n$. Here a_n is any real number and n is any whole number. Such polynomials have the standard form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \tag{5.2.1}$$

Typically, we arrange terms of polynomials in descending order based on the degree of each term. The leading coefficient is the coefficient of the variable with the highest power, in this case, a_n.

Example 5.2.4

Write in standard form:

$$3x - 4x^2 + 5x^3 + 7 - 2x^4.$$

Solution:

Since terms are separated by addition, write the following:

$$\begin{aligned} &3x - 4x^2 + 5x^3 + 7 - 2x^4 \\ &= 3x + (-4x^2) + 5x^3 + 7 + (-2x^4) \end{aligned}$$

In this form, we can see that the subtraction in the original corresponds to negative coefficients. Because addition is commutative, we can write the terms in descending order based on the degree of each term as follows:

$$\begin{aligned} &= (-2x^4) + 5x^3 + (-4x^2) + 3x + 7 \\ &= -2x^4 + 5x^3 - 4x^2 + 3x + 7 \end{aligned}$$

Answer:

$$-2x^4 + 5x^3 - 4x^2 + 3x + 7$$

We can further classify polynomials with one variable by their degree as follows:

Table 5.2.4

Polynomial	Name
5	**Constant** (degree 0)
$2x + 1$	**Linear** (degree 1)
$3x^2 + 5x - 3$	**Quadratic** (degree 2)
$x^3 + x^2 + x + 1$	**Cubic** (degree 3)
$7x^4 + 3x^3 - 7x + 8$	Fourth-degree polynomial

In this text, we call any polynomial of degree $n \geq 4$ an nth-degree polynomial. In other words, if the degree is 4, we call the polynomial a fourth-degree polynomial. If the degree is 5, we call it a fifth-degree polynomial, and so on.

Evaluating Polynomials

Given the values for the variables in a polynomial, we can substitute and simplify using the order of operations.

Example 5.2.5

Evaluate:

$3x - 1$, where $x = -\frac{3}{2}$.

Solution:

First, replace the variable with parentheses and then substitute the given value.

Answer:

$-\frac{11}{2}$

Example 5.2.6

Evaluate:

$3x^2 + 2x - 1$, where $x = -1$.

Solution:

Answer:

0

Example 5.2.7

Evaluate:

$-2a^2b + ab^2 - 7$, where $a = 3$ and $b = -2$.

Solution:

Answer:

41

Example 5.2.8

The volume of a sphere in cubic units is given by the formula $V = \frac{4}{3}\pi r^3$, where r is the radius. Calculate the volume of a sphere with radius $r = \frac{3}{2}$ meters.

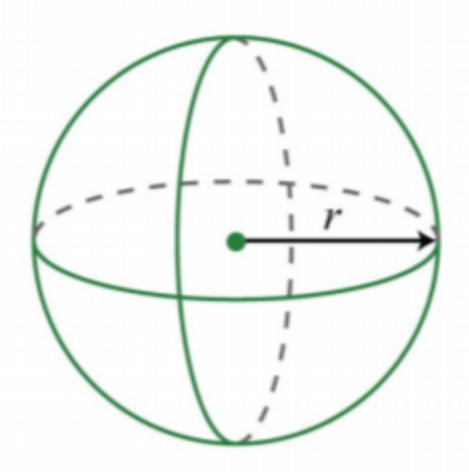


Figure 5.2.1

Solution:

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi\left(\frac{3}{2}\right)^3 \\ &= \frac{4}{3}\pi\frac{3^3}{2^3} \\ &= \frac{\overset{1}{\cancel{4}}}{\cancel{3}}\pi\frac{\overset{9}{\cancel{27}}}{\cancel{8}} \\ &= \frac{9}{2}\pi \end{aligned}$$

Answer:

$\frac{9}{2}\pi$ cubic meters

Exercise 5.2.1

Evaluate:

$x^3 - x^2 + 4x - 2$, where $x = -3$.

Answer available at the end of this section.

Polynomial Functions

Polynomial functions with one variable are functions that can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 \tag{5.2.2}$$

,

where a_n is any real number and n is any whole number. Some examples of the different classes of polynomial functions are listed below:

Table 5.2.5

Polynomial function	Name
$f(x) = 5$	Constant function (degree 0)
$f(x) = -2x + 1$	Linear function (degree 1)
$f(x) = 5x^2 + 4x - 3$	Quadratic function (degree 2)
$f(x) = x^3 - 1$	Cubic function (degree 3)
$f(x) = 4x^5 + 3x^4 - 7$	**Polynomial function**

Since there are no restrictions on the values for x, the domain of any polynomial function consists of all real numbers.

Example 5.2.9

Calculate:

$$f(5), \text{ given } f(x) = -2x^2 + 5x + 10.$$

Solution:

Recall that the function notation $f(5)$ indicates we should evaluate the function when $x = 5$. Replace every instance of the variable x with the value 5.

Answer:

$f(5) = -15$

Example 5.2.10

Calculate:

$$f(-1), \text{ given } f(x) = -x^3 + 2x^2 - 4x + 1.$$

Solution:

Replace the variable x with -1.

$$\begin{aligned} f(-1) &= -(-1)^3 + 2(-1)^2 - 4(-1) + 1 \\ &= -(-1) + 2 \cdot 1 + 4 + 1 \\ &= 1 + 2 + 4 + 1 \\ &= 8 \end{aligned}$$

Answer:

$$f(-1) = 8$$

Exercise 5.2.2

Given $g(x) = x^3 - 2x^2 - x - 4$, calculate $g(-1)$.

Answer available at the end of this section.

Key Takeaways

- Polynomials are special algebraic expressions where the terms are the products of real numbers and variables with whole number exponents.
- The degree of a polynomial with one variable is the largest exponent of the variable found in any term.
- The terms of a polynomial are typically arranged in descending order based on the degree of each term.
- When evaluating a polynomial, it is a good practice to replace all variables with parentheses and then substitute the appropriate values.
- All polynomials are functions.

Exercise 5.2.3 Definitions

Classify the given polynomial as linear, quadratic, or cubic.

1. $2x + 1$
2. $x^2 + 7x + 2$
3. $2 - 3x^2 + x$
4. $4x$
5. $x^2 - x^3 + x + 1$
6. $5 - 10x^3$

Answer available at the end of this section.

Exercise 5.2.4 Definitions

Classify the given polynomial as a monomial, binomial, or trinomial and state the degree.

1. $x^3 - 1$
2. x^2y^2
3. $x - x^5 + 1$
4. $x^2 + 3x - 1$
5. $5ab^4$
6. $13x - 12$
7. $-5x^3 + 2x + 1$
8. $8x^2 - 9$
9. $4x^5 - 5x^3 + 6x$
10. $8x^4 - x^5 + 2x - 3$
11. $9x + 7$
12. $x^5 + x^4 + x^3 + x^2 - x + 1$
13. $6x - 1 + 5x^4 - 8$
14. $4x - 3x^2 + 3$
15. 7
16. x^2
17. $4x^2y - 3x^3y^3 + xy^3$
18. $a^3b^2 - 6ab$
19. a^3b^3
20. $x^2y - y^2x$
21. $xy - 3$
22. $a^5bc^2 + 3a^9 - 5a^4b^3c$
23. $-3x^{10}y^2z - xy^{12}z + 9x^{13} + 30$
24. $7x^0$

Answer available at the end of this section.

Exercise 5.2.5 Definitions

Write the following polynomials in standard form.

1. $1-6x+7x^2$
2. $x-9x^2-8$
3. $7-x^3+x^7-x^2+x-5x^5$
4. $a^3-a^9+6a^5-a+3-a^4$

Answer available at the end of this section.

Exercise 5.2.6 Evaluating Polynomials

1. Fill in the following chart:

x	x^2
-3	
-2	
0	
2	
3	

Figure 5.2.2

2. Fill in the following chart:

x	x^3
-3	
-2	
0	
2	
3	

Figure 5.2.3

Answer available at the end of this section.

Exercise 5.2.7 Evaluating Polynomials

Evaluate.

1. $2x - 3$, where $x = 3$
2. $x^2 - 3x + 5$, where $x = -2$
3. $-12x + 13$, where $x = -13$
4. $-x^2 + 5x - 1$, where $x = -12$
5. $-2x^2 + 3x - 5$, where $x = 0$
6. $8x^5 - 27x^3 + 81x - 17$, where $x = 0$
7. $y^3 - 2y + 1$, where $y = -2$
8. $y^4 + 2y^2 - 32$, where $y = 2$
9. $a^3 + 2a^2 + a - 3$, where $a = -3$
10. $x^3 - x^2$, where $x = 5$
11. $34x^2 - 12x + 36$, where $x = -23$
12. $58x^2 - 14x + 12$, where $x = 4$
13. $x^2y + xy^2$, where $x = 2$ and $y = -3$
14. $2a^5b - ab^4 + a^2b^2$, where $a = -1$ and $b = -2$
15. $a^2 - b^2$, where $a = 5$ and $b = -6$
16. $a^2 - b^2$, where $a = 34$ and $b = -14$
17. $a^3 - b^3$, where $a = -2$ and $b = 3$
18. $a^3 + b^3$, where $a = 5$ and $b = -5$

Answer available at the end of this section.

Exercise 5.2.8 Evaluating Polynomials

For each problem, evaluate $b^2 - 4ac$, given the following values.

1. $a = -1, b = 2$, and $c = -1$
2. $a = 2, b = -2$, and $c = 12$
3. $a = 3, b = -5, c = 0$
4. $a = 1, b = 0$, and $c = -4$
5. $a = 14, b = -4$, and $c = 2$
6. $a = 1, b = 5$, and $c = 6$

Answer available at the end of this section.

Exercise 5.2.9 Evaluating Polynomials

The volume of a sphere in cubic units is given by the formula $V = \frac{4}{3}\pi r^3$, where r is the radius. For each problem, calculate the volume of a sphere given the following radii.

1. $r = 3$ centimeters
2. $r = 1$ centimeter
3. $r = \frac{1}{2}$ feet
4. $r = \frac{3}{2}$ feet
5. $r = 0.15$ in
6. $r = 1.3$ inches

Answer available at the end of this section.

Exercise 5.2.10 Evaluating Polynomials

The height in feet of a projectile launched vertically from the ground with an initial velocity v_0 in feet per second is given by the formula $h = -16t^2 + v_0 t$, where t represents time in seconds. For each problem, calculate the height of the projectile given the following initial velocity and times.

1. $v_0 = 64$ feet/second, at times $t = 0, 1, 2, 3, 4$ seconds
2. $v_0 = 80$ feet/second, at times $t = 0, 1, 2, 2.5, 3, 4, 5$ seconds

Answer available at the end of this section.

Exercise 5.2.11 Evaluating Polynomials

The stopping distance of a car, taking into account an average reaction time, can be estimated with the formula $d = 0.05v^2 + 1.5$, where d is in feet and v is the speed in miles per hour. For each problem, calculate the stopping distance of a car traveling at the given speeds.

1. 20 miles per hour
2. 40 miles per hour
3. 80 miles per hour
4. 100 miles per hour

Answer available at the end of this section.

Exercise 5.2.12 Polynomial Functions

Given the linear function $f(x) = \frac{2}{3}x + 6$, evaluate each of the following.

1. $f(-6)$
2. $f(-3)$
3. $f(0)$
4. $f(3)$
5. Find x when $f(x) = 10$.
6. Find x when $f(x) = -4$.

Answer available at the end of this section.

Exercise 5.2.13 Polynomial Functions

Given the quadratic function $f(x) = 2x^2 - 3x + 5$, evaluate each of the following.

1. $f(-2)$
2. $f(-1)$
3. $f(0)$
4. $f(2)$

Answer available at the end of this section.

Exercise 5.2.14 Polynomial Functions

Given the cubic function $g(x) = x^3 - x^2 + x - 1$, evaluate each of the following.

1. $g(-2)$
2. $g(-1)$
3. $g(0)$
4. $g(1)$

Answer available at the end of this section.

Exercise 5.2.15 Polynomial Functions

The height in feet of a projectile launched vertically from the ground with an initial velocity of 128 feet per second is given by the function $h(t) = -16t^2 + 128t$, where t is in seconds. Calculate and interpret the following.

1. $h(0)$
2. $h(12)$
3. $h(1)$
4. $h(3)$
5. $h(4)$
6. $h(5)$
7. $h(7)$
8. $h(8)$

Answer available at the end of this section.

Exercise 5.2.16 Discussion Board Topics

1. Find and share some graphs of polynomial functions.
2. Explain how to convert feet per second into miles per hour.
3. Find and share the names of fourth-degree, fifth-degree, and higher polynomials.

Answer available at the end of this section.

Answers

Exercise 5.2.1

-50

Exercise 5.2.2

$g(-1) = -6$

Exercise 5.2.3 **Definitions**

1. Linear
3. Quadratic
5. Cubic

Exercise 5.2.4 **Definitions**

1. Binomial; degree 3
3. Trinomial; degree 5
5. Monomial; degree 5
7. Trinomial; degree 3
9. Trinomial; degree 5
11. Binomial; degree 1
13. Not a polynomial
15. Monomial; degree 0
17. Trinomial; degree 6
19. Monomial; degree 6
21. Binomial; degree 2
23. Polynomial; degree 14

Exercise 5.2.5 **Definitions**

1. $7x^2 - 6x + 1$

3. $x^7 - 5x^5 - x^3 - x^2 + x + 7$

Exercise 5.2.6 **Evaluating Polynomials**

1.

x	x^2
-3	9
-2	4
0	0
2	4
3	9

Figure 5.2.4

Exercise 5.2.7 **Evaluating Polynomials**

1. 3
3. $\frac{1}{2}$
5. -5
7. -3
9. -15
11. $\frac{7}{6}$
13. 6
15. -11
17. -35

Exercise 5.2.8 **Evaluating Polynomials**

1. 0
3. 25
5. 14

Exercise 5.2.9 **Evaluating Polynomials**

1. 36π cubic centimeters
3. $\frac{\pi}{6}$ cubic feet
5. 0.014 cubic inches

Exercise 5.2.10 **Evaluating Polynomials**

1.

Table 5.2.6

Time	Height
$t = 0$ seconds	$h = 0$ feet
$t = 1$ second	$h = 48$ feet
$t = 2$ seconds	$h = 64$ feet
$t = 3$ seconds	$h = 48$ feet
$t = 4$ seconds	$h = 0$ feet

Exercise 5.2.11 **Evaluating Polynomials**

1. 21.5 feet
3. 321.5 feet

Exercise 5.2.12 **Polynomial Functions**

1. 2
3. 6
5. $x = 6$

Exercise $5.2.13$ **Polynomial Functions**

1. 19
3. 5

Exercise $5.2.14$ **Polynomial Functions**

1. -15
3. -1

Exercise $5.2.15$ **Polynomial Functions**

1. The projectile is launched from the ground.
3. The projectile is 112 feet above the ground 1 second after launch.
5. The projectile is 256 feet above the ground 4 seconds after launch.
7. The projectile is 112 feet above the ground 7 seconds after launch.

Exercise $5.2.16$ **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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5.3: Adding and Subtracting Polynomials

Learning Objectives

- Add polynomials.
- Subtract polynomials.
- Add and subtract polynomial functions

Adding Polynomials

Recall that we combine like terms, or terms with the same variable part, as a means to simplify expressions. To do this, add the coefficients of the terms to obtain a single term with the same variable part. For example,

$$5x^2 + 8x^2 = 13x^2 \tag{5.3.1}$$

Notice that the variable part, x^2, does not change. This, in addition to the commutative and associative properties of addition, allows us to add polynomials.

Example 5.3.1

Add:

$$3x + (4x - 5).$$

Solution:

The property $+(a + b) = a + b$, which was derived using the distributive property, allows us to remove the parentheses so that we can add like terms.

Answer:

$7x - 5$

Example 5.3.2

Add:

$$(3x^2 + 3x + 5) + (2x^2 - x - 2).$$

Solution:

Remove the parentheses and then combine like terms.

Answer:

$5x^2 + 2x + 3$

Example 5.3.3

Add:

$$(-5x^2y - 2xy^2 + 7xy) + (4x^2y + 7xy^2 - 3xy).$$

Solution:

Remember that the variable parts have to be exactly the same before we can add the coefficients.

$$\begin{aligned}&(-5x^2y - 2xy^2 + 7xy) + (4x^2y + 7xy^2 - 3xy)\\ &= \underline{-5x^2y} \underline{\underline{-2xy^2}} + \underline{\underline{\underline{7xy}}} + \underline{4x^2y} + \underline{\underline{7xy^2}} - \underline{\underline{\underline{3xy}}}\\ &= -x^2y + 5xy^2 + 4xy\end{aligned}$$

Answer:

$$-x^2y + 5xy^2 + 4xy$$

It is common practice to present the terms of the simplified polynomial expression in descending order based on their degree. In other words, we typically present polynomials in standard form, with terms in order from highest to lowest degree.

Example 5.3.4

Add:

$$(a - 4a^3 + a^5 - 8) + (-9a^5 + a^4 - 7a + 5 + a^3).$$

Solution:

Answer:

$-8a^5 + a^4 - 3a^3 - 6a - 3$

Exercise 5.3.1

Add:

$$(6 - 5x^3 + x^2 - x) + (x^2 + x + 6x^3 - 1).$$

Answer available at the end of this section.

Subtracting Polynomials

When subtracting polynomials, we see that the parentheses become very important. Recall that the distributive property allowed us to derive the following:

$$-(a + b) = -a - b \tag{5.3.2}$$

In other words, when subtracting an algebraic expression, we remove the parentheses by subtracting each term.

Example 5.3.5

Subtract:

$$10x - (3x + 5).$$

Solution:

Subtract each term within the parentheses and then combine like terms.

Answer:

$7x - 5$

Subtracting a quantity is equivalent to multiplying it by -1.

Example 5.3.6

Subtract:

$$(3x^2 + 3x + 5) - (2x^2 - x - 2).$$

Solution:

Distribute the -1, remove the parentheses, and then combine like terms.

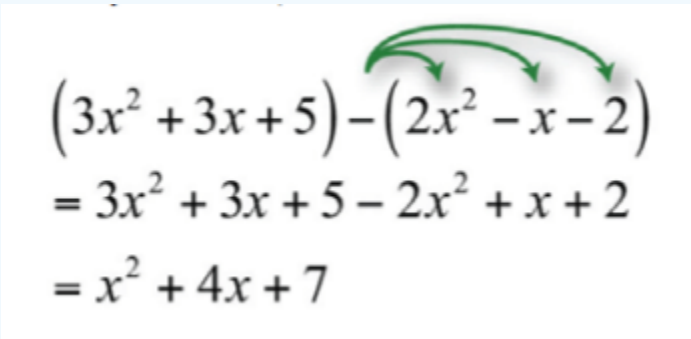


Figure 5.3.1

Answer:

$$x^2 + 4x + 7$$

Multiplying the terms of a polynomial by -1 changes all the signs.

Example 5.3.7

Subtract:

$$(-5x^3 - 2x^2 + 7) - (4x^3 + 7x^2 - 3x + 2).$$

Solution:

Distribute the -1, remove the parentheses, and then combine like terms.

Answer:

$-9x^3 - 9x^2 + 3x + 5$

Example 5.3.8

Subtract $6x^2 - 3x - 1$ from $2x^2 + 5x - 2$.

Solution

Since subtraction is not commutative, we must take care to set up the difference correctly. First, write the quantity $(2x^2 + 5x - 2)$; from this, subtract the quantity $(6x^2 - 3x - 1)$.

Answer:

$-4x^2 + 8x - 1$

Example 5.3.9

Simplify:

$$(2x^2 - 3x + 5) - (x^2 - 3x + 1) + (5x^2 - 4x - 8).$$

Solution:

Apply the distributive property, remove the parentheses, and then combine like terms.

Answer:

$6x^2 - 4x - 4$

Exercise 5.3.2

Subtract:

$$(8x^2y - 5xy^2 + 6) - (x^2y + 2xy^2 - 1).$$

Answer available at the end of this section.

Adding and Subtracting Polynomial Functions

We use function notation to indicate addition and subtraction of functions as follows:

Table 5.3.1

Addition of functions:	$(f + g)(x) = f(x) + g(x)$
Subtraction of functions:	$(f - g)(x) = f(x) - g(x)$

When using function notation, be careful to group the entire function and add or subtract accordingly.

Example 5.3.10

Calculate:

$(f + g)(x)$, given $f(x) = -x^2 - 3x + 5$ and $g(x) = 3x^2 + 2x + 1$.

Solution:

The notation $(f + g)(x)$ indicates that you should add the functions $f(x) + g(x)$ and collect like terms.

$$\begin{aligned} f(x)+g(x) &= (-x^2-3x+5)+(3x^2+2x+1) \\ &= -x^2-3x+5+3x^2+2x+1 \\ &= 2x^2-x+6 \end{aligned}$$

Answer:

$$(f+g)(x) = 2x^2-x+6$$

Example 5.3.11

Calculate:

$(f-g)(x)$, given $f(x)=2x-3$ and $g(x)=-2x^2+2x+5$.

Solution:

The notation $(f-g)(x)$ indicates that you should subtract the functions $f(x)-g(x)$:

Answer:

$(f-g)(x) = 2x^2-8$

We may be asked to evaluate the sum or difference of two functions. We have the option to first find the sum or difference and use the resulting function to evaluate for the given variable, or to first evaluate each function and then find the sum or difference.

Example 5.3.12

Calculate:

$(f-g)(5)$, given $f(x)=x^2+x-7$ and $g(x)=4x+10$.

Solution:

First, find $(f-g)(x)=f(x)-g(x)$.

Therefore,

$(f-g)(x) = x^2-3x-17$

Next, substitute 5 for the variable x.

Answer:

$(f-g)(5) = -7$

Alternate Solution:

Since $(f-g)(5)=f(5)-g(5)$, we can find $f(5)$ and $g(5)$ and then subtract the results.

$$\begin{aligned} f(x) &= x^2+x-7 & g(x) &= 4x+10 \\ f(5) &= (5)^2+(5)-7 & g(5) &= 4(5)+10 \\ &= 25+5-7 & &= 20+10 \\ &= 23 & &= 30 \end{aligned}$$

Therefore, we have

Answer:

$(f-g)(5)=-7$

Key Takeaways

- When adding polynomials, remove the associated parentheses and then combine like terms.
- When subtracting polynomials, distribute the -1 and subtract all the terms before removing the parentheses and combining like terms.
- The notation $(f+g)(x)$ indicates that you add the functions.
- The notation $(f-g)(x)$ indicates that you subtract the functions

Exercise 5.3.3 Addition of Polynomials

Add.

1. $(2x+1)+(-x+7)$
2. $(-6x+5)+(3x-1)$
3. $(\frac{2}{3}x+\frac{1}{2})+(\frac{1}{3}x-2)$
4. $(\frac{1}{3}x-\frac{3}{4})+(\frac{5}{6}x+\frac{1}{8})$
5. $(2x+1)+(x-3)+(5x-2)$
6. $(2x-8)+(-3x^2+7x-5)$
7. $(x^2-3x+7)+(3x^2-8x-5)$
8. $(-5x^2-1+x)+(-x+7x^2-9)$
9. $(\frac{1}{2}x^2-\frac{1}{3}x+\frac{1}{6})+(-\frac{3}{2}x^2+\frac{2}{3}x-1)$
10. $(-\frac{3}{5}x2+\frac{1}{4}x-6)+(2x^2-\frac{3}{8}x+\frac{5}{2})$
11. $(x^2+5)+(3x^2-2x+1)+(x^2+x-3)$
12. $(a^3-a^2+a-8)+(a^3+a^2+6a-2)$
13. $(a^3-8)+(-3a^3+5a^2-2)$
14. $(4a^5+5a^3-a)+(3a^4-2a^2+7)$
15. $(2x^2+5x-12)+(7x-5)$
16. $(3x+5)+(x^2-x+1)+(x^3+2x^2-3x+6)$
17. $(6x^5-7x^3+x^2-15)+(x^4+2x^3-6x+12)$
18. $(1+7x-5x^3+4x^4)+(-3x^3+5-x^2+x)$
19. $(x^2y^2-7xy+7)+(4x^2y^2-3xy-8)$
20. $(x^2+xy-y^2)+(7x^2-5xy+2y^2)$
21. $(2x^2+3xy-7y^2)+(-5x^2-3xy+8y^2)$
22. $(a^2b^2-100)+(2a^2b^2-3ab+20)$
23. $(ab^2-3a^2b+ab-3)+(-2a^2b+ab^2-7ab-1)$
24. $(10a^2b-7ab+8ab^2)+(6a^2b-ab+5ab^2)$
25. Find the sum of $2x+8$ and $7x-1$.
26. Find the sum of $13x-15$ and $16x+110$.
27. Find the sum of $x^2-10x+8$ and $5x^2-2x-6$.
28. Find the sum of $a^2-5a+10$ and $-9a^2+7a-11$.
29. Find the sum of x^2y^2-xy+6 and x^2y^2+xy-7.
30. Find the sum of $x^2-9xy+7y^2$ and $-3x^2-3xy+7y^2$.

Answer available at the end of this section.

Exercise 5.3.4 Subtraction of Polynomials

Subtract.

1. $(5x-3)-(2x-1)$

2. $(-4x+1)-(7x+10)$
3. $(\frac{1}{4}x-\frac{3}{4})-(\frac{3}{4}x+\frac{1}{8})$
4. $(-\frac{3}{5}x+\frac{3}{7})-(\frac{2}{5}x-\frac{3}{2})$
5. $(x^2+7x-5)-(4x^2-5x+1)$
6. $(-6x^2+3x-12)-(-6x^2+3x-12)$
7. $(-3x^3+4x-8)-(-x^2+4x+10)$
8. $(\frac{1}{2}x^2+\frac{1}{3}x-\frac{3}{4})-(\frac{3}{2}x^2-\frac{1}{6}x+\frac{1}{2})$
9. $(\frac{5}{9}x^2+\frac{1}{5}x-\frac{1}{3})-(\frac{1}{3}x^2+\frac{3}{10}x+\frac{5}{9})$
10. $(a^3-4a^2+3a-7)-(7a^3-2a^2-6a+9)$
11. $(3a^3+5a^2-2)-(a^3-a+8)$
12. $(5x^5+4x^3+x^2-6)-(4x^4-3x^3-x+3)$
13. $(3-5x-x^3+5x^4)-(-5x^3+2-x^2-7x)$
14. $(x^5-6x^3+9x)-(4x^4+2x^2-5)$
15. $(2x^2y^2-4xy+9)-(3x^2y^2-3xy-5)$
16. $(x^2+xy-y^2)-(x^2+xy-y^2)$
17. $(2x^2+3xy-7y^2)-(-5x^2-3xy+8y^2)$
18. $(ab^2-3a^2b+ab-3)-(-2a^2b+ab^2-7ab-1)$
19. $(10a^2b-7ab+8ab^2)-(6a^2b-ab+5ab^2)$
20. $(10a^2b^2+5ab-6)-(5a^2b^2+5ab-6)$
21. Subtract $3x+1$ from $5x-9$.
22. Subtract $x^2-5x+10$ from x^2+5x-5.
23. Find the difference of $3x-7$ and $8x+6$.
24. Find the difference of $2x^2+3x-5$ and x^2-9.
25. The cost in dollars of producing customized coffee mugs with a company logo is given by the formula $C=150+0.10x$, where x is the number of cups produced. The revenue from selling the cups in the company store is given by $R=10x-0.05x^2$, where x is the number of units sold.
    a. Find a formula for the profit. (profit = revenue – cost)
    b. Find the profit from producing and selling 100 mugs in the company store.
26. The cost in dollars of producing sweat shirts is given by the formula $C=10q+1200$, where C is the cost and q represents the quantity produced. The revenue generated by selling the sweat shirts for \$37 each is given by $R=37q$, where q represents the quantity sold. Determine the profit generated if 125 sweat shirts are produced and sold.
27. The outer radius of a washer is 3 times the radius of the hole.

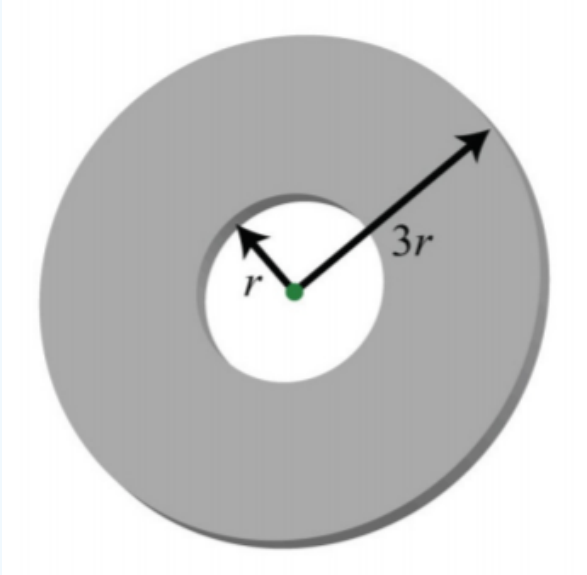


Figure $5.3.2$

a. Derive a formula for the area of the face of the washer.
b. What is the area of the washer if the hole has a diameter of 10 millimeters?

28. Derive a formula for the surface area of the following rectangular solid.

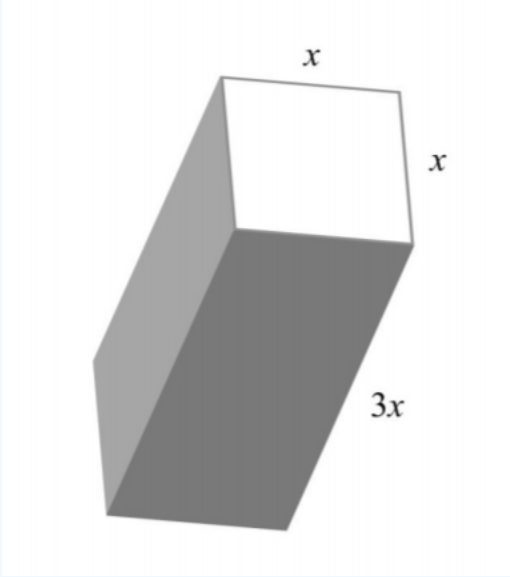


Figure 5.3.3

Answer available at the end of this section.

Exercise 5.3.5 Addition and Subtraction of Polynomial

Simplify.

1. $(2x+3)-(5x-8)+(x-7)$
2. $(3x-5)-(7x-11)-(5x+2)$
3. $(3x-2)-(4x-1)+(x+7)$
4. $(5x-3)-(2x+1)-(x-1)$
5. $(5x^2-3x+2)-(x^2+x-4)+(7x^2-2x-6)$
6. $(-2x^3+x^2-8)-(3x^2+x-6)-(2x-1)$
7. $(2x-7)-(x^2+3x-7)+(6x-1)$
8. $(6x^2-10x+13)+(4x^2-9)-(9-x^2)$
9. $(a^2-b^2)-(2a^2+3ab-4b^2)+(5ab-1)$
10. $(a^2-3ab+b^2)-(a^2+\ b^2)-(3ab-5)$
11. $(\frac{1}{2}x^2-\frac{3}{4}x+\frac{1}{4})-(\frac{3}{2}x-\frac{3}{4})+(\frac{5}{4}x-\frac{1}{2})$
12. $(\frac{9}{5}x^2-\frac{1}{3}x+2)-(\frac{3}{10}x^2-\frac{4}{5})-(x+\frac{5}{2})$

Answer available at the end of this section.

Exercise 5.3.6 Addition and Subtraction of Polynomial Functions

Find $(f+g)(x)$ and $(f-g)(x)$, given the following functions.

1. $f(x)=4x-1$ and $g(x)=-3x+1$
2. $f(x)=-x+5$ and $g(x)=2x-3$
3. $f(x)=3x^2-5x+7$ and $g(x)=-2x^2+5x-1$
4. $f(x)=x^3+2x^2-6x+2$ and $g(x)=2x^3+2x^2-5x-1$
5. $f(x)=\frac{1}{2}x+\frac{1}{3}$ and $g(x)=\frac{1}{5}x^2-\frac{3}{2}x+\frac{1}{6}$
6. $f(x)=x^2-5x+\frac{1}{3}$ and $g(x)=\frac{2}{3}x^2-x-\frac{1}{2}$

Answer available at the end of this section.

Exercise 5.3.7 Addition and Subtraction of Polynomial Functions

Given $f(x) = 2x - 3$ and $g(x) = x^2 + 3x - 1$, find the following.

1. $(f + g)(x)$
2. $(g + f)(x)$
3. $(f - g)(x)$
4. $(g - f)(x)$
5. $(g + g)(x)$
6. $(f + g)(3)$
7. $(f + g)(-2)$
8. $(f + g)(0)$
9. $(f - g)(0)$
10. $(f - g)(-2)$
11. $(g - f)(-2)$
12. $(g - f)(12)$

Answer available at the end of this section.

Exercise 5.3.8 Addition and Subtraction of Polynomial Functions

Given $f(x) = 5x^2 - 3x + 2$ and $g(x) = 2x^2 + 6x - 4$, find the following.

1. $(f + g)(x)$
2. $(g + f)(x)$
3. $(f - g)(x)$
4. $(g - f)(x)$
5. $(f + g)(-2)$
6. $(f - g)(-2)$
7. $(f + g)(0)$
8. $(f - g)(0)$

Answer available at the end of this section.

Answers

Exercise 5.3.1

$x^3 + 2x^2 + 5$

Exercise 5.3.2

$7x^2y - 7xy^2 + 7$

Exercise 5.3.3 **Addition of Polynomials**

1. $x + 8$
3. $x - \frac{3}{2}$
5. $8x - 4$
7. $4x^2 - 11x + 2$
9. $-x^2 + \frac{1}{3}x - \frac{5}{6}$
11. $5x^2 - x + 3$
13. $-2a^3 + 5a^2 - 10$
15. $2x^2 + 12x - 17$
17. $6x^5 + x^4 - 5x^3 + x^2 - 6x - 3$
19. $5x^2y^2 - 10xy - 1$
21. $-3x^2 + y^2$
23. $-5a^2b + 2ab^2 - 6ab - 4$
25. $9x + 7$
27. $6x^2 - 12x + 2$
29. $2x^2y^2 - 1$

Exercise 5.3.4 **Subtraction of Polynomials**

1. $3x - 2$
3. $-\frac{1}{2}x - \frac{7}{8}$
5. $-3x^2 + 12x - 6$
7. $-3x^3 + x^2 - 18$
9. $\frac{2}{9}x^2 - \frac{1}{10}x - \frac{8}{9}$
11. $2a^3 + 5a^2 + a - 10$
13. $5x^4 + 4x^3 + x^2 + 2x + 1$
15. $-x^2y^2 - xy + 14$
17. $7x^2 + 6xy - 15y^2$
19. $4a^2b + 3ab^2 - 6ab$
21. $2x - 10$
23. $-5x - 13$
25. a. $P = -0.05x^2 + 9.9x - 150$; b. $340
27. a. $A = 8\pi r^2$; b. 628.32 square millimeters

Exercise 5.3.5 **Addition and Subtraction of Polynomial**

1. $-2x + 4$
3. 6
5. $11x^2 - 6x$
7. $-x^2 + 5x - 1$
9. $-a^2 + 2ab + 3b^2 - 1$
11. $12x^2 - x + 12$

Exercise 5.3.6 **Addition and Subtraction of Polynomial Functions**

1. $(f + g)(x) = x$ and $(f - g)(x) = 7x - 2$
3. $(f + g)(x) = x^2 + 6$ and $(f - g)(x) = 5x^2 - 10x + 8$
5. $(f + g)(x) = \frac{1}{5}x^2 - x + \frac{1}{2}$ and $(f - g)(x) = -\frac{1}{5}x^2 + 2x + \frac{1}{6}$

Exercise 5.3.7 **Addition and Subtraction of Polynomial Functions**

1. $(f + g)(x) = x^2 + 5x - 4$
3. $(f - g)(x) = -x^2 - x - 2$
5. $(g + g)(x) = 2x^2 + 6x - 2$
7. $(f + g)(-2) = -10$
9. $(f - g)(0) = -2$
11. $(g - f)(-2) = 4$

Exercise 5.3.8 **Addition and Subtraction of Polynomial Functions**

1. $(f + g)(x) = 7x^2 + 3x - 2$
3. $(f - g)(x) = 3x^2 - 9x + 6$
5. $(f + g)(-2) = 20$
7. $(f + g)(0) = -2$

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5.4: Multiplying Polynomials

Learning Objectives

- Multiply a polynomial by a monomial.
- Multiply a polynomial by a binomial.
- Multiply a polynomial by any size polynomial.
- Recognize and calculate special products.
- Multiply polynomial functions.

Multiplying by a Monomial

Recall the product rule for exponents: if m and n are positive integers, then

$$x^m \cdot x^n = x^{m+n} \tag{5.4.1}$$

In other words, when multiplying two expressions with the same base, add the exponents. This rule applies when multiplying a monomial by a monomial. To find the product of monomials, multiply the coefficients and add the exponents of variable factors with the same base. For example,

$$\begin{aligned} 3x \cdot 5x^2 &= 3 \cdot 5 \cdot x^1 \cdot x^2 && \textit{Commutative property} \\ &= 15x^{1+2} && \textit{Product rule for exponents} \\ &= 15x^3 \end{aligned}$$

To multiply a polynomial by a monomial, apply the distributive property and then simplify each term.

Example 5.4.1

Multiply:

$$-5x(4x-2).$$

Solution:

In this case, multiply the monomial, $-5x$, by the binomial, $4x-2$. Apply the distributive property and then simplify.

$$\begin{aligned} &-5x(4x-2) \\ &= (-5x) \cdot 4x + (-5x) \cdot (-2) \\ &= -20x^2 + 10x \end{aligned}$$

Figure 5.4.1

Answer:

$$-20x^2 + 10x$$

Example 5.4.2

Multiply:

$$2x^2(3x^2 - 5x + 1).$$

Solution:

Apply the distributive property and then simplify.

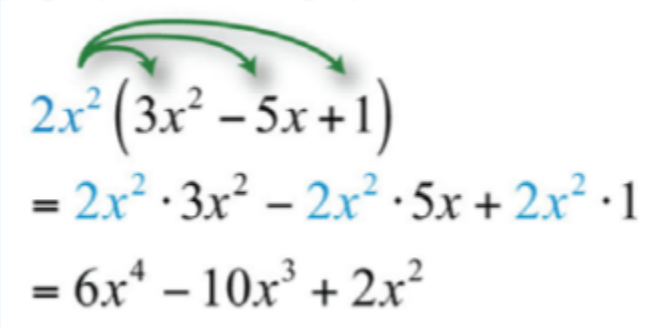


Figure 5.4.2

Answer:

$$6x^4 - 10x^3 + 2x^2$$

Example 5.4.3

Multiply:

$$-3ab^2(a^2b^3 + 2a^3b - 6ab - 4).$$

Solution:

Answer:

To summarize, multiplying a polynomial by a monomial involves the distributive property and the product rule for exponents. Multiply all of the terms of the polynomial by the monomial. For each term, multiply the coefficients and add exponents of variables where the bases are the same.

Exercise 5.4.1

Multiply:

$$-5x^2y(2xy^2 - 3xy + 6x^2y - 1).$$

Answer available at the end of this section.

Multiplying by a Binomial

In the same way that we used the distributive property to find the product of a monomial and a binomial, we will use it to find the product of two binomials.

$$\begin{aligned}(a+b)(c+d) &\\ &= (a+b)\cdot c + (a+b)\cdot d \\ &= ac+bc+ad+bd \\ &= ac+ad+bc+bd\end{aligned} \tag{5.4.2}$$

Here we apply the distributive property multiple times to produce the final result. This same result is obtained in one step if we apply the distributive property to a and b separately as follows:

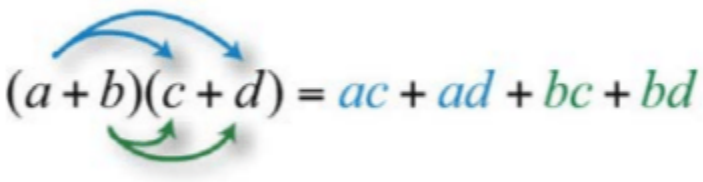


Figure 5.4.3

This is often called the FOIL method. We add the products of the first terms of each binomial ac, the $o$uter terms ad, the $i$nner terms bc, and finally the last terms bd. This mnemonic device only works for products of binomials; hence it is best to just remember that the distributive property applies.

Example 5.4.4

Multiply:

$$(2x+3)(5x-2).$$

Solution:

Distribute $2x$ and then distribute 3.

Simplify by combining like terms.

$= 10x^2 + 11x - 6$

Answer:

$$10x^2 + 11x - 6$$

Example 5.4.5

Multiply:

$$(\tfrac{1}{2}x - \tfrac{1}{4})(\tfrac{1}{2}x + \tfrac{1}{4}).$$

Solution:

Distribute $\frac{1}{2}x$ and then distribute $-\frac{1}{4}$.

$$\begin{aligned}(\frac{1}{2}x-\frac{1}{4})(\frac{1}{2}x+\frac{1}{4}) &= \frac{1}{2}x\frac{1}{2}x+\frac{1}{2}x\cdot\frac{1}{4}+\left(-\frac{1}{4}\right)\cdot\frac{1}{2}x+\left(-\frac{1}{4}\right)\cdot\frac{1}{4} \\ &= \frac{1}{4}x^2+\frac{1}{8}x-\frac{1}{8}x-\frac{1}{16} \\ &= \frac{1}{4}x^2-\frac{1}{16}\end{aligned}$$

Answer:

$$\frac{1}{4}x^2 - \frac{1}{16}$$

Example 5.4.6

Multiply:

$$(3y^2 - 1)(2y + 1).$$

Solution:

Answer:

$6y^3 + 3y^2 - 2y - 1$

After applying the distributive property, combine any like terms.

Example 5.4.7

Multiply:

$$(x^2 - 5)(3x^2 - 2x + 2).$$

Solution:

After multiplying each term of the trinomial by x^2 and -5, simplify.

Answer:

$3x^4 - 2x^3 - 13x^2 + 10x - 10$

Example 5.4.8

Multiply:

$$(2x - 1)^3.$$

Solution:

Perform one product at a time.

$$
\begin{aligned}
(2x-1)^3 &= (2x-1)\underbrace{(2x-1)(2x-1)}_{\text{multiply first}} \\
&= (2x-1)\underbrace{(4x^2-2x-2x+1)}_{\text{simplify}} \\
&= \underbrace{(2x-1)(4x^2-4x+1)}_{\text{multiply second}} \\
&= 8x^3-8x^2+2x-4x^2+4x-1 \\
&= 8x^3-12x^2+6x-1
\end{aligned}
$$

Figure 5.4.4

Answer:

$$8x^3 - 12x^2 + 6x - 1$$

At this point, it is worth pointing out a common mistake:

$$(2x-1)^3 \neq (2x)^3 - (1)^3$$

The confusion comes from the product to a power rule of exponents, where we apply the power to all factors. Since there are two terms within the parentheses, that rule does not apply. Care should be taken to understand what is different in the following two examples:

$$(xy)^2 = x^2y^2 \quad \checkmark$$
$$(x+y)^2 \neq x^2+y^2 \quad x$$

Exercise 5.4.2

Multiply:

$$(2x-3)(7x^2-5x+4).$$

Answer available at the end of this section.

Product of Polynomials

When multiplying polynomials, we apply the distributive property many times. Multiply all of the terms of each polynomial and then combine like terms.

Example 5.4.9

Multiply:

$$(2x^2+x-3)(x^2-2x+5).$$

Solution:

Multiply each term of the first trinomial by each term of the second trinomial and then combine like terms.

Aligning like terms in columns, as we have here, aids in the simplification process

Answer:

$2x^4 - 3x^3 + 5x^2 + 11x - 15$

Notice that when multiplying a trinomial by a trinomial, we obtain nine terms before simplifying. In fact, when multiplying an n -term polynomial by an m-term polynomial, we will obtain $n \times m$ terms. In the previous example, we were asked to multiply and found that

$$(2x^2 + x - 3)(x^2 - 2x + 5) = 2x^4 - 3x^3 + 5x^2 + 11x - 15$$

Because it is easy to make a small calculation error, it is a good practice to trace through the steps mentally to verify that the operations were performed correctly. Alternatively, we can check by evaluating any value for x in both expressions to verify that the results are the same. Here we choose $x = 2$:

$$\begin{aligned}(2x^2 + x - 3)(x^2 - 2x + 5) &= (2(2)^2 + (2) - 3)((2)^2 - 2(2) + 5) \\ &= (8 + 2 - 3)(4 - 4 + 5) \\ &= (7)(5) \\ &= 35\end{aligned}$$

Because the results could coincidentally be the same, a check by evaluating does not necessarily prove that we have multiplied correctly. However, after verifying a few values, we can be fairly confident that the product is correct.

Multiply:

$$(x^2 - 2x - 3)^2.$$

Answer

$x^4 - 4x^3 - 2x^2 + 12x + 9$

Special Products

In this section, the goal is to recognize certain special products that occur often in our study of algebra. We will develop three formulas that will be very useful as we move along. The three should be memorized. We begin by considering the following two calculations:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= a^2 + ab + ba + b^2 \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2\end{aligned} \qquad \begin{aligned}(a-b)^2 &= (a-b)(a-b) \\ &= a^2 - ab - ba + b^2 \\ &= a^2 - ab - ab + b^2 \\ &= a^2 - 2ab + b^2\end{aligned}$$

This leads us to two formulas that describe perfect square trinomials:

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{5.4.3}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \tag{5.4.4}$$

We can use these formulas to quickly square a binomial.

Example 5.4.10

Multiply:

$$(3x+5)^2.$$

Solution:

Here $a=3x$ and $b=5$. Apply the formula:

$$\begin{array}{rl} (a+b)^2= & a^2 \quad +2 \quad a \quad b \quad + \quad b^2 \\ & \downarrow \qquad\quad \downarrow \quad \downarrow \qquad \downarrow \\ (3x+5)^2 &= (3x)^2+2\cdot(3x)(5)+(5)^2 \\ &= 9x^2+30x+25 \end{array}$$

Answer:

$$9x^2+30x+25$$

This process should become routine enough to be performed mentally.

Example 5.4.11

Multiply:

$$(x-4)^2.$$

Solution:

Here $a=x$ and $b=4$. Apply the appropriate formula as follows:

$$\begin{array}{rl} (a-b)^2= & a^2 \quad -2 \quad a \quad b \quad + \quad b^2 \\ & \downarrow \qquad\quad \downarrow \quad \downarrow \qquad \downarrow \\ (x-4)^2 &= (x)^2-2\cdot(x)(4)+(4)^2 \\ &= x^2-8x+16 \end{array}$$

Answer:

$$x^2-8x+16$$

Our third special product follows:

$$\begin{aligned} (a+b)(a-b) &= a^2-ab+ba-b^2 \\ &= a^2 \color{red}{-ab+ab} -b^2 \\ &= a^2-b^2 \end{aligned}$$

This product is called difference of squares:

$$(a+b)(a-b)=a^2-b^2 \tag{5.4.5}$$

The binomials $(a+b)$ and $(a-b)$ are called conjugate binomials. Therefore, when conjugate binomials are multiplied, the middle term eliminates, and the product is itself a binomial.

Example 5.4.12

Multiply:

$$(7x+4)(7x-4).$$

Solution:

Answer:

$49x^2-16$

Exercise 5.4.4

Multiply:

$$(-5x+2)^2.$$

Answer available at the end of this section.

Multiplying Polynomial Functions

We use function notation to indicate multiplication as follows:

Table 5.4.1

Multiplication of functions:	$(f \cdot g)(x) = f(x) \cdot g(x)$

Example 5.4.13

Calculate:

$(f \cdot g)(x)$, given $f(x) = 5x^2$ and $g(x) = -x^2 + 2x - 3$.

Solution:

Multiply all terms of the trinomial by the monomial function $f(x)$.

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x) \\ &= 5x^2 \cdot (-x^2 + 2x - 3) \\ &= -5x^4 + 10x^3 - 15x^2\end{aligned}$$

Answer:

$$(f \cdot g)(x) = -5x^4 + 10x^3 - 15x^2$$

Example 5.4.14

Calculate:

$(f \cdot g)(-1)$, given $f(x) = -x + 3$ and $g(x) = 4x^2 - 3x + 6$.

Solution:

First, determine $(f \cdot g)(x)$.

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x) \\ &= (-x+3)(4x^2-3x+6) \\ &= -4x^3+3x^2-6x+12x^2-9x+18 \\ &= -4x^3+15x^2-15x+18\end{aligned}$$

We have

\((f\cdot g)(x) = -4x^{3}+15x^{2}-15x+18

Next, substitute -1 for the variable x.

$$\begin{aligned}(f \cdot g)(-1) &= -4(-1)^3+15(-1)^2-15(-1)+18 \\ &= -4 \cdot (-1)+15 \cdot 1+15+18 \\ &= 4+15+15+18 \\ &= 52\end{aligned}$$

Answer:

$$(f \cdot g)(-1) = 52$$

Because $(f \cdot g)(-1) = f(-1) \cdot g(-1)$, we could alternatively calculate $f(-1)$ and $g(-1)$ separately and then multiply the results (try this as an exercise). However, if we were asked to evaluate multiple values for the function $(f \cdot g)(x)$, it would be best to first determine the general form, as we have in the previous example.

Key Takeaways

- To multiply a polynomial by a monomial, apply the distributive property and then simplify each of the resulting terms.
- To multiply polynomials, multiply each term in the first polynomial with each term in the second polynomial. Then combine like terms.
- The product of an n-term polynomial and an m-term polynomial results in an $m \times n$ term polynomial before like terms are combined.
- Check results by evaluating values in the original expression and in your answer to verify that the results are the same.
- Use the formulas for special products to quickly multiply binomials that occur often in algebra.

Exercise 5.4.5 Product of a Monomial and a Polynomial

Multiply.

1. $5x(-3x^2y)$
2. $(-2x^3y^2)(-3xy^4)$
3. $\frac{1}{2}(4x-3)$
4. $-\frac{3}{4}(\frac{2}{3}x-6)$
5. $3x(5x-2)$
6. $-4x(2x-1)$
7. $x^2(3x+2)$
8. $-6x^2(5x+3)$
9. $2ab(4a-2b)$
10. $5a^2b(a^2-b^2)$
11. $6x^2y^3(-3x^3y+xy^2)$
12. $3ab^3(-5ab^3+6a^2b)$
13. $-\frac{1}{2}x^2y(4xy-10)$
14. $-3x^4y^2(3x^8y^3)$
15. $2x^2(-5x^3)(3x^4)$
16. $4ab(a^2b^3c)(a^4b^2c^4)$

17. $-2(5x^2-3x+4)$
18. $\frac{4}{5}(25x^2-50xy+5y^2)$
19. $3x(5x^2-2x+3)$
20. $-x(x^2+x-1)$
21. $x^2(3x^2-5x-7)$
22. $x^3(-4x^2-7x+9)$
23. $\frac{1}{4}x^4(8x^3-2x^2+\frac{1}{2}x-5)$
24. $-\frac{1}{3}x^3(\frac{3}{2}x^5-\frac{2}{3}x^3+\frac{9}{2}x-1)$
25. $a^2b(a^2-3ab+b^2)$
26. $6a^2bc^3(2a-3b+c^2)$
27. $\frac{2}{3}xy^2(9x^3y-27xy+3xy^3)$
28. $-3x^2y^2(12x^2-10xy-6y^2)$
29. Find the product of $3x$ and $2x^2-3x+5$.
30. Find the product of $-8y$ and $y^2-2y+12$.
31. Find the product of $-4x$ and $x^4-3x^3+2x^2-7x+8$.
32. Find the product of $3xy^2$ and $-2x^2y+4xy-xy^2$.

Answer available at the end of this section.

Exercise 5.4.6 Product of a Binomial and a Polynomial

Multiply.

1. $(3x-2)(x+4)$
2. $(x+2)(x-3)$
3. $(x-1)(x+1)$
4. $(3x-1)(3x+1)$
5. $(2x-5)(x+3)$
6. $(5x-2)(3x+4)$
7. $(-3x+1)(x-1)$
8. $(x+5)(-x+1)$
9. $(y-\frac{2}{3})(y+\frac{2}{3})$
10. $(\frac{1}{2}x+\frac{1}{3})(\frac{3}{2}x-\frac{2}{3})$
11. $(\frac{3}{4}x+\frac{1}{5})(\frac{1}{4}x+\frac{2}{5})$
12. $(\frac{1}{5}x+\frac{3}{10})(\frac{3}{5}x-\frac{5}{2})$
13. $(y^2-2)(y+2)$
14. $(y^3-1)(y^2+2)$
15. $(a^2-b^2)(a^2+b^2)$
16. $(a^2-3b)^2$
17. $(x-5)(2x^2+3x+4)$
18. $(3x-1)(x^2-4x+7)$
19. $(2x-3)(4x^2+6x+9)$
20. $(5x+1)(25x^2-5x+1)$
21. $(x-\frac{1}{2})(3x^2+4x-1)$
22. $(\frac{1}{3}x-\frac{1}{4})(3x^2+9x-3)$
23. $(x+3)^3$
24. $(x-2)^3$
25. $(3x-1)^3$
26. $(2x+y)^3$
27. $(5x-2)(2x^3-4x^2+3x-2)$
28. $(x^2-2)(x^3-2x^2+x+1)$

Answer available at the end of this section.

Exercise 5.4.7 Product of Polynomials

Multiply.

1. $(x^2 - x + 1)(x^2 + 2x + 1)$
2. $(3x^2 - 2x - 1)(2x^2 + 3x - 4)$
3. $(2x^2 - 3x + 5)(x^2 + 5x - 1)$
4. $(a + b + c)(a - b - c)$
5. $(a + 2b - c)^2$
6. $(x + y + z)^2$
7. $(x - 3)^4$
8. $(x + y)^4$
9. Find the volume of a rectangular solid with sides measuring $x, x + 2$, and $x + 4$ units.
10. Find the volume of a cube where each side measures $x - 5$ units.

Answer available at the end of this section.

Exercise 5.4.8 Special Products

Multiply.

1. $(x + 2)^2$
2. $(x - 3)^2$
3. $(2x + 5)^2$
4. $(3x - 7)^2$
5. $(-x + 2)^2$
6. $(-9x + 1)^2$
7. $(a + 6)^2$
8. $(2a - 3b)^2$
9. $(\frac{2}{3}x + \frac{3}{4})^2$
10. $(\frac{1}{2}x - \frac{3}{5})^2$
11. $(x^2 + 2)^2$
12. $(x^2 + y^2)^2$
13. $(x + 4)(x - 4)$
14. $(2x + 1)(2x - 1)$
15. $(5x + 3)(5x - 3)$
16. $(\frac{1}{5}x - \frac{1}{3})(\frac{1}{5}x + \frac{1}{3})$
17. $(\frac{3}{2}x + \frac{2}{5})(\frac{3}{2}x - \frac{2}{5})$
18. $(2x - 3y)(2x + 3y)$
19. $(4x - y)(4x + y)$
20. $(a^3 - b^3)(a^3 + b^3)$

21. A box is made by cutting out the corners and folding up the edges of a square piece of cardboard. A template for a cardboard box with a height of 2 inches is given. Find a formula for the volume, if the initial piece of cardboard is a square with sides measuring x inches.

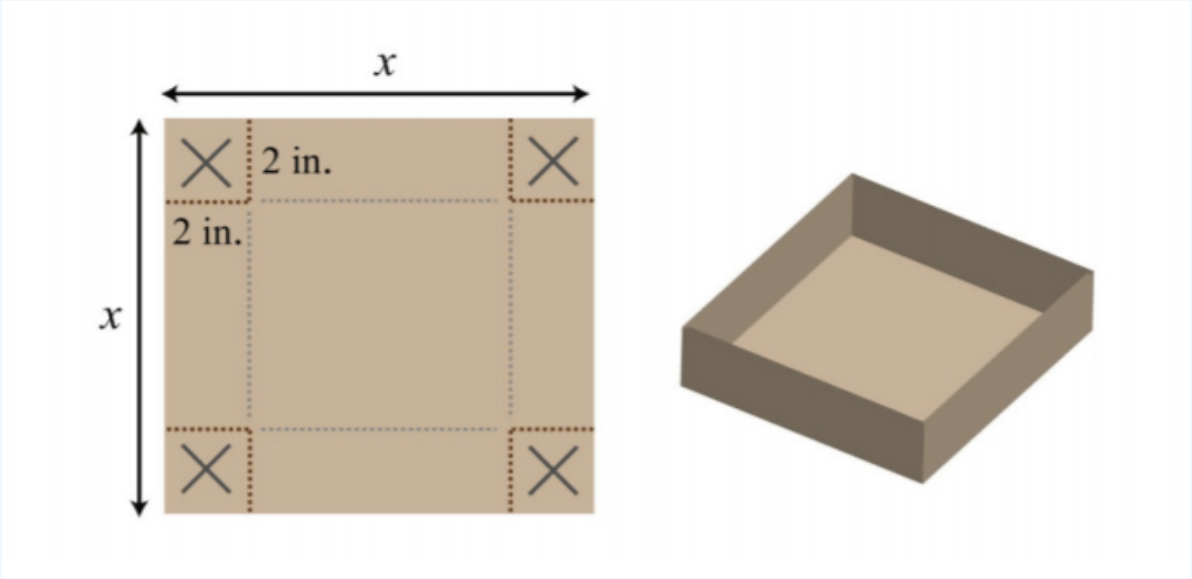


Figure $5.4.5$

22. A template for a cardboard box with a height of x inches is given. Find a formula for the volume, if the initial piece of cardboard is a square with sides measuring 12 inches.

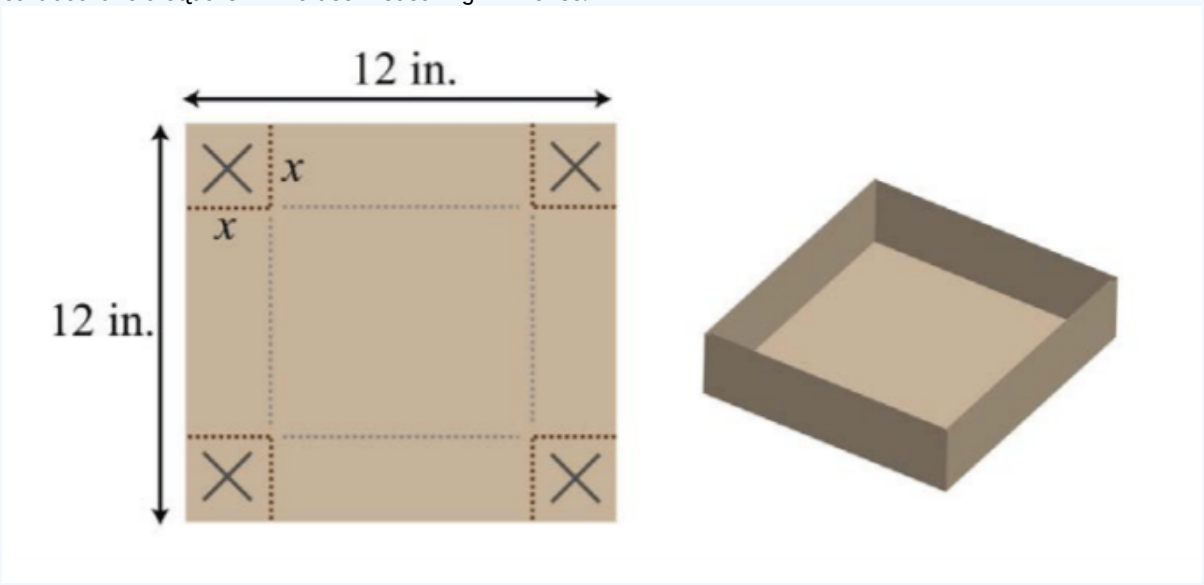


Figure $5.4.6$

Answer available at the end of this section.

Exercise $5.4.9$ Multiplying Polynomial Functions

For each problem, calculate $(f \cdot g)(x)$, given the functions.

1. $f(x) = 8x$ and $g(x) = 3x - 5$
2. $f(x) = x^2$ and $g(x) = -5x + 1$
3. $f(x) = x - 7$ and $g(x) = 6x - 1$
4. $f(x) = 5x + 3$ and $g(x) = x^2 + 2x - 3$

5. $f(x) = x^2 + 6x - 3$ and $g(x) = 2x^2 - 3x + 5$
6. $f(x) = 3x^2 - x + 1$ and $g(x) = -x^2 + 2x - 1$

Answer available at the end of this section.

Exercise 5.4.10 Multiplying Polynomial Functions

Given $f(x) = 2x - 3$ and $g(x) = 3x - 1$, find the following

1. $(f \cdot g)(x)$
2. $(g \cdot f)(x)$
3. $(f \cdot g)(0)$
4. $(f \cdot g)(-1)$
5. $(f \cdot g)(1)$
6. $(f \cdot g)(\frac{1}{2})$

Answer available at the end of this section.

Exercise 5.4.11 Multiplying Polynomial Functions

Given $f(x) = 5x - 1$ and $g(x) = 2x^2 - 4x + 5$, find the following.

1. $(f \cdot g)(x)$
2. $(g \cdot f)(x)$
3. $(f \cdot g)(0)$
4. $(f \cdot g)(-1)$
5. $(f \cdot g)(1)$
6. $(f \cdot g)(\frac{1}{2})$
7. $(f \cdot f)(x)$
8. $(g \cdot g)(x)$

Answer available at the end of this section.

Exercise 5.4.12 Discussion Board Topics

1. Explain why $(x + y)^2 \neq x^2 + y2$.
2. Explain how to quickly multiply a binomial with its conjugate. Give an example.
3. What are the advantages and disadvantages of using the mnemonic device FOIL?

Answer available at the end of this section.

Answers

Exercise 5.4.1

$-10x^3y^3 + 15x^3y^2 - 30x^4y^2 + 5x^2y$

Exercise 5.4.2

$14x^3 - 31x^2 + 23x - 12$

Exercise 5.4.4

$25x^2 - 20x + 4$

Exercise 5.4.5 **Product of a Monomial and a Polynomial**

1. $-15x^3y$
3. $2x - \frac{3}{2}$
5. $15x^2 - 6x$
7. $3x^3 + 2x^2$

9. $8a^2b - 4ab^2$
11. $-18x^5y^4 + 6x^3y^5$
13. $-2x^3y^2 + 5x^2y$
15. $-30x^9$
17. $-10x^2 + 6x - 8$
19. $15x^3 - 6x^2 + 9x$
21. $3x^4 - 5x^3 - 7x^2$
23. $2x^7 - \frac{1}{2}x^6 + \frac{1}{8}x^5 - \frac{5}{4}x^4$
25. $a^4b - 3a^3b^2 + a^2b^3$
27. $6x^4y^3 - 18x^2y^3 + 2x^2y^5$
29. $6x^3 - 9x^2 + 15x$
31. $-4x^5 + 12x^4 - 8x^3 + 28x^2 - 32x$

Exercise 5.4.6 **Product of a Binomial and a Polynomial**

1. $3x^2 + 10x - 8$
3. $x^2 - 1$
5. $2x^2 + x - 15$
7. $-3x^2 + 4x - 1$
9. $y^2 - \frac{4}{9}$
11. $\frac{3}{16}x^2 + \frac{7}{20}x + \frac{2}{25}$
13. $y^3 + 2y^2 - 2y - 4$
15. $a^4 - b^4$
17. $2x^3 - 7x^2 - 11x - 20$
19. $8x^3 - 27$
21. $3x^3 + \frac{5}{2}x^2 - 3x + 12$
23. $x^3 + 9x^2 + 27x + 27$
25. $27x^3 - 27x^2 + 9x - 1$
27. $10x^4 - 24x^3 + 23x^2 - 16x + 4$

Exercise 5.4.7 **Product of Polynomials**

1. $x^4 + x^3 + x + 1$
3. $2x^4 + 7x^3 - 12x^2 + 28x - 5$
5. $a^2 + 4ab - 2ac + 4b^2 - 4bc + c^2$
7. $x^4 - 12x^3 + 54x^2 - 108x + 81$
9. $x^3 + 6x^2 + 8x$

Exercise 5.4.8 **Special Products**

1. $x^2 + 4x + 4$
3. $4x^2 + 20x + 25$
5. $x^2 - 4x + 4$
7. $a^2 + 12a + 36$
9. $\frac{4}{9}x^2 + x + \frac{9}{16}$
11. $x^4 + 4x^2 + 4$
13. $x^2 - 16$
15. $25x^2 - 9$
17. $\frac{9}{4}x^2 - \frac{4}{25}$
19. $16x^2 - y^2$
21. $V = 2x^2 - 16x + 32$ cubic inches

Exercise 5.4.9 **Multiplying Polynomial Functions**

1. $(f \cdot g)(x) = 24x^2 - 40x$
3. $(f \cdot g)(x) = 6x^2 - 43x + 7$
5. $(f \cdot g)(x) = 2x^4 + 9x^3 - 19x^2 + 39x - 15$

Exercise 5.4.10 **Multiplying Polynomial Functions**

1. $(f \cdot g)(x) = 6x^2 - 11x + 3$
3. $(f \cdot g)(0) = 3$
5. $(f \cdot g)(1) = -2$

Exercise 5.4.11 **Multiplying Polynomial Functions**

1. $(f \cdot g)(x) = 10x^3 - 22x^2 + 29x - 5$
3. $(f \cdot g)(0) = -5$
5. $(f \cdot g)(1) = 12$
7. $(f \cdot f)(x) = 25x^2 - 10x + 1$

Exercise 5.4.12 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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5.5: Dividing Polynomials

Learning Objectives

- Divide by a monomial.
- Divide by a polynomial using the division algorithm.
- Divide polynomial functions.

Dividing by a Polynomial

Recall the quotient rule for exponents: if x is nonzero and m and n are positive integers, then

$$\frac{x^m}{x^n} = x^{m-n} \tag{5.5.1}$$

In other words, when dividing two expressions with the same base, subtract the exponents. This rule applies when dividing a monomial by a monomial. In this section, we will assume that all variables in the denominator are nonzero.

Example 5.5.1

Divide:

$$\frac{28y^3}{7y}.$$

Solution:

Divide the coefficients and subtract the exponents of the variable y.

$$\begin{aligned}\frac{28y^3}{7y} &= \frac{28}{7}y^{3-1} \\ &= 4y^2\end{aligned}$$

Answer:

$$4y^2$$

Example 5.5.2

Divide:

$$\frac{24x^7y^5}{8x^3y^2}.$$

Solution:

Divide the coefficients and apply the quotient rule by subtracting the exponents of the like bases.

$$\begin{aligned}\frac{24x^7y^5}{8x^3y^2} &= \frac{24}{8}x^{7-3}y^{5-2} \\ &= 3x^4y^3\end{aligned}$$

Answer:

$$3x^4y^3$$

When dividing a polynomial by a monomial, we may treat the monomial as a common denominator and break up the fraction using the following property:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \tag{5.5.2}$$

Applying this property results in terms that can be treated as quotients of monomials.

Example 5.5.3

Divide:

$\frac{-5x^4+25x^3-15x^2}{5x^2}$.

Solution:

Break up the fraction by dividing each term in the numerator by the monomial in the denominator and then simplify each term.

Answer:

$-x^2 + 5x - 3 \cdot 1$

Check your division by multiplying the answer, the quotient, by the monomial in the denominator, the divisor, to see if you obtain the original numerator, the dividend.

$$\frac{dividend}{divisor} = quotient \tag{5.5.3}$$

or

$$dividend = divisor \cdot quotient \tag{5.5.4}$$

$$\begin{aligned} 5x^2 \cdot (-x^2 + 5x - 3) &= 5x^2 \cdot (-x^2) + 5x^2 \cdot 5x - 5x^2 \cdot 3 \\ &= -5x^4 + 25x^3 - 15x^2 \quad \checkmark \end{aligned}$$

Example 5.5.4

Divide:

$\frac{9a^4b-7a^3b^2+3a^2b}{-3a^2b}$.

Solution:

Answer:

$-3a^2 + \frac{7}{3}ab - 1$. The check is optional and is left to the reader.

Exercise 5.5.1

$(16x^5 - 8x^4 + 5x^3 + 2x^2) \div (2x^2)$.

Answer available at the end of this section.

Dividing by a Polynomial

The same technique outlined for dividing by a monomial ***does not*** work for polynomials with two or more terms in the denominator. In this section, we will outline a process called polynomial long division, which is based on the division algorithm for real numbers. For the sake of clarity, we will assume that all expressions in the denominator are nonzero.

Example 5.5.5

Divide:

$$\frac{x^3+3x^2-8x-4}{x-2}.$$

Solution:

Here $x-2$ is the divisor and x^3+3x^2-8x-4 is the dividend.

Step 1: To determine the first term of the quotient, divide the leading term of the dividend by the leading term of the divisor.

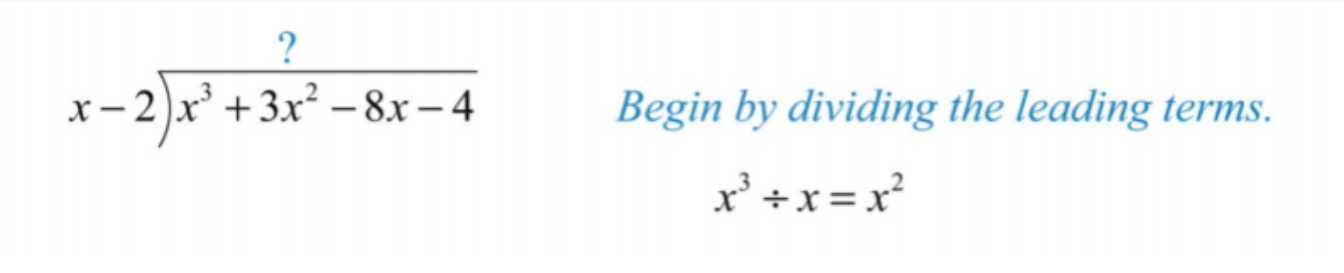


Figure 5.5.1

Step 2: Multiply the first term of the quotient by the divisor, remembering to distribute, and line up like terms with the dividend.

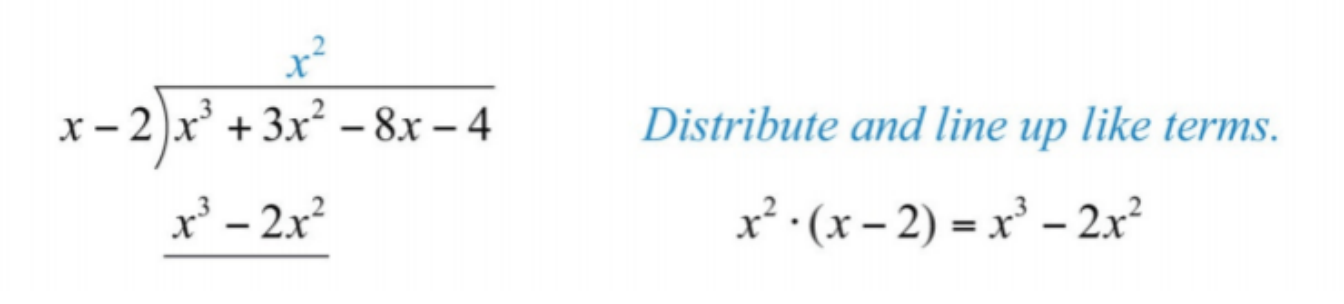


Figure 5.5.2

Step 3: Subtract the resulting quantity from the dividend. Take care to subtract both terms.

$$\begin{array}{r} x^2 \\ x-2\overline{)\,x^3+3x^2-8x-4} \\ -(x^3-2x^2) \\ \hline 5x^2 \end{array}$$

Subtract.

$$-(x^3-2x^2)=-x^3+2x^2$$

Figure 5.5.3

Step 4: Bring down the remaining terms and repeat the process from step 1.

$$\begin{array}{r} x^2 \quad ? \\ x-2\overline{)\,x^3+3x^2-8x-4} \\ -(x^3-2x^2) \\ \hline 5x^2-8x-4 \end{array}$$

Bring down the remaining terms.

Figure 5.5.4

Notice that the leading term is eliminated and that the result has a degree that is one less than the dividend. The complete process is illustrated below:

$$\begin{array}{r} 9x^2-12x+16 \\ 3x+4\overline{)\,27x^3+0x^2+0x+64} \\ -(27x^3+36x^2) \\ \hline -36x^2+0x+64 \\ -(-36x^2-48x) \\ \hline 48x+64 \\ -(48x+64) \\ \hline 0 \end{array}$$

Figure 5.5.5

Polynomial long division ends when the degree of the remainder is less than the degree of the divisor. Here the remainder is 0. Therefore, the binomial divides the polynomial evenly and the answer is the quotient shown above the division line.

$$\frac{x^3+3x^2-8x-4}{x-2} = x^2+5x+2$$

To check the answer, multiply the divisor by the quotient to see if you obtain the dividend:

$$x^3+3x^2-8x-4 = (x-2)(x^2+5x+2)$$

Answer:

$$x^2+5x+2$$

Next, we demonstrate the case where there is a nonzero remainder.

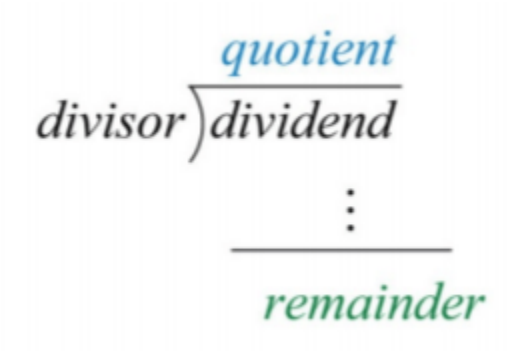


Figure $5.5.6$

Just as with real numbers, the final answer adds the fraction where the remainder is the numerator and the divisor is the denominator to the quotient. In general, when dividing we have

$$\frac{dividend}{divisor} = quotient + \frac{remainder}{divisor} \tag{5.5.5}$$

If we multiply both sides by the divisor we obtain

$$dividend = quotient \times divisor + remainder \tag{5.5.6}$$

Example $5.5.6$

Divide:

$$\frac{6x^2-5x+3}{2x-1}$$

Solution:

Since the denominator is a binomial, begin by setting up polynomial long division.

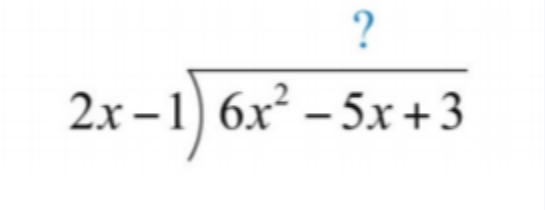


Figure $5.5.7$

To start, determine what monomial times $2x-1$ results in a leading term $6x^2$. This is the quotient of the given leading terms: $(6x^2) \div (2x) = 3x$. Multiply $3x$ times the divisor $2x-1$ and line up the result with like terms of the dividend.

$$\begin{array}{r} 3x \\ 2x-1\overline{)\,6x^2-5x+3} \\ 6x^2-3x \end{array}$$

Figure 5.5.8

Subtract the result from the dividend and bring down the constant term $+3$.

$$\begin{array}{r} 3x\ \ ? \\ 2x-1\overline{)\,6x^2-5x+3} \\ -(6x^2-3x) \\ \hline -2x+3 \end{array}$$

Figure 5.5.9

Subtracting eliminates the leading term and $-5x-(-3x) = -5x+3x = -2x$. The quotient of $-2x$ and $2x$ is -1. Multiply $2x-1$ by -1 and line up the result.

$$\begin{array}{r} 3x-1 \\ 2x-1\overline{)\,6x^2-5x+3} \\ -(6x^2-3x) \\ \hline -2x+3 \\ -2x+1 \end{array}$$

Figure 5.5.10

Subtract again and notice that we are left with a remainder.

$$
\begin{array}{r}
3x-1 \\
2x-1\overline{)\,6x^2-5x+3} \\
-(6x^2-3x) \\
\hline
-2x+3 \\
-(-2x+1) \\
\hline
2
\end{array}
$$

Figure 5.5.11

The constant term 2 has degree 0, and thus the division ends. We may write

$$\frac{6x^2-5x+3}{2x-1} = 3x-1+\frac{2}{2x-1}$$

Answer:

$3x-1+\frac{2}{2x-1}$. To check that this result is correct, we multiply as follows:

$$\begin{aligned} quotient \times divisor + remainder &= (3x-1)(2x-1)+2 \\ &= 6x^2-3x-2x+1+2 \\ &= 6x^2-5x+3 = dividend \quad \checkmark \end{aligned}$$

Occasionally, some of the powers of the variables appear to be missing within a polynomial. This can lead to errors when lining up like terms. Therefore, when first learning how to divide polynomials using long division, fill in the missing terms with zero coefficients, called placeholders.

Example 5.5.7

Divide:

$$\frac{27x^3+64}{3x+4}.$$

Solution:

Notice that the binomial in the numerator does not have terms with degree 2 or 1. The division is simplified if we rewrite the expression with placeholders:

$$27x^3+64 = 27x^3+0x^2+0x+64$$

Set up polynomial long division:

$$3x+4\overline{)27x^3+0x^2+0x+64}\quad ?$$

Figure 5.5.12

We begin with $27x^3 \div 3x = 9x^2$ and work the rest of the division algorithm.

$$\begin{array}{r} 9x^2-12x+16 \\ 3x+4\overline{)\ 27x^3+0x^2+0x+64} \\ \underline{-(27x^3+36x^2)} \\ -36x^2+\ 0x+64 \\ \underline{-(-36x^2-48x)} \\ 48x+64 \\ \underline{-(48x+64)} \\ 0 \end{array}$$

Figure 5.5.13

Answer:

$$9x^2-12x+16$$

Example 5.5.8

Divide:

$$\frac{3x^4-2x^3+6x^2+23x-7}{x^2-2x+5}.$$

Solution:

$$x^2-2x+5\overline{)3x^4-2x^3+6x^2+23x-7}\quad ?$$

Figure 5.5.14

Begin the process by dividing the leading terms to determine the leading term of the quotient $3x^4 \div x^2 = 3x^2$. Take care to distribute and line up the like terms. Continue the process until the remainder has a degree less than 2.

$$
\begin{array}{r}
3x^2+4x-1 \\
x^2-2x+5\overline{)\,3x^4-2x^3+6x^2+23x-7} \\
\underline{-(3x^4-6x^3+15x^2)} \\
4x^3-9x^2+23x-7 \\
\underline{-(4x^3-8x^2+20x)} \\
-x^2+3x-7 \\
\underline{-(-x^2+2x-5)} \\
x-2
\end{array}
$$

Figure 5.5.15

The remainder is $x-2$. Write the answer with the remainder:

$$\frac{3x^4-2x^3+6x^2+23x-7}{x^2-2x+5}=3x^2+4x-1+\frac{x-2}{x^2-2x+5}$$

Answer:

$$3x^2+4x-1+\frac{x-2}{x^2-2x+5}$$

Polynomial long division takes time and practice to master. Work lots of problems and remember that you may check your answers by multiplying the quotient by the divisor (and adding the remainder if present) to obtain the dividend.

Exercise 5.5.2

Divide:

$$\frac{20x^4-32x^3+7x^2+8x-10}{5x-3}.$$

Answer available at the end of this section.

Dividing Polynomial Functions

We may use function notation to indicate division as follows:

Table 5.5.1

Division of functions:	$(f/g)(x)=\frac{f(x)}{g(x)}$

The quotient of two polynomial functions does not necessarily have a domain of all real numbers. The values for x that make the function in the denominator 0 are restricted from the domain. This will be discussed in more detail at a later time. For now, assume all functions in the denominator are nonzero.

Example 5.5.9

Calculate:

$$(f/g)(x) \text{ given } f(x) = 6x^5 - 36x^4 + 12x^3 - 6x^2 \text{ and } g(x) = -6x^2.$$

Solution:

The notation indicates that we should divide:

$$\begin{aligned}(f/g)(x) &= \frac{f(x)}{g(x)} \\ &= \frac{6x^5 - 36x^4 + 12x^3 - 6x^2}{-6x^2} \\ &= \frac{6x^5}{-6x^2} - \frac{36x^4}{-6x^2} + \frac{12x^3}{-6x^2} - \frac{6x^2}{-6x^2} \\ &= -1x^{5-2} + 6x^{4-2} - 2x^{3-2} + 1x^{2-2} \\ &= -x^3 + 6x^2 - 2x + 1\end{aligned}$$

Answer:

$$(f/g)(x) = -x^3 + 6x^2 - 2x + 1$$

Example 5.5.10

Calculate:

$$(f/g)(-1), \text{ given } f(x) = -3x^3 + 7x^2 - 11x - 1 \text{ and } g(x) = 3x - 1.$$

Solution:

First, determine $(f/g)(x)$.

$$(f/g)(x) = \tfrac{f(x)}{g(x)} = \tfrac{-3x^3+7x^2-11x-1}{3x-1}$$

$$\begin{array}{r} -x^2+2x-3 \\ 3x-1\overline{)\,-3x^3+7x^2-11x-1} \\ \underline{-(-3x^3+x^2)} \\ 6x^2-11x-1 \\ \underline{-(6x^2-2x)} \\ -9x-1 \\ \underline{-(-9x+3)} \\ -4 \end{array}$$

Figure 5.5.16

Therefore,

$$(f/g)(x) = -x^2 + 2x - 3 - \frac{4}{3x-1}$$

Substitute -1 for the variable x.

Answer:

$(f/g)(-1) = -5$

Key Takeaways

- When dividing by a monomial, divide all terms in the numerator by the monomial and then simplify each term. To simplify each term, divide the coefficients and apply the quotient rule for exponents.
- When dividing a polynomial by another polynomial, apply the division algorithm.
- To check the answer after dividing, multiply the divisor by the quotient and add the remainder (if necessary) to obtain the dividend.
- It is a good practice to include placeholders when performing polynomial long division.

Exercise 5.5.3 Dividing by a Monomial

Divide.

1. $\frac{81y^5}{9y^2}$
2. $\frac{36y^9}{9y^3}$
3. $\frac{52x^2y}{4xy}$
4. $\frac{24xy^5}{2xy^4}$
5. $\frac{25x^2y^5z^3}{5xyz}$
6. $\frac{-77x^4y^9z^2}{2x^3y^3z}$
7. $\frac{125a^3b^2c}{-10abc}$
8. $\frac{36a^2b^5c^5}{-6a^2b^2c^3}$
9. $\frac{9x^2+27x-3}{3}$
10. $\frac{10x^3-5x^2+40x-15}{5}$
11. $\frac{20x^3-10x^2+30x}{2x}$
12. $\frac{10x^4+8x^2-6x}{24x}$
13. $\frac{-6x^5-9x^3+3x}{-3x}$
14. $\frac{36a^{12}-6a^9+12a^5}{-12a^5}$
15. $\frac{-12x^5+18x^3-6x^2}{-6x^2}$
16. $\frac{-49a^8+7a^5-21a^3}{7a^3}$
17. $\frac{9x^7-6x^4+12x^3-x^2}{3x^2}$
18. $\frac{8x^9+16x^7-24x^4+8x^3}{-8x^3}$
19. $\frac{16a^7-32a^6+20a^5-a^4}{4a^4}$
20. $\frac{5a^6+2a^5+6a^3-12a^2}{3a^2}$
21. $\frac{-4x^2y^3+16x^7y^8-8x^2y^5}{-4x^2y^3}$
22. $\frac{100a^{10}b^{30}c^5-50a^{20}b^5c^{40}+20a^5b^{20}c^{10}}{10a^5b^5c^5}$
23. Find the quotient of $-36x^9y^7$ and $2x^8y^5$.
24. Find the quotient of $144x^3y^{10}z^2$ and $-12x^3y^5z$.
25. Find the quotient of $3a^4 - 18a^3 + 27a^2$ and $3a^2$.
26. Find the quotient of $64a^2bc^3 - 16a^5bc^7$ and $4a^2bc^3$.

Answer available at the end of this section.

Exercise 5.5.4 Dividing by a Polynomial

Divide.

1. $(2x^2-5x-3)\div(x-3)$
2. $(3x^2+5x-2)\div(x+2)$
3. $(6x^2+11x+3)\div(3x+1)$
4. $(8x^2-14x+3)\div(2x-3)$
5. $\frac{x^3-x^2-2x-12}{x-3}$
6. $\frac{2x^3+11x^2+4x-5}{x+5}$
7. $\frac{2x^3-x^2-4x+3}{2x+3}$
8. $\frac{-15x^3-14x^2+23x-6}{5x-2}$
9. $\frac{14x^4-9x^3+22x^2+4x-1}{7x-1}$
10. $\frac{8x^5+16x^4-8x^3-5x^2-21x+10}{2x+5}$
11. $\frac{x^2+8x+17}{x+5}$
12. $\frac{2x^2-5x+5}{x-2}$
13. $\frac{6x^2-13x+9}{-2x+1}$
14. $\frac{-12x^2+x+1}{3x+2}$
15. $\frac{x^3+9x^2+19x+1}{x+4}$
16. $\frac{2x^3-13x^2+17x-11}{x-5}$
17. $\frac{9x^3-12x^2+16x-15}{3x-2}$
18. $\frac{3x^4-8x^3+5x^2-5x+9}{x-2}$
19. $(6x^5-13x^4+4x^3-3x^2+13x-2)\div(3x+1)$
20. $(8x^5-22x^4+19x^3-20x^2+23x-3)\div(2x-3)$
21. $\frac{5x^5+12x^4+12x^3-7x^2-19x+3}{x^2+2x+3}$
22. $\frac{6x^5-17x^4+5x^3+16x^2-7x-3}{2x^2-3x-1}$
23. $\frac{x^5+7x^4-x^3-7x^2-49x+9}{x^2+7x-1}$
24. $\frac{5x^6-6x^4-4x^2+x+2}{5x^2-1}$
25. $\frac{x^3-27}{x-3}$
26. $\frac{8x^3+125}{2x+5}$
27. $(15x^5-9x^4-20x^3+12x^2+15x-9)\div(5x-3)$
28. $(2x^6-5x^5-4x^4+10x^3+6x^2-17x+5)\div(2x-5)$
29. $\frac{x^5-2x^3+3x-1}{x-1}$
30. $\frac{x^4-3x^2+5x-1}{3x+2}$
31. $\frac{a^2-4}{a+2}$
32. $\frac{a^5+1}{a^5+1}$
33. $\frac{a^6-1}{a-1}$
34. $\frac{x^5-1}{x-1}$
35. $\frac{x^5+x^4+6x^3+12x^2-4}{x^2+x-1}$
36. $\frac{50x^6-30x^5-5x^4+15x^3-5x+1}{5x^2-3x+2}$
37. $\frac{5x^5-15x^3+25x^2-5}{5x}$
38. $\frac{-36x^6+12x^4-6x^2}{6x^2}$
39. $\frac{150x^5y^2z^{15}-10x^3y^6z^5+4x^3y^2z^4}{10x^3y^2z^5}$
40. $\frac{27m^6+9m^4-81m^2+1}{9m^2}$
41. Divide $3x^6-2x^5+27x^4-18x^3-6x^2+7x-10$ by $3x-2$.
42. Divide $8x^6+4x^5-14x^4-5x^3+x^2-2x-3$ by $2x+1$.

Answer available at the end of this section.

Exercise 5.5.5 Dividing Polynomial Functions

Calculate $(f/g)(x)$, given the functions.

1. $f(x) = 40x^8$ and $g(x) = 10x^5$
2. $f(x) = 54x^5$ and $g(x) = 9x^3$
3. $f(x) = 12x^2 + 24x - 15$ and $g(x) = 2x + 5$
4. $f(x) = -8x^2 + 30x - 7$ and $g(x) = 2x - 7$
5. $f(x) = 18x^2 - 36x + 5$ and $g(x) = 3x - 5$
6. $f(x) = -7x^2 + 29x - 6$ and $g(x) = 7x - 1$
7. $f(x) = 10x^3 - 9x^2 + 27x - 10$ and $g(x) = 5x - 2$
8. $f(x) = 15x^3 + 28x^2 - 11x + 56$ and $g(x) = 3x + 8$
9. $f(x) = 2x^4 + 5x^3 - 11x^2 - 19x + 20$ and $g(x) = x^2 + x - 5$
10. $f(x) = 4x^4 - 12x^3 - 20x^2 + 26x - 3$ and $g(x) = 2x^2 + 2x - 3$

Answer available at the end of this section.

Exercise 5.5.6 Dividing Polynomial Functions

Given $f(x) = 6x^3 + 4x^2 - 11x + 3$ and $g(x) = 3x - 1$, find the following.

1. $(f/g)(x)$
2. $(f/g)(-1)$
3. $(f/g)(0)$
4. $(f/g)(1)$

Answer available at the end of this section.

Exercise 5.5.7 Dividing Polynomial Functions

Given $f(x) = 5x^3 - 13x^2 + 7x + 3$ and $g(x) = x - 2$, find the following.

1. $(f/g)(x)$
2. $(f/g)(-3)$
3. $(f/g)(0)$
4. $(f/g)(7)$

Answer available at the end of this section.

Exercise 5.5.8 Discussion Board Topics

1. How do you use the distributive property when dividing a polynomial by a monomial?
2. Compare long division of real numbers with polynomial long division. Provide an example of each.

Answer available at the end of this section.

Answers

Exercise 5.5.1

$8x^3 - 4x^2 + \frac{5}{2}x + 1$

Exercise 5.5.2

$4x^3 - 4x^2 - x + 1 - \frac{7}{5}x - 3$

Exercise 5.5.3 **Dividing by a Monomial**

1. $9y^3$
3. $13x$
5. $5xy^4z^2$
7. $-\frac{25}{2}a^2b$
9. $3x^2 + 9x - 1$

11. $10x^2 - 5x + 15$
13. $2x^4 + 3x^2 - 1$
15. $2x^3 - 3x + 1$
17. $3x^5 - 2x^2 + 4x - \frac{1}{3}$
19. $4a^3 - 8a^2 + 5a - \frac{1}{4}$
21. $-4x^5y^5 + 2y^2 + 1$
23. $-18xy^2$
25. $a^2 - 6a + 9$

Exercise 5.5.4 **Dividing by a Polynomial**

1. $2x + 1$
3. $2x + 3$
5. $x^2 + 2x + 4$
7. $x^2 - 2x + 1$
9. $2x^3 - x^2 + 3x + 1$
11. $x + 3 + \frac{2}{x+5}$
13. $-3x + 5 + \frac{4}{-2x+1}$
15. $x^2 + 5x - 1 + \frac{5}{x+4}$
17. $3x^2 - 2x + 4 - \frac{7}{3x-2}$
19. $2x^4 - 5x^3 + 3x^2 - 2x + 5 - \frac{7}{3x+1}$
21. $5x^3 + 2x^2 - 7x + 1$
23. $x^3 - 7 + \frac{2}{x^2+7x-1}$
25. $x^2 + 3x + 9$
27. $3x^4 - 4x^2 + 3$
29. $x^4 + x^3 - x^2 - x + 2 + \frac{1}{x-1}$
31. $a - 2$
33. $a^5 + a^4 + a^3 + a^2 + a + 1$
35. $x^3 + 7x + 5 + \frac{2x+1}{x^2+x-1}$
37. $x^4 - 3x^2 + 5x - \frac{1}{x}$
39. $15x^2z^{10} - y^4 + \frac{2}{5z}$
41. $x^5 + 9x^3 - 2x + 1 - \frac{8}{3x-2}$

Exercise 5.5.5 **Dividing Polynomial Functions**

1. $(f/g)(x) = 4x^3$
3. $(f/g)(x) = 6x - 3$
5. $(f/g)(x) = 6x - 2 - \frac{5}{3x-5}$
7. $(f/g)(x) = 2x^2 - x + 5$
9. $(f/g)(x) = 2x^2 + 3x - 4$

Exercise 5.5.6 **Dividing Polynomial Functions**

1. $(f/g)(x) = 2x^2 + 2x - 3$
3. $(f/g)(0) = -3$

Exercise 5.5.7 **Dividing Polynomial Functions**

1. $(f/g)(x) = 5x^2 - 3x + 1 + \frac{5}{x-2}$
3. $(f/g)(0) = -\frac{3}{2}$

Exercise 5.5.8 **Discussion Board Topics**

1. Answers may vary

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5.6: Negative Exponents

Learning Objectives

- Simplify expressions with negative integer exponents.
- Work with scientific notation.

Negative Exponents

In this section, we define what it means to have negative integer exponents. We begin with the following equivalent fractions:

$$\frac{1}{8} = \frac{4}{32}$$

Notice that $4, 8$, and 32 are all powers of 2. Hence we can write $4 = 2^2, 8 = 2^3, and 32 = 2^5$.

$$\frac{1}{2^3} = \frac{1}{8} = \frac{4}{32} = \frac{2^2}{2^5}$$

If the exponent of the term in the denominator is larger than the exponent of the term in the numerator, then the application of the quotient rule for exponents results in a negative exponent. In this case, we have the following:

$$\frac{1}{2^3} = \frac{1}{8} = \frac{4}{32} = \frac{2^2}{2^5} = 2^{2-5} = 2^{-3}$$

We conclude that $2^{-3} = \frac{1}{2}^3$. This is true in general and leads to the definition of negative exponents. Given any integer n and $x \neq 0$, then

$$x^{-n} = \frac{1}{x^n} \tag{5.6.1}$$

Here $x \neq 0$ because \frac{1}{0}\) is undefined. For clarity, in this section, assume all variables are nonzero.

Simplifying expressions with negative exponents requires that we rewrite the expression with positive exponents.

Example 5.6.1

Simplify:

$$10^{-2}.$$

Solution:

$$\begin{aligned} 10^{-2} &= \frac{1}{10^2} \\ &= \frac{1}{100} \end{aligned}$$

Answer:

$$\frac{1}{100}$$

Example 5.6.2

Simplify:

$(-3)^{-1}$.

Solution:

$$\begin{aligned}(-3)^{-1} &= \frac{1}{(-3)^1} \\ &= -\frac{1}{3}\end{aligned}$$

Answer:

$-\frac{1}{3}$

Example 5.6.3

Simplify:

$\frac{1}{y^{-3}}$.

Solution:

$$\begin{aligned}\frac{1}{y^{-3}} &= \frac{1}{\frac{1}{y^3}} \\ &= 1 \cdot \frac{y^3}{1} \\ &= y^3\end{aligned}$$

Answer:

y^3

At this point we highlight two very important examples,

Simplify $(2x)^{-1}$	*Simplify* $2x^{-1}$
$(2x)^{-1} = \frac{1}{(2x)^1}$ $= \frac{1}{2x}$	$2x^{-1} = 2 \cdot \frac{1}{x^1}$ $= \frac{2}{x}$

Figure 5.6.1

If the grouped quantity is raised to a negative exponent, then apply the definition and write the entire grouped quantity in the denominator. If there is no grouping, then apply the definition only to the base preceding the exponent.

Example 5.6.4

Simplify:

$(2ab)^{-3}$.

Solution:

First, apply the definition of –3 as an exponent and then apply the power of a product rule.

$$\begin{aligned}(2ab)^{-3} &= \frac{1}{(2ab)^3} && \textit{Apply the negative exponent.}\\ &= \frac{1}{2^3a^3b^3} && \textit{Apply the power rule for a product.}\\ &= \frac{1}{8a^3b^3}\end{aligned}$$

Answer:

$\frac{1}{8a^3b^3}$

Example 5.6.5

Simplify:

$$(-3xy^3)^{-2}.$$

Solution:

$$\begin{aligned}(-3xy^3)^{-2} &= \frac{1}{(-3xy^3)^2}\\ &= \frac{1}{(-3)^2x^2(y^3)^2}\\ &= \frac{1}{9x^2y^6}\end{aligned}$$

Answer:

$\frac{1}{9x^2y^6}$

Example 5.6.6

Simplify:

$$\frac{x^{-3}}{y^{-4}}.$$

Solution:

$$\frac{x^{-3}}{y^{-4}} = \frac{\frac{1}{x^3}}{\frac{1}{y^4}} = \frac{1}{x^3}\cdot\frac{y^4}{1} = \frac{y^4}{x^3}$$

Answer:

$\frac{y^4}{x^3}$

The previous example suggests a property of quotients with negative exponents. If given any integers m and n, where $x \neq 0$ and $y \neq 0$, then

$$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n} \tag{5.6.2}$$

In other words, negative exponents in the numerator can be written as positive exponents in the denominator, and negative exponents in the denominator can be written as positive exponents in the numerator.

Example 5.6.7

Simplify:

$$\frac{-2x^{-5}y^3}{z^{-2}}.$$

Solution:

Take care with the coefficient -2; recognize that this is the base and that the exponent is actually $+1$: $-2=(-2)^1$. Hence the rules of negative exponents do not apply to this coefficient; leave it in the numerator.

$$\begin{aligned}\frac{-2x^{-5}y^3}{z^{-2}} &= \frac{-2x^{-5}y^3}{z^{-2}} \\ &= \frac{-2y^3z^2}{x^5}\end{aligned}$$

Answer:

$$\frac{-2y^3z^2}{x^5}$$

Example 5.6.8

Simplify:

$$\frac{(-3x^{-4})^{-3}}{y^{-2}}.$$

Solution:

$$\begin{aligned}\frac{(-3x^{-4})^{-3}}{y^{-2}} &= \frac{(-3)^{-3}(x^{-4})^{-3}}{y^{-2}} && \textit{Apply the product to a power rule.} \\ &= \frac{(-3)^{-3}x^{12}}{y^{-2}} && \textit{Power rule} \\ &= \frac{x^{12}y^2}{(-3)^3} && \textit{Negative exponents} \\ &= \frac{x^{12}y^2}{-27} \\ &- \frac{x^{12}y^2}{27}\end{aligned}$$

Answer:

$$-\frac{x^{12}y^2}{27}$$

Example 5.6.9

Simplify:

$$\frac{(3x^2)^{-4}}{(-2y^{-1}z^3)^{-2}}.$$

Solution:

$$\begin{aligned}\frac{(3x^2)^{-4}}{(-2y^{-1}z^3)^{-2}} &= \frac{3^{-4}(x^2)^{-4}}{(-2)^{-2}(y^{-1})^{-2}(z^3)^{-2}} && \textit{Product to a power rule}\\ &= \frac{3^{-4}x^{-8}}{(-2)^{-2}y^2z^{-6}} && \textit{Power rule}\\ &= \frac{(-2)^2z^6}{3^4x^8y^2} && \textit{Negative exponents}\\ &= \frac{4z^6}{81x^8y^2}\end{aligned}$$

Answer:

$\frac{4z^6}{81x^8y^2}$

Example 5.6.10

Simplify:

$\frac{(5x^2y)^3}{x^{-5}y^{-3}}$.

Solution:

First, apply the power of a product rule and then the quotient rule.

$$\frac{(5x^2)^3}{x^{-5}y^{-3}} = \frac{5^3x^6y^3}{x^{-5}y^{-3}} = 5^3x^{6-(-5)}y^{3-(-3)} = 5^3x^{6+5}y^{3+3} = 125x^{11}y^6$$

Answer:

$125x^{11}y^6$

To summarize, we have the following rules for negative integer exponents with nonzero bases:

Table 5.6.1

Negative exponents:	$x^{-n} = \frac{1}{x^n}$
Quotients with negative exponents:	$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$

Exercise 5.6.1

Simplify:

$\frac{(-5xy^{-3})^{-2}}{5x^4y^{-4}}$.

Answer available at the end of this section.

Scientific Notation

Real numbers expressed in scientific notation have the form

$$a \times 10^n$$

where n is an integer and $1 \leq a < 10$. This form is particularly useful when the numbers are very large or very small. For example,

$$9,460,000,000,000,000m = 9.46 \times 10^{15}m \qquad \textit{One light year}$$
$$0.000000000025m = 2.5 \times 10^{-11}m \qquad \textit{Radius of a hydrogen atom}$$

It is cumbersome to write all the zeros in both of these cases. Scientific notation is an alternative, compact representation of these numbers. The factor 10^n indicates the power of 10 to multiply the coefficient by to convert back to decimal form:

$$9.46\times10^{15} = 9.46\times1,\overbrace{000,000,000,000,000}^{15\ zeros} = 9,460,000,000,000,000$$

Figure $5.6.2$

This is equivalent to moving the decimal in the coefficient fifteen places to the right. A negative exponent indicates that the number is very small:

$$2.5\times10^{-11} = 2.5\times\frac{1}{10^{11}} = \frac{2.5}{\underbrace{100,000,000,000}_{11\ zeros}} = 0.000000000025$$

Figure $5.6.3$

This is equivalent to moving the decimal in the coefficient eleven places to the left.

Converting a decimal number to scientific notation involves moving the decimal as well. Consider all of the equivalent forms of 0.00563 with factors of 10 that follow:

$$\begin{aligned} 0.00563 &= 0.0563 \times 10^{-1} \\ &= 0.563 \times 10^{-2} \\ &= 5.63 \times 10^{-3} \\ &= 56.3 \times 10^{-4} \\ &= 563 \times 10^{-5} \end{aligned}$$

While all of these are equal, 5.63×10^{-3} is the only form considered to be expressed in scientific notation. This is because the coefficient 5.63 is between 1 and 10 as required by the definition. Notice that we can convert 5.63×10^{-3} back to decimal form, as a check, by moving the decimal to the left three places.

Example $5.6.11$

Write $1,075,000,000,000$ using scientific notation.

Solution:

Here we count twelve decimal places to the left of the decimal point to obtain the number 1.075.

$$1,075,000,000,000 = 1.075 \times 10^{12}$$

Answer:

$$1.075 \times 10^{12}$$

Example 5.6.12

Write 0.000003045 using scientific notation.

Solution:

Here we count six decimal places to the right to obtain 3.045.

$$0.000003045 = 3.045 \times 10^{-6}$$

Answer:

$$3.045 \times 10^{-6}$$

Often we will need to perform operations when using numbers in scientific notation. All the rules of exponents developed so far also apply to numbers in scientific notation.

Example 5.6.13

Multiply:

$$(4.36 \times 10^{-5})(5.3 \times 10^{12}).$$

Solution:

Use the fact that multiplication is commutative and apply the product rule for exponents.

$$\begin{aligned}(4.36 \times 10^{-5})(5.3 \times 10^{12}) &= (4.36 \cdot 5.30) \times (10^{-5} \cdot 10^{12}) \\ &= 23.108 \times 10^{-5+12} \\ &= 2.3108 \times 10^{1} \times 10^{7} \\ &= 2.3108 \times 10^{1+7} \\ &= 2.3108 \times 10^{8}\end{aligned}$$

Answer:

$$2.3108 \times 10^{8}$$

Example 5.6.14

Divide:

$$(3.24 \times 10^{8}) \div (9.0 \times 10^{-3}).$$

Solution:

$$\begin{aligned}\frac{(3.24 \times 10^{8})}{(9.0 \times 10^{-3})} &= \left(\frac{3.24}{9.0}\right) \times \left(\frac{10^{8}}{10^{-3}}\right) \\ &= 0.36 \times 10^{8-(-3)} \\ &= 0.36 \times 10^{8+3} \\ &= 3.6 \times 10^{-1} \times 10^{11} \\ &= 3.6 \times 10^{-1+11} \\ &= 3.6 \times 10^{10}\end{aligned}$$

Answer:

$$3.6 \times 10^{10}$$

Example 5.6.15

The speed of light is approximately 6.7×10^8 miles per hour. Express this speed in miles per second.

Solution:

A unit analysis indicates that we must divide the number by $3,600$.

$$\begin{aligned} 6.7 \times 10^8\ mph &= \frac{6.7 \times 10^8 miles}{1\ \cancel{hour}} \cdot \left(\frac{1\ \cancel{hour}}{60\ \cancel{minutes}}\right) \cdot \left(\frac{1\ \cancel{minutes}}{60 seconds}\right) \\ &= \frac{6.7 \times 10^8 miles}{3600 seconds} \\ &= \left(\frac{6.7}{3600}\right) \times 10^8 \\ &\approx 0.0019 \times 10^8 \qquad \textit{Rounded to two significant digits} \\ &= 1.9 \times 10^{-3} \times 10^8 \\ &= 1.9 \times 10^{-3+8} \\ &= 1.9 \times 10^5 \end{aligned}$$

Answer:

The speed of light is approximately 1.9×10^5 miles per second.

Example 5.6.16

By what factor is the radius of the sun larger than the radius of earth?

$$\begin{aligned} 6,300,000m &= 6.3 \times 10^6 m \qquad \textit{Radius of Earth} \\ 700,000,000m &= 7.0 \times 10^8 m \qquad \textit{Radius of the Sun} \end{aligned}$$

Solution:

We want to find the number that when multiplied times the radius of earth equals the radius of the sun.

$$\begin{aligned} n \cdot \textit{radius of the Earth} &= \textit{radius of the Sun} \\ n &= \frac{\textit{radius of the Sun}}{\textit{radius of the Earth}} \end{aligned}$$

Therefore,

$$\begin{aligned} n &= \frac{7.0 \times 10^8 m}{6.3 \times 10^6 m} \\ &= \frac{7.0}{6.3} \times \frac{10^8}{10^6} \\ &\approx 1.1 \times 10^{8-6} \\ &= 1.1 \times 10^2 \\ &= 110 \end{aligned}$$

Exercise 5.6.2

Divide:

$$(6.75 \times 10^{-8}) \div (9 \times 10^{-17}).$$

Answer available at the end of this section.

Key Takeaways

- Expressions with negative exponents in the numerator can be rewritten as expressions with positive exponents in the denominator.
- Expressions with negative exponents in the denominator can be rewritten as expressions with positive exponents in the numerator.
- Take care to distinguish negative coefficients from negative exponents.
- Scientific notation is particularly useful when working with numbers that are very large or very small.

Exercise 5.6.3 Negative Exponents

Simplify. (Assume variables are nonzero.)

1. 5^{-1}
2. 5^{-2}
3. $(-7)^{-1}$
4. -7^{-1}
5. $\frac{1}{2}^{-3}$
6. $\frac{5}{3}^{-2}$
7. $(\frac{3}{5})^{-2}$
8. $(\frac{1}{2})^{-5}$
9. $(-\frac{2}{3})^{-4}$
10. $(-\frac{1}{3})^{-3}$
11. x^{-4}
12. y^{-1}
13. $3x^{-5}$
14. $(3x)^{-5}$
15. $\frac{1}{y^{-3}}$
16. $\frac{5}{2}x^{-1}$
17. $\frac{x^{-1}}{y^{-2}}$
18. $\frac{1}{(x-y)^{-4}}$
19. $\frac{x^2y^{-3}}{z^{-5}}$
20. $\frac{x}{y^{-3}}$
21. $(ab)^{-1}$
22. $\frac{1}{(ab)^{-1}}$
23. $-5x^{-3}y^2z^{-4}$
24. $\frac{3}{-2x^3y^{-5}z}$
25. $3x^{-4}y^2 \cdot 2x^{-1}y^3$
26. $-10a^2b^3 \cdot 2a^{-8}b^{-10}$
27. $(2a^{-3})^{-2}$
28. $(-3x^2)^{-1}$
29. $(5a^2b^{-3}c)^{-2}$
30. $(7r^3s^{-5}t)^{-3}$
31. $(-2r^2s^0t^{-3})^{-1}$
32. $(2xy^{-3}z^2)^{-3}$
33. $(-5a^2b^{-3}c^0)^4$
34. $(-x^{-2}y^3z^{-4})^{-7}$
35. $(\frac{1}{2}x^{-3})^{-5}$
36. $(2xy^2)^{-2}$

37. $(x^2y^{-1})^{-4}$
38. $(-3a^2bc^5)^{-5}$
39. $(\frac{20x^{-3}y^2}{5yz^{-1}})^{-1}$
40. $(\frac{4r^5s^{-3}t^4}{2r^3st^0})^{-3}$
41. $(\frac{2xy^3z^{-1}}{y^2z^3})^{-3}$
42. $(-\frac{3a^2bc}{ab^0c^4})^2$
43. $(\frac{-xyz}{x^4y^{-2}z^3})^{-4}$
44. $(-\frac{125x^{-3}y^4z^{-5}}{5x^2y^4(x+y)^3})^0$
45. $(x^n)^{-2}$
46. $(x^ny^n)^{-2}$

Answer available at the end of this section.

Exercise 5.6.4 Negative Exponents

The value in dollars of a new MP3 player can be estimated by using the formula $V = 100(t+1)^{-1}$, where t is the number of years after purchase.

1. How much was the MP3 player worth new?
2. How much will the MP3 player be worth in 1 year?
3. How much will the MP3 player be worth in 4 years?
4. How much will the MP3 player be worth in 9 years?
5. How much will the MP3 player be worth in 99 years?
6. According to the formula, will the MP3 ever be worthless? Explain.

Answer available at the end of this section.

Exercise 5.6.5 Scientific Notation

Convert to a decimal number.

1. 9.3×10^9
2. 1.004×10^4
3. 6.08×10^{10}
4. 3.042×10^7
5. 4.01×10^{-7}
6. 1.0×10^{-10}
7. 9.9×10^{-3}
8. 7.0011×10^{-5}

Answer available at the end of this section.

Exercise 5.6.6 Scientific Notation

Rewrite using scientific notation.

1. $500,000,000$
2. $407,300,000,000,000$
3. $9,740,000$
4. $100,230$
5. 0.0000123
6. 0.000012
7. 0.000000010034

8. 0.99071

Answer available at the end of this section.

Exercise $5.6.7$ Scientific Notation

Perform the indicated operations.

1. $(3 \times 10^5)(9 \times 10^4)$
2. $(8 \times 10^{-22})(2 \times 10^{-12})$
3. $(2.1 \times 10^{-19})(3.0 \times 10^8)$
4. $(4.32 \times 10^7)(1.50 \times 10^{-18})$
5. $9.12 \times 10^{-9} 3.2 \times 10^{10}$
6. $1.15 \times 10^9 2.3 \times 10^{-11}$
7. $1.004 \times 10^{-8} 2.008 \times 10^{-14}$
8. $3.276 \times 10^{25} 5.2 \times 10^{15}$
9. $59,000,000,000,000 \times 0.000032$
10. $0.0000000000432 \times 0.0000000000673$
11. $1,030,000,000,000,000,000 \div 2,000,000$
12. $6,045,000,000,000,000,000 \div 0.00000005$
13. The population density of earth refers to the number of people per square mile of land area. If the total land area on earth is 5.751×10^7 square miles and the population in 2007 was estimated to be 6.67×10^9 people, then calculate the population density of earth at that time.
14. In 2008 the population of New York City was estimated to be 8.364 million people. The total land area is 305 square miles. Calculate the population density of New York City.
15. The mass of earth is 5.97×10^{24} kilograms and the mass of the moon is 7.35×10^{22} kilograms. By what factor is the mass of earth greater than the mass of the moon?
16. The mass of the sun is 1.99×10^{30} kilograms and the mass of earth is 5.97×10^{24} kilograms. By what factor is the mass of the sun greater than the mass of earth? Express your answer in scientific notation.
17. The radius of the sun is 4.322×10^5 miles and the average distance from earth to the moon is 2.392×10^5 miles. By what factor is the radius of the sun larger than the average distance from earth to the moon?
18. One light year, 9.461×10^{15} meters, is the distance that light travels in a vacuum in one year. If the distance to the nearest star to our sun, Proxima Centauri, is estimated to be 3.991×10^{16} meters, then calculate the number of years it would take light to travel that distance.
19. It is estimated that there are about 1 million ants per person on the planet. If the world population was estimated to be 6.67 billion people in 2007, then estimate the world ant population at that time.
20. The sun moves around the center of the galaxy in a nearly circular orbit. The distance from the center of our galaxy to the sun is approximately $26,000$ light years. What is the circumference of the orbit of the sun around the galaxy in meters?
21. Water weighs approximately 18 grams per mole. If one mole is about 6×10^{23} molecules, then approximate the weight of each molecule of water.
22. A gigabyte is 1×10^9 bytes and a megabyte is 1×10^6 bytes. If the average song in the MP3 format consumes about 4.5 megabytes of storage, then how many songs will fit on a 4-gigabyte memory card?

Answer available at the end of this section.

Answers

Exercise $5.6.1$

$\frac{y^{10}}{125x^6}$

Exercise $5.6.2$

7.5×10^8

Exercise $5.6.3$ **Negative Exponents**

1. $\frac{1}{5}$
3. $-\frac{1}{7}$
5. 8
7. $\frac{25}{9}$

9. $\frac{81}{16}$
11. $\frac{1}{x^4}$
13. $3x^5$
15. y^3
17. $\frac{y^2}{x}$
19. $\frac{x^2z^5}{y^3}$
21. $\frac{1}{ab}$
23. $\frac{-5y^2}{x^3z^4}$
25. $\frac{6y^5}{x^5}$
27. $\frac{a^6}{4}$
29. $\frac{b^6}{25a^4c^2}$
31. $-\frac{t^3}{2r^2}$
33. $\frac{625a^8}{b^{12}}$
35. $32x^{15}$
37. $\frac{y^4}{x^8}$
39. $\frac{x^3}{4yz}$
41. $\frac{z^{12}}{8x^3y^3}$
43. $\frac{x^{12}z^8}{y^{12}}$
45. $\frac{1}{x^{2n}}$

Exercise $5.6.4$ **Negative Exponents**

1. $\$100$
3. $\$20$
5. $\$1$

Exercise $5.6.5$ **Scientific Notation**

1. $9,300,000,000$
3. $60,800,000,000$
5. 0.000000401
7. 0.0099

Exercise $5.6.6$ **Scientific Notation**

1. 5×10^8
3. 9.74×10^6
5. 1.23×10^{-5}
7. 1.0034×10^{-8}

Exercise $5.6.7$ **Scientific Notation**

1. 2.7×10^{10}
3. 6.3×10^{-11}
5. 2.85×10^{-19}
7. 5×10^5
9. 1.888×10^9
11. 5.15×10^{11}
13. About 116 people per square mile
15. 81.2
17. 1.807
19. 6.67×10^{15} ants
21. 3×10^{-23} grams

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5.E: Review Exercises and Sample Exam

Review Exercises

Exercise $5.E.1$ Rules of Exponents

Simplify.

1. $7^3 \cdot 7^6$
2. $5^9 5^6$
3. $y^5 \cdot y^2 \cdot y^3$
4. $x^3 y^2 \cdot xy^3$
5. $-5a^3b^2c \cdot 6a^2bc^2$
6. $\frac{55x^2yz}{55xyz^2}$
7. $(\frac{-3a^2b^4}{2c^3})^2$
8. $(-2a^3b^4c^4)^3$
9. $-5x^3y^0(z^2)^3 \cdot 2x^4(y^3)^2z$
10. $(-25x^6y^5z)^0$
11. Each side of a square measures $5x^2$ units. Find the area of the square in terms of x.
12. Each side of a cube measures $2x^3$ units. Find the volume of the cube in terms of x.

