# COLLEGE ALGEBRA FOR LIBERAL ARTS AND HUMANITIES



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# Algebra for Humans

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# 1: EQUATIONS AND INEQUALITIES


1.1: Linear and Absolute Value Equations

1.2: Quadratic Equations and Complex Numbers

1.3: Rational and Radical Equations

1.4: Linear and Absolute Value Inequalities


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## Contributors

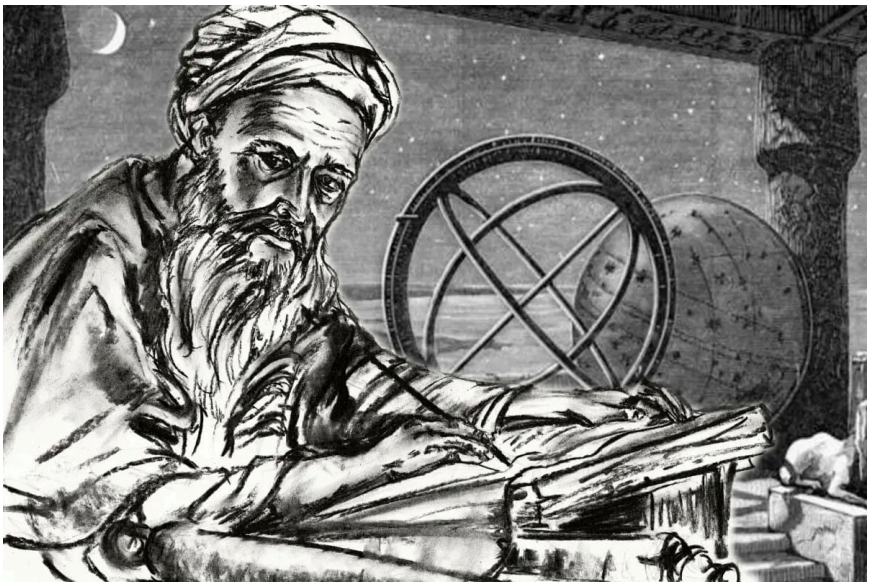
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# 1.1: Linear and Absolute Value Equations

**Learning Objectives**

- Solve linear equations in one variable algebraically.
- Identify when an equation is conditional, a contradiction or an identity.
- Determine the number of solutions that a given linear equation will have.
- Solve absolute value equations.
- Express the solutions in set form.



## *Calculation by Restoration and Balancing*

(Arabic: الكتاب المختصر في حساب الجبر والمقابلة, *al-Kitāb al-Mukhtaṣar fī Ḥisāb al-Jabr wal-Muqābalah*; or Latin: *Liber Algebræ et Almucabola*), commonly abbreviated ***Al-Jabr*** or ***Algebra*** (Arabic: الجبر), is an Arabic mathematical treatise on algebra written in Baghdad around 820 C.E. by the Persian polymath Muhammad Al-Khwarizmi. It was a landmark work in the history of mathematics, with its title being the ultimate etymology of the word "algebra".

## Translating Words into Algebraic Expressions and Equations

In the language of algebra we often make statements that require careful translation into algebraic symbols to ensure that we are performing the correct operations.

Here are some of the most common words and phrases that we will employ in this course.

- A **Sum** means to perform the operation of **addition.** The symbol is denoted by $+$
- A **Difference** means to perform the operation of **subtraction**. The symbol is denoted by $-$
- A **Product** means to perform the operation of **multiplication**. The symbol is denoted by $\times$ or to avoid confusion with the letter $x$, we often use a dot $\cdot$
- A **Quotient** means to perform the operation of **division**. The symbol is denoted by $\div$ or we may also use a fraction bar $\frac{}{}$ or

In algebra, when we say "*a number*", we are referring to an unknown value and we use a letter to represent this value in a mathematical expression or equation. The most common variable is of course the letter $x$. However, we will also use other letters such as: $y$, $z$, $u$, $v$, $w$, $p$, $q$, $r$, $s$, $t$. Generally, we try to use lowercase letters only, and we avoid using the letter $o$ since it may appear to be the number $0$.

In many formulas, some numbers may be specified as given values, these are called "*constants*", and we traditionally use letters such as: $a$, $b$, $c$, and $d$.

Although it is too early in this text to present the details. We will reserve the letters: $e$, $f$, $g$, $h$ and $i$ for very specific purposes.

### Example 1.1.1: Translating Phrases into Mathematical Expressions

Translate the following into mathematical expressions. Use the letter $x$ as the variable.

**A.** *The sum of a number and six.*

**B.** *The difference of seven and a number.*

**C.** *The product of four and a number.*

**D.** *The quotient of a number and two.*

**Solution**

**A.** Sum means to add, so we will add the variable $x$ and 6.

$$x + 6 \tag{1.1.1}$$

*note: since the operation of addition is "commutative", we may also write this as* $6 + x$

**B.** Difference means to subtract (in the reverse order in which it is written), so we will subtract the variable $x$ from 7.

$$7 - x \tag{1.1.2}$$

*note: since the operation of subtraction is "NOT commutative", we may NOT write this as* $x - 7$

**C.** Product means to multiply, so we will multiply 4 and the variable $x$

$$4 \cdot x \tag{1.1.3}$$

*note: when a known value such as 4 is written next to a variable, the operation of multiplication is implied, hence we may remove the* $\cdot$ *and simply write the result as* $4x$

*Furthermore, since the operation of multiplication is "commutative", we may also write this as* $x \cdot 4$*. In fact, it is technically permitted to write this expression as* $x4$*. However, most humans find it to be awkward and so it is typically avoided.*

**D.** Quotient means to divide (in the order in which it is written), so we will divide the variable $x$ by 2.

$$x \div 2 \tag{1.1.4}$$

or we may use a fraction bar, and write this as

$$\frac{x}{2} \tag{1.1.5}$$

or like this

$$x/2 \tag{1.1.6}$$

*note: since the operation of division is "NOT commutative", we may NOT write this as* $2 \div x$ or $\frac{2}{x}$ or $2/x$

Just as many human languages have several different words with the same meaning. In the language of math, we have several different ways to describe the same operation.

**Here are some of the most useful ways to describe the 4-basic arithmetic operations.**

| Addition | Subtraction | Multiplication | Division |
|---|---|---|---|
| $+$ | $-$ | $\cdot$ or $\times$ | $\frac{\square}{\square}$ or $\square/\square$ or $\div$ |
| Add | Subtract | Multiply | Divide |
| Sum | Difference | Product | Quotient |
| More than | Less than | Times | Per |
| Plus | Minus | Twice (multiply by 2) | Ratio |
| Increased by | Decreased by | Double (multiply by 2) | Half (divide by 2) |
| In addtion to | Take away from | Triple (multiply by 3) | Third (divide by 3) |

## Expression vs. Equation

The example above consists of numbers, variables and mathematical operations. You may notice a common symbol in mathematics that does not occur in the previous example. The equal sign

An arrangement of symbols that DO NOT contain the $=$ sign is called an **expression.**

An **equation** is a mathematical statement in which two expressions are connected with an equal sign. We can think of an equation as a "proposition" or a "claim". When we read it, it is our job to either determine if the statement is TRUE or FALSE, or if possible, find the value or values of the variable that make the statement true.

$$\begin{aligned} 1+2=3 &\quad \text{this statement is TRUE} \\ 2+3=4 &\quad \text{this statement is FALSE} \\ x+3=5 &\quad \text{this statement is TRUE only when the value of } x \text{ is } 2 \end{aligned}$$

When translating phrases into equations, look for the following keywords: **is, equals, equal to, is equal to, is equivalent to**

All of these are translated into the symbol $=$

### Example 1.1.2: Translating Statements into Equations

Translate the following into mathematical expressions. Use the letter $x$ as the variable.

**A.** *Seven more than three times a number is fifteen.*

**B.** *Half of the sum of a number and four is equal to eight.*

**C.** *Six equals four less than twice a number.*

**Solution**

**A.** "more than" is a synonym for a **sum**, it means to add. Three times a number means to multiply the variable by 3. Finally, "is" means equals.

So, we will add 7 to the variable expression $3x$ and set it equal to 15.

$$3x + 7 = 15 \tag{1.1.7}$$

*note: since the operation of addition is "commutative", we may also write this as* $7 + 3x = 15$

**B.** "half" means to **divide** by 2. The **sum**, as we know, means to add. Finally, "is equal to" is self evident as an equality sign.

So, we will first write the sum of a variable $x$ and 4, then divide this expression by 2, and set it equal to 8.

$$\frac{x+4}{2} = 8 \tag{1.1.8}$$

*note: we should be careful with the expression in the numerator, it is often helpful to use parenthesis like this* $\frac{(x+4)}{2} = 8$

*this can avoid confusing it, or incorrectly writing the result as* $\frac{x}{2} + 4 = 8$

**C.** Sometimes it is helpful to read the last part of a statement first. "twice a number" means to multiply the variable by 2, and "less than" means to subtract in the opposite order as the words appear from left to right. In other words, "four less than twice a number" is translated into the expression $\backslash 2 \cdot x - 4$the variable $x$. Finally, the word "equals" clearly states that 6 is equal to the previously described expression.

$$6 = 2 \cdot x - 4 \tag{1.1.9}$$

We may omit the $\cdot$ and simply writ e the resulting equation as: $6 = 2x - 4$

*note: An equation of the form* $A = B$ *has the same meaning as the equation* $B = A$*, therefore we may also write this equation as:* $2x - 4 = 6$

**You DO it! 1.1.1**

Translate the following statements into either an expression or an equation.

**A.** Eight *more than one third of a number.*

**B.** *The sum of twice a number and five is nine.*

**C.** *Seven is equal to four less than six times a number.*

*Answer available at the end of this section.*

## Solving Linear Equations in One Variable

A *linear equation,* in one variable, is an equation in which the only power of the variable is 1. Linear equations in one variable *can be written* the form $ax + b = 0$ and are solved using basic algebraic operations. We begin by classifying linear equations in one variable as one of three types: identity, conditional, or inconsistent.

- A **conditional equation** is true for only some values of the variable. For example, if we are to solve the equation $5x + 2 = 3x - 6$, we have the following:

$$\begin{aligned} 5x + 2 &= 3x - 6 \\ 2x &= -8 \\ x &= -4 \end{aligned}$$

The solution set consists of one number: $-4$. It is the only solution and, therefore, we have solved a conditional equation.

- An **inconsistent equation** *(also known as a contradiction)* results in a false statement. For example, if we are to solve $5x - 15 = 5(x - 4)$, we have the following:

$$\begin{aligned} 5x - 15 &= 5x - 20 \\ 5x - 15 - 5x &= 5x - 20 - 5x \\ -15 &\neq -20 \end{aligned}$$

Indeed, $-15 \neq -20$. There is no solution because this is an inconsistent equation.

- An **identity equation** is true for all values of the variable. Here is an example of an identity equation:

$$3x = 2x + x$$

The *solution set* consists of all values that make the equation true. For this equation, the solution set is all real numbers because any real number substituted for $x$ will make the equation true.

Solving linear equations in one variable involves the fundamental properties of equality and basic algebraic operations. A brief review of those operations follows.

### Definition: Linear Equation in One Variable

A linear equation in one variable can be written in the form

$$ax + b = 0 \tag{1.1.10}$$

where $a$, called the "coefficient", and $b$, called the "constant term" are real numbers.

### How to: Solve a Linear Equation in One Variable

The following steps are used to manipulate an equation to determine the unknown value of the variable. You may interchange the order of the first two steps, it will depend on specific equation that you are given:

1. **SIMPLIFY:** Apply the distributive property as needed: $A(B + C) = AB + AC$. If possible, combine like terms on each side of the equation.
2. **BALANCE:** Add, subtract, multiply, or divide an equation by a number or an expression as long as we do the same thing to both sides of the equal sign. Note that we cannot divide by zero.
3. **ISOLATE:** Move all variable terms to one side of the equation.
4. **RESOLVE:** When the variable is multiplied by a coefficient in the final stage, multiply both sides of the equation by the reciprocal of the coefficient.

### Example 1.1.3: Solving an Equation in One Variable

Solve the following equation: $2x + 19 = 7$

**Solution**

This equation can be written in the form $ax + b = 0$ by subtracting 7 from both sides.

$$\begin{aligned} 2x + 19 &= 7 \\ 2x + 12 &= 0 \qquad \text{Subtract 7 from both sides} \\ 2x &= -12 \qquad \text{Subtract 12 from both sides (or add } -12) \\ x &= -6 \qquad \text{Multiply both sides by } \frac{1}{2} \text{ (or divide by 2)} \end{aligned}$$

The solution is $-6$.

However, in many cases, it may be easier to solve the equation in its original by simply performing the necessary algebraic operations required to isolate the variable.

$$
\begin{aligned}
2x + 19 &= 7 \\
2x &= -12 \qquad \text{Subtract 19 from both sides} \\
x &= 6 \qquad \text{Multiply both sides by } \frac{1}{2}
\end{aligned}
$$

The solution is $-6$.

Either method is acceptable and both result in the same solution.

## YOU do it! 1.1.2

Solve the linear equation: $2x + 1 = -9$

*Answer available at the end of this section.*

## Example 1.1.4: Solving When the Variable Appears on Both Sides

Solve the following equation: $4(x - 3) + 12 = 15 - 5(x + 6)$

**Solution**

Apply standard algebraic properties.

$$
\begin{aligned}
4(x - 3) + 12 &= 15 - 5(x + 6) \\
4x - 12 + 12 &= 15 - 5x - 30 \qquad \text{Apply the distributive property} \\
4x &= -15 - 5x \qquad \text{Combine like terms} \\
9x &= -15 \qquad \text{Place x terms on one side and simplify} \\
x &= -\frac{15}{9} \qquad \text{Multiply both sides by } \frac{1}{9} \text{ , or divide by 9} \\
x &= -\frac{5}{3}
\end{aligned}
$$

**Analysis**

This problem requires the distributive property to be applied twice, and then the properties of algebra are used to reach the final line, $x = -\frac{5}{3}$.

The solution is $-\frac{5}{3}$

## You DO it! 1.1.3

Solve the equation in one variable: $-2(3x - 1) + x = 14 - x$

*Answer available at the end of this section.*

## Example 1.1.5

Solve: $6 - 3(4x - 1) = 4x - 7$.

*Answer available at the end of this section.*

## Example 1.1.6

Solve: $\frac{5}{3}x + 2 = -8$

**Solution**

Isolate the variable term using the addition property of equality, and then multiply both sides of the equation by the reciprocal of the coefficient $\frac{5}{3}$.

$$\begin{aligned} \frac{5}{3}x + 2 &= -8 \\ \frac{5}{3}x + 2 - 2 &= -8 - 2 \quad \textit{Subtract 2 on both sides.} \\ \frac{5}{3}x &= -10 \\ \frac{3}{5}\cdot\frac{5}{3}x &= \frac{3}{\cancel{5}}\cdot\left(\frac{\overset{-2}{\cancel{-10}}}{}\right) \quad \textit{Multiply both sides by } \frac{3}{5}. \\ 1x &= 3\cdot(-2) \\ x &= -6 \end{aligned} \tag{1.1.11}$$

**Answer**:

The solution is $-6$

## You Do it! 1.1.4

Solve: $\frac{2}{3}x + \frac{1}{2} = -\frac{5}{6}$.

*Answer available at the end of this section.*

## Keep Your BALANCE!

Remember that in order to maintain equivalent equations, we must perform the **same** operation on both sides of the equal sign.

## Example 1.1.7:

Solve: $-\frac{1}{2}(x-2) + 3 = 4(1-2x)$.

**Solution**

Simplify the linear expressions on either side of the equal sign first.

$$\begin{aligned} -\frac{1}{2}(x-2) + 3 &= 4(1-2x) && \textit{Multiply both sides by 2} \\ -1(x-2) + 3 &= 8(1-2x) && \textit{Distribute} \\ -1x + 2 + 3 &= 8 - 16x && \textit{Combine same - side like terms.} \\ -x + 5 &= 8 - 16x && \textit{Add } 16x \textit{ to both sides.} \\ -x + 5 + 16x &= 8 - 16x + 16x && \textit{Simplify.} \\ 15x + 5 &= 8 && \textit{Subtract 5 from both sides.} \\ 15x + 5 - 5 &= 8 - 5 && \textit{Divide both sides by 15 and simplify.} \\ x &= \frac{3}{15} = \frac{1}{5} \end{aligned}$$

**Answer**:

The solution is $\frac{1}{5}$.

There are three different types of equations. Up to this point, we have been solving **conditional equations**[135]. These are equations that are true for particular values.

An **identity**[136] is an equation that is true for all possible values of the variable. For example,

$$x = x \quad Identity$$

has a solution set consisting of all real numbers, $\mathbb{R}$.

A **contradiction**[137] is an equation that is never true and thus has no solutions. For example,

$$x + 1 = x \quad Contradiction$$

has no solution. We use the empty set, $\emptyset$, to indicate that there are no solutions. Sometimes we say the solution "does not exist" and we type DNE.

If the end result of solving an equation is a true statement, like $0 = 0$, then the equation is an identity and any real number is a solution. If solving results in a false statement, like $0 = 1$, then the equation is a contradiction and there is no solution.

## Example 1.1.8

Determine if the given equation is conditional, and if so, find the solution. Otherwise, indicate whether the equation is an identity or a contradiction.

$3x + 7 = 3x - 6$

**Solution**

This equation can be written in the form $ax + b = 0$ by subtracting $3x$ from both sides.

$$\begin{aligned} 3x + 7 &= 3x - 6 \\ 0x + 7 &= -6 \quad \text{Subtract 3x from both sides} \\ 7 &= -6 \end{aligned}$$

This is a contradiction. There is NO solution. DNE (we say "the solution does not exist")

## Example 1.1.9

Determine if the given equation is conditional, and if so, find the solution. Otherwise, indicate whether the equation is an identity or a contradiction.

$10 - x = x - 2(x - 5)$

**Solution**

This equation can be written in the form $ax + b = 0$ by first applying the distributive property on the right hand side.

$$\begin{aligned} 10 - x &= x - 2(x - 5) \\ 10 - x &= x - 2x + 10 \quad \text{Distribute by multiplying the terms inside the parenthesis by -2} \\ 10 - x &= -1x + 10 \quad \text{Combine like terms} \\ 10 &= 10 \quad \text{Add 1x to both sides} \end{aligned}$$

This is an identity. There are infinitely solutions. The solution set is the set of all real numbers.

**You DO it! 1.1.5**

Determine if the given equation is conditional, and if so, find the solution. Otherwise, indicate whether the equation is an identity or a contradiction.

$-4x = 6 + 5(x - 3)$

*Answer available at the end of this section.*

## Equivalent Equations, Solutions and Solution Sets.

When we solve an equation, we often arrive at a resolution (the last line of the process) that looks something like $x = 1$. In this case it is common to say that "the solution is $x = 1$". However, to be precise, this is just another equation that is "equivalent" to the original equation that was given.

A solution to an equation is ONLY the numeric part of the resolved equation which make the original statement TRUE. Therefore, the proper way to express the solution in the above equation is to say "the solution is 1"

We may also describe the solution "set" by placing braces around the numeric result. We say "the solution set is $\{1\}$.

It is unlikely that an instructor, such as myself, would penalize you for saying that the solution is $x = 1$, when you are asked to solve an equation. However, when you are asked for a "solution set" you should make a special effort to present the answer using the proper notation.

## Understanding Absolute Value

In its basic form $|x|$, the **absolute value**, is one of our fundamental algebraic operations. The absolute value is commonly thought of as providing the distance that a specified number is from zero on a number line. Intuitively, for whatever the inside value is, the result is the value without regard to sign.

**Definition: Absolute Value**

The **absolute value** can be defined as a piecewise equation

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{1.1.12}$$

**Example 1.1.10: Determine the Numbers with a Prescribed Distance from Zero**

Describe all values of $x$ with a distance of 6 from 0.

**Solution**

We want the distance between $x$ and 0 to be equal to 6. We can draw a number line, such as the one in Figure 1.1.5, to represent the condition to be satisfied.

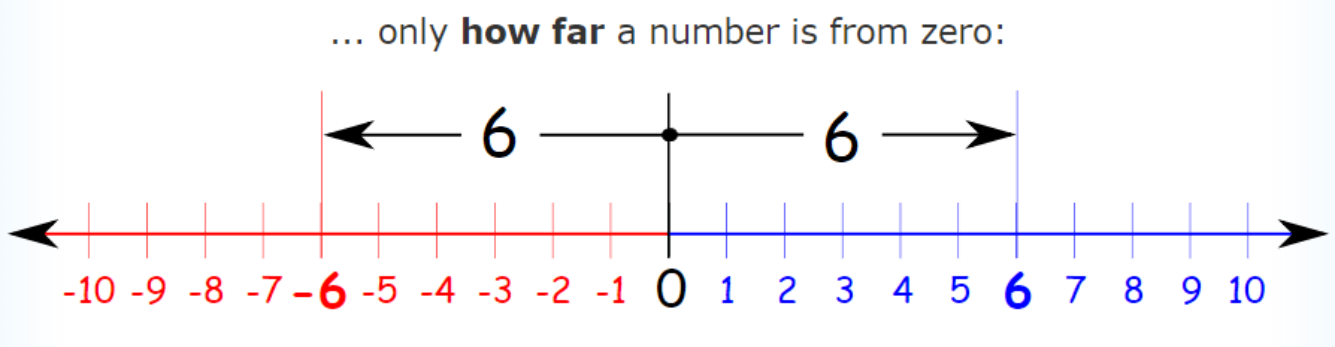



Figure 1.1.5: Number line describing the distance of 6 away from 0.

The distance from $6$ to $0$ can be represented using the absolute value as $|x| = 6$.

$$|x| = 6$$
$$x = -6 \quad \text{or} \quad x = 6$$

## YOU do it! 1.1.6

Describe all values of $x$ for which the distance from 0 is 3.

*Answer available at the end of this section.*

## Example 1.1.12: Determine the Numbers with a Prescribed Distance from another Number

Describe all values $x$ with a distance of 4 from the number 5.

**Solution**

We want the distance between $x$ and 5 to be equal to 4. We can draw a number line, such as the one in , to represent the condition to be satisfied.

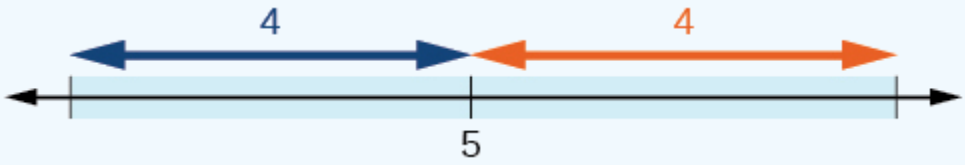



Figure 1.1.2: Number line describing the points with a distance of 4 units away from 5.

The distance from $x$ to 5 can be represented using the absolute value as $|x - 5|$. We want the values of $x$ that satisfy the condition $|x - 5| = 4$.

Note that

$$\begin{aligned} x - 5 &= -4 \quad \text{OR} \quad & x - 5 &= 4 \\ x &= -4 + 5 & x &= 4 + 5 \\ x &= 1 & x &= 9 \end{aligned}$$

So $|x - 5| = 4$ has two possible solutions $x = 1$ or $x = 9$. We may also write $x = 1, 9$

However, often mathematicians will prefer to use set notation when there is more than one solution. So, we say the solution is the set $\{1, 9\}$

**You DO it! 1.1.7**

Describe all values $x$ with a distance of 3 from the number 2.

*Answer available at the end of this section.*

## Solving an Absolute Value Equation

Next, we will learn how to solve an **absolute value equation**. To solve an equation such as $|2x - 6| = 8$, we notice that the absolute value will be equal to $8$ if the quantity inside the absolute value bars is $8$ or $-8$. This leads to two different equations we can solve independently.

$$\begin{aligned} 2x - 6 &= 8 \\ 2x &= 14 \\ x &= 7 \end{aligned}$$

$$\text{OR} \tag{1.1.13}$$

$$\begin{aligned} 2x - 6 &= -8 \\ 2x &= -2 \\ x &= -1 \end{aligned}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

**ABSOLUTE VALUE EQUATIONS**

For an algebraic expression $A$ and a real number $k$, an equation of the form $|A| = k$, with $k > 0$, may have TWO possible solutions when $A = -k$ or $A = k$.

If $k = 0$, the equation $|A| = k$ has ONE solution, $A = 0$.

If $k < 0$, the equation $|A| = k$ has NO solution.

An **absolute value equation** in the form $|ax + b| = c$ has the following properties:

- If $c < 0$,$|ax + b| = c$ has no solution.
- If $c = 0$,$|ax + b| = c$ has one solution.
- If $c > 0$,$|ax + b| = c$ has two solutions.

**How to**

**Given an absolute value equation, solve it.**

1. Isolate the absolute value expression on one side of the equal sign.
2. If $c > 0$, write and solve two equations: $ax + b = c$ or $ax + b = -c$.

*note: in this description, the word "or" means BOTH. Hence, we may interpret it as having the same meaning as the word "and".*

## Soup OR Salad?

A simple way to understand the word "OR" is to think of a restaurant that offers 3 choices with the purchase of any meal.

**1.** Soup

**2.** Salad

**3.** Both Soup and Salad

While it is very uncommon, this restaurant could simplify their menu by simply stating that you may choose "Soup OR Salad".

Then, if you wanted both soup and salad, you could respond by saying "*yes, please*" and you would choosing BOTH! This is because the logical disjunction defined by "A or B" is defined to be true when A is satisfied (soup), B is satisfied (salad), or both A and B are satisfied (soup and salad).




### Example 1.1.13: Solving Absolute Value Equations

Solve the following absolute value equations. Give the solutions in set form.

a. $|6x+4|=8$
b. $|3x+4|=-9$
c. $|3x-5|-4=6$
d. $|-5x+10|=0$

**Solution**

a. $|6x+4|=8$

Write two equations and solve each:

$$\begin{aligned} 6x+4&=8 \\ 6x&=4 \\ x&=\frac{2}{3} \end{aligned}$$

$$\text{OR} \tag{1.1.14}$$

$$\begin{aligned} 6x+4 &= -8 \\ 6x &= -12 \\ x &= -2 \end{aligned}$$

The solution set is $\{\frac{2}{3}, -2\}$.

b. $|3x+4| = -9$

There is no solution as an absolute value cannot be negative.

The solution set is empty. $\{\quad\}$

We may also say that the solution "does not exist" and we abbreviate this as DNE.

c. $|3x-5|-4 = 6$

Isolate the absolute value expression and then write two equations.

$$\begin{aligned} |3x-5|-4 &= 6 \\ |3x-5| &= 10 \\ 3x-5 &= 10 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

OR

$$\begin{aligned} 3x-5 &= -10 \\ 3x &= -5 \\ x &= \frac{5}{3} \end{aligned}$$

There are two solutions: $5$ or $-\frac{5}{3}$.

The solution set is $\{5, -\frac{5}{3}\}$.

d. $|-5x+10| = 0$

The equation is set equal to zero, so we have to write only one equation.

$$\begin{aligned} -5x+10 &= 0 \\ -5x &= -10 \\ x &= 2 \end{aligned}$$

The solution set is $\{2\}$

**You DO it! 1.1.8**

Find the solution set for the absolute value equation: $|1-4x|+8 = 13$.

*Answer available at the end of this section.*

## Key Concepts

- We can solve linear equations in one variable in the form $ax+b=0$ using standard algebraic properties.
- Given a linear equation in the form $ax+b=0$, the equation is conditional when $a \neq 0$
- Given a linear equation in the form $ax+b=0$, the equation is a contradiction when $a=0$ and $b \neq 0$
- Given a linear equation in the form $ax+b=0$, the equation is an identity when $a=0$ and $b=0$

- A conditional linear equation has exactly one solution.
- A contradiction has NO solution.
- An identity has infinitely many solutions.
- The absolute value $|x|$ measures the distance on the real number line from $x$ to $0$.
- For an algebraic expression $A$, an equation of the form $|A| = k$ has TWO solutions if $k > 0$.
- For an algebraic expression $A$, an equation of the form $|A| = 0$ has ONE solution.
- For an algebraic expression $A$, an equation of the form $|A| = k$ has NO solution if $k < 0$.

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## Answers

**You DO it!** 1.1.1

Add texts here.

A. $8 + \frac{x}{3}$

B. $2x + 5 = 9$

C. $7 = 6x - 4$

**YOU do it!** 1.1.2

$x = -5$

The solution is $-5$

**You DO it!** 1.1.3

The solution is $-3$

**Example** 1.1.5

$x = 1$

**You Do it!** 1.1.4

$x = -2$

**You DO it!** 1.1.5

This is a conditional linear equation. It can be written in the form $ax + b = 0$ where $a \neq 0$.

There is ONE solution, $x = 1$

**YOU do it!** 1.1.6

$|x| = 3$

$x = -3$ or $x = 3$

**You DO it!** 1.1.7

$|x - 2| = 3$

$x = -1$ or $x = 5$

The solution is the set $\{-1, 5\}$

**You DO it!** 1.1.8

$\{-1, \frac{3}{2}\}$

---

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# 1.2: Quadratic Equations and Complex Numbers

**Learning Objectives**

- Solve quadratic equations by factoring.
- Solve quadratic equations by the square root property.
- Solve quadratic equations by completing the square.
- Solve quadratic equations by using the quadratic formula.
- Express complex numbers using the imaginary unit $i$.
- Use the discriminant to determine the number and type of solutions for a given quadratic equation.

## Solving Quadratic Equations by Factoring

**Definition: Quadratic Equation**

Any second-degree polynomial equation that can be written in the form

$$ax^2 + bx + c = 0 \tag{1.2.1}$$

where $a, b, c$ are real numbers, and $a \neq 0$ is called a **quadratic equation.**

For example, equations such as $2x^2 + 3x - 1 = 0$ , $x^2 - 4 = 0$ , $-3x^2 + 5x = 0$and are quadratic equations. They are used in countless ways in the fields of engineering, architecture, finance, biological science, and, of course, mathematics.

Often the easiest method of solving a quadratic equation is **factoring**. Factoring means finding expressions that can be multiplied together to give the expression on one side of the equation.

If a quadratic equation can be factored, it is written as a product of linear terms. Solving by factoring depends on the zero-product property, which states that if $a \cdot b = 0$, then $a = 0$ or $b = 0$, where a and b are real numbers or algebraic expressions. In other words, if the product of two numbers or two expressions equals zero, then one of the numbers or one of the expressions must equal zero because zero multiplied by anything equals zero.

Multiplying the factors expands the equation to a string of terms separated by plus or minus signs. So, in that sense, the operation of multiplication undoes the operation of factoring. For example, expand the factored expression $(x - 2)(x + 3)$ by multiplying the two factors together.

$$\begin{aligned}(x-2)(x+3) &= x^2 + 3x - 2x - 6 \\ &= x^2 + x - 6\end{aligned}$$

The product is a quadratic expression. Set equal to zero, $x^2 + x - 6 = 0$ is a quadratic equation. If we were to factor the equation, we would get back the factors we multiplied.

The process of factoring a quadratic equation depends on the leading coefficient, whether it is $1$ or another integer. We will look at both situations; but first, we want to confirm that the equation is written in standard form, $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are real numbers, and $a \neq 0$. The equation $x^2 + x - 6 = 0$ is in standard form.

We can use the zero-product property to solve quadratic equations in which we first have to factor out the greatest common factor(GCF), and for equations that have special factoring formulas as well, such as the difference of squares, both of which we will see later in this section.

### ZERO-PRODUCT PROPERTY AND QUADRATIC EQUATIONS

The **zero-product property** states

$$\text{If } A \cdot B = 0, \text{ then } A = 0 \text{ or } B = 0,$$

where $A$ and $B$ are real numbers or algebraic expressions.

A quadratic equation is an equation containing a second-degree polynomial; for example

$$ax^2 + bx + c = 0 \tag{1.2.2}$$

where $a$, $b$, and $c$ are real numbers, and if $a \neq 0$, it is in standard form.

## Factoring and Solving Quadratics with a Leading Coefficient of 1

In the quadratic equation $x^2 + x - 6 = 0$, the leading coefficient, or the coefficient of $x^2$, is $1$. We have one method of factoring quadratic equations in this form.

### How to: Factor a quadratic equation with the leading coefficient of 1

1. Find two numbers whose product equals $c$ and whose sum equals $b$.
2. Use those numbers to write two factors of the form $(x + k)$ or $(x - k)$, where k is one of the numbers found in step 1. Use the numbers exactly as they are. In other words, if the two numbers are $1$ and $-2$, the factors are $(x + 1)(x - 2)$.
3. Solve using the zero-product property by setting each factor equal to zero and solving for the variable.

### Example 1.2.1: Solving a Quadratic with Leading Coefficient of 1

Factor and solve the equation: $x^2 + x - 6 = 0$.

**Solution**

To factor $x^2 + x - 6 = 0$, we look for two numbers whose product equals $-6$ and whose sum equals $1$. Begin by looking at the possible factors of $-6$.

$$1 \cdot (-6)$$

$$(-6) \cdot 1$$

$$2 \cdot (-3)$$

$$3 \cdot (-2)$$

The last pair, $3 \cdot (-2)$ sums to $1$, so these are the numbers. Note that only one pair of numbers will work. Then, write the factors.

$$(x - 2)(x + 3) = 0$$

To solve this equation, we use the zero-product property. Set each factor equal to zero and solve.

$$\begin{aligned} (x - 2)(x + 3) &= 0 \\ (x - 2) &= 0 \\ x &= 2 \\ (x + 3) &= 0 \\ x &= -3 \end{aligned}$$

The two solutions are $2$ and $-3$.

### You DO it! 1.2.1

Factor and solve the quadratic equation: $x^2 - 5x - 6 = 0$.

*Answer available at the end of this section.*

### Example 1.2.2: Solve the Quadratic Equation by Factoring

Solve the quadratic equation by factoring: $x^2 + 8x + 15 = 0$.

**Solution**

Find two numbers whose product equals $15$ and whose sum equals $8$. List the factors of $15$.

$$1 \cdot 15$$

$$3 \cdot 5$$

$$(-1) \cdot (-15)$$

$$(-3) \cdot (-5)$$

The numbers that add to $8$ are $3$ and $5$. Then, write the factors, set each factor equal to zero, and solve.

$$\begin{aligned} (x+3)(x+5) &= 0 \\ (x+3) &= 0 \\ x &= -3 \\ (x+5) &= 0 \\ x &= -5 \end{aligned}$$

The solutions are $-3$ and $-5$.

### You Do it! 1.2.2

Solve the quadratic equation by factoring: $x^2 - 4x - 21 = 0$.

*Answer available at the end of this section.*

### Example 1.2.3: Using Zero-Product Property to Solve a Quadratic Equation

Solve the difference of squares equation using the zero-product property: $x^2 - 9 = 0$.

**Solution**

Recognizing that the equation represents the difference of squares, we can write the two factors by taking the square root of each term, using a minus sign as the operator in one factor and a plus sign as the operator in the other. Solve using the zero-factor property.

$$
\begin{aligned}
x^2 - 9 &= 0 \\
(x-3)(x+3) &= 0 \\
x - 3 &= 0 \\
x &= 3 \\
(x+3) &= 0 \\
x &= -3
\end{aligned}
$$

The solutions are $3$ and $-3$.

**You DO it! 1.2.3**

Solve by factoring: $x^2 - 25 = 0$.

*Answer available at the end of this section.*

## Factoring and Solving Quadratic Equations with a Leading Coefficient other than 1

When the leading coefficient is not $1$, we factor a quadratic equation using the method called *grouping*, which requires four terms.

**Grouping: Steps for factoring quadratic equations**

With the equation in standard form, let's review the grouping procedures

1. With the quadratic in standard form, $ax^2 + bx + c = 0$, multiply $a \cdot c$.
2. Find two numbers whose product equals $ac$ and whose sum equals $b$.
3. Rewrite the equation replacing the $bx$ term with two terms using the numbers found in step $1$ as coefficients of $x$.
4. Factor the first two terms and then factor the last two terms. The expressions in parentheses must be exactly the same to use grouping.
5. Factor out the expression in parentheses.
6. Set the expressions equal to zero and solve for the variable.

**Example 1.2.4: Solving a Quadratic Equation Using Grouping**

Use grouping to factor and solve the quadratic equation: $4x^2 + 15x + 9 = 0$.

**Solution**

First, multiply $ac : 4(9) = 36$. Then list the factors of $36$.

$$1 \cdot 36$$

$$2 \cdot 18$$

$$3 \cdot 12$$

$$4 \cdot 9$$

$$6 \cdot 6$$

The only pair of factors that sums to $15$ is $3 + 12$. Rewrite the equation replacing the b term, $15x$, with two terms using $3$ and $12$ as coefficients of $x$. Factor the first two terms, and then factor the last two terms.

$$
\begin{aligned}
4x^2+3x+12x+9&=0\\
x(4x+3)+3(4x+3)&=0\\
(4x+3)(x+3)&=0 \qquad \text{Solve using the zero-product property}\\
(4x+3)&=3\\
x&=-\frac{3}{4}\\
(x+3)&=0\\
x&=-3
\end{aligned}
$$

The solutions are $-\frac{3}{4}$, and $-3$.

### You DO it! 1.2.4

Solve using factoring by grouping: $12x^2+11x+2=0$.

*Answer available at the end of this section.*

### Example 1.2.5: Solving a Higher Degree Polynomial Equation by Factoring

Solve the equation by factoring: $-3x^3-5x^2-2x=0$.

**Solution**

This equation does not look like a quadratic, as the highest power is 3, not 2. Recall that the first thing we want to do when solving any equation is to factor out the GCF, if one exists. And it does here. We can factor out $-x$ from all of the terms and then proceed with grouping.

$$
\begin{aligned}
-3x^3-5x^2-2x&=0\\
-x(3x^2+5x+2)&=0\\
-x(3x^2+3x+2x+2)&=0 \qquad \text{Use grouping on the expression in parentheses}\\
-x[3x(x+1)+2(x+1)]&=0\\
-x(3x+2)(x+1)&=0
\end{aligned}
$$

Now, we use the zero-product property. Notice that we have three factors.

$$
\begin{aligned}
-x&=0\\
x&=0\\
3x+2&=0\\
x&=-\frac{2}{3}\\
x+1&=0\\
x&=-1
\end{aligned}
$$

The solutions are $0$, $-\frac{2}{3}$, and $-1$.

### You DO it! 1.2.5

Solve by factoring: $x^3+11x^2+10x=0$.

*Answer available at the end of this section.*

## Using the Square Root Property

When there is no linear term in the equation, another method of solving a quadratic equation is by using the **square root**

**property**, in which we isolate the $x^2$ term and take the square root of the number on the other side of the equals sign. Keep in mind that sometimes we may have to manipulate the equation to isolate the $x^2$ term so that the square root property can be used.

### THE SQUARE ROOT PROPERTY

With the $x^2$ term isolated, the square root property states that:

$$\text{if } x^2 = k, \text{ then } x = \pm\sqrt{k}$$

where $k$ is a nonzero real number.

### Howto: Given a quadratic equation with an $x^2$ term but no $x$ term, use the square root property to solve it

1. Isolate the $x^2$ term on one side of the equal sign.
2. Take the square root of both sides of the equation, putting a $\pm$ sign before the expression on the side opposite the squared term.
3. Simplify the numbers on the side with the $\pm$ sign.

### Example 1.2.6: Solving a Simple Quadratic Equation Using the Square Root Property

Solve the quadratic using the square root property: $x^2 = 8$.

**Solution**

Take the square root of both sides, and then simplify the radical. Remember to use a $\pm$ sign before the radical symbol.

$$\begin{aligned} x^2 &= 8 \\ x &= \pm\sqrt{8} \\ &= \pm 2\sqrt{2} \end{aligned}$$

The solutions are $2\sqrt{2}, -2\sqrt{2}$

### Example 1.2.7: Solving a Quadratic Equation Using the Square Root Property

Solve the quadratic equation: $4x^2 + 1 = 7$.

**Solution**

First, isolate the $x^2$ term. Then take the square root of both sides.

$$\begin{aligned} 4x^2 + 1 &= 7 \\ 4x^2 &= 6 \\ x^2 &= \frac{6}{4} \\ x &= \pm\frac{\sqrt{6}}{2} \end{aligned}$$

The solutions are $\frac{\sqrt{6}}{2}$, and $-\frac{\sqrt{6}}{2}$.

**You DO it! 1.2.6**

Solve the quadratic equation using the square root property: $3(x-4)^2 = 15$.

*Answer available at the end of this section.*

## Completing the Square

Not all quadratic equations can be factored or can be solved in their original form using the square root property. In these cases, we may use a method for solving a **quadratic equation** known as **completing the square**. Using this method, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property. To complete the square, the leading coefficient, $a$, must equal $1$. If it does not, then divide the entire equation by $a$. Then, we can use the following procedures to solve a quadratic equation by completing the square.

We will use the example $x^2 + 4x + 1 = 0$ to illustrate each step.

Given a quadratic equation that cannot be factored, and with $a = 1$, first add or subtract the constant term to the right sign of the equal sign.

$$
\begin{aligned}
x^2 + 4x + 1 &= 0 \\
x^2 + 4x &= -1 \qquad \text{Multiply the b term by } \frac{1}{2} \text{ and square it.} \\
\frac{1}{2}(4) &= 2 \\
2^2 &= 4 \qquad \text{Add 4 to both sides of the equal sign and simplify the right side. We have} \\
x^2 + 4x + 4 &= -1 + 4 \\
x^2 + 4x + 4 &= 3 \qquad \text{The left side of the equation can now be factored as a perfect square.} \\
(x+2)^2 &= 3 \\
\sqrt{(x+2)^2} &= \pm\sqrt{3} \qquad \text{Use the square root property and solve.} \\
\sqrt{(x+2)^2} &= \pm\sqrt{3} \\
x + 2 &= \pm\sqrt{3} \\
x &= -2 \pm \sqrt{3}
\end{aligned}
$$

The solutions are $-2 + \sqrt{3}$, and $-2 - \sqrt{3}$.

**Example 1.2.8: Solving a Quadratic by Completing the Square**

Solve the quadratic equation by completing the square: $x^2 - 3x - 5 = 0$.

**Solution**

First, move the constant term to the right side of the equal sign.

$$x^2 - 3x = 5 \qquad \text{Then, take } \frac{1}{2} \text{ of the b term and square it.}$$

$$\frac{1}{2}(-3) = -\frac{3}{2}$$

$$\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$x^2 - 3x + \left(-\frac{3}{2}\right)^2 = 5 + \left(-\frac{3}{2}\right)^2 \qquad \text{Add the result to both sides of the equal sign.}$$

$$x^2 - 3x + \frac{9}{4} = 5 + \frac{9}{4}$$

Factor the left side as a perfect square and simplify the right side.

$$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4}$$

$$(x - \frac{3}{2}) = \pm\frac{\sqrt{29}}{2} \qquad \text{Use the square root property and solve.}$$

$$x = \frac{3}{2} \pm \frac{\sqrt{29}}{2}$$

The solutions are $\frac{3}{2} + \frac{\sqrt{29}}{2}$, and $\frac{3}{2} - \frac{\sqrt{29}}{2}$

**You DO it! 1.2.7**

Solve by completing the square: $x^2 - 6x = 13$.

*Answer available at the end of this section.*

## Using the Quadratic Formula

The fourth method of solving a **quadratic equation** is by using the **quadratic formula**, a formula that will solve all quadratic equations. Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses when inserting a negative number.

We can derive the quadratic formula by **completing the square**. We will assume that the leading coefficient is positive; if it is negative, we can multiply the equation by $-1$ and obtain a positive a. Given $ax^2 + bx + c = 0, a \neq 0$, we will complete the square as follows:

First, move the constant term to the right side of the equal sign:

$$ax^2 + bx = -c$$

As we want the leading coefficient to equal $1$, divide through by $a$:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Then, find $\frac{1}{2}$ of the middle term, and add $(\frac{1}{2}\frac{b}{a})^2 = \frac{b^2}{4a^2}$ to both sides of the equal sign:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

Next, write the left side as a perfect square. Find the common denominator of the right side and write it as a single fraction:

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

Now, use the square root property, which gives

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Finally, add $-\frac{b}{2a}$ to both sides of the equation and combine the terms on the right side. Thus,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## THE QUADRATIC FORMULA

Written in standard form, $ax^2 + bx + c = 0$, any quadratic equation can be solved using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.2.3}$$

where $a$, $b$, and $c$ are real numbers and $a \neq 0$.

## How to

**Given a quadratic equation, solve it using the quadratic formula**

1. Make sure the equation is in standard form: $ax^2 + bx + c = 0$.
2. Make note of the values of the coefficients and constant term, $a$, $b$, and $c$.
3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula.
4. Calculate and solve.

## Example 1.2.9: Solve the Quadratic Equation Using the Quadratic Formula

Solve the quadratic equation: $x^2 + 5x + 1 = 0$.

**Solution**

Identify the coefficients: $a = 1, b = 5, c = 1$. Then use the quadratic formula.

$$\begin{aligned} x &= \frac{-(5) \pm \sqrt{(5)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{-5 \pm \sqrt{25 - 4}}{2} \\ &= \frac{-5 \pm \sqrt{21}}{2} \end{aligned}$$

## You DO it! 1.2.8

Solve the quadratic equation using the quadratic formula: $9x^2 + 3x - 2 = 0$.

*Answer available at the end of this section.*

## Expressing Square Roots of Negative Numbers as Multiples of $i$

We know how to find the square root of any positive real number. In a similar way, we can find the square root of any negative number. The difference is that the root is not a real number. If the value of the radicand is negative, the root is said to be an imaginary number.

The **imaginary unit** $i$ is defined as the principle square root of $-1$.

$$\sqrt{-1} = i \tag{1.2.4}$$

So, using properties of radicals,

$$i^2 = (\sqrt{-1})^2 = -1 \tag{1.2.5}$$

We can write the square root of any negative number as a multiple of $i$. Consider the square root of $-49$.

$$\begin{aligned}\sqrt{-49} &= \sqrt{49 \cdot (-1)} \\ &= \sqrt{49}\sqrt{-1} \\ &= 7i\end{aligned}$$

We use $7i$ and not $-7i$ because the principal root of $49$ is the positive root.

### How to: Simplify a Square Root using the imaginary Unit $i$

1. Write $\sqrt{-a}$ as $\sqrt{a}\sqrt{-1}$.
2. Re-write $\sqrt{-1}$ as $i$.
3. Express the result $\sqrt{a} \cdot i$ in simplest form.

### Example 1.2.10 Simplifying Square Roots with Negatives under the Radical

Simplify the following square roots.

a. $\sqrt{-25}$

b. $-\sqrt{-4}$

c. $\sqrt{(-6)^2}$

d. $-\sqrt{-5}$

**Solution**

a. $\sqrt{-25} = \sqrt{25} \cdot \sqrt{-1} = 5i$

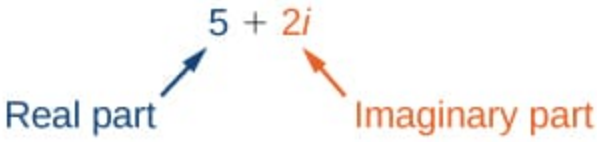
b. $-\sqrt{-4} - \sqrt{4} \cdot \sqrt{-1} = -2i$

c. $\sqrt{(-6)^2} = \sqrt{36} = 6$

d. $\sqrt{-5} = \sqrt{5} \cdot \sqrt{-1} = \sqrt{5}i$ or we may prefer to write this as $i\sqrt{5}$.

## Understanding Complex Numbers

A complex number is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written $a + bi$ where $a$ is the real part and $b$ is the imaginary part. For example, $5 + 2i$ is a complex number. So, too, is $3 + 4i\sqrt{3}$.




Imaginary numbers differ from real numbers in that a squared imaginary number produces a negative real number. Recall that when a positive real number is squared, the result is a positive real number and when a negative real number is squared, the result is also a positive real number. Complex numbers consist of real and imaginary numbers.

### Definition: Imaginary and Complex Numbers

An **imaginary number** is any even root of a negative number. The principal square root of $-1$ is called the *imaginary unit* $i$ $= \sqrt{-1}$

A **complex number** is any number of the form $a + bi$ where $a$ and $b$ are real numbers.

- $a$ is called the *real part* of the complex number.
- $b$ is called the *imaginary part* of the complex number.

If $a = 0$ and $b \neq 0$, the complex number is called a **pure imaginary number**.

If $b = 0$, then the complex number $a + bi$ is a real number, also known as a **pure real number**.

### Example 1.2.11: Expressing a Complex Number in Standard Form

Express $\sqrt{-9}$ in standard form.

**Solution**

$$\begin{aligned}\sqrt{-9} &= \sqrt{9}\sqrt{-1} \\ &= 3i\end{aligned}$$

In standard form, this is $0 + 3i$. This is a pure imaginary number.

### YOU Do it! 1.2.9

Express $\sqrt{-24}$ in standard form.

*Answer available at the end of this section.*

### Example 1.2.12: Expressing a Complex Number in Standard Form

a. $\sqrt{81}$

b. $-\sqrt{-100}$

c. $\sqrt{4} + \sqrt{-16}$

d. $-\sqrt{-9} + \sqrt{1}$

**Solution**

a. $\sqrt{81} = 9$ in standard form this is written as $9 + 0i$. This is a pure real number.

b. $-\sqrt{-100} = -10i$ in standard form this is written as $0 - 10i$. This is a pure imaginary number.

c. $\sqrt{4} + \sqrt{-16} = 2 + 4i$ is already in standard form of a complex number.

d. $-\sqrt{-9} + \sqrt{1} = -3i + 1$ in standard form this is written as $1 - 3i$. A complex (not-real) number.

**Example 1.2.13**

Use the quadratic formula to solve $x^2 + x + 2 = 0$

**Solution**

First, we identify the coefficients: $a = 1$,$b = 1$, and $c = 2$.

Substitute these values into the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{1 - 8}}{2} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2} \end{aligned}$$

## The Discriminant

The **quadratic formula** not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the **discriminant**, or the expression under the radical, $b^2 - 4ac$. The discriminant tells us whether the solutions are real numbers or complex numbers, and how many solutions of each type to expect. Table 1.2.1 relates the value of the discriminant to the solutions of a quadratic equation.

*Table* 1.2.1

| Value of Discriminant | Results |
|---|---|
| $b^2 - 4ac = 0$ | One rational solution (double solution) |
| $b^2 - 4ac > 0$, perfect square | Two rational solutions |
| $b^2 - 4ac > 0$, not a perfect square | Two irrational solutions |
| $b^2 - 4ac < 0$ | Two complex (not real) solutions |

**THE DISCRIMINANT**

For $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are real numbers, the **discriminant** is the expression under the radical in the quadratic formula: $b^2 - 4ac$. It tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

### Example 1.2.14: Using the Discriminant to Find the Nature of the Solutions to a Quadratic Equation

Use the discriminant to find the nature of the solutions to the following quadratic equations:

a. $x^2+4x+4=0$
b. $8x^2+14x+3=0$
c. $3x^2-5x-2=0$
d. $3x^2-10x+15=0$

**Solution**

Calculate the discriminant $b^2-4ac$ for each equation and state the expected type of solutions.

a.

$x^2+4x+4=0$

$b^2-4ac=(4)^2-4(1)(4)=0$ There will be one rational double solution.

b.

$8x^2+14x+3=0$

$b^2-4ac=(14)^2-4(8)(3)=100$ As $100$ is a perfect square, there will be two rational solutions.

c.

$3x^2-5x-2=0$

$b^2-4ac=(-5)^2-4(3)(-2)=49$ As $49$ is a perfect square, there will be two rational solutions.

d.

$3x^2-10x+15=0$

$b^2-4ac=(-10)^2-4(3)(15)=-80$ There will be two complex (not real) solutions.

## Key Equations

| | |
|---|---|
| The Quadratic Formula | $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ |
| The imaginary unit | $i=\sqrt{-1}$ |

## Key Concepts

- Many quadratic equations can be solved by factoring when the equation has a leading coefficient of $1$ or if the equation is a difference of squares. The zero-factor property is then used to find solutions.
- Many quadratic equations with a leading coefficient other than $1$ can be solved by factoring using the grouping method.
- Another method for solving quadratics is the square root property. The variable is squared. We isolate the squared term and take the square root of both sides of the equation. The solution will yield a positive and negative solution.
- Completing the square is a method of solving quadratic equations when the equation cannot be factored.
- A highly dependable method for solving quadratic equations is the quadratic formula, based on the coefficients and the constant term in the equation.

- The discriminant is used to indicate the nature of the roots that the quadratic equation will yield: real or complex, rational or irrational, and how many of each.

## Answers

**You DO it!** 1.2.1

$(x-6)(x+1)=0$, $x=6$, $x=-1$

**You Do it!** 1.2.2

$(x-7)(x+3)=0$, $x=7$, $x=-3$

**You DO it!** 1.2.3

$(x+5)(x-5)=0, x=-5, x=5$

**You DO it!** 1.2.4

$(3x+2)(4x+1)=0$, $x=-\frac{2}{3}$, $x=-\frac{1}{4}$

**You DO it!** 1.2.5

$x=0, x=-10, x=-1$

**You DO it!** 1.2.6

$x=4\pm\sqrt{5}$

**You DO it!** 1.2.7

$x=3\pm\sqrt{22}$

**You DO it!** 1.2.8

$x=-\frac{2}{3}, x=\frac{1}{3}$

**YOU Do it!** 1.2.9

$\sqrt{-24}=0+2i\sqrt{6}$

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# 1.3: Rational and Radical Equations

**Learning Objectives**

- Solve rational equations.
- Solve radical equations.

## Solving Other Types of Equations

We have solved linear equations, rational equations, and quadratic equations using several methods. However, there are many other types of equations, and we will investigate a few more types in this section. We will look at rational and radical equations. Solving any equation, however, employs the same basic algebraic rules. We will learn some new techniques as they apply to certain equations, but the algebra never changes.

## Solving a Rational Equation

**Definition: Rational Equation**

A *rational equation* contains at least one rational expression where the variable appears in at least one of the denominators.

In this section, we look at rational equations that, after some manipulation, result in a linear equation. If an equation contains at least one rational expression, it is a considered a **rational equation**. Recall that a **rational number** is the ratio of two numbers, such as $\frac{2}{3}$ or $\frac{7}{2}$. A rational expression is the ratio, or quotient, of two polynomials. Here are three examples.

$$\frac{x+1}{x^2-4}$$

$$\frac{1}{x-3}$$

or

$$\frac{4}{x^2+x-2}$$

Rational equations have a variable in the denominator in at least one of the terms. Our goal is to perform algebraic operations so that the variables appear in the numerator. In fact, we will eliminate all denominators by multiplying both sides of the equation by the **least common denominator** (LCD). Finding the LCD is identifying an expression that contains the highest power of all of the factors in all of the denominators. We do this because when the equation is multiplied by the LCD, the common factors in the LCD and in each denominator will equal one and will cancel out.

**Example 1.3.1: Solving a Rational Equation**

Solve the rational equation:

$$\frac{7}{2x}-\frac{5}{3x}=\frac{22}{3}$$

**Solution**

We have three denominators; $2x$,$3x$, and $3$. The LCD must contain $2x$,$3x$, and $3$. An LCD of $6x$ contains all three denominators. In other words, each denominator can be divided evenly into the LCD. Next, multiply both sides of the equation by the LCD $6x$.

$$(6x)\left[\frac{7}{2x}-\frac{5}{3x}\right]=\left[\frac{22}{3}\right](6x)$$
$$(6x)\left(\frac{7}{2x}\right)-(6x)\left(\frac{5}{3x}\right)=\left(\frac{22}{3}\right)(6x) \quad \text{Use the distributive property. Cancel out the common factors}$$
$$3(7)-2(5)=22(2x) \quad \text{Multiply remaining factors by each numerator.}$$
$$21-10=44x$$
$$11=44x$$
$$\frac{11}{44}=x$$
$$\frac{1}{4}=x$$

A common mistake made when solving rational equations involves finding the LCD when one of the denominators is a binomial—two terms added or subtracted—such as $(x+1)$. Always consider a binomial as an individual factor—the terms cannot be separated. For example, suppose a problem has three terms and the denominators are $x$, $x-1$, and $3x-3$. First, factor all denominators. We then have $x$, $(x-1)$, and $3(x-1)$ as the denominators. (Note the parentheses placed around the second denominator.) Only the last two denominators have a common factor of $(x-1)$. The x in the first denominator is separate from the $x$ in the $(x-1)$ denominators. An effective way to remember this is to write factored and binomial denominators in parentheses, and consider each parentheses as a separate unit or a separate factor. The LCD in this instance is found by multiplying together the $x$, one factor of $(x-1)$, and the 3. Thus, the LCD is the following:

$$x(x-1)3=3x(x-1)$$

So, both sides of the equation would be multiplied by $3x(x-1)$. Leave the LCD in factored form, as this makes it easier to see how each denominator in the problem cancels out.

Another example is a problem with two denominators, such as $x$ and $x^2+2x$. Once the second denominator is factored as $x^2+2x=x(x+2)$, there is a common factor of $x$ in both denominators and the LCD is $x(x+2)$.

Sometimes we have a rational equation in the form of a proportion; that is, when one fraction equals another fraction and there are no other terms in the equation.

$$\frac{a}{b}=\frac{c}{d} \tag{1.3.1}$$

We can use another method of solving this type of equation without finding the LCD: cross-multiplication. We multiply terms by crossing over the equal sign.

$$\text{If } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a}{b}\times\frac{c}{d}.$$

Multiply a(d) and b(c), which results in $ad=bc$.

Any value that makes a denominator in the original expression equal to zero must be excluded from the possible solutions. This value is called an **exclusion.**

### How to: Given a rational equation, solve it.

1. Factor all denominators in the equation.
2. Find and exclude values that set each denominator equal to zero.
3. Find the LCD.
4. Multiply the whole equation by the LCD. If the LCD is correct, there will be no denominators left.
5. Solve the remaining equation.

6. Make sure to plug in all possible solutions back into the original equation to avoid producing zero in a denominator

## Example 1.3.2: Solving a Rational Equation without Factoring

Solve the following rational equation:

$$\frac{2}{x} - \frac{3}{2} = \frac{7}{2x}$$

**Solution**

We have three denominators: $x$, $2$, and $2x$. No factoring is required. The product of the first two denominators is equal to the third denominator, so, the LCD is $2x$. Only one value is excluded from a solution set, $0$. Next, multiply the whole equation (both sides of the equal sign) by $2x$.

$$\begin{aligned} 2x\left[\frac{2}{x} - \frac{3}{2}\right] &= \left[\frac{7}{2x}\right](2x) \\ 2x\left(\frac{2}{x}\right) - 2x\left(\frac{3}{2}\right) &= \left(\frac{7}{2x}\right)(2x) \qquad \text{Distribute } 2x \\ 2(2) - 3x &= 7 \qquad \text{Denominators cancel out.} \\ 4 - 3x &= 7 \\ -3x &= 3 \\ x &= -1 \text{ or } \{-1\} \end{aligned}$$

The proposed solution is $-1$, which is not an excluded value, so the solution set contains one number, $x = -1$, or $\{-1\}$ written in set notation.

## You DO it! 1.3.1

Solve the rational equation:

$$\frac{2}{3x} = \frac{1}{4} - \frac{1}{6x}$$

*Answer available at the end of this section.*

## Example 1.3.3: Solving a Rational Equation by Factoring the Denominator

Solve the following rational equation:

$$\frac{1}{x} = \frac{1}{10} - \frac{3}{4x}$$

**Solution**

First find the common denominator. The three denominators in factored form are $x, 10 = 2 \cdot 5$, and $4x = 2 \cdot 2 \cdot x$. The smallest expression that is divisible by each one of the denominators is $20x$. Only $x = 0$ is an excluded value. Multiply the whole equation by $20x$.

$$\begin{aligned} 20x\left(\frac{1}{x}\right) &= \left(\frac{1}{10} - \frac{3}{4x}\right)20x \\ 20 &= 2x - 15 \\ 35 &= 2x \\ \frac{35}{2} &= x \end{aligned}$$

The solution is $\frac{35}{2}$.

## You DO it! 1.3.2

Solve the rational equation:

$$-\frac{5}{2x}+\frac{3}{4x}=-\frac{7}{4}$$

*Answer available at the end of this section.*

## Example 1.3.4: Solving Rational Equations with a Binomial in the Denominator

Solve the following rational equations and state the excluded values:

a. $\frac{3}{x-6}=\frac{5}{x}$
b. $\frac{x}{x-3}=\frac{5}{x-3}-\frac{1}{2}$
c. $\frac{x}{x-2}=\frac{5}{x-2}-\frac{1}{2}$

**Solution**

a.

The denominators $x$ and $x-6$ have nothing in common. Therefore, the LCD is the product $x(x-6)$. However, for this problem, we can cross-multiply.

$$\begin{aligned}\frac{3}{x-6}&=\frac{5}{x}\\ 3x&=5(x-6) \qquad \text{Distribute.}\\ 3x&=5x-30\\ -2x&=-30\\ x&=15\end{aligned}$$

The solution is $15$. The excluded values are $6$ and $0$.

b.

The LCD is $2(x-3)$. Multiply both sides of the equation by $2(x-3)$.

$$\begin{aligned}2(x-3)\left[\frac{x}{x-3}\right]&=\left[\frac{5}{x-3}-\frac{1}{2}\right]2(x-3)\\ \frac{2(x-3)x}{x-3}&=\frac{2(x-3)5}{x-3}-\frac{2(x-3)}{2}\\ 2x&=10-(x-3)\\ 2x&=13-x\\ 3x&=13\\ x&=\frac{13}{3}\end{aligned}$$

The solution is $\frac{13}{3}$. The excluded value is $3$.

c.

The least common denominator is $2(x-2)$. Multiply both sides of the equation by $x(x-2)$.

$$2(x-2)\left[\frac{x}{x-2}\right] = \left[\frac{5}{x-2} - \frac{1}{2}\right] 2(x-2)$$
$$2x = 10 - (x-2)$$
$$2x = 12 - x$$
$$3x = 12$$
$$x = 4$$

The solution is $4$. The excluded value is $2$.

### You DO it! 1.3.3

Solve $\frac{-3}{2x+1} = \frac{4}{3x+1}$. State the excluded values.

*Answer available at the end of this section.*

### Example 1.3.5: Solving a Rational Equation with Factored Denominators and Stating Excluded Values

Solve the rational equation after factoring the denominators: $\frac{2}{x+1} - \frac{1}{x-1} = \frac{2x}{x^2-1}$. State the excluded values.

**Solution**

We must factor the denominator $x^2 - 1$. We recognize this as the difference of squares, and factor it as $(x-1)(x+1)$. Thus, the LCD that contains each denominator is $(x-1)(x+1)$. Multiply the whole equation by the LCD, cancel out the denominators, and solve the remaining equation.

$$(x+1)(x-1)\left[\frac{2}{x+1} - \frac{1}{x-1}\right] = \left[\frac{2x}{x^2-1}\right](x+1)(x-1)$$
$$2(x-1) - (x+1) = 2x$$
$$2x - 2 - x - 1 = 2x \text{ Distribute the negative sign}$$
$$-3 - x = 0$$
$$x = -3$$

The solution is $-3$. The excluded values are $1$ and $-1$.

### YOU Do It! 1.3.4

Solve the rational equation:

$$\frac{2}{x-2} + \frac{1}{x+1} = \frac{1}{x^2-x-2}$$

*Answer available at the end of this section.*

## Solving Rational Equations Resulting in a Quadratic

Earlier, we solved rational equations. Sometimes, solving a rational equation results in a quadratic. When this happens, we continue the solution by simplifying the quadratic equation by one of the methods we have seen. It may turn out that there is no solution.

## Example 1.3.6: Solving a Rational Equation Leading to a Quadratic

Solve the following rational equation: $\frac{1}{x+3}-\frac{1}{x+4}=\frac{1}{2}$

### Solution

We want to find the LCD. None of the denominators cannot be factored further. So, the LCD is $2(x+3)(x+4)$.

Next, we multiply the whole equation by the LCD.

$$\begin{aligned} 2(x+3)(x+4)\left(\frac{1}{x+3}-\frac{1}{x+4}\right) &= \left(\frac{1}{2}\right)2(x+3)(x+4) \\ 2(x+4)-2(x+3) &= (x+3)(x+4) \\ 2x+8-2x-6 &= x^2+7x+12 \\ 2 &= x^2+7x+12 \\ x^2+7x+10 &= 0 \\ (x+5)(x+2) &= 0 \\ x &= -5 \\ x &= -2 \end{aligned}$$

The excluded values are $x=-3$ and $x=-4$

Therefore, neither of our candidates are excluded. Hence the solutions are $x=-5,-2$.

The solution set is $\{-5,-2\}$

## Example 1.3.7: Solving a Rational Equation Leading to a Quadratic

Solve the following rational equation: $\frac{-4x}{x-1}+\frac{4}{x+1}=\frac{-8}{x^2-1}$

**Solution**

We want all denominators in factored form to find the LCD. Two of the denominators cannot be factored further. However, $x^2-1=(x+1)(x-1)$. Then, the LCD is $(x+1)(x-1)$. Next, we multiply the whole equation by the LCD.

$$\begin{aligned} (x+1)(x-1)\left(\frac{-4x}{x-1}+\frac{4}{x+1}\right) &= \left(\frac{-8}{x^2-1}\right)(x+1)(x-1) \\ -4x(x+1)+4(x-1) &= -8 \\ -4x^2-4x+4x-4 &= -8 \\ -4x^2+4 &= 0 \\ -4(x^2-1) &= 0 \\ -4(x+1)(x-1) &= 0 \\ x &= -1 \\ x &= 1 \end{aligned}$$

In this case, either solution produces a zero in the denominator in the original equation. Thus, there is no solution.

## YOU Do it! 1.3.5

Solve $\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^2-2x}$

*Answer available at the end of this section.*

# Solving Radical Equations

**Radical equations** are equations that contain variables in the **radicand** (the expression under a radical symbol), such as

$$\sqrt{3x+18}=x$$

$$\sqrt{x+3}=x-3$$

$$\sqrt{x+5}-\sqrt{x-3}=2$$

Radical equations may have one or more radical terms, and are solved by eliminating each radical, one at a time. We have to be careful when solving radical equations, as it is not unusual to find **extraneous solutions**, roots that are not, in fact, solutions to the equation. These solutions are not due to a mistake in the solving method, but result from the process of raising both sides of an equation to a power. However, checking each answer in the original equation will confirm the true solutions.

## Definition: Radical Equation

An equation containing variable terms in the radicand is called a **radical equation.**

## How to: Given a radical equation, solve it

1. Isolate the radical expression on one side of the equal sign. Put all remaining terms on the other side.
2. If the radical is a square root, then square both sides of the equation. If it is a cube root, then raise both sides of the equation to the third power. In other words, for an $n^{th}$ root radical, raise both sides to the $n^{th}$ power. Doing so eliminates the radical symbol.
3. Solve the remaining equation.
4. If a radical term still remains, repeat steps 1–2.
5. Confirm solutions by substituting them into the original equation.

## Example 1.3.8: Solving an Equation with One Radical

Solve $\sqrt{15-2x}=x$.

**Solution**

The radical is already isolated on the left side of the equal side, so proceed to square both sides.

$$\begin{aligned}\sqrt{15-2x}&=x\\ \left(\sqrt{15-2x}\right)^2&=(x)^2\\ 15-2x&=x^2\end{aligned}$$

We see that the remaining equation is a quadratic. Set it equal to zero and solve.

$$\begin{aligned}0&=x^2+2x-15\\ 0&=(x+5)(x-3)\\ x&=-5\\ x&=3\end{aligned}$$

The proposed solutions are $-5$ and $3$. Let us check each solution back in the original equation. First, check $x=-5$.

$$\begin{aligned}\sqrt{15-2x} &= x\\ \sqrt{15-2(-5)} &= -5\\ \sqrt{25} &= -5\\ 5 &\neq -5\end{aligned}$$

This is an extraneous solution. While no mistake was made solving the equation, we found a solution that does not satisfy the original equation.

Check $x = 3$.

$$\begin{aligned}\sqrt{15-2x} &= x\\ \sqrt{15-2(3)} &= 3\\ \sqrt{9} &= 3\\ 3 &= 3\end{aligned}$$

The solution is $3$.

## You DO it! 1.3.6

Solve the radical equation: $\sqrt{x+3} = 3x - 1$

*Answer available at the end of this section.*

## Example 1.3.9: Solving a Radical Equation Containing Two Radicals

Solve $\sqrt{2x+3} + \sqrt{x-2} = 4$

**Solution**

As this equation contains two radicals, we isolate one radical, eliminate it, and then isolate the second radical.

$$\sqrt{2x+3} + \sqrt{x-2} = 4$$

$$\begin{aligned}\sqrt{2x+3} &= 4 - \sqrt{x-2} && \text{Subtract } \sqrt{x-2} \text{ from both sides}\\ \left(\sqrt{2x+3}\right)^2 &= \left(4 - \sqrt{x-2}\right)^2 && \text{Square both sides}\end{aligned}$$

Use the perfect square formula to expand the right side: $(a-b)^2 = a^2 - 2ab + b^2$.

$$\begin{aligned}2x+3 &= (4)^2 - 2(4)\sqrt{x-2} + (\sqrt{x-2})^2\\ 2x+3 &= 16 - 8\sqrt{x-2} + (x-2)\\ 2x+3 &= 14 + x - 8\sqrt{x-2} && \text{Combine like terms}\\ x-11 &= -8\sqrt{x-2} && \text{Isolate the second radical}\\ (x-11)^2 &= (-8\sqrt{x-2})^2 && \text{Square both sides}\\ x^2 - 22x + 121 &= 64(x-2)\end{aligned}$$

Now that both radicals have been eliminated, set the quadratic equal to zero and solve.

$$\begin{aligned}x^2 - 22x + 121 &= 64x - 128\\ x^2 - 86x + 249 &= 0\\ (x-3)(x-83) &= 0\\ x &= 3\\ x &= 83\end{aligned}$$

The proposed solutions are $3$ and $83$. Check each solution in the original equation.

$$\begin{aligned}\sqrt{2x+3}+\sqrt{x-2}&=4\\ \sqrt{2x+3}&=4-\sqrt{x-2}\\ \sqrt{2(3)+3}&=4-\sqrt{(3)-2}\\ \sqrt{9}&=4-\sqrt{1}\\ 3&=3\end{aligned}$$

One solution is $3$.

Check $x=83$.

$$\begin{aligned}\sqrt{2x+3}+\sqrt{x-2}&=4\\ \sqrt{2x+3}&=4-\sqrt{x-2}\\ \sqrt{2(83)+3}&=4-\sqrt{(83)-2}\\ \sqrt{169}&=4-\sqrt{81}\\ 13&\neq -5\end{aligned}$$

The only solution is $3$. We see that $x=83$ is an extraneous solution.

### You DO it! 1.3.6

Solve the equation with two radicals: $\sqrt{3x+7}+\sqrt{x+2}=1$

*Answer available at the end of this section.*

## Key Concepts

- Rational equations can be solved by finding the LCD and multiplying both sides by this expression.
- Check for excluded values. These are values that may appear to be correct, however when you replace the variable in the orignal equation with one of these values, you obtain a zero in the denominator which is undefined.
- We can solve radical equations by isolating the radical and raising both sides of the equation to a power that matches the index.
- Solving a rational or a radical equation may also lead to a quadratic equation or an equation in quadratic form.
- Check for extraneous solutions. These are values that may appear to be correct, however when you replace the variable in the orignal equation with one of these values, you obtain a false statement.

*Answer available at the end of this section.*

## Answers

**You DO it!** 1.3.1

$x=\frac{10}{3}$

**You DO it!** 1.3.2

$x=1$

**You DO it!** 1.3.3

$x=-\frac{7}{17}$. Excluded values are $x=-12$ and $x=-13$.

**YOU Do It!** 1.3.4

$x=\frac{1}{3}$

**YOU Do it!** 1.3.5

The solution is $x=-1$

$x = 0$ is an excluded value, and therefore it is not a solution.

**You DO it!** $1.3.6$

$x = 1$, extraneous solution $x = -\frac{2}{9}$

**You DO it!** $1.3.6$

$x = -2$, extraneous solution $x = -1$

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# 1.4: Linear and Absolute Value Inequalities

**Learning Objectives**

- Use interval notation.
- Use properties of inequalities.
- Solve inequalities in one variable algebraically.
- Solve absolute value equations and inequalities.

## Using Interval Notation

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.

- We can use a number line as shown in Figure $1.4.1$. The blue ray begins at $x = 4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to $4$.




Figure $1.4.1$

- We can use **set-builder notation**: $\{x|x \geq 4\}$, which translates to "all real numbers $x$ such that $x$ is greater than or equal to $4$." Notice that braces are used to indicate a set and the vertical line that appears after the first $x$ variable is to be read "such that".
- The third method is **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $[4, \infty)$. This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

The main concept to remember is that parentheses represent solutions greater than or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled." A few examples of an interval, or a set of numbers in which a solution falls, are $[-2, 6)$, or all numbers between $-2$ and $6$, including $-2$, but not including $6$; $(-1, 0)$, all real numbers between, but not including $-1$ and $0$; and $(-\infty, 1]$, all real numbers less than or equal to $1$. Table $1.4.1$ outlines the possibilities.

*Table* $1.4.1$

| Set Description | Set-Builder Notation | Interval Notation |
|---|---|---|
| All real numbers between $a$ and $b$, but not including $a$ or $b$ | $\{x\|a < x < b\}$ | $(a, b)$ |
| All real numbers greater than $a$, but not including $a$ | $\{x\|x > a\}$ | $(a, \infty)$ |
| All real numbers less than $b$, but not including $b$ | $\{x\|x < b\}$ | $(-\infty, b)$ |
| All real numbers greater than $a$, including $a$ | $\{x\|x \geq a\}$ | $[a, \infty)$ |
| All real numbers less than $b$, including $b$ | $\{x\|x \leq b\}$ | $(-\infty, b]$ |
| All real numbers between $a$ and $b$, including $a$ | $\{x\|a \leq x < b\}$ | $[a, b)$ |
| All real numbers between $a$ and $b$, including $b$ | $\{x\|a < x \leq b\}$ | $(a, b]$ |
| All real numbers between $a$ and $b$, including $a$ and $b$ | $\{x\|a \leq x \leq b\}$ | $[a, b]$ |
| All real numbers less than $a$ or greater than $b$ | $\{x\|x < a \text{ and } x > b\}$ | $(-\infty, a) \cup (b, \infty)$ |
| All real numbers | $\{x\|x \text{ is all real numbers}\}$ | $(-\infty, \infty)$ |

### Example 1.4.1: Using Interval Notation to Express All Real Numbers Greater Than or Equal to a

Use interval notation to indicate all real numbers greater than or equal to $-2$.

**Solution**

Use a bracket on the left of $-2$ and parentheses after infinity: $[-2, \infty)$. The bracket indicates that $-2$ is included in the set with all real numbers greater than $-2$ to infinity.

### Try It! 1.4.1

Use interval notation to indicate all real numbers between and including $-3$ and $5$.

*Answer available at the end of this section.*

### Example 1.4.2: Using Interval Notation to Express All Real Numbers Less Than or Equal to a or Greater Than or Equal to b

Write the interval expressing all real numbers less than or equal to $-1$ or greater than or equal to $1$.

**Solution**

We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to $1$. So, this interval begins at $-\infty$ and ends at $-1$, which is written as $(-\infty, -1]$.

The second interval must show all real numbers greater than or equal to $1$, which is written as $[1, \infty)$. However, we want to combine these two sets. We accomplish this by inserting the union symbol, $\cup$, between the two intervals.

$$(-\infty, -1] \cup [1, \infty)$$

### Try It! 1.4.2

Express all real numbers less than $-2$ or greater than or equal to $3$ in interval notation.

*Answer available at the end of this section.*

## Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the **addition property** and the **multiplication property** to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

### PROPERTIES OF INEQUALITIES

**Addition Property**

- If $a < b$, then $a + c < b + c$.

**Multiplication Property**

- If $a < b$ and $c > 0$, then $ac < bc$.

- If $a < b$ and $c < 0$, then $ac > bc$.

These properties also apply to $a \leq b$, $a > b$, and $a \geq b$.

## Example 1.4.3: Demonstrating the Addition Property

Illustrate the addition property for inequalities by solving each of the following. Give the solution sets using interval notation.

a. $x - 15 < 4$
b. $6 \geq x - 1$
c. $x + 9 \geq 7$

**Solution**

The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

a.

$$\begin{aligned} x - 15 &< 4 \\ x - 15 + 15 &< 4 + 15 \\ x &< 19 \end{aligned}$$

*We read this as "$x$ is less than 19."*

$$(-\infty, 19) \tag{1.4.1}$$

b.

$$\begin{aligned} 6 &\geq x - 1 \\ 6 + 1 &\geq x - 1 + 1 \\ 7 &\geq x \end{aligned}$$

*Note: because the variable is on the right hand side of the inequality. We read this as "$x$ is less than or equal to 7."*

$$(-\infty, 7] \tag{1.4.2}$$

c.

$$\begin{aligned} x + 9 &\geq 7 \\ x + 9 - 9 &\geq 7 - 9 \\ x &\geq -2 \end{aligned}$$

*We read this as "$x$ is greater than or equal to -2."*

$$[-2, \infty) \tag{1.4.3}$$

## Try It! 1.4.3

Solve $3x - 2 < 1$ and give the solution in set-builder notation.

*Answer available at the end of this section.*

## Example 1.4.4: Demonstrating the Multiplication Property

Illustrate the multiplication property for inequalities by solving each of the following. GIve the solutions sing interval notation.

a. $3x < 6$
b. $-2x - 1 \geq 5$
c. $5 - x < 10$

**Solution**

a.

$$\begin{aligned} 3x &< 6 \\ \frac{1}{3}(3x) &< (6)\frac{1}{3} \\ x &< 2 \end{aligned}$$

$$\text{write the result using interval notation} \quad (-\infty, 2) \tag{1.4.4}$$

b.

$$\begin{aligned} -2x - 1 &\geq 5 \\ -2x &\geq 6 \\ \left(-\frac{1}{2}\right)(-2) &\geq (6)\left(-\frac{1}{2}\right) \qquad \text{Multiply by } \left(-\frac{1}{2}\right) \\ x &\leq -3 \qquad \text{Reverse the inequality.} \end{aligned}$$

$$\text{write the result using interval notation} \quad (-\infty, -3] \tag{1.4.5}$$

c.

$$\begin{aligned} 5 - x &< 10 \\ -x &< 5 \\ (-1)(-x) &< (5)(-1) \qquad \text{Multiply by } -1 \\ x &> -5 \qquad \text{Reverse the inequality.} \end{aligned}$$

$$\text{write the result using interval notation} \quad (-5, \infty) \tag{1.4.6}$$

**Try It! 1.4.4**

Solve $4x + 7 \geq 2x - 3$ and give the solution set as an interval.

*Answer available at the end of this section.*

## Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations; we combine like terms and perform operations. To solve, we isolate the variable.

**Example 1.4.5: Solving an Inequality Algebraically**

Solve the inequality: $13 - 7x \geq 10x - 4$ and state the solution set using interval notation, then read the result.

**Solution**

Solving this inequality is similar to solving an equation up until the last step.

$$\begin{aligned} 13 - 7x &\geq 10x - 4 \\ 13 - 17x &\geq -4 \qquad \text{Move variable terms to one side of the inequality} \\ -17x &\geq -17 \qquad \text{Isolate the variable term} \\ x &\leq 1 \qquad \text{Dividing both sides by -17 reverses the inequality.} \end{aligned}$$

The solution set is given by the interval $(-\infty, 1]$, we read this as *"all real numbers less than or equal to* 1".

### Try It! 1.4.5

Solve the inequality and express the result using interval notation: $-x + 4 < \frac{1}{2}x + 1$.

*Answer available at the end of this section.*

### Example 1.4.6: Solving an Inequality with Fractions

Solve the following inequality and give the solution set in interval notation: $-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x$.

**Solution**

We begin solving in the same way we do when solving an equation.

$$-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x$$

$$-\frac{3}{4}x - \frac{2}{3}x \geq -\frac{5}{8} \qquad \text{Put variable terms on one side.}$$

$$-\frac{9}{12}x - \frac{8}{12}x \geq -\frac{5}{8} \qquad \text{Write fractions with common denominator.}$$

$$-\frac{17}{12}x \geq -\frac{5}{8}$$

$$x \leq -\frac{5}{8}\left(-\frac{12}{17}\right) \qquad \text{Multiplying by a negative number reverses the inequality.}$$

$$x \leq \frac{15}{34}$$

The solution set is the interval $\left(-\infty, \frac{15}{34}\right]$.

### Try It! 1.4.6

Solve the inequality and give the solution as an interval: $-\frac{5}{6}x \leq \frac{3}{4} + \frac{8}{3}x$.

*Answer available at the end of this section.*

## Understanding Compound Inequalities

A **compound inequality** is a statement of two or more* inequalities joined together by the compounding words "AND" or "OR". A statement such as $4 < x$ and $x \leq 6$.. Another such statement is $x < -3$ or $x > 5$..

**While it is possible to have statements involving more than two inequalities, such as* $x > -1$ *and* $x \geq 2$ *or* $x < 5$. *The resulting solution sets of these types of propositions would depend upon the interpretation of the implied parenthesis that may not appear. For example, we could read this as "*$x > -1$ *and* $x \geq 2$" *or "*$x < 5$*", on the other hand we could read it as "*$x > -1$*" and "*$x \geq 2$ *or* $x < 5$*". For this reason we shall save this deeper discussion for a course in formal logic and thus restrict our study of compound inequalities to exactly TWO inequalities joined together by one compound word at a time.*

### Example 1.4.7: Solving a compound inequality AND

Solve the compound inequality and give the solution using interval notation: $5x-1>-7$ and $2x\leq 6$

**Solution**

$$\begin{array}{lll} 5x-1>-7 & \text{and} & 2x\leq 6 \\ 5x>-7+1 & \text{and} & \frac{1}{2}x\leq \frac{6}{2} \\ 5x>-6 & \text{and} & x\leq 3 \\ x>\frac{-6}{5} & \text{and} & x\leq 3 \end{array}$$

We read this as " all real numbers greater than $-\frac{6}{5}$ AND less than or equal to $3$.

Hence, the solution set is the interval $(-\frac{6}{5},3]$

### Example 1.4.1: Solving a compound inequality OR

Solve the compound inequality and give the solution using interval notation: $-6x>8$ or $x+5>9$

**Solution**

$$\begin{array}{lll} -6x>8 & \text{or} & x+5>9 \\ \frac{-6x}{-6}>\frac{8}{-6} & \text{or} & x+5-5>9-5 \\ x<\frac{8}{-6} & \text{or} & x>4 \\ x<-\frac{4}{3} & \text{or} & x>4 \end{array}$$

We read this as " all real numbers less than $-\frac{4}{3}$ OR greater than $4$.

Hence, the solution set is the interval $(-\infty,-\frac{4}{3})U(4,\infty)$

## Continued Inequalities

A **continued inequality** is a statement in which the implied compound is the word "AND". These inequalities are written as one statement such as $-1\leq x<2$. We sometimes refer to these as "sandwich" inequalities because in their simplest form, the variable is sandwiched between two real numbers. There are two common methods used to solve continued inequalities:

1. Separating them into two separate inequalities joined by the word "AND"

2. Leaving the compound inequality intact and performing operations on all three parts at the same time.

We will illustrate both methods.

### Example 1.4.7: Solving a Continued Inequality

Solve the continued inequality: $3\leq 2x+2<6$

**Solution**

The first method is to write two separate inequalities: $3 \leq 2x + 2$ and $2x + 2 < 6$. We solve them independently.

$$\begin{array}{ll} 3 \leq 2x + 2 \quad \text{and} & 2x + 2 < 6 \\ 1 \leq 2x & 2x < 4 \\ \frac{1}{2} \leq x & x < 2 \end{array}$$

Then, we can rewrite the solution as a compound inequality, the same way the problem began.

$$\frac{1}{2} \leq x < 2$$

In interval notation, the solution is written as $[\frac{1}{2},\ 2)$.

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.

$$\begin{array}{ll} 3 \leq 2x + 2 < 6 & \\ 1 \leq 2x < 4 & \text{Isolate the variable term, and subtract 2 from all three parts.} \\ \frac{1}{2} \leq x < 2 & \text{Divide through all three parts by 2.} \end{array}$$

We get the same solution: $[\frac{1}{2},\ 2)$.

## Try It! 1.4.7

Solve the continued inequality: $4 < 2x - 8 \leq 10$.

*Answer available at the end of this section.*

## Example 1.4.8: Solving a Continued Inequality with the Variable in All Three Parts

Solve the continued inequality with variables in all three parts: $5x - 10 < 7x - 2 < 3 + x$.

**Solution**

Let's try the first method. Write two inequalities:

$$\begin{array}{ll} 5x - 10 < 7x - 2 \quad \text{and} & 7x - 2 < 3 + x \\ -2x - 10 < -2 & 6x - 2 < 3 \\ -2x < 8 & 6x < 5 \\ x > -4 & x < \frac{5}{6} \end{array}$$

The solution set is $-4 < x < \frac{5}{6}$ or in interval notation $\left(-4, \frac{5}{6}\right)$. Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right, as they appear on a number line. See Figure 1.4.2.

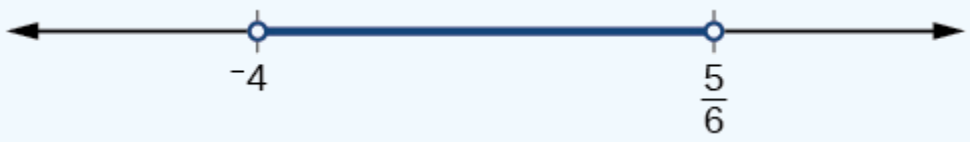



Figure 1.4.2

**Try It! 1.4.8**

Solve the continued inequality: $3y < 4 - 5y < 5 + 3y$.

*Answer available at the end of this section.*

## Summary of Compound Inequalities using AND the Solution Sets as Intervals

Let **a, b** be real numbers such that **a<b**.

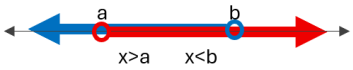
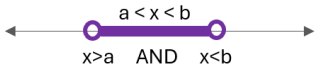
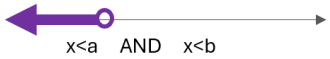
| Inequalities on a number line | Compound Inequality on a number line | Interval Solution |
|---|---|---|
|  |  | DNE |
|  |  | ( a, b ) |
|  |  | (-∞, a ) |
|  |  | ( b, ∞) |

Figure 1.4.3

## Summary of Compound Inequalities using OR the Solution Sets as Intervals

Let **a, b** be real numbers such that **a<b**.

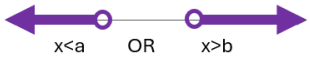
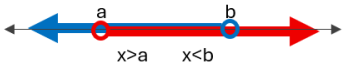
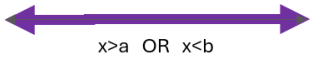
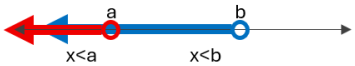
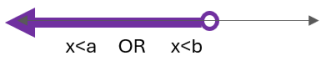
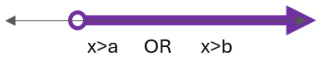
| Inequalities on a number line | Compound Inequality on a number line | Compound Interval |
|---|---|---|
|   |   | (-∞, a) U ( b, ∞) |
|   |   | ( -∞, ∞ ) |
|   |   | (-∞, b ) |
|   |   | ( a, ∞) |

Figure 1.4.4

Note: The above compound inequality summaries can be extended to $\leq$ and $\geq$ by adjusting the interval notation symbols on the endpoints accordingly.

## Understanding Absolute Value

In its basic form $|x|$, the **absolute value**, is one of our fundamental algebraic operations. The absolute value is commonly thought of as providing the distance that a specified number is from zero on a number line. Intuitively, for whatever the inside value is, the result is the value without regard to sign.

### Definition: Absolute Value

The **absolute value** can be defined as a piecewise equation

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{1.4.7}$$

### Example 1.4.9: Determine a Number within a Prescribed Distance

Describe all values of $x$ with a distance of 6 from 0.

**Solution**

We want the distance between $x$ and 0 to be equal to 6. We can draw a number line, such as the one in Figure 1.4.5, to represent the condition to be satisfied.

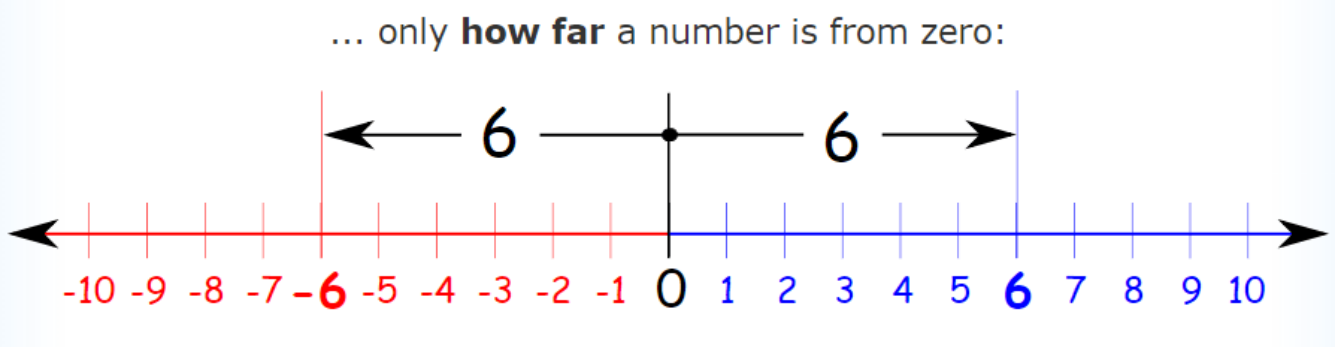



Figure 1.4.5: Number line describing the distance of 6 away from 0.

The distance from $6$ to $0$ can be represented using the absolute value as $|x| = 6$.

$$|x| = 6$$
$$x = -6 \qquad x = 6$$

**Exercise 1.4.9**

Describe all values of $x$ for which the distance from 0 is 3.

*Answer available at the end of this section.*

## Solving an Absolute Value Equation

Next, we will learn how to solve an **absolute value equation**. To solve an equation such as $|2x - 6| = 8$, we notice that the absolute value will be equal to $8$ if the quantity inside the absolute value bars is $8$ or $-8$. This leads to two different equations we can solve independently.

$$2x - 6 = 8$$
$$2x = 14$$
$$x = 7$$

OR

$$2x - 6 = -8$$
$$2x = -2$$
$$x = -1$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

**ABSOLUTE VALUE EQUATIONS**

For real number expressions $A$ and $B$, an equation of the form $|A| = B$, with $B > 0$, will have TWO solutions when $A = B$ or $A = -B$.

If $B = 0$, the equation $|A| = B$ has ONE solution, $A = 0$.

If $B < 0$, the equation $|A| = B$ has no solution.

An **absolute value equation** in the form $|ax+b|=c$ has the following properties:

- If $c<0$,$|ax+b|=c$ has no solution.
- If $c=0$,$|ax+b|=c$ has one solution.
- If $c>0$,$|ax+b|=c$ has two solutions.

How to

**Given an absolute value equation, solve it.**

1. Isolate the absolute value expression on one side of the equal sign.
2. If $c>0$, write and solve two equations: $ax+b=c$ and $ax+b=-c$.

Example 1.4.10: Solving Absolute Value Equations

Solve the following absolute value equations:

a. $|6x+4|=8$
b. $|3x+4|=-9$
c. $|3x-5|-4=6$
d. $|-5x+10|=0$

**Solution**

a. $|6x+4|=8$

Write two equations and solve each:

$$\begin{aligned} 6x+4&=8\\ 6x&=4\\ x&=\frac{2}{3}\end{aligned}$$

OR

$$\begin{aligned} 6x+4&=-8\\ 6x&=-12\\ x&=-2\end{aligned}$$

The two solutions are $\frac{2}{3}$ and $-2$.

b. $|3x+4|=-9$

There is no solution as an absolute value cannot be negative.

c. $|3x-5|-4=6$

Isolate the absolute value expression and then write two equations.

$$\begin{aligned} |3x-5|-4&=6\\ |3x-5|&=10\\ 3x-5&=10\\ 3x&=15\\ x&=5\end{aligned}$$

OR

$$3x - 5 = -10$$
$$3x = -5$$
$$x = \frac{5}{3}$$

There are two solutions: $5$, and $-\frac{5}{3}$.

d. $|-5x + 10| = 0$

The equation is set equal to zero, so we have to write only one equation.

$$-5x + 10 = 0$$
$$-5x = -10$$
$$x = 2$$

There is only one solution, namely $2$.

**Exercise 1.4.10**

Solve the absolute value equation: $|1 - 4x| + 8 = 13$.

*Answer available at the end of this section.*

## Solving Absolute Value Inequalities

As we know, the absolute value of a quantity is a positive number or zero. From the origin, a point located at $(-x,\ 0)$ has an absolute value of $x$, as it is $x$ units away. Consider absolute value as the distance from one point to another point. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

An absolute value inequality is an equation of the form

$$|A| < B, \quad |A| \leq B, \quad |A| > B, \quad \text{or} \quad |A| \geq B,$$

Where $A$, and sometimes $B$, represents an algebraic expression dependent on a variable $x$. Solving the inequality means finding the set of all $x$-values that satisfy the problem. Usually this set will be an interval or the union of two intervals and will include a range of values.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two equations. The advantage of the algebraic approach is that solutions are exact, as precise solutions are sometimes difficult to read from a graph.

To solve absolute value inequalities, just as with absolute value equations, we write two inequalities and then solve them independently.

**Definition: Absolute Value Inequalities**

For an algebraic expression $X$ and $k > 0$, an **absolute value inequality** is an inequality of the form

$$|X| < k \text{ is equivalent to } X > -k \text{ and } X < k$$
$$|X| > k \text{ is equivalent to } X < -k \text{ or } X > k$$

These statements also apply to $|X| \leq k$ and $|X| \geq k$.

## Example 1.4.11: Determining a Number within a Prescribed Distance

Describe all values $x$ for which the distance from the number $5$ is at most $4$.

**Solution**

We want the distance between $x$ and $5$ to be less than or equal to $4$. We can draw a number line, such as in Figure $1.4.6$, to represent the condition to be satisfied.

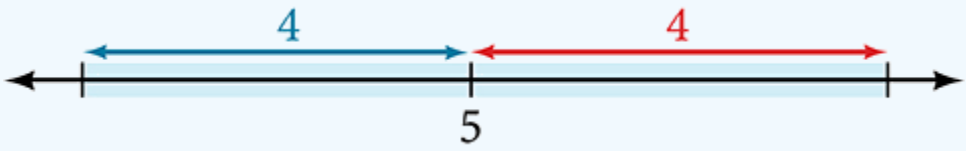



Figure $1.4.6$

The distance from $x$ to $5$ can be represented using an absolute value symbol, $|x-5|$. Write the values of $x$ that satisfy the condition as an absolute value inequality.

$$|x-5| \leq 4$$

We need to write two inequalities as there are always two solutions to an absolute value equation.

$$\begin{array}{lll} x-5 \leq 4 & \text{and} & x-5 \geq -4 \\ x \leq 9 & & x \geq 1 \end{array}$$

If the solution set is $x \leq 9$ and $x \geq 1$, then the solution set is an interval including all real numbers between and including 1 and 9.

So $|x-5| \leq 4$ is equivalent to $[1,9]$ in interval notation.

## Try It! 1.4.11

Describe all $x$-values within a distance of $3$ from the number $2$.

*Answer available at the end of this section.*

## Example 1.4.12: Solving an Absolute Value Inequality

Solve $|x-1| \leq 3$.

**Solution**

$$\begin{aligned} |x-1| &\leq 3 \\ -3 \leq x-1 &\leq 3 \\ -2 \leq x &\leq 4 \\ [-2,4] & \end{aligned}$$

## Example 1.4.13: Solve Absolute Value Inequalities

Given the equation $-\frac{1}{2}|4x-5|+3<0$

**Solution**

We begin by isolating the absolute value.

$$-\tfrac{1}{2}|4x-5| < -3 \qquad \text{Multiply both sides by } -2, \text{ and reverse the inequality.}$$
$$|4x-5| > 6$$

Next, we solve for the equality $|4x-5| > 6$.

$$\begin{array}{lcl} 4x-5<-6 & & 4x-5>6 \\ 4x<-1 & \text{or} & 4x>11 \\ x<-\frac{1}{4} & & x>\frac{11}{4} \end{array}$$

The solution is the disjoint interval

$(-\infty, -\frac{1}{4}) U (\frac{11}{4}, \infty)$

**Try It! 1.4.12**

Solve $-2|k-4| \leq -6$.

*Answer available at the end of this section.*

## Answers

**Try It!** 1.4.1
$[-3, 5]$

**Try It!** 1.4.2
$(-\infty, -2) \cup [3, \infty)$

**Try It!** 1.4.3
$x|x < 1$

**Try It!** 1.4.4
$[5, \infty)$

**Try It!** 1.4.5
$(2, \infty)$

**Try It!** 1.4.6
$\left[-\frac{3}{14}, \infty\right)$

**Try It!** 1.4.7
$6 < x \leq 9$ or $(6, 9]$

**Try It!** 1.4.8
$(-\frac{1}{8}, \frac{1}{2})$

**Exercise** 1.4.9
$|x| = 3$
$x = -3$ or $x = 3$

**Exercise** 1.4.10
$x = -1, x = \frac{3}{2}$

**Try It!** 1.4.11
As an absolute value inequality $|x-2| \leq 3$
As a compound inequality ( implied "and") $-1 \leq x \leq 5$
As a compound inequality ( explicit ) $x \geq -1$ and $x \leq 5$
As an interval $[-1, 5]$

**Try It!** 1.4.12

$k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty, 1] \cup [7, \infty)$.

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# 2: EQUATIONS IN TWO OR MORE VARIABLES

2.1: Rectangular Coordinates and Graphing
2.2: Equations of Lines
2.3: Models and Applications
2.4: Systems of Linear Equations - Two Variables
2.5: Systems of Linear Equations - Three Variables
2.6: Solving Systems with Gaussian Elimination

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# 2.1: Rectangular Coordinates and Graphing

**Learning Objectives**

- Plot ordered pairs in a Cartesian coordinate system.
- Identify the four quadrants.
- Identify the origin and intercepts in the plane.

## René Descartes (1596 - 1650)



An old story describes how seventeenth-century philosopher/mathematician René Descartes invented the system that has become the foundation of algebra while sick in bed. According to the story, Descartes was staring at a fly crawling on the ceiling when he realized that he could describe the fly's location in relation to the perpendicular lines formed by the adjacent walls of his room. He viewed the perpendicular lines as horizontal and vertical axes. Further, by dividing each axis into equal unit lengths, Descartes saw that it was possible to locate any object in a two-dimensional plane using just two numbers—the displacement from the horizontal axis and the displacement from the vertical axis.

While there is evidence that ideas similar to Descartes' grid system existed centuries earlier, it was Descartes who introduced the components that comprise the Cartesian coordinate system, a grid system having perpendicular axes. Descartes named the horizontal axis the $x$-axis and the vertical axis the $y$-axis.

## The Cartesian coordinate system

Also called the *rectangular coordinate system*, is a two-dimensional plane consisting of the $x$-axis and the $y$-axis. Perpendicular to each other, the axes divide the plane into four sections. Each section is called a quadrant; the quadrants are numbered counterclockwise as shown in Figure $2.1.1$.

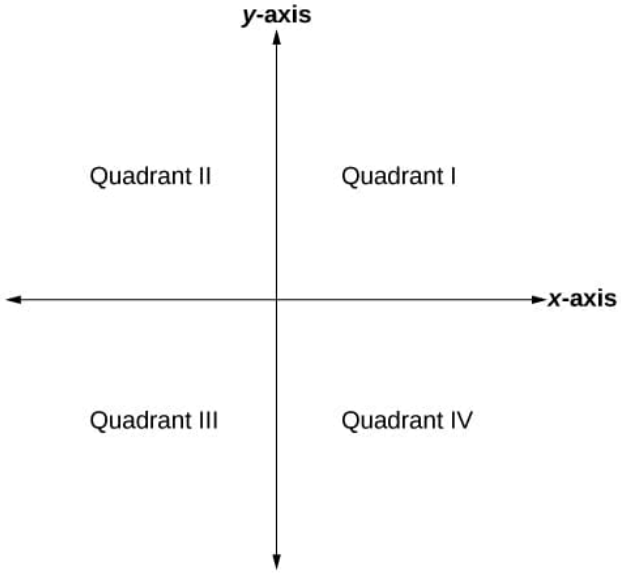



Figure 2.1.1

The center of the plane is the point at which the two axes cross. It is known as the origin, or point $(0, 0)$. From the origin, each axis is further divided into equal units: increasing, positive numbers to the right on the $x$-axis and up the $y$-axis; decreasing, negative numbers to the left on the $x$-axis and down the $y$-axis. The axes extend to positive and negative infinity as shown by the arrowheads in Figure 2.1.2.

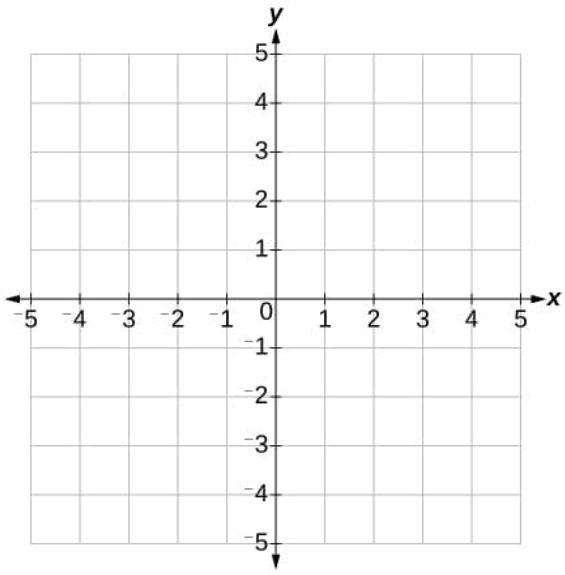



Figure 2.1.2

Each point in the plane is identified by its $x$-coordinate, or horizontal displacement from the origin, and its $y$-coordinate, or vertical displacement from the origin. Together, we write them as an ordered pair indicating the combined distance from the origin in the form $(x, y)$. An ordered pair is also known as a coordinate pair because it consists of $x$- and $y$-coordinates. For example, we can represent the point $(3, -1)$ in the plane by moving three units to the right of the origin in the horizontal direction, and one unit down in the vertical direction. See Figure 2.1.3.

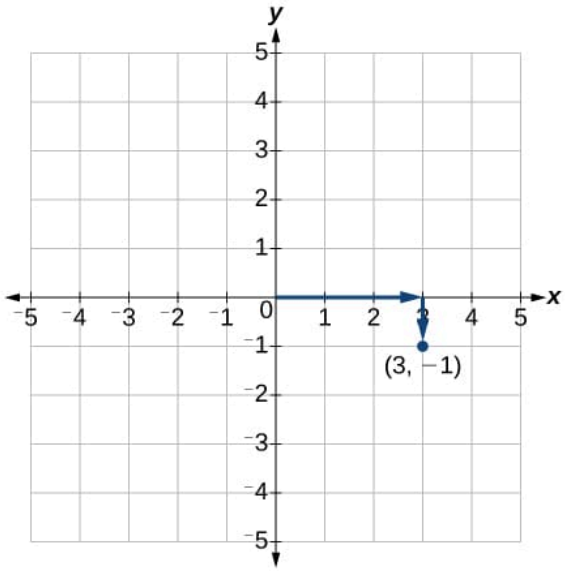



Figure 2.1.3

When dividing the axes into equally spaced increments, note that the $x$-axis may be considered separately from the $y$-axis. In other words, while the $x$-axis may be divided and labeled according to consecutive integers, the $y$-axis may be divided and labeled by increments of $2$, or $10$, or $100$. In fact, the axes may represent other units, such as years against the balance in a savings account, or quantity against cost, and so on. Consider the rectangular coordinate system primarily as a method for showing the relationship between two quantities.

## Cartesian (Rectangular) Coordinate System

A two-dimensional space, called the "plane"

- The $x$-axis is the horizontal axis. (also called the "abscissa")
- The $y$-axis is the vertical axis. (also called the "ordinate")
- The center point (0,0) is called the "origin".

A point in the plane is defined as an ordered pair, $(x, y)$, such that $x$ is determined by its horizontal distance from the origin and $y$ is determined by its vertical distance from the origin. We call this space the **Rectangular Coordinate System**.

## Example 2.1.1: Plotting Points in a Rectangular Coordinate System

Plot the points $(-2, 4)$, $(3, 3)$, and $(0, -3)$ in the plane.

**Solution**

To plot the point $(-2, 4)$, begin at the origin. The $x$-coordinate is $-2$, so move two units to the left. The $y$-coordinate is $4$, so then move four units up in the positive $y$ direction.

To plot the point $(3, 3)$, begin again at the origin. The $x$-coordinate is $3$, so move three units to the right. The $y$-coordinate is also $3$, so move three units up in the positive $y$ direction.

To plot the point $(0, -3)$, begin again at the origin. The $x$-coordinate is $0$. This tells us not to move in either direction along the $x$-axis. The $y$-coordinate is $-3$, so move three units down in the negative $y$ direction. See the graph in Figure 2.1.5.

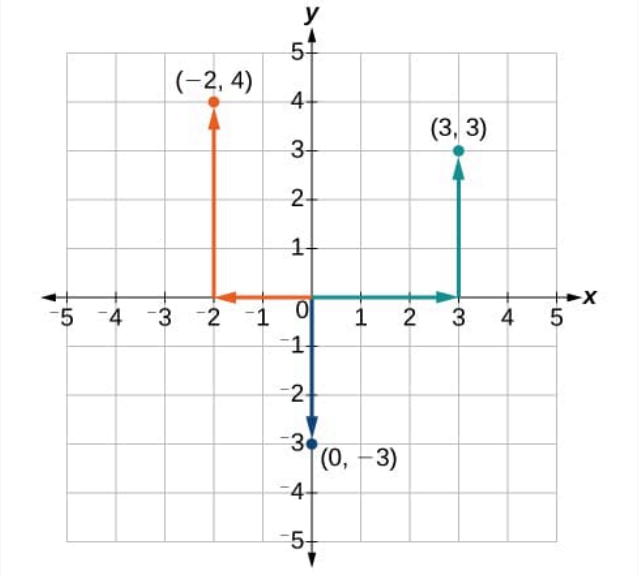
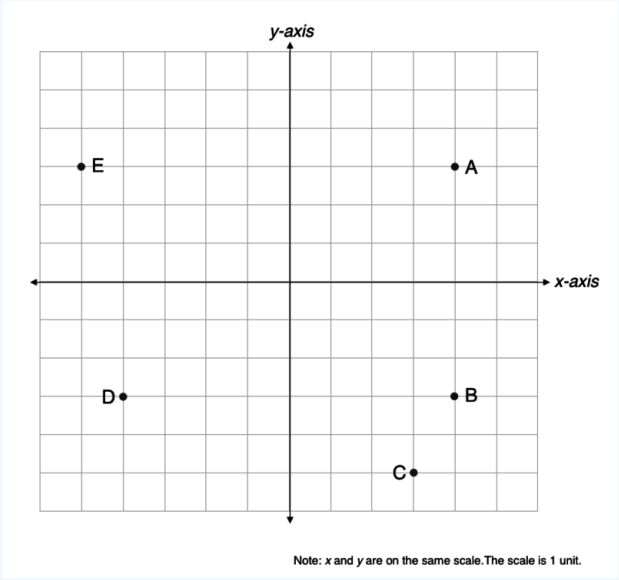



Figure 2.1.5

## You Do it! 2.1.1

Identify the ordered pairs for each point shown in the plane.