Answer available at the end of this section.

Exercise $5.E.2$ Introduction to Polynomials

Classify the given polynomial as a monomial, binomial, or trinomial and state the degree.

1. $8a^3 - 1$
2. $5y^2 - y + 1$
3. $-12ab^2$
4. 10

Answer available at the end of this section.

Exercise $5.E.3$ Introduction to Polynomials

Write the following polynomials in standard form.

1. $7 - x^2 - 5x$
2. $5x^2 - 1 - 3x + 2x^3$

Answer available at the end of this section.

Exercise $5.E.4$ Introduction to Polynomials

Evaluate.

1. $2x^2 - x + 1$, where $x = -3$
2. $\frac{1}{2}x - \frac{3}{4}$, where $x = \frac{1}{3}$
3. $b^2 - 4ac$, where $a = -\frac{1}{2}, b = -3$, and $c = -\frac{3}{2}$

4. $a^2 - b^2$, where $a = -\frac{1}{2}$ and $b = -\frac{1}{3}$
5. $a^3 - b^3$, where $a = -2$ and $b = -1$
6. $xy^2 - 2x^2y$, where $x = -3$ and $y = -1$
7. Given $f(x) = 3x^2 - 5x + 2$, find $f(-2)$.
8. Given $g(x) = x^3 - x^2 + x - 1$, find $g(-1)$.
9. The surface area of a rectangular solid is given by the formula $SA = 2lw + 2wh + 2lh$, where l, w, and h represent the length, width, and height, respectively. If the length of a rectangular solid measures 2 units, the width measures 3 units, and the height measures 5 units, then calculate the surface area.
10. The surface area of a sphere is given by the formula $SA = 4\pi r^2$, where r represents the radius of the sphere. If a sphere has a radius of 5 units, then calculate the surface area.

Answer available at the end of this section.

Exercise $5.E.5$ Adding and Subtracting Polynomials

Perform the operations.

1. $(3x - 4) + (9x - 1)$
2. $(13x - 19) + (16x + 12)$
3. $(7x^2 - x + 9) + (x^2 - 5x + 6)$
4. $(6x^2y - 5xy^2 - 3) + (-2x^2y + 3xy^2 + 1)$
5. $(4y + 7) - (6y - 2) + (10y - 1)$
6. $(5y^2 - 3y + 1) - (8y^2 + 6y - 11)$
7. $(7x^2y^2 - 3xy + 6) - (6x^2y^2 + 2xy - 1)$
8. $(a^3 - b^3) - (a^3 + 1) - (b^3 - 1)$
9. $(x^5 - x^3 + x - 1) - (x^4 - x^2 + 5)$
10. $(5x^3 - 4x^2 + x - 3) - (5x^3 - 3) + (4x^2 - x)$
11. Subtract $2x - 1$ from $9x + 8$.
12. Subtract $3x^2 - 10x - 2$ from $5x^2 + x - 5$.
13. Given $f(x) = 3x^2 - x + 5$ and $g(x) = x^2 - 9$, find $(f + g)(x)$.
14. Given $f(x) = 3x^2 - x + 5$ and $g(x) = x^2 - 9$, find $(f - g)(x)$.
15. Given $f(x) = 3x^2 - x + 5$ and $g(x) = x^2 - 9$, find $(f + g)(-2)$.
16. Given $f(x) = 3x^2 - x + 5$ and $g(x) = x^2 - 9$, find $(f - g)(-2)$.

Answer available at the end of this section.

Exercise $5.E.6$ Multiplying Polynomials

Multiply.

1. $6x^2(-5x^4)$
2. $3ab^2(7a^2b)$
3. $2y(5y - 12)$
4. $-3x(3x^2 - x + 2)$
5. $x^2y(2x^2y - 5xy^2 + 2)$
6. $-4ab(a^2 - 8ab + b^2)$
7. $(x - 8)(x + 5)$
8. $(2y - 5)(2y + 5)$
9. $(3x - 1)^2$
10. $(3x - 1)^3$
11. $(2x - 1)(5x^2 - 3x + 1)$
12. $(x^2 + 3)(x^3 - 2x - 1)$
13. $(5y + 7)^2$
14. $(y^2 - 1)^2$
15. Find the product of $x^2 - 1$ and $x^2 + 1$.

16. Find the product of $32x^2y$ and $10x - 30y + 2$.
17. Given $f(x) = 7x - 2$ and $g(x) = x^2 - 3x + 1$, find $(f \cdot g)(x)$.
18. Given $f(x) = x - 5$ and $g(x) = x^2 - 9$, find $(f \cdot g)(x)$.
19. Given $f(x) = 7x - 2$ and $g(x) = x^2 - 3x + 1$, find $(f \cdot g)(-1)$.
20. Given $f(x) = x - 5$ and $g(x) = x^2 - 9$, find $(f \cdot g)(-1)$.

Answer available at the end of this section.

Exercise $5.E.7$ Dividing Polynomials

Divide.

1. $\frac{7y^2-14y+28}{7}$
2. $\frac{12x^5-30x^3+6x}{6x}$
3. $\frac{4a^2b-16ab^2-4ab}{-4ab}$
4. $\frac{6a^6-24a^4+5a^2}{3a^2}$
5. $(10x^2 - 19x + 6) \div (2x - 3)$
6. $(2x^3 - 5x^2 + 5x - 6) \div (x - 2)$
7. $\frac{10x^4-21x^3-16x^2+23x-20}{2x-5}$
8. $\frac{x^5-3x^4-28x^3+61x^2-12x+36}{x-6}$
9. $\frac{10x^3-55x^2+72x-4}{2x-7}$
10. $\frac{3x^4+19x^3+3x^2-16x-11}{3x+1}$
11. $\frac{5x^4+4x^3-5x^2+21x+21}{5x+4}$
12. $\frac{x^4-4}{x-4}$
13. $\frac{2x^4+10x^3-23x^2-15x+30}{2x^2-3}$
14. $\frac{7x^4-17x^3+17x^2-11x+2}{x^2-2x+1}$
15. Given $f(x) = x^3 - 4x + 1$ and $g(x) = x - 1$, find $(f/g)(x)$.
16. Given $f(x) = x^5 - 32$ and $g(x) = x - 2$, find $(f/g)(x)$.
17. Given $f(x) = x^3 - 4x + 1$ and $g(x) = x - 1$, find $(f/g)(2)$.
18. Given $f(x) = x^5 - 32$ and $g(x) = x - 2$, find $(f/g)(0)$.

Answer available at the end of this section.

Exercise $5.E.8$ Negative Exponents

Simplify.

1. $(-10)^{-2}$
2. -10^{-2}
3. $5x^{-3}$
4. $(5x)^{-3}$
5. $\frac{1}{7y^{-3}}$
6. $3x^{-4}y^{-2}$
7. $\frac{-2a^2b^{-5}}{c^{-8}}$
8. $(-5x^2yz^{-1})^{-2}$
9. $(-2x^{-3}y^0z^2)^{-3}$
10. $(\frac{-10a^5b^3c^2}{5ab^2c^2})^{-1}$
11. $(\frac{a^2b^{-4}c^0}{2a^4b^{-3}c})^{-3}$

Answer available at the end of this section.

Exercise $5.E.9$ Negative Exponents

The value in dollars of a new laptop computer can be estimated by using the formula $V = 1200(t+1)^{-1}$, where t represents the number of years after the purchase.

1. Estimate the value of the laptop when it is $1\frac{1}{2}$ years old.
2. What was the laptop worth new?

Answer available at the end of this section.

Exercise $5.E.10$ Negative Exponents

Rewrite using scientific notation.

1. $2,030,000,000$
2. 0.00000004011

Answer available at the end of this section.

Exercise $5.E.11$ Negative Exponents

Perform the indicated operations.

1. $(5.2 \times 10^{12})(1.8 \times 10^{-3})$
2. $(9.2 \times 10^{-4})(6.3 \times 10^{22})$
3. $\frac{4 \times 10^{16}}{8 \times 10^{-7}}$
4. $\frac{9 \times 10^{-30}}{4 \times 10^{-10}}$
5. $5,000,000,000,000 \times 0.0000023$
6. $\frac{0.0003}{120{,}000{,}000{,}000{,}000}$

Answer available at the end of this section.

Simple Exam

Exercise $5.E.12$

Simplify.

1. $-5x^3(2x^2y)$
2. $(x^2)^4 \cdot x^3 \cdot x$
3. $\frac{(-2x^2y^3)^2}{x^2y}$
4. a. $(-5)^0$
   b. -5^0

Answer available at the end of this section.

Exercise $5.E.13$

Evaluate.

1. $2x^2 - x + 5$, where $x = -5$

2. $a^2 - b^2$, where $a = 4$ and $b = -3$

Answer available at the end of this section.

Exercise $5.E.14$

Perform the operations.

1. $(3x^2 - 4x + 5) + (-7x^2 + 9x - 2)$
2. $(8x^2 - 5x + 1) - (10x^2 + 2x - 1)$
3. $(\frac{3}{5}a - \frac{1}{2}) - (\frac{2}{3}a^2 + \frac{2}{3}a - \frac{2}{9}) + (\frac{1}{15}a - \frac{5}{18})$
4. $2x^2(2x^3 - 3x^2 - 4x + 5)$
5. $(2x - 3)(x + 5)$
6. $(x - 1)^3$
7. $\frac{81x^5y^2z}{-3x^3yz}$
8. $\frac{10x^9 - 15x^5 + 5x^2}{-5x^2}$
9. $\frac{x^3 - 5x^2 + 7x - 2}{x - 2}$
10. $\frac{6x^4 - x^3 - 13x^2 - 2x - 1}{2x - 1}$

Answer available at the end of this section.

Exercise $5.E.15$

Simplify.

1. 2^{-3}
2. $-5x^{-2}$
3. $(2x^4y^{-3}z)^{-2}$
4. $(\frac{-2a^3b^{-5}c^{-2}}{ab^{-3}c^2})^{-3}$
5. Subtract $5x^2y - 4xy^2 + 1$ from $10x^2y - 6xy^2 + 2$.
6. If each side of a cube measures $4x4$ units, calculate the volume in terms of x.
7. The height of a projectile in feet is given by the formula $h = -16t^2 + 96t + 10$, where t represents time in seconds. Calculate the height of the projectile at $1\frac{1}{2}$ seconds.
8. The cost in dollars of producing custom t-shirts is given by the formula $C = 120 + 3.50x$, where x represents the number of t-shirts produced. The revenue generated by selling the t-shirts for \$$6.50$ each is given by the formula $R = 6.50x$, where x represents the number of t-shirts sold.
   a. Find a formula for the profit. (profit = revenue – cost)
   b. Use the formula to calculate the profit from producing and selling 150 t-shirts.
9. The total volume of water in earth's oceans, seas, and bays is estimated to be 4.73×10^{19} cubic feet. By what factor is the volume of the moon, 7.76×10^{20} cubic feet, larger than the volume of earth's oceans? Round to the nearest tenth.

Answer available at the end of this section.

Answers

Exercise $5.E.1$ Rules of Exponents

1. 7^9
3. y^{10}
5. $-30a^5b^3c^3$
7. $\frac{9a^4b^8}{4c^6}$
9. $-10x^7y^6z^7$
11. $A = 25x^4$

Exercise $5.E.2$ Introduction to Polynomials

1. Binomial; degree 3
3. Monomial; degree 3

Exercise $5.E.3$ **Introduction to Polynomials**

1. $-x^2-5x+7$

Exercise $5.E.4$ **Introduction to Polynomials**

1. 22
3. 6
5. -7
7. $f(-2)=24$
9. 62 square units

Exercise $5.E.5$ **Adding and Subtracting Polynomials**

1. $12x-5$
3. $8x^2-6x+15$
5. $8y+8$
7. $x^2y^2-5xy+7$
9. $x^5-x^4-x^3+x^2+x-6$
11. $7x+9$
13. $(f+g)(x)=4x^2-x-4$
15. $(f+g)(-2)=14$

Exercise $5.E.6$ **Multiplying Polynomials**

1. $-30x^6$
3. $10y^2-24y$
5. $2x^4y^2-5x^3y^3+2x^2y$
7. $x^2-3x-40$
9. $9x^2-6x+1$
11. $10x^3-11x^2+5x-1$
13. $25y^2+70y+49$
15. x^4-1
17. $(f\cdot g)(x)=7x^3-23x^2+13x-2$
19. $(f\cdot g)(-1)=-45$

Exercise $5.E.7$ **Dividing Polynomials**

1. y^2-2y+4
3. $-a+4b+1$
5. $5x-2$
7. $5x^3+2x^2-3x+4$
9. $5x^2-10x+1+\frac{3}{2x-7}$
11. $x^3-x+5+\frac{1}{5x+4}$
13. $x^2+5x-10$
15. $(f/g)(x)=x^2+x-3-\frac{2}{x-1}$
17. $(f/g)(2)=1$

Exercise $5.E.8$ **Negative Exponents**

1. $\frac{1}{100}$
3. $\frac{5}{x^3}$
5. $\frac{y^3}{7}$
7. $\frac{-2a^2c^8}{b^5}$
9. $\frac{-x^9}{8z^6}$
11. $8a^6b^3c^3$

Exercise $5.E.9$ **Negative Exponents**

2. $\$1,200$

Exercise $5.E.10$ **Negative Exponents**

2. 5.796×10^{19}

Exercise $5.E.11$ **Negative Exponents**

2. 5.796×10^{19}
4. 2.25×10^{-20}
6. 2.5×10^{-18}

Exercise $5.E.12$

1. $-10x^5y$
3. $4x^2y^5$

Exercise $5.E.13$

1. 60

Exercise $5.E.14$

1. $-4x^2+5x+3$

3. $-\frac{2}{3}a^2-\frac{5}{9}$
5. $2x^2+7x-15$
7. $-27x^2y$
9. x^2-3x+1

Exercise $5.E.15$

1. $\frac{1}{8}$
3. $\frac{y^6}{4x^8z^2}$
5. $5x^2y-2xy^2+1$
7. 118 feet
9. 16.4

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6: FACTORING AND SOLVING BY FACTORING

6.1: Introduction to Factoring

6.2: Factoring Trinomials of the Form x^2+bx+c

6.3: Factoring Trinomials of the Form ax^2+bx+c

6.4: Factoring Special Binomials

6.5: General Guidelines for Factoring Polynomials

6.6: Solving Equations by Factoring

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6.E: Review Exercises and Sample Exam

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6.1: Introduction to Factoring

Learning Objectives

- Determine the greatest common factor (GCF) of natural numbers.
- Determine the GCF of monomials.
- Factor out the GCF of a polynomial.
- Factor a four-term polynomial by grouping.

GCF of Natural Numbers

The process of writing a number or expression as a product is called factoring. If we write $60 = 5 \cdot 12$, we say that the product $5 \cdot 12$ is a factorization of 60 and that 5 and 12 are factors. Typically, there are many ways to factor a number. For example,

$$\begin{aligned} 60 &= 6 \cdot 10 \\ 60 &= 2 \cdot 30 \qquad \textit{Factorizations of } 60 \\ 60 &= 4 \cdot 3 \cdot 5 \end{aligned}$$

Recall that a prime number is defined as a natural number with exactly two natural number factors, 1 and itself. The first ten prime numbers follow:

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ \ldots$$

Any natural number greater than 1 can be uniquely written as a product of prime numbers. This product is called the prime factorization. The prime factorization of 60 can be determined by continuing to factor until only a product of prime numbers remains.

$$\begin{aligned} 60 &= 2 \cdot 30 \\ &= 2 \cdot 2 \cdot 15 \\ &= 2 \cdot 2 \cdot 3 \cdot 5 \end{aligned}$$

Since the prime factorization is unique, it does not matter how we choose to initially factor the number; the end result will be the same. The prime factorization of 60 follows:

$$60 = 2^2 \cdot 3 \cdot 5 \qquad \textit{The prime factorization of } 60$$

Recall that the greatest common factor (GCF) of any two natural numbers is the product of all the common prime factors.

Example 6.1.1

Find the GCF of 60 and 140.

Solution:

First, determine the prime factorizations of both integers.

$$\begin{aligned} 140 &= 10 \cdot 14 & 60 &= 6 \cdot 10 \\ &= 2 \cdot 5 \cdot 2 \cdot 7 & &= 2 \cdot 3 \cdot 2 \cdot 5 \\ &= 2^2 \cdot 5 \cdot 7 & &= 2^2 \cdot 3 \cdot 5 \end{aligned}$$

The product of the common prime factors is $2^2 \cdot 5$; hence the GCF$(60, 140) = 2^2 \cdot 5 = 20$. To see that it is the greatest common factor, we can write the following:

$$140 = 20 \cdot 7$$
$$60 = 20 \cdot 3$$

Answer:

The greatest common factor of 60 and 140 is 20.

Example 6.1.2

Find the GCF of 504 and 1,080.

Solution:

First, determine the prime factorizations of both integers.

$$\begin{aligned} 504 &= 9 \cdot 56 \\ &= 3 \cdot 3 \cdot 7 \cdot 8 \\ &= 3 \cdot 3 \cdot 7 \cdot 2 \cdot 2 \cdot 2 \\ &= 2^3 \cdot 3^2 \cdot 7 \end{aligned} \qquad \begin{aligned} 1080 &= 10 \cdot 108 \\ &= 2 \cdot 5 \cdot 9 \cdot 12 \\ &= 2 \cdot 5 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 3 \\ &= 2^3 \cdot 3^3 \cdot 5 \end{aligned}$$

The product of the common prime factors is $2^3 \cdot 3^2$.

The GCF$(504, 1080) = 2^3 \cdot 3^2 = 72$. Note that we multiplied the common prime factors with the smallest exponent.

$$5040 = 72 \cdot 7$$
$$1080 = 72 \cdot 15$$

The numbers 7 and 15 share no common natural number factor other than 1; we say that they are relatively prime.

Answer:

The greatest common factor of 504 and $1,080$ is 72.

GCF of Monomials

We next consider factorizations of monomials. For example, $6x$ and x^4 are factors of $6x^5$ because $6x^5 = 6x \cdot x^4$. Typically, there are many ways to factor a monomial. Some factorizations of $6x^5$ follow:

$$\begin{aligned} 6x^5 &= 2x^3 \cdot 3x^2 \\ 6x^5 &= 6x^4 \cdot x \\ 6x^5 &= 2x \cdot 3x \cdot x^3 \end{aligned} \qquad \textit{Factorizations of } 6x^5$$

Given two or more monomials, it will be useful to find the greatest common monomial factor of each. For example, consider $6x^5y^3z$ and $8x^2y^3z^2$. The variable part of these two monomials look very much like the prime factorization of natural numbers and, in fact, can be treated the same way.

Steps for finding the GCF of monomials are outlined in the following example.

Example 6.1.3

Find the GCF of $6x^5y^3z$ and $8x^2y^3z^2$.

Solution:

Step 1: Find the GCF of the coefficients.

$$6x^5y^3z \quad \text{and} \quad 8x^2y^3z^2$$

In this case, the GCF$(6, 8) = 2$.

Step 2: Determine the common variable factors with smallest exponents.

$$6x^5y^3z \quad \text{and} \quad 8x^2y^3z^2$$

In this case, the common variables with the smallest exponents are $x^2, y^3, andz^1$.

Step 3: The GCF of the monomials is the product of the common variable factors and the GCF of the coefficients. Therefore,

$$\text{GCF } (6x^5y^3z, 8x^2y^3z^2) = 2 \cdot x^2 \cdot y^3 \cdot z$$

Answer:

$$2x^2y^3z$$

It is worth pointing out that the GCF in the previous example divides both expressions evenly:

$$\frac{6x^5y^3z}{2x^2y^3z} = 3x^3 \quad \text{and} \quad \frac{8x^2y^3z^2}{2x^2y^3z} = 4z$$

Furthermore, we can write the following:

$$6x^5y^3z = 2x^2y^3z \cdot 3x^3 \quad \text{and} \quad 8x^2y^3z^2 = 2x^2y^3z \cdot 4z$$

The factors $3x^3$ and $4z$ share no common monomial factors other than 1; they are relatively prime.

Example 6.1.4

Determine the GCF of the following expressions:

$$30x^6y \text{ and } 18x^4y^2z.$$

Solution:

The prime factorizations of the coefficients are

$$30 = 2 \cdot 3 \cdot 5$$
$$18 = 2 \cdot 3 \cdot 3$$

Thus the GCF$(30, 18) = 2 \cdot 3 = 6$. Next, consider the variable part:

$$30x^6y \quad \text{and} \quad 18x^4y^2z$$

The variable factors in common are x^4 and y. The factor z is not in common and we have

$$\text{GCF } = 6 \cdot x^4 \cdot y$$

Answer:

$$6x^4y$$

Example 6.1.5

Determine the GCF of the following three expressions:

$$12a^5b^2(a+b)^5 \ , \ 60a^4b^3c(a+b)^3, \text{ and } 24a^2b^7c^3(a+b)^2.$$

Solution:

First, determine the GCF of the coefficients.

$$12 = 2^2 \cdot 3$$
$$60 = 2^2 \cdot 3 \cdot 5$$
$$24 = 2^3 \cdot 3$$

The GCF $(12, 60, 24) = 2^2 \cdot 3 = 12$. Next, determine the common factors of the variable part:

$$12a^5b^2(a+b)^5 \quad \text{and} \quad 60a^4b^3c(a+b)^3 \quad \text{and} \quad 24a^2b^7c^3(a+b)^2$$

The variable factors in common are a^2, b^2, and $(a+b)^2$. Therefore,

$$\text{GCF } = 12 \cdot a^2 \cdot b^2 \cdot (a+b)^2$$

Answer:

$12a^2b^2(a+b)^2$. Note that the variable c is not common to all three expressions and thus is not included in the GCF.

Exercise 6.1.1

Determine the GCF of the following:

$$60x^4y^3(x+2y)^7, 45x^2y^5(x+2y)^4, \text{ and } 30x^7y^7(x+2y)^3.$$

Answer available at the end of this section.

Factoring out the GCF

We have seen that application of the distributive property is the key to multiplying polynomials. The process of factoring a polynomial involves using the distributive property in reverse to write each polynomial as a product of polynomial factors.

$$a(b+c) = ab+ac \quad \textit{Multiplying}$$
$$ab+ac = a(b+c) \quad \textit{Factoring}$$

To demonstrate this idea, we multiply and factor side by side. Factoring utilizes the GCF of the terms.

Table 6.1.1

Multiplying	Factoring
$3(5x+1) = 3{\cdot}5x+3{\cdot}1$ $= 15x+3$	$15x+3 = 3{\cdot}5x+3{\cdot}1$ $= 3(5x+1)$
$2x^2(3x^3+4) = 2x^2{\cdot}3x^3+2x^2{\cdot}4$ $= 6x^5+8x^2$	$6x^5+8x^2 = 2x^2{\cdot}3x^3+2x^2{\cdot}4$ $= 2x^2(3x^3+4)$

In the previous example, we see that the distributive property allows us to write the polynomial $6x^5+8x^2$ as a product of the two factors $2x^2$ and $(3x^3+4)$. Note that in this case, $2x^2$ is the GCF of the terms of the polynomial:

$$\text{GCF } 6x^5, 8x^2) = 2x^2$$

Factoring out the GCF involves rewriting a polynomial as a product where a factor is the GCF of all of its terms:

$$15x+3=3(5x+1) \qquad \textit{Factoring out the GCF}$$
$$6x^5+8x^2=2x^2(3x^3+4)$$

The steps for factoring out the GCF of a polynomial are outlined in the following example.

Example 6.1.6

Factor out the GCF:

$$7x^4+21x^3-14x^2.$$

Solution:

Step 1: Identify the GCF of all the terms. In this case, the GCF$(7,21,14)=7$, and the common variable factor with the smallest exponent is x^2. The GCF of the polynomial is $7x^2$.

$$7x^4+21x^3-14x^2=7x^2(\quad ?\quad)$$

Step 2: Determine the terms of the missing factor by dividing each term of the original expression by the GCF. (**This step is usually performed mentally**.)

$$\tfrac{7x^4}{7x^2}=x^2 \qquad \tfrac{21x^3}{7x^2}=3x \qquad \tfrac{-14x^2}{7x^2}=-2$$

Step 3: Apply the distributive property (in reverse) using the terms found in the previous step.

$$7x^4+21x^3-14x^2=7x^2(x^2+3x-2)$$

Step 4: As a check, multiply using the distributive property to verify that the product equals the original expression. (**This step is optional and can be performed mentally**.)

$$\begin{aligned}7x^2(x^2+3x-2)&=7x^2\cdot x^2+7x^2\cdot 3x-7x^2\cdot 2\\&=7x^4+21x^3-14x^2 \quad \checkmark\end{aligned}$$

Answer:

$$7x^2(x^2+3x-2)$$

Example 6.1.7

Factor out the GCF:

$$48a-16b+4c$$

Solution:

There are no variable factors in common and the GCF$(48,16,4)=4$.

$$\begin{aligned}48a-16b+4c&=4(\quad ?\quad)\\&=4(12a-4b+c)\end{aligned}$$

Answer:

$$4(12a-4b+c)$$

Example 6.1.8

Factor out the GCF:

$$25x^3 + 15x^2 + 5x$$

Solution:

The GCF$(25, 15, 5) = 5$, and the common variable factor with smallest exponents is x^1. The GCF of all the terms is $5x$.

$$\begin{aligned} 25x^3 + 15x^2 + 5x &= 5x(\quad ? \quad) \\ &= 5x(5x^2 + 3x + 1) \end{aligned}$$

Answer:

$$5x(5x^2 + 3x + 1)$$

If the GCF is the same as one of the terms, then, after the GCF is factored out, a constant term 1 will remain. In the previous example, we can see that $\frac{5x}{5x} = 1$. The importance of remembering the constant term becomes clear when performing the check using the distributive property:

$$\begin{aligned} 5x(5x^2 + 3x + 1) &= 5x \cdot 5x^2 + 5x \cdot 3x + 5x \cdot 1 \\ &= 25x^3 + 15x^2 + 5x \quad \checkmark \end{aligned}$$

The constant term 1 allows us to obtain the same original expression after we distribute.

Example 6.1.9

Factor out the GCF:

$$15x^6y^4 + 10x^5y^3z^2 - 20xy^6z^3.$$

Solution:

The GCF$(10, 15, 20) = 5$, and the common variables with smallest exponent are x^1 and y^3. Therefore, the GCF of the terms is $5xy^3$. The first term does not have a variable factor of z and thus cannot be a part of the greatest common factor. If we divide each term by $5xy^3$, we obtain

$$\frac{15x^6y^4}{5xy^3} = 3x^5y \qquad \frac{10x^5y^3z^2}{5xy^3} = 2x^4z^2 \qquad \frac{-20xy^6z^3}{5xy^3} = -4y^3z^3$$

and can write

$$\begin{aligned} 15x^6y^4 + 10x^5y^3z^2 - 20xy^6z^3 &= 5xy^3(\quad ? \quad) \\ &= 5xy^3(3x^5y + 2x^4z^2 - 4y^3z^3) \end{aligned}$$

Answer:

$$5xy^3(3x^5y + 2x^4z^2 - 4y^3z^3)$$

Example 6.1.10

Factor out the GCF:

$$24a^6b^2c^5 + 8a^7b^5c$$

Solution:

The GCF$(24, 8) = 8$, and the variable factors with smallest exponents are a^6, b^2, and c. Therefore, the GCF of all the terms is $8a^6b^2c$.

$$\begin{aligned} 24a^6b^2c^5 + 8a^7b^5c &= 8a^6b^2c(\quad ?\quad) \\ &= 8a^6b^2c(3c^4 + ab^3) \end{aligned}$$

Answer:

$$8a^6b^2c(3c^4 + ab^3)$$

Of course, not all polynomials with integer coefficients can be factored as a product of polynomials with integer coefficients other than 1 and itself. If this is the case, then we say that it is a prime polynomial.

Example 6.1.11

Factor:

$$3x - 5$$

Solution:

Prime: there are no polynomial factors other than 1 and itself.

Answer:

Prime

Exercise 6.1.2

Factor out the GCF:

$$16x^4y^3 - 8x^2y^5 - 4x^2y.$$

Answer available at the end of this section.

Factor by Grouping

In this section, we outline a technique for factoring polynomials with four terms. First, review some preliminary examples where the terms have a common binomial factor.

Example 6.1.12

Factor:

$$5x(x - 3) + 2(x - 3).$$

Solution:

This expression is a binomial with terms $5x(x - 3)$ and $2(x - 3)$. In this case, $(x - 3)$ is a common factor. Begin by factoring this common factor out:

$$5x(x-3)+2(x-3) = (x-3)(\quad ?\quad)$$

To determine the terms of the remaining factor, divide each term by $(x-3)$:

$$\frac{5x(x-3)}{(x-3)} = 5x \qquad \frac{2(x-3)}{(x-3)} = 2$$

This step is typically performed mentally. We have

$$\begin{aligned} 5x(x-3)+2(x-3) &= (x-3)(\quad ?\quad) \\ &= (x-3)(5x+2) \end{aligned}$$

Answer:

$$(x-3)(5x+2)$$

Recall that 1 is always a common factor. If the GCF is the same as a term, then the factor 1 remains after we factor out that GCF.

Example 6.1.13

Factor:

$$3x(4x+1)-(4x+1).$$

Solution:

Rewrite the second term $-(4x+1)$ as $-1(4x+1)$ and then factor out the common binomial factor $(4x+1)$.

$$\begin{aligned} 3x(4x+1)-(4x+1) &= 3x(4x+1)-1(4x+1) \\ &= (4x+1)(\quad ?\quad) \\ &= (4x+1)(3x-1) \end{aligned}$$

Answer:

$$(4x+1)(3x-1)$$

Remember that the goal for this section is to develop a technique that enables us to factor polynomials with four terms into a product of binomials. The intermediate step of this process looks like the previous two examples. For example, we wish to factor

$$5x^2-15x+2x-6$$

Begin by grouping the first two terms and the last two terms. Then factor out the GCF of each grouping:

$$\underbrace{5x^2-15x}_{group} \quad + \quad \underbrace{2x-6}_{group}$$

$$= 5x(x-3)+2(x-3)$$

In this form, it is a binomial with a common binomial factor, (x–3).

$$\begin{aligned} &= (x-3)(\quad ?\quad) \\ &= (x-3)(5x+2) \end{aligned}$$

The steps that follow outline a technique for factoring four-term polynomials called factor by grouping.

Example 6.1.14

Factor:

$$2x^3 + 4x^2 + 3x + 6.$$

Solution:

Group terms in such a way as to obtain a binomial with common factors.

Step 1: Group the first two and last two terms and then factor out the GCF of each.

$$\underbrace{2x^3 + 4x^2}_{group} \quad + \quad \underbrace{3x + 6}_{group}$$

The GCF of the first two terms is $2x^2$, and the GCF of the second two terms is 3.

$$\begin{aligned} 2x^3 + 4x^2 + 3x + 6 = & \underbrace{2x^3 + 4x^2}_{group} \quad + \quad \underbrace{3x + 6}_{group} \\ & = 2x^2(\quad ?\quad) + 3(\quad ?\quad) \\ & = 2x^2(x + 2) + 3(x + 2) \end{aligned}$$

Step 2: At this point, the polynomial is a binomial. Factor out any factors common to both terms. Here $(x + 2)$ is a common factor.

$$\begin{aligned} & = 2x^2(x + 2) + 3(x + 2) \\ & = (x + 2)(\quad ?\quad) \\ & = (x + 2)(2x^2 + 3) \end{aligned}$$

Step 3: Optional check: multiply to verify that we obtain the original expression.

$$\begin{aligned} (x + 2)(2x^2 + 3) &= 2x^3 + 3x + 4x^2 + 6 \\ &= 2x^3 + 4x^2 + 3x + 6 \quad \checkmark \end{aligned}$$

Answer:

$$(x + 2)(2x^2 + 3)$$

Example 6.1.15

Factor:

$$2a^3 - 3a^2 + 2a - 3.$$

Solution:

The GCF of the first two terms is a^2 and the GCF of the second two terms is 1.

$$\begin{aligned} 2a^3 - 3a^2 + 2a - 3 = & \underbrace{2a^3 - 3a^2}_{group} \quad + \quad \underbrace{2a - 3}_{group} \\ & = a^2(\quad ?\quad) + 1(\quad ?\quad) \\ & = a^2(2a - 3) + 1(2a - 3) \\ & = (2a - 3)(\quad ?\quad) \\ & = (2a - 3)(a^2 + 1) \end{aligned}$$

Answer:

$2a - 3)(a^2 + 1)$. The check is left to the reader.

Example 6.1.16

Factor:

$$6x^4 - 24x^3 - 5x + 20.$$

Solution:

The GCF for the first group is $6x^3$. We have to choose 5 or -5 to factor out of the second group.

$$\underbrace{6x^4 - 24x^3}_{group} \quad - \quad \underbrace{5x + 20}_{group}$$

$$\begin{aligned} &= 6x^3(x-4)+5(-x+4) && x \\ &= 6x^3(x-4)-5(x-4) && \checkmark \end{aligned}$$

Factoring out a $+5$ does not result in a common binomial factor. If we choose to factor out -5, then we obtain a common binomial factor and can proceed. Note that when factoring out a negative number, we change the signs of the factored terms.

$$6x^4 - 24x^3 - 5x + 20 = \underbrace{6x^4 - 24x^3}_{group} \quad - \quad \underbrace{5x + 20}_{group}$$

$$\begin{aligned} &= 6x^3(\quad ?\quad) - 5(\quad ?\quad) \\ &= 6x^3(x-4) - 5(x-4) \\ &= (x-4)(\quad ?\quad) \\ &= (x-4)(6x^3 - 5) \end{aligned}$$

Answer:

$(x-4)(6x^3-5)$. The check is left to the reader.

Note

The sign of the leading coefficient in the second grouping usually indicates whether or not to factor out a negative factor. If that coefficient is positive, factor out a positive factor. If it is negative, factor out a negative factor.

When all the terms of a polynomial have a GCF other than 1, it is a best practice to factor that out before factoring by grouping.

Example 6.1.17

Factor:

$$3y^4 + 9y^2 - 6y^3 - 18y.$$

Solution:

Here we notice that the greatest common factor of all the terms is $3y$. Begin by factoring out the GCF and then factor the result by grouping.

$$
\begin{aligned}
&3y^4+9y^2-6y^3-18y \\
&=3y[y^3+3y-2y^2-6] \qquad \textit{Factor out the GCF.} \\
&=3y[y(y^2+3)-2(y^2+3)] \quad \textit{Factor by grouping.} \\
&=3y[(y^2+3)(y-2)] \\
&=3y(y^2+3)(y-2)
\end{aligned}
$$

Answer:

$$3y(y^2+3)(y-2)$$

Sometimes we must first rearrange the terms in order to obtain a common factor.

Example 6.1.18

Factor:

$$ab-2a^2b+a^3-2b^3.$$

Solution:

Simply factoring the GCF out of the first group and last group does not yield a common binomial factor.

$$\underbrace{ab-2a^2b}_{group} + \underbrace{a^3-2b^3}_{group}$$

$$=ab(1-2a)+1(a^3-2b^3)$$

We must rearrange the terms, searching for a grouping that produces a common factor. In this example, we have a workable grouping if we switch the terms a^3 and ab.

$$ab-2a^2b+a^3-2b^3 = \underbrace{a^3-2a^2b}_{group} + \underbrace{ab-2b^3}_{group}$$

$$
\begin{aligned}
&=a^2(a-2b)+b(a-2b) \\
&=(a-2b)(a^2+b)
\end{aligned}
$$

Answer:

$$(a-2b)(a^2+b)$$

Not all factorable four-term polynomials can be factored with this technique. For example,

$$3x^3+5x^2-x+2$$

This four-term polynomial cannot be grouped in any way as to produce a common binomial factor. Despite this, the polynomial is not prime and can be written as a product of polynomials. It can be factored as follows:

$$3x^3+5x^2-x+2=(x+2)(3x^2-x+1)$$

Factoring such polynomials is something that we will learn to do as we move further along in our study of algebra. For now, we will limit our attempt to factor four-term polynomials to using the factor by grouping technique.

Exercise 6.1.3

Factor:

$$x^3 - x^2y - xy + y^2$$

Answer available at the end of this section.

Key Takeaways

- To find the greatest common factor (GCF) of any collection of natural numbers, first find the prime factorization of each. The GCF is the product of all the common prime factors.
- The GCF of two or more monomials is the product of the GCF of the coefficients and the common variable factors with the smallest power.
- If the terms of a polynomial have a greatest common factor, then factor out that GCF using the distributive property. Divide each term of the polynomial by the GCF to determine the terms of the remaining factor.
- Some four-term polynomials can be factored by grouping the first two terms and the last two terms. Factor out the GCF of each group and then factor out the common binomial factor.
- When factoring by grouping, you sometimes have to rearrange the terms to find a common binomial factor. After factoring out the GCF, the remaining binomial factors must be the same for the technique to work.
- Not all polynomials can be factored as the product of polynomials with integer coefficients. In this case, we call it a prime polynomial.

Exercise 6.1.4 GCF of Natural Numbers

Give the prime factorization of each number and determine the GCF.

1. $18, 24$
2. $45, 75$
3. $72, 60$
4. $168, 175$
5. $144, 245$
6. $15, 50, 60$
7. $14, 63, 70$
8. $12, 48, 125$
9. $60, 72, 900$
10. $252, 336, 360$

Answer available at the end of this section.

Exercise 6.1.5 GCF of Variable Expressions

Determine the GCF of all the terms.

1. $15x, 30$
2. $14x, 21$
3. $45x^4, 8x^3$
4. $36x^5, 35y^2$
5. $6x, 27x, 36x$
6. $12x^3, 4x^2, 6x$
7. $12x^2y, 60xy^3$
8. $7ab^2, 2a^2b, 3a^3b^3$
9. $6a^2b^2, 18a^3b^2, 9ab^2$

10. $15x(x+2), 9(x+2)$
11. $20x(2x-1), 16(2x-1)$
12. $20x^3(x+y)^5, 10x^5(x+y)^2$

Answer available at the end of this section.

Exercise 6.1.6 Factoring out the GCF

Given the GCF, determine the missing factor.

1. $25x^2 + 10x = 5x(\quad ? \quad)$
2. $12y^5 + 7y^2 = y^2(\quad ? \quad)$
3. $22x^4 - 121x^2 + 11x = 11x(\quad ? \quad)$
4. $30y^3 - 45y^2 - 3y = 3y(\quad ? \quad)$
5. $36a^5b^7 - 60a^6b^5 = 12a^5b^5(\quad ? \quad)$
6. $24x^2y + 48xy^2 - 12xy = 12xy(\quad ? \quad)$

Answer available at the end of this section.

Exercise 6.1.7 Factoring out the GCF

Factor out the GCF.

1. $4x - 8$
2. $27x - 9$
3. $3x - 18$
4. $5x - 10$
5. $25x - 16$
6. $72x - 35$
7. $15x^2 + 30x$
8. $14a^2 - 7a$
9. $30a^5 - 10a^2$
10. $8x^4 - 16x^2$
11. $5x^6 + x^3$
12. $3x^7 - 9x^5$
13. $18a^2 + 30a - 6$
14. $24a^2 - 36a - 12$
15. $27x^3 - 6x^2 + 3x$
16. $8x^3 - 12x^2 + 2x$
17. $9x^4 + 18x^3 - 3x^2$
18. $12y^4 - 16y^3 + 20y^2$
19. $7x^5 - 21x^3 - 14x^2 + 28x$
20. $36y^{10} + 12y^8 - 18y^4 - 6y^3$
21. $12x^5y^2 - 8x^3y$
22. $125a^8b^4c^3 - 25a^2b^3c^3$
23. $6x^4y^3 - 4x^3y^2 + 8x^2y$
24. $15x^4y^2 - 30x^3y^3 + 15x^2y^4$
25. $81x^7y^6z^2 - 18x^2y^8z^4 + 9x^2y^5z^2$
26. $4x^5y^4z^9 + 26x^5y^3z^4 - 14x^6y^8z^5$
27. $2x(x-3) + 5(x-3)$
28. $3x(2x+1) - 4(2x+1)$
29. $5x(5x+2) - (5x+2)$
30. $2x(3x+4) + (3x+4)$
31. $x^2(4x-7) - 5(4x-7)$
32. $(x+6) - 3x^2(x+6)$

33. $(a+b)^2-3a(a+b)^2$
34. $(ab+2)^3+3ab(ab+2)^3$
35. $7x(x+7)^5+14x^2(x+7)^5$
36. $36x^5(3x+2)^4-12x^3(3x+2)^4$

Answer available at the end of this section.

Exercise 6.1.8 Factoring out the GCF

Are the following factored correctly? Check by multiplying.

1. $4x^2-16x=4x(x-4)$
2. $3a^3-3a=3a(a^2)$
3. $3x^3-5x^6=x^3(3-x^2)$
4. $5x^3-10x^4+15x^5=5x^3(1-2x+3x^2)$
5. $x^3-x^2+x=x(x^2-x)$
6. $12x^4y^3-16x^5y^2+8x^6y^7=4x^4y^2(3y-4x+2x^2y^5)$

Answer available at the end of this section.

Exercise 6.1.9 Factoring out the GCF

Use polynomial long division to show that the given factor divides the polynomial evenly.

1. Show that $(x-1)$ is a factor of $(2x^3-5x^2+4x-1)$.
2. Show that $(x+3)$ is a factor of $(3x^3+7x^2-4x+6)$.
3. Show that $(3x-2)$ is a factor of $(3x^3+4x^2-7x+2)$.
4. Show that $(2x+1)$ is a factor of $(2x^3-5x^2+x+2)$.
5. The height in feet of an object tossed into the air is given by the function $h(t)=-16t^2+32t$, where t is the time in seconds after it is tossed. Write the function in factored form.
6. The height in feet of an object dropped from a 16-foot ladder is given by the function $h(t)=-16t^2+16$, where t is the time in seconds after it is tossed. Write the function in factored form.
7. The surface area of a cylinder is given by the formula $SA=2\pi r^2+2\pi rh$, where r represents the radius of the base and h is the height of the cylinder. Express this formula in factored form.

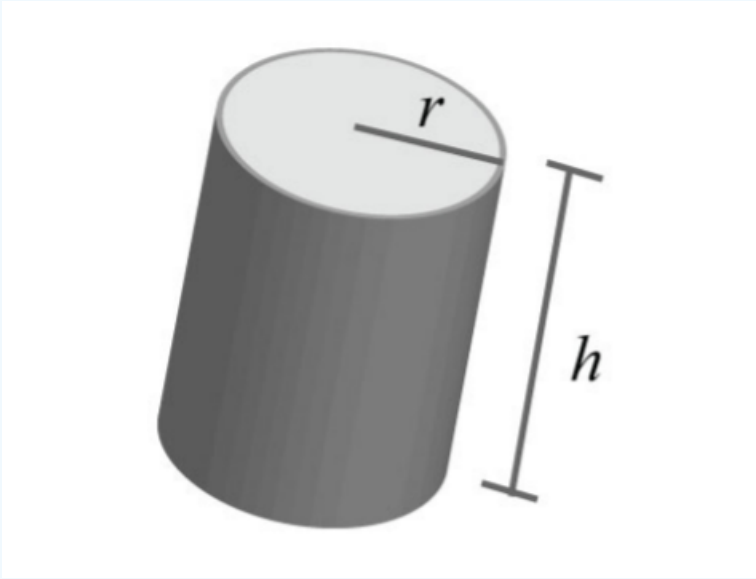


Figure 6.1.1

8. The surface area of a cone is given by the formula $SA = \pi r^2 + \pi rs$, where r represents the radius of the base and s represents the slant height. Express this formula in factored form.

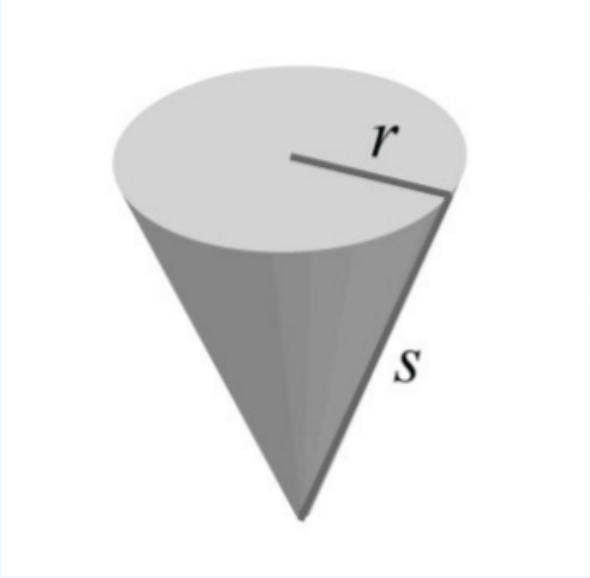


Figure 6.1.2

Answer available at the end of this section.

Exercise 6.1.10 Factor by Grouping

Factor by grouping.

1. $x^2 - 10x + 2x - 20$
2. $x^2 - 6x - 3x + 18$
3. $x^3 + 2x^2 + 2x + 4$
4. $x^3 - 3x^2 + 5x - 15$
5. $x^3 + 7x^2 - 2x - 14$
6. $2x^3 + 2x^2 - x - 1$
7. $x^3 - 5x^2 + 4x - 20$
8. $6x^3 - 3x^2 + 2x - 1$
9. $9x^3 - 6x^2 - 3x + 2$
10. $2x^4 - x^3 - 6x + 3$
11. $x^5 + x^3 + 2x^2 + 2$
12. $6x^5 - 4x^3 - 9x^2 + 6$
13. $3a^3b + 3ab^2 + 2a^2 + 2b$
14. $2a^3 + 2ab^3 - 3a^2b - 3b^4$
15. $2a^3 - a^2b^2 - 2ab + b^3$
16. $a^4 - 3a^3b^2 + ab^2 - 3b^4$
17. $3a^2b^4 - 6b^3 - a^2b + 2$
18. $3x^3 + 2y^3 + x^3y^3 + 6$
19. $-3x^3 - 5y^3 + x^3y^3 + 15$
20. $2x^3y^3 + 2 - y^3 - 4x^3$
21. $3x^2 - y^3 + xy^2 - 3xy$
22. $2x^2 + y^3 - 2xy - xy^2$

Answer available at the end of this section.

Exercise 6.1.11 Factor by Grouping

Factor out the GCF first and then factor by grouping.

1. $5x^2 - 35x - 15x + 105$
2. $12x^2 - 30x - 12x + 30$
3. $2x^3 + 6x^2 - 10x - 30$
4. $6x^3 - 3x^2 - 42x + 21$
5. $4x^4 + 4x^3 - 12x^2 - 12x$
6. $-9x^4 + 6x^3 - 45x^2 + 30x$
7. $-12x^5 + 4x^4 + 6x^3 - 2x^2$
8. $24x^5 - 36x^4 + 8x^3 - 12x^2$
9. $24a^3b^2 - 60a^3b + 40ab^2 - 100ab$
10. $a^4b^2 - 2a^3b^3 + a^2b^3 - 2ab^4$

Answer available at the end of this section.

Exercise 6.1.12 Discussion Board Topics

1. Research the Euclidean algorithm for finding the GCF of two natural numbers. Give an example that illustrates the steps.
2. Research and discuss the contributions of Euclid of Alexandria.
3. Explain what factoring is and give an example.
4. Is $5x(x + 2) - 3(x + 2)$ fully factored? Explain.
5. Make up a factoring problem of your own and provide the answer. Post the problem and the solution on the discussion board.

Answer available at the end of this section.

Answers

Exercise 6.1.1

$15x^2y^3(x + 2y)^3$

Exercise 6.1.2

$4x^2y(4x^2y^2 - 2y^4 - 1)$

Exercise 6.1.3

$(x - y)(x^2 - y)$

Exercise 6.1.4 **GCF of Natural Numbers**

1. $18 = 2 \cdot 32, 24 = 23 \cdot 3$, GCF $= 6$
3. $72 = 23 \cdot 32, 60 = 22 \cdot 3 \cdot 5$, GCF $= 12$
5. $144 = 24 \cdot 32, 245 = 5 \cdot 72$, GCF $= 1$
7. $14 = 2 \cdot 7, 63 = 32 \cdot 7, 70 = 2 \cdot 5 \cdot 7$, GCF $= 7$
9. $60 = 22 \cdot 3 \cdot 5, 72 = 23 \cdot 32, 900 = 22 \cdot 32 \cdot 52$, GCF $= 12$

Exercise 6.1.5 **GCF of Variable Expressions**

1. 15
3. x^3
5. $3x$
7. $12xy$
9. $3ab^2$
11. $4(2x - 1)$

Exercise 6.1.6 **Factoring out the GCF**

1. $(5x + 2)$
3. $(2x^3 - 11x + 1)$
5. $(3b^2 - 5a)$

Exercise 6.1.7 **Factoring out the GCF**

1. $4(x - 2)$
3. $3(x - 6)$

5. Prime
7. $15x(x+2)$
9. $10a^2(3a^3-1)$
11. $x^3(5x^3+1)$
13. $6(3a^2+5a-1)$
15. $3x(9x^2-2x+1)$
17. $3x^2(3x^2+6x-1)$
19. $7x(x^4-3x^2-2x+4)$
21. $4x^3y(3x^2y-2)$
23. $2x^2y(3x^2y^2-2xy+4)$
25. $9x^2y^5z^2(9x^5y-2y^3z^2+1)$
27. $(x-3)(2x+5)$
29. $(5x+2)(5x-1)$
31. $(4x-7)(x^2-5)$
33. $(a+b)^2(1-3a)$
35. $7x(x+7)^5(1+2x)$

Exercise 6.1.8 **Factoring out the GCF**

1. Yes
3. No
5. No

Exercise 6.1.9 **Factoring out the GCF**

5. $h(t)=-16t(t-2)$
7. $SA=2\pi r(r+h)$

Exercise 6.1.10 **Factor by Grouping**

1. $(x-10)(x+2)$
3. $(x+2)(x^2+2)$
5. $(x+7)(x^2-2)$
7. $(x-5)(x^2+4)$
9. $(3x-2)(3x^2-1)$
11. $(x^2+1)(x^3+2)$
13. $(a^2+b)(3ab+2)$
15. $(a^2-b)(2a-b^2)$
17. $(3b^3-1)(a^2b-2)$
19. $(x^3-5)(y^3-3)$
21. $(x-y)(3x+y^2)$

Exercise 6.1.11 **Factor by Grouping**

1. $5(x-7)(x-3)$
3. $2(x+3)(x^2-5)$
5. $4x(x+1)(x^2-3)$
7. $-2x^2(3x-1)(2x^2-1)$
9. $4ab(3a^2+5)(2b-5)$

Exercise 6.1.12 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary
5. Answers may vary

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6.2: Factoring Trinomials of the Form x²+bx+c

Learning Objectives

- Factor trinomials of the form $x^2 + bx + c$.
- Factor trinomials using the AC method.

Factoring Trinomials of the Form $x^2 + bx + c$

Some trinomials of the form $x^2 + bx + c$ can be factored as a product of binomials. For example,

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

We can verify this factorization by multiplying:

$$\begin{aligned}(x + 2)(x + 5) &= x^2 + 5x + 2x + 10\\ &= x^2 + 7x + 10 \quad \checkmark\end{aligned}$$

Factoring trinomials requires that we work the distributive process in reverse. Notice that the product of the first terms of each binomial is equal to the first term of the trinomial.

$$x \cdot x = x^2$$

The middle term of the trinomial, $7x$, is the sum of the products of the outer and inner terms of the binomials:

$$5x + 2x = 7x$$

And the product of the last terms of each binomial is equal to the last term of the trinomial.

$$2 \cdot 5 = 10$$

This can be visually interpreted as follows:

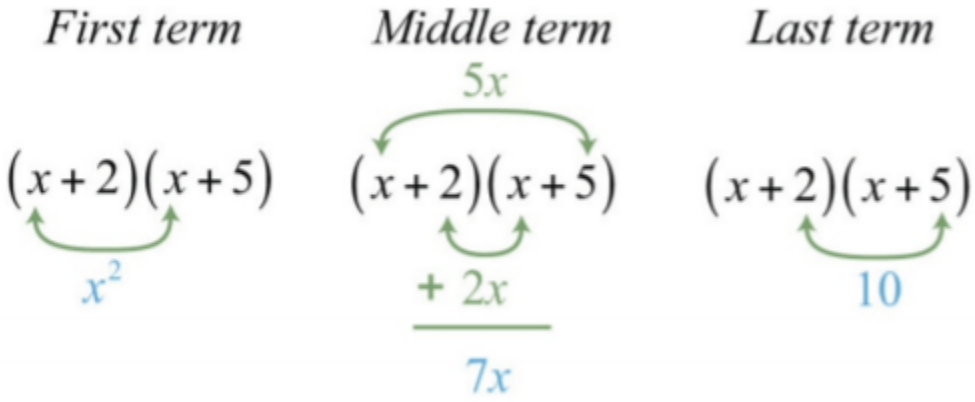


Figure 6.2.1

If a trinomial of this type factors, then these relationships will be true:

$$\begin{aligned} x^2 + bx + c &= (x + m)(x + n) \\ &= x^2 + nx + mx + mn \\ &= x^2 + (n + m)x + mn \end{aligned}$$

This gives us

$$b = n + m \quad \text{and} \quad c = mn \tag{6.2.1}$$

In short, if the leading coefficient of a factorable trinomial is one, then the factors of the last term must add up to the coefficient of the middle term. This observation is the key to factoring trinomials using the technique known astrial and error (or guess and check). The steps are outlined in the following example.

Example 6.2.1

Factor:

$$x^2 + 7x + 12$$

Solution:

Note that the polynomial to be factored has three terms; it is a trinomial with a leading coefficient of 1. Use trial and error to factor as follows:

Step 1: Write two sets of blank parentheses. If a trinomial of this form factors, then it will factor into two linear binomial factors.

$$x^2 + 7x + 12 = (\quad)(\quad)$$

Step 2: Write the factors of the first term in the first space of each set of parentheses. In this case, factor $x^2 = x \cdot x$.

$$x^2 + 7x + 12 = (x \quad)(x \quad)$$

Step 3: Determine the factors of the last term whose sum equals the coefficient of the middle term. To do this, list all of the factorizations of 12 and search for factors whose sum equals the coefficient of the middle term, 7.

$$\begin{aligned} 12 &= 1 \cdot 12 & &\rightarrow 1 + 12 = 13 \\ &= 2 \cdot 6 & &\rightarrow 2 + 6 = 8 \\ &= 3 \cdot 4 & &\rightarrow 3 + 4 = 7 \end{aligned}$$

Choose $12 = 3 \cdot 4$ because $3 + 4 = 7$.

Step 4: Write in the last term of each binomial using the factors determined in the previous step.

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

Step 5: Check by multiplying the two binomials.

$$\begin{aligned} (x + 3)(x + 4) &= x^2 + 4x + 3x + 12 \\ &= x^2 + 7x + 12 \quad \checkmark \end{aligned}$$

Answer:

$$(x + 3)(x + 4)$$

Since multiplication is commutative, the order of the factors does not matter.

$$\begin{aligned} x^2 + 7x + 12 &= (x + 3)(x + 4) \\ &= (x + 4)(x + 3) \end{aligned}$$

If the last term of the trinomial is positive, then either both of the constant factors must be negative or both must be positive. Therefore, when looking at the list of factorizations of the last term, we are searching for sums that are equal to the coefficient of the middle term.

Example 6.2.2

Factor:

$$x^2 - 9x + 20.$$

Solution:

First, factor $x^2 = x \cdot x$.

$$x^2 - 9x + 20 = (x \quad ?)(x \quad ?)$$

Next, determine which factors of 20 add up to -9:

$$\begin{aligned} 20 &= 1 \cdot 20 \\ &\quad - 2 \cdot 10 \\ &= 4 \cdot 5 \end{aligned}$$

In this case, choose -4 and -5 because $(-4)(-5) = +20$ and $-4 + (-5) = -9$.

$$x^2 - 9x + 20 = (x - 4)(x - 5)$$

Check.

$$\begin{aligned} (x-4)(x-5) &= x^2 - 5x - 4x + 20 \\ &= x^2 - 9x + 20 \quad \checkmark \end{aligned}$$

Answer:

$$(x - 4)(x - 5)$$

If the last term of the trinomial is negative, then one of its factors must be negative. In this case, search the list of factorizations of the last term for differences that equal the coefficient of the middle term.

Example 6.2.3

Factor:

$$x^2 - 4x - 12.$$

Solution:

Begin by factoring the first term $x^2 = x \cdot x$.

$$x^2 - 4x - 12 = (x \quad ?)(x \quad ?)$$

The factors of 12 are listed below. In this example, we are looking for factors whose difference is -4.

Here choose the factors 2 and -6 because the coefficient of the middle term, -4, is obtained if we add $2 + (-6)$.

$x^2 - 4x - 12 = (x + 2)(x - 6)$

Multiply to check.

Answer:

$(x+2)(x-6)$

Often our first guess will not produce a correct factorization. This process may require repeated trials. For this reason, the check is very important and is not optional.

Example 6.2.4

Factor:

$$x^2+5x-6.$$

Solution:

The first term of this trinomial x^2 factors as $x \cdot x$.

\(x^{2}+5x-6=(x\quad ?)(x\quad ?)

Consider the factors of 6:

$$\begin{aligned} 6 &= 1 \cdot 6 \\ &= 2 \cdot 3 \end{aligned}$$

Suppose we choose the factors 2 and 3 because $2+3=5$, the coefficient of the middle term. Then we have the following incorrect factorization:

$$x^2+5x-6 \stackrel{?}{=} (x+2)(x+3)$$

When we multiply to check, we find the error.

$$\begin{aligned} (x+2)(x+3) &= x^2+3x+2x+6 \\ &= x^2+5x+6 \qquad x \end{aligned}$$

In this case, the middle term is correct but the last term is not. Since the last term in the original expression is negative, we need to choose factors that are opposite in sign. Therefore, we must try again. This time we choose the factors -1 and 6 because $-1+6=5$.

$$x^2+5x-6=(x-1)(x+6)$$

Now the check shows that this factorization is correct.

Answer:

$(x-1)(x+6)$

If we choose the factors wisely, then we can reduce much of the guesswork in this process. However, if a guess is not correct, do not get discouraged; just try a different set of factors.

Example 6.2.5

Factor:

$$x^2+3x+20.$$

Solution:

$$x^2+3x+20=(x\quad ?)(x\quad ?)$$
$$\begin{aligned}20&=1\cdot 20\\&=2\cdot 10\\&=4\cdot 5\end{aligned}$$

Here there are no factors of 20 whose sum is 3. Therefore, the original trinomial cannot be factored as a product of two binomials. This trinomial is prime.

Answer:

Prime

Exercise 6.2.1

Factor:

$$x^2-13x-30.$$

Answer available at the end of this section.

The techniques described can also be used to factor trinomials with more than one variable.

Example 6.2.6

Factor:

$$x^2-14xy-72y^2$$

Solution:

The first term x^2 factors as $x\cdot x$.

$$x^2+xy-72y^2=(x\quad ?)(x\quad ?)$$

Next, look for factors of the coefficient of the last term, 72, whose sum is -14.

$$\begin{aligned}72&=1\cdot 72\\&=2\cdot 36\\&=3\cdot 24\\&=4\cdot 18 \qquad \rightarrow 4+(-18)=-14\\&=6\cdot 12\\&=8\cdot 9\end{aligned}$$

Therefore, the coefficient of the last term can be factored $-72=4(-18)$, where $4+(-18)=-14$. Because the last term has a variable factor of y^2, factor $72y^2$ as $4y(-18y)$ and try the following factorization:

$$x^2-14xy-72y^2=(x+4y)(x-18y)$$

Multiply to check.

$$\begin{aligned}(x+4y)(x-18y)&=x^2-18xy+4xy-72y^2\\&=x^2-14xy-72y^2\quad\checkmark\end{aligned}$$

Visually, we have the following:

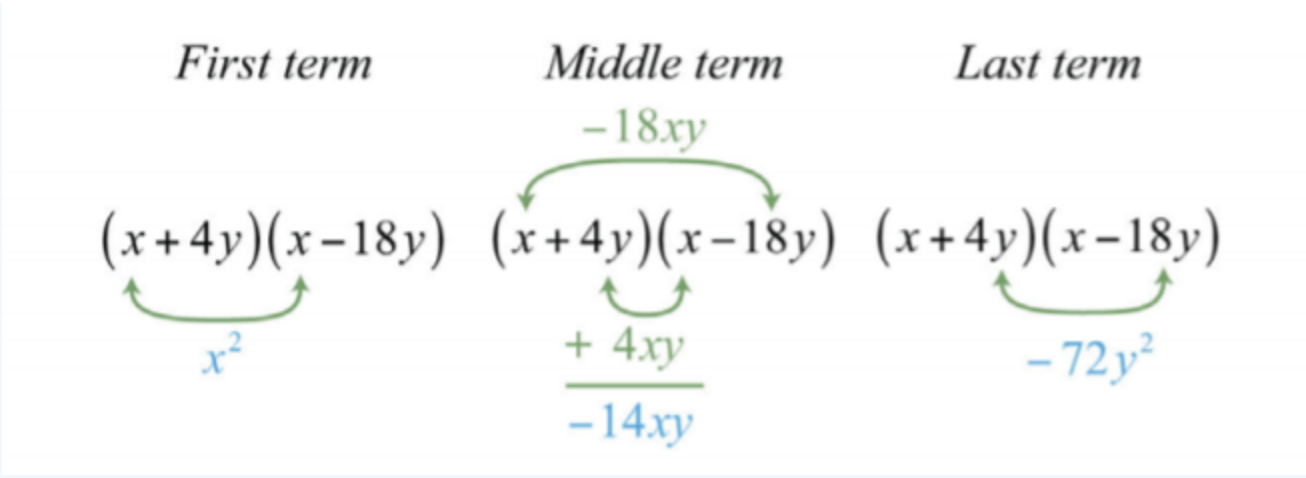


Figure 6.2.2

Answer:

$$(x + 4y)(x - 18y)$$

Exercise 6.2.2

Factor:

$$x^2y^2 + 9xy - 10.$$

Answer available at the end of this section.

Factoring Using the AC Method

An alternate technique for factoring trinomials, called the AC method, makes use of the grouping method for factoring four-term polynomials. If a trinomial in the form $ax^2 + bx + c$ can be factored, then the middle term, bx, can be replaced with two terms with coefficients whose sum is b and product ac. This substitution results in an equivalent expression with four terms that can be factored by grouping. The steps are outlined in the following example.

Example 6.2.7

Factor using the AC method:

$$x^2 - x - 30.$$

Solution:

In this example $a = 1, b = -1$, and $c = -30$.

Step 1: Determine the product ac.

Step 2: Find factors of ac whose sum equals the coefficient of the middle term, b.

$$\begin{aligned} -30 &= 1(-30) \\ &= 2(-15) \\ &= 3(-10) \\ &= 5(-5) \qquad \rightarrow 5 + (-6) = -1 \end{aligned}$$

We can see that the sum of the factors 5 and -6 is equal to the coefficient of the middle term, -1.

Step 3: Use the factors as coefficients for the terms that replace the middle term. Here $-x = -6x + 5x$. Write

$$x^2 - x - 30 = x^2 - 6x + 5x - 30$$

Step 4: Factor the equivalent expression by grouping.

$$x^2 - x - 30 = \underbrace{x^2 - 6x}_{group} + \underbrace{5x - 20}_{group}$$

$$= x(x-6) + 5(x-6)$$
$$= (x-6)(x+5)$$

Answer:

$$(x-6)(x+5)$$

Notice that the AC method is consistent with the trial and error method. Both methods require that $b = m + n$, where $c = mn$. In the example above, $-30 = (-6)(5)$ and $-1 = (-6) + 5$. The only difference between the methods, when the leading coefficient is 1 , is in the process used to obtain the final factorization.

Example 6.2.8

Factor:

$$y^2 - 14x + 48.$$

Solution:

Here $ac = 48$ and we search for factors whose sum is -14.

$$\begin{aligned} 48 &= 1(48) \\ &= 2(24) \\ &= 3(16) \\ &= 4(12) \\ &= 6(8) \qquad \rightarrow -6 + (-8) = -14 \end{aligned}$$

Therefore, $-14x = -6x - 8x$. Substitute the new terms and factor by grouping.

$$\begin{aligned} y^2 - 14x + 48 &= x^2 - 6x - 8x + 48 \\ &= x(x-6) - 8(x-6) \\ &= (x-6)(x-8) \end{aligned}$$

Answer:

$(x-6)(x-8)$. The check is left to the reader.

At this point, it is recommended that the reader stop and factor as many trinomials of the form $x^2 + bx + c$ as time allows before moving on to the next section. Factoring trinomials is one of the more important skills that we learn in this course and should be mastered.

Key Takeaways

- Factor a trinomial by systematically guessing what factors give two binomials whose product is the original trinomial.
- If a trinomial of the form $x^2 + bx + c$ factors into the product of two binomials, then the coefficient of the middle term is the sum of factors of the last term.

- Not all trinomials can be factored as the product of binomials with integer coefficients. In this case, we call it a prime trinomial.
- Factoring is one of the more important skills required in algebra. For this reason, you should practice working as many problems as it takes to become proficient.

Exercise 6.2.3 Factoring Trinomials with Leading Coefficient 1

Are the following factored correctly? Check by multiplying.

1. $x^2+5x-6=(x+2)(x+3)$
2. $x^2+6x+16=(x+8)(x-2)$
3. $y^2+2y-8=(y+4)(y-2)$
4. $y^2-10y+21=(y-3)(y-7)$
5. $a^2-10a+25=(a-5)^2$
6. $a^2+6a+9=(a-3)^2$
7. $x^2+10x-25=(x+5)(x-5)$
8. $x^2+5x+14=(x-2)(x+7)$
9. $y^2+50y-600=(y+60)(y-10)$
10. $y^2-3y+2=(y-2)(y-1)$

Answer available at the end of this section.

Exercise 6.2.4 Factoring Trinomials with Leading Coefficient 1

Factor.

1. x^2+6x+8
2. x^2+4x+3
3. x^2+3x+2
4. x^2+x-2
5. $x^2+3x-10$
6. $x^2-2x-35$
7. $x^2-13x+12$
8. $x^2-15x+36$
9. $x^2-12x+36$
10. $x^2+18x+81$
11. x^2-2x+1
12. $x^2-18x+81$
13. x^2+5x+5
14. x^2-4x+6
15. $x^2-20x+91$
16. $x^2+20x+91$
17. $x^2-2x-48$
18. $x^2+16x+48$
19. $x^2+22x+48$
20. $x^2+22x-48$
21. $y^2+7y+12$
22. $y^2+8y-20$
23. $y^2-16y+60$
24. $y^2-31y-32$
25. $a^2-11a-42$
26. $a^2-14a-51$
27. $a^2+26a+25$
28. $a^2-22a+120$
29. a^2+4a-1
30. a^2-6a+2

31. $y^2 - 14x + 40$
32. $y^2 - 4y - 96$
33. $x^2 - 2xy + y^2$
34. $x^2 + 2xy + y^2$
35. $x^2 - 16xy + 15y^2$
36. $x^2 - 4xy - 32y^2$
37. $x^2 + 2xy - 15y^2$
38. $x^2 - 12xy + 32y^2$
39. $x^2y^2 - 6xy + 9$
40. $x^2y^2 + 25xy - 26$
41. $a^2 + 4ab + 4b^2$
42. $a^2 - 19ab - 20b^2$
43. $a^2 - ab - 12b^2$
44. $a^2 - 10ab - 56b^2$
45. $a^2b^2 - 2ab - 15$
46. $a^2b^2 - 10ab + 24$
47. The area of a square is given by the function $A(x) = x^2 - 14x + 49$, where x is measured in meters. Rewrite this function in factored form.
48. The area of a square is given by the function $A(x) = x^2 + 16x + 64$, where x is measured in meters. Rewrite this function in factored form.

Answer available at the end of this section.

Exercise 6.2.5 Factor Using the AC Method

Factor using the AC method.

1. $x^2 + 5x - 14$
2. $x^2 + 2x - 48$
3. $x^2 - 9x + 8$
4. $x^2 - 14x + 24$
5. $x^2 - x - 72$
6. $x^2 - x - 90$
7. $y^2 - 8y + 16$
8. $y^2 + 16y + 64$
9. $x^2 + 4x + 12$
10. $x^2 + 5x - 8$
11. $x^2 + 3xy - 18y2$
12. $x^2 - 15xy + 50y^2$

Answer available at the end of this section.

Exercise 6.2.6 Discussion Board Topics

1. Create your own trinomial of the form $x^2 + bx + c$ that factors. Share it along with the solution on the discussion board.
2. Write out your own list of steps for factoring a trinomial of the form $x^2 + bx + c$ and share your steps on the discussion board.
3. Create a trinomial that does not factor and share it along with an explanation of why it does not factor.

Answer available at the end of this section.

Answers

Exercise 6.2.1

$(x + 2)(x - 15)$

Exercise 6.2.2

$(xy-1)(xy+10)$

Exercise 6.2.3 **Factoring Trinomials with Leading Coefficient 1**

1. No
3. Yes
5. Yes
7. No
9. Yes

Exercise 6.2.4 **Factoring Trinomials with Leading Coefficient 1**

1. $(x+2)(x+4)$
3. $(x+1)(x+2)$
5. $(x-2)(x+5)$
7. $(x-1)(x-12)$
9. $(x-6)^2$
11. $(x-1)^2$
13. Prime
15. $(x-7)(x-13)$
17. $(x+6)(x-8)$
19. Prime
21. $(y+3)(y+4)$
23. $(y-6)(y-10)$
25. $(a+3)(a-14)$
27. $(a+1)(a+25)$
29. Prime
31. $(y-10)(y-4)$
33. $(x-y)^2$
35. $(x-15y)(x-y)$
37. $(x+5y)(x-3y)$
39. $(xy-3)^2$
41. $(a+2b)^2$
43. $(a+3b)(a-4b)$
45. $(ab+3)(ab-5)$
45. $A(x)=(x-7)^2$

Exercise 6.2.5 **Factor Using the AC Method**

1. $(x+7)(x-2)$
3. $(x-8)(x-1)$
5. $(x-9)(x+8)$
7. $(y-4)^2$
9. Prime
11. $(x+6y)(x-3y)$

Exercise 6.2.6 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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6.3: Factoring Trinomials of the Form ax²+bx+c

Learning Objectives

- Factor trinomials of the form $ax^2 + bx + c$.
- Factor trinomials with a common factor.

Factoring Trinomials of the Form $ax^2 + bx + c$

Factoring trinomials of the form $ax^2 + bx + c$ can be challenging because the middle term is affected by the factors of both a and c. To illustrate this, consider the following factored trinomial:

$$10x^2 + 17x + 3 = (2x + 3)(5x + 1)$$

We can multiply to verify that this is the correct factorization.

$$\begin{aligned}(2x + 3)(5x + 1) &= 10x^2 + 2x + 15x + 3\\ &= 10x^2 + 17x + 3 \quad \checkmark\end{aligned}$$

As we have seen before, the product of the first terms of each binomial is equal to the first term of the trinomial. The middle term of the trinomial is the sum of the products of the outer and inner terms of the binomials. The product of the last terms of each binomial is equal to the last term of the trinomial. Visually, we have the following:

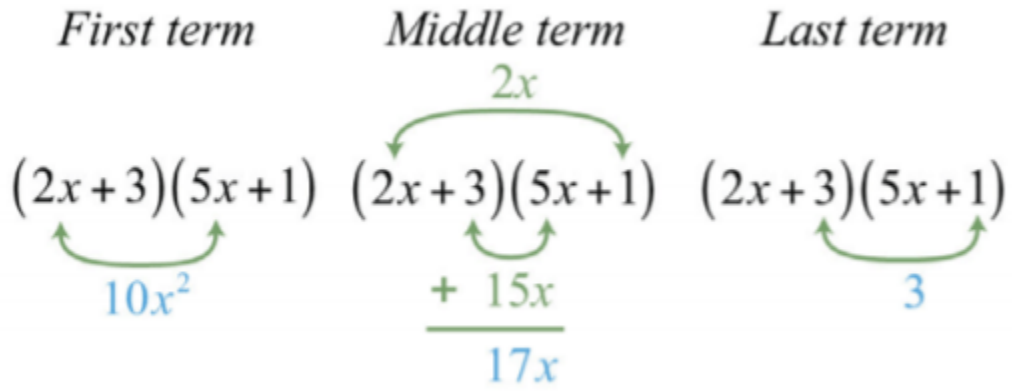


Figure 6.3.1

In general,

$$\begin{aligned}ax^2+bx+c &= (px + m)(qx + n)\\ &= pqx^2 + pnx + qmx + mn\\ &= pqx^2+(pn + qm)x+mn\end{aligned}$$

This gives us,

$$a = pq \quad \text{and} \quad b = pn + qm, \quad \text{where} \quad c = mn \tag{6.3.1}$$

In short, when the leading coefficient of a trinomial is something other than 1, there will be more to consider when determining the factors using the trial and error method. The key lies in the understanding of how the middle term is obtained. Multiply $(2x + 5)(3x + 7)$ and carefully follow the formation of the middle term.

$$(2x+5)(3x+7) = 2x\cdot 3x \quad \underbrace{+2x\cdot 7+5\cdot 3x}_{middle\ term} \quad +5\cdot 7$$

$$= 6x^2 + 14x + 15x + 35$$
$$= 6x^2 + 29x + 35$$

If we think of the FOIL method for multiplying binomials, then the middle term results from the sum of the inner product and the outer product. In this case, $14x + 15x = 29x$, as illustrated below:

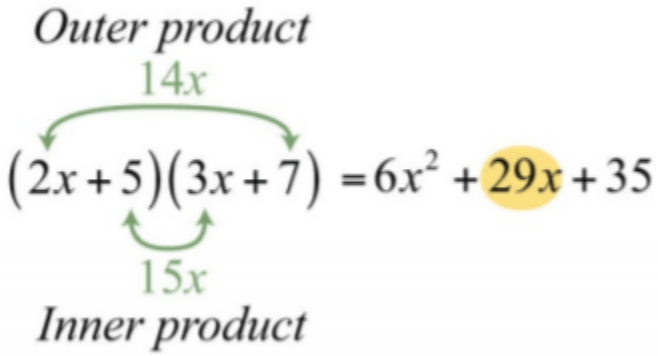


Figure 6.3.2

For this reason, we need to look for products of the factors of the first and last terms whose sum is equal to the coefficient of the middle term. For example, to factor $6x^2 + 29x + 35$, look at the factors of 6 and 35.

$$\begin{aligned} 6 &= 1\cdot 6 & 35 &= 1\cdot 35 \\ &= 2\cdot 3 & &= 5\cdot 7 \end{aligned}$$

The combination that produces the coefficient of the middle term is $2\cdot 7 + 3\cdot 5 = 14 + 15 = 29$. Make sure that the outer terms have coefficients 2 and 7, and that the inner terms have coefficients 5 and 3. Use this information to factor the trinomial:

$$\begin{aligned} 6x^2 + 29x + 35 &= (2x \quad ?)(3x \quad ?) \\ &= (2x+5)(3x+7) \end{aligned}$$

Example 6.3.1

Factor:

$$3x^2 + 7x + 2.$$

Solution:

Since the leading coefficient and the last term are both prime, there is only one way to factor each.

$$3 = 1\cdot 3 \quad \text{and} \quad 2 = 1\cdot 2$$

Begin by writing the factors of the first term, $3x^2$, as follows:

$$3x^2 + 7x + 2 = (x \quad ?)(3x \quad ?)$$

The middle and last term are both positive; therefore, the factors of 2 are chosen as positive numbers. In this case, the only choice is in which grouping to place these factors.

$$(x+1)(3x+2) \quad \text{or} \quad (x+2)(3x+1)$$

Determine which grouping is correct by multiplying each expression.

$$\begin{aligned}(x+1)(3x+2) &= 3x^2+2x+3x+2\\ &= 3x^2+5x+2 \quad x\\ (x+2)(3x+1) &= 3x^2+x+6x+2\\ &= 3x^2+7x+2 \quad \checkmark\end{aligned}$$

Notice that these products differ only in their middle terms. Also, notice that the middle term is the sum of the inner and outer product, as illustrated below:

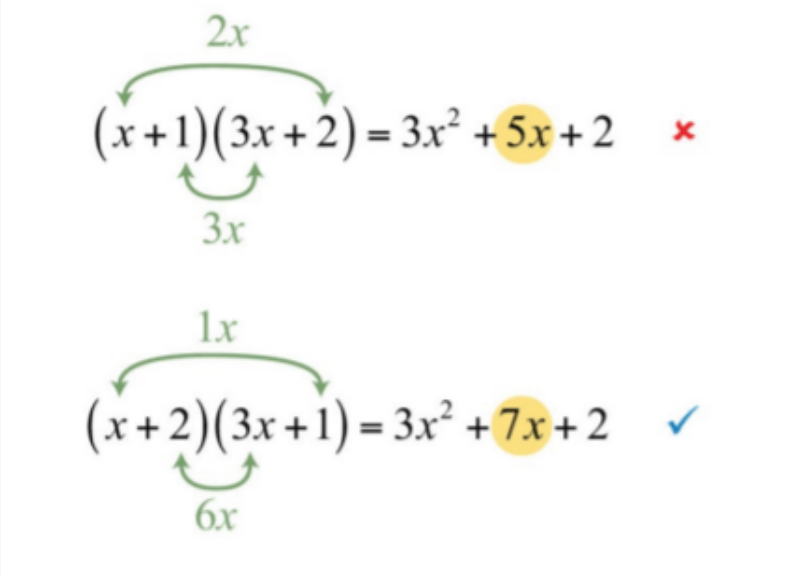


Figure 6.3.3

Answer:

$$(x+2)(3x+1)$$

Example 6.3.2

Factor:

$$12x^2+38x+20.$$

Solution:

First, consider the factors of the first and last terms.

$$\begin{aligned}12 &= 1\cdot 12 & 20 &= 1\cdot 20\\ &= 2\cdot 6 & &= 2\cdot 10\\ &= 3\cdot 4 & &= 4\cdot 5\end{aligned}$$

We search for products of factors whose sum equals the coefficient of the middle term, 38. For brevity, the thought process is illustrated starting with the factors 2 and 6. Factoring begins at this point with the first term.

$$12x^2+38x+20 = (2x \quad ?)(6x \quad ?)$$

We search for factors of 20 that along with the factors of 12 produce a middle term of 38x

Factors of 20	*Possible*	*factorization*
$1 \cdot 20$	$(2x+1)(6x+20)$	*middle term* $\Rightarrow 46x$
$1 \cdot 20$	$(2x+20)(6x+1)$	*middle term* $\Rightarrow 122x$
$2 \cdot 10$	$(2x+2)(6x+10)$	*middle term* $\Rightarrow 32x$
$2 \cdot 10$	$(2x+10)(6x+2)$	*middle term* $\Rightarrow 64x$
$4 \cdot 5$	$(2x+4)(6x+5)$	*middle term* $\Rightarrow 34x$
$4 \cdot 5$	$(2x+5)(6x+4)$	*middle term* $\Rightarrow 38x$ ✓

Here the last combination produces a middle term of $38x$.

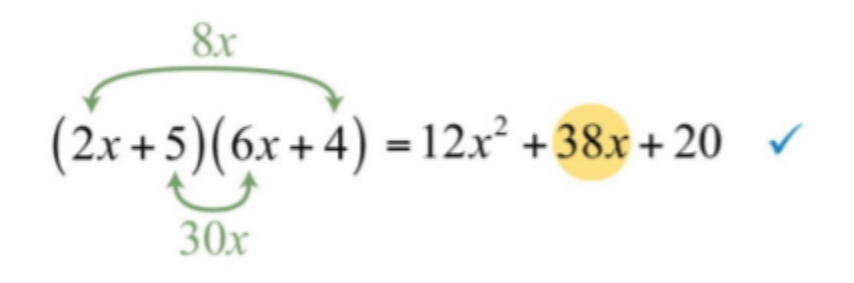


Figure 6.3.4

Answer:

$$(2x+5)(6x+4)$$

Example 6.3.3

Factor:

$$10x^2 - 23x + 6.$$

Solution

First, consider the factors of the first and last terms.

$$\begin{aligned} 10 &= 1 \cdot 10 \\ &= 2 \cdot 5 \end{aligned} \qquad \begin{aligned} 6 &= 1 \cdot 6 \\ &= 2 \cdot 3 \end{aligned}$$

We are searching for products of factors whose sum equals the coefficient of the middle term, -23. Factoring begins at this point with two sets of blank parentheses:

$$10x^2 - 23x + 6 = (\quad)(\quad)$$

Since the last term is positive and the middle term is negative, we know that both factors of the last term must be negative. Here we list all possible combinations with the factors of $10x^2 = 2x \cdot 5x$.

$$10x^2 - 23x + 6 = (2x \quad ?)(5x \quad ?)$$

$$\begin{aligned} &(2x-1)(5x-6) && \textit{middle term} \Rightarrow -17x \\ &(2x-6)(5x-1) && \textit{middle term} \Rightarrow -32x \\ &(2x-2)(5x-3) && \textit{middle term} \Rightarrow -16x \\ &(2x-3)(5x-2) && \textit{middle term} \Rightarrow -19x \end{aligned}$$

There is no combination that produces a middle term of $-23x$. We then move on to the factors of $10x^2 = 10x \cdot x$ and list all possible combinations:

$$10x^2 - 23x + 6 = (10x \quad ?)(x \quad ?)$$

$$\begin{aligned}
&(10x-1)(x-6) \quad middle\ term \Rightarrow -61x \\
&(10x-6)(x-1) \quad middle\ term \Rightarrow -162x \\
&(10x-2)(x-3) \quad middle\ term \Rightarrow -32x \\
&(10x-3)(x-2) \quad middle\ term \Rightarrow -23x \quad \checkmark
\end{aligned}$$

And we can write

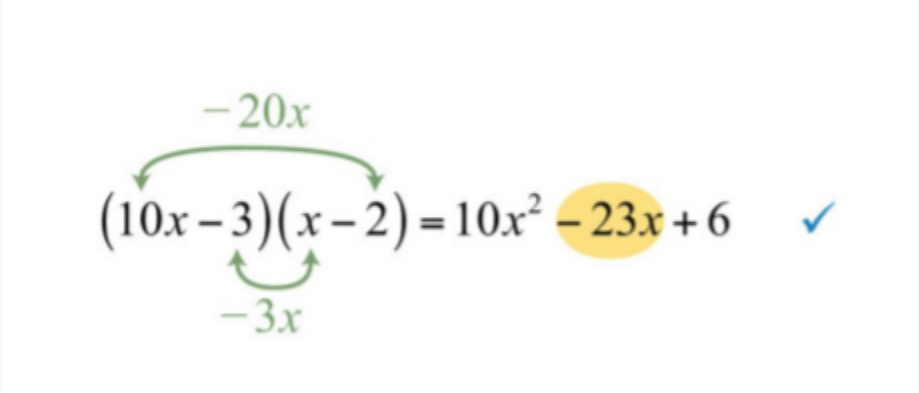


Figure 6.3.5

Answer:

$(10x-3)(x-2)$. The complete check is left to the reader.

We can reduce much of the guesswork involved in factoring trinomials if we consider all of the factors of the first and last terms and their products.

Example 6.3.4

Factor:

$$5x^2 + 38x - 16.$$

Solution:

We begin with the factors of 5 and 16.

$$\begin{aligned}
5 &= 1 \cdot 5 \\
16 &= 1 \cdot 16 \\
&= 2 \cdot 8 \\
&= 4 \cdot 4
\end{aligned}$$

Since the leading coefficient is prime, we can begin with the following:

$$5x^2 + 38x - 16 = (x \quad ?)(5x \quad ?)$$

We look for products of the factors of 5 and 16 that could possibly add to 38.

Factors of 16	*Possible products*	
$1 \cdot 16$	$1 \cdot 1$ *and* $5 \cdot 16$	*products* $\Rightarrow$ 1 *and* 80
$1 \cdot 16$	$1 \cdot 16$ *and* $5 \cdot 1$	*products* $\Rightarrow$ 16 *and* 5
$2 \cdot 8$	$1 \cdot 2$ *and* $5 \cdot 8$	*products* $\Rightarrow$ 2 *and* 40 ✓
$2 \cdot 8$	$1 \cdot 8$ *and* $5 \cdot 2$	*products* $\Rightarrow$ 8 *and* 10
$4 \cdot 4$	$1 \cdot 4$ *and* $5 \cdot 4$	*products* $\Rightarrow$ 4 *and* 20

Since the last term is negative, we must look for factors with opposite signs. Here we can see that the products 2 and 40 add up to 38 if they have opposite signs:

$$1 \cdot (-2) + 5 \cdot 8 = -2 + 40 = 38$$

Therefore, use -2 and 8 as the factors of 16, making sure that the inner and outer products are $-2x$ and $40x$:

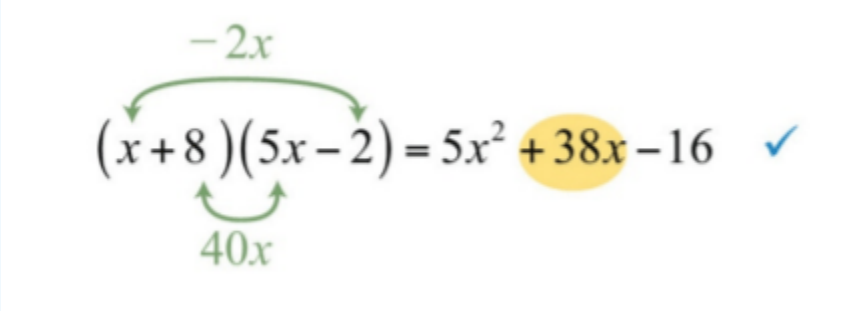


Figure 6.3.6

Answer:

$(x + 8)(5x - 2)$. The complete check is left to the reader.

After lots of practice, the process described in the previous example can be performed mentally.

Exercise 6.3.1

Factor:

$$12x^2 - 31x - 30$$

Answer available at the end of this section.

When given trinomials with multiple variables, the process is similar.

Example 6.3.5

Factor:

$$9x^2 + 30xy + 25y^2.$$

Solution:

Search for factors of the first and last terms such that the sum of the inner and outer products equals the middle term.

$$\begin{aligned} 9x^2 &= 1x \cdot 9x \\ &= 3x \cdot 3x \end{aligned} \qquad \begin{aligned} 25y^2 &= 1y \cdot 25y \\ &= 5y \cdot 5y \end{aligned}$$

Add the following products to obtain the middle term: $3x \cdot 5y + 3x \cdot 5y = 30xy$.

$$\begin{aligned} 9x^2 + 30xy + 25y^2 &= (3x \quad)(3x \quad) \\ &= (3x + 5y)(3x + 5y) \\ &= (3x + 5y)^2 \end{aligned}$$

In this example, we have a perfect square trinomial. Check.

$$\begin{aligned} (3x + 5y)^2 &= 9x^2 + 2 \cdot 3x \cdot 5y + 25y^2 \\ &= 9x^2 + 30xy + 25y^2 \quad \checkmark \end{aligned}$$

Answer:

$$(3x + 5y)^2$$

Exercise 6.3.2

Factor:

$$16x^2 - 24xy + 9y^2.$$

Answer available at the end of this section.

Factoring Trinomials with Common Factors

It is a good practice to first factor out the GCF, if there is one. Doing this produces a trinomial factor with smaller coefficients. As we have seen, trinomials with smaller coefficients require much less effort to factor. This commonly overlooked step is worth identifying early.

Example 6.3.6

Factor:

$$12x^2 - 27x + 6.$$

Solution:

Begin by factoring out the GCF.

$$12x^2 - 27x + 6 = 3(4x^2 - 9x + 2)$$

After factoring out 3, the coefficients of the resulting trinomial are smaller and have fewer factors.

$$\begin{aligned} 4 &= 1 \cdot 4 \quad 2 = 1 \cdot 2 \\ &= 2 \cdot 2 \end{aligned}$$

After some thought, we can see that the combination that gives the coefficient of the middle term is $4(-2) + 1(-1) = -8 - 1 = -9$.

$$\begin{aligned} 3(4x^2 - 9x + 2) &= 3(4x \quad ?)(x \quad ?) \\ &= 3(4x - 1)(x - 2) \end{aligned}$$

Check.

$$\begin{aligned} 3(4x-1)(x-2) &= 3(4x^2-8x-x+2) \\ &= 3(4x^2-9x+2) \\ &= 12x^2-27x+6 \quad \checkmark \end{aligned}$$

The factor 3 is part of the factored form of the original expression; be sure to include it in the answer.

Answer:

$$3(4x-1)(x-2)$$

It is a good practice to consistently work with trinomials where the leading coefficient is positive.

Example 6.3.7

Factor:

$$-x^2+2x+15.$$

Solution

In this example, the leading coefficient is -1. Before beginning the factoring process, factor out the -1:

$$-x^2+2x+15 = -1(x^2-2x-15)$$

At this point, factor the remaining trinomial as usual, remembering to write the -1 as a factor in your final answer. Because $3+(-5)=-2$, use 3 and 5 as the factors of 15.

$$\begin{aligned} -x^2+2x=15 &= -1(x^2-2x-15) \\ &= -1(x\quad)(x\quad) \\ &= -(x+3)(x-5) \end{aligned}$$

Answer:

$-1(x+3)(x-5)$. The check is left to the reader.

Example 6.3.8

Factor:

$$-60a^2-5a+30$$

Solution

The GCF of all the terms is 5. However, in this case factor out -5 because this produces a trinomial factor where the leading coefficient is positive.

$$-60a^2-5a+30 = -5(12a^2+a-6)$$

Focus on the factors of 12 and 6 that combine to give the middle coefficient, 1.

$$\begin{aligned} 12 &= 1\cdot 12 \\ &= 2\cdot 6 \\ &= 3\cdot 4 \end{aligned} \qquad \begin{aligned} 6 &= 1\cdot 6 \\ &= 2\cdot 3 \end{aligned}$$

After much thought, we find that $3\cdot 3-4\cdot 2=9-8=1$. Factor the remaining trinomial.

$$\begin{aligned} -60a^2 - 5a + 30 &= -5(12a^2 + a - 6) \\ &= -5(4a \quad)(3a \quad) \\ &= -5(4a + 3)(3a - 2) \end{aligned}$$

Answer:

$-5(4a + 3)(3a - 2)$. The check is left to the reader.

Exercise 6.3.3

Factor:

$$24 + 2x - x^2.$$

Answer available at the end of this section.

Factoring Using the AC Method

In this section, we factor trinomials of the form $ax^2 + bx + c$ using the AC method described previously.

Example 6.3.9

Factor using the AC method:

$$18x^2 - 21x + 5.$$

Solution:

Here $a = 18, b = -21$, and $c = 5$.

$$\begin{aligned} ac &= 18(5) \\ &= 90 \end{aligned}$$

Factor 90 and search for factors whose sum is -21.

$$\begin{aligned} 90 &= 1(90) \\ &= 2(45) \\ &= 3(30) \\ &= 5(18) \\ &= 6(15) \quad \checkmark \\ &= 9(10) \end{aligned}$$

In this case, the sum of the factors -6 and -15 equals the middle coefficient, -21. Therefore, $-21x = -6x - 15x$, and we can write

$$18x^2 - 21x + 5 = 18x^2 - 6x - 15x + 5$$

Factor the equivalent expression by grouping.

$$\begin{aligned} 18x^2 - 21x + 5 &= 18x^2 - 6x - 15x + 5 \\ &= 6x(3x - 1) - 5(3x - 1) \\ &= (3x - 1)(6x - 5) \end{aligned}$$

Answer:

$$(3x - 1)(6x - 5)$$

Example 6.3.10

Factor using the AC method: $9x^2 - 61x - 14$.

Solution:

Here $a = 9, b = -61$, and $c = -14$.

We factor -126 as follows:

$$\begin{aligned} -126 &= 1(-126) \\ &= 2(-63) \quad \checkmark \\ &= 3(-42) \\ &= 6(-21) \\ &= 7(-18) \\ &= 9(-14) \end{aligned}$$

The sum of factors 2 and -63 equals the middle coefficient, -61. Replace $-61x$ with $2x - 63x$:

$$\begin{aligned} 9x^2 - 61x - 14 &= 9x^2 + 2x - 63x - 14 \quad \textit{Rearrange the terms.} \\ &= 9x^2 - 63x + 2x - 14 \quad \textit{Factor by grouping.} \\ &= 9x(x - 7) + 2(x - 7) \\ &= (x - 7)(9x + 2) \end{aligned}$$

Answer:

$(x - 7)(9x + 2)$. The check is left to the reader.

Key Takeaways

- If a trinomial of the form $ax^2 + bx + c$ factors into the product of two binomials, then the coefficient of the middle term will be the sum of certain products of factors of the first and last terms.
- If the trinomial has a greatest common factor, then it is a best practice to first factor out the GCF before attempting to factor it into a product of binomials.
- If the leading coefficient of a trinomial is negative, then it is a best practice to factor that negative factor out before attempting to factor the trinomial.
- Factoring trinomials of the form $ax^2 + bx + c$ takes lots of practice and patience. It is extremely important to take the time to become proficient by working lots of exercises.

Exercise 6.3.4 Factoring Trinomials

Factor.

1. $3x^2 - 14x - 5$
2. $5x^2 + 7x + 2$
3. $2x^2 + 5x - 3$
4. $2x^2 + 13x - 7$
5. $2x^2 + 9x - 5$
6. $7x^2 + 20x - 3$
7. $7x^2 - 46x - 21$
8. $3x^2 + x - 2$
9. $5x^2 + 34x - 7$
10. $5x^2 - 28x - 12$
11. $9x^2 - 12x + 4$
12. $4x^2 - 20x + 25$

13. $49x^2 + 14x + 1$
14. $25x^2 - 10x + 1$
15. $2x^2 + 7x + 16$
16. $6x^2 - 19x - 10$
17. $27x^2 + 66x - 16$
18. $12x^2 - 88x - 15$
19. $12y^2 - 8y + 1$
20. $16y^2 - 66y - 27$
21. $9x^2 - 12xy + 4y^2$
22. $25x^2 + 40x + 16$
23. $15x^2 - 26xy + 8y^2$
24. $12a^2 - 4ab - 5b^2$
25. $4x^2y^2 + 16xy - 9$
26. $20x^2y^2 + 4xy - 7$
27. The area of a rectangle is given by the function $A(x) = 3x^2 - 10x + 3$, where x is measured in meters. Rewrite this function in factored form.
28. The area of a rectangle is given by the function $A(x) = 10x^2 - 59x - 6$, where x is measured in meters. Rewrite this function in factored form.

Answer available at the end of this section.

Exercise 6.3.5 Factoring Trinomials with Common Factors

Factor.

1. $6x^2 - 20x - 16$
2. $45x^2 + 27x - 18$
3. $20x^2 - 20x + 5$
4. $3x^2 + 39x - 90$
5. $16x^2 + 26x - 10$
6. $54x^2 - 15x + 6$
7. $45x^2 - 45x - 20$
8. $90x^2 + 300x + 250$
9. $40x^2 - 36xy + 8y^2$
10. $24a^2b^2 + 18ab - 81$
11. $6x^2y^2 + 46xy + 28$
12. $2x^5 + 44x^4 + 144x^3$
13. $5x^3 - 65x^2 + 60x$
14. $15a^4b^2 - 25a^3b - 10a^2$
15. $6a^4b + 2a^3b^2 - 4a^2b^3$
16. $20a^3b^2 - 60a^2b^3 + 45ab^4$

Answer available at the end of this section.

Exercise 6.3.6 Factoring Trinomials with Common Factors

Factor out -1 and then factor further.

1. $-x^2 - 4x + 21$
2. $-x^2 + x + 12$
3. $-x^2 + 15x - 56$
4. $-x^2 + x + 72$
5. $-y^2 + 10y - 25$
6. $-y^2 - 16y - 64$
7. $36 - 9a - a^2$

8. $72 - 6a - a^2$
9. $32 + 4x - x^2$
10. $200 + 10x - x^2$

Answer available at the end of this section.

Exercise 6.3.7 Factoring Trinomials with Common Factors

Factor out a negative common factor first and then factor further if possible.

1. $-8x^2 + 6x + 9$
2. $-4x^2 + 28x - 49$
3. $-18x^2 - 6x + 4$
4. $2 + 4x - 30x^2$
5. $15 + 39x - 18x^2$
6. $90 + 45x - 10x^2$
7. $-2x^2 + 26x + 28$
8. $-18x^3 - 51x^2 + 9x$
9. $-3x^2y^2 + 18xy^2 - 24y^2$
10. $-16a^4 + 16a^3b - 4a^2b^2$
11. The height in feet of a projectile launched from a tower is given by the function $h(t) = -16t^2 + 64t + 80$, where t represents the number of seconds after launch. Rewrite the given function in factored form.
12. The height in feet of a projectile launched from a tower is given by the function $h(t) = -16t^2 + 64t + 192$, where t represents the number of seconds after launch. Rewrite the given function in factored form.

Answer available at the end of this section.

Exercise 6.3.8 Factoring Using the AC Method

Factor using the AC method.

1. $2x^2 + 5x - 7$
2. $3x^2 + 7x - 10$
3. $4x^2 - 25x + 6$
4. $16x^2 - 38x - 5$
5. $6x^2 + 23x - 18$
6. $8x^2 + 10x - 25$
7. $4x^2 + 28x + 40$
8. $-6x^2 - 3x + 30$
9. $12x^2 - 56xy + 60y^2$
10. $20x^2 + 80xy + 35y^2$

Answer available at the end of this section.

Exercise 6.3.9 Discussion Board Topics

1. Create your own trinomial of the form $ax^2 + bx + c$ that factors. Share it, along with the solution, on the discussion board.
2. Write out your own list of steps for factoring a trinomial of the form $ax^2 + bx + c$ and share it on the discussion board.
3. Create a trinomial of the form $ax^2 + bx + c$ that does not factor and share it along with the reason why it does not factor.

Answer available at the end of this section.

Answers

Exercise 6.3.1

$(3x-10)(4x+3)$

Exercise 6.3.2

$(4x-3y)^2$

Exercise 6.3.3

$-1(x-6)(x+4)$

Exercise 6.3.4 **Factoring Trinomials**

1. $(x-5)(3x+1)$
3. $(x+3)(2x-1)$
5. $(x+5)(2x-1)$
7. $(x-7)(7x+3)$
9. $(x+7)(5x-1)$
11. $(3x-2)^2$
13. $(7x+1)^2$
15. Prime
17. $(3x+8)(9x-2)$
19. $(6y-1)(2y-1)$
21. $(3x-2y)^2$
23. $(3x-4y)(5x-2y)$
25. $(2xy-1)(2xy+9)$
27. $A(x)=(3x-1)(x-3)$

Exercise 6.3.5 **Factoring Trinomials with Common Factors**

1. $2(x-4)(3x+2)$
3. $5(2x-1)^2$
5. $2(8x^2+13x-5)$
7. $5(3x-4)(3x+1)$
9. $4(5x-2y)(2x-y)$
11. $2(xy+7)(3xy+2)$
13. $5x(x-12)(x-1)$
15. $2a^2b(3a-2b)(a+b)$

Exercise 6.3.6 **Factoring Trinomials with Common Factors**

1. $-1(x-3)(x+7)$
3. $-1(x-7)(x-8)$
5. $-1(y-5)^2$
7. $-1(a-3)(a+12)$
9. $-1(x-8)(x+4)$

Exercise 6.3.7 **Factoring Trinomials with Common Factors**

1. $-(2x-3)(4x+3)$
3. $-2(3x-1)(3x+2)$
5. $-3(2x-5)(3x+1)$
7. $-2(x-14)(x+1)$
9. $-3y^2(x-4)(x-2)$
11. $h(t)=-16(t+1)(t-5)$

Exercise 6.3.8 **Factoring Using the AC Method**

1. $(x-1)(2x+7)$
3. $(x-6)(4x-1)$
5. $(2x+9)(3x-2)$
7. $4(x+2)(x+5)$
9. $4(x-3y)(3x-5y)$

Exercise 6.3.9 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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6.4: Factoring Special Binomials

Learning Objectives

- Factor binomials that are differences of squares.
- Factor binomials that are sums and differences of cubes.

Difference of Squares

A binomial is a polynomial with two terms. We begin with our first special binomial called difference of squares:

$$a^2 - b^2 = (a + b)(a - b) \tag{6.4.1}$$

To verify the above formula, multiply:

$$\begin{aligned}(a + b)(a - b) &= a^2 - ab + ba - b^2 \\ &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2\end{aligned}$$

We use this formula to factor certain special binomials.

Example 6.4.1

Factor:

$$x^2 - 16.$$

Solution:

Step 1: Identify the binomial as difference of squares and determine the square factors of each term.

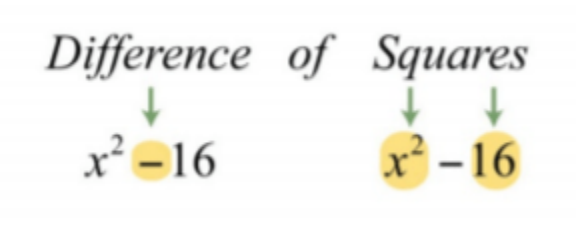


Figure 6.4.1

Here we can write

$$x^2 - 16 = (x)^2 - (4)^2$$

The terms are squares of x and 4. Hence $a = x$ and $b = 4$.

Step 2: Substitute into the difference of squares formula.

$$\begin{aligned}a^2 - b^2 &= (a + b)(a - b) \\ &\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ x^2 - 16 &= (x + 4)(x - 4)\end{aligned}$$

Step 3: Multiply to check. This step is optional.

Answer:

$(x+4)(x-4)$

It is worth taking some extra time at this point to review all of the squares of integers from 1 to 12.

$$\begin{aligned} 1^2 &= 1 & 7^2 &= 49 \\ 2^2 &= 4 & 8^2 &= 64 \\ 3^2 &= 9 & 9^2 &= 81 \\ 4^2 &= 16 & 10^2 &= 100 \\ 5^2 &= 25 & 11^2 &= 121 \\ 6^2 &= 36 & 12^2 &= 144 \end{aligned}$$

Recognizing these perfect square integers helps speed the factoring process.

Example 6.4.2

Factor:

$$9x^2 - 121.$$

Solution:

The subtraction indicates that this is a difference. Furthermore, we recognize that the terms are squares.

$$9x^2 - 121 = (3x)^2 - (11)^2$$

In this case, $a = 3x$ and $b = 11$. Substitute into the formula for difference of squares.

$$\begin{aligned} a^2 - b^2 &= (a+b)(a-b) \\ 9x^2 - 121 &= (3x+11)(3x-11) \end{aligned}$$

Answer:

$$(3x+11)(3x-11)$$

It may be the case that the terms of the binomial have a common factor. If so, it will be difficult to identify the perfect squares until we first factor out the GCF.

Example 6.4.3

Factor:

$$12y^2 - 75.$$

Solution:

The terms are not perfect squares. However, notice that they do have a common factor. First, factor out the GCF, 3.

$$12y^2 - 75 = 3(4y^2 - 25)$$

The resulting binomial factor is a difference of squares with $a = 2y$ and $b = 5$.

$$\begin{aligned} 12y^2 - 75 &= 3(4y^2 - 25) \\ &= 3(2y+5)(2y-5) \end{aligned}$$

Answer:

$$3(2y+5)(2y-5)$$

Example 6.4.4

Factor:

$$49x^2 - 100y^2.$$

Solution:

Here we have a binomial with two variables and recognize that it is a difference of squares.

$$49x^2 - 100y^2 = (7x)^2 - (10y)^2$$

Therefore, $a = 7x$ and $b = 10y$. Substitute into the formula for difference of squares.

$$\begin{aligned} a^2 - b^2 &= (a+b)(a-b) \\ 49x^2 - 100y^2 &= (7x+10y)(7x-10y) \end{aligned}$$

Answer:

$$(7x+10y)(7x-10y)$$

Exercise 6.4.1

Factor:

$$36x^2 - 1.$$

Answer available at the end of this section.

Given any real number b, a polynomial of the form $x^2 + b^2$ is prime.

Furthermore, the sum of squares $a^2 + b^2$ does not have a general factored equivalent. Care should be taken not to confuse this with a perfect square trinomial:

$$\begin{aligned} (a+b)^2 &= (a+b)(a+b) \\ &= a^2 + ab + ba + b^2 \\ &= a^2 + 2ab + b^2 \end{aligned}$$

Therefore,

$$(a+b)^2 \neq a^2 + b^2 \tag{6.4.2}$$

When the degree of the special binomial is greater than two, we may need to apply the difference of squares formula multiple times. A polynomial is completely factored when none of the factors can be factored any further.

Example 6.4.5

Factor completely:

$$x^4 - 16.$$

Solution:

First, identify what is being squared:

$$x^4 - 16 = (\quad)^2 - (\quad)^2$$

To do this, recall the power rule for exponents, $(x^m)^n = x^{mn}$. When exponents are raised to a power, multiply them. With this in mind, determine that $(x^2)^2 = x^4$ and write

$$x^4 - 16 = (x^2)^2 - (4)^2$$

Therefore, $a = x^2$ and $b = 4$. Substitute into the formula for difference of squares.

$$x^4 - 16 = (x^2 + 4)(x^2 - 4)$$

At this point, notice that the factor $(x^2 - 4)$ is itself a difference of two squares and thus can be further factored using $a = x$ and $b = 2$. The factor $(x^2 + 4)$ is a sum of squares, which cannot be factored using real numbers.

$$\begin{aligned} x^4 - 16 &= (x^2 + 4)(x^2 - 4) \\ &= (x^2 + 4)(x + 2)(x - 2) \end{aligned}$$

Answer:

$$(x^2 + 4)(x + 2)(x - 2)$$

Exercise 6.4.2

Factor completely:

$$81x^4 - 1.$$

Answer available at the end of this section.

Sum and Difference of Cubes

Two other special binomials of interest are the sum and difference of cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \tag{6.4.3}$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \tag{6.4.4}$$

We can verify these formulas by multiplying:

$$\begin{aligned} (a + b)(a^2 - ab + b^2) &= a^3 - a^2b + ab^2 - ab^2 + b^3 \\ &= a^3 + b^3 \quad \checkmark \\ (a - b)(a^2 + ab + b^2) &= a^3 + a^2b + ab^2 = a^2b - ab^2 - b^3 \\ &= a^3 - b^3 \quad \checkmark \end{aligned}$$

The process for factoring the sum and difference of cubes is very similar to that for the difference of squares. We first identify a and b and then substitute into the appropriate formula. The separate formulas for sum and difference of cubes allow us to always choose a and b to be positive.

Example 6.4.6

Factor:

$$x^3 + 8.$$

Solution:

The plus sign and the fact that the terms are cubes indicate to us that this is a sum of cubes.

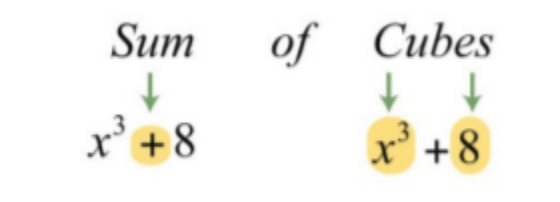


Figure 6.4.2

Next, identify what is being cubed.

$$x^3 + 8 = (x)^3 + (2)^3$$

In this case, $a = x$ and $b = 2$. Substitute into the sum of cubes formula.

$$\begin{aligned} a^3 + b^3 &= (a+b)(a^2 - a\ b + \ b^2) \\ &\quad\ \downarrow\ \downarrow\ \downarrow\ \ \downarrow\ \downarrow\ \ \downarrow \\ x^3 + 8 &= (x+2)(x^2 - x \cdot 2 + 2^2) \\ &= (x+2)(x^2 - 2x + 4) \end{aligned}$$

The resulting trinomial is prime and the factoring is complete. We can check this factorization by multiplying.

$$\begin{aligned} (x+2)(x^2 - 2x + 4) &= x^3 - 2x^2 + 4x + 2x^2 - 4x + 8 \\ &= x^3 - \cancel{2x^2} + \cancel{4x} + \cancel{2x^2} - \cancel{4x} + 8 \\ &= x^3 + 8 \quad \checkmark \end{aligned}$$

Answer:

$$(x-2)(x^2 - 2x + 4)$$

It is helpful to review the perfect cubes of integers from 1 to 12. This will aid you in identifying sums and differences of cubes.

$$\begin{aligned} 1^3 &= 1 & y^3 &= 343 \\ 2^3 &= 8 & 8^3 &= 512 \\ 3^3 &= 27 & 9^3 &= 729 \\ 4^3 &= 64 & 10^3 &= 1000 \\ 5^3 &= 125 & 11^3 &= 1331 \\ 6^3 &= 216 & 12^3 &= 1728 \end{aligned}$$

Example 6.4.7

Factor:

$$y^3 - 125.$$

Solution:

In this case, we have a difference of cubes.

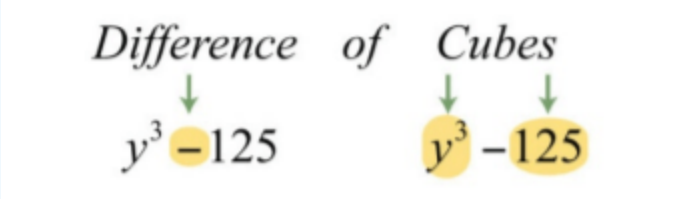


Figure 6.4.3

We can write

$$y^3 - 125 = (y)^3 - (5)^3$$

Substitute $a = y$ and $b = 5$ into the formula for difference of cubes.

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2 + a\ b + \ b^2) \\ y^3 - 1258 &= (y-5)(y^2 + y \cdot 5 + 5^2) \\ &= (y-5)(y^2 + 5y + 25) \end{aligned}$$

Answer:

$$(y-5)(y^2 + 5y + 25)$$

Always look for common factors when factoring. If the terms of the binomial have a GCF other than 1, then factor that out first.

Example 6.4.8

Factor:

$$54x^4 + 128x.$$

Solution:

Begin by factoring out the GCF $2x$.

$$54x^4 + 128x = 2x(27x^3 + 64)$$

The resulting binomial factor is a sum of cubes, where $a = 3x$ and $b = 4$.

$$\begin{aligned} 54x^4 + 128x &= 2x(27x^3 + 64) \\ &= 2x(3x+4)((3x)^2 - (3x)(4) + 4^2) \\ &= 2x(3x+4)(9x^2 - 12x + 16) \end{aligned}$$

Answer:

$$2x(3x+4)(9x^2 - 12x + 16)$$

Example 6.4.9

Factor:

$$x^3y^3 - 1.$$

Solution

This binomial is a difference of cubes with two variables. Identify what is being cubed.

$$x^3y^3 - 1 = (xy)^3 - (1)^3$$

Here $a = xy$ and $b = 1$. Substitute into the appropriate formula and simplify.

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \\ x^3y^3 - 1 &= (xy-1)((xy)^2 + xy \cdot 1 + 1^2) \\ &= (xy-1)(x^2y^2 + xy + 1) \end{aligned}$$

Answer:

$$(xy-1)(x^2y^2 + xy + 1)$$

Exercise 6.4.3

Factor:

$$8x^3 + 343.$$

Answer available at the end of this section.

When factoring, always look for resulting factors to factor further.

Example 6.4.10

Factor completely:

$$x^6 - 64.$$

Solution:

When confronted with a binomial that is both a difference of squares and cubes, as this is, make it a rule to factor using difference of squares first.

$$x^6 - 64 = (x^3)^2 - (8)^2$$

Therefore, $a = x^3$ and $b = 8$. Substitute into the difference of squares formula.

$$x^6 - 64 = (x^3 + 8)(x^3 - 8)$$

The resulting two binomial factors are a sum and a difference of cubes. Each can be factored further.

$$\begin{aligned} x^3 + 8 &= (x+2)(x^2 - 2x + 2^2) \\ x^3 - 8 &= (x-2)(x^2 + 2x + 2^2) \end{aligned}$$

Therefore, we have

$$\begin{aligned} x^6 - 64 &= \overbrace{(x^3+8)}^{} \cdot \overbrace{(x^3-8)}^{} \\ &= \overbrace{(x+2)(x^2-2x+4)}^{} \cdot \overbrace{(x-2)(x^2+2x+4)}^{} \end{aligned}$$

The trinomial factors are prime and the expression is completely factored.

Answer:

$$(x+2)(x^2-2x+4)(x-2)(x^2+2x+4)$$

As an exercise, factor the previous example as a difference of cubes first and then compare the results. Why do you think we make it a rule to factor using difference of squares first?

Exercise 6.4.4

Factor:

$$x^6 - y^6.$$

Answer available at the end of this section.

Key Takeaways

- When factoring special binomials, the first step is to identify it as a sum or difference. Once we identify the binomial, we then determine the values of a and b and substitute into the appropriate formula.
- The formulas for all of the special binomials should be memorized. In addition, to help facilitate the identification of special binomials, memorize the squares and cubes of integers up to at least 12.
- If a binomial is both a difference of squares and cubes, then first factor it as a difference of squares.

Exercise 6.4.5 Difference of Squares

Factor completely.

1. $x^2 - 9$
2. $x^2 - 100$
3. $y^2 - 36$
4. $y^2 - 144$
5. $x^2 + 4$
6. $x^2 - 5$
7. $m^2 - 81$
8. $m^2 - 64$
9. $16x^2 - 9$
10. $25x^2 - 4$
11. $144x^2 - 1$
12. $9x^2 - 121$
13. $4y^2 - 81$
14. $100y^2 - 49$
15. $9 - 4x^2$
16. $100 - x^2$
17. $1 - y^2$
18. $25 - 9y^2$
19. $-3x^2 + 75$
20. $-16x^2 + 25$
21. $2x^2 - 72$
22. $20x^3 - 45x$
23. $-48x + 27x^3$
24. $36x^2 - 100$
25. $x^2 - y^2$
26. $25x^2 - 9y^2$
27. $a^2 - 4b^2$
28. $a^2b^2 - 36$
29. $4x^2y^2 - 1$
30. $x^2y^2 - 25$
31. $2a^3 - 8ab^2$
32. $3a^3b^4 - 75ab^2$

33. $-100xy^3 + 4x^3y$
34. $-18x^3y^3 + 32xy$
35. $(x+1)^2 - y^2$
36. $x^2 - (y-2)^2$
37. $(x-3)^2 - (y+3)^2$
38. $(x^2+2)^2 - (x-1)^2$
39. $(x^2-1)^2 - (2x+3)^2$
40. $x^4 - 1$
41. $x^4 - y^4$
42. $16x^4 - 81$
43. $a^4b^4 - 16$
44. $a^4 - 16b^4$
45. $x^8 - 1$
46. $25x^8 - 1$
47. $a^8 - b^2$
48. $a^4 - 9$
49. $x^8 - y^8$
50. $81x^8 - 1$
51. The height of a projectile dropped from a 64-foot tower is given by the function $h(t) = -16t^2 + 64$, where t represents the time in seconds after it is dropped. Rewrite this function in factored form. (Hint: Factor out -16 first.)
52. The height of a projectile dropped from a 36-foot tower is given by the function $h(t) = -16t^2 + 36$, where t represents the time in seconds after it is dropped. Rewrite this function in factored form.

Answer available at the end of this section.

Exercise 6.4.6 Sum and Difference of Cubes

Factor completely.

1. $x^3 - 1$
2. $x^3 + 1$
3. $y^3 - 27$
4. $y^3 - 8$
5. $8y^3 + 1$
6. $27y^3 - 1$
7. $64a^3 - 125$
8. $8a^3 + 27$
9. $a^3 + 216$
10. $a^3 - 125$
11. $x^3 - 1000$
12. $343m^3 - 1$
13. $512n^3 + 1$
14. $8x^3 + 343$
15. $40y^3 - 135$
16. $27y^3 + 729$
17. $27y^3 - 64$
18. $x^3 + 3$
19. $5x^3 + 1$
20. $1 - y^3$
21. $27 - 1,000y^3$
22. $343 + 125a^3$
23. $x^3 - y^3$
24. $x^3 + y^3$
25. $x^3y^3 + 125$
26. $8x^3y^3 - 27$
27. $27a^3 - 8b^3$

28. $16x^3 - 250y^3$
29. $128x^3 + 2y^3$
30. $54x^3 - 2y^3$
31. $3a^4b - 24ab^4$
32. $a^3b^3c^3 - 1$
33. $(x+1)^3 - 8$
34. $8x^3 - (x-5)^3$
35. $(x-2)^3 + (x+2)^3$
36. $(a+3)^3 + (b-3)^3$
37. $x^6 - 1$
38. $x^6 + 1$
39. $64a^6 - 1$
40. $a^6 - 81b^6$
41. $x^6 - y^6$
42. $x^6 + y^6$

Answer available at the end of this section.

Exercise 6.4.7 Discussion Board Topics

1. If a binomial falls into both categories, difference of squares and difference of cubes, which would be best to factor it as, and why? Create an example that illustrates this situation and factor it using both formulas.
2. What can be said about the degrees of the factors of a polynomial? Give an example.
3. Make up your own difference of squares factoring exercise and provide the answer. Explain how you solved it.
4. Make up your own sum or difference of cubes factoring exercise and provide the answer. Explain how you solved it.

Answer available at the end of this section.

Answers

Exercise 6.4.1

$(6x+1)(6x-1)$

Exercise 6.4.2

$(9x^2+1)(3x+1)(3x-1)$

Exercise 6.4.3

$(2x+7)(4x^2-14x+49)$

Exercise 6.4.4

$(x+y)(x^2-xy+y2)(x-y)(x^2+xy+y2)$

Exercise 6.4.5 **Difference of Squares**

1. $(x+3)(x-3)$
3. $(y+6)(y-6)$
5. Prime
7. $(m+9)(m-9)$
9. $(4x+3)(4x-3)$
11. $(12x+1)(12x-1)$
13. $(2y+9)(2y-9)$
15. $(3+2x)(3-2x)$
17. $(1+y)(1-y)$
19. $-3(x+5)(x-5)$
21. $2(x+6)(x-6)$
23. $3x(3x+4)(3x-4)$
25. $(x+y)(x-y)$
27. $(a+2b)(a-2b)$
29. $(2xy+1)(2xy-1)$
31. $2a(a+2b)(a-2b)$
33. $4xy(x+5y)(x-5y)$
35. $(x+1+y)(x+1-y)$

37. $(x+y)(x-y-6)$
39. $(x^2+2x+2)(x^2-2x-4)$
41. $(x^2+y^2)(x+y)(x-y)$
43. $(a^2b^2+4)(ab+2)(ab-2)$
45. $(x^4+1)(x^2+1)(x+1)(x-1)$
47. $(a^4+b)(a^4-b)$
49. $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$
51. $h(t)=-16(t+2)(t-2)$

Exercise 6.4.6 **Sum and Difference of Cubes**

1. $(x-1)(x^2+x+1)$
3. $(y-3)(y^2+3y+9)$
5. $(2y+1)(4y^2-2y+1)$
7. $(4a-5)(16a^2+20a+25)$
9. $(a+6)(a^2-6a+36)$
11. $(x-10)(x^2+10x+100)$
13. $(8n+1)(64n^2-8n+1)$
15. $5(2y-3)(4y^2+6y+9)$
17. $(3y-4)(9y^2+12y+16)$
19. Prime
21. $(3-10y)(9+30y+100y^2)$
23. $(x-y)(x^2+xy+y^2)$
25. $(xy+5)(x^2y^2-5xy+25)$
27. $(3a-2b)(9a^2+6ab+4b^2)$
29. $2(4x+y)(16x^2-4xy+y^2)$
31. $3ab(a-2b)(a^2+2ab+4b^2)$
33. $(x-1)(x^2+4x+7)$
35. $2x(x^2+12)$
37. $(x+1)(x^2-x+1)(x-1)(x^2+x+1)$
39. $(2a+1)(4a^2-2a+1)(2a-1)(4a^2+2a+1)$
41. $(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$

Exercise 6.4.7 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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6.5: General Guidelines for Factoring Polynomials

Learning Objectives

- Develop a general strategy for factoring polynomials.

General Factoring Strategy

We have learned various techniques for factoring polynomials with up to four terms. The challenge is to identify the type of polynomial and then decide which method to apply. The following outlines a **general guideline for factoring** polynomials:

1. Check for common factors. If the terms have common factors, then factor out the greatest common factor (GCF) and look at the resulting polynomial factors to factor further.
2. Determine the number of terms in the polynomial.
   a. Factor four-term polynomials by grouping.
   b. Factor trinomials (three terms) using "trial and error" or the AC method.
   c. Factor binomials (two terms) using the following special products:

Table 6.5.1

Difference of Squares:	$a^2 - b^2 + (a + b)(a - b)$
Sum of Squares:	$a^2 + b^2$ no general formula
Difference of Cubes:	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
Sum of Cubes:	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

3. Look for factors that can be factored further.
4. Check by multiplying.

Note

- If a binomial is both a difference of squares and a difference of cubes, then first factor it as difference of squares and then as a sum and difference of cubes to obtain a more complete factorization.
- Not all polynomials with integer coefficients factor. When this is the case, we say that the polynomial is prime.

If an expression has a GCF, then factor this out first. Doing so is often overlooked and typically results in factors that are easier to work with. Also, look for the resulting factors to factor further; many factoring problems require more than one step. A polynomial is completely factored when none of the factors can be factored further.

Example 6.5.1

Factor:

$$6x^4 - 3x^3 - 24x^2 + 12x.$$

Solution:

This four-term polynomial has a GCF of $3x$. Factor this out first.

$$6x^4 - 3x^3 - 24x^2 + 12x = 3x(2x^3 - x^2 - 8x + 4)$$

Now factor the resulting four-term polynomial by grouping.

$$6x^4 - 3x^3 - 24x^2 + 12x = 3x \quad \underbrace{(2x^3 - x^2}_{group} \quad \underbrace{-8x + 4)}_{group}$$

$$\begin{aligned} &= 3x(x^2(2x-1) - 4(2x-1)) \\ &= 3x((2x-1)(x^2-4)) \\ &= 3x(2x-1)(x^2-4) \end{aligned}$$

The factor $(x^2 - 4)$ is a difference of squares and can be factored further.

$$\begin{aligned} &= 3x(2x-1)(x^2-4) \\ &= 3x(2x-1)(x+2)(x-2) \end{aligned}$$

Answer:

$$3x(2x-1)(x+2)(x-2)$$

Example 6.5.2

Factor:

$$18x^3y - 60x^2y + 50xy.$$

Solution:

This trinomial has a GCF of $2xy$. Factor this out first.

$$18x^3y - 60x^2y + 50xy = 2xy(9x^2 - 30x + 25)$$

The trinomial factor can be factored further using the trial and error method. Use the factors $9 = 3 \cdot 3$ and $25 = (-5) \cdot (-5)$. These combine to generate the correct coefficient for the middle term: $3(-5) + 3(-5) = -15 - 15 = -30$.

$$\begin{aligned} 18x^3y - 60x^2y + 50xy &= 2xy(9x^2 - 30x + 25) \\ &= 2xy(3x \quad ?)(3x \quad ?) \\ &= 2xy(3x-5)(3x-5) \\ &= 2xy(3x-5)^2 \end{aligned}$$

Check.

$$\begin{aligned} 2xy(3x-5)^2 &= 2xy(9x^2 - 2 \cdot 3x \cdot 5 + 5^2) \\ &= 2xy(9x^2 - 30x + 25) \\ &= 18x^3y - 60x^2y + 50xy \quad \checkmark \end{aligned}$$

Answer:

$$2xy(3x-5)^2$$

Example 6.5.3

Factor:

$$5a^3b^4 + 10a^2b^3 - 75ab^2.$$

Solution:

This trinomial has a GCF of 5ab2. Factor this out first.

$$5a^3b^4 + 10a^2b^3 - 75ab^2 = 5ab^2(a^2b^2 + 2ab - 15)$$

The resulting trinomial factor can be factored as follows:

$$\begin{aligned} 5a^3b^4 + 10a^2b^3 - 75ab^2 &= 5ab^2(a^2b^2 + 2ab - 15) \\ &= 5ab^2(ab \quad ?)(ab \quad ?) \\ &= 5ab^2(ab + 5)(ab - 3) \end{aligned}$$

Answer:

$$5ab^2(ab + 5)(ab - 3)$$

Exercise 6.5.1

Factor:

$$3x^3y - 12x^2y^2 + 12xy^3.$$

Answer available at the end of this section.

Example 6.5.4

Factor:

$$16y^4 - 1.$$

Solution:

This binomial does not have a GCF. Therefore, begin factoring by identifying it as a difference of squares.

$$16y^4 - 1 = (4y^2)^2 - (1)^2$$

Here $a = 4y^2$ and $b = 1$. Substitute into the formula for difference of squares.

The factor $(4y^2 + 1)$ is a sum of squares and is prime. However, $(4y^2 - 1)$ is a difference of squares and can be factored further.

$$\begin{aligned} 16y^4 - 1 &= (4y^2 + 1)(4y^2 - 1) \\ &= (4y^2 + 1)(2y + 1)(2y - 1) \end{aligned}$$

Answer:

$$(4y^2 + 1)(2y + 1)(2y - 1)$$

Example 6.5.5

Factor:

$$x^6 - 64y^6.$$

Solution:

This binomial is a difference of squares and a difference of cubes. When this is the case, first factor it as a difference of squares.

$$x^6 - 64y^6 = (x^3)^2 - (8y^3)^2$$

We can write

$$x^6 - 64y^6 = (x^3 + 8y^3)(x^3 - 8y^3)$$

Each factor can be further factored either as a sum or difference of cubes, respectively.

$$x^3 + 8y^3 = (x)^3 + (2y)^3 = (x + 2y)(x^2 - 2xy + 4y^2)$$
$$x^3 - 8y^3 = (x)^3 - (2y)^3 = (x - 2y)(x^2 + 2xy + 4y^2)$$

Therefore,

$$\begin{aligned} x^6 - 64y^6 &= (x^3 + 8y^3)(x^3 - 8y^3) \\ &= (x + 2y)(x^2 - 2xy + 4y^2)(x - 2y)(x^2 + 2xy + 4y^2) \end{aligned}$$

Answer:

$$(x + 2y)(x^2 - 2xy + 4y^2)(x - 2y)(x^2 + 2xy + 4y^2)$$

Example 6.5.6

First, identify this expression as a difference of squares.

$$x^2 - (2x - 1)^2 = (x)^2 - (2x - 1)^2$$

Solution:

Here use $a = x$ and $b = 2x - 1$ in the formula for a difference of squares.

$$\begin{aligned} a^2 - b^2 &= (a + b)(a - b) \\ x^2 - (2x - 1)^2 &= [(x + (2x - 1))][(x - (2x - 1))] \\ &= (x + 2x - 1)(x - 2x + 1) \\ &= (3x - 1)(-x + 1) \end{aligned}$$

Answer:

$$(3x - 1)(-x + 1)$$

Exercise 6.5.2

Factor:

$$x^4 + 2x^3 + 27x + 54.$$

Answer available at the end of this section.

Key Takeaways

- Use the polynomial type to determine the method used to factor it.
- It is a best practice to look for and factor out the greatest common factor (GCF) first. This will facilitate further factoring and simplify the process. Be sure to include the GCF as a factor in the final answer.

- Look for resulting factors to factor further. It is often the case that factoring requires more than one step.
- If a binomial can be considered as both a difference of squares and a difference of cubes, then first factor it as a difference of squares. This results in a more complete factorization.

Exercise 6.5.3 Mixed Factoring

Factor completely.

1. $2x^5y^2 - 12x^4y^3$
2. $18x^5y^3 - 6x^4y^5$
3. $5x^2 + 20x - 25$
4. $4x^2 + 10x - 6$
5. $24x^3 - 30x^2 - 9x$
6. $30x^3 - 65x^2 + 10x$
7. $6x^3 + 27x^2 - 9x$
8. $21x^3 + 49x^2 - 28x$
9. $5x^3 - 30x^2 - 15x + 90$
10. $6x^4 + 24x^3 - 2x^2 - 8x$
11. $x^4 - 6x^3 + 8x - 48$
12. $x^4 - 5x^3 + 27x - 135$
13. $4x^3 - 4x^2 - 9x + 9$
14. $50x^3 + 25x^2 - 32x - 16$
15. $2x^3 + 250$
16. $3x^5 - 81x^2$
17. $2x^5 - 162x^3$
18. $4x^4 - 36$
19. $x^4 + 16$
20. $x^3 + 9$
21. $72 - 2x^2$
22. $5x^4 - 25x^2$
23. $7x^3 - 14x$
24. $36x^2 - 12x + 1$
25. $25x^2 + 10x + 1$
26. $250x^3 + 200x^4 + 40x^5$
27. $-7x^2 + 19x + 6$
28. $-8x^4 + 40x^3 - 50x^2$
29. $a^4 - 16$
30. $16a^4 - 81b^4$
31. $y^5 + y^4 - y - 1$
32. $4y^5 + 2y^4 - 4y^2 - 2y$
33. $3x^8 - 192x^2$
34. $4x^7 + 4x$
35. $4x^2 - 19xy + 12y^2$
36. $16x^2 - 66xy - 27y^2$
37. $5x^5 - 3x^4 - 5x^3 + 3x^2$
38. $4a^2b^2 - 4a^2 - 9b^2 + 9$
39. $15a^2 - 4ab - 4b^2$
40. $6a^2 - 25ab + 4b^2$
41. $6x^2 + 5xy + 6y^2$
42. $9x^2 + 5xy - 12y^2$
43. $(3x - 1)^2 - 64$
44. $(x - 5)^2 - (x - 2)^2$
45. $(x + 1)^3 + 8$
46. $(x - 4)^3 - 27$
47. $(2x - 1)^2 - (2x - 1) - 12$
48. $(x - 4)^2 + 5(x - 4) + 6$
49. $a^3b - 10a^2b^2 + 25ab^3$

50. $2a^3b^2 - 12a^2b + 18a$
51. $15a^2b^2 - 57ab - 12$
52. $-60x^3 + 4x^2 + 24x$
53. $-24x^5 + 78x^3 - 54x$
54. $9y^6 - 13y^4 + 4y^2$
55. $36 - 15a - 6a^2$
56. $60ab^2 + 5a^2b^2 - 5a^3b^2$
57. $x^4 - 1$
58. $16x^4 - 64$
59. $x^8 - 1$
60. $81x^8 - 1$
61. $x^{16} - 1$
62. $x^{12} - 1$
63. $54x^6 - 216x^4 - 2x^3 + 8x$
64. $4a^4 - 4a^2b^2 - a^2 + b^2$
65. $32y^3 + 32y^2 - 18y - 18$
66. $3a^3 + a^2b - 12ab - 4b^2$
67. $18m^2 - 21mn - 9n^2$
68. $5m^2n^2 + 10mn - 15$
69. The volume of a certain rectangular solid is given by the function $V(x) = x^3 - 2x^2 - 3x$. Write the function in its factored form.
70. The volume of a certain right circular cylinder is given by the function $V(x) = 4\pi x^3 - 4\pi x^2 + \pi x$. Write the function in its factored form.

Answer available at the end of this section.

Exercise 6.5.4 Discussion Board

1. First, factor the trinomial $24x^2 - 28x - 40$. Then factor out the GCF. Discuss the significance of factoring out the GCF first. Do you obtain the same result?
2. Discuss a plan for factoring polynomial expressions on an exam. What should you be looking for and what should you be expecting?

Answer available at the end of this section.

Answers

Exercise 6.5.1

$3xy(x - 2y)^2$

Exercise 6.5.2

$(x + 2)(x + 3)(x^2 - 3x + 9)$

Exercise 6.5.3 **Mixed Factoring**

1. $2x^4y^2(x - 6y)$
3. $5(x - 1)(x + 5)$
5. $3x(2x - 3)(4x + 1)$
7. $3x(2x^2 + 9x - 3)$
9. $5(x - 6)(x^2 - 3)$
11. $(x - 6)(x + 2)(x^2 - 2x + 4)$
13. $(x - 1)(2x - 3)(2x + 3)$
15. $2(x + 5)(x^2 - 5x + 25)$
17. $2x^3(x + 9)(x - 9)$
19. Prime
21. $2(6 + x)(6 - x)$
23. $7x(x^2 - 2)$
25. $(5x + 1)^2$
27. $-(x - 3)(7x + 2)$
29. $(a^2 + 4)(a + 2)(a - 2)$

31. $(y^2+1)(y-1)(y+1)^2$
33. $3x^2(x+2)(x^2-2x+4)(x-2)(x^2+2x+4)$
35. $(x-4y)(4x-3y)$
37. $x^2(5x-3)(x+1)(x-1)$
39. $(3a-2b)(5a+2b)$
41. Prime
43. $3(x-3)(3x+7)$
45. $(x+3)(x^2+3)$
47. $2(x+1)(2x-5)$
49. $ab(a-5b)^2$
51. $3(ab-4)(5ab+1)$
53. $-6x(x+1)(x-1)(2x+3)(2x-3)$
55. $-3(a+4)(2a-3)$
57. $(x^2+1)(x+1)(x-1)$
59. $(x^4+1)(x^2+1)(x+1)(x-1)$
61. $(x^8+1)(x^4+1)(x^2+1)(x+1)(x-1)$
63. $2x(x+2)(x-2)(3x-1)(9x^2+3x+1)$
65. $2(y+1)(4y-3)(4y+3)$
67. $3(2m-3n)(3m+n)$
69. $V(x)=x(x+1)(x-3)$

Exercise $6.5.4$ **Discussion Board**

1. Answer may vary

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6.6: Solving Equations by Factoring

Learning Objectives

- Verify solutions to quadratic equations.
- Solve quadratic equations by factoring.
- Determine a quadratic equation with given solutions.
- Solve polynomial equations by factoring.

Solving Quadratic Equations by Factoring

Learning how to solve equations is one of our main goals in algebra. Up to this point, we have solved linear equations, which are of degree 1. In this section, we will learn a technique that can be used to solve certain equations of degree 2. A quadratic equation is any equation that can be written in the standard form

$$ax^2 + bx + c = 0 \tag{6.6.1}$$

,

where a, b, and c are real numbers and $a \neq 0$. The following are some examples of quadratic equations, all of which will be solved in this section:

$$x^2 + x - 6 = 0$$

$$4x^2 - 9 = 0$$

$$2x^2 + 10x + 20 = -3x + 5$$

A solution of a quadratic equation in standard form is called a root. Quadratic equations can have two real solutions, one real solution, or no real solution. The quadratic equation $x^2 + x - 6 = 0$ has two solutions, namely, $x = -3$ and $x = 2$.

Example 6.6.1

Verify that $x = -3$ and $x = 2$ are solutions to $x^2 + x - 6 = 0$.

Solution:

To verify solutions, substitute the values for x and then simplify to see if a true statement results.

$$\begin{aligned} Check: x = -3 \qquad & \qquad Check: x = 2 \\ x^2 + x - 6 = 0 \qquad & \qquad x^2 + x - 6 = 0 \\ (-3)^2 + (-3) - 6 = 0 \qquad & \qquad (2)^2 + (2) - 6 = 0 \\ 9 - 3 - 6 = 0 \qquad & \qquad 4 + 2 - 6 = 0 \\ 0 = 0 \quad \checkmark \qquad & \qquad 0 = 0 \quad \checkmark \end{aligned}$$

Answer:

Both values produce true statements. Therefore, they are both solutions to the equation.

Our goal is to develop algebraic techniques for finding solutions to quadratic equations. The first technique requires the zero-product property:

If $a \cdot b = 0$, then $a = 0$ or $b = 0$

In other words, if any product is equal to zero, then one or both of the variable factors must be equal to zero.

Example 6.6.2

Solve:

$$(x-8)(x+7)=0.$$

Solution:

This equation consists of a product of two quantities equal to zero; therefore, the zero-product property applies. One or both of the quantities must be zero.

$$\begin{array}{ccc} (x-8)=0 & \text{or} & (x+7)=0 \\ x-8+8=0+8 & & x+7-7=0-7 \\ x=8 & & x=-7 \end{array}$$

To verify that these are solutions, substitute them for the variable x.

$$\begin{array}{rr} Check: x=8 & Check: x=-7 \\ (x-8)(x+7)=0 & (x-8)(x+7)=0 \\ (8-8)(8+7)=0 & (-7+8)(-7+7)=0 \\ (0)(15)=0 & (1)(0)=0 \\ 0=0 \quad \checkmark & 0=0 \quad \checkmark \end{array}$$

Notice that each solution produces a factor that is equal to zero.

Answer:

The solutions are 8 and -7.

The quadratic equation may not be given in its factored form.

Example 6.6.3

Solve:

$$x^2+3x-10=0.$$

Solution:

The goal is to produce a product that is equal to zero. We can do that by factoring the trinomial on the left side of the equation.

$$\begin{array}{ll} x^2+3x-10=0 & \\ (x \quad ?)(x \quad ?)=0 & -10=5(-2) \\ & \text{and } 3=5+(-2) \\ (x+5)(x-2)=0 & \end{array}$$

Next, apply the zero-product property and set each factor equal to zero.

$$x+5=0 \quad \text{or} \quad x-2=0$$

This leaves us with two linear equations, each of which can be solved for x.

$$\begin{aligned} x+5-5&=0-5 \quad & x-2+2&=0+2 \\ x&=-5 & x&=2 \end{aligned}$$

Check the solutions by substituting into the original equation to verify that we obtain true statements.

$$\begin{aligned} Check: x=-5 \qquad & \qquad Check: x=2 \\ x^2+3x-10&=0 & x^2+3x-10&=0 \\ (-5)^2+3(-5)-10&=0 & (2)^2+3(2)-10&=0 \\ 25-15-10&=0 & 4+6-10&=0 \\ 0&=0 \quad \checkmark & 0&=0 \quad \checkmark \end{aligned}$$

Answer:

The solutions are -5 and 2.

Using the zero-product property after factoring a quadratic equation in standard form is the key to this technique. However, the quadratic equation may not be given in standard form, and so there may be some preliminary steps before factoring. The steps required to solve by factoring are outlined in the following example.

Example 6.6.4

Solve:

$$2x^2+10x+20=-3x+5.$$

Solution:

Step 1: Express the quadratic equation in standard form. For the zero-product property to apply, the quadratic expression must be equal to zero. Use the addition and subtraction properties of equality to combine opposite-side like terms and obtain zero on one side of the equation. In this example, add $3x$ to and subtract 5 from both sides.

$$\begin{aligned} 2x^2+10x+20+3x&=-3x+5+3x \\ 2x^2+13x+20&=5 \\ 2x^2+13x+20-5&=5-5 \\ 2x^2+13x+15&=0 \end{aligned}$$

Step 2: Factor the quadratic expression.

$$(2x+3)(x+5)=0$$

Step 3: Apply the zero-product property and set each variable factor equal to zero.

$$2x+3=0 \quad \text{or} \quad x+5=0$$

Step 4: Solve the resulting linear equations.

$$\begin{aligned} 2x+3&=0 \quad \text{or} \quad & x+5&=0 \\ 2x&=-3 & x&=-5 \\ \tfrac{2x}{2}&=\tfrac{-3}{2} & & \\ x&=-\tfrac{3}{2} & & \end{aligned}$$

Answer:

The solutions are -5 and $-\frac{3}{2}$. The check is optional.

Example 6.6.5

Solve:

$$9x^2 + 1 = 6x.$$

Solution:

Write this in standard form by subtracting \(6x\ from both sides.

$$\begin{aligned} 9x^2 + 1 - 6x &= 6x - 6x \\ 9x^2 - 6x + 1 &= 0 \end{aligned}$$

Once the equation is in standard form, equal to zero, factor.

$$(3x - 1)(3x - 1) = 0$$

This is a perfect square trinomial. Hence setting each factor equal to zero results in a repeated solution.

$$\begin{array}{ccc} 3x - 1 = 0 & \text{or} & 3x - 1 = 0 \\ 3x = 1 & & 3x = 1 \\ x = \frac{1}{3} & & x = \frac{1}{3} \end{array}$$

A repeated solution is called a double root and does not have to be written twice.

Answer:

The solution is $\frac{1}{3}$.

Exercise 6.6.1

Solve:

$$x^2 - 3x = 28.$$

Answer available at the end of this section.

Not all quadratic equations in standard form are trinomials. We often encounter binomials.

Example 6.6.6

Solve:

$$x^2 - 9 = 0.$$

Solution:

This quadratic equation is given in standard form, where the binomial on the left side is a difference of squares. Factor as follows:

$$\begin{aligned} x^2 - 9 &= 0 \\ (x + 3)(x - 3) &= 0 \end{aligned}$$

Next, set each factor equal to zero and solve.

$$x+3=0 \quad \text{or} \quad x-3=0$$
$$x=-3 \qquad\qquad x=3$$

Answer:

The solutions are 3 and -3, which can also be written as ± 3.

Example 6.6.7

Solve:

$$5x^2=15x$$

Solution:

By inspection, we see that $x=0$ is a solution to this quadratic equation. Since dividing by zero is undefined, we want to avoid dividing both sides of this equation by x. In general, we wish to avoid dividing both sides of any equation by a variable or an expression containing a variable. We will discuss this in more detail later. The first step is to rewrite this equation in standard form with zero on one side.

$$\begin{aligned} 5x^2 &= 15x \\ 5x^2-15x &= 15x-15x \quad \textit{Subtract } 15x \textit{ from both sides.} \\ 5x^2-15x &= 0 \end{aligned}$$

Next, factor the expression. Notice that the binomial on the left has a GCF of $5x$.

$$\begin{aligned} 5x^2-15x &= 0 \\ 5x(x-3) &= 0 \end{aligned}$$

Set each factor to equal to zero.

$$\begin{aligned} 5x=0 \quad &\text{or} \quad x-3=0 \\ \tfrac{5x}{5}=\tfrac{0}{5} \quad &\qquad\quad x=3 \\ x=0 \quad & \end{aligned}$$

Answer:

The solutions are 0 and 3.

Example 6.6.8

Solve:

$$(2x+1)(x+5)=11.$$

Solution:

This quadratic equation appears to be factored; hence it might be tempting to set each factor equal to 11. However, this would lead to incorrect results. We must rewrite the equation in standard form, equal to zero, so that we can apply the zero-product property.

$$\begin{aligned} (2x+1)(x+5) &= 11 \\ 2x^2+10x+x+5 &= 11 \\ 2x^2+11x+5-11 &= 11-11 \\ 2x^2+11x-6 &= 0 \end{aligned}$$

Once it is in standard form, we can factor and then set each factor equal to zero.

$$(2x-1)(x+6)=0$$

$$\begin{array}{ll} 2x-1=0 & \text{or} \quad x+6=0 \\ 2x=1 & x=-6 \\ x=\frac{1}{2} & \end{array}$$

Answer:

The solutions are $\frac{1}{2}$ and -6.

Example 6.6.9

Solve:

$$15x^2-25x+10=0.$$

Solution:

We begin by factoring out the GCF of 5. Then factor the resulting trinomial.

$$\begin{array}{c} 15x^2-25x+10=0 \\ 5(3x^2-5x+2)=0 \\ 5(3x-2)(x-1)=0 \end{array}$$

Next, we set each ***variable factor*** equal to zero and solve for x.

$$\begin{array}{ll} 3x-2=0 & \text{or} \quad x-1=0 \\ 3x=2 & x=1 \\ x=\frac{2}{3} & \end{array}$$

Notice that the factor 5 is not a variable factor and thus did not contribute to the solution set.

Answer:

The solutions are $\frac{2}{3}$ and 1.

Example 6.6.10

Factor:

$$52x^2+76x-13=0.$$

Solution:

Clear the fractions by multiplying both sides of the equation by the LCD, which is equal to 6.

$$\begin{array}{c} \frac{5}{2}x^2+\frac{7}{6}x-\frac{1}{3}=0 \\ 6\cdot\left(\frac{5}{2}x^2+\frac{7}{6}x-\frac{1}{3}\right)=6\cdot(0) \\ 15x^2+7x-2=0 \end{array}$$

At this point, we have an equivalent equation with integer coefficients and can factor as usual. Begin with the factors of 15 and 2.

$$15 = 1 \cdot 15 \quad 2 = 1 \cdot 2$$
$$= 3 \cdot 5$$

The coefficient of the middle term is $7 = 3(-1) + 5(2)$. Factor as follows:

$$15x^2 + 7x - 2 = 0$$
$$(3x + 2)(5x - 1) = 0$$

Set each factor equal to zero and solve.

$$3x + 2 = 0 \quad \text{or} \quad 5x - 1 = 0$$
$$3x = -2 \qquad\qquad 5x = 1$$
$$x = -\tfrac{2}{3} \qquad\qquad x = \tfrac{1}{5}$$

Answer:

The solutions are $-\frac{2}{3}$ and $\frac{1}{5}$.

Exercise 6.6.2

Solve:

$$4x^2 - 9 = 0.$$

Answer available at the end of this section.

Finding Equations with Given Solutions

The zero-product property states,

$$\text{If } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0$$

And, in fact, the converse is true as well:

$$\text{If } a = 0 \text{ or } b = 0, \text{ then } ab = 0$$

When this is the case, we can write the following:

$$a \cdot b = 0, \text{ if and only if } a = 0 \text{ or } b = 0$$

We use this property to find equations, given the solutions. To do this, the steps for solving by factoring are performed in reverse.

Example 6.6.11

Find a quadratic equation with solutions -7 and 2.

Solution:

Given the solutions, we can determine two linear factors.

$$x = -7 \quad \text{or} \quad x = 2$$
$$x + 7 = 0 \qquad x - 2 = 0$$

The product of these linear factors is equal to zero when $x = -7$ or $x = 2$:

$$(x+7)(x-2)=0$$

Multiply the binomials and present the equation in standard form.

$$\begin{aligned} x^2-2x+7x-14&=0\\ x^2+5x-14&=0 \end{aligned}$$

Answer:

$$x^2+5x-14=0.$$

We may check our equation by substituting the given answers to see if we obtain a true statement. Also, the equation found aboveis not unique and so the check becomes essential when our equation looks different from someone else's. This is left as an exercise.

Example 6.6.12

Find a quadratic equation with integer coefficients, given solutions $\frac{1}{2}$ and $-\frac{3}{4}$.

Solution:

To avoid fractional coefficients, we first clear the fractions by multiplying both sides by the denominator.

$$\begin{array}{ccc} x=\frac{1}{2} & \text{or} & x=-\frac{3}{4}\\ 2\cdot x=2\cdot\frac{1}{2} & & 4\cdot x=-\frac{3}{4}\cdot 4\\ 2x=1 & & 4x=-3\\ 2x-1=0 & & 4x+3=0 \end{array}$$

Apply the zero-product property and multiply.

$$\begin{aligned} (2x-1)(4x+3)&=0\\ 8x^2+6x-4x-3&=0\\ 8x^2+2x-3&=0 \end{aligned}$$

Answer:

$$8x^2+2x-3=0$$

Exercise 6.6.3

Find a quadratic equation with integer coefficients, given solutions -1 and $\frac{2}{3}$.

Answer available at the end of this section.

Solving Polynomial Equations by Factoring

The zero-product property is true for any number of factors that make up an equation. If an expression is equal to zero and can be factored into linear factors, then we will be able to set each factor equal to zero and solve for each equation.

Example 6.6.13

Solve:

$$3x(x-5)(3x-2)=0.$$

Solution:

Set each variable factor equal to zero and solve.

$$\begin{array}{ccccc} 3x=0 & \text{or} & x-5=0 & \text{or} & 3x-2=0 \\ \frac{3x}{3}=\frac{0}{3} & & x=5 & & \frac{3x}{3}=\frac{2}{3} \\ x=0 & & & & x=\frac{2}{3} \end{array}$$

Answer:

The solutions are $0, 5$, and $\frac{2}{3}$.

Of course, we cannot expect the equation to be given in factored form.

Example 6.6.14

Solve:

$$x^3+2x^2-9x-18=0.$$

Solution

Begin by factoring the left side completely.

$$\begin{array}{rl} x^3+2x^2-9x-18=0 & \textit{Factor by grouping.} \\ x^2(x+2)-9(x+2)=0 & \\ (x+2)(x^2-9)=0 & \textit{Factor as a difference of squares.} \\ (x+2)(x+3)(x-3)=0 & \end{array}$$

Set each factor equal to zero and solve.

$$\begin{array}{ccccc} x+2=0 & \text{or} & x+3=0 & \text{or} & x-3=0 \\ x=-2 & & x=-3 & & x=3 \end{array}$$

Answer:

The solutions are $-2, -3$, and 3.

Notice that the degree of the polynomial is 3 and we obtained three solutions. In general, for any polynomial equation with one variable of degree n, the fundamental theorem of algebra guarantees n real solutions or fewer. We have seen that many polynomials do not factor. This does not imply that equations involving these unfactorable polynomials do not have real solutions. In fact, many polynomial equations that do not factor do have real solutions. We will learn how to solve these types of equations as we continue in our study of algebra.

Exercise 6.6.4

Solve:

$$-10x^3-18x^2+4x=0.$$

Answer available at the end of this section.

Key Takeaways

- A polynomial can have at most a number of solutions equal to its degree. Therefore, quadratic equations can have up to two real solutions.
- To solve a quadratic equation, first write it in standard form. Once the quadratic expression is equal to zero, factor it and then set each variable factor equal to zero. The solutions to the resulting linear equations are the solutions to the quadratic equation.
- Not all quadratic equations can be solved by factoring. We will learn how to solve quadratic equations that do not factor later in the course.
- To find a quadratic equation with given solutions, perform the process of solving by factoring in reverse.
- If any polynomial is factored into linear factors and is set to zero, then we can determine the solutions by setting each variable factor equal to zero and solving each individually.

Exercise 6.6.5 Solutions to Quadratic Equations

Determine whether the given set of values are solutions to the quadratic equation.

1. $-3, 5; x^2 - 2x - 15 = 0$
2. $7, -1; x^2 - 6x - 7 = 0$
3. $-\frac{1}{2}, \frac{1}{2}; x^2 - 14 = 0$
4. $-\frac{3}{4}, \frac{3}{4}; x^2 - 916 = 0$
5. $-3, 2; x^2 - x - 6 = 0$
6. $-5, 1; x^2 - 4x - 5 = 0$

Answer available at the end of this section.

Exercise 6.6.6 Solutions to Quadratic Equations

Solve.

1. $(x - 3)(x + 2) = 0$
2. $(x + 5)(x + 1) = 0$
3. $(2x - 1)(x - 4) = 0$
4. $(3x + 1)(3x - 1) = 0$
5. $(x - 2)^2 = 0$
6. $(5x + 3)^2 = 0$
7. $7x(x - 5) = 0$
8. $-2x(2x - 3) = 0$
9. $(x - 12)(x + 34) = 0$
10. $(x + 58)(x - 38) = 0$
11. $(14x + 12)(16x - 23) = 0$
12. $(15x - 3)^2 = 0$
13. $-5(x + 1)(x - 2) = 0$
14. $12(x - 7)(x - 6) = 0$
15. $(x + 5)(x - 1) = 0$
16. $(x + 5)(x + 1) = 0$
17. $-2(3x - 2)(2x + 5) = 0$
18. $5(7x - 8)^2 = 0$

Answer available at the end of this section.

Exercise 6.6.7 Solve by Factoring

Solve.

1. $x^2 - x - 6 = 0$
2. $x^2 + 3x - 10 = 0$
3. $y^2 - 10y + 24 = 0$
4. $y^2 + 6y - 27 = 0$
5. $x^2 - 14x + 40 = 0$
6. $x^2 + 14x + 49 = 0$
7. $x^2 - 10x + 25 = 0$
8. $3x^2 + 2x - 1 = 0$
9. $5x^2 - 9x - 2 = 0$
10. $7y^2 + 20y - 3 = 0$
11. $9x^2 - 42x + 49 = 0$
12. $25x^2 + 30x + 9 = 0$
13. $2y^2 + y - 3 = 0$
14. $7x^2 - 11x - 6 = 0$
15. $2x^2 = -15x + 8$
16. $8x - 5 = 3x^2$
17. $x^2 - 36 = 0$
18. $x^2 - 100 = 0$
19. $4x^2 - 81 = 0$
20. $49x^2 - 4 = 0$
21. $x^2 = 4$
22. $9y^2 = 1$
23. $16y^2 = 25$
24. $36x^2 = 25$
25. $4x^2 - 36 = 0$
26. $2x^2 - 18 = 0$
27. $10x^2 + 20x = 0$
28. $-3x^2 + 6x = 0$
29. $25x^2 = 50x$
30. $x^2 = 0$
31. $(x + 1)^2 - 25 = 0$
32. $(x - 2)^2 - 36 = 0$
33. $5x(x - 4) = -4 + x$
34. $(x - 1)(x - 10) = 22$
35. $(x - 3)(x - 5) = 24$
36. $-2x(x - 9) = x + 21$
37. $(x + 1)(6x + 1) = 2x$
38. $(x - 2)(x + 12) = 15x$
39. $(x + 1)(x + 2) = 2(x + 16)$
40. $(x - 9)(2x + 3) = 2(x - 9)$

Answer available at the end of this section.

Exercise 6.6.8 Solve by Factoring

Clear the fractions by first multiplying both sides by the LCD and then solve.

1. $115x^2 + 13x + 25 = 0$
2. $114x^2 - 12x + 37 = 0$
3. $32x^2 - 23 = 0$
4. $52x^2 - 110 = 0$
5. $314x^2 - 212 = 0$
6. $13x^2 - 15x = 0$
7. $132x^2 - 12x + 2 = 0$
8. $13x^2 + 56x - 12 = 0$
9. The sides of a square measure $x + 3$ units. If the area is 25 square units, then find x.

10. The height of a triangle is 2 units more than its base. If the area is 40 square units, then find the length of the base.
11. The sides of a right triangle have measures that are consecutive integers. Find the length of the hypotenuse. (Hint: The hypotenuse is the longest side. Apply the Pythagorean theorem.)
12. The profit in dollars generated by producing and selling x custom lamps is given by the function $P(x) = -10x^2 + 800x - 12000$. How many lamps must be sold and produced to break even? (Hint: We break even when the profit is zero.)

Answer available at the end of this section.

Exercise 6.6.9 Solve by Factoring

Assuming dry road conditions and average reaction times, the safe stopping distance, d in feet of an average car is given using the formula $d = 120v^2 + v$, where v represents the speed of the car in miles per hour. For each problem below, given the stopping distance, determine the safe speed.

1. 15 feet
2. 40 feet
3. 75 feet
4. 120 feet

Answer available at the end of this section.

Exercise 6.6.10 Finding Equations with Given Solutions

Find a quadratic equation with integer coefficients, given the following solutions.

1. $-3, 1$
2. $-5, 3$
3. $-10, -3$
4. $-7, -4$
5. $-1, 0$
6. $0, \frac{3}{5}$
7. $-2, 2$
8. $-\frac{1}{2}, \frac{1}{2}$
9. $-4, \frac{1}{3}$
10. $\frac{2}{3}, \frac{2}{5}$
11. $-\frac{1}{5}, -\frac{2}{3}$
12. $-\frac{3}{2}, \frac{3}{4}$
13. 3, double root
14. -5, double root

Answer available at the end of this section.

Exercise 6.6.11 Solving Polynomial Equations

Solve.

1. $7x(x + 5)(x - 9) = 0$
2. $(x - 1)(x - 2)(x - 3) = 0$
3. $-2x(x - 10)(x - 1) = 0$
4. $8x(x - 4)^2 = 0$
5. $4(x + 3)(x - 2)(x + 1) = 0$
6. $-2(3x + 1)(3x - 1)(x - 1)(x + 1) = 0$
7. $x^3 - x^2 - 2x = 0$
8. $2x^3 + 5x^2 - 3x = 0$

9. $5x^3 - 15x^2 + 10x = 0$
10. $-2x^3 + 2x^2 + 12x = 0$
11. $3x^3 - 27x = 0$
12. $-2x^3 + 8x = 0$
13. $x^3 + x^2 - x - 1 = 0$
14. $x^3 + 2x^2 - 16x - 32 = 0$
15. $8x^3 - 4x^2 - 18x + 9 = 0$
16. $12x^3 = 27x$

Answer available at the end of this section.

Exercise 6.6.12 Discussion Board Topics

1. Explain why $2(x + 5)(x - 5) = 0$ has two solutions and $2x(x + 5)(x - 5) = 0$ has three solutions.
2. Make up your own quadratic equation and post it and the solutions on the discussion board.
3. Explain, in your own words, how to solve a quadratic equation in standard form.

Answer available at the end of this section.

Answers

Exercise 6.6.1

$x = -4$ or $x = 7$

Exercise 6.6.2

$-\frac{3}{2}$ and $\frac{3}{2}$

Exercise 6.6.3

$3x^2 + x - 2 = 0$

Exercise 6.6.4

$-2, 0, \frac{1}{5}$

Exercise 6.6.5 **Solutions to Quadratic Equations**

1. Yes
3. Yes
5. No

Exercise 6.6.6 **Solutions to Quadratic Equations**

1. $-2, 3$
3. $\frac{1}{2}, 4$
5. 2
7. $0, 5$
9. $-\frac{3}{4}, \frac{1}{2}$
11. $-2, 4$
13. $-1, 2$
15. $-5, 1$
17. $-\frac{5}{2}, \frac{2}{3}$

Exercise 6.6.7 **Solve by Factoring**

1. $-2, 3$
3. $4, 6$
5. $4, 10$
7. 5
9. $-\frac{1}{5}, 2$
11. $\frac{7}{3}$
13. $-\frac{3}{2}, 1$
15. $-8, \frac{1}{2}$
17. $-6, 6$
19. $-\frac{9}{2}, \frac{9}{2}$
21. $-2, 2$
23. $-\frac{5}{4}, \frac{5}{4}$
25. $-3, 3$

27. $-2, 0$
29. $0, 2$
31. $-6, 4$
33. $\frac{1}{5}, 4$
35. $-1, 9$
37. $-\frac{1}{2}, -\frac{1}{3}$
39. $-6, 5$

Exercise $6.6.8$ **Solve by Factoring**

1. $-3, -2$
3. $-\frac{2}{3}, \frac{2}{3}$
5. ± 7
7. 8
9. 2 units
11. 5 units

Exercise $6.6.9$ **Solve by Factoring**

1. 10 miles per hour
3. 30 miles per hour

Exercise $6.6.10$ **Finding Equations with Given Solutions**

1. $x^2 + 2x - 3 = 0$
3. $x^2 + 13x + 30 = 0$
5. $x^2 + x = 0$
7. $x^2 - 4 = 0$
9. $3x^2 + 11x - 4 = 0$
11. $15x^2 + 13x + 2 = 0$
13. $x^2 - 6x + 9 = 0$

Exercise $6.6.11$ **Solving Polynomial Equations**

1. $-5, 0, 9$
3. $0, 1, 10$
5. $-3, -1, 2$
7. $-1, 0, 2$
9. $0, 1, 2$
11. $-3, 0, 3$
13. $-1, 1$
15. $-\frac{3}{2}, \frac{1}{2}, \frac{3}{2}$

Exercise $6.6.12$ **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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6.7: Applications Involving Quadratic Equations

Learning Objectives

- Set up and solve applications involving relationships between real numbers.
- Set up and solve applications involving geometric relationships involving area and the Pythagorean theorem.
- Set up and solve applications involving the height of projectiles.

Number Problems

The algebraic setups of the word problems that we have previously encountered led to linear equations. When we translate the applications to algebraic setups in this section, the setups lead to quadratic equations. Just as before, we want to avoid relying on the "guess and check" method for solving applications. Using algebra to solve problems simplifies the process and is more reliable.

Example 6.7.1

One integer is 4 less than twice another integer, and their product is 96. Set up an algebraic equation and solve it to find the two integers.

Solution:

First, identify the variables. Avoid two variables by using the relationship between the two unknowns.

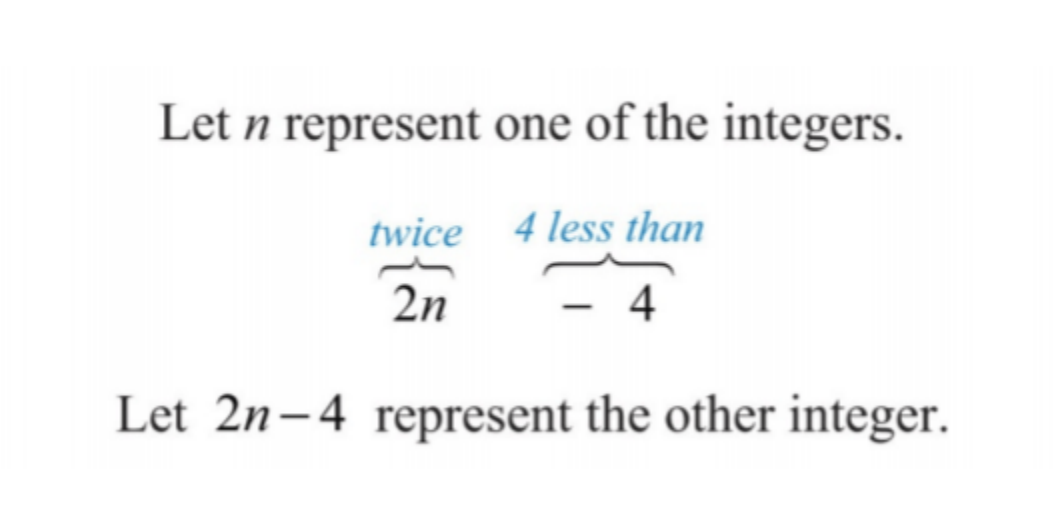


Figure 6.7.1

The key phrase, "their product is 96," indicates that we should multiply and set the product equal to 96.

$$n \cdot (2n - 4) = 96$$

Once we have the problem translated to a mathematical equation, we then solve. In this case, we can solve by factoring. The first step is to write the equation in standard form:

$$\begin{aligned} n \cdot (2n-4) &= 96 && \textit{Distribute } n. \\ 2n^2 - 4n &= 96 && \textit{Subtract 96 from both sides.} \\ 2n^2 - 4n - 96 &= 0 \end{aligned}$$

Next, factor completely and set each variable factor equal to zero.

$$\begin{aligned} 2n^2 - 4n - 96 &= 0 && \textit{Factor out the GCF, 2.} \\ 2(n^2 - 2n - 48) &= 0 && \textit{Factor the resulting trinomial.} \\ 2(n+6)(n-8) &= 0 && \textit{Set each variable factor equal to zero.} \end{aligned}$$

$$\begin{aligned} n + 6 &= 0 \quad \text{or} \quad n - 8 = 0 \\ n &= -6 \qquad\quad n = 8 \end{aligned}$$

The problem calls for two integers whose product is $+96$. The product of two positive numbers is positive and the product of two negative numbers is positive. Hence we can have two sets of solutions. Use $2n-4$ to determine the other integers.

$$\begin{aligned} n &= -6 & n &= 8 \\ 2n-4 &= 2(-6) - 4 & 2n-4 &= 2(8) - 4 \\ &= -12 - 4 & &= 16 - 4 \\ &= -16 & &= 12 \end{aligned}$$

Answer:

Two sets of integers solve this problem: $\{8, 12\}$ and $\{-6, -16\}$. Notice that $(8)(12) = 96$ and $(-6)(-16) = 96$; our solutions check out.

With quadratic equations, we often obtain two solutions for the identified unknown. Although it may be the case that both are solutions to the equation, they may not be solutions to the problem. If a solution does not solve the original application, then we disregard it.

Recall that consecutive odd and even integers both are separated by two units.

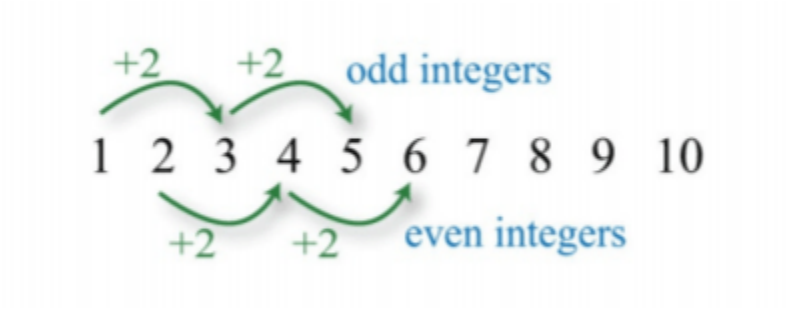


Figure 6.7.2

Example 6.7.2

The product of two consecutive positive odd integers is 99. Find the integers.

Solution:

Let n represent the first positive odd integer.

Let $n+2$ represent the next positive odd integer.

The key phase, “product...is 99,” indicates that we should multiply and set the product equal to 99.

$$n \cdot (n+2) = 99$$

Rewrite the quadratic equation in standard form and solve by factoring.

$$\begin{aligned} n^2 + 2n &= 99 \\ n^2 + 2n - 99 &= 0 \\ (n-9)(n+11) &= 0 \end{aligned}$$

$$\begin{array}{lll} n-9=0 & \text{or} & n+11=0 \\ n=9 & & n=-11 \end{array}$$

Because the problem asks for positive integers, $n = 9$ is the only solution. Back substitute to determine the next odd integer.

$$\begin{aligned} n+2 &= 9+2 \\ &= 11 \end{aligned}$$

Answer:

The consecutive positive odd integers are 9 and 11.

Example 6.7.3

Given two consecutive positive odd integers, the product of the larger and twice the smaller is equal to 70. Find the integers.

Solution:

Let n represent the smaller positive odd integer.

Let $n+2$ represent the next positive odd integer.

The key phrase “twice the smaller” can be translated to $2n$. The phrase “product...is 70” indicates that we should multiply this by the larger odd integer and set the product equal to 70.

$$(n+2) \cdot 2n = 70$$

Solve by factoring.

$$\begin{array}{ll} (n+2) \cdot 2n = 70 & \textit{Distribute.} \\ 2n^2 + 4n = 70 & \textit{Subtract 70 from both sides.} \\ 2n^2 + 4n - 70 = 0 & \textit{Factor out the GCF, 2.} \\ 2(n^2 + 2n - 35) = 0 & \textit{Factor the resulting trinomial.} \\ 2(n-5)(n+7) = 0 & \textit{Set each variable factor equal to zero.} \end{array}$$

$$\begin{array}{lll} n-5=0 & \text{or} & n+7=0 \\ n=5 & & n=-7 \end{array}$$

Because the problem asks for positive integers, $n = 5$ is the only solution.

Back substitute into $n+2$ to determine the next odd integer.

$$\begin{aligned} n+2 &= 5+2 \\ &= 7 \end{aligned}$$

Answer:

The positive odd integers are 5 and 7.

Exercise 6.7.1

The product of two consecutive positive even integers is 168. Find the integers.

Answer available at the end of this section.

Geometry Problems

When working with geometry problems, it is helpful to draw a picture. Below are some area formulas that you are expected to know. (Recall that $\pi \approx 3.14$.)

Table 6.7.1

Area of a rectangle:	$A = l \cdot w$
Area of a square:	$A = s^2$
Area of a triangle:	$A = \frac{1}{2}bh$
Area of a circle:	$A = \pi r^2$

Example 6.7.4

The floor of a rectangular room has a length that is 4 feet more than twice its width. If the total area of the floor is 240 square feet, then find the dimensions of the floor.

Solution:

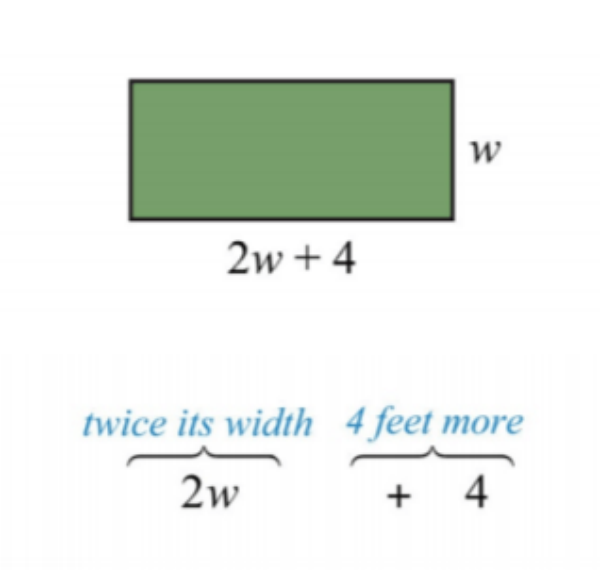


Figure 6.7.3

Use the formula $A = l \cdot w$ and the fact that the area is 240 square feet to set up an algebraic equation.

$$A = l \cdot w$$
$$240 = (2w + 4) \cdot w$$

Solve by factoring.

$$w-10=0 \quad \text{or} \quad w+12=0$$
$$w=10 \qquad\qquad w=-12$$

At this point we have two possibilities for the width of the rectangle. However, since a negative width is not defined, choose the positive solution, $w=10$. Back substitute to find the length.

$$\begin{aligned} 2w+4 &= 2(10)+4 \\ &= 20+4 \\ &= 24 \end{aligned}$$

Answer:

The width is 10 feet and the length is 24 feet.

It is important to include the correct units in the final presentation of the answer. In the previous example, it would not make much sense to say the width is 10. Make sure to indicate that the width is 10 feet.

Example 6.7.5

The height of a triangle is 3 inches less than twice the length of its base. If the total area of the triangle is 7 square inches, then find the lengths of the base and height.

Solution:

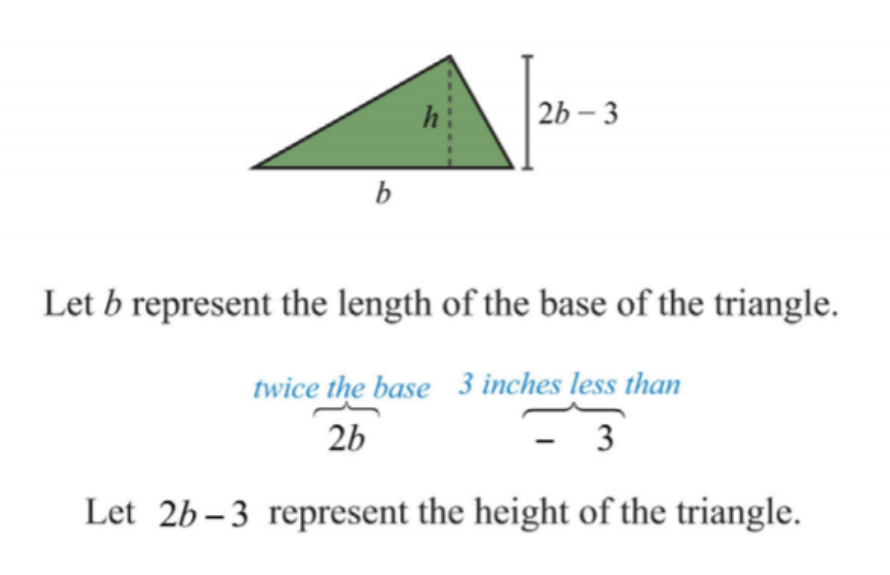


Figure 6.7.4

Use the formula $A=\frac{1}{2}bh$ and the fact that the area is 7 square inches to set up an algebraic equation.

$$\begin{aligned} A &= \frac{1}{2}b\cdot h \\ 7 &= \frac{1}{2}b(2b-3) \end{aligned}$$

To avoid fractional coefficients, multiply both sides by 2 and then rewrite the quadratic equation in standard form.

Factor and then set each factor equal to zero.

$$\begin{aligned} 2b-7=0 \quad &\text{or} \quad b+2=0 \\ 2b=7 \quad &\quad\quad b=-2 \\ b=\tfrac{7}{2} \quad &\end{aligned}$$

In this case, disregard the negative answer; the length of the base is $\frac{7}{2}$ inches long. Use $2b-3$ to determine the height of the triangle.

Answer:

The base measures $\frac{7}{2}=3\frac{1}{2}$ inches and the height is 4 inches.

Exercise 6.7.1

The base of a triangle is 5 units less than twice the height. If the area is 75 square units, then what is the length of the base and height?

Answer available at the end of this section.

Recall that a right triangle is a triangle where one of the angles measures $90°$. The side opposite of the right angle is the longest side of the triangle and is called the hypotenuse. The Pythagorean theorem gives us a relationship between the legs and hypotenuse of any right triangle, where a and b are the lengths of the legs and c is the length of the hypotenuse:

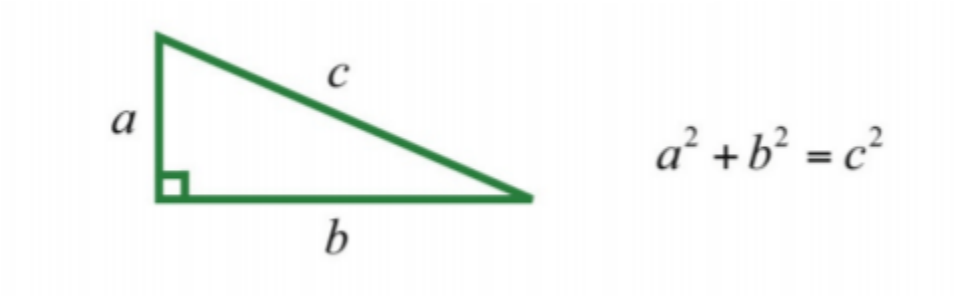


Figure 6.7.5

Given certain relationships, we use this theorem when determining the lengths of sides of right triangles.

Example 6.7.6

The hypotenuse of a right triangle is 10 inches. If the short leg is 2 inches less than the long leg, then find the lengths of the legs.

Solution:

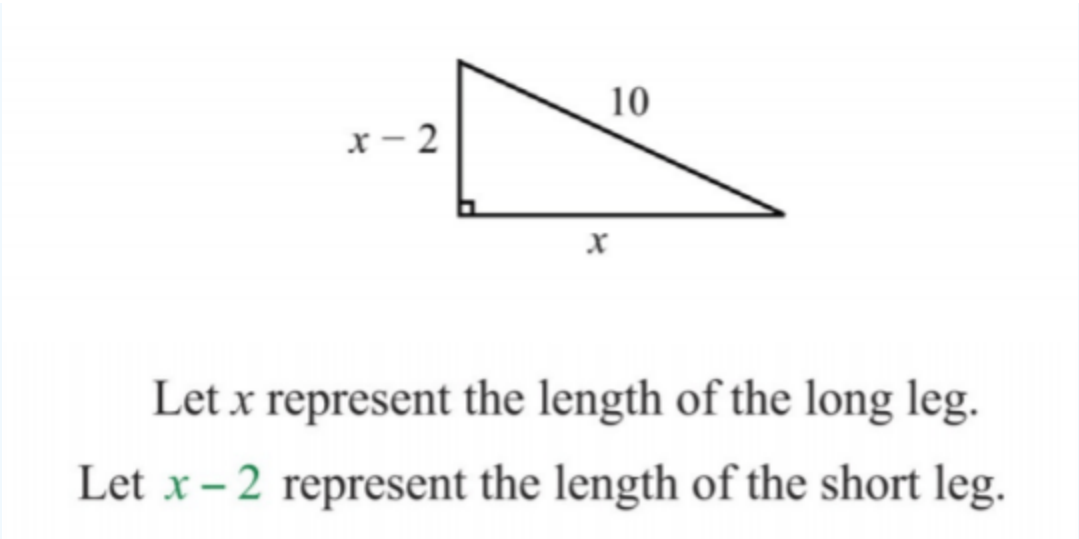


Figure 6.7.6

Given that the hypotenuse measures 10 inches, substitute its value into the Pythagorean theorem and obtain a quadratic equation in terms of x.

$$a^2 + b^2 = c^2$$
$$(x-2)^2 + x^2 = 10^2$$

Multiply and rewrite the equation in standard form.

$$(x-2)^2 + x^2 = 10^2$$
$$x^2 - 4x + 4 + x^2 = 100$$
$$2x^2 - 4x - 96 = 0$$

Once it is in standard form, factor and set each variable factor equal to zero.

$$2x^2 - 4x - 96 = 0$$
$$2(x^2 - 2x - 48) = 0$$
$$2(x+6)(x-8) = 0$$

$$x + 6 = 0 \quad \text{or} \quad x - 8 = 0$$
$$x = -6 \qquad\qquad x = 8$$

Because lengths cannot be negative, disregard the negative answer. In this case, the long leg measures 8 inches. Use $x - 2$ to determine the length of the short leg.

$$x - 2 = 8 - 2$$
$$= 6$$

Answer:

The short leg measures 6 inches and the long leg measures 8 inches.

Example 6.7.7

One leg of a right triangle measures 3 centimeters. The hypotenuse of the right triangle measures 3 centimeters less than twice the length of the unknown leg. Find the measure of all the sides of the triangle.

Solution:

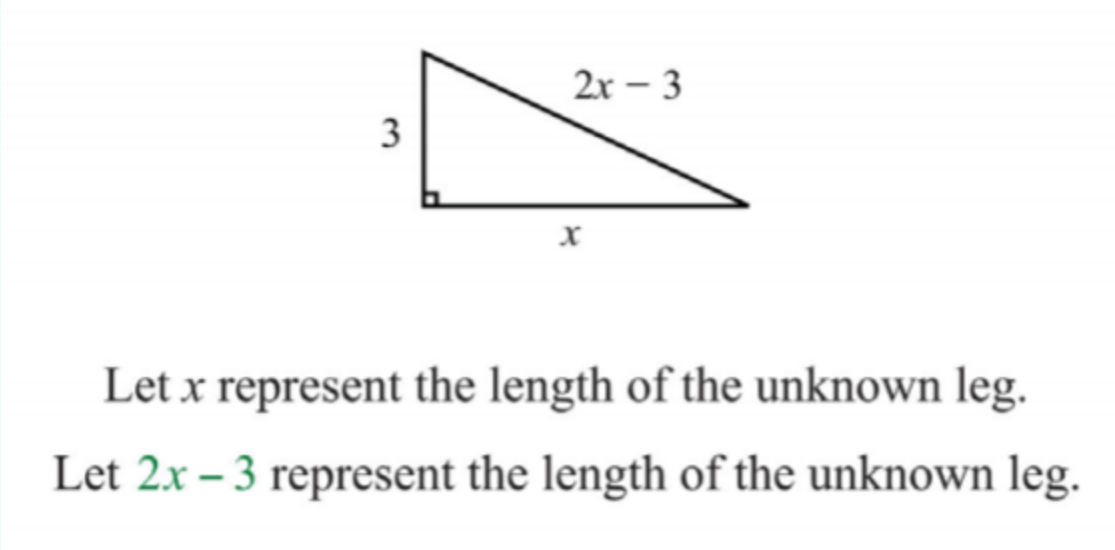


Figure 6.7.7

To set up an algebraic equation, we use the Pythagorean theorem.

$$a^2 + b^2 = c^2$$
$$3^2 + x^2 = (2x - 3)^2$$

Solve by factoring.

$$3^2 + x^2 = (2x - 3)^2$$
$$9 + x^2 = 4x^2 - 12x + 9$$
$$0 = 3x^2 - 12x$$
$$0 = 3x(x - 4)$$

$$3x = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = 0 \qquad\qquad x = 4$$

Disregard 0. The length of the unknown leg is 4 centimeters. Use $2x - 3$ to determine the length of the hypotenuse.

Answer:

The sides of the triangle measure 3 centimeters, 4 centimeters, and 5 centimeters.

Exercise 6.7.2

The hypotenuse of a right triangle measures 13 units. If one leg is 2 units more than twice that of the other, then find the length of each leg.

Answer available at the end of this section.

Projectile Problems

The height of an object launched upward, ignoring the effects of air resistance, can be modeled with the following formula:

$$\text{height} = -\frac{1}{2}gt^2 + v_0 t + s_0 \tag{6.7.1}$$

Using function notation, which is more appropriate, we have

$$h(t) = -\frac{1}{2}gt^2 + v_0t + s_0 \tag{6.7.2}$$

With this formula, the height can be calculated at any given time t after the object is launched. The coefficients represent the following:

Table 6.7.2

$-\frac{1}{2}g$	The letter g represents the acceleration due to gravity.
v_0	"v-naught" represents the initial velocity of the object.
s_0	"s-naught" represents the initial height from which the object is launched.

We consider only problems where the acceleration due to gravity can be expressed as $g = 32$ ft/sec^2. Therefore, in this section time will be measured in seconds and the height in feet. Certainly though, the formula is valid using units other than these.

Example 6.7.8

The height of a projectile launched upward at a speed of 32 feet/second from a height of 128 feet is given by the function $h(t) = -16t^2 + 32t + 128$. How long does it take to hit the ground?

Solution:

An inefficient method for finding the time to hit the ground is to simply start guessing at times and evaluating. To do this, construct a chart.

t	h		*Points*
0	128	$h(0) = -16(0)^2 + 32(0) + 128 = 128$	(0, 128)
1	144	$h(1) = -16(1)^2 + 32(1) + 128 = 144$	(1, 144)
2	128	$h(2) = -16(2)^2 + 32(2) + 128 = 128$	(2, 128)
3	80	$h(3) = -16(3)^2 + 32(3) + 128 = 80$	(3, 80)
4	0	$h(4) = -16(4)^2 + 32(4) + 128 = 0$	(4, 0)

Figure 6.7.8

Use the table to sketch the height of the projectile over time.

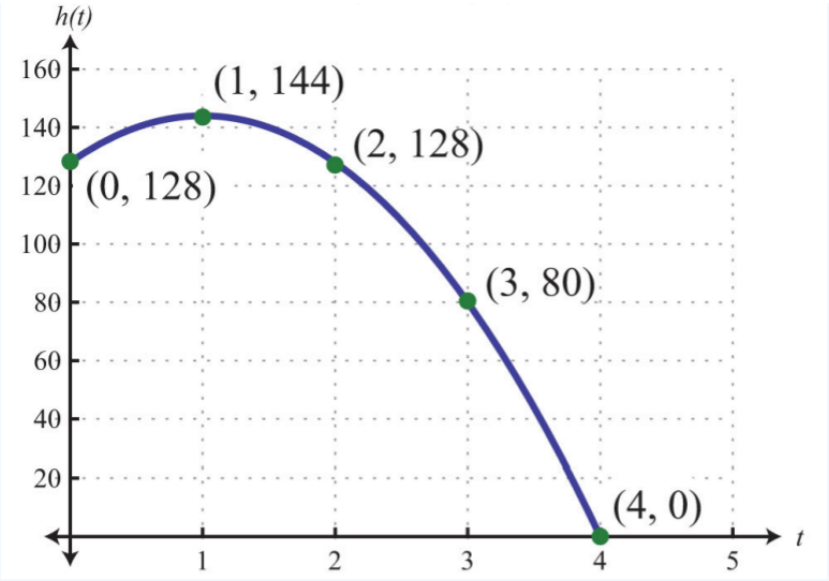


Figure 6.7.9

We see that at 4 seconds, the projectile hits the ground. Note that when this occurs, the height is equal to 0. Now we need to solve this problem algebraically. To find the solution algebraically, use the fact that the height is 0 when the object hits the ground. We need to find the time, t, when $h(t) = 0$.

$$h(t) = -16t^2 + 32t + 128 \downarrow 0 = -16t^2 + 32t + 128$$

Solve the equation by factoring

$$\begin{aligned} 0 &= -16t^2 + 32t + 128 \\ 0 &= -16(t^2 - 2t - 8) \\ 0 &= -16(t-4)(t+2) \end{aligned}$$

Now set each variable factor to zero.

$$\begin{aligned} t - 4 = 0 \quad &\text{or} \quad t + 2 = 0 \\ t = 4 \quad &\qquad t = -2 \end{aligned}$$

As expected, the projectile hits the ground at $t = 4$ seconds. Disregard -2 as a solution because negative time is not defined.

Answer:

The projectile hits the ground 4 seconds after it is launched.

Example 6.7.9

The height of a certain book dropped from the top of a 144 foot building is given by $h(t) = -16t^2 + 144$. How long does it take to hit the ground?

Solution:

Find the time t when the height $h(t) = 0$.

$$\begin{aligned} 0 &= -16t^2 + 144 \\ 0 &= -16(t^2 - 9) \\ 0 &= -16(t+3)(t-3) \end{aligned}$$

$$t+3=0 \quad \text{or} \quad t-3=0$$
$$t=-3 \qquad\qquad t=3$$

Answer:

The book takes 3 seconds to hit the ground when dropped from the top of a 144-foot building.

Exercise $6.7.3$

The height of a projectile, shot straight up into the air from the ground, is given by $h(t)=-16t^2+80t$. How long does it take to come back down to the ground?

Answer available at the end of this section.

Key Takeaways

- It is best to translate a word problem to a mathematical setup and then solve using algebra. Avoid using the "guess and check" method of solving applications in this section.
- When solving applications, check that your solutions make sense in the context of the question. For example, if you wish to find the length of the base of a triangle, then you would disregard any negative solutions.
- It is important to identify each variable and state in a sentence what each variable represents. It is often helpful to draw a picture.

Exercise $6.7.4$ Number Problems

Set up an algebraic equation and then solve.

1. One integer is five times another. If the product of the two integers is 80, then find the integers.
2. One integer is four times another. If the product of the two integers is 36, then find the integers.
3. An integer is one more than four times another. If the product of the two integers is 39, then find the integers.
4. An integer is 3 more than another. If the product of the two integers is 130, then find the integers.
5. An integer is 2 less than twice another. If the product of the two integers is 220, then find the integers.
6. An integer is 3 more than twice another. If the product of the two integers is 90, then find the integers.
7. One integer is 2 units more than another. If the product of the two integers is equal to five times the larger, then find the two integers.
8. A positive integer is 1 less than twice another. If the product of the two integers is equal to fifteen times the smaller, then find the two integers.
9. A positive integer is 3 more than twice a smaller positive integer. If the product of the two integers is equal to six times the larger, then find the integers.
10. One positive integer is 3 more than another. If the product of the two integers is equal to twelve times the smaller, then find the integers.
11. An integer is 3 more than another. If the product of the two integers is equal to 2 more than four times their sum, then find the integers.
12. An integer is 5 more than another. If the product of the two integers is equal to 2 more than twice their sum, then find the integers.
13. The product of two consecutive positive even integers is 120. Find the integers.
14. The product of two consecutive positive odd integers is 99. Find the integers.
15. The product of two consecutive positive integers is 110. Find the integers.
16. The product of two consecutive positive integers is 42. Find the integers.
17. The product of two consecutive positive odd integers is equal to 1 less than seven times the sum of the integers. Find the integers.
18. The product of two consecutive positive even integers is equal to 22 more than eleven times the sum of the integers. Find the integers.
19. The sum of the squares of two consecutive positive odd integers is 74. Find the integers.

20. The sum of the squares of two consecutive positive even integers is 100. Find the integers.
21. The sum of the squares of two consecutive positive integers is 265. Find the integers.
22. The sum of the squares of two consecutive positive integers is 181. Find the integers.
23. For two consecutive positive odd integers, the product of twice the smaller and the larger is 126. Find the integers.
24. For two consecutive positive even integers, the product of the smaller and twice the larger is 160. Find the integers.

Answer available at the end of this section.

Exercise $6.7.5$ Geometry Problems

Set up an algebraic equation and then solve.

1. The width of a rectangle is 7 feet less than its length. If the area of the rectangle is 170 square feet, then find the length and width.
2. The length of a rectangle is 2 feet more than its width. If the area of the rectangle is 48 square feet, then find the length and width.
3. The width of a rectangle is 3 units less than the length. If the area is 70 square units, then find the dimensions of the rectangle.
4. The width of a rectangle measures one half of the length. If the area is 72 square feet, then find the dimensions of the rectangle.
5. The length of a rectangle is twice that of its width. If the area of the rectangle is 72 square inches, then find the length and width.
6. The length of a rectangle is three times that of its width. If the area of the rectangle is 75 square centimeters, then find the length and width.
7. The length of a rectangle is 2 inches more than its width. The area of the rectangle is equal to 12 inches more than three times the perimeter. Find the length and width of the rectangle.
8. The length of a rectangle is 3 meters more than twice the width. The area of the rectangle is equal to 10 meters less than three times the perimeter. Find the length and width of the rectangle.
9. A uniform border is to be placed around an 8-inch-by-10-inch picture. If the total area including the border must be 224 square inches, then how wide should the border be?

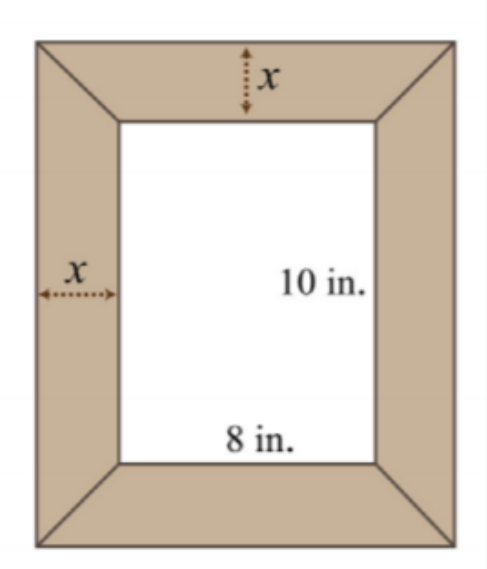


Figure $6.7.10$

10. A 2-foot brick border is constructed around a square cement slab. If the total area, including the border, is 121 square feet, then what are the dimensions of the slab?

11. The area of a picture frame including a 2-inch wide border is 99 square inches. If the width of the inner area is 2 inches more than its length, then find the dimensions of the inner area.

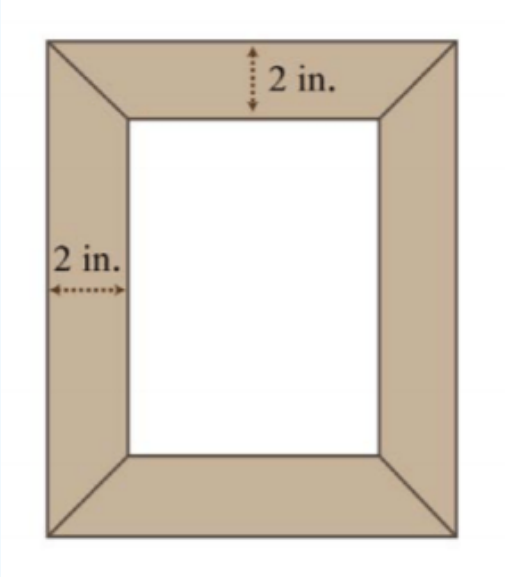


Figure $6.7.11$

12. A box can be made by cutting out the corners and folding up the edges of a square sheet of cardboard. A template for a cardboard box with a height of 2 inches is given. What is the length of each side of the cardboard sheet if the volume of the box is to be 50 cubic inches?

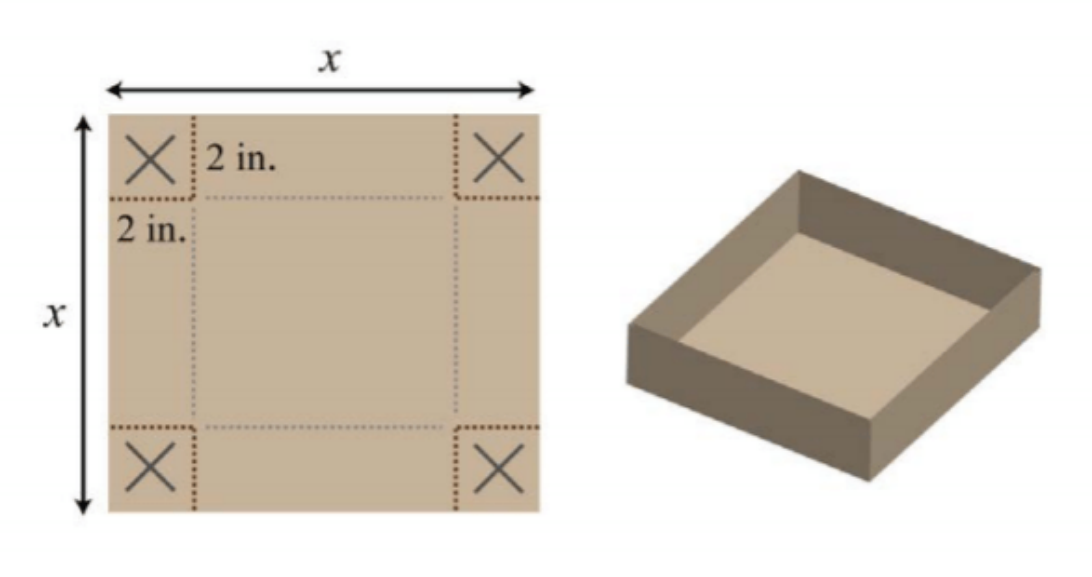


Figure $6.7.12$

13. The height of a triangle is 3 inches more than the length of its base. If the area of the triangle is 44 square inches, then find the length of its base and height.

14. The height of a triangle is 4 units less than the length of the base. If the area of the triangle is 48 square units, then find the length of its base and height.

15. The base of a triangle is twice that of its height. If the area is 36 square centimeters, then find the length of its base and height.

16. The height of a triangle is three times the length of its base. If the area is $73\frac{1}{2}$ square feet, then find the length of the base and height.

17. The height of a triangle is 1 unit more than the length of its base. If the area is 5 units more than four times the height, then find the length of the base and height of the triangle.

18. The base of a triangle is 4 times that of its height. If the area is 3 units more than five times the height, then find the length of the base and height of the triangle.

19. The diagonal of a rectangle measures 5 inches. If the length is 1 inch more than its width, then find the dimensions of the rectangle.

20. The diagonal of a rectangle measures 10 inches. If the width is 2 inches less than the length, then find the area of the rectangle.

21. If the sides of a right triangle are consecutive even integers, then what are their measures?

22. The hypotenuse of a right triangle is 13 units. If the length of one leg is 2 more than twice the other, then what are their lengths?

23. The shortest leg of a right triangle measures 9 centimeters and the hypotenuse measures 3 centimeters more than the longer leg. Find the length of the hypotenuse.

24. The long leg of a right triangle measures 24 centimeters and the hypotenuse measures 4 centimeters more three times the short leg. Find the length of the hypotenuse.

Answer available at the end of this section.

Exercise $6.7.6$ Projectile Problems

Set up an algebraic equation and then solve.

1. The height of a projectile launched upward at a speed of 32 feet/second from a height of 48 feet is given by the function \(h(t)=–16t^{2}+32t+48. How long will it take the projectile to hit the ground?
2. The height of a projectile launched upward at a speed of 16 feet/second from a height of 192 feet is given by the function $h(t) = -16t^2 + 16t + 192$. How long will it take to hit the ground?
3. An object launched upward at a speed of 64 feet/second from a height of 80 feet. How long will it take the projectile to hit the ground?
4. An object launched upward at a speed of 128 feet/second from a height of 144 feet. How long will it take the projectile to hit the ground?
5. The height of an object dropped from the top of a 64-foot building is given by $h(t) = -16t^2 + 64$. How long will it take the object to hit the ground?
6. The height of an object dropped from an airplane at $1,600$ feet is given by $h(t) = -16t^2 + 1,600$. How long will it take the object to hit the ground?
7. An object is dropped from a ladder at a height of 16 feet. How long will it take to hit the ground?
8. An object is dropped from a 144-foot building. How long will it take to hit the ground?
9. The height of a projectile, shot straight up into the air from the ground at 128 feet/second, is given by $h(t) = -16t^2 + 128t$. How long does it take to come back down to the ground?
10. A baseball, tossed up into the air from the ground at 32 feet/second, is given by $h(t) = -16t^2 + 32t$. How long does it take to come back down to the ground?
11. How long will it take a baseball thrown into the air at 48 feet/second to come back down to the ground?
12. A football is kicked up into the air at 80 feet/second. Calculate how long will it hang in the air.

Answer available at the end of this section.

Exercise 6.7.7 Discussion Board Topics

1. Research and discuss the life of Pythagoras.
2. If the sides of a square are doubled, then by what factor is the area increased? Why?
3. Design your own geometry problem involving the area of a rectangle or triangle. Post the question and a complete solution on the discussion board.
4. Write down your strategy for setting up and solving word problems. Share your strategy on the discussion board.

Answer available at the end of this section.

Answers

Exercise 6.7.1

The positive even integers are 12 and 14.

Exercise 6.7.1

The height is 10 units and the base is 15 units.

Exercise 6.7.2

The two legs measure 5 units and 12 units.

Exercise 6.7.3

It will take 5 seconds to come back down to the ground.

Exercise 6.7.4 **Number Problems**

1. $\{4, 20\}$ or $\{-4, -20\}$
3. $3, 13$
5. $\{11, 20\}$ or $\{-22, -10\}$
7. $\{5, 7\}$ or $\{-2, 0\}$
9. $6, 15$
11. $\{7, 10\}$ or $\{-2, 1\}$
13. $10, 12$
15. $10, 11$
17. $13, 15$
19. $5, 7$
21. $11, 12$
23. $7, 9$

Exercise 6.7.5 **Geometry Problems**

1. Length: 17 feet; width: 10 feet
3. Length: 10 units; width: 7 units
5. Length: 12 inches; width: 6 inches
7. Length: 14 inches; width: 12 inches
9. 3 inches
11. 5 inches by 7 inches
13. Base: 8 inches; height: 11 inches
15. Base: 12 centimeters; height: 6 centimeters
17. Base: 9 units; height: 10 units
19. 3 inches by 4 inches
21. 6 units, 8 units, and 10 units
23. 15 centimeters

Exercise 6.7.6 **Projectile Problems**

1. 3 seconds
3. 5 seconds
5. 2 seconds
7. 1 second
9. 8 seconds
11. 3 seconds

Exercise 6.7.7 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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6.E: Review Exercises and Sample Exam

Review Exercises

Exercise $6.E.1$ Introduction to Factoring

Determine the missing factor.

1. $12x^3 - 24x^2 + 4x = 4x(\quad ? \quad)$
2. $10y^4 - 35y^3 - 5y^2 = 5y^2(\quad ? \quad)$
3. $-18a^5 + 9a^4 - 27a^3 = -9a^3(\quad ? \quad)$
4. $-21x^2y + 7xy^2 - 49xy = -7xy(\quad ? \quad)$

Answer available at the end of this section.

Exercise $6.E.2$ Introduction to Factoring

Factor out the GCF.

1. $22x^2 + 11x$
2. $15y^4 - 5y^3$
3. $18a^3 - 12a^2 + 30a$
4. $12a^5 + 20a^3 - 4a$
5. $9x^3y^2 - 18x^2y^2 + 27xy^2$
6. $16a^5b^5c - 8a^3b^6 + 24a^3b^2c$

Answer available at the end of this section.

Exercise $6.E.3$ Introduction to Factoring

Factor by grouping.

1. $x^2 + 2x - 5x - 10$
2. $2x^2 - 2x - 3x + 3$
3. $x^3 + 5x^2 - 3x - 15$
4. $x^3 - 6x^2 + x - 6$
5. $x^3 - x^2y - 2x + 2y$
6. $a^2b^2 - 2a^3 + 6ab - 3b^3$

Answer available at the end of this section.

Exercise $6.E.4$ Factoring Trinomials of the Form $x^2 + bx + c$

Are the following factored correctly? Check by multiplying.

1. $x^2 + 5x + 6 = (x + 6)(x - 1)$
2. $x^2 + 3x - 10 = (x + 5)(x - 2)$
3. $x^2 + 6x + 9 = (x + 3)^2$
4. $x^2 - 6x - 9 = (x - 3)(x + 3)$

Answer available at the end of this section.

Exercise $6.E.5$ Factoring Trinomials of the Form $x^2 + bx + c$

Factor.

1. $x^2 - 13x - 14$
2. $x^2 + 13x + 12$
3. $y^2 + 10y + 25$
4. $y^2 - 20y + 100$
5. $a^2 - 8a - 48$
6. $b^2 - 18b + 45$
7. $x^2 + 2x + 24$
8. $x^2 - 10x - 16$
9. $a^2 + ab - 2b^2$
10. $a^2b^2 + 5ab - 50$

Answer available at the end of this section.

Exercise $6.E.6$ Factoring Trinomials of the Form $ax^2 + bx + c$

Factor.

1. $5x^2 - 27x - 18$
2. $3x^2 - 14x + 8$
3. $4x^2 - 28x + 49$
4. $9x^2 + 48x + 64$
5. $6x^2 - 29x - 9$
6. $8x^2 + 6x + 9$
7. $60x^2 - 65x + 15$
8. $16x^2 - 40x + 16$
9. $6x^3 - 10x^2y + 4xy^2$
10. $10x^3y - 82x^2y^2 + 16xy^3$
11. $-y^2 + 9y + 36$
12. $-a^2 - 7a + 98$
13. $16 + 142x - 18x^2$
14. $45 - 132x - 60x^2$

Answer available at the end of this section.

Exercise $6.E.7$ Factoring Special Binomials

Factor completely.

1. $x^2 - 81$
2. $25x^2 - 36$
3. $4x^2 - 49$
4. $81x^2 - 1$
5. $x^2 - 64y^2$
6. $100x^2y^2 - 1$
7. $16x^4 - y^4$
8. $x^4 - 81y^4$
9. $8x^3 - 125$
10. $27 + y^3$

11. $54x^4y - 2xy^4$
12. $3x^4y^2 + 24xy^5$
13. $64x^6 - y^6$
14. $x^6 + 1$

Answer available at the end of this section.

Exercise $6.E.8$ General Guidelines for Factoring Polynomials

Factor completely.

1. $8x^3 - 4x^2 + 20x$
2. $50a^4b^4c + 5a^3b^5c^2$
3. $x^3 - 12x^2 - x + 12$
4. $a^3 - 2a^2 - 3ab + 6b$
5. $-y^2 - 15y + 16$
6. $x^2 - 18x + 72$
7. $144x^2 - 25$
8. $3x^4 - 48$
9. $20x^2 - 41x - 9$
10. $24x^2 + 14x - 20$
11. $a^4b - 343ab^4$
12. $32x^7y^2 + 4xy^8$

Answer available at the end of this section.

Exercise $6.E.9$ Solving Equations by Factoring

Solve.

1. $(x - 9)(x + 10) = 0$
2. $-3x(x + 8) = 0$
3. $6(x + 1)(x - 1) = 0$
4. $(x - 12)(x + 4)(2x - 1) = 0$
5. $x^2 + 5x - 50 = 0$
6. $3x^2 - 13x + 4 = 0$
7. $3x^2 - 12 = 0$
8. $16x^2 - 9 = 0$
9. $(x - 2)(x + 6) = 20$
10. $2(x - 2)(x + 3) = 7x - 9$
11. $52x^2 - 203x = 0$
12. $23x^2 - 512x + 124 = 0$

Answer available at the end of this section.

Exercise $6.E.10$ Solving Equations by Factoring

Find a quadratic equation with integer coefficients, given the following solutions.

1. $-7, 6$
2. $0, -10$
3. $-\frac{1}{9}, \frac{1}{2}$
4. $\pm\frac{3}{2}$

Answer available at the end of this section.

Exercise $6.E.11$ Applications Involving Quadratic Equations

Set up an algebraic equation and then solve the following.

1. An integer is 4 less than twice another. If the product of the two integers is 96, then find the integers.
2. The sum of the squares of two consecutive positive even integers is 52. Find the integers.
3. A 20-foot ladder leaning against a wall reaches a height that is 4 feet more than the distance from the wall to the base of the ladder. How high does the ladder reach?
4. The height of an object dropped from the top of a 196-foot building is given by $h(t) = -16t^2 + 196$, where t represents the number of seconds after the object has been released. How long will it take the object to hit the ground?
5. The length of a rectangle is 1 centimeter less than three times the width. If the area is 70 square centimeters, then find the dimensions of the rectangle.
6. The base of a triangle is 4 centimeters more than twice the height. If the area of the triangle is 80 square centimeters, then find the measure of the base.

Answer available at the end of this section.

Sample Exam

Exercise $6.E.12$

1. Determine the GCF of the terms $25a^2b^2c, 50ab^4$, and $35a^3b^3c^2$.
2. Determine the missing factor: $24x^2y^3 - 16x^3y^2 + 8x^2y = 8x^2y($ $?$ $)$.

Answer available at the end of this section.

Exercise $6.E.13$

Factor.

1. $12x^5 - 15x^4 + 3x^2$
2. $x^3 - 4x^2 - 2x + 8$
3. $x^2 - 7x + 12$
4. $9x^2 - 12x + 4$
5. $x^2 - 81$
6. $x^3 + 27y^3$

Answer available at the end of this section.

Exercise $6.E.14$

Factor completely.

1. $x^3 + 2x^2 - 4x - 8$
2. $x^4 - 1$
3. $-6x^3 + 20x^2 - 6x$
4. $x^6 - 1$

Answer available at the end of this section.

Exercise $6.E.15$

Solve.

1. $(2x+1)(x-7)=0$
2. $3x(4x-3)(x+1)=0$
3. $x^2-64=0$
4. $x^2+4x-12=0$
5. $23x^2+89x-16=0$
6. $(x-5)(x-3)=-1$
7. $3x(x+3)=14x+2$
8. $(3x+1)(3x+2)=9x+3$

Answer available at the end of this section.

Exercise $6.E.16$

For each problem, set up an algebraic equation and then solve.

1. An integer is 4 less than twice another. If the product of the two integers is 70, then find the integers.
2. The sum of the squares of two consecutive positive odd integers is 130. Find the integers.
3. The length of a rectangle is 4 feet more than twice its width. If the area is 160 square feet, then find the dimensions of the rectangle.
4. The height of a triangle is 6 centimeters less than four times the length of its base. If the area measures 27 square centimeters, then what is the height of the triangle?
5. The height of a projectile launched upward at a speed of 64 feet/second from a height of 36 feet is given by the function $h(t)=-16t^2+64t+36$. How long will it take the projectile to hit the ground?

Answer available at the end of this section.

Answers

Exercise $6.E.1$ Introduction to Factoring

1. $(3x^2-6x+1)$
3. $(2a^2-a+3)$

Exercise $6.E.2$ Introduction to Factoring

1. $11x(2x+1)$
3. $6a(3a^2-2a+5)$
5. $9xy2(x^2-2x+3)$

Exercise $6.E.3$ Introduction to Factoring

1. $(x+2)(x-5)$
3. $(x+5)(x^2-3)$
5. $(x-y)(x^2-2)$

Exercise $6.E.4$ Factoring Trinomials of the Form x^2+bx+c

1. No
3. Yes

Exercise $6.E.5$ Factoring Trinomials of the Form x^2+bx+c

1. $(x-14)(x+1)$
3. $(y+5)^2$
5. $(a-12)(a+4)$
7. Prime
9. $(a-b)(a+2b)$

Exercise $6.E.6$ Factoring Trinomials of the Form ax^2+bx+c

1. $(5x+3)(x-6)$
3. $(2x-7)^2$
5. Prime

7. $5(3x-1)(4x-3)$
9. $2x(3x-2y)(x-y)$
11. $-1(y-12)(y+3)$
13. $-2(9x+1)(x-8)$

Exercise $6.E.7$ **Factoring Special Binomials**

1. $(x+9)(x-9)$
3. $(2x+7)(2x-7)$
5. $(x+8y)(x-8y)$
7. $(4x^2+y^2)(2x+y)(2x-y)$
9. $(2x-5)(4x^2+10x+25)$
11. $2xy(3x-y)(9x^2+3xy+y^2)$
13. $(2x+y)(4x^2-2xy+y^2)(2x-y)(4x^2+2xy+y^2)$

Exercise $6.E.8$ **General Guidelines for Factoring Polynomials**

1. $4x(2x^2-x+5)$
3. $(x-12)(x+1)(x-1)$
5. $-1(y+16)(y-1)$
7. $(12x+5)(12x-5)$
9. $(4x-9)(5x+1)$
11. $ab(a-7b)(a^2+7ab+49b^2)$

Exercise $6.E.9$ **Solving Equations by Factoring**

1. $9, -10$
3. $-1, 1$
5. $-10, 5$
7. ± 2
9. $-8, 4$
11. $0, \frac{8}{3}$

Exercise $6.E.10$ **Solving Equations by Factoring**

1. $x^2+x-42=0$
3. $18x^2-7x-1=0$

Exercise $6.E.11$ **Applications Involving Quadratic Equations**

1. $\{8, 12\}$ or $\{-6, -16\}$
3. 16 feet
5. Length: 14 centimeters; width: 5 centimeters

Exercise $6.E.12$

1. $5ab^2$

Exercise $6.E.13$

1. $3x^2(4x^3-5x^2+1)$
3. $(x-4)(x-3)$
5. $(x+9)(x-9)$

Exercise $6.E.14$

1. $(x+2)^2(x-2)$
3. $-2x(3x-1)(x-3)$

Exercise $6.E.15$

1. $-\frac{1}{2}, 7$
3. ± 8
5. $-\frac{3}{2}, \frac{1}{6}$
7. $-\frac{1}{3}, 2$

Exercise $6.E.16$

1. $\{7, 10\}$ or $\{-14, -5\}$
3. Width: 8 feet; length: 20 feet
5. $4\frac{1}{2}$ sec

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7: RATIONAL EXPRESSIONS AND EQUATIONS

7.1: Simplifying Rational Expressions

7.2: Multiplying and Dividing Rational Expressions

7.3: Adding and Subtracting Rational Expressions

7.4: Complex Rational Expressions

7.5: Solving Rational Equations

7.6: Applications of Rational Equations

7.7: Variation

7.8: 7.E Review Exercises and Sample Exam

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7.1: Simplifying Rational Expressions

Learning Objectives

- Determine the restrictions to the domain of a rational expression.
- Simplify rational expressions.
- Simplify expressions with opposite binomial factors.
- Simplify and evaluate rational functions.

Rational Expressions, Evaluation, and Restrictions

A rational number, or fraction $\frac{a}{b}$, is a real number defined as a quotient of two integers ***a*** and ***b***, where $b \neq 0$. Similarly, we define a rational expression, or algebraic fraction $\frac{P}{Q}$, as the quotient of two polynomials ***P*** and ***Q***, where $Q \neq 0$. Some examples of rational expressions follow:

The example $\frac{x+3}{x-5}$ consists of linear expressions in both the numerator and denominator. Because the denominator contains a variable, this expression is not defined for all values of ***x***.

Evaluate $\frac{x+3}{x-5}$ for the set of ***x***-values $\{-3, 4, 5\}$.

Solution:

Substitute the values in for ***x***.

$$\begin{array}{ccc} x=-3 & x=4 & x=5 \\ \frac{x+3}{x-5}=\frac{(-3)+3}{(-3)-5} & \frac{x+3}{x-5}=\frac{(4)+3}{(4)-5} & \frac{x+3}{x-5}=\frac{(5)+3}{(5)-5} \\ =\frac{0}{-8} & =\frac{7}{-1} & =\frac{8}{0}\ Undefined \\ =0 & =-7 & \end{array}$$

Answer:

When $x=-3$, the value of the rational expression is 0; when $x=4$, the value of the rational expression is -7; and when $x=5$, the value of the rational expression is undefined.

This example illustrates that variables are restricted to values that do not make the denominator equal to 0. The domain of a rational expression is the set of real numbers for which it is defined, and restrictions are the real numbers for which the expression is not defined. We often express the domain of a rational expression in terms of its restrictions.

Example 7.1.2

Find the domain of the following:

$$\frac{x+7}{2x^2+x-6}$$

Solution:

In this example, the numerator $x+7$ is a linear expression and the denominator $2x^2+x-6$ is a quadratic expression. If we factor the denominator, then we will obtain an equivalent expression.

$$\frac{x+7}{2x^2+x-6} = \frac{x+7}{(2x-3)(x+2)}$$

Because rational expressions are undefined when the denominator is 0, we wish to find the values for ***x*** that make it 0. To do this, apply the zero product property. Set each factor in the denominator equal to 0 and solve.

$$(2x-3)(x+2) = 0$$

$$\begin{aligned} 2x-3 &= 0 \quad \text{or} \quad x+2 = 0 \\ 2x &= 3 \qquad\qquad x = -2 \\ x &= \tfrac{3}{2} \end{aligned}$$

We conclude that the original expression is defined for any real number except $\frac{3}{2}$ and -2. These two values are the restrictions to the domain. It is important to note that -7 is ***not*** a restriction to the domain because the expression is defined as 0 when the numerator is 0.

$$\begin{aligned} \frac{x+7}{2x^2+x-6} &= \frac{(-7)+7}{2(-7)^2+(-7)-6} \\ &= \frac{0}{98-7-6} \\ &= \frac{0}{85} \\ &= 0 \end{aligned}$$

Answer:

The domain consists of any real number ***x***, where $x \neq \frac{3}{2}$ and $x \neq -2$.

We can express the domain of the previous example using notation as follows:

$Set-builder\ notation$	$Interval\ notation$
$\{x \mid x \neq -2, \frac{3}{2}\}$	$(-\infty,-2) \cup (-2,\frac{3}{2}) \cup (\frac{3}{2},\infty)$

The restrictions to the domain of a rational expression are determined by the denominator. Ignore the numerator when finding those restrictions.

Example 7.1.3

Determine the domain:

$$\frac{x^4+x^3-2x^2-x}{x^2-1}$$

Solution:

To find the restrictions to the domain, set the denominator equal to 0 and solve:

$$\begin{aligned} x^2-1 &= 0 \\ (x+1)(x-1) &= 0 \end{aligned}$$

$$\begin{aligned} x+1 &= 0 \quad \text{or} \quad x-1 = 0 \\ x &= -1 \qquad\quad x = 1 \end{aligned}$$

These two values cause the denominator to be 0. Hence they are restricted from the domain.

Answer:

The domain consists of any real number ***x***, where $x \neq \pm 1$.

Example 7.1.4

Determine the domain:

$$\frac{x^2-25}{4}$$

Solution:

There is no variable in the denominator and thus no restriction to the domain.

Answer:

The domain consists of all real numbers, **R**.

Simplifying Rational Expressions

When simplifying fractions, look for common factors that cancel. For example,

$$\frac{12}{60} = \frac{1 \cdot \cancel{12}}{5 \cdot \cancel{12}} = \frac{1}{5}$$

We say that the fraction $\frac{12}{60}$ is equivalent to $\frac{1}{5}$. Fractions are in simplest form if the numerator and denominator share no common factor other than 1. Similarly, when working with rational expressions, look for factors to cancel. For example,

$$\frac{x+4}{(x-3)(x+4)} = \frac{1 \cdot \cancel{(x+4)}}{(x-3)\cancel{(x+4)}} = \frac{1}{x-3}$$

The resulting rational expression is equivalent if it shares the same domain. Therefore, we must make note of the restrictions and write

$$\frac{x+4}{(x-3)(x+4)} = \frac{1}{x-3}, \text{ where } x \neq 3 \text{ and } x \neq -4$$

In words, $\frac{x+4}{(x-3)(x+4)}$ is equivalent to $\frac{1}{x-3}$, if $x \neq 3$ and $x \neq -4$. We can verify this by choosing a few values with which to evaluate both expressions to see if the results are the same. Here we choose $x = 7$ and evaluate as follows:

$$\begin{aligned} \frac{x+4}{(x-3)(x+4)} &= \frac{1}{x-3} \\ \frac{(7)+4}{(7-3)(7+4)} &= \frac{1}{(7)-3} \\ \frac{11}{(4)(11)} &= \frac{1}{4} \\ \frac{1}{4} &= \frac{1}{4} \quad \checkmark \end{aligned}$$

It is important to state the restrictions before simplifying rational expressions because the simplified expression may be defined for restrictions of the original. In this case, the expressions are not equivalent. Here –4 is defined for the simplified equivalent but not for the original, as illustrated below:

$$\begin{aligned} \frac{x+4}{(x-3)(x+4)} &= \frac{1}{x-3} \\ \frac{(-4)+4}{(-4-3)(-4+4)} &= \frac{1}{(-4)-3} \\ \frac{0}{(-7)(0)} &= \frac{1}{-7} \\ \frac{0}{0} &= -\frac{1}{7} \quad x \end{aligned}$$

Example 7.1.5

Simplify and state the restriction:

$$\frac{25x^2}{15x^3}$$

Solution:

In this example, the expression is undefined when x is 0.

$$\frac{25x^2}{15x^3} = \frac{25(0)^2}{15(0)^3} = \frac{0}{0} \quad Undefined$$

Therefore, the domain consists of all real numbers x, where $x \neq 0$. With this understanding, we can cancel common factors.

$$\begin{aligned} \frac{25x^2}{15x^3} &= \frac{5 \cdot \cancel{5x^2}}{3x \cdot \cancel{5x^2}} \\ &= \frac{5}{3x} \end{aligned}$$

Answer:

$\frac{5}{3x}$, where $x \neq 0$

Example 7.1.6

State the restrictions and simplify:

$$\frac{3x(x-5)}{(2x-1)(x-5)}$$

Solution:

To determine the restrictions, set the denominator equal to 0 and solve.

$$(2x+1)(x-5) = 0$$

$$\begin{aligned} 2x+1 &= 0 \quad \text{or} \quad x-5 = 0 \\ 2x &= -1 \qquad\quad x = 5 \\ x &= -\tfrac{1}{2} \end{aligned}$$

The domain consists of all real numbers except for $-\frac{1}{2}$ and 5. Next, we find an equivalent expression by canceling common factors.

$$\begin{aligned} \frac{3x(x-5)}{(2x+1)(x-5)} &= \frac{3x\ \cancel{(x-5)}}{(2x+1)\ \cancel{(x-5)}} \\ &= \frac{3x}{2x+1} \end{aligned}$$

Answer:

$\frac{3x}{2x+1}$, where $x \neq -\frac{1}{2}$ and $x \neq 5$

Typically, rational expressions are not given in factored form. If this is the case, factor first and then cancel. The steps are outlined in the following example.

Example 7.1.7

State the restrictions and simplify:

$$\frac{3x+6}{x^2+x-2}$$

Solution:

Step 1: Completely factor the numerator and denominator.

$$\frac{3x+6}{x^2+x-2} = \frac{3(x+2)}{(x-1)(x+2)}$$

Step 2: Determine the restrictions to the domain. To do this, set the denominator equal to 0 and solve.

$$(x-1)(x+2) = 0$$

$$\begin{array}{rlll} x-1 & = 0 & \text{or} & x+2 = 0 \\ x & = 1 & & x = -2 \end{array}$$

The domain consists of all real numbers except -2 and 1.

Step 3: Cancel common factors, if any.

$$\begin{aligned} \frac{3x+6}{x^2+x-2} &= \frac{3\,\cancel{(x+2)}}{(x-1)\,\cancel{(x+2)}} \\ &= \frac{3}{x-1} \end{aligned}$$

Answer:

$\frac{3}{x-1}$, where $x \neq 1$ and $x \neq -2$

Example 7.1.8

State the restrictions and simplify:

Solution:

First, factor the numerator and denominator.

Any value of ***x*** that results in a value of 0 in the denominator is a restriction. By inspection, we determine that the domain consists of all real numbers except 4 and 3. Next, cancel common factors.

$$\begin{aligned} &= \frac{\cancel{(x-3)}\,(x+10)}{(x-4)\,\cancel{(x-3)}} \\ &= \frac{x+10}{x-4} \end{aligned}$$

Answer:

$\frac{x+10}{x-4}$, where $x \neq 3$ and $x \neq 4$

It is important to remember that we can only cancel factors of a product. A common mistake is to cancel terms. For example,

State the restrictions and simplify:

$$\frac{x^2-16}{5x^2-20x}$$

Answer

$\frac{x+4}{5x}$, where $x \neq 0$ and $x \neq 4$

In some examples, we will make a broad assumption that the denominator is nonzero. When we make that assumption, we do not need to determine the restrictions.

Example 7.1.9

Simplify:

(Assume all denominators are nonzero.)

Solution:

Factor the numerator by grouping. Factor the denominator using the formula for a difference of squares.

Next, cancel common factors.

$$\begin{aligned} &= \frac{\cancel{(x+y)}\,(y-3)}{\cancel{(x+y)}\,(x-y)} \\ &= \frac{y-3}{x-y} \end{aligned}$$

Answer:

$$\frac{y-3}{x-y}$$

Opposite Binomial Factors

Recall that the opposite of the real number ***a*** is ***–a***. Similarly, we can define the opposite of a polynomial ***P*** to be ***–P***. We first consider the opposite of the binomial $a-b$:

$$-(a-b) = -a+b = b-a \tag{7.1.1}$$

This leads us to the opposite binomial property:

$$-(a-b) = (b-a) \tag{7.1.2}$$

This is equivalent to factoring out a -1.

$$(b-a) = -1(a-b) \tag{7.1.3}$$

If $a \neq b$, then we can divide both sides by $(a-b)$ and obtain the following:

$$\frac{b-a}{a-b} = -1$$

Example 7.1.10

State the restrictions and simplify:

$\frac{3-x}{x-3}$

Solution:

By inspection, we can see that the denominator is 0 if $x = 3$. Therefore, 3 is the restriction to the domain. Apply the opposite binomial property to the numerator and then cancel.

Answer:

$\frac{3-x}{x-3}$, where $x \neq 3$

Since addition is commutative, we have

$$(a + b) = (b + a) \tag{7.1.4}$$

or

$$\frac{b+a}{a+b} = 1$$

Take care not to confuse this with the opposite binomial property. Also, it is important to recall that

$$\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}$$

In other words, show a negative fraction by placing the negative sign in the numerator, in front of the fraction bar, or in the denominator. Generally, negative denominators are avoided.

Example 7.1.11

Simplify and state the restrictions:

$$\frac{4-x^2}{x^2+3x-10}$$

Solution:

Begin by factoring the numerator and denominator.

$$\begin{aligned} \frac{4-x^2}{x^2+3x-10} &= \frac{(2+x)(2-x)}{(x-2)(x+5)} && \textit{The restrictions are } x \neq 2 \textit{ and } x \neq -5. \\ &= \frac{(2+x)\cdot(-1)\,\cancel{(x-2)}}{\cancel{(x-2)}\,(x+5)} && \textit{Apply the opposite binomial property, then cancel.} \\ &= \frac{(2+x)\cdot(-1)}{(x+5)} \\ &= -\frac{2+x}{x+5} \quad \text{or} \quad = -\frac{x+2}{x+5} \end{aligned}$$

Answer:

$-\frac{x+2}{x+5}$, where $x \neq 2$ and $x \neq -5$

Exercise 7.1.2

Simplify and state the restrictions:

Answer available at the end of this section.

Rational Functions

Rational functions have the form

$$r(x) = \frac{p(x)}{q(x)} \tag{7.1.5}$$

,

where *p(x)* and *q(x)* are polynomials and *q(x)*≠0. The domain of a rational function consists of all real numbers ***x*** such that the denominator *q(x)*≠0.

Example 7.1.12

a. Simplify: $r(x) = \frac{2x^2+5x-3}{6x^2+18x}$.
b. State the domain.
c. Calculate $r(-2)$.

Solution:

a. To simplify the rational function, first factor and then cancel.

$$\begin{aligned} r(x) &= \frac{2x^2+5x-3}{6x^2+18x} \\ &= \frac{(2x-1)\cancel{(x+3)}}{6x\cancel{(x+3)}} \\ &= \frac{2x-1}{6x} \end{aligned}$$

b. To determine the restrictions, set the denominator of the original function equal to 0 and solve.

$$\begin{aligned} 6x^2+18x &= 0 \\ 6x(x+3) &= 0 \end{aligned}$$

$$\begin{aligned} 6x = 0 \quad &\text{or} \quad x+3 = 0 \\ x = 0 \quad &\quad\;\; x = -3 \end{aligned}$$

The domain consists of all real numbers ***x***, where $x \neq 0$ and $x \neq -3$.

c. Since -2 is not a restriction, substitute it for the variable ***x*** using the simplified form.

$$\begin{aligned} r(x) &= \frac{2x-1}{6x} \\ r(-2) &= \frac{2(-2)-1}{6(-2)} \\ &= \frac{-4-1}{-12} \\ &= \frac{-5}{-12} \\ &= \frac{5}{12} \end{aligned}$$

Answer:

a. $\frac{2x-1}{6x}$ b. The domain is all real numbers except 0 and -3. c. $r(-2) = \frac{5}{12}$

If a cost function $C(x)$ represents the cost of producing ***x*** units, then the average cost $c(x)$ is the cost divided by the number of units produced.

$$\textit{Average cost}: c(x) = \frac{C(x)}{x} \tag{7.1.6}$$

Example 7.1.13

The cost in dollars of producing t-shirts with a company logo is given by $C(x) = 7x + 200$, where ***x*** represents the number of shirts produced. Determine the average cost of producing

a. 40 t-shirts
b. 250 t-shirts
c. 1,000 t-shirts

Solution:

Set up a function representing the average cost.

$$c(x) = \frac{C(x)}{x} = \frac{7x+200}{x}$$

Next, calculate c(40), c(250), and c(1000).

$$c(40) = \frac{7(40) + 200}{40} = \frac{280 + 200}{40} = \frac{480}{40} = 12.00$$
$$c(250) = \frac{7(250) + 200}{250} = \frac{1750 + 200}{250} = \frac{1950}{250} = 7.80$$
$$c(1000) = \frac{7(1000) + 200}{1000} = \frac{7000 + 200}{1000} = \frac{7200}{1000} = 7.20$$

Answer:

a. If 40 t-shirts are produced, then the average cost per t-shirt is $12.00.
b. If 250 t-shirts are produced, then the average cost per t-shirt is $7.80.
c. If 1,000 t-shirts are produced, then the average cost per t-shirt is $7.20.

Key Takeaways

- Rational expressions usually are not defined for all real numbers. The real numbers that give a value of 0 in the denominator are not part of the domain. These values are called restrictions.
- Simplifying rational expressions is similar to simplifying fractions. First, factor the numerator and denominator and then cancel the common factors. Rational expressions are simplified if there are no common factors other than 1 in the numerator and the denominator.
- Simplified rational expressions are equivalent for values in the domain of the original expression. Be sure to state the restrictions if the denominators are not assumed to be nonzero.
- Use the opposite binomial property to cancel binomial factors that involve subtraction. Use $-(a-b) = b-a$ to replace factors that will then cancel. Do not confuse this with factors that involve addition, such as $(a+b) = (b+a)$.

Exercise 7.1.3 Rational Expressions

Evaluate for the given set of *x*-values.

1. $5x; -1, 0, 1$
2. $\frac{4x^3}{x^2}; -1, 0, 1$
3. $\frac{1}{x+9}; -10, -9, 0$
4. $\frac{x+6}{x-5}; -6, 0, 5$
5. $\frac{3x(x-2)}{2x-1}; 0, \frac{1}{2}, 2$
6. $\frac{9x^2-1}{x-7}; 0, \frac{1}{3}, 7$

7. $5x^2 - 9; -3, 0, 3$
8. $\frac{x^2-2}{5x^2-3x-10}; -5, -4, 5$
9. Fill in the following chart:

x	$\frac{1}{x}$
-3	
-2	
0	
2	
3	

Figure 7.1.1

10. Fill in the following chart:

x	$\frac{1}{x-2}$
-1	
0	
1	
2	
3	

Figure 7.1.2

11. Fill in the following chart:

x	$\frac{1}{x+2}$
-4	
-3	
-2	
-1	
0	

Figure 7.1.3

12. Fill in the following chart:

x	$\frac{1}{x}-2$
-2	
-1	
0	
1	
2	

Figure 7.1.4

Answer available at the end of this section.

Exercise 7.1.4 Rational Expressions

An object's weight depends on its height above the surface of earth. If an object weighs 120 pounds on the surface of earth, then its weight in pounds, *W*, *x* miles above the surface is approximated by the formula

$$W = \frac{120 \cdot 4000^2}{(4000+x)^2}$$

For each problem below, approximate the weight of a 120-pound object at the given height above the surface of earth. (1 mile = 5,280 feet)

1. 100 miles

2. 1,000 miles
3. 44,350 feet
4. 90,000 feet

Answer available at the end of this section.

Exercise 7.1.5 Rational Expressions

The price to earnings ratio (P/E) is a metric used to compare the valuations of similar publicly traded companies. The P/E ratio is calculated using the stock price and the earnings per share (EPS) over the previous 12-month period as follows:

$$\text{P/E}=\text{price per share earnings per share}$$

If each share of a company stock is priced at $22.40, then calculate the P/E ratio given the following values for the earnings per share.

1. $1.40
2. $1.21
3. What happens to the P/E ratio when earnings decrease?
4. What happens to the P/E ratio when earnings increase?

Answer available at the end of this section.

Exercise 7.1.6 Rational Expressions

State the restrictions to the domain.