*Answer available at the end of this section.*

It is often useful and convenient to be able to identify the quadrant in which a point lies without actually drawing a graph. By inspecting the sign of each coordinate, we can immediately determine the quadrant in which the point lies.

$$Quadrant \quad II \qquad Quadrant \quad I \tag{2.1.1}$$

$$(-,+) \qquad (+,+) \tag{2.1.2}$$

$$Quadrant \quad III \qquad Quadrant \quad IV \tag{2.1.3}$$

$$(-,-) \qquad (+,-) \tag{2.1.4}$$

## Example 2.1.2

Identify the corresponding quadrant in which each point lies.

A. $(3,6)$

B. $(5,-7)$

C. $(-9,-1)$

D. $(-5,6)$

**Solution**

A. Since both coordinates are positive, this point lies in Quadrant I.

B. Since the first coordinate is positive and the second coordinate is negative, this point lies in Quadrant IV.

C. Since both coordinates are negative, this point lies in Quadrant III

D. Since the first coordinate is negative and the second coordinate is positive, this point lies in Quadrant II.

In the same way that we identify the quadrant in which a point lies without actually drawing a graph. We can identify intercepts (points that lie on the $x$ or $y$ axis, or both) by simply looking for the value of $0$ in either the $x$ or $y$ position in the given ordered pair.

$$x-intercepts \tag{2.1.5}$$

$$(-,0) \qquad (+,0) \tag{2.1.6}$$

$$y-intercepts \tag{2.1.7}$$

$$(0,-) \qquad (0,+) \tag{2.1.8}$$

$$x-intercept \quad and \quad y-intercept \tag{2.1.9}$$

$$(0,0) \tag{2.1.10}$$

*Note: the origin $(0,0)$ is the ONLY point in the plane that is both an $x$ and a $y$ intercept.*

## Example 2.1.3

Determine if the given point is an $x$-intercept, a $y$-intercept, or both.

A. $(-4,0)$

B. $(0, 8)$

C. $(0, 0)$

D. $(5, 0)$

**Solution**

A. Since a $0$ appears as the $y$ coordinate in the ordered pair, this point is an $x$-intercept.

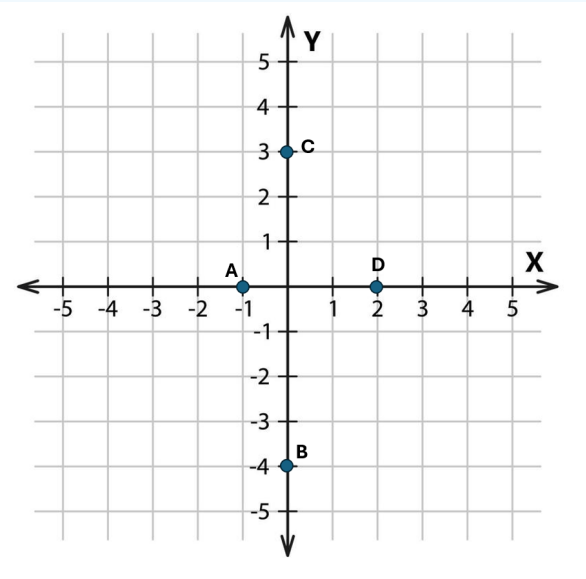
B. Since a $0$ appears as the $x$ coordinate in the ordered pair, this point is a $y$-intercept.

C. Since a $0$ appears as both coordinates in the ordered pair, this point is the origin. Hence, it is both an $x$-intercept, and a $y$ -intercept.

D. Since a $0$ appears as the $y$ coordinate in the ordered pair, this point is an $x$-intercept.

## You Do it! 2.1.2

Indicate the type of intercept for each point shown and state the coordinates as ordered pairs.




*Answer available at the end of this section.*

Ordered pairs are often organized into a table of values. It is traditional to list the $x$ values in the first column, and list the $y$ values in the second column.

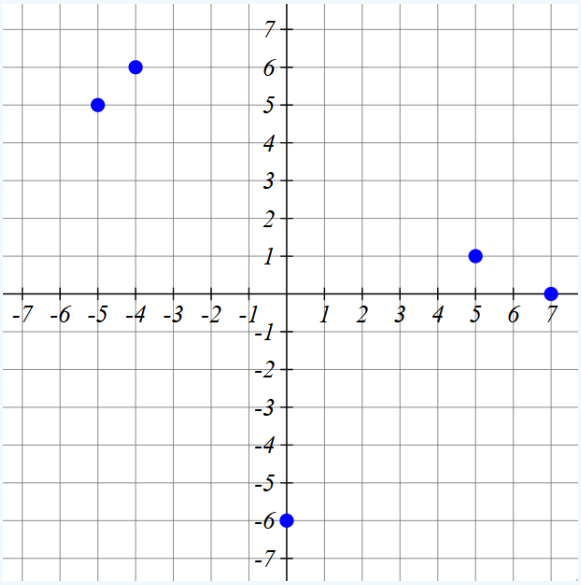
## Example 2.1.4

Plot the ordered pairs given in the table as points in the plane.

| $x$ | $y$ |
|---|---|
| $-5$ | $5$ |
| $-4$ | $6$ |

| | |
|---|---|
| 0 | $-6$ |
| 5 | 1 |
| 7 | 0 |

**Solution**



## Key Concepts

- The rectangular coordinate system is also called the "Cartesian" coordinate system, or the graphing "plane".
- A point is also called an "ordered pair".
- The origin is the point (0,0).
- An x intercept is any point having zero as the y-coordinate.
- A y intercept is any point having zero as the x-coordinate.

## Answers

**You Do it!** 2.1.1

A $(4, 3)$
B $(4, -3)$
C $(3, -5)$
D $(-4, -3)$
E $(-5, 3)$

**You Do it!** 2.1.2

A. x - intercept $(-1, 0)$
B. y - intercept $(0, -4)$
C. y - intercept $(0, 3)$
D. x - intercept $(2, 0)$

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# 2.2: Equations of Lines

**Learning Objectives**

- Find the slope of a line passing through two points.
- Find the equation of a line in slope-intercept form.
- Find the equation of a line in point-slope form.
- Find the equation of a line in standard form.

## Equations of Lines

**Definition: Standard Form of a Linear Equation in Two Variables**

A linear equation in TWO variables can be written in the following **standard** form

$$Ax + By = C \tag{2.2.1}$$

where $A$ , $B$ and $C$ are real numbers, where $A$ and $B$ are NOT BOTH equal to zero.

**Example 2.2.1: Identifying the intercepts of a Line Given an Equation in Standard Form**

Identify the intercepts of the line given by the equation $2x + 3y = 6$.

**Solution**

To find an $x$ - intercept, we must substitute the value of $0$ for the $y$ variable.

$$\begin{aligned} 2x + 3y &= 6 \\ 2x + 3(0) &= 6 \\ 2x + 0 &= 6 \\ 2x &= 6 \\ x &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

Hence, the $x$ - intercept is $(3, 0)$

Similarly, to find the $y$ - intercept, we must substitute the value of $0$ for the $x$ variable.

$$\begin{aligned} 2x + 3y &= 6 \\ 2(0) + 3y &= 6 \\ 0 + 3y &= 6 \\ 3y &= 6 \\ y &= \frac{6}{3} \\ y &= 2 \end{aligned}$$

Therefore, the $y$ - intercept is $(0, 2)$

**Analysis**

The $x$-intercept is the point at which the line crosses the $x$-axis. On the $x$-axis, $y = 0$.

The $y$-intercept is the point at which the line crosses the $y$-axis. On the $y$-axis, $x = 0$.

To visualize this line, simply plot these two points and connect the dots with a straight line.

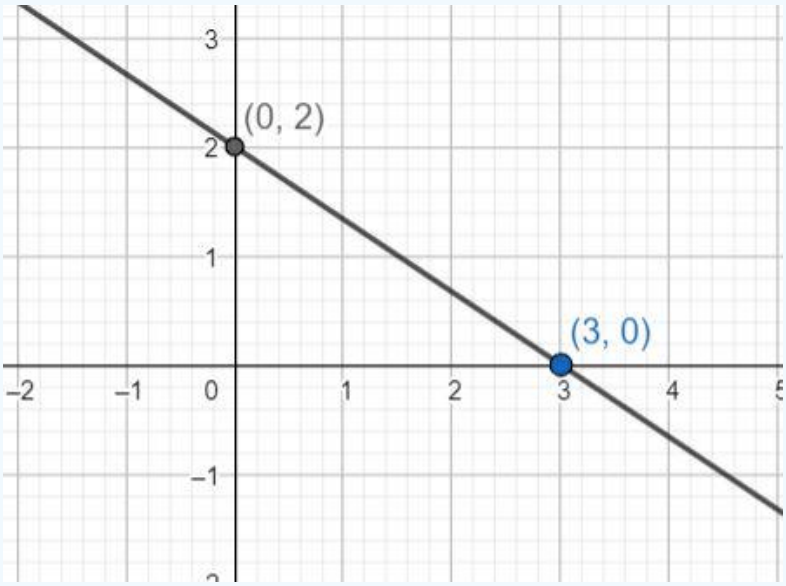



Figure 2.2.1

The **slope** of a line refers to the ratio of the vertical change in $y$ over the horizontal change in $x$ between any two points on a line. It indicates the direction in which a line slants as well as its steepness. Slope is sometimes described as rise over run.

$$m = \frac{\Delta y}{\Delta x} \tag{2.2.2}$$

where $\Delta y$ is the vertical displacement and $\Delta x$ is the horizontal displacement.

**Definition: Slope of a Line**

The **slope** of a line, $m$, represents the change in $y$ divided by the change in $x$. Given two points, $(x_1, y_1)$ and $(x_2, y_2)$, the following formula determines the slope of a line containing these points:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.2.3}$$

## Calculating and Interpreting Slope

In the examples we have seen so far, we have had the slope provided for us. However, we often need to calculate the slope given input and output values. Given two values for the input, $x_1$ and $x_2$, and two corresponding values for the output, $y_1$ and $y_2$—which can be represented by a set of points, $(x_1, y_1)$ and $(x_2, y_2)$—we can calculate the slope $m$, as follows

$$\begin{aligned} m &= \frac{\text{change in y values (rise)}}{\text{change in x values (run)}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \end{aligned}$$

Figure 2.2.3 indicates how the slope of the line between the points, $(x_1, y_1)$ and $(x_2, y_2)$, is calculated. Recall that the slope

measures steepness. The greater the absolute value of the slope, the steeper the line is.

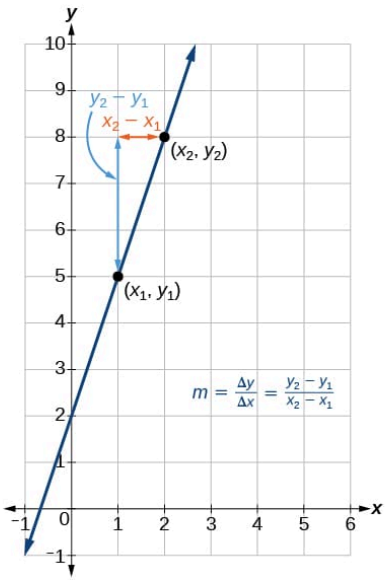



Figure 2.2.3: The slope of a function is calculated by the change in $y$ divided by the change in $x$. It does not matter which coordinate is used as the $(x_2, y_2)$ and which is the $(x_1, y_1)$,as long as each calculation is started with the elements from the same coordinate pair.

## Note: Alternate Formula to Calculate Slope

The slope, $m$,of a line may also be calculated according to the following:

$$m = \frac{y_1 - y_2}{x_1 - x_2} \tag{2.2.4}$$

where $(x_1, y_1)$ and $(x_2, y_2)$ are any two points on the line.

## Example 2.2.2: Finding the Slope of a Line Given Two Points

Find the slope of a line that passes through the points $(2, -1)$ and $(-6, 3)$.

**Solution**

We substitute the $y$-values and the $x$-values into the formula.

$$\begin{aligned} m &= \frac{3 - (-1)}{-6 - 2} \\ &= \frac{3 + 1}{-8} \\ &= \frac{4}{-8} \\ &= -\frac{4}{8} \\ &= -\frac{1}{2} \end{aligned}$$

The slope is $-\frac{1}{2}$

**Analysis**

It does not matter which point is called $(x_1, y_1)$ or $(x_2, y_2)$. As long as we are consistent with the order of the $y$ terms and the order of the $x$ terms in the numerator and denominator, the calculation will yield the same result.

### Exercise 2.2.1

Find the slope of the line that passes through the points $(-2, 8)$ and $(1, 4)$.

*Answer available at the end of this section.*

### Example 2.2.3: Identifying the Slope and y-intercept of a Line Given an Equation

Identify the slope and $y$-intercept, given the equation $y = 2x - 4$.

**Solution**

As the line is in $y = mx + b$ form, the given line has a slope of $2$. The $y$-intercept is $(0, -4)$.

**Analysis**

The slope is an integer, so we can interpret this as a unit ratio $\frac{2}{1}$ when we are thinking of the behavior of the graph.

## The Slope-Intercept Form

Given the slope and the $y$-intercept of a line, we can find the equation of the line using the slope-intercept form.

$$y = mx + b \tag{2.2.5}$$

This is an important equation, as it will be used in several topics throughout college algebra.

If the slope is positive, the line slants to the right. If the slope is negative, the line slants to the left. As the slope increases, the line becomes steeper. Some examples are shown in Figure 2.2.2. The lines indicate the following slopes: $m = -3$, $m = 2$, and $m = \frac{1}{3}$.

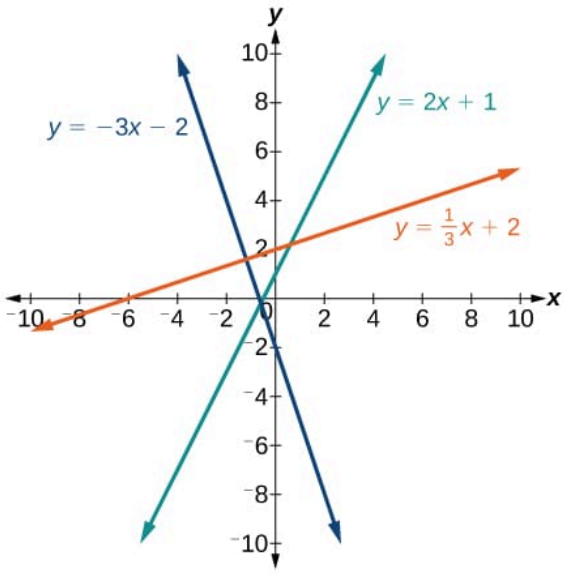



Figure 2.2.2

### The Slope-Intercept Form

Given the slope, $m$ and the $y$-intercept $(0, b)$, the equation of a line in **slope-intercept form** is given by:

$$y = mx + b \tag{2.2.6}$$

### Example 2.2.4: Finding the Equation of a Line Given the Slope and One Point

Write the equation of the line with slope $-3$ and passing through the point $(0, 8)$. Write the final equation in slope-intercept form.

**Solution**

Using the form

$$y = mx + b \tag{2.2.7}$$

Substitute $-3$ for $m$ and since the given point $(0, 8)$ is the $y$ -intercept, we can subsitute the value $8$ for $b$

$$\begin{aligned} y &= mx + b \\ y &= -3x + 8 \end{aligned}$$

### Exercise 2.2.2

Find the equation of the line in slope-intercept form passing through the point $(0, -5)$, having a slope of $4$.

*Answer available at the end of this section.*

## The Point-Slope Form of a Linear Equation

Up until now, we have been using the slope-intercept form of a linear equation to describe linear functions. Here, we will learn another way to write a linear function, the **point-slope form**.

$$y - y_1 = m(x - x_1) \tag{2.2.8}$$

The point-slope form is derived from the slope formula.

$$\begin{aligned} m &= \frac{y - y_1}{x - x_1} && \text{assuming } x \neq x_1 \\ m(x - x_1) &= \frac{y - y_1}{x - x_1}(x - x_1) && \text{Multiply both sides by } (x - x_1). \\ m(x - x_1) &= y - y_1 && \text{Simplify} \\ y - y_1 &= m(x - x_1) && \text{Rearrange} \end{aligned}$$

Keep in mind that the slope-intercept form and the point-slope form can be used to describe the same function. We can move from one form to another using basic algebra. For example, suppose we are given an equation in point-slope form, $y - 4 = -\frac{1}{2}(x - 6)$. We can convert it to the slope-intercept form as shown.

$$\begin{aligned} y - 4 &= -\frac{1}{2}(x - 6) \\ y - 4 &= -\frac{1}{2}x + 3 && \text{Distribute the } -\frac{1}{2}. \\ y &= -\frac{1}{2}x + 7 && \text{Add } 4 \text{ to each side.} \end{aligned}$$

Therefore, the same line can be described in slope-intercept form as $y = \frac{1}{2}x + 7$.

### Point-Slope Form of a Linear Equation

The **point-slope** form of a linear equation is given by:

$$y - y_1 = m(x - x_1) \tag{2.2.9}$$

where $m$ is the slope, $x_1$ and $y_1$ are the $x$ and $y$ coordinates of a specific point through which the line passes.

## Writing the Equation of a Line Using a Point and the Slope

The point-slope form is particularly useful if we know one point and the slope of a line. Suppose, for example, we are told that a line has a slope of 2 and passes through the point $(4, 1)$. We know that $m = 2$ and that $x_1 = 4$ and $y_1 = 1$. We can substitute these values into the general point-slope equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - 4) \end{aligned}$$

If we wanted to then rewrite the equation in slope-intercept form, we apply algebraic techniques.

$$\begin{aligned} y - 1 &= 2(x - 4) & \\ y - 1 &= 2x - 8 & \text{Distribute the 2.} \\ y &= 2x - 7 & \text{Add 1 to each side.} \end{aligned}$$

Both equations, $y - 1 = 2(x - 4)$ and $y = 2x–7$, describe the same line. See Figure 2.2.4.

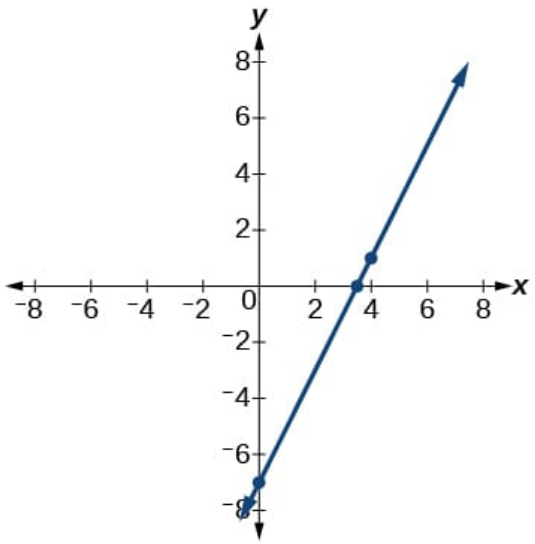



Figure 2.2.4

### Example 2.2.5: Writing Linear Equations Using a Point and the Slope

Write the point-slope form of an equation of a line with a slope of 3 that passes through the point $(6, –1)$. Then rewrite it in the slope-intercept form.

**Solution**

Let's figure out what we know from the given information. The slope is 3, so $m = 3$. We also know one point, so we know $x_1 = 6$ and $y_1 = –1$. Now we can substitute these values into the general point-slope equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-1) &= 3(x - 6) && \text{Substitute known values.} \\ y + 1 &= 3(x - 6) && \text{Distribute } -1 \text{ to find point-slope form.} \end{aligned}$$

Then we use algebra to find the slope-intercept form.

$$\begin{aligned} y + 1 &= 3(x - 6) \\ y + 1 &= 3x - 18 && \text{Distribute 3.} \\ y &= 3x - 19 && \text{Simplify to slope-intercept form.} \end{aligned}$$

**Try it! 2.2.3**

Write the point-slope form of an equation of a line with a slope of –2 that passes through the point $(-2, 2)$. Then rewrite it in the slope-intercept form.

*Answer available at the end of this section.*

## Writing the Equation of a Line Using Two Points

The point-slope form of an equation is also useful if we know any two points through which a line passes. Suppose, for example, we know that a line passes through the points $(0, 1)$ and $(3, 2)$. We can use the coordinates of the two points to find the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 1}{3 - 0} \\ &= \frac{1}{3} \end{aligned}$$

Now we can use the slope we found and the coordinates of one of the points to find the equation for the line. Let use $(0, 1)$ for our point.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{1}{3}(x - 0) \end{aligned}$$

As before, we can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y - 1 &= \frac{1}{3}(x - 0) \\ y - 1 &= \frac{1}{3}x && \text{Distribute the } \frac{1}{3}. \\ y &= \frac{1}{3}x + 1 && \text{Add 1 to each side.} \end{aligned}$$

Both equations describe the line shown in Figure 2.2.5.

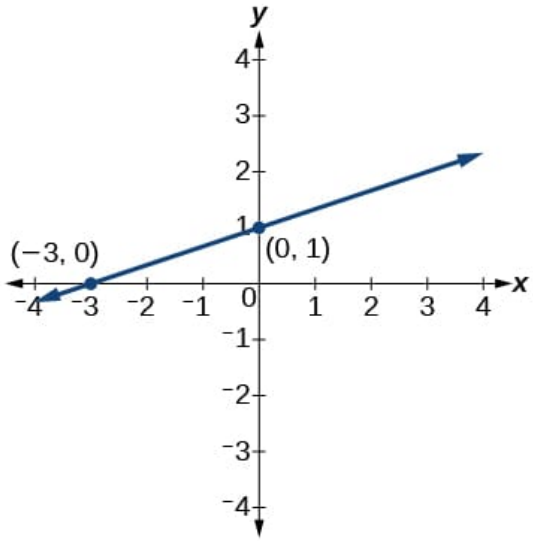



Figure 2.2.5

## Example 2.2.6: Writing Linear Equations Using Two Points

Write the point-slope form of an equation of a line that passes through the points $(5, 1)$ and $(8, 7)$. Then rewrite it in the slope-intercept form.

Let's begin by finding the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{7 - 1}{8 - 5} \\ &= \frac{6}{3} \\ &= 2 \end{aligned}$$

So $m = 2$. Next, we substitute the slope and the coordinates for one of the points into the general point-slope equation. We can choose either point, but we will use $(5, 1)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - 5) \end{aligned}$$

The point-slope equation of the line is $y_2 – 1 = 2(x_2 – 5)$. To rewrite the equation in slope-intercept form, we use algebra.

$$\begin{aligned} y - 1 &= 2(x - 5) \\ y - 1 &= 2x - 10 \\ y &= 2x - 9 \end{aligned}$$

The slope-intercept equation of the line is $y = 2x – 9$.

## Try It! 2.2.4

Write the point-slope form of an equation of a line that passes through the points $(–1, 2)$ and $(2, –7)$. Then rewrite it in the slope-intercept form.

*Answer available at the end of this section.*

### Example 2.2.7: Finding the Equation of a Line Passing Through Two Points

Find the equation of the line passing through the points $(3, 4)$ and $(0, -3)$. Write the final equation in slope-intercept form.

**Solution**

First, we calculate the slope using the slope formula and two points.

$$m = \frac{-3-4}{0-3}$$
$$m = \frac{-7}{-3}$$
$$m = \frac{7}{3}$$

Next, we use the point-slope formula with the slope of $\frac{7}{3}$, and either point. Let's pick the point $(3, 4)$ for $(x_1, y_1)$.

$$y - 4 = \frac{7}{3}(x - 3)$$
$$y - 4 = \frac{7}{3}x - 7$$
$$y = \frac{7}{3}x - 3$$

In slope-intercept form, the equation is written as $y = \frac{7}{3}x - 3$

**Analysis**

To prove that either point can be used, let us use the second point $(0, -3)$ and see if we get the same equation.

$$y - (-3) = \frac{7}{3}(x - 0)$$
$$y + 3 = \frac{7}{3}x$$
$$y = \frac{7}{3}x - 3$$

We see that the same line will be obtained using either point. This makes sense because we used both points to calculate the slope.

## Standard Form of a Line

Recall the definition of a linear equation in TWO variables. When $A$, $B$, and $C$ are integers, this equation is called the **standard form** of a line.

$$Ax + By = C \tag{2.2.10}$$

The $x$- and $y$-terms are on one side of the equal sign and the constant term is on the other side.

### Example 2.2.6: Finding the Equation of a Line and Writing It in Standard Form

Find the equation of the line with $m = -6$ and passing through the point $\left(\frac{1}{4}, -2\right)$. Write the equation in standard form.

**Solution**

We begin using the point-slope formula.

$$y-(-2)=-6\left(x-\frac{1}{4}\right)$$
$$y+2=-6x+\frac{3}{2}$$

From here, we multiply through by $2$, as no fractions are permitted in standard form, and then move both variables to the left aside of the equal sign and move the constants to the right.

$$2(y+2)=\left(-6x+\frac{3}{2}\right)2$$
$$2y+4=-12x+3$$
$$12x+2y=-1$$

This equation is now written in standard form.

**Exercise 2.2.3**

Find the equation of the line in standard form with slope $m=-\frac{1}{3}$ and passing through the point $(1,13)$.

*Answer available at the end of this section.*

## Vertical and Horizontal Lines

Recall the definition of a linear equation in TWO variables requires that $A$ , $B$ and $C$ are real numbers, where $A$ and $B$ are NOT BOTH equal to zero.

$$Ax+By=C \tag{2.2.11}$$

The equations of vertical and horizontal lines occur when ONE of the values $A$ or $B$ is equal to zero.

When $B=0$, the equation represents a **vertical line** and is given by:

$$Ax=C \tag{2.2.12}$$

We simplify this into the following form:

**Definition: Vertical Line**

The equation

$$x=h \tag{2.2.13}$$

where $h$ is a constant, represents a vertical line with an $x$- intercept at $(h,0)$.

The slope of a vertical line is undefined, and regardless of the $y$-value of any point on the line, the $x$-coordinate of every point will be $h$. We can read the equation $x=h$, as "$x$ is ALWAYS equal to $h$."

Suppose that we want to find the equation of a line containing the following points: $(-3,-5)$,$(-3,1)$,$(-3,3)$, and $(-3,5)$. First, we will find the slope.

$$m=\frac{5-3}{-3-(-3)}=\frac{2}{0}$$

Zero in the denominator means that the slope is undefined and, therefore, we cannot use the point-slope, or the slope intercept formulas. However, we can plot the points.

Notice that the y-coordinates vary, however ALL of the $x$-coordinates are the same. This is a vertical line through $x=-3$. See

Figure 2.2.3.

Similarly,

When $A = 0$, the equation represents a **horizontal line** and is given by:

$$By = C \tag{2.2.14}$$

We simplify this into the following form:

**Definition: Horizontal Line**

The equation

$$y = k \tag{2.2.15}$$

where $k$ is a constant, represents a horizontal line with a $y$- intercept at $(0, k)$.

The slope of a horizontal line is zero, and for every $x$-value of any point on the line, the $y$-coordinate will always be $k$. We can read the equation $y = k$, as "$y$ is ALWAYS equal to $k$."

Suppose we want to find the equation of a line that contains the following set of points: $(-2, -2)$,$(0, -2)$,$(3, -2)$, and $(5, -2)$. We can use the point-slope formula. First, we find the slope using any two points on the line.

$$\begin{aligned} m &= \frac{-2-(-2)}{0-(-2)} \\ &= \frac{0}{2} \\ &= 0 \end{aligned}$$

Use any point for $(x_1, y_1)$ in the formula, provided that $y_1 = -2$. In particular, we can use the $y$-intercept $(0, -2)$

$$\begin{aligned} y-(-2) &= 0(x-0) \\ y+2 &= 0 \\ y &= -2 \end{aligned}$$

The graph is a horizontal line through $y = -2$. Notice that the x-coordinates vary, however ALL of the y-coordinates are the same. See Figure 2.2.6.

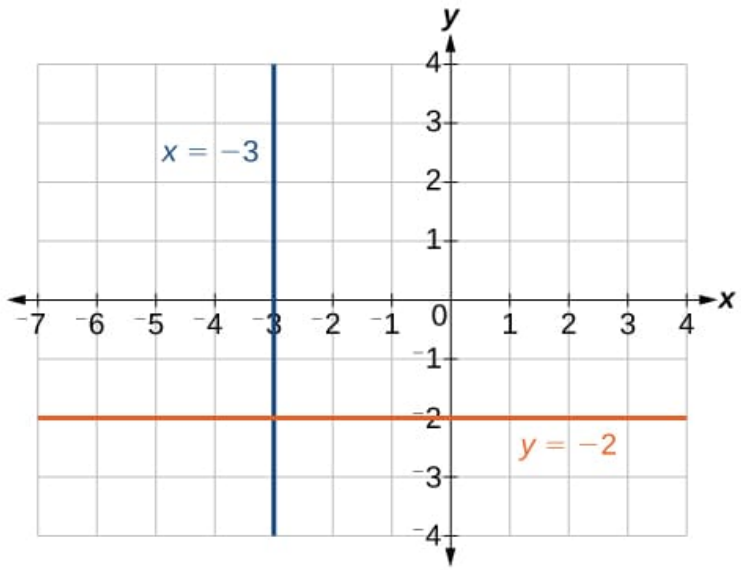



Figure 2.2.6: The line x = –3 is a vertical line. The line y = –2 is a horizontal line

### Example 2.2.7: Finding the Equation of a Line Passing Through the Given Points

Find the equation of the line passing through the given points: $(1, -3)$ and $(1, 4)$.

**Solution**

The $x$-coordinate of both points is $1$. Therefore, we have a vertical line, $x = 1$.

### Exercise 2.2.4

Find the equation of the line passing through $(-5, 2)$ and $(2, 2)$.

*Answer available at the end of this section.*

## Key Concepts

- Given two points, we can find the slope of a line using the slope formula. See Example.
- We can identify the slope and $y$-intercept of an equation in slope-intercept form. See Example.
- We can find the equation of a line given the slope and a point. See Example.
- We can also find the equation of a line given two points. Find the slope and use the point-slope formula. See Example.
- The standard form of a line has no fractions. See Example.
- Horizontal lines have a slope of zero and are defined as $y = c$, where $c$ is a constant.
- Vertical lines have an undefined slope (zero in the denominator), and are defined as $x = c$, where $c$ is a constant. See Example.

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### Answers

**Exercise** 2.2.1

$m = -\frac{4}{3}$

**Exercise** 2.2.2

$y = 4x - 5$

$y - 2 = -2(x + 2)$ ; $y = -2x - 2$

**Try It!** 2.2.4

$y - 2 = -3(x + 1)$; or $y + 7 = -3(x - 2)$; both of these forms smplify to: $y = -3x - 1$.

**Exercise** 2.2.3

$x + 3y = 2$

**Exercise** 2.2.4

Horizontal line: $y = 2$

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# 2.3: Models and Applications

**Learning Objectives**

- Simplify Rates and Unit Rates
- Solve Linear Direct Variation Applications
- Set up a linear equation to solve a real-world application.
- Use a formula to solve a real-world application.

## Rates

We now introduce the concept of *rate*, a special type of ratio.

**Rate**

A *rate* is a quotient of two measurements with different units.

The physical interpretation of a rate in terms of its units is an important skill.

**Example 1:**

An automobile travels 224 miles on 12 gallons of gasoline. Express the ratio distance traveled to gas consumption as a fraction reduced to lowest terms. Write a short sentence explaining the physical significance of your solution. Include units in your description.

**Solution**

Place miles traveled over gallons of gasoline consumed and reduce.

$$\begin{aligned} \frac{224 \text{ mi}}{12 \text{ gal}} &= \frac{56 \cdot 4 \text{ mi}}{3 \cdot 4 \text{ gal}} && \text{Factor.} \\ &= \frac{56 \cdot \cancel{4} \text{ mi}}{3 \cdot \cancel{4} \text{ gal}} && \text{Cancel common factor.} \\ &= \frac{56 \text{ mi}}{3 \text{ gal}} \end{aligned}$$

Thus, the rate is 56 miles to 3 gallons of gasoline. In plain-speak, this means that the automobile travels 56 miles on 3 gallons of gasoline.

## Unit Rates

When making comparisons, it is helpful to have a rate in a form where the denominator is 1. Such rates are given a special name.

**Unit Rate**

A *unit rate* is a rate whose denominator is 1.

## Example 2:

Herman drives 120 miles in 4 hours. Find his average rate of speed.

**Solution**

Place the distance traveled over the time it takes to drive that distance.

$$\begin{aligned}\frac{120 \text{ miles}}{4 \text{ hours}} &= \frac{30 \text{ miles}}{1 \text{ hour}} \quad \text{Divide: } 120/4 = 30.\\ &= 30 \text{ miles/hour}\end{aligned}$$

Hence, Herman's average rate of speed is 30 miles per hour.

## Example 3:

Aditya works 8.5 hours and receives $95 in pay. What is his hourly pay rate?

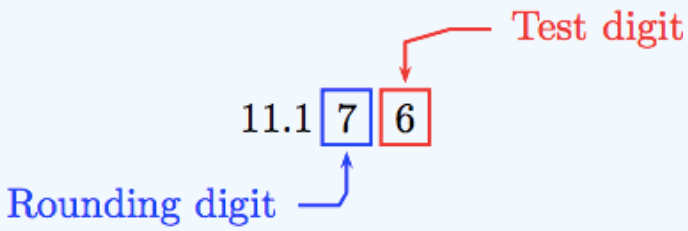
**Solution**

Let's place money earned over hours worked to get the following rate:

$$\frac{95 \text{ dollars}}{8.5 \text{ hours}}$$

We will get a much better idea of Aditya's salary rate if we express the rate with a denominator of 1. To do so, divide. Push the decimal in the divisor to the far right, then move the decimal an equal number of places in the dividend. As we are dealing with dollars and cents, we will round our answer to the nearest hundredth.

$$\begin{array}{r} 11.176 \\ 85\overline{)950.000} \\ 85 \\ \hline 100 \\ 85 \\ \hline 15\,0 \\ 8\,5 \\ \hline 6\,50 \\ 5\,95 \\ \hline 550 \\ 510 \\ \hline 40 \end{array}$$




Because the test digit is greater than or equal to 5, we add 1 to the rounding digit and truncate; i.e., 95/8.5 ≈ 11.18. Hence,

$$\begin{aligned}\frac{95 \text{ dollars}}{8.5 \text{ hours}} &= \frac{11.18 \text{ dollars}}{1 \text{ hour}} \quad \text{Divide: } 95/8.5 \approx 11.18.\\ &= 11.18 \text{ dollars/hour.}\end{aligned}$$

That is, his pay rate is 11.18 dollars per hour.

Example 4:

One automobile travels 422 miles on 15 gallons of gasoline. A second automobile travels 354 miles on 13 gallons of gasoline. Which automobile gets the better gas mileage?

**Solution**

Decimal division (rounded to the nearest tenth) reveals the better gas mileage.

| | |
|---|---|
| In the case of the first automobile, we get the following rate: $$\frac{422 \text{ mi}}{15\text{gal}}$$ | I |
| Divide. $$\begin{array}{r} 28.13 \\ 15\overline{)422.00} \\ \underline{30} \\ 122 \\ \underline{120} \\ 20 \\ \underline{15} \\ 50 \\ \underline{45} \\ 5 \end{array}$$ | D |
| To the nearest tenth, 28.1. | T |

In the case of the first automobile, the mileage rate is 28.1 mi/1 gal, which can be read “28.1 miles per gallon.” In the case of the second automobile, the mileage rate is 27.2 mi/1 gal, which can be read “27.2 miles per gallon.” Therefore, the first automobile gets the better gas mileage.

## Solving Linear Direct Variation Applications

A used-car company has just offered their best candidate, Nicole, a position in sales. The position offers 16% commission on her sales. Her earnings depend on the amount of her sales. For instance, if she sells a vehicle for $4,600, she will earn $736. She wants to evaluate the offer, but she is not sure how. In this section, we will look at relationships, such as this one, between earnings, sales, and commission rate.

Nicole’s earnings can be found by multiplying her sales by her commission. The formula $e = 0.16s$ tells us her earnings, $e$, come from the product of 0.16, her commission, and the sale price of the vehicle. If we create a table, we observe that as the sales price increases, the earnings increase as well, which should be intuitive. See **Table 5.8.1**.

| $s$, sales price | $e = 0.16s$ | Interpretation |
|---|---|---|
| $9,200 | $e = 0.16(9,200) = 1,472$ | A sale of a $9,200 vehicle results in $1472 earnings. |
| $4,600 | $e = 0.16(4,600) = 736$ | A sale of a $4,600 vehicle results in $736 earnings. |
| $18,400 | $e = 0.16(18,400) = 2,944$ | A sale of a $18,400 vehicle results in $2944 earnings. |

**Table 5.8.1**

Notice that earnings are a multiple of sales. As sales increase, earnings increase in a predictable way. Double the sales of the vehicle from $4,600 to $9,200, and we double the earnings from $736 to $1,472. As the input increases, the output increases as a multiple of the input. A relationship in which one quantity is a constant multiplied by another quantity is called **direct variation**. Each variable in this type of relationship **varies directly** with the other.

Figure 5.8.1 represents the data for Nicole's potential earnings. We say that earnings vary directly with the sales price of the car. The formula $y = kx$ is used for direct variation. The value $k$ is a nonzero constant greater than zero and is called the **constant of variation**. In this case, $k = 0.16$ and $n = 1$. We saw functions like this one when we discussed power functions.

Figure 5.8.1

## A General Note: DIRECT VARIATION

If $x$ and $y$ are related by an equation of the form

$$y = kx$$

then we say that the relationship is **direct variation** and $y$ **varies directly** with, or is proportional to $x$. In direct variation relationships, there is a nonzero constant ratio $k = \frac{y}{x}$, where $k$ is called the **constant of variation**, which help defines the relationship between the variables.

## Given a description of a direct variation problem, solve for an unknown.

1. Identify the units for the variable, $x$,and the units for the variable, $y$.
2. Determine the constant of variation. You may need to divide $y$ by the specified power of $x$ to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

## Example 5:

Solving a Direct Variation Problem

The number of kilometers $y$ varies directly with the number of miles $x$. If $y = 100$ km, then $x = 60$ mi, find the number of miles $500$ that is equivalent to kilometers.

**Solution**

The general linear formula for direct variation is $y = kx$. The constant can be found by dividing $y$ by $x$.

$$k = \frac{y}{x}$$

$$= \frac{100}{60}$$

$$= \frac{5}{3}$$

Now use the constant to write an equation that represents this relationship.

$$y = \frac{5}{3}x$$

Substitute $y = 500$ and solve for $x$.

$$500 = \frac{5}{3}x$$

$$1500 = 5x$$

$$x = \frac{1500}{5}$$

$$x = 300$$

**Answer:** 500 kilometers is equivalent to 300 miles.

## Q&A

**Do the graphs of all linear direct variation equations pass through the origin?**

*YES. Linear Direct variation equations are always of the form $y = kx$, since $0 = k * 0$ for any constant of variation $k$, all of these graphs pass through $(0, 0)$.*

## Example 6:

The mass of an object in kilograms varies directly with the weight in pounds. If $y = 100$ kg, then $x = 220$ lbs, find the mass in kilograms of an object that weights 55 lbs.

**Solution**

$k = \frac{100}{220}$

$k = \frac{5}{11}$

$y = \frac{5}{11}x$

since $x = 55$

$y = \frac{5}{11}(55)$

$y = 25$

**Answer**

An object that weights 55 lbs has a mass of 25 kilograms.

# Using Rates and Direct Variation in a Formula

## Example 7:

It takes Andrew $30\ min$ to drive to work in the morning. He drives home using the same route, but it takes 10 minutes longer, and he averages $10\ mi/h$ less than in the morning. How far does Andrew drive to work?

**Solution**

This is a distance problem, so we can use the formula $d = rt$, where distance equals rate multiplied by time. Note that when rate is given in $mi/h$, time must be expressed in hours. Consistent units of measurement are key to obtaining a correct solution.

First, we identify the known and unknown quantities. Andrew's morning drive to work takes $30\ min$, or $\frac{1}{2}\ h$ at rate $r$. His drive home takes $40\ min$, or $\frac{2}{3}\ h$, and his speed averages $10\ mi/h$ less than the morning drive. Both trips cover distance $d$. A table, such as Table 2.3.2, is often helpful for keeping track of information in these types of problems.

*Table* 2.3.2

| | $d$ | $r$ | $t$ |
|---|---|---|---|
| To Work | $d$ | $r$ | $\frac{1}{2}$ |
| To Home | $d$ | $r-10$ | $\frac{2}{3}$ |

Write two equations, one for each trip.

$$d = r\left(\frac{1}{2}\right) \qquad \text{To work}$$

$$d = (r-10)\left(\frac{2}{3}\right) \qquad \text{To home}$$

As both equations equal the same distance, we set them equal to each other and solve for $r$.

$$\begin{aligned} r\left(\frac{1}{2}\right) &= (r-10)\left(\frac{2}{3}\right) \\ \frac{1}{2}r &= \frac{2}{3}r - \frac{20}{3} \\ \frac{1}{2}r - \frac{2}{3}r &= -\frac{20}{3} \\ -\frac{1}{6}r &= -\frac{20}{3} \\ r &= -\frac{20}{3}(-6) \\ r &= 40 \end{aligned}$$

We have solved for the rate of speed to work, $40\ mph$. Substituting $40$ into the rate on the return trip yields $30mi/h$. Now we can answer the question. Substitute the rate back into either equation and solve for $d$.

$$\begin{aligned} d &= 40\left(\frac{1}{2}\right) \\ &= 20 \end{aligned}$$

The distance between home and work is $20\ mi$.

**Analysis**

Note that we could have cleared the fractions in the equation by multiplying both sides of the equation by the LCD to solve for $r$.

$$\begin{aligned} r\left(\frac{1}{2}\right) &= (r-10)\left(\frac{2}{3}\right) \\ 6 \times r\left(\frac{1}{2}\right) &= 6 \times (r-10)\left(\frac{2}{3}\right) \\ 3r &= 4(r-10) \\ 3r &= 4r-40 \\ r &= 40 \end{aligned}$$

## Building Linear Models

When **modeling** scenarios with linear equations and solving problems involving quantities with a **constant rate of change**, we typically follow the 5 step method.

We list the strategy here, and then use it to solve some problems.

### A PROBLEM-SOLVING STRATEGY.

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for and choose a variable to represent that quantity.
3. **Translate** into an equation. It may be helpful to restate the problem, then translate into an algebraic equation.
4. **Solve** the equation using algebraic techniques.
5. **Check** the answer in the problem and make sure it makes sense. Clearly state your result using appropriate units.

### Example 8:

Emily is a college student who plans to spend a summer in Seattle. She has saved $3,600 for her trip and anticipates spending $400 each day on food, lodging and activities. How many days can Emily remain in Seattle before her money runs out?

There are two changing quantities: time and money. The amount of money she has remaining while on vacation *depends* on how long she stays.

We can use this information to define our variables, including units.

- Dependent: M, money remaining, in dollars
- Independent: t, time, in days

So, the amount of money M, remaining depends on the number of days t

We can also identify the initial value and the rate of change.

- Initial Value: She saved $3,600, so 3600 is the initial value for M.
- Rate of Change: She anticipates spending $400 each day, so –400 is the slope.

Notice that the unit of dollars per day matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each day.

Since this student is spending the same amount of money each day, the **rate of change is constant**,

so we can start with the linear model $M = mt + b$.

Then we can substitute $m = -400$ for the slope and $b = 3600$ for the initial value.

$M = -400t + 3600$

**Solution**

To find the t-intercept, we set the amount of money to zero, and solve for the time.

0=−400t+3600

400t = 3600

t=9

**Answer**:

The t-intercept is $(9, 0)$ we say that Emily will have no money left after 9 days.

## Using a Given Intercept to Build a Model

Some real-world problems provide the y-intercept, which is the constant or initial value. Once the y-intercept is known, the x-intercept can be calculated.

Data can be used to construct functions that model real-world applications. Once an equation that fits given data is determined, we can use the equation to make certain predictions; this is called **mathematical modeling**[35].

Example 9:

Suppose that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is $1,000. She plans to pay $250 per month until her balance is $0. The y-intercept is the initial amount of her debt, or $1,000. The rate of change, or slope, is -$250 per month. We can then use the slope-intercept form and the given information to develop a linear model, $y = mx + b$

**Solution**

$y = -250x + 1000$

Now we can set y equal to 0, and solve for x to find the x-intercept.

$0 = -250x + 1000$

$250x = 1000$

$x = 4$

The $x$-intercept is the number of months it takes her to reach a balance of $0.

**Answer**:

The x-intercept is 4 months, so it will take Hannah four months to pay off her loan.

## Using Two Given Pairs of Values to Build a Model

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear model. We might sometimes instead be asked to evaluate the linear model at a given value or set the equation of the linear model equal to a specified value.

## Example 10:

Using a Linear Model to Investigate a Town's Population

A town's population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. Assume this trend continues.

1. Predict the population in 2013.
2. Identify the year in which the population will reach 15,000.

**Solution**

The two changing quantities are the population size and time. While we could use the actual year value as the input quantity, doing so tends to lead to very cumbersome equations because the y-intercept would correspond to the year 0, more than 2000 years ago!

To make computation a little nicer, we will define our input as the number of years since 2004:

- Independent: t, years since 2004
- Dependent: P, the town's population

To predict the population in 2013 (t=9), we would first need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the number of years that would produce a population of 15,000. To write an equation, we need the initial value and the rate of change, or slope.

To determine the rate of change, we will use the change in output per change in input.

m=change in population/change in time

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to t=0, giving the point (0,6200). Notice that through our clever choice of variable definition, we have "given" ourselves the y-intercept of the function. The year 2009 would correspond to t=5, giving the point (5,8100).

The two coordinate pairs are (0,6200) and (5,8100). Recall that we encountered examples in which we were provided two points earlier in the chapter. We can use these values to calculate the slope.

$m = \frac{8100-6200}{5-0}$

= 380 people per year

We already know the y-intercept of the line, so we can immediately write the equation:

P=380t+6200

To predict the population in 2013, we evaluate our equation at t=9.

P=380(9)+6,200

P=9,620

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set P=15000 and solve for t.

15000=380t+6200

8800 = 380t

t≈23.158

**Answer**: Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

## Modeling Cost Applications

Data can be used to construct functions that model real-world applications. Once an equation that fits given data is determined, we can use the equation to make certain predictions; this is called **mathematical modeling**[35].

### Example 11:

The cost of a daily truck rental is $\$48.00$, plus an additional $\$0.45$ for every mile driven. Write a function that gives the cost of the daily truck rental and use it to determine the total cost of renting the truck for a day and driving it $60$ miles.

**Solution**

The total cost of the truck rental depends on the number of miles driven. If we let $x$ represent the number of miles driven, then $0.45x$ represents the variable cost of renting the truck. Use this and the fixed cost, $\$48.00$, to write a function that models the total cost,

$$C = 0.45x + 48$$

Use this function to calculate the cost of the rental when $x = 60$ miles.

$$\begin{aligned} C &= 0.45(60) + 48 \\ &= 27 + 48 \\ &= 75 \end{aligned}$$

**Answer**:

The total cost of renting the truck for the day and driving it $60$ miles would be $\$75$.

We can use the model $C = 0.45x + 48$ to answer many more questions. For example, how many miles can be driven to keep the cost of the rental at most $\$66$? To answer this question, set up an inequality that expresses the cost less than or equal to $\$66$.

$$\begin{aligned} C &\leq \$66 \\ 0.45x + 48 &\leq 66 \end{aligned}$$

Solve for $x$ to determine the number of miles that can be driven.

$$\begin{aligned} 0.45x + 48 &\leq 66 \\ 0.45x &\leq 18 \\ x &\leq 40 \end{aligned}$$

To limit the rental cost to $\$66$, the truck can be driven $40$ miles or less.

## Modeling Depreciation of Value

### Example 12:

A company purchased a new piece of equipment for $\$12,000$. Four years later it was valued at $\$9,000$ dollars. Use this data to construct a linear function that models the value of the piece of equipment over time.

**Solution**

The value of the item depends on the number of years after it was purchased. Therefore, the age of the piece of equipment is the independent variable. Use ordered pairs where the $x$-values represent the age and the $y$-values represent the corresponding value.

$$(age, value)$$

From the problem, we can determine two ordered pairs. Purchased new $(age = 0)$, the item cost $\$12,000$, and $4$ years later the item was valued at $\$9,000$. Therefore, we can write the following two $(age, value)$ ordered pairs:

$$(0,\ 12000) \quad \text{and} \quad (4,\ 9000)$$

Use these two ordered pairs to construct a linear model. Begin by finding the slope $m$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{9000 - 12000}{4 - 0} \\ &= \frac{-3000}{4} \\ &= -750 \end{aligned}$$

Here we have $m = -750$. The ordered pair $(0, 12000)$ gives the $y$-intercept; therefore, $b = 12000$.

$$\begin{aligned} y &= mx + b \\ y &= -750x + 12000 \end{aligned}$$

Lastly, write this model as a function which gives the value of the piece of equipment over time. Choose the function name $V$, for value, and the variable $t$ instead of $x$ to represent time in years.

$$V = -750t + 12000$$

**Answer**:

$V = -750t + 12,000$

The equation $V = -750t + 12,000$ called a **linear depreciation model**[36]. It uses a linear equation to expresses the declining value of an item over time. Using this to determine the value of the item between the given data points is called **interpolation**[37]. For example, we can use the equation to determine the value of the item where $t = 2$,

$$\begin{aligned} V &= -750(2) + 12,000 \\ &= 10,500 \end{aligned}$$

This shows that the item was worth $\$10,500$ two years after it was purchased. Using this model to predict the value outside the given data points is called **extrapolation**[38]. For example, we can use the equation to determine the value of the item when $t = 10$:

$$\begin{aligned} V &= -750(10) + 12,000 \\ &= -7,500 + 12,000 \\ &= 4,500 \end{aligned}$$

The model predicts that the piece of equipment will be worth $\$4,500$ ten years after it is purchased.

Figure 2.3.5

## Business Models

In a business application, revenue results from the sale of a number of items. For example, if an item can be sold for $\$150$ and we let $n$ represent the number of units sold, then we can form the following **revenue equation**[39]:

$$R = 150n$$

Use this function to determine the revenue generated from selling $n = 100$ units,

$$R = 150(100) = 15,000$$

The function shows that the revenue generated from selling $100$ items is $\$15,000$. Typically, selling items does not represent the entire story. There are a number of costs associated with the generation of revenue. For example, if there is a one-time set up fee of $\$5,280$ and each item cost $\$62$ to produce, then we can form the following **cost equation**[40]:

$$C = 62n + 5,280$$

Here $n$ represents the number of items produced. Use this to determine the cost associated with producing $n = 100$ units:

$$C = 62(100) + 5,280 = 11,480$$

This shows that the cost associated with producing 100 items is $11,480. Profit is revenue less costs:

$$\begin{aligned} Profit &= Revenue - Cost \\ &= 15,000 - 11,480 \\ &= 3,520 \end{aligned}$$

Therefore, the profit generated by producing and selling $100$ items is $\$3,520$. In general, given a revenue equation$R$ and a cost equation $C$, we can form a **profit equation**[41] by subtracting as follows:

$$P = R - C$$

## Example 13:

The cost in dollars of producing n items is given by the formula $C = 62n + 5,280$. The revenue in dollars is given by $R = 150n$ , where $n$ represents the number items sold. Write an equation that gives the profit generated by producing and selling n items. Use this equation to determine how many items must be produced and sold in order to earn a profit of at least $\$7,000$.

**Solution**

Obtain the profit equation by subtracting the cost equation from the revenue equation.

$$\begin{aligned} P &= R - C \\ &= 150n - (62n + 5,280) \\ &= 150n - 62n - 5,280 \\ &= 88n - 5,280 \end{aligned}$$

Therefore, $P(n) = 88n + 5,280$ models the profit. To determine the number of items that must be produced and sold to profit at least $\$7,000$, solve the following:

$$\begin{aligned} P &\geq 7,000 \\ 88n - 5,280 &\geq 7,000 \\ 88n &\geq 12,280 \\ n &\geq 139.5 \end{aligned}$$

Round up because the number of units produced and sold must be an integer. To see this, calculate the profit where $n$ is $139$ and $140$ units.

$$\begin{aligned} P &= 88(139) - 5,280 = 6,952 \\ P &= 88(140) - 5,280 = 7,040 \end{aligned}$$

**Answer**:

$140$ or more items must be produced and sold in order to earn a profit of at least $\$7,000$.

Sometimes the costs exceed the revenue, in which case, the profit will be negative. For example, use the profit equation of the previous example, $P = 88n - 5,280$, to calculate the profit generated where $n = 50$.

$$P = 88(50) - 5,280 = -880$$

This indicates that when $50$ units are produced and sold the corresponding profit is a loss of $\$880$.

It is often important to determine how many items must be produced and sold to break even. To break even means to neither have a gain nor a loss; in this case, the profit will be equal to zero. To determine the **breakeven point**[42], set the profit function equal to zero and solve:

$$\begin{aligned} P &= 88n - 5,280 \\ 0 &= 88n - 5,280 \\ 5,280 &= 88n \\ 60 &= n \end{aligned}$$

Therefore, $60$ items must be produced and sold to break even.

Figure $2.3.6$

**You Try it.**

Custom t-shirts can be sold for $\$6.50$ each. In addition to an initial setup fee of $\$120$, each t-shirt cost $\$3.50$ to produce.

a. Write an equation that models the revenue and an equation that models the cost.
b. Determine the equation that models the profit and use it to determine the profit from producing and selling $150$ t-shirts.
c. Calculate the number of t-shirts that must be sold to break even.

*Answer available at the end of this section.*

## Key Concepts

- A rate is a quotient of two measurements with different units.
- There are many known formulas that can be used to solve applications. Distance problems, for example, are solved using the $d = rt$ formula. See Example.A linear equation can be used to solve for an unknown in a number problem. See Example.
- Applications can be written as mathematical problems by identifying known quantities and assigning a variable to unknown quantities. See Example.
- Given two points we can find the equation of a line.
- To find an equation of a line, first use the given information to determine the slope. Then use the slope and a point on the line to find the equation using point-slope form.
- To construct a linear equation that models a real-world application, first identify the dependent and independent variables. Next, find two ordered pairs that describe the given situation. Use these two ordered pairs to construct a linear function by finding the slope and $y$-intercept.
-

### Answers

**You Try it.**

a. Revenue: $R = 6.50x$; cost: $C = 3.50x + 120$;
b. profit: $P = 3x - 120$; profit from producing and selling 150 t-shirts: $\$330$
c. $40$

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# 2.4: Systems of Linear Equations - Two Variables

**Learning Objectives**

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.
- Solve applications using a system of linear equations.