1. $\frac{1}{3x}$
2. $\frac{3x^2}{7x^5}$
3. $\frac{3x(x+1)}{x+4}$
4. $\frac{2x^2(x-3)}{x-1}$
5. $\frac{1}{5x-1}$
6. $\frac{x-2}{3x-2}$
7. $\frac{x-9}{5x(x-2)}$
8. $\frac{1}{(x-3)(x+6)}$
9. $\frac{x}{1-x^2}$
10. $\frac{x^2-9}{x^2-36}$
11. $\frac{1}{2x(x+3)(2x-1)}$
12. $\frac{x-3}{(3x-1)(2x+3)}$
13. $\frac{4x(2x+1)}{12x^2+x-1}$
14. $\frac{x-5}{3x^2-15x}$

Answer available at the end of this section.

Exercise 7.1.7 Simplifying Rational Expressions

State the restrictions and then simplify.

1. $\frac{5x^2}{20x^3}$
2. $\frac{12x^6}{60x}$
3. $\frac{3x^2(x-2)}{9x(x-2)}$
4. $\frac{20(x-3)(x-5)}{6(x-3)(x+1)}$

5. $\frac{6x^2(x-8)}{36x(x+9)(x-8)}$
6. $\frac{16x^2-1}{(4x+1)^2}$
7. $\frac{9x^2-6x+1}{(3x-1)^2}$
8. $\frac{x-7}{x^2-49}$
9. $\frac{x^2-64}{x^2+8x}$
10. $\frac{x+10}{x^2-100}$
11. $\frac{2x^3-12x^2}{5x^2-30x}$
12. $\frac{30x^5+60x^4}{2x^3-8x}$
13. $\frac{2x-12}{x^2+x-6}$
14. $\frac{x^2-x-6}{3x^2-8x-3}$
15. $\frac{6x^2-25x+25}{3x^2+16x-35}$
16. $\frac{3x^2+4x-1}{5x^2-9}$
17. $\frac{x^2-10x+21}{x^2-4x-21}$
18. $\frac{x^3-1}{x^2-1}$
19. $\frac{x^3+8}{x^2-4}$
20. $\frac{x^4-1}{6x^2-4}$

Answer available at the end of this section.

Exercise 7.1.8 Simplifying Rational Expressions with Opposite Binomial Factors

State the restrictions and then simplify.

1. $\frac{x-9}{9-x}$
2. $\frac{3x-2}{2-3x}$
3. $\frac{x+6}{6+x}$
4. $\frac{3x+1}{1+3x}$
5. $\frac{(2x-5)(x-7)}{(7-x)(2x-1)}$
6. $\frac{(3x+2)(x+5)}{(x-5)(2+3x)}$
7. $\frac{x^2-4}{(2-x)^2}$
8. $\frac{16-9x^2}{(3x+4)^2}$
9. $\frac{4x^2(10-x)}{3x^3-300x}$
10. $-\frac{2x+1}{4x^3-49x}$
11. $\frac{2x^2-7x-4}{1-4x^2}$
12. $\frac{9x^2-4}{4x-6x^2}$
13. $\frac{x^2-5x-14}{7-15x+2x^2}$
14. $\frac{2x^3+x^2-2x-1}{1+x-2x^2}$
15. $\frac{x^3+2x-3x^2-6}{2+x^2}$
16. $\frac{27+x^3}{x^2+6x+9}$
17. $\frac{64-x^3}{x^2-8x+16}$
18. $\frac{x^2+4}{4-x^2}$

Answer available at the end of this section.

Exercise 7.1.9 Simplifying Rational Expressions with Opposite Binomial Factors

Simplify. (Assume all denominators are nonzero.)

1. $-\frac{15x^3y^2}{5xy^2(x+y)}$
2. $\frac{14x^7y^2(x-2y)^4}{7x^8y(x-2y)^2}$

3. $\frac{y+x}{x^2-y^2}$
4. $\frac{y-x}{x^2-y^2}$
5. $\frac{x^2-y^2}{(x-y)^2}$
6. $\frac{a^2-ab-6b^2}{a^2-6ab+9b^2}$
7. $\frac{2a^2-11a+12}{-32+2a^2}$
8. $\frac{a^2b-3a^2}{3a^2-3ab}$
9. $\frac{xy^2-x+y^3-y}{x-xy^2}$
10. $\frac{x^3-xy^2-x^2y+y}{3x^2-2xy+y^2}$
11. $\frac{x^3-27}{x^2+3x+9}$
12. $\frac{x^2-x+1}{x^3+1}$

Answer available at the end of this section.

Exercise 7.1.10 Rational Functions

Calculate the following.

1. $f(x)=\frac{5x}{x-3}; f(0), f(2), f(4)$
2. $f(x)=\frac{x+7}{x^2+1}; f(-1), f(0), f(1)$
3. $g(x)=\frac{x^3}{(x-2)^2}; g(0), g(2), g(-2)$
4. $g(x)=\frac{x^2-9}{9-x^2}; g(-2), g(0), g(2)$
5. $g(x)=\frac{x^3}{x^2+1}; g(-1), g(0), g(1)$
6. $g(x)=\frac{5x+1}{x^2-25}; g(-\frac{1}{5}), g(-1), g(-5)$

Answer available at the end of this section.

Exercise 7.1.11 Rational Functions

State the restrictions to the domain and then simplify.

1. $f(x)=-\frac{3x^2-6x}{x^2+4x+4}$
2. $f(x)=\frac{x^2+6x+9}{2x^2+5x-3}$
3. $g(x)=\frac{9-x}{x^2-81}$
4. $g(x)=\frac{x^3-27}{3-x}$
5. $g(x)=\frac{3x-15}{10-2x}$
6. $g(x)=\frac{25-5x}{4x-20}$
7. The cost in dollars of producing coffee mugs with a company logo is given by $C(x)=x+40$, where x represents the number of mugs produced. Calculate the average cost of producing 100 mugs and the average cost of producing 500 mugs.
8. The cost in dollars of renting a moving truck for the day is given by $C(x)=0.45x+90$, where x represents the number of miles driven. Calculate the average cost per mile if the truck is driven 250 miles in one day.
9. The cost in dollars of producing sweat shirts with a custom design on the back is given by $C(x)=1200+(12-0.05x)x$, where x represents the number of sweat shirts produced. Calculate the average cost of producing 150 custom sweat shirts.
10. The cost in dollars of producing a custom injected molded part is given by $C(x)=500+(3-0.001x)x$, where x represents the number of parts produced. Calculate the average cost of producing 1,000 custom parts.

Answer available at the end of this section.

Exercise 7.1.12 Discussion Board

1. Explain why $\frac{b-a}{a-b} = -1$ and illustrate this fact by substituting some numbers for the variables.
2. Explain why $\frac{b+a}{a+b} = 1$ and illustrate this fact by substituting some numbers for the variables.
3. Explain why we cannot cancel x in the expression $\frac{x}{x+1}$.

Answer available at the end of this section.

Answers

Exercise 7.1.2

$\frac{-2x+3}{x+5}$**, where** $x \neq \pm 5$

Exercise 7.1.3 **Rational Expressions**

1. -5, undefined, 5
3. -1, undefined, $\frac{1}{9}$
5. 0, undefined, 0
7. Undefined, $-\frac{5}{9}$, undefined
9.

x	$\frac{1}{x}$
-3	$-1/3$
-2	$-1/2$
0	undefined
2	$1/2$
3	$1/3$

Figure 7.1.5

11.

x	$\frac{1}{x+2}$
-4	$-1/2$
-3	-1
-2	undefined
-1	1
0	$1/2$

Figure 7.1.6

Exercise 7.1.4 **Rational Expressions**

1. 114 pounds
3. 119.5 pounds

Exercise 7.1.5 **Rational Expressions**

1. 16
3. The P/E ratio increases.

Exercise 7.1.6 **Rational Expressions**

1. $x \neq 0$
3. $x \neq -4$
5. $x \neq \frac{1}{5}$
7. $x \neq 0$ and $x \neq 2$
9. $x \neq \pm 1$
11. $x \neq 0$, $x \neq -3$, and $x \neq \frac{1}{2}$
13. $x \neq -\frac{1}{3}$ and $x \neq \frac{1}{4}$

Exercise 7.1.7 **Simplifying Rational Expressions**

1. $\frac{1}{4x}; x \neq 0$
3. $\frac{x}{3}; x \neq 0, 2$
5. $\frac{x}{6(x+9)}; x \neq 0, -9, 8$
7. $1; x \neq \frac{1}{3}$
9. $\frac{x-8}{x}; x \neq 0, -8$
11. $\frac{2x}{5}; x \neq 0, 6$
13. $\frac{2x-12}{x^2+x-6}$ $x \neq -2, \frac{3}{2}$
15. $\frac{2x-5}{x+7}; x \neq -7, \frac{5}{3}$
17. $\frac{x-3}{x+3}; x \neq -3, 7$
19. $\frac{x^2-2x+4}{x-2}; x \neq \pm 2$

Exercise 7.1.8 **Simplifying Rational Expressions with Opposite Binomial Factors**

1. $-1; x \neq 9$
3. $1; x \neq -6$
5. $\frac{-2x-5}{2x-1}; x \neq \frac{1}{2}, 7$
7. $\frac{x+2}{x-2}; x \neq 2$
9. $-\frac{4x}{3(x+10)}; x \neq \pm 10, 0$
11. $\frac{x-4}{1-2x}; x \neq \pm \frac{1}{2}$
13. $\frac{x+2}{2x-1}; x \neq \frac{1}{2}, 7$
15. $x - 3$; none
17. $\frac{-16+4x+x^2}{x-4}; x \neq 4$

Exercise 7.1.9 **Simplifying Rational Expressions with Opposite Binomial Factors**

1. $-\frac{3x^2}{x+y}$
3. $\frac{1}{x-y}$

5. $\frac{x+y}{x-y}$
7. $\frac{2a-3}{2(a+4)}$
9. $-\frac{x+y}{x}$
11. $x-3$

Exercise 7.1.10 **Rational Functions**

1. f(0)=0, f(2)=–10, f(4)=20
3. g(0)=0, g(2) undefined, g(–2)=$-\frac{1}{2}$
5. g(–1)=$-\frac{1}{2}$, g(0)=0, g(1)=$\frac{1}{2}$

Exercise 7.1.11 **Rational Functions**

1. $f(x)=-\frac{3x}{x+2}; x\neq -2$
3. $g(x)=-\frac{1}{x+9}; x\neq \pm 9$
5. $g(x)=-\frac{3}{2}; x\neq 5$
7. The average cost of producing 100 mugs is $1.40 per mug. The average cost of producing 500 mugs is $1.08 per mug.
9. $12.50

Exercise 7.1.12 **Discussion Board**

1. Answers may vary
3. Answers may vary

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7.2: Multiplying and Dividing Rational Expressions

Learning Objectives

- Multiply rational expressions.
- Divide rational expressions.
- Multiply and divide rational functions.

Multiplying Rational Expressions

When multiplying fractions, we can multiply the numerators and denominators together and then reduce, as illustrated:

$$\frac{3}{5}\cdot\frac{5}{9}=\frac{3\cdot 5}{5\cdot 9}=\frac{\overset{1}{\cancel{3}}\cdot\overset{1}{\cancel{5}}}{\underset{1}{\cancel{5}}\cdot\underset{3}{\cancel{9}}}=\frac{1}{3}$$

Multiplying rational expressions is performed in a similar manner. For example,

$$\frac{y}{x}\cdot\frac{x}{y^2}=\frac{y\cdot x}{x\cdot y^2}=\frac{\overset{1}{\cancel{y}}\cdot\overset{1}{\cancel{x}}}{\underset{1}{\cancel{x}}\cdot\underset{3}{\cancel{y^2}}}=\frac{1}{y}$$

In general, given polynomials P, Q, R, and S, where $Q\neq 0$ and $S\neq 0$, we have

$$\frac{P}{Q}\cdot\frac{R}{S}=\frac{PR}{QS} \tag{7.2.1}$$

In this section, assume that all variable expressions in the denominator are nonzero unless otherwise stated.

Example 7.2.1

Multiply:

$$\frac{12x^2}{5y^3}\cdot\frac{20y^4}{6x^3}$$

Solution:

Multiply numerators and denominators and then cancel common factors.

$$\begin{aligned}\frac{12x^2}{5y^3}\cdot\frac{20y^4}{6x^3}&=\frac{240x^2y^4}{30x^3y^3} && \textit{Multiply.}\\ &=\frac{\overset{8}{\cancel{240}}\ \overset{1}{\cancel{x^2}}\ \overset{y}{\cancel{y^4}}}{\underset{1}{\cancel{30}}\ \underset{x}{\cancel{x^3}}\ \underset{1}{\cancel{y^3}}} && \textit{Cancel.}\\ &=\frac{8y}{x}\end{aligned}$$

Answer:

$$\frac{8y}{x}$$

Example 7.2.2

Multiply:

$$\frac{x-3}{x+5}\cdot\frac{x+5}{x+7}$$

Solution:

Leave the product in factored form and cancel the common factors.

$$\begin{aligned}\frac{x-3}{x+5}\cdot\frac{x+5}{x+7} &= \frac{(x-3)\cdot\cancel{(x+5)}}{\cancel{(x+5)}\cdot(x+7)}\\ &= \frac{x-3}{x+7}\end{aligned}$$

Answer:

$$\frac{x-3}{x+7}$$

Example 7.2.3

Multiply:

$$\frac{15x^2y^3}{(2x-1)}\cdot\frac{x(2x-1)}{3x^2y(x+3)}$$

Solution:

Leave the polynomials in the numerator and denominator factored so that we can cancel the factors. In other words, do not apply the distributive property.

$$\begin{aligned}\frac{15x^2y^3}{(2x-1)}\cdot\frac{x(2x-1)}{3x^2y(x+3)} &= \frac{15x^3y^3(2x-1)}{3x^2y(2x-1)(x+3)} && \textit{Multiply.}\\ &= \frac{\cancel{15}^{5}\,\cancel{x^3}^{x}\,\cancel{y^3}^{y^2}\,\cancel{(2x-1)}^{1}}{\cancel{3}\,\cancel{x^2}\,\cancel{y}\,\cancel{(2x-1)}\,(x+3)} && \textit{Cancel.}\\ &= \frac{5xy^2}{x+3}\end{aligned}$$

Answer:

$$\frac{5xy^2}{x+3}$$

Typically, rational expressions will not be given in factored form. In this case, first factor all numerators and denominators completely. Next, multiply and cancel any common factors, if there are any.

Example 7.2.4

Multiply:

$$\frac{x+5}{x-5}\cdot\frac{x-5}{x^2-25}$$

Solution

Factor the denominator $x^2 - 25$ as a difference of squares. Then multiply and cancel.

$$\begin{aligned} \frac{x+5}{x-5} \cdot \frac{x-5}{x^2-25} &= \frac{x+5}{x-5} \cdot \frac{x-5}{(x+5)(x-5)} && \textit{Factor.} \\ &= \frac{\overset{1}{\cancel{(x+5)}}\ \overset{1}{\cancel{(x-5)}}}{\cancel{(x-5)}\ \cancel{(x+5)}\ (x-5)} && \textit{Cancel.} \\ &= \frac{1}{x-5} \end{aligned}$$

Keep in mind that 1 is always a factor; so when the entire numerator cancels out, make sure to write the factor 1.

Answer:

$$\frac{1}{x-5}$$

Example 7.2.5

Multiply:

$$\frac{x^2+3x+2}{x^2-5x+6} \cdot \frac{x^2-7x+12}{x^2+8x+7} \tag{7.2.2}$$

Solution:

$$\begin{aligned} \frac{x^2+3x+2}{x^2-5x+6} \cdot \frac{x^2-7x+12}{x^2+8x+7} &= \frac{(x+1)(x+2)}{(x-2)(x-3)} \cdot \frac{(x-3)(x-4)}{(x+1)(x+7)} \\ &= \frac{(x+1)(x+2)(x-3)(x-4)}{(x-2)(x-3)(x+1)(x+7)} \\ &= \frac{(x+2)(x-4)}{(x-2)(x+7)} \end{aligned} \tag{7.2.3}$$

It is a best practice to leave the final answer in factored form.

Answer:

$$\frac{(x+2)(x-4)}{(x-2)(x+7)}$$

Example 7.2.6

Multiply:

$$\frac{-2x^2+x+3}{x^2+2x-8} \cdot \frac{3x-6}{x^2+x}. \tag{7.2.4}$$

Solution:

The trinomial $-2x^2 + x + 3$ in the numerator has a negative leading coefficient. Recall that it is a best practice to first factor out a -1 and then factor the resulting trinomial.

$$\begin{aligned}\frac{-2x^2+x+3}{x^2+2x-8}\cdot\frac{3x-6}{x^2+x} &= \frac{-1\left(2x^2-x-3\right)}{x^2+2x-8}\cdot\frac{3(x-2)}{x(x+1)}\\ &= \frac{-1(2x-3)(x+1)}{(x-2)(x+4)}\cdot\frac{3(x-2)}{x(x+1)}\\ &= \frac{-1(2x-3)(x+1)\cdot 3(x-2)}{(x-2)(x+4)\cdot x(x+1)}\\ &= \frac{-1(2x-3)\cdot 3}{(x+4)\cdot x}\\ &= \frac{-3(2x-3)}{x(x+4)} \quad \text{or} \quad = -\frac{3(2x-3)}{x(x+4)}\end{aligned} \tag{7.2.5}$$

Answer:

$$-\frac{3(2x-3)}{x(x+4)}$$

Example 7.2.7

Multiply:

$$\frac{7-x}{x^2+3x}\cdot\frac{x^2+10x+21}{x^2-49}$$

Solution:

We replace $7-x$ with $-1(x-7)$ so that we can cancel this factor.

$$\begin{aligned}\frac{7-x}{x^2+3x}\cdot\frac{x^2+10x+21}{x^2-49} &= \frac{-1(x-7)}{x(x+3)}\cdot\frac{(x+3)(x+7)}{(x+7)(x-7)}\\ &= \frac{-1\ \cancel{(x-7)}\ \cancel{(x+3)}\ \cancel{(x+7)}}{x\ \cancel{(x+3)}\ \cancel{(x+7)}\ \cancel{(x-7)}}\\ &= \frac{-1}{x}\\ &= -\frac{1}{x}\end{aligned}$$

Answer:

$$-\frac{1}{x}$$

Exercise 7.2.1

Multiply:

$$\frac{x^2-64}{8-x}\cdot\frac{x+x^2}{x^2+9x+8}$$

Answer available at the end of this section.

Dividing Rational Expressions

To divide two fractions, we multiply by the reciprocal of the divisor, as illustrated:

$$\frac{5}{8} \div \frac{1}{2} = \frac{5}{8} \cdot \frac{2}{1} = \frac{5 \cdot \cancel{2}^{1}}{\cancel{8}^{4} \cdot 1} = \frac{5}{4}$$

Dividing rational expressions is performed in a similar manner. For example,

$$\frac{x}{y^2} \div \frac{1}{y} = \frac{x}{y^2} \cdot \frac{y}{1} = \frac{x \cdot \cancel{y}^{1}}{\cancel{y^2}^{y} \cdot 1} = \frac{x}{y}$$

In general, given polynomials ***P***, ***Q***, ***R***, and **S**, where $Q \neq 0$, $R \neq 0$, and $S \neq 0$, we have

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{PS}{QR} \tag{7.2.6}$$

Example 7.2.8

Divide:

$$\frac{8x^5y}{25z^6} \div \frac{20xy^4}{15z^3}$$

Solution:

First, multiply by the reciprocal of the divisor and then cancel.

$$\begin{aligned} \frac{8x^5y}{25z^6} \div \frac{20xy^4}{15z^3} &= \frac{8x^5y}{25z^6} \cdot \frac{15z^3}{20xy^4} && \textit{Multiply by the reciprocal of the divisor.} \\ &= \frac{120x^5yz^3}{500xy^4z^6} \\ &= \frac{\cancel{120}^{6}\,\cancel{x^5}^{x^4}\,\cancel{y}\,\cancel{z^3}}{\cancel{500}^{25}\,\cancel{x}\,\cancel{y^4}^{y^3}\,\cancel{z^6}^{z^3}} && \textit{Cancel.} \\ &= \frac{6x^4}{25y^3z^3} \end{aligned}$$

Answer:

$$\frac{6x^4}{25y^3z^3}$$

Example 7.2.9

Divide:

$$\frac{x+2}{x^2-4} \div \frac{x+3}{x-2}$$

Solution:

After multiplying by the reciprocal of the divisor, factor and cancel.

$$\begin{aligned}\frac{x+2}{x^2-4} \div \frac{x+3}{x-2} &= \frac{x+2}{x^2-4} \cdot \frac{x-2}{x+3} && \textit{Multiply by the reciprocal of the divisor.}\\ &= \frac{(x+2)}{(x+2)(x-2)} \cdot \frac{(x-2)}{(x+3)} && \textit{Factor.}\\ &= \frac{\cancel{(x+2)}\ \cancel{(x-2)}}{\cancel{(x+2)}\ \cancel{(x-2)}\ (x+3)} && \textit{Cancel.}\\ &= \frac{1}{x+3}\end{aligned}$$

Answer:

$$\frac{1}{x+3}$$

Example 7.2.10

Divide:

$$\frac{x^2-6x-16}{x^2+4x-21} \div \frac{x^2+9x+14}{x^2-8x+15} \tag{7.2.7}$$

Solution:

Begin by multiplying by the reciprocal of the divisor. After doing so, factor and cancel.

$$\begin{aligned}\frac{x^2-6x-16}{x^2+4x-21} \div \frac{x^2+9x+14}{x^2-8x+15} &= \frac{x^2-6x-16}{x^2+4x-21} \cdot \frac{x^2-8x+15}{x^2+9x+14}\\ &= \frac{(x+2)(x-8)}{(x-3)(x+7)} \cdot \frac{(x-3)(x-5)}{(x+2)(x+7)}\\ &= \frac{(x+2)(x-8)(x-3)(x-5)}{(x-3)(x+7)(x+2)(x+7)}\\ &= \frac{(x-8)(x-5)}{(x+7)^2}\end{aligned} \tag{7.2.8}$$

Answer:

$$\frac{(x-8)(x-5)}{(x+7)^2}$$

Example 7.2.11

Divide: $\dfrac{9-4x^2}{x+2} \div (2x-3)$

Solution:

Just as we do with fractions, think of the divisor $(2x-3)$ as an algebraic fraction over 1.

$$\begin{aligned}\frac{9-4x^2}{x+2} \div (2x-3) &= \frac{9-4x^2}{x+2} \div \frac{2x-3}{1}\\ &= \frac{(3+2x)(3-2x)}{x+2} \cdot \frac{1}{2x-3}\\ &= \frac{(3+2x)\cdot(-1)(2x-3)}{(x+2)(2x-3)}\\ &= -\frac{3+2x}{x+2} \text{ or } = -\frac{2x+3}{x+2}\end{aligned}$$

Answer:

$$-\frac{2x+3}{x+2}$$

Exercise 7.2.2

Divide: $\frac{4x^2+7x-2}{25x^2} \div \frac{1-4x}{100x^4}$

Answer available at the end of this section.

Multiplying and Dividing Rational Functions

The product and quotient of two rational functions can be simplified using the techniques described in this section. The restrictions to the domain of a product consist of the restrictions of each function.

Example 7.2.12

Calculate $(f \cdot g)(x)$ and determine the restrictions to the domain.

$$f(x) = \frac{16x^2-1}{5x}, g(x) = \frac{1}{1-4x} \tag{7.2.9}$$

Solution:

In this case, the domain of $f(x)$ consists of all real numbers except 0, and the domain of $g(x)$ consists of all real numbers except $\frac{1}{4}$. Therefore, the domain of the product consists of all real numbers except 0 and $\frac{1}{4}$. Multiply the functions and then simplify the result.

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x) \\ &= \frac{16x^2-1}{5x} \cdot \frac{1}{1-4x} \\ &= \frac{(4x+1)(4x-1)}{5x} \cdot \frac{1}{1-4x} \\ &= \frac{(4x+1)(4x-1)}{5x \cdot (-1)(4x-1)} \\ &= \frac{4x+1}{-5x} \\ &= -\frac{4x+1}{5x}\end{aligned} \tag{7.2.10}$$

Answer:

$$(f \cdot g)(x) = -\frac{4x+1}{5x}, \text{ where } x \neq 0, \frac{1}{4}$$

The restrictions to the domain of a quotient will consist of the restrictions of each function as well as the restrictions on the reciprocal of the divisor.

Example 7.2.13

Calculate $(f/g)(x)$ and determine the restrictions.

$$f(x)=\frac{x^2-64}{x^2-11x+24}, g(x)=\frac{x+8}{x-3} \tag{7.2.11}$$

Solution:

$$\begin{aligned}(f/g)(x) &= f(x)\div g(x)\\ &= \frac{x^2-64}{x^2-11x+24}\div\frac{x+8}{x-3}\\ &= \frac{x^2-64}{x^2-11x+24}\cdot\frac{x-3}{x+8}\\ &= \frac{(x+8)(x-8)}{(x-3)(x-8)}\cdot\frac{(x-3)}{(x+8)}\\ &= \frac{(x+8)(x-8)(x-3)}{(x-3)(x-8)(x+8)}\\ &= \frac{1}{1}\\ &= 1\end{aligned} \tag{7.2.12}$$

In this case, the domain of $f(x)$ consists of all real numbers except 3 and 8, and the domain of $g(x)$ consists all real numbers except 3. In addition, the reciprocal of $g(x)$ has a restriction of –8. Therefore, the domain of this quotient consists of all real numbers except 3, 8, and –8.

Answer:

$(f/g)(x)=1$, where $x\neq 3,8,-8$

Key Takeaways

- After multiplying rational expressions, factor both the numerator and denominator and then cancel common factors. Make note of the restrictions to the domain. The values that give a value of 0 in the denominator are the restrictions.
- To divide rational expressions, multiply by the reciprocal of the divisor.
- The restrictions to the domain of a product consist of the restrictions to the domain of each factor.
- The restrictions to the domain of a quotient consist of the restrictions to the domain of each rational expression as well as the restrictions on the reciprocal of the divisor.

Exercise 7.2.2 Multiplying Rational Expressions

Multiply. (Assume all denominators are nonzero.)

1. $\frac{2x}{3}\cdot\frac{9}{4x^2}$
2. $-\frac{5x}{3y}\cdot\frac{y^2}{25x}$
3. $\frac{5x^2}{2y}\cdot\frac{4y^2}{15x^3}$
4. $\frac{16a^4}{7b^2}\cdot\frac{49b^3}{2a^3}$
5. $\frac{x-6}{12x^3}\cdot\frac{24x^2}{x-6}$
6. $\frac{x+10}{2x-1}\cdot\frac{x-2}{x+10}$

7. $\frac{(y-1)^2}{y+1} \cdot \frac{1}{y-1}$

8. $\frac{y^2-9}{y+3} \cdot \frac{2y-3}{y-3}$

9. $\frac{2a-5}{a-5} \cdot \frac{2a+5}{4a^2-25}$

10. $\frac{2a^2-9a+4}{a^2-16} \cdot (a^2+4a)$

11. $\frac{2x^2+3x-2}{(2x-1)^2} \cdot \frac{2x}{x+2}$

12. $\frac{9x^2+19x+2}{4-x^2} \cdot \frac{x^2-4x+4}{9x^2-8x-1}$

13. $\frac{x^2+8x+16}{16-x^2} \cdot \frac{x^2-3x-4}{x^2+5x+4}$

14. $\frac{x^2-x-2}{x^2+8x+7} \cdot \frac{x^2+2x-15}{x^2-5x+6}$

15. $\frac{x+1}{x-3} \cdot \frac{3-x}{x+5}$

16. $\frac{2x-1}{x-1} \cdot \frac{x+6}{1-2x}$

17. $\frac{9+x}{3x+1} \cdot \frac{3}{x+9}$

18. $\frac{1}{2+5x} \cdot \frac{5x+2}{5x}$

19. $\frac{100-y^2}{y-10} \cdot \frac{25y^2}{y+10}$

20. $\frac{3y^3}{6y-5} \cdot \frac{36y^2-25}{5+6y}$

21. $\frac{3a^2+14a-5}{a^2+1} \cdot \frac{3a+1}{1-9a^2}$

22. $\frac{4a^2-16a}{4a-1} \cdot \frac{1-16a^2}{4a^2-15a-4}$

23. $\frac{x+9}{-x^2+14x-45} \cdot (x^2-81)$

24. $\frac{1}{2+5x} \cdot (25x^2+20x+4)$

25. $\frac{x^2+x-6}{3x^2+15x+18} \cdot \frac{2x^2-8}{x^2-4x+4}$

26. $\frac{5x^2-4x-1}{5x^2-6x+1} \cdot \frac{25x^2-10x+1}{3-75x^2}$

Answer available at the end of this section.

Exercise 7.2.3 Dividing Rational Expressions

Divide. (Assume all denominators are nonzero.)

1. $\frac{5x}{8} \div \frac{15x^2}{4}$
2. $\frac{3}{8y} \div \frac{15}{2y^2}$
3. $\frac{\frac{5x^9}{3y^3}}{\frac{25x^{10}}{9y^5}}$
4. $\frac{\frac{12x^4y^2}{21z^5}}{\frac{6x^3y^2}{7z^3}}$
5. $\frac{(x-4)^2}{30x^4} \div \frac{x-4}{15x}$
6. $\frac{5y^4}{10(3y-5)^2} \div \frac{10y^5}{2(3y-5)^3}$
7. $\frac{x^2-9}{5x} \div (x-3)$
8. $\frac{y^2-64}{8y} \div (8+y)$
9. $\frac{(a-8)^2}{2a^2+10a} \div \frac{a-8}{a}$
10. $\frac{2}{4a^2b^3(a-2b)} \div 12ab(a-2b)^5$
11. $\frac{x^2+7x+10}{x^2+4x+4} \div \frac{1}{x^2-4}$
12. $\frac{2x^2-x-1}{2x^2-3x+1} \div \frac{1}{4x^2-1}$
13. $\frac{y+1}{y^2-3y} \div \frac{y^2-1}{y^2-6y+9}$
14. $\frac{9-a^2}{a^2-8a+15} \div \frac{2a^2-10a}{a^2-10a+25}$
15. $\frac{a^2-3a-18}{2a^2-11a-6} \div \frac{a^2+a-6}{2a^2-a-1}$
16. $\frac{y^2-7y+10}{y^2+5y-14} \div \frac{2y^2-9y-5}{y^2+14y+49}$
17. $\frac{6y^2+y-1}{4y^2+4y+1} \div \frac{3y^2+2y-1}{2y^2-7y-4}$

18. $\dfrac{x^2-7x-18}{x^2+8x+12} \div \dfrac{x^2-81}{x^2+12x+36}$

19. $\dfrac{4a^2-b^2}{b+2a} \div (b-2a)^2$

20. $\dfrac{x^2-y^2}{y+x} \div (y-x)^2$

21. $\dfrac{5y^2(y-3)}{4x^3} \div \dfrac{25y(3-y)}{2x^2}$

22. $\dfrac{15x^3}{3(y+7)} \div \dfrac{25x^6}{9(7+y)^2}$

23. $\dfrac{3x+4}{x-8} \div \dfrac{7x}{8-x}$

24. $\dfrac{3x-2}{2x+1} \div \dfrac{2-3x}{3x}$

25. $\dfrac{(7x-1)^2}{4x+1} \div \dfrac{28x^2-11x+1}{1-4x}$

26. $\dfrac{4x}{(x+2)^2} \div \dfrac{2-x}{x^2-4}$

27. $\dfrac{a^2-b^2}{a} \div (b-a)^2$

28. $\dfrac{(a-2b)^2}{2b} \div (2b^2+ab-a^2)$

29. $\dfrac{x^2-6x+9}{x^2+7x+12} \div \dfrac{9-x^2}{x^2+8x+16}$

30. $\dfrac{2x^2-9x-5}{25-x^2} \div \dfrac{1-4x+4x^2}{-2x^2-9x+5}$

31. $\dfrac{3x^2-16x+5}{100-4x^2} \div \dfrac{9x^2-6x+1}{3x^2+14x-5}$

32. $\dfrac{10x^2-25x-15}{x^2-6x+9} \div \dfrac{9-x^2}{x^2+6x+9}$

Answer available at the end of this section.

Exercise 7.2.4 Dividing Rational Expressions

Recall that multiplication and division are to be performed in the order they appear from left to right. Simplify the following.

1. $\dfrac{1}{x^2} \cdot \dfrac{x-1}{x+3} \div \dfrac{x-1}{x^3}$
2. $\dfrac{x-7}{x+9} \cdot \dfrac{1}{x^3} \div \dfrac{x-7}{x}$
3. $\dfrac{x+1}{x-2} \div \dfrac{x}{x-5} \cdot \dfrac{x^2}{x+1}$
4. $\dfrac{x+4}{2x+5} \div \dfrac{x-3}{2x+5} \cdot \dfrac{x+4}{x-3}$
5. $\dfrac{2x-1}{x+1} \div \dfrac{x-4}{x^2+1} \cdot \dfrac{x-4}{2x-1}$

6. $\dfrac{4x^2-1}{3x+2} \div \dfrac{2x-1}{x+5} \cdot \dfrac{3x+2}{2x+1}$

Answer available at the end of this section.

Exercise 7.2.5 Multiplying and Dividing Rational Functions

Calculate $(f \cdot g)(x)$ and determine the restrictions to the domain.

1. $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x-1}$
2. $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = x^2 - 1$
3. $f(x) = \dfrac{3x+2}{x+2}$ and $g(x) = \dfrac{x^2-4}{(3x+2)^2}$
4. $f(x) = \dfrac{(1-3x)}{2x-6}$ and $g(x) = \dfrac{(x-6)^2}{9x^2-1}$
5. $f(x) = \dfrac{25x^2-1}{x^2+6x+9}$ and $g(x) = \dfrac{x^2-9}{5x+1}$
6. $f(x) = \dfrac{x^2-49}{2x^2+13x-7}$ and $g(x) = \dfrac{4x^2-4x+1}{7-x}$

Answer available at the end of this section.

Exercise 7.2.6 Multiplying and Dividing Rational Functions

Calculate $(f/g)(x)$ and state the restrictions.

1. $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x-2}{x-1}$
2. $f(x) = \dfrac{(5x+3)^2}{x^2}$ and $g(x) = \dfrac{5x+3}{6-x}$
3. $f(x) = \dfrac{5-x}{(x-8)^2}$ and $g(x) = \dfrac{x^2-2}{5x-8}$
4. $f(x) = \dfrac{x^2-2x-1}{5x^2-3x-10}$ and $g(x) = \dfrac{2x^2-5x-3}{x^2-7x+12}$
5. $f(x) = \dfrac{3x^2+11x-4}{9x^2-6x+1}$ and $g(x) = \dfrac{x^2-2x+1}{3x^2-4x+1}$
6. $f(x) = \dfrac{36-x^2}{x^2+12x+36}$ and $g(x) = \dfrac{x^2-12x+3}{6x^2+4x-12}$

Answer available at the end of this section.

Exercise 7.2.7 Discussion Board Topics

1. In the history of fractions, who is credited for the first use of the fraction bar?
2. How did the ancient Egyptians use fractions?
3. Explain why $x = 7$ is a restriction to $\dfrac{1}{x} \div \dfrac{x-7}{x-2}$.

Answer available at the end of this section.

Answers

Exercise 7.2.1

$-x$

Exercise 7.2.2

$-4x^3 - 8x^2$

Exercise 7.2.2 **Multiplying Rational Expressions**

1. $\frac{3}{2x}$
3. $\frac{2y}{3x}$
5. $\frac{2}{x}$
7. $\frac{y-1}{y+1}$
9. $\frac{1}{a-5}$
11. $\frac{2x}{2x-1}$
13. -1
15. $-\frac{x+1}{x+5}$
17. $\frac{3}{3x+1}$
19. $-25y^2$
21. $-\frac{a+5}{a^2+1}$
23. $-\frac{(x+9)^2}{x-5}$
25. $\frac{2}{3}$

Exercise 7.2.3 **Dividing Rational Expressions**

1. $\frac{1}{6x}$
3. $\frac{3y^2}{5x}$
5. $\frac{x-4}{2x3}$
7. $\frac{x+3}{5x}$
9. $\frac{a-8}{2(a+5)}$
11. $(x+5)(x-2)$
13. $\frac{y-3}{y(y-1)}$
15. $\frac{a-1}{a-2}$
17. $\frac{y-4}{y+1}$
19. $\frac{1}{2a-b}$
21. $-\frac{y}{10x}$
23. $-\frac{3x+4}{7x}$
25. $-\frac{7x-1}{4x+1}$
27. $\frac{a+b}{a(a-b)}$
29. $-\frac{(x-3)(x+4)}{(x+3)^2}$
31. $-\frac{1}{4}$

Exercise 7.2.4 **Dividing Rational Expressions**

1. $\frac{x}{x+3}$
3. $\frac{x(x-5)}{x-2}$
5. $\frac{x^2+1}{x+1}$

Exercise 7.2.5 **Multiplying and Dividing Rational Functions**

1. $(f \cdot g)(x) = \frac{1}{x(x-1)}; x \neq 0, 1$
3. $(f \cdot g)(x) = \frac{x-2}{3x+2}; x \neq -2, -\frac{2}{3}$
5. $(f \cdot g)(x) = \frac{(x-3)}{(5x-1)x+3}; x \neq -3, -\frac{1}{5}$

Exercise 7.2.6 **Multiplying and Dividing Rational Functions**

1. $(f/g)(x) = \frac{x-1}{x(x-2)}; x \neq 0, 1, 2$

3. $(f/g)(x) = -\dfrac{1}{(x-8)(x+5)}; x \neq \pm 5, 8$

5. $(f/g)(x) = \dfrac{(x+4)}{(x-1)}; x \neq \dfrac{1}{3}, 1$

Exercise 7.2.7 **Discussion Board Topics**

1. Answer may vary

3. Answer may vary

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7.3: Adding and Subtracting Rational Expressions

Learning Objectives

- Add and subtract rational expressions with common denominators.
- Add and subtract rational expressions with unlike denominators.
- Add and subtract rational functions.

Adding and Subtracting with Common Denominators

Adding and subtracting rational expressions is similar to adding and subtracting fractions. Recall that if the denominators are the same, we can add or subtract the numerators and write the result over the common denominator.

$$\begin{aligned}\frac{3}{13}+\frac{7}{13} &= \frac{3+7}{13} \\ &= \frac{10}{13}\end{aligned}$$

When working with rational expressions, the common denominator will be a polynomial. In general, given polynomials ***P***, ***Q***, and ***R***, where $Q \neq 0$, we have the following:

$$\frac{P}{Q} \pm \frac{R}{Q} = \frac{P \pm R}{Q} \tag{7.3.1}$$

In this section, assume that all variable factors in the denominator are nonzero.

Example 7.3.1

Add:

$$\frac{3}{y}+\frac{7}{y}$$

Solution:

Add the numerators 3 and 7, and write the result over the common denominator, ***y***.

$$\begin{aligned}\frac{3}{y}+\frac{7}{y} &= \frac{3+7}{y} \\ &= \frac{10}{y}\end{aligned}$$

Answer:

$$\frac{10}{y}$$

Example 7.3.2

Subtract:

$$\frac{x-5}{2x-1}-\frac{1}{2x-1}$$

Solution:

Subtract the numerators $x-5$ and 1, and write the result over the common denominator, $2x-1$.

$$\begin{aligned}\frac{x-5}{2x-1}-\frac{1}{2x-1}&=\frac{x-5-1}{2x-1} && \textit{Simplify the numerator.}\\ &=\frac{x-6}{2x-1}\end{aligned}$$

Answer:

$$\frac{x-6}{2x-1}$$

Example 7.3.3

Subtract:

$$\frac{2x+7}{(x+5)(x-3)}-\frac{x+10}{(x+5)(x-3)}$$

Solution:

We use parentheses to remind us to subtract the entire numerator of the second rational expression.

$$\begin{aligned}\frac{2x+7}{(x+5)(x-3)}-\frac{x+10}{(x+5)(x-3)}&=\frac{(2x+7)-(x+10)}{(x+5)(x-3)} && \textit{Simplify the numerator.}\\ &=\frac{2x+7-x-10}{(x+5)(x-3)} && \textit{Leave the denominator factored.}\\ &=\frac{\cancel{x-3}^{\,1}}{(x+5)\,\cancel{(x-3)}} && \textit{Cancel common factors.}\\ &=\frac{1}{x+5}\end{aligned}$$

Answer:

$$\frac{1}{x+5}$$

Example 7.3.4

Simplify:

$$\frac{2x^2+10x+3}{x^2-36}-\frac{x^2+6x+5}{x^2-36}+\frac{x-4}{x^2-36}$$

Solution:

Subtract and add the numerators. Make use of parentheses and write the result over the common denominator, x^2-36.

$$\begin{aligned}\frac{2x+7}{(x+5)(x-3)}-\frac{x+10}{(x+5)(x-3)} &= \frac{(2x+7)-(x+10)}{(x+5)(x-3)}\\ &= \frac{2x+7-x-10}{(x+5)(x-3)}\\ &= \frac{x-3}{(x+5)(x-3)}\\ &= \frac{1}{x+5}\end{aligned} \tag{7.3.2}$$

Answer:

$\frac{x-1}{x-6}$

Exercise 7.3.1

Subtract: $\frac{x^2+1}{2x^2-7x-4}-\frac{x^2-2x}{2x^2-7x-4}$.

Answer available at the end of this section.

Adding and Subtracting with Unlike Denominators

To add rational expressions with unlike denominators, first find equivalent expressions with common denominators. Do this just as you have with fractions. If the denominators of fractions are relatively prime, then the least common denominator (LCD) is their product. For example,

$$\frac{1}{3}+\frac{1}{5} \Rightarrow \text{LCD} = 3 \cdot 5 = 15$$

Multiply each fraction by the appropriate form of 1 to obtain equivalent fractions with a common denominator.

$$\begin{aligned}\frac{1}{3}+\frac{1}{5} &= \frac{1\cdot 5}{3\cdot 5}+\frac{1\cdot 3}{5\cdot 3} \qquad \textit{Equivalent fractions with a common denominator}\\ &= \frac{5}{15}+\frac{3}{15}\\ &= \frac{5+3}{15}\\ &= \frac{8}{15}\end{aligned}$$

The process of adding and subtracting rational expressions is similar. In general, given polynomials ***P***, ***Q***, ***R***, and ***S***, where $Q \neq 0$ and $S \neq 0$, we have the following:

$$\frac{P}{Q} \pm \frac{R}{S} = \frac{PS \pm QR}{QS} \tag{7.3.3}$$

In this section, assume that all variable factors in the denominator are nonzero.

Example 7.3.5

Add:

$$\frac{1}{x}+\frac{1}{y}$$

Solution:

In this example, the $LCD = xy$. To obtain equivalent terms with this common denominator, multiply the first term by $\frac{y}{y}$ and the second term by $\frac{x}{x}$.

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{1}{x} \cdot \frac{y}{y} + \frac{1}{y} \cdot \frac{x}{x} \\ &= \frac{y}{xy} + \frac{x}{xy} && \textit{Equivalent terms with a common denominator.} \\ &= \frac{y+x}{xy} && \textit{Add the numerators and place the result over the common denominator, } xy. \end{aligned}$$

Answer:

$$\frac{y+x}{xy}$$

Example 7.3.6

Subtract:

$$\frac{1}{y} - \frac{1}{y-3}$$

Solution:

Since the $LCD = y(y-3)$, multiply the first term by 1 in the form of $\frac{(y-3)}{(y-3)}$ and the second term by $\frac{y}{y}$.

$$\begin{aligned} \frac{1}{y} - \frac{1}{y-3} &= \frac{1}{y} \cdot \frac{(y-3)}{(y-3)} - \frac{1}{y-3} \cdot \frac{y}{y} \\ &= \frac{(y-3)}{y(y-3)} - \frac{y}{y(y-3)} \\ &= \frac{y-3-y}{y(y-3)} \\ &= \frac{-3}{y(y-3)} \quad \text{or} \quad = -\frac{3}{y(y-3)} \end{aligned}$$

Answer:

$$\frac{-3}{y(y-3)}$$

It is not always the case that the LCD is the product of the given denominators. Typically, the denominators are not relatively prime; thus determining the LCD requires some thought. Begin by factoring all denominators. The LCD is the product of all factors with the highest power. For example, given

$$\frac{1}{x^3(x+2)(x-3)} \quad \text{and} \quad \frac{1}{x(x+2)^2}$$

there are three base factors in the denominator: $x, (x+2)$, and $(x-3)$. The highest powers of these factors are $x^3, (x+2)^2$, and $(x-3)^1$. Therefore,

$$\text{LCD} = x^3(x+2)^2(x-3)$$

The general steps for adding or subtracting rational expressions are illustrated in the following example.

Example 7.3.7

Subtract: $\dfrac{x}{x^2+4x+3}-\dfrac{3}{x^2-4x-5}$.

Solution:

Step 1: Factor all denominators to determine the LCD.

$$\begin{aligned}&\frac{x}{x^2+4x+3}-\frac{3}{x^2-4x-5}\\=&\frac{x}{(x+1)(x+3)}-\frac{3}{(x+1)(x-5)}\end{aligned}\tag{7.3.4}$$

The LCD is $(x+1)(x+3)(x-5)$.

Step 2: Multiply by the appropriate factors to obtain equivalent terms with a common denominator. To do this, multiply the first term by $\dfrac{(x-5)}{(x-5)}$ and the second term by $\dfrac{(x+3)}{(x+3)}$.

$$\begin{aligned}&=\frac{x}{(x+1)(x+3)}\cdot\frac{(x-5)}{(x-5)}-\frac{3}{(x+1)(x-5)}\cdot\frac{(x+3)}{(x+3)}\\&=\frac{x(x-5)}{(x+1)(x+3)(x-5)}-\frac{3(x+3)}{(x+1)(x+3)(x-5)}\end{aligned}$$

Step 3: Add or subtract the numerators and place the result over the common denominator.

$$=\frac{x(x-5)-3(x+3)}{(x+1)(x+3)(x-5)}$$

Step 4: Simplify the resulting algebraic fraction.

$$\begin{aligned}&=\frac{x(x-5)-3(x+3)}{(x+1)(x+3)(x-5)}\\&=\frac{x^2-5x-3x-9}{(x+1)(x+3)(x-5)}\\&=\frac{x^2-8x-9}{(x+1)(x+3)(x-5)}\\&=\frac{(x+1)(x-9)}{(x+1)(x+3)(x-5)}\\&=\frac{(x-9)}{(x+3)(x-5)}\end{aligned}$$

Answer:

$$\frac{(x-9)}{(x+3)(x-5)}$$

Example 7.3.8

Subtract: $\dfrac{x^2-9x+18}{x^2-13x+36}-\dfrac{x}{x-4}$.

Solution:

It is best not to factor the numerator, $x^2-9x+18$, because we will most likely need to simplify after we subtract.

$$
\begin{aligned}
&\frac{x^2-9x+18}{x^2-13x+36}-\frac{x}{x-4}\\
=&\frac{x^2-9x+18}{(x-4)(x-9)}-\frac{x}{x-4}\\
=&\frac{x^2-9x+18}{(x-4)(x-9)}-\frac{x}{(x-4)}\cdot\frac{(x-9)}{(x-9)}\\
=&\frac{x^2-9x+18}{(x-4)(x-9)}-\frac{x(x-9)}{(x-4)(x-9)}\\
=&\frac{x^2-9x+18-x(x-9)}{(x-4)(x-9)}\\
=&\frac{x^2-9x+18-x^2+9x}{(x-4)(x-9)}\\
=&\frac{18}{(x-4)(x-9)}
\end{aligned}
$$

Answer:

$$\frac{18}{(x-4)(x-9)}$$

Example 7.3.9

Subtract:

$$\frac{1}{x^2-4}-\frac{1}{2-x}$$

Solution:

First, factor the denominators and determine the LCD. Notice how the opposite binomial property is applied to obtain a more workable denominator.

$$
\begin{aligned}
\frac{1}{x^2-4}-\frac{1}{2-x}&=\frac{1}{(x+2)(x-2)}-\frac{1}{-1(x-2)}\\
&=\frac{1}{(x+2)(x-2)}+\frac{1}{(x-2)}
\end{aligned}
$$

The LCD is $(x+2)(x-2)$. Multiply the second term by 1 in the form of $\frac{(x+2)}{(x+2)}$.

$$
\begin{aligned}
&=\frac{1}{(x+2)(x-2)}+\frac{1}{(x-2)}\cdot\frac{(x+2)}{(x+2)}\\
&=\frac{1}{(x+2)(x-2)}+\frac{x+2}{(x-2)(x+2)}
\end{aligned}
$$

Now that we have equivalent terms with a common denominator, add the numerators and write the result over the common denominator.

$$
\begin{aligned}
&=\frac{1}{(x+2)(x-2)}+\frac{x+2}{(x-2)(x+2)}\\
&=\frac{1+x+2}{(x+2)(x-2)}\\
&=\frac{x+3}{(x+2)(x-2)}
\end{aligned}
$$

Answer:

$$\frac{x+3}{(x+2)(x-2)}$$

Example 7.3.10

Simplify:

$$\frac{y-1}{y+1}-\frac{y+1}{y-1}+\frac{y^2-5}{y^2-1}$$

Solution:

Begin by factoring the denominator.

$$\begin{aligned}&\frac{y-1}{y+1}-\frac{y+1}{y-1}+\frac{y^2-5}{y^2-1}\\ =&\frac{y-1}{y+1}-\frac{y+1}{y-1}+\frac{y^2-5}{(y+1)(y-1)}\end{aligned}$$

We can see that the LCD is $(y+1)(y-1)$. Find equivalent fractions with this denominator.

$$\begin{aligned}&=\frac{(y-1)}{(y+1)}\cdot\frac{(y-1)}{(y-1)}-\frac{(y+1)}{(y-1)}\cdot\frac{(y+1)}{(y+1)}+\frac{y^2-5}{(y+1)(y-1)}\\ &=\frac{y^2-2y+1}{(y+1)(y-1)}-\frac{y^2+2y+1}{(y+1)(y-1)}+\frac{y^2-5}{(y+1)(y-1)}\end{aligned}$$

Next, subtract and add the numerators and place the result over the common denominator.

$$=\frac{(y^2-2y+1)-(y^2+2y+1)+(y^2-5)}{(y+1)(y-1)}$$

Finish by simplifying the resulting rational expression.

$$\begin{aligned}&=\frac{y^2-2y+1-y^2-2y-1+y^2-5}{(y+1)(y-1)}\\ &=\frac{y^2-4y-5}{(y+1)(y-1)}\\ &=\frac{(y+1)(y-5)}{(y+1)(y-1)}\\ &=\frac{(y-5)}{(y-1)}\end{aligned}$$

Answer:

$$\frac{y-5}{y-1}$$

Exercise 7.3.2

Simplify:

$$-\frac{2x^2-1+x}{1+x}-\frac{5}{1-x}$$

Answer available at the end of this section.

Rational expressions are sometimes expressed using negative exponents. In this case, apply the rules for negative exponents before simplifying the expression.

Example 7.3.11

Simplify:

$$y^{-2}+(y-1)^{-1}$$

Solution:

Recall that $x^{-n}=\dfrac{1}{x^n}$. We begin by rewriting the negative exponents as rational expressions.

$$\begin{aligned}
&y^{-2}+(y-1)^{-1}\\
=&\frac{1}{y^2}+\frac{1}{(y-1)^1} && \textit{Replace negative exponents.}\\
=&\frac{1}{y^2}\cdot\frac{(y-1)}{(y-1)}+\frac{1}{(y-1)}\cdot\frac{y^2}{y^2} && \textit{Multiply by factors to obtain equivalent}\\
& && \textit{expressions with a common denominator.}\\
=&\frac{(y-1)}{y^2(y-1)}+\frac{y^2}{y^2(y-1)}\\
=&\frac{(y-1)+y^2}{y^2(y-1)} && \textit{Add and simplify.}\\
=&\frac{y^2+y-1}{y^2(y-1)} && \textit{The trinomial does not factor.}
\end{aligned}$$

Answer:

$$\frac{y^2+y-1}{y^2(y-1)}$$

Adding and Subtracting Rational Functions

We can simplify sums or differences of rational functions using the techniques learned in this section. The restrictions of the result consist of the restrictions to the domains of each function.

Example 7.3.12

Calculate $(f+g)(x)$, given $f(x)=\dfrac{1}{x+3}$ and $g(x)=\dfrac{1}{x-2}$, and state the restrictions.

Solution:

$$\begin{aligned}
(f+g)(x)&=f(x)+g(x)\\
&=\frac{1}{x+3}+\frac{1}{x-2}\\
&=\frac{1}{x+3}\cdot\frac{(x-2)}{(x-2)}+\frac{1}{x-2}\cdot\frac{(x+3)}{(x+3)}\\
&=\frac{x-2}{(x+3)(x-2)}+\frac{x+3}{(x-2)(x+3)}\\
&=\frac{x-2+x+3}{(x+3)(x-2)}\\
&=\frac{2x+1}{(x+3)(x-2)}
\end{aligned}$$

Here the domain of ***f*** consists of all real numbers except -3, and the domain of ***g*** consists of all real numbers except 2. Therefore, the domain of ***f* + *g*** consists of all real numbers except -3 and 2.

Answer:

$$\frac{2x+1}{(x+3)(x-2)}, \text{ where } x \neq -3, 2$$

Example 7.3.13

Calculate $(f-g)(x)$, given $f(x)=\frac{x(x-1)}{x^2-25}$ and $g(x)=\frac{x-3}{x-5}$, and state the restrictions to the domain.

Solution:

$$\begin{aligned}
(f-g)(x) &= f(x)-g(x) \\
&= \frac{x(x-1)}{x^2-25}-\frac{x-3}{x-5} \\
&= \frac{x(x-1)}{(x+5)(x-5)}-\frac{(x-3)}{(x-5)}\cdot\frac{(x+5)}{(x+5)} \\
&= \frac{x(x-1)-(x-3)(x+5)}{(x+5)(x-5)} \\
&= \frac{x^2-x-(x^2+5x-3x-15)}{(x+5)(x-5)} \\
&= \frac{x^2-x-(x^2+2x-15)}{(x+5)(x-5)} \\
&= \frac{x^2-x-x^2-2x+15}{(x+5)(x-5)} \\
&= \frac{-3x+15}{(x+5)(x-5)} \\
&= \frac{-3\cancel{(x-5)}}{(x+5)\cancel{(x-5)}} \\
&= \frac{-3}{x+5}
\end{aligned}$$

The domain of ***f*** consists of all real numbers except 5 and -5, and the domain of ***g*** consists of all real numbers except 5. Therefore, the domain of ***f* – *g*** consists of all real numbers except -5 and 5.

Answer:

$$\frac{-3}{x+5}, \text{ where } x \neq \pm 5$$

Key Takeaways

- When adding or subtracting rational expressions with a common denominator, add or subtract the expressions in the numerator and write the result over the common denominator.
- To find equivalent rational expressions with a common denominator, first factor all denominators and determine the least common multiple. Then multiply numerator and denominator of each term by the appropriate factor to obtain a common denominator. Finally, add or subtract the expressions in the numerator and write the result over the common denominator.
- The restrictions to the domain of a sum or difference of rational functions consist of the restrictions to the domains of each function.

Exercise 7.3.3 Adding and Subtracting with Common Denominators

Simplify. (Assume all denominators are nonzero.)

1. $\frac{3}{x}+\frac{7}{x}$
2. $\frac{9}{x}-\frac{10}{x}$
3. $\frac{x}{y}-\frac{3}{y}$
4. $\frac{4}{x-3}+\frac{6}{x-3}$
5. $\frac{7}{2x-1}-\frac{x}{2x-1}$
6. $\frac{8}{3x-8}-\frac{3x}{3x-8}$
7. $\frac{2}{x-9}+\frac{x-11}{x-9}$
8. $\frac{y+2}{2y+3}-\frac{y+3}{2y+3}$
9. $\frac{2x-3}{4x-1}-\frac{x-4}{4x-1}$
10. $\frac{2x}{x-1}-\frac{3x+4}{x-1}+\frac{x-2}{x-1}$
11. $\frac{1}{3y}-\frac{2y-9}{3y}-\frac{13-5y}{3y}$
12. $\frac{-3y+2}{5y-10}+\frac{y+7}{5y-10}-\frac{3y+4}{5y-10}$
13. $\frac{x}{(x+1)(x-3)}-\frac{3}{(x+1)(x-3)}$
14. $\frac{3x+5}{(2x-1)(x-6)}-\frac{x+6}{(2x-1)(x-6)}$
15. $\frac{x}{x^2-36}+\frac{6}{x^2-36}$
16. $\frac{x}{x^2-81}-\frac{9}{x^2-81}$
17. $\frac{x^2+2}{x^2+3x-28}+\frac{x-2}{2x^2+3x-28}$
18. $\frac{x^2}{x^2-x-3}-\frac{3-x^2}{x^2-x-3}$

Answer available at the end of this section.

Exercise 7.3.4 Adding and Subtracting with Unlike Denominators

Simplify. (Assume all denominators are nonzero.)

1. $\frac{1}{2}+\frac{1}{3x}$
2. $\frac{1}{5x^2}-\frac{1}{x}$
3. $\frac{1}{12y^2}+\frac{3}{10y^3}$
4. $\frac{1}{x}-\frac{1}{2y}$
5. $\frac{1}{y}-2$
6. $\frac{3}{y+2}-4$
7. $\frac{2}{x+4}+2$
8. $\frac{2}{y}-\frac{1}{y^2}$
9. $\frac{3}{x+1}+\frac{1}{x}$
10. $\frac{1}{x-1}-\frac{2}{x}$
11. $\frac{1}{x-3}+\frac{1}{x+5}$
12. $\frac{1}{x+2}-\frac{1}{x-3}$
13. $\frac{x}{x+1}-\frac{2}{x-2}$
14. $\frac{2x-3}{x+5}-\frac{x}{x-3}$
15. $\frac{y+1}{y-1}+\frac{y-1}{y+1}$
16. $\frac{3y-1}{3y}-\frac{y+4}{y-2}$
17. $\frac{2x-5}{2x+5}-\frac{2x+5}{2x-5}$
18. $\frac{2}{2x-1}-\frac{2x+1}{1-2x}$
19. $\frac{3x+4}{x-8}-\frac{2}{8-x}$

20. $\frac{1}{y-1}+\frac{1}{1-y}$

21. $\frac{2x^2}{x^2-9}+\frac{x+15}{9-x^2}$

22. $\frac{x}{x+3}+\frac{1}{x-3}-\frac{1}{5}-\frac{x}{(x+3)(x-3)}$

23. $\frac{2x}{3x-1}-\frac{1}{3x+1}+\frac{2(x-1)}{(3x-1)(3x+1)}$

24. $\frac{4x}{2x+1}-\frac{x}{x-5}+\frac{16x-3}{(2x+1)(x-5)}$

25. $\frac{x}{3x}+\frac{2}{x-2}+\frac{4}{3x(x-2)}$

26. $-\frac{2x}{x+6}-\frac{3x}{6-x}-\frac{18(x-2)}{(x+6)(x-6)}$

27. $\frac{x}{x+5}-\frac{1}{x-7}-\frac{25-7x}{(x+5)(x-7)}$

28. $\frac{x}{x^2}-\frac{2}{x-3}+\frac{2}{x-3}$

29. $\frac{1}{x+5}-\frac{x^2}{x^2-25}$

30. $\frac{5x-2}{x^2-4}-\frac{2}{x-2}$

31. $\frac{1}{x+1}-\frac{6x-3}{x^2-7x-8}$

32. $\frac{3x}{9x^2-16}-\frac{1}{3x+4}$

33. $\frac{2x}{x^2-1}+\frac{1}{x^2+x}$

34. $\frac{x(4x-1)}{2x^2}+\frac{7}{x-4}-\frac{x}{4+x}$

35. $\frac{3x^2}{3x^2+5x-2}-\frac{2x}{3x-1}$

36. $\frac{2x}{x-4}-\frac{11x+4}{x^2-2x-8}$

37. $\frac{x}{2x+1}+\frac{6x-24}{2x^2-7x-4}$

38. $\frac{1}{x^2-x-6}+\frac{1}{x^2-3x-10}$

39. $\frac{x}{x^2+4x+3}-\frac{3}{x^2-4x-5}$

40. $\frac{y+1}{2y^2+5y-3}-\frac{y}{4y^2-1}$

41. $\dfrac{y-1}{y^2-25}-\dfrac{2}{y^2-10y+25}$

42. $\dfrac{3x^2+2}{4x^2-2x-8}-\dfrac{1}{2x-4}$

43. $\dfrac{4x^2+2}{8x^2-6x-7}-\dfrac{2}{8x-7}$

44. $\dfrac{a}{4-a+a^2}-\dfrac{9a+18}{a^2-13a+36}$

45. $\dfrac{3a-12}{a^2-8a+16}-\dfrac{a+2}{4-a}$

46. $\dfrac{a^2-14}{2a^2-7a-4}-\dfrac{5}{1+2a}$

47. $\dfrac{1}{x+3}-\dfrac{x}{x^2-6x+9}+\dfrac{3}{x^2-9}$

48. $\dfrac{3x}{x+7}-\dfrac{2x}{x-2}+\dfrac{23x-10}{x^2+5x-14}$

49. $\dfrac{x+3}{x-1}+\dfrac{x-1}{x+2}-\dfrac{x(x+11)}{x^2+x-2}$

50. $-\dfrac{2x}{3x+1}-\dfrac{4}{x-2}+\dfrac{4(x+5)}{3x^2-5x-2}$

51. $\dfrac{x-1}{4x-1}-\dfrac{x+3}{2x+3}-\dfrac{3(x+5)}{8x^2+10x-3}$

52. $\dfrac{3x}{2x-3}-\dfrac{2}{2x+3}-\dfrac{6x^2-5x-9}{4x^2-9}$

53. $\dfrac{1}{y+1}+\dfrac{1}{y}+\dfrac{2}{y^2-1}$

54. $\dfrac{1}{y}-\dfrac{1}{y+1}+\dfrac{1}{y-1}$

55. $5^{-2}+2^{-1}$

56. $6^{-1}+4^{-2}$

57. $x^{-1}+y^{-1}$

58. $x^{-2}-y^{-1}$

59. $(2x-1)^{-1}-x^{-2}$

60. $(x-4)^{-1}-(x+1)^{-1}$

61. $3x^2(x-1)^{-1}-2x$

62. $2(y-1)^{-2}-(y-1)^{-1}$

Answer available at the end of this section.

Exercise 7.3.5 Adding and Subtracting Rational Functions

Calculate $(f+g)(x)$ and $(f-g)(x)$ and state the restrictions to the domain.

1. $f(x)=\dfrac{1}{3x}$ and $g(x)=\dfrac{1}{x-2}$
2. $f(x)=\dfrac{1}{x-1}$ and $g(x)=\dfrac{1}{x+5}$
3. $f(x)=\dfrac{x}{x-4}$ and $g(x)=\dfrac{1}{4-x}$
4. $f(x)=\dfrac{x}{x-5}$ and $g(x)=\dfrac{1}{2x-3}$
5. $f(x)=\dfrac{x-1}{x^2-4}$ and $g(x)=\dfrac{4}{x^2-6x-16}$
6. $f(x)=\dfrac{5}{x+2}$ and $g(x)=\dfrac{3}{x+4}$

Answer available at the end of this section.

Exercise 7.3.6 Adding and Subtracting Rational Functions

Calculate $(f+f)(x)$ and state the restrictions to the domain.

1. $f(x)=\dfrac{1}{x}$
2. $f(x)=\dfrac{1}{2x}$
3. $f(x)=\dfrac{x}{2x-1}$
4. $f(x)=\dfrac{1}{x+2}$

Answer available at the end of this section.

Exercise 7.3.7 Discussion Board

1. Explain to a classmate why this is incorrect: $\dfrac{1}{x^2}+\dfrac{2}{x^2}=\dfrac{3}{2x^2}$.
2. Explain to a classmate how to find the common denominator when adding algebraic expressions. Give an example.

Answer available at the end of this section.

Answers

Exercise 7.3.1

$$\frac{1}{x-4}$$

Exercise 7.3.2

$$\frac{x+3}{x-1}$$

Exercise 7.3.3 **Adding and Subtracting with Common Denominators**

1. $\frac{10}{x}$
3. $\frac{x-3}{y}$
5. $\frac{7-x}{2x-1}$
7. 1
9. $\frac{x+1}{4x-1}$
11. $\frac{y-1}{y}$
13. $\frac{1}{x+1}$
15. $\frac{1}{x-6}$
17. $\frac{x+5}{x+7}$

Exercise 7.3.4 **Adding and Subtracting with Unlike Denominators**

1. $\frac{3x+2}{6x}$
3. $\frac{5y+18}{60y^3}$
5. $\frac{1-2y}{y}$
7. $\frac{2(x+5)}{x+4}$
9. $\frac{4x+1}{x(x+1)}$
11. $\frac{2(x+1)}{(x-3)(x+5)}$
13. $\frac{x^2-4x-2}{(x-2)(x+1)}$
15. $\frac{2(y2+1)}{(y+1)(y-1)}$
17. $-\frac{40x}{(2x+5)(2x-5)}$
19. $\frac{3(x+2)}{x-8}$
21. $\frac{2x+5}{x+3}$
23. $\frac{2x+1}{3x+1}$
25. $\frac{x^2+4x+4}{3x(x-2)}$
27. $\frac{x-6}{x-7}$
29. $\frac{x-5-x^2}{(x+5)(x-5)}$
31. $-\frac{5}{x-8}$
33. $\frac{2x-1}{x(x-1)}$
35. $\frac{x(x-4)}{(x+2)(3x-1)}$
37. $\frac{x+6}{2x+1}$
39. $\frac{x-9}{(x-5)(x+3)}$
41. $\frac{y^2-8y-5}{(y+5)(y-5)^2}$
43. $\frac{4x}{x+1}$
45. $\frac{a+5}{a-4}$
47. $-\frac{6x}{(x+3)(x-3)^2}$
49. $\frac{x-7}{x+2}$
51. $-\frac{x-5}{4x-1}$
53. $\frac{2y-1}{y(y-1)}$
55. $\frac{27}{50}$
57. $\frac{x+y}{xy}$

59. $\frac{(x-1)^2}{x^2(2x-1)}$

61. $\frac{x(x+2)}{x-1}$

Exercise 7.3.5 **Adding and Subtracting Rational Functions**

1. $(f+g)(x)=\frac{2(2x-1)}{3x(x-2)}; (f-g)(x)=-\frac{2(x+1)}{3x(x-2)}; x\neq 0,2$

3. $(f+g)(x)=\frac{x-1}{x-4}; (f-g)(x)=\frac{x+1}{x-4}; x\neq 4$

5. $(f+g)(x)=\frac{x(x-5)}{(x+2)(x-2)(x-8)}; (f-g)(x)=\frac{x^2-13x+16}{(x+2)(x-2)(x-8)}; x\neq -2,2,8$

Exercise 7.3.6 **Adding and Subtracting Rational Functions**

1. $(f+f)(x)=\frac{2}{x}; x\neq 0$

3. $(f+f)(x)=\frac{2x}{2x-1}; x\neq \frac{1}{2}$

Exercise 7.3.7 **Discussion Board**

1. Answer may vary

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7.4: Complex Rational Expressions

Learning Objectives

- Simplify complex rational expressions by multiplying the numerator by the reciprocal of the divisor.
- Simplify complex rational expressions by multiplying numerator and denominator by the least common denominator (LCD).

Definitions

A complex fraction is a fraction where the numerator or denominator consists of one or more fractions. For example,

$$\frac{\frac{3}{4}}{\frac{1}{2}}$$

Simplifying such a fraction requires us to find an equivalent fraction with integer numerator and denominator. One way to do this is to divide. Recall that dividing fractions involves multiplying by the reciprocal of the divisor.

$$\frac{\frac{3}{4}}{\frac{1}{2}} = \frac{3}{\cancel{4}_{2}} \cdot \frac{\cancel{2}^{1}}{1} = \frac{3}{2} \qquad Method\ 1:\ using\ division$$

An alternative method for simplifying this complex fraction involves multiplying both the numerator and denominator by the LCD of all the given fractions. In this case, the LCD = 4.

$$\frac{\frac{3}{4} \cdot 4}{\frac{1}{2} \cdot 4} = \frac{3}{2} \qquad Method\ 2:\ using\ the\ LCD$$

A complex rational expression is defined as a rational expression that contains one or more rational expressions in the numerator or denominator or both. For example,

$$\frac{\frac{1}{2} + \frac{1}{x}}{\frac{1}{4} - \frac{1}{x^2}}$$

We simplify a complex rational expression by finding an equivalent fraction where the numerator and denominator are polynomials. As illustrated above, there are two methods for simplifying complex rational expressions, and we will outline the steps for both methods. For the sake of clarity, assume that variable expressions used as denominators are nonzero.

Method 1: Simplify Using Division

We begin our discussion on simplifying complex rational expressions using division. Before we can multiply by the reciprocal of the divisor, we must simplify the numerator and denominator separately. The goal is to first obtain single algebraic fractions in the numerator and the denominator. The steps for simplifying a complex algebraic fraction are illustrated in the following example.

Example 7.4.1

Simplify:

$$\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}}$$

Solution:

Step 1: Simplify the numerator and denominator. The goal is to obtain a single algebraic fraction divided by another single algebraic fraction. In this example, find equivalent terms with a common denominator in both the numerator and denominator before adding and subtracting.

$$\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}} = \frac{\frac{1}{2}\cdot\frac{x}{x}+\frac{1}{x}\cdot\frac{2}{2}}{\frac{1}{4}\cdot\frac{x^2}{x^2}-\frac{1}{x^2}\cdot\frac{4}{4}}$$

$$= \frac{\frac{x}{2x}+\frac{2}{2x}}{\frac{x^2}{4x^2}-\frac{4}{4x^2}} \qquad \textit{Equivalent fractions with common denominators}$$

$$= \frac{\frac{x+2}{2x}}{\frac{x^2-4}{4x^2}} \qquad \text{Add the fractions in the numerator and denominator.}$$

At this point we have a single algebraic fraction divided by a single algebraic fraction.

Step 2: Multiply the numerator by the reciprocal of the divisor.

$$\frac{\frac{x+2}{2x}}{\frac{x^2-4}{4x^2}} = \frac{x+2}{2x}\cdot\frac{4x^2}{x^2-4}$$

Step 3: Factor all numerators and denominators completely.

$$= \frac{x+2}{2x}\cdot\frac{4x^2}{(x+2)(x-2)}$$

Step 4: Cancel all common factors.

$$= \frac{4x^2\cdot(x+2)}{2x(x+2)(x-2)}$$

$$= \frac{\overset{2x}{\cancel{4x^2}}\cdot\cancel{(x+2)}}{\cancel{2x}\ \cancel{(x+2)}\ (x-2)}$$

$$= \frac{2x}{(x-2)}$$

Answer:

$$\frac{2x}{x-2}$$

Example 7.4.2

Simplify: $\dfrac{\dfrac{1}{x}-\dfrac{1}{x-2}}{\dfrac{4}{x^2-2x}}$

Solution:

$$\begin{aligned}\frac{\frac{1}{x}-\frac{1}{x-2}}{\frac{4}{x^2-2x}} &= \frac{\frac{1}{x}\cdot\frac{(x-2)}{(x-2)}-\frac{1}{x-2}\cdot\frac{x}{x}}{\frac{4}{x^2-2x}} \\ &= \frac{\frac{x-2}{x(x-2)}-\frac{x}{x(x-2)}}{\frac{4}{x^2-2x}} \\ &= \frac{\frac{x-2-x}{x(x-2)}}{\frac{4}{x^2-2x}} \\ &= \frac{\frac{-2}{x(x-2)}}{\frac{4}{x^2-2x}} \\ &= \frac{-2}{x(x-2)}\cdot\frac{x^2-2x}{4} \\ &= \frac{-1}{\cancel{x}^2\cdot\cancel{x}(x-2)} \\ &= \frac{-1}{2}\end{aligned}$$

Answer:

$$-\frac{1}{2}$$

Example 7.4.3

Simplify:

$$\frac{1-\dfrac{4}{x}-\dfrac{21}{x^2}}{1-\dfrac{2}{x}-\dfrac{15}{x^2}}$$

Solution:

The LCD of the rational expressions in both the numerator and denominator is x^2. Multiply by the appropriate factors to obtain equivalent terms with this as the denominator and then subtract.

$$\begin{aligned}\frac{1-\frac{4}{x}-\frac{21}{x^2}}{1-\frac{2}{x}-\frac{15}{x^2}} &= \frac{\frac{1}{1}\cdot\frac{x^2}{x^2}-\frac{4}{x}\cdot\frac{x}{x}-\frac{21}{x^2}}{\frac{1}{1}\cdot\frac{x^2}{x^2}-\frac{2}{x}\cdot\frac{x}{x}-\frac{15}{x^2}} \\ &= \frac{\frac{x^2}{x^2}-\frac{4x}{x^2}-\frac{21}{x^2}}{\frac{x^2}{x^2}-\frac{2x}{x^2}-\frac{15}{x^2}} \\ &= \frac{\frac{x^2-4x-21}{x^2}}{\frac{x^2-2x-15}{x^2}}\end{aligned}$$

We now have a single rational expression divided by another single rational expression. Next, multiply the numerator by the reciprocal of the divisor and then factor and cancel.

Answer:

$$\frac{x-7}{x-5}$$

Example 7.4.4

Simplify:

$$\frac{1-\frac{1}{x^2}}{\frac{1}{x}-1}$$

Solution:

$$\begin{aligned}\frac{1-\frac{1}{x^2}}{\frac{1}{x}-1} &= \frac{\frac{1}{1}\cdot\frac{x^2}{x^2}-\frac{1}{x^2}}{\frac{1}{x}-\frac{1}{1}\cdot\frac{x}{x}} \\ &= \frac{\frac{x^2-1}{x^2}}{\frac{1-x}{x}} \\ &= \frac{x^2-1}{x^2}\cdot\frac{x}{1-x} \\ &= \frac{(x+1)\cancel{(x-1)}}{\cancel{x^2}\,x}\cdot\frac{\cancel{x}}{-1\cdot\cancel{(x-1)}} \\ &= \frac{x+1}{-1\cdot x} \\ &= -\frac{x+1}{x}\end{aligned}$$

Answer:

$$-\frac{x+1}{x}$$

Exercise 7.4.1

Simplify:

$$\frac{\frac{1}{81}-\frac{1}{x^2}}{\frac{1}{9}+\frac{1}{x}}$$

Answer available at the end of this section.

Method 2: Simplify Using the LCD

An alternative method for simplifying complex rational expressions involves clearing the fractions by multiplying the expression by a special form of 1. In this method, multiply the numerator and denominator by the least common denominator (LCD) of all given fractions.

Example 7.4.5

Simplify:

$$\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}}$$

Solution:

Step 1: Determine the LCD of all the fractions in the numerator and denominator. In this case, the denominators of the given fractions are $2, x, 4$, and x^2. Therefore, the LCD is $4x^2$.

Step 2: Multiply the numerator and denominator by the LCD. This step should clear the fractions in both the numerator and denominator.

$$\begin{aligned}\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}} &= \frac{\left(\frac{1}{2}+\frac{1}{x}\right)\cdot 4x^2}{\left(\frac{1}{4}-\frac{1}{x^2}\right)\cdot 4x^2} && \textit{Distribute.}\\ &= \frac{\frac{1}{2}\cdot 4x^2+\frac{1}{x}\cdot 4x^2}{\frac{1}{4}\cdot 4x^2-\frac{1}{x^2}\cdot 4x^2} && \textit{Cancel.}\\ &= \frac{2x^2+4x}{x^2-4}\end{aligned}$$

This leaves us with a single algebraic fraction with a polynomial in the numerator and in the denominator.

Step 3: Factor the numerator and denominator completely.

$$\begin{aligned}&= \frac{2x^2+4x}{x^2-4}\\ &= \frac{2x(x+2)}{(x+2)(x-2)}\end{aligned}$$

Step 4: Cancel all common factors.

$$\begin{aligned}&= \frac{2x\ \cancel{(x+2)}}{\cancel{(x+2)}\ (x-2)}\\ &= \frac{2x}{x-2}\end{aligned}$$

Answer:

$$\frac{2x}{x-2}$$

Note

This was the same problem that we began this section with, and the results here are the same. It is worth taking the time to compare the steps involved using both methods on the same problem.

Example 7.4.6

Simplify:

$$\frac{1-\frac{2}{x}-\frac{15}{x^2}}{3-\frac{14}{x}-\frac{5}{x^2}}$$

Solution:

Considering all of the denominators, we find that the LCD is x^2. Therefore, multiply the numerator and denominator by x^2:

At this point, we have a rational expression that can be simplified by factoring and then canceling the common factors.

$$= \frac{(x+3)\cancel{(x-5)}}{(3x+1)\cancel{(x-5)}} \quad \textit{Cancel.}$$
$$= \frac{x+3}{3x+1}$$

Answer:

$$\frac{x+3}{3x+1}$$

It is important to point out that multiplying the numerator and denominator by the same nonzero factor is equivalent to multiplying by 1 and does not change the problem. Because $\frac{x^2}{x^2}=1$, we can multiply the numerator and denominator by x^2 in the previous example and obtain an equivalent expression.

Example 7.4.7

Simplify:

$$\frac{\frac{1}{x+1}+\frac{3}{x-3}}{\frac{2}{x-3}-\frac{1}{x+1}}$$

Solution:

The LCM of all the denominators is $(x+1)(x-3)$. Begin by multiplying the numerator and denominator by these factors.

$$\begin{aligned}
\frac{\frac{1}{x+1}+\frac{3}{x-3}}{\frac{2}{x-3}-\frac{1}{x+1}} &= \frac{\left(\frac{1}{x+1}+\frac{3}{x-3}\right)\cdot(x+1)(x-3)}{\left(\frac{2}{x-3}-\frac{1}{x+1}\right)\cdot(x+1)(x-3)} && \textit{Distribute.}\\
&= \frac{\frac{1\cancel{(x+1)}\,(x-3)}{\cancel{x+1}}+\frac{3(x+1)\,\cancel{(x-3)}}{\cancel{x-3}}}{\frac{2(x+1)\,\cancel{(x-3)}}{\cancel{x-3}}-\frac{1\,\cancel{(x+1)}\,(x-3)}{\cancel{x+1}}} && \textit{Cancel.}\\
&= \frac{(x-3)+3(x+1)}{2(x+1)-1(x-3)} && \textit{Simplify.}\\
&= \frac{x-3+3x+3}{2x+2-x+3}\\
&= \frac{4x}{x+5}
\end{aligned}$$

Answer:

$$\frac{4x}{x+5}$$

Exercise 7.4.2

Simplify:

$$\frac{\frac{1}{y}-\frac{1}{4}}{\frac{1}{16}-\frac{1}{y^2}}$$

Answer available at the end of this section.

Key Takeaways

- Complex rational expressions can be simplified into equivalent expressions with a polynomial numerator and polynomial denominator.
- One method of simplifying a complex rational expression requires us to first write the numerator and denominator as a single algebraic fraction. Then multiply the numerator by the reciprocal of the divisor and simplify the result.
- Another method for simplifying a complex rational expression requires that we multiply it by a special form of 1. Multiply the numerator and denominator by the LCM of all the denominators as a means to clear the fractions. After doing this, simplify the remaining rational expression.
- An algebraic fraction is reduced to lowest terms if the numerator and denominator are polynomials that share no common factors other than 1.

Exercise 7.4.3 Complex Rational Expressions

Simplify. (Assume all denominators are nonzero.)

1. $\dfrac{\frac{1}{2}}{\frac{5}{4}}$
2. $\dfrac{\frac{7}{8}}{\frac{5}{4}}$
3. $\dfrac{\frac{10}{3}}{\frac{20}{9}}$
4. $-\dfrac{\frac{4}{21}}{\frac{8}{7}}$
5. $\dfrac{\frac{2}{3}}{\frac{5}{6}}$
6. $\dfrac{\frac{7}{4}}{\frac{14}{3}}$
7. $\dfrac{1-\frac{3}{2}}{\frac{5}{4}-\frac{1}{3}}$
8. $\dfrac{\frac{1}{2}-5}{\frac{1}{2}+\frac{1}{3}}$
9. $\dfrac{1+\frac{3}{2}}{1-\frac{1}{4}}$
10. $\dfrac{2-\frac{1}{2}}{1+\frac{3}{4}}$
11. $\dfrac{\frac{5x^2}{x+1}}{\frac{25x}{x+1}}$
12. $\dfrac{\frac{7+x}{7x}}{\frac{x+7}{14x^2}}$

13. $\dfrac{\dfrac{3y}{x}}{\dfrac{y^2}{x-1}}$

14. $\dfrac{\dfrac{5a^2}{b-1}}{\dfrac{15a^3}{(b-1)^2}}$

15. $\dfrac{1+\dfrac{1}{x}}{2-\dfrac{1}{x}}$

16. $\dfrac{\dfrac{2}{x}+1}{3-\dfrac{1}{x}}$

17. $\dfrac{\dfrac{2}{3y}-4}{6-\dfrac{1}{y}}$

18. $\dfrac{\dfrac{5}{y-12}}{\dfrac{10-y}{y^2}}$

19. $\dfrac{\dfrac{1}{5}-\dfrac{1}{x}}{\dfrac{1}{25}-\dfrac{1}{x^2}}$

20. $\dfrac{\dfrac{1}{x}+\dfrac{1}{5}}{\dfrac{1}{25}-\dfrac{1}{x^2}}$

21. $\dfrac{\dfrac{1}{x}-\dfrac{1}{3}}{\dfrac{1}{9}-\dfrac{1}{x^2}}$

22. $\dfrac{\dfrac{1}{4}+\dfrac{1}{x}}{\dfrac{1}{x^2}-\dfrac{1}{16}}$

23. $\dfrac{16-\dfrac{1}{x^2}}{\dfrac{1}{x}-4}$

24. $\dfrac{2-\dfrac{1}{y}}{1-\dfrac{1}{4y^2}}$

25. $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{y^2}-\frac{1}{x^2}}$

26. $\dfrac{\frac{1}{2x}-\frac{4}{3}}{\frac{1}{4x^2}-\frac{16}{9}}$

27. $\dfrac{\frac{2}{25}-\frac{1}{2x^2}}{\frac{1}{5}-\frac{1}{2x}}$

28. $\dfrac{\frac{4}{25}-\frac{1}{4x^2}}{\frac{1}{5}+\frac{1}{4x}}$

29. $\dfrac{\frac{1}{y}-\frac{1}{x}}{4-\frac{2}{xy}}$

30. $\dfrac{\frac{1}{ab}+2}{\frac{1}{a}+\frac{1}{b}}$

31. $\dfrac{\frac{1}{y}+\frac{1}{x}}{xy}$

32. $\dfrac{\frac{3}{x}}{\frac{1}{3}-\frac{1}{x}}$

33. $\dfrac{1-\frac{4}{x}-\frac{21}{x^2}}{1-\frac{2}{x}-\frac{15}{x^2}}$

34. $\dfrac{1-\frac{3}{x}-\frac{4}{x^2}}{1-\frac{16}{x^2}}$

35. $\dfrac{3-\frac{1}{2x}-\frac{1}{2x^2}}{2-\frac{2}{x}+\frac{1}{2x^2}}$

36. $\dfrac{\frac{1}{2}-\frac{5}{x}+\frac{12}{x^2}}{\frac{1}{2}-\frac{6}{x}+\frac{18}{x^2}}$

37. $\dfrac{\dfrac{1}{x}-\dfrac{4}{3x^2}}{3-\dfrac{8}{x}+\dfrac{16}{3x^2}}$

38. $\dfrac{1+\dfrac{3}{10x}-\dfrac{1}{10x^2}}{\dfrac{3}{5}-\dfrac{1}{10x}-\dfrac{1}{5x^2}}$

39. $\dfrac{x-1}{1+\dfrac{4}{x}}-\dfrac{5}{x^2}$

40. $2-\dfrac{5}{2x}-\dfrac{3x}{24x+3}$

41. $\dfrac{\dfrac{1}{x-3}+\dfrac{2}{x}}{\dfrac{1}{x}-\dfrac{3}{x-3}}$

42. $\dfrac{\dfrac{14}{x}-5+\dfrac{1}{x^2}}{\dfrac{1}{x^2}+\dfrac{13}{x}-10}$

43. $\dfrac{\dfrac{1}{x+5}+\dfrac{4}{x-2}}{\dfrac{2}{x-2}-\dfrac{1}{x+5}}$

44. $\dfrac{\dfrac{3}{x-1}-\dfrac{2}{x+3}}{\dfrac{2}{x+3}+\dfrac{1}{x-3}}$

45. $\dfrac{\dfrac{x}{x+1}-\dfrac{2}{x+3}}{\dfrac{x}{3x+4}+\dfrac{1}{x+1}}$

46. $\dfrac{\dfrac{x}{x-9}+\dfrac{2}{x+1}}{\dfrac{x}{7x-9}-\dfrac{1}{x+1}}$

47. $\dfrac{\dfrac{x}{3x+2}-\dfrac{1}{x+2}}{\dfrac{x}{x+2}-\dfrac{2}{x+2}}$

48. $\dfrac{\dfrac{x}{x-4}+\dfrac{1}{x+2}}{\dfrac{x}{3x+4}+\dfrac{1}{x+2}}$

49. $\dfrac{\dfrac{a}{3-8b^3}}{\dfrac{2}{\dfrac{7}{a}-\dfrac{2}{b}}}$

50. $\dfrac{\dfrac{27a}{3+b^3}}{\dfrac{\dfrac{a}{b}}{3a+b}}$

51. $\dfrac{\dfrac{1}{b^3}+\dfrac{1}{a^3}}{\dfrac{1}{b}+\dfrac{1}{a}}$

52. $\dfrac{\dfrac{1}{b^3}-\dfrac{1}{a^3}}{\dfrac{1}{a}-\dfrac{1}{b}}$

53. $\dfrac{\dfrac{x^2+y^2}{\dfrac{x}{y}+2}}{\dfrac{x^2-y^2}{\dfrac{2}{xy}}}$

54. $\dfrac{x}{\dfrac{y+4+4yx}{xy+3+2yx}}$

55. $1+\dfrac{1}{1+\dfrac{1}{2}}$

56. $2-\dfrac{1}{1+\dfrac{1}{3}}$

57. $\dfrac{1}{1+\dfrac{1}{1+x}}$

58. $\dfrac{x+\dfrac{1}{x}}{1-\dfrac{1}{x}+1}$

59. $\dfrac{1-\dfrac{1}{x}}{x-\dfrac{1}{x}}$

60. $\dfrac{\dfrac{1}{x}-x}{x-\dfrac{1}{x^2}}$

Answer available at the end of this section.

Exercise 7.4.4 Discussion Board Topics

1. Choose a problem from this exercise set and clearly work it out on paper, explaining each step in words. Scan your page and post it on the discussion board.

2. Explain why we need to simplify the numerator and denominator to a single algebraic fraction before multiplying by the reciprocal of the divisor.
3. Two methods for simplifying complex rational expressions have been presented in this section. Which of the two methods do you feel is more efficient, and why?

Answer available at the end of this section.

Answers

Exercise 7.4.1

$\frac{x-9}{9x}$

Exercise 7.4.2

$-\frac{4y}{y+4}$

Exercise 7.4.3 **Complex Rational Expressions**

1. $\frac{2}{5}$
3. $\frac{3}{2}$
5. $\frac{4}{5}$
7. $-\frac{6}{11}$
9. $\frac{10}{3}$
11. $\frac{x}{5}$
13. $\frac{3(x-1)}{yx}$
15. $\frac{x+1}{2x-1}$
17. $-\frac{2}{3}$
19. $\frac{5x}{x+5}$
21. $-\frac{3x}{x+3}$
23. $-\frac{4x+1}{x}$
25. $\frac{xy}{x-y}$
27. $\frac{2x+5}{5x}$
29. $\frac{x-y}{4xy-2}$
31. $\frac{x+y}{y^2x^2}$
33. $\frac{x-7}{x-5}$
35. $\frac{3x+1}{2x-1}$
37. $\frac{1}{3x-4}$

39. $\frac{x(x-1)}{x+4}-\frac{5}{x^2}$

41. $\frac{3x-6}{-2x-3}$

43. $\frac{5x+18}{x+12}$

45. $\frac{(x-1)(3x+4)}{(x+2)(x+3)}$

47. $\frac{x+1}{3x+2}$

49. $\frac{7b-2a}{2b\left(3-8b^3\right)}$

51. $\frac{a^2-ba+b^2}{b^2a^2}$

53. $\frac{2\left(x^2+y^2\right)}{x(x+2y)\left(x^2-y^2\right)}$

55. $\frac{5}{3}$

57. $\frac{x+1}{x+2}$

59. $\frac{1}{x+1}$

Exercise 7.4.4 **Discussion Board Topics**

1. Answers may vary
3. Answers may vary

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7.5: Solving Rational Equations

Learning Objectives

- Solve rational equations.
- Solve literal equations, or formulas, involving rational expressions.

Solving Rational Equations

A rational equation is an equation containing at least one rational expression. Rational expressions typically contain a variable in the denominator. For this reason, we will take care to ensure that the denominator is not 0 by making note of restrictions and checking our solutions.

Solve rational equations by clearing the fractions by multiplying both sides of the equation by the least common denominator (LCD).

Example 7.5.1

Solve:

$$\frac{5}{x}-\frac{1}{3}=\frac{1}{x}$$

Solution:

We first make a note that $x \neq 0$ and then multiply both sides by the LCD, $3x$:

$$\begin{aligned} 3x\cdot\left(\frac{5}{x}-\frac{1}{3}\right) &= 3x\cdot\left(\frac{1}{x}\right) && \textit{Multiply both sides by the LCD.} \\ 3x\cdot\frac{5}{x}-3x\cdot\frac{1}{3} &= 3x\cdot\frac{1}{x} && \textit{Distribute} \\ 15-x &= 3 && \textit{Simplify, and then solve.} \\ -x &= -12 \\ x &= 12 \end{aligned} \tag{7.5.1}$$

Check your answer by substituting 12 for x to see if you obtain a true statement.

$$\begin{aligned} \frac{5}{x}-\frac{1}{3} &= \frac{1}{x} \\ \frac{5}{12}-\frac{1}{3} &= \frac{1}{12} \\ \frac{5}{12}-\frac{4}{12} &= \frac{1}{12} \\ \frac{1}{12} &= \frac{1}{12} \quad \checkmark \end{aligned}$$

Answer:

The solution is 12.

After multiplying both sides of the previous example by the LCD, we were left with a linear equation to solve. This is not always the case; sometimes we will be left with a quadratic equation.

Example 7.5.2

Solve:

$$2-\frac{1}{x(x+1)}=\frac{3}{x+1}$$

Solution:

In this example, there are two restrictions, $x\neq 0$ and $x\neq -1$. Begin by multiplying both sides by the LCD, $x(x+1)$.

$$\begin{aligned} x(x+1)\cdot\left(2-\frac{1}{x(x+1)}\right) &= x(x+1)\cdot\frac{3}{x+1} \\ x(x+1)\cdot 2-\frac{x(x+1)\cdot 1}{x(x+1)} &= \frac{x(x+1)\cdot 3}{x+1} \\ 2x(x+1)-1 &= 3x \\ 2x^2+2x-1 &= 3x \end{aligned} \tag{7.5.2}$$

After distributing and dividing out the common factors, a quadratic equation remains. To solve it, rewrite it in standard form, factor, and then set each factor equal to 0.

$$\begin{aligned} 2x+1 &= 0 \quad \text{or} \quad x-1=0 \\ 2x &= -1 \qquad\quad x=1 \\ x &= -\tfrac{1}{2} \end{aligned}$$

Check to see if these values solve the original equation.

$$2-\frac{1}{x(x+1)}=\frac{3}{x+1}$$

$$\begin{array}{ll} Check\ x=-\frac{1}{2} & Check\ x=1 \\ 2-\frac{1}{(-\frac{1}{2})((-\frac{1}{2})+1)}=\frac{3}{(-\frac{1}{2})+1} & 2-\frac{1}{1(1+1)=}\frac{3}{1+1} \\ 2-\frac{1}{(-\frac{1}{4})}=\frac{3}{\frac{1}{2}} & \frac{4}{2}-\frac{1}{2}=\frac{3}{2} \\ 2+1\cdot\frac{4}{1}=3\cdot\frac{2}{1} & \frac{3}{2}=\frac{3}{2}\quad\checkmark \\ 2+4=6 & \\ 6=6\quad\checkmark & \end{array}$$

Answer:

The solutions are $-\frac{1}{2}$ and 1.

Up to this point, all of the possible solutions have solved the original equation. However, this may not always be the case. Multiplying both sides of an equation by variable factors may lead to extraneous solutions, which are solutions that do not solve the original equation. A complete list of steps for solving a rational equation is outlined in the following example.

Example 7.5.3

Solve:

$$\frac{x}{x+2}+\frac{2}{x^2+5x+6}=\frac{5}{x+3}$$

Solution:

Step 1: Factor all denominators and determine the LCD.

$$\frac{x}{x+2}+\frac{2}{x^2+5x+6}=\frac{5}{x+3}$$
$$\frac{x}{(x+2)}+\frac{2}{(x+2)(x+3)}=\frac{5}{(x+3)}$$

The LCD is $(x+2)(x+3)$.

Step 2: Identify the restrictions. In this case, they are $x \neq -2$ and $x \neq -3$.

Step 3: Multiply both sides of the equation by the LCD. Distribute carefully and then simplify.

$$(x+2)(x+3)\cdot\left(\frac{x}{(x+2)}+\frac{2}{(x+2)(x+3)}\right)=(x+2)(x+3)\cdot\frac{5}{(x+3)}$$
$$\frac{x\cdot\cancel{(x+2)}\,(x+3)}{\cancel{(x+2)}}+\frac{2\cdot\cancel{(x+2)}\,\cancel{(x+3)}}{\cancel{(x+2)}\,\cancel{(x+3)}}=\frac{5\cdot(x+2)\,\cancel{(x+3)}}{\cancel{(x+3)}}$$
$$x(x+3)+2=5(x+2)$$
$$x^2+3x+2=5x+10$$

Step 4: Solve the resulting equation. Here the result is a quadratic equation. Rewrite it in standard form, factor, and then set each factor equal to 0.

$$x+2=0 \quad \text{or} \quad x-4=0$$
$$x=-2 \qquad\qquad x=4$$

Step 5: Check for extraneous solutions. Always substitute into the original equation, or the factored equivalent. In this case, choose the factored equivalent to check:

$$\frac{x}{(x+2)}+\frac{2}{(x+2)(x+3)}=\frac{5}{(x+3)}$$

$Check\ x=-2$

$$\frac{-2}{(-2+2)}+\frac{2}{(-2+2)(-2+3)}=\frac{5}{(-2+3)}$$
$$-\frac{2}{0}+\frac{2}{0(1)}=\frac{5}{1} \quad x$$
$Undefined\ terms!$

$Check\ x=4$

$$\frac{4}{(4+2)}+\frac{2}{(4+2)(4+3)}=\frac{5}{(4+3)}$$
$$\frac{4}{6}+\frac{2}{6\cdot 7}=\frac{5}{7}$$
$$\frac{2}{3}+\frac{1}{21}=\frac{5}{7}$$
$$\frac{14}{21}+\frac{1}{21}=\frac{5}{7}$$
$$\frac{15}{21}=\frac{5}{7}$$
$$\frac{5}{7}=\frac{5}{7} \quad \checkmark$$

Here -2 is an extraneous solution and is not included in the solution set. It is important to note that -2 is a restriction.

Answer:

The solution is 4.

If this process produces a solution that happens to be a restriction, then disregard it as an extraneous solution.

Exercise 7.5.1

Try this! Solve: $\frac{x}{x-5}+\frac{3}{x+2}=\frac{7x}{x^2-3x-10}$.

Answer available at the end of this section.

YouTube Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Sometimes all potential solutions are extraneous, in which case we say that there is no solution to the original equation. In the next two examples, we demonstrate two ways in which a rational equation can have no solutions.

Example 7.5.4

Solve:

$$\frac{3x}{x^2-4} - \frac{2}{x+2} = \frac{1}{x+2}$$

Solution:

To identify the LCD, first factor the denominators.

$$\frac{3x}{x^2-4} - \frac{2}{x+2} = \frac{1}{x+2}$$
$$\frac{3x}{(x+2)(x-2)} - \frac{2}{(x+2)} = \frac{1}{(x+2)}$$

Multiply both sides by the least common denominator (LCD), $(x+2)(x-2)$, distributing carefully.

$$\begin{aligned} \frac{3x\cdot(x+2)(x-2)}{(x+2)(x-2)} - \frac{2\cdot(x+2)(x-2)}{(x+2)} &= \frac{1\cdot(x+2)(x-2)}{(x+2)} \\ 3x - 2(x-2) &= 1(x-2) \\ 3x - 2x + 4 &= x - 2 \\ x + 4 &= x - 2 \\ 4 &= -2 \text{ False} \end{aligned} \tag{7.5.3}$$

The equation is a contradiction and thus has no solution.

Answer:

No solution, $\emptyset$

Example 7.5.5

Solve:

$$\frac{x}{x-4} - \frac{4}{x+5} = \frac{36}{x^2+x-20}$$

Solution:

First, factor the denominators.

$$\frac{x}{(x-4)} - \frac{4}{(x+5)} = \frac{36}{(x-4)(x+5)}$$

Take note that the restrictions are $x \neq 4$ and $x \neq -5$. To clear the fractions, multiply by the LCD, $(x-4)(x+5)$.

$$\frac{x\cdot(x-4)(x+5)}{(x-4)}-\frac{4\cdot(x-4)(x+5)}{(x+5)}=\frac{36\cdot(x-4)(x+5)}{(x-4)(x+5)}$$
$$\begin{aligned} x(x+5)-4(x-4)&=36\\ x^2+5x-4x+16&=36\\ x^2+x-20&=0\\ (x-4)(x+5)&=0 \end{aligned} \tag{7.5.4}$$

$$\begin{aligned} x-4=0 \quad &\text{or} \quad x+5=0\\ x=4 \quad &\quad\;\; x=-5 \end{aligned} \tag{7.5.5}$$

Both of these values are restrictions of the original equation; hence both are extraneous.

Answer:

No solution, $\emptyset$

Exercise 7.5.2

$$\text{Try this! Solve: } \frac{1}{x+1}+\frac{x}{x-3}=\frac{4x}{x^2-2x-3}. \tag{7.5.6}$$

Answer available at the end of this section.

It is important to point out that this technique for clearing algebraic fractions only works for equations. Do not try to clear algebraic fractions when simplifying expressions. As a reminder, we have

Expression	*Equation*
$\frac{1}{x}+\frac{x}{2x+1}$	$\frac{1}{x}+\frac{x}{2x+1}=0$

Expressions are to be simplified and equations are to be solved. If we multiply the expression by the LCD, $x(2x+1)$, we obtain another expression that is not equivalent.

Incorrect	*Correct*
$\frac{1}{x}+\frac{x}{2x+1}$	$\frac{1}{x}+\frac{x}{2x+1}=0$
$\neq x(2x+1)\cdot\left(\frac{1}{x}+\frac{x}{2x+1}\right)$	$x(2x+1)\cdot\left(\frac{1}{x}+\frac{x}{2x+1}\right)=x(2x+1)\cdot 0$
$=2x+1+x^2$ x	$2x+1+x^2=0$
	$x^2+2x+1=0$ ✓

Literal Equations

Literal equations, or formulas, are often rational equations. Hence the techniques described in this section can be used to solve for particular variables. Assume that all variable expressions in the denominator are nonzero.

Example 7.5.6

Solve for x:

$$z=\frac{x-5}{y}$$

Solution:

The goal is to isolate ***x***. Assuming that ***y*** is nonzero, multiply both sides by ***y*** and then add 5 to both sides.

$$\begin{aligned} y \cdot z &= y \cdot \frac{x-5}{y} \\ yz &= x-5 \\ yz+5 &= x-5+5 \\ yz+5 &= x \end{aligned} \tag{7.5.7}$$

Answer:

$x = yz + 5$

Example 7.5.7

Solve for c:

$$\tfrac{1}{c} = \tfrac{1}{a} + \tfrac{1}{b}$$

Solution:

In this example, the goal is to isolate c. We begin by multiplying both sides by the LCD, $a \cdot b \cdot c$, distributing carefully.

$$\begin{aligned} \frac{1}{c} &= \frac{1}{a} + \frac{1}{b} \\ abc \cdot \frac{1}{c} &= abc \cdot \frac{1}{a} + abc \cdot \frac{1}{b} \\ ab &= bc + ac \end{aligned}$$

On the right side of the equation, factor out c.

$$ab = c(b+a)$$

Next, divide both sides of the equation by the quantity $(b+a)$.

$$\begin{aligned} \tfrac{ab}{(b+a)} &= \tfrac{c(b+a)}{(b+a)} \\ \tfrac{ab}{b+a} &= c \end{aligned}$$

Answer:

$$c = \tfrac{ab}{b+a}$$

Exercise 7.5.3

Solve for y:

$$x = \tfrac{y+1}{y-1}$$

Answer available at the end of this section.

Key Takeaways

- Begin solving rational equations by multiplying both sides by the LCD. The resulting equivalent equation can be solved using the techniques learned up to this point.
- Multiplying both sides of a rational equation by a variable expression introduces the possibility of extraneous solutions. Therefore, we must check the solutions against the set of restrictions. If a solution is a restriction, then it is not part of the domain and is extraneous.
- When multiplying both sides of an equation by an expression, distribute carefully and multiply each term by that expression.

- If all of the resulting solutions are extraneous, then the original equation has no solutions.

Exercise 7.5.4 Rational Equations

Solve.

1. $\frac{1}{2}+\frac{1}{x}=\frac{1}{8}$
2. $\frac{1}{3}-\frac{1}{x}=\frac{2}{9}$
3. $\frac{1}{3x}-\frac{2}{3}=\frac{1}{x}$
4. $\frac{2}{5x}-\frac{1}{x}=\frac{3}{10}$
5. $\frac{1}{2x+1}=5$
6. $\frac{3}{3x-1}+4=5$
7. $\frac{2x-3}{x+5}=\frac{2}{x+5}$
8. $\frac{5x}{2x-1}=\frac{x-1}{2x-1}$
9. $\frac{5}{x-7}=\frac{6}{x-9}$
10. $\frac{5}{x+5}=\frac{3}{x+1}$
11. $\frac{x}{6}-\frac{6}{x}=0$
12. $\frac{5x+x}{5}=-2$
13. $\frac{x}{x+12}=\frac{2}{x}$
14. $\frac{2x}{x+5}=\frac{1}{6-x}$
15. $\frac{1}{x}+\frac{x}{2x+1}=0$
16. $\frac{9x}{3x-1}-\frac{4}{x}=0$
17. $1-\frac{2}{x}=\frac{48}{x^2}$
18. $2-\frac{9}{x}=\frac{5}{x^2}$
19. $1+\frac{12}{x}=\frac{12}{x-2}$
20. $1-\frac{3x-5}{x(3x-4)}=-\frac{1}{x}$
21. $\frac{x}{2}=\frac{14}{x+3}$
22. $\frac{3x}{2}=\frac{x+1}{3-x}$
23. $6=\frac{-3x+3}{x-1}$
24. $\frac{1}{2x-2}=2+\frac{6(4-x)}{x-2}$
25. $2+\frac{2x}{x-3}=\frac{3(x-1)}{x-3}$
26. $\frac{x}{x-1}+\frac{1}{6x-1}=\frac{x(x-1)}{(6x-1)}$
27. $\frac{12}{x^2-81}=\frac{1}{x+9}-\frac{2}{x-9}$
28. $\frac{14}{x^2-49}=\frac{2}{x-7}-\frac{3}{x+7}$
29. $\frac{6x}{x+3}+\frac{4}{x-3}=\frac{3x}{x^2-9}$
30. $\frac{3x}{x+2}-\frac{17}{x-2}=-\frac{48}{x^2-4}$
31. $x^{-1}+3=0$
32. $4^{-y}-1=0$
33. $y^{-2}-4=0$
34. $9x^{-2}-1=0$
35. $3(x-1)^{-1}+5=0$
36. $5-2(3x+1)^{-1}=0$
37. $3+2x^{-3}=2x^{-3}$
38. $\frac{1}{x}=\frac{1}{x+1}$
39. $\frac{x}{x+1}=\frac{x+1}{x}$
40. $\frac{3x-1}{3x}=\frac{x}{x+3}$
41. $\frac{4x-7}{x-5}=\frac{3x-2}{x-5}$
42. $\frac{x}{x^2-9}=\frac{1}{x-3}$
43. $\frac{3x+4}{x-8}-\frac{2}{8-x}=1$
44. $\frac{1}{x}=\frac{6}{x(x+3)}$
45. $\frac{3}{x}=\frac{1}{x+1}+\frac{13}{x(x+1)}$
46. $\frac{x}{x-1}-\frac{3}{4x-1}=\frac{9x}{(4x-1)(x-1)}$
47. $\frac{1}{x-4}+\frac{x}{x-2}=2x^2-6x+8$

48. $\frac{x}{x-5}+\frac{x-1}{x^2-11x+30}=\frac{5}{x-6}$
49. $\frac{x}{x+1}-\frac{6}{5x^2+4x-1}=-\frac{5}{5x-1}$
50. $\frac{-8x^2-4}{x-12}+\frac{2(x+2)}{x^2+4x-60}=\frac{1}{x+2}$
51. $\frac{x}{x+2}-\frac{20}{x^2-6-x}=\frac{-4}{x-3}$
52. $\frac{x+7}{x-1}+\frac{x-1}{x+1}=\frac{4}{x^2-1}$
53. $\frac{x-1}{x-3}+\frac{x-3}{x-1}=-\frac{x+5}{x-3}$
54. $\frac{x-2}{x-5}-\frac{x-5}{x-2}=\frac{8-x}{x-5}$
55. $\frac{x+7}{x-2}-\frac{81}{x^2+5x-14}=\frac{9}{x+7}$
56. $\frac{x}{x-6}+1=\frac{5x+30}{36-x^2}$
57. $\frac{2x}{x+1}-\frac{4}{4x-3}=-\frac{7}{4x^2+x-3}$
58. $\frac{x-5}{x-10}+\frac{5}{x-5}=-\frac{5x}{x^2-15x+50}$
59. $\frac{5}{x^2+5x+4}+\frac{x+1}{x^2+3x-4}=\frac{5}{x^2-1}$
60. $\frac{1}{x^2-2x-63}+\frac{x-9x^2+10}{x+21}=\frac{1}{x^2-6x-27}$
61. $\frac{4}{x^2-4}+\frac{2(x-2)}{x^2-4x-12}=\frac{x+2}{x^2-8x+12}$
62. $\frac{x+2}{x^2-5x+4}+\frac{x+2}{x^2+x-2}=\frac{x-1}{x^2-2x-8}$
63. $\frac{6x}{x-1}-\frac{11x+1}{2x^2-x-1}=\frac{6x}{2x+1}$
64. $\frac{8x}{2x-3}+\frac{4x^2}{x^2-7x+6}=\frac{1}{x-2}$

Answer available at the end of this section.

Exercise 7.5.5 Literal Equations

Solve for the indicated variable.

1. Solve for r: $t=\frac{D}{r}$
2. Solve for b: $h=\frac{2A}{b}$
3. Solve for P: $t=\frac{I}{Pr}$
4. Solve for π: $r=\frac{C}{2\pi}$
5. Solve for c: $\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$
6. Solve for y: $m=\frac{y-y_1}{x-x_1}$
7. Solve for w: $P=2(l+w)$
8. Solve for t: $A=P(1+rt)$
9. Solve for m: $s=\frac{1}{n+m}$
10. Solve for S: $h=S2\pi r-r$
11. Solve for x: $y=\frac{x}{x+2}$
12. Solve for x: $y=2x+15x$
13. Solve for R: $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$
14. Solve for S_1: $\frac{1}{f}=\frac{1}{S_1}+\frac{1}{S_2}$

Answer available at the end of this section.

Exercise 7.5.6 Discussion Board

1. Explain why multiplying both sides of an equation by the LCD sometimes produces extraneous solutions.
2. Explain the connection between the technique of cross multiplication and multiplying both sides of a rational equation by the LCD.
3. Explain how we can tell the difference between a rational expression and a rational equation. How do we treat them differently?

Answer available at the end of this section.

Answers

Exercise 7.5.1

-3

Exercise 7.5.2

$\emptyset$

Exercise 7.5.3

$y = \frac{1+x}{x-1}$

Exercise 7.5.4 **Rational Equations**

1. $-\frac{8}{3}$
3. -1
5. $-\frac{2}{5}$
7. $\frac{5}{2}$
9. -3
11. $-6, 6$
13. $-4, 6$
15. -1
17. $-6, 8$
19. $-4, 6$
21. $-7, 4$
23. $\emptyset$
25. $\emptyset$
27. -39
29. $\frac{4}{3}, \frac{3}{2}$
31. $-\frac{1}{3}$
33. $-\frac{1}{2}, \frac{1}{2}$
35. $\frac{2}{5}$
37. $\emptyset$
39. $-\frac{1}{2}$
41. $\emptyset$
43. -7
45. 5
47. -1
49. $\emptyset$
51. -4
53. $\frac{5}{3}$
55. $\emptyset$
57. $\frac{1}{2}$
59. $-6, 4$
61. 10
63. $\frac{1}{3}$

Exercise 7.5.5 **Literal Equations**

1. $r = \frac{D}{t}$
3. $P = \frac{I}{tr}$
5. $c = \frac{ab}{b-a}$
7. $w = \frac{P}{2} - l$
9. $m = \frac{1}{s} - n$
11. $x = -\frac{2y}{y-1}$
13. $R = \frac{R_1 R_2}{R_1 + R_2}$

Exercise 7.5.6 **Discussion Board**

1. Answers may vary
3. Answers may vary

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7.6: Applications of Rational Equations

Learning Objectives

- Solve applications involving relationships between real numbers.
- Solve applications involving uniform motion (distance problems).
- Solve work-rate applications.

Number Problems

Recall that the reciprocal of a nonzero number n is $\frac{1}{n}$. For example, the reciprocal of 5 is $\frac{1}{5}$ and $5 \cdot \frac{1}{5} = 1$. In this section, the applications will often involve the key word "reciprocal." When this is the case, we will see that the algebraic setup results in a rational equation.

Example 7.6.1

A positive integer is 4 less than another. The sum of the reciprocals of the two positive integers is $\frac{10}{21}$. Find the two integers.

Solution:

Begin by assigning variables to the unknowns.

Let n represent the larger positive integer.

Let $n-4$ represent the smallest positive integer.

Next, use the reciprocals $\frac{1}{n}$ and $\frac{1}{n-4}$ to translate the sentences into an algebraic equation.

$$\begin{array}{lll} \textit{the sum of the reciprocals} & \textit{is} & \\ \frac{1}{n}+\frac{1}{n-4} & = & \frac{10}{21} \end{array}$$

We can solve this rational expression by multiplying both sides of the equation by the least common denominator (LCD). In this case, the LCD is $21n(n-4)$.

$$\begin{array}{rll} \frac{1}{n}+\frac{1}{n-4}=\frac{10}{21} & & \\ 21n(n-4)\cdot\left(\frac{1}{n}+\frac{1}{n-4}\right)=21n(n-4)\cdot\left(\frac{10}{21}\right) & & \textit{Multiply both sides by the LCD.} \\ 21n(n-4)\cdot\frac{1}{n}+21n(n-4)\cdot\frac{1}{n-4}=21n(n-4)\cdot\left(\frac{10}{21}\right) & & \textit{Distribute and then cancel.} \\ 21(n-4)+21n=10n(n-4) & & \end{array}$$

Solve the resulting quadratic equation.

$$\begin{array}{rlll} 5n-6 & =0 & \text{or} & n-7=0 \\ 5n & =6 & & n=7 \\ n & =\frac{6}{5} & & \end{array}$$

The question calls for integers and the only integer solution is $n=7$. Hence disregard $\frac{6}{5}$. Use the expression $n-4$ to find the smaller integer.

$$n-4=7-4=3$$

Answer:

The two positive integers are 3 and 7. The check is left to the reader.

Example 7.6.2

A positive integer is 4 less than another. If the reciprocal of the smaller integer is subtracted from twice the reciprocal of the larger, then the result is $\frac{1}{30}$. Find the two integers.

Solution:

Let n represent the larger positive integer.

Let $n-4$ represent the smaller positive integer.

Set up an algebraic equation.

Solve this rational expression by multiplying both sides by the LCD. The LCD is $30n(n-4)$.

$$\frac{2}{n}-\frac{1}{n-4}=\frac{1}{30}$$

$$30n(n-4)\cdot\left(\frac{2}{n}-\frac{1}{n-4}\right)=30n(n-4)\cdot\left(\frac{1}{30}\right)$$

$$30n(n-4)\cdot\frac{2}{n}-30n(n-4)\cdot\frac{1}{n-4}=30n(n-4)\cdot\left(\frac{1}{30}\right)$$

$$\begin{aligned} n-10 &= 0 \quad \text{or} \quad n-24 = 0 \\ n &= 10 \qquad\quad n = 24 \end{aligned}$$

Here we have two viable possibilities for the larger integer. For this reason, we will we have two solutions to this problem.

If $n=10$, then $n-4=10-4=6$.

If $n=24$, then $n-4=24-4=20$.

As a check, perform the operations indicated in the problem.

$$\begin{aligned} &\text{Check } 6 \text{ and } 10. \\ 2\left(\tfrac{1}{10}\right)-\tfrac{1}{6} &= \tfrac{1}{5}-\tfrac{1}{6} \\ &= \tfrac{6}{30}-\tfrac{5}{30} \\ &= \tfrac{1}{30} \quad \checkmark \end{aligned} \qquad \begin{aligned} &\text{Check } 20 \text{ and } 24. \\ 2\left(\tfrac{1}{24}\right)-\tfrac{1}{20} &= \tfrac{1}{12}-\tfrac{1}{20} \\ &= \tfrac{5}{60}-\tfrac{3}{60} \\ &= \tfrac{1}{30} \quad \checkmark \end{aligned}$$

Answer:

Two sets of positive integers solve this problem: $\{6,10\}$ and $\{20,24\}$.

Exercise 7.6.1

The difference between the reciprocals of two consecutive positive odd integers is $\frac{2}{15}$. Find the integers.

Answer available at the end of this section.

Uniform Motion Problems

Uniform motion problems, also referred to as distance problems, involve the formula

$$D = rt \tag{7.6.1}$$

where the distance, D, is given as the product of the average rate, r, and the time, t, traveled at that rate. If we divide both sides by the average rate, r, then we obtain the formula

$$t = \frac{D}{r} \tag{7.6.2}$$

For this reason, when the unknown quantity is time, the algebraic setup for distance problems often results in a rational equation. Similarly, when the unknown quantity is the rate, the setup also may result in a rational equation.

We begin any uniform motion problem by first organizing our data with a chart. Use this information to set up an algebraic equation that models the application.

Example 7.6.5

Mary spent the first 120 miles of her road trip in traffic. When the traffic cleared, she was able to drive twice as fast for the remaining 300 miles. If the total trip took 9 hours, then how fast was she moving in traffic?

Solution:

First, identify the unknown quantity and organize the data.

Let x represent Mary's average speed (miles per hour) in traffic.

Let $2x$ represent her average speed after the traffic cleared.

	Distance =	***Rate*** ×	***Time***
In traffic	120 mi	x	
Clear traffic	300 mi	$2x$	
Total			9 hours

Figure 7.6.1

To avoid introducing two more variables for the time column, use the formula $t = \frac{D}{r}$. Here the time for each leg of the trip is calculated as follows:

$$\begin{aligned} \textit{Time spent in traffic}: \; t &= \tfrac{D}{r} = \tfrac{120}{x} \\ \textit{Time clear of traffic}: \; t &= \tfrac{D}{r} = \tfrac{300}{2x} \end{aligned}$$

Use these expressions to complete the chart.

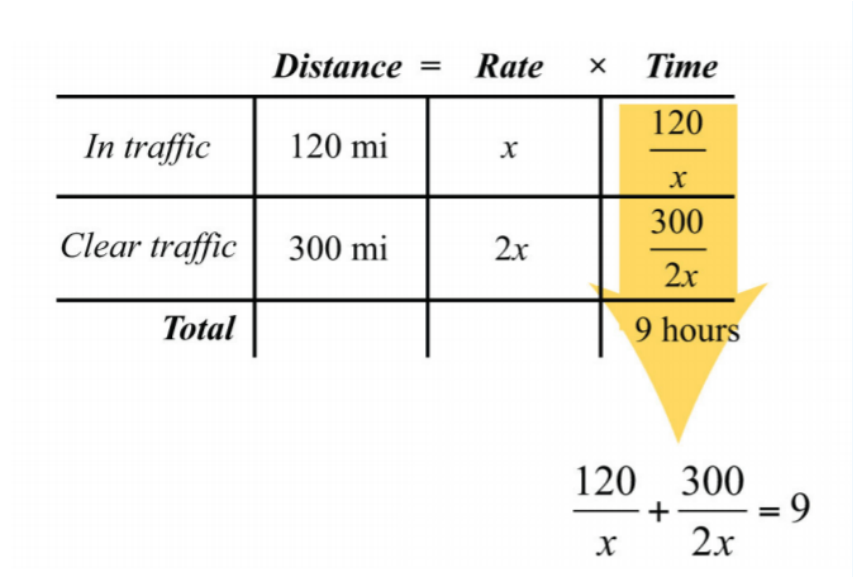


Figure 7.6.2

The algebraic setup is defined by the time column. Add the times for each leg of the trip to obtain a total of 9 hours:

$$\overbrace{\frac{120}{x}}^{time\ spent\ in\ traffic} + \overbrace{\frac{300}{2x}}^{time\ clear\ of\ traffic} = \overbrace{9}^{total\ time\ of\ trip}$$

We begin solving this equation by first multiplying both sides by the LCD, $2x$.

$$\begin{aligned} \frac{120}{x}+\frac{300}{2x} &= 9 \\ 2x\cdot\left(\frac{120}{x}+\frac{300}{2x}\right) &= 2x\cdot 9 \\ 2x\cdot\frac{120}{x}+2x\cdot\frac{300}{2x} &= 2x\cdot 9 \\ 240+300 &= 18x \\ 540 &= 18x \\ 30 &= x \end{aligned}$$

Answer:

Mary averaged 30 miles per hour in traffic.

Example 7.6.6

A passenger train can travel, on average, 20 miles per hour faster than a freight train. If the passenger train covers 390 miles in the same time it takes the freight train to cover 270 miles, then how fast is each train?

Solution:

First, identify the unknown quantities and organize the data.

Let x represent the average speed of the freight train.

Let $x+20$ represent the speed of the passenger train.

Next, organize the given data in a chart.

	Distance =	Rate ×	Time
Passenger Train	390 mi	$x+20$	
Freight Train	270 mi	x	
Total			

Figure 7.6.3

Use the formula $t = \frac{D}{r}$ to fill in the time column for each train.

$$\begin{aligned} \textit{Passenger train}: \; t &= \tfrac{D}{r} = \tfrac{390}{x+20} \\ \textit{Freight train}: \; t &= \tfrac{D}{r} = \tfrac{270}{x} \end{aligned}$$

	Distance =	Rate ×	Time
Passenger Train	390 mi	$x+20$	$\frac{390}{x+20}$
Freight Train	270 mi	x	$\frac{270}{x}$
Total			

Figure 7.6.4

Because the trains travel the same amount of time, finish the algebraic setup by equating the expressions that represent the times:

$$\tfrac{390}{x+20} = \tfrac{270}{x}$$

Solve this equation by first multiplying both sides by the LCD, $x(x+20)$.

$$\begin{aligned} x(x+20)\cdot\left(\frac{390}{x+20}\right) &= x(x+20)\cdot\left(\frac{270}{x}\right) \\ 390x &= 270(x+20) \\ 390x &= 270x + 5400 \\ 120x &= 5400 \\ x &= 45 \end{aligned}$$

Use $x+20$ to find the speed of the passenger train.

$$x + 20 = 45 + 20 = 65$$

Answer:

The speed of the passenger train is 65 miles per hour and the speed of the freight train is 45 miles per hour.

Example 7.6.7

Brett lives on the river 8 miles upstream from town. When the current is 2 miles per hour, he can row his boat downstream to town for supplies and back in 3 hours. What is his average rowing speed in still water?

Solution:

Let x represent Brett's average rowing speed in still water.

Rowing downstream, the current increases his speed, and his rate is $x+2$ miles per hour. Rowing upstream, the current decreases his speed, and his rate is $x-2$ miles per hour. Begin by organizing the data in the following chart:

	Distance =	Rate ×	Time
Trip downstream	8 mi	$x+2$	
Trip upstream	8 mi	$x-2$	
Total			3 hours

Figure 7.6.5

Use the formula $t=\frac{D}{r}$ to fill in the time column for each leg of the trip.

$$\textit{Trip downstream}: \; t=\frac{D}{r}=\frac{8}{x+2}$$
$$\textit{Trip upstream}: \; t=\frac{D}{r}=\frac{8}{x-2}$$

	Distance =	Rate ×	Time
Trip downstream	8 mi	$x+2$	$\frac{8}{x+2}$
Trip upstream	8 mi	$x-2$	$\frac{8}{x-2}$
Total			3 hours

Figure 7.6.6

The algebraic setup is defined by the time column. Add the times for each leg of the trip to obtain a total of 3 hours:

$$\underbrace{\frac{8}{x+2}}_{time\ rowing\ downstream} + \underbrace{\frac{8}{x-2}}_{time\ rowing\ upstream} = \underbrace{3}_{total\ time\ of\ trip}$$

Solve this equation by first multiplying both sides by the LCD, $(x+2)(x-2)$.

Next, solve the resulting quadratic equation.

$$\begin{aligned} 3x+2 &= 0 \quad \text{or} \quad x-6=0 \\ 3x &= -2 \qquad\quad x=6 \\ x &= \tfrac{-2}{3} \end{aligned}$$

Use only the positive solution, $x=6$ miles per hour.

Answer:

His rowing speed is 6 miles per hour.

Exercise 7.6.2

Dwayne drove 18 miles to the airport to pick up his father and then returned home. On the return trip he was able to drive an average of 15 miles per hour faster than he did on the trip there. If the total driving time was 1 hour, then what was his average speed driving to the airport?

Answer available at the end of this section.

Work-Rate Problems

The rate at which a task can be performed is called a work rate.

Example 7.6.8

For example, if a painter can paint a room in 8 hours, then the task is to paint the room, and we can write

$$\underbrace{\frac{1\text{ task}}{8\text{ hours}}}_{Work\ rate}$$

In other words, the painter can complete $\frac{1}{8}$ of the task per hour.

a. Calculate the fraction of the task that one painter can accomplish in 2, 4 and 8 hours

b. If an apprentice would take 10 hours to paint the room, how long would it take for the apprentice and the master painter to paint the room.

Solution

a. If he works for less than 8 hours, then he will perform a fraction of the task. For example,

$$\begin{aligned} work\ rate \times time &= work\ completed \\ \tfrac{1}{8} \times 2\text{hrs} &= \tfrac{1}{4} \qquad One-quarter\ of\ the\ room\ painted \\ \tfrac{1}{8} \times 4\text{hrs} &= \tfrac{1}{2} \qquad One-half\ of\ the\ room\ painted \\ \tfrac{1}{8} \times 8\text{hrs} &= 1 \qquad One\ whole\ room\ painted \end{aligned}$$

Obtain the amount of the task completed by multiplying the work rate by the amount of time the painter works. Typically, work-rate problems involve people working together to complete tasks. When this is the case, we can organize the data in a chart, just as we have done with distance problems.

b. Suppose an apprentice painter can paint the same room by himself in 10 hours. Then we say that he can complete $\frac{1}{10}$ of the task per hour.

Let t represent the time it takes both of the painters, working together, to paint the room.

	Amount of Task Completed	=	Work - Rate	×	Time
Painter			$\frac{1}{8}$		t
Apprentice			$\frac{1}{10}$		t
Total	1				

Figure $7.6.7$

To complete the chart, multiply the work rate by the time for each person.

The portion of the room each can paint adds to a total of 1 task completed.

This is represented by the equation obtained from the first column of the chart:

	Amount of Task Completed	=	Work - Rate	×	Time
Painter	$\frac{1}{8}t$		$\frac{1}{8}$		t
Apprentice	$\frac{1}{10}t$		$\frac{1}{10}$		t
Total	1				

$$\frac{1}{8}t+\frac{1}{10}t=1$$

Figure $7.6.8$

This setup results in a rational equation that can be solved for t by multiplying both sides by the LCD, 40.

$$\begin{aligned}
\frac{1}{8}t+\frac{1}{10}t &= 1\\
40\cdot\left(\frac{1}{8}t+\frac{1}{10}t\right) &= 40\cdot 1\\
40\cdot\frac{t}{8}+40\cdot\frac{t}{10} &= 40\cdot 1\\
5t+4t &= 40\\
9t &= 40\\
t &= \frac{40}{9}\\
t &= 4\frac{4}{9}
\end{aligned}$$

Therefore, the two painters, working together, complete the task in $4\frac{4}{9}$ hours.

In general, we have the following work-rate formula:

$$\frac{1}{t_1}t+\frac{1}{t_2}t=1 \tag{7.6.3}$$

Here $\frac{1}{t_1}$ and $\frac{1}{t_2}$ are the individual work rates and t is the time it takes to complete one task working together. If we factor out the time, t, and then divide both sides by t, we obtain an equivalent work-rate formula:

$$\begin{aligned}
&\tfrac{1}{t_1}t+\tfrac{1}{t_2}t=1\\
&t\left(\tfrac{1}{t_1}+\tfrac{1}{t_2}\right)=1\\
&\tfrac{1}{t_1}+\tfrac{1}{t_2}=\tfrac{1}{t}
\end{aligned}$$

In summary, we have the following equivalent work-rate formulas:

$$\textit{Work rate formulas :}$$
$$\tfrac{1}{t_1}t \quad \text{or} \quad \tfrac{t}{t_1}+\tfrac{t}{t_2}=1 \quad \text{or} \quad \tfrac{1}{t_1}+\tfrac{1}{t_2}=\tfrac{1}{t}$$

Example 7.6.9

Working alone, Billy's dad can complete the yard work in 3 hours. If Billy helps his dad, then the yard work takes 2 hours. How long would it take Billy working alone to complete the yard work?

Solution:

The given information tells us that Billy's dad has an individual work rate of $\frac{1}{3}$ task per hour. If we let x represent the time it takes Billy working alone to complete the yard work, then Billy's individual work rate is $\frac{1}{x}$, and we can write

	Amount of Task Completed	=	Work - Rate	×	Time
Billy's Dad			$\frac{1}{3}$		2
Billy			$\frac{1}{x}$		2
Total	1				

Figure 7.6.9

Working together, they can complete the task in 2 hours. Multiply the individual work rates by 2 hours to fill in the chart.

	Amount of Task Completed	=	Work - Rate	×	Time
Billy's Dad	$\frac{1}{3}\cdot 2$		$\frac{1}{3}$		2
Billy	$\frac{1}{x}\cdot 2$		$\frac{1}{x}$		2
Total	1				

$$\frac{1}{3}\cdot 2+\frac{1}{x}\cdot 2=1$$

Figure 7.6.10

The amount of the task each completes will total 1 completed task. To solve for x, we first multiply both sides by the LCD, $3x$.

$$\begin{aligned}\frac{1}{3}\cdot 2+\frac{1}{x}\cdot 2&=1\\ 3x\cdot\left(\frac{2}{3}+\frac{2}{x}\right)&=3x\cdot 1\\ 3x\cdot\frac{2}{3}+3x\cdot\frac{2}{x}&=3x\cdot 1\\ 2x+6&=3x\\ 6&=x\end{aligned}$$

Answer:

It takes Billy 6 hours to complete the yard work alone.

Of course, the unit of time for the work rate need not always be in hours.

Example 7.6.10

Working together, two construction crews can build a shed in 5 days. Working separately, the less experienced crew takes twice as long to build a shed than the more experienced crew. Working separately, how long does it take each crew to build a shed?

Solution:

Let x represent the time it takes the more experienced crew to build a shed.

Let $2x$ represent the time it takes the less experienced crew to build a shed.

Working together, the job is completed in 5 days. This gives the following setup:

	Amount of Task Completed	= Work - Rate	× Time
Experienced Crew	$\frac{1}{x}\cdot 5$	$\frac{1}{x}$	5 days
Inexperienced Crew	$\frac{1}{2x}\cdot 5$	$\frac{1}{2x}$	5 days
Total	1		

Figure 7.6.11

The first column in the chart gives us an algebraic equation that models the problem:

$$\frac{1}{x}\cdot 5+\frac{1}{2x}\cdot 5=1$$
$$\frac{5}{x}+\frac{5}{2x}=1$$

Solve the equation by multiplying both sides by $2x$.

$$2x\cdot\left(\frac{5}{x}+\frac{5}{2x}\right)=2x\cdot 1$$
$$2x\cdot\frac{5}{x}+2x\cdot\frac{5}{2x}=2x\cdot 1$$
$$10+5=2x$$
$$15=2x$$
$$\frac{15}{2}=x \quad \text{or} \quad x=7\frac{1}{2}\text{ days}$$

To determine the time it takes the less experienced crew, we use $2x$:

$$2x = 2\left(\frac{15}{2}\right)$$
$$= 15 \text{ days}$$

Answer:

Working separately, the experienced crew takes $7\frac{1}{2}$ days to build a shed, and the less experienced crew takes 15 days to build a shed.

Exercise 7.6.3

Joe's garden hose fills the pool in 12 hours. His neighbor has a thinner hose that fills the pool in 15 hours. How long will it take to fill the pool using both hoses?

Answer available at the end of this section.

Key Takeaways

- In this section, all of the steps outlined for solving general word problems apply. Look for the new key word "reciprocal," which indicates that you should write the quantity in the denominator of a fraction with numerator 1.
- When solving distance problems where the time element is unknown, use the equivalent form of the uniform motion formula, $t = \frac{D}{r}$, to avoid introducing more variables.
- When solving work-rate problems, multiply the individual work rate by the time to obtain the portion of the task completed. The sum of the portions of the task results in the total amount of work completed.

Exercise 7.6.4 Number Problems

Use algebra to solve the following applications.

1. A positive integer is twice another. The sum of the reciprocals of the two positive integers is $\frac{3}{10}$. Find the two integers.
2. A positive integer is twice another. The sum of the reciprocals of the two positive integers is $\frac{3}{12}$. Find the two integers.
3. A positive integer is twice another. The difference of the reciprocals of the two positive integers is $\frac{1}{8}$. Find the two integers.
4. A positive integer is twice another. The difference of the reciprocals of the two positive integers is $\frac{1}{18}$. Find the two integers.
5. A positive integer is 2 less than another. If the sum of the reciprocal of the smaller and twice the reciprocal of the larger is $\frac{5}{12}$, then find the two integers.
6. A positive integer is 2 more than another. If the sum of the reciprocal of the smaller and twice the reciprocal of the larger is $\frac{17}{35}$, then find the two integers.
7. The sum of the reciprocals of two consecutive positive even integers is $\frac{11}{60}$. Find the two even integers.
8. The sum of the reciprocals of two consecutive positive odd integers is $\frac{16}{63}$. Find the integers.
9. The difference of the reciprocals of two consecutive positive even integers is $\frac{1}{24}$. Find the two even integers.
10. The difference of the reciprocals of two consecutive positive odd integers is $\frac{2}{99}$. Find the integers.
11. If 3 times the reciprocal of the larger of two consecutive integers is subtracted from 2 times the reciprocal of the smaller, then the result is $\frac{1}{2}$. Find the two integers.
12. If 3 times the reciprocal of the smaller of two consecutive integers is subtracted from 7 times the reciprocal of the larger, then the result is $\frac{1}{2}$. Find the two integers.
13. A positive integer is 5 less than another. If the reciprocal of the smaller integer is subtracted from 3 times the reciprocal of the larger, then the result is $\frac{1}{12}$. Find the two integers.
14. A positive integer is 6 less than another. If the reciprocal of the smaller integer is subtracted from 10 times the reciprocal of the larger, then the result is $\frac{3}{7}$. Find the two integers.

Answer available at the end of this section.

Exercise 7.6.5 Uniform Motion Problems

Use algebra to solve the following applications.

1. James can jog twice as fast as he can walk. He was able to jog the first 9 miles to his grandmother's house, but then he tired and walked the remaining 1.5 miles. If the total trip took 2 hours, then what was his average jogging speed?
2. On a business trip, an executive traveled 720 miles by jet aircraft and then another 80 miles by helicopter. If the jet averaged 3 times the speed of the helicopter and the total trip took 4 hours, then what was the average speed of the jet?
3. Sally was able to drive an average of 20 miles per hour faster in her car after the traffic cleared. She drove 23 miles in traffic before it cleared and then drove another 99 miles. If the total trip took 2 hours, then what was her average speed in traffic?
4. Harry traveled 15 miles on the bus and then another 72 miles on a train. If the train was 18 miles per hour faster than the bus and the total trip took 2 hours, then what was the average speed of the train?
5. A bus averages 6 miles per hour faster than a trolley. If the bus travels 90 miles in the same time it takes the trolley to travel 75 miles, then what is the speed of each?
6. A passenger car averages 16 miles per hour faster than the bus. If the bus travels 56 miles in the same time it takes the passenger car to travel 84 miles, then what is the speed of each?
7. A light aircraft travels 2 miles per hour less than twice as fast as a passenger car. If the passenger car can travel 231 miles in the same time it takes the aircraft to travel 455 miles, then what is the average speed of each?
8. Mary can run 1 mile per hour more than twice as fast as Bill can walk. If Bill can walk 3 miles in the same time it takes Mary to run 7.2 miles, then what is Bill's average walking speed?
9. An airplane traveling with a 20-mile-per-hour tailwind covers 270 miles. On the return trip against the wind, it covers 190 miles in the same amount of time. What is the speed of the airplane in still air?
10. A jet airliner traveling with a 30-mile-per-hour tailwind covers 525 miles in the same amount of time it is able to travel 495 miles after the tailwind eases to 10 miles per hour. What is the speed of the airliner in still air?
11. A boat averages 16 miles per hour in still water. With the current, the boat can travel 95 miles in the same time it travels 65 miles against it. What is the speed of the current?
12. A river tour boat averages 7 miles per hour in still water. If the total 24-mile tour down river and 24 miles back takes 7 hours, then how fast is the river current?
13. If the river current flows at an average 3 miles per hour, then a tour boat makes the 9-mile tour downstream with the current and back the 9 miles against the current in 4 hours. What is the average speed of the boat in still water?
14. Jane rowed her canoe against a 1-mile-per-hour current upstream 12 miles and then returned the 12 miles back downstream. If the total trip took 5 hours, then at what speed can Jane row in still water?
15. Jose drove 15 miles to pick up his sister and then returned home. On the return trip, he was able to average 15 miles per hour faster than he did on the trip to pick her up. If the total trip took 1 hour, then what was Jose's average speed on the return trip?
16. Barry drove the 24 miles to town and then back in 1 hour. On the return trip, he was able to average 14 miles per hour faster than he averaged on the trip to town. What was his average speed on the trip to town?
17. Jerry paddled his kayak upstream against a 1-mile-per-hour current for 12 miles. The return trip downstream with the 1 -mile-per-hour current took 1 hour less time. How fast can Jerry paddle the kayak in still water?
18. It takes a light aircraft 1 hour more time to fly 360 miles against a 30- mile-per-hour head wind than it does to fly the same distance with it. What is the speed of the aircraft in calm air?

Answer available at the end of this section.

Exercise 7.6.6 Work-Rate Problems

Use algebra to solve the following applications.

1. James can paint the office by himself in 7 hours. Manny paints the office in 10 hours. How long will it take them to paint the office working together?
2. Barry can lay a brick driveway by himself in 12 hours. Robert does the same job in 10 hours. How long will it take them to lay the brick driveway working together?
3. Jerry can detail a car by himself in 50 minutes. Sally does the same job in 1 hour. How long will it take them to detail a car working together?

4. Jose can build a small shed by himself in 26 hours. Alex builds the same small shed in 2 days. How long would it take them to build the shed working together?
5. Allison can complete a sales route by herself in 6 hours. Working with an associate, she completes the route in 4 hours. How long would it take her associate to complete the route by herself?
6. James can prepare and paint a house by himself in 5 days. Working with his brother, Bryan, they can do it in 3 days. How long would it take Bryan to prepare and paint the house by himself?
7. Joe can assemble a computer by himself in 1 hour. Working with an assistant, he can assemble a computer in 40 minutes. How long would it take his assistant to assemble a computer working alone?
8. The teacher's assistant can grade class homework assignments by herself in 1 hour. If the teacher helps, then the grading can be completed in 20 minutes. How long would it take the teacher to grade the papers working alone?
9. A larger pipe fills a water tank twice as fast as a smaller pipe. When both pipes are used, they fill the tank in 5 hours. If the larger pipe is left off, then how long would it take the smaller pipe to fill the tank?
10. A newer printer can print twice as fast as an older printer. If both printers working together can print a batch of flyers in 45 minutes, then how long would it take the newer printer to print the batch working alone?
11. Working alone, Henry takes 9 hours longer than Mary to clean the carpets in the entire office. Working together, they clean the carpets in 6 hours. How long would it take Mary to clean the office carpets if Henry were not there to help?
12. Working alone, Monique takes 4 hours longer than Audrey to record the inventory of the entire shop. Working together, they take inventory in 1.5 hours. How long would it take Audrey to record the inventory working alone?
13. Jerry can lay a tile floor in 3 hours less time than Jake. If they work together, the floor takes 2 hours. How long would it take Jerry to lay the floor by himself?
14. Jeremy can build a model airplane in 5 hours less time than his brother. Working together, they need 6 hours to build the plane. How long would it take Jeremy to build the model airplane working alone?
15. Harry can paint a shed by himself in 6 hours. Jeremy can paint the same shed by himself in 8 hours. How long will it take them to paint two sheds working together?
16. Joe assembles a computer by himself in 1 hour. Working with an assistant, he can assemble 10 computers in 6 hours. How long would it take his assistant to assemble 1 computer working alone?
17. Jerry can lay a tile floor in 3 hours, and his assistant can do the same job in 4 hours. If Jerry starts the job and his assistant joins him 1 hour later, then how long will it take to lay the floor?
18. Working alone, Monique takes 6 hours to record the inventory of the entire shop, while it takes Audrey only 4 hours to do the same job. How long will it take them working together if Monique leaves 2 hours early?

Answer available at the end of this section.

Answers

Exercise $7.6.1$

The integers are 3 and 5.

Exercise $7.6.2$

His average speed driving to the airport was 30 miles per hour.

Exercise $7.6.3$

It will take both hoses $6\frac{2}{3}$ hours to fill the pool.

Exercise $7.6.4$ **Number Problems**

1. $\{5, 10\}$
3. $\{4, 8\}$
5. $\{6, 8\}$
7. $\{10, 12\}$
9. $\{6, 8\}$
11. $\{1, 2\}$ or $\{-4, -3\}$
13. $\{4, 9\}$ or $\{15, 20\}$

Exercise $7.6.5$ **Uniform Motion Problems**

1. 6 miles per hour
3. 46 miles per hour
5. Trolley: 30 miles per hour; bus: 36 miles per hour
7. Passenger car: 66 miles per hour; aircraft: 130 miles per hour
9. 115 miles per hour
11. 3 miles per hour
13. 6 miles per hour
15. 40 miles per hour

17. 5 miles per hour

Exercise $7.6.6$ **Work-Rate Problems**

1. $4\frac{2}{17}$ hours
3. $27\frac{3}{11}$ minutes
5. 12 hours
7. 2 hours
9. 15 hours
11. 9 hours
13. 3 hours
15. $6\frac{6}{7}$ hours
17. $2\frac{1}{7}$ hours

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7.7: Variation

Learning Objectives

- Solve applications involving direct variation.
- Solve applications involving inverse variation.
- Solve applications involving joint variation.

Direct Variation

Consider a freight train moving at a constant speed of 30 miles per hour. The equation that expresses the distance traveled at that speed in terms of time is given by

$$D = 30t$$

After 1 hour the train has traveled 30 miles, after 2 hours the train has traveled 60 miles, and so on. We can construct a chart and graph this relation.

Table 7.7.1

Time, t, in hours	Distance $D = 30t$
0	0
1	30
2	60
3	90
4	120

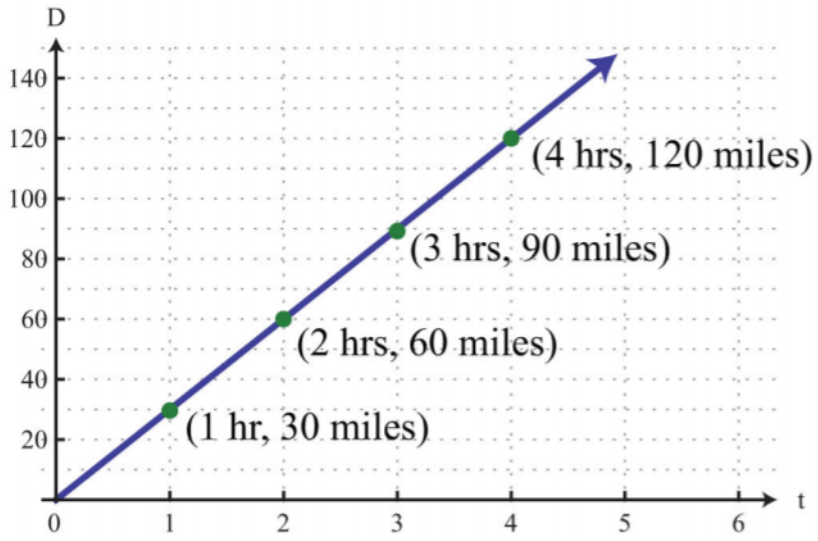


Figure 7.7.1

In this example, we can see that the distance varies over time as the product of the constant rate, 30 miles per hour, and the variable, t. This relationship is described as direct variation and 30 is called the **variation constant**. In addition, if we divide both sides of $D = 30t$ by t we have

$$\frac{D}{t} = 30$$

In this form, it is reasonable to say that D is proportional to t, where 30 is the constant of proportionality. In general, we have

Table 7.7.2

Key Words	Translation
"y **varies directly** as x"	$y = kx$
"y is directly proportional to x"	
"y is **proportional** to x"	

Here k is nonzero and is called the constant of variation or the constant of proportionality.

Example 7.7.1

The circumference of a circle is directly proportional to its diameter, and the constant of proportionality is π. If the circumference is measured to be 20 inches, then what is the radius of the circle?

Solution:

Let C represent the circumference of the circle.

Let d represent the diameter of a circle.

Use the fact that "the circumference is directly proportional to the diameter" to write an equation that relates the two variables.

$$C = kd$$

We are given that "the constant of proportionality is π," or $k = \pi$. Therefore, we write

$$C = \pi d$$

Now use this formula to find d when the circumference is 20 inches.

$$\begin{aligned} 20 &= \pi d \\ \frac{20}{\pi} &= \frac{\pi d}{\pi} \\ \frac{20}{\pi} &= d \end{aligned}$$

The radius of the circle, r, is one-half of its diameter.

$$\begin{aligned} r &= \frac{d}{2} \\ &= \frac{\frac{20}{\pi}}{2} \\ &= \frac{20}{\pi} \cdot \frac{1}{2} \\ &= \frac{10}{\pi} \end{aligned}$$

Answer:

The radius is $\frac{10}{\pi}$ inches, or approximately 3.18 inches

Typically, we will not be given the constant of variation. Instead, we will be given information from which it can be determined.

Example 7.7.2

An object's weight on earth varies directly to its weight on the moon. If a man weighs 180 pounds on earth, then he will weigh 30 pounds on the moon. Set up an algebraic equation that expresses the weight on earth in terms of the weight on the moon and use it to determine the weight of a woman on the moon if she weighs 120 pounds on earth.

Solution:

Let y represent the weight on Earth.

Let x represent the weight on the Moon.

We are given that the "weight on earth varies directly to the weight on the moon."

$$y = kx$$

To find the constant of variation k, use the given information. A 180-pound man on earth weighs 30 pounds on the moon, or $y = 180$ when $x = 30$.

$$180 = k \cdot 30$$

Solve for k.

$$\begin{aligned} \frac{180}{30} &= k \\ 6 &= k \end{aligned}$$

Next, set up a formula that models the given information.

$$y = 6x$$

This implies that a person's weight on earth is 6 times her weight on the moon. To answer the question, use the woman's weight on earth, $y = 120$ pounds, and solve for x.

$$\begin{aligned} 120 &= 6x \\ \frac{120}{6} &= x \\ 20 &= x \end{aligned}$$

Answer:

The woman weighs 20 pounds on the mood.

Inverse Variation

Next, consider the relationship between time and rate,

$$r = \frac{D}{t}$$

If we wish to travel a fixed distance, then we can determine the average speed required to travel that distance in a given amount of time. For example, if we wish to drive 240 miles in 4 hours, we can determine the required average speed as follows:

$$r = \frac{240}{4} = 60$$

The average speed required to drive 240 miles in 4 hours is 60 miles per hour. If we wish to drive the 240 miles in 5 hours, then determine the required speed using a similar equation:

$$r = \frac{240}{5} = 48$$

In this case, we would only have to average 48 miles per hour. We can make a chart and view this relationship on a graph.

Table 7.7.3

Time t in hours	Speed $r = \frac{240}{t}$
2	120
3	80
4	60
5	48

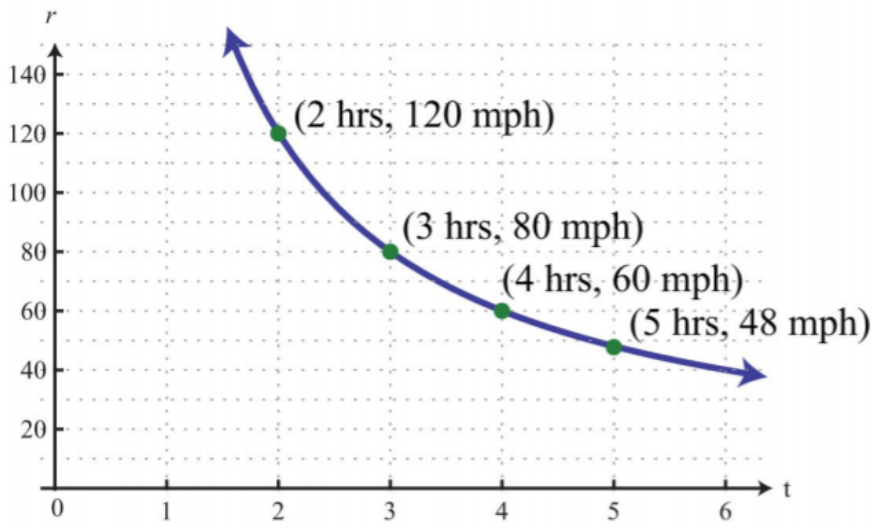


Figure 7.7.2

This is an example of an inverse relationship. We say that r is inversely proportional to the time t, where 240 is the constant of proportionality. In general, we have

Table 7.7.4

Key Words	Translation
"y varies inversely as x"	$y = \frac{k}{x}$
"y is inversely proportional to x"	

Again, k is nonzero and is called the **constant of variation** or the **constant of proportionality**.

Example 7.7.3

If y varies inversely as x and $y = 5$ when $x = 2$, then find the constant of proportionality and an equation that relates the two variables.

Solution:

If we let k represent the constant of proportionality, then the statement "y varies inversely as x" can be written as follows:

$$y = \frac{k}{x}$$

Use the given information, $y = 5$ when $x = 2$, to find k.

$$5 = \frac{k}{2}$$

Solve for k.

Misplaced &

Therefore, the formula that models the problem is

$$y = \frac{10}{x}$$

Answer:

The constant of proportionality is 10, and the equation is $y = \frac{10}{x}$.

Example 7.7.4

The weight of an object varies inversely as the square of its distance from the center of earth. If an object weighs 100 pounds on the surface of earth (approximately $4,000$ miles from the center), then how much will it weigh at $1,000$ miles above earth's surface?

Solution:

Let w represent weight of the object.

Let d represent the object's distance from the center of Earth.

Since "w varies inversely as the square of d," we can write

$$w = \frac{k}{d^2}$$

Use the given information to find k. An object weighs 100 pounds on the surface of earth, approximately $4,000$ miles from the center. In other words, $w = 100$ when $d = 4,000$:

$$100 = \frac{k}{(4000)^2}$$

Solve for k:

$$\begin{aligned} (4,000)^2 \cdot 100 &= (4,000)^2 \cdot \frac{k}{(4,000)^2} \\ 1,600,000,000 &= k \\ 1.6 \times 10^9 &= k \end{aligned}$$

Therefore, we can model the problem with the following formula:

$$w = \frac{1.6\times10^9}{d^2}$$

To use the formula to find the weight, we need the distance from the center of earth. Since the object is $1,000$ miles above the surface, find the distance from the center of earth by adding $4,000$ miles:

$$d = 4,000 + 1,000 = 5,000 \text{ miles}$$

To answer the question, use the formula with $d = 5,000$.

$$\begin{aligned} y &= \frac{1.6 \times 10^9}{(5,000)^2} \\ &= \frac{1.6 \times 10^9}{25,000,000} \\ &= \frac{1.6 \times 10^9}{2.5 \times 10^7} \\ &= 0.64 \times 10^2 \\ &= 64 \end{aligned}$$

Answer:

The object will weigh 64 pounds at a distance $1,000$ miles above the surface of earth.

Joint Variation

Lastly, we define relationships between multiple variables. In general, we have

Table 7.7.5

Vocabulary	Translation
"y varies jointly as x and z"	$y = kxz$
"y is jointly proportional to x and z"	

Here k is nonzero and is called the **constant of variation** or the **constant of proportionality**.

Example 7.7.5

The area of an ellipse varies jointly as a, half of the ellipse's major axis, and b, half of the ellipse's minor axis. If the area of an ellipse is $300\pi \text{ cm}^2$, where $a = 10$ cm and $b = 30$ cm, then what is the constant of proportionality? Give a formula for the area of an ellipse.

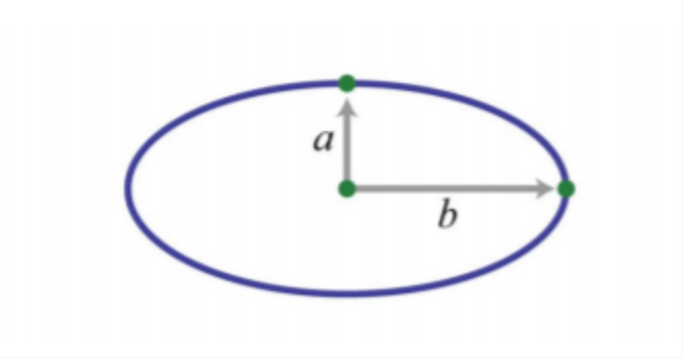


Figure 7.7.3

Solution:

If we let A represent the area of an ellipse, then we can use the statement "area varies jointly as a and b" to write

$$A = kab$$

To find the constant of variation, k, use the fact that the area is 300π when $a = 10$ and $b = 30$.

$$\begin{aligned} 300\pi &= k(10)(30) \\ 300\pi &= 300k \\ \pi &= k \end{aligned}$$

Therefore, the formula for the area of an ellipse is

$$A = \pi ab$$

Answer:

The constant of proportionality is π, and the formula for the area is $A = ab\pi$.

Exercise 7.7.1

Given that y varies directly as the square of x and inversely to z, where $y = 2$ when $x = 3$ and $z = 27$, find y when $x = 2$ and $z = 16$.

Answer available at the end of this section.

Key Takeaways

- The setup of variation problems usually requires multiple steps. First, identify the key words to set up an equation and then use the given information to find the constant of variation k. After determining the constant of variation, write a formula that models the problem. Once a formula is found, use it to answer the question.

Exercise 7.7.2 Variation Problems

Translate the following sentences into a mathematical formula.

1. The distance, D, an automobile can travel is directly proportional to the time, t, that it travels at a constant speed.
2. The extension of a hanging spring, d, is directly proportional to the weight, w, attached to it.
3. An automobile's breaking distance, d, is directly proportional to the square of the automobile's speed, v.
4. The volume, V, of a sphere varies directly as the cube of its radius, r.

5. The volume, V, of a given mass of gas is inversely proportional to the pressure, p, exerted on it.
6. The intensity, I, of light from a light source is inversely proportional to the square of the distance, d, from the source.
7. Every particle of matter in the universe attracts every other particle with a force, F, that is directly proportional to the product of the masses, m_1 and m_2, of the particles and inversely proportional to the square of the distance, d, between them.
8. Simple interest, I, is jointly proportional to the annual interest rate, r, and the time, t, in years a fixed amount of money is invested.
9. The period, T, of a pendulum is directly proportional to the square root of its length, L.
10. The time, t, it takes an object to fall is directly proportional to the square root of the distance, d, it falls.

Answer available at the end of this section.

Exercise 7.7.3 Variation Problems

Construct a mathematical model given the following.

1. y varies directly as x, and $y = 30$ when $x = 6$.
2. y varies directly as x, and $y = 52$ when $x = 4$.
3. y is directly proportional to x, and $y = 12$ when $x = 3$.
4. y is directly proportional to x, and $y = 120$ when $x = 20$.
5. y varies directly as x, and $y = 14$ when $x = 10$.
6. y varies directly as x, and $y = 2$ when $x = 8$.
7. y varies inversely as x, and $y = 5$ when $x = 7$.
8. y varies inversely as x, and $y = 12$ when $x = 2$.
9. y is inversely proportional to x, and $y = 3$ when $x = 9$.
10. y is inversely proportional to x, and $y = 21$ when $x = 3$.
11. y varies inversely as x, and $y = 2$ when $x = \frac{1}{8}$.
12. y varies inversely as x, and $y = \frac{3}{2}$ when $x = \frac{1}{9}$.
13. y varies jointly as x and z, where $y = 8$ when $x = 4$ and $z = \frac{1}{2}$.
14. y varies jointly as x and z, where $y = 24$ when $x = \frac{1}{3}$ and $z = 9$.
15. y is jointly proportional to x and z, where $y = 2$ when $x = 1$ and $z = 3$.
16. y is jointly proportional to x and z, where $y = 15$ when $x = 3$ and $z = 7$.
17. y varies jointly as x and z, where $y = \frac{2}{3}$ when $x = \frac{1}{2}$ and $z = 12$.
18. y varies jointly as x and z, where $y = 5$ when $x = \frac{3}{2}$ and $z = \frac{2}{9}$.
19. y varies directly as the square of x, where $y = 45$ when $x = 3$.
20. y varies directly as the square of x, where $y = 3$ when $x = \frac{1}{2}$.
21. y is inversely proportional to the square of x, where $y = 27$ when $x = \frac{1}{3}$.
22. y is inversely proportional to the square of x, where $y = 9$ when $x = \frac{2}{3}$.
23. y varies jointly as x and the square of z, where $y = 54$ when $x = 2$ and $z = 3$.
24. y varies jointly as x and the square of z, where $y = 6$ when $x = \frac{1}{4}$ and $z = \frac{2}{3}$.
25. y varies jointly as x and z and inversely as the square of w, where $y = 30$ when $x = 8, z = 3$, and $w = 2$
26. y varies jointly as x and z and inversely as the square of w, where $y = 5$ when $x = 1, z = 3$, and $w = \frac{1}{2}$.
27. y varies directly as the square root of x and inversely as z, where $y = 12$ when $x = 9$ and $z = 5$.
28. y varies directly as the square root of x and inversely as the square of z, where $y = 15$ when $x = 25$ and $z = 2$.
29. y varies directly as the square of x and inversely as z and the square of w, where $y = 14$ when $x = 4, w = 2$, and $z = 2$.
30. y varies directly as the square root of x and inversely as z and the square of w, where $y = 27$ when $x = 9, w = \frac{1}{2}$, and $z = 4$.

Answer available at the end of this section.

Exercise 7.7.4 Variation Problems

Applications involving variation.

1. Revenue in dollars is directly proportional to the number of branded sweat shirts sold. If the revenue earned from selling 25 sweat shirts is $\$318.75$, then determine the revenue if 30 sweat shirts are sold.
2. The sales tax on the purchase of a new car varies directly as the price of the car. If an $\$18,000$ new car is purchased, then the sales tax is $\$1,350$. How much sales tax is charged if the new car is priced at $\$22,000$?
3. The price of a share of common stock in a company is directly proportional to the earnings per share (EPS) of the previous 12 months. If the price of a share of common stock in a company is $\$22.55$ and the EPS is published to be $\$1.10$, then determine the value of the stock if the EPS increases by $\$0.20$.
4. The distance traveled on a road trip varies directly with the time spent on the road. If a 126-mile trip can be made in 3 hours, then what distance can be traveled in 4 hours?
5. The circumference of a circle is directly proportional to its radius. If the circumference of a circle with radius 7 centimeters is measured as 14π centimeters, then find the constant of proportionality.
6. The area of circle varies directly as the square of its radius. If the area of a circle with radius 7 centimeters is determined to be 49π square centimeters, then find the constant of proportionality.
7. The surface area of a sphere varies directly as the square of its radius. When the radius of a sphere measures 2 meters, the surface area measures 16π square meters. Find the surface area of a sphere with radius 3 meters.
8. The volume of a sphere varies directly as the cube of its radius. When the radius of a sphere measures 3 meters, the volume is 36π cubic meters. Find the volume of a sphere with radius 1 meter.
9. With a fixed height, the volume of a cone is directly proportional to the square of the radius at the base. When the radius at the base measures 10 centimeters, the volume is 200 cubic centimeters. Determine the volume of the cone if the radius of the base is halved.
10. The distance, d, an object in free fall drops varies directly with the square of the time, t, that it has been falling. If an object in free fall drops 36 feet in 1.5 seconds, then how far will it have fallen in 3 seconds?

Answer available at the end of this section.

Exercise 7.7.5 Variation Problems

Hooke's law suggests that the extension of a hanging spring is directly proportional to the weight attached to it. The constant of variation is called the spring constant.

1. If a hanging spring is stretched 5 inches when a 20-pound weight is attached to it, then determine its spring constant.
2. If a hanging spring is stretched 3 centimeters when a 2-kilogram weight is attached to it, then determine the spring constant.
3. If a hanging spring is stretched 3 inches when a 2-pound weight is attached, then how far will it stretch with a 5-pound weight attached?
4. If a hanging spring is stretched 6 centimeters when a 4-kilogram weight is attached to it, then how far will it stretch with a 2-kilogram weight attached?

Answer available at the end of this section.

Exercise 7.7.6 Variation Problems

The breaking distance of an automobile is directly proportional to the square of its speed.

1. If it takes 36 feet to stop a particular automobile moving at a speed of 30 miles per hour, then how much breaking distance is required if the speed is 35 miles per hour?
2. After an accident, it was determined that it took a driver 80 feet to stop his car. In an experiment under similar conditions, it takes 45 feet to stop the car moving at a speed of 30 miles per hour. Estimate how fast the driver was moving before the accident.

Answer available at the end of this section.

Exercise 7.7.7 Variation Problems

Boyle's law states that if the temperature remains constant, the volume, V, of a given mass of gas is inversely proportional to the pressure, p, exerted on it.

1. A balloon is filled to a volume of 216 cubic inches on a diving boat under 1 atmosphere of pressure. If the balloon is taken underwater approximately 33 feet, where the pressure measures 2 atmospheres, then what is the volume of the balloon?
2. If a balloon is filled to 216 cubic inches under a pressure of 3 atmospheres at a depth of 66 feet, then what would the volume be at the surface, where the pressure is 1 atmosphere?
3. To balance a seesaw, the distance from the fulcrum that a person must sit is inversely proportional to his weight. If a 72-pound boy is sitting 3 feet from the fulcrum, then how far from the fulcrum must a 54- pound boy sit to balance the seesaw?
4. The current, I, in an electrical conductor is inversely proportional to its resistance, R. If the current is $\frac{1}{4}$ ampere when the resistance is 100 ohms, then what is the current when the resistance is 150 ohms?
5. The number of men, represented by y, needed to lay a cobblestone driveway is directly proportional to the area, A, of the driveway and inversely proportional to the amount of time, t, allowed to complete the job. Typically, 3 men can lay $1,200$ square feet of cobblestone in 4 hours. How many men will be required to lay $2,400$ square feet of cobblestone given 6 hours?
6. The volume of a right circular cylinder varies jointly as the square of its radius and its height. A right circular cylinder with a 3-centimeter radius and a height of 4 centimeters has a volume of 36π cubic centimeters. Find a formula for the volume of a right circular cylinder in terms of its radius and height.
7. The period, T, of a pendulum is directly proportional to the square root of its length, L. If the length of a pendulum is 1 meter, then the period is approximately 2 seconds. Approximate the period of a pendulum that is 0.5 meter in length.
8. The time, t, it takes an object to fall is directly proportional to the square root of the distance, d, it falls. An object dropped from 4 feet will take $\frac{1}{2}$ second to hit the ground. How long will it take an object dropped from 16 feet to hit the ground?

Answer available at the end of this section.

Exercise 7.7.8 Variation Problems

Newton's universal law of gravitation states that every particle of matter in the universe attracts every other particle with a force, F, that is directly proportional to the product of the masses, m_1 and m_2, of the particles and inversely proportional to the square of the distance, d, between them. The constant of proportionality is called the gravitational constant.

1. If two objects with masses 50 kilograms and 100 kilograms are $\frac{1}{2}$ meter apart, then they produce approximately 1.34×10^{-6} newtons (N) of force. Calculate the gravitational constant.
2. Use the gravitational constant from the previous exercise to write a formula that approximates the force, F, in newtons between two masses m_1 and m_2, expressed in kilograms, given the distance d between them in meters.
3. Calculate the force in newtons between earth and the moon, given that the mass of the moon is approximately 7.3×10^{22} kilograms, the mass of earth is approximately 6.0×10^{24} kilograms, and the distance between them is on average 1.5×10^{11} meters.
4. Calculate the force in newtons between earth and the sun, given that the mass of the sun is approximately 2.0×10^{30} kilograms, the mass of earth is approximately 6.0×10^{24} kilograms, and the distance between them is on average 3.85×10^{8} meters.
5. If y varies directly as the square of x, then how does y change if x is doubled?
6. If y varies inversely as square of t, then how does y change if t is doubled?
7. If y varies directly as the square of x and inversely as the square of t, then how does y change if both x and t are doubled?

Answer available at the end of this section.

Answers

Exercise 7.7.1

$\frac{3}{2}$

Exercise 7.7.2 **Variation Problems**

1. $D = kt$
3. $d = kv^2$
5. $V = \frac{k}{p}$
7. $F = \frac{km_1 \cdot m_2}{d^2}$
9. $T = k\sqrt{L}$

Exercise 7.7.3 **Variation Problems**

1. $y = 5x$
3. $y = 4x$
5. $y = \frac{7}{5}x$
7. $y = \frac{35}{x}$
9. $y = \frac{27}{x}$
11. $y = \frac{1}{4x}$
13. $y = 4xz$
15. $y = \frac{2}{3}xz$
17. $y = \frac{1}{9}xz$
19. $y = 5x^2$
21. $y = 3x^2$
23. $y = 3xz^2$
25. $y = \frac{5xz}{w^2}$
27. $y = \frac{20x}{\sqrt{z}}$
29. $y = \frac{7x^2}{w^2z}$

Exercise 7.7.4 **Variation Problems**

1. $382.50
3. $26.65
5. 2π
7. 36π square meters
9. 50 cubic centimeters

Exercise 7.7.5 **Variation Problems**

1. $\frac{1}{4}$
3. 7.5 inches

Exercise 7.7.6 **Variation Problems**

1. 49 feet

Exercise 7.7.7 **Variation Problems**

1. 108 cubic inches
3. 4 feet
5. 4 men
7. 1.4 seconds

Exercise 7.7.8 **Variation Problems**

1. $6.7 \times 10^{-11}\ \frac{Nm^2}{kg^2}$
3. 1.98×10^{20} N
5. y changes by a factor of 4
7. y remains unchanged

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7.8: 7.E Review Exercises and Sample Exam

Review Exercises

Exercise 7.8.1 Simplifying Rational Expressions

Evaluate for the given set of x-values.

1. $\frac{25}{2x^2}$; $\{-5, 0, 5\}$
2. $\frac{x-4}{2x-1}$; $\{\frac{1}{2}, 2, 4\}$
3. $\frac{1}{x^2+9}$; $\{-3, 0, 3\}$
4. $\frac{x+3}{x^2-9}$; $\{-3, 0, 3\}$

Answer available at the end of this section.

Exercise 7.8.2 Simplifying Rational Expressions

State the restrictions to the domain.

1. $\frac{5}{x}$
2. $\frac{1}{x(3x+1)}$
3. $\frac{x+2}{x^2-25}$
4. $\frac{x-1}{(x-1)(2x-3)}$

Answer available at the end of this section.

Exercise 7.8.3 Simplifying Rational Expressions

State the restrictions and simplify.

1. $\frac{x-8}{x^2-64}$
2. $\frac{3x^2+9x}{2x^3-18x}$
3. $\frac{x^2-5x-24}{x^2-3x-40}$
4. $\frac{2x^2+9x-5}{4x^2-1}$
5. $\frac{x^2-144}{12-x}$
6. $\frac{8x^2-10x-3}{9-4x^2}$
7. Given $f(x) = \frac{x-3}{x^2+9}$, find $f(-3)$, $f(0)$, and $f(3)$.
8. Simplify $g(x) = \frac{x^2-2x-24}{2x^2-9x-18}$ and state the restrictions.

Answer available at the end of this section.

Exercise 7.8.4 Multiplying and Dividing Rational Expressions

Multiply. (Assume all denominators are nonzero.)

1. $\frac{3x^5}{x-3} \cdot \frac{x-3}{9x^2}$
2. $\frac{12y^2}{y^3(2y-1)} \cdot \frac{(2y-1)}{3y}$
3. $\frac{3x^2}{x-2} \cdot \frac{x^2-4x+4}{5x^3}$

4. $\frac{x^2-8x+15}{9x^5} \cdot \frac{12x^2}{x-3}$
5. $\frac{x^2-36}{x^2-x-30} \cdot \frac{2x^2+10x}{x^2+5x-6}$
6. $\frac{9x^2+11x+2}{4-81x^2} \cdot \frac{9x-2}{(x+1)^2}$

Answer available at the end of this section.

Exercise 7.8.5 Multiplying and Dividing Rational Expressions

Divide. (Assume all denominators are nonzero.)

1. $\frac{9x^2-25}{5x^3} \div \frac{3x+5}{15x^4}$
2. $\frac{4x^2}{4x^2-1} \div \frac{2x^2}{x-1}$
3. $\frac{3x^2-13x-10}{x^2-x-20} \div \frac{9x^2+12x+4}{x^2+8x+16}$
4. $\frac{2x^2+xy-y^2}{x^2+2xy+y^2} \div \frac{4x^2-y^2}{3x^2+2xy-y^2}$
5. $\frac{2x^2-6x-20}{8x^2+17x+2} \div (8x^2-39x-5)$
6. $\frac{12x^2-27x^4}{15x^4+10x^3} \div (3x^2+x-2)$
7. $\frac{25y^2-15y}{4(y-2)} \cdot \frac{1}{5y-1} \div \frac{10y}{2(y-2)^2}$
8. $\frac{10x^4}{1-36x^2} \div \frac{5x^2}{6x^2-7x+1} \cdot x-12x$
9. Given $f(x)=\frac{1}{6x^2-9x+5}$ and $g(x)=\frac{x^2+3x-10}{4x^2+5x-6}$, calculate $(f \cdot g)(x)$ and state the restrictions.
10. Given $f(x)=\frac{x+7}{5x-1}$ and $g(x)=\frac{x^2-49}{25x^2-5x}$, calculate $(f/g)(x)$ and state the restrictions.

Answer available at the end of this section.

Exercise 7.8.6 Adding and Subtracting Rational Expressions

Simplify. (Assume all denominators are nonzero.)

1. $\frac{5x}{y}-\frac{3}{y}$
2. $\frac{x}{x^2-x-6}-\frac{3}{x^2-x-6}$
3. $\frac{2x}{2x+1}+\frac{1}{x-5}$
4. $\frac{3}{x-7}+\frac{1-2x}{x^2}$
5. $\frac{7x}{4x^2-9x+2}-\frac{2}{x-2}$
6. $\frac{5}{x-5}+\frac{20-9x}{2x^2-15x+25}$
7. $\frac{x}{x-5}-\frac{2}{x-3}-\frac{5(x-3)}{x^2-8x+15}$
8. $\frac{3x}{2x-1}-\frac{x-4}{x+4}+\frac{12(2-x)}{2x^2+7x-4}$
9. $\frac{1}{x^2+8x-9}-\frac{1}{x^2+11x+18}$
10. $\frac{4}{x^2+13x+36}+\frac{3}{x^2+6x-27}$
11. $\frac{y+1}{y+2}-\frac{1}{2-y}+\frac{2y}{y^2-4}$
12. $\frac{1}{y-11}-\frac{y-2}{y^2-1}$
13. Given $f(x)=x+12x-5$ and $g(x)=\frac{x}{x+1}$, calculate $(f+g)(x)$ and state the restrictions.
14. Given $f(x)=x+13x$ and $g(x)=\frac{2}{x-8}$, calculate $(f-g)(x)$ and state the restrictions.

Answer available at the end of this section.

Exercise 7.8.7 Complex Fractions

Simplify.

1. $\frac{4-\frac{2}{x}}{\frac{2x-1}{3x}}$

2. $\frac{\frac{1}{3}-\frac{1}{3y}}{\frac{1}{5}-\frac{1}{5y}}$
3. $\frac{\frac{1}{6}+\frac{1}{x}}{\frac{1}{36}-\frac{1}{x^2}}$
4. $\frac{\frac{1}{100}-\frac{1}{x^2}}{\frac{1}{10}-\frac{1}{x}}$
5. $\frac{\frac{x}{x+3}-\frac{2}{x+1}}{\frac{x}{x+4}+\frac{1}{x+3}}$
6. $\frac{\frac{3x-1}{x-5}}{5x+2-\frac{2}{x}}$
7. $\frac{1-12x+35x^2}{1-25x^2}$
8. $2-15x+\frac{25x^2}{2x-5}$

Answer available at the end of this section.

Exercise 7.8.8 Solving Rational Equations

Solve.

1. $\frac{6}{x-6}=\frac{2}{2x-1}$
2. $\frac{x}{x-6}=\frac{x+2}{x-2}$
3. $\frac{1}{3x}-\frac{2}{9}=\frac{1}{x}$
4. $\frac{2}{x-5}+\frac{3}{5}=\frac{1}{x-5}$
5. $\frac{x}{x-5}+\frac{4}{x+5}=-10x^2-25$
6. $\frac{2x-12}{2x+3}=\frac{2-3x^2}{2x^2+3x}$
7. $\frac{x+1}{2(x-2)}+\frac{x-6}{x}=1$
8. $\frac{5x+2}{x+1}-\frac{x}{x+4}=4$
9. $\frac{x}{x+5}+\frac{1}{x-4}=\frac{4x-7}{x^2+x-20}$
10. $\frac{2}{3x-1}+\frac{x}{2x+1}=\frac{2(3-4x)}{6x^2+x-1}$
11. $\frac{x}{x-1}+\frac{1}{x+1}=\frac{2x}{x^2-1}$
12. $\frac{2x}{x+5}-\frac{1}{2x-3}=\frac{4-7x^2}{x^2+7x-15}$
13. Solve for a: $\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$.
14. Solve for y: $x=\frac{2}{y}-\frac{1}{3y}$.

Answer available at the end of this section.

Exercise 7.8.9 Applications of Rational Equations

Use algebra to solve the following applications.

1. A positive integer is twice another. The sum of the reciprocals of the two positive integers is $\frac{1}{4}$. Find the two integers.
2. If the reciprocal of the smaller of two consecutive integers is subtracted from three times the reciprocal of the larger, the result is $\frac{3}{10}$. Find the integers.
3. Mary can jog, on average, 2 miles per hour faster than her husband, James. James can jog 6.6 miles in the same amount of time it takes Mary to jog 9 miles. How fast, on average, can Mary jog?
4. Billy traveled 140 miles to visit his grandmother on the bus and then drove the 140 miles back in a rental car. The bus averages 14 miles per hour slower than the car. If the total time spent traveling was 4.5 hours, then what was the average speed of the bus?
5. Jerry takes twice as long as Manny to assemble a skateboard. If they work together, they can assemble a skateboard in 6 minutes. How long would it take Manny to assemble the skateboard without Jerry's help?
6. Working alone, Joe completes the yard work in 30 minutes. It takes Mike 45 minutes to complete work on the same yard. How long would it take them working together?

Answer available at the end of this section.

Exercise 7.8.10 Variation

Construct a mathematical model given the following.

1. y varies directly with x, and $y = 12$ when $x = 4$.
2. y varies inversely as x, and $y = 2$ when $x = 5$.
3. y is jointly proportional to x and z, where $y = 36$ when $x = 3$ and $z = 4$.
4. y is directly proportional to the square of x and inversely proportional to z, where $y = 20$ when $x = 2$ and $z = 5$.
5. The distance an object in free fall drops varies directly with the square of the time that it has been falling. It is observed that an object falls 16 feet in 1 second. Find an equation that models the distance an object will fall and use it to determine how far it will fall in 2 seconds.
6. The weight of an object varies inversely as the square of its distance from the center of earth. If an object weighs 180 pounds on the surface of earth (approximately $4,000$ miles from the center), then how much will it weigh at $2,000$ miles above earth's surface?

Answer available at the end of this section.

Sample Exam

Exercise 7.8.11

Simplify and state the restrictions.

1. $\frac{15x^3(3x-1)^2}{3x(3x-1)}$
2. $\frac{x^2-144}{x^2+12x}$
3. $\frac{x^2+x-12}{2x^2+7x-4}$
4. $\frac{9-x^2}{(x-3)^2}$

Answer available at the end of this section.

Exercise 7.8.12

Simplify. (Assume all variables in the denominator are positive.)

1. $\frac{5x}{x^2-25} \cdot \frac{x-5}{25x^2}$
2. $\frac{x^2+x-6}{x^2-4x+4} \cdot \frac{3x^2-5x-2}{x^2-9}$
3. $\frac{x^2-4x-12}{12x^2} \div \frac{x-6}{6x}$
4. $\frac{2x^2-7x-4}{6x^2-24x} \div \frac{2x^2+7x+3}{10x^2+30x}$
5. $\frac{1}{x-5} + \frac{1}{x+5}$
6. $\frac{x}{x+1} - \frac{8}{2-x} - \frac{12x}{x^2-x-2}$
7. $\frac{\frac{1}{y}+\frac{1}{x}}{\frac{1}{y^2}-\frac{1}{x^2}}$
8. $\frac{1-6x+9x^2}{2-5x-3x^2}$
9. Given $f(x) = \frac{x^2-81}{(4x-3)^2}$ and $g(x) = \frac{4x-3}{x-9}$, calculate $(f \cdot g)(x)$ and state the restrictions.
10. Given $f(x) = \frac{x}{x-5}$ and $g(x) = \frac{1}{3x-5}$, calculate $(f - g)(x)$ and state the restrictions.

Answer available at the end of this section.

Exercise 7.8.13

Solve.

1. $\frac{1}{3}+\frac{1}{x}=2$
2. $\frac{1}{x-5}=\frac{3}{2x-3}$
3. $1-9x+20x^2=0$
4. $\frac{x+2}{x-2}+\frac{1}{x+2}=\frac{4(x+1)}{x^2-4}$
5. $\frac{x}{x-2}-\frac{1}{x-3}=\frac{3x-10}{x^2-5x+6}$
6. $\frac{5}{x+4}-\frac{x}{4-x}=\frac{9x-4}{x^2-16}$
7. Solve for r:$P=\frac{120}{1+3r}$.

Answer available at the end of this section.

Exercise 7.8.14

Set up an algebraic equation and then solve.

1. An integer is three times another. The sum of the reciprocals of the two integers is $\frac{1}{3}$. Find the two integers.
2. Working alone, Joe can paint the room in 6 hours. If Manny helps, then together they can paint the room in 2 hours. How long would it take Manny to paint the room by himself?
3. A river tour boat averages 6 miles per hour in still water. With the current, the boat can travel 17 miles in the same time it can travel 7 miles against the current. What is the speed of the current?
4. The breaking distance of an automobile is directly proportional to the square of its speed. Under optimal conditions, a certain automobile moving at 35 miles per hour can break to a stop in 25 feet. Find an equation that models the breaking distance under optimal conditions and use it to determine the breaking distance if the automobile is moving 28 miles per hour.

Answer available at the end of this section.

Answers

Exercise 7.8.1 Simplifying Rational Expressions

1. $\frac{1}{2}$, undefined, $\frac{1}{2}$
3. $\frac{1}{18}, \frac{1}{9}, \frac{1}{18}$

Exercise 7.8.2 Simplifying Rational Expressions

1. $x\neq 0$
3. $x\neq \pm 5$

Exercise 7.8.3 Simplifying Rational Expressions

1. $\frac{1}{x+8}$; $x\neq \pm 8$
3. $\frac{x+3}{x+5}$; $x\neq -5, 8$
5. $-(x+12)$; $x\neq 12$
7. $f(-3)=-\frac{1}{3}, f(0)=-\frac{1}{3}, f(3)=0$

Exercise 7.8.4 Multiplying and Dividing Rational Expressions

1. $\frac{x^3}{3}$
3. $\frac{3(x-2)}{5x}$
5. $\frac{2x}{x-1}$

Exercise 7.8.5 Multiplying and Dividing Rational Expressions

1. $3x(3x-5)$
3. $\frac{x+4}{3x+2}$
5. $\frac{2}{(8x+1)^2}$
7. $\frac{5y^2-13y+6}{4(5y-1)}$
9. $(f\cdot g)(x)=\frac{(4x+3)(x-2)}{x+2}$; $x\neq -5, -2, 34$

Exercise 7.8.6 Adding and Subtracting Rational Expressions

1. $\frac{5x-3}{y}$
3. $\frac{2x^2-8x+1}{(2x+1)(x-5)}$
5. $-\frac{1}{4x-1}$
7. $\frac{x-5}{x-3}$
9. $\frac{3}{(x-1)(x+2)(x+9)}$

11. $\frac{y}{y-2}$
13. $(f+g)(x)=\frac{3x^2-3x+1}{(2x-5)(x+1)}$; $x\neq -1, \frac{5}{2}$

Exercise 7.8.7 **Complex Fractions**

1. 6
3. $\frac{6x}{x-6}$
5. $\frac{(x-3)(x+4)}{(x+1)(x+2)}$
7. $-\frac{7x-1}{5x+1}$

Exercise 7.8.8 **Solving Rational Equations**

1. $-\frac{3}{5}$
3. -3
5. $-10, 1$
7. $3, 8$
9. 3
11. $\varnothing$
13. $a=\frac{bc}{b+c}$

Exercise 7.8.9 **Applications of Rational Equations**

1. $6, 12$
3. 7.5 miles per hour
5. 9 minutes

Exercise 7.8.10 **Variation**

1. $y=3x$
3. $y=3xz$
5. $d=16t^2$; 64 feet

Exercise 7.8.11

1. $\frac{5x^2(3x-1)^2}{x-1}$; $x\neq 1$
3. $\frac{x-3}{2x-1}$; $x\neq -4, \frac{1}{2}$

Exercise 7.8.12

1. $\frac{1}{5x(x+5)}$
3. $\frac{x+2}{2x}$
5. $\frac{2x}{(x-5)(x+5)}$
7. $\frac{xy}{x-y}$
9. $(f\cdot g)(x)=\frac{x+9}{4x-3}$; $x\neq \frac{3}{4}, 9$

Exercise 7.8.13

1. $\frac{3}{5}$
3. $\frac{1}{4}, \frac{1}{5}$
5. 4
7. $r=40P-\frac{1}{3}$

Exercise 7.8.14

2. 3 hours
4. $y=\frac{1}{49}x^2$; 16 feet

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8: RADICAL EXPRESSIONS AND EQUATIONS

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8.1: Radicals

Learning Objectives

- Find square roots.
- Find cube roots.
- Find *n* th roots.
- Simplify expressions using the product and quotient rules for radicals.

Square Roots

The square root of a number is that number that when multiplied by itself yields the original number. For example, 4 is a square root of 16, because $4^2 = 16$. Since $(-4)^2 = 16$, we can say that –4 is a square root of 16 as well. Every positive real number has two square roots, one positive and one negative. For this reason, we use the radical sign $\sqrt{}$ to denote the principal (nonnegative) square root and a negative sign in front of the radical $-\sqrt{}$ to denote the negative square root.

Zero is the only real number with one square root.

$$\sqrt{0} = 0 \quad \text{because} \quad 0^2 = 0 \tag{8.1.1}$$

If the radicand, the number inside the radical sign, is nonnegative and can be factored as the square of another nonnegative number, then the square root of the number is apparent. In this case, we have the following property:

$$\sqrt{a^2} = a, \quad \text{if} \quad a \geq 0 \tag{8.1.2}$$

Example 8.1.1

Find the square root.

a. $\sqrt{36}$
b. $\sqrt{144}$
c. $\sqrt{0.04}$
d. $\sqrt{\frac{1}{9}}$

Solution:

a. $\sqrt{36} = \sqrt{6^2} = 6$
b. $\sqrt{144} = \sqrt{12^2} = 12$
c. $\sqrt{0.04} = \sqrt{0.02^2} = 0.02$
d. $\sqrt{\frac{1}{9}} = \sqrt{(\frac{1}{3})^2} = \frac{1}{3}$

Example 8.1.2

Find the negative square root.

a. $-\sqrt{4}$
b. $-\sqrt{1}$

Solution:

a. $-\sqrt{4} = -\sqrt{2^2} = -2$
b. $-\sqrt{1} = -\sqrt{1^2} = -1$

The radicand may not always be a perfect square. If a positive integer is not a perfect square, then its square root will be irrational. For example, $\sqrt{2}$ is an irrational number and can be approximated on most calculators using the square root button.

$$\sqrt{2} \approx 1.414 \quad \text{because} \quad 1.414\text{^}2 \approx 2$$

Next, consider the square root of a negative number. To determine the square root of –9, you must find a number that when squared results in –9:

$$\sqrt{-9} = ? \quad \text{or} \quad (?)^2 = -9$$

However, any real number squared always results in a positive number:

$$(3)^2 = 9 \quad \text{and} \quad (-3)^2 = 9$$

The square root of a negative number is currently left undefined. For now, we will state that $\sqrt{-9}$ is not a real a number.

Cube Roots

The cube root of a number is that number that when multiplied by itself three times yields the original number. Furthermore, we denote a cube root using the symbol $\sqrt[3]{}$, where 3 is called the index. For example,

$$\sqrt[3]{125} = 5, \quad \text{because} \quad 5^3 = 125$$

The product of three equal factors will be positive if the factor is positive and negative if the factor is negative. For this reason, any real number will have only one real cube root. Hence the technicalities associated with the principal root do not apply. For example,

$$\sqrt[3]{-125} = -5, \quad \text{because} \quad (-5)^3 = -125$$

In general, given any real number ***a***, we have the following property:

$$\sqrt[3]{a^3} = a \tag{8.1.3}$$

When simplifying cube roots, look for factors that are perfect cubes.

Example 8.1.3

Find the cube root.

a. $\sqrt[3]{27}$
b. $\sqrt[3]{64}$
c. $\sqrt[3]{0}$
d. $\sqrt[3]{\frac{1}{8}}$

Solution:

a. $\sqrt[3]{27} = \sqrt[3]{3^3} = 3$
b. $\sqrt[3]{64} = \sqrt[3]{4^3} = 4$
c. $\sqrt[3]{0} = \sqrt[3]{0^3} = 0$
d. $\sqrt[3]{\frac{1}{8}} = \sqrt[3]{(\frac{1}{2})^3} = \frac{1}{2}$

Example 8.1.4

Find the cube root.

a. $\sqrt[3]{-8}$
b. $\sqrt[3]{-1}$
c. $\sqrt[3]{-\frac{1}{27}}$

Solution:

a. $\sqrt[3]{-8} = \sqrt[3]{(-2)^3} = -2$
b. $\sqrt[3]{-1} = \sqrt[3]{(-1)^3} = -1$
c. $\sqrt[3]{-\frac{1}{27}} = \sqrt[3]{(-\frac{1}{3})^3} = -\frac{1}{3}$

It may be the case that the radicand is not a perfect cube. If an integer is not a perfect cube, then its cube root will be irrational. For example, $\sqrt[3]{2}$ is an irrational number which can be approximated on most calculators using the root button. Depending on the calculator, we typically type in the index prior to pushing the button and then the radicand as follows:

$$3 \quad \sqrt[x]{y} \quad 2 \quad =$$

Therefore we have

$$\sqrt[3]{2} \approx 1.260, \qquad \text{because} \qquad 1.260\text{^}3 \approx 2$$

n th Roots

For any integer n≥2, we define the ***n*** th root of a positive real number as that number that when raised to the ***n*** th power yields the original number. Given any nonnegative real number ***a***, we have the following property:

$$\sqrt[n]{a^n} = a, \qquad \text{if} \qquad a \geq 0 \tag{8.1.4}$$

Here *n* is called the **index** and a^n is called the **radicand**. Furthermore, we can refer to the entire expression \sqrt[n]{a}\) as a radical. When the index is an integer greater than 3, we say “fourth root”, “fifth root”, and so on. The ***n*** th root of any number is apparent if we can write the radicand with an exponent equal to the index.

Example 8.1.5

Find the ***n*** th root.

a. $\sqrt[4]{81}$
b. $\sqrt[5]{32}$
c. $\sqrt[7]{1}$
d. $\sqrt[4]{\frac{1}{16}}$

Solution:

a. $\sqrt[4]{81} = \sqrt[4]{3^4} = 3$
b. $\sqrt[5]{32} = \sqrt[5]{2^5} = 2$
c. $\sqrt[7]{1} = \sqrt[7]{1^7} = 1$
d. $\sqrt[4]{\frac{1}{16}} = \sqrt[4]{(\frac{1}{2})^4} = \frac{1}{2}$

If the index is n=2, then the radical indicates a square root and it is customary to write the radical without the index, as illustrated below:

$$\sqrt[2]{a} = \sqrt{a} \tag{8.1.5}$$

We have already taken care to define the principal square root of a number. At this point, we extend this idea to ***n*** th roots when ***n*** is even. For example, 3 is a fourth root of 81, because $3^4 = 81$. And since $(-3)^4 = 81$, we can say that –3 is a fourth root of 81 as well. Hence we use the radical sign $\sqrt[n]{}$ to denote the principal (nonnegative) ***n*** th root when ***n*** is even. In this case, for any real number ***a***, we use the following property:

$$\sqrt[n]{a^n} = |a| \qquad \textit{When } n \textit{ is even} \tag{8.1.6}$$

For example,

$$\sqrt[4]{81} = \sqrt[4]{3^4} = |3| = 3$$
$$\sqrt[4]{81} = \sqrt[4]{(-3)^4} = |-3| = 3$$

The negative ***n*** th root, when ***n*** is even, will be denoted using a negative sign in front of the radical $-\sqrt[n]{}$.

$$-\sqrt[4]{81} = -\sqrt[4]{3^4} = -3$$

We have seen that the square root of a negative number is not real because any real number, when squared, will result in a positive number. In fact, a similar problem arises for any even index:

$$\sqrt[4]{-81} = ? \qquad \text{or} \qquad (?)^4 = -81$$

Here the fourth root of –81 is not a real number because the fourth power of any real number is always positive.

$$\left.\begin{matrix}\sqrt{-4} \\ \sqrt[4]{-81}\end{matrix}\right\} \quad \textit{These radicals are not real numbers.}$$

Example 8.1.6

Simplify

a. $\sqrt[4]{-16}$
b. $-\sqrt[4]{16}$

Solution:

a. The radicand is negative and the index is even. Therefore, there is no real number that when raised to the fourth power is –16.

$$\sqrt[4]{-16} \qquad \textit{Not a real number}$$

b. Here the radicand is positive. Furthermore, $16 = 2^4$, and we can simplify as follows:

$$-\sqrt[4]{16} = -\sqrt[4]{2^4} = -2$$

When ***n*** is odd, the same problems do not occur. The product of an odd number of positive factors is positive and the product of an odd number of negative factors is negative. Hence when the index ***n*** is odd, there is only one real ***n*** th root for any real number ***a***. And we have the following property:

$$\sqrt[n]{a^n} = a \qquad \textit{When } n \textit{ is odd} \tag{8.1.7}$$

Example 8.1.7

Find the ***n*** th root.

a. $\sqrt[5]{-32}$
b. $\sqrt[7]{-1}$

Solution:

a. $\sqrt[5]{-32} = \sqrt[5]{(-2)^5} = -2$

b. $\sqrt[7]{-1} = \sqrt[7]{(-1)^7} = -1$

Exercise 8.1.1

Find the fourth root:

$$\sqrt[4]{625}$$

Answer available at the end of this section.

Summary

When ***n* is odd**, the nth root is **positive or negative** depending on the sign of the radicand.

When ***n* is even**, the ***n*** th root is **positive or not real** depending on the sign of the radicand.

$\sqrt[4]{16} = \sqrt[4]{(-2)^4} = |-2| = 2$

$\sqrt[4]{-16}$ *The radical is not a real number.*

Simplifying Using the Product and Quotient Rule for Radicals

It will not always be the case that the radicand is a perfect power of the given index. If not, we use the following two properties to simplify them. If ***a*** and ***b*** represent positive real numbers, then we have

Table 8.1.1

Product rule for radicals:	$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
Quotient rule for radicals:	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

A radical is simplified if it does not contain any factor that can be written as a perfect power of the index.

Example 8.1.8

Simplify:

$$\sqrt{12}$$

Solution:

Here 12 can be written as $4 \cdot 3$, where 4 is a perfect square.

$$\begin{aligned}\sqrt{12} &= \sqrt{4 \cdot 3} && \textit{Apply the product rule for radicals.}\\ &= \sqrt{4} \cdot \sqrt{3} && \textit{Simplify.}\\ &= 2 \cdot \sqrt{3}\end{aligned}$$

We can verify our answer on a calculator:

$$\sqrt{12} \approx 3.46 \quad \text{and} \quad 2 \cdot \sqrt{3} \approx 3.46$$

Also, it is worth noting that

$$3.46\text{^}2 \approx 12$$

Answer:

$2\sqrt{3}$

Example 8.1.9

Simplify:

$$\sqrt{135}$$

Solution:

Begin by finding the largest perfect square factor of 135.

$$\begin{aligned}135 &= 3^3 \cdot 5\\ &= 3^2 \cdot 3 \cdot 5\\ &= 9 \cdot 15\end{aligned}$$

Therefore,

$$\begin{aligned}\sqrt{135} &= \sqrt{9 \cdot 15} && \textit{Apply the product rule for radicals.}\\ &= \sqrt{9} \cdot \sqrt{15} && \textit{Simplify.}\\ &= 3 \cdot \sqrt{15}\end{aligned}$$

Answer:

$3\sqrt{15}$

Example 8.1.10

Simplify:

$$\sqrt{\tfrac{50}{121}}$$

Solution:

Begin by finding the prime factorizations of both 50 and 121. This will enable us to easily determine the largest perfect square factors.

$$\begin{aligned}50 &= 5^2 \cdot 2\\ 121 &= 11^2\end{aligned}$$

Therefore,

$$\begin{aligned}\sqrt{\frac{50}{121}} &= \sqrt{\frac{5^2 \cdot 2}{11^2}} && \textit{Apply the product and quotient rule for radicals.}\\ &= \frac{\sqrt{5^2} \cdot \sqrt{2}}{\sqrt{11^2}} && \textit{Simplify.}\\ &= \frac{5 \cdot \sqrt{2}}{11}\end{aligned}$$

Answer:

$$\frac{5 \cdot \sqrt{2}}{11}$$

Example 8.1.11

Simplify:

$$\sqrt[3]{162}$$

Solution:

Use the prime factorization of 162 to find the largest perfect cube factor:

$$\begin{aligned}162 &= 3^4 \cdot 2\\ &= 3^3 \cdot 3 \cdot 2\end{aligned}$$

Replace the radicand with this factorization and then apply the product rule for radicals.

$$\begin{aligned}\sqrt[3]{162} &= \sqrt[3]{3^3 \cdot 3 \cdot 2} && \textit{Apply the product rule for radicals.}\\ &= \sqrt[3]{3^3} \cdot \sqrt[3]{3 \cdot 2} && \textit{Simplify.}\\ &= 3 \cdot \sqrt[3]{6}\end{aligned}$$

We can verify our answer on a calculator.

$$\sqrt[3]{162} \approx 5.451 \quad \text{and} \quad 3 \cdot \sqrt[3]{6} \approx 5.451$$

Answer:

$$3\sqrt[3]{6}$$

Exercise 8.1.2

Simplify:

$$2\sqrt[3]{96}$$

Answer available at the end of this section.

Example 8.1.12

Simplify:

$$\sqrt[5]{-96}$$

Solution:

Here we note that the index is odd and the radicand is negative; hence the result will be negative. We can factor the radicand as follows:

Then simplify:

Answer:

Example 8.1.13

Simplify:

$$\sqrt[3]{-\frac{8}{64}}$$

Solution

In this case, consider the equivalent fraction with $-8=(-2)^3$ in the numerator and then simplify.

$$\begin{aligned}\sqrt[3]{-\frac{8}{64}} &= \sqrt[3]{\frac{-8}{64}} && \textit{Apply the quotient rule for radicals.}\\ &= \frac{\sqrt[3]{(-2)^3}}{\sqrt[3]{4^3}} && \textit{Simplify.}\\ &= \frac{-2}{4}\\ &= -\frac{1}{2}\end{aligned}$$

Answer:

$$-\tfrac{1}{2}$$

Exercise 8.1.3

Simplify:

$$\sqrt[3]{-108}$$

Answer available at the end of this section.

Key Takeaways

- The square root of a number is that number that when multiplied by itself yields the original number. When the radicand *a* is positive, \(\sqrt{a^{2}=a\). When the radicand is negative, the result is not a real number.
- The cube root of a number is that number that when used as a factor with itself three times yields the original number. The cube root may be positive or negative depending on the sign of the radicand. Therefore, for any real number *a*, we have the property $\sqrt[3]{a^3}=a$.
- When working with *n* th roots, *n* determines the definition that applies. We use $\sqrt[n]{a^n}=a$ when *n* is odd and $\sqrt[n]{a^n}=|a|$ when *n* is even. When *n* is even, the negative *n* th root is denoted with a negative sign in front of the radical sign.
- To simplify square roots, look for the largest perfect square factor of the radicand and then apply the product or quotient rule for radicals.
- To simplify cube roots, look for the largest perfect cube factor of the radicand and then apply the product or quotient rule for radicals.
- To simplify *n* th roots, look for the factors that have a power that is equal to the index *n* and then apply the product or quotient rule for radicals. Typically, the process is streamlined if you work with the prime factorization of the radicand.

Exercise 8.1.4 radicals

Simplify.

1. $\sqrt{81}$
2. $\sqrt{100}$
3. $\sqrt{64}$
4. $\sqrt{121}$
5. $\sqrt{0}$
6. $\sqrt{1}$
7. $\sqrt{0.25}$
8. $\sqrt{0.01}$
9. $\sqrt{1.21}$
10. $\sqrt{2.25}$
11. $\sqrt{14}$
12. $\sqrt{136}$
13. $\sqrt{\frac{25}{16}}$
14. $\sqrt{\frac{9}{25}}$
15. $\sqrt{-25}$
16. $\sqrt{-9}$
17. $-\sqrt{36}$
18. $-\sqrt{81}$
19. $-\sqrt{100}$
20. $-\sqrt{1}$
21. $\sqrt[3]{27}$
22. $\sqrt[3]{125}$
23. $\sqrt[3]{64}$
24. $\sqrt[3]{8}$
25. $\sqrt[3]{\frac{1}{1}}$
26. $\sqrt[3]{\frac{1}{64}}$
27. $\sqrt[3]{\frac{8}{27}}$
28. $\sqrt[3]{\frac{64}{125}}$
29. $\sqrt[3]{0.001}$
30. $\sqrt[3]{1,000}$
31. $\sqrt[3]{-1}$
32. $\sqrt[3]{-8}$
33. $\sqrt[3]{-27}$
34. $\sqrt[3]{-64}$
35. $\sqrt[3]{-18}$
36. $-\sqrt[3]{\frac{27}{64}}$
37. $-\sqrt[3]{\frac{8}{27}}$
38. $-\sqrt[3]{\frac{1}{125}}$
39. $\sqrt[4]{81}$
40. $\sqrt[4]{625}$
41. $\sqrt[4]{16}$
42. $\sqrt[4]{10,000}$
43. $\sqrt[5]{32}$
44. $\sqrt[5]{1}$
45. $\sqrt[5]{243}$
46. $\sqrt[5]{100,000}$
47. $-\sqrt[4]{16}$
48. $-\sqrt[6]{1}$
49. $\sqrt[5]{-32}$

50. $\sqrt[5]{-1}$
51. $\sqrt{-1}$
52. $\sqrt[4]{-16}$
53. $-5\sqrt[3]{-27}$
54. $-2\sqrt[3]{-8}$
55. $5\sqrt[3]{-1,000}$
56. $3\sqrt[5]{-243}$
57. $10\sqrt[4]{-16}$
58. $2\sqrt[6]{-64}$
59. $\sqrt{325}$
60. $\sqrt{64}$
61. $2\sqrt[3]{27}$
62. $8\sqrt[3]{243}$
63. $-7\sqrt[3]{8}$
64. $-4\sqrt[4]{625}$
65. $6\sqrt[5]{100,000}$
66. $5\sqrt[7]{128}$

Answer available at the end of this section.

Exercise 8.1.5 simplifying radicals

Simplify.

1. $\sqrt{32}$
2. $\sqrt{250}$
3. $\sqrt{80}$
4. $\sqrt{150}$
5. $\sqrt{160}$
6. \(\sqrt{60}\
7. $\sqrt{175}$
8. $\sqrt{216}$
9. $\sqrt{5112}$
10. $\sqrt{10135}$
11. $\sqrt{\frac{50}{49}}$
12. $-\sqrt{2120}$
13. $-3\sqrt{162}$
14. $\sqrt{89}$
15. $\sqrt{\frac{45}{121}}$
16. $\sqrt{\frac{9}{681}}$
17. $\sqrt[3]{54}$
18. $\sqrt[3]{24}$
19. $\sqrt[3]{48}$
20. $\sqrt[3]{81}$
21. $\sqrt[3]{40}$
22. $\sqrt[3]{120}$
23. $\sqrt[3]{162}$
24. $\sqrt[3]{500}$
25. $\sqrt[3]{\frac{54}{125}}$
26. $\sqrt[3]{\frac{40}{343}}$
27. $5\sqrt[3]{-48}$
28. $2\sqrt[3]{-108}$
29. $8\sqrt[4]{96}$
30. $7\sqrt[4]{162}$

31. $\sqrt[5]{160}$
32. $\sqrt[5]{486}$
33. $\sqrt[5]{\frac{224}{243}}$
34. $\sqrt[5]{532}$

Answer available at the end of this section.

Exercise 8.1.6 simplifying radicals

Simplify. Give the exact answer and the approximate answer rounded to the nearest hundredth.

1. $\sqrt{8}$
2. $\sqrt{200}$
3. $\sqrt{45}$
4. $\sqrt{72}$
5. $3\sqrt{4}$
6. $\sqrt{59}$
7. $\sqrt{3225}$
8. $\sqrt{4849}$
9. $\sqrt[3]{80}$
10. $\sqrt[3]{320}$
11. $\sqrt[3]{48}$
12. $\sqrt[3]{270}$

Answer available at the end of this section.

Exercise 8.1.7 simplifying radicals

Rewrite the following as a radical expression with coefficient 1.

1. $2\sqrt{15}$
2. $3\sqrt{7}$
3. $5\sqrt{10}$
4. $10\sqrt{3}$
5. $2\sqrt[3]{7}$
6. $3\sqrt[3]{6}$
7. $2\sqrt[4]{5}$
8. $3\sqrt[4]{2}$
9. The formula for the area *A* of a square is $A = s^2$. If the area is 18 square units, then what is the length of each side?
10. Calculate the length of a side of a square with an area of 60 square centimeters.
11. The formula for the volume *V* of a cube is $V = s^3$. If the volume of a cube is 112 cubic units, then what is the length of each side?
12. Calculate the length of a side of a cube with a volume of 54 cubic centimeters.

Answer available at the end of this section.

Exercise 8.1.8 discussion board

1. Explain why there are two square roots for any nonzero real number.
2. Explain why there is only one cube root for any real number.
3. What is the square root of 1, and what is the cube root of 1? Explain why.
4. Explain why $\sqrt{-1}$ is not a real number and why $\sqrt[3]{-1}$ is a real number.

Answer available at the end of this section.

Answers

Exercise 8.1.1

5

Exercise 8.1.2

$4\sqrt[3]{12}$

Exercise 8.1.3

$-3\sqrt[3]{4}$

Exercise 8.1.4 **radicals**

1. 9
3. 8
5. 0
7. 0.5
9. 1.1
11. $\frac{1}{2}$
13. $\frac{5}{4}$
15. Not a real number
17. –6
19. –10
21. 3
23. 4
25. $\frac{1}{2}$
27. $\frac{2}{3}$
29. 0.1
31. –1
33. –3
35. $-\frac{1}{2}$
37. $-\frac{2}{3}$
39. 3
41. 2
43. 2
45. 3
47. –2
49. –2
51. Not a real number
53. 15
55. –50
57. Not a real number
59. 15
61. 6
63. –14
65. 60

Exercise 8.1.5 **simplifying radicals**

1. $4\sqrt{2}$
3. $4\sqrt{5}$
5. $4\sqrt{10}$
7. $5\sqrt{7}$
9. $6\sqrt{142}$
11. $\frac{5\sqrt{2}}{7}$
13. $-27\sqrt{2}$
15. $\frac{3\sqrt{5}}{11}$
17. $3\sqrt[3]{2}$
19. $2\sqrt[3]{6}$
21. $2\sqrt[3]{5}$
23. $3\sqrt[3]{6}$
25. $\frac{3\sqrt[3]{2}}{5}$
27. $-10\sqrt[3]{6}$

29. $16\sqrt[4]{6}$
31. $2\sqrt[3]{5}$
33. $\frac{2\sqrt[3]{7}}{3}$

Exercise $8.1.6$ **simplifying radicals**

1. $2\sqrt{2} \approx 2.83$
3. $3\sqrt{5} \approx 6.71$
5. $3\sqrt{2} \approx 0.87$
7. $\frac{4\sqrt{2}}{5} \approx 1.13$
9. $2\sqrt[3]{10} \approx 4.31$
11. $2\sqrt[3]{6} \approx 3.63$

Exercise $8.1.7$ **simplifying radicals**

1. $\sqrt{60}$
3. $\sqrt{250}$
5. $\sqrt[3]{56}$
7. $\sqrt[4]{80}$
9. $3\sqrt{2}$ units
11. $2\sqrt[3]{14}$ units

Exercise $8.1.8$ **discussion board**

1. Answers may vary
3. Answers may vary

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8.2: Simplifying Radical Expressions

Learning Objectives

- Simplify radical expressions using the product and quotient rule for radicals.
- Use formulas involving radicals.
- Evaluate given square root and cube root functions.

Simplifying Radical Expressions

An algebraic expression that contains radicals is called a radical expression. We use the product and quotient rules to simplify them.

Example 8.2.1

Simplify:

$$\sqrt[3]{8y^3}$$

Solution:

Use the fact that $\sqrt[n]{a^n} = a$ when ***n*** is odd.

$$\begin{aligned} \sqrt[3]{8y^3} &= \sqrt[3]{2^3 \cdot y^3} && \textit{Apply the product rule for radicals.} \\ &= \sqrt[3]{2^3} \cdot \sqrt[3]{y^3} && \textit{Simplify.} \\ &= 2y \end{aligned}$$

Answer:

$$2y$$

Example 8.2.2

Simplify:

$$\sqrt{9x^2}$$

Solution:

The square root has index 2; use the fact that $\sqrt[n]{a^n} = a$ when ***n*** is even.

$$\begin{aligned} \sqrt{9x^2} &= \sqrt{3^2 x^2} && \textit{Apply the product rule for radicals.} \\ &= \sqrt{3^2} \cdot \sqrt{x^2} && \textit{Simplify.} \\ &= 3|x| \end{aligned}$$

Since ***x*** is a variable, it may represent a negative number. Thus we need to ensure that the result is positive by including the absolute value operator.

Answer:

$$3|x|$$

Note

Typically, at this point beginning algebra texts note that all variables are assumed to be positive. If this is the case, then ***x*** in the previous example is positive and the absolute value operator is not needed. The example can be simplified as follows:

$$\sqrt{9x^2} = \sqrt{3^2x^2} = \sqrt{3^2} \cdot \sqrt{x^2} = 3x$$

In this section, we will assume that all variables are positive. This allows us to focus on calculating ***n*** th roots without the technicalities associated with the principal ***n*** th root problem. For this reason, we will use the following property for the rest of the section:

$$\sqrt[n]{a^n} = a, \text{ if } a \geq 0 \; n \text{ th root}$$

When simplifying radical expressions, look for factors with powers that match the index.

Example 8.2.3

Simplify:

$$\sqrt{18x^3y^4}$$

Solution:

Begin by determining the square factors of $18, x^3$, and y^4.

$$\begin{aligned} 18 &= 2 \cdot 3^2 \\ x^3 &= x^2 \cdot x \qquad \textit{Square factors} \\ y^4 &= \left(y^2\right)^2 \end{aligned}$$

Make these substitutions and then apply the product rule for radicals and simplify.

$$\begin{aligned} \sqrt{18x^3y^4} &= \sqrt{2 \cdot 3^2 \cdot x^2 \cdot x \cdot \left(y^2\right)^2} && \textit{Apply the product rule for radicals.} \\ &= \sqrt{3^2} \cdot \sqrt{x^2} \cdot \sqrt{\left(y^2\right)^2} \cdot \sqrt{2x} && \textit{Simplify.} \\ &= 3 \cdot x \cdot y^2 \cdot \sqrt{2x} \\ &= 3xy^2\sqrt{2x} \end{aligned}$$

Answer:

$$3xy^2\sqrt{2x}$$

Example 8.2.4

Simplify:

$$\sqrt{4a^5b^6}$$

Solution:

Begin by determining the square factors of $4, a^5$, and b^6.

$$\begin{aligned} 4 &= 2^2 \\ a^5 &= a^2 \cdot a^2 \cdot a = \left(a^2\right)^2 \cdot a \qquad \textit{Square factors} \\ b^6 &= b^3 \cdot b^3 = \left(b^3\right)^2 \end{aligned}$$

Make these substitutions and then apply the product rule for radicals and simplify.

$$\begin{aligned}\sqrt{\frac{4a^5}{b^6}} &= \sqrt{\frac{2^2\left(a^2\right)^2\cdot a}{\left(b^3\right)^2}} && \textit{Apply the product and quotient rule for radicals.}\\ &= \frac{\sqrt{2^2}\cdot\sqrt{\left(a^2\right)^2}\cdot\sqrt{a}}{\sqrt{\left(b^3\right)^2}} && \textit{Simplify.}\\ &= \frac{2a^2\sqrt{a}}{b^3}\end{aligned}$$

Answer:

$$\frac{2a^2\sqrt{a}}{b^3}$$

Example 8.2.5

Simplify:

$$\sqrt[3]{80x^5y^7}$$

Solution:

Begin by determining the cubic factors of $80, x^5$, and y^7.

$$\begin{aligned}80 &= 2^4\cdot 5 = 2^3\cdot 2\cdot 5\\ x^5 &= x^3\cdot x^2 && \textit{Cubic factors}\\ y^7 &= y^6\cdot y = \left(y^2\right)^3\cdot y\end{aligned}$$

Make these substitutions and then apply the product rule for radicals and simplify.

$$\begin{aligned}\sqrt[3]{80x^5y^7} &= \sqrt[3]{2^3\cdot 2\cdot 5\cdot x^3\cdot x^2\cdot\left(y^2\right)^3\cdot y} && \textit{Apply the product rule for radicals.}\\ &= \sqrt[3]{2^3}\cdot\sqrt[3]{x^3}\cdot\sqrt[3]{\left(y^2\right)^3}\cdot\sqrt[3]{2\cdot 5\cdot x^2\cdot y} && \textit{Simplify.}\\ &= 2\cdot x\cdot y^2\cdot\sqrt[3]{10x^2y}\\ &= 2xy^2\sqrt[3]{10x^2y}\end{aligned}$$

Answer:

$$2xy^2\sqrt[3]{10x^2y}$$

Example 8.2.6

Simplify:

$$\sqrt[3]{9x^6y^3z^9}$$

Solution:

The coefficient $9 = 3^2$ and thus does not have any perfect cube factors. It will be left as the only remaining radicand because all of the other factors are cubes, as illustrated below:

$$\begin{aligned} x^6 &= \left(x^2\right)^3 \\ y^3 &= (y)^3 \qquad \textit{Cubic factors} \\ z^9 &= \left(z^3\right)^3 \end{aligned}$$

Replace the variables with these equivalents, apply the product and quotient rule for radicals, and then simplify.

$$\begin{aligned} \sqrt[3]{\frac{9x^6}{y^3z^9}} &= \sqrt[3]{\frac{3^2 \cdot \left(x^2\right)^3}{y^3 \cdot \left(z^3\right)^3}} \\ &= \frac{\sqrt[3]{3^2} \cdot \sqrt[3]{\left(x^2\right)^3}}{\sqrt[3]{y^3} \cdot \sqrt[3]{\left(z^3\right)^3}} \\ &= \frac{\sqrt[3]{3^2} \cdot x^2}{y \cdot z^3} \\ &= \frac{\sqrt[3]{9} \cdot x^2}{y \cdot z^3} \end{aligned}$$

Answer:

$$\frac{\sqrt[3]{9} \cdot x^2}{y \cdot z^3}$$

Example 8.2.7

Simplify:

$$\sqrt[4]{81a^4b^5}$$

Solution:

Determine all factors that can be written as perfect powers of 4. Here it is important to see that $b^5 = b^4 \cdot b$. Hence the factor b will be left inside the radical.

$$\begin{aligned} \sqrt[4]{81a^4b^5} &= \sqrt[4]{3^4 \cdot a^4 \cdot b^4 \cdot b} \\ &= \sqrt[4]{3^4} \cdot \sqrt[4]{a^4} \cdot \sqrt[4]{b^4} \cdot \sqrt[4]{b} \\ &= 3 \cdot a \cdot b \cdot \sqrt[4]{b} \end{aligned}$$

Answer:

$$3ab\sqrt[4]{b}$$

Example 8.2.8

Simplify:

Solution:

Notice that the variable factor ***x*** cannot be written as a power of 5 and thus will be left inside the radical. In addition, for $y^6 = y^5 \cdot y$; the factor ***y*** will be left inside the radical as well.

Answer:

$$-2yz\sqrt[5]{x^3y}$$

Exercise 8.2.1

Simplify:

$$\sqrt{192x^6y^7z^{12}}$$

(Assume all variables are positive.)

Answer available at the end of this section.

Note

To easily simplify an ***n*** th root, we can divide the powers by the index.

$\sqrt{a^6} = a^3$, which is $a^{6\div 2} = a^3$ $\sqrt[3]{b^6} = b^2$, which is $b^{6\div 3} = b^2$ $\sqrt[6]{c^6} = c$, which is $c^{6\div 6} = c^1$

If the index does not divide into the power evenly, then we can use the quotient and remainder to simplify. For example,

$\sqrt{a^5} = a^2 \cdot \sqrt{a}$, which is $a^{5\div 2} = a^2_{r\,1}$ $\sqrt[3]{b^5} = b \cdot \sqrt[3]{b^2}$, which is $b^{5\div 3} = b^1_{r\,2}$ $\sqrt[5]{c^{14}} = c^2 \cdot \sqrt[5]{c^4}$, which is $c^{14\div 5} = c^2_{r\,4}$

The quotient is the exponent of the factor outside of the radical, and the remainder is the exponent of the factor left inside the radical.

Formulas Involving Radicals

We next review the distance formula. Given two points (x_1, y_1) and (x_2, y_2),

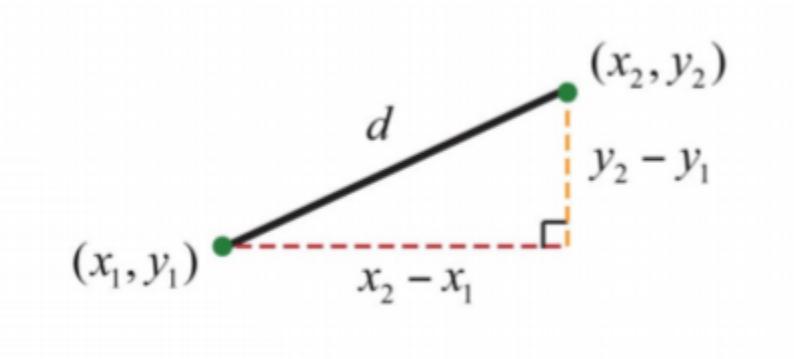


Figure 8.2.1

The distance, ***d***, between them is given by the following formula:

Distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{8.2.1}$$

Recall that this formula was derived from the Pythagorean theorem.

Example 8.2.9

Calculate the distance between $(-4, 7)$ and $(2, 1)$.

Solution:

Use the distance formula with the following points.

$$\begin{array}{ll}(x_1, y_1) & (x_2, y_2)\\ (-4,7) & (2,1)\end{array}$$

It is a good practice to include the formula in its general form before substituting values for the variables; this improves readability and reduces the probability of making errors.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2-(-4))^2 + (1-7)^2} \\ &= \sqrt{(2+4)^2 + (1-7)^2} \\ &= \sqrt{(6)^2 + (-6)^2} \\ &= \sqrt{72} \\ &= \sqrt{36 \cdot 2} \\ &= 6\sqrt{2} \end{aligned}$$

Answer:

$6\sqrt{2}$ units

Example 8.2.10

The period, ***T***, of a pendulum in seconds is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}} \tag{8.2.2}$$

where ***L*** represents the length of the pendulum in feet. If the length of a pendulum measures 6 feet, then calculate the period rounded off to the nearest tenth of a second.

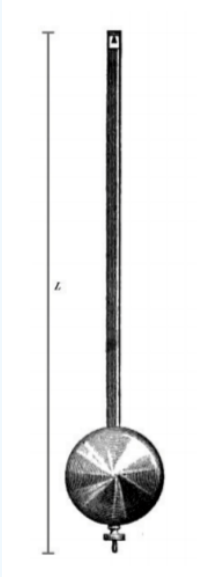


Figure 8.2.2

Solution:

Substitute 6 for ***L*** and then simplify.

$$\begin{aligned} T &= 2\pi\sqrt{\frac{L}{32}} \\ &= 2\pi\sqrt{\frac{6}{32}} && \textit{Reduce.} \\ &= 2\pi\sqrt{\frac{3}{16}} && \textit{Apply the quotient rule for radicals.} \\ &= 2\pi\frac{\sqrt{3}}{\sqrt{16}} && \textit{Simplify.} \\ &= \frac{2\pi\sqrt{3}}{4} && \textit{Use a calculator.} \\ &\approx 2.7 \end{aligned}$$

Answer:

The period is approximately 2.7 seconds.

Square Root and Cube Root Functions

We begin with the square root function:

$$f(x) = \sqrt{x}$$

We know that the square root is not a real number when the radicand ***x*** is negative. Therefore, we conclude that the domain consists of all real numbers greater than or equal to 0. Here we choose 0 and some positive values for ***x***, calculate the corresponding ***y***-values, and plot the resulting ordered pairs.

x	$y = f(x)$		*Ordered pairs*
0	0	$f(0) = \sqrt{0} = 0$	(0, 0)
1	1	$f(1) = \sqrt{1} = 1$	(1, 1)
4	2	$f(4) = \sqrt{4} = 2$	(4, 2)
9	3	$f(9) = \sqrt{9} = 3$	(9, 3)
16	4	$f(16) = \sqrt{16} = 4$	(16, 4)

Figure 8.2.3

After plotting the points, we can then sketch the graph of the square root function.

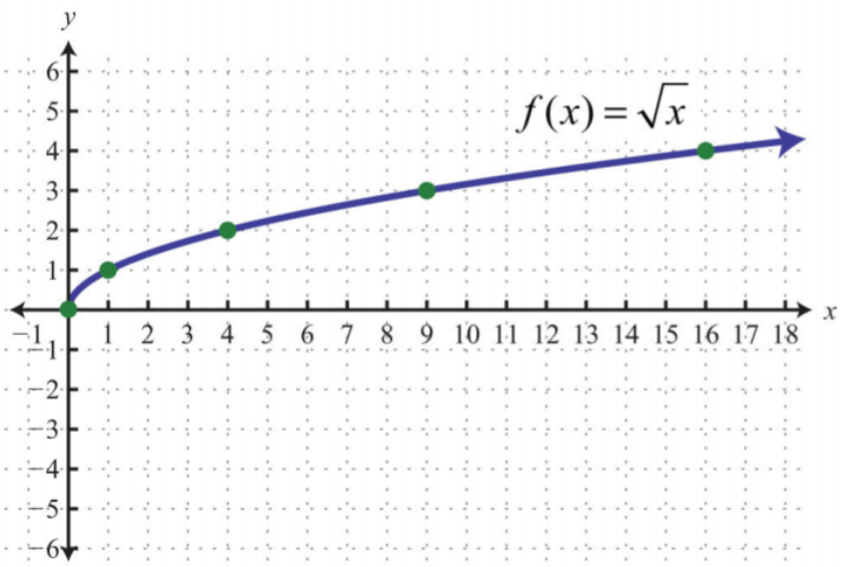


Figure 8.2.4

Example 8.2.11

Given the function $f(x) = \sqrt{x+2}$, find *f(–2)*, *f(2)*, and *f(6)*.

Solution:

Replace ***x*** with each of the given values.

$$f(x) = \sqrt{x+2}$$

$$f(-2) = \sqrt{-2+2} = \sqrt{0} = 0$$
$$f(2) = \sqrt{2+2} = \sqrt{4} = 2$$
$$f(6) = \sqrt{6+2} = \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$$

Answer:

$f(-2) = 0, f(2) = 2$, and $f(6) = 2\sqrt{2}$

Next, consider the cube root function:

$$f(x) = \sqrt[3]{x}$$

Since the cube root could be either negative or positive, we conclude that the domain consists of all real numbers. For completeness, choose some positive and negative values for ***x***, as well as 0, and then calculate the corresponding ***y***-values

x	$y = f(x)$		Ordered pairs
-8	-2	$f(-8) = \sqrt[3]{-8} = -2$	$(-8, -2)$
-1	-1	$f(-1) = \sqrt[3]{-1} = -1$	$(-1, -1)$
0	0	$f(0) = \sqrt[3]{0} = 0$	$(0, 0)$
1	1	$f(1) = \sqrt[3]{1} = 1$	$(1, 1)$
8	2	$f(8) = \sqrt[3]{8} = 2$	$(8, 2)$

Figure 8.2.5

Plot the points and sketch the graph of the cube root function.

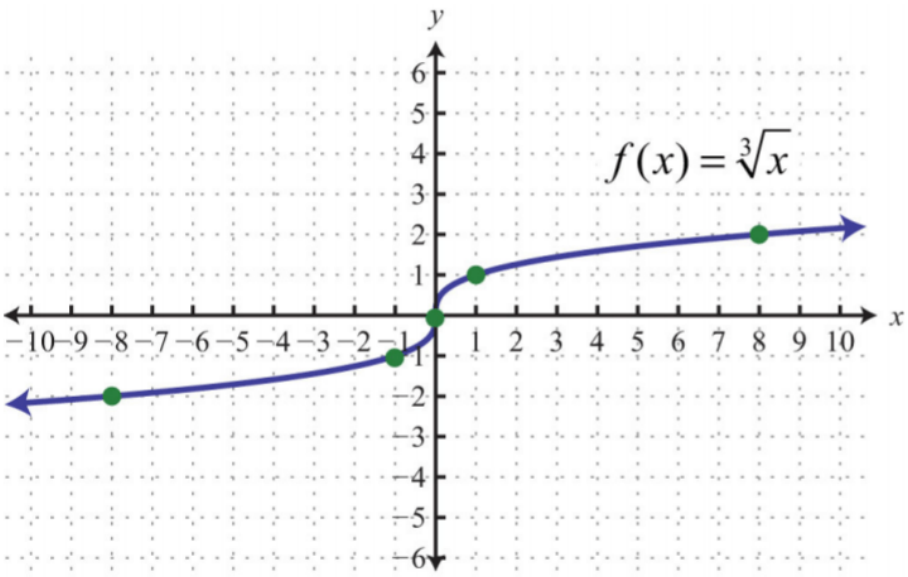


Figure 8.2.6

Example 8.2.12

Given the function $g(x) = \sqrt[3]{x-1}$, find *g(−7)*, *g(0)*, and *g(55)*.

Solution:

Replace ***x*** with each of the given values.

$$g(x) = \sqrt[3]{x-1}$$

Answer:

$g(-7) = -2, g(0) = -1$, and $g(55) = 3\sqrt[3]{2}$

Key Takeaways

- In beginning algebra, we typically assume that all variable expressions within the radical are positive. This allows us to focus on simplifying radicals without the technical issues associated with the principal nth root.

- To simplify radical expressions, look for factors of the radicand with powers that match the index. If found, they can be simplified by applying the product and quotient rules for radicals, as well as the property $\sqrt[n]{a^n} = a$, where a is positive.

Exercise 8.2.2 simplifying radical expressions

Simplify. (Assume all variables represent positive numbers.)

1. $\sqrt{36a^2}$
2. $\sqrt{121b^2}$
3. $\sqrt{x^2y^2}$
4. $\sqrt{25x^2y^2z^2}$
5. $\sqrt{180x^3}$
6. $\sqrt{150y^3}$
7. $\sqrt{49a^3b^2}$
8. $\sqrt{4a^4b^3c}$
9. $\sqrt{45x^5y^3}$
10. $\sqrt{50x^6y^4}$
11. $\sqrt{64r^2s^6t^5}$
12. $\sqrt{144r^8s^6t^2}$
13. $\sqrt{(x+1)^2}$
14. $\sqrt{(2x+3)^2}$
15. $\sqrt{4(3x-1)^2}$
16. $\sqrt{9(2x+3)^2}$
17. $\sqrt{9x^35y^2}$
18. $\sqrt{4x^59y^4}$
19. $\sqrt{m^736n^4}$
20. $\sqrt{147m^9n^6}$
21. $\sqrt{2r^2s^525t^4}$
22. $\sqrt{36r^5s^2t^6}$
23. $\sqrt[3]{27a^3}$
24. $\sqrt[3]{125b^3}$
25. $\sqrt[3]{250x^4y^3}$
26. $\sqrt[3]{162a^3b^5}$
27. $\sqrt[3]{64x^3y^6z^9}$
28. $\sqrt[3]{216x^{12}y^3}$
29. $\sqrt[3]{8x^3y^4}$
30. $\sqrt[3]{27x^5y^3}$
31. $\sqrt[3]{a^4b^5c^6}$
32. $\sqrt[3]{a^7b^5c^3}$
33. $\sqrt[3]{8x^427y^3}$
34. $\sqrt[3]{x^5125y^6}$
35. $\sqrt[3]{360r^5s^{12}t^{13}}$
36. $\sqrt[3]{540r^3s^2t^9}$
37. $\sqrt[4]{81x^4}$
38. $\sqrt[4]{x^4y^4}$
39. $\sqrt[4]{16x^4y^8}$
40. $\sqrt[4]{81x^{12}y^4}$
41. $\sqrt[4]{a^4b^5c^6}$
42. $\sqrt[4]{54a^6c^8}$
43. $\sqrt[4]{128x^6}$
44. $\sqrt[4]{243y^7}$
45. $\sqrt[5]{32m^{10}n^5}$
46. $\sqrt[5]{37m^9n^{10}}$
47. $-3\sqrt{4x^2}$
48. $7\sqrt{9y^2}$
49. $-5\sqrt{x4x^2y}$

50. $-3\sqrt{y^{16}x^3y^2}$
51. $12ab\sqrt{a^5b^3}$
52. $6a^2b^9\sqrt{a^7b^2}$
53. $2x\sqrt[3]{8x^6}$
54. $-5x^2\sqrt[3]{27x^3}$
55. $2ab\sqrt[3]{-8a^4b^5}$
56. $5a^2b\sqrt[3]{-27a^3b^3}$

Answer available at the end of this section.

Exercise 8.2.3 simplifying radical expressions

Rewrite the following as a radical expression with coefficient 1.

1. $5\sqrt{2x}$
2. $2\sqrt{3y}$
3. $2x\sqrt{3}$
4. $3y\sqrt{2}$
5. $ab\sqrt{10a}$
6. $2ab\sqrt{2a}$
7. $m^2n\sqrt{mn}$
8. $2m^2n^3\sqrt{3n}$
9. $5\sqrt[3]{2x}$
10. $3\sqrt[3]{5y}$
11. $2x\sqrt[3]{3}$
12. $3y\sqrt[3]{2}$

Answer available at the end of this section.

Exercise 8.2.4 simplifying radical expressions

Assume that the variable could represent any real number and then simplify.

1. $\sqrt{4x^2}$
2. $\sqrt{25y^2}$
3. $\sqrt[3]{8y^3}$
4. $\sqrt[3]{125a^3}$
5. $\sqrt[4]{64x^4}$
6. $\sqrt[4]{81y^4}$
7. $\sqrt{36a^4}$
8. $\sqrt{100a^8}$
9. $\sqrt{4a^6}$
10. $\sqrt{a^{10}}$
11. $\sqrt{18a^4b^5}$
12. $\sqrt{48a^5b^3}$
13. $\sqrt[6]{128x^6y^8}$
14. $\sqrt[6]{a^6b^7c^8}$

Answer available at the end of this section.

Exercise 8.2.5 formulas involving radicals

The *y* -intercepts for any graph will have the form (0, y), where y is a real number. Therefore, to find *y* -intercepts, set $x = 0$ and solve for *y*. Find the *y* -intercepts for the following.

1. $y = \sqrt{x+4} - 1$
2. $y = \sqrt{x+1} - 3$
3. $y = \sqrt[3]{x-1} + 2$
4. $y = \sqrt[3]{x+1} - 3$

Answer available at the end of this section.

Exercise 8.2.6 formulas involving radicals

Use the distance formula to calculate the distance between the given two points.

1. $(5, -7)$ and $(3, -8)$
2. $(-9, 7)$ and $(-8, 4)$
3. $(-3, -4)$ and $(3, -6)$
4. $(-5, -2)$ and $(1, -6)$
5. $(-1, 1)$ and $(-4, 10)$
6. $(8, -3)$ and $(2, -12)$

Answer available at the end of this section.

Exercise 8.2.7 formulas involving radicals

Factor the radicand and then simplify. (Assume that all expressions are positive.)

1. $\sqrt{x^2 - 6x + 9}$
2. $\sqrt{x^2 - 10x + 25}$
3. $\sqrt{4x^2 + 12x + 9}$
4. $\sqrt{9x^2 + 6x + 1}$
5. The speed of a vehicle before the brakes were applied can be estimated by the length of the skid marks left on the road. On dry pavement, the speed, *v*, in miles per hour can be estimated by the formula $v = \sqrt{5d}$, where *d* represents the length of the skid marks in feet. Estimate the speed of a vehicle before applying the brakes on dry pavement if the skid marks left behind measure 36 feet.
6. The radius, *r*, of a sphere can be calculated using the formula $r = \sqrt[3]{\frac{3V}{4\pi}}$, where *V* represents the sphere's volume. What is the radius of a sphere if the volume is 36π cubic centimeters?

Answer available at the end of this section.

Exercise 8.2.8 formulas involving radicals

The period, *T*, of a pendulum in seconds is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where *L* represents the length in feet. Calculate the period, given the following lengths. Give the exact value and the approximate value rounded off to the nearest tenth of a second.

1. 8 feet
2. 32 feet
3. 1/2 foot
4. 1/8 foot

Answer available at the end of this section.

Exercise 8.2.9 formulas involving radicals

The time, *t*, in seconds that an object is in free fall is given by the formula

$$s = 16 \cdot t^2$$

where s represents the distance it has fallen in feet. Calculate the time it takes an object to fall, given the following distances. Give the exact value and the approximate value rounded off to the nearest tenth of a second.

1. 48 feet
2. 80 feet
3. 192 feet
4. 288 feet

Answer available at the end of this section.

Exercise 8.2.10 radical functions

Given the function, calculate the following.