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.



Figure 2.4.1

## Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. **A system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$\begin{aligned} 2x+y &= 15 \\ 3x–y &= 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair $(4,7)$ is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

$$2(4)+(7)=15 \text{ True}$$
$$3(4)-(7)=5 \text{ True}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A consistent system of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same *y*-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different *y*-intercepts. There are no points common to both lines; hence, there is no solution to the system.

## TYPES OF LINEAR SYSTEMS

There are three types of systems of linear equations in two variables, and three types of solutions.

- An independent system has exactly one solution pair $(x, y)$. The point where the two lines intersect is the only solution.
- An inconsistent system has no solution. Notice that the two lines are parallel and will never intersect.
- A dependent system has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

Figure 2.4.2 compares graphical representations of each type of system.

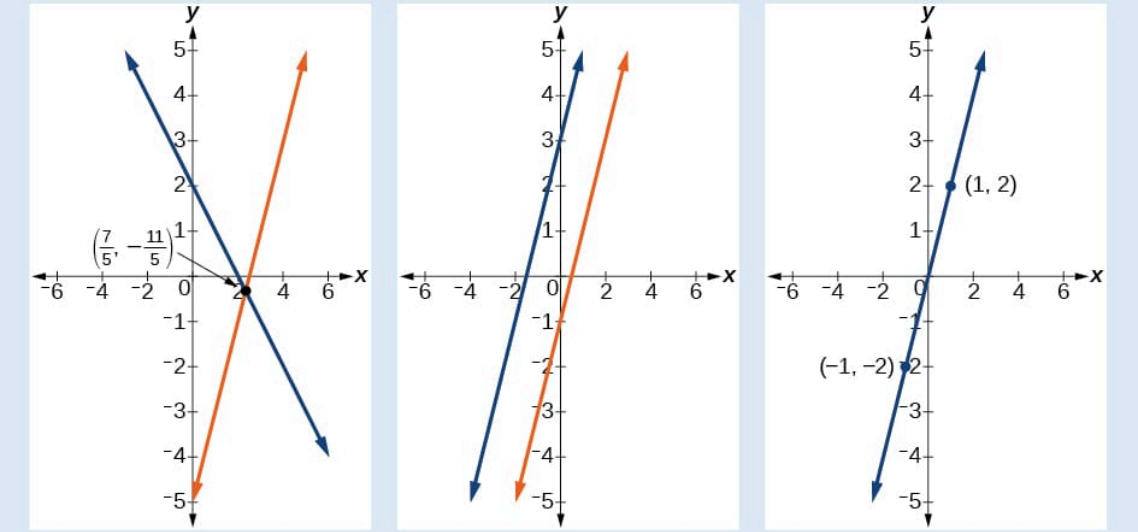



Figure 2.4.2

## Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

## Example 2.4.1: Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

$$x + 3y = 8$$
$$2x - 9 = y$$

**Solution**

Substitute the ordered pair $(5, 1)$ into both equations.

$$\begin{aligned} (5) + 3(1) &= 8 \\ 8 &= 8 \text{ True} \\ 2(5) - 9 &= (1) \\ 1 &= 1 \text{ True} \end{aligned}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

**Analysis**

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure 2.4.3.

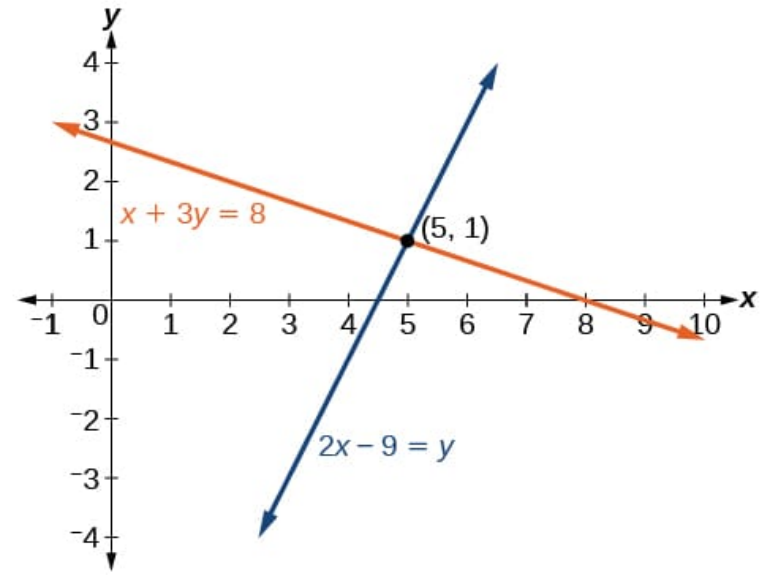



Figure 2.4.3

## Exercise 2.4.1

Determine whether the ordered pair $(8, 5)$ is a solution to the following system.

$$5x - 4y = 20$$
$$2x + 1 = 3y$$

*Answer available at the end of this section.*

## Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

## Example 2.4.2: Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

**Solution**

Solve the first equation for $y$.

$$\begin{aligned} 2x + y &= -8 \\ y &= -2x - 8 \end{aligned}$$

Solve the second equation for $y$.

$$\begin{aligned} x - y &= -1 \\ y &= x + 1 \end{aligned}$$

Graph both equations on the same set of axes as in Figure 2.4.4.

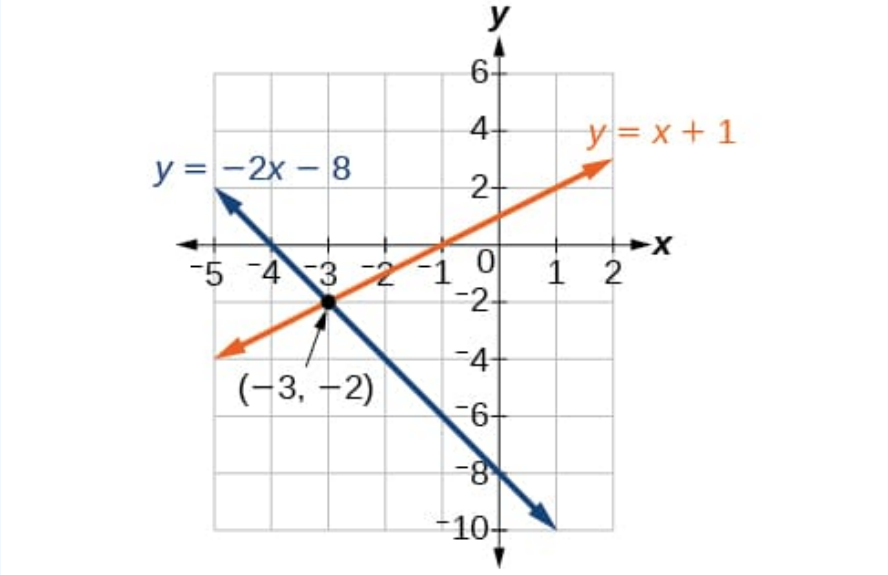



Figure 2.4.4

The lines appear to intersect at the point $(-3, -2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned} 2(-3) + (-2) &= -8 \\ -8 &= -8 \text{ True} \\ (-3) - (-2) &= -1 \\ -1 &= -1 \text{ True} \end{aligned}$$

The solution to the system is the ordered pair $(-3, -2)$,so the system is independent.

## Exercise 2.4.2

Solve the following system of equations by graphing.

$$\begin{aligned} 2x - 5y &= -25 \\ -4x + 5y &= 35 \end{aligned}$$

*Answer available at the end of this section.*

Q&A

**Can graphing be used if the system is inconsistent or dependent?**

*Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.*

## Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a **system of linear equations** that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

How to: Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Example 2.4.3: Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$\begin{aligned} -x+y&=-5 \\ 2x-5y&=1 \end{aligned}$$

**Solution**

First, we will solve the first equation for $y$.

$$\begin{aligned} -x+y&=-5 \\ y&=x-5 \end{aligned}$$

Now we can substitute the expression $x-5$ for $y$ in the second equation.

$$\begin{aligned} 2x-5y&=1 \\ 2x-5(x-5)&=1 \\ 2x-5x+25&=1 \\ -3x&=-24 \\ x&=8 \end{aligned}$$

Now, we substitute $x=8$ into the first equation and solve for $y$.

$$\begin{aligned} -(8)+y&=-5 \\ y&=3 \end{aligned}$$

Our solution is $(8, 3)$.

Check the solution by substituting $(8, 3)$ into both equations.

$$-x + y = -5$$
$$-(8) + (3) = -5 \text{ True}$$
$$2x - 5y = 1$$
$$2(8) - 5(3) = 1 \text{ True}$$

### Exercise 2.4.3

Solve the following system of equations by substitution.

$$x = y + 3$$
$$4 = 3x - 2y$$

*Answer available at the end of this section.*

### Q&A

**Can the substitution method be used to solve any linear system in two variables?**

*Yes, but the method works best if one of the equations contains a coefficient of $1$ or $-1$ so that we do not have to deal with fractions.*

## Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the addition method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

### How to: Given a system of equations, solve using the addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

### Example 2.4.4: Solving a System by the Addition Method

Solve the given system of equations by addition.

$$x + 2y = -1$$
$$-x + y = 3$$

**Solution**

Both equations are already set equal to a constant. Notice that the coefficient of $x$ in the second equation, $-1$, is the opposite of the coefficient of $x$ in the first equation, $1$. We can add the two equations to eliminate $x$ without needing to multiply by a constant.

$$\begin{aligned} x+2y &= -1 \\ \underline{-x+y &= 3} \\ 3y &= 2 \end{aligned}$$

Now that we have eliminated $x$, we can solve the resulting equation for $y$.

$$\begin{aligned} 3y &= 2 \\ y &= \frac{2}{3} \end{aligned}$$

Then, we substitute this value for $y$ into one of the original equations and solve for $x$.

$$\begin{aligned} -x+y &= 3 \\ -x+\frac{2}{3} &= 3 \\ -x &= 3-\frac{2}{3} \\ -x &= \frac{7}{3} \\ x &= -\frac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned} x+2y &= -1 \\ \left(-\frac{7}{3}\right)+2\left(\frac{2}{3}\right) &= \\ -\frac{7}{3}+\frac{4}{3} &= -\frac{3}{3} \\ -1 &= -1 \qquad \text{True} \end{aligned}$$

**Analysis**

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure $2.4.6$ to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

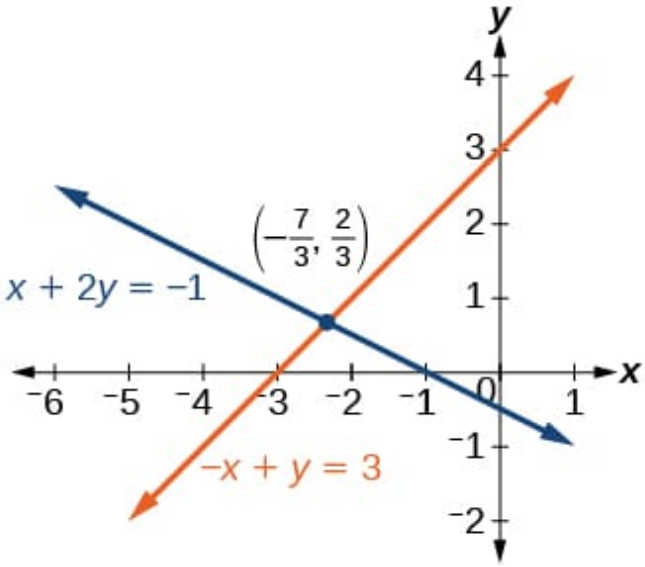



Figure 2.4.6

## Example 2.4.5: Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

$$\begin{aligned} 3x+5y &= -11 \\ x-2y &= 11 \end{aligned}$$

**Solution**

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has $x$. So if we multiply the second equation by $-3$,the *x*-terms will add to zero.

$$\begin{aligned} x-2y &= 11 \\ -3(x-2y) &= -3(11) \qquad \text{Multiply both sides by } -3. \\ -3x+6y &= -33 \qquad \text{Use the distributive property.} \end{aligned}$$

Now, let's add them.

$$\begin{aligned} 3x+5y &= -11 \\ \underline{-3x+6y} &= \underline{-33} \\ 11y &= -44 \\ y &= -4 \end{aligned}$$

For the last step, we substitute $y=-4$ into one of the original equations and solve for $x$.

$$\begin{aligned} 3x+5y &= -11 \\ 3x+5(-4) &= -11 \\ 3x-20 &= -11 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Our solution is the ordered pair $(3,-4)$. See Figure 2.4.7. Check the solution in the original second equation.

$$\begin{aligned} x-2y &= 11 \\ (3)-2(-4) &= 3+8 \\ &= 11 \qquad \text{True} \end{aligned}$$

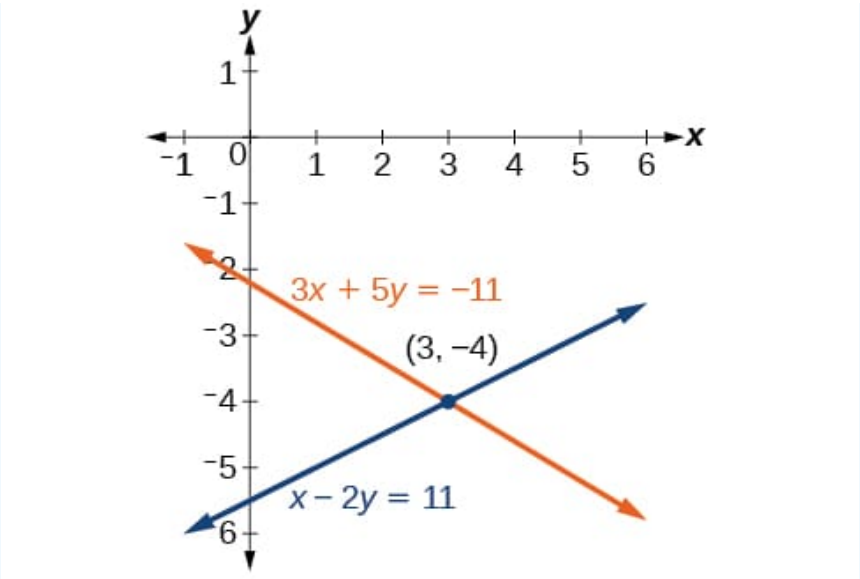



Figure 2.4.7

## Exercise 2.4.4

Solve the system of equations by addition.

$$\begin{aligned} 2x - 7y &= 2 \\ 3x + y &= -20 \end{aligned}$$

*Answer available at the end of this section.*

## Example 2.4.6: Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.

$$\begin{aligned} 2x + 3y &= -16 \\ 5x - 10y &= 30 \end{aligned}$$

**Solution**

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate $x$ by multiplying the first equation by $-5$ and the second equation by $2$.

$$\begin{aligned} -5(2x + 3y) &= -5(-16) \\ -10x - 15y &= 80 \\ 2(5x - 10y) &= 2(30) \\ 10x - 20y &= 60 \end{aligned}$$

Then, we add the two equations together.

$$\begin{aligned} -10x - 15y &= 80 \\ \underline{10x - 20y} &= \underline{60} \\ -35y &= 140 \\ y &= -4 \end{aligned}$$

Substitute $y = -4$ into the original first equation.

$$2x + 3(-4) = -16$$
$$2x - 12 = -16$$
$$2x = -4$$
$$x = -2$$

The solution is $(-2, -4)$. Check it in the other equation.

$$5x - 10y = 30$$
$$5(-2) - 10(-4) = 30$$
$$-10 + 40 = 30$$
$$30 = 30$$

See Figure 2.4.8.

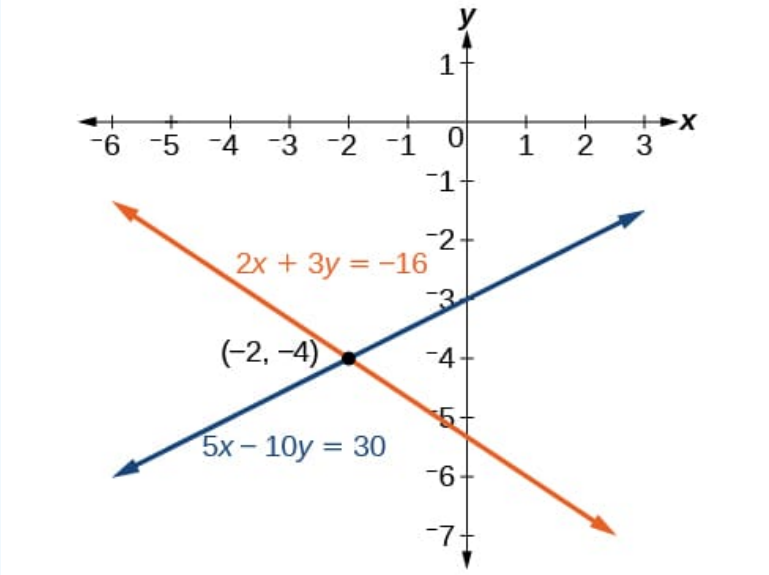



Figure 2.4.8

## Example 2.4.7: Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.

$$\frac{x}{3} + \frac{y}{6} = 3$$
$$\frac{x}{2} - \frac{y}{4} = 1$$

**Solution**

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$$6\left(\frac{x}{3} + \frac{y}{6}\right) = 6(3)$$
$$2x + y = 18$$
$$4\left(\frac{x}{2} - \frac{y}{4}\right) = 4(1)$$
$$2x - y = 4$$

Now multiply the second equation by $-1$ so that we can eliminate the *x*-variable.

$$-1(2x - y) = -1(4)$$
$$-2x + y = -4$$

Add the two equations to eliminate the $x$-variable and solve the resulting equation.

$$\begin{aligned} 2x+y &= 18 \\ -2x+y &= -4 \\ 2y &= 14 \\ y &= 7 \end{aligned}$$

Substitute $y=7$ into the first equation.

$$\begin{aligned} 2x+(7) &= 18 \\ 2x &= 11 \\ x &= \frac{11}{2} \\ &= 7.5 \end{aligned}$$

The solution is $\left(\frac{11}{2}, 7\right)$. Check it in the other equation.

$$\begin{aligned} \frac{x}{2}-\frac{y}{4} &= 1 \\ \frac{\frac{11}{2}}{2}-\frac{7}{4} &= 1 \\ \frac{11}{4}-\frac{7}{4} &= 1 \\ \frac{4}{4} &= 1 \end{aligned}$$

## Exercise 2.4.5

Solve the system of equations by addition.

$$\begin{aligned} 2x+3y &= 8 \\ 3x+5y &= 10 \end{aligned}$$

*Answer available at the end of this section.*

## Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12=0$.

## Example 2.4.8: Solving an Inconsistent System of Equations

Solve the following system of equations.

$$\begin{aligned} x &= 9-2y \\ x+2y &= 13 \end{aligned}$$

**Solution**

We can approach this problem in two ways. Because one equation is already solved for $x$, the most obvious step is to use substitution.

$$\begin{aligned} x + 2y &= 13 \\ (9 - 2y) + 2y &= 13 \\ 9 + 0y &= 13 \\ 9 &= 13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$\begin{aligned} x &= 9 - 2y \\ 2y &= -x + 9 \\ y &= -\frac{1}{2}x + \frac{9}{2} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

$$\begin{aligned} x + 2y &= 13 \\ 2y &= -x + 13 \\ y &= -\frac{1}{2}x + \frac{13}{2} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different $y$-intercepts. Therefore, the lines are parallel and do not intersect.

$$\begin{aligned} y &= -\frac{1}{2}x + \frac{9}{2} \\ y &= -\frac{1}{2}x + \frac{13}{2} \end{aligned}$$

**Analysis**

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in Figure $2.4.9$.

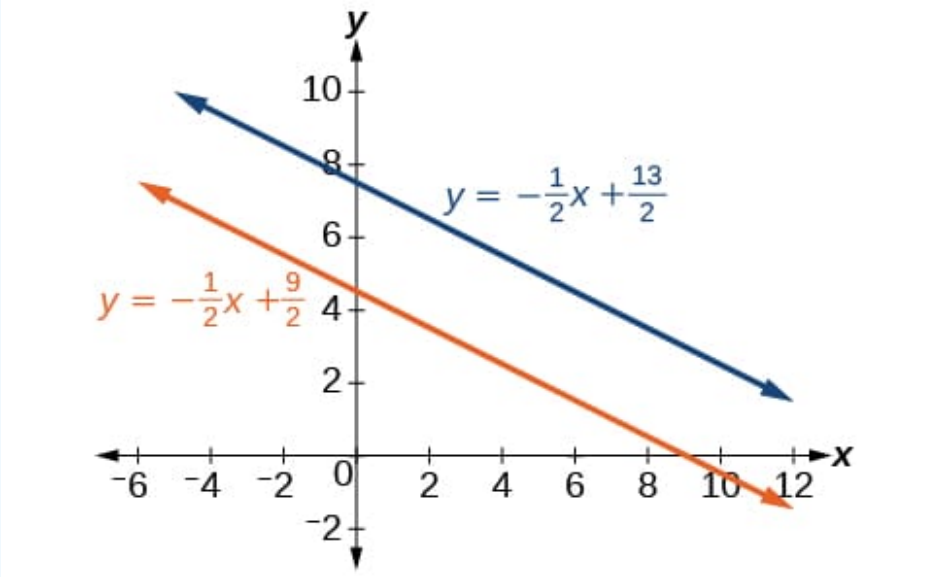



Figure $2.4.9$

### Exercise 2.4.6

Solve the following system of equations in two variables.

$$2y - 2x = 2$$
$$2y - 2x = 6$$

*Answer available at the end of this section.*

## Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0 = 0$.

### Example 2.4.9: Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the **addition method**.

$$x + 3y = 2$$
$$3x + 9y = 6$$

**Solution**

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating $x$. If we multiply both sides of the first equation by $-3$, then we will be able to eliminate the x-variable.

$$x + 3y = 2$$
$$(-3)(x + 3y) = (-3)(2)$$
$$-3x - 9y = -6$$

Now add the equations.

$$\begin{aligned} -3x - 9y &= -6 \\ +\ 3x + 9y &= \underline{6} \\ 0 &= 0 \end{aligned}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

**Analysis**

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.

$$\begin{aligned} x + 3y &= 2 \\ 3y &= -x + 2 \\ y &= -\frac{1}{3}x + \frac{2}{3} \\ 3x + 9y &= 6 \\ 9y &= -3x + 6 \\ y &= -\frac{3}{9}x + \frac{6}{9} \\ y &= -\frac{1}{3}x + \frac{2}{3} \end{aligned}$$

See Figure 2.4.10. Notice the results are the same. The general solution to the system is $\left(x, -\frac{1}{3}x + \frac{2}{3}\right)$.

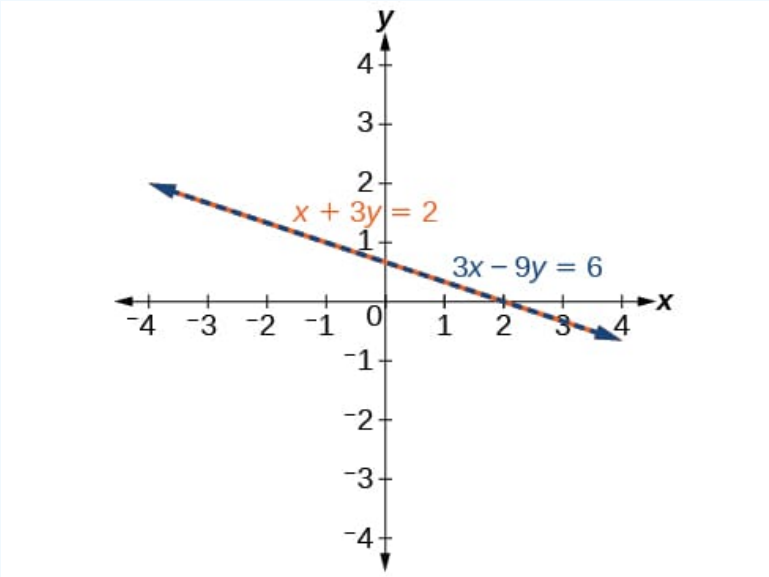



Figure 2.4.10

**Exercise 2.4.7**

Solve the following system of equations in two variables.

$$y - 2x = 5$$
$$-3y + 6x = -15$$

*Answer available at the end of this section.*

## Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's revenue function is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where $x$=quantity and $p$=price. The revenue function is shown in orange in Figure 2.4.11.

The cost function is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in Figure 2.4.11. The $x$-axis represents quantity in hundreds of units. The $y$-axis represents either cost or revenue in hundreds of dollars.

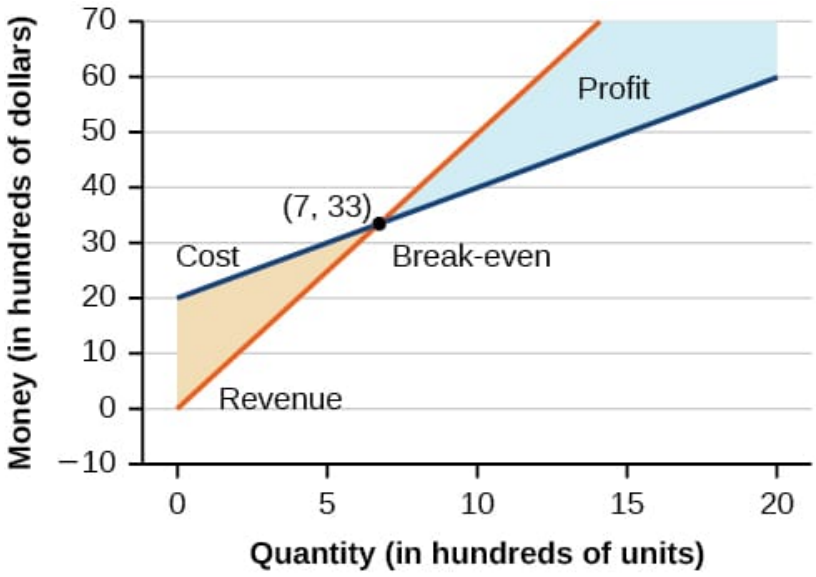



Figure 2.4.11

The point at which the two lines intersect is called the break-even point. We can see from the graph that if $700$ units are produced, the cost is $\$3,300$ and the revenue is also $\$3,300$. In other words, the company breaks even if they produce and sell $700$ units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The profit function is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

## Example 2.4.10: Finding the Break-Even Point

A company plans to produce and sell "widgets". The cost to produce each widget is 85 cents plus additional fixed costs of 35 thousand dollars per month.

They sell each widget for $1.55. How many widgets must be produced and sold each month in order to break even.

**Solution**

The cost equation is $C = 0.85x + 35,000$ and the revenue equation is $R = 1.55x$

Since cost and revenue are both measured using the same units "dollars", we can rewrite both equations using $y$ to replace the variables $C$ and $R$ in the cost and revenue equations respectively.

$$\begin{aligned} y &= 0.85x + 35,000 \\ y &= 1.55x \end{aligned}$$

Substitute the expression $0.85x + 35,000$ from the first equation into the second equation and solve for $x$.

$$\begin{aligned} 0.85x + 35,000 &= 1.55x \\ 35,000 &= 0.7x \\ 50,000 &= x \end{aligned}$$

Then, we substitute $x = 50,000$ into either the cost equation or the revenue equation.

$$1.55(50,000) = 77,500$$

The break-even point is $(50,000, 77,500)$.

Therefore, this company must produce and sell 50,000 widgets each month in order to break even.

**Analysis**

The cost to produce $50,000$ units is $\$77,500$, and the revenue from the sales of $50,000$ units is also $\$77,500$. To make a profit, the business must produce and sell more than $50,000$ units. See Figure $2.4.12$.

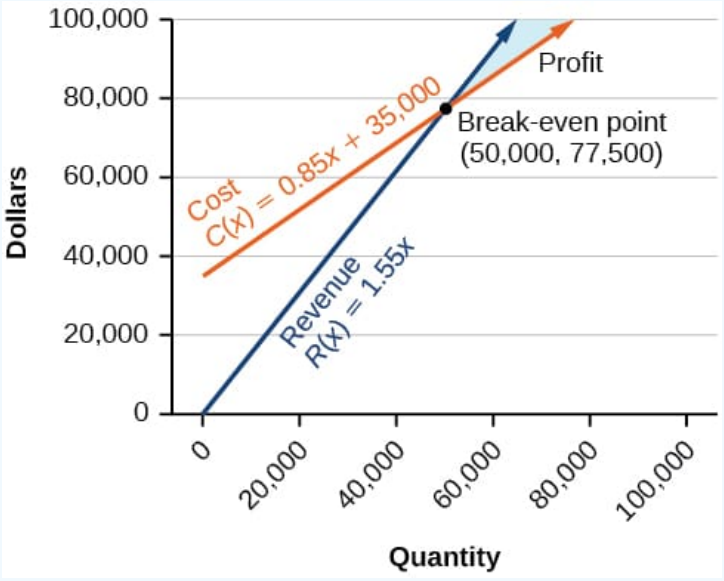



Figure $2.4.12$

We see from the graph in Figure $2.4.13$ that the profit equation has a negative value until $x = 50,000$, when the graph crosses the $x$-axis. Then, the graph emerges into positive $y$-values and continues on this linear path. This illustrates that the break-even point for businesses occurs when the profit is $0$. The area to the left of the break-even point represents the business operating at a loss. Conversely, the area to the right of the break-even point represents the operation producing a positive profit.

The profit equation is found using the formula $P = R - C$.

$$\begin{aligned} P &= 1.55x - (0.85x + 35,000) \\ &= 0.7x - 35,000 \end{aligned}$$

The profit is given by the equation $P = 0.7x - 35,000$.

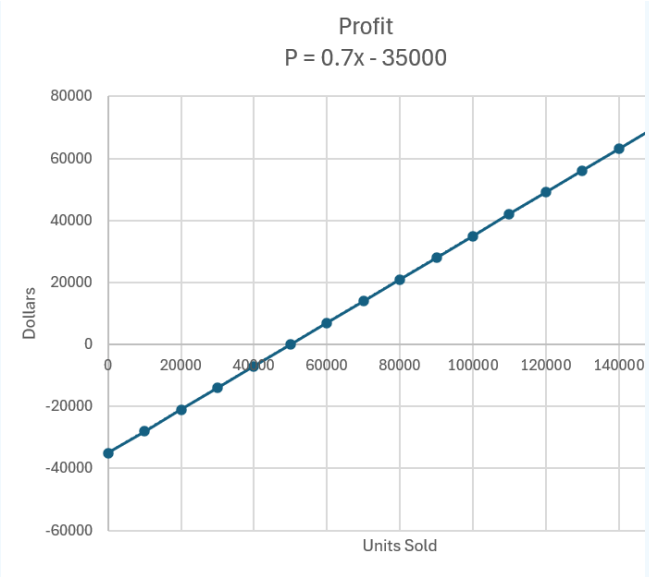



Figure 2.4.13

## Example 2.4.11: Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2,000$ and the total gate revenue is $\$70,000$. How many children and how many adults bought tickets?

**Solution**

Let $c$ = the number of children and $a$ = the number of adults in attendance.

The total number of people is $2,000$. We can use this to write an equation for the number of people at the circus that day.

$$c + a = 2,000$$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c$. The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a$. The total revenue is $\$70,000$. We can use this to write an equation for the revenue.

$$25c + 50a = 70,000$$

We now have a system of linear equations in two variables.

$$c + a = 2,000$$

$$25c + 50a = 70,000$$

In the first equation, the coefficient of both variables is $1$. We can quickly solve the first equation for either $c$ or $a$. We will solve for $a$.

$$\begin{aligned} c + a &= 2,000 \\ a &= 2,000 - c \end{aligned}$$

Substitute the expression $2,000 - c$ in the second equation fora a and solve for $c$.

$$\begin{aligned} 25c + 50(2,000 - c) &= 70,000 \\ 25c + 100,000 - 50c &= 70,000 \\ -25c &= -30,000 \\ c &= 1,200 \end{aligned}$$

Substitute $c = 1,200$ into the first equation to solve for $a$.

$$1,200 + a = 2,000$$
$$a = 800$$

We find that $1,200$ children and $800$ adults bought tickets to the circus that day.

### Exercise 2.4.8

Tickets to a school play cost $\$4.00$ for children and $\$12.00$ for adults. If $1,650$ tickets to "A Christmas Carol" were sold for a total of, $\$14,200$, how many of each type of tickets were purchased for this play?

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with systems of linear equations.

- Solving Systems of Equations Using Substitution
- Solving Systems of Equations Using Elimination
- Applications of Systems of Equations

## Key Concepts

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See Example 2.4.1.
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See Example 2.4.2.
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See Example 2.4.3.
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See Example 2.4.4.
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See Example 2.4.5, Example 2.4.6, and Example 2.4.7.
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See Example 2.4.8.
- The solution to a system of dependent equations will always be true because both equations describe the same line. See Example 2.4.9.
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See Example 2.4.10 and Example 2.4.11.

## Answers

**Exercise** 2.4.1

Not a solution.

**Exercise** 2.4.2

The solution to the system is the ordered pair $(-5, 3)$.

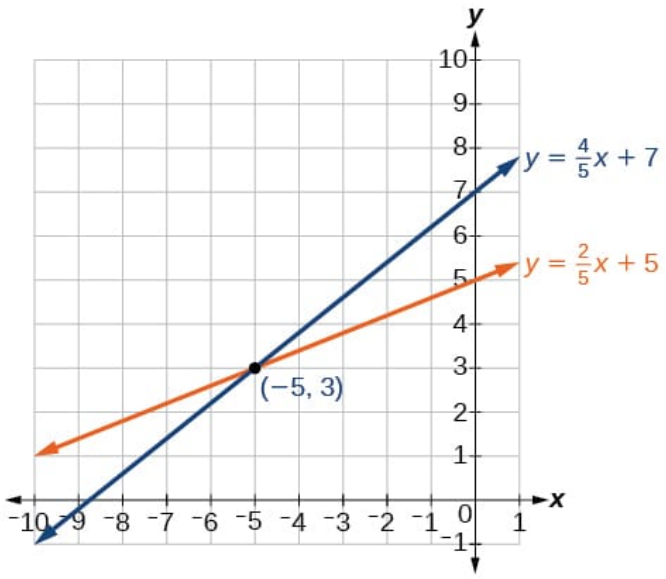



Figure 2.4.5

**Exercise** 2.4.3

$(-2, -5)$

**Exercise** 2.4.4

$(-6, -2)$

**Exercise** 2.4.5

$(10, -4)$

**Exercise** 2.4.6

No solution. It is an inconsistent system.

**Exercise** 2.4.7

The system is dependent so there are infinite solutions of the form $(x, 2x + 5)$.

**Exercise** 2.4.8

700 children, 950 adults

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# 2.5: Systems of Linear Equations - Three Variables

**Learning Objectives**

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

Don received an inheritance of $\$120,000$ that he divided into three parts and invested in three ways: in a money-market fund paying $3\%$ annual interest; in municipal bonds paying $4\%$ annual interest; and in mutual funds paying $7\%$ annual interest. John invested $\$40,000$ more in municipal funds than in municipal bonds. He earned $\$6700$ in interest the first year. How much did John invest in each type of fund?



Figure 2.5.1: (credit: "Elembis," Wikimedia Commons)

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

## Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution $(x, y, z)$, which we call an ordered triple. A system in upper triangular form looks like the following:

$$\begin{aligned} Ax + By + Cz &= D \\ Ey + Fz &= G \\ Hz &= K \end{aligned}$$

The third equation can be solved for $z$,and then we back-substitute to find $y$ and $x$. To write the system in upper triangular form,

we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

The **solution set** to a three-by-three system is an ordered triple $(x, y, z)$. Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

## NUMBER OF POSSIBLE SolutionS

Figure 2.5.2 and Figure 2.5.3 illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a solution set consisting of an ordered triple $(x, y, z)$. Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.

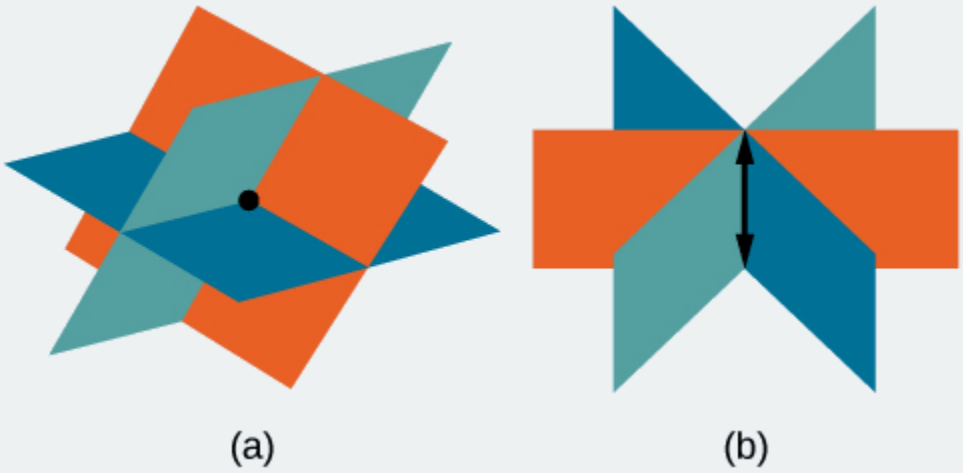



Figure 2.5.2: (a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.

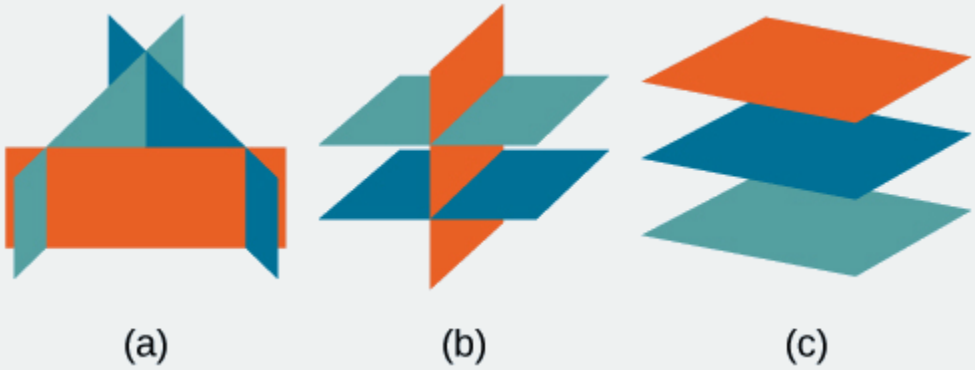



Figure 2.5.3: All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

## Example 2.5.1: Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$\begin{aligned} x + y + z &= 2 \\ 6x - 4y + 5z &= 31 \\ 5x + 2y + 2z &= 13 \end{aligned}$$

**Solution**

We will check each equation by substituting in the values of the ordered triple for $x, y$, and $z$.

$$\begin{array}{rrr} 6x - 4y + 5z = 31 & 5x + 2y + 2z = 13 \\ x + y + z = 2 \quad 6(3) - 4(-2) + 5(1) = 31 & 5(3) + 2(-2) + 2(1) = 13 \\ (3) + (-2) + (1) = 2 \quad 18 + 8 + 5 = 31 & 15 - 4 + 2 = 13 \\ \text{True} \quad \text{True} & \text{True} \end{array}$$

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

## How to: Given a linear system of three equations, solve for three unknowns

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

## Example 2.5.2: Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$\begin{aligned} x - 2y + 3z &= 9 \ (1) \\ -x + 3y - z &= -6 \ (2) \\ 2x - 5y + 5z &= 17 \ (3) \end{aligned}$$

**Solution**

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate $x$ by adding equations (1) and (2).

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ \underline{-x + 3y - z} &\underline{= -6 \quad (2)} \\ y + 2z &= 3 \quad (4) \end{aligned}$$

The second step is multiplying equation (1) by $-2$ and adding the result to equation (3). These two steps will eliminate the variable $x$.

$$\begin{aligned} -2x + 4y - 6z &= -18 \ (1) \quad \text{multiplied by } -2 \\ \underline{2x - 5y + 5z} &\underline{= 17 \ (3)} \\ -y - z &= -1 \ (5) \end{aligned}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for $z$ by adding the two equations.

$$y + 2z = 3 \ (4)$$
$$\underline{-y - z = -1} \ (5)$$
$$z = 2 \ (6)$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$x - 2y + 3z = 9 \ (1)$$
$$y + 2z = 3 \ (4)$$
$$z = 2 \ (6)$$

Next, we back-substitute $z = 2$ into equation (4) and solve for $y$.

$$y + 2(2) = 3$$
$$y + 4 = 3$$
$$y = -1$$

Finally, we can back-substitute $z = 2$ and $y = -1$ into equation (1). This will yield the solution for $x$.

$$x - 2(-1) + 3(2) = 9$$
$$x + 2 + 6 = 9$$
$$x = 1$$

The solution is the ordered triple $(1, -1, 2)$. See Figure $2.5.4$.

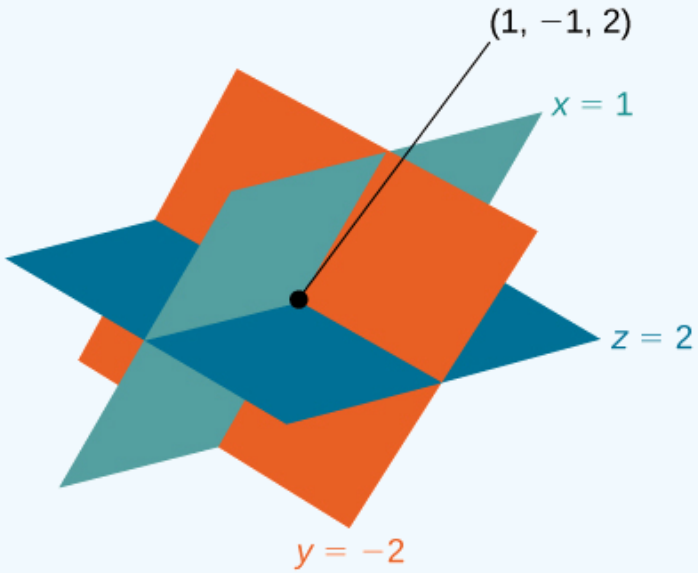



Figure $2.5.4$

## Example $2.5.3$: Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, Don invested his inheritance of $\$120,000$ in three different funds: part in a money-market fund paying $3\%$ interest annually; part in municipal bonds paying $4\%$ annually; and the rest in mutual funds paying $7\%$ annually. Don invested $\$40,000$ more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $\$6700$. How much did he invest in each type of fund?

**Solution**

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$x = \text{amount invested in money-market fund}$$
$$y = \text{amount invested in municipal bonds}$$
$$z = \text{amount invested in mutual funds}$$

The first equation indicates that the sum of the three principal amounts is $\$120,000$.

$$x + y + z = 120,000$$

We form the second equation according to the information that Don invested $\$40,000$ more in mutual funds than he invested in municipal bonds.

$$z = y + 40,000$$

The third equation shows that the total amount of interest earned from each fund equals $\$6700$.

$$0.03x + 0.04y + 0.07z = 6700$$

Then, we write the three equations as a system.

$$\begin{aligned} x + y + z &= 120,000 \\ -y + z &= 40,000 \\ 0.03x + 0.04y + 0.07z &= 6700 \end{aligned}$$

To make the calculations simpler, we can multiply the third equation by $100$. Thus,

$$\begin{aligned} x + y + z &= 120,000 \quad (1) \\ -y + z &= 40,000 \quad (2) \\ 3x + 4y + 7z &= 670,000 \quad (3) \end{aligned}$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.

$$\begin{aligned} x + y + z &= 120,000 \\ 3x + 4y + 7z &= 670,000 \\ -y + z &= 40,000 \end{aligned}$$

Step 2. Multiply equation (1) by $-3$ and add to equation (2). Write the result as row 2.

$$\begin{aligned} x + y + z &= 120,000 \\ y + 4z &= 31,000 \\ -y + z &= 40,000 \end{aligned}$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).

$$\begin{aligned} x + y + z &= 120,000 \\ y + 4z &= 310,000 \\ 5z &= 350,000 \end{aligned}$$

Step 4. Solve for $z$ in equation (3). Back-substitute that value in equation (2) and solve for $y$. Then, back-substitute the values for $z$ and $y$ into equation (1) and solve for $x$.

$$5z = 350,000$$
$$z = 70,000$$

$$y + 4(70,000) = 310,000$$
$$y = 30,000$$

$$x + 30,000 + 70,000 = 120,000$$
$$x = 20,000$$

John invested $\$20,000$ in a money-market fund, $\$30,000$ in municipal bonds, and $\$70,000$ in mutual funds.

**Exercise 2.5.1**

Solve the system of equations in three variables.

$$2x + y - 2z = -1$$
$$3x - 3y - z = 5$$
$$x - 2y + 3z = 6$$

*Answer available at the end of this section.*

## Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3 = 7$ or some other contradiction.

**Example 2.5.4: Solving an Inconsistent System of Three Equations in Three Variables**

Solve the following system.

$$x - 3y + z = 4 \quad (2.5.1)$$
$$-x + 2y - 5z = 3 \quad (2.5.2)$$
$$5x - 13y + 13z = 8 \quad (2.5.3)$$

**Solution**

Looking at the coefficients of $x$, we can see that we can eliminate $x$ by adding Equation 2.5.1 to Equation 2.5.2.

$$x - 3y + z = 4 \quad (1)$$
$$\underline{-x + 2y - 5z = 3 \quad (2)}$$
$$-y - 4z = 7 \quad (4)$$

Next, we multiply equation (1) by $-5$ and add it to equation (3).

$$-5x + 15y - 5z = -20 \quad (1) \quad \text{multiplied by } -5$$
$$\underline{5x - 13y + 13z = 8 \quad (3)}$$
$$2y + 8z = -12 \quad (5)$$

Then, we multiply equation (4) by 2 and add it to equation (5).

$$\begin{aligned} -2y-8z &= 14 \quad (4) \quad \text{multiplied by } 2 \\ \underline{2y+8z} &= \underline{-12} \quad (5) \\ 0 &= 2 \end{aligned}$$

The final equation $0 = 2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

**Analysis**

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

### Exercise 2.5.2

Solve the system of three equations in three variables.

$$\begin{aligned} x+y+z &= 2 \\ y-3z &= 1 \\ 2x+y+5z &= 0 \end{aligned}$$

*Answer available at the end of this section.*

## Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

### Example 2.5.5: Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$$\begin{aligned} x-y+z &= 0 \quad (1) \\ 4x+2y-6z &= 0 \quad (2) \\ 2x+y-3z &= 0 \quad (3) \end{aligned}$$

**Solution**

First, we can multiply equation (3) by $-2$ and add it to equation (2).

$$\begin{aligned} -4x-2y+6z &= 0 \quad (3) \quad \text{multiplied by } -2 \\ \underline{4x+2y-6z} &= \underline{0} \quad (2) \\ 0 &= 0 \end{aligned}$$

We do not need to proceed any further. The result we get is an identity, $0 = 0$, which tells us that this system has an infinite number of solutions.

This is called a **dependent system**, and we can find general expressions for the solutions in terms of one of the variables.

In this example, we choose the variable $z$ to use a parameter. This means that the solutions for $x$ and $y$ will depend on $z$.

Adding equations (1) and (3), we have

$$\begin{aligned} 2x + y - 3z &= 0 \\ x - y + z &= 0 \\ 3x - 2z &= 0 \end{aligned}$$

We then solve the resulting equation for $x$.

$$\begin{aligned} 3x - 2z &= 0 \\ x &= \frac{2}{3}z \end{aligned}$$

We back-substitute the expression for $x$ into one of the equations and solve for $y$.

$$\begin{aligned} 2(\frac{2}{3}z) + y - 3z &= 0 \\ \frac{4}{3}x + y - 3z &= 0 \\ y &= 3z - \frac{4}{3}z \\ y &= \frac{5}{3}z \end{aligned}$$

So the general solution is $\left(\frac{2}{3}z, \frac{5}{3}z, z\right)$. In this solution, $z$ can be any real number. The values of $x$ and $y$ are dependent on a value selected for $z$. Since there are infinitley many such choices for $z$, we can see that there infinitely many ordered triples $(x, y, z$ that will be solutions to the system. Here are just a few...

If $z = 3$, then $x = (\frac{2}{3})(3) = 2$, and $y = (\frac{5}{3})(3) = 5$. Therefore , $(2, 5, 3)$ is a solution.

If $z = -3$, then $x = (\frac{2}{3})(-3) = -2$, and $y = (\frac{5}{3})(-3) = -5$. Therefore , $(-2, -5, -3)$ is a solution.

If $z = 6$, then $x = (\frac{2}{3})(6) = 4$, and $y = (\frac{5}{3})(6) = 10$. Therefore , $(4, 10, 6)$ is a solution.

**Analysis**

Equation (3) $2x + y - 3z = 0$ and equation (2) $4x + 2y - 6z = 0$ are the equivalent, so they define the same plane shown as $P_2$ in Figure $2.5.5$

Equation (1) $x - y + z = 0$ is shown as $P_1$. These planes intersect to form a line $L$. The solution set is infinite, as all points along the intersection line will satisfy the system of three equations.

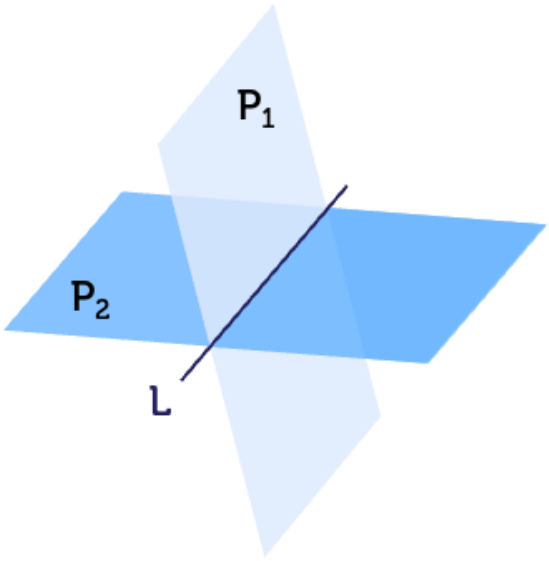



Figure 2.5.5

**Q&A: Does the general solution to a dependent system always have to be written in terms of $z$?**

No, you can write the general solution in terms of any of the variables, but it is common to write it in terms of $z$ and if needed $x$ and $y$.

**Exercise 2.5.3:**

Solve the following dependent system. Use $z$ as the parameter.

$$\begin{aligned} 3x + 2y - 8z &= 8 \\ -8x + 4y - 20z &= -24 \\ -2x + 2y - 6z &= -4 \end{aligned}$$

*Answer available at the end of this section.*

**Key Concepts**

- A solution set is an ordered triple {(x,y,z)} that represents the intersection of three planes in space. See Example 2.5.1.
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See Example 2.5.2.
- Systems of three equations in three variables are useful for solving many different types of real-world problems. See Example 2.5.3.
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See Example 2.5.4.
- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See Example 2.5.5.
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

# Contributors

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## Answers

**Exercise** $2.5.1$

$(1, -1, 1)$

**Exercise** $2.5.2$

No solution.

**Exercise** $2.5.3$**:**

An infinite number of solutions of the form $(4 - 2z, 2 + z, z)$.

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# 2.6: Solving Systems with Gaussian Elimination

**Learning Objectives**

- Write the augmented matrix of a system of equations.
- Write the system of equations from an augmented matrix.
- Perform row operations on a matrix.
- Solve a system of linear equations using matrices.

Carl Friedrich Gauss lived during the late $18^{th}$ century and early $19^{th}$ century, but he is still considered one of the most prolific mathematicians in history. His contributions to the science of mathematics and physics span fields such as algebra, number theory, analysis, differential geometry, astronomy, and optics, among others. His discoveries regarding matrix theory changed the way mathematicians have worked for the last two centuries.



Figure 2.6.1: German mathematician Carl Friedrich Gauss (1777–1855).

We first encountered Gaussian elimination in Systems of Linear Equations: Two Variables. In this section, we will revisit this technique for solving systems, this time using matrices.

## Writing the Augmented Matrix of a System of Equations

A matrix can serve as a device for representing and solving a system of equations. To express a system in matrix form, we extract the coefficients of the variables and the constants, and these become the entries of the matrix. We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equal signs. When a system is written in this form, we call it an **augmented matrix**.

For example, consider the following $2 \times 2$ system of equations.

$$\begin{aligned} 3x + 4y &= 7 \\ 4x - 2y &= 5 \end{aligned}$$

We can write this system as an augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 4 & 7 \\ 4 & -2 & 5 \end{array}\right]$$

We can also write a matrix containing just the coefficients. This is called the **coefficient matrix.**

$$\begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix}$$

A three-by-three **system of equations** such as

$$\begin{aligned} 3x - y - z &= 0 \\ x + y &= 5 \\ 2x - 3z &= 2 \end{aligned}$$

has a coefficient matrix

$$\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & -3 \end{bmatrix}$$

and is represented by the augmented matrix

$$\left[\begin{array}{cccc} 3 & -1 & -1 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & 0 & -3 & 2 \end{array}\right]$$

Notice that the matrix is written so that the variables line up in their own columns: $x$-terms go in the first column, $y$-terms in the second column, and $z$-terms in the third column. It is very important that each equation is written in standard form $ax + by + cz = d$ so that the variables line up. When there is a missing variable term in an equation, the coefficient is $0$.

### How to: Given a system of equations, write an augmented matrix

1. Write the coefficients of the $x$-terms as the numbers down the first column.
2. Write the coefficients of the $y$-terms as the numbers down the second column.
3. If there are $z$-terms, write the coefficients as the numbers down the third column.
4. Draw a vertical line and write the constants to the right of the line.

### Example 2.6.1: Writing the Augmented Matrix for a System of Equations

Write the augmented matrix for the given system of equations.

$$\begin{aligned} x + 2y - z &= 3 \\ 2x - y + 2z &= 6 \\ x - 3y + 3z &= 4 \end{aligned}$$

**Solution**

The augmented matrix displays the coefficients of the variables, and an additional column for the constants.

$$\left[\begin{array}{cccc} 1 & 2 & -1 & 3 \\ 2 & -1 & 2 & 6 \\ 1 & -3 & 3 & 4 \end{array}\right]$$

### Exercise 2.6.1

Write the augmented matrix of the given system of equations.

$$\begin{aligned} 4x - 3y &= 11 \\ 3x + 2y &= 4 \end{aligned}$$

*Answer available at the end of this section.*

## Writing a System of Equations from an Augmented Matrix

We can use augmented matrices to help us solve systems of equations because they simplify operations when the systems are not encumbered by the variables. However, it is important to understand how to move back and forth between formats in order to make finding solutions smoother and more intuitive. Here, we will use the information in an augmented matrix to write the system of equations in standard form.

### Example 2.6.2: Writing a System of Equations from an Augmented Matrix Form

Find the system of equations from the augmented matrix.

$$\begin{bmatrix} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ -3 & 5 & 4 & 6 \end{bmatrix}$$

**Solution**

When the columns represent the variables $x$, $y$, and $z$,

$$\begin{aligned} x - 3y - 5z &= -2 \\ 2x - 5y - 4z &= 5 \\ -3x + 5y + 4z &= 6 \end{aligned}$$

### Exercise 2.6.2

Write the system of equations from the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & -1 & 3 & 1 \\ 0 & 1 & 1 & -9 \end{bmatrix}$$

*Answer available at the end of this section.*

# Performing Row Operations on a Matrix

Now that we can write systems of equations in augmented matrix form, we will examine the various **row operations** that can be performed on a matrix, such as addition, multiplication by a constant, and interchanging rows.

Performing row operations on a matrix is the method we use for solving a system of equations. In order to solve the system of equations, we want to convert the matrix to **row-echelon** form, in which there are ones down the **main diagonal** from the upper left corner to the lower right corner, and zeros in every position below the main diagonal as shown.

$$\text{Row-echelon form } \begin{bmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}$$

We use row operations corresponding to equation operations to obtain a new matrix that is **row-equivalent** in a simpler form. Here are the guidelines to obtaining row-echelon form.

1. In any nonzero row, the first nonzero number is a $1$. It is called a *leading* $1$.
2. Any all-zero rows are placed at the bottom on the matrix.
3. Any leading $1$ is below and to the right of a previous leading $1$.

4. Any column containing a leading $1$ has zeros in all other positions in the column.

To solve a system of equations we can perform the following row operations to convert the **coefficient matrix** to row-echelon form and do back-substitution to find the solution.

1. Interchange rows. (Notation: $R_i \leftrightarrow R_j$)
2. Multiply a row by a constant. (Notation: $cR_i$)
3. Add the product of a row multiplied by a constant to another row. (Notation: $R_i + cR_j$)

Each of the row operations corresponds to the operations we have already learned to solve systems of equations in three variables. With these operations, there are some key moves that will quickly achieve the goal of writing a matrix in row-echelon form. To obtain a matrix in row-echelon form for finding solutions, we use Gaussian elimination, a method that uses row operations to obtain a $1$ as the first entry so that row $1$ can be used to convert the remaining rows.

## GAUSSIAN ELIMINATION

The Gaussian elimination method refers to a strategy used to obtain the row-echelon form of a matrix. The goal is to write matrix $A$ with the number $1$ as the entry down the main diagonal and have all zeros below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{After\ Gaussian\ elimination} A = \begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

The first step of the Gaussian strategy includes obtaining a $1$ as the first entry, so that row $1$ may be used to alter the rows below.

## How to: Given an augmented matrix, perform row operations to achieve row-echelon form

1. The first equation should have a leading coefficient of $1$. Interchange rows or multiply by a constant, if necessary.
2. Use row operations to obtain zeros down the first column below the first entry of $1$.
3. Use row operations to obtain a $1$ in row 2, column 2.
4. Use row operations to obtain zeros down column 2, below the entry of 1.
5. Use row operations to obtain a $1$ in row 3, column 3.
6. Continue this process for all rows until there is a \(1 in every entry down the main diagonal and there are only zeros below.
7. If any rows contain all zeros, place them at the bottom.

## Example $2.6.3$: Solving a $2 \times 2$ System by Gaussian Elimination

Solve the given system by Gaussian elimination.

$$\begin{aligned} 2x + 3y &= 6 \\ x - y &= \frac{1}{2} \end{aligned}$$

**Solution**

First, we write this as an augmented matrix.

$$\begin{bmatrix} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{bmatrix}$$

We want a $1$ in row 1, column 1. This can be accomplished by interchanging row 1 and row 2.

$$R_1 \leftrightarrow R_2 \rightarrow \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 2 & 3 & 6 \end{bmatrix}$$

We now have a $1$ as the first entry in row 1, column 1. Now let's obtain a $0$ in row 2, column 1. This can be accomplished by multiplying row 1 by $-2$, and then adding the result to row 2.

$$-2R_1 + R_2 = R_2 \rightarrow \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 5 & 5 \end{bmatrix}$$

We only have one more step, to multiply row 2 by $\frac{1}{5}$.

$$\frac{1}{5}R_2 = R_2 \rightarrow \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

Use back-substitution. The second row of the matrix represents $y = 1$. Back-substitute $y = 1$ into the first equation.

$$\begin{aligned} x - (1) &= \frac{1}{2} \\ x &= \frac{3}{2} \end{aligned}$$

The solution is the point $\left(\frac{3}{2}, 1\right)$.

## Exercise 2.6.3

Solve the given system by Gaussian elimination.

$$\begin{aligned} 4x + 3y &= 11 \\ x - 3y &= -1 \end{aligned}$$

*Answer available at the end of this section.*

## Example 2.6.4: Using Gaussian Elimination to Solve a System of Equations

Use **Gaussian elimination** to solve the given $2 \times 2$ **system of equations**.

$$\begin{aligned} 2x + y &= 1 \\ 4x + 2y &= 6 \end{aligned}$$

**Solution**

Write the system as an **augmented matrix**.

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 6 \end{bmatrix}$$

Obtain a $1$ in row 1, column 1. This can be accomplished by multiplying the first row by $\frac{1}{2}$.

$$\frac{1}{2}R_1 = R_1 \rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 4 & 2 & 6 \end{bmatrix}$$

Next, we want a $0$ in row 2, column 1. Multiply row 1 by $-4$ and add row 1 to row 2.

$$-4R_1 + R_2 = R_2 \rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 4 \end{bmatrix}$$

The second row represents the equation $0 = 4$. Therefore, the system is inconsistent and has no solution.

## Example 2.6.5: Solving a Dependent System

Solve the system of equations.

$$\begin{aligned} 3x + 4y &= 12 \\ 6x + 8y &= 24 \end{aligned}$$

**Solution**

Perform **row operations** on the augmented matrix to try and achieve **row-echelon form**.

$$A = \begin{bmatrix} 3 & 4 & 12 \\ 6 & 8 & 24 \end{bmatrix}$$

$$-\frac{1}{2}R_2 + R_1 = R_1 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 6 & 8 & 24 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 = \begin{bmatrix} 6 & 8 & 24 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix ends up with all zeros in the last row: $0y = 0$. Thus, there are an infinite number of solutions and the system is classified as dependent. To find the generic solution, return to one of the original equations and solve for $y$.

$$\begin{aligned} 3x + 4y &= 12 \\ 4y &= 12 - 3x \\ y &= 3 - \frac{3}{4}x \end{aligned}$$

So the solution to this system is $\left(x, 3 - \frac{3}{4}x\right)$.

## Example 2.6.6: Performing Row Operations on a $3 \times 3$ Augmented Matrix to Obtain Row-Echelon Form

Perform row operations on the given matrix to obtain row-echelon form.

$$\begin{bmatrix} 1 & -3 & 4 & 3 \\ 2 & -5 & 6 & 6 \\ -3 & 3 & 4 & 6 \end{bmatrix}$$

**Solution**

The first row already has a $1$ in row 1, column 1. The next step is to multiply row 1 by $-2$ and add it to row 2. Then replace row 2 with the result.

$$-2R_1 + R_2 = R_2 \begin{bmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ -3 & 3 & 4 & 6 \end{bmatrix}$$

Next, obtain a zero in row 3, column 1.

$$3R_1 + R_3 = R_3 \begin{bmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & -6 & 16 & 15 \end{bmatrix}$$

Next, obtain a zero in row 3, column 2.

$$6R_2 + R_3 = R_3 \begin{bmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 15 \end{bmatrix}$$

The last step is to obtain a 1 in row 3, column 3.

$$\frac{1}{4}R_3 = R_3 \begin{bmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & \frac{15}{4} \end{bmatrix}$$

### Exercise 2.6.4

Write the system of equations in row-echelon form.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

*Answer available at the end of this section.*

## Solving a System of Linear Equations Using Matrices

We have seen how to write a **system of equations** with an **augmented matrix**, and then how to use row operations and back-substitution to obtain row-echelon form. Now, we will take **row-echelon form** a step farther to solve a $3$ by $3$ system of linear equations. The general idea is to eliminate all but one variable using row operations and then back-substitute to solve for the other variables.

### Example 2.6.7: Solving a System of Linear Equations Using Matrices

Solve the system of linear equations using matrices.

$$\begin{aligned} x - y + z &= 8 \\ 2x + 3y - z &= -2 \\ 3x - 2y - 9z &= 9 \end{aligned}$$

**Solution**

First, we write the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{bmatrix}$$

Next, we perform row operations to obtain row-echelon form.

$$-2R_1 + R_2 = R_2 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 3 & -2 & -9 & 9 \end{bmatrix}$$

$$-3R_1 + R_3 = R_3 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 0 & 1 & -12 & -15 \end{bmatrix}$$

The easiest way to obtain a $1$ in row 2 of column 1 is to interchange $R_2$ and $R_3$.

$$Interchange\ R_2\ and\ R_3 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 5 & -3 & -18 \end{bmatrix}$$

Then

$$-5R_2 + R_3 = R_3 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{bmatrix}$$

$$-\frac{1}{57}R_3 = R_3 \rightarrow \begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The last matrix represents the equivalent system.

$$\begin{aligned} x - y + z &= 8 \\ y - 12z &= -15 \\ z &= 1 \end{aligned}$$

Using back-substitution, we obtain the solution as $(4, -3, 1)$.

## Example 2.6.8: Solving a Dependent System of Linear Equations Using Matrices

Solve the following system of linear equations using matrices.

$$\begin{aligned} -x - 2y + z &= -1 \\ 2x + 3y &= 2 \\ y - 2z &= 0 \end{aligned}$$

**Solution**

Write the augmented matrix.

$$\begin{bmatrix} -1 & -2 & 1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

First, multiply row 1 by $-1$ to get a $1$ in row 1, column 1. Then, perform **row operations** to obtain row-echelon form.

$$-R_1 \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 2 & 3 & 0 & 2 \end{bmatrix}$$

$$-2R_1 + R_3 = R_3 \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix}$$

$$R_2 + R_3 = R_3 \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix represents the following system.

$$\begin{aligned} x + 2y - z &= 1 \\ y - 2z &= 0 \\ 0 &= 0 \end{aligned}$$

We see by the identity $0 = 0$ that this is a dependent system with an infinite number of solutions. We then find the generic solution. By solving the second equation for $y$ and substituting it into the first equation we can solve for $z$ in terms of $x$.

$$\begin{aligned} x + 2y - z &= 1 \\ y &= 2z \\ x + 2(2z) - z &= 1 \\ x + 3z &= 1 \\ z &= \frac{1 - x}{3} \end{aligned}$$

Now we substitute the expression for $z$ into the second equation to solve for $y$ in terms of $x$.

$$\begin{aligned} y - 2z &= 0 \\ z &= \frac{1 - x}{3} \\ y - 2\left(\frac{1 - x}{3}\right) &= 0 \\ y &= \frac{2 - 2x}{3} \end{aligned}$$

The generic solution is $\left(x, \frac{2 - 2x}{3}, \frac{1 - x}{3}\right)$.

## Exercise 2.6.5

Solve the system using matrices.

$$\begin{aligned} x + 4y - z &= 4 \\ 2x + 5y + 8z &= 1 \\ 5x + 3y - 3z &= 1 \end{aligned}$$

*Answer available at the end of this section.*

## Q&A: Can any system of linear equations be solved by Gaussian elimination?

*Yes, a system of linear equations of any size can be solved by Gaussian elimination.*

## How to: Given a system of equations, solve with matrices using a calculator

1. Save the augmented matrix as a matrix variable $[A], [B], [C], \ldots$.
2. Use the **ref(** function in the calculator, calling up each matrix variable as needed.

## Example 2.6.9A: Solving Systems of Equations with Matrices Using a Calculator

Solve the system of equations.

$$5x + 3y + 9z = -1$$
$$-2x + 3y - z = -2$$
$$-x - 4y + 5z = 1$$

**Solution**

Write the augmented matrix for the system of equations.

$$\begin{bmatrix} 5 & 3 & 9 & -1 \\ -2 & 3 & -1 & -2 \\ -1 & -4 & 5 & 1 \end{bmatrix}$$

On the matrix page of the calculator, enter the augmented matrix above as the matrix variable $[A]$.

$$[A] = \begin{bmatrix} 5 & 3 & 9 & -1 \\ -2 & 3 & -1 & -2 \\ -1 & -4 & 5 & 1 \end{bmatrix}$$

Use the **ref(** function in the calculator, calling up the matrix variable $[A]$.

**ref([A])**

**Evaluate**

Erroneous nesting of equation structures

Using back-substitution, the solution is $\left(\frac{61}{187}, -\frac{92}{187}, -\frac{24}{187}\right)$.

## Example $2.6.9B$: Applying $2 \times 2$ Matrices to Finance

Carolyn invests a total of $\$12,000$ in two municipal bonds, one paying $10.5$ interest and the other paying $12$ interest. The annual interest earned on the two investments last year was $\$1,335$. How much was invested at each rate?

**Solution**

We have a system of two equations in two variables. Let $x =$ the amount invested at $10.5$ interest, and $y =$ the amount invested at $12$ interest.

$$x + y = 12,000$$
$$0.105x + 0.12y = 1,335$$

As a matrix, we have

$$\begin{bmatrix} 1 & 1 & 12,000 \\ 0.105 & 0.12 & 1,335 \end{bmatrix}$$

Multiply row 1 by $-0.105$ and add the result to row 2.

$$\begin{bmatrix} 1 & 1 & 12,000 \\ 0 & 0.015 & 75 \end{bmatrix}$$

Then,

$$0.015y = 75$$
$$y = 5,000$$

So $12,000 - 5,000 = 7,000$.

Thus, $\$5,000$ was invested at $12$ interest and $\$7,000$ at $10.5$ interest.

## Example 2.6.10: Applying $3 \times 3$ Matrices to Finance

Ava invests a total of $\$10,000$ in three accounts, one paying $5$ interest, another paying $8$ interest, and the third paying $9$ interest. The annual interest earned on the three investments last year was $\$770$. The amount invested at $9$ was twice the amount invested at $5$. How much was invested at each rate?

**Solution**

We have a system of three equations in three variables. Let $x$ be the amount invested at $5$ interest, let $y$ be the amount invested at $8$ interest, and let $z$ be the amount invested at $9$ interest. Thus,

$$\begin{aligned} x + y + z &= 10,000 \\ 0.05x + 0.08y + 0.09z &= 770 \\ 2x - z &= 0 \end{aligned}$$

As a matrix, we have

$$\begin{bmatrix} 1 & 1 & 1 & 10,000 \\ 0.05 & 0.08 & 0.09 & 770 \\ 2 & 0 & -1 & 0 \end{bmatrix}$$

Now, we perform Gaussian elimination to achieve row-echelon form.

$$-0.05R_1 + R_2 = R_2 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 10,000 \\ 0 & 0.03 & 0.04 & 270 \\ 2 & 0 & -1 & 0 \end{bmatrix}$$

$$-2R_1 + R_3 = R_3 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 10,000 \\ 0 & 0.03 & 0.04 & 270 \\ 0 & -2 & -3 & -20,000 \end{bmatrix}$$

$$\frac{1}{0.03}R_2 = R_2 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 10,000 \\ 0 & 1 & \frac{4}{3} & 9,000 \\ 0 & -2 & -3 & -20,000 \end{bmatrix}$$

$$2R_2 + R_3 = R_3 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 10,000 \\ 0 & 1 & \frac{4}{3} & 9,000 \\ 0 & 0 & -\frac{1}{3} & -2,000 \end{bmatrix}$$

The third row tells us $-\frac{1}{3}z = -2,000$; thus $z = 6,000$.

The second row tells us $y + \frac{4}{3}z = 9,000$. Substituting $z = 6,000$,we get

$$\begin{aligned} y + \frac{4}{3}(6,000) &= 9,000 \\ y + 8,000 &= 9,000 \\ y &= 1,000 \end{aligned}$$

The first row tells us $x + y + z = 10,000$. Substituting $y = 1,000$ and $z = 6,000$,we get

$$\begin{aligned} x + 1,000 + 6,000 &= 10,000 \\ x &= 3,000 \end{aligned}$$

The answer is $\$3,000$ invested at $5$ interest, $\$1,000$ invested at $8$, and $\$6,000$ invested at $9$ interest.

### Exercise 2.6.6

A small shoe company took out a loan of $\$1,500,000$ to expand their inventory. Part of the money was borrowed at $7$, part was borrowed at $8$, and part was borrowed at $10$. The amount borrowed at $10$ was four times the amount borrowed at $7$, and the annual interest on all three loans was $\$130,500$. Use matrices to find the amount borrowed at each rate.

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with solving systems of linear equations using Gaussian elimination.

- Solve a System of Two Equations Using an Augmented Matrix
- Solve a System of Three Equations Using an Augmented Matrix
- Augmented Matrices on the Calculator

## Key Concepts

- An augmented matrix is one that contains the coefficients and constants of a system of equations. See Example $2.6.1$.
- A matrix augmented with the constant column can be represented as the original system of equations. See Example $2.6.2$.
- Row operations include multiplying a row by a constant, adding one row to another row, and interchanging rows.
- We can use Gaussian elimination to solve a system of equations. See Example $2.6.3$, Example $2.6.4$, and Example $2.6.5$.
- Row operations are performed on matrices to obtain row-echelon form. See Example $2.6.6$.
- To solve a system of equations, write it in augmented matrix form. Perform row operations to obtain row-echelon form. Back-substitute to find the solutions. See Example $2.6.7$ and Example $2.6.8$.
- A calculator can be used to solve systems of equations using matrices. See Example $2.6.9$.
- Many real-world problems can be solved using augmented matrices. See Example $2.6.10$ and Example $2.6.11$.

## Answers

**Exercise** $2.6.1$

$$\begin{bmatrix} 4 & -3 & 11 \\ 3 & 2 & 4 \end{bmatrix}$$

**Exercise** $2.6.2$

$$\begin{aligned} x - y + z &= 5 \\ 2x - y + 3z &= 1 \\ y + z &= -9 \end{aligned}$$

**Exercise** $2.6.3$

$(2, 1)$

**Exercise** $2.6.4$

$$\begin{bmatrix} 1 & -\frac{5}{2} & \frac{5}{2} & \frac{17}{2} \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

**Exercise** $2.6.5$

$(1, 1, 1)$

**Exercise** $2.6.6$

$\$150,000$ at $7$, $\$750,000$ at $8$, $\$600,000$ at $10$

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# 3: FUNCTIONS, GRAPHS, AND TRANSFORMATIONS


3.1: Functions and Function Notation

3.2: Domain and Range

3.3: Rates of Change and Behavior of Graphs

3.4: Transformations of Functions

3.5: Composition of Functions

3.6: Inverse Functions


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# 3.1: Functions and Function Notation

**Learning Objectives**

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

The weight of a growing child increases with age. An airplane changes altitude over time after it departs from the runway. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

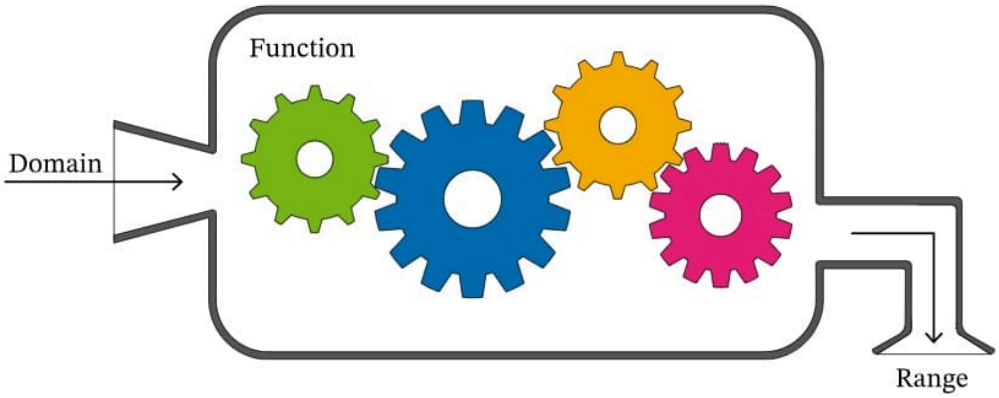
## Determining Whether a Relation Represents a Function

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the **domain** and the set of the second components of each ordered pair is called the **range**. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\} \tag{1.1.1}$$

The domain is $\{1,2,3,4,5\}$. The range is $\{2,4,6,8,10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter $x$. Each value in the range is also known as an **output** value, or **dependent variable**, and is often labeled lowercase letter $y$.




A function $f$ is a relation that assigns a single value in the range to each value in the domain. In other words, no $x$-values are repeated. For our example that relates the first five **natural numbers** to numbers double their values, this relation is a function because each element in the domain, {1, 2, 3, 4, 5}, is paired with exactly one element in the range, $\{2,4,6,8,10\}$.

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\} \tag{1.1.2}$$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1,2,3,4,5\}$. For example, the term "odd" corresponds to three values from the range, $\{1,3,5\}$, and the term "even" corresponds to two values from the

range, $\{2, 4\}$. This violates the definition of a function, so this relation is not a function.

Figure $3.1.1$ compares relations that are functions and not functions.

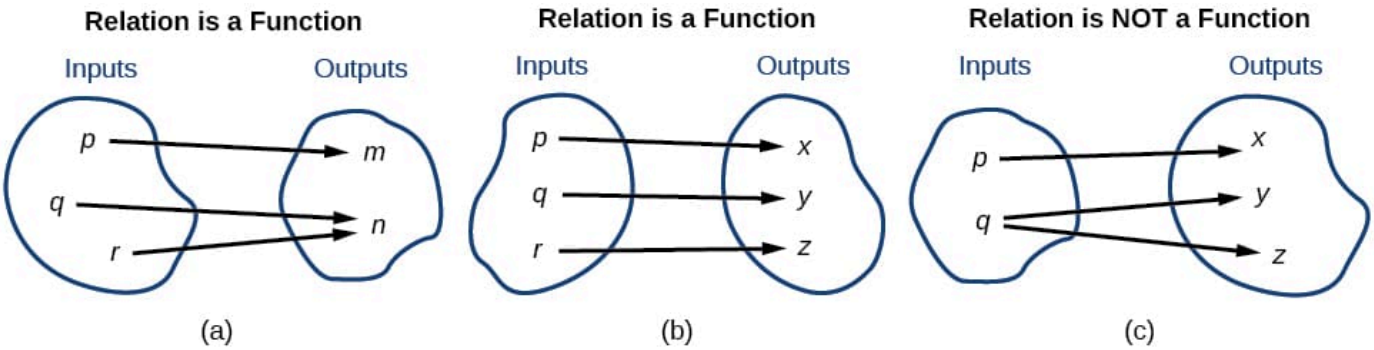



Figure $3.1.1$: (a) This relationship is a function because each input is associated with a single output. Note that input $q$ and $r$ both give output $n$. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input $q$ is associated with two different outputs.

### Definition: Function

A **function** is a relation in which each possible input value corresponds to exactly one output value.

We say "the output is a function of the input."

The set of all **input** values is called the **domain**, and the set of all **output** values is called the **range**.

### How To: Given a relationship between two quantities, determine whether the relationship is a function

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

### Example $3.1.1$: Determining If Menu Price Lists Are Functions

At a popular Television Burger Shop, the menu is very simple, it contains a small list of items and their associated prices. See Figure $3.1.2$

a. Are the prices a function of the menu items?
b. Are the menu items a function of their prices?




Figure 3.1.2: A menu of prices from a Burger Shop.

**Solution**

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See Figure 3.1.3.




Figure 3.1.3: Association of menu items to prices.

Each item on the menu has only one price, so the price is a function of the item.




Figure 3.1.4: Association of the prices to menu items.

b. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value will have more than one output associated with it. See Figure 3.1.4.Therefore, the menu items are not a function of price.

## Example 3.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the letter grade corresponds to a percentage. See Table 3.1.1:

a. Is the letter grade a function of the percent grade?

b. Is the percent grade a function of the grade point average?

*Table 3.1.1: Class Grades*

| **Percent grade** | 0–54 | 55 – 69 | 70 – 79 | 80 – 89 | 90 – 100 |
|---|---|---|---|---|---|
| **Letter grade** | F | D | C | B | A |

**Solution**

For any percent grade earned, there is an associated letter grade, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is only ONE specified letter grade.

In the grading system given, there is a range of percent grades that correspond to the same letter grade. For example, students who receive a letter grade of B could have a variety of percent grades ranging from 80 to 89. Thus, percent grade is not a function of the letter grade.

## Exercise 3.1.2

Table 3.1.2 lists the batting averages of 10 of the greatest major league baseball players in history.

*Table 3.1.2: Greatest baseball players.*

| Player | Average |
|---|---|
| Babe Ruth | 0.342 |
| Willie Mays | 0.301 |
| Ty Cobb | 0.366 |
| Shohei Otani | 0.298 |
| Hank Aaron | 0.305 |
| Lou Gehrig | 0.340 |
| Barry Bonds | 0.298 |
| Derek Jeter | 0.310 |
| Ted Williams | 0.344 |
| Joe Jackson | 0.356 |

a. Is the batting average a function of the player name?
b. Is the player name a function of the batting average?

*Answer a available at the end of this section.*

## Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions. Typically, the letters $f$, $g$ and $h$ are often used to represent functions just as we use $x$, $y$ and $z$ to represent numbers and $A$, $B$ and $C$ to represent sets.

To represent "weight is a function of age," we start by identifying the descriptive variables $w$ for weight and $a$ for age. Next, we select the letter $f$ to use as the name for this function.

$$\begin{aligned} &w \text{ is } f \text{ of } a && \text{We name the function } f\text{; weight is a function of age.} \\ &w = f(a) && \text{We use parentheses to indicate the input } a \text{ ; and the equal sign to indicate the output } w. \\ &f(a) && \text{We read this expression as " } f \text{ of } a.\text{" ; it has the same meaning as the value } w. \end{aligned} \tag{3.1.1}$$

The parentheses indicate that age is input into the function; they do not indicate multiplication. Recall that the letter $f$ is NOT a variable, in fact $f$ is NOT a numerical value, it is just the name of a function; it would not make mathematical sense to multiply a name by a number.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means "first add $a$ and $b$, and the result is the input for the function $f$." Whereas, $f(a)+f(b)$ means "first input $a$ and $b$ into the function, and then add the results." The operations must be performed in this order to obtain the correct result.

### Function Notation

The notation $y = f(x)$ defines a function named $f$. This is read as "$y$ is a function of $x$." The letter $x$ represents the input value, or independent variable. The letter $y$, or $f(x)$, represents the output value, or dependent variable.

## Example 3.1.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

**Solution**

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function $f$, we write $\text{days} = f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

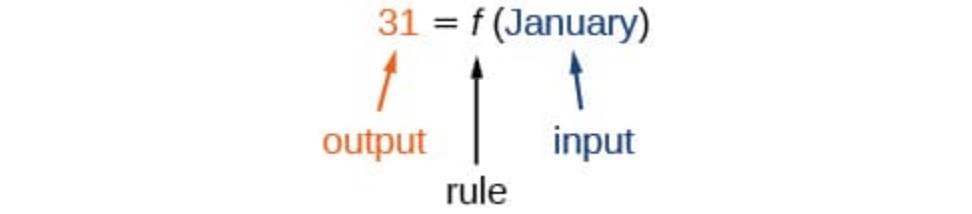



Figure 3.1.5: The function $31 = f(January)$ where 31 is the output, f is the rule, and January is the input.

For example, $f(\text{March}) = 31$, because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, $d$ (the output), is dependent on the name of the month, $m$ (the input).

**Analysis**

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

## Example 3.1.3B: Interpreting Function Notation

A function $n = f(y)$ gives the number of people, $n$, in a town in year $y$. What does $f(2025) = 3000$ represent?

**Solution**

When we read $f(2025) = 3000$, we see that the input year is 2025. The value for the output, the number of people $(n)$, is 3000. Remember, $n = f(y)$. The statement $f(2025) = 3000$ tells us that in the year 2025 there were 3000 police officers in the town.

## Exercise 3.1.3

Use function notation to express the weight $w$ of a baby in pounds as a function $f$ of its age in months $m$. Interpret the meaning of $f(9) = 21$

*Answer available at the end of this section.*

## Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "$y$ is a function of $x$?"

*Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as $f$, which is a rule or procedure, and the output y we get by applying $f$ to a particular input $x$. This is why we usually use notation such as $y = f(x), P = W(d)$, and so on.*

## Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table 3.1.3 lists the input number of each month ($\text{January} = 1$, $\text{February} = 2$, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function $f$ where $D = f(m)$ identifies months by an integer rather than by name.

*Table 3.1.3: Months and number of days per month.*

| **Month number,** $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Days in month,** $D$ | 31 | 28 | 31 | 30 | 31 | 30 | 31 | 31 | 30 | 31 | 30 | 31 |

Table 3.1.4 defines a function $Q = g(n)$ Remember, this notation tells us that $g$ is the name of the function that takes the input $n$ and gives the output $Q$.

*Table 3.1.4: Function $Q = g(n)$*

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $Q$ | 8 | 6 | 7 | 6 | 8 |

Table 3.1.5 displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

*Table 3.1.5: Age of children and their corresponding heights.*

| **Age in years,** $a$ | 5 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| **Height in inches,** $h$ | 40 | 42 | 44 | 47 | 50 | 52 | 54 |

### How To: Given a table of input and output values, determine whether the table represents a function

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

### Example 3.1.5: Identifying Tables that Represent Functions

Which table, Table 3.1.6, Table 3.1.7, or Table 3.1.8, represents a function (if any)?

*Table 3.1.6*

| Input | Output |
|---|---|
| 2 | 1 |
| 5 | 3 |

| Input | Output |
|---|---|
| 8 | 6 |
| 1 | 2 |
| 3 | 4 |

*Table* 3.1.7

| Input | Output |
|---|---|
| -3 | 5 |
| 0 | 1 |
| 4 | 5 |
| -2 | 0 |
| 1 | 9 |

*Table* 3.1.8

| Input | Output |
|---|---|
| 1 | 0 |
| 5 | 2 |
| 4 | 3 |
| 2 | 1 |
| 5 | 4 |

**Solution**

Table 3.1.6 and Table 3.1.7 define functions. In both, each input value corresponds to exactly one output value. Table 3.1.8 does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table 3.1.6 can be represented by writing

$$f(2) = 1,\ f(5) = 3,\ f(8) = 6,\ f(1) = 2, \text{ and } f(3) = 4$$

Similarly, the statements

$$g(-3) = 5,\ g(0) = 1,\ g(4) = 5,\ g(-2) = 0, \text{ and } g(1) = 9$$

represent the function in Table 3.1.7. note: we use a different letter to indicate that the function is not the same as the function $f$ seen in Table 3.1.6.

Table 3.1.8 cannot be expressed using function notation because it does not represent a function.

## Exercise 3.1.5

Does Table 3.1.9 represent a function?

*Table* 3.1.9

| Input | Output |
|---|---|
| 1 | 10 |
| 2 | 100 |
| 3 | 1000 |
| 4 | 10000 |
| 5 | 100000 |

*Answer available at the end of this section.*

# Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

## Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

### How To: Given the formula for a function, evaluate.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

### Example 3.1.6A: Evaluating Functions at Specific Values

1. Given the function $f(x) = x^2 + 3x - 4$, find the following values.

a. $f(2)$
b. $f(a)$
c. $f(a+h)$
d. Evaluate $\frac{f(a+h)-f(a)}{h}$

**Solution**

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} \\ &= \frac{(2ah + h^2 + 3h)}{h} \\ &= \frac{h(2a + h + 3)}{h} && \text{Factor out h.} \\ &= 2a + h + 3 && \text{Simplify.} \end{aligned}$$

## Example 3.1.6B: Evaluating Functions

Given the function $g(x) = -x^2 + 2x$, evaluate $g(-4)$.

**Solution**

To evaluate $g(-4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} g(x) &= -x^2 + 2x \\ f(-4) &= -(-4)^2 + 2(-4) \\ &= -16 + 8 \\ &= -8 \end{aligned}$$

Therefore, for an input of -4, we have an output of -8.

## Exercise 3.1.6

Given the function $g(x) = \sqrt{x - 4}$, evaluate $g(20)$.

*Answer available at the end of this section.*

## Example 3.1.7: Solving Functions

Given the function $f(x) = x^2 + 2x$, solve $f(x) = 3$.

**Solution**

$$\begin{aligned} f(x) &= 3 \\ x^2 + 2x &= 3 \quad \text{Substitute the original function} \\ x^2 + 2x - 3 &= 0 \quad \text{Subtract 3 from each side.} \\ (x+3)(x-1) &= 0 \quad \text{Factor.} \end{aligned}$$

If $(x+3)(x-1) = 0$, either $(x+3) = 0$ or $(x-1) = 0$ (or both of them equal $0$). We will set each factor equal to $0$ and solve for $p$ in each case.

$$(x+3) = 0,\ x = -3$$

$$(x-1) = 0,\ x = 1$$

This gives us two solutions. The output $h(p) = 3$ when the input is either $x = 1$ or $x = -3$.

### Exercise 3.1.7

Given the function $g(x) = \sqrt{x+6}$, solve $g(x) = 2$.

*Answer available at the end of this section.*

## Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in **equation** form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n + 6p = 12$ expresses a functional relationship between $n$ and $p$. We can rewrite it to decide if $p$ is a function of $n$.

### How to: Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves only the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

### Example 3.1.8A: Finding an Equation of a Function

Express the relationship $2x + 6y = 12$ as a function $y = f(x)$, if possible.

**Solution**

To express the relationship in this form, we need to be able to write the relationship where $p$ is a function of $n$, which means writing it as $p = [\text{expression involving } n]$.

$$\begin{aligned} 2x + 6y &= 12 \\ 6y &= 12 - 2x \quad \text{Subtract 2x from both sides.} \\ y &= \frac{12 - 2x}{6} \quad \text{Divide both sides by 6 and simplify.} \\ y &= \frac{12}{6} - \frac{2x}{6} \\ y &= 2 - \frac{1}{3}x \end{aligned}$$

Therefore, $y$ as a function of $x$ is written as

$$y = f(x) = 2 - \frac{1}{3}x$$

**Analysis**

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

### Example $3.1.8B$: Expressing the Equation of a Circle as a Function

Does the equation $x^2 + y^2 = 1$ represent a function with $x$ as input and $y$ as output? If so, express the relationship as a function $y = f(x)$.

**Solution**

First we subtract $x^2$ from both sides.

$$y^2 = 1 - x^2$$

We now try to solve for $y$ in this equation.

$$y = \pm\sqrt{1 - x^2}$$

$$\text{so, } y = \sqrt{1 - x^2} \text{ and } y = -\sqrt{1 - x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$ .

### Exercise $3.1.8$

If $x - 8y^3 = 0$, express $y$ as a function of $x$.

*Answer available at the end of this section.*

### Q & A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only $x$ that equals $y$. However, each $x$ does determine a unique value for $y$, and there are mathematical procedures by which $y$ can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for $y$ as a function of $x$, even though the formula cannot be written explicitly.

## Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 6 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for about 9 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table (Table $3.1.10$).

*Table* 3.1.10

| Pet Memory | span in hours |
|---|---|
| Puppy | 0.0083 |
| Adult Dog | 0.15 |
| Cat | 16 |
| Goldfish | 2190 |
| Beta Fish | 4380 |

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function $P$. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function $P$ at the input value of "goldfish." We would write $P(goldfish) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

### How To: Given a function represented by a table, identify specific output and input values

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

### Example 3.1.9: Evaluating and Solving a Tabular Function

Using Table 3.1.11,

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

*Table* 3.1.11

| $n$ | 1 | 2 | **3** | 4 | 5 |
|---|---|---|---|---|---|
| $g(n)$ | 8 | 6 | **7** | 6 | 8 |

**Solution**

a. Evaluating $g(3)$ means determining the output value of the function $g$ for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. Table 3.1.12 shows two solutions: $x = 2$ or $x = 4$. We typically write this soution as $x = 2, 4$.

*Table* 3.1.12

| $n$ | 1 | **2** | 3 | **4** | 5 |
|---|---|---|---|---|---|
| $g(n)$ | 8 | **6** | 7 | **6** | 8 |

When we input 2 into the function $g$, our output is 6. When we input 4 into the function $g$, our output is also 6.

### Exercise 3.1.1

Using Table 3.1.13 of the function $f$, shown below.

a. Evaluate $f(1)$

b. Solve $f(x) = 4$

| $x$ | $f(x)$ |
|---|---|
| -3 | 5 |
| 0 | 1 |
| 4 | 3 |
| -2 | 4 |
| 1 | 9 |

Table 3.1.13

*Answers available at the end of this section.*

## Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

### Example 3.1.10: Reading Function Values from a Graph

Given the graph in Figure 3.1.7,

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

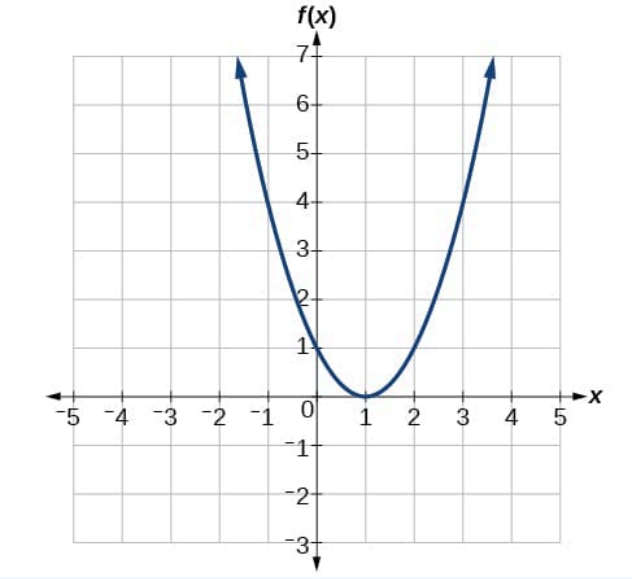
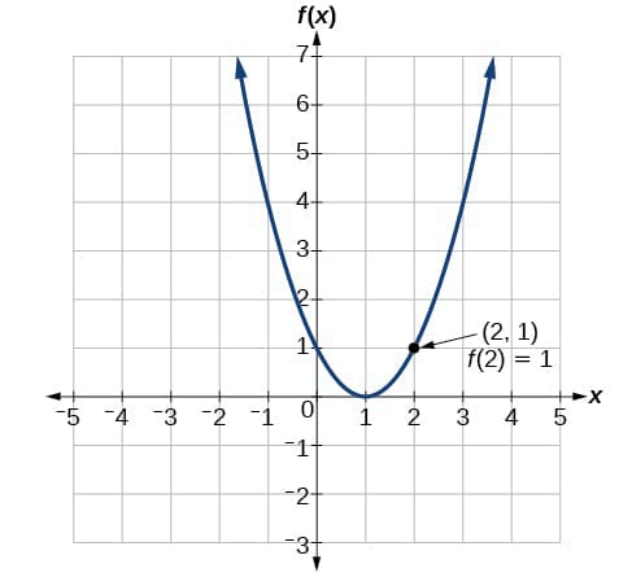



Figure 3.1.7: Graph of a positive parabola centered at $(1, 0)$.

**Solution**

To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure 3.1.8.




3.1.8: Graph of a positive parabola centered at $(1, 0)$ with the labeled point $(2, 1)$ where $f(2) = 1$.

To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: –1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is –1 or 3, the output is 4. See Figure 3.1.9.

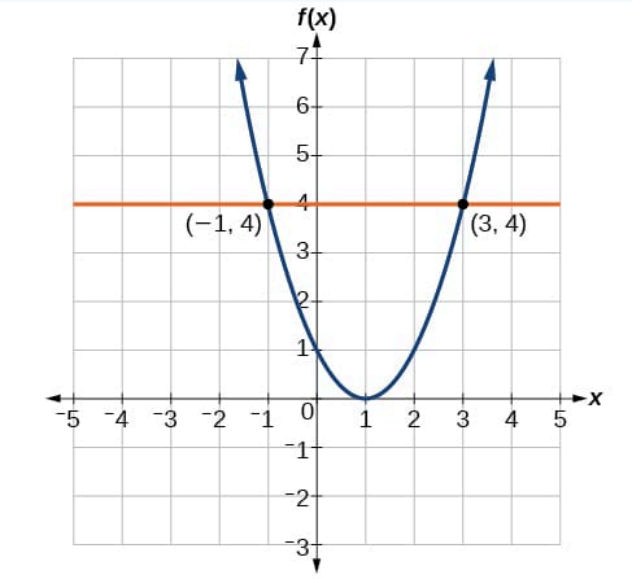



Figure 3.1.9: Graph of an upward-facing parabola with a vertex at $(0, 1)$ and labeled points at $(-1, 4)$ and $(3, 4)$. A line at $y = 4$ intersects the parabola at the labeled points.

## Exercise 3.1.10

Given the graph in Figure 3.1.7, solve $f(x) = 1$.

*Answer available at the end of this section.*

# Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value $x$ and the output $y$, and we say $y$ is a function of $x$, or $y = f(x)$ when the function is named $f$. The graph of the function is the set of all points $(x, y)$ in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure $3.1.10$ tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points $(x, y)$ satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

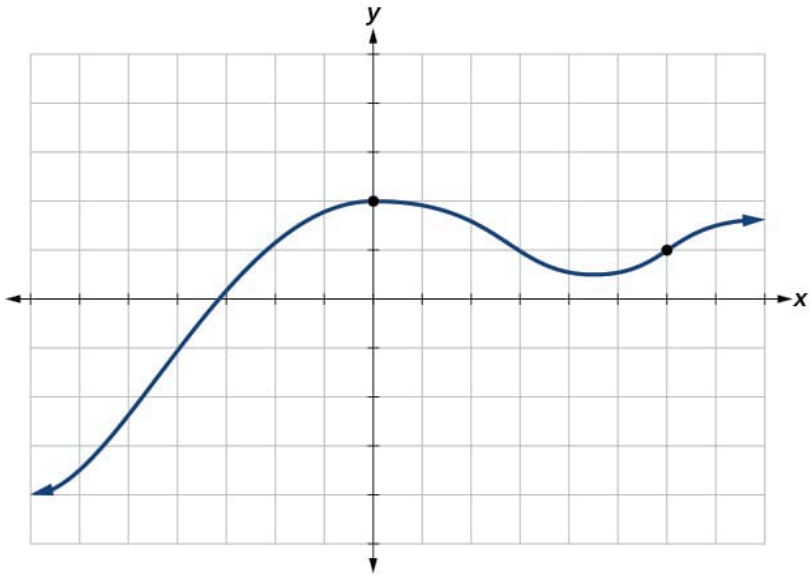



Figure $3.1.10$: Graph of a polynomial.

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. See Figure $3.1.11$**.**

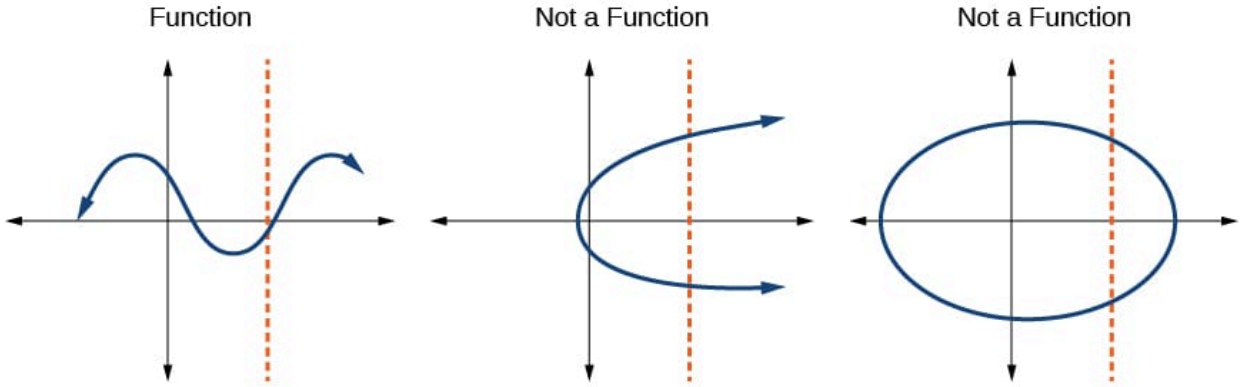



Figure $3.1.11$: Three graphs visually showing what is and is not a function.

**How to: Given a graph, use the vertical line test to determine if the graph represents a function**

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

## Example 3.1.12: Applying the Vertical Line Test

Which of the graphs in Figure 3.1.12 represent(s) a function $y = f(x)$?

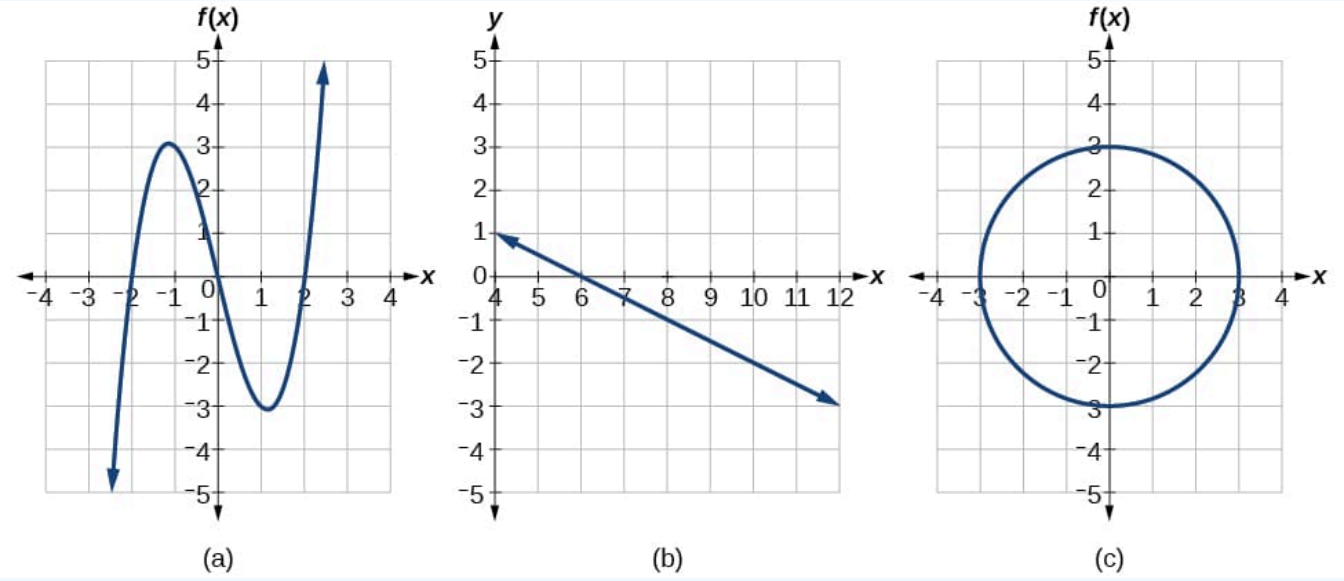



Figure 3.1.12: Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

**Solution**

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure 3.1.12. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in Figure 3.1.13.

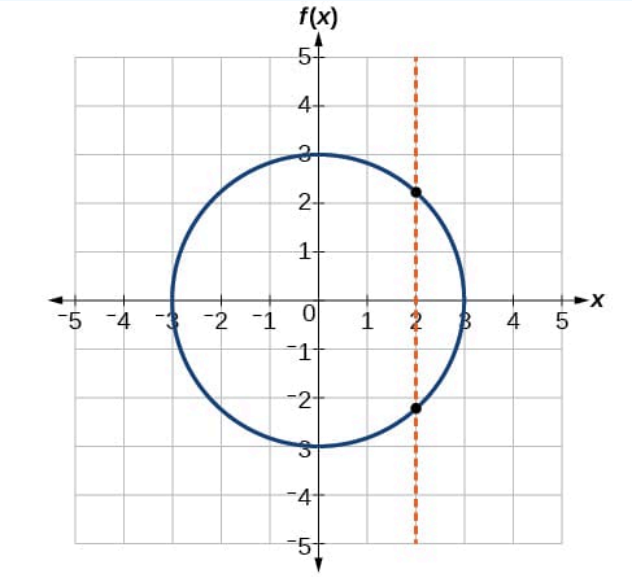



Figure 3.1.13: Graph of a circle.

## Exercise 3.1.12

Does the graph in Figure 3.1.14 represent a function?

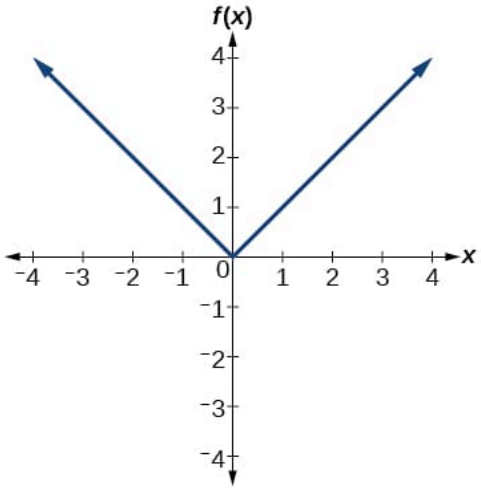



Figure 3.1.14: Graph of absolute value function.

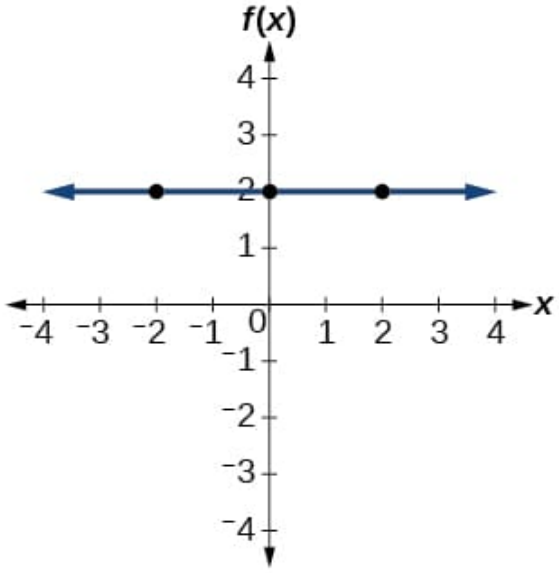
*Answer available at the end of this section.*

## Identifying Parent Functions

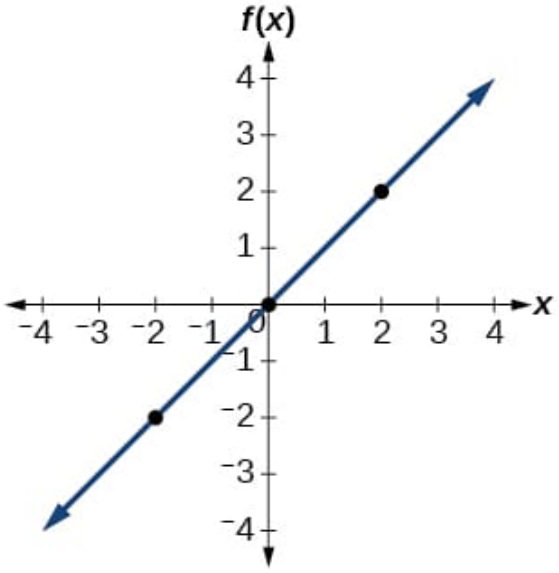
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our “parent" functions, or sometimes we refer to them as "mother functions". They form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will use these like a toolkit of functions in which we will create combinations of these functions, analyze their graphs by using transformations frequently throughout this book. It will be very helpful if we can recognize these basic functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table 3.1.14.