1. $f(x) = \sqrt{x-1}$, find *f(1)*, *f(2)*, and *f(5)*
2. $f(x) = \sqrt{x+5}$, find *f(−5)*, *f(−1)*, and *f(20)*
3. $f(x) = \sqrt{x}+3$, find *f(0)*, *f(1)*, and *f(16)*
4. $f(x) = \sqrt{x}-5$, find *f(0)*, *f(1)*, and *f(25)*
5. $g(x) = \sqrt[3]{x}$, find *g(−1)*, *g(0)*, and *g(1)*
6. $g(x) = \sqrt[3]{x+7}$, find *g(−15)*, *g(−7)*, and *g(20)*
7. $g(x) = \sqrt[3]{x}-2$, find *g(−1)*, *g(0)*, and *g(8)*
8. $g(x) = \(\sqrt[3]{x-1}+2$, find *g(0)*, *g(2)*, and *g(9)*

Answer available at the end of this section.

Exercise 8.2.11 radical functions

For each function, fill in the table.

1. $f(x) = \sqrt{x+1}$

x	$f(x)$
-1	
0	
3	
8	
15	

Figure 8.2.7

2. $f(x) = \sqrt{x-2}$

x	$f(x)$
2	
3	
6	
11	
18	

Figure 8.2.8

3. $f(x) = \sqrt[3]{x} + 1$

x	$f(x)$
-8	
-1	
0	
1	
8	

Figure $8.2.9$

4. $f(x) = \sqrt[3]{x+2}$

x	$f(x)$
-10	
-3	
-2	
-1	
6	

Figure $8.2.10$

Answer available at the end of this section.

Exercise $8.2.12$ discussion board

1. Give a value for x such that $\sqrt{x^2} \neq x$. Explain why it is important to assume that the variables represent positive numbers.
2. Research and discuss the accomplishments of Christoph Rudolff. What is he credited for?
3. Research and discuss the methods used for calculating square roots before the common use of electronic calculators.
4. What is a surd, and where does the word come from?

Answer available at the end of this section.

Answers

Exercise 8.2.1

$8x^3y^3z^6\sqrt{3y}$

Exercise 8.2.2 **simplifying radical expressions**

1. $6a$
3. xy
5. $6x\sqrt{5x}$
7. $7ab\sqrt{a}$
9. $3x^2y\sqrt{5xy}$
11. $8rs^3t^2\sqrt{t}$
13. $x+1$
15. $2(3x-1)$
17. $3xy\sqrt{5x}$
19. $6n^2m^3\sqrt{m}$
21. $5\sqrt{2s}rt^2s^2$
23. $3a$
25. $5xy\sqrt[3]{2x}$
27. $4xy^2z^3$
29. $2xy\sqrt[3]{y}$
31. $abc^2\sqrt[3]{ab^2}$
33. $6yx\sqrt[3]{x}$
35. $2rs^4t^4\sqrt[3]{45r^2t}$
37. $3x$
39. $2xy^2$
41. $abc\sqrt[4]{bc^2}$
43. $2x\sqrt[4]{8x^2}$
45. 2^m2^n
47. $-6x$
49. $-10x\sqrt{xy}$
51. $12a^3b^2\sqrt{ab}$
53. $4x^3$
55. $-4a^2b^2\sqrt[3]{ab^2}$

Exercise 8.2.3 **simplifying radical expressions**

1. $\sqrt{50x}$
3. $\sqrt{12x^2}$
5. $\sqrt{10a^3b^2}$
7. $\sqrt{m^5n^3}$
9. $\sqrt[3]{250x}$
11. $\sqrt[3]{24x^3}$

Exercise 8.2.4 **simplifying radical expressions**

1. $2|x|$
3. $2y$
5. $2|x|$
7. $6a^2$
9. $2|a^3|$
11. $3a^2b^2\sqrt{2b}$
13. $2|xy|\sqrt[6]{2y^2}$

Exercise 8.2.5 **formulas involving radicals**

1. $(0,1)$
3. $(0,1)$

Exercise 8.2.6 **formulas involving radicals**

1. $\sqrt{5}$
3. $2\sqrt{10}$
5. $3\sqrt{10}$

Exercise 8.2.7 **formulas involving radicals**

1. $x-3$
3. $2x+3$
5. 30 miles per hour

Exercise 8.2.8 **formulas involving radicals**

1. $\pi \approx 3.1$ seconds
3. $\frac{\pi}{4} \approx 0.8$ seconds

Exercise $8.2.9$ **formulas involving radicals**

1. $\sqrt{3} \approx 1.7$ seconds
3. $2\sqrt{3} \approx 3.5$ seconds

Exercise $8.2.10$ **radical functions**

1. $f(1) = 0, f(2) = 1$, and $f(5) = 2$
3. $f(0) = 3, f(1) = 4$, and $f(16) = 7$
5. $g(-1) = -1, g(0) = 0$, and $g(1) = 1$
7. $g(-1) = -3, g(0) = -2$, and $g(8) = 0$

Exercise $8.2.11$ **radical functions**

1.

x	$f(x)$
-1	0
0	1
3	2
8	3
15	4

Figure $8.2.11$

3.

x	$f(x)$
-8	-1
-1	0
0	1
1	2
8	3

Figure $8.2.12$

Exercise $8.2.12$ **discussion board**

1. Answers may vary
3. Answers may vary

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8.3: Adding and Subtracting Radical Expressions

Learning Objectives

- Add and subtract like radicals.
- Simplify radical expressions involving like radicals.

Adding and Subtracting Radical Expressions

Adding and subtracting radical expressions is similar to adding and subtracting like terms. Radicals are considered to be like radicals, or similar radicals, when they share the same index and radicand. For example, the terms $3\sqrt{5}$ and $4\sqrt{5}$ contain like radicals and can be added using the distributive property as follows:

$$\begin{aligned} 3\sqrt{5}+4\sqrt{5} &= (3+4)\sqrt{5} \\ &= 7\sqrt{5} \end{aligned}$$

Typically, we do not show the step involving the distributive property and simply write

$$3\sqrt{5}+4\sqrt{5}=7\sqrt{5}$$

When adding terms with like radicals, add only the coefficients; the radical part remains the same.

Example 8.3.1

Add:

$$3\sqrt{2}+2\sqrt{2}$$

Solution:

The terms contain like radicals; therefore, add the coefficients.

$$3\sqrt{2}+2\sqrt{2}=5\sqrt{2}$$

Answer:

$$5\sqrt{2}$$

Subtraction is performed in a similar manner.

Example 8.3.2

Subtract:

$$2\sqrt{7}-3\sqrt{7}$$

Solution:

$$\begin{aligned} 2\sqrt{7}-3\sqrt{7} &= (2-3)\sqrt{7} \\ &= -1\sqrt{7} \\ &= -\sqrt{7} \end{aligned}$$

Answer:

$-\sqrt{7}$

If the radicand and the index are not exactly the same, then the radicals are not similar and we cannot combine them.

Example 8.3.3

Simplify: $10\sqrt{5}+6\sqrt{2}-9\sqrt{5}-7\sqrt{2}$

Solution:

We cannot simplify any further because $\sqrt{5}$ and $\sqrt{2}$ are not like radicals; the radicands are not the same.

Answer:

$$\sqrt{5}-\sqrt{2}$$

Note

Caution

It is important to point out that $\sqrt{5}-\sqrt{2}\neq\sqrt{5-2}$. We can verify this by calculating the value of each side with a calculator.

$$\begin{aligned}\sqrt{5}-\sqrt{2}&\approx 0.82\\ \sqrt{5-2}=\sqrt{3}&\approx 1.73\end{aligned}$$

In general, note that $\sqrt[n]{a}\pm\sqrt[n]{b}\neq\sqrt[n]{a\pm b}$.

Example 8.3.4

Simplify: $3\sqrt[3]{6}+2\sqrt{6}-\sqrt[3]{6}-3\sqrt{6}$

Solution:

$$\begin{aligned}3\sqrt[3]{6}+2\sqrt{6}-\sqrt[3]{6}-3\sqrt{6}&=\underbrace{3\sqrt[3]{6}-\sqrt[3]{6}}+\underbrace{2\sqrt{6}-3\sqrt{6}}\\ &=2\sqrt[3]{6}-\sqrt{6}\end{aligned}$$

We cannot simplify any further because $\sqrt[3]{6}$ and $\sqrt{6}$ are not like radicals; the indices are not the same.

Answer:

$$2\sqrt[3]{6}-\sqrt{6}$$

Often we will have to simplify before we can identify the like radicals within the terms.

Example 8.3.5

Subtract:

$$\sqrt{12}-\sqrt{48}$$

Solution:

At first glance, the radicals do not appear to be similar. However, after simplifying completely, we will see that we can combine them.

$$\begin{aligned}\sqrt{12}-\sqrt{48} &= \sqrt{4\cdot 3}-\sqrt{16\cdot 3}\\ &= 2\sqrt{3}-4\sqrt{3}\\ &= -2\sqrt{3}\end{aligned}$$

Answer:

$-2\sqrt{3}$

Example 8.3.6

Simplify: $\sqrt{20}+\sqrt{27}-3\sqrt{5}-2\sqrt{12}$

Solution:

$$\begin{aligned}\sqrt{20}+\sqrt{27}-3\sqrt{5}-2\sqrt{12} &= \sqrt{4\cdot 5}+\sqrt{9\cdot 3}-3\sqrt{5}-2\sqrt{4\cdot 3}\\ &= 2\sqrt{5}+3\sqrt{3}-3\sqrt{5}-2\cdot 2\sqrt{3}\\ &= 2\sqrt{5}-3\sqrt{5}+3\sqrt{3}-4\sqrt{3}\\ &= -\sqrt{5}-\sqrt{3}\end{aligned}$$

Answer:

$$-\sqrt{5}-\sqrt{3}$$

Exercise 8.3.1

Subtract:

$$2\sqrt{50}-6\sqrt{8}$$

Answer available at the end of this section.

Next, we work with radical expressions involving variables. In this section, assume all radicands containing variable expressions are not negative.

Example 8.3.7

Simplify: $-6\sqrt[3]{2x}-\sqrt[3]{3x}+7\sqrt[3]{2x}$

Solution:

$$\begin{aligned}-6\sqrt[3]{2x}-\sqrt[3]{3x}+7\sqrt[3]{2x} &= \underbrace{-6\sqrt[3]{2x}+7\sqrt[3]{2x}}-\sqrt[3]{3x}\\ &= \sqrt[3]{2x}-\sqrt[3]{3x}\end{aligned}$$

We cannot combine any further because the remaining radical expressions do not share the same radicand; they are not like radicals. Note that $\sqrt[3]{2x}-\sqrt[3]{3x}\neq\sqrt[3]{2x-3x}$.

Answer:

$$\sqrt[3]{2x}-\sqrt[3]{3x}$$

We will often find the need to subtract a radical expression with multiple terms. If this is the case, remember to apply the distributive property before combining like terms.

Example 8.3.8

Simplify: $(9\sqrt{x}-2\sqrt{y})-(10\sqrt{x}+7\sqrt{y})$.

Solution:

$$\begin{aligned}(9\sqrt{x}-2\sqrt{y})-(10\sqrt{x}+7\sqrt{y}) &= 9\sqrt{x}-2\sqrt{y}-10\sqrt{x}-7\sqrt{y} && \text{Distribute.}\\ &= 9\sqrt{x}-10\sqrt{x}-2\sqrt{y}-7\sqrt{y} && \text{Combine}\\ &= -\sqrt{x}-9\sqrt{y} && \text{like terms.}\end{aligned}$$

Answer:

$$-\sqrt{x}-9\sqrt{y}$$

Until we simplify, it is often unclear which terms involving radicals are similar.

Example 8.3.9

Simplify: $5\sqrt[3]{2y}-(\sqrt[3]{54y}-\sqrt[3]{16})$

Solution

$$\begin{aligned}5\sqrt[3]{2y}-(\sqrt[3]{54y}-\sqrt[3]{16}) &= 5\sqrt[3]{2y}-\sqrt[3]{54y}+\sqrt[3]{16} && \text{Distribute.}\\ &= 5\sqrt[3]{2y}-\sqrt[3]{3^3\cdot 2y}+\sqrt[3]{2^3\cdot 2} && \text{Simplify.}\\ &= 5\sqrt[3]{2y}-3\sqrt[3]{2y}+2\sqrt[3]{2} && \text{Combine lik}\\ &= 2\sqrt[3]{2y}+2\sqrt[3]{2}\end{aligned}$$

Answer

$2\sqrt[3]{2y}+2\sqrt[3]{2}$

Example 8.3.10

Simplify:

$$2a\sqrt{125a^2b}-a^2\sqrt{80b}+4\sqrt{20a^4b}$$

Solution:

$$\begin{aligned}& 2a\sqrt{125a^2b}-a^2\sqrt{80b}+4\sqrt{20a^4b}\\ =\ & 2a\sqrt{25\cdot 5\cdot a^2\cdot b}-a^2\sqrt{16\cdot 5\cdot b}+4\sqrt{4\cdot 5\cdot (a^2)^2 b}\\ =\ & 2a\cdot 5\cdot a\sqrt{5b}-a^2\cdot 4\sqrt{5b}+4\cdot 2\cdot a^2\sqrt{5b} && \text{Simplify.}\\ =\ & 10a^2\sqrt{5b}-4a^2\sqrt{5b}+8a^2\sqrt{5b} && \text{Combine like terms.}\\ =\ & 14a^2\sqrt{5b}\end{aligned}$$

Answer:

$14a^2\sqrt{5b}$

Exercise 8.3.2

Simplify:

$$\sqrt{45x^3} - (\sqrt{20x^3 - \sqrt{80x}})$$

Answer available at the end of this section.

Tip

Take careful note of the differences between products and sums within a radical.

Table 8.3.1

Products
$\sqrt{x^2y^2} = xy$
$\sqrt[3]{x^3y^3} = xy$

The property $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ says that we can simplify radicals when the operation in the radicand is multiplication. There is no corresponding property for addition.

Key Takeaways

- Add and subtract terms that contain like radicals just as you do like terms. If the index and radicand are exactly the same, then the radicals are similar and can be combined. This involves adding or subtracting only the coefficients; the radical part remains the same.
- Simplify each radical completely before combining like terms.

Exercise 8.3.3 adding and subtracting like radicals

Simplify.

1. $9\sqrt{3} + 5\sqrt{3}$
2. $12\sqrt{6} + 3\sqrt{6}$
3. $4\sqrt{5} - 7\sqrt{5}$
4. $3\sqrt{10} - 8\sqrt{10}$
5. $\sqrt{6} - 4\sqrt{6} + 2\sqrt{6}$
6. $5\sqrt{10} - 15\sqrt{10} - 2\sqrt{10}$
7. $13\sqrt{7} - 6\sqrt{2} - 5\sqrt{7} + 5\sqrt{2}$
8. $10\sqrt{13} - 12\sqrt{15} + 5\sqrt{13} - 18\sqrt{15}$
9. $6\sqrt{5} - (4\sqrt{3} - 3\sqrt{5})$
10. $-12\sqrt{2} - (6\sqrt{6} + \sqrt{2})$
11. $(2\sqrt{5} - 3\sqrt{10}) - (\sqrt{10} + 3\sqrt{5})$
12. $(-8\sqrt{3} + 6\sqrt{15}) - (\sqrt{3} - \sqrt{15})$
13. $4\sqrt[3]{6} - 3\sqrt[3]{5} + 6\sqrt[3]{6}$
14. $\sqrt[3]{10} + 5\sqrt[3]{10} - 4\sqrt[3]{10}$
15. $(7\sqrt[3]{9} - 4\sqrt[3]{3}) - (\sqrt[3]{9} - 3\sqrt[3]{3})$
16. $(-8\sqrt[3]{5} + \sqrt[3]{25}) - (2\sqrt[3]{5} + 6\sqrt[3]{25})$

Answer available at the end of this section.

Exercise 8.3.4 adding and subtracting like radicals

Simplify. (Assume all radicands containing variable expressions are positive.)

1. $9\sqrt{x}+7\sqrt{x}$
2. $-8\sqrt{y}+4\sqrt{y}$
3. $7\sqrt{xy}-3\sqrt{xy}+\sqrt{xy}$
4. $10\sqrt{y^2x}-12\sqrt{y^2x}-2\sqrt{y^2x}$
5. $2\sqrt{ab}-5\sqrt{a}+6\sqrt{ab}-10\sqrt{a}$
6. $-3\sqrt{xy}+6\sqrt{y}-4\sqrt{xy}-7\sqrt{y}$
7. $5\sqrt{xy}-(3\sqrt{xy}-7\sqrt{xy})$
8. $-8\sqrt{ab}-(2\sqrt{ab}-4\sqrt{ab})$
9. $(3\sqrt{2x}-\sqrt{3x})-(\sqrt{2x}-7\sqrt{3x})$
10. $(\sqrt{y}-4\sqrt{2y})-(\sqrt{y}-5\sqrt{2y})$
11. $5\sqrt[3]{x}-12\sqrt[3]{x}$
12. $-2\sqrt[3]{y}-3\sqrt[3]{y}$
13. $a\sqrt[5]{3b}+4a\sqrt[5]{3b}-a\sqrt[5]{3b}$
14. $-8\sqrt[4]{ab}+3\sqrt[4]{ab}-2\sqrt[4]{ab}$
15. $6\sqrt{2a}-4\sqrt[3]{2a}+7\sqrt{2a}-\sqrt[3]{2a}$
16. $4\sqrt[5]{3a}+\sqrt[3]{3a}-9\sqrt[5]{3a}+\sqrt[3]{3a}$
17. $(\sqrt[4]{4xy}-\sqrt[3]{xy})-(2\sqrt[4]{4xy}-\sqrt[3]{xy})$
18. $(5\sqrt[6]{6y}-\sqrt{5y})-(2\sqrt[6]{6y}+\sqrt{3y})$

Answer available at the end of this section.

Exercise 8.3.5 adding and subtracting rational expressions

Simplify.

1. $\sqrt{75}-\sqrt{12}$
2. $\sqrt{24}-\sqrt{54}$
3. $\sqrt{32}+\sqrt{27}-\sqrt{8}$
4. $\sqrt{20}+\sqrt{48}-\sqrt{45}$
5. $\sqrt{28}-\sqrt{27}+\sqrt{63}-\sqrt{12}$
6. $\sqrt{90}+\sqrt{24}-\sqrt{40}-\sqrt{54}$
7. $\sqrt{45}-\sqrt{80}+\sqrt{245}-\sqrt{5}$
8. $\sqrt{108}+\sqrt{48}-\sqrt{75}-\sqrt{3}$
9. $4\sqrt{2}-(\sqrt{27}-\sqrt{72})$
10. $-\sqrt{35}-(\sqrt{20}-\sqrt{50})$
11. $\sqrt[3]{16}-\sqrt[3]{54}$
12. $\sqrt[3]{81}-\sqrt[3]{24}$
13. $\sqrt[3]{135}+\sqrt[3]{40}-\sqrt[3]{5}$
14. $\sqrt[3]{108}-\sqrt[3]{32}-\sqrt[3]{4}$
15. $2\sqrt{27}-2\sqrt{12}$
16. $3\sqrt{50}-4\sqrt{32}$
17. $3\sqrt{243}-2\sqrt{18}-\sqrt{48}$
18. $6\sqrt{216}-2\sqrt{24}-2\sqrt{96}$
19. $2\sqrt{18}-3\sqrt{75}-2\sqrt{98}+4\sqrt{48}$
20. $\sqrt{245}-\sqrt{12}+\sqrt{220}-\sqrt{108}$
21. $(2\sqrt{363}-3\sqrt{96})-(7\sqrt{12}-2\sqrt{54})$
22. $(2\sqrt{288}+3\sqrt{360})-(2\sqrt{72}-7\sqrt{40})$
23. $3\sqrt[3]{54}+5\sqrt[3]{250}-4\sqrt[3]{16}$
24. $4\sqrt[3]{162}-2\sqrt[3]{384}-3\sqrt[3]{750}$

Answer available at the end of this section.

Exercise 8.3.6 adding and subtracting rational expressions

Simplify. (Assume all radicands containing variable expressions are positive.)

1. $\sqrt{81b}+\sqrt{4b}$
2. $\sqrt{100a}+\sqrt{a}$
3. $\sqrt{9a^2b}-\sqrt{36a^2b}$
4. $\sqrt{50a^2}-\sqrt{18a^2}$
5. $\sqrt{49x}-\sqrt{9y}+\sqrt{x}-\sqrt{4y}$
6. $\sqrt{9x}+\sqrt{64y}-\sqrt{25x}-\sqrt{y}$
7. $7\sqrt{8x}-(3\sqrt{16y}-2\sqrt{18x})$
8. $2\sqrt{64y}-(3\sqrt{32y}-\sqrt{81y})$
9. $2\sqrt{9m2n}-5\sqrt{m2n}+\sqrt{m2n}$
10. $4\sqrt{18n^2m}-2\sqrt{n^8m}+\sqrt{n^2m}$
11. $\sqrt{4x^2y}-\sqrt{9xy^2}-\sqrt{16x^2y}+\sqrt{y^2x}$
12. $\sqrt{32x^2y^2}+\sqrt{12x^2y}-\sqrt{18x^2y^2}-\sqrt{27x^2y}$
13. $(\sqrt{9x^2y}-\sqrt{16y})-(\sqrt{49x^2y}-4\sqrt{y})$
14. $(\sqrt{72x^2y^2}-\sqrt{18x^2y})-(\sqrt{50x^2y^2}+\sqrt{x^2y})$
15. $\sqrt{12m^4n}-m\sqrt{75m^2n}+2\sqrt{27m^4n}$
16. $5n\sqrt{27mn^2}+2\sqrt{12mn^4}-n\sqrt{3mn^2}$
17. $2\sqrt{27a^3b}-a\sqrt{48ab}-a\sqrt{144a^3b}$
18. $\sqrt{298a^4b}-2a\sqrt{162a^2b}+a\sqrt{200b}$
19. $\sqrt[3]{125a}-\sqrt[3]{27a}$
20. $\sqrt[3]{1000a^2}-\sqrt[3]{64a^2}$
21. $2x\sqrt[3]{54x}-2\sqrt[3]{16x^4}+5\sqrt[3]{2x^4}$
22. $\sqrt[3]{54x^3}-\sqrt[3]{250x^6}+x2\sqrt[3]{2}$
23. $\sqrt[4]{16y^2}+\sqrt[4]{81y^2}$
24. $\sqrt[5]{32y^4}-\sqrt[5]{y^4}$
25. $\sqrt[4]{32a^3}-\sqrt[4]{162a^3}+5\sqrt[4]{2a^3}$
26. $\sqrt[4]{80a^4b}+\sqrt[4]{5a^4b}-a\sqrt[4]{5b}$
27. $\sqrt[3]{27x^3}+\sqrt[3]{8x}-\sqrt[3]{125x^3}$
28. $\sqrt[3]{24x}-\sqrt[3]{128x}-\sqrt[3]{81x}$
29. $\sqrt[3]{27x^4y}-\sqrt[3]{8xy^3}+x\sqrt[3]{64xy}-y\sqrt[3]{x}$
30. $\sqrt[3]{125xy^3}+\sqrt[3]{8x^3y}-\sqrt[3]{216xy^3}+10x\sqrt[3]{y}$
31. $(\sqrt[3]{162x^4y}-\sqrt[3]{250x^4y^2})-(\sqrt[3]{2x^4y^2}-\sqrt[3]{384x^4y})$
32. $(\sqrt[5]{32x^2y^6}-\sqrt[5]{243x^6y^2})-(\sqrt[5]{x^2y^6}-x\sqrt[5]{xy^2})$

Answer available at the end of this section.

Exercise 8.3.7 discussion board

1. Choose values for *x* and *y* and use a calculator to show that $\sqrt{x+y}\neq\sqrt{x}+\sqrt{y}$.
2. Choose values for *x* and *y* and use a calculator to show that $\sqrt{x^2+y^2}\neq x+y$.

Answer available at the end of this section.

Answers

Exercise 8.3.1

$-2\sqrt{2}$

Exercise 8.3.2

$x\sqrt{5x}+4\sqrt{5x}$

Exercise 8.3.3 **adding and subtracting like radicals**

1. $14\sqrt{3}$
3. $-2\sqrt{5}$

5. $-\sqrt{6}$
7. $8\sqrt{7}-\sqrt{2}$
9. $9\sqrt{5}-4\sqrt{3}$
11. $-\sqrt{5}-4\sqrt{10}$
13. $10\sqrt[3]{6}-3\sqrt[3]{5}$
15. $6\sqrt[3]{9}-\sqrt[3]{3}$

Exercise 8.3.4 **adding and subtracting like radicals**

1. $16\sqrt{x}$
3. $5\sqrt{xy}$
5. $8\sqrt{ab}-15\sqrt{a}$
7. $9\sqrt{xy}$
9. $2\sqrt{2x}+6\sqrt{3x}$
11. $-7\sqrt[3]{x}$
13. $4a\sqrt[5]{3b}$
15. $13\sqrt{2a}-5\sqrt[3]{2a}$
17. $-\sqrt[4]{4xy}$

Exercise 8.3.5 **adding and subtracting rational expressions**

1. $3\sqrt{3}$
3. $2\sqrt{2}+3\sqrt{3}$
5. $5\sqrt{7}-5\sqrt{3}$
7. $5\sqrt{5}$
9. $10\sqrt{2}-3\sqrt{3}$
11. $\sqrt[3]{-2}$
13. $4\sqrt[3]{5}$
15. $2\sqrt{3}$
17. $23\sqrt{3}-6\sqrt{2}$
19. $-8\sqrt{2}+\sqrt{3}$
21. $8\sqrt{3}-6\sqrt{6}$
23. $26\sqrt[3]{2}$

Exercise 8.3.6 **adding and subtracting rational expressions**

1. $11\sqrt{b}$
3. $-3a\sqrt{b}$
5. $8\sqrt{x}-5\sqrt{y}$
7. $20\sqrt{2x}-12\sqrt{y}$
9. $2\sqrt{2}\sqrt{mn}$
11. $-2x\sqrt{y}-2y\sqrt{x}$
13. $-4x\sqrt{y}$
15. $3\sqrt{3}m^2\sqrt{n}$
17. $2a\sqrt{3ab}-12a^2\sqrt{ab}$
19. $2\sqrt[3]{a}$
21. $7x\sqrt[3]{2x}$
23. $5\sqrt{y}$
25. $4\sqrt[4]{2a^3}$
27. $-2x+2\sqrt[3]{x}$
29. $7x\sqrt[3]{xy}-3y\sqrt[3]{x}$
31. $7x\sqrt[3]{6xy}-6x\sqrt[3]{2xy^2}$

Exercise 8.3.7 **discussion board**

1. Answers may vary

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8.4: Multiplying and Dividing Radical Expressions

Learning Objectives

- Multiply radical expressions.
- Divide radical expressions.
- Rationalize the denominator.

Multiplying Radical Expressions

When multiplying radical expressions with the same index, we use the product rule for radicals. If ***a*** and ***b*** represent positive real numbers,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b} \tag{8.4.1}$$

Example 8.4.1

Multiply:

$$\sqrt{2} \cdot \sqrt{6}$$

Solution:

This problem is a product of two square roots. Apply the product rule for radicals and then simplify.

$$\begin{aligned} \sqrt{2} \cdot \sqrt{6} &= \sqrt{2 \cdot 6} && \textit{Multiply the radicands.} \\ &= \sqrt{12} && \textit{Simplify.} \\ &= \sqrt{2^2 \cdot 3} \\ &= 2\sqrt{3} \end{aligned}$$

Answer:

$$2\sqrt{3}$$

Example 8.4.2

Multiply:

$$\sqrt[3]{9} \cdot \sqrt[3]{6}$$

Solution:

This problem is a product of cube roots. Apply the product rule for radicals and then simplify.

$$\begin{aligned} \sqrt[3]{9} \cdot \sqrt[3]{6} &= \sqrt[3]{9 \cdot 6} && \textit{Multiply the radiands.} \\ &= \sqrt[3]{54} && \textit{Simplify.} \\ &= \sqrt[3]{3^3 \cdot 2} \\ &= 3\sqrt[3]{2} \end{aligned}$$

Answer:

$$3\sqrt[3]{2}$$

Often there will be coefficients in front of the radicals.

Example 8.4.3

Multiply:

$$2\sqrt{3} \cdot 5\sqrt{2}$$

Solution:

Using the product rule for radicals and the fact that multiplication is commutative, we can multiply the coefficients and the radicands as follows.

$$\begin{aligned} 2\sqrt{3} \cdot 5\sqrt{2} &= 2 \cdot 5 \cdot \sqrt{3} \cdot \sqrt{2} && \textit{Multiplication is commutative.} \\ &= 10 \cdot \sqrt{6} && \textit{Multiply the coefficients and the radicands.} \\ &= 10\sqrt{6} \end{aligned}$$

Typically, the first step involving the application of the commutative property is not shown.

Answer:

$$10\sqrt{6}$$

Example 8.4.4

Multiply:

Solution:

Answer:

$$-30x$$

Use the distributive property when multiplying rational expressions with more than one term.

Example 8.4.5

Multiply:

Solution:

Apply the distributive property and multiply each term by $4\sqrt{3}$.

Answer:

Multiply:

Solution:

Apply the distributive property and then simplify the result.

Answer:

The process for multiplying radical expressions with multiple terms is the same process used when multiplying polynomials. Apply the distributive property, simplify each radical, and then combine like terms.

Multiply:

$$(\sqrt{5}+2)(\sqrt{5}-4)$$

Solution:

Begin by applying the distributive property.

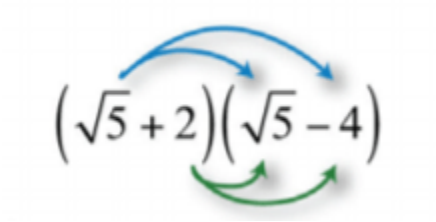


Figure 8.4.1

Answer:

$-3-2\sqrt{5}$

Example 8.4.8

Multiply:

$$(3\sqrt{x}-\sqrt{y})^2$$

Solution:

Answer:

$9x-6\sqrt{xy}+y$

Exercise 8.4.1

Multiply:

$$(2\sqrt{3}+5\sqrt{2})(\sqrt{3}-2\sqrt{6})$$

Answer available at the end of this section.

The expressions $(a+b)$ and $(a-b)$ are called conjugates. When multiplying conjugates, the sum of the products of the inner and outer terms results in 0.

Example 8.4.9

Multiply:

$$(\sqrt{2}+\sqrt{5})(\sqrt{2}-\sqrt{5})$$

Solution:

Apply the distributive property and then combine like terms.

Answer:

-3

It is important to note that when multiplying conjugate radical expressions, we obtain a rational expression. This is true in general and is often used in our study of algebra.

$$\begin{aligned}(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) &= \sqrt{a^2}-\sqrt{ab}+\sqrt{ab}-\sqrt{b^2}\\ &= a-b\end{aligned}$$

Therefore, for nonnegative real numbers ***a*** and ***b***, we have the following property:

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b \tag{8.4.2}$$

Dividing Radical Expressions (Rationalizing the Denominator)

To divide radical expressions with the same index, we use the quotient rule for radicals. If ***a*** and ***b*** represent nonnegative numbers, where $b \neq 0$, then we have

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \tag{8.4.3}$$

Example 8.4.10

Divide:

$$\frac{\sqrt{80}}{\sqrt{10}}$$

Solution:

In this case, we can see that 10 and 80 have common factors. If we apply the quotient rule for radicals and write it as a single square root, we will be able to reduce the fractional radicand.

$$\begin{aligned}\frac{\sqrt{80}}{\sqrt{10}} &= \sqrt{\frac{80}{10}} && \textit{Apply the quotient rule for radicals and reduce the radicand.}\\ &= \sqrt{8} && \textit{Simplify.}\\ &= \sqrt{4\cdot 2}\\ &= 2\sqrt{2}\end{aligned}$$

Answer:

$$2\sqrt{2}$$

Example 8.4.11

Divide:

$$\frac{\sqrt{16x^5y^4}}{\sqrt{2xy}}$$

Solution:

$$\begin{aligned}\frac{\sqrt{16x^5y^4}}{\sqrt{2xy}} &= \sqrt{\frac{16x^5y^4}{2xy}} && \textit{Apply the quotient rule for radicals and cancel.}\\ &= \sqrt{8x^4y^3} && \textit{Simplify.}\\ &= \sqrt{4\cdot 2\cdot (x^2)^2\cdot y^2\cdot y}\\ &= 2x^2y\sqrt{2y}\end{aligned}$$

Answer:

$$2x^2y\sqrt{2y}$$

Example 8.4.12

Divide:

$$\frac{\sqrt[3]{54a^3b^5}}{\sqrt[3]{16a^2b^2}}$$

Solution:

$$\begin{aligned}\frac{\sqrt[3]{54a^3b^5}}{\sqrt[3]{16a^2b^2}} &= \sqrt[3]{\frac{54\cdot a^3b^5}{16a^2b^2}} && \textit{Apply the quotient rule for radicals and then cancel.}\\ &= \sqrt[3]{\frac{27ab^3}{8}} && \textit{Replace 27 and 8 with their prime factorizations.}\\ &= \sqrt[3]{\frac{3^3ab^3}{2^3}} && \textit{Simplify.}\\ &= \frac{\sqrt[3]{3^3ab^3}}{\sqrt[3]{2^3}}\\ &= \frac{3b\sqrt[3]{a}}{2}\end{aligned}$$

Answer:

$$\frac{3b\sqrt[3]{a}}{2}$$

When the divisor of a radical expression contains a radical, it is a common practice to find an equivalent expression where the denominator is a rational number. Finding such an equivalent expression is called rationalizing the denominator.

Radical expression	*Rational denominator*
$\frac{1}{\sqrt{3}}$	$=\frac{\sqrt{3}}{3}$

To do this, multiply the fraction by a special form of 1 so that the radicand in the denominator can be written with a power that matches the index. After doing this, simplify and eliminate the radical in the denominator. For example,

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3^2}} = \frac{\sqrt{3}}{3}$$

Remember, to obtain an equivalent expression, you must multiply the numerator and denominator by the exact same nonzero factor.

Example 8.4.13

Rationalize the denominator:

$$\frac{\sqrt{3}}{\sqrt{2}}$$

Solution:

The goal is to find an equivalent expression without a radical in the denominator. In this example, multiply by 1 in the form $\frac{\sqrt{2}}{\sqrt{2}}$.

$$\begin{aligned}\frac{\sqrt{3}}{\sqrt{2}} &= \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} && \textit{Multiply by } \frac{\sqrt{2}}{\sqrt{2}}.\\ &= \frac{\sqrt{6}}{\sqrt{2^2}} && \textit{Simplify.}\\ &= \frac{\sqrt{6}}{2} && \textit{Rational denominator.}\end{aligned}$$

Answer:

$$\frac{\sqrt{6}}{2}$$

Example 8.4.14

Rationalize the denominator:

$$\frac{1}{2\sqrt{3x}}$$

Solution:

The radicand in the denominator determines the factors that you need to use to rationalize it. In this example, multiply by 1 in the form $\frac{1}{2\sqrt{3x}}$.

$$\begin{aligned}\frac{1}{2\sqrt{3x}} &= \frac{1}{2\sqrt{3x}} \cdot \frac{\sqrt{3x}}{\sqrt{3x}} && \textit{Multiply by } \frac{\sqrt{3x}}{\sqrt{3x}}.\\ &= \frac{\sqrt{3x}}{2\sqrt{3^2x^2}} && \textit{Simplify.}\\ &= \frac{\sqrt{3x}}{2 \cdot 3x}\\ &= \frac{\sqrt{3x}}{6x}\end{aligned}$$

Answer:

$$\frac{\sqrt{3x}}{6x}$$

Typically, we will find the need to reduce, or cancel, after rationalizing the denominator.

Example 8.4.15

Rationalize the denominator:

$$\frac{5\sqrt{2}}{\sqrt{5ab}}$$

Solution

In this example, we will multiply by 1 in the form $\frac{\sqrt{5ab}}{\sqrt{5ab}}$.

$$\begin{aligned}\frac{5\sqrt{2}}{\sqrt{5ab}} &= \frac{5\sqrt{2}}{\sqrt{5ab}} \cdot \frac{\sqrt{5ab}}{\sqrt{5ab}} \\ &= \frac{5\sqrt{10ab}}{\sqrt{25a^2b^2}} \qquad Simplify. \\ &= \frac{5\sqrt{10ab}}{5ab} \qquad Cancel. \\ &= \frac{\sqrt{10ab}}{ab}\end{aligned}$$

Notice that ***a*** and ***b*** do not cancel in this example. Do not cancel factors inside a radical with those that are outside.

Answer:

$$\frac{\sqrt{10ab}}{ab}$$

Exercise 8.4.2

Rationalize the denominator:

$$\sqrt{\frac{4a}{3b}}$$

Answer available at the end of this section.

Up to this point, we have seen that multiplying a numerator and a denominator by a square root with the exact same radicand results in a rational denominator. In general, this is true only when the denominator contains a square root. However, this is not the case for a cube root. For example,

$$\frac{1}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{\sqrt[3]{x}}{\sqrt[3]{x^2}}$$

Note that multiplying by the same factor in the denominator does not rationalize it. In this case, if we multiply by 1 in the form of $\frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}}$, then we can write the radicand in the denominator as a power of 3. Simplifying the result then yields a rationalized denominator. For example,

$$\frac{1}{\sqrt[3]{x}} = \frac{1}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{\sqrt[3]{x^2}}{x}$$

Therefore, to rationalize the denominator of radical expressions with one radical term in the denominator, begin by factoring the radicand of the denominator. The factors of this radicand and the index determine what we should multiply by. Multiply numerator and denominator by the ***n*** th root of factors that produce ***n*** th powers of all the factors in the radicand of the denominator.

Example 8.4.16

Rationalize the denominator:

$$\frac{1}{\sqrt[3]{25}}$$

Solution:

The radical in the denominator is equivalent to $\sqrt[3]{5^2}$. To rationalize the denominator, it should be $\sqrt[3]{5^3}$. To obtain this, we need one more factor of 5. Therefore, multiply by 1 in the form of $\frac{\sqrt[3]{5}}{\sqrt[3]{5}}$.

$$\begin{aligned}\frac{1}{\sqrt[3]{25}} &= \frac{1}{\sqrt[3]{5^2}} \cdot \frac{\sqrt[3]{5}}{\sqrt[3]{5}} && \textit{Multiply by the cube root of factors that result in powers of 3.}\\ &= \frac{\sqrt[3]{5}}{\sqrt[3]{5^3}} && \textit{Simplify.}\\ &= \frac{\sqrt[3]{5}}{5}\end{aligned}$$

Answer:

$$\frac{\sqrt[3]{5}}{5}$$

Example 8.4.17

Rationalize the denominator:

$$\sqrt[3]{\frac{27a}{2b^2}}$$

Solution:

In this example, we will multiply by 1 in the form $\frac{\sqrt[3]{2^2b}}{\sqrt[3]{2^2b}}$.

$$\begin{aligned}\sqrt[3]{\frac{27a}{2b^2}} &= \frac{\sqrt[3]{3^3a}}{\sqrt[3]{2b^2}} && \textit{Apply the quotient rule for radicals.}\\ &= \frac{3\sqrt[3]{a}}{\sqrt[3]{2b^2}} \cdot \frac{\sqrt[3]{2^2b}}{\sqrt[3]{2^2b}} && \textit{Multiply by the cube root of factors that result in powers of 3.}\\ &= \frac{3\sqrt[3]{2^2ab}}{\sqrt[3]{2^3b^3}} && \textit{Simplify.}\\ &= \frac{3\sqrt[3]{4ab}}{2b}\end{aligned}$$

Answer:

$$\frac{3\sqrt[3]{4ab}}{2b}$$

Example 8.4.18

Rationalize the denominator:

$$\frac{1}{\sqrt[5]{4x^3}}$$

Solution:

In this example, we will multiply by 1 in the form $\frac{\sqrt[5]{2^3x^2}}{\sqrt[5]{2^3x^2}}$.

$$\begin{aligned}\frac{1}{\sqrt[5]{4x^3}} &= \frac{1}{\sqrt[5]{2^2x^3}} \cdot \frac{\sqrt[5]{2^3x^2}}{\sqrt[5]{2^3x^2}} && \textit{Multiply by the fifth root of factors that result in powers of 5.}\\ &= \frac{\sqrt[5]{2^3x^5}}{\sqrt[5]{2^5x^5}} && \textit{Simplify.}\\ &= \frac{\sqrt[5]{8x^2}}{2x}\end{aligned}$$

Answer:

$$\frac{\sqrt[5]{8x^2}}{2x}$$

When two terms involving square roots appear in the denominator, we can rationalize it using a very special technique. This technique involves multiplying the numerator and the denominator of the fraction by the conjugate of the denominator. Recall that multiplying a radical expression by its conjugate produces a rational number.

Example 8.4.19

Rationalize the denominator:

$$\frac{1}{\sqrt{3}-\sqrt{2}}$$

Solution:

In this example, the conjugate of the denominator is $\sqrt{3}+\sqrt{2}$. Therefore, multiply by 1 in the form $\frac{(\sqrt{3}+\sqrt{2})}{(\sqrt{3}+\sqrt{2})}$.

$$\begin{aligned}\frac{1}{\sqrt{3}-\sqrt{2}} &= \frac{1}{(\sqrt{3}-\sqrt{2})}\frac{(\sqrt{3}+\sqrt{2})}{(\sqrt{3}+\sqrt{2})} && \textit{Multiply numerator and denominator by the conjugate of the denominator.}\\ &= \frac{\sqrt{3}+\sqrt{2}}{\sqrt{9}+\sqrt{6}-\sqrt{6}-\sqrt{4}} && \textit{Simplify.}\\ &= \frac{\sqrt{3}+\sqrt{2}}{3-2}\\ &= \frac{\sqrt{3}+\sqrt{2}}{1}\\ &= \sqrt{3}+\sqrt{2}\end{aligned}$$

Answer:

$$\sqrt{3}+\sqrt{2}$$

Notice that the terms involving the square root in the denominator are eliminated by multiplying by the conjugate. We can use the property $\frac{(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})} = a-b$ to expedite the process of multiplying the expressions in the denominator.

Example 8.4.20

Rationalize the denominator:

$$\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}+\sqrt{6}}$$

Solution:

Multiply by 1 in the form $\frac{(\sqrt{2}-\sqrt{6})}{(\sqrt{2}-\sqrt{6})}$

Answer:

$-2+\sqrt{3}$

Example 8.4.21

Rationalize the denominator:

$$\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

Solution:

In this example, we will multiply by 1 in the form $\frac{(\sqrt{x}-\sqrt{y})}{(\sqrt{x}-\sqrt{y})}$.

Answer:

Exercise 8.4.3

Rationalize the denominator:

$$\frac{3\sqrt{5}+5}{2\sqrt{5}-3}$$

Answer available at the end of this section.

Key Takeaways

- To multiply two single-term radical expressions, multiply the coefficients and multiply the radicands. If possible, simplify the result.
- Apply the distributive property when multiplying radical expressions with multiple terms. Then simplify and combine all like radicals.
- Multiplying a two-term radical expression involving square roots by its conjugate results in a rational expression.
- It is common practice to write radical expressions without radicals in the denominator. The process of finding such an equivalent expression is called rationalizing the denominator.
- If an expression has one term in the denominator involving a radical, then rationalize it by multiplying numerator and denominator by the nth root of factors of the radicand so that their powers equal the index.
- If a radical expression has two terms in the denominator involving square roots, then rationalize it by multiplying the numerator and denominator by its conjugate.

Exercise 8.4.4 multiplying radical expressions

Multiply. (Assume all variables are nonnegative.)

1. $\sqrt{3}\cdot\sqrt{5}$
2. $\sqrt{7}\cdot\sqrt{3}$
3. $\sqrt{2}\cdot\sqrt{6}$
4. $\sqrt{5}\cdot\sqrt{15}$
5. $\sqrt{7}\cdot\sqrt{7}$
6. $\sqrt{12}\cdot\sqrt{12}$
7. $2\sqrt{5}\cdot 7\sqrt{10}$
8. $3\sqrt{15}\cdot 2\sqrt{6}$
9. $(2\sqrt{5})^2$
10. $(6\sqrt{2})^2$
11. $\sqrt{2x}\cdot\sqrt{2x}$
12. $\sqrt{5y}\cdot\sqrt{5y}$
13. $\sqrt{3a}\cdot\sqrt{12}$
14. $\sqrt{3a}\cdot\sqrt{2a}$
15. $4\sqrt{2x}\cdot 3\sqrt{6x}$
16. $5\sqrt{10y}\cdot 2\sqrt{2y}$
17. $\sqrt[3]{5}\cdot\sqrt[3]{25}$
18. $\sqrt[3]{4}\cdot\sqrt[3]{2}$
19. $\sqrt[3]{4}\cdot\sqrt[3]{10}$

20. $\sqrt[3]{18} \cdot \sqrt[3]{6}$
21. $(5\sqrt[3]{9})(2\sqrt[3]{6})$
22. $(2\sqrt[3]{4})(3\sqrt[3]{4})$
23. $(2\sqrt[3]{2})^3$
24. $(3\sqrt[3]{4})^3$
25. $\sqrt[3]{3a^2} \cdot \sqrt[3]{9a}$
26. $\sqrt[3]{7b} \cdot \sqrt[3]{49b^2}$
27. $\sqrt[3]{6x^2} \cdot \sqrt[3]{4x^2}$
28. $\sqrt[3]{12y} \cdot \sqrt[3]{9y^2}$
29. $\sqrt[3]{20x^2y} \cdot \sqrt[3]{10x^2y^2}$
30. $\sqrt[3]{63xy} \cdot \sqrt[3]{12x^4y^2}$
31. $\sqrt{5}(3-\sqrt{5})$
32. $\sqrt{2}(\sqrt{3}-\sqrt{2})$
33. $3\sqrt{7}(2\sqrt{7}-\sqrt{3})$
34. $2\sqrt{5}(6-3\sqrt{10})$
35. $\sqrt{6}(\sqrt{3}-\sqrt{2})$
36. $\sqrt{15}(\sqrt{5}+\sqrt{3})$
37. $\sqrt{x}(\sqrt{x}+\sqrt{xy})$
38. $\sqrt{y}(\sqrt{xy}+\sqrt{y})$
39. $\sqrt{2ab}(\sqrt{14a}-2\sqrt{10b})$
40. $\sqrt{6ab}(5\sqrt{2a}-\sqrt{3b})$
41. $(\sqrt{2}-\sqrt{5})(\sqrt{3}+\sqrt{7})$
42. $(\sqrt{3}+\sqrt{2})(\sqrt{5}-\sqrt{7})$
43. $(2\sqrt{3}-4)(3\sqrt{6}+1)$
44. $(5-2\sqrt{6})(7-2\sqrt{3})$
45. $(\sqrt{5}-\sqrt{3})^2$
46. $(\sqrt{7}-\sqrt{2})^2$
47. $(2\sqrt{3}+\sqrt{2})(2\sqrt{3}-\sqrt{2})$
48. $(\sqrt{2}+3\sqrt{7})(\sqrt{2}-3\sqrt{7})$
49. $(\sqrt{a}-\sqrt{2b})^2$
50. $(\sqrt{ab}+1)^2$
51. What are the perimeter and area of a rectangle with length of $5\sqrt{3}$ centimeters and width of $3\sqrt{2}$ centimeters?
52. What are the perimeter and area of a rectangle with length of $2\sqrt{6}$ centimeters and width of $\sqrt{3}$ centimeters?
53. If the base of a triangle measures $6\sqrt{2}$ meters and the height measures $3\sqrt{2}$ meters, then what is the area?
54. If the base of a triangle measures $6\sqrt{3}$ meters and the height measures $3\sqrt{6}$ meters, then what is the area?

Answer available at the end of this section.

Exercise 8.4.5 dividing radical expressions

Divide.

1. $\frac{\sqrt{75}}{\sqrt{3}}$
2. $\frac{\sqrt{360}}{\sqrt{10}}$
3. $\frac{\sqrt{72}}{\sqrt{75}}$
4. $\frac{\sqrt{90}}{\sqrt{98}}$
5. $\frac{\sqrt{90x^5}}{\sqrt{2x}}$
6. $\frac{\sqrt{96y^3}}{\sqrt{3y}}$
7. $\frac{\sqrt{162x^7y^5}}{\sqrt{2xy}}$
8. $\frac{\sqrt{363x^4y^9}}{\sqrt{3xy}}$
9. $\frac{\sqrt[3]{16a^5b^2}}{\sqrt[3]{2a^2b^2}}$
10. $\frac{\sqrt[3]{192a^2b^7}}{\sqrt[3]{2a^2b^2}}$

Answer available at the end of this section.

Exercise 8.4.6 dividing radical expressions

Rationalize the denominator

1. $\sqrt{\frac{1}{5}}$
2. $\sqrt{\frac{1}{6}}$
3. $\frac{\sqrt{2}}{\sqrt{3}}$
4. $\frac{\sqrt{3}}{\sqrt{7}}$
5. $\sqrt{\frac{52}{10}}$
6. $\sqrt{\frac{3}{56}}$
7. $\frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}}$
8. $\frac{\sqrt{6}-\sqrt{2}}{\sqrt{2}}$
9. $\sqrt{\frac{1}{7x}}$
10. $\sqrt{\frac{1}{3y}}$
11. $a\sqrt{\frac{5}{ab}}$
12. \(\3 b^{2} \sqrt{\frac{23}{a b}})
13. 236−−√3
14. 147√3
15. 14x−−√3
16. 13y2−−−−√3
17. 9x⋅2√39xy2−−−−−√3
18. 5y2⋅x−−√35x2y−−−−−√3
19. 3a2 3a2b2−−−−−√3
20. 25n3 25m2n−−−−−−√3
21. 327x2y−−−−−√5
22. 216xy2−−−−−−√5
23. ab9a3b−−−−√5
24. abcab2c3−−−−−√5
25. 310−−√−3
26. 26√−2
27. 15√+3√
28. 17√−2√
29. 3√3√+6√
30. 5√5√+15−−√
31. 105−35√
32. −22√4−32√
33. 3√+5√3√−5√
34. 10−−√−2√10−−√+2√
35. 23√−32√43√+2√
36. 65√+225√−2√
37. x+y√x−y√
38. x−y√x+y√
39. a√−b√a√+b√
40. ab−−√+2√ab−−√−2√
41. x−−√5−2x−−√ 106. 1x−−√−y

Answer available at the end of this section.

Exercise 8.4.7 discussion

1. Research and discuss some of the reasons why it is a common practice to rationalize the denominator.
2. Explain in your own words how to rationalize the denominator.

Answer available at the end of this section.

Answers

Exercise 8.4.1

$6-12\sqrt{2}+5\sqrt{6}-20\sqrt{3}$

Exercise 8.4.2

$\frac{2\sqrt{3ab}}{3b}$

Exercise 8.4.3

$\frac{45+19\sqrt{5}}{11}$

Exercise 8.4.4 **multiplying radical expressions**

1. $\sqrt{15}$
3. $2\sqrt{3}$
5. 7
7. $70\sqrt{2}$
9. 20
11. $2x$
13. $6\sqrt{a}$
15. $24x\sqrt{3}$
17. 5
19. $2\sqrt[3]{5}$
21. $30\sqrt[3]{2}$
23. 16
25. $3a$
27. $2\sqrt[3]{3}\left(x^2\right)^{\frac{2}{3}}$
29. $2xy\sqrt[3]{25x}$
31. $3\sqrt{5}-5$
33. $42-3\sqrt{21}$
35. $3\sqrt{2}-2\sqrt{3}$
37. $x+x\sqrt{y}$
39. $\sqrt{2}(\sqrt{14}\sqrt{a}-2\sqrt{10}\sqrt{b})\sqrt{ab}$
41. $\sqrt{6}+\sqrt{14}-\sqrt{15}-\sqrt{35}$
43. $18\sqrt{2}+2\sqrt{3}-12\sqrt{6}-4$
45. $8-2\sqrt{15}$
47. 10
49. $(\sqrt{a}-\sqrt{2b})^2$
51. Perimeter: $(10\sqrt{3}+6\sqrt{2})$ centimeters; area: $15\sqrt{6}$ square centimeters
53. 18 square meters

Exercise 8.4.5 **dividing radical expressions**

1. 5
3. $\frac{2\sqrt{6}}{5}$
5. $3\sqrt{5}x^2$
7. $9x^3y^2$
9. $2a$

Exercise 8.4.6 **dividing radical expressions**

1. $\frac{\sqrt{5}}{5}$
3. $\frac{\sqrt{6}}{3}$
5. $\frac{\sqrt{130}}{5}$
7. $\frac{3-\sqrt{15}}{3}$
9. $\frac{\sqrt{7x}}{7x}$
11. $a\frac{\sqrt{5ab}}{ab}$
13. 6√33
15. 2x2−−−−√32x

17. 3 6x2y−−−−−√3y
19. 9ab−−−√32b
21. 9x3y4−−−−−−√5xy
23. 27a2b4−−−−−−√53
25. 310−−√+9
27. 5√−3√2
29. −1+2√
31. −5−35√2
33. −4−15−−√
35. 15−76√23
37. x2+2xy√+yx2−y
39. a−2ab−−√+ba−b
41. 5x−−√+2x25−4x

Exercise 8.4.7 **discussion**

1. Answer may vary

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8.5: Rational Exponents

Learning Objectives

- Write expressions with rational exponents in radical form.
- Write radical expressions with rational exponents.
- Perform operations and simplify expressions with rational exponents.
- Perform operations on radicals with different indices.

Definition of Rational Exponents

So far, exponents have been limited to integers. In this section, we will define what rational (or fractional) exponents mean and how to work with them. All of the rules for exponents developed up to this point apply. In particular, recall the product rule for exponents. Given any rational numbers ***m*** and ***n***, then

$$x^m \cdot x^n = x^{m+n} \tag{8.5.1}$$

For example, if we have an exponent of $\frac{1}{2}$, then the product rule for exponents implies the following:

$$5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5$$

Here $5^{\frac{1}{2}}$ is one of two equal factors of 5; hence it is a square root of 5, and we can write

$$5^{1/2} = \sqrt{5}$$

Furthermore, we can see that $2^{\frac{1}{3}}$ is one of three equal factors of 2.

$$2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = 2^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 2^{\frac{3}{3}} = 2^1 = 2$$

Therefore, $2^{\frac{1}{3}}$ is the cube root of 2, and we can write

$$2^{1/3} = \sqrt[3]{2}$$

This is true in general, given any nonzero real number ***a***,

$$a^{1/n} = \sqrt[n]{a} \tag{8.5.2}$$

In other words, the denominator of a fractional exponent determines the index of an ***n***th root.

Example 8.5.1

Rewrite as a radical.

a. $7^{1/2}$
b. $7^{1/3}$

Solution:

a. $7^{1/2} = \sqrt{7}$

b. $7^{1/3} = \sqrt[3]{7}$

Example 8.5.2

Rewrite as a radical and then simplify.

a. $81^{1/2}$
b. $81^{1/4}$

Solution:

a. $81^{1/2} = \sqrt{81} = 9$
b. $81^{1/4} = \sqrt[4]{81} = \sqrt[4]{3^4} = 3$

Example 8.5.3

Rewrite as a radical and then simplify.

a. $(125x^3)^{1/3}$
b. $(-32y^{10})^{1/5}$

Solution:

a.

$$\begin{aligned}(125x^3)^{1/3} &= \sqrt[3]{125x^3} \\ &= \sqrt[3]{5^3x^3} \\ &= 5x\end{aligned}$$

b.

Next, consider fractional exponents where the numerator is an integer other than 1. For example, consider the following:

$5^{\frac{2}{3}} \cdot 5^{\frac{2}{3}} \cdot 5^{\frac{2}{3}} = 5^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = 5^{\frac{6}{3}} = 5^2$

This shows that $5^{2/3}$ is one of three equal factors of 5^2. In other words, $5^{2/3}$ is the cube root of 5^2 and we can write:

$$5^{2/3} = \sqrt[3]{5^2}$$

In general, given any real number ***a***,

$$a^{m/n} = \sqrt[n]{a^m} \tag{8.5.3}$$

An expression with a rational exponent is equivalent to a radical where the denominator is the index and the numerator is the exponent. Any radical expression can be written with a rational exponent, which we call exponential form.

Radical form		*Exponential form*
$\sqrt[5]{x^2}$	$=$	$x^{2/5}$

Example 8.5.4

Rewrite as a radical.

a. $7^{2/5}$
b. $2^{3/4}$

Solution:

a. $7^{2/5} = \sqrt[5]{7^2} = \sqrt[5]{49}$
b. $2^{3/4} = \sqrt[4]{2^3} = \sqrt[4]{8}$

Example 8.5.5

Rewrite as a radical and then simplify

a. $8^{2/3}$
b. $(32)^{3/5}$

Solution:

a.

$$\begin{aligned} 8^{2/3} &= \sqrt[3]{8^2} \\ &= \sqrt[3]{64} \\ &= \sqrt[3]{4^3} \\ &= 4 \end{aligned}$$

b. We can often avoid very large integers by working with their prime factorization.

$$\begin{aligned} (32)^{3/5} &= \sqrt[5]{(32)^3} && \textit{Replace } 32 \textit{ with } 2^5. \\ &= \sqrt[5]{(2^5)^3} && \textit{Apply the power rule for exponents.} \\ &= \sqrt[5]{2^{15}} && 15 \div 5 = 3, \textit{ so } 2^{15} = (2^3)^5. \\ &= \sqrt[5]{(2^3)^5} && \textit{Simplify.} \\ &= 2^3 \\ &= 8 \end{aligned}$$

Given a radical expression, we will be asked to find the equivalent in exponential form. Assume all variables are positive.

Example 8.5.6

Rewrite using rational exponents:

$$\sqrt[3]{x^2}$$

Solution:

Here the index is 3 and the power is 2. We can write

$$\sqrt[3]{x^2} = x^{2/3}$$

Answer:

$$x^{2/3}$$

Example 8.5.7

Rewrite using rational exponents:

$$\sqrt[6]{y^3}$$

Solution:

Here the index is 6 and the power is 3. We can write

$$\begin{aligned}\sqrt[6]{y^3} &= y^{3/6} \\ &= y^{1/2}\end{aligned}$$

Answer:

$$y^{1/2}$$

It is important to note that the following are equivalent.

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m \tag{8.5.4}$$

In other words, it does not matter if we apply the power first or the root first. For example, we can apply the power before the root:

$$27^{2/3} = \sqrt[3]{27^2} = \sqrt[3]{(3^3)^2} = \sqrt[3]{3^6} = 3^2 = 9$$

Or we can apply the ***n***th root before the power:

$$27^{2/3} = (\sqrt[3]{27})^2 = \left(\sqrt[3]{3^3}\right)^2 = 3^2 = 9$$

The results are the same.

Example 8.5.8

Rewrite as a radical and then simplify:

$$(-8)^{2/3}$$

Solution:

Here the index is 3 and the power is 2. We can write

$$(-8)^{2/3} = (\sqrt[3]{-8})^2 = (-2)^2 = 4$$

Answer:

$$4$$

Exercise 8.5.1

Rewrite as a radical and then simplify:

$$25^{3/2}$$

Answer available at the end of this section.

Some calculators have a caret button ^. If so, we can calculate approximations for radicals using it and rational exponents. For example, to calculate $\sqrt{2} = 2^{1/2} = 2 \wedge (1/2) \approx 1.414$, we would type

$$2 \quad \wedge \quad (\quad 1 \quad \div \quad 2 \quad) \quad =$$

To calculate $\sqrt[3]{2^2} = 2^{2/3} = 2 \wedge (2/3) = \approx 1.587$, we would type

$$2 \wedge (2 \div 3) =$$

Operations Using the Rules of Exponents

In this section, we review all of the rules of exponents, which extend to include rational exponents. If given any rational numbers ***m*** and ***n***, then we have

Table 8.5.1

Product rule:	$x^m \cdot x^n = x^{m+n}$
Quotient rule:	$\frac{x^m}{x^n} = x^{m-n}, x \neq 0$
Power rule:	$(x^m)^n = x^{m \cdot n}$
Power rule for a product:	$(xy)^n = x^n y^n$
Power rule for a quotient:	$(\frac{x}{y})^n = \frac{x^n}{y^n}, y \neq 0$
Negative exponents:	$x^{-n} = \frac{1}{x^n}$
Zero exponent:	$x^0 = 1, x \neq 0$

These rules allow us to perform operations with rational exponents.

Example 8.5.9

Simplify:

$$2^{\frac{2}{3}} \cdot 2^{\frac{1}{6}}$$

Solution:

$$\begin{aligned} 2^{\frac{2}{3}} \cdot 2^{\frac{1}{6}} &= 2^{\frac{2}{3}+\frac{1}{6}} && \textit{Apply the product rule } x^m \cdot x^n = x^{m+n}. \\ &= 2^{\frac{4}{6}+\frac{1}{6}} && \textit{Find equivalent fractions with a common denominator and then add.} \\ &= 2^{\frac{5}{6}} \end{aligned}$$

Answer:

$$2^{\frac{5}{6}}$$

Example 8.5.10

Simplify:

$$\frac{x^{1/2}}{x^{1/3}}$$

Solution:

$$\begin{aligned}\frac{x^{1/2}}{x^{1/3}} &= x^{\frac{1}{2}-\frac{1}{3}} && \textit{Apply the quotient rule } \frac{x^m}{x^n} = x^{m-n}.\\ &= x^{\frac{3}{6}-\frac{2}{6}} && \textit{Find equivalent fractions with a common denominator and then subtract.}\\ &= x^{\frac{1}{6}}\end{aligned}$$

Answer:

$$x^{\frac{1}{6}}$$

Example 8.5.11

Simplify:

$$\left(y^{3/4}\right)^{2/3}$$

Solution:

$$\begin{aligned}\left(y^{3/4}\right)^{2/3} &= y^{\frac{3}{4}\cdot\frac{2}{3}} && \textit{Apply the power rule } (x^m)^n = x^{m\cdot n}.\\ &= y^{\frac{6}{12}} && \textit{Multiply the exponents and reduce.}\\ &= y^{\frac{1}{2}}\end{aligned}$$

Answer:

$$y^{\frac{1}{2}}$$

Example 8.5.12

Simplify:

$$\left(16a^4b^8\right)^{3/4}$$

Solution:

$$\begin{aligned}\left(16a^4b^8\right)^{3/4} &= \left(2^4a^4b^8\right)^{3/4} && \textit{Rewrite } 16 \textit{ as } 2^4.\\ &= \left(2^4\right)^{3/4}\left(a^4\right)^{3/4}\left(b^8\right)^{3/4} && \textit{Apply the power rule for a product } (xy)^n = x^ny^n.\\ &= 2^{4\cdot\frac{3}{4}}a^{4\cdot\frac{3}{4}}b^{8\cdot\frac{3}{4}} && \textit{Apply the power rule to each factor.}\\ &= 2^3a^3b^6 && \textit{Simplify.}\\ &= 8a^3b^6\end{aligned}$$

Answer:

$$8a^3b^6$$

Example 8.5.13

Simplify:

Solution:

Answer:

$$\frac{1}{125}$$

Exercise 8.5.2

Simplify:

$$(8a^{3/4}b^3)^{2/3}a^{1/3}$$

Answer available at the end of this section.

Radical Expressions with Different Indices

To apply the product or quotient rule for radicals, the indices of the radicals involved must be the same. If the indices are different, then first rewrite the radicals in exponential form and then apply the rules for exponents.

Example 8.5.14

Multiply:

$$\sqrt{2}\cdot\sqrt[3]{2}$$

Solution:

In this example, the index of each radical factor is different. Hence the product rule for radicals does not apply. Begin by converting the radicals into an equivalent form using rational exponents. Then apply the product rule for exponents.

$$\begin{aligned}\sqrt{2}\cdot\sqrt[3]{2} &= 2^{\frac{1}{2}}\cdot 2^{\frac{1}{3}} && \textit{Equivalents using rational exponents.}\\ &= 2^{\frac{1}{2}+\frac{1}{3}} && \textit{Apply the product rule for exponents.}\\ &= 2^{\frac{3+2}{6}}\\ &= 2^{\frac{5}{6}}\\ &= \sqrt[6]{2^5}\end{aligned}$$

Answer:

$$\sqrt[6]{2^5}$$

Example 8.5.15

Divide:

$$\frac{\sqrt[3]{4}}{\sqrt[5]{2}}$$

Solution:

In this example, the index of the radical in the numerator is different from the index of the radical in the denominator. Hence the quotient rule for radicals does not apply. Begin by converting the radicals into an equivalent form using rational exponents and then apply the quotient rule for exponents.

$$\begin{aligned}\frac{\sqrt[3]{4}}{\sqrt[5]{2}} &= \frac{\sqrt[3]{2^2}}{\sqrt[5]{2}} \\ &= \frac{2^{\frac{2}{3}}}{2^{\frac{1}{5}}} && \textit{Equivalents using rational exponents.} \\ &= 2^{\frac{2}{3}-\frac{1}{5}} && \textit{Apply the quotient rule for exponents.} \\ &= 2^{\frac{10-3}{15}} \\ &= 2^{\frac{7}{15}} \\ &= \sqrt[15]{2^7}\end{aligned}$$

Answer:

$$\sqrt[15]{2^7}$$

Example 8.5.16

Simplify:

$$\sqrt{\sqrt[3]{4}}$$

Solution:

Here the radicand of the square root is a cube root. After rewriting this expression using rational exponents, we will see that the power rule for exponents applies.

$$\begin{aligned}\sqrt{\sqrt[3]{4}} &= \sqrt{\sqrt[3]{2^2}} \\ &= \left(2^{2/3}\right)^{1/2} && \textit{Equivalents using rational exponents.} \\ &= 2^{\frac{3}{2}\cdot\frac{1}{2}} && \textit{Apply the power rule for exponents.} \\ &= 2^{\frac{1}{3}} \\ &= \sqrt[3]{2}\end{aligned}$$

Answer:

$$\sqrt[3]{2}$$

Key Takeaways

- When converting fractional exponents to radicals, use the numerator as the power and the denominator as the index of the radical.
- All the rules of exponents apply to expressions with rational exponents.

Exercise 8.5.3 Rational Exponents

Express using rational exponents.

1. $\sqrt{6}$
2. $\sqrt{10}$
3. $\sqrt[3]{11}$
4. $\sqrt[4]{2}$
5. $\sqrt[3]{5^2}$
6. $\sqrt[4]{2^3}$
7. $\sqrt[5]{x}$

8. $\sqrt[6]{x}$
9. $\sqrt[6]{x^7}$
10. $\sqrt[5]{x^4}$

Answer available at the end of this section.

Exercise 8.5.4 Rational Exponents

Express in radical form.

1. $2^{1/2}$
2. $5^{1/3}$
3. $7^{2/3}$
4. $2^{3/5}$
5. $x^{3/4}$
6. $x^{5/6}$
7. $x^{-1/2}$
8. $x^{-3/4}$
9. $(\frac{1}{x})^{-1/3}$
10. $(\frac{1}{x})^{-3/5}$

Answer available at the end of this section.

Exercise 8.5.5 Rational Exponents

Write as a radical and then simplify.

1. $25^{1/2}$
2. $36^{1/2}$
3. $121^{1/2}$
4. $144^{1/2}$
5. $(\frac{1}{4})^{\frac{1}{2}}$
6. $(\frac{4}{9})^{\frac{1}{2}}$
7. $(4)^{-\frac{1}{2}}$
8. $(9)^{-\frac{1}{2}}$
9. $(\frac{1}{4})^{-\frac{1}{2}}$
10. $(\frac{1}{16})^{-1/2}$
11. $8^{1/3}$
12. $125^{1/3}$
13. $(\frac{1}{27})^{\frac{1}{3}}$
14. $(\frac{8}{125})^{1/3}$
15. $(-27)^{\frac{1}{3}}$
16. $(-64)^{1/3}$
17. $16^{1/4}$
18. $625^{1/4}$
19. $81^{-1/4}$
20. $16^{-1/4}$
21. $100,000^{1/5}$
22. $(-32)^{1/5}$
23. $(\frac{1}{32})^{\frac{1}{5}}$
24. $(\frac{1}{243})^{1/5}$
25. $9^{3/2}$
26. $4^{3/2}$
27. $8^{5/3}$

28. $27^{2/3}$
29. $16^{3/2}$
30. $32^{2/5}$
31. $(\frac{1}{16})^{3/4}$
32. $(\frac{1}{81})^{3/4}$
33. $(-27)^{2/3}$
34. $(-27)^{4/3}$
35. $(-32)^{3/5}$
36. $(-32)^{4/5}$

Answer available at the end of this section.

Exercise 8.5.6 Rational Exponents

Use a calculator to approximate an answer rounded to the nearest hundredth.

1. $2^{3/4}$
2. $3^{2/3}$
3. $5^{1/5}$
4. $7^{1/7}$
5. $(-9)^{3/2}$
6. $-9^{3/2}$
7. Explain why $(-4)^{(3/2)}$ gives an error on a calculator and $-4^{(3/2)}$ gives an answer of –8.
8. Marcy received a text message from Mark asking her how old she was. In response, Marcy texted back "$125^{(2/3)}$ years old." Help Mark determine how old Marcy is.

Answer available at the end of this section.

Exercise 8.5.7 Rational Exponents

Perform the operations and simplify. Leave answers in exponential form.

1. $2^{2/3} \cdot 2^{4/3}$
2. $3^{3/2} \cdot 3^{1/2}$
3. $5^{1/2} \cdot 5^{1/3}$
4. $2^{1/6} \cdot 2^{3/4}$
5. $y^{1/4} \cdot y^{2/5}$
6. $x^{1/2} \cdot x^{1/4}$
7. $\frac{5^{\frac{7}{3}}}{5^{\frac{1}{3}}}$
8. $\frac{2^{9/2}}{2^{1/2}}$
9. $\frac{2a^{\frac{2}{3}}}{a^{\frac{1}{6}}}$
10. $\frac{3b^{\frac{1}{2}}}{b^{\frac{1}{3}}}$
11. $\left(8^{\frac{1}{2}}\right)^{\frac{2}{3}}$
12. $(3^6)^{2/3}$
13. $\left(x^{\frac{2}{3}}\right)^{\frac{1}{2}}$
14. $\left(y^{\frac{3}{4}}\right)^{\frac{4}{5}}$
15. $\left(\frac{4x^2}{y^4}\right)^{\frac{1}{2}}$
16. $\left(\frac{9x^6}{y^2}\right)^{\frac{1}{2}}$
17. $\left(\frac{2x^{\frac{1}{3}}}{y^{\frac{2}{3}}}\right)^3$
18. $\left(\frac{8x^{\frac{3}{2}}}{y^{\frac{1}{2}}}\right)^2$

19. $\left(\frac{a^{\frac{3}{4}}}{a^{\frac{1}{2}}}\right)^{\frac{4}{3}}$
20. $\left(\frac{b^{\frac{4}{5}}}{b^{\frac{1}{10}}}\right)^{\frac{10}{3}}$
21. $\left(\frac{4x^{\frac{2}{3}}}{y^4}\right)^{\frac{1}{2}}$
22. $\left(\frac{27x^{\frac{3}{4}}}{y^9}\right)^{\frac{1}{3}}$
23. $y^{\frac{1}{2}} \cdot \frac{y^{\frac{2}{3}}}{y^{\frac{1}{6}}}$
24. $x^{\frac{2}{5}} \cdot \frac{x^{\frac{1}{2}}}{x^{\frac{1}{10}}}$
25. $\frac{xy}{x^{\frac{1}{2}}y^{\frac{1}{3}}}$
26. $\frac{x^{\frac{5}{4}}y}{xy^{\frac{2}{5}}}$
27. $\frac{49a^{\frac{5}{7}}b^{\frac{3}{2}}}{7a^{\frac{3}{7}}b^{\frac{1}{4}}}$
28. $\frac{16a^{\frac{5}{6}}b^{\frac{5}{4}}}{8a^{\frac{1}{2}}b^{\frac{2}{3}}}$
29. $\left(\frac{9x^{\frac{2}{3}}}{y^6}\right)^{\frac{3}{2}} x^{\frac{1}{2}}y$
30. $\left(\frac{125x^3}{y^{\frac{3}{5}}}\right)^{\frac{2}{3}} xy^{\frac{1}{3}}$
31. $\frac{\left(27a^{\frac{1}{4}}b^{\frac{3}{2}}\right)^{\frac{2}{3}}}{a^{\frac{1}{6}}b^{\frac{1}{2}}}$
32. $\frac{\left(25a^{\frac{2}{3}}b^{\frac{4}{3}}\right)^{\frac{3}{2}}}{a^{\frac{1}{6}}b^{\frac{1}{3}}}$

Answer available at the end of this section.

Exercise 8.5.8 Mixed Indices

Perform the operations.

1. $\sqrt[3]{9} \cdot \sqrt[5]{3}$
2. $\sqrt{5} \cdot \sqrt[5]{25}$
3. $\sqrt{x} \cdot \sqrt[3]{x}$
4. $\sqrt{y} \cdot \sqrt[4]{y}$
5. $\sqrt[3]{x^2} \cdot \sqrt[4]{x}$
6. $\sqrt[5]{x^3} \cdot \sqrt[3]{x}$
7. $\frac{\sqrt[3]{100}}{\sqrt{10}}$
8. $\frac{\sqrt[3]{16}}{\sqrt[3]{4}}$
9. $\frac{\sqrt[3]{a^2}}{\sqrt{a}}$
10. $\frac{\sqrt{a}}{\sqrt[5]{b^4}}$
11. $\frac{\sqrt[3]{b}}{\sqrt[3]{x^2}}$
12. $\frac{\sqrt[4]{x^3}}{\sqrt[5]{x^3}}$
13. $\sqrt[5]{\sqrt{16}}$
14. $\sqrt[3]{\sqrt{9}}$
15. $\sqrt[3]{\sqrt[5]{2}}$
16. $\sqrt[3]{\sqrt[5]{5}}$
17. $\sqrt[3]{\sqrt{7}}$
18. $\sqrt[3]{\sqrt{3}}$

Answer available at the end of this section.

Exercise 8.5.9 Discussion Board

1. Who is credited for devising the notation for rational exponents? What are some of his other accomplishments?
2. When using text, it is best to communicate *n*th roots using rational exponents. Give an example.

Answer available at the end of this section.

Answers

Exercise 8.5.1

125

Exercise 8.5.2

$4a^{5/6}b^2$

Exercise 8.5.3 **Rational Exponents**

1. $6^{1/2}$
3. $11^{1/3}$
5. $5^{2/3}$
7. $x^{1/5}$
9. $x^{7/6}$

Exercise 8.5.4 **Rational Exponents**

1. $\sqrt{2}$
3. $\sqrt[3]{7^2}$
5. $\sqrt[4]{x^3}$
7. $\sqrt{\frac{1}{x}}$
9. $\sqrt[3]{x}$

Exercise 8.5.5 **Rational Exponents**

1. 5
3. 11
5. $\frac{1}{2}$
7. $\frac{1}{2}$
9. 2
11. 2
13. $\frac{1}{3}$
15. −3
17. 2
19. $\frac{1}{3}$
21. 10
23. $\frac{1}{2}$
25. 27
27. 32
29. 64
31. $\frac{1}{8}$
33. 9
35. −8

Exercise 8.5.6 **Rational Exponents**

1. 1.68
3. 1.38
5. Not a real number
7. In the first expression, the square root of a negative number creates an error condition on the calculator. The square root of a negative number is not real. In the second expression, because of the order of operations, the negative sign is applied to the answer after 4 is raised to the (3/2) power.

Exercise 8.5.7 **Rational Exponents**

1. 4
3. $5^{5/6}$
5. $y^{13/20}$
7. 25
9. $2a^{1/2}$
11. 2
13. $x^{1/3}$

15. $\frac{2x}{y^2}$
17. $\frac{8x}{y^2}$
19 $a^{1/3}$
21. $\frac{2x^{\frac{1}{3}}}{y^3}$
23. y
25. $x^{\frac{1}{2}}y^{\frac{2}{3}}$
27. $7a^{\frac{2}{7}}b^{\frac{5}{4}}$
29. $\frac{27x^{\frac{3}{2}}}{y^3}$
31. $9b^{1/2}$

Exercise $8.5.8$ **Mixed Indices**

1. $\sqrt[15]{3^13}$
3. x
5. $\sqrt[13]{x^{11}}$
7. $\sqrt[6]{10}$
9. $\sqrt[6]{a}$
11. $\sqrt[15]{x}$
13. $\sqrt[5]{4}$
15. $\sqrt[15]{2}$
17. $\sqrt[6]{7}$

Exercise $8.5.9$ **Discussion Board**

1. Answers may vary

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8.6: Solving Radical Equations

Learning Objectives

- Solve equations involving square roots.
- Solve equations involving cube roots

Radical Equations

A radical equation is any equation that contains one or more radicals with a variable in the radicand. Following are some examples of radical equations, all of which will be solved in this section:

$$\sqrt{x-1}=5$$
$$\sqrt{2x-5}+4=x$$
$$\sqrt[3]{x^2+4}-2=0$$

We begin with the squaring property of equality; given real numbers ***a*** and ***b***, we have the following:

$$\text{If } a=b, \text{ then } a^2=b^2$$

In other words, equality is retained if we square both sides of an equation.

$$\begin{aligned} -3=-3 \quad \Rightarrow \quad (-3)^2 &= (-3)^2 \\ 9 &= 9 \ \checkmark \end{aligned}$$

The converse, on the other hand, is not necessarily true:

This is important because we will use this property to solve radical equations. Consider a very simple radical equation that can be solved by inspection:

$\sqrt{x}=3$

Here we can see that $x=9$ is a solution. To solve this equation algebraically, make use of the squaring property of equality and the fact that ((\sqrt{a})^{2}=\sqrt{a^{2}}=a\) when ***a*** is positive. Eliminate the square root by squaring both sides of the equation as follows:

$$\begin{aligned} (\sqrt{x})^2 &= (3)^2 \\ x &= 9 \end{aligned}$$

As a check, we can see that $\sqrt{9}=3$ as expected. Because the converse of the squaring property of equality is not necessarily true, solutions to the squared equation may not be solutions to the original. Hence squaring both sides of an equation introduces the possibility of extraneous solutions, or solutions that do not solve the original equation. For this reason, we must check the answers that result from squaring both sides of an equation.

Example 8.6.1

Solve:

$$\sqrt{x-1}=5$$

Solution:

We can eliminate the square root by applying the squaring property of equality.

Next, we must check.

$$\begin{aligned}\sqrt{26-1} &= 5\\ \sqrt{25} &= 5\\ 5 &= 5 \ \checkmark\end{aligned}$$

Answer:

The solution is 26.

Example 8.6.2

Solve:

Solution:

Begin by squaring both sides of the equation.

You are left with a quadratic equation that can be solved by factoring.

$$\begin{aligned} x+5=0 \quad &\text{or} \quad x-1=0\\ x=-5 \quad &\qquad x=1\end{aligned}$$

Since you squared both sides, you must check your solutions.

After checking, you can see that $x=-5$ was extraneous; it did not solve the original radical equation. Disregard that answer. This leaves $x=1$ as the only solution.

Answer:

The solution is $x=1$.

In the previous two examples, notice that the radical is isolated on one side of the equation. Typically, this is not the case. The steps for solving radical equations involving square roots are outlined in the following example.

Example 8.6.3

Solve:

$$\sqrt{2x-5}+4=x$$

Solution:

Step 1: Isolate the square root. Begin by subtracting 4 from both sides of the equation.

Step 2: Square both sides. Squaring both sides eliminates the square root.

Step 3: Solve the resulting equation. Here you are left with a quadratic equation that can be solved by factoring.

$$\begin{aligned} x-3=0 \quad &\text{or} \quad x-7=0\\ x=3 \quad &\qquad x=7\end{aligned}$$

Step 4: Check the solutions in the original equation. Squaring both sides introduces the possibility of extraneous solutions; hence the check is required.

Check $x=3$	Check $x=7$
$\sqrt{2x-5}+4=x$	$\sqrt{2x-5}+4=x$
$\sqrt{2(3)-5}+4=3$	$\sqrt{2(7)-5}+4=7$
$\sqrt{6-5}+4=3$	$\sqrt{14-5}+4=7$
$\sqrt{1}+4=3$	$\sqrt{9}+4=7$
$1+4=3$	$3+4=7$
$5=3$ x	$7=7$ ✓

After checking, we can see that $x=3$ is an extraneous root; it does not solve the original radical equation. This leaves $x=7$ as the only solution.

Answer:

The solution is $x=7$.

Example 8.6.4

Solve:

Solution:

Begin by isolating the term with the radical

Despite the fact that the term on the left side has a coefficient, it is still considered isolated. Recall that terms are separated by addition or subtraction operators.

$$\begin{aligned} 3\sqrt{x+1} &= 2x \\ (3\sqrt{x+1})^2 &= (2x)^2 \qquad \textit{Square both sides.} \\ 9(x+1) &= 4x^2 \end{aligned}$$

Solve the resulting quadratic equation.

$$\begin{aligned} 4x+3 &= 0 \quad \text{or} \quad x-3=0 \\ 4x &= -3 \qquad\quad x=3 \\ x &= -\tfrac{3}{4} \end{aligned}$$

Since we squared both sides, we must check our solutions.

After checking, we can see that $x=-\frac{3}{4}$ was extraneous.

Answer:

The solution is 3.

Sometimes both of the possible solutions are extraneous.

Example 8.6.5

Solve:

Solution:

Begin by isolating the radical.

Since we squared both sides, we must check our solutions.

Since both possible solutions are extraneous, the equation has no solution.

Answer:

No solution, Ø

The squaring property of equality extends to any positive integer power ***n***. Given real numbers ***a*** and ***b***, we have the following:

$$\text{If } a = b, \text{ then } a^n = b^n$$

This is often referred to as the power property of equality. Use this property, along with the fact that $(\sqrt[n]{a})^n = \sqrt[n]{a^n} = a$, when ***a*** is positive, to solve radical equations with indices greater than 2.

Example 8.6.6

Solve:

$$\sqrt[3]{x^2+4} - 2 = 0$$

Solution:

Isolate the radical and then cube both sides of the equation.

$$\begin{aligned} x+2 &= 0 \quad \text{or} \quad x-2=0 \\ x &= -2 \qquad\quad x=2 \end{aligned}$$

Check.

$$\begin{aligned} \text{Check } x=-2 \qquad & \qquad \text{Check } x=2 \\ \sqrt[3]{x^2+4}-2=0 \qquad & \qquad \sqrt[3]{x^2+4}-2=0 \\ \sqrt[3]{(-2)^2+4}-2=0 \qquad & \qquad \sqrt[3]{(2)^2+4}-2=0 \\ \sqrt[3]{4+4}-2=0 \qquad & \qquad \sqrt[3]{4+4}-2=0 \\ \sqrt[3]{8}-2=0 \qquad & \qquad \sqrt[3]{8}-2=0 \\ 2-2=0 \qquad & \qquad 2-2=0 \\ 0=0 \ \checkmark \qquad & \qquad 0=0 \ \checkmark \end{aligned}$$

Answer:

Solve:

$$\sqrt{2x-1} + 2 = x$$

Answer

$x = 5$ ($x = 1$ is extraneous)

It may be the case that the equation has two radical expressions.

Example 8.6.7

Solve:

$$\sqrt{3x-4}=\sqrt{2x+9}$$

Solution:

Both radicals are considered isolated on separate sides of the equation.

Check $x=13$.

$$\begin{aligned}\sqrt{3x-4}&=\sqrt{2x+9}\\ \sqrt{3(13)-4}&=\sqrt{2(13)+9}\\ \sqrt{39-4}&=\sqrt{26+9}\\ \sqrt{35}&=\sqrt{35}\quad\checkmark\end{aligned}$$

Answer:

The solution is 13.

Example 8.6.8

Solve:

$$\sqrt[3]{x^2+x-14}=\sqrt[3]{x+50}$$

Solution:

Eliminate the radicals by cubing both sides.

$$\begin{aligned}x+8&=0 \quad \text{or} \quad x-8=0\\ x&=-8 \qquad\quad x=8\end{aligned}$$

Check.

$$\begin{aligned}\text{Check } x=-8\\ \sqrt[3]{x^2+x-14}&=\sqrt[3]{x+50}\\ \sqrt[3]{(-8)^2+(-8)-14}&=\sqrt[3]{(-8)+50}\\ \sqrt[3]{64-8-14}&=\sqrt[3]{42}\\ \sqrt[3]{42}&=\sqrt[3]{42}\quad\checkmark\end{aligned}$$

$$\begin{aligned}\text{Check } x=8\\ \sqrt[3]{x^2+x-14}&=\sqrt[3]{x+50}\\ \sqrt[3]{(8)^2+(8)-14}&=\sqrt[3]{(8)+50}\\ \sqrt[3]{64+8-14}&=\sqrt[3]{58}\\ \sqrt[3]{58}&=\sqrt[3]{58}\quad\checkmark\end{aligned}$$

Answer:

We will learn how to solve some of the more advanced radical equations in the next course, Intermediate Algebra.

Solve:

$$\sqrt{3x+1}=\sqrt{2x-3}$$

Answer

No solution for x

Key Takeaways

- Solve equations involving square roots by first isolating the radical and then squaring both sides. Squaring a square root eliminates the radical, leaving us with an equation that can be solved using the techniques learned earlier in our study of algebra. However, squaring both sides of an equation introduces the possibility of extraneous solutions, so check your answers in the original equation.
- Solve equations involving cube roots by first isolating the radical and then cubing both sides. This eliminates the radical and results in an equation that may be solved with techniques you have already mastered.

Exercise 8.6.3 Solving Radical Equations

Solve.

1. $\sqrt{x}=2$
2. $\sqrt{x}=7$
3. $\sqrt{x}+7=8$
4. $\sqrt{x}+4=9$
5. $\sqrt{x}+6=3$
6. $\sqrt{x}+2=1$
7. $5\sqrt{x}-1=0$
8. $3\sqrt{x}-2=0$
9. $\sqrt{x-3}=3$
10. $\sqrt{x+5}=6$
11. $\sqrt{3x+1}=2$
12. $\sqrt{5x-4}=4$
13. $\sqrt{7x+4}+6=11$
14. $\sqrt{3x-5}+9=14$
15. $\sqrt{2x-1}-3=0$
16. $\sqrt{3x+1}-2=0$
17. $\sqrt[3]{x}=2$
18. $\sqrt[3]{x}=5$
19. $\sqrt[3]{2x+9}=3$
20. $\sqrt[3]{4x-11}=1$
21. $\sqrt[3]{5x+7}+3=1$
22. $\sqrt[3]{3x-6}+5=2$
23. $2\sqrt[3]{x+2}-1=0$
24. $2\sqrt[3]{2x-3}-1=0$
25. $\sqrt{8x+11}=\sqrt{3x+1}$
26. $2\sqrt{3x-4}=\sqrt{2(3x+1)}$
27. $\sqrt{2(x+10)}=\sqrt{7x-15}$
28. $\sqrt{5(x-4)}=\sqrt{x+4}$
29. $\sqrt[3]{5x-2}=\sqrt[3]{4x}$
30. $\sqrt[3]{9(x-1)}=\sqrt[3]{3(x+7)}$
31. $\sqrt[3]{3x+1}=\sqrt[3]{2(x-1)}$
32. $\sqrt[3]{9x}=\sqrt[3]{3(x-6)}$
33. $\sqrt{4x+21}=x$
34. $\sqrt{8x+9}=x$
35. $\sqrt{4(2x-3)}=x$
36. $\sqrt{3(4x-9)}=x$
37. $2\sqrt{x-1}=x$
38. $3\sqrt{2x-9}=x$
39. $\sqrt{9x+9}=x+1$

40. $\sqrt{3x+10}=x+4$
41. $\sqrt{x-1}=x-3$
42. $\sqrt{2x-5}=x-4$
43. $\sqrt{16-3x}=x-6$
44. $\sqrt{7-3x}=x-3$
45. $3\sqrt{2x+10}=x+9$
46. $2\sqrt{2x+5}=x+4$
47. $3\sqrt{x-1}-1=x$
48. $2\sqrt{2x+2}-1=x$
49. $\sqrt{10x+41}-5=x$
50. $\sqrt{6(x+3)}-3=x$
51. $\sqrt{8x^2-4x+1}=2x$
52. $\sqrt{18x^2-6x+1}=3x$
53. $5\sqrt{x+2}=x+8$
54. $4\sqrt{2(x+1)}=x+7$
55. $\sqrt{x^2-25}=x$
56. $\sqrt{x^2+9}=x$
57. $3+\sqrt{6x-11}=x$
58. $2+\sqrt{9x-8}=x$
59. $\sqrt{4x+25}-x=7$
60. $\sqrt{8x+73}-x=10$
61. $2\sqrt{4x+3}-3=2x$
62. $2\sqrt{6x+3}-3=3x$
63. $2x-4=\sqrt{14-10x}$
64. $3x-6=3\sqrt{3-24x}$
65. $\sqrt[3]{x^2-24}=1$
66. $\sqrt[3]{x^2-54}=3$
67. $\sqrt[3]{x^2+6x}+1=4$
68. $\sqrt[3]{x^2+2x}+5=7$
69. $\sqrt[3]{25x^2-10x-7}=-2$
70. $\sqrt[3]{9x^2-12x-23}=-3$
71. $\sqrt{2x^2-15x+25}=\sqrt{(x+5)(x-5)}$
72. $\sqrt{x^2-4x+4}=\sqrt{x(5-x)}$
73. $\sqrt[3]{2\left(x^2+3x-20\right)}=\sqrt[3]{(x+3)^2}$
74. $\sqrt[3]{3x^2+3x+40}=\sqrt[3]{(x-5)^2}$
75. $x^{1/2}-10=0$
76. $x^{1/2}-6=0$
77. $x^{1/3}+2=0$
78. $x^{1/3}+4=0$
79. $(x-1)^{1/2}-3=0$
80. $(x+2)^{1/2}-6=0$
81. $(2x-1)^{1/3}+3=0$
82. $(3x-1)^{1/3}-2=0$
83. $(4x+15)^{1/2}-2x=0$
84. $(3x+2)^{1/2}-3x=0$
85. $(2x+12)^{1/2}-x=6$
86. $(4x+36)^{1/2}-x=9$
87. $2(5x+26)^{1/2}=x+10$
88. $3(x-1)^{1/2}=x+1$
89. The square root of 1 less than twice a number is equal to 2 less than the number. Find the number.
90. The square root of 4 less than twice a number is equal to 6 less than the number. Find the number.
91. The square root of twice a number is equal to one-half of that number. Find the number.
92. The square root of twice a number is equal to one-third of that number. Find the number.
93. The distance, *d*, measured in miles, a person can see an object is given by the formula \(d=\sqrt{\frac{3h}{2}\) where *h* represents the person's height above sea level, measured in feet. How high must a person be to see an object 5 miles away?

94. The current, *I*, measured in amperes, is given by the formula $I = \sqrt{\frac{P}{R}}$ where *P* is the power usage, measured in watts, and *R* is the resistance, measured in ohms. If a light bulb requires 1/2 ampere of current and uses 60 watts of power, then what is the resistance of the bulb?

Answer available at the end of this section.

Exercise 8.6.4 Solving Radical Equations

The period, *T*, of a pendulum in seconds is given by the formula

$$T = 2\pi\sqrt{L/32}$$

where *L* represents the length in feet. For each problem below, calculate the length of a pendulum, given the period. Give the exact value and the approximate value rounded off to the nearest tenth of a foot.

1. 1 second
2. 2 seconds
3. $\frac{1}{2}$ second
4. $\frac{1}{3}$ second

Answer available at the end of this section.

Exercise 8.6.5 Solving Radical Equations

The time, *t*, in seconds an object is in free fall is given by the formula

$$s = 16 \cdot t^2$$

where s represents the distance in feet the object has fallen. For each problem below, calculate the distance an object falls, given the amount of time.

1. 1 second
2. 2 seconds
3. $\frac{1}{2}$ second
4. $\frac{1}{4}$ second

Answer available at the end of this section.

Exercise 8.6.6 Solving Radical Equations

The *x*-intercepts for any graph have the form $(x, 0)$, where *x* is a real number. Therefore, to find *x*-intercepts, set $y = 0$ and solve for *x*. Find the *x*-intercepts for each of the following.

1. $y = \sqrt{x-3} - 1$
2. $y = \sqrt{x+2} - 3$
3. $y = \sqrt[3]{x-1} + 2$
4. $y = \sqrt[3]{x+1} - 3$

Answer available at the end of this section.

Exercise 8.6.7 Discussion Board

1. Discuss reasons why we sometimes obtain extraneous solutions when solving radical equations. Are there ever any conditions where we do not need to check for extraneous solutions? Why?

Answer available at the end of this section.

Answers

Exercise 8.6.3 Solving Radical Equations

1. 4
3. 1
5. $\varnothing$
7. $\frac{1}{25}$
9. 12
11. 1
13. 3
15. 5
17. 8
19. 9
21. -3
23. $-\frac{15}{8}$
25. $\varnothing$
27. 7
29. 2
31. -3
33. 7
35. $2, 6$
37. 2
39. $-1, 8$
41. 5
43. $\varnothing$
45. $-3, 3$
47. $2, 5$
49. $4, -4$
51. $\frac{1}{2}$
53. $2, 7$
55. $\varnothing$
57. 10
59. $-6, -4$
61. $-\frac{1}{2}, \frac{3}{2}$
63. $\varnothing$
65. $-5, 5$
67. $-9, 3$
69. $\frac{1}{5}$
71. $5, 10$
73. $-7, 7$
75. 100
77. -8
79. 10
81. -13
83. $\frac{5}{2}$
85. $-6, -4$
87. $-2, 2$
89. 5
91. 8
93. $16\frac{2}{3}$ feet

Exercise 8.6.4 Solving Radical Equations

1. $\frac{8}{\pi^2} \approx 0.8$ foot

3. $\frac{2}{\pi^2} \approx 0.2$ foot

Exercise $8.6.5$ **Solving Radical Equations**

1. 16 feet
3. 4 feet

Exercise $8.6.6$ **Solving Radical Equations**

1. $(4, 0)$
3. $(-7, 0)$

Exercise $8.6.7$ **Discussion Board**

1. Answers may vary

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8.E: Review Exercises and Sample Exam

Review Exercises

(Assume all variables represent nonnegative numbers.)

Exercise $8.E.1$ Radicals

Simplify.

1. $\sqrt{36}$
2. $\sqrt{425}$
3. $\sqrt{-16}$
4. $-\sqrt{9}$
5. $\sqrt[3]{125}$
6. $3\sqrt[3]{-8}$
7. $\sqrt[3]{\frac{1}{64}}$
8. $-5\sqrt[3]{-27}$
9. $\sqrt{40}$
10. $-3\sqrt{50}$
11. $\sqrt{\frac{98}{81}}$
12. $\sqrt{\frac{11}{21}}$
13. $5\sqrt[3]{192}$
14. $2\sqrt[3]{-54}$

Answer available at the end of this section.

Exercise $8.E.2$ Simplifying Radical Expressions

Simplify.

1. $\sqrt{49x^2}$
2. $\sqrt{25a^2b^2}$
3. $\sqrt{75x^3y^2}$
4. $\sqrt{200m^4n^3}$
5. $\sqrt{\frac{18x^3}{25y^2}}$
6. $\sqrt{\frac{108x^3}{49y^4}}$
7. $\sqrt[3]{216x^3}$
8. $\sqrt[3]{-125x^6y^3}$
9. $\sqrt[3]{27a^7b^5c^3}$
10. $\sqrt[3]{120x^9y^4}$

Answer available at the end of this section.

Exercise $8.E.3$ Simplifying Radical Expressions

Use the distance formula to calculate the distance between the given two points.

1. $(5,-8)$ and $(2,-10)$

2. $(-7,-1)$ and $(-6,1)$
3. $(-10,-1)$ and $(0,-5)$
4. $(5,-1)$ and $(-2,-2)$

Answer available at the end of this section.

Exercise $8.E.4$ Adding and Subtracting Radical Expressions

Simplify.

1. $8\sqrt{3}+3\sqrt{3}$
2. $12\sqrt{10}-2\sqrt{10}$
3. $14\sqrt{3}+5\sqrt{2}-5\sqrt{3}-6\sqrt{2}$
4. $22\sqrt{ab}-5\sqrt{ab}+7\sqrt{ab}-2\sqrt{ab}$
5. $7\sqrt{x}-(3\sqrt{x}+2\sqrt{y})$
6. $(8y\sqrt{x}-7x\sqrt{y})-(5x\sqrt{y}-12y\sqrt{x})$
7. $\sqrt{45}+\sqrt{12}-\sqrt{20}-\sqrt{75}$
8. $\sqrt{24}-\sqrt{32}+\sqrt{54}-2\sqrt{32}$
9. $2\sqrt{3x^2}+\sqrt{45x}-x\sqrt{27}+\sqrt{20x}$
10. $\sqrt{56a^2b}+\sqrt{8a^2b^2}-\sqrt{224a^2b}-a\sqrt{18b^2}$
11. $5y\sqrt{4x^2y}-(x\sqrt{16y^3}-2\sqrt{9x^2y^3})$
12. $(2b\sqrt{9a^2c}-3a\sqrt{16b^2c})-(\sqrt{64a^2b^2c}-9b\sqrt{a^2c})$
13. $\sqrt[3]{216x}-\sqrt[3]{125xy}-\sqrt[3]{8x}$
14. $\sqrt[3]{128x^3}-2x\sqrt[3]{54}+3\sqrt[3]{2x^3}$
15. $\sqrt[3]{8x^3y}-2x\sqrt[3]{8y}+\sqrt[3]{27x^3y}+x\sqrt[3]{y}$
16. $\sqrt[3]{27a^3b}-3\sqrt[3]{8ab^3}+a\sqrt[3]{64b}-b\sqrt[3]{a}$

Answer available at the end of this section.

Exercise $8.E.5$ Multiplying and Dividing Radical Expressions

Multiply.

1. $\sqrt{3}\cdot\sqrt{6}$
2. $(3\sqrt{5})^2$
3. $\sqrt{2}(\sqrt{3}-\sqrt{6})$
4. $(\sqrt{2}-\sqrt{6})^2$
5. $(1-\sqrt{5})(1+\sqrt{5})$
6. $(2\sqrt{3}+\sqrt{5})(3\sqrt{2}-2\sqrt{5})$
7. $\sqrt[3]{2a^2}\cdot\sqrt[3]{4a}$
8. $\sqrt[3]{25a^2b}\cdot\sqrt[3]{5a^2b^2}$

Answer available at the end of this section.

Exercise $8.E.6$ Multiplying and Dividing Radical Expressions

Divide.

1. $\frac{\sqrt{72}}{\sqrt{4}}$
2. $10\frac{\sqrt{48}}{\sqrt{64}}$
3. $\frac{\sqrt{98x^4y^2}}{\sqrt{36x^2}}$
4. $\frac{\sqrt[3]{81x^6y^7}}{\sqrt[3]{8y^3}}$

Answer available at the end of this section.

Exercise $8.E.7$ Multiplying and Dividing Radical Expressions

Rationalize the denominator.

1. $\frac{2}{\sqrt{7}}$
2. $\frac{\sqrt{6}}{\sqrt{3}}$
3. $\sqrt{\frac{14}{2x}}$
4. $\sqrt{\frac{12}{15}}$
5. $\sqrt[3]{\frac{1}{2x^2}}$
6. $\sqrt[3]{\frac{5a^2b^5}{ab^2}}$
7. $\frac{1}{\sqrt{3}-\sqrt{2}}$
8. $\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}+\sqrt{6}}$

Answer available at the end of this section.

Exercise $8.E.8$ Rational Exponents

Express in radical form.

1. $7^{1/2}$
2. $3^{2/3}$
3. $x^{4/5}$
4. $y^{-3/4}$

Answer available at the end of this section.

Exercise $8.E.9$Rational Exponents

Write as a radical and then simplify.

1. $4^{1/2}$
2. $50^{1/2}$
3. $4^{2/3}$
4. $81^{1/3}$
5. $(\frac{1}{4})^{3/2}$
6. $(\frac{12}{16})^{-1/3}$

Answer available at the end of this section.

Exercise $8.E.10$ Rational Exponent

Perform the operations and simplify. Leave answers in exponential form.

1. $3^{1/2} \cdot 3^{3/2}$
2. $2^{1/2} \cdot 2^{1/3}$
3. $\frac{4^{\frac{3}{2}}}{4^{\frac{1}{2}}}$
4. $\frac{9^{\frac{3}{4}}}{9^{\frac{1}{4}}}$

5. $(36x^4y^2)^{\frac{1}{2}}$
6. $(8x^6y^9)^{1/3}$
7. $\left(\frac{a^{\frac{4}{3}}}{a^{\frac{1}{2}}}\right)^{\frac{2}{5}}$
8. $\left(\frac{16x^{\frac{4}{3}}}{y^2}\right)^{\frac{1}{2}}$

Answer available at the end of this section.

Exercise $8.E.11$ Solving Radical Equations

Solve.

1. $\sqrt{x}=5$
2. $\sqrt{2x-1}=3$
3. $\sqrt{x-8}+2=5$
4. $\sqrt{3x-5}-1=11$
5. $\sqrt{5x-3}=\sqrt{2x+15}$
6. $\sqrt{8x-15}=x$
7. $\sqrt{x+41}=x-1$
8. $\sqrt{7-3x}=x-3$
9. $2(x+1)=\sqrt{2(x+1)}$
10. $\sqrt{x(x+6)}=4$
11. $\sqrt[3]{x(3x+10)}=2$
12. $\sqrt[3]{2x^2-x}+4=5$
13. $\sqrt[3]{3(x+4)(x+1)}=\sqrt[3]{5x+37}$
14. $\sqrt[3]{3x^2-9x+24}=\sqrt[3]{(x+2)^2}$
15. $y^{1/2}-3=0$
16. $y^{1/3}+3=0$
17. $(x-5)^{1/2}-2=0$
18. $(2x-1)^{1/3}-5=0$

Answer available at the end of this section.

Sample Exam

In Exercises 12-16, assume all variables represent nonnegative numbers.

Exercise $8.E.12$

Simplify.

1. a. $\sqrt{100}$
   b. $\sqrt{-100}$
   c. $-\sqrt{100}$

Answer available at the end of this section.

Exercise $8.E.13$

Simplify.

1. g

a. $\sqrt[3]{27}$
b. $\sqrt[3]{-27}$
c. $-\sqrt[3]{27}$

2. $\sqrt{\frac{128}{25}}$
3. $\sqrt[3]{\frac{192}{125}}$
4. $5\sqrt{12x^2y^3z}$
5. $2\sqrt[3]{50x^2y^3z^5}$

Answer available at the end of this section.

Exercise $8.E.14$

Perform the operations.

1. $5\sqrt{24}-\sqrt{108}+\sqrt{96}-3\sqrt{27}$
2. $3\sqrt{8x^2y}-\left(x\sqrt{200y}-\sqrt{18x^2y}\right)$
3. $2\sqrt{ab}\cdot(3\sqrt{2a}-\sqrt{b})$
4. $(\sqrt{x}-\sqrt{2y})^2$

Answer available at the end of this section.

Exercise $8.E.15$

Rationalize the denominator.

1. $\frac{10}{\sqrt{2x}}$
2. $\sqrt[3]{\frac{1}{4xy^2}}$
3. $\frac{1}{\sqrt{x}+5}$
4. $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$

Answer available at the end of this section.

Exercise $8.E.16$

Perform the operations and simplify. Leave answers in exponential form.

1. $2^{\frac{2}{3}}\cdot 2^{\frac{1}{6}}$
2. $\frac{10^{\frac{4}{5}}}{10^{\frac{1}{3}}}$
3. $\left(121a^4b^2\right)^{\frac{1}{2}}$
4. $\frac{\left(9y^{\frac{1}{3}}x^6\right)^{\frac{1}{2}}}{y^{\frac{1}{6}}}$

Answer available at the end of this section.

Exercise $8.E.17$

Solve.

1. $\sqrt{x}-7=0$
2. $\sqrt{3x+5}=1$

3. $\sqrt{2x-1}+2=x$
4. $3\sqrt{1-10x}=x-4$
5. $\sqrt{(2x+1)(3x+2)}=\sqrt{3(2x+1)}$
6. $\sqrt[3]{x(2x-15)}=3$
7. The period, *T*, of a pendulum in seconds is given the formula $T=2\pi\sqrt{L/32}$, where *L* represents the length in feet. Calculate the length of a pendulum if the period is $1^{1/2}$ seconds. Round off to the nearest tenth.

Answer available at the end of this section.

Answers

Exercise $8.E.1$ **Radicals**

1. 6
3. Not a real number
5. 5
7. $\frac{1}{4}$
9. $2\sqrt{10}$
11. $\frac{7\sqrt{2}}{9}$
13. $20\sqrt[3]{3}$

Exercise $8.E.2$ **Simplifying Radical Expressions**

1. $7x$
3. $5xy\sqrt{3x}$
5. $\frac{3x\sqrt{2x}}{5y}$
7. $6x$
9. $3a^2bc\sqrt[3]{ab^2}$

Exercise $8.E.3$ **Simplifying Radical Expressions**

1. $\sqrt{13}$
3. $2\sqrt{29}$

Exercise $8.E.4$ **Adding and Subtracting Radical Expressions**

1. $11\sqrt{3}$
3. $9\sqrt{3}-\sqrt{2}$
5. $4\sqrt{x}-2\sqrt{y}$
7. $\sqrt{5}-3\sqrt{3}$
9. $-\sqrt{3}x+5\sqrt{5}\sqrt{x}$
11. $12xy\sqrt{y}$
13. $4\sqrt[3]{x}-5\sqrt[3]{xy}$
15. $2x\sqrt[3]{y}$

Exercise $8.E.5$ **Multiplying and Dividing Radical Expressions**

1. $3\sqrt{2}$
3. $\sqrt{6}-2\sqrt{3}$
5. -4
7. $2a$

Exercise $8.E.6$ **Multiplying and Dividing Radical Expressions**

1. $3\sqrt{2}$
3. $\frac{7xy\sqrt{2}}{6}$

Exercise $8.E.7$ **Multiplying and Dividing Radical Expressions**

1. $\frac{2\sqrt{7}}{7}$
3. $\frac{\sqrt{7}\sqrt{x}}{x}$
5. $\frac{2^{\frac{2}{3}}\sqrt[3]{x}}{2x}$
7. $\sqrt{3}+\sqrt{2}$

Exercise $8.E.8$ **Rational Exponents**

1. $\sqrt{7}$
3. $\sqrt[5]{x^4}$

Exercise $8.E.9$**Rational Exponents**

1. 2
3. $2\sqrt[3]{2}$
5. $\frac{1}{8}$

Exercise $8.E.10$ **Rational Exponent**

1. 9
3. 4
5. $6x^2y$
7. $a^{1/3}$

Exercise $8.E.11$ **Solving Radical Equations**

1. 25
3. 17
5. 6
7. 8
9. $-\frac{1}{2}, -1$
11. $\frac{2}{3}, -4$
13. $-5, \frac{5}{3}$
15. 9
17. 9

Exercise $8.E.12$

1. a. 10 b. Not a real number c. -10

Exercise $8.E.13$

2. $\frac{8\sqrt{2}}{5}$
4. $10xy\sqrt{3yz}$

Exercise $8.E.14$

1. $14\sqrt{6} - 15\sqrt{3}$
3. $6a\sqrt{2b} - 2b\sqrt{a}$

Exercise $8.E.15$

1. $\frac{5\sqrt{2x}}{x}$
3. $\frac{\sqrt{x}-5}{x-25}$

Exercise $8.E.16$

1. $2^{5/6}$
3. $11a^2b$

Exercise $8.E.17$

1. 49
3. 5
5. $-\frac{1}{2}, \frac{1}{3}$
7. 1.8 feet

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9: SOLVING QUADRATIC EQUATIONS AND GRAPHING PARABOLAS

9.1: Extracting Square Roots

9.2: Completing the Square

9.3: Quadratic Formula

9.4: Guidelines for Solving Quadratic Equations and Applications

9.5: Graphing Parabolas

9.6: Introduction to Complex Numbers and Complex Solutions

9.E: Review Exercises and Sample Exam

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9.1: Extracting Square Roots

Learning Objectives

- Solve quadratic equations by extracting square roots.

Extracting Square Roots

Recall that a quadratic equation is in **standard form** if it is equal to 0:

$$ax^2 + bx + c = 0 \tag{9.1.1}$$

where ***a***, ***b***, and ***c*** are real numbers and $a \neq 0$. A solution to such an equation is called a **root**. Quadratic equations can have two real solutions, one real solution, or no real solution. If the quadratic expression on the left factors, then we can solve it by factoring. A review of the steps used to solve by factoring follow:

Step 1: Express the quadratic equation in standard form.

Step 2: Factor the quadratic expression.

Step 3: Apply the zero-product property and set each variable factor equal to 0.

Step 4: Solve the resulting linear equations.

For example, we can solve $x^2 - 4 = 0$ by factoring as follows:

The two solutions are –2 and 2. The goal in this section is to develop an alternative method that can be used to easily solve equations where ***b*** = 0, giving the form

$$ax^2 + c = 0 \tag{9.1.2}$$

The equation $x^2 - 4 = 0$ is in this form and can be solved by first isolating x^2.

If we take the square root of both sides of this equation, we obtain the following:

$$\sqrt{x^2} = \sqrt{4}$$
$$|x| = 2$$

Here we see that $x = -2$ and $x = 2$ are solutions to the resulting equation. In general, this describes the **square root property**; for any real number ***k***,

$$\text{If } x^2 = k, \text{ then } x = \pm\sqrt{k} \tag{9.1.3}$$

The notation "±" is read "plus or minus" and is used as compact notation that indicates two solutions. Hence the statement $x = \pm\sqrt{k}$ indicates that $x = -\sqrt{k}$or $x = +\sqrt{k}$. Applying the square root property as a means of solving a quadratic equation is called **extracting the roots**.

Example 9.1.1

Solve:

$$x^2 - 25 = 0$$

Solution:

Begin by isolating the square.

Next, apply the square root property.

$$x^2 = 25$$
$$x = \pm\sqrt{25}$$
$$x = \pm 5$$

Answer:

The solutions are –5 and 5. The check is left to the reader.

Certainly, the previous example could have been solved just as easily by factoring. However, it demonstrates a technique that can be used to solve equations in this form that do not factor.

Example 9.1.2

Solve:

$$x^2 - 5 = 0$$

Solution:

Notice that the quadratic expression on the left does not factor. We can extract the roots if we first isolate the leading term, x^2.

Apply the square root property.

$$x = \pm\sqrt{5}$$

For completeness, check that these two real solutions solve the original quadratic equation. Generally, the check is optional.

Check $x = -\sqrt{5}$	Check $x = \sqrt{5}$
$x^2 - 5 = 0$	$x^2 - 5 = 0$
$(-\sqrt{5})^2 - 5 = 0$	$(\sqrt{5})^2 - 5 = 0$
$5 - 5 = 0$	$5 - 5 = 0$
$0 = 0$✓	$0 = 0$✓

Answer:
The solutions are $-\sqrt{5}$ and $\sqrt{5}$.

Example 9.1.3

Solve

Solution

Begin by isolating x^2.

Apply the square root property and then simplify.

$$x = \pm\sqrt{\frac{45}{4}}$$
$$x = \pm\frac{\sqrt{9\cdot 5}}{\sqrt{4}}$$
$$x = \pm\frac{3\sqrt{5}}{2}$$

Answer:

The solutions are $-\frac{3\sqrt{5}}{2}$ and $\frac{3\sqrt{5}}{2}$.

Sometimes quadratic equations have no real solution.

Example 9.1.4

Solve:

$$x^2 + 9 = 0$$

Solution

Begin by isolating x^2.

After applying the square root property, we are left with the square root of a negative number. Therefore, there is no real solution to this equation.

Answer:

No real solution

Reverse this process to find equations with given solutions of the form **±*k***.

Find an equation with solutions $-2\sqrt{3}$ and $2\sqrt{3}$.

Solution:

Begin by squaring both sides of the following equation:

$$x = \pm 2\sqrt{3}$$
$$x^2 = (\pm 2\sqrt{3})^2$$
$$x^2 = 4 \cdot 3$$
$$x^2 = 12$$

Lastly, subtract 12 from both sides and present the equation in standard form.

Answer:

$x^2 - 12 = 0$

Exercise 9.1.1

Solve:

$$9x^2 - 8 = 0.$$

Answer available at the end of this section.

Consider solving the following equation:

$$(x+2)^2 = 25$$

To solve this equation by factoring, first square x+2 and then put it in standard form, equal to zero, by subtracting 25 from both sides.

Factor and then apply the zero-product property.

$$x^2 + 4x - 21 = 0$$
$$(x+7)(x-3) = 0$$

$$\begin{array}{rcl} x+7=0 & \text{or} & x-3=0 \\ x=-7 & & x=3 \end{array}$$

The two solutions are -7 and 3.

When an equation is in this form, we can obtain the solutions in fewer steps by extracting the roots.

Example 9.1.6

Solve:

$$(x=2)^2 = 25.$$

Solution

Solve by extracting the roots.

$$\begin{array}{rll} (x+2)^2 & = 25 & \textit{Apply the square root property.} \\ x+2 & = \pm\sqrt{25} & \textit{Simplify.} \\ x+2 & = \pm 5 & \\ x & = -2 \pm 5 & \end{array}$$

At this point, separate the “plus or minus” into two equations and simplify each individually.

$$\begin{array}{rcl} x=-2-5 & \text{or } x=-2+5 \\ x=-7 & x=3 \end{array}$$

Answer:

The solutions are -7 and 3.

In addition to fewer steps, this method allows us to solve equations that do not factor.

Example 9.1.7

Solve:

$$(3x+3)^2 - 27 = 0$$

Solution

Begin by isolating the square.

Next, extract the roots and simplify.

$$(3x+3)^2 = 27$$
$$3x+3 = \pm\sqrt{27}$$
$$3x+3 = \pm\sqrt{9\cdot 3}$$
$$3x+3 = \pm 3\sqrt{3}$$

Solve for x.

Answer:

The solutions are $-1-\sqrt{3}$ and $-1+\sqrt{3}$

Solve:

$$9(2x-1)^2 - 8 = 0$$

Solution:

Begin by isolating the square factor.

Apply the square root property and solve.

Answer:

The solutions are $\frac{3-2\sqrt{2}}{6}$ and $\frac{3+2\sqrt{2}}{6}$

Exercise 9.1.2

Solve:

$$3(x-5)^2 - 2 = 0$$

Answer available at the end of this section.

Example 9.1.9

The length of a rectangle is twice its width. If the diagonal measures 2 feet, then find the dimensions of the rectangle.

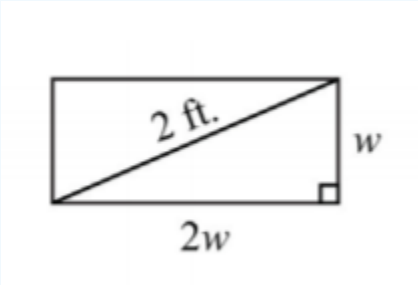


Figure 9.1.1

Solution

Let *w* represent the width.

Let *2w* represent the length.

The diagonal of any rectangle forms two right triangles. Thus the Pythagorean theorem applies. The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse:

$$(\text{leg})^2 + (\text{leg})^2 = \text{ hypotenuse }^2$$
$$(2w)^2 + (w)^2 = (2)^2$$

Solve.

$$\begin{aligned}(2w)^2 + (w)^2 &= (2)^2 \\ 4w^2 + w^2 &= 4 \\ 5w^2 &= 4 \qquad \textit{Isolate } w^2. \\ \frac{5w^2}{5} &= \frac{4}{5} \\ w^2 &= \frac{4}{5} \qquad \textit{Extract the roots.} \\ w &= \pm\sqrt{\frac{4}{5}} \\ w &= \pm\frac{2}{\sqrt{5}}\end{aligned}$$

Here we obtain two solutions, $w = -\frac{2}{\sqrt{5}}$ and $w = \frac{2}{\sqrt{5}}$. Since the problem asked for a length of a rectangle, we disregard the negative answer. Furthermore, we will rationalize the denominator and present our solutions without any radicals in the denominator.

$$\begin{aligned}w &= \frac{2}{\sqrt{5}} \qquad \textit{Rationalize the denominator.} \\ &= \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} \\ &= \frac{2\sqrt{5}}{\sqrt{25}} \\ &= \frac{2\sqrt{5}}{5}\end{aligned}$$

Back substitute to find the length.

$$\begin{aligned}l &= 2w \\ &= 2\left(\frac{2\sqrt{5}}{5}\right) \\ &= \frac{4\sqrt{5}}{5}\end{aligned}$$

Answer:

The length of the rectangle is $\frac{4\sqrt{5}}{5}$ feet and the width is $\frac{2\sqrt{5}}{5}$ feet.

Key Takeaways

- Solve equations of the form $ax^2 + c = 0$ by extracting the roots.
- Extracting roots involves isolating the square and then applying the square root property. After applying the square root property, you have two linear equations that each can be solved. Be sure to simplify all radical expressions and rationalize the denominator if necessary.

Exercise 9.1.3 extracting square roots

Solve by factoring and then solve by extracting roots. Check answers.

1. $x^2 - 36 = 0$
2. $x^2 - 81 = 0$
3. $4y^2 - 9 = 0$
4. $9y^2 - 25 = 0$
5. $(x-2)^2 - 1 = 0$
6. $(x+1)^2 - 4 = 0$
7. $4(y-2)^2 - 9 = 0$
8. $9(y+1)^2 - 4 = 0$
9. $-3(t-1)^2 + 12 = 0$
10. $-2(t+1)^2 + 8 = 0$
11. $(x-5)^2 - 25 = 0$
12. $(x+2)^2 - 4 = 0$

Answer available at the end of this section.

Exercise 9.1.4 extracting square roots

Solve by extracting the roots.

1. $x^2 = 16$
2. $x^2 = 1$
3. $y^2 = 9$
4. $y^2 = 64$
5. $x^2 = 14$
6. $x^2 = 19$
7. $y^2 = 0.25$
8. $y^2 = 0.04$
9. $x^2 = 12$
10. $x^2 = 18$
11. $16x^2 = 9$
12. $4x^2 = 25$
13. $2t^2 = 1$
14. $3t^2 = 2$
15. $x^2 - 100 = 0$
16. $x^2 - 121 = 0$
17. $y^2 + 4 = 0$
18. $y^2 + 1 = 0$
19. $x^2 - 49 = 0$
20. $x^2 - 925 = 0$
21. $y^2 - 0.09 = 0$
22. $y^2 - 0.81 = 0$
23. $x^2 - 7 = 0$
24. $x^2 - 2 = 0$
25. $x^2 - 8 = 0$
26. $t^2 - 18 = 0$
27. $x^2 + 8 = 0$
28. $x^2 + 125 = 0$
29. $16x^2 - 27 = 0$
30. $9x^2 - 8 = 0$
31. $2y^2 - 3 = 0$
32. $5y^2 - 2 = 0$
33. $3x^2 - 1 = 0$

34. $6x^2 - 3 = 0$
35. $(x + 7)^2 - 4 = 0$
36. $(x + 9)^2 - 36 = 0$
37. $(2y - 3)^2 - 81 = 0$
38. $(2y + 1)^2 - 25 = 0$
39. $(x - 5)^2 - 20 = 0$
40. $(x + 1)^2 - 28 = 0$
41. $(3t + 2)^2 - 6 = 0$
42. $(3t - 5)^2 - 10 = 0$
43. $4(y + 2)^2 - 3 = 0$
44. $9(y - 7)^2 - 5 = 0$
45. $4(3x + 1)^2 - 27 = 0$
46. $9(2x - 3)^2 - 8 = 0$
47. $2(3x - 1)^2 + 3 = 0$
48. $5(2x - 1)^2 - 3 = 0$
49. $3(y - 23)^2 - 32 = 0$
50. $2(3y - 13)^2 - 52 = 0$

Answer available at the end of this section.

Exercise 9.1.5 extracting square roots

Find a quadratic equation in standard form with the following solutions.

1. ±7
2. ±13
3. $\pm\sqrt{7}$
4. $\pm\sqrt{3}$
5. $\pm 3\sqrt{5}$
6. $\pm 5\sqrt{2}$
7. $1 \pm \sqrt{2}$
8. $2 \pm \sqrt{3}$

Answer available at the end of this section.

Exercise 9.1.6 extracting square roots

Solve and round off the solutions to the nearest hundredth.

1. $9x(x + 2) = 18x + 1$
2. $x^2 = 10(x^2 - 2) - 5$
3. $(x + 3)(x - 7) = 11 - 4x$
4. $(x - 4)(x - 3) = 66 - 7x$
5. $(x - 2)^2 = 67 - 4x$
6. $(x + 3)^2 = 6x + 59$
7. $(2x + 1)(x + 3) - (x + 7) = (x + 3)^2$
8. $(3x - 1)(x + 4) = 2x(x + 6) - (x - 3)$

Answer available at the end of this section.

Exercise 9.1.7 extracting square roots

Set up an algebraic equation and use it to solve the following.

1. If 9 is subtracted from 4 times the square of a number, then the result is 3. Find the number.
2. If 20 is subtracted from the square of a number, then the result is 4. Find the number.
3. If 1 is added to 3 times the square of a number, then the result is 2. Find the number.
4. If 3 is added to 2 times the square of a number, then the result is 12. Find the number.
5. If a square has an area of 8 square centimeters, then find the length of each side.
6. If a circle has an area of 32π square centimeters, then find the length of the radius.
7. The volume of a right circular cone is 36π cubic centimeters when the height is 6 centimeters. Find the radius of the cone. (The volume of a right circular cone is given by $V = 13\pi r^2 h$.)
8. The surface area of a sphere is 75π square centimeters. Find the radius of the sphere. (The surface area of a sphere is given by SA=$4\pi r^2$.)
9. The length of a rectangle is 6 times its width. If the area is 96 square inches, then find the dimensions of the rectangle.
10. The base of a triangle is twice its height. If the area is 16 square centimeters, then find the length of its base.
11. A square has an area of 36 square units. By what equal amount will the sides have to be increased to create a square with double the given area?
12. A circle has an area of 25π square units. By what amount will the radius have to be increased to create a circle with double the given area?
13. If the sides of a square measure 1 unit, then find the length of the diagonal.
14. If the sides of a square measure 2 units, then find the length of the diagonal.
15. The diagonal of a square measures 5 inches. Find the length of each side.
16. The diagonal of a square measures 3 inches. Find the length of each side.
17. The length of a rectangle is twice its width. If the diagonal measures 10 feet, then find the dimensions of the rectangle.
18. The length of a rectangle is twice its width. If the diagonal measures 8 feet, then find the dimensions of the rectangle.
19. The length of a rectangle is 3 times its width. If the diagonal measures 5 meters, then find the dimensions of the rectangle.
20. The length of a rectangle is 3 times its width. If the diagonal measures 2 feet, then find the dimensions of the rectangle.
21. The height in feet of an object dropped from a 9-foot ladder is given by h(t) $= -16t^2 + 9$, where t represents the time in seconds after the object has been dropped. How long does it take the object to hit the ground? (Hint: The height is 0 when the object hits the ground.)
22. The height in feet of an object dropped from a 20-foot platform is given by h(t) $= -16t^2 + 20$, where t represents the time in seconds after the object has been dropped. How long does it take the object to hit the ground?
23. The height in feet of an object dropped from the top of a 144-foot building is given by h(t) $= -16t^2 + 144$, where t is measured in seconds.
    a. How long will it take to reach half of the distance to the ground, 72 feet?
    b. How long will it take to travel the rest of the distance to the ground? Round off to the nearest hundredth of a second.
24. The height in feet of an object dropped from an airplane at 1,600 feet is given by h(t) $= -16t^2 + 1,600$, where t is in seconds.
    a. How long will it take to reach half of the distance to the ground?
    b. How long will it take to travel the rest of the distance to the ground? Round off to the nearest hundredth of a second.
25. Create an equation of your own that can be solved by extracting the root. Share it, along with the solution, on the discussion board.
26. Explain why the technique of extracting roots greatly expands our ability to solve quadratic equations.
27. Explain in your own words how to solve by extracting the roots. 106. Derive a formula for the diagonal of a square in terms of its sides.

Answer available at the end of this section.

Answers

Exercise $9.1.1$

$x = -\frac{2\sqrt{2}}{3}$ or $x = \frac{2\sqrt{2}}{3}$

Exercise $9.1.2$

$15 \pm 6\sqrt{3}$

Exercise $9.1.3$ **extracting square roots**

1. $-6, 6$
3. $-\frac{3}{2}, \frac{3}{2}$
5. $1, 3$
7. $\frac{1}{2}, \frac{7}{2}$

9. $-1, 3$
11. $0, 10$

Exercise $9.1.4$ **extracting square roots**

1. ± 4
3. ± 3
5. $\pm \frac{1}{2}$
7. ± 0.5
9. $\pm 2\sqrt{3}$
11. $\pm \frac{3}{4}$
13. $\pm \sqrt{\frac{1}{2}}$
15. ± 10
17. No real solution
19. $\pm \frac{2}{3}$
21. ± 0.3
23. $\pm \sqrt{7}$
25. $\pm 2\sqrt{2}$
27. No real solution
29. $\pm \frac{3\sqrt{3}}{4}$
31. $\pm 6\sqrt{2}$
33. $\pm 3\sqrt{3}$
35. $-9, -5$
37. $-3, 6$
39. $5 \pm 2\sqrt{5}$
41. $\pm \frac{\sqrt{6}-2}{3}$
43. $\pm \frac{\sqrt{3}}{2} - 2$
45. $\pm \frac{3\sqrt{3}-2}{6}$
47. No real solution
49. $\pm \frac{4\sqrt{6}}{3} + 23$

Exercise $9.1.5$ **extracting square roots**

1. $x^2 - 49 = 0$
3. $x^2 - 7 = 0$
5. $x^2 - 45 = 0$
7. $x^2 - 2x - 1 = 0$

Exercise $9.1.6$ **extracting square roots**

1. ± 0.33
3. ± 5.66
5. ± 7.94
7. ± 3.61

Exercise $9.1.7$ **extracting square roots**

1. $-\sqrt{3}$ or $\sqrt{3}$
3. $-3\sqrt{3}$ or $3\sqrt{3}$
5. $2\sqrt{2}$ centimeters
7. $3\sqrt{2}$ centimeters
9. Length: 24 inches; width: 4 inches
11. $-6 + 6\sqrt{2} \approx 2.49$ units
13. $\sqrt{2}$ units
15. $52\sqrt{2}$ inches
17. Length: $4\sqrt{5}$ feet; width: $2\sqrt{5}$ feet
19. Length: $310 - -\sqrt{2}$ meters; width: $10 - -\sqrt{2}$ meters
21. $\frac{3}{4}$ second
23. a. 2.12 seconds b. 0.88 second
25. Answers may vary

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9.2: Completing the Square

Learning Objectives

- Solve quadratic equations by completing the square.

Completing the Square

In this section, we will devise a method for rewriting any quadratic equation of the form

$$ax^2 + bx + c = 0 \tag{9.2.1}$$

in the form

$$(x - p)^2 = q \tag{9.2.2}$$

This process is called completing the square. As we have seen, quadratic equations in this form can easily be solved by extracting roots. We begin by examining perfect square trinomials:

$$\begin{array}{rccccc} (x+3)^2 = & x^2 & + & 6x & + & 9 \\ & & & \downarrow & & \uparrow \\ & & & \left(\frac{6}{2}\right)^2 = (3)^2 = & & 9 \end{array}$$

The last term, 9, is the square of one-half of the coefficient of ***x***. In general, this is true for any perfect square trinomial of the form $x^2 + bx + c$.

$$\begin{aligned} \left(x + \frac{b}{2}\right)^2 &= x^2 + 2 \cdot \frac{b}{2}x + \left(\frac{b}{2}\right)^2 \\ &= x^2 + bx + \left(\frac{b}{2}\right)^2 \end{aligned}$$

In other words, any trinomial of the form $x^2 + bx + c$ will be a perfect square trinomial if

$$c = \left(\frac{b}{2}\right)^2 \tag{9.2.3}$$

Note

It is important to point out that the leading coefficient must be equal to 1 for this to be true.

Example 9.2.1

Complete the square $x^2 + 8x + \ ? \ = (x + \ ?)^2$

Solution:

In this example, the coefficient of the middle term ***b*** = 8, so find the value that completes the square as follows:

$$\left(\tfrac{b}{2}\right)^2 = \left(\tfrac{8}{2}\right)^2 = (4)^2 = 16$$

The value that completes the square is 16.

$$x^2+8x+16=(x+4)(x+4)$$
$$=(x+4)^2$$

Answer:

$x^2+8x+16=(x+4)^2$

Example 9.2.2

Complete the square $x^2+3x+\ ?\ =(x+\ ?)^2$

Solution

Here ***b*** = 3, so find the value that will complete the square as follows:

$$\left(\tfrac{b}{2}\right)^2=\left(\tfrac{3}{2}\right)^2=\tfrac{9}{4}$$

The value 9/4 completes the square:

$$x^2+3x+\frac{9}{4}=\left(x+\frac{3}{2}\right)\left(x+\frac{3}{2}\right)$$
$$=\left(x+\frac{3}{2}\right)^2$$

Answer:

$x^2+3x+94=(x+\frac{3}{2})^2$

We can use this technique to solve quadratic equations. The idea is to take any quadratic equation in standard form and complete the square so that we can solve it by extracting roots. The following are general steps for solving a quadratic equation with a leading coefficient of 1 in standard form by completing the square.

Example 9.2.3

Solve by completing the square: $x^2+14x+46=0$.

Solution:

Step 1: Add or subtract the constant term to obtain the equation in the form $x^2+bx\ \ =c$. In this example, subtract 46 to move it to the right side of the equation.

Step 2: Use $\left(\tfrac{b}{2}\right)^2$ to determine the value that completes the square. Here ***b*** = 14:

$\left(\tfrac{b}{2}\right)^2=\left(\tfrac{14}{2}\right)^2=(7)^2=49$

Step 3: Add $\left(\tfrac{b}{2}\right)^2$ to both sides of the equation and complete the square.

Step 4: Solve by extracting roots.

Answer:

The solutions are $-7-\sqrt{3}$ or $-7+\sqrt{3}$. The check is optional.

Example 9.2.4

Solve by completing the square: $x^2 - 18x + 72 = 0$

Solution:

Begin by subtracting 72 from both sides.

Next, find the value that completes the square using ***b*** = –18.

$\left(\frac{b}{2}\right)^2 = \left(\frac{-18}{2}\right)^2 = (-9)^2 = 81$

To complete the square, add 81 to both sides, complete the square, and then solve by extracting the roots.

At this point, separate the “plus or minus” into two equations and solve each.

$$\begin{array}{lll} x = 9 - 3 & \text{or} & x = 9 + 3 \\ x = 6 & & x = 12 \end{array}$$

Answer:

The solutions are 6 and 12.

Note that in the previous example the solutions are integers. If this is the case, then the original equation will factor.

If it factors, we can solve it by factoring. However, not all quadratic equations will factor.

Solve by completing the square: $x^2 + 10x + 1 = 0$.

Solution:

Begin by subtracting 1 from both sides of the equation.

Here ***b*** = 10, and we determine the value that completes the square as follows:

$\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2 = (5)^2 = 25$

To complete the square, add 25 to both sides of the equation.

$$\begin{aligned} x^2 + 10x &= -1 \\ x^2 + 10x + 25 &= -1 + 25 \\ x^2 + 10x + 25 &= 24 \end{aligned}$$

Factor and then solve by extracting roots.

Answer:

The solutions are $-5 - 2\sqrt{6}$ and $-5 + 2\sqrt{6}$

Sometimes quadratic equations do not have real solutions.

Solve by completing the square: $x^2 - 2x + 3 = 0$.

Solution:

Begin by subtracting 3 from both sides of the equation.

Here ***b*** = –2, and we have

$\left(\frac{b}{2}\right)^2 = \left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$

Therefore,

At this point we see that extracting the root leads to the square root of a negative number.

Answer:

No real solution

Exercise 9.2.1

Solve by completing the square: $x^2 - 2x - 27 = 0$.

Answer available at the end of this section.

The coefficient of ***x*** is not always divisible by 2.

Example 9.2.7

Solve by completing the square: $x^2 + 3x - 2 = 0$

Solution:

Begin by adding 2 to both sides.

$$x^2 + 3x - 2 = 0$$
$$x^2 + 3x = 2$$

Use ***b*** = 3 to find the value that completes the square:

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

To complete the square, add 9/4 to both sides of the equation.

$$x^2 + 3x = 2$$
$$x^2 + 3x + \frac{9}{4} = 2 + \frac{9}{4}$$
$$\left(x + \frac{3}{2}\right)\left(x + \frac{3}{2}\right) = \frac{8}{4} + \frac{9}{4}$$
$$\left(x + \frac{3}{2}\right)^2 = \frac{17}{4}$$

Solve by extracting roots.

Answer:

The solutions are $\frac{-3 \pm \sqrt{17}}{2}$

So far, all of the examples have had a leading coefficient of 1. The formula $\left(\frac{b}{2}\right)^2$ determines the value that completes the square only if the leading coefficient is 1. If this is not the case, then simply divide both sides by the leading coefficient.

Solve by completing the square: $2x^2 + 5x - 1 = 0$.

Solution:

Notice that the leading coefficient is 2. Therefore, divide both sides by 2 before beginning the steps required to solve by completing the square.

$$\begin{aligned} \tfrac{2x^2+5x-1}{2} &= \tfrac{0}{2} \\ \tfrac{2x^2}{2} + \tfrac{5x}{2} - \tfrac{1}{2} &= 0 \\ x^2 + \tfrac{5}{2}x - \tfrac{1}{2} &= 0 \end{aligned}$$

Begin by adding 1/2 to both sides of the equation.

$$\begin{aligned} x^2 + \tfrac{5}{2}x - \tfrac{1}{2} &= 0 \\ x^2 + \tfrac{5}{2}x &= \tfrac{1}{2} \end{aligned}$$

Here ***b*** = 5/2, and we can find the value that completes the square as follows:

$$\left(\tfrac{b}{2}\right)^2 = \left(\tfrac{\frac{5}{2}}{2}\right)^2 = \left(\tfrac{5}{2} \cdot \tfrac{1}{2}\right)^2 = \left(\tfrac{5}{4}\right)^2 = \tfrac{25}{16}$$

To complete the square, add 25/16 to both sides of the equation.

$$\begin{aligned} x^2 + \frac{5}{2}x &= \frac{1}{2} \\ x^2 + \frac{5}{2}x + \frac{25}{16} &= \frac{1}{2} + \frac{25}{16} \\ \left(x + \frac{5}{4}\right)\left(x + \frac{5}{4}\right) &= \frac{8}{16} + \frac{25}{16} \\ \left(x + \frac{5}{4}\right)^2 &= \frac{33}{16} \end{aligned}$$

Next, solve by extracting roots.

Answer:

The solutions are $\frac{-5 \pm \sqrt{33}}{4}$

Solve $2x^2 - 2x - 3 = 0$.

Answer

$\frac{1 \pm \sqrt{13}}{4}$

Key Takeaways

- Solve any quadratic equation by completing the square.
- You can apply the square root property to solve an equation if you can first convert the equation to the form $(x - p)^2 = q$.
- To complete the square, first make sure the equation is in the form $x^2 + bx \ = c$. Then add the value $(\frac{b}{2})^2$ to both sides and factor.
- The process for completing the square always works, but it may lead to some tedious calculations with fractions. This is the case when the middle term, *b*, is not divisible by 2.

Exercise 9.2.3 completing the square

Complete the square.

1. $x^2 + 6x + \ ? \ = (x + \ ?)^2$
2. $x^2 + 8x + \ ? \ = (x + \ ?)^2$
3. $x^2 - 2x + \ ? \ = (x - \ ?)^2$
4. $x^2 - 4x + \ ? \ = (x - \ ?)^2$
5. $x^2 + 7x + \ ? \ = (x + \ ?)^2$
6. $x^2 + 3x + \ ? \ = (x + \ ?)^2$
7. $x^2 + 23x + \ ? \ = (x + \ ?)^2$
8. $x^2 + 45x + \ ? \ = (x + \ ?)^2$
9. $x^2 + 34x + \ ? \ = (x + \ ?)^2$
10. $x^2 + 53x + \ ? \ = (x + \ ?)^2$

Answer available at the end of this section.

Exercise 9.2.4

Solve by factoring and then solve by completing the square. Check answers.

1. $x^2 + 2x - 8 = 0$
2. $x^2 - 8x + 15 = 0$
3. $y^2 + 2y - 24 = 0$
4. $y^2 - 12y + 11 = 0$
5. $t^2 + 3t - 28 = 0$
6. $t^2 - 7t + 10 = 0$
7. $2x^2 + 3x - 2 = 0$
8. $3x^2 - x - 2 = 0$
9. $2y^2 - y - 1 = 0$
10. $2y^2 + 7y - 4 = 0$

Answer available at the end of this section.

Exercise 9.2.5

Solve by completing the square.

1. $x^2 + 6x - 1 = 0$
2. $x^2 + 8x + 10 = 0$
3. $x^2 - 2x - 7 = 0$
4. $x^2 - 6x - 3 = 0$
5. $x^2 - 2x + 4 = 0$
6. $x^2 - 4x + 9 = 0$
7. $t^2 + 10t - 75 = 0$
8. $t^2 + 12t - 108 = 0$
9. $x^2 - 4x - 1 = 15$
10. $x^2 - 12x + 8 = -10$
11. $y^2 - 20y = -25$
12. $y^2 + 18y = -53$
13. $x^2 - 0.6x - 0.27 = 0$
14. $x^2 - 1.6x - 0.8 = 0$
15. $x^2 - 23x - 13 = 0$
16. $x^2 - 45x - 15 = 0$

17. $x^2 + x - 1 = 0$
18. $x^2 + x - 3 = 0$
19. $y^2 + 3y - 2 = 0$
20. $y^2 + 5y - 3 = 0$
21. $x^2 + 3x + 5 = 0$
22. $x^2 + x + 1 = 0$
23. $x^2 - 7x + 112 = 0$
24. $x^2 - 9x + 32 = 0$
25. $t^2 - 12t - 1 = 0$
26. $t^2 - 13t - 2 = 0$
27. $x^2 - 1.7x - 0.0875 = 0$
28. $x^2 + 3.3x - 1.2775 = 0$
29. $4x^2 - 8x - 1 = 0$
30. $2x^2 - 4x - 3 = 0$
31. $3x^2 + 6x + 1 = 0$
32. $5x^2 + 10x + 2 = 0$
33. $3x^2 + 2x - 3 = 0$
34. $5x^2 + 2x - 5 = 0$
35. $4x^2 - 12x - 15 = 0$
36. $2x^2 + 4x - 43 = 0$
37. $2x^2 - 4x + 10 = 0$
38. $6x^2 - 24x + 42 = 0$
39. $2x^2 - x - 2 = 0$
40. $2x^2 + 3x - 1 = 0$
41. $3x^2 + 2x - 2 = 0$
42. $3x^2 - x - 1 = 0$
43. $x(x + 1) - 11(x - 2) = 0$
44. $(x + 1)(x + 7) - 4(3x + 2) = 0$
45. $y^2 = (2y + 3)(y - 1) - 2(y - 1)$
46. $(2y + 5)(y - 5) - y(y - 8) = -24$
47. $(t + 2)^2 = 3(3t + 1)$
48. $(3t + 2)(t - 4) - (t - 8) = 1 - 10t$

Answer available at the end of this section.

Exercise 9.2.6

Solve by completing the square and round off the solutions to the nearest hundredth

1. $(2x - 1)^2 = 2x$
2. $(3x - 2)^2 = 5 - 15x$
3. $(2x + 1)(3x + 1) = 9x + 4$
4. $(3x + 1)(4x - 1) = 17x - 4$
5. $9x(x - 1) - 2(2x - 1) = -4x$
6. $(6x + 1)2 - 6(6x + 1) = 0$

Answer available at the end of this section.

Exercise 9.2.7 discussion board

1. Research and discuss the Hindu method for completing the square.
2. Explain why the technique for completing the square described in this section requires that the leading coefficient be equal to 1.

Answer available at the end of this section.

Answers

Exercise 9.2.1

$x = 1 \pm 2\sqrt{7}$

Exercise 9.2.3 **completing the square**

1. $x^2 + 6x + 9 = (x + 3)^2$
3. $x^2 - 2x + 1 = (x - 1)^2$
5. $x^2 + 7x + 494 = (x + 72)^2$
7. $x^2 + 23x + 19 = (x + 13)^2$
9. $x^2 + 34x + 964 = (x + 38)^2$

Exercise 9.2.4

1. –4, 2
3. –6, 4
5. –7, 4
7. 1/2, –2
9. –1/2, 1

Exercise 9.2.5

1. $-3 \pm \sqrt{10}$
3. $1 \pm 2\sqrt{2}$
5. No real solution
7. –15, 5
9. $2(1 \pm \sqrt{5})$
11. $5(2 \pm \sqrt{3})$
13. –0.3, 0.9
15. –1/3, 1
17. $\frac{-1 \pm \sqrt{5}}{2}$
19. $\frac{-3 \pm \sqrt{17}}{2}$
21. No real solution
23. $= \frac{7}{2} \pm i\frac{\sqrt{399}}{2}$
25. $6 \pm \sqrt{37}$
27. –0.05, 1.75
29. $\frac{2 \pm \sqrt{5}}{2}$
31. $\frac{-3 \pm \sqrt{6}}{3}$
33. $\frac{-1 \pm \sqrt{10}}{3}$
35. $\frac{3 \pm 2\sqrt{6}}{2}$
37. No real solution
39. $\frac{1 \pm \sqrt{17}}{4}$
41. $\frac{-1 \pm \sqrt{7}}{3}$
43. $5 \pm \sqrt{3}$
45. $1 \pm 5\sqrt{2}$
47. $\frac{5 \pm \sqrt{21}}{2}$

Exercise 9.2.6

1. 0.19, 1.31
3. –0.45, 1.12
5. 0.33, 0.67

Exercise 9.2.7 **discussion board**

1. Answers may vary

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9.3: Quadratic Formula

Learning Objectives

- Solve quadratic equations with real solutions using the quadratic formula.

The Quadratic Formula

In this section, we will develop a formula that gives the solutions to any quadratic equation in standard form. To do this, we begin with a general quadratic equation in standard form and solve for ***x*** by completing the square. Here ***a***, ***b***, and ***c*** are real numbers and $a \neq 0$:

$$\begin{aligned} ax^2+bx+c &= 0 \\ \frac{ax^2+bx+c}{a} &= \frac{0}{a} && \textit{Divide both sides by } a. \\ x^2+\frac{b}{a}x+\frac{c}{a} &= 0 && \textit{Subtract } \tfrac{c}{a} \textit{ from both sides.} \\ x^2+\frac{b}{a}x &= -\frac{c}{a} \end{aligned}$$

Determine the constant that completes the square: take the coefficient of ***x***, divide it by 2, and then square it.

$$\left(\frac{b/a}{2}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} \tag{9.3.1}$$

Add this to both sides of the equation and factor.

Solve by extracting roots.

This derivation gives us a formula that solves any quadratic equation in standard form. Given $ax^2+bx+c=0$, where ***a***, ***b***, and ***c*** are real numbers and $a \neq 0$, then the solutions can be calculated using the quadratic formula:

Consider the quadratic equation $2x^2-7x+3=0$. It can be solved by factoring as follows:

$$\begin{aligned} 2x-1 &= 0 \quad \text{or } x-3=0 \\ 2x &= 1 \qquad\quad x=3 \\ x &= \tfrac{1}{2} \end{aligned}$$

The solutions are 1/2 and 3. The following example shows that we can obtain the same results using the quadratic formula.

Example 9.3.1

Solve using the quadratic formula:

$$2x^2-7x+3=0.$$

Solution

Begin by identifying ***a***, ***b***, and ***c*** as the coefficients of each term.

Substitute these values into the quadratic formula and then simplify.

Separate the “plus or minus” into two equations and simplify each individually.

$$x = \frac{7-5}{4} \quad \text{or} \quad x = \frac{7+5}{4}$$
$$x = \frac{2}{4} \qquad\qquad x = \frac{12}{4}$$
$$x = \frac{1}{2} \qquad\qquad x = 3$$

Answer:

The solutions are 1/2 and 3.

Of course, if the quadratic factors, then it is a best practice to solve it by factoring. However, not all quadratic polynomials factor; nevertheless, the quadratic formula provides us with a means to solve such equations.

Example 9.3.2

Solve using the quadratic formula:

$$5x^2 + 2x - 1 = 0.$$

Solution:

Begin by identifying ***a***, ***b***, and ***c***.

$$a = 5 \qquad b = 2 \qquad c = -1$$

Substitute these values into the quadratic formula.

Answer:

The solutions are $\frac{-1 \pm \sqrt{6}}{5}$

Often terms are missing. When this is the case, use 0 as the coefficient.

Solve using the quadratic formula:

$$x^2 - 18 = 0.$$

Solution:

Think of this equation with the following coefficients:

$$1x^2 + 0x - 18 = 0$$

Here

$$a = 1 \qquad b = 0 \qquad c = -18$$

Substitute these values into the quadratic formula.

Answer:

The solutions are $\pm 3\sqrt{2}$

Since the coefficient of ***x*** was 0, we could have solved the equation by extracting the roots. As an exercise, solve the previous example using this method and verify that the results are the same.

Solve using the quadratic formula:

$$9x^2 - 12x + 4 = 0.$$

Solution:

In this case,

Substitute these values into the quadratic formula and then simplify.

In this example, notice that the radicand of the square root is 0. This results in only one solution to this quadratic equation. Normally, we expect two solutions. When we find only one solution, the solution is called a double root. If we solve this equation by factoring, then the solution appears twice.

$$\begin{aligned} 3x - 2 &= 0 \quad \text{or} \quad 3x - 2 = 0 \\ 3x &= 2 \qquad\qquad 3x = 2 \\ x &= \tfrac{2}{3} \qquad\qquad x = \tfrac{2}{3} \end{aligned}$$

Answer: $\frac{2}{3}$, double root

Example 9.3.5

Solve using the quadratic formula:

$$x^2 + x + 1 = 0.$$

Solution:

In this case,

$$a = 1 \qquad b = 1 \qquad c = 1$$

Substitute these values into the quadratic formula.

The solution involves the square root of a negative number; hence the solutions are not real. This quadratic equation has two non real solutions and will be discussed in further detail as we continue in our study of algebra. For now, simply state that the equation does not have real solutions.

Answer:

No real solutions

Solve $x^2 - 2x - 2 = 0$.

Answer

$1 \pm \sqrt{3}$

It is important to place the quadratic equation in standard form before using the quadratic formula.

Example 9.3.6

Solve using the quadratic formula:

$$(2x+1)(2x-1)=24x+8$$

Solution:

Begin by using the distributive property to expand the left side and combining like terms to obtain an equation in standard form, equal to 0.

Once the equation is in standard form, identify ***a***, ***b***, and ***c***. Here

Substitute these values into the quadratic formula and then simplify.

Answer:

The solutions are $\frac{6\pm 3\sqrt{5}}{2}$

Solve $3x(x-2)=1$

Answer

$\frac{3\pm 2\sqrt{3}}{3}$

Key Takeaways

- Use the quadratic formula to solve any quadratic equation in standard form.
- To solve any quadratic equation, first rewrite in standard form, $ax^2+bx+c=0$, substitute the appropriate coefficients into the quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, and then simplify.

Exercise 9.3.3 quadratic formula

Identify the coefficients a, b, and c used in the quadratic formula. Do not solve.

1. $x^2-x+5=0$
2. $x^2-3x-1=0$
3. $3x^2-10=0$
4. $-y^2+5=0$
5. $5t^2-7t=0$
6. $-y^2+y=0$
7. $-x^2+x=-6$
8. $-2x^2-x=-15$
9. $(3x+1)(2x+5)=19x+4$
10. $(4x+1)(2x+1)=16x+4$

Answer available at the end of this section.

Exercise 9.3.4 quadratic formula

Solve by factoring and then solve using the quadratic formula. Check answers.

1. $x^2 - 10x + 24 = 0$
2. $x^2 - 3x - 18 = 0$
3. $t^2 + 6t + 5 = 0$
4. $t^2 + 9t + 14 = 0$
5. $2x^2 - 7x - 4 = 0$
6. $3x^2 - x - 2 = 0$
7. $-2x^2 - x + 3 = 0$
8. $-6x^2 + x + 1 = 0$
9. $y^2 - 2y + 1 = 0$
10. $y^2 - 1 = 0$

Answer available at the end of this section.

Exercise 9.3.5 quadratic formula

Use the quadratic formula to solve the following.

1. $x^2 - 6x + 4 = 0$
2. $x^2 - 4x + 1 = 0$
3. $x^2 + 2x - 5 = 0$
4. $x^2 + 4x - 6 = 0$
5. $t^2 - 4t - 1 = 0$
6. $t^2 - 8t - 2 = 0$
7. $-y^2 + y + 1 = 0$
8. $-y^2 - 3y + 2 = 0$
9. $-x^2 + 16x - 62 = 0$
10. $-x^2 + 14x - 46 = 0$
11. $2t^2 - 4t - 3 = 0$
12. $4t^2 - 8t - 1 = 0$
13. $-4y^2 + 12y - 9 = 0$
14. $-25x^2 + 10x - 1 = 0$
15. $3x^2 + 6x + 2 = 0$
16. $5x^2 + 10x + 2 = 0$
17. $9t^2 + 6t - 11 = 0$
18. $8t^2 + 8t + 1 = 0$
19. $x^2 - 2 = 0$
20. $x^2 - 18 = 0$
21. $9x^2 - 3 = 0$
22. $2x^2 - 5 = 0$
23. $y^2 + 9 = 0$
24. $y^2 + 1 = 0$
25. $2x^2 = 0$
26. $x^2 = 0$
27. $-2y^2 + 5y = 0$
28. \(-3y^{2] + 7y = 0\)
29. $t^2 - t = 0$
30. \(t^{2] + 2 t = 0\)
31. $x^2 - 0.6x - 0.27 = 0$
32. $x^2 - 1.6x - 0.8 = 0$
33. $y^2 - 1.4y - 0.15 = 0$
34. $y^2 - 3.6y + 2.03 = 0$
35. $12t^2 + 5t + 32 = 0$
36. $-t^2 + 3t - 34 = 0$
37. $3y^2 + 12y - 13 = 0$
38. $-2y^2 + 13y + 12 = 0$
39. $2x^2 - 10x + 3 = 4$

40. $3x^2 + 6x + 1 = 8$
41. $-2y^2 = 3(y - 1)$
42. $3y^2 = 5(2y - 1)$
43. $(t + 1)^2 = 2t + 7$
44. $(2t - 1)^2 = 73 - 4t$
45. $(x + 5)(x - 1) = 2x + 1$
46. $(x + 7)(x - 2) = 3(x + 1)$
47. $x(x + 5) = 3(x - 1)$
48. $x(x + 4) = -7$
49. $(5x + 3)(5x - 3) - 10(x - 1) = 0$
50. $(3x + 4)(3x - 1) - 33x = -20$
51. $27y(y + 1) + 2(3y - 2) = 0$
52. $8(4y^2 + 3) - 3(28y - 1) = 0$
53. $(x + 2)^2 - 2(x + 7) = 4(x + 1)$
54. $(x + 3)^2 - 10(x + 5) = -2(x + 1)$

Answer available at the end of this section.

Exercise $9.3.6$ discussion board

1. When talking about a quadratic equation in standard form, $ax^2 + bx + c = 0$, why is it necessary to state that a≠0? What happens if a is equal to 0?
2. Research and discuss the history of the quadratic formula and solutions to quadratic equations.

Answer available at the end of this section.

Answers

Exercise $9.3.3$ quadratic formula

1. a=1, b=−1, and c=5
3. a=3, b=0, and c=−10
5. a=5, b=−7, and c=0
7. a=−1, b=1, and c=6
9. a=6, b=−2, and c=1

Exercise $9.3.4$ quadratic formula

1. 4, 6
3. −5, −1
5. −1/2, 4
7. −3/2, 1
9. 1, double root

Exercise $9.3.5$ quadratic formula

1. $3 \pm \sqrt{5}$
3. $-1 \pm \sqrt{6}$
5. $2 \pm \sqrt{5}$
7. $\frac{1 \pm \sqrt{5}}{2}$
9. $8 \pm \sqrt{2}$
11. $\frac{2 \pm \sqrt{10}}{2}$
13. $\frac{3}{2}$, double root
15. $\frac{-3 \pm \sqrt{3}}{3}$
17. $\frac{-1 \pm 2\sqrt{3}}{3}$
19. $\pm\sqrt{2}$
21. $\pm\sqrt{\frac{1}{3}}$
23. No real solutions
25. 0, double root
27. $0, \frac{5}{2}$
29. $0, 1$
31. $-0.3, 0.9$

33. $-0.1, 1.5$
35. No real solutions
37. $\frac{-6\pm5\sqrt{3}}{3}$
39. $\frac{5\pm3\sqrt{3}}{2}$
41. $\frac{-3\pm\sqrt{33}}{4}$
43. $\pm\sqrt{6}$
45. $-1\pm\sqrt{7}$
47. No real solutions
49. $\frac{1}{5}$, double root
51. $-\frac{4}{3}, \frac{1}{9}$
53. $1\pm2\sqrt{10}$

Exercise $9.3.6$ **discussion board**

1. Answers may vary

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9.4: Guidelines for Solving Quadratic Equations and Applications

Learning Objectives

- Use the discriminant to determine the number and type of solutions to any quadratic equation.
- Develop a general strategy for solving quadratic equations.
- Solve applications involving quadratic equations.

Discriminant

If given a quadratic equation in standard form, $ax^2 + bx + c = 0$, where ***a***, ***b***, and ***c*** are real numbers and a≠0, then the solutions can be calculated using the quadratic formula:

The solutions are rational, irrational, or not real. We can determine the type and number of solutions by studying the discriminant, the expression inside the radical, $b^2 - 4ac$. If the value of this expression is negative, then the equation has no real solutions. If the discriminant is positive, then we have two real solutions. And if the discriminant is 0, then we have one real solution.

Determine the type and number of solutions:

$$x^2 - 10x + 30 = 0.$$

Solution:

We begin by identifying ***a***, ***b***, and ***c***. Here

Substitute these values into the discriminant and simplify.

Since the discriminant is negative, we conclude that there are no real solutions.

Answer:

No real solution

If we use the quadratic formula in the previous example, we find that a negative radicand stops the process of simplification and shows that there is no real solution.

We will study quadratic equations with no real solutions as we progress in our study of algebra.

Example 9.4.2

Determine the type and number of solutions:

$$7x^2 - 10x + 1 = 0$$

Solution:

Here

Substitute these values into the discriminant:

Since the discriminant is positive, we can conclude that there are two real solutions.

Answer:

Two real solutions

If we use the quadratic formula in the previous example, we find that a positive radicand in the quadratic formula leads to two real solutions.

The two real solutions are $\frac{5-3\sqrt{2}}{7}$ and $\frac{5+3\sqrt{2}}{7}$. Note that these solutions are irrational; we can approximate the values on a calculator.

Example 9.4.3

Determine the type and number of solutions:

$$2x^2 - 7x - 4 = 0$$

Solution:

In this example,

Substitute these values into the discriminant and simplify.

Since the discriminant is positive, we conclude that there are two real solutions. Furthermore, since the discriminant is a perfect square, we obtain two rational solutions.

Answer:

Two real solutions

We could solve the previous quadratic equation using the quadratic formula as follows:

$$\begin{array}{lll} x = \frac{7-9}{4} & \text{or} & x = \frac{7+9}{4} \\ x = \frac{-2}{4} & & x = \frac{16}{4} \\ x = -\frac{1}{2} & & x = 4 \end{array}$$

Note

If the discriminant is a perfect square, then we could have factored the original equation.

$$\begin{array}{rll} 2x+1 &= 0 & \text{or} \quad x-4=0 \\ 2x &= -1 & x = 4 \\ x &= -\frac{1}{2} & \end{array}$$

Given the special condition where the discriminant is 0, we obtain only one solution, a double root.

Example 9.4.4

Determine the type and number of solutions:

$$9x^2 - 6x + 1 = 0$$

Solution:

Here a=9, b=−6, and c=1, and we have

Since the discriminant is 0, we conclude that there is only one real solution, a double root.

Answer:

One real solution

Since 0 is a perfect square, we can solve the equation above by factoring.

$$\begin{aligned} 3x - 1 &= 0 \quad \text{or} \quad 3x - 1 = 0 \\ 3x &= 1 \qquad\qquad 3x = 1 \\ x &= \tfrac{1}{3} \qquad\quad x = \tfrac{1}{3} \end{aligned}$$

Here 1/3 is a solution that occurs twice; it is a double root.

In summary, if given any quadratic equation in standard form, $ax^2 + bx + c = 0$, where ***a***, ***b***, and ***c*** are real numbers and a≠0, then we have the following:

Table 9.4.1

Positive discriminant:	$b^2 - 4ac > 0$	Two real solutions
Zero discriminant:	$b^2 - 4ac = 0$	One real solution
Negative discriminant:	$b^2 - 4ac < 0$	No real solution

As we will see, knowing the number and type of solutions ahead of time helps us determine which method is best for solving a quadratic equation.

Exercise 9.4.1

Determine the number and type of solutions:

$$3x^2 - 5x + 4 = 0$$

Answer available at the end of this section.

General Guidelines for Solving Quadratic Equations

Use the coefficients of a quadratic equation to help decide which method is most appropriate for solving it. While the quadratic formula always works, it is sometimes not the most efficient method. Given any quadratic equation in standard form, $ax^2 + bx + c = 0$, general guidelines for determining the method for solving it follow:

1. If **c** = 0, then factor out the GCF and solve by factoring.
2. If **b** = 0, then solve by extracting the roots.
3. If **a**, **b**, and **c** are all nonzero, then determine the value for the discriminant, $b^2 - 4ac$:
   a. If the discriminant is a perfect square, then solve by factoring.
   b. If the discriminant is not a perfect square, then solve using the quadratic formula.
   c. If the discriminant is positive, we obtain two real solutions.
   d. If the discriminant is negative, then there is no real solution.

Example 9.4.5

Solve:

$$15x^2 - 5x = 0$$

Solution:

In this case, ***c*** = 0 and we can solve by factoring out the GCF.

$$\begin{array}{lll} 5x = 0 & \text{or} & 3x - 1 = 0 \\ x = 0 & & 3x = 1 \\ & & x = \frac{1}{3} \end{array}$$

Answer:

The solutions are 0 and $\frac{1}{3}$.

Example 9.4.6

Solve:

$$3x^2 - 5 = 0$$

Solution:

In this case, ***b*** = 0 and we can solve by extracting the roots.

Answer:

The solutions are $\pm\frac{\sqrt{15}}{3}$

Solve:

$$9x^2 - 6x - 7 = 0$$

Solution:

Begin by identifying ***a***, ***b***, and ***c*** as the coefficients of each term. Here

Substitute these values into the discriminant and then simplify.

Since the discriminant is positive and not a perfect square, use the quadratic formula and expect two real solutions.

Answer:

The solutions are $= \frac{1\pm2\sqrt{2}}{3}$

Solve:

$$4x(x - 2) = -7$$

Solution:

Begin by rewriting the quadratic equation in standard form.

Here

Substitute these values into the discriminant and then simplify.

Since the discriminant is negative, the solutions are not real numbers.

Answer:

No real solution

Solve:

$$(3x+5)(3x+7)=6x+10$$

Solution:

Begin by rewriting the quadratic equation in standard form.

$$\begin{aligned}(3x+5)(3x+7)&=6x+10\\9x^2+21x+15x+35&=6x+10\\9x^2+36x+35&=6x+10\\9x^2+30x+25&=0\end{aligned}$$

Substitute ***a***=9, ***b***=30, and ***c***=25 into the discriminant.

Since the discriminant is 0, solve by factoring and expect one real solution, a double root.

$$\begin{aligned}9x^2+30x+25&=0\\(3x+5)(3x+5)&=0\end{aligned}$$

$$\begin{aligned}3x+5=0 \quad &\text{or} \quad 3x+5=0\\3x=-5 \quad &\quad\;\; 3x=-5\\x=-\tfrac{5}{3} \quad &\quad\;\; x=-\tfrac{5}{3}\end{aligned}$$

Answer:

The solution is $-\frac{5}{3}$

Exercise 9.4.2

Solve:

$$5x^2+2x-7=2x-3$$

Answer available at the end of this section.

Applications Involving Quadratic Equations

In this section, the algebraic setups usually consist of a quadratic equation where the solutions may not be integers.

Example 9.4.10

The height of a triangle is 2 inches less than twice the length of its base. If the total area of the triangle is 11 square inches, then find the lengths of the base and height. Round answers to the nearest hundredth

Solution:

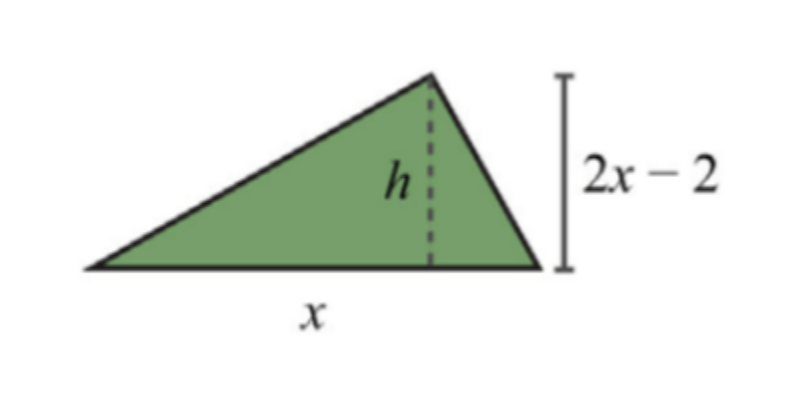


Figure 9.4.1

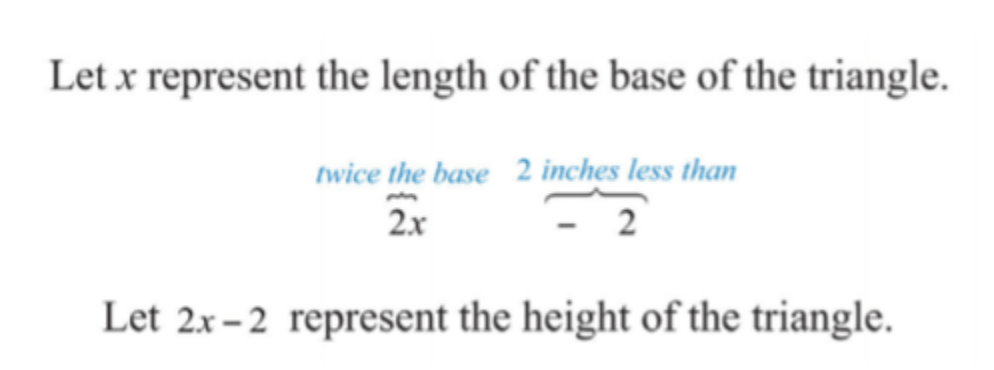


Figure 9.4.2

Use the formula $A = \frac{1}{2}bh$ and the fact that the area is 11 square inches to set up an algebraic equation.

$$A = \tfrac{1}{2}b \cdot h$$
$$11 = \tfrac{1}{2}x(2x - 2)$$

To rewrite this quadratic equation in standard form, first distribute $\frac{1}{2}x$.

$$11 = \tfrac{1}{2}x(2x - 2)$$
$$11 = x^2 - x$$
$$0 = x^2 - x - 11$$

Use the coefficients, ***a*** = 1, ***b*** = −1, and ***c*** = −11, to determine the type of solutions.

Since the discriminant is positive, expect two real solutions.

In this problem, disregard the negative solution and consider only the positive solution.

$$x = \tfrac{1+3\sqrt{5}}{2}$$

Back substitute to find the height.

Answer:

The base measures $\frac{-1+3\sqrt{5}}{2} \approx 3.85$ inches and the height is $-1+3\sqrt{5} \approx 5.71$ inches.

The sum of the squares of two consecutive positive integers is 481. Find the integers.

Solution:

Let *n* represent the first positive integer.

Let *n+1* represent the next positive integer.

The algebraic setup follows:

$$\overbrace{n^2 + (n+1)^2}^{\text{The sum of the squares of...}} \quad \overbrace{= 481}^{\text{...is 481}}$$

Figure 9.4.3

Rewrite the quadratic equation in standard form.

When the coefficients are large, sometimes it is less work to use the quadratic formula instead of trying to factor it. In this case, *a=1*, *b=1*, and *c=–240*. Substitute into the quadratic formula and then simplify.

$$\begin{aligned} n &= \tfrac{-1-31}{2} & n &= \tfrac{-1+31}{2} \\ n &= \tfrac{-32}{2} & n &= \tfrac{30}{2} \\ n &= -16 & n &= 15 \end{aligned}$$

Since the problem calls for positive integers, disregard the negative solution and choose ***n*** = 15.

$$\begin{aligned} n+1 &= 15+1 \\ &= 16 \end{aligned}$$

Answer:

The positive integers are 15 and 16.

Key Takeaways

- Determine the number and type of solutions to any quadratic equation in standard form using the discriminant, b^2-4ac. If the discriminant is negative, then the solutions are not real. If the discriminant is positive, then the solutions are real. If the discriminant is 0, then there is only one solution, a double root.
- Choose the appropriate method for solving a quadratic equation based on the value of its discriminant. While the quadratic formula will solve any quadratic equation, it may not be the most efficient method.
- When solving applications, use the key words and phrases to set up an algebraic equation that models the problem. In this section, the setup typically involves a quadratic equation.

Exercise 9.4.3 use the discriminant

Calculate the discriminant and use it to determine the number and type of solutions. Do not solve.

1. $x^2 + 2x + 3 = 0$
2. $x^2 - 2x - 3 = 0$
3. $3x^2 - 1x - 2 = 0$
4. $3x^2 - 1x + 2 = 0$
5. $9y^2 + 2 = 0$
6. $9y^2 - 2 = 0$
7. $5x^2 + x = 0$
8. $5x^2 - x = 0$
9. $12x^2 - 2x + 52 = 0$
10. $12x^2 - x - 12 = 0$
11. $-x^2 - 2x + 4 = 0$
12. $-x^2 - 4x + 2 = 0$
13. $4t^2 - 20t + 25 = 0$
14. $9t^2 - 6t + 1 = 0$

Answer available at the end of this section.

Exercise 9.4.4 solving

Choose the appropriate method to solve the following.

1. $x^2 - 2x - 3 = 0$
2. $x^2 + 2x + 3 = 0$
3. $3x^2 - x - 2 = 0$
4. $3x^2 - x + 2 = 0$
5. $9y^2 + 2 = 0$
6. $9y^2 - 2 = 0$
7. $5x^2 + x = 0$
8. $5x^2 - x = 0$
9. $12x^2 - 2x + 52 = 0$
10. $12x^2 - x - 12 = 0$
11. $-x^2 - 2x + 4 = 0$
12. $-x^2 - 4x + 2 = 0$
13. $4t^2 - 20t + 25 = 0$
14. $9t^2 - 6t + 1 = 0$
15. $y^2 - 4y - 1 = 0$
16. $y^2 - 6y - 3 = 0$
17. $25x^2 + 1 = 0$
18. $36x^2 + 4 = 0$
19. $5t^2 - 4 = 0$
20. $2t^2 - 9 = 0$
21. $12x^2 - 94x + 1 = 0$
22. $3x^2 + 12x - 16 = 0$
23. $36y^2 = 2y$
24. $50y^2 = -10y$
25. $x(x - 6) = -29$
26. $x(x - 4) = -16$
27. $4y(y + 1) = 5$
28. $2y(y + 2) = 3$
29. $-3x^2 = 2x + 1$
30. $3x^2 + 4x = -2$
31. $6(x + 1)2 = 11x + 7$
32. $2(x + 2)2 = 7x + 11$
33. $9t^2 = 4(3t - 1)$
34. $5t(5t - 6) = -9$
35. $(x + 1)(x + 7) = 3$

36. $(x-5)(x+7)=14$

Answer available at the end of this section.

Exercise 9.4.5 applications number problems

Set up an algebraic equation and use it to solve the following.

1. A positive real number is 2 less than another. When 4 times the larger is added to the square of the smaller, the result is 49. Find the numbers.
2. A positive real number is 1 more than another. When twice the smaller is subtracted from the square of the larger, the result is 4. Find the numbers.
3. A positive real number is 6 less than another. If the sum of the squares of the two numbers is 38, then find the numbers.
4. A positive real number is 1 more than twice another. If 4 times the smaller number is subtracted from the square of the larger, then the result is 21. Find the numbers.

Answer available at the end of this section.

Exercise 9.4.6 applications geometry problems

Round off your answers to the nearest hundredth.

1. The area of a rectangle is 60 square inches. If the length is 3 times the width, then find the dimensions of the rectangle.
2. The area of a rectangle is 6 square feet. If the length is 2 feet more than the width, then find the dimensions of the rectangle.
3. The area of a rectangle is 27 square meters. If the length is 6 meters less than 3 times the width, then find the dimensions of the rectangle.
4. The area of a triangle is 48 square inches. If the base is 2 times the height, then find the length of the base.
5. The area of a triangle is 14 square feet. If the base is 4 feet more than 2 times the height, then find the length of the base and the height.
6. The area of a triangle is 8 square meters. If the base is 4 meters less than the height, then find the length of the base and the height.
7. The perimeter of a rectangle is 54 centimeters and the area is 180 square centimeters. Find the dimensions of the rectangle.
8. The perimeter of a rectangle is 50 inches and the area is 126 square inches. Find the dimensions of the rectangle.
9. George maintains a successful 6-meter-by-8-meter garden. Next season he plans on doubling the planting area by increasing the width and height by an equal amount. By how much must he increase the length and width?
10. A uniform brick border is to be constructed around a 6-foot-by-8-foot garden. If the total area of the garden, including the border, is to be 100 square feet, then find the width of the brick border.

Answer available at the end of this section.

Exercise 9.4.7 applications pythagorean theorem

1. If the sides of a square measure $\sqrt{106}$ units, then find the length of the diagonal.
2. If the diagonal of a square measures $\sqrt{310}$ units, then find the length of each side.
3. The diagonal of a rectangle measures $\sqrt{63}$ inches. If the width is 4 inches less than the length, then find the dimensions of the rectangle.
4. The diagonal of a rectangle measures $\sqrt{23}$ inches. If the width is 2 inches less than the length, then find the dimensions of the rectangle.
5. The top of a 20-foot ladder, leaning against a building, reaches a height of 18 feet. How far is the base of the ladder from the wall? Round off to the nearest hundredth.
6. To safely use a ladder, the base should be placed about 1/4 of the ladder's length away from the wall. If a 20-foot ladder is to be safely used, then how high against a building will the top of the ladder reach? Round off to the nearest hundredth.

7. The diagonal of a television monitor measures 32 inches. If the monitor has a 3:2 aspect ratio, then determine its length and width. Round off to the nearest hundredth.
8. The diagonal of a television monitor measures 52 inches. If the monitor has a 16:9 aspect ratio, then determine its length and width. Round off to the nearest hundredth.

Answer available at the end of this section.

Exercise 9.4.8 applications business problems

1. The profit in dollars of running an assembly line that produces custom uniforms each day is given by the function $P(t) = -40t^2 + 960t - 4,000$, where *t* represents the number of hours the line is in operation.
   a. Calculate the profit on running the assembly line for 10 hours a day.
   b. Calculate the number of hours the assembly line should run in order to break even. Round off to the nearest tenth of an hour.
2. The profit in dollars generated by producing and selling *x* custom lamps is given by the function $P(x) = -10x^2 + 800x - 12,000$.
   a. Calculate the profit on the production and sale of 35 lamps.
   b. Calculate the number of lamps that must be sold to profit $3,000.
3. If $1,200 is invested in an account earning an annual interest rate *r*, then the amount *A* that is in the account at the end of 2 years is given by the formula $A = 1,200(1 + r)^2$. If at the end of 2 years the amount in the account is $1,335.63, then what was the interest rate?
4. A manufacturing company has determined that the daily revenue, *R*, in thousands of dollars depends on the number, *n*, of palettes of product sold according to the formula $R = 12n - 0.6n^2$. Determine the number of palettes that must be sold in order to maintain revenues at $60,000 per day.

Answer available at the end of this section.

Exercise 9.4.9 applications projectile problems

1. The height of a projectile launched upward at a speed of 32 feet/second from a height of 128 feet is given by the function $h(t) = -16t^2 + 32t + 128$.
   a. What is the height of the projectile at 1/2 second?
   b. At what time after launch will the projectile reach a height of 128 feet?
2. The height of a projectile launched upward at a speed of 16 feet/second from a height of 192 feet is given by the function $h(t) = -16t^2 + 16t + 192$.
   a. What is the height of the projectile at 3/2 seconds?
   b. At what time will the projectile reach 128 feet?
3. The height of an object dropped from the top of a 144-foot building is given by $h(t) = -16t^2 + 144$. How long will it take to reach a point halfway to the ground?
4. The height of a projectile shot straight up into the air at 80 feet/second from the ground is given by $h(t) = -16t^2 + 80t$. At what time will the projectile reach 95 feet?

Answer available at the end of this section.

Exercise 9.4.10 applications discussion board

1. Discuss the strategy of always using the quadratic formula to solve quadratic equations.
2. List all of the methods that we have learned so far to solve quadratic equations. Discuss the pros and cons of each.

Answer available at the end of this section.

Answers

Exercise 9.4.1

No real solution

Exercise 9.4.2

$\pm\frac{2\sqrt{5}}{5}$

Exercise 9.4.3 **use the discriminant**

1. –8, no real solution
3. 25, two real solutions
5. –72, no real solution
7. 1, two real solutions
9. –1, no real solution
11. 20, two real solutions
13. 0, one real solution

Exercise 9.4.4 **solving**

1. $-1, 3$
3. $-\frac{2}{3}, 1$
5. No real solution
7. $-\frac{1}{5}, 0$
9. No real solution
11. $-1 \pm \sqrt{5}$
13. $\frac{5}{2}$
15. $2 \pm \sqrt{5}$
17. No real solution
19. $\pm\frac{2\sqrt{5}}{5}$
21. $\frac{1}{2}, 4$
23. $0, \frac{1}{18}$
25. No real solution
27. $\frac{-1\pm\sqrt{6}}{2}$
29. No real solution
31. $-\frac{1}{2}, \frac{1}{3}$
33. $\frac{2}{3}$
35. $-4 \pm 2\sqrt{3}$

Exercise 9.4.5 **applications number problems**

1. $3\sqrt{5}$ and \(3\sqrt{5}-2)
3. \(\sqrt{10\pm 3\)

Exercise 9.4.6 **applications geometry problems**

1. Length: 13.42 inches; width: 4.47 inches
3. Length: 6.48 meters; width: 4.16 meters
5. Height: 2.87 feet; base: 9.74 feet
7. Length: 15 centimeters; width: 12 centimeters
9. 2.85 meters

Exercise 9.4.7 **applications pythagorean theorem**

1. $\sqrt{203}$ units
3. Length: $2 + \sqrt{52}$ inches; width: $-2 + \sqrt{52}$ inches
5. $2\sqrt{19} \approx 8.72$ feet
7. Length: 26.63 inches; width: 17.75 inches

Exercise 9.4.8 **applications business problems**

1. a. $1,600; b. 5.4 hours and 18.6 hours
3. 5.5%

Exercise 9.4.9 **applications projectile problems**

1. a. 140 feet; b. 0 seconds and 2 seconds
3. 2.12 seconds

Exercise 9.4.10 **applications discussion board**

1. Answers may vary

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9.5: Graphing Parabolas

Learning Objectives

- Graph a parabola.
- Find the intercepts and vertex of a parabola.
- Find the vertex of a parabola by completing the square.

The Graph of a Quadratic Equation

We know that any linear equation with two variables can be written in the form $y = mx + b$ and that its graph is a line. In this section, we will see that any quadratic equation of the form $y = ax^2 + bx + c$ has a curved graph called a parabola.

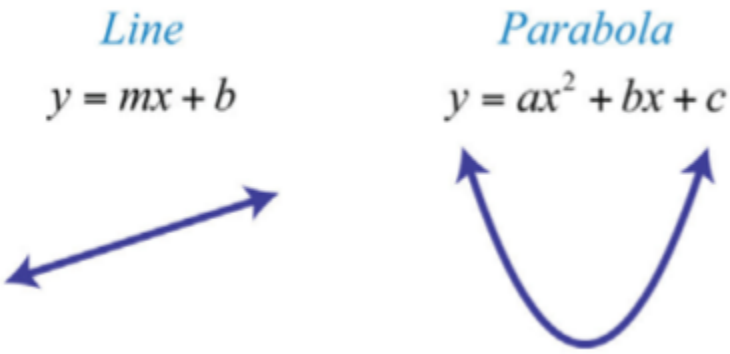


Figure 9.5.1

Two points determine any line. However, since a parabola is curved, we should find more than two points. In this text, we will determine at least five points as a means to produce an acceptable sketch. To begin, we graph our first parabola by plotting points. Given a quadratic equation of the form $y = ax^2 + bx + c$, ***x*** is the independent variable and ***y*** is the dependent variable. Choose some values for ***x*** and then determine the corresponding ***y***-values. Then plot the points and sketch the graph.

Example 9.5.1

Graph by plotting points:

$$y = x^2 - 2x - 3$$

Solution:

In this example, choose the ***x***-values {–2, –1, 0, 1, 2, 3, 4} and calculate the corresponding ***y***-values.

x	y		Points
-2	5	$y=(-2)^2-2(-2)-3=4+4-3=5$	$(-2, 5)$
-1	0	$y=(-1)^2-2(-1)-3=1+2-3=0$	$(-1, 0)$
0	-3	$y=(0)^2-2(0)-3=0-0-3=-3$	$(0, -3)$
1	-4	$y=(1)^2-2(1)-3=1-2-3=-4$	$(1, -4)$
2	-3	$y=(2)^2-2(2)-3=4-4-3=-3$	$(2, -3)$
3	0	$y=(3)^2-2(3)-3=9-6-3=0$	$(3, 0)$
4	5	$y=(4)^2-2(4)-3=16-8-3=5$	$(4, 5)$

Figure 9.5.2

Plot these points and determine the shape of the graph.

Answer:

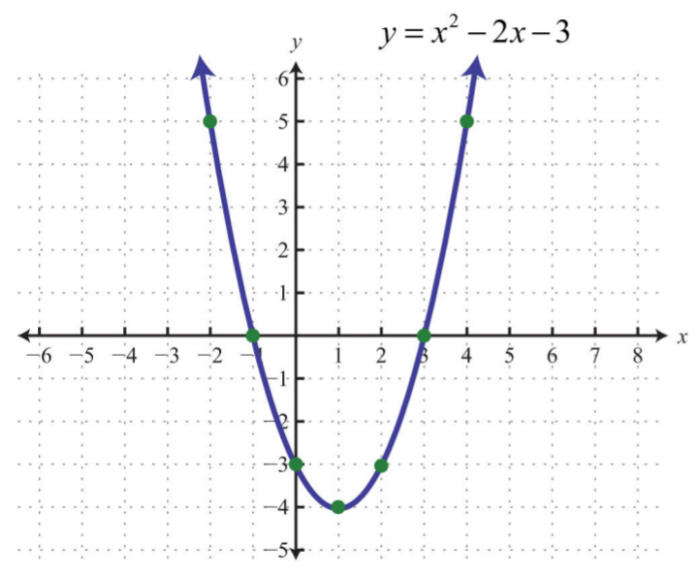


Figure 9.5.3

When graphing, we want to include certain special points in the graph. The ***y***-intercept is the point where the graph intersects the ***y***-axis. The ***x***-intercepts are the points where the graph intersects the ***x***-axis. The vertex is the point that defines the minimum or maximum of the graph. Lastly, the line of symmetry(also called the axis of symmetry) is the vertical line through the vertex, about which the parabola is symmetric.

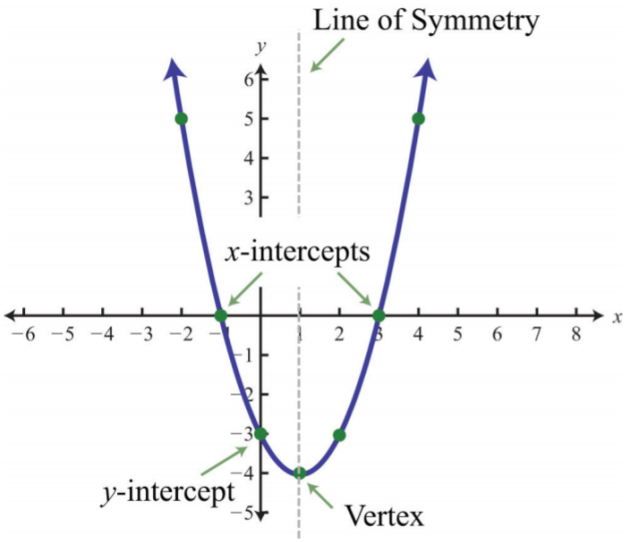


Figure 9.5.4

For any parabola, we will find the vertex and ***y***-intercept. In addition, if the ***x***-intercepts exist, then we will want to determine those as well. Guessing at the ***x***-values of these special points is not practical; therefore, we will develop techniques that will facilitate finding them. Many of these techniques will be used extensively as we progress in our study of algebra. Given a quadratic equation of the form $y = ax^2 + bx + c$, find the ***y***-intercept by setting x=0 and solving. In general, $y = a(0)^2 + b(0) + c = c$, and we have

\(\color{Cerulean}{y-intercept}

\[(0,c]\)

Next, recall that the ***x***-intercepts, if they exist, can be found by setting y=0. Doing this, we have $0 = a^2 + bx + c$, which has general solutions given by the quadratic formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Therefore, the x-intercepts have this general form:

$$x - intercepts$$

Using the fact that a parabola is symmetric, we can determine the vertical line of symmetry using the ***x***-intercepts. To do this, we find the ***x***-value midway between the ***x***-intercepts by taking an average as follows:

Therefore, the line of symmetry is the vertical line:

$$Line\ of\ symmetry$$

$$x = -\frac{b}{2a} \tag{9.5.1}$$

We can use the line of symmetry to find the ***x***-value of the vertex. The steps for graphing a parabola are outlined in the following example.

Example 9.5.2

Graph:

Solution

Step 1: Determine the ***y***-intercept. To do this, set ***x*** = 0 and solve for ***y***.

The ***y***-intercept is (0, 3).

Step 2: Determine the ***x***-intercepts. To do this, set ***y*** = 0 and solve for ***x***.

$$\begin{aligned} x+3 &= 0 \quad \text{or} \quad x-1=0 \\ x &= -3 \qquad\quad x=1 \end{aligned}$$

Here when ***y*** = 0, we obtain two solutions. There are two ***x***-intercepts, (–3, 0) and (1, 0).

Step 3: Determine the vertex. One way to do this is to use the equation for the line of symmetry, $x = \frac{-b}{2a}$, to find the ***x***-value of the vertex. In this example, ***a*** = –1 and ***b*** = –2:

Substitute –1 into the original equation to find the corresponding ***y***-value.

The vertex is (–1, 4).

Step 4: Determine extra points so that we have at least five points to plot. In this example, one other point will suffice. Choose ***x*** = –2 and find the corresponding ***y***-value.

x	y		*Point*
-2	3	$y=-(-2)^2-2(-2)+3=-4+4+3=3$	$(-2, 3)$

Figure 9.5.5

Our fifth point is (-2,3).

Step 5: Plot the points and sketch the graph. To recap, the points that we have found are

Table 9.5.1

y-intercept:	(0,3)
x-intercept:	(-3,0) and (1,0)
Vertex:	(-1,4)
Extra point:	(-2,3)

Answer:

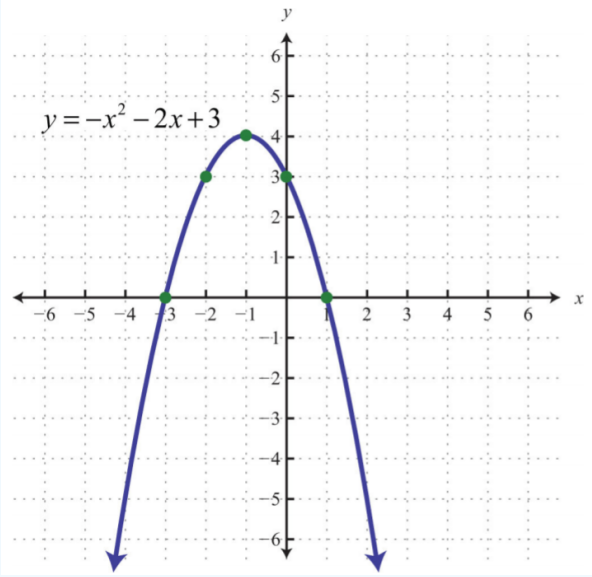


Figure 9.5.6

The parabola opens downward. In general, use the leading coefficient to determine whether the parabola opens upward or downward. If the leading coefficient is negative, as in the previous example, then the parabola opens downward. If the leading coefficient is positive, then the parabola opens upward.

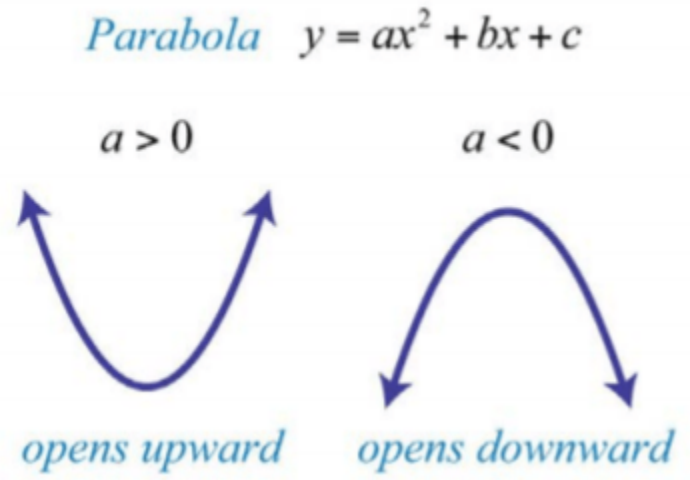


Figure 9.5.7

All quadratic equations of the form $y = ax^2 + bx + c$ have parabolic graphs with ***y***-intercept (0, ***c***). However, not all parabolas have ***x*** intercepts.

Example 9.5.3

Graph:

$$y = 2x^2 + 4x + 5$$

Solution:

Because the leading coefficient 2 is positive, note that the parabola opens upward. Here ***c*** = 5 and the ***y***-intercept is (0, 5). To find the ***x***-intercepts, set ***y*** = 0.

$$y = 2x^2 + 4x + 5$$
$$0 = 2x^2 + 4x + 5$$

In this case, ***a*** = 2, ***b*** = 4, and ***c*** = 5. Use the discriminant to determine the number and type of solutions.

Since the discriminant is negative, we conclude that there are no real solutions. Because there are no real solutions, there are no ***x***-intercepts. Next, we determine the ***x***-value of the vertex.

Given that the ***x***-value of the vertex is –1, substitute into the original equation to find the corresponding ***y***-value.

$$\begin{aligned} y &= 2x^2 + 4x + 5 \\ &= 2(-1)^2 + 4(-1) + 5 \\ &= 2 - 4 + 5 \\ &= 3 \end{aligned}$$

The vertex is (–1, 3). So far, we have only two points. To determine three more, choose some ***x***-values on either side of the line of symmetry, ***x*** = –1. Here we choose ***x***-values –3, –2, and 1.

x	y		*Points*
-3	11	$y = 2(-3)^2 + 4(-3) + 5 = 18 - 12 + 5 = 11$	$(-3, 11)$
-2	5	$y = 2(-2)^2 + 4(-2) + 5 = 8 - 8 + 5 = 5$	$(-2, 5)$
1	11	$y = 2(1)^2 + 4(1) + 5 = 2 + 4 + 5 = 11$	$(1, 11)$

Figure 9.5.8

To summarize, we have

Table 9.5.2

y-intercept:	(0,5)
x-intercepts:	None
Vertex:	(-1,3)
Extra points	(-3,11), (-2,5), (1,11)

Plot the points and sketch the graph.

Answer:

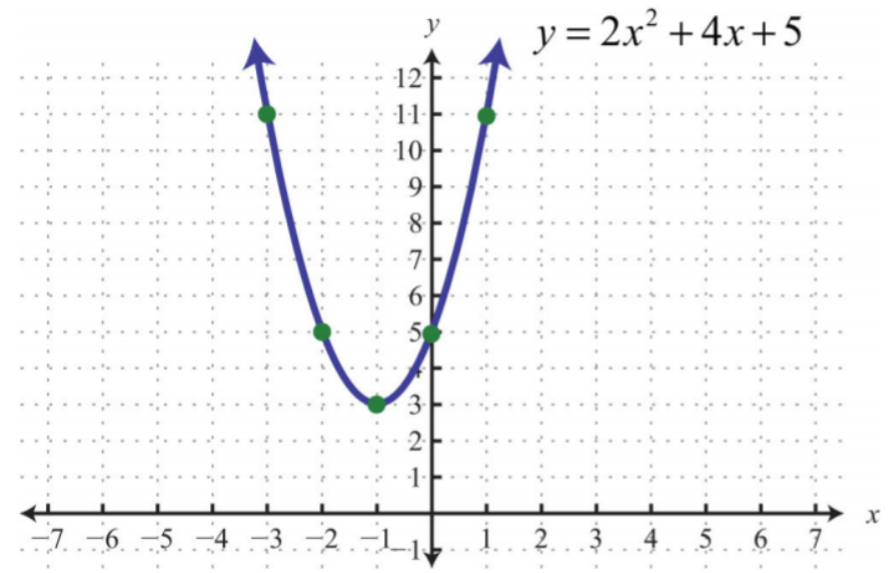


Figure 9.5.9

Example 9.5.4

Graph:

Solution

Note that ***a*** = –2: the parabola opens downward. Since ***c*** = –18, the ***y***-intercept is (0, –18). To find the ***x***-intercepts, set ***y*** = 0.

Solve by factoring.

$$\begin{aligned} x-3=0 \quad &\text{or} \quad x-3=0 \\ x=3 \quad &\qquad\quad x=3 \end{aligned}$$

Here ***x*** = 3 is a double root, so there is only one ***x***-intercept, (3, 0). From the original equation, ***a*** = –2, ***b*** = 12, and ***c*** = –18. The ***x***-value of the vertex can be calculated as follows:

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(12)}{2(-2)} \\ &= \frac{-12}{-4} \\ &= 3 \end{aligned}$$

Given that the ***x***-value of the vertex is 3, substitute into the original equation to find the corresponding ***y***-value.

Therefore, the vertex is (3, 0), which happens to be the same point as the ***x***-intercept. So far, we have only two points. To determine three more, choose some ***x***-values on either side of the line of symmetry, ***x*** = 3 in this case.

Choose ***x***-values 1, 5, and 6.

Figure 9.5.10

To summarize, we have

Table 9.5.3

y-intercept:	(0, -18)
x-intercepts:	(3, 0)
Vertex:	(3, 0)
Extra points:	(1, -8), (5, -8), (6, -18)

Plot the points and sketch the graph.

Answer:

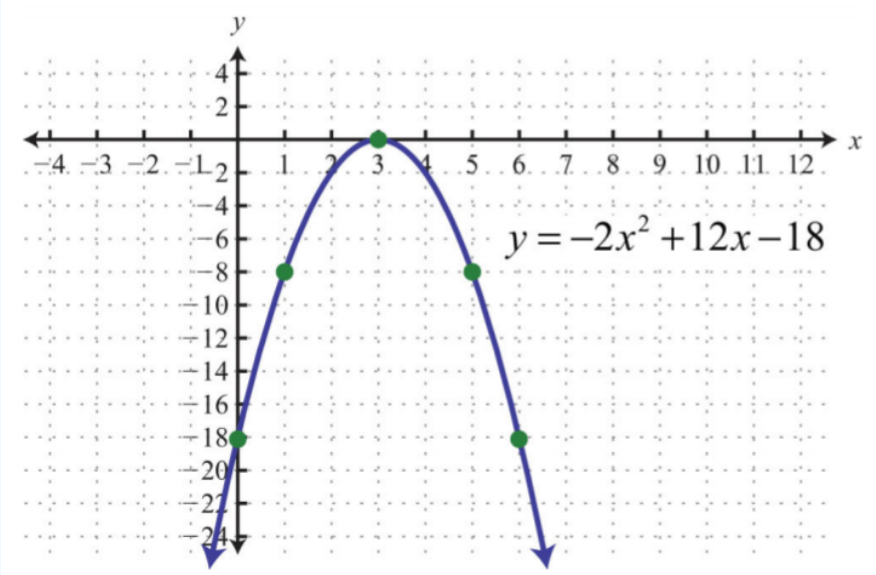


Figure 9.5.11

Example 9.5.5

Graph:

Solution:

Since ***a*** = 1, the parabola opens upward. Furthermore, ***c*** = –1, so the ***y***-intercept is (0, –1). To find the ***x***-intercepts, set ***y*** = 0.

In this case, solve using the quadratic formula with ***a*** = 1, ***b*** = –2, and ***c*** = –1.

Here we obtain two real solutions for ***x***, and thus there are two ***x***-intercepts:

$(1-\sqrt{2},0) \quad \text{and} \quad (1+\sqrt{2},0)$

Approximate values using a calculator:

$$(-0.41,0) \quad \text{and} \quad (2.41,0)$$

Use the approximate answers to place the ordered pair on the graph.

However, we will present the exact ***x***-intercepts on the graph. Next, find the vertex.

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(-2)}{2(1)} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

Given that the ***x***-value of the vertex is 1, substitute into the original equation to find the corresponding ***y***-value.

The vertex is (1, -2). We need one more point.

Figure $9.5.12$

To summarize, we have

Table $9.5.4$

y-intercept:	(0, -1)
x-intercepts:	$(1-\sqrt{2},0)$ and $(1+\sqrt{2},0)$
Vertex:	$(1,-2)$
Extra point:	$(2,-1)$

Plot the points and sketch the graph.

Answer:

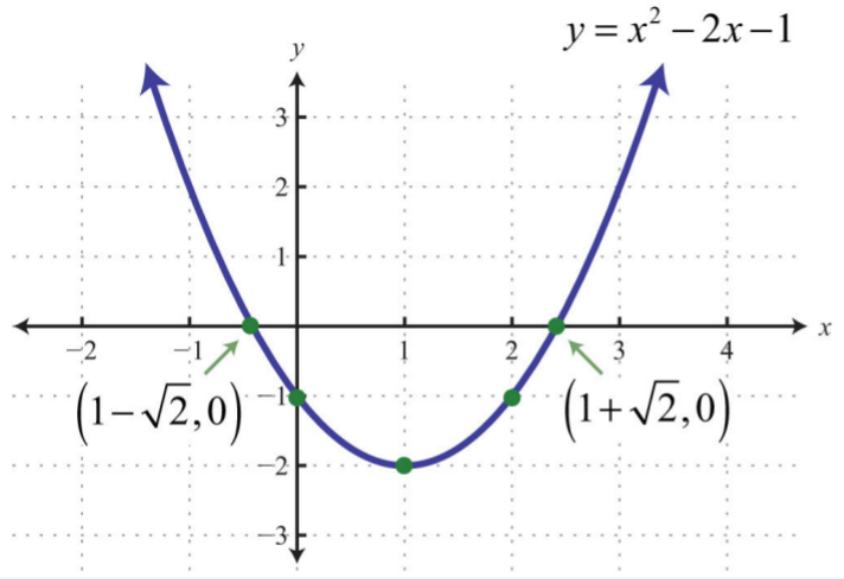


Figure $9.5.13$

Exercise 9.5.1

Graph:

$$y = 9x^2 - 5$$

Answer available at the end of this section.

Finding the Maximum and Minimum

It is often useful to find the maximum and/or minimum values of functions that model real-life applications. To find these important values given a quadratic function, we use the vertex. If the leading coefficient ***a*** is positive, then the parabola opens upward and there will be a minimum ***y***-value. If the leading coefficient ***a*** is negative, then the parabola opens downward and there will be a maximum ***y***-value.

Example 9.5.6

Determine the maximum or minimum:

$$y = -4x^2 + 24x - 35$$

Solution:

Since ***a*** = −4, we know that the parabola opens downward and there will be a maximum ***y***-value. To find it, we first find the ***x***-value of the vertex.

$$\begin{aligned} x &= -\frac{b}{2a} && x - value\ of\ the\ vertex. \\ &= -\frac{24}{2(-4)} && Substitute\ a = -4\ and\ b = 24. \\ &= -\frac{24}{-8} && Simplify. \\ &= 3 \end{aligned}$$

The ***x***-value of the vertex is 3. Substitute this value into the original equation to find the corresponding ***y***-value.

The vertex is (3, 1). Therefore, the maximum ***y***-value is 1, which occurs when ***x*** = 3, as illustrated below:

Figure 9.5.15

Note

The graph is not required to answer this question.

Answer:

The maximum is 1.

Example 9.5.7

Determine the maximum or minimum:

$$y = 4x^2 - 32x + 62$$

Solution:

Since ***a*** = +4, the parabola opens upward and there is a minimum ***y***-value. Begin by finding the ***x***-value of the vertex.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{-32}{2(4)} && \textit{Substitute } a = 4 \textit{ and } b = -32. \\ &= -\frac{-32}{8} && \textit{Simplify.} \\ &= 4 \end{aligned}$$

Substitute ***x*** = 4 into the original equation to find the corresponding ***y***-value.

The vertex is (4, –2). Therefore, the minimum ***y***-value of –2 occurs when ***x*** = 4, as illustrated below:

Figure 9.5.16

Answer:

The minimum is -2.

Exercise 9.5.2

Determine the maximum or minimum:

$$y = (x-3)^2 - 9$$

Answer available at the end of this section.

A parabola, opening upward or downward (as opposed to sideways), defines a function and extends indefinitely to the right and left as indicated by the arrows. Therefore, the domain (the set of ***x***-values) consists of all real numbers. However, the range (the set of ***y***-values) is bounded by the ***y***-value of the vertex.

Example 9.5.8

Determine the domain and range:

$$y = x^2 - 4x + 3$$

Solution:

First, note that since a=1 is positive, the parabola opens upward. Hence there will be a minimum ***y***-value. To find that value, find the ***x***-value of the vertex:

$$x = -\tfrac{b}{2a} = -\tfrac{-4}{2(1)} = 2$$

Then substitute into the equation to find the corresponding ***y***-value.

The vertex is (2, –1). The range consists of the set of ***y***-values greater than or equal to the minimum ***y***-value –1.

Figure 9.5.17

Answer:

Domain: **R** = $(-\infty, \infty)$
; Range: $[-1, \infty)$

Example 9.5.9

The height in feet of a projectile is given by the function $h(t) = -16t^2 + 72t$, where *t* represents the time in seconds after launch. What is the maximum height reached by the projectile?

Solution:

Here a=–16, and the parabola opens downward. Therefore, the ***y***-value of the vertex determines the maximum height. Begin by finding the ***x***-value of the vertex:

$$x = -\frac{b}{2a} = -\frac{72}{2(-16)} = \frac{72}{32} = \frac{9}{4}$$

The maximum height will occur in 9/4 = 2¼ seconds. Substitute this time into the function to determine the height attained.

$$\begin{aligned} h\left(\frac{9}{4}\right) &= -16\left(\frac{9}{4}\right)^2 + 72\left(\frac{9}{4}\right) \\ &= -16\left(\frac{81}{16}\right) + 72\left(\frac{9}{4}\right) \\ &= -81 + 162 \\ &= 81 \end{aligned}$$

Answer:

The maximum height of the projectile is 81 feet.

Finding the Vertex by Completing the Square

In this section, we demonstrate an alternate approach for finding the vertex. Any quadratic equation $y = ax^2 + bx + c$ can be rewritten in the form

$$y = a(x - h)^2 + k \tag{9.5.2}$$

In this form, the vertex is *(h, k)*.

Example 9.5.10

Determine the vertex:

$$y = -4(x - 3)^2 + 1$$

Solution:

When the equation is in this form, we can read the vertex directly from the equation.

$$\begin{aligned} y = \ & a \ (x - h)^2 + k \\ & \downarrow \quad \downarrow \\ y = & -4(x - 3)^2 + 1 \end{aligned}$$

Here ***h***=3 and ***k***=1.

Answer:

The vertex is (3, 1).

Example 9.5.11

Determine the vertex:

$$y = 2(x+3)^2 - 2$$

Solution:

Rewrite the equation as follows before determining ***h*** and ***k***.

$$\begin{array}{rcl} y = & a\ (x & -h)^2 \quad + \quad k \\ & \downarrow & \quad\ \downarrow \\ y = & 2(x & -(-3))^2 + (-2) \end{array}$$

Here ***h***=-3 and ***k***=-2.

Answer:

The vertex is (-3,-2).

Often the equation is not given in this form. To obtain this form, complete the square.

Example 9.5.12

Rewrite in $y = a(x-h)^2 + k$ form and determine the vertex: $y = x^2 + 4x + 9$.

Solution:

Begin by making room for the constant term that completes the square.

$$\begin{aligned} y &= x^2 + 4x + 9 \\ &= x^2 + 4x + \_\_ + 9 - \_\_ \end{aligned}$$

The idea is to add and subtract the value that completes the square, $\frac{b^2}{2}$, and then factor. In this case, add and subtract $\frac{4^2}{2} = 2^2 = 4$.

$$\begin{aligned} y &= x^2 + 4x + 9 && \textit{Add and subtract 4.} \\ &= \underbrace{x^2 + 4x + 4}_{\text{factor}} + 9 - 4 && \textit{Factor.} \\ &= (x+2)(x+2) + 5 \\ &= (x+2)^2 + 5 \end{aligned}$$

Adding and subtracting the same value within an expression does not change it. Doing so is equivalent to adding 0. Once the equation is in this form, we can easily determine the vertex.

$$\begin{array}{rcl} y = & a(x & -h)^2 \ + \ k \\ & \downarrow & \ \downarrow \\ y = & (x & -(-2))^2 + 5 \end{array}$$

Here ***h***=-2 and ***k***=5.

Answer:

The vertex is (-2, 5).

If there is a leading coefficient other than 1, then we must first factor out the leading coefficient from the first two terms of the trinomial.

Example 9.5.13

Rewrite in $y = a(x-h)^2 + k$ form and determine the vertex: $y = 2x^2 - 4x + 8$.

Solution:

Since ***a*** = 2, factor this out of the first two terms in order to complete the square. Leave room inside the parentheses to add a constant term.

Now use –2 to determine the value that completes the square. In this case, $\frac{(-2)^2}{2} = \backslash((-1)^2 = 1$. Add and subtract 1 and factor as follows:

In this form, we can easily determine the vertex.

$$\begin{array}{c} y = a(x-h)^2 + k \\ \downarrow \quad \downarrow \\ y = 2(x-1)^2 + 6 \end{array}$$

Here ***h***=1 and ***k***=6.

Answer:

The vertex is (1, 6).

Exercise 9.5.3

Rewrite in $y = a(x-h)^2 + k$ form and determine the vertex:

$$y = -2x^2 - 12x + 3.$$

Answer available at the end of this section.

Key Takeaways

- The graph of any quadratic equation $y = ax^2 + bx + c$, where *a*, *b*, and *c* are real numbers and $a \neq 0$, is called a parabola.
- When graphing parabolas, find the vertex and *y*-intercept. If the *x*-intercepts exist, find those as well. Also, be sure to find ordered pair solutions on either side of the line of symmetry, $x = \frac{-b^2}{a}$.
- Use the leading coefficient, *a*, to determine if a parabola opens upward or downward. If *a* is positive, then it opens upward. If *a* is negative, then it opens downward.
- The vertex of any parabola has an x-value equal to $x = \frac{-b^2}{a}$. After finding the *x*-value of the vertex, substitute it into the original equation to find the corresponding *y*-value. This *y*-value is a maximum if the parabola opens downward, and it is a minimum if the parabola opens upward.
- The domain of a parabola opening upward or downward consists of all real numbers. The range is bounded by the *y*-value of the vertex.
- An alternate approach to finding the vertex is to rewrite the quadratic equation in the form $y = a(x-h)^2 + k$. When in this form, the vertex is *(h, k)* and can be read directly from the equation. To obtain this form, take $y = ax^2 + bx + c$ and complete the square.

Exercise 9.5.4 the graph of quadratic equations

Does the parabola open upward or downward? Explain.

1. $y = x^2 - 9x + 20$
2. $y = x^2 - 12x + 32$
3. $y = -2x^2 + 5x + 12$
4. $y = -6x^2 + 13x - 6$
5. $y = 64 - x^2$
6. $y = -3x + 9x^2$

Answer available at the end of this section.

Exercise 9.5.5 the graph of quadratic equations

Determine the *x*- and *y*-intercepts.

1. $y = x^2 + 4x - 12$
2. $y = x^2 - 13x + 12$
3. $y = 2x^2 + 5x - 3$
4. $y = 3x^2 - 4x - 4$
5. $y = -5x^2 - 3x + 2$
6. $y = -6x^2 + 11x - 4$
7. $y = 4x^2 - 25$
8. $y = 9x^2 - 49$
9. $y = x^2 - x + 1$
10. $y = 5x^2 + 15x$

Answer available at the end of this section.

Exercise 9.5.6 the graph of quadratic equations

Find the vertex and the line of symmetry.

1. $y = -x^2 + 10x - 34$
2. $y = -x^2 - 6x + 1$
3. $y = -4x^2 + 12x - 7$
4. $y = -9x^2 + 6x + 2$
5. $y = 4x^2 - 1$
6. $y = x^2 - 16$

Answer available at the end of this section.

Exercise 9.5.7 the graph of quadratic equations

Graph. Find the vertex and the *y*-intercept. In addition, find the *x*-intercepts if they exist.

1. $y = x^2 - 2x - 8$
2. $y = x^2 - 4x - 5$
3. $y = -x^2 + 4x + 12$
4. $y = -x^2 - 2x + 15$
5. $y = x^2 - 10x$
6. $y = x^2 + 8x$

7. $y = x^2 - 9$
8. $y = x^2 - 25$
9. $y = 1 - x^2$
10. $y = 4 - x^2$
11. $y = x^2 - 2x + 1$
12. $y = x^2 + 4x + 4$
13. $y = -4x^2 + 12x - 9$
14. $y = -4x^2 - 4x + 3$
15. $y = x^2 - 2$
16. $y = x^2 - 3$
17. $y = -4x^2 + 4x - 3$
18. $y = 4x^2 + 4x + 3$
19. $y = x^2 - 2x - 2$
20. $y = x^2 - 6x + 6$
21. $y = -2x^2 + 6x - 3$
22. $y = -4x^2 + 4x + 1$
23. $y = x^2 + 3x + 4$
24. $y = -x^2 + 3x - 4$
25. $y = -2x^2 + 3$
26. $y = -2x^2 - 1$
27. $y = 2x^2 + 4x - 3$
28. $y = 3x^2 + 2x - 2$

Answer available at the end of this section.

Exercise 9.5.8 maximum or minimum

Determine the maximum or minimum *y*-value.

1. $y = -x^2 - 6x + 1$
2. $y = -x^2 - 4x + 8$
3. $y = 25x^2 - 10x + 5$
4. $y = 16x^2 - 24x + 7$
5. \(y=–x^{2}
6. $y = 1 - 9x^2$
7. $y = 20x - 10x^2$
8. $y = 12x + 4x^2$
9. $y = 3x^2 - 4x - 2$
10. $y = 6x^2 - 8x + 5$

Answer available at the end of this section.

Exercise 9.5.9 maximum or minimum

Given the following quadratic functions, determine the domain and range.

1. $f(x) = 3x^2 + 30x + 50$
2. $f(x) = 5x^2 - 10x + 1$
3. $g(x) = -2x^2 + 4x + 1$
4. $g(x) = -7x^2 - 14x - 9$
5. The height in feet reached by a baseball tossed upward at a speed of 48 feet/second from the ground is given by the function $h(t) = -16t^2 + 48t$, where *t* represents time in seconds. What is the baseball's maximum height and how long will it take to attain that height?
6. The height of a projectile launched straight up from a mound is given by the function $h(t) = -16t^2 + 96t + 4$, where *t* represents seconds after launch. What is the maximum height?

7. The profit in dollars generated by producing and selling x custom lamps is given by the function $P(x) = -10x^2 + 800x - 12,000$. What is the maximum profit?
8. The revenue in dollars generated from selling a particular item is modeled by the formula $R(x) = 100x - 0.0025x^2$, where x represents the number of units sold. What number of units must be sold to maximize revenue?
9. The average number of hits to a radio station website is modeled by the formula $f(x) = 450t^2 - 3,600t + 8,000$, where t represents the number of hours since 8:00 a.m. At what hour of the day is the number of hits to the website at a minimum?
10. The value in dollars of a new car is modeled by the formula $V(t) = 125t^2 - 3,000t + 22,000$, where t represents the number of years since it was purchased. Determine the minimum value of the car.
11. The daily production costs in dollars of a textile manufacturing company producing custom uniforms is modeled by the formula $C(x) = 0.02x^2 - 20x + 10,000$, where x represents the number of uniforms produced.
    a. How many uniforms should be produced to minimize the daily production costs?
    b. What is the minimum daily production cost?
12. The area of a certain rectangular pen is given by the formula $A = 14w - w^2$, where w represents the width in feet. Determine the width that produces the maximum area.

Answer available at the end of this section.

Exercise 9.5.10 vertex by completing the square

Determine the vertex.

1. $y = -(x-5)^2 + 3$
2. $y = -2(x-1)^2 + 7$
3. $y = 5(x+1)^2 + 6$
4. $y = 3(x+4)^2 + 10$
5. $y = -5(x+8)^2 - 1$
6. $y = (x+2)^2 - 5$

Answer available at the end of this section.

Exercise 9.5.11 vertex by completing the square

Rewrite in $y = a(x-h)^2 + k$ form and determine the vertex.

1. $y = x^2 - 14x + 24$
2. $y = x^2 - 12x + 40$
3. $y = x^2 + 4x - 12$
4. $y = x^2 + 6x - 1$
5. $y = 2x^2 - 12x - 3$
6. $y = 3x^2 - 6x + 5$
7. $y = -x^2 + 16x + 17$
8. $y = -x^2 + 10x$

Answer available at the end of this section.

Exercise 9.5.12 vertex by completing the square

Graph.

1. $y = x^2 - 1$
2. $y = x^2 + 1$
3. $y = (x-1)^2$
4. $y = (x+1)^2$

5. $y = (x-4)^2 - 9$
6. $y = (x-1)^2 - 4$
7. $y = -2(x+1)^2 + 8$
8. $y = -3(x+2)^2 + 12$
9. $y = -5(x-1)^2$
10. $y = -(x+2)^2$
11. $y = -4(x-1)^2 - 2$
12. $y = 9(x+1)^2 + 2$
13. $y = (x+5)^2 - 15$
14. $y = 2(x-5)^2 - 3$
15. $y = -2(x-4)^2 + 22$
16. $y = 2(x+3)^2 - 13$

Answer available at the end of this section.

Exercise 9.5.13 discussion board

1. Write down your plan for graphing a parabola on an exam. What will you be looking for and how will you present your answer? Share your plan on the discussion board.
2. Why is any parabola that opens upward or downward a function? Explain to a classmate how to determine the domain and range.

Answer available at the end of this section.

Answers

Exercise 9.5.1

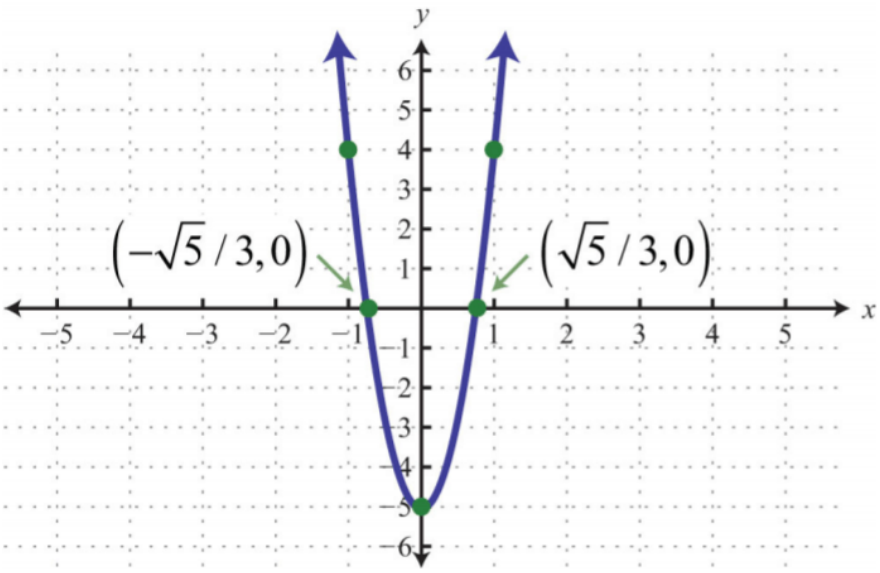


Figure 9.5.14

Exercise 9.5.2

The minimum is –9.

Exercise 9.5.3

$y = -2(x+3)^2 + 21$; vertex: $(-3, 21)$

Exercise 9.5.4 **the graph of quadratic equations**

1. Upward
3. Downward
5. Downward

Exercise 9.5.5 **the graph of quadratic equations**

1. *x*-intercepts: $(-6,0),(2,0)$; *y*-intercept: $(0,-12)$
3. *x*-intercepts: $(-3,0),(\frac{1}{2},0)$; *y*-intercept: $(0,-3)$
5. *x*-intercepts: $(-1,0),(\frac{2}{5},0)$; *y*-intercept: $(0,2)$
7. *x*-intercepts: $(-\frac{5}{2},0),(\frac{5}{2},0)$; *y*-intercept: $(0,-25)$
9. *x*-intercepts: none; *y*-intercept: $(0,1)$

Exercise $9.5.6$ **the graph of quadratic equations**

1. Vertex: $(5,-9)$; line of symmetry: $x=5$
3. Vertex: $(\frac{3}{2},2)$; line of symmetry: $x=\frac{3}{2}$
5. Vertex: $(0,-1)$; line of symmetry: $x=0$

Exercise $9.5.7$ **the graph of quadratic equations**

1.

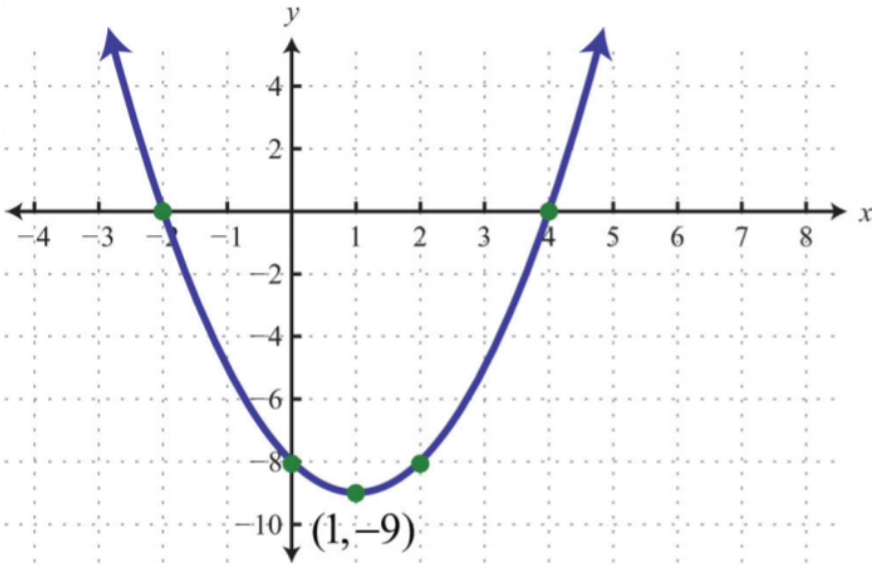


Figure $9.5.18$

3.

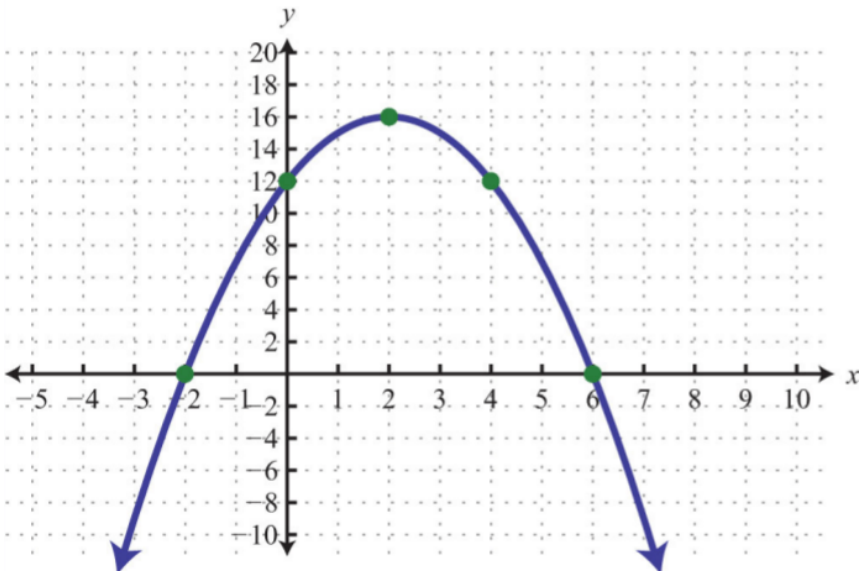


Figure $9.5.19$

5.

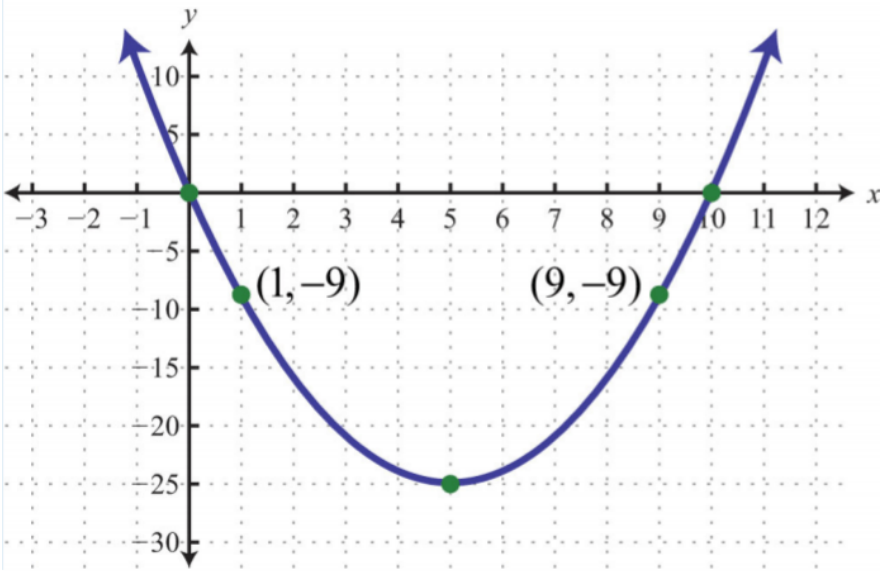


Figure 9.5.20

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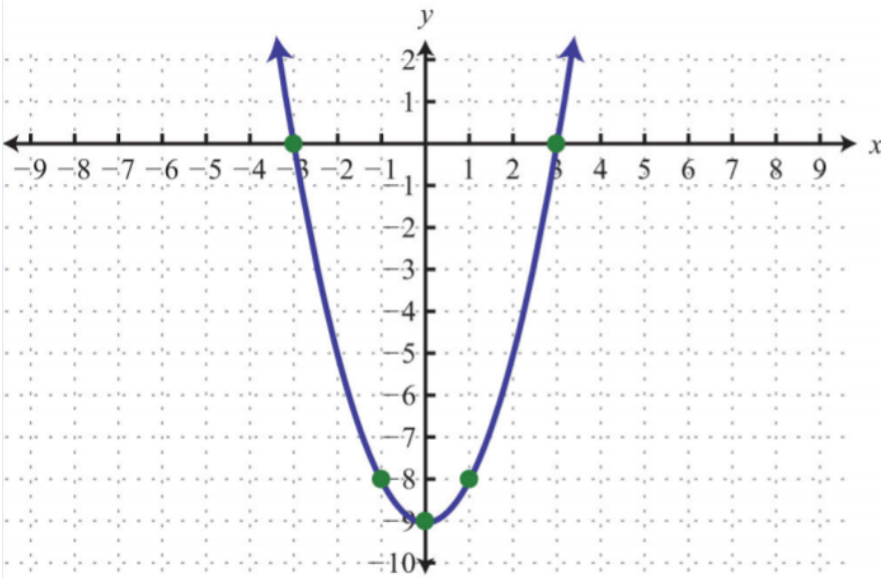


Figure 9.5.21

9.

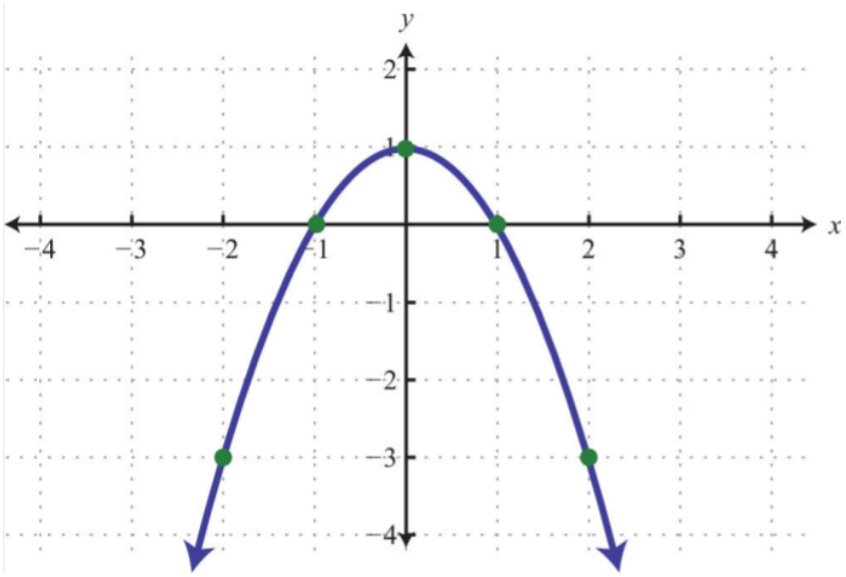


Figure 9.5.22

11.

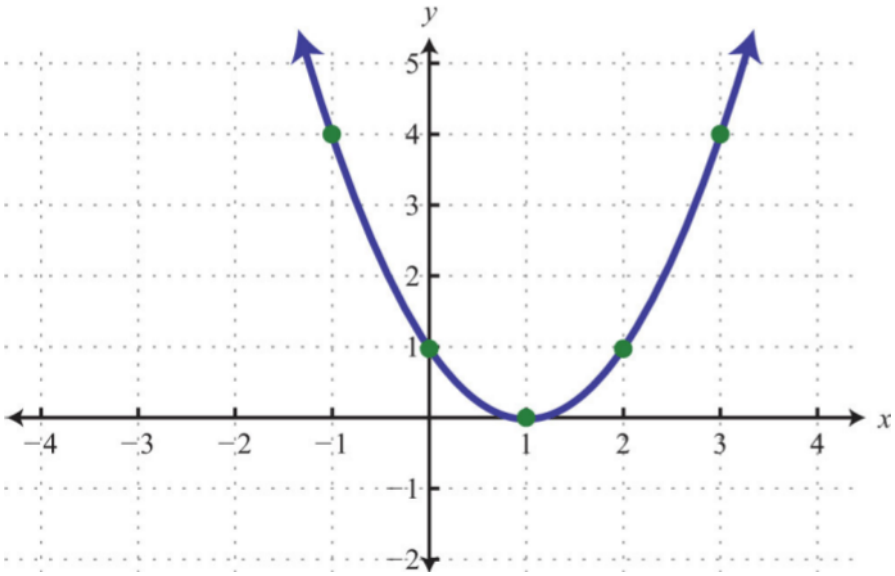


Figure 9.5.23

13.

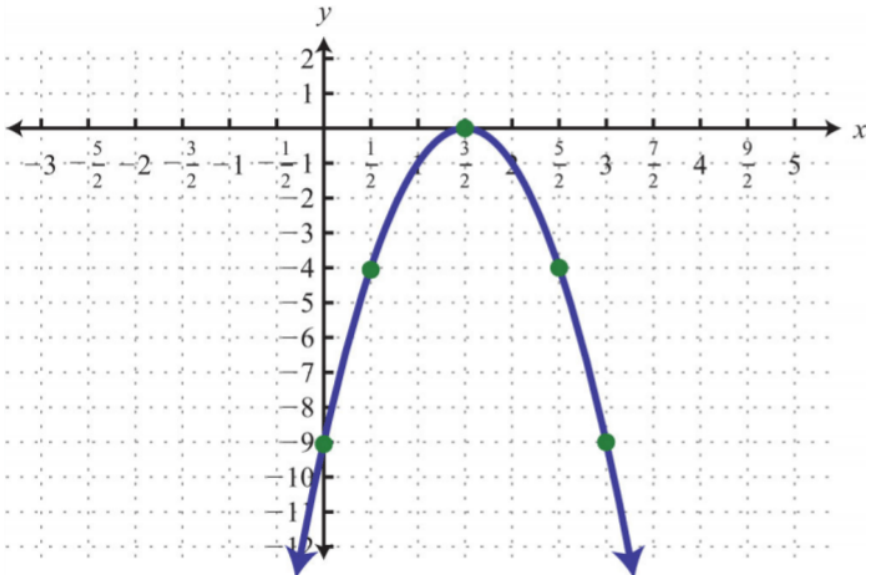


Figure 9.5.24

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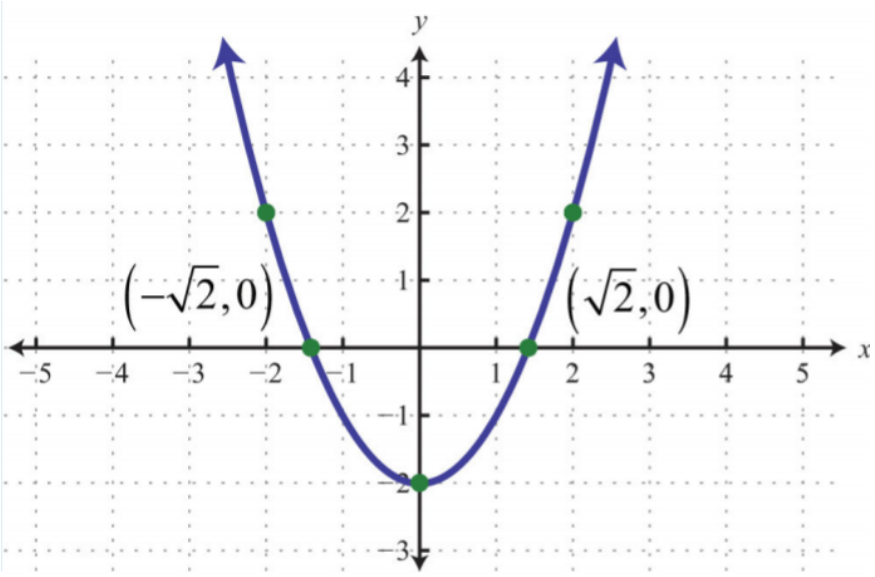


Figure 9.5.25

17.

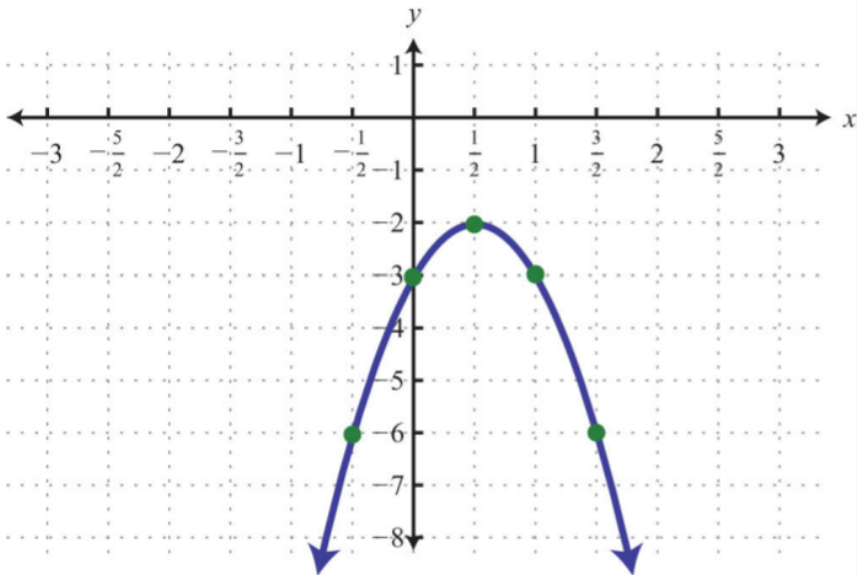


Figure 9.5.26

19.