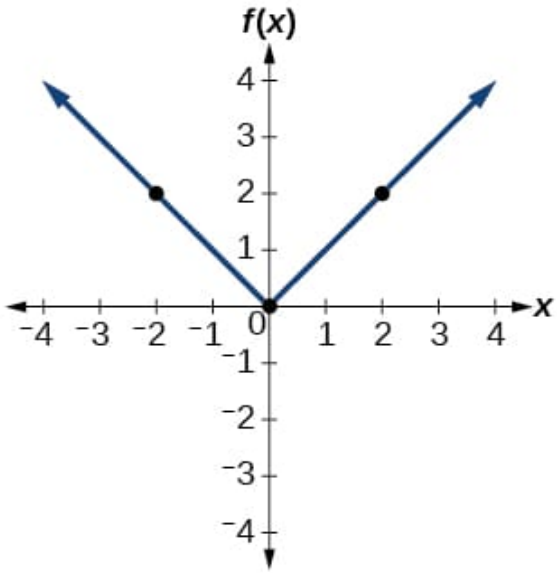
*Table 3.1.14: Parent Functions*

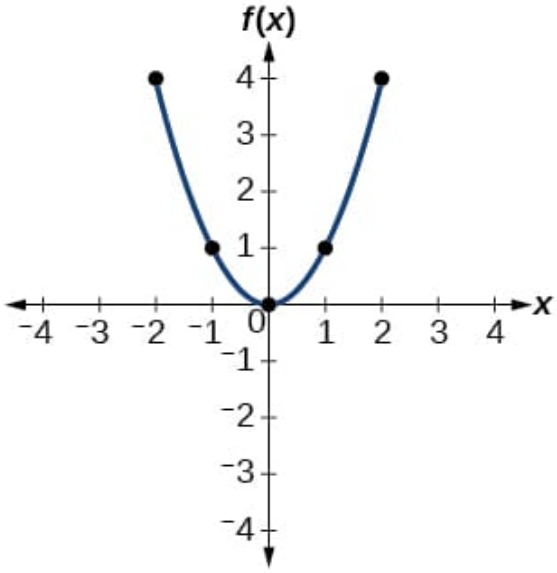
| Name | Function | Graph |
|---|---|---|
| Constant | $f(x) = c$ where $c$ is a constant |   |

| x | f(x) |
|---|---|
| −2 | 2 |
| 0 | 2 |
| 2 | 2 |

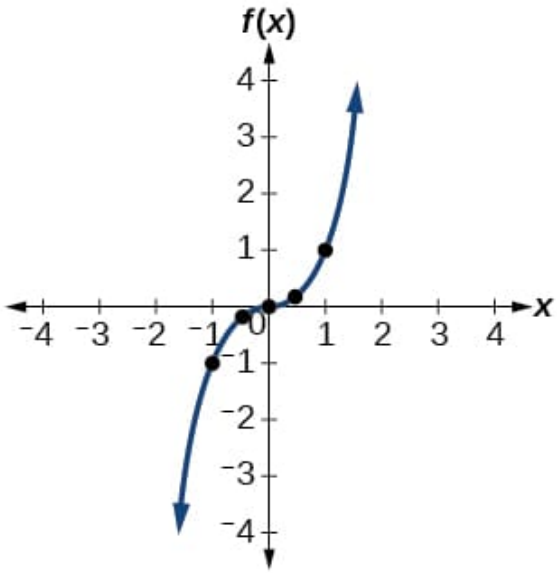
| Name | Function | Graph |
|---|---|---|
| Identity | $f(x) = x$ |   |

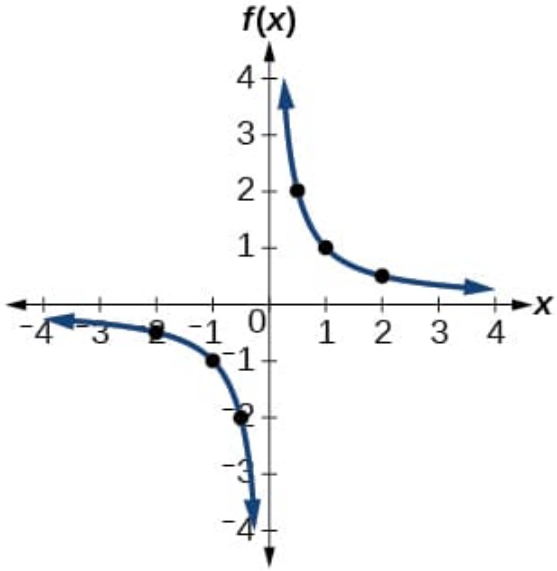
| x | f(x) |
|---|---|
| −2 | −2 |
| 0 | 0 |
| 2 | 2 |

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Absolute Value</td><td>$f(x) = |x|$</td><td> 

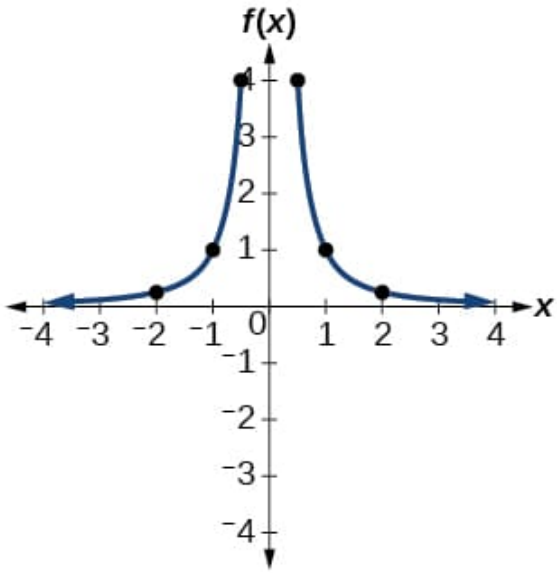
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>2</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
<tr><td>Quadratic</td><td>$f(x) = x^2$</td><td> 

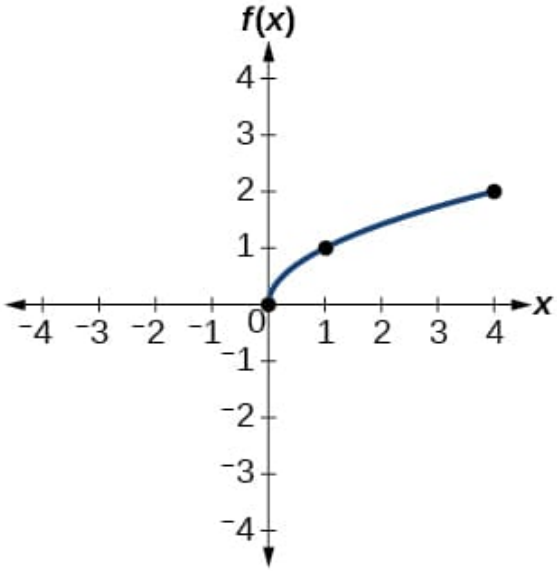
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>4</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>4</td></tr>
</table></td></tr>
</table>

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Cubic</td><td>$f(x) = x^3$</td><td> 

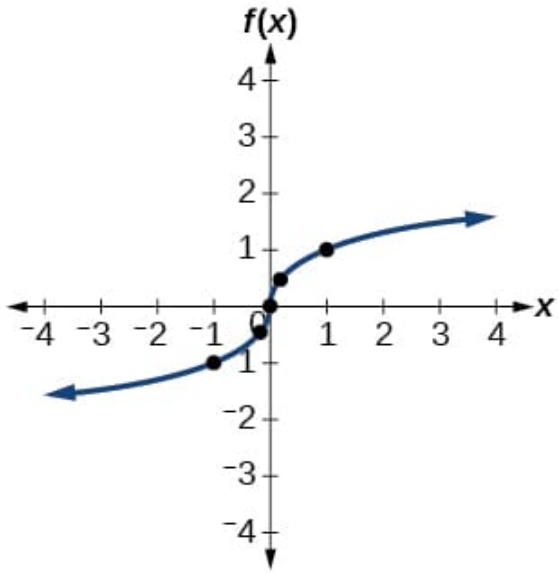
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−0.125</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.5</td><td>0.125</td></tr>
<tr><td>1</td><td>1</td></tr>
</table></td></tr>
<tr><td>reciprocal</td><td>$f(x) = \frac{1}{x}$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−0.5</td></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−2</td></tr>
<tr><td>0.5</td><td>2</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.5</td></tr>
</table></td></tr>
</table>

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Reciprocal squared</td><td>$f(x) = \frac{1}{x^2}$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>0.25</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>−0.5</td><td>4</td></tr>
<tr><td>0.5</td><td>4</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.25</td></tr>
</table></td></tr>
<tr><td>Square root</td><td>$f(x) = \sqrt{x}$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>4</td><td>2</td></tr>
</table></td></tr>
</table>

| Name | Function | Graph |
|---|---|---|
| Cube root | $f(x) = \sqrt[3]{x}$ |  |



| x | f(x) |
|---|---|
| −1 | −1 |
| −0.125 | −0.5 |
| 0 | 0 |
| 0.125 | 0.5 |
| 1 | 1 |

## Key Equations

- Constant function $f(x) = c$, where $c$ is a constant
- Identity function $f(x) = x$
- Absolute value function $f(x) = |x|$
- Quadratic function $f(x) = x^2$
- Cubic function $f(x) = x^3$
- Reciprocal function $f(x) = \dfrac{1}{x}$
- Reciprocal squared function $f(x) = \frac{1}{x^2}$
- Square root function $f(x) = \sqrt{x}$
- Cube root function $f(x) = 3\sqrt{x}$

## Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output.
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$.
- In tabular form, a function can be represented by rows or columns that relate to input and output values.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value.
- To solve for a specific function value, we determine the input values that yield the specific output value.
- An algebraic form of a function can be written from an equation.
- Input and output values of a function can be identified from a table.
- Relating input values to output values on a graph is another way to evaluate a function.
- A function is one-to-one if each output value corresponds to only one input value.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point.
- The graph of a one-to-one function passes the horizontal line test.

## Footnotes

1 http://www.baseball-almanac.com/lege.../lisn100.shtml. Accessed 3/24/2014.
2 www.kgbanswers.com/how-long-i...y-span/4221590. Accessed 3/24/2014.

# Glossary

**dependent variable**
an output variable

**domain**
the set of all possible input values for a relation

**function**
a relation in which each input value yields a unique output value

**horizontal line test**
a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

**independent variable**
an input variable

**input**
each object or value in a domain that relates to another object or value by a relationship known as a function

**one-to-one function**
a function for which each value of the output is associated with a unique input value

**output**
each object or value in the range that is produced when an input value is entered into a function

**range**
the set of output values that result from the input values in a relation

**relation**
a set of ordered pairs

**vertical line test**
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

---

## Answers

**Exercise** 3.1.2

Yes (since each player has a unique name, the second column would always be a function of the name, no matter what the values are)

No. (Since two players have a 0.298 average (Barry Bonds and Shohei Otani), the player name is NOT a function of the batting average.)

**Exercise** 3.1.3

$w = f(m)$

$f(9) = 21$ means that a 9 month old baby would weight 21 lbs.

**Exercise** 3.1.5

yes. In fact, since this is an easily recognizable pattern, the function is $f(x) = 10^x$

**Exercise** 3.1.6

$g(20) = \sqrt{20 - 4} = \sqrt{16} = 4$

**Exercise** 3.1.7

$x = -2$

**Exercise** 3.1.8

$y = f(x) = \frac{\sqrt[3]{x}}{2}$

**Exercise** 3.1.1

a. $g(1) = 9$

b. $x = -2$

**Exercise** 3.1.10

$x = 0$ or $x = 2$

**Exercise** 3.1.12

yes

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# 3.2: Domain and Range

**Learning Objectives**

- Find the domain of a function defined by a set of ordered pairs.
- Find the domain of a function defined by an equation.
- Find the domain and range of a function defined by a graph.

## Understanding Domain and Range as Sets

In Functions and Function Notation, we were introduced to the concepts of **domain and range**. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products (Figure 3.2.2).

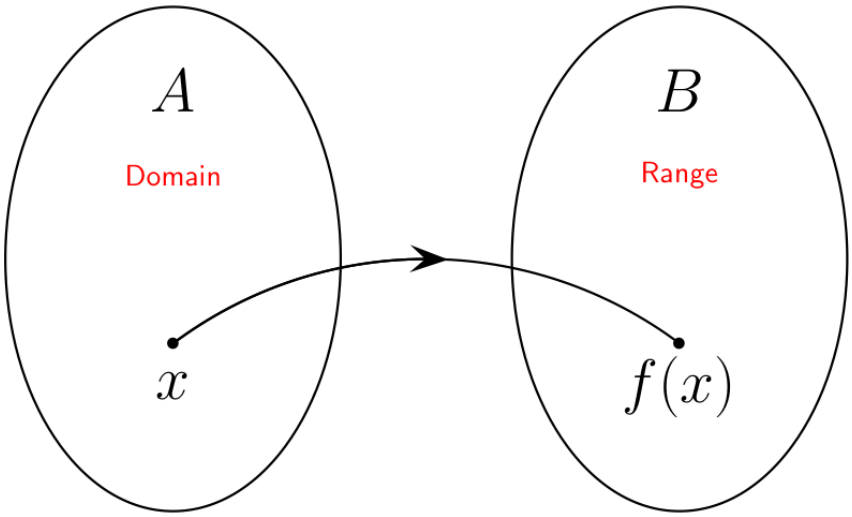



Figure 3.2.2: Diagram of how a function relates two sets.

**Example 3.2.1**

Find the domain of the following function: $\{(2,10),(3,10),(4,20),(5,30),(6,40)\}$.

**Solution**

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

\[\{2,3,4,5,6\} \nonumber\

**Exercse 3.2.1**

Find the domain of the function:

$$\{(-5,4),(0,0),(5,-4),(10,-8),(15,-12)\}$$

*Answer available at the end of this section.*

**How To: Given a function written in equation form, find the domain.**

- Identify the input values.
- Identify any restrictions on the input and exclude those values from the domain.
- Write the domain in interval form, if possible.

## Understanding Interval Notation

We can write the **domain and range** in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [ when the set includes the endpoint and a parenthesis ( to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, ( or ), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [ or ], are used to indicate that an endpoint is included, called inclusive.

See Figure 3.2.3 for a summary of interval notation.

| Inequality | Interval Notation | Graph on Number Line | Description |
|---|---|---|---|
| $x > a$ | $(a, \infty)$ |  | $x$ is greater than $a$ |
| $x < a$ | $(-\infty, a)$ |  | $x$ is less than $a$ |
| $x \geq a$ | $[a, \infty)$ |  | $x$ is greater than or equal to $a$ |
| $x \leq a$ | $(-\infty, a]$ |  | $x$ is less than or equal to $a$ |
| $a < x < b$ | $(a, b)$ |  | $x$ is strictly between $a$ and $b$ |
| $a \leq x < b$ | $[a, b)$ |  | $x$ is between $a$ and $b$, to include $a$ |
| $a < x \leq b$ | $(a, b]$ |  | $x$ is between $a$ and $b$, to include $b$ |
| $a \leq x \leq b$ | $[a, b]$ |  | $x$ is between $a$ and $b$, to include $a$ and $b$ |

Figure 3.2.3: Summary of interval notation.

# Finding the Domain of a Function Defined by an Equation

## Example 3.2.2: Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$.

**Solution**

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty, \infty)$.

## Exercse 3.2.2

Find the domain of the function:

$$f(x) = 5 - x + x^3$$

*Answer available at the end of this section.*

## How to: Given a function written in an equation form that includes a fraction, find the domain

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

## Example 3.2.3: Finding the Domain of a Function Involving a Denominator

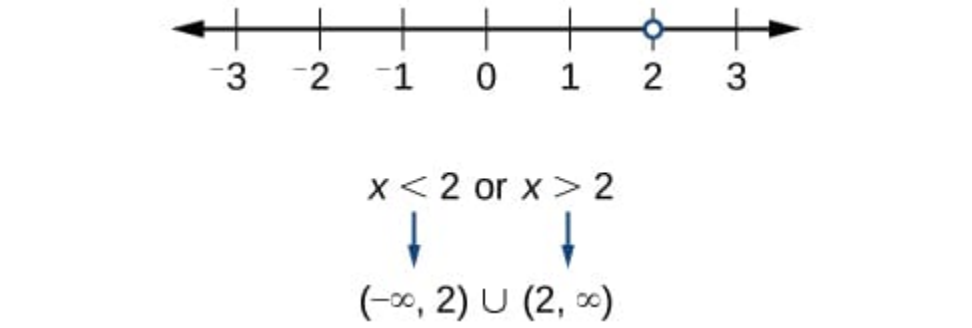
Find the domain of the function $f(x) = \dfrac{x+1}{2-x}$.

**Solution**

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2 - x &= 0 \\ -x &= -2 \\ x &= 2 \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty, 2) \cup (2, \infty)$.




In interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

## Exercse 3.2.3

Find the domain of the function:

$$f(x) = \frac{1+4x}{2x-1}$$

*Answer available at the end of this section.*

## How To: Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

## Example 3.2.4: Finding the Domain of a Function with an Even Root

Find the domain of the function:

$$f(x) = \sqrt{7-x}.$$

**Solution**

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7-x &\geq 0 \\ -x &\geq -7 \\ x &\leq 7 \end{aligned}$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

**Exercse 3.2.4**

Find the domain of the function

$$f(x) = \sqrt{5+2x}.$$

*Answer available at the end of this section.*

**Q&A: Can there be functions in which the domain and range do not intersect at all?**

*Yes. For example, the function $f(x) = -\dfrac{1}{\sqrt{x}}$ has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.*

## Using Formal Set Notation (Set Builder)

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in set-builder notation. For example, $\{x|10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar $|$ is read as "such that," so we would read $\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure 3.2.4 compares inequality notation, set-builder notation, and interval notation.

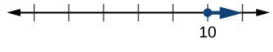
| | Inequality Notation | Set-builder Notation | Interval Notation |
|---|---|---|---|
|   | $5 < h \leq 10$ | $\lvert h \mid 5 < h \leq 10 \rvert$ | $(5, 10]$ |
|   | $5 \leq h < 10$ | $\lvert h \mid 5 \leq h < 10 \rvert$ | $[5, 10)$ |
|   | $5 < h < 10$ | $\lvert h \mid 5 < h < 10 \rvert$ | $(5, 10)$ |
|   | $h < 10$ | $\lvert h \mid h < 10 \rvert$ | $(-\infty, 10)$ |
|   | $h \geq 10$ | $\lvert h \mid h \geq 10 \rvert$ | $[10, \infty)$ |
|   | All real numbers | $\mathbb{R}$ | $(-\infty, \infty)$ |

Figure 3.2.4: Summary of notations for inequalities, set-builder, and interval.

To combine two intervals using inequality notation or set-builder notation, we use the word “or.” As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets$\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x||x| \geq 3\} = (-\infty, -3] \cup [3, \infty) \tag{3.2.1}$$

## Set-Builder Notation and Interval Notation

*Set-builder notation* is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|\text{ statement about x}\}$ which is read as, “the set of all x such that the statement about x is true.” For example,

$$\{x|4 < x \leq 12\}$$

*Interval notation* is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$




**Given a line graph, describe the set of values using interval notation.**

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [ with each end value to be included in the set (solid dot) or ( for each excluded end value (open dot).
3. At the right end of each interval, use ] with each end value to be included in the set (filled dot) or ) for each excluded end value (open dot).
4. Use the union symbol $\cup$ to combine all intervals into one set.

## Example 3.2.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure 3.2.5 using inequality notation, set-builder notation, and interval notation.

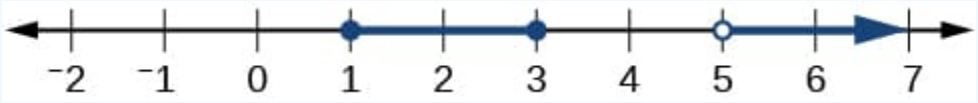



Figure 3.2.5: Line graph of $1 \leq x \leq 3$ or $x > 5$.

**Solution**

To describe the values, $x$, included in the intervals shown, we would say, "$x$ is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality

$$1 \leq x \leq 3 \text{ or } x > 5$$

Set-builder Notation

$$\{x | 1 \leq x \leq 3 \text{ or } x > 5\}$$

Interval notation

$$[1, 3] \cup (5, \infty)$$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

## Exercse 3.2.5

Given Figure 3.2.6, specify the graphed set in

a. words
b. set-builder notation
c. interval notation

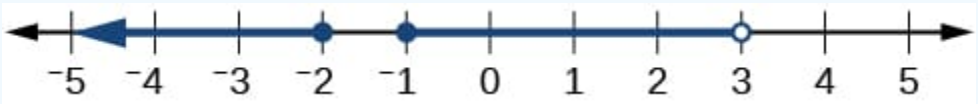



Figure 3.2.6: Line graph of $-2 \leq x$, $-1 \leq x < 3$.leq}x\), $-1 \leq x < 3$.

*Answer a available at the end of this section.*

## Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure 3.2.7.

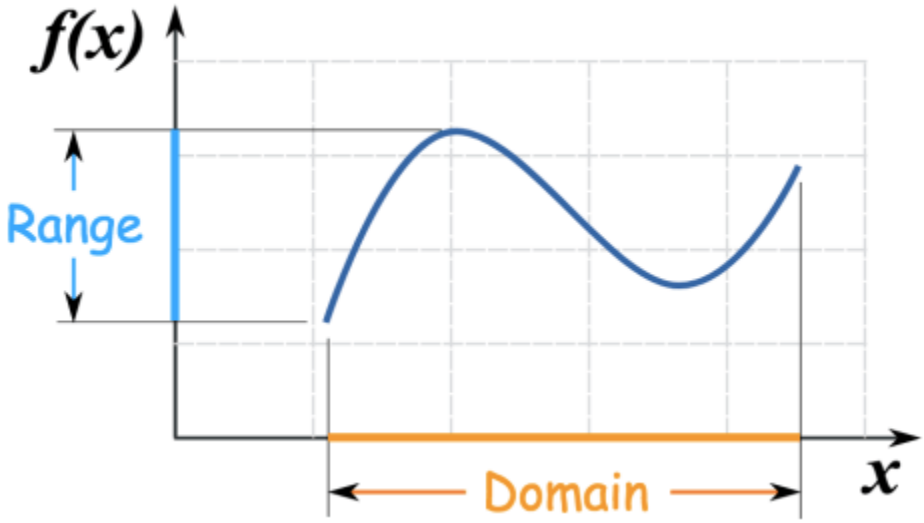



Figure 3.2.7: Graph of a function that shows the set of all x-values is the domain and the set of all y-values is the range

***Note: The domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.***

**Example 3.2.6: Finding Domain and Range from a Graph**

Find the domain and range of the function f whose graph is shown below.

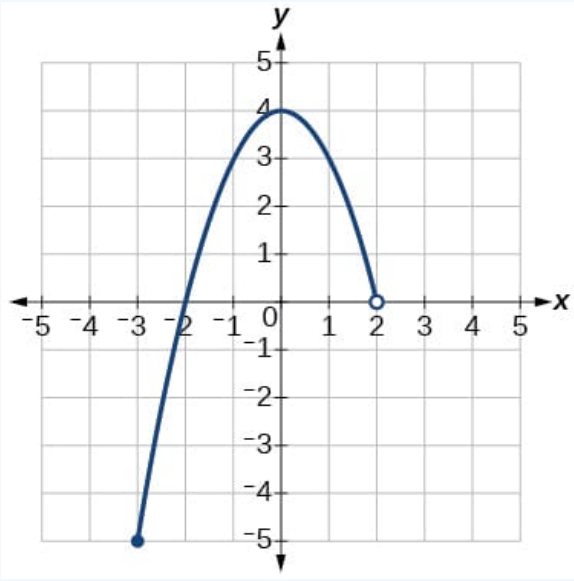



Figure 3.2.8: Graph of a restricted quadratic function.

**Solution**

We can observe that the horizontal extent of the graph is –3 to 2, there is a closed point (solid dot) at $(-3,-5)$ and an open "hole" at $(2,0)$, so the domain of f is $[-3,2)$.

The vertical extent of the graph is from –5 to 4, there are closed points (solid dot or line/curve of the graph) at $(-3,-5)$ and $(0,4)$ , so the range is $[-5,4]$. See Figure 3.2.9.

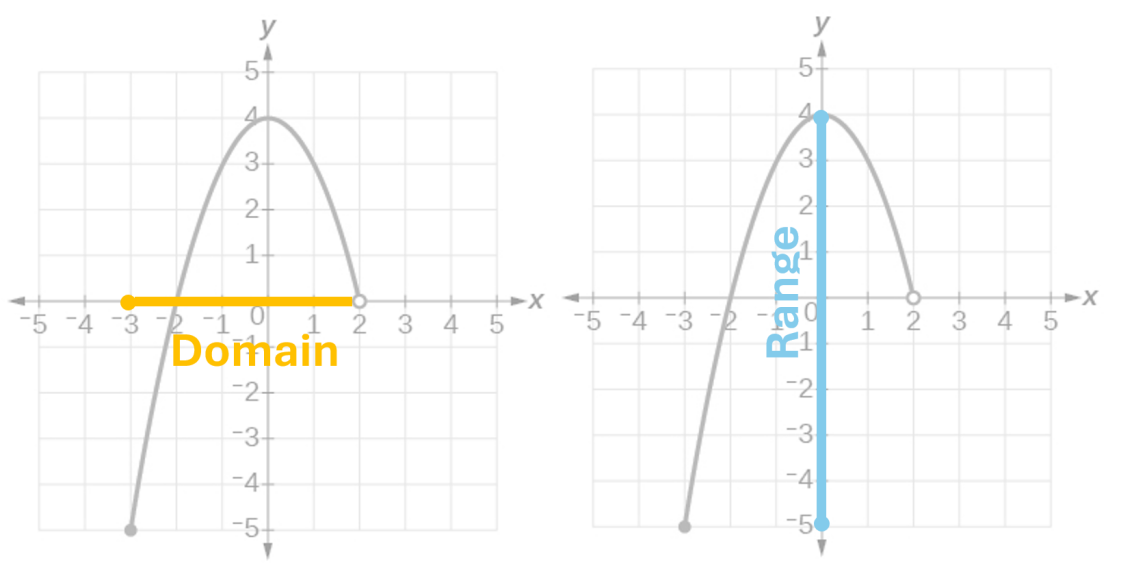
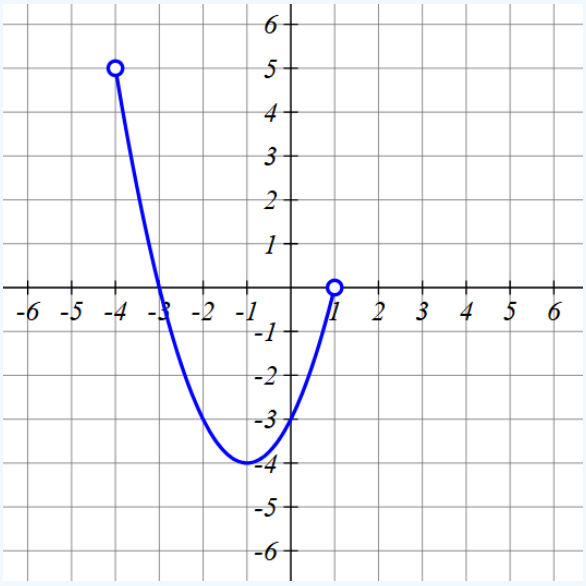



Figure 3.2.9: Graph of the previous function shows the domain and range

## Exercise 3.2.6

Find the domain and range of the function f whose graph is shown below.



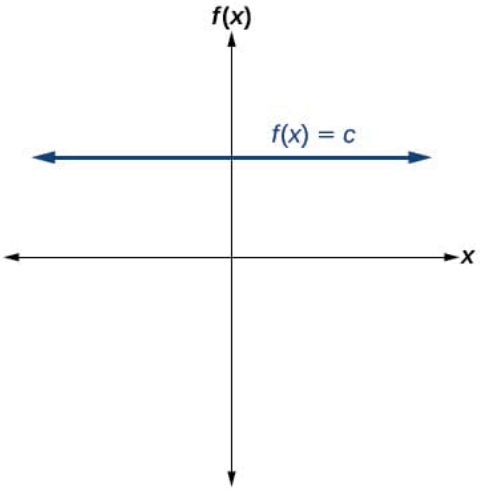
*Answer available at the end of this section.*

**Can a function's domain and range be the same?**

*Yes. For example, the domain and range of the cube root function are both the set of all real numbers.*

## Finding Domains and Ranges of the Toolkit Functions

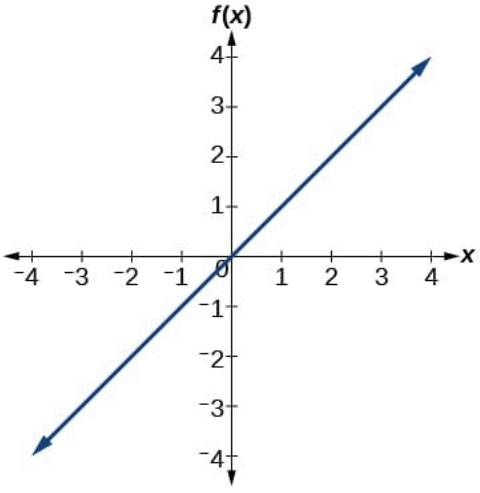
We will now return to our set of toolkit functions to determine the domain and range of each.




Domain: $(-\infty, \infty)$
Range: $[c,c]$

Figure 3.2.12: Constant function $f(x) = c$.

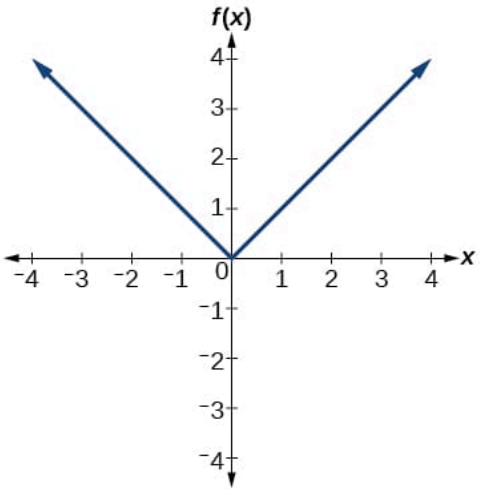
For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant $c$, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with $c$.




Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

***Figure*** 3.2.13***:*** *Identity function f(x)=x.*

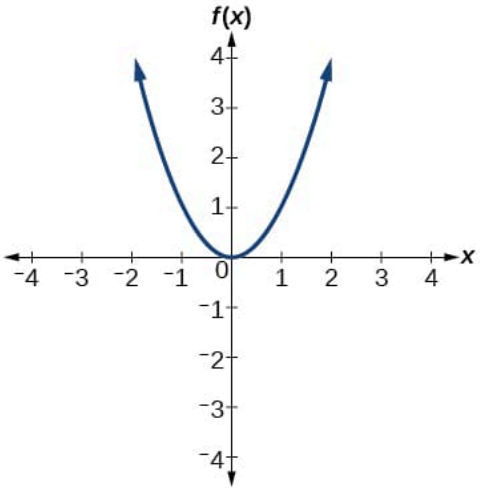
For the **identity function** $f(x) = x$, there is no restriction on $x$. Both the domain and range are the set of all real numbers.




Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 3.2.14: Absolute function $f(x) = |x|$.

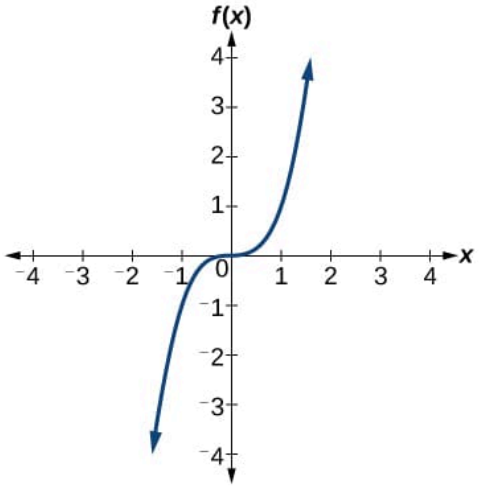
For the **absolute value function** $f(x) = |x|$, there is no restriction on $x$. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.




Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 3.2.15: Quadratic function $f(x) = x^2$.

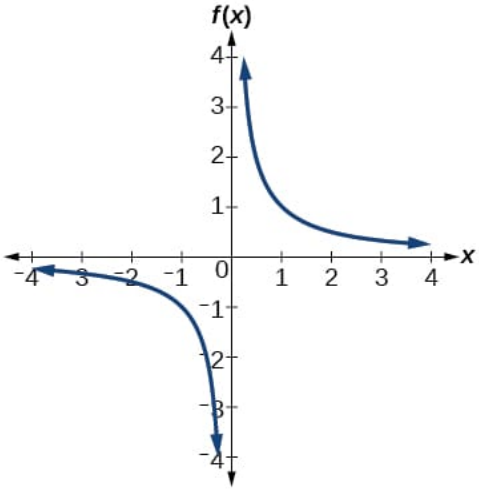
For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.




Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 3.2.16: Cubic function $f(x) - x^3$.

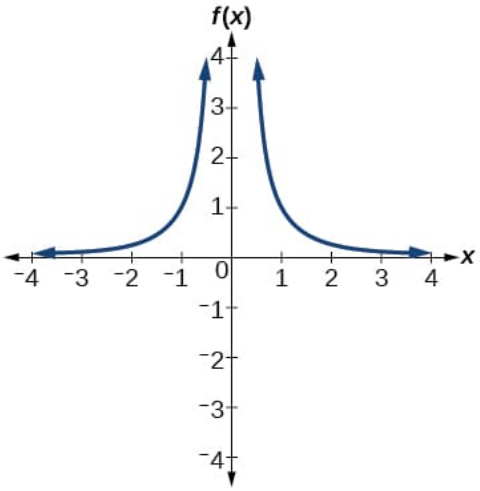
For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.




Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Figure 3.2.17: Reciprocal function $f(x) = \frac{1}{x}$.

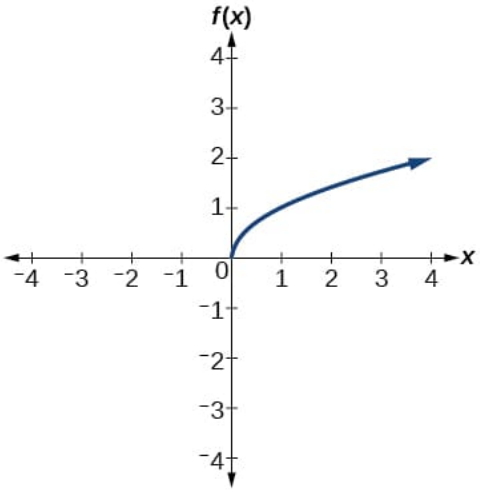
For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x|x \neq 0\}$,the set of all real numbers that are not zero.




Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

Figure 3.2.18: Reciprocal squared function $f(x) = \frac{1}{x^2}$

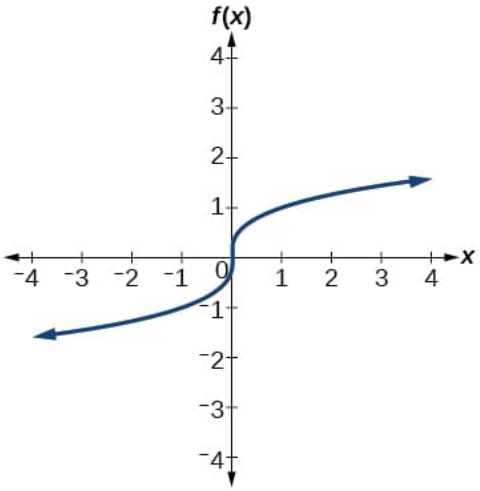
For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.




Domain: $[0, \infty)$
Range: $[0, \infty)$

***Figure*** 3.2.19***:*** *Square root function* $f(x) = \sqrt{(x)}$.

For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number $x$ is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us $x$.




Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 3.2.20: Cube root function $f(x) = \sqrt[3]{x}$.

For the **cube root function** $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

**How To:**

**Given the formula for a function, determine the domain and range.**

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

### Example 3.2.7: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

**Solution**

We cannot evaluate the function at –1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

### Example 3.2.8: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

**Solution**

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$x + 4 \geq 0$ when $x \geq -4$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as $x$ increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

**Analysis**

Figure $3.2.19$ represents the function $f$.

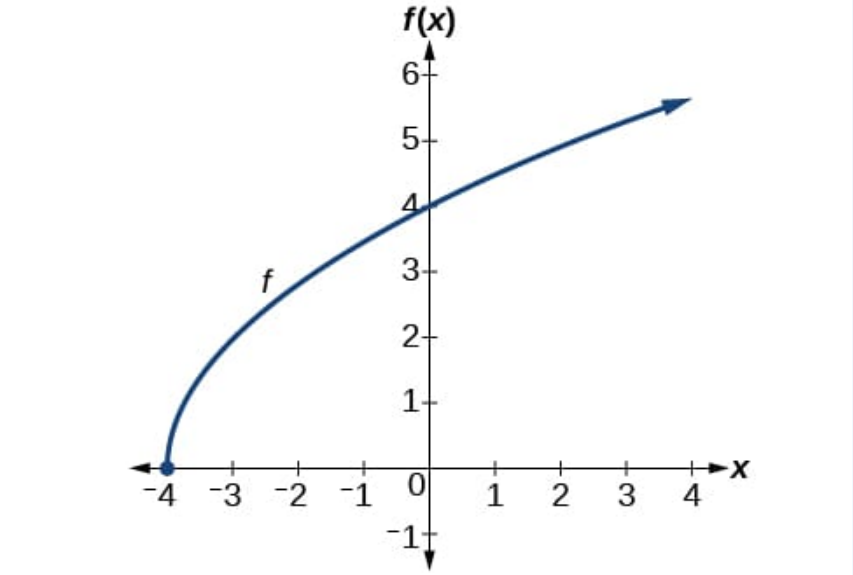



Figure $3.2.19$: Graph of a square root function at $(-4, 0)$.

**Exercise $3.2.7$**

Find the domain and range of

$f(x) = \sqrt{-2 - x}$.

*Answer available at the end of this section.*

## Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.
- An understanding of toolkit functions can be used to find the domain and range of related functions.

## Glossary

**interval notation**

a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

**set-builder notation**

a method of describing a set by a rule that all of its members obey; it takes the form {x| statement about x}

## Answers

**Exercse** 3.2.1

$\{-5, 0, 5, 10, 15\}$

**Exercse** 3.2.2

$(-\infty, \infty)$

**Exercse** 3.2.3

$$(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$$

**Exercse** 3.2.4

$$[-2.5, \infty)$$

**Exercse** 3.2.5

Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;

$\{x | x \leq -2 or -1 \leq x < 3\}$

$(-\infty, -2] \cup [-1, 3)$

**Exercise** 3.2.6

Domain: $(-4, 1)$

Range: $[-4, 5)$

**Exercise** 3.2.7

domain: $(-\infty, -2]$

range: $[0, \infty)$

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# 3.3: Rates of Change and Behavior of Graphs

**Learning Objectives**

- Find the average rate of change of a function.
- Interpret average rate of change as the slope of a secant line.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local and global extrema.

Gasoline costs have experienced some wild fluctuations over the last several decades. Table 3.3.1 lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

*Table* 3.3.1

| $y$ | 2015 | 2016 | 2017 | 2018 | 2019 | 2020 | 2021 | 2022 |
|---|---|---|---|---|---|---|---|---|
| $C(y)$ | 4.29 | 4.62 | 4.84 | 5.30 | 4.41 | 4.84 | 3.58 | 3.68 |

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from $2.31 to $3.68, an increase of $1.37. While this is interesting, it might be more useful to look at how much the price changed per year. In this section, we will investigate changes such as these.

## Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in Table 3.3.1 did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \text{Slope of the Secant Line between } (x_1, y_1) \text{ and } (x_2, y_2) \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter $\Delta$ (delta) signifies the change in a quantity; we read the ratio as "delta-$y$ over delta-$x$" or "the change in $y$ divided by the change in $x$." Occasionally we write $\Delta f$ instead of $\Delta y$, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by $1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7\text{years}} \approx 0.196 \text{ dollars per year.} \tag{3.3.1}$$

On average, the price of gas increased by about 19.6¢ each year. Other examples of rates of change include:

- The price of gasoline decreasing by 22 cents per year.
- A population of coyotes increasing by 40 coyotes per week.
- A car traveling 68 miles per hour.

- A vehicle fuel efficiency of 27 miles per gallon.

### Definition: Rate of Change

A **rate of change** describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{3.3.2}$$

### How To...

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values $x_1$ and $x_2$.

1. Calculate the difference $y_2 - y_1 = \Delta y$.
2. Calculate the difference $x_2 - x_1 = \Delta x$.
3. Find the ratio $\frac{\Delta y}{\Delta x}$.

### Example 3.3.1: Computing an Average Rate of Change

Using the data in Table 3.3.1, find the average rate of change of the price of gasoline between 2017 and 2019.

**Solution**

In 2017, the price of gasoline was \$4.84. In 2019, the cost was \$4.41. The average rate of change is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$2.41 - \$2.84}{2019 - 2017} \\ &= \frac{-\$0.43}{2 \text{ years}} \\ &= -\$0.22 \text{ per year}\end{aligned}$$

**Analysis**

Note that a decrease is expressed by a negative change or "negative increase." A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

### Exercise 3.3.1

Using the data in Table 3.3.1, find the average rate of change between 2015 and 2020.

*Solution available at the end of this section.*

## Example 3.3.2: Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in Figure 3.3.1, find the *slope of the secant line* between the points $(-1, g(-1))$ and $2, g(2))$

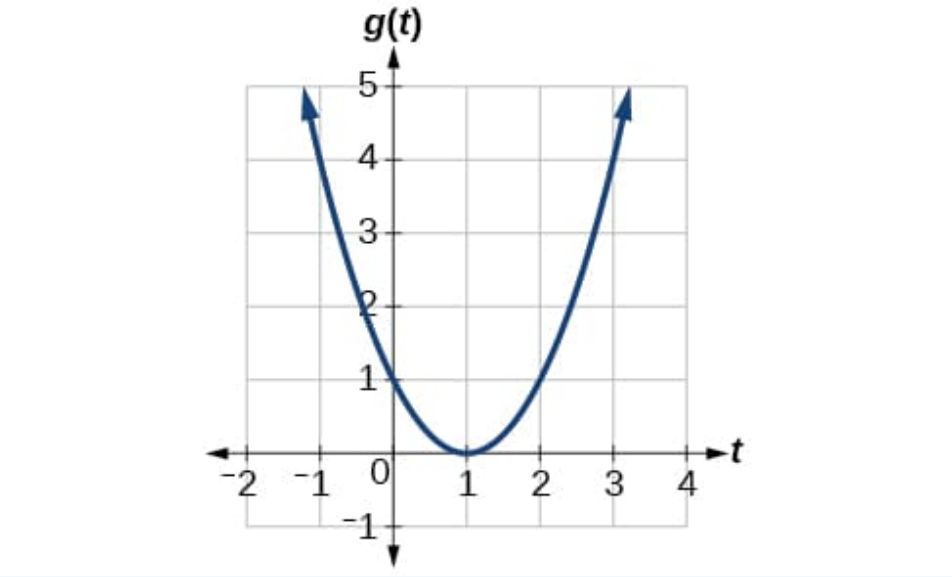



Figure 3.3.1: Graph of a parabola.

**Solution**

At $t = -1$, Figure 3.3.2 shows $g(-1) = 4$. At $t = 2$,the graph shows $g(2) = 1$.

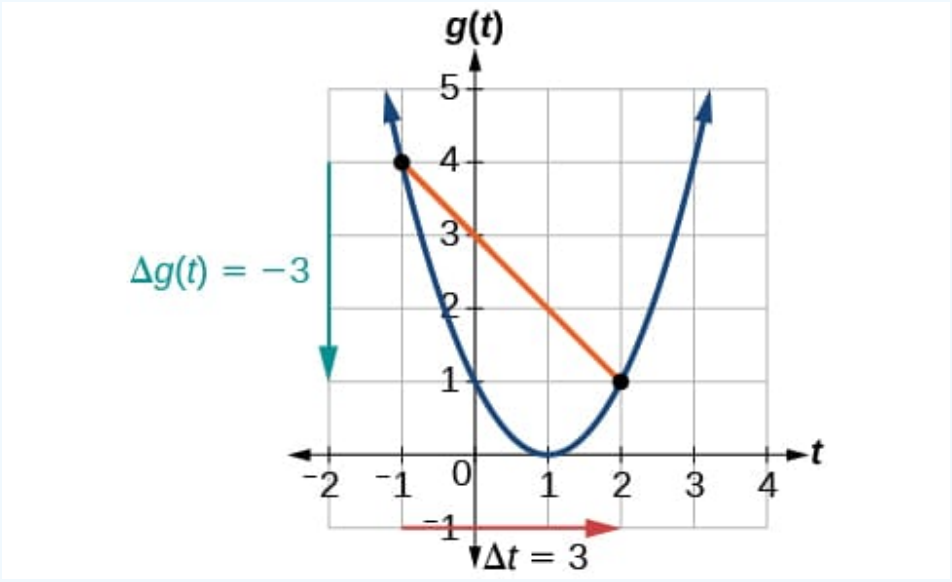



Figure 3.3.2: Graph of a parabola with a line from points (-1, 4) and (2, 1) to show the changes for g(t) and t.

The horizontal change $\Delta t = 3$ is shown by the red arrow, and the vertical change $\Delta g(t) = -3$ is shown by the turquoise arrow. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)} = \frac{-3}{3} = -1 \tag{3.3.3}$$

**Analysis**

Note that the order in which we evaluate is very important. For example, if we evaluated $\frac{y_2 - y_1}{x_1 - x_2}$, we will not get the correct answer.

You must decide which point will have subscript 1 and which point will have a subscript 2, then BE CONSISTENT when evaluating by subtracting in the SAME ORDER each time.

## Example 3.3.3: Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in Table 3.3.2. Find her average speed over the first 6 hours.

*Table* 3.3.2

| $t$ **(hours)** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $D(t)$ **(miles)** | 10 | 55 | 90 | 153 | 214 | 240 | 292 | 300 |

**Solution**

Here, the average speed is the average rate of change. She traveled 292 miles in 6 hours, for an average speed of

$$\begin{aligned} \frac{292 - 10}{6 - 0} &= \frac{282}{6} \\ &= 47 \end{aligned}$$

The average speed is about 47 miles per hour.

**Analysis**

Because the speed is not constant, the average speed depends on the interval chosen. For the interval $[2, 3]$, the average speed is 63 miles per hour.

## Example 3.3.4: Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x) = x^2 - 3x + 1$ on the interval $[2, 4]$.

**Solution**

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2 - 3(2) + 1 \\ &= 4 - 6 + 1 \\ &= -1 \end{aligned}$$

$$\begin{aligned} f(4) &= 4^2 - 3(4) + 1 \\ &= 16 - 12 + 1 \\ &= 5 \end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned} \text{AROC} &= \frac{f(4)-f(2)}{4-2} \\ &= \frac{5-(-1)}{4-2} \\ &= \frac{5+1}{2} \\ &= 3 \end{aligned}$$

### Exercise 3.3.2

Find the average rate of change of $f(x)=\frac{1}{3x+1}$ on the interval $[0,1]$.

*Solution available at the end of this section.*

### Example 3.3.5: Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t)=t^2+3t+1$ on the interval $[a,a+h]$. The answer will be an expression involving $a$.

**Solution**

We use the average rate of change formula.

$$\begin{aligned} \text{Average rate of change} &= \frac{g(a+h)-g(a)}{a+h-a} && \text{Evaluate.} \\ &= \frac{(a^2+2ah+h^2+3a+3h+1)-(a^2+3a+1)}{h} && \text{Simplify.} \\ &= \frac{h^2+2ah+3h}{h} && \text{Simplify and factor.} \\ &= \frac{h(h+2a+3)}{h} && \text{Divide by the common factor h} \\ &= h+2a+3 \end{aligned}$$

This result tells us the average rate of change between $t_1=a$ and any other point $t_2=a+h$ is equal to $h+2a+3$

For example, on the interval $[1,1+h]$, the average rate of change would be $h+2(1)+3=h+5$ where $h$ represents the horizontal distance between $t_1$ and $t_2$

### Exercise 3.3.3

Find the average rate of change of $f(x)=x^2+2x-8$ on the interval $[a,a+5]$.

*Solution available at the end of this section.*

## Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. Figure 3.3.3 shows examples of increasing and decreasing intervals on a function.

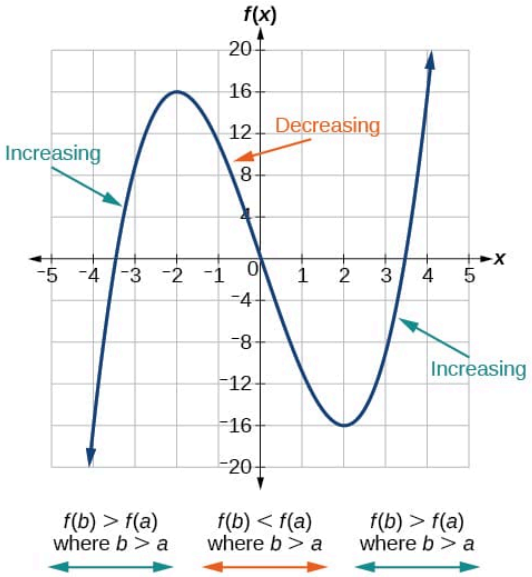



Figure 3.3.3: The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2, 2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is "local minima." Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (Fun Fact: The singular form of extrema is "extremum")

Often, the term local is replaced by the term **relative**.

A function that is neither increasing nor decreasing on an interval is called **constant**. The graph would be a horizontal line.

A function is also neither increasing nor decreasing at the extrema.

For the function whose graph is shown in Figure 3.3.4, the local maximum is 16, and it occurs at $x = -2$. The local minimum is –16 and it occurs at $x = 2$.

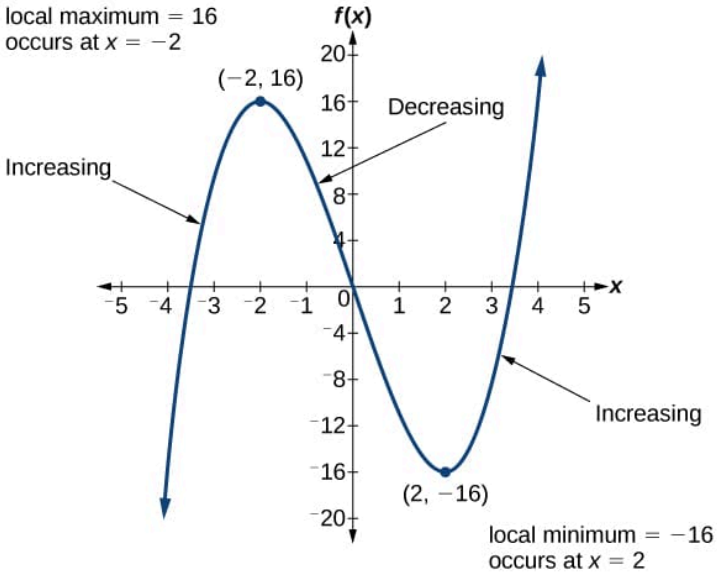



Figure 3.3.4: Graph of a polynomial that shows the increasing and decreasing intervals and local maximum.maximum

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its

highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. Figure 3.3.5 illustrates these ideas for a local maximum.

*Note: We must speak of these values as local extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.*

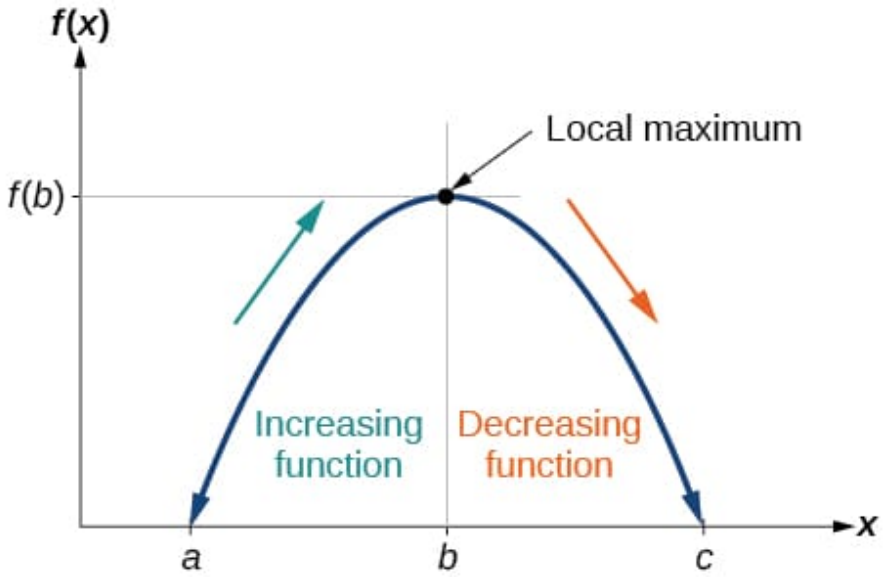



Figure 3.3.5: Definition of a local maximum

These observations lead us to the formal definitions of increasing, decreasing and constant and a formal definition of local extrema.

### Increasing, Decreasing, Constant.

- A function $f$ is **increasing** on an open interval if $f(b) > f(a)$ for every $a$, $b$ interval where $b > a$.
- A function $f$ is **decreasing** on an open interval if $f(b) < f(a)$ for every $a$, $b$ interval where $b > a$.
- A function $f$ is **constant** on open interval if $f(b) = f(a)$ for every $a$, $b$ interval where $b > a$.

### Local Minima and Local Maxima

- A function $f$ has a **local maximum** of $f(b)$ at $x = b$ in an open interval $(a, c)$ if $f(b)$ is greater than or equal to $f(x)$ for every $x$ in the interval ($x$ does not equal $b$)
- Similarly, $f$ has a **local minimum** of $f(b)$ at $x = b$ in $(a, c)$ if $f(b)$ is less than or equal to $f(x)$ for every $x$ in the interval ($x$ does not equal $b$)

### Example 3.3.7 Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in Figure 3.3.6, identify the intervals on which the function appears to be increasing.

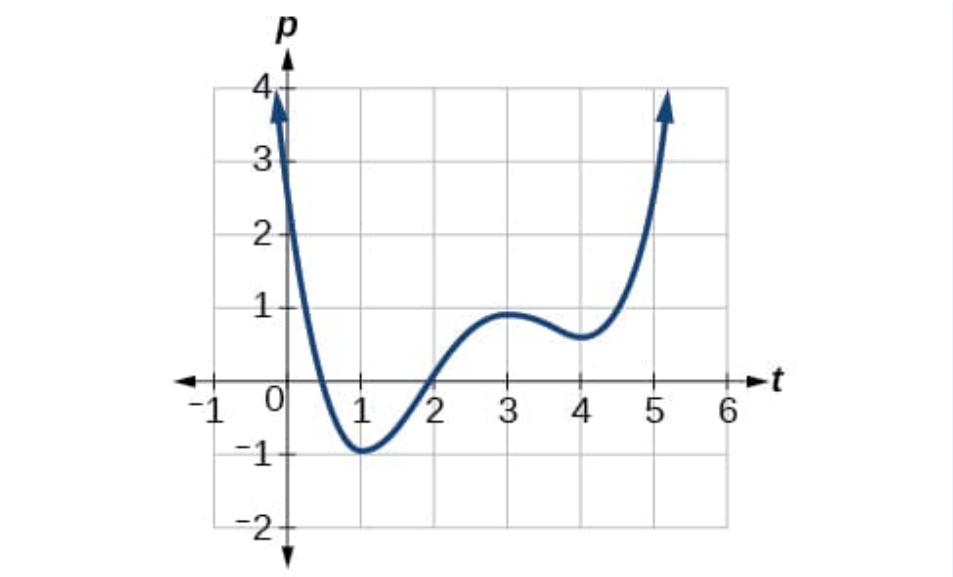



Figure 3.3.6: Graph of a polynomial.