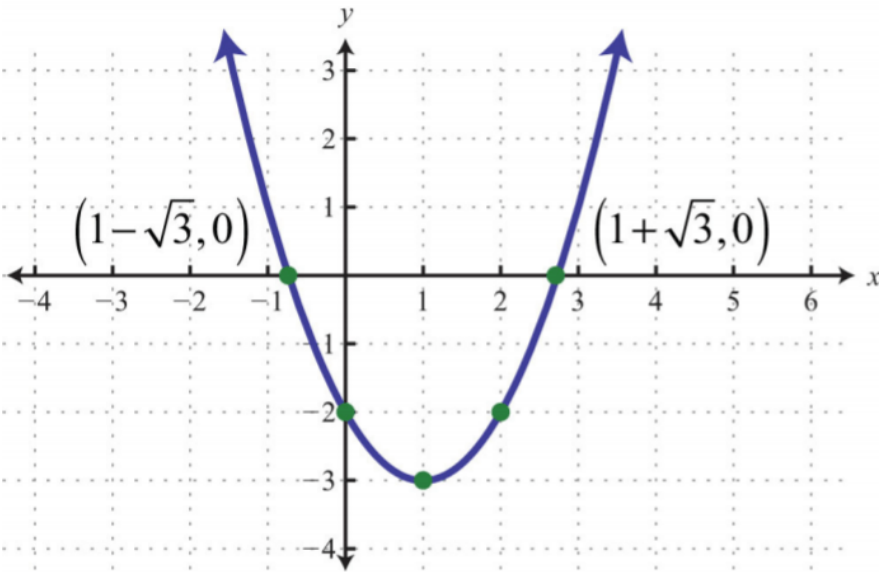


Figure 9.5.27

21.

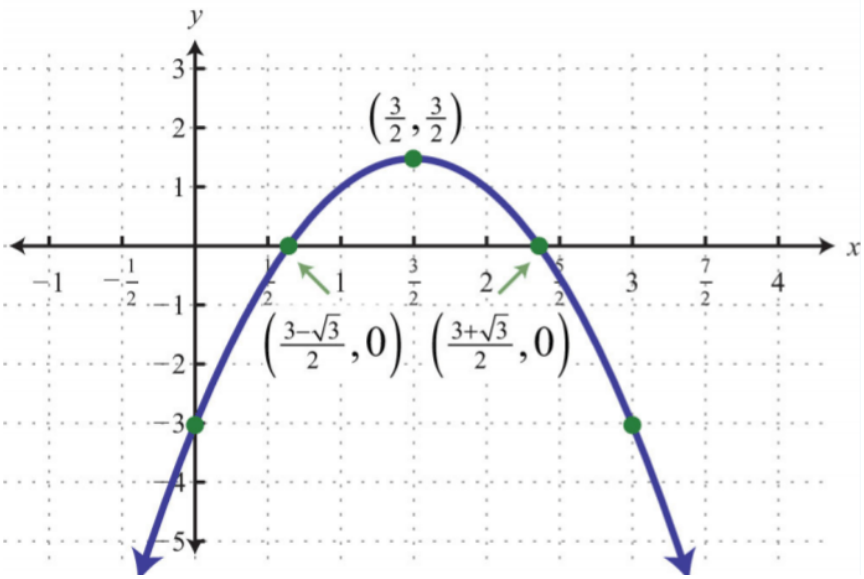


Figure 9.5.28

23.

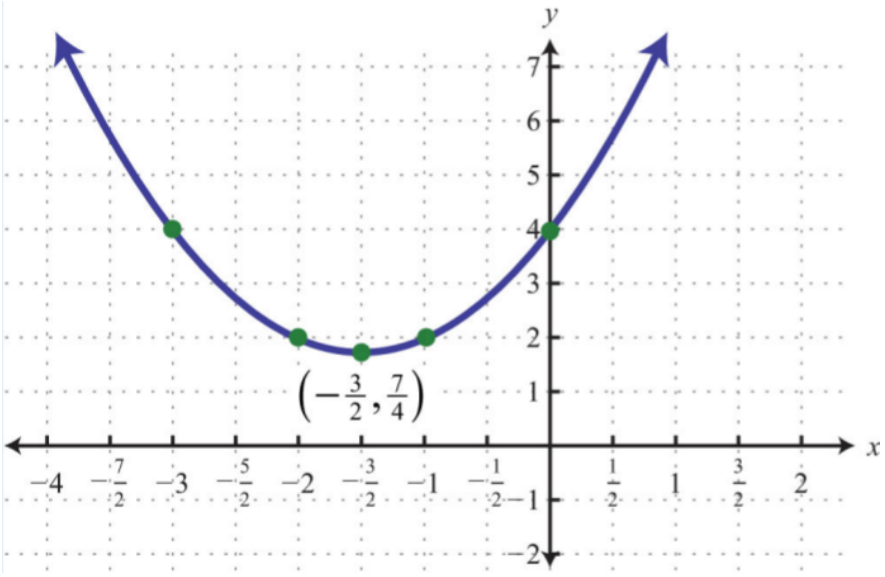


Figure 9.5.29

25.

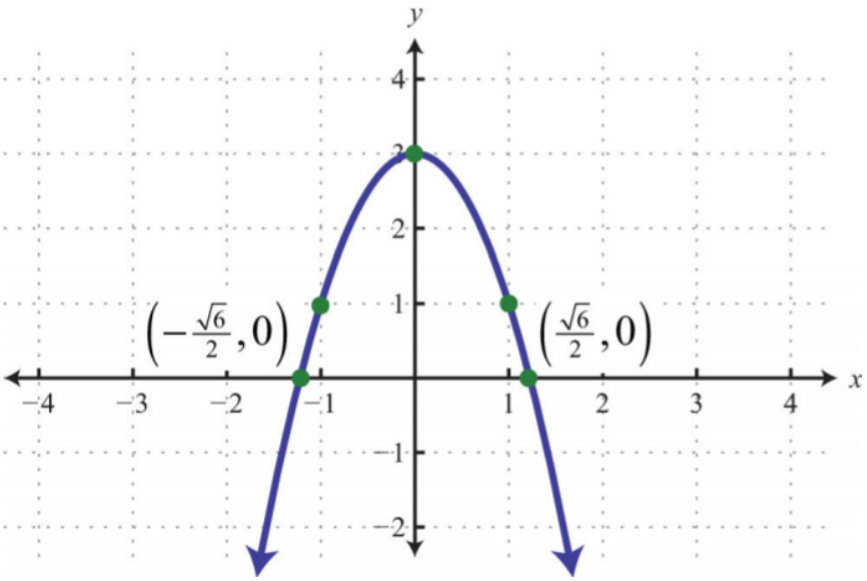


Figure 9.5.30

27.

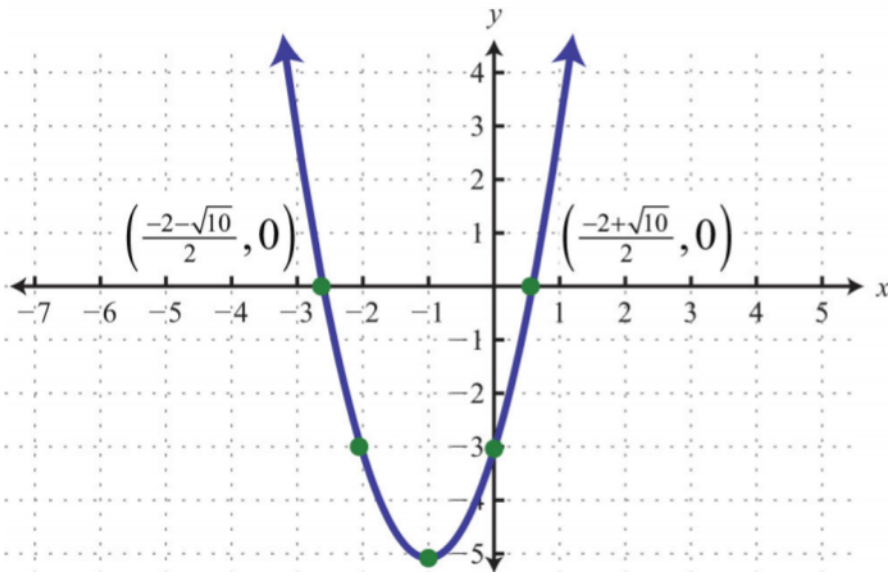


Figure 9.5.31

Exercise 9.5.8 **maximum or minimum**

1. Maximum: $y = 10$
3. Minimum: $y = 4$
5. Maximum: $y = 0$
7. Maximum: $y = 10$
9. Minimum: $y = -\frac{10}{3}$

Exercise 9.5.9 **maximum or minimum**

1. Domain: R; range: $[-25, \infty)$
3. Domain: R; range: $(-\infty, 3]$
5. The maximum height of 36 feet occurs after 1.5 seconds.
7. $4,000
9. 12:00 p.m.
11. a. 500 uniforms; b. $5,000

Exercise 9.5.10 **vertex by completing the square**

1. $(5, 3)$
3. $(-1, 6)$
5. $(-8, -1)$

Exercise 9.5.11 **vertex by completing the square**

1. $y = (x-7)^2 - 25$; vertex: $(7, -25)$
3. $y = (x+2)^2 - 16$; vertex: $(-2, -16)$
5. $y = 2(x-3)^2 - 21$; vertex: $(3, -21)$
7. $y = -(x-8)^2 + 81$; vertex: $(8, 81)$

Exercise $9.5.12$ **vertex by completing the square**

1.

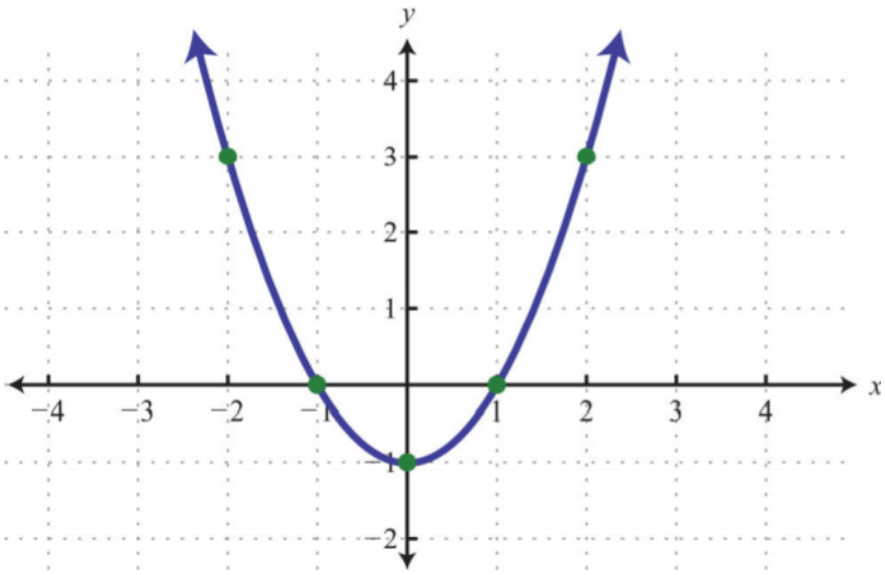


Figure $9.5.32$

3.

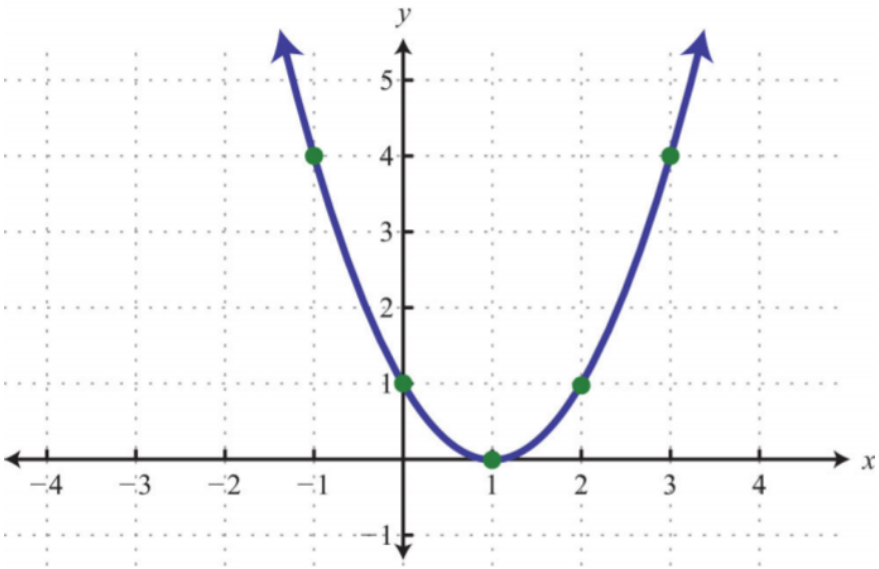


Figure $9.5.33$

5.

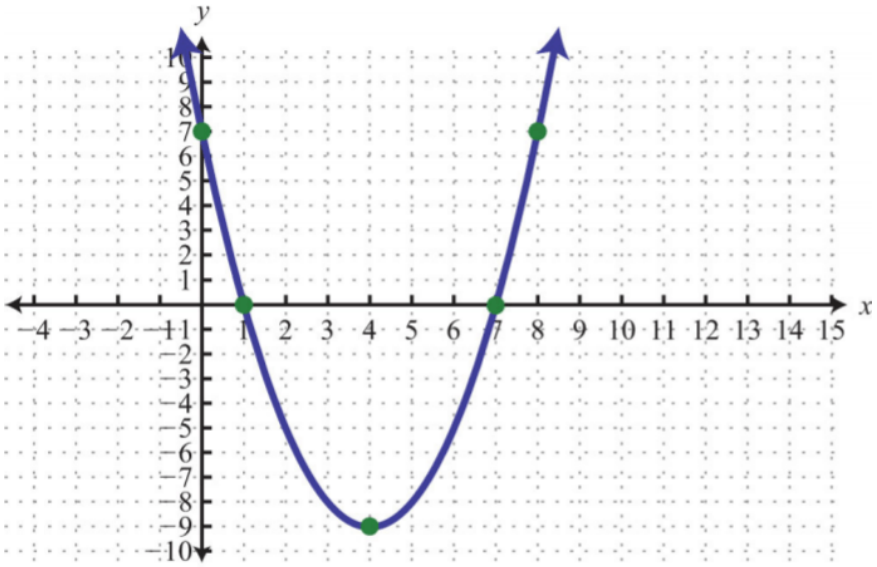


Figure 9.5.34

7.

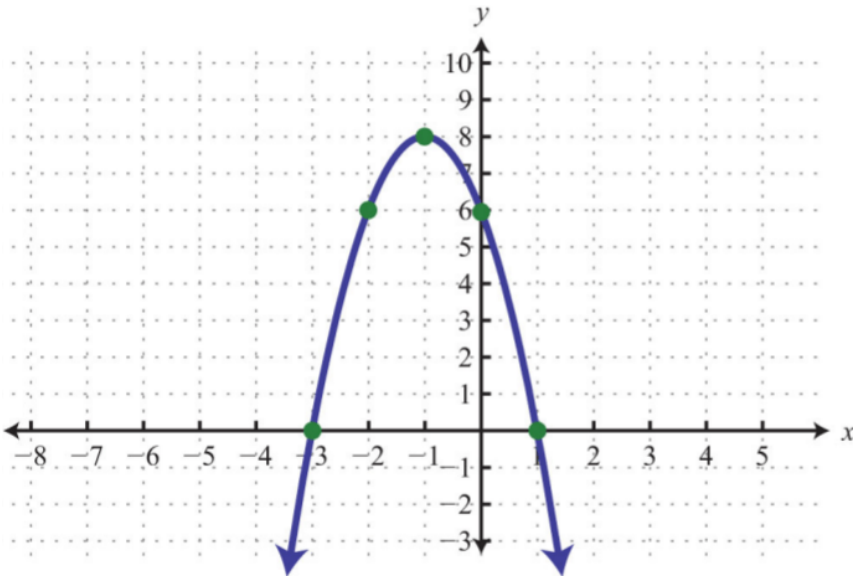


Figure 9.5.35

9.

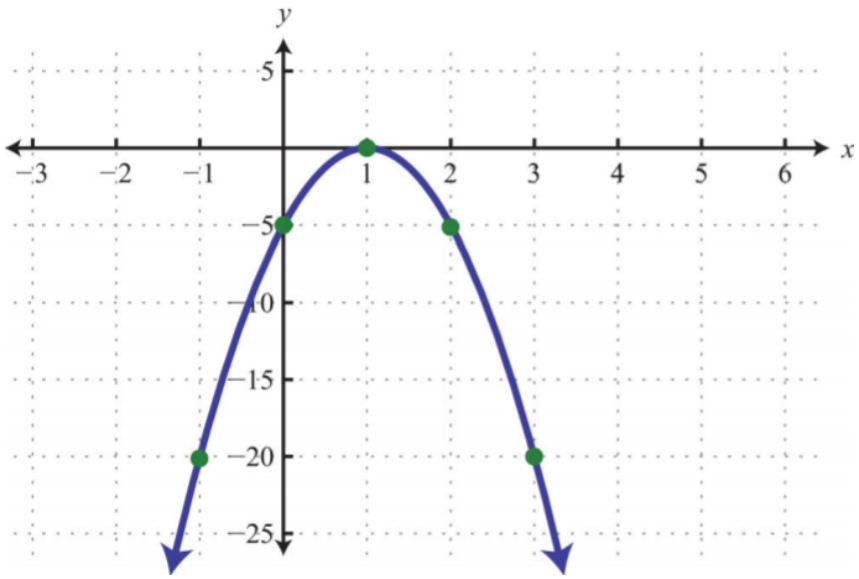


Figure 9.5.36

11.

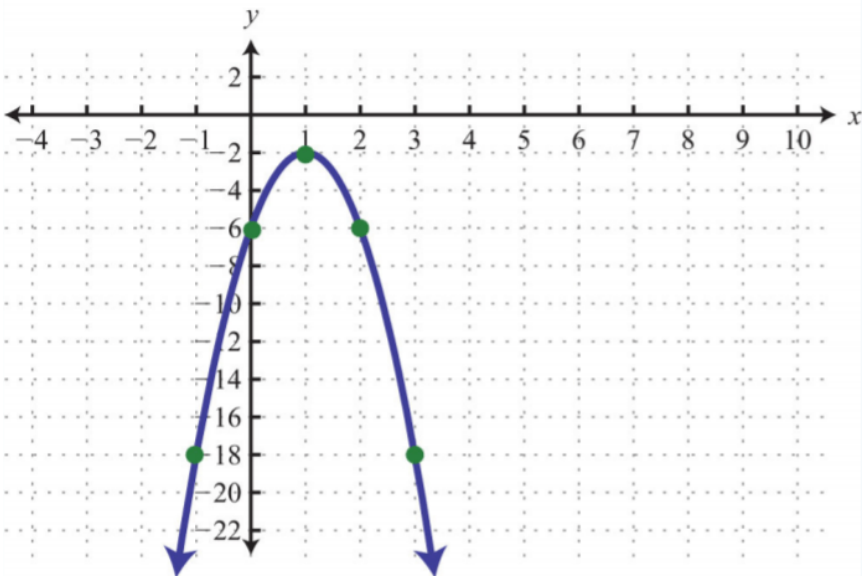


Figure 9.5.37

13.

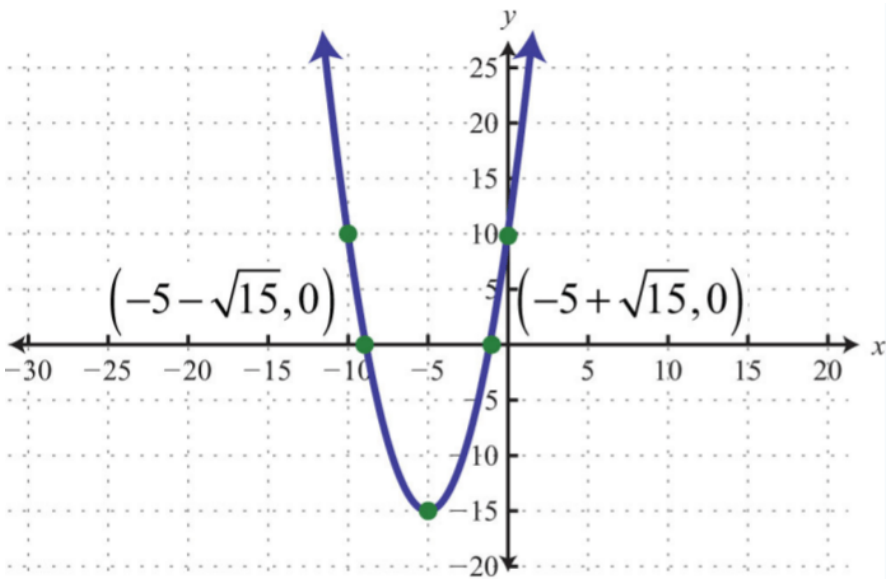


Figure 9.5.38

15.

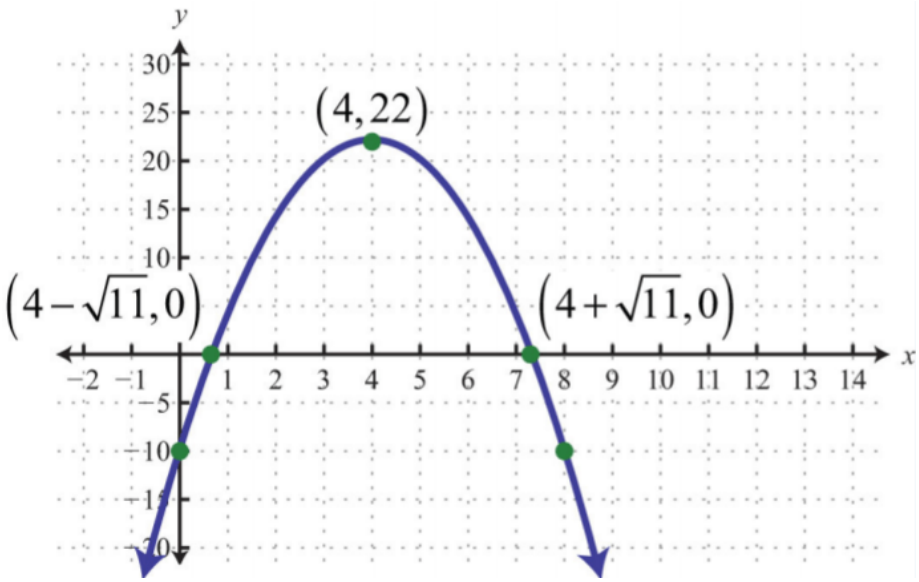


Figure 9.5.39

Exercise 9.5.13 **discussion board**

1. Answers may vary

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9.6: Introduction to Complex Numbers and Complex Solutions

Learning Objectives

- Perform operations with complex numbers.
- Solve quadratic equations with complex solutions.

Introduction to Complex Numbers

Up to this point, the square root of a negative number has been left undefined. For example, we know that $\sqrt{-9}$ is not a real a number.

$$\sqrt{-9} = ? \quad \text{or} \quad (?)^2 = -9$$

There is no real number that when squared results in a negative number. We begin the resolution of this issue by defining the imaginary unit, i, as the square root of –1.

$$i = \sqrt{-1} \quad \text{and} \quad i^2 = -1 \tag{9.6.1}$$

To express a square root of a negative number in terms of the imaginary unit ***i***, we use the following property, where ***a*** represents any nonnegative real number:

With this we can write

If $\sqrt{-9} = 3i$, then we would expect that 3***i*** squared equals: -9:

Therefore, the square root of any negative real number can be written in terms of the imaginary unit. Such numbers are often called imaginary numbers.

Rewrite in terms of the imaginary unit ***i***.

a. $\sqrt{-4}$
b. $\sqrt{-5}$
c. $\sqrt{-8}$

Solution:

a. $\sqrt{-4} = \sqrt{-1 \cdot 4} = \sqrt{-1} \cdot \sqrt{4} = i \cdot 2 = 2i$

b. $\sqrt{-5} = \sqrt{-1 \cdot 5} = \sqrt{-1} \cdot \sqrt{5} = i \cdot \sqrt{5} = i\sqrt{5}$

c. $\sqrt{-8} = \sqrt{-1 \cdot 4 \cdot 2} = \sqrt{-1} \cdot \sqrt{4}\sqrt{2} = i \cdot 2 \cdot \sqrt{2} = 2i\sqrt{2}$

Note

When an imaginary number involves a radical, place i in front of the radical. Consider the following:

$$2i\sqrt{2} = 2\sqrt{2}i$$

Since multiplication is commutative, these numbers are equivalent. However, in the form $2\sqrt{2}i$, the imaginary unit ***i*** is often misinterpreted to be part of the radicand. To avoid this confusion, it is a best practice to place the ***i*** in front of the radical and use $2i\sqrt{2}$.

A complex number is any number of the form

$$a + bi \tag{9.6.2}$$

where ***a*** and ***b*** are real numbers. Here ***a*** is called the real part and ***b*** is called the imaginary part. For example, $3 - 4i$ is a complex number with a real part, 3, and an imaginary part, –4. It is important to note that any real number is also a complex number. For example, the real number 5 is also a complex number because it can be written as $5 + 0i$ with a real part of 5 and an imaginary part of 0. Hence the set of real numbers, denoted **R**, is a subset of the set of complex numbers, denoted **C**.

Adding and subtracting complex numbers is similar to adding and subtracting like terms. Add or subtract the real parts and then the imaginary parts.

Example 9.6.2

Add:

$$(3 - 4i) + (2 + 5i)$$

Solution:

Add the real parts and then add the imaginary parts.

Answer:

$5 + i$

To subtract complex numbers, subtract the real parts and subtract the imaginary parts. This is consistent with the use of the distributive property.

Subtract:

$$(3 - 4i) - (2 + 5i)$$

Solution:

Distribute the negative one and then combine like terms

Answer:

$1 - 9i$

The distributive property also applies when multiplying complex numbers. Make use of the fact that $i^2 = -1$ to resolve the result into standard form: $a + bi$.

Multiply:

$$5i(3 - 4i)$$

Solution:

Begin by applying the distributive property.

Answer:

$20 + 15i$

Multiply:

$$(3 - 4i)(4 + 5i)$$

Solution

Answer:

$32 - i$

Given a complex number $a + bi$, its complex conjugate is $a - bi$. We next explore the product of complex conjugates.

Multiply:

$$(3 - 4i)(3 + 4i)$$

Solution:

Answer:

25

In general, the product of complex conjugates follows:

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - a \cdot bi + bi \cdot a - b^2 i^2 \\ &= a^2 - abi + abi - b^2(-1) \\ &= a^2 + b^2\end{aligned}$$

Note that the result does not involve the imaginary unit; hence the result is real. This leads us to the very useful property:

$$(a + bi)(a - bi) = a^2 + b^2 \tag{9.6.3}$$

To divide complex numbers, we apply the technique used to rationalize the denominator. Multiply the numerator and denominator (dividend and divisor) by the conjugate of the denominator. The result can then be resolved into standard form, $a + bi$.

Example 9.6.7

Divide:

$$\frac{1}{1-2i}$$

Solution:

In this example, the conjugate of the denominator is $1 + 2i$. Multiply by 1 in the form $\frac{(1+2i)}{(1+2i)}$.

To express this complex number in standard form, write each term over the common denominator 5.

$$\begin{aligned}\frac{1+2i}{5} &= \frac{1}{5}+\frac{2i}{5}\\ &= \frac{1}{5}+\frac{2}{5}i\end{aligned}$$

Answer:

$\frac{1}{5}+\frac{2}{5}i$

Example 9.6.8

Divide:

$$\frac{3-4i}{3+2i}.$$

Solution:

Answer:

$\frac{1}{13}-\frac{18}{13}i$

Exercise 9.6.1

Divide:

$$\frac{5+5i}{1-3i}$$

Answer available at the end of this section.

Quadratic Equations with Complex Solutions

Now that complex numbers are defined, we can complete our study of solutions to quadratic equations. Often solutions to quadratic equations are not real.

Example 9.6.9

Solve using the quadratic formula:

$$x^2-2x+5=0$$

Solution:

Begin by identifying ***a***, ***b***, and ***c***. Here

Substitute these values into the quadratic formula and then simplify.

Check these solutions by substituting them into the original equation.

$Check\ x = 1 - 2i$	$Check\ x = 1 + 2i$
$x^2 - 2x + 5 = 0$	$x^2 - 2x + 5 = 0$
$(1-2i)^2 - 2(1-2i) + 5 = 0$	$(1+2i)^2 - 2(1+2i) + 5 = 0$
$1 - 4i + 4i^2 - 2 + 4i + 5 = 0$	$1 + 4i + 4i^2 - 2 - 4i + 5 = 0$
$4i^2 + 4 = 0$	$4i^2 + 4 = 0$
$4 - 1 + 4 = 0$	$4 - 1 + 4 = 0$
$-4 + 4 = 0$ ✓	$-4 + 4 = 0$ ✓

Answer:

The solutions are $1 - 2i$ and $1 + 2i$.

The equation may not be given in standard form. The general steps for solving using the quadratic formula are outlined in the following example.

Example 9.6.10

Solve:

$$(2x + 1)(x - 3) = x - 8$$

Solution:

Step 1: Write the quadratic equation in standard form.

Step 2: Identify ***a***, ***b***, and ***c*** for use in the quadratic formula. Here

Step 3: Substitute the appropriate values into the quadratic formula and then simplify.

Answer:

The solution is $\frac{3}{2} \pm \frac{1}{2}i$. The check is optional.

Solve:

$$x(x + 2) = -19$$

Solution:

Begin by rewriting the equation in standard form.

Here *a=1*, *b=2*, and *c=19*. Substitute these values into the quadratic formula.

Answer:

The solutions are $-1 - 3i\sqrt{2}$ and $-1 + 3i\sqrt{2}$.

Note

Consider the following:

Both numbers are equivalent and $-1+3\sqrt{2}i$ is in standard form, where the real part is –1 and the imaginary part is $3\sqrt{2}$. However, this number is often expressed as $-1+3i\sqrt{2}$, even though this expression is not in standard form. Again, this is done to avoid the possibility of misinterpreting the imaginary unit as part of the radicand.

Solve:

$$(2x+3)(x+5)=5x+4$$

Answer

$-4\pm 6i\sqrt{2}=-2\pmi\sqrt{\frac{3}{2}}$

Key Takeaways

- The result of adding, subtracting, multiplying, and dividing complex numbers is a complex number.
- Use complex numbers to describe solutions to quadratic equations that are not real.

Exercise 9.6.3 introduction to complex numbers

Rewrite in terms of *i*.

1. $\sqrt{-64}$
2. $\sqrt{-81}$
3. $\sqrt{-20}$
4. $\sqrt{-18}$
5. $\sqrt{-50}$
6. $\sqrt{-48}$
7. $\sqrt{-45}$
8. $\sqrt{-8}$
9. $\sqrt{-14}$
10. $\sqrt{-29}$

Answer available at the end of this section.

Exercise 9.6.4 introduction to complex numbers

Perform the operations.

1. $(3+5i)+(7-4i)$
2. $(6-7i)+(-5-2i)$
3. $(-8-3i)+(5+2i)$
4. $(-10+15i)+(15-20i)$
5. $(12+34i)+(16-18i)$
6. $(25-16i)+(110-32i)$
7. $(5+2i)-(8-3i)$
8. $(7-i)-(-6-9i)$
9. $(-9-5i)-(8+12i)$
10. $(-11+2i)-(13-7i)$
11. $(114+32i)-(47-34i)$
12. $(38-13i)-(12-12i)$
13. $2i(7-4i)$

14. $6i(1-2i)$
15. $-2i(3-4i)$
16. $-5i(2-i)$
17. $(2+i)(2-3i)$
18. $(3-5i)(1-2i)$
19. $(1-i)(8-9i)$
20. $(1+5i)(5+2i)$
21. $(4+3i)^2$
22. $(2-5i)^2$
23. $(4-2i)(4+2i)$
24. $(6+5i)(6-5i)$
25. $(12+23i)(13-12i)$
26. $(23-13i)(12-32i)$
27. $15+4i$
28. $13-4i$
29. $\frac{20i}{1-3i}$
30. $\frac{10i}{1-2i}$
31. $\frac{10-5i}{3-i}$
32. $\frac{4-2i}{2-2i}$
33. $\frac{5+10i}{3+4i}$
34. $\frac{2-4i}{5+3i}$
35. $\frac{1+2i}{2-3i}$
36. $\frac{3-i}{4-5i}$

Answer available at the end of this section.

Exercise 9.6.5 complex roots

Solve by extracting the roots and then solve by using the quadratic formula. Check answers.

1. $x^2+9=0$
2. $x^2+1=0$
3. $4t^2+25=0$
4. $9t^2+4=0$
5. $4y^2+3=0$
6. $9y^2+5=0$
7. $3x^2+2=0$
8. $5x^2+3=0$
9. $(x+1)^2+4=0$
10. $(x+3)^2+9=0$

Answer available at the end of this section.

Exercise 9.6.6 complex roots

Solve using the quadratic formula.

1. $x^2-2x+10=0$
2. $x^2-4x+13=0$
3. $x^2+4x+6=0$
4. $x^2+2x+9=0$
5. $y^2-6y+17=0$
6. $y^2-2y+19=0$
7. $t^2-5t+10=0$

8. $t^2 + 3t + 4 = 0$
9. $-x^2 + 10x - 29 = 0$
10. $-x^2 + 6x - 10 = 0$
11. $-y^2 - y - 2 = 0$
12. $-y^2 + 3y - 5 = 0$
13. $-2x^2 + 10x - 17 = 0$
14. $-8x^2 + 20x - 13 = 0$
15. $3y^2 - 2y + 4 = 0$
16. $5y^2 - 4y + 3 = 0$
17. $2x^2 + 3x + 2 = 0$
18. $4x^2 + 2x + 1 = 0$
19. $2x^2 - 12x + 14 = 0$
20. $3x^2 - 23x + 13 = 0$
21. $2x(x - 1) = -1$
22. $x(2x + 5) = 3x - 5$
23. $3t(t - 2) + 4 = 0$
24. $5t(t - 1) = t - 4$
25. $(2x + 3)^2 = 16x + 4$
26. $(2y + 5)^2 - 12(y + 1) = 0$
27. $-3(y + 3)(y - 5) = 5y + 46$
28. $-2(y - 4)(y + 1) = 3y + 10$
29. $9x(x - 1) + 3(x + 2) = 1$
30. $5x(x + 2) - 6(2x - 1) = 5$
31. $3(t - 1) - 2t(t - 2) = 6t$
32. $3(t - 3) - t(t - 5) = 7t$
33. $(2x + 3)(2x - 3) - 5(x^2 + 1) = -9$
34. $5(x + 1)(x - 1) - 3x^2 = -8$

Answer available at the end of this section.

Exercise 9.6.7 discussion board

1. Explore the powers of *i*. Share your discoveries on the discussion board.
2. Research and discuss the rich history of imaginary numbers.
3. Research and discuss real-world applications involving complex numbers.

Answer available at the end of this section.

Answers

Exercise 9.6.1

$-1 + 2i$

Exercise 9.6.3 **introduction to complex numbers**

1. $8i$
3. $2i\sqrt{5}$
5. $5i\sqrt{2}$
7. $-3i\sqrt{5}$
9. $i\sqrt{14}$

Exercise 9.6.4 **introduction to complex numbers**

1. $10 + i$
3. $-3 - i$
5. $28 + 16i$
7. $-3 + 5i$
9. $-17 - 17i$
11. $67 + 66i$
13. $8 + 14i$
15. $-8 - 6i$

17. $7-4i$
19. $-1-17i$
21. $7+24i$
23. 20
25. $-140-892i$
27. $15+4i$
29. $-6+2i$
31. $\frac{7-i}{2}$
33. $\frac{11+2i}{5}$
35. $\frac{-4+7i}{13}$

Exercise $9.6.5$ **complex roots**

1. $\pm 3i$
3. $\pm\frac{5}{2}i$
5. $\pm i\frac{\sqrt{3}}{2}$
7. \(±i\sqrt{\frac{2}{3}}
9. $-1\pm 2i$

Exercise $9.6.6$ **complex roots**

1. $1\pm 3i$
3. $-2\pm i\sqrt{2}$
5. $3\pm 2\sqrt{2}i$
7. $\frac{5}{2}\pm i\frac{\sqrt{15}}{2}$
9. $5\pm 2i$
11. $\frac{-1}{2}\pm i\frac{\sqrt{7}}{2}$
13. $\frac{5}{2}\pm i\frac{3}{2}$
15. $\frac{1}{3}\pm i\frac{\sqrt{11}}{3}$
17. $\frac{-3}{4}\pm i\frac{\sqrt{7}}{4}$
19. $3\pm\sqrt{2}$
21. $\frac{1}{2}\pm i\frac{1}{2}$
23. $1\pm i\frac{\sqrt{3}}{3}$
25. $\frac{1}{2}\pm i$
27. $\frac{1}{6}\pm i\frac{\sqrt{11}}{6}$
29. $\frac{1}{3}\pm i\frac{2}{3}$
31. $\frac{1}{4}\pm i\frac{\sqrt{23}}{4}$
33. $\pm i\sqrt{\frac{3}{2}}$

Exercise $9.6.7$ **discussion board**

1. Answers may vary
3. Answers may vary

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9.E: Review Exercises and Sample Exam

Review Exercises

Exercise $9.E.1$ extracting square roots

Solve by extracting the roots.

1. $x^2 - 16 = 0$
2. $y^2 = 94$
3. $x^2 - 27 = 0$
4. $x^2 + 27 = 0$
5. $3y^2 - 25 = 0$
6. $9x^2 - 2 = 0$
7. $(x-5)^2 - 9 = 0$
8. $(2x-1)^2 - 1 = 0$
9. $16(x-6)^2 - 3 = 0$
10. $2(x+3)^2 - 5 = 0$
11. $(x+3)(x-2) = x + 12$
12. $(x+2)(5x-1) = 9x - 1$

Answer available at the end of this section.

Exercise $9.E.2$ extracting square roots

Find a quadratic equation in standard form with the given solutions.

1. $\pm\sqrt{2}$
2. $\pm 2\sqrt{5}$

Answer available at the end of this section.

Exercise $9.E.3$ completing the square

Complete the square.

1. $x^2 - 6x + ? = (x - ?)^2$
2. $x^2 - x + ? = (x - ?)^2$

Answer available at the end of this section.

Exercise $9.E.4$ completing the square

Solve by completing the square.

1. $x^2 - 12x + 1 = 0$
2. $x^2 + 8x + 3 = 0$
3. $y^2 - 4y - 14 = 0$
4. $y^2 - 2y - 74 = 0$
5. $x^2 + 5x - 1 = 0$

6. $x^2 - 7x - 2 = 0$
7. $2x^2 + x - 3 = 0$
8. $5x^2 + 9x - 2 = 0$
9. $2x^2 - 16x + 5 = 0$
10. $3x^2 - 6x + 1 = 0$
11. $2y^2 + 10y + 1 = 0$
12. $5y^2 + y - 3 = 0$
13. $x(x + 9) = 5x + 8$
14. $(2x + 5)(x + 2) = 8x + 7$

Answer available at the end of this section.

Exercise $9.E.5$ quadratic formula

Identify the coefficients a, b, and c used in the quadratic formula. Do not solve.

1. $x^2 - x + 4 = 0$
2. $-x^2 + 5x - 14 = 0$
3. $x^2 - 5 = 0$
4. $6x^2 + x = 0$

Answer available at the end of this section.

Exercise $9.E.6$ quadratic formula

Use the quadratic formula to solve the following.

1. $x^2 - 6x + 6 = 0$
2. $x^2 + 10x + 23 = 0$
3. $3y^2 - y - 1 = 0$
4. $2y^2 - 3y + 5 = 0$
5. $5x^2 - 36 = 0$
6. $7x^2 + 2x = 0$
7. $-x^2 + 5x + 1 = 0$
8. $-4x^2 - 2x + 1 = 0$
9. $t^2 - 12t - 288 = 0$
10. $t^2 - 44t + 484 = 0$
11. $(x - 3)^2 - 2x = 47$
12. $9x(x + 1) - 5 = 3x$

Answer available at the end of this section.

Exercise $9.E.7$ Guidelines for Solving Quadratic Equations and Applications

Use the discriminant to determine the number and type of solutions.

1. $-x^2 + 5x + 1 = 0$
2. $-x^2 + x - 1 = 0$
3. $4x^2 - 4x + 1 = 0$
4. $9x^2 - 4 = 0$

Answer available at the end of this section.

Exercise $9.E.8$ Guidelines for Solving Quadratic Equations and Applications

Solve using any method.

1. $x^2 + 4x - 60 = 0$
2. $9x^2 + 7x = 0$
3. $25t^2 - 1 = 0$
4. $t^2 + 16 = 0$
5. $x^2 - x - 3 = 0$
6. $9x^2 + 12x + 1 = 0$
7. $4(x-1)^2 - 27 = 0$
8. $(3x+5)^2 - 4 = 0$
9. $(x-2)(x+3) = 6$
10. $x(x-5) = 12$
11. $(x+1)(x-8) + 28 = 3x$
12. $(9x-2)(x+4) = 28x - 9$

Answer available at the end of this section.

Exercise $9.E.9$ Guidelines for Solving Quadratic Equations and Applications

Set up an algebraic equation and use it to solve the following.

1. The length of a rectangle is 2 inches less than twice the width. If the area measures 25 square inches, then find the dimensions of the rectangle. Round off to the nearest hundredth.
2. An 18-foot ladder leaning against a building reaches a height of 17 feet. How far is the base of the ladder from the wall? Round to the nearest tenth of a foot.
3. The value in dollars of a new car is modeled by the function $V(t) = 125t^2 - 3,000t + 22,000$, where t represents the number of years since it was purchased. Determine the age of the car when its value is \$22,000.
4. The height in feet reached by a baseball tossed upward at a speed of 48 feet/second from the ground is given by the function $h(t) = -16t^2 + 48t$, where t represents time in seconds. At what time will the baseball reach a height of 16 feet?

Answer available at the end of this section.

Exercise $9.E.10$ graphing parabolas

Determine the *x*- and *y*-intercepts.

1. $y = 2x^2 + 5x - 3$
2. $y = x^2 - 12$
3. $y = 5x^2 - x + 2$
4. $y = -x^2 + 10x - 25$

Answer available at the end of this section.

Exercise $9.E.11$ graphing parabolas

Find the vertex and the line of symmetry.

1. $y = x^2 - 6x + 1$
2. $y = -x^2 + 8x - 1$
3. $y = x^2 + 3x - 1$
4. $y = 9x^2 - 1$

Answer available at the end of this section.

Exercise $9.E.12$ graphing parabolas

Graph. Find the vertex and the y-intercept. In addition, find the x-intercepts if they exist.

1. $y = x^2 + 8x + 12$
2. $y = -x^2 - 6x + 7$
3. $y = -2x^2 - 4$
4. $y = x^2 + 4x$
5. $y = 4x^2 - 4x + 1$
6. $y = -2x^2$
7. $y = -2x^2 + 8x - 7$
8. $y = 3x^2 - 1$

Answer available at the end of this section.

Exercise $9.E.13$ graphing parabolas

Determine the maximum or minimum y-value.

1. $y = x^2 - 10x + 1$
2. $y = -x^2 + 12x - 1$
3. $y = -5x^2 + 6x$
4. $y = 2x^2 - x - 1$
5. The value in dollars of a new car is modeled by the function $V(t) = 125t^2 - 3,000t + 22,000$, where *t* represents the number of years since it was purchased. Determine the age of the car when its value is at a minimum.
6. The height in feet reached by a baseball tossed upward at a speed of 48 feet/second from the ground is given by the function $h(t) = -16t^2 + 48t$, where *t* represents time in seconds. What is the maximum height of the baseball?

Answer available at the end of this section.

Exercise $9.E.14$ introduction to complex numbers and complex solutions

Rewrite in terms of *i*.

1. $\sqrt{-36}$
2. $\sqrt{-40}$
3. $\sqrt{-\frac{8}{25}}$
4. $-\sqrt{-19}$

Answer available at the end of this section.

Exercise $9.E.15$ introduction to complex numbers and complex solutions

Perform the operations.

1. $(2 - 5i) + (3 + 4i)$
2. $(6 - 7i) - (12 - 3i)$
3. $(2 - 3i)(5 + i)$
4. $4 - i^2 - 3i$

Answer available at the end of this section.

Exercise $9.E.16$ introduction to complex numbers and complex solutions

Solve.

1. $9x^2 + 25 = 0$
2. $3x^2 + 1 = 0$
3. $y^2 - y + 5 = 0$
4. $y^2 + 2y + 4$
5. $4x(x + 2) + 5 = 8x$
6. $2(x + 2)(x + 3) = 3(x^2 + 13)$

Answer available at the end of this section.

Sample Exam

Exercise $9.E.17$

Solve by extracting the roots.

1. $4x^2 - 9 = 0$
2. $(4x + 1)^2 - 5 = 0$

Answer available at the end of this section.

Exercise $9.E.18$

Solve by completing the square.

1. $x^2 + 10x + 19 = 0$
2. $x^2 - x - 1 = 0$

Answer available at the end of this section.

Exercise $9.E.19$

Solve using the quadratic formula.

1. $-2x^2 + x + 3 = 0$
2. $x^2 + 6x - 31 = 0$

Answer available at the end of this section.

Exercise $9.E.20$

Solve using any method.

1. $(5x + 1)(x + 1) = 1$
2. $(x + 5)(x - 5) = 65$
3. $x(x + 3) = -2$

4. $2(x-2)^2-6=3x^2$

Answer available at the end of this section.

Exercise $9.E.21$

Set up an algebraic equation and solve.

1. The length of a rectangle is twice its width. If the diagonal measures $6\sqrt{5}$ centimeters, then find the dimensions of the rectangle.
2. The height in feet reached by a model rocket launched from a platform is given by the function $h(t)=-16t^2+256t+3$, where *t* represents time in seconds after launch. At what time will the rocket reach 451 feet?

Answer available at the end of this section.

Exercise $9.E.22$

Graph. Find the vertex and the y-intercept. In addition, find the x-intercepts if they exist.

1. $y=2x^2-4x-6$
2. $y=-x^2+4x-4$
3. $y=4x^2-9$
4. $y=x^2+2x-1$
5. Determine the maximum or minimum y-value: $y=-3x^2+12x-15$.
6. Determine the x- and y-intercepts: $y=x^2+x+4$.
7. Determine the domain and range: $y=25x^2-10x+1$.
8. The height in feet reached by a model rocket launched from a platform is given by the function $h(t)=-16t^2+256t+3$, where *t* represents time in seconds after launch. What is the maximum height attained by the rocket.
9. A bicycle manufacturing company has determined that the weekly revenue in dollars can be modeled by the formula $R=200n-n^2$, where *n* represents the number of bicycles produced and sold. How many bicycles does the company have to produce and sell in order to maximize revenue?
10. Rewrite in terms of *i*: $\sqrt{-60}$.
11. Divide: $\frac{4-2i}{4+2i}$.

Answer available at the end of this section.

Exercise $9.E.23$

Solve.

1. $25x^2+3=0$
2. $-2x^2+5x-1=0$

Answer available at the end of this section.

Answers

Exercise $9.E.1$ extracting square roots

1. ± 16
3. $\pm 3\sqrt{3}$
5. $\pm\frac{5\sqrt{3}}{3}$
7. $2, 8$
9. $6\pm\frac{\sqrt{3}}{4}$
11. $\pm 3\sqrt{2}$

Exercise $9.E.2$ **extracting square roots**

1. $x^2 - 2 = 0$

Exercise $9.E.3$ **completing the square**

1. $x^2 - 6x + 9 = (x - 3)^2$

Exercise $9.E.4$ **completing the square**

1. $6 \pm \sqrt{35}$
3. $2 \pm 3\sqrt{2}$
5. $\frac{-5 \pm \sqrt{29}}{2}$
7. $\frac{-3}{2}, 1$
9. $\frac{8 \pm 3\sqrt{6}}{2}$
11. $\frac{-5 \pm \sqrt{23}}{2}$
13. $-2 \pm 2\sqrt{3}$

Exercise $9.E.5$ **quadratic formula**

1. $a = 1, b = -1$, and $c = 4$
3. $a = 1, b = 0$, and $c = -5$

Exercise $9.E.6$ **quadratic formula**

1. $3 \pm \sqrt{3}$
3. $\frac{1 \pm \sqrt{13}}{6}$
5. $\pm \frac{6\sqrt{5}}{5}$
7. $\frac{5 \pm \sqrt{29}}{2}$
9. $-12, 24$
11. $4 \pm 3\sqrt{6}$

Exercise $9.E.7$ **Guidelines for Solving Quadratic Equations and Applications**

1. Two real solutions
3. One real solution

Exercise $9.E.8$ **Guidelines for Solving Quadratic Equations and Applications**

1. $-10, 6$
3. $\pm \frac{1}{5}$
5. $\frac{1 \pm \sqrt{13}}{2}$
7. $1 \pm \frac{3\sqrt{3}}{2}$
9. $-4, 3$
11. $5 \pm \sqrt{5}$

Exercise $9.E.9$ **Guidelines for Solving Quadratic Equations and Applications**

1. Length: 6.14 inches; width: 4.07 inches
3. It is worth $22,000 new and when it is 24 years old.

Exercise $9.E.10$ **graphing parabolas**

1. x-intercepts: $(-3, 0), (\frac{1}{2}, 0)$; y-intercept: $(0, -3)$
3. x-intercepts: none; y-intercept: $(0, 2)$

Exercise $9.E.11$ **graphing parabolas**

1. Vertex: $(3, -8)$; line of symmetry: $x = 3$
3. Vertex: $(-\frac{3}{2}, -\frac{13}{4})$; line of symmetry: $x = -32$

Exercise $9.E.12$ **graphing parabolas**

1.

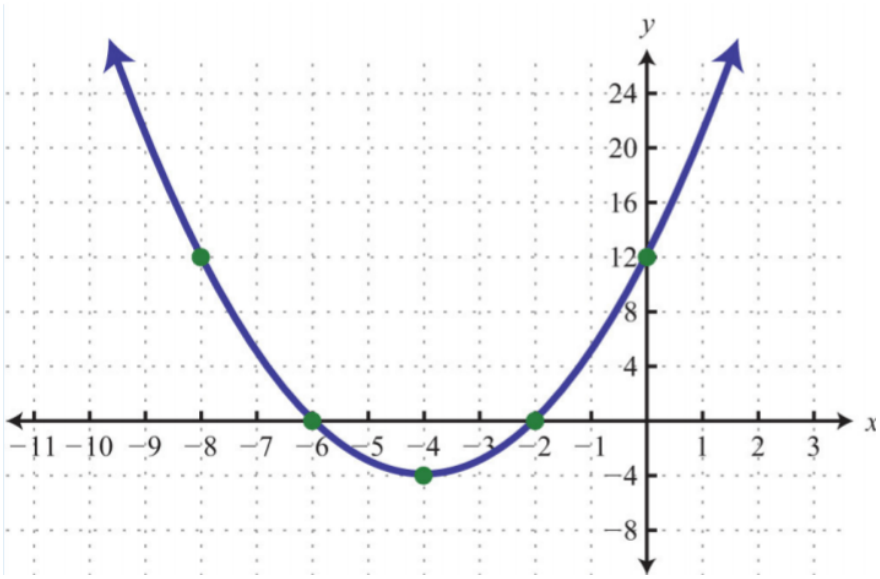


Figure 9.E.1

3.

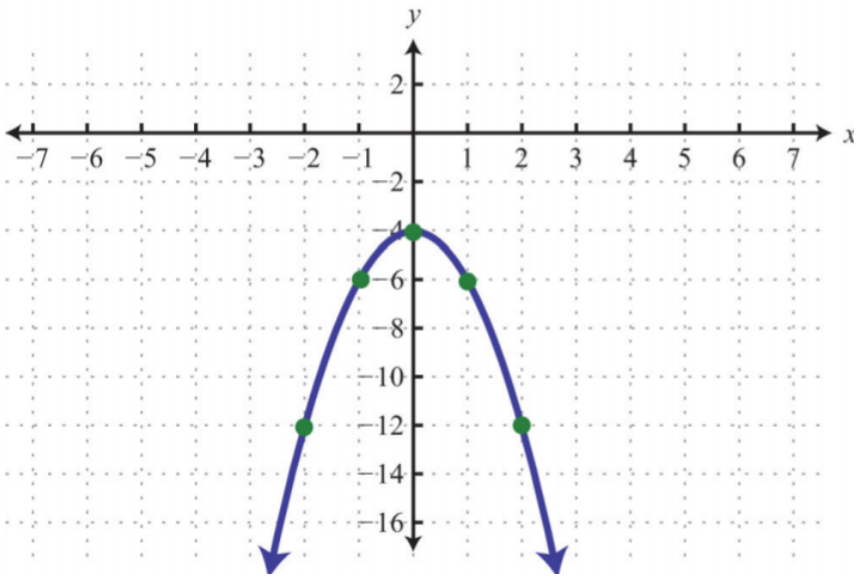


Figure 9.E.2

5.

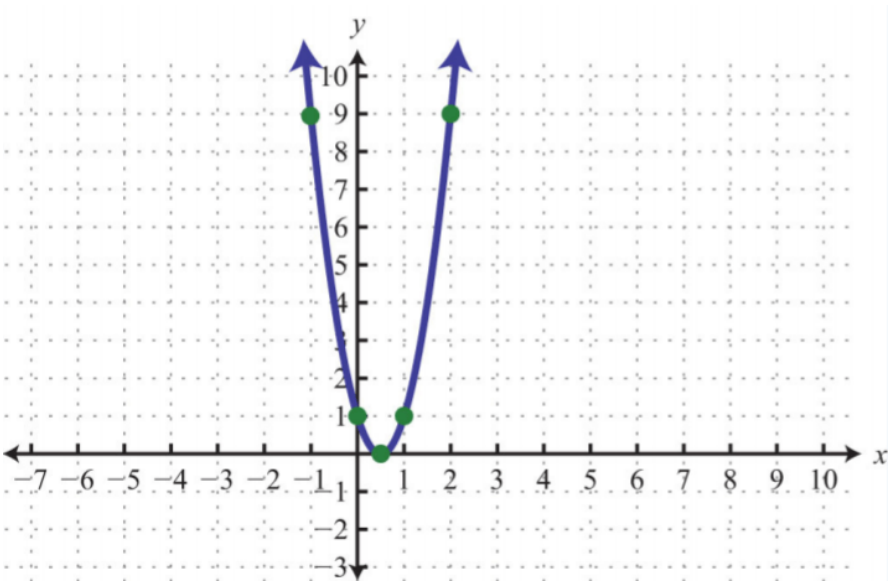


Figure 9.E.3

7.

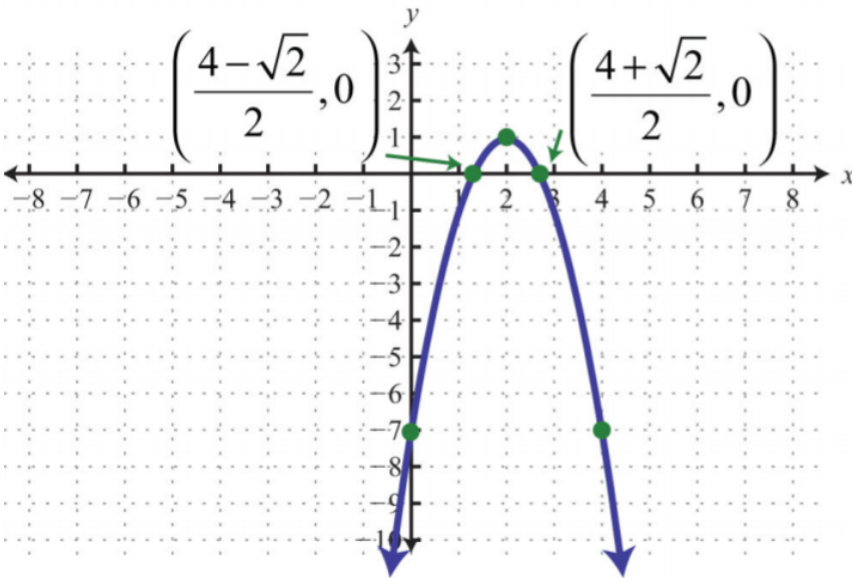


Figure 9.E.4

Exercise $9.E.13$ **graphing parabolas**

1. Minimum: $y=-24$
3. Maximum: $y=\frac{9}{5}$
5. The car will have a minimum value 12 years after it is purchased.

Exercise $9.E.14$ **introduction to complex numbers and complex solutions**

1. 6i
3. $\frac{2\sqrt{2}i}{5}$

Exercise $9.E.15$ **introduction to complex numbers and complex solutions**

1. $5-i$
3. $13-13i$

Exercise $9.E.16$ **introduction to complex numbers and complex solutions**

1. $\pm\frac{3}{3}$
3. $\frac{1}{2}\pm i\frac{\sqrt{19}}{2}$
5. $\pm i\frac{\sqrt{5}}{2}$

Exercise $9.E.17$

1. $\pm\frac{3}{2}$

Exercise $9.E.18$

1. $-5 \pm \sqrt{6}$

Exercise $9.E.19$

1. $-1, \frac{3}{2}$

Exercise $9.E.20$

1. $-\frac{6}{5}, 0$
3. $-2, -1$

Exercise $9.E.21$

1. Length: 12 centimeters; width: 6 centimeters

Exercise $9.E.22$

1.

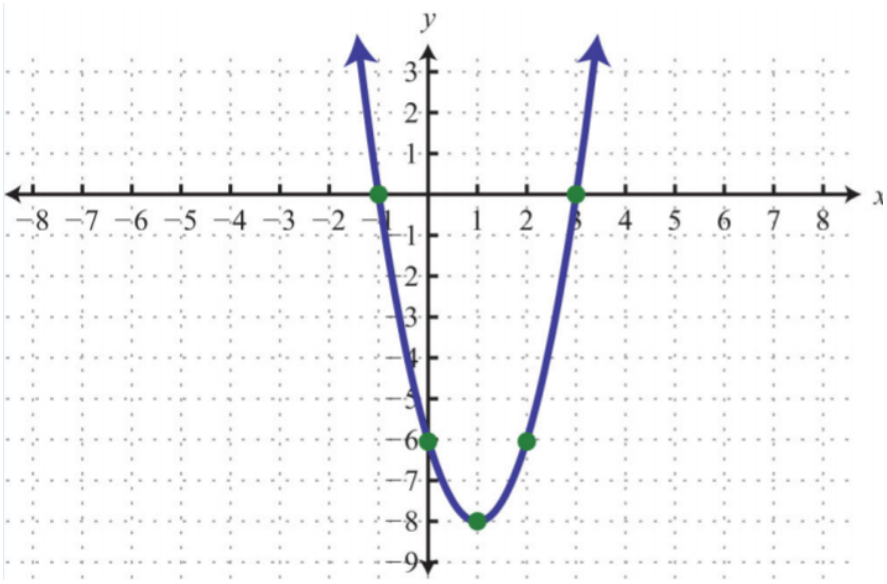


Figure 9.E.5

3.

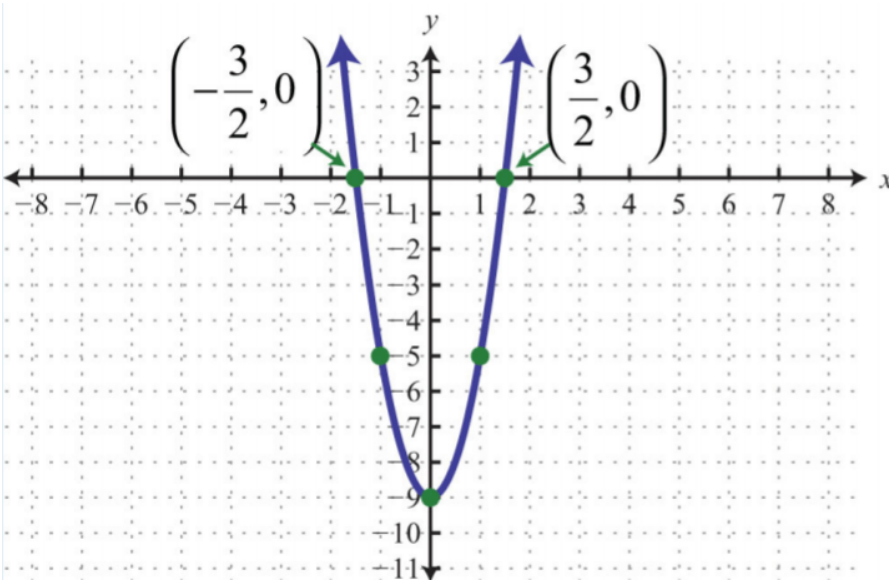


Figure 9.E.6

5. Maximum: $y = -3$
7. Domain: R; range: $[0, \infty)$
9. To maximize revenue, the company needs to produce and sell 100 bicycles a week.
11. $\frac{3}{5} - i\frac{4}{5}$

Exercise $9.E.23$

2. $\frac{5 \pm \sqrt{17}}{4}$

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10: APPENDIX - GEOMETRIC FIGURES

10.1: Plane

10.2: Solid

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10.1: Plane

Area (***A***) is measured in square units, perimeter (***P***) is measured in units, and circumference (***C***) is measured in units.

Square

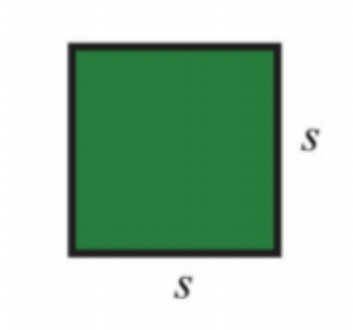


Figure 10.1.1

$$P = 4s \tag{10.1.1}$$

$$A = s^2 \tag{10.1.2}$$

Rectangle

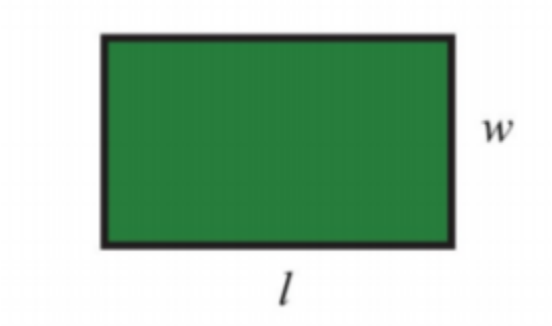


Figure 10.1.2

$$P = 2l + 2w \tag{10.1.3}$$

$$A = lw \tag{10.1.4}$$

Parallelogram

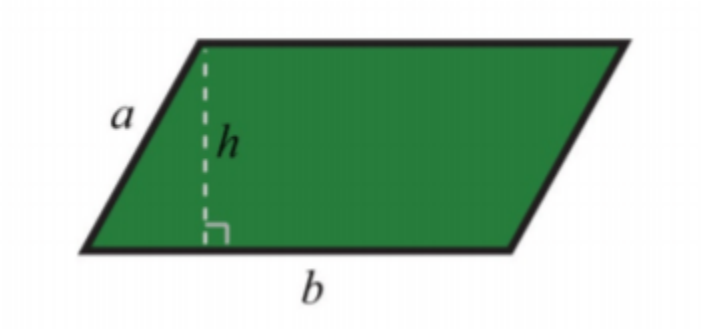


Figure 10.1.3

$$P = 2a + 2b \tag{10.1.5}$$

$$A = bh \tag{10.1.6}$$

Trapezoid

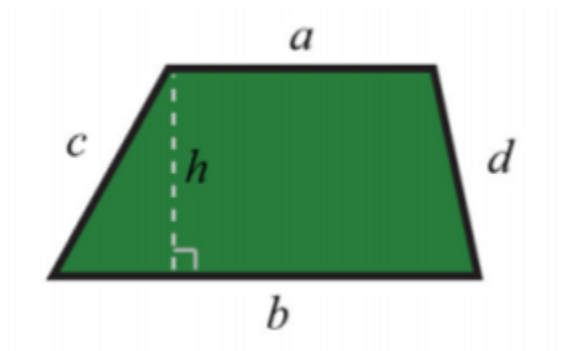


Figure 10.1.4

$$P = a + b + c + d \tag{10.1.7}$$

$$A = \frac{1}{2}h(a + b) \tag{10.1.8}$$

Triangle

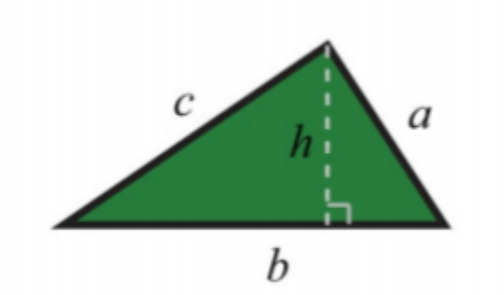


Figure 10.1.5

$$P = a + b + c \tag{10.1.9}$$

$$A = \frac{1}{2}bh \tag{10.1.10}$$

Circle

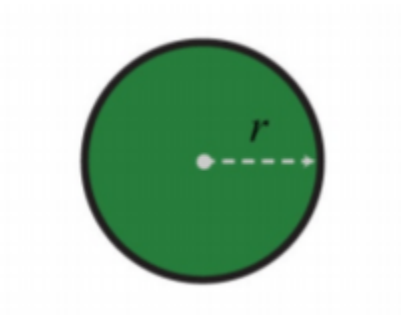


Figure 10.1.6

$$C = 2\pi \tag{10.1.11}$$

$$r = \pi r^2 \tag{10.1.12}$$

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10.2: Solid

Volume (***V***) is measured in cubic units and surface area (***SA***) is measured in square units.

Cube

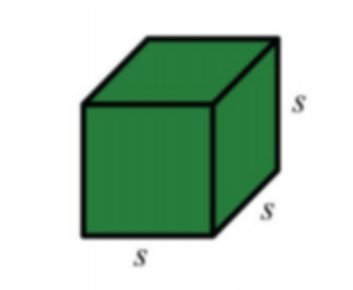


Figure 10.2.1

$$SA = 6s^2 \tag{10.2.1}$$

$$V = s^3 \tag{10.2.2}$$

Rectangular Solid

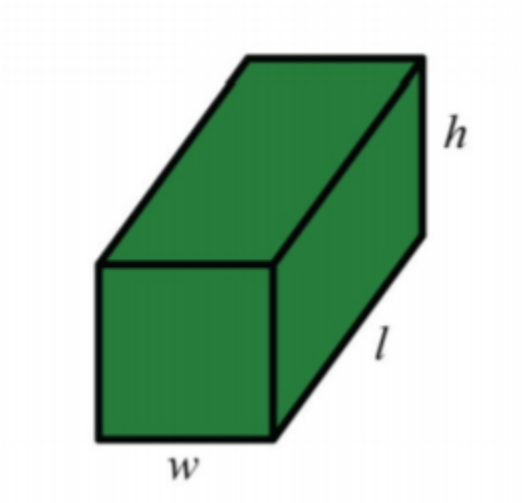


Figure 10.2.2

$$SA = 2lw + 2lh + 2wh \tag{10.2.3}$$

$$V = lwh \tag{10.2.4}$$

Right Circular Cylinder

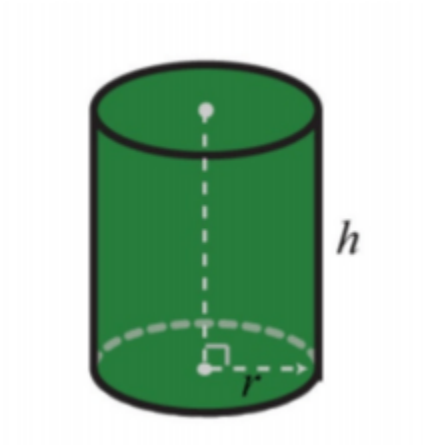


Figure 10.2.3

$$SA = 2\pi r^2 + 2\pi rh \tag{10.2.5}$$

$$V = \pi r^2 h \tag{10.2.6}$$

Right Circular Cone

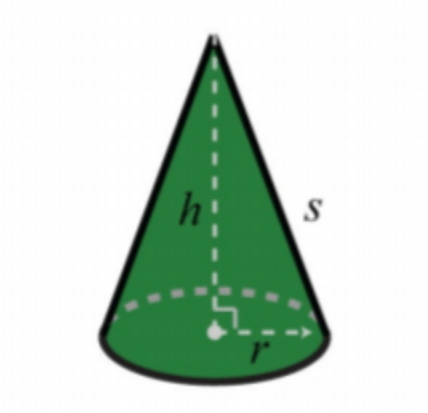


Figure 10.2.4

$$SA = \pi r^2 + \pi rs \tag{10.2.7}$$

$$V = \frac{1}{3}\pi r^2 h \tag{10.2.8}$$

Sphere

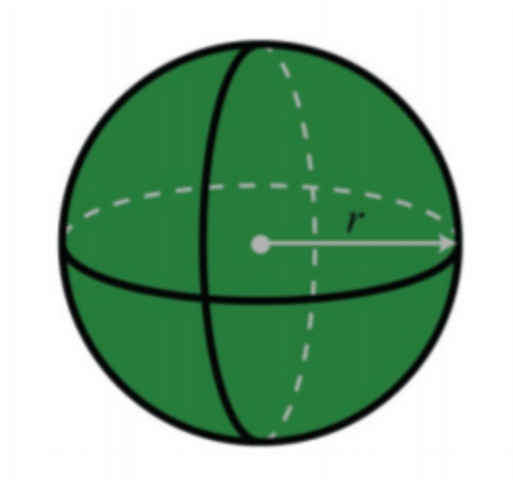


Figure 10.2.5

$$SA = 4\pi r^2 \tag{10.2.9}$$

$$V = \frac{4}{3}\pi r^3 \tag{10.2.10}$$

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GLOSSARY

A

absolute value | The absolute value of a number is the distance from the graph of the number to zero on a number line.

absolute value | The absolute value of a number is the distance from the graph of the number to zero on a number line.

AC method | Method for factoring trinomials by replacing the middle term with two terms that allow us to factor the resulting four-term polynomial by grouping.

AC method | Method for factoring trinomials by replacing the middle term with two terms that allow us to factor the resulting four-term polynomial by grouping.

add polynomials | The process of combining all like terms of two or more polynomials.

add polynomials | The process of combining all like terms of two or more polynomials.

addition property of equations | If *A*, *B*, *C*, and *D* are algebraic expressions, where *A* = *B* and *C* = *D*, then *A* + *C* = *B* + *D*.

addition property of equations | If *A*, *B*, *C*, and *D* are algebraic expressions, where *A* = *B* and *C* = *D*, then *A* + *C* = *B* + *D*.

Additive identity property | Given any real number *a*, $a+0=0+a=a$.

Additive identity property | Given any real number *a*, $a+0=0+a=a$.

Additive inverse property | Given any real number *a*, $a+(-a)=(-a)+a=0$.

Additive inverse property | Given any real number *a*, $a+(-a)=(-a)+a=0$.

algebraic expressions | Combinations of variables and numbers along with mathematical operations used to generalize specific arithmetic operations.

algebraic expressions | Combinations of variables and numbers along with mathematical operations used to generalize specific arithmetic operations.

algebraic fraction | Term used when referring to a rational expression.

algebraic fraction | Term used when referring to a rational expression.

Area of a circle | $A=\pi r^2$, where *r* represents the radius and the constant $\pi \approx 3.14$.

Area of a circle | $A=\pi r^2$, where *r* represents the radius and the constant $\pi \approx 3.14$.

Area of a rectangle | $A=lw$, where *l* represents the length and *w* represents the width.

Area of a rectangle | $A=lw$, where *l* represents the length and *w* represents the width.

Area of a square | $A=s^2$, where s represents the length of each side.

Area of a square | $A=s^2$, where s represents the length of each side.

Area of a triangle | $A=\frac{1}{2}bh$, where *b* represents the length of the base and *h* represents the height.

Area of a triangle | $A=\frac{1}{2}bh$, where *b* represents the length of the base and *h* represents the height.

Associative property | Given real numbers *a*, *b* and *c*, $(a+b)+c=a+(b+c)$.

Associative property | Given any real numbers *a*, *b*, and *c*, $(a \cdot b) \cdot c=a \cdot (b \cdot c)$.

Associative property | Given real numbers *a*, *b* and *c*, $(a+b)+c=a+(b+c)$.

Associative property | Given any real numbers *a*, *b*, and *c*, $(a \cdot b) \cdot c=a \cdot (b \cdot c)$.

asterisk | The symbol (*) that indicates multiplication within text-based applications.

asterisk | The symbol (*) that indicates multiplication within text-based applications.

average | Used in reference to the arithmetic mean.

average | Used in reference to the arithmetic mean.

average cost | The total cost divided by the number of units produced, which can be represented by $c(x)=\frac{C(x)}{x}$, where $C(x)$ is a cost function.

average cost | The total cost divided by the number of units produced, which can be represented by $c(x)=\frac{C(x)}{x}$, where $C(x)$ is a cost function.

axis of symmetry | A term used when referencing the line of symmetry.

axis of symmetry | A term used when referencing the line of symmetry.

B

Back substitute | Once a value is found for a variable, substitute it back into one of the original equations, or their equivalent equations, to determine the corresponding value of the other variable.

Back substitute | Once a value is found for a variable, substitute it back into one of the original equations, or their equivalent equations, to determine the corresponding value of the other variable.

back substituting | The process of finding the answers to other unknowns after one has been found.

back substituting | The process of finding the answers to other unknowns after one has been found.

Binomial | Polynomial with two terms.

Binomial | Polynomial with two terms.

C

caret | The symbol ^ that indicates exponents on many calculators, $a^n = a\hat{}n$.

caret | The symbol ^ that indicates exponents on many calculators, $a^n = a\hat{}n$.

Cartesian coordinate system | Used in honor of René Descartes when referring to the rectangular coordinate system.

Cartesian coordinate system | Used in honor of René Descartes when referring to the rectangular coordinate system.

check by evaluating | We can be fairly certain that we have multiplied the polynomials correctly if we check that a few values evaluate to the same results in the original expression and in the answer.

check by evaluating | We can be fairly certain that we have multiplied the polynomials correctly if we check that a few values evaluate to the same results in the original expression and in the answer.

circumference | The perimeter of a circle given by $C = 2\pi r$, where *r* represents the radius of the circle and $\pi \approx 3.14159$.

circumference | The perimeter of a circle given by $C = 2\pi r$, where *r* represents the radius of the circle and $\pi \approx 3.14159$.

collinear | Describes points that lie on the same line.

collinear | Describes points that lie on the same line.

combining like terms | Adding or subtracting like terms within an algebraic expression to obtain a single term with the same variable part.

combining like terms | Adding or subtracting like terms within an algebraic expression to obtain a single term with the same variable part.

common denominator | A denominator that is shared by more than one fraction.

common denominator | A denominator that is shared by more than one fraction.

common factor | A factor that is shared by more than one real number.

common factor | A factor that is shared by more than one real number.

Commutative property | Given real numbers *a* and *b*, $a + b = b + a$.

Commutative property | Given any real numbers *a* and *b*, $a \cdot b = b \cdot a$.

Commutative property | Given real numbers *a* and *b*, $a + b = b + a$.

Commutative property | Given any real numbers *a* and *b*, $a \cdot b = b \cdot a$.

completing the square | The process of rewriting a quadratic equation in the form $(x - p)^2 = q$.

completing the square | The process of rewriting a quadratic equation in the form $(x - p)^2 = q$.

complex conjugate | Two complex numbers whose real parts are the same and imaginary parts are opposite. If given $a + bi$, then its complex conjugate is $a - bi$.

complex conjugate | Two complex numbers whose real parts are the same and imaginary parts are opposite. If given $a + bi$, then its complex conjugate is $a - bi$.

complex fraction | A fraction where the numerator or denominator consists of one or more fractions.

complex fraction | A fraction where the numerator or denominator consists of one or more fractions.

complex fraction | A fraction where the numerator or denominator consists of one or more fractions.

complex fraction | A fraction where the numerator or denominator consists of one or more fractions.

complex number | Numbers of the form $a + bi$, where *a* and *b* are real numbers.

complex number | Numbers of the form $a + bi$, where *a* and *b* are real numbers.

complex rational expression | A rational expression where the numerator or denominator consists of one or more rational expressions.

complex rational expression | A rational expression where the numerator or denominator consists of one or more rational expressions.

compound inequalities | Two or more inequalities in one statement joined by the word “and” or by the word “or.”

compound inequalities | Two or more inequalities in one statement joined by the word “and” or by the word “or.”

conjugate binomials | The binomials $(a+b)$ and $(a-b)$.

conjugate binomials | The binomials $(a+b)$ and $(a-b)$.

conjugates | The factors $(a+b)$ and $(a-b)$ are conjugates.

conjugates | The factors $(a+b)$ and $(a-b)$ are conjugates.

Constant function | A polynomial function with degree 0.

Constant function | A polynomial function with degree 0.

constant of proportionality | Used when referring to the constant of variation.

constant of proportionality | Used when referring to the constant of variation.

constant term | A term written without a variable factor.

constant term | A term written without a variable factor.

contradiction | An equation that is never true and has no solution.

contradiction | An equation that is never true and has no solution.

cross canceling | Cancelling common factors in the numerator and the denominator of fractions before multiplying.

cross canceling | Cancelling common factors in the numerator and the denominator of fractions before multiplying.

cross multiplication | If $\frac{a}{b}=\frac{c}{d}$, then $ad=bc$.

cross multiplication | If $\frac{a}{b}=\frac{c}{d}$, then $ad=bc$.

cube | The result when the exponent of any real number is 3.

cube | The result when the exponent of any real number is 3.

cube root function | The function $f(x)=\sqrt[3]{x}$.

cube root function | The function $f(x)=\sqrt[3]{x}$.

Cubic function | A polynomial function with degree 3.

Cubic function | A polynomial function with degree 3.

D

decimal | A real number expressed using the decimal system.

decimal | A real number expressed using the decimal system.

degree of a polynomial | The largest degree of all of its terms.

degree of a polynomial | The largest degree of all of its terms.

degree of a term | The exponent of the variable; if there is more than one variable in the term, the degree of the term is the sum their exponents.

degree of a term | The exponent of the variable; if there is more than one variable in the term, the degree of the term is the sum their exponents.

dependent system | A system that consists of equivalent equations with infinitely many ordered pair solutions, denoted by (*x*, *mx* + *b*).

dependent system | A system that consists of equivalent equations with infinitely many ordered pair solutions, denoted by (*x*, *mx* + *b*).

dependent variable | The variable whose value is determined by the value of the independent variable. Usually we think of the *y*-value as the dependent variable.

dependent variable | The variable whose value is determined by the value of the independent variable. Usually we think of the *y*-value as the dependent variable.

difference of cubes | $a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$, where *a* and *b* represent algebraic expressions.

difference of cubes | $a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$, where *a* and *b* represent algebraic expressions.

difference of squares | $a^2-b^2=(a+b)(a-b)$, where *a* and *b* represent algebraic expressions.

difference of squares | $a^2-b^2=(a+b)(a-b)$, where *a* and *b* represent algebraic expressions.

difference of squares | $a^2-b^2=(a+b)(a-b)$, where *a* and *b* represent algebraic expressions.

difference of squares | $a^2-b^2=(a+b)(a-b)$, where *a* and *b* represent algebraic expressions.

direct variation | Describes two quantities *x* and *y* that are constant multiples of each other: $y=kx$.

direct variation | Describes two quantities *x* and *y* that are constant multiples of each other: $y=kx$.

directly proportional | Used when referring to direct variation.

directly proportional | Used when referring to direct variation.

discriminant | The algebraic expression $b^2 - 4ac$.

discriminant | The algebraic expression $b^2 - 4ac$.

discriminant | The expression inside the radical of the quadratic formula, $b^2 - 4ac$.

discriminant | The expression inside the radical of the quadratic formula, $b^2 - 4ac$.

distance formula | Given two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$, calculate the distance *d* between them using the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

distance formula | Given two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$, calculate the distance *d* between them using the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Distance formula | Given two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$, calculate the distance *d* between them using the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Distance formula | Given two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$, calculate the distance *d* between them using the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

distance formula for a number line | The distance between any two real numbers *a* and *b* on a number line can be calculated using the formula $d = |b - a|$.

distance formula for a number line | The distance between any two real numbers *a* and *b* on a number line can be calculated using the formula $d = |b - a|$.

distributive property | Given any real numbers *a*, *b*, and *c*, $a\,(b + c) = ab + ac$ or $(b + c)\,a = ba + ca$.

distributive property | Given any real numbers *a*, *b*, and *c*, $a\,(b + c) = ab + ac$ or $(b + c)\,a = ba + ca$.

dividend | The numerator of a quotient.

dividend | The numerator of a quotient.

divisor | The denominator of a quotient.

divisor | The denominator of a quotient.

double root | A root that is repeated twice.

double root | A root that is repeated twice.

double-negative property | The opposite of a negative number is positive: –(–*a*) = *a*.

double-negative property | The opposite of a negative number is positive: –(–*a*) = *a*.

E

elimination (or addition) method | A means of solving a system by adding equivalent equations in such a way as to eliminate a variable.

elimination (or addition) method | A means of solving a system by adding equivalent equations in such a way as to eliminate a variable.

empty set | A subset with no elements, denoted ∅ or { }.

empty set | A subset with no elements, denoted ∅ or { }.

equality relationship | Express equality with the symbol =. If two quantities are not equal, use the symbol $\neq$.

equality relationship | Express equality with the symbol =. If two quantities are not equal, use the symbol $\neq$.

equivalent equations | Equations with the same solution set.

equivalent equations | Equations with the same solution set.

Equivalent fractions | Two equal fractions expressed using different numerators and denominators.

Equivalent fractions | Two equal fractions expressed using different numerators and denominators.

equivalent inequality | Inequalities that share the same solution set.

equivalent inequality | Inequalities that share the same solution set.

equivalent system | A system consisting of equivalent equations that share the same solution set.

equivalent system | A system consisting of equivalent equations that share the same solution set.

evaluating | The process of performing the operations of an algebraic expression for given values of the variables.

evaluating | The process of performing the operations of an algebraic expression for given values of the variables.

even integers | Integers that are divisible by two or are multiples of two.

even integers | Integers that are divisible by two or are multiples of two.

exponent | The positive integer *n* in the exponential notation a^n that indicates the number of times the base is used as a factor.

exponent | The positive integer *n* in the exponential notation a^n that indicates the number of times the base is used as a factor.

exponential form | An equivalent expression written using a rational exponent.

exponential form | An equivalent expression written using a rational exponent.

exponential form | An equivalent expression written using a rational exponent.

exponential form | An equivalent expression written using a rational exponent.

exponential notation | The compact notation $ax^2 + bx + c = 0.$ used when a factor is repeated multiple times.

exponential notation | The compact notation $ax^2 + bx + c = 0.$ used when a factor is repeated multiple times.

extracting the roots | Applying the square root property as a means of solving a quadratic equation.

extracting the roots | Applying the square root property as a means of solving a quadratic equation.

extraneous solutions | A solution that does not solve the original equation.

extraneous solutions | A solution that does not solve the original equation.

extraneous solutions | A solution that does not solve the original equation.

extraneous solutions | A solution that does not solve the original equation.

F

factor by grouping | A technique for factoring polynomials with four terms.

factor by grouping | A technique for factoring polynomials with four terms.

factoring a polynomial | The process of rewriting a polynomial as a product of polynomial factors.

factoring a polynomial | The process of rewriting a polynomial as a product of polynomial factors.

Factoring out the GCF | The process of rewriting a polynomial as a product using the GCF of all of its terms.

Factoring out the GCF | The process of rewriting a polynomial as a product using the GCF of all of its terms.

factors | Any of the numbers or expressions that form a product.

factors | Any of the numbers or expressions that form a product.

factors | Any of the numbers or expressions that form a product.

factors | Any of the numbers or expressions that form a product.

FOIL | When multiplying binomials we apply the distributive property multiple times in such a way as to multiply the first terms, outer terms, inner terms, and last terms.

FOIL | When multiplying binomials we apply the distributive property multiple times in such a way as to multiply the first terms, outer terms, inner terms, and last terms.

formulas | A reusable mathematical model using algebraic expressions to describe a common application.

formulas | A reusable mathematical model using algebraic expressions to describe a common application.

functions | Relations where every *x*-value corresponds to exactly one *y*-value. With the definition comes new notation: $f(x) = y$, which is read "*f* of *x* is equal to *y*."

functions | Relations where every *x*-value corresponds to exactly one *y*-value. With the definition comes new notation: $f(x) = y$, which is read "*f* of *x* is equal to *y*."

fundamental theorem of algebra | Guarantees that there will be as many (or fewer) real solutions to a polynomial with one variable as its degree.

fundamental theorem of algebra | Guarantees that there will be as many (or fewer) real solutions to a polynomial with one variable as its degree.

G

GCF of a polynomial | The greatest common factor of all the terms of the polynomial.

GCF of a polynomial | The greatest common factor of all the terms of the polynomial.

GCF of monomials | The product of the GCF of the coefficients and all common variable factors.

GCF of monomials | The product of the GCF of the coefficients and all common variable factors.

graph | A point on the number line associated with a coordinate.

graph | A point on the number line associated with a coordinate.

graph | A point on the number line associated with a coordinate.

graph | A point on the number line associated with a coordinate.

graphing method | A means of solving a system by graphing the equations on the same set of axes and determining where they intersect.

graphing method | A means of solving a system by graphing the equations on the same set of axes and determining where they intersect.

greatest common factor (GCF) | The largest shared factor of any number of integers.

greatest common factor (GCF) | The largest shared factor of any number of integers.

greatest common factor (GCF) | The product of all the common prime factors.

greatest common factor (GCF) | The product of all the common prime factors.

grouping symbols | Parentheses, brackets, braces, and the fraction bar are the common symbols used to group expressions and mathematical operations within a computation.

grouping symbols | Parentheses, brackets, braces, and the fraction bar are the common symbols used to group expressions and mathematical operations within a computation.

guess and check | Used when referring to the trial and error method for factoring trinomials.

guess and check | Used when referring to the trial and error method for factoring trinomials.

H

horizontal line | Any line whose equation can be written in the form $y = k$, where k is a real number.

horizontal line | Any line whose equation can be written in the form $y = k$, where k is a real number.

I

identity | An equation that is true for all possible values.

identity | An equation that is true for all possible values.

imaginary numbers | The square roots of any negative real numbers.

imaginary numbers | The square roots of any negative real numbers.

imaginary part | The real number b of a complex number $a + bi$.

imaginary part | The real number b of a complex number $a + bi$.

imaginary unit | Defined as $i = \sqrt{-1}$ and $i^2 = -1$.

imaginary unit | Defined as $i = \sqrt{-1}$ and $i^2 = -1$.

inclusive inequalities | Use the symbol $\leq$ to express quantities that are “less than or equal to” and $\geq$ for quantities that are “greater than or equal to” each other.

inclusive inequalities | Use the symbol $\leq$ to express quantities that are “less than or equal to” and $\geq$ for quantities that are “greater than or equal to” each other.

inconsistent systems | A system with no simultaneous solution.

inconsistent systems | A system with no simultaneous solution.

independent systems | A system of equations with one ordered pair solution (x, y).

independent systems | A system of equations with one ordered pair solution (x, y).

indeterminate | A quotient such as $\frac{0}{0}$, which is a quantity that is uncertain or ambiguous.

indeterminate | A quotient such as $\frac{0}{0}$, which is a quantity that is uncertain or ambiguous.

index | The positive integer n in the notation $\sqrt[n]{}$ that is used to indicate an nth root.

index | The positive integer n in the notation $\sqrt[n]{}$ that is used to indicate an nth root.

infinity | The symbol (∞) indicates the interval is unbounded to the right.

infinity | The symbol (∞) indicates the interval is unbounded to the right.

integers | The set of positive and negative whole numbers combined with zero {..., –3, –2, –1, 0, 1, 2, 3, ...}.

integers | The set of positive and negative whole numbers combined with zero {..., –3, –2, –1, 0, 1, 2, 3, ...}.

interest and money problems | Applications involving simple interest and money.

interest and money problems | Applications involving simple interest and money.

intersection | The set formed by the shared values of the individual solution sets that is indicated by the logical use of the word “and,” denoted

with the symbol ∩.

intersection | The set formed by the shared values of the individual solution sets that is indicated by the logical use of the word “and,” denoted with the symbol ∩.

interval notation | A textual system of expressing solutions to an algebraic inequality.

interval notation | A textual system of expressing solutions to an algebraic inequality.

inversely proportional | Used when referring to inverse variation.

inversely proportional | Used when referring to inverse variation.

Irrational numbers | Numbers that cannot be written as a ratio of two integers.

Irrational numbers | Numbers that cannot be written as a ratio of two integers.

J

jointly proportional | Used when referring to joint variation.

jointly proportional | Used when referring to joint variation.

L

leading coefficient | The coefficient of the term with the largest degree.

leading coefficient | The coefficient of the term with the largest degree.

least common denominator (LCD) | The least common multiple of a set of denominators.

least common denominator (LCD) | The least common multiple of a set of denominators.

least common multiple (LCM) | The smallest number that is evenly divisible by a set of numbers.

least common multiple (LCM) | The smallest number that is evenly divisible by a set of numbers.

line graph | A set of related data values graphed on a coordinate plane and connected by line segments.

line graph | A set of related data values graphed on a coordinate plane and connected by line segments.

linear equation with one variable | An equation that can be written in the general form $ax + b = 0$, where *a* and *b* are real numbers and $a \neq 0$.

linear equation with one variable | An equation that can be written in the general form $ax + b = 0$, where *a* and *b* are real numbers and $a \neq 0$.

linear equation with two variables | An equation with two variables that can be written in the standard form $ax + by = c$, where the real numbers *a* and *b* are not both zero.

linear equation with two variables | An equation with two variables that can be written in the standard form $ax + by = c$, where the real numbers *a* and *b* are not both zero.

linear function | Any function that can be written in the form *f*(*x*) = *mx* + *b*.

linear function | Any function that can be written in the form *f*(*x*) = *mx* + *b*.

Linear function | A polynomial function with degree 1.

Linear function | A polynomial function with degree 1.

linear inequality | A mathematical statement relating a linear expression as either less than or greater than another.

linear inequality | A mathematical statement relating a linear expression as either less than or greater than another.

linear inequality with two variables | An inequality relating linear expressions with two variables. The solution set is a region defining half of the plane.

linear inequality with two variables | An inequality relating linear expressions with two variables. The solution set is a region defining half of the plane.

linear systems | In this section, we restrict our study to systems of two linear equations with two variables.

linear systems | In this section, we restrict our study to systems of two linear equations with two variables.

literal equations | A formula that summarizes whole classes of problems.

literal equations | A formula that summarizes whole classes of problems.

M

midpoint | Given two points, (x_1, y_1) and (x_2, y_2), the midpoint is an ordered pair given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

midpoint | Given two points, (x_1, y_1) and (x_2, y_2), the midpoint is an ordered pair given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

mixed number | A number that represents the sum of a whole number and a fraction.

mixed number | A number that represents the sum of a whole number and a fraction.

Mixture problems | Applications involving a mixture of amounts usually given as a percentage of some total.

Mixture problems | Applications involving a mixture of amounts usually given as a percentage of some total.

Monomial | Polynomial with one term.

Monomial | Polynomial with one term.

Multiplicative identity property | Given any real number *a*, $a \cdot 1 = 1 \cdot a = a$.

Multiplicative identity property | Given any real number *a*, $a \cdot 1 = 1 \cdot a = a$.

N

natural (or counting) numbers | The set of counting numbers {1, 2, 3, 4, 5, ...}.

natural (or counting) numbers | The set of counting numbers {1, 2, 3, 4, 5, ...}.

negative exponents | $x^{-n} = \frac{1}{x^n}$, given any integer *n*, where *x* is nonzero.

negative exponents | $x^{-n} = \frac{1}{x^n}$, given any integer *n*, where *x* is nonzero.

negative infinity | The symbol (−∞) indicates the interval is unbounded to the left.

negative infinity | The symbol (−∞) indicates the interval is unbounded to the left.

nth root | The number that, when raised to the *n*th power, yields the original number.

nth root | The number that, when raised to the *n*th power, yields the original number.

O

odd integers | Integers that are not divisible by 2.

odd integers | Integers that are not divisible by 2.

opposite | Real numbers whose graphs are on opposite sides of the origin with the same distance to the origin.

opposite | Real numbers whose graphs are on opposite sides of the origin with the same distance to the origin.

opposite binomial property | If given a binomial $a - b$, then the opposite is $-(a - b) = b - a$.

opposite binomial property | If given a binomial $a - b$, then the opposite is $-(a - b) = b - a$.

opposite reciprocals | Two real numbers whose product is –1. Given a real number $\frac{a}{b}$, the opposite reciprocal is $-\frac{b}{a}$.

opposite reciprocals | Two real numbers whose product is –1. Given a real number $\frac{a}{b}$, the opposite reciprocal is $-\frac{b}{a}$.

opposite-side like terms | Like terms of an equation on opposite sides of the equal sign.

opposite-side like terms | Like terms of an equation on opposite sides of the equal sign.

order | To ensure a single correct result, perform mathematical operations in a specific order.

order | To ensure a single correct result, perform mathematical operations in a specific order.

origin | The point on the number line that represtents zero.

origin | The point on the number line that represtents zero.

origin | The point where the *x*- and *y*-axes cross, denoted by (0, 0).

origin | The point where the *x*- and *y*-axes cross, denoted by (0, 0).

P

parabola | The graph of any quadratic equation $y = ax^2 + bx + c$, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

parabola | The graph of any quadratic equation $y = ax^2 + bx + c$, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

Parallel lines | Lines in the same plane that do not intersect; their slopes are the same.

Parallel lines | Lines in the same plane that do not intersect; their slopes are the same.

percent | A representation of a number as part of 100: $N\% = \frac{N}{100}$.

percent | A representation of a number as part of 100: $N\% = \frac{N}{100}$.

perfect cube | The result of cubing an integer.

perfect cube | The result of cubing an integer.

perfect square | The result of squaring an integer.

perfect square | The result of squaring an integer.

perfect square trinomials | The trinomials obtained by squaring the binomials $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$.

perfect square trinomials | The trinomials obtained by squaring the binomials $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$.

perimeter | The sum of the lengths of all the outside edges of a polygon.

perimeter | The sum of the lengths of all the outside edges of a polygon.

Perimeter of a rectangle | $P = 2l + 2w$, where *l* represents the length and *w* represents the width.

Perimeter of a rectangle | $P = 2l + 2w$, where *l* represents the length and *w* represents the width.

Perimeter of a square | $P = 4s$, where *s* represents the length of a side.

Perimeter of a square | $P = 4s$, where *s* represents the length of a side.

Perimeter of a triangle | $P = a + b + c$, where *a*, *b*, and *c* each represents the length of a different side.

Perimeter of a triangle | $P = a + b + c$, where *a*, *b*, and *c* each represents the length of a different side.

pie chart | A circular graph divided into sectors whose area is proportional to the relative size of the ratio of the part to the total.

pie chart | A circular graph divided into sectors whose area is proportional to the relative size of the ratio of the part to the total.

placeholders | Terms with zero coefficients used to fill in all missing exponents within a polynomial.

placeholders | Terms with zero coefficients used to fill in all missing exponents within a polynomial.

point-slope form of a line | Any nonvertical line can be written in the form $y - y_1 = m(x - x_1)$, where *m* is the slope and (x_1, y_1) is any point on the line.

point-slope form of a line | Any nonvertical line can be written in the form $y - y_1 = m(x - x_1)$, where *m* is the slope and (x_1, y_1) is any point on the line.

polynomial | An algebraic expression consisting of terms with real number coefficients and variables with whole number exponents.

polynomial | An algebraic expression consisting of terms with real number coefficients and variables with whole number exponents.

Polynomial | An algebraic expression consisting of terms with real number coefficients and variables with whole number exponents.

Polynomial | An algebraic expression consisting of terms with real number coefficients and variables with whole number exponents.

polynomial long division | The process of dividing two polynomials using the division algorithm.

polynomial long division | The process of dividing two polynomials using the division algorithm.

polynomials with one variable | A polynomial where each term has the form $a_n x^n$, where a_n is any real number and *n* is any whole number.

polynomials with one variable | A polynomial where each term has the form $a_n x^n$, where a_n is any real number and *n* is any whole number.

power property of equality | Given any positive integer *n* and real numbers *a* and *b*, where $a = b$, then $a^n = b^n$.

power property of equality | Given any positive integer *n* and real numbers *a* and *b*, where $a = b$, then $a^n = b^n$.

power rule for a product | $(xy)^n = x^n y^n$; if a product is raised to a power, then apply that power to each factor in the product.

power rule for a product | $(xy)^n = x^n y^n$; if a product is raised to a power, then apply that power to each factor in the product.

power rule for a quotient | $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$; if a quotient is raised to a power, then apply that power to the numerator and the denominator.

power rule for a quotient | $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$; if a quotient is raised to a power, then apply that power to the numerator and the denominator.

power rule for exponents | $(x^m)^n = x^{mn}$; a power raised to a power can be simplified by multiplying the exponents.

power rule for exponents | $(x^m)^n = x^{mn}$; a power raised to a power can be simplified by multiplying the exponents.

prime factorization | The unique factorization of a natural number written as a product of primes.

prime factorization | The unique factorization of a natural number written as a product of primes.

prime factorization | The unique factorization of a natural number written as a product of primes.

prime factorization | The unique factorization of a natural number written as a product of primes.

prime number | Integers greater than 1 that are divisible only by 1 and itself.

prime number | Integers greater than 1 that are divisible only by 1 and itself.

prime polynomial | A polynomial with integer coefficients that cannot be factored as a product of polynomials with integer coefficients other than 1 and itself.

prime polynomial | A polynomial with integer coefficients that cannot be factored as a product of polynomials with integer coefficients other than 1 and itself.

principal (nonnegative) nth root | The positive *n*th root when *n* is even.

principal (nonnegative) nth root | The positive *n*th root when *n* is even.

principal (nonnegative) square root | The non-negative square root.

principal (nonnegative) square root | The non-negative square root.

principal (nonnegative) square root | The positive square root of a real number, denoted with the symbol $\sqrt{}$.

principal (nonnegative) square root | The positive square root of a real number, denoted with the symbol $\sqrt{}$.

product of complex conjugates | The real number that results from multiplying complex conjugates: $(a+bi)(a-bi)=a^2+b^2$.

product of complex conjugates | The real number that results from multiplying complex conjugates: $(a+bi)(a-bi)=a^2+b^2$.

product rule for exponents | $x^m \cdot x^n = x^{m+n}$; the product of two expressions with the same base can be simplified by adding the exponents.

product rule for exponents | $x^m \cdot x^n = x^{m+n}$; the product of two expressions with the same base can be simplified by adding the exponents.

Product rule for radicals | $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, where *a* and *b* represent positive real numbers.

Product rule for radicals | $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, where *a* and *b* represent positive real numbers.

properties of equality | Properties that allow us to obtain equivalent equations by adding, subtracting, multiplying, and dividing both sides of an equation by nonzero real numbers.

properties of equality | Properties that allow us to obtain equivalent equations by adding, subtracting, multiplying, and dividing both sides of an equation by nonzero real numbers.

properties of inequalities | Properties used to obtain equivalent inequalities and used as a means to solve them.

properties of inequalities | Properties used to obtain equivalent inequalities and used as a means to solve them.

proportion | A statement of the equality of two ratios.

proportion | A statement of the equality of two ratios.

Pythagorean theorem | Given any right triangle with legs measuring *a* and *b* units and hypotenuse measuring *c* units, then $a^2+b^2=c^2$.

Pythagorean theorem | Given any right triangle with legs measuring *a* and *b* units and hypotenuse measuring *c* units, then $a^2+b^2=c^2$.

Pythagorean theorem | Given any right triangle with legs measuring *a* and *b* units and hypotenuse measuring *c* units, then $a^2+b^2=c^2$.

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Pythagorean theorem | Given any right triangle with legs measuring *a* and *b* units and hypotenuse measuring *c* units, then $a^2+b^2=c^2$.

Q

quadrants | The four regions of a rectangular coordinate plane partly bounded by the *x*- and *y*-axes and numbered using the roman numerals I, II, III, and IV.

quadrants | The four regions of a rectangular coordinate plane partly bounded by the *x*- and *y*-axes and numbered using the roman numerals I, II, III, and IV.

quadratic formula | The formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, which gives the solutions to any quadratic equation in the form $ax^2+bx+c=0$, where *a*, *b*, and *c* are real numbers and $a\neq 0$.

quadratic formula | The formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, which gives the solutions to any quadratic equation in the form $ax^2+bx+c=0$, where *a*, *b*, and *c* are real numbers and $a\neq 0$.

Quadratic function | A polynomial function with degree 2.

Quadratic function | A polynomial function with degree 2.

quotient | The result after dividing.

quotient | The result after dividing.

quotient rule for exponents | $\frac{x^m}{x^n}=x^{m-n}$; the quotient of two expressions with the same base can be simplified by subtracting the exponents.

quotient rule for exponents | $\frac{x^m}{x^n}=x^{m-n}$; the quotient of two expressions with the same base can be simplified by subtracting the exponents.

Quotient rule for radicals | $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, where *a* and *b* represent positive real numbers.

Quotient rule for radicals | $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, where *a* and *b* represent positive real numbers.

quotients with negative exponents | $\frac{x^{-n}}{y^{-m}}=\frac{y^m}{x^n}$, given any integers *m* and *n*, where $x\neq 0$ and $y\neq 0$.

quotients with negative exponents | $\frac{x^{-n}}{y^{-m}}=\frac{y^m}{x^n}$, given any integers *m* and *n*, where $x\neq 0$ and $y\neq 0$.

R

radical | Used when referring to an expression of the form $\sqrt[n]{a}$.

radical | Used when referring to an expression of the form $\sqrt[n]{a}$.

radical equation | Any equation that contains one or more radicals with a variable in the radicand.

radical equation | Any equation that contains one or more radicals with a variable in the radicand.

radical expression | An algebraic expression that contains radicals.

radical expression | An algebraic expression that contains radicals.

radicand | The expression *a* within a radical sign, $\sqrt[n]{a}$.

radicand | The expression *a* within a radical sign, $\sqrt[n]{a}$.

radicand | The expression *a* within a radical sign, $\sqrt[n]{a}$.

radicand | The expression *a* within a radical sign, $\sqrt[n]{a}$.

range | The set of second components of a relation. The *y*-values define the range in relations consisting of points (*x*, *y*) in the rectangular coordinate plane.

range | The set of second components of a relation. The *y*-values define the range in relations consisting of points (*x*, *y*) in the rectangular coordinate plane.

rate | A ratio where the units for the numerator and the denominator are different.

rate | A ratio where the units for the numerator and the denominator are different.

ratio | Relationship between two numbers or quantities usually expressed as a quotient.

ratio | Relationship between two numbers or quantities usually expressed as a quotient.

ratio | Relationship between two numbers or quantities usually expressed as a quotient.

ratio | Relationship between two numbers or quantities usually expressed as a quotient.

rational (or fractional) exponents | The fractional exponent *m*/*n* that indicates a radical with index *n* and exponent *m*: $a^{m/n} = \sqrt[n]{a^m}$.

rational (or fractional) exponents | The fractional exponent *m*/*n* that indicates a radical with index *n* and exponent *m*: $a^{m/n} = \sqrt[n]{a^m}$.

rational equation | An equation containing at least one rational expression.

rational equation | An equation containing at least one rational expression.

Rational numbers | Numbers of the form $\frac{a}{b}$, where *a* and *b* are integers and *b* is nonzero.

Rational numbers | Numbers of the form $\frac{a}{b}$, where *a* and *b* are integers and *b* is nonzero.

rationalizing the denominator | The process of determining an equivalent radical expression with a rational denominator.

rationalizing the denominator | The process of determining an equivalent radical expression with a rational denominator.

real numbers | The set of all rational and irrational numbers.

real numbers | The set of all rational and irrational numbers.

reciprocal | The reciprocal of a nonzero number *n* is 1/*n*.

reciprocal | The reciprocal of a nonzero number *n* is 1/*n*.

reciprocals | The reciprocal of a nonzero number *n* is 1/*n*.

reciprocals | The reciprocal of a nonzero number *n* is 1/*n*.

reducing | The process of finding equivalent fractions by dividing the numerator and the denominator by common factors.

reducing | The process of finding equivalent fractions by dividing the numerator and the denominator by common factors.

reducing to lowest terms | Finding equivalent fractions where the numerator and the denominator share no common integer factor other than 1.

reducing to lowest terms | Finding equivalent fractions where the numerator and the denominator share no common integer factor other than 1.

relatively prime | Expressions that share no common factors other than 1.

relatively prime | Expressions that share no common factors other than 1.

remainder | The expression that is left after the division algorithm ends.

remainder | The expression that is left after the division algorithm ends.

restrictions | The set of real numbers for which a rational expression is not defined.

restrictions | The set of real numbers for which a rational expression is not defined.

root | A solution to a quadratic equation in standard form.

root | A solution to a quadratic equation in standard form.

root | A solution to a quadratic equation in standard form.

root | A solution to a quadratic equation in standard form.

round off | A means of approximating decimals with a specified number of significant digits.

round off | A means of approximating decimals with a specified number of significant digits.

run | The horizontal change between any two points on a line.

run | The horizontal change between any two points on a line.

S

same-side like terms | Like terms of an equation on the same side of the equal sign.

same-side like terms | Like terms of an equation on the same side of the equal sign.

satisfy the equation | After replacing the variable with a solution and simplifying, it produces a true statement.

satisfy the equation | After replacing the variable with a solution and simplifying, it produces a true statement.

scale factor | The reduced ratio of any two corresponding sides of similar triangles.

scale factor | The reduced ratio of any two corresponding sides of similar triangles.

scientific notation | Real numbers expressed in the form $a \times 10^n$, where *n* is an integer and $1 \leq a < 10$.

scientific notation | Real numbers expressed in the form $a \times 10^n$, where *n* is an integer and $1 \leq a < 10$.

set-builder notation | A system for describing sets using familiar mathematical notation.

set-builder notation | A system for describing sets using familiar mathematical notation.

similar radicals | Term used when referring to like radicals.

similar radicals | Term used when referring to like radicals.

similar terms | Used when referring to like terms.

similar terms | Used when referring to like terms.

similar triangles | Triangles with the same shape but not necessarily the same size. The measures of corresponding angles are equal and the corresponding sides are proportional.

similar triangles | Triangles with the same shape but not necessarily the same size. The measures of corresponding angles are equal and the corresponding sides are proportional.

Simple interest | Modeled by the formula $I = prt$, where *p* represents the principal amount invested at an annual interest rate *r* for *t* years.

Simple interest | Modeled by the formula $I = prt$, where *p* represents the principal amount invested at an annual interest rate *r* for *t* years.

simplified | A radical where the radicand does not consist of any factor that can be written as a perfect power of the index.

simplified | A radical where the radicand does not consist of any factor that can be written as a perfect power of the index.

simplifying the expression | The process of combining like terms until the expression contains no more similar terms.

simplifying the expression | The process of combining like terms until the expression contains no more similar terms.

simultaneous solution | Used when referring to a solution of a system of equations.

simultaneous solution | Used when referring to a solution of a system of equations.

slope formula | Given two points (x_1, y_1) and (x_2, y_2), then the slope of the line is given by the formula $m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$.

slope formula | Given two points (x_1, y_1) and (x_2, y_2), then the slope of the line is given by the formula $m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$.

slope-intercept form | Any nonvertical line can be written in the form $y = mx + b$, where *m* is the slope and (0, *b*) is the *y*-intercept.

slope-intercept form | Any nonvertical line can be written in the form $y = mx + b$, where *m* is the slope and (0, *b*) is the *y*-intercept.

solution | Any value that can replace the variable in an equation to produce a true statement.

solution | Any value that can replace the variable in an equation to produce a true statement.

solution to a linear inequality | A real number that produces a true statement when its value is substituted for the variable.

solution to a linear inequality | A real number that produces a true statement when its value is substituted for the variable.

solve by factoring | The process of solving an equation that is equal to zero by factoring it and then setting each variable factor equal to zero.

solve by factoring | The process of solving an equation that is equal to zero by factoring it and then setting each variable factor equal to zero.

square | The result when the exponent of any real number is 2.

square | The result when the exponent of any real number is 2.

square root | The number that, when multiplied by itself, yields the original number.

square root | The number that, when multiplied by itself, yields the original number.

square root function | The function $f(x) = \sqrt{x}$.

square root function | The function $f(x) = \sqrt{x}$.

square root property | For any real number *k*, if $x^2 = k$, then $x = \pm\sqrt{k}$.

square root property | For any real number *k*, if $x^2 = k$, then $x = \pm\sqrt{k}$.

squaring property of equality | Given real numbers *a* and *b*, where $a = b$, then $a^2 = b^2$.

squaring property of equality | Given real numbers *a* and *b*, where $a = b$, then $a^2 = b^2$.

standard form | A quadratic equation written in the form $ax^2 + bx + c = 0$.

standard form | A quadratic equation written in the form $ax^2 + bx + c = 0$.

standard form | Any quadratic equation in the form $ax^2 + bx + c = 0$, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

standard form | Any quadratic equation in the form $ax^2 + bx + c = 0$, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

Strict inequalities | Express ordering relationships using the symbol < for “less than” and > for “greater than.”

Strict inequalities | Express ordering relationships using the symbol < for “less than” and > for “greater than.”

substitute | The act of replacing a variable with an equivalent quantity.

substitute | The act of replacing a variable with an equivalent quantity.

substitution method | A means of solving a linear system by solving for one of the variables and substituting the result into the other equation.

substitution method | A means of solving a linear system by solving for one of the variables and substituting the result into the other equation.

subtracting polynomials | The process of subtracting all the terms of one polynomial from another and combining like terms.

subtracting polynomials | The process of subtracting all the terms of one polynomial from another and combining like terms.

sum of squares | $a^2 + b^2$ does not have a general factored equivalent.

sum of squares | $a^2 + b^2$ does not have a general factored equivalent.

symmetric property | Allows you to solve for the variable on either side of the equal sign, because $5 = x$ is equivalent to $x = 5$.

symmetric property | Allows you to solve for the variable on either side of the equal sign, because $5 = x$ is equivalent to $x = 5$.

system of linear inequalities | A set of two or more linear inequalities that define the conditions to be considered simultaneously.

system of linear inequalities | A set of two or more linear inequalities that define the conditions to be considered simultaneously.

T

test points | A point not on the boundary of the linear inequality used as a means to determine in which half-plane the solutions lie.

test points | A point not on the boundary of the linear inequality used as a means to determine in which half-plane the solutions lie.

Trinomial | Polynomial with three terms.

Trinomial | Polynomial with three terms.

U

undefined | A quotient such as $\frac{5}{0}$, which is left without meaning and is not assigned an interpretation.

undefined | A quotient such as $\frac{5}{0}$, which is left without meaning and is not assigned an interpretation.

Uniform motion | Described by the formula $D = rt$, where the distance *D* is given as the product of the average rate *r* and the time *t* traveled at that rate.

Uniform motion | Described by the formula $D = rt$, where the distance *D* is given as the product of the average rate *r* and the time *t* traveled at that rate.

Uniform motion | Described by the formula $D = rt$, where the distance, *D*, is given as the product of the average rate, *r*, and the time, *t*, traveled at that rate.

Uniform motion | Described by the formula $D = rt$, where the distance, *D*, is given as the product of the average rate, *r*, and the time, *t*, traveled at that rate.

uniform motion problems | Applications relating distance, average rate, and time.

uniform motion problems | Applications relating distance, average rate, and time.

union | The set formed by joining the individual solution sets indicated by the logical use of the word “or” and denoted with the symbol ∪.

union | The set formed by joining the individual solution sets indicated by the logical use of the word “or” and denoted with the symbol ∪.

unit cost | The price of each unit.

unit cost | The price of each unit.

unlike denominators | Denominators of fractions that are not the same.

unlike denominators | Denominators of fractions that are not the same.

V

varies inversely | Describes two quantities *x* and *y*, where one variable is directly proportional to the reciprocal of the other: $y = \frac{k}{x}$.

varies inversely | Describes two quantities *x* and *y*, where one variable is directly proportional to the reciprocal of the other: $y = \frac{k}{x}$.

varies jointly | Describes a quantity *y* that varies directly as the product of two other quantities *x* and *z*: $y = kxz$.

varies jointly | Describes a quantity *y* that varies directly as the product of two other quantities *x* and *z*: $y = kxz$.

vertical line | Any line whose equation can be written in the form *x* = *k*, where *k* is a real number.

vertical line | Any line whose equation can be written in the form *x* = *k*, where *k* is a real number.

vertical line test | If a vertical line intersects a graph more than once, then the graph does not represent a function.

vertical line test | If a vertical line intersects a graph more than once, then the graph does not represent a function.

W

whole numbers | The set of natural numbers combined with zero {0, 1, 2, 3, 4, 5, ...}.

whole numbers | The set of natural numbers combined with zero {0, 1, 2, 3, 4, 5, ...}.

work rate | The rate at which a task can be performed.

work rate | The rate at which a task can be performed.

work-rate formula | $\frac{1}{t_1} \cdot t + \frac{1}{t_2} \cdot t = 1$, where $\frac{1}{t_1}$ and $\frac{1}{t_2}$ are the individual work rates and *t* is the time it takes to complete the task working together.

work-rate formula | $\frac{1}{t_1} \cdot t + \frac{1}{t_2} \cdot t = 1$, where $\frac{1}{t_1}$ and $\frac{1}{t_2}$ are the individual work rates and *t* is the time it takes to complete the task working together.

Y

***y*-intercept** | The point (or points) where a graph intersects the *y*-axis, expressed as an ordered pair (0, *y*).

***y*-intercept** | The point (or points) where a graph intersects the *y*-axis, expressed as an ordered pair (0, *y*).

Z

zero as an exponent | $x^0 = 1$; any nonzero base raised to the 0 power is defined to be 1.

zero as an exponent | $x^0 = 1$; any nonzero base raised to the 0 power is defined to be 1.

Zero factor property | Given any real number *a*, $a \cdot 0 = 0 \cdot a = 0$.

Zero factor property | Given any real number *a*, $a \cdot 0 = 0 \cdot a = 0$.

zero-product property | Any product is equal to zero if and only if at least one of the factors is zero.

zero-product property | Any product is equal to zero if and only if at least one of the factors is zero.

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