**Solution**

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

**Analysis**

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

## Example 3.3.8

The graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ is shown in Figure 3.3.7

Estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

*Solution available at the end of this section.*

## Example 3.3.9: Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown in Figure 3.3.8, find all local maxima and minima.

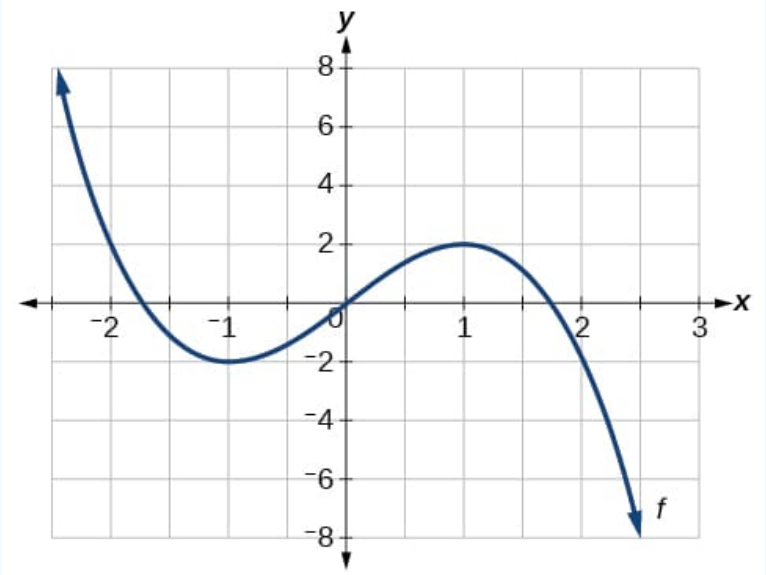



Figure 3.3.8: Graph of a polynomial.

**Solution**

Observe the graph of $f$. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$.The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is –2.

## Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively. To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function.

Often, the term absolute is replaced by the term **global**.

(Figure 3.3.9).

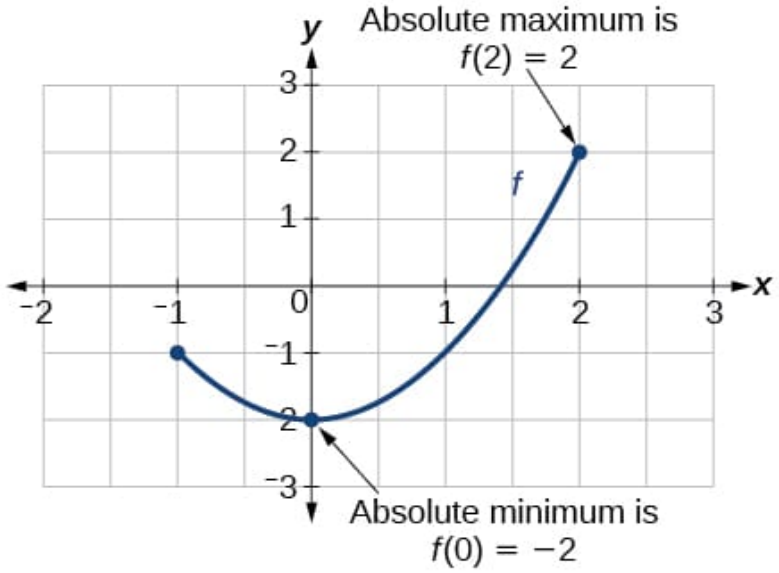



Figure 3.3.9: Graph of a segment of a parabola with an absolute minimum of $-2$ at $x = 0$and absolute maximum of $2$ at $x = 2$.

Not every function has an absolute maximum or minimum value. For example the function $f(x) = x^2$ does NOT have an absolute maximum.

## Absolute Maxima and Minima

- The **absolute maximum** of $f$ at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all $x$ in the domain of $f$.
- The **absolute minimum** of $f$ at $x = d$ is $f(d)$ where $f(d) \leq f(x)$ for all $x$ in the domain of $f$.

## Example 3.3.10: Finding Absolute Maxima and Minima from a Graph

For the function f shown in Figure 3.3.10, find all absolute maxima and minima.

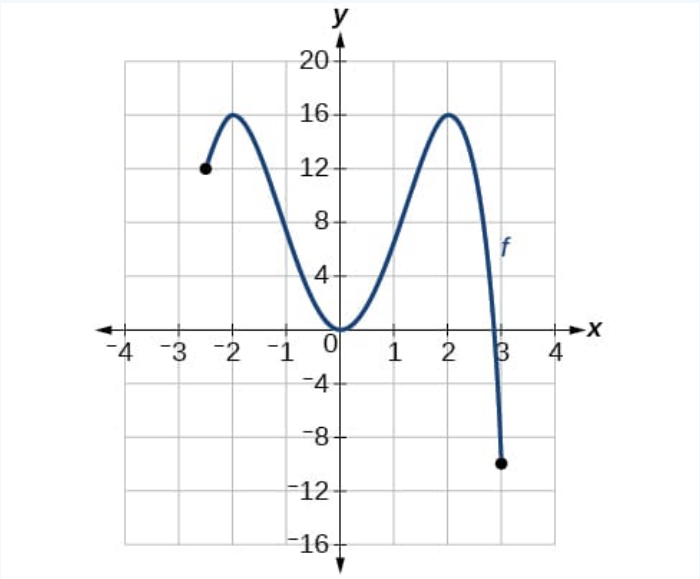



Figure 3.3.10: Graph of a polynomial.

**Solution**

Observe the graph of $f$. The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the y-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at x=3, because it is the lowest point on the domain of the function's graph. The absolute minimum is the y-coordinate at x=3,which is–10.

### Example 3.3.11: Finding Absolute Maxima and Minima from a Graph

For the function f shown in Figure 3.3.11, find all absolute maxima and minima.

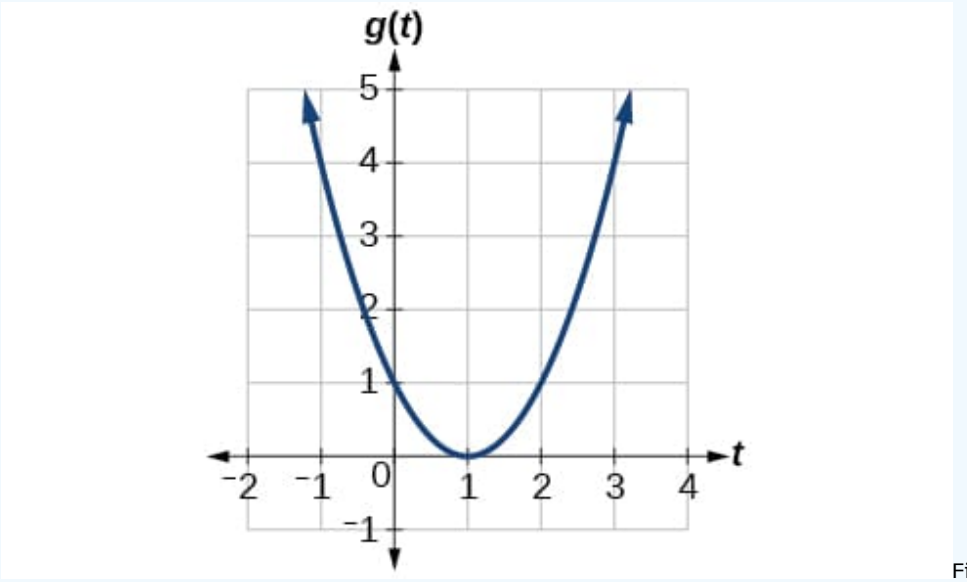



Figure 3.3.11: Graph of a quadratic function.

**Solution**

Observe the graph of $f$T, the implied domain is $(-oo, oo)$.

The **absolute maximum does not exist** because there is no point on this graph for which the value of $f(x)$ is greater than all other values for every $x$ in the implied domain.

The graph attains an absolute minimum is the y-coordinate at $x = 1$, which is $y = 0$, because $f(1) = 0$ and \(0 ≤ f(x)\) for all possible $x$ in the domain of the function.

**Answers**

The absolute minimum is $0$ at $x = 1$.

The absolute maximum does not exist, we would enter DNE for this part of the answer.

## Key Equation

- Average rate of change: $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

# Key Concepts

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See Example.
- Identifying points that mark the interval on a graph can be used to find the average rate of change. See Example.
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See Example.
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See Example and Example.
- The average rate of change can sometimes be determined as an expression. See Example.
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See Example.
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.
- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See Example and Example.
- The highest and lowest points on a graph indicate the maxima and minima. See Example.

## Answers

**Exercise** 3.3.1

$\frac{\$4.84 - \$4.29}{5 \text{ years}} = \frac{\$0.55}{5 \text{ years}} = \$0.11 \text{per year.}$

**Exercise** 3.3.2

AROC = $-\frac{3}{4}$

**Exercise** 3.3.3

AROC = $2a + 7$

**Example** 3.3.8

The local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.

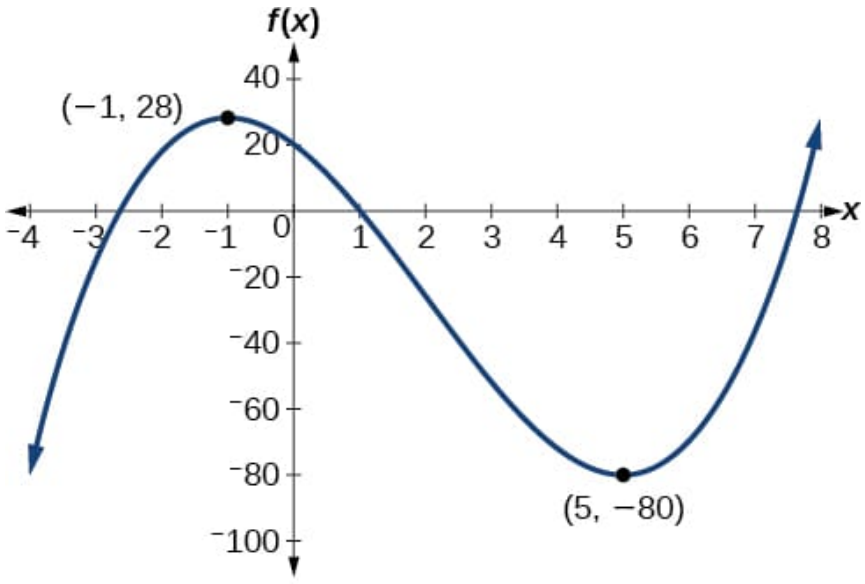



Figure 3.3.7 *Graph of a polynomial with a local maximum at (-1, 28) and local minimum at (5, -80).*

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# 3.4: Transformations of Functions

**Learning Objectives**

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Graph functions using compressions and stretches.
- Combine transformations.
- Determine whether a function is even, odd, or neither.

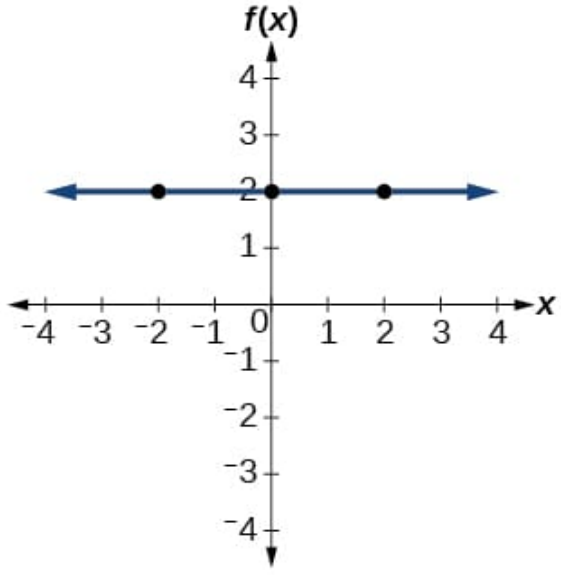
Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit or "parent" functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

## Parent Functions

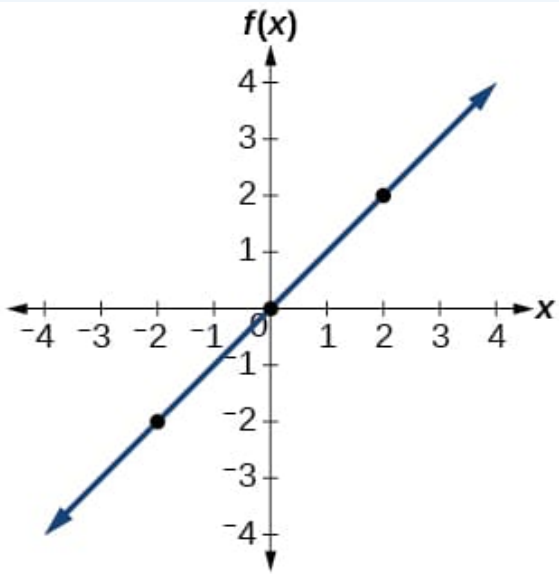
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our “toolkit functions,” which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

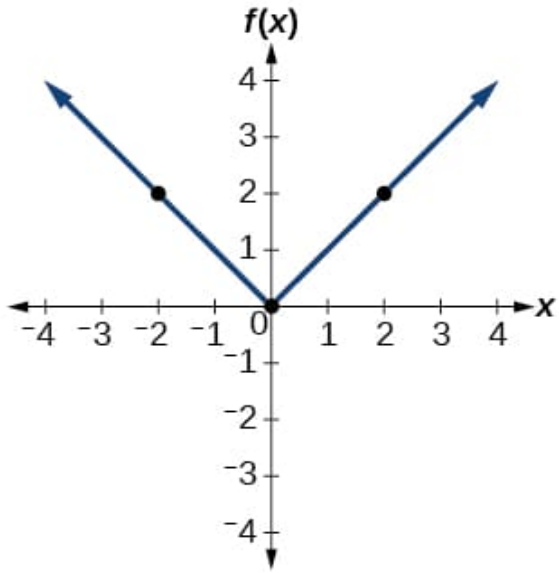
We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table 3.4.14.

*Table 3.4.14: Toolkit Functions*

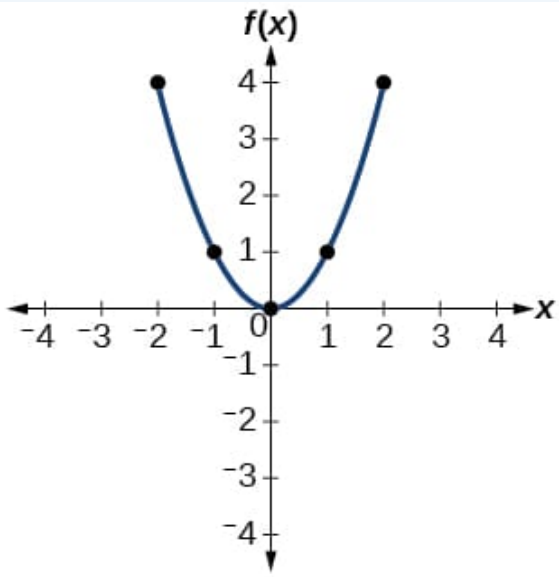
| Name | Function | Graph |
|---|---|---|
| Constant | $f(x) = c$ where $c$ is a constant |   |

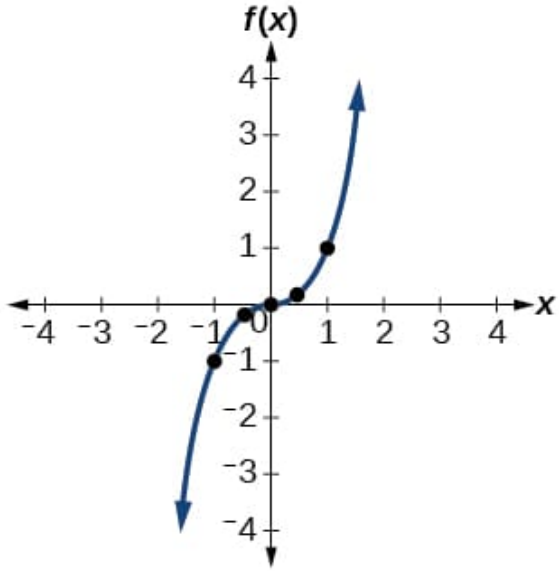
| x | f(x) |
|---|---|
| −2 | 2 |
| 0 | 2 |
| 2 | 2 |

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Identity</td><td>$f(x) = x$</td><td> 

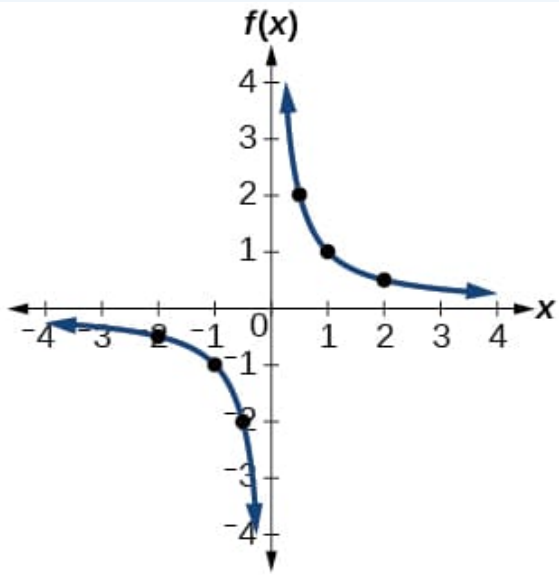
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−2</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
<tr><td>Absolute Value</td><td>$f(x) = |x|$</td><td> 

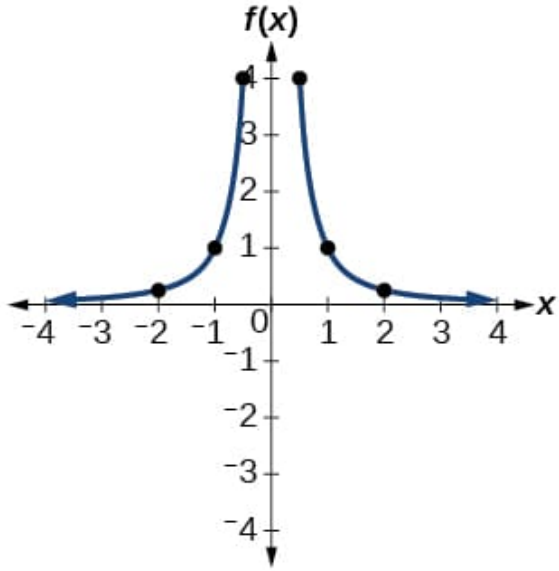
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>2</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
</table>

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Quadratic</td><td>$f(x) = x^2$</td><td> 

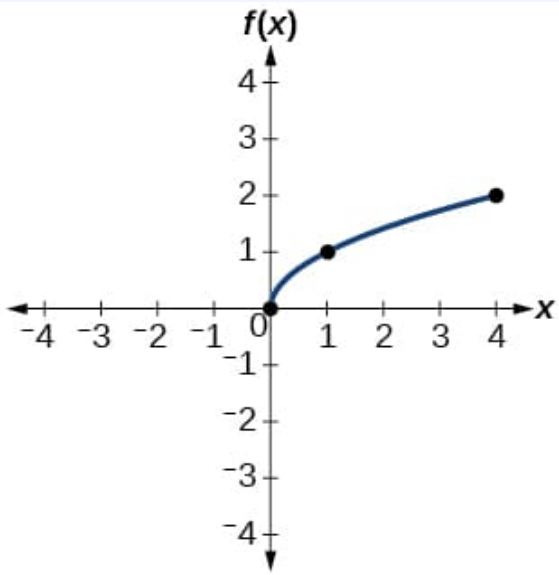
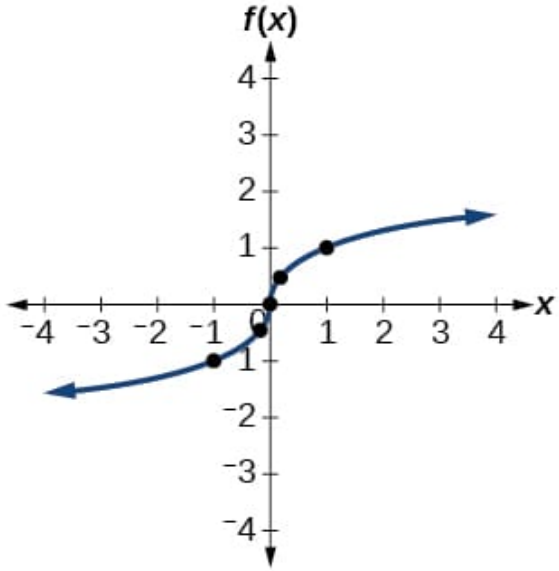
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>4</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>4</td></tr>
</table></td></tr>
<tr><td>Cubic</td><td>$f(x) = x^3$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−0.125</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.5</td><td>0.125</td></tr>
<tr><td>1</td><td>1</td></tr>
</table></td></tr>
</table>

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Reciprocal</td><td>$f(x) = \frac{1}{x}$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−0.5</td></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−2</td></tr>
<tr><td>0.5</td><td>2</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.5</td></tr>
</table></td></tr>
<tr><td>Reciprocal squared</td><td>$f(x) = \frac{1}{x^2}$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>0.25</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>−0.5</td><td>4</td></tr>
<tr><td>0.5</td><td>4</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.25</td></tr>
</table></td></tr>
</table>

| Name | Function | Graph |
|---|---|---|
| Square root | $f(x) = \sqrt{x}$ |   |
| Cube root | $f(x) = \sqrt[3]{x}$ |   |

| x | f(x) |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 4 | 2 |

| x | f(x) |
|---|---|
| −1 | −1 |
| −0.125 | −0.5 |
| 0 | 0 |
| 0.125 | 0.5 |
| 1 | 1 |

## Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically $k$ units. See Figure $3.4.2$ for an example.

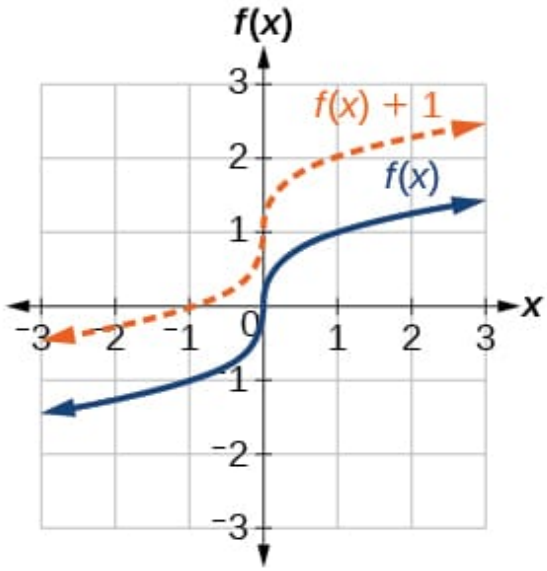



Figure 3.4.2: Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of $y$ is replaced by $y + k$, so the $y$-value increases or decreases depending on the value of $k$. The result is a shift upward or downward.

## Definition: Vertical Shift

Given a function $f(x)$, a new function $y = f(x) + k$, where $k$ is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by $k$ units. If $k$ is positive, the graph will shift up. If $k$ is negative, the graph will shift down.

## Example 3.4.1: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 3.4.2. Create a table for the function $y = f(x) - 3$.

*Table* 3.4.2

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

**Solution**

The formula $y = f(x) - 3$ tells us that we can find the new output values by subtracting 3 from the output values of $f$.

For example:

$$\begin{aligned} f(2) &= 1 && \text{Given} \\ y &= f(x) - 3 && \text{Given Transformation} \\ y(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $y = f(x) - 3$ as shown in Table 3.4.3.

*Table* 3.4.3

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|

| $y$ | -2 | 0 | 4 | 8 |
|---|---|---|---|---|

**Analysis**

Notice that with a vertical shift, the input values stay the same and only the output values change.

**Exercise 3.4.1**

The function $h(t) = -4.9t^2 + 30t$ gives the height $h$ of a ball (in meters) thrown upward from the ground after $t$ seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

*Answer available at the end of this section.*

## Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 3.4.4.

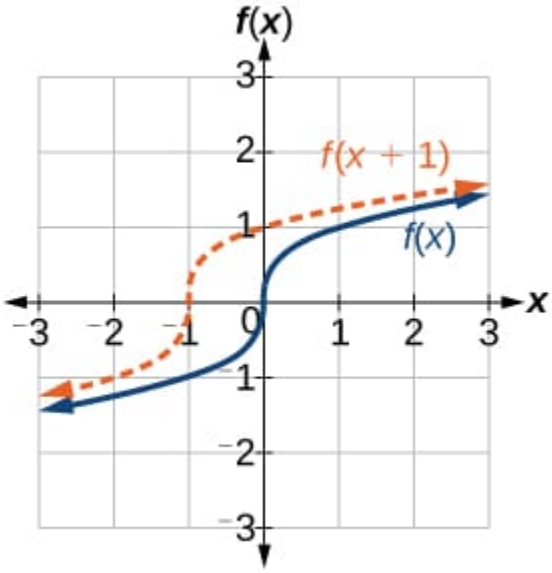



Figure 3.4.4: Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = -1$ shifts the graph to the left, that is, towards negative values of $x$.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in $f$.

**Definition: Horizontal Shift**

Given a function $f$, a new function $y = f(x-h)$, where $h$ is a constant, is a **horizontal shift** of the function $f$. If $h$ is positive, the graph will shift right. If $h$ is negative, the graph will shift left.

## Example 3.4.2: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 3.4.4. Create a table for the function $g(x) = f(x-3)$.

*Table* 3.4.4

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

**Solution**

The formula $g(x) = f(x-3)$ tells us that the output values of $g$ are the same as the output value of $f$ when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function $g$, we will need an input value that is 3 larger. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function $f$.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create Table 3.4.5.

*Table* 3.4.5

| $x$ | 5 | 7 | 9 | 11 |
|---|---|---|---|---|
| $x-3$ | 2 | 4 | 6 | 8 |
| $f(x)$ | 1 | 3 | 7 | 11 |
| $g(x)$ | 1 | 3 | 7 | 11 |

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, $x$, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

**Analysis**

Figure 3.4.6 represents both of the functions. We can see the horizontal shift in each point.

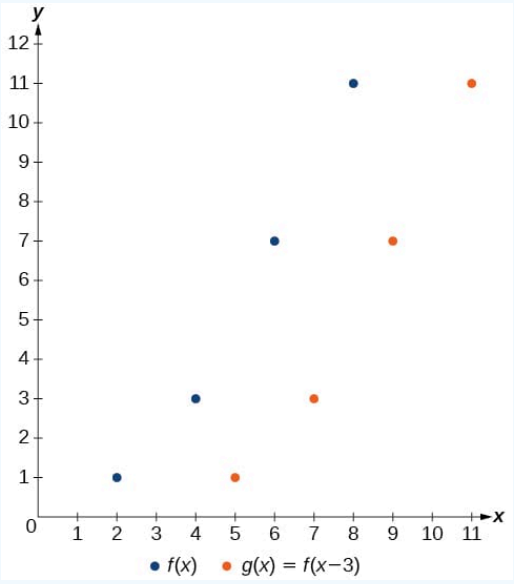



Figure 3.4.6: Graph of the points from the previous table for $f(x)$ and $g(x) = f(x-3)$

## Example 3.4.3: Identifying a Horizontal Shift of a Toolkit Function

Figure 3.4.7 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

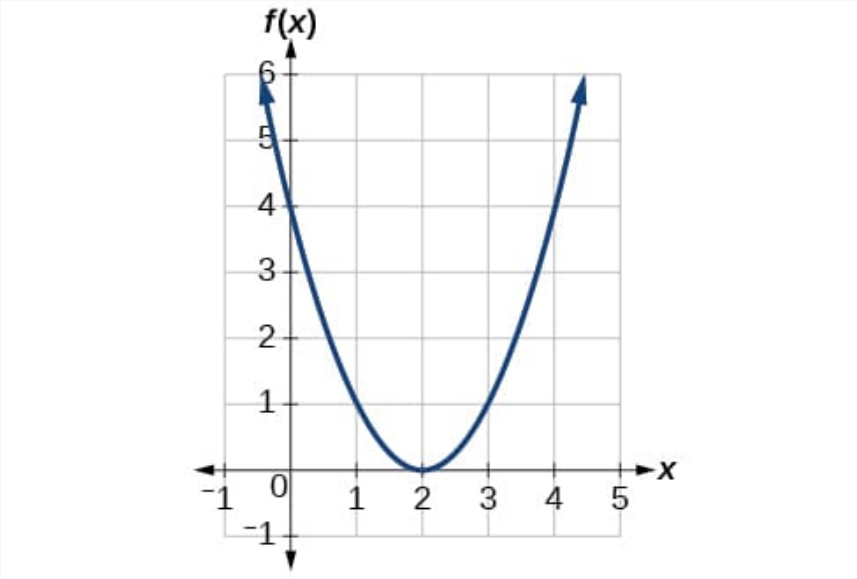



Figure 3.4.7: Graph of a parabola

**Solution**

Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the $x$-values are shifted to the right 2 units. The vertex used to be at $(0, 0)$, but now the vertex is at $(2, 0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x - 2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the $x$-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x - 2)$.

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x - 2) \\ g(x) &= f(x - 2) = (x - 2)^2 \end{aligned}$$

**Analysis**

To determine whether the shift is $+2$ or $-2$, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function $f$, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x - 2) = f(0) = 0$. For this to work, we will need to subtract 2 units from our input values.

## Exercise 3.4.2

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

*Answer available at the end of this section.*

# Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output $(y-)$ axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input $(x-)$ axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

How To...

**Given a function and both a vertical and a horizontal shift, sketch the graph.**

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

## Example 3.4.4: Graphing Combined Vertical and Horizontal Shifts

Given $f(x)=|x|$, sketch a graph of $y=f(x+1)-3$.

**Solution**

The function $f$ is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of $y=f(x+1)-3$ has transformed $f$ in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 3.4.9.

Let us follow one point of the graph of $f(x)=|x|$.

- The point $(0,0)$ is transformed first by shifting left 1 unit:$(0,0)\rightarrow(-1,0)$
- The point $(-1,0)$ is transformed next by shifting down 3 units:$(-1,0)\rightarrow(-1,-3)$

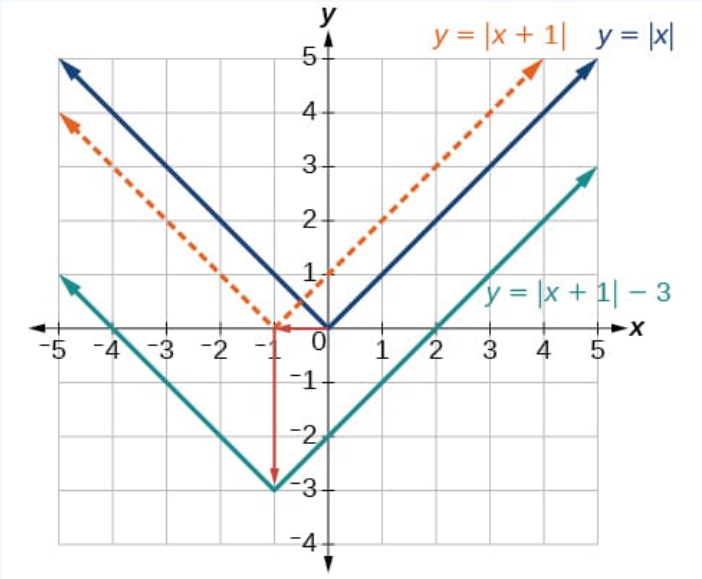



Figure 3.4.9: Graph of an absolute function, $y=|x|$, and how it was transformed to $y=|x+1|-3$

Figure 3.4.10 shows the graph of $h$.

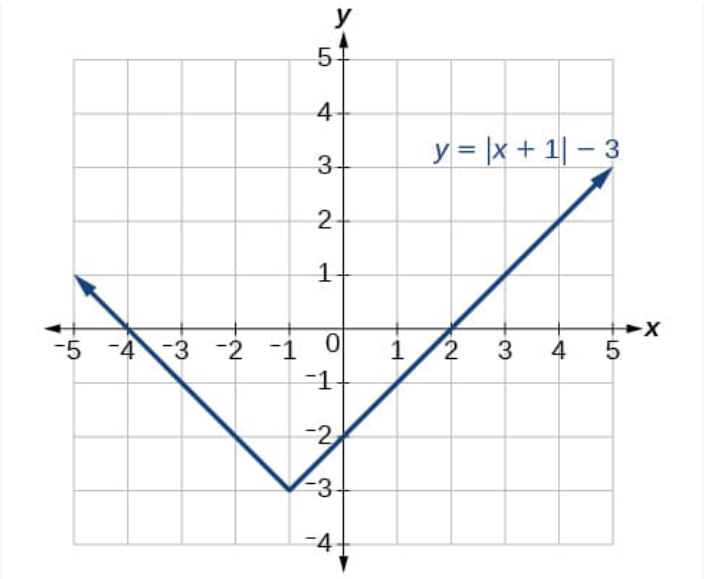



Figure 3.4.10: The final function $y = |x + 1| - 3$.

## Example 3.4.5: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure 3.4.12, which is a transformation of the toolkit square root function.

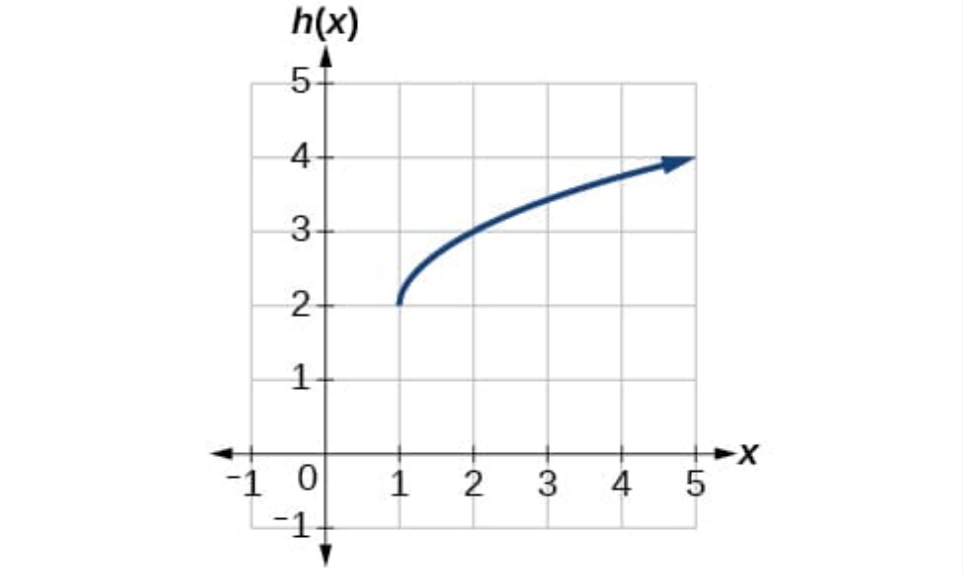



Figure 3.4.12: Graph of a square root function transposed right one unit and up 2.

**Solution**

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x - 1) + 2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x - 1} + 2$$

**Analysis**

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

**Exercise 3.4.3**

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

*Answer available at the end of this section.*

## Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 3.4.13**.**

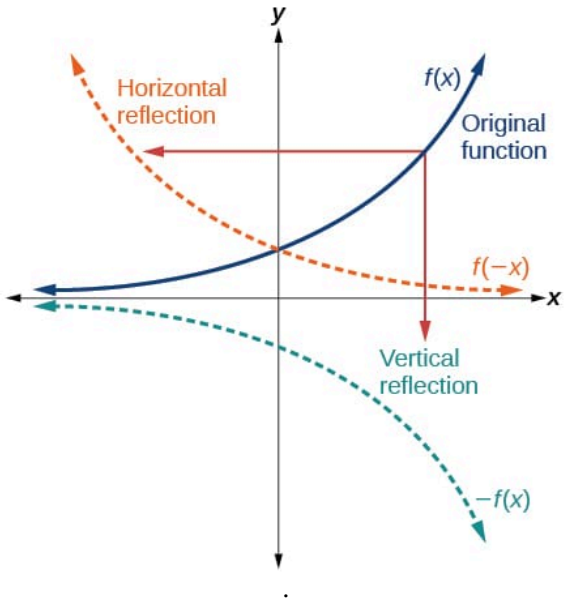



Figure 3.4.13: Graph of the vertical and horizontal reflection of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

**Definitions: Reflections**

Given a function $f(x)$, a new function $y = -f(x)$ is a **vertical reflection** of the function $f(x)$, called a reflection about (or over, or across) the $x$-axis.

Given a function $f(x)$, a new function $y = f(-x)$ is a **horizontal reflection** of the function $f(x)$, called a reflection about the $y$-axis.

**How To...**

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the y-axis.

## Example 3.4.6: Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

**Solution**

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure 3.4.14.

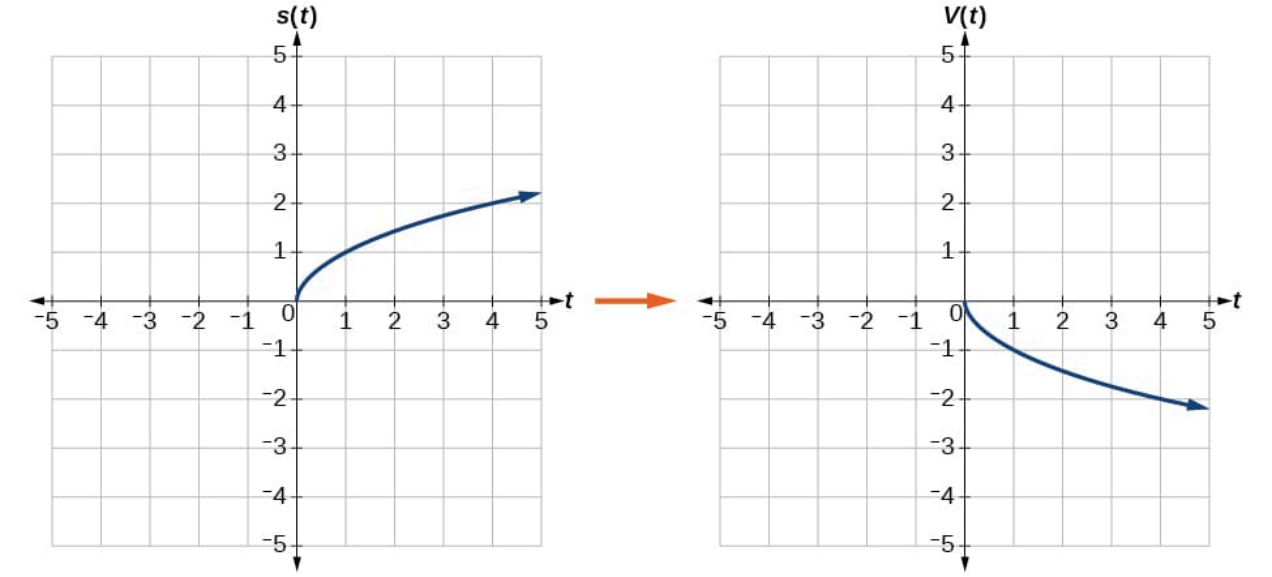



Figure 3.4.14: Graph of the vertical reflection of the square root function.

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 3.4.15.

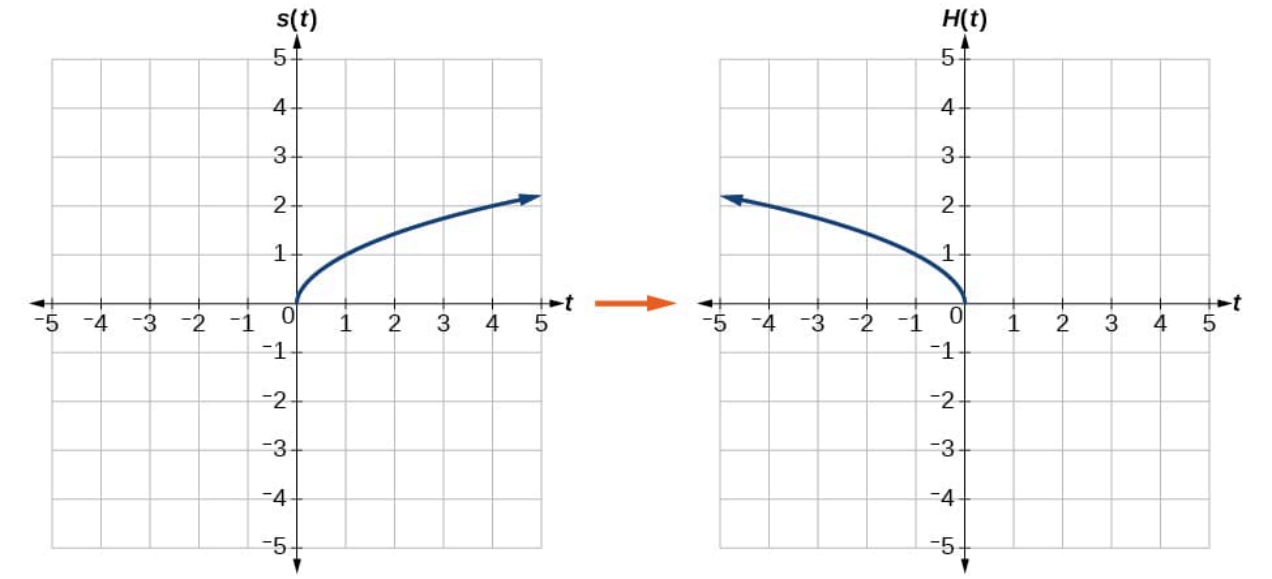



Figure 3.4.15: Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

## Exercise 3.4.4

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

*Answer available at the end of this section.*

## Example 3.4.7: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table 3.4.6. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

*Table* 3.4.6

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 1 | 3 | 7 | 11 |

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 3.4.7.

*Table* 3.4.7

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $g(x)$ | -1 | -3 | -7 | -11 |

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 3.4.8.

*Table* 3.4.8

| $x$ | -2 | -4 | -6 | -8 |
|---|---|---|---|---|
| $h(x)$ | 1 | 3 | 7 | 11 |

**Exercise 3.4.5**

A function $f(x)$ is given as Table 3.4.9. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

*Table* 3.4.9

| $x$ | -2 | 0 | 2 | 4 |
|---|---|---|---|---|
| $f(x)$ | 5 | 10 | 15 | 20 |

*Answer available at the end of this section.*

**Exercise 3.4.6**

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

*Answer available at the end of this section.*

# Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

## Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 3.4.23 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

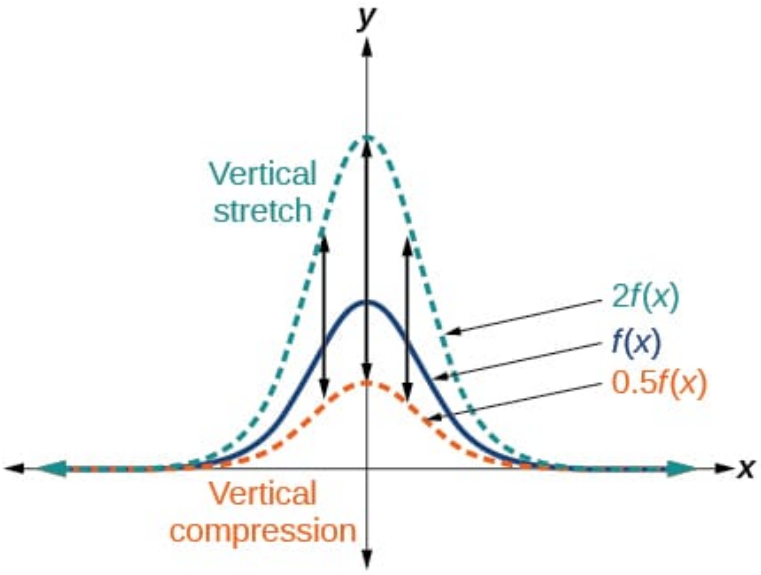



Figure 3.4.23: Vertical stretch and compression

## Definitions: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = cf(x)$, where $c$ is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $c > 1$, then the graph will be stretched vertically by a factor of $c$
- If $0 < c < 1$, then the graph will be compressed vertically by a factor of $\frac{1}{c}$
- If $c < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

## How To...

Given a function, graph its vertical stretch.

a. Identify the value of $c$.
b. Multiply all range values by $c$
c. If $c > 1$, the graph is stretched by a factor of $c$.
d. If $0 < c < 1$, the graph is compressed by a factor of $c$.
e. If $c < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

## Exercise 3.4.7

A function $f$ is given as Table 3.4.14. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

*Table* 3.4.14

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(x)$ | 12 | 16 | 20 | 0 |

*Answer available at the end of this section.*

### Example 3.4.8: Recognizing a Vertical Stretch

The graph in Figure 3.4.26 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

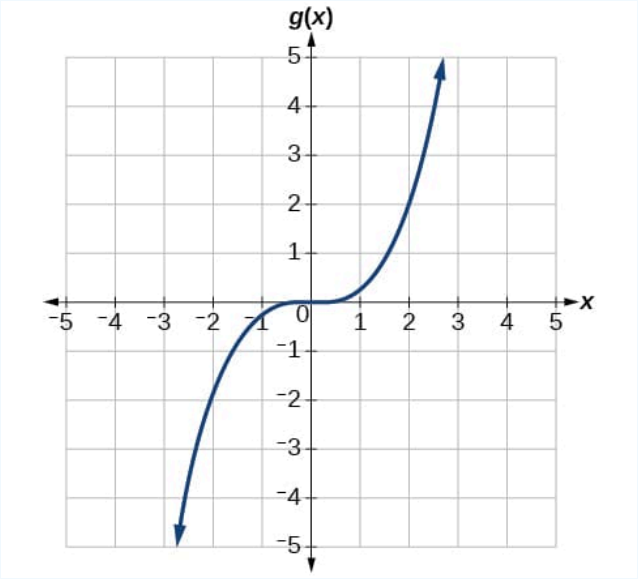



Figure 3.4.26: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of $g$ are $\frac{1}{4}$ the outputs of the function $f$ because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for $g$ by using the definition of the function $f$.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3. \tag{3.4.1}$$

### Exercise 3.4.8

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

*Answer available at the end of this section.*

## Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

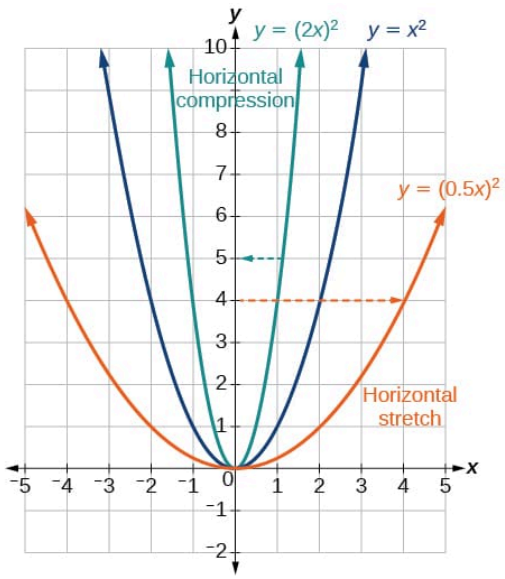



Figure 3.4.27: Graph of the vertical stretch and compression of $x^2$.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 3.4.27. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

## Definitions: Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(cx)$, where $c$ is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $c > 1$, then the graph will be compressed by a factor of $c$.
- If $0 < c < 1$, then the graph will be stretched by a factor of $\frac{1}{c}$.
- If $c < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

## How To...

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(cx)$ where $c > 1$ for a compression or $0 < c < 1$ for a stretch.

## Example 3.4.9: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 3.4.30.

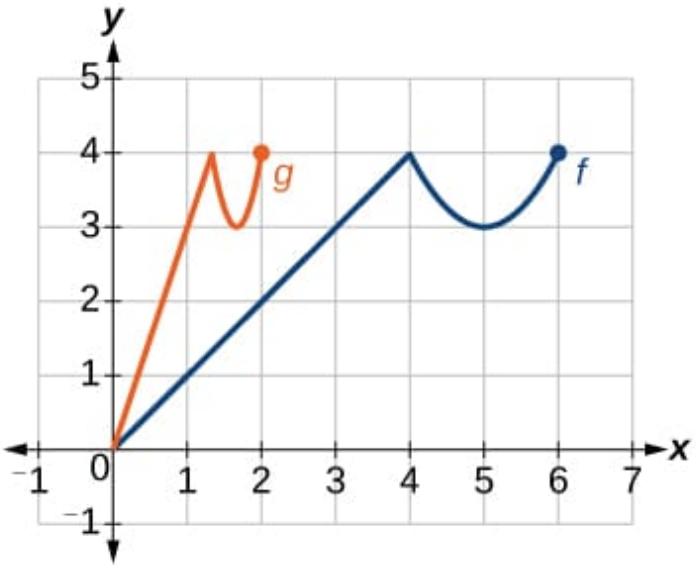



Figure 3.4.30: Graph of $f(x)$ being vertically compressed to $g(x)$.

**Solution**

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

**Analysis**

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

**Exercise 3.4.9**

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

*Answer available at the end of this section.*

## Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x) + 3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x + 3)$, for example, we have to think about how the inputs to the function $g$ relate to the inputs to the function $f$. Suppose we know $f(7) = 12$. What input to $g$ would produce that output? In other words, what value of $x$ will allow $g(x) = f(2x + 3) = 12$? We would need $2x + 3 = 7$. To solve for $x$, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p)=f(b(x+\frac{p}{b}))$$

Let's work through an example.

$$f(x)=(2x+4)^2$$

We can factor out a 2.

$$f(x)=(2(x+2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

## Combining Transformations: Order May Matter!

- When combining a scalar and a vertical transformation of the form $cf(x)+k$, first vertically stretch (or compress) and then vertically shift by $k$.
- When combining a scalar and a horizontal transformation of the form $f(cx+h)$, first horizontally shift by $h$ and then horizontally stretch (or compress).
- When combining a scalar and a horizontal transformation of the form $f(c(x+h))$, first horizontally stretch (or compress) and then horizontally shift by $h$.
- When combining a reflection and a vertical transformation of the form $-f(x)+k$, first vertically reflect and then vertically shift by $k$.
- When combining a reflection and a horizontal transformation of the form $f(-x+h)$, first horizontally shift by $h$ and then horizontally reflect.
- When combining a reflection and a horizontal transformation of the form $f(-(x+h))$, first horizontally reflect and then horizontally shift by $h$.
- Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

## Example 3.4.10: Finding a Triple Transformation of a Tabular Function

Given Table 3.4.18 for the function $f(x)$, create a table of values for the function $g(x)=2f(3x)+1$.

*Table* 3.4.18

| $x$ | 6 | 12 | 18 | 24 |
|---|---|---|---|---|
| $f(x)$ | 10 | 14 | 15 | 17 |

**Solution**

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each $x$-value by $\frac{1}{3}$.See Table 3.4.19.

*Table* 3.4.19

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $f(3x)$ | 10 | 14 | 15 | 17 |

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 3.4.20.

*Table* 3.4.20

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $2f(3x)$ | 20 | 28 | 30 | 34 |

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 3.4.21.

*Table* 3.4.21

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $g(x) = 2f(3x) + 1$ | 21 | 29 | 31 | 35 |

## Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in Figure 3.4.3.

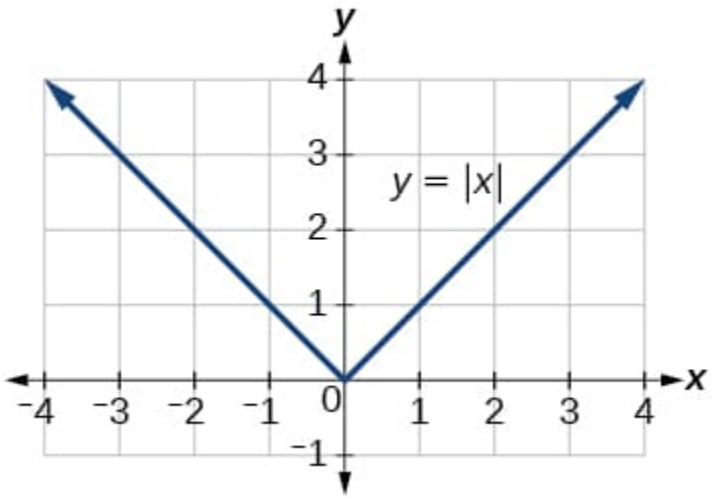



Figure 3.4.3: Graph of an absolute function.

Figure 3.4.3 shows the graph of $y = 2|x–3| + 4$. The graph of $y = |x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3, 4)$ for this transformed function.

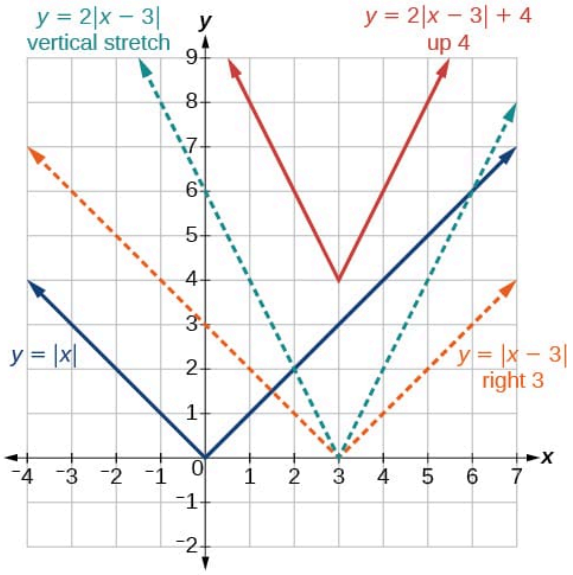



Figure 3.4.4: Graph of the different types of transformations for an absolute function.

## Example 3.4.11: Writing an Equation for an Absolute Value Function

Write an equation for the function graphed in Figure 3.4.5.

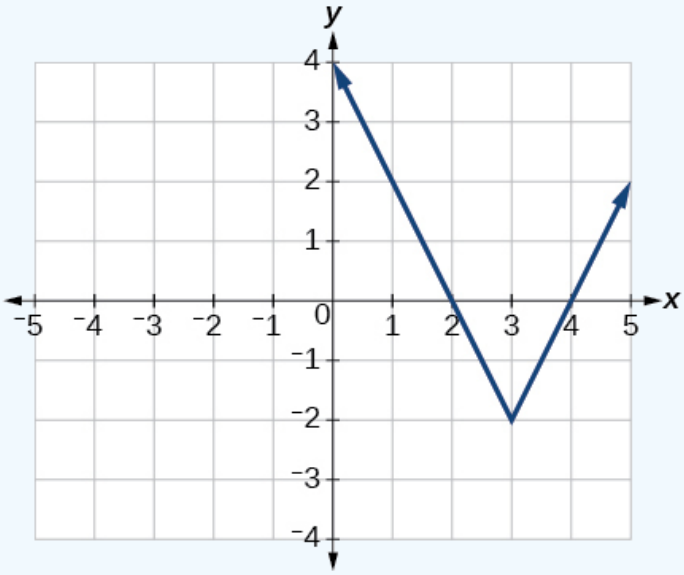



Figure 3.4.5: Graph of an absolute function.

**Solution**

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See Figure 3.4.6.

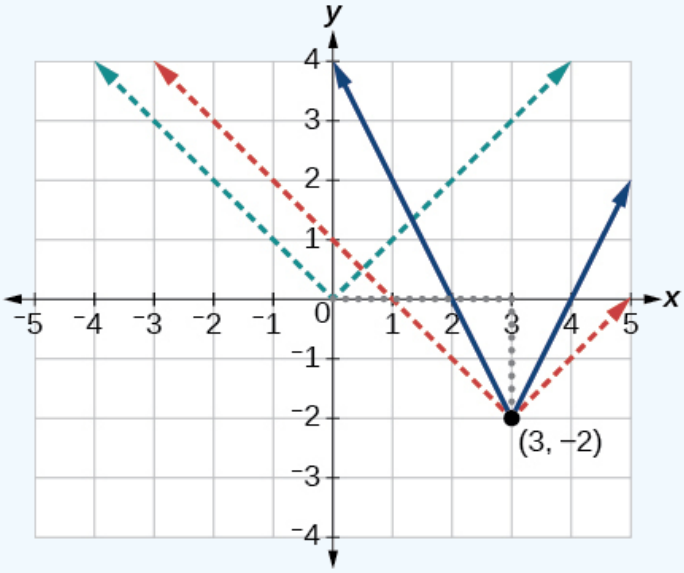



Figure 3.4.6: Graph of two transformations for an absolute function at $(3, -2)$.

We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in Figure 3.4.7.

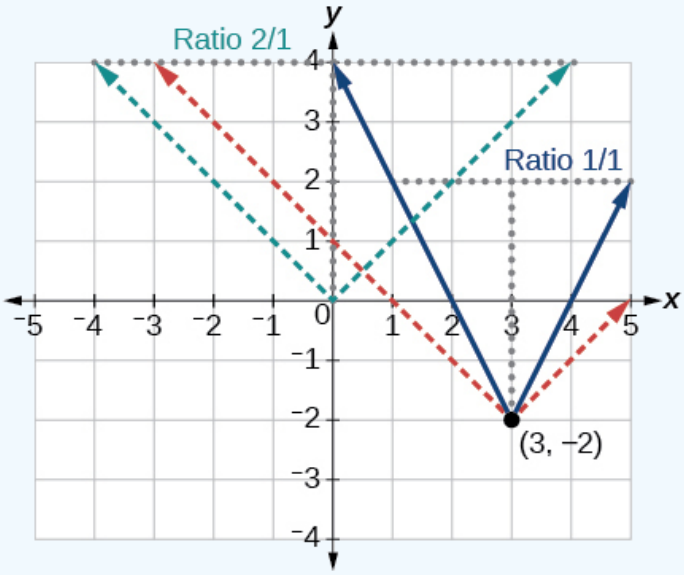



Figure 3.4.7: Graph of two transformations for an absolute function at $(3, -2)$ and the ratios between the two different transformations.

From this information we can write the equation

$$\begin{aligned} f(x) &= 2|x-3|-2, && \text{treating the stretch as a vertial stretch, or} \\ f(x) &= |2(x-3)|-2, && \text{treating the stretch as a horizontal compression.} \end{aligned}$$

**Analysis**

Note that these equations are algebraically equivalent—the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression.

**Q & A**

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

*Answer available at the end of this section.*

**Exercise 3.4.10**

Write the equation for the absolute value function that is horizontally shifted left 2 units, then reflected about the $x$ axis, and vertically shifted up 3 units.

*Answer available at the end of this section.*

**Q & A**

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

*Answer available at the end of this section.*

## Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. Functions whose graphs are symmetric about the y-axis are called **even functions**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph, as shown in Figure 3.4.21.

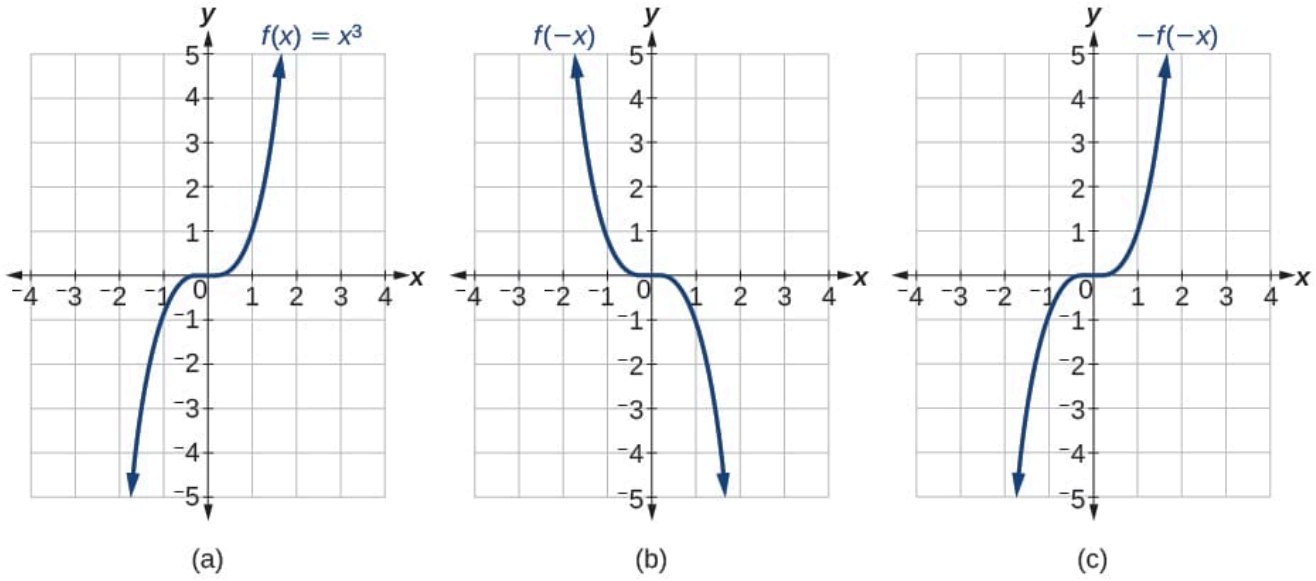



Figure 3.4.21: (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor

odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

### Definitions: Even and Odd Functions

A function is called an **even function** if for every input $x$

$f(x) = f(-x)$

The graph of an even function is symmetric about the y-axis.

A function is called an **odd function** if for every input $x$

$f(x) = -f(-x)$

The graph of an odd function is symmetric about the origin.

### How To...

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

### Example 3.4.12: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

**Solution**

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

**Analysis**

Consider the graph of $f$ in Figure 3.4.22. Notice that the graph is symmetric about the origin. For every point $(x, y)$ on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of $f$, and the corresponding point $(-1, -3)$ is also on the graph.

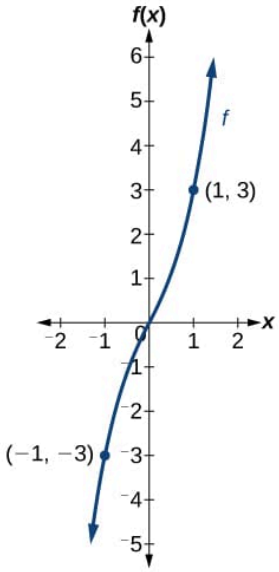



Figure 3.4.22: Graph of $f(x)$ with labeled points at $(1, 3)$ and $(-1, -3)$.

**Exercise 3.4.11**

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

*Answer available at the end of this section.*

## Key Equations

- Vertical shift $y = f(x) + k$ (up for $k > 0$)
- Horizontal shift $y = f(x - h)$(right) for $h > 0$
- Vertical reflection $y = -f(x)$
- Horizontal reflection $y = f(-x)$
- Vertical stretch $y = cf(x)$ (c>0 )
- Vertical compression $y = cf(x)$ (0<c<1)
- Horizontal stretch $y = f(cx)(0 < c < 1)$
- Horizontal compression $y = f(cx)$ (c>1)

## Key Concepts

- A function can be shifted vertically by adding a constant to the output.
- A function can be shifted horizontally by adding a constant to the input.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts.
- Vertical and horizontal shifts are often combined.
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y-axis. A graph can be reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly.
- A function presented as an equation can be reflected by applying transformations one at a time.
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $-f(x) = f(-x)$.
- A function can be odd, even, or neither.

- A function can be compressed or stretched vertically by multiplying the output by a constant.
- A function can be compressed or stretched horizontally by multiplying the input by a constant.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order.

## Glossary

**even function**

a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y-axis

**horizontal compression**
a transformation that compresses a function's graph horizontally, by multiplying the input by a constant c>1

**horizontal reflection**
a transformation that reflects a function's graph across the y-axis by multiplying the input by –1

**horizontal shift**
a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input

**horizontal stretch**
a transformation that stretches a function's graph horizontally by multiplying the input by a constant 0<c<1

**odd function**
a function whose graph is unchanged by combined horizontal and vertical reflection, $-f(x) = f(-x)$, and is symmetric about the origin

**vertical compression**
a function transformation that compresses the function's graph vertically by multiplying the output by a constant 0<c<1

**vertical reflection**
a transformation that reflects a function's graph across the x-axis by multiplying the output by –1

**vertical shift**
a transformation that shifts a function's graph up or down by adding a positive or negative constant to the output

**vertical stretch**
a transformation that stretches a function's graph vertically by multiplying the output by a constant c>1

## Answers

**Exercise** 3.4.1

$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$

**Exercise** 3.4.2

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

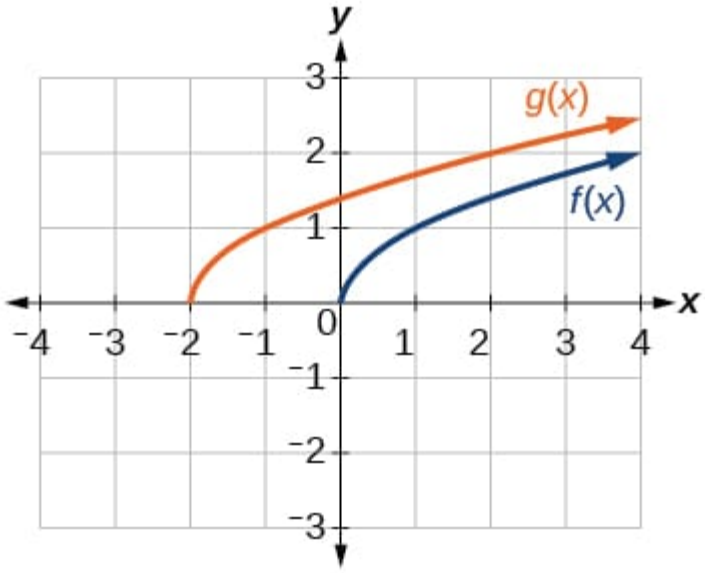



Figure 3.4.8

**Exercise** 3.4.3

$$g(x) = \frac{1}{x-1} + 1$$

**Exercise** 3.4.4

a.

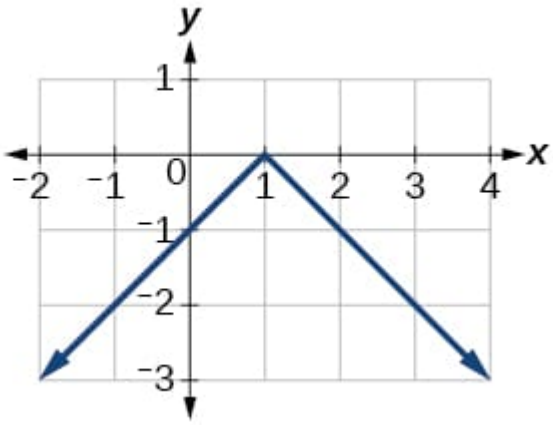



Figure 3.4.16: Graph of a vertically reflected absolute function.

b.

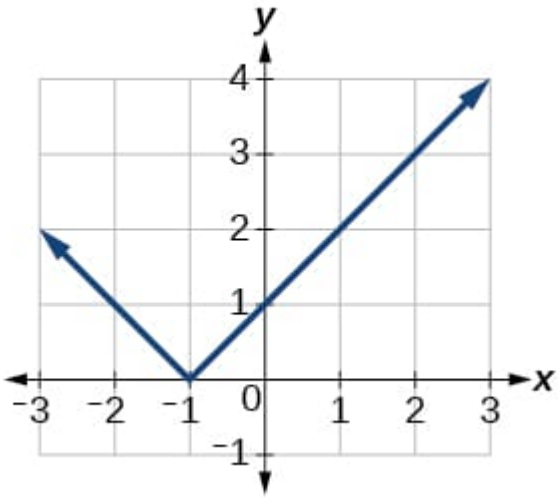



Figure 3.4.17: Graph of an absolute function translated one unit left.

**Exercise** 3.4.5

**a.** $g(x) = -f(x)$

*Table* 3.4.10

| $x$ | -2 | 0 | 2 | 4 |
|---|---|---|---|---|
| $g(x)$ | -5 | -10 | -15 | -20 |

**b.** $h(x) = f(-x)$

*Table* 3.4.11

| $x$ | -2 | 0 | 2 | -4 |
|---|---|---|---|---|
| $h(x)$ | 15 | 10 | 5 | 20 |

**Exercise** 3.4.6

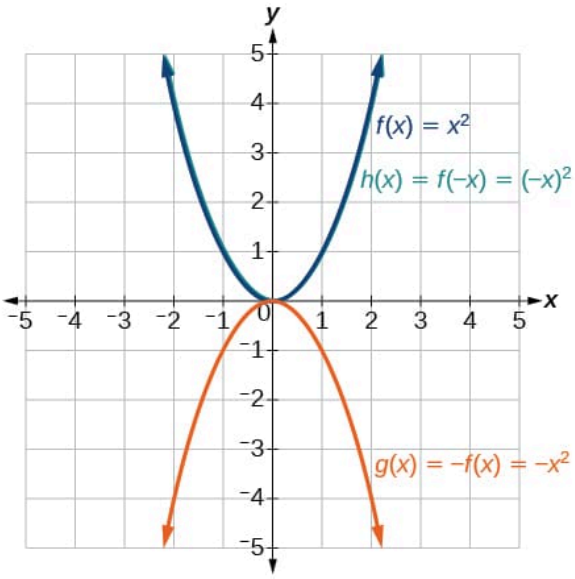



Figure 3.4.20: Graph of $x^2$ and its reflections.

Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

**Exercise** 3.4.7

*Table* 3.4.15

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $g(x)$ | 9 | 12 | 15 | 0 |

**Exercise** 3.4.8

$g(x) = 3x - 2$

**Exercise** 3.4.9

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

**Q & A**

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for $x$ and $f(x)$.

$$f(x) = a|x - 3| - 2$$

Now substituting in the point $(1, 2)$

$$\begin{aligned} 2 &= a|1 - 3| - 2 \\ 4 &= 2a \\ a &= 2 \end{aligned}$$

**Exercise** 3.4.10

$f(x) = -|x + 2| + 3$

**Q & A**

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (Figure 3.4.8).

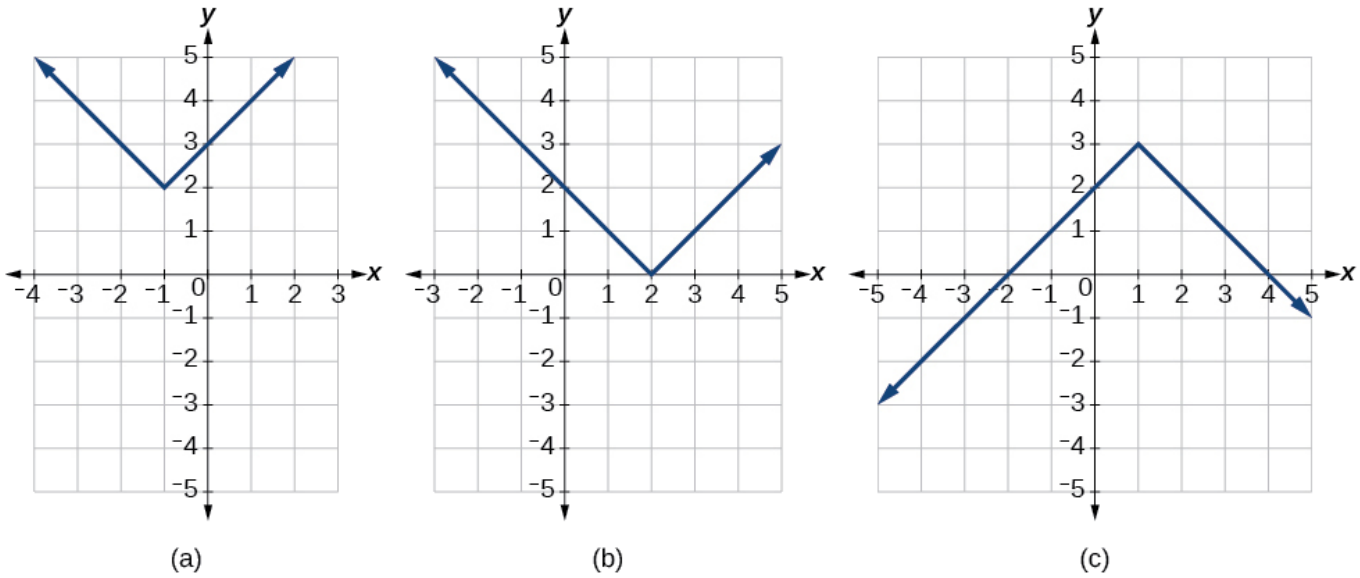



Figure 3.4.8: (a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

**Exercise** 3.4.11

even

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# 3.5: Composition of Functions

**Learning Objectives**

- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.

## Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of **combining functions** so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$f \circ g(x) = f(g(x)) \tag{3.5.1}$$

We read the left-hand side as"$f$ composed with $g$ at $x$," and the right-hand side as"$f$ of $g$ of $x$."The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function $g$ takes the input $x$ first and yields an output $g(x)$. Then the function $f$ takes $g(x)$ as an input and yields an output $f(g(x))$.

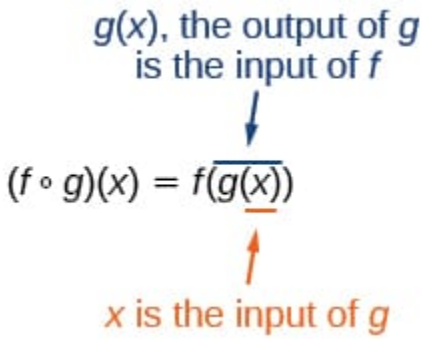



Figure 3.5.1: Explanation of the composite function.

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all $x$. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

## Definition: Composition

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input $x$ and functions $f$ and $g$, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)) \tag{3.5.2}$$

It is important to realize that the product of functions $fg$ is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

## Other Compositions

Similarly, we can form other types of compositions as follows:

$$(g \circ f)(x) = g(f(x)) \tag{3.5.3}$$

$$(f \circ f)(x) = f(f(x)) \tag{3.5.4}$$

$$(g \circ g)(x) = g(g(x)) \tag{3.5.5}$$

## Example 3.5.1: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is **commutative**.

$$f(x) = 2x + 1 \quad g(x) = 3x - 4$$

**Solution**

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= 2(3x - 4) + 1 \\ &= 6x - 12 + 1 \\ &= 6x - 11 \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= 3(2x + 1) - 4 \\ &= 6x + 3 - 4 \\ &= 6x - 1 \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is **not** commutative.

## Example 3.5.2: Finding Composite Functions

Given the functions $f(x) = x^2 + 6x$ and $g(x) = x + 2$. Find the compositions and simplify the resulting expressions.

1. $(f \circ g)(x)$
2. $(g \circ f)(x)$
3. $(f \circ f)(x)$
4. $(g \circ g)(x)$

**Solutions**

By definition

1.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= (x+2)^2 + 6(x+2) \\ &= x^2 + 4x + 4 + 6x + 12 \\ &= x^2 + 10x + 16\end{aligned}$$

Therefore, $(f \circ g)(x) = x^2 + 10x + 16$

2.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= (x^2 + 6x) + 2) \\ &= x^2 + 6x + 2\end{aligned}$$

Therefore, $(g \circ f)(x) = x^2 + 6x + 2$

3.

$$\begin{aligned}(f \circ f)(x) &= f(f(x)) \\ &= (x^2 + 6x)^2 + 6(x^2 + 6x) \\ &= x^2(x+6)^2 + 6x^2 + 36x \\ &= x^2(x^2 + 12x + 36) + 6x^2 + 36x \\ &= x^4 + 12x^3 + 36x^2 + 6x^2 + 36x \\ &= x^4 + 12x^3 + 42x^2 + 36x\end{aligned}$$

Therefore, $(f \circ f)(x) = x^4 + 12x^3 + 42x^2 + 36x$

4.

$$\begin{aligned}(g \circ g)(x) &= g(g(x)) \\ &= (x+2) + 2 \\ &= x + 4\end{aligned}$$

Therefore, $(g \circ g)(x) = x + 4$

## Example 3.5.3: Investigating the Order of a Composition by Evaluating

Suppose $f(x) = x^2 + 6x$ and $g(x) = x + 2$, evaluate $(f \circ g)(1)$ and $(g \circ f)(1)$, discuss the results.

**Solution**

Since we already have the composite function $(f \circ g)(x) = x^2 + 10x + 16$ obtained in the previous example, we can simply evaluate this function at $x = 1$

$(f \circ g)(1) = (1)^2 + 10(1) + 16 = 27$

Next, to evaluate $(g \circ f)(1)$, we can use the definition

$(g \circ f)(1) = g(f(1))$

First we evaluate $f(1) = (1)^2 + 6(1) = 7$

Next, we use the output of $f$ as the input for $g$, so now we must evaluate $g(7)$

$$g(x) = x + 2$$

$$g(7) = (7) + 2$$
$$= 9$$

Therefore, $(g \circ f)(1) = 9$

**Discussion**

Since $(f \circ g)(1) = 27$ and $(g \circ f)(1) = 9$, we have confirmed that composition is **not** commutative.

### Question/Answer

Are there any situations where $f(g(x))$ and $g(f(x))$ would be equal?

Yes. There are several possible cases in which this happens.

1. When $f(x) = g(x)$

2. When BOTH functions are of the form $y = x + b$

3. When BOTH functions are of the form $y = ax$

4. When BOTH functions are of the form $y = x^n$

5. When $f$ and $g$ are inverses* of each other.

**Later in this chapter, we will discuss "inverse" functions. When these compositions are equal, then $f$ and $g$ have a special relationship to each other that we will learn more about...stay tuned.*

## Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(x) = x^2 - x$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

### Example 3.5.4: Evaluating a Composition of Functions with a Numerical Input

Given $f(x) = x^2 - x$ and $g(x) = 3x + 2$, evaluate $f(g(1))$.

**Solution**

Because the inside expression is $g(1)$, we start by evaluating $g(x)$ at $x = 1$.

$$g(1) = 3(1) + 2$$
$$g(1) = 5$$

Then $f(g(1)) = f(5)$, so we evaluate $f(x)$ at an input of 5.

$$f(g(1)) = f(5)$$
$$f(g(1)) = 5^2 - 5$$
$$f(g(1)) = 20$$

### YOU Do it! 3.5.1

Given $f(x) = x^2 - x$ and $g(x) = 3x + 2$, evaluate

a. $g(f(2))$
b. $g(f(-2))$

*Answer a available at the end of this section.*

## Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

### Example 3.5.5: Using a Table to Evaluate a Composite Function

Using Table 3.5.1, evaluate $f(g(3))$ and $g(f(3))$.

*Table* 3.5.1

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 1 | 6 | 3 |
| 2 | 8 | 5 |
| 3 | 3 | 2 |
| 4 | 1 | 7 |

**Solution**

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function $g : g(3) = 2$. We can then use that result as the input to the function $f$, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function $f$, we find that $f(2) = 8$.

$$g(3) = 2$$

$$f(g(3)) = f(2) = 8$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for $g$, we can evaluate

$$g(f(3)) = g(3) = 2$$

Table 3.5.2 shows the composite functions $f \circ g$ and $g \circ f$ as tables.

*Table* 3.5.2

| $x$ | $g(x)$ | $f(g(x))$ | $f(x)$ | $g(f(x))$ |
|---|---|---|---|---|
| 3 | 2 | 8 | 3 | 2 |

## You DO it! 3.5.2

Using Table 3.5.1, evaluate $f(g(1))$ and $g(f(4))$.

*Answer available at the end of this section.*

## Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

## How To ...

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

## Example 3.5.6: Using a Graph to Evaluate a Composite Function

Using Figure 3.5.2, evaluate $f(g(1))$.

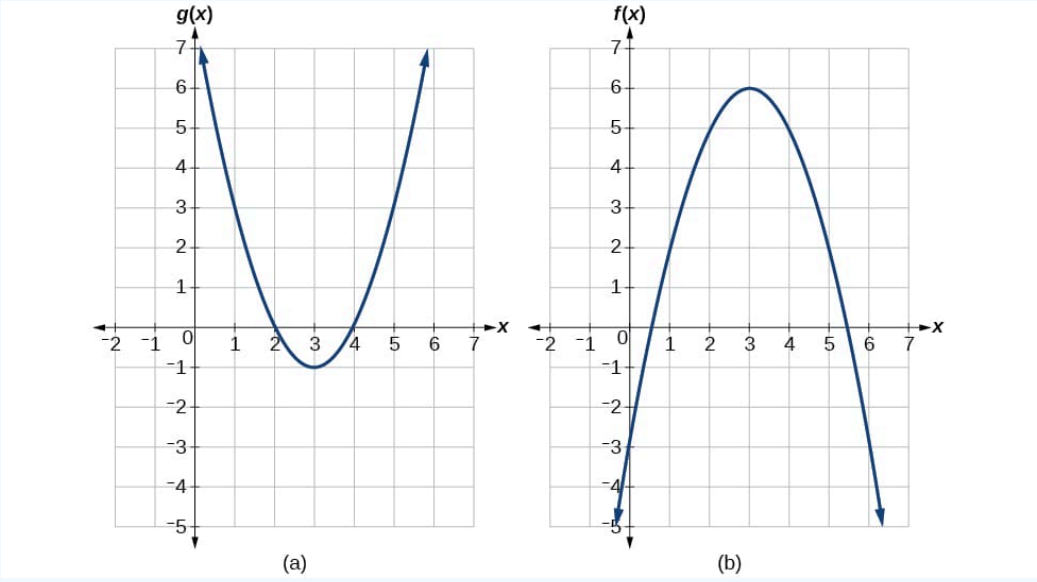



Figure 3.5.2: Two graphs of a positive and negative parabola.

**Solution**

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure 3.5.3.

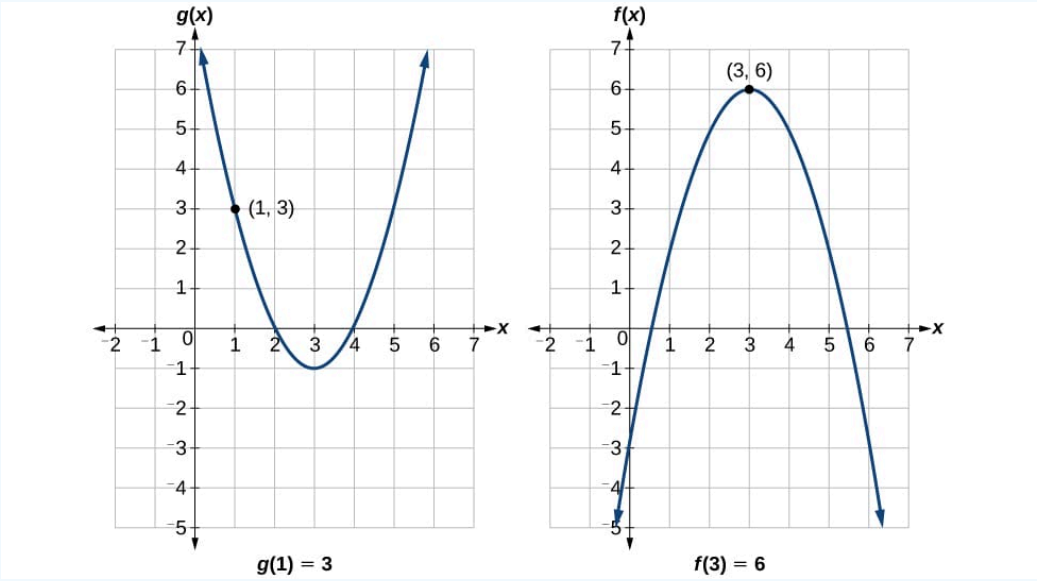



Figure 3.5.3: Two graphs of a positive parabola $g(x)$ and a negative parabola $f(x)$. The following points are plotted: $g(1) = 3$ and $f(3) = 6$.

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function $f$.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

**Analysis**

Figure 3.5.4 shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

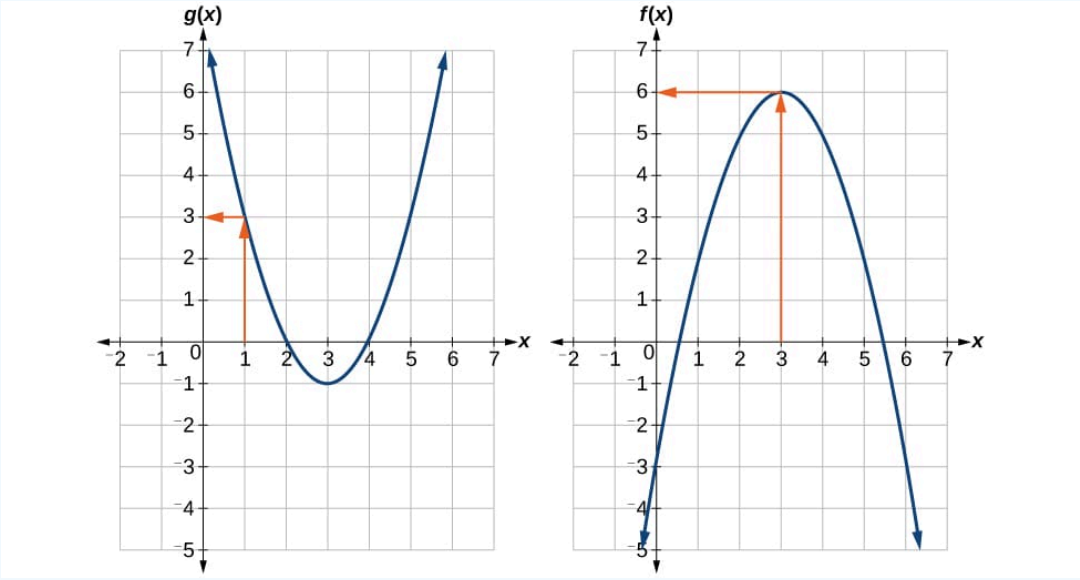



Figure 3.5.4: Two graphs of a positive and negative parabola.

**You DO it! 3.5.3**

Using Figure 3.5.4, evaluate $g(f(2))$.

*Answer available at the end of this section.*

## Finding the Domain of a Composite Function

The **domain of a composite function** such as $f \circ g$ is dependent on the domain of $g$ and the domain of $f$. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions $f$ and $g$ separately. If we write the composite function for an input $x$ as $f(g(x))$, we can see right away that $x$ must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of $f$, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of $g$ that produce outputs from $g$ belonging to the domain of $f$. Note that the domain of $f$ composed with $g$ is the set of all $x$ such that $x$ is in the domain of $g$ and g(x)\) is in the domain of $f$.

**Definition: Domain of a Composite Function $f \circ g$**

The **domain of a composite function** $f(g(x))$ is the set of those inputs $x$ in the domain of $g$ for which $g(x)$ is in the domain of $f$.

**How To...**

Given a function composition $f(g(x))$, determine its domain.

1. Find the domain of $g$.
2. Find the domain of $f$.
3. Find those inputs $x$ in the domain of $g$ for which $g(x)$ is in the domain of $f$. That is, exclude those inputs $x$ from the domain of $g$ for which $g(x)$ is not in the domain of $f$. The resulting set is the domain of $f \circ g$.

**Example 3.5.7: Finding the Domain of a Composite Function**

Find the domain of $(f \circ g)(x)$

$$\text{where } f(x) = \frac{5}{x-1} \text{ and } g(x) = \frac{4}{3x-2}$$

**Solution**

The domain of $g(x)$ is all real numbers such that $x \neq \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of $f$ consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of $x$ for which $g(x) = 1$.

$$\begin{aligned} \frac{4}{3x-2} &= 1 \\ 4 &= 3x-2 \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty)$$

## Example 3.5.8: Finding the Domain of a Composite Function Involving Radicals

Find the domain of $(f \circ g)(x)$

$$\text{where } f(x) = \sqrt{2 - x} \text{ and } g(x) = \sqrt{x + 3}$$

**Solution**

The domain of $g$ is $[-3, \infty)$.

Now we check the domain of the composite function

$$(f \circ g)(x) = \sqrt{2 - \sqrt{x + 3}}$$

For $(f \circ g)(x) = \sqrt{2 - \sqrt{x + 3}}$, since the radicand of a square root must be positive.

We have

$$2 - \sqrt{x + 3} \geq 0 \tag{3.5.6}$$

$$-\sqrt{x + 3} \geq -2 \tag{3.5.7}$$

$$\sqrt{x + 3} \leq 2 \tag{3.5.8}$$

$$x + 3 \leq 4 \tag{3.5.9}$$

$$x \leq 1 \tag{3.5.10}$$

Now, we have to consider the compound inequality $x \leq 1$ and $x \geq -3$.

This gives a domain of $[-3, 1]$.

**Analysis**

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of $f$ , though they must be in the domain of $g$.

## You DO it! 3.5.4

Find the domain of $(f \circ g)(x)$

$$\text{where } f(x) = \frac{1}{x - 2} \text{ and } g(x) = \sqrt{x + 4}$$

*Answer available at the end of this section.*

As we discussed previously, the **domain of a composite function** such as $f \circ g$ is dependent on the domain of $g$ and the domain of $f$. Similarly, when we compute the composition $g \circ f$ we must first know the domains of the functions $f$ and $g$ separately. If we write the composite function for an input $x$ as $g(f(x))$, we can see right away that $x$ must be a member of the domain of f in order

for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $f(x)$ must be a member of the domain of $g$, otherwise the second function evaluation in $g(f(x))$ cannot be completed, and the expression is still undefined.

### Definition: Domain of a Composite Function $g \circ f$

The **domain of a composite function** $g(f(x))$ is the set of those inputs $x$ in the domain of $f$ for which $f(x)$ is in the domain of $g$.

### How To...

Given a function composition $g(f(x))$, determine its domain.

1. Find the domain of $f$.
2. Find the domain of $g$.
3. Find those inputs $x$ in the domain of $f$ for which $f(x)$ is in the domain of $g$. That is, exclude those inputs $x$ from the domain of $f$ for which $f(x)$ is not in the domain of $g$. The resulting set is the domain of $g \circ f$.

### Example 3.5.9

Find the domain of $(g \circ f)(x)$

$$\text{where } f(x) = -\frac{2}{x+5} \text{ and } g(x) = \frac{1}{x-4}$$

**Solution**

The domain of $f$ consists of all real numbers except $x = -5$, since that input value would cause us to divide by 0. Likewise, the domain of $g$ consists of all real numbers except 4. So we need to exclude from the domain of $f(x)$ that value of $x$ for which $f(x) = 2$.

$$\begin{aligned} -\frac{2}{x+5} &= 4 \\ -2 &= 4x + 20 \\ -18 &= 4x \\ -\frac{9}{2} &= x \end{aligned}$$

So the domain of $g \circ f$ is the set of all real numbers except $-\frac{9}{2}$ and $4$.

This gives us the following exclusions:

$$x \neq -\frac{9}{2} \text{ or } x \neq 4$$

We can write the domain in interval notation as:

$$\left(-\infty, -\frac{9}{2}\right) \cup \left(-\frac{9}{2}, 4\right) \cup (4, \infty)$$

### You DO it! 3.5.5

Find the domain of $(g \circ f)(x)$

$$\text{where } f(x) = \sqrt{6x+8} \text{ and } g(x) = 5x - 2$$

*Answer available at the end of this section.*

## Key Equation

- Composite function $(f \circ g)(x) = f(g(x))$
- Composite function $(g \circ f)(x) = g(f(x))$

## Key Concepts

- We can perform algebraic operations on functions. See Example.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See Example and Example.
- The order of function composition must be considered when interpreting the meaning of composite functions. See Example.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See Example.
- A composite function can be evaluated from a graph. See Example.
- A composite function can be evaluated from a formula. See Example.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See Example and Example.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See Example.

## Glossary

**composite function**

the new function formed by function composition, when the output of one function is used as the input of another

### Answers

**YOU Do it!** 3.5.1

8

20

**You DO it!** 3.5.2

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

**You DO it!** 3.5.3

$g(f(2)) = g(5) = 3$

**You DO it!** 3.5.4

$[-4, 0) \cup (0, \infty)$

**You DO it!** 3.5.5

$\left[-\frac{4}{3}, \infty\right)$

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# 3.6: Inverse Functions

**Learning Objectives**

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure 3.6.1 provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

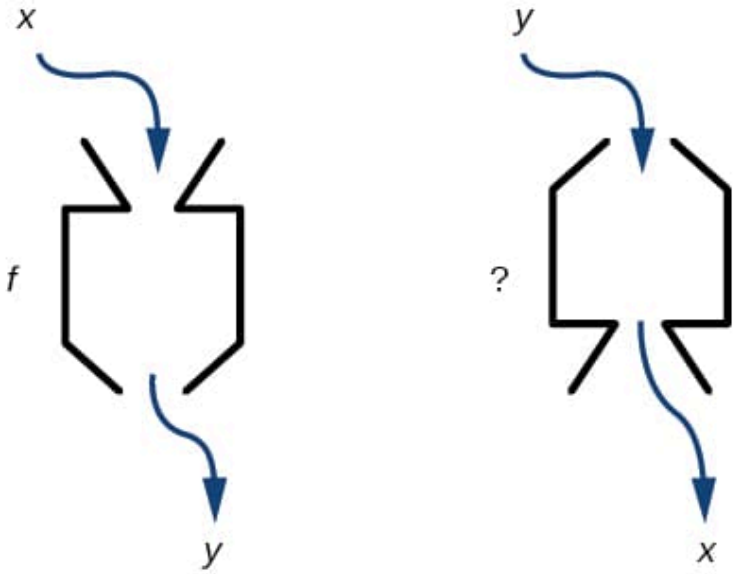



***Figure** 3.6.1**:** Can a function "machine" operate in reverse?*

## Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to more than one input value. For example, in a basic quadratic function $f(x) = x^2$, the output value $y = 4$ corresponds to BOTH $x = 2$ and $x = -2$.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table 3.6.13.

*Table 3.6.13: Letter grades and decimal equivalents.*

| Letter Grade | Grade Point Average |
|---|---|
| A | 4.0 |
| B | 3.0 |

| Letter Grade | Grade Point Average |
|---|---|
| C | 2.0 |
| D | 1.0 |

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

### Definition 3.6.1: One-to-One Function

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

### Example 3.6.11: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

**Solution**

A circle of radius $r$ has a unique area measure given by $A = \pi r^2$, so for any input, $r$, there is only one output, $A$. The area is a function of radius$r$.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure $A$ is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

### Exercise 3.6.11$A$

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

*Answer available at the end of this section.*

### Exercise 3.6.11$B$

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

*Answer available at the end of this section.*

## Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

**How to: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function**

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

**Example 3.6.13: Applying the Horizontal Line Test**

Consider the functions shown in Figure 3.6.12$a$ and Figure 3.6.12$b$. Are either of the functions one-to-one?

**Solution**

The function in Figure 3.6.12$a$ is not one-to-one. The horizontal line shown in Figure 3.6.15 intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

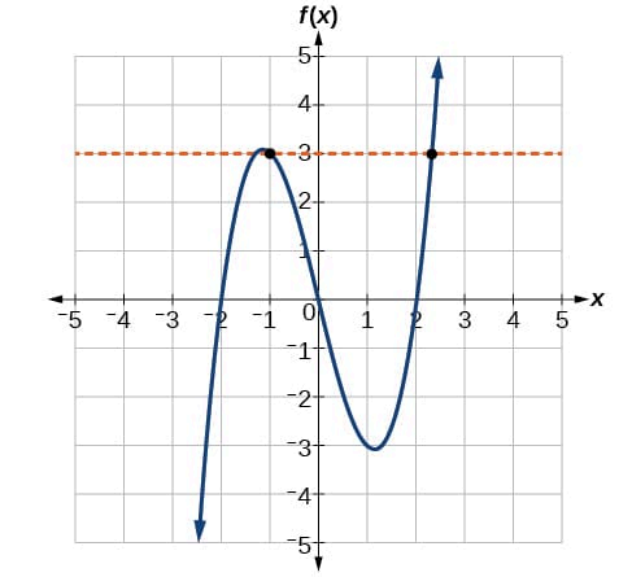



Figure 3.6.15: Graph of a polynomial with a horizontal line crossing through 2 points

The function in Figure 3.6.12$b$ is one-to-one. Any horizontal line will intersect a diagonal line at most once.

**Exercise 3.6.13**

Is the graph shown in Figure 3.6.13 one-to-one?

*Answer available at the end of this section.*

## Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the **Celsius** scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees **Fahrenheit** to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32) \tag{3.6.1}$$

and substitutes 75 for $F$ to calculate

$$\frac{5}{9}(75-32) \approx 24^{\circ} \tag{3.6.2}$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure 3.6.2 for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

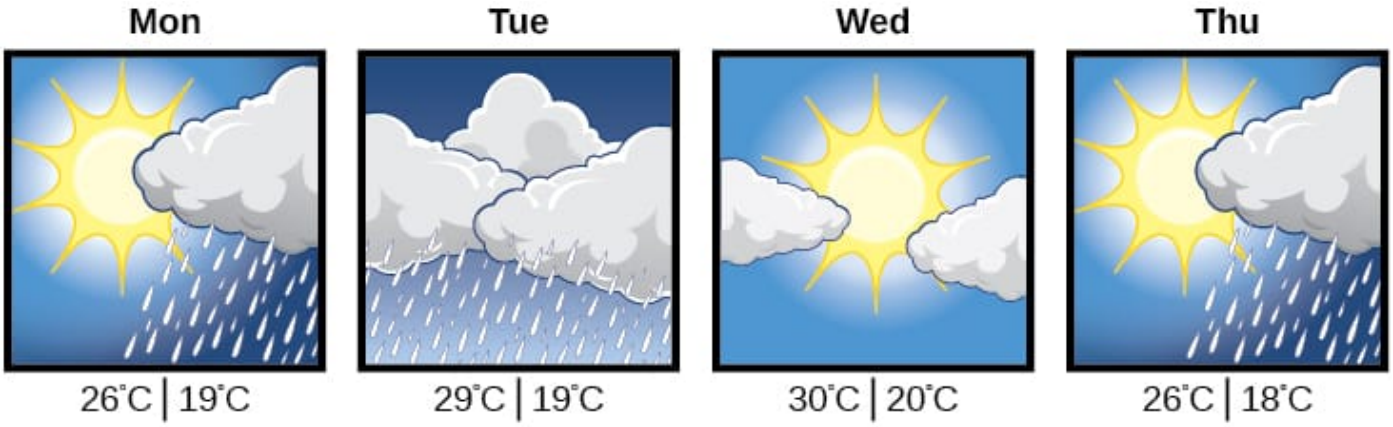



Figure 3.6.2: A forecast of Monday's through Thursday's weather.

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for $F$ after substituting a value for $C$. For example, to convert 26 degrees Celsius, she could write

$$26 = \frac{5}{9}(F-32) \tag{3.6.3}$$

$$26 \cdot \frac{9}{5} = F - 32 \tag{3.6.4}$$

$$F = 26 \cdot \frac{9}{5} + 32 \approx 79 \tag{3.6.5}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "$f$ inverse of $x$." The raised –1 is part of the notation. It is not an exponent; it does not imply a power of –1 . In other words, $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of $f$ and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number $a$, so $f^{-1} \circ f$ equals the identity function, that is,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x \tag{3.6.6}$$

This holds for all $x$ in the domain of $f$. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a **reciprocal**, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x \tag{3.6.7}$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x \tag{3.6.8}$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

## Definition 3.6.2: Inverse Function

Given any *one-to-one* function $f(x) = y$, a function $f^{-1}(x)$ is called an **inverse function** of $f$ if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all $x$ in the domain of $f$. It also follows that $f(f^{-1}(x)) = x$ for all $x$ in the domain of $f^{-1}$ if $f^{-1}$ is the inverse of $f$.

The notation $f^{-1}$ is read "$f$ inverse." Like any other function, we can use any variable name as the input for $f^{-1}$, so we will often write $f^{-1}(x)$, which we read as "$f$ inverse of $x$." Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)} \tag{3.6.9}$$

and not all functions have inverses.

## Example 3.6.1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

**Solution**

The inverse function reverses the input and output quantities, so if

$$\begin{aligned} f(2) = 4, \text{ then } f^{-1}(4) = 2; \\ f(5) = 12, \text{ then } f^{-1}(12) = 5 \end{aligned} \tag{3.6.10}$$

.

Alternatively, if we want to name the inverse function $g$, then $g(4) = 2$ and $g(12) = 5$.

**Analysis**

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See Table 3.6.1.

*Table* 3.6.1

| $(x, f(x))$ | $(x, g(x))$ |
|---|---|
| $(2, 4)$ | $(4, 2)$ |
| $(5, 12)$ | $(12, 5)$ |

## Exercise 3.6.1

Given that $f^{-1}(6) = 2$, what are the corresponding input and output values of the original function $f$?

*Answer available at the end of this section.*

### How To: Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

1. Determine whether $f(g(x)) = x$ or $g(f(x)) = x$.
2. If both statements are true, then $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

### Example 3.6.2: Testing Inverse Relationships Algebraically

If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$, is $g = f^{-1}$?

**Solution**

$$g(f(x)) = \frac{1}{(\frac{1}{x+2}) - 2} \tag{3.6.11}$$
$$= x + 2 - 2 \tag{3.6.12}$$
$$= x \tag{3.6.13}$$

so

$$g = f^{-1} \text{ and } f = g^{-1} \tag{3.6.14}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$f(g(x)) = \frac{1}{\frac{1}{x} - 2 + 2} \tag{3.6.15}$$
$$= \frac{1}{\frac{1}{x}} \tag{3.6.16}$$
$$= x \tag{3.6.17}$$

**Analysis**

Notice the inverse operations are in reverse order of the operations from the original function.

### Exercise 3.6.2

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

*Answer available at the end of this section.*

### Example 3.6.3: Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

**Solution**

$$f(g(x)) = \frac{x^3}{27} \neq x \tag{3.6.18}$$

No, the functions are not inverses.

**Analysis**

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x} = x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

**Exercise 3.6.3**

If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

*Answer available at the end of this section.*

## Finding Domain and Range of Inverse Functions

The outputs of the function $f$ are the inputs to $f^{-1}$, so the range of $f$ is also the domain of $f^{-1}$. Likewise, because the inputs to $f$ are the outputs of $f^{-1}$, the domain of $f$ is the range of $f^{-1}$. We can visualize the situation as in Figure 3.6.3.

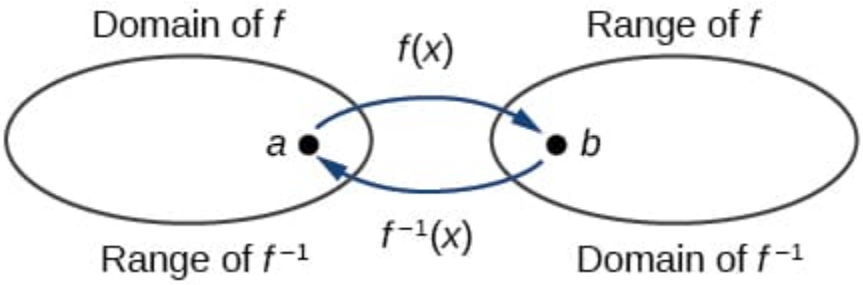



***Figure** 3.6.3**:** Domain and range of a function and its inverse.*

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its range limited to $[0, \infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x-1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of $f$ = range of $f^{-1} = [1, \infty)$.
- The domain of $f^{-1}$ = range of $f = [0, \infty)$.

**Is it possible for a function to have more than one inverse?**

*No. If two supposedly different functions, say, $g$ and $h$, both meet the definition of being inverses of another function $f$, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.*

## Note: Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

## How To: Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

## Example 3.6.4: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 3.6.2. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

*Table* 3.6.2

| Constant | Identity | Quadratic | Cubic | Reciprocal |
|---|---|---|---|---|
| $f(x)=c$ | $f(x)=x$ | $f(x)=x^2$ | $f(x)=x^3$ | $f(x)=\frac{1}{x}$ |
| Reciprocal squared | Cube Root | Square Root | Absolute Value | |
| $f(x)=\frac{1}{x^2}$ | $f(x)=\sqrt[3]{x}$ | $f(x)=\sqrt{x}$ | $f(x)=\lvert x\rvert$ | |

**Solution**

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0,\infty)$,where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0,\infty)$.

**Analysis**

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 3.6.4. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.

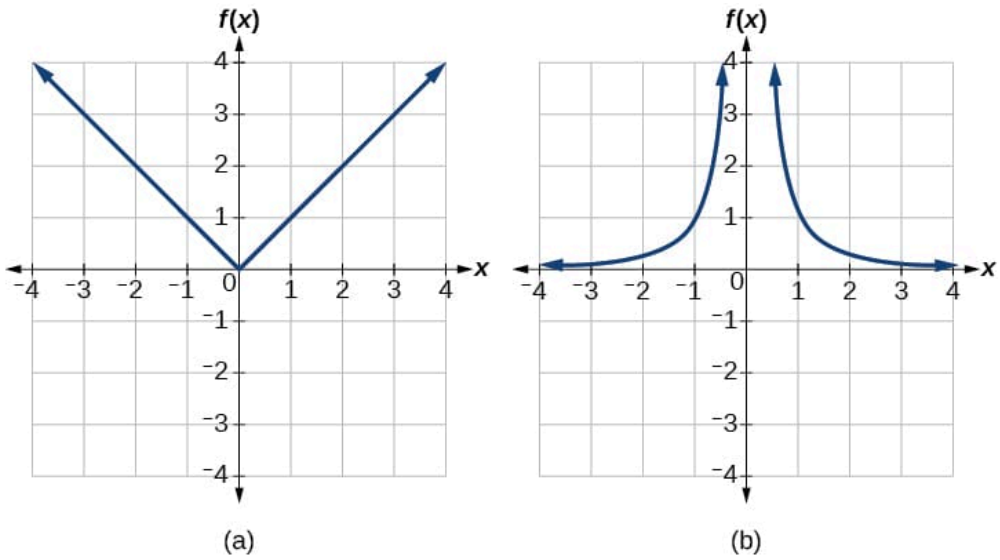



***Figure*** 3.6.4***:*** *(a) Absolute value (b) Reciprocal squared*



3.6.4**:** The domain of function $f$ is $(1, \infty)$ and the range of function $f$ is $(-\infty, -2)$. Find the domain and range of the inverse function.

**Solution**

The domain of function $f^{-1}$ is $(-\infty, -2)$ and the range of function $f^{-1}$ is $(1, \infty)$.

# Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

## Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

### Example 3.6.5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table 3.6.3, showing distance in miles that a car has traveled in $t$ minutes. Find and interpret $f^{-1}(70)$

*Table* 3.6.3

| $t$ **(minutes)** | 30 | 50 | 70 | 90 |
|---|---|---|---|---|
| $f(t)$ **(miles)** | 20 | 40 | 60 | 70 |

The inverse function takes an output of $f$ and returns an input for $f$. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function $f$, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value $a$ so that $f(a) = 70$. In this case, we are looking for a $t$ so that $f(t) = 70$, which is when $t = 90$.

### Exercise 3.6.5

Using Table 3.6.4, find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

*Table* 3.6.4

| $t$ **(minutes)** | 30 | 50 | 60 | 70 | 90 |
|---|---|---|---|---|---|
| $f(t)$ **(miles)** | 20 | 40 | 50 | 60 | 70 |

*Answer available at the end of this section.*

## Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in **Functions and Function Notation** that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.




**Given the graph of a function, evaluate its inverse at specific points.**

1. Find the desired input on the y-axis of the given graph.
2. Read the inverse function's output from the x-axis of the given graph.

### Example 3.6.6: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 3.6.5. Find $g(3)$ and $g^{-1}(3)$.

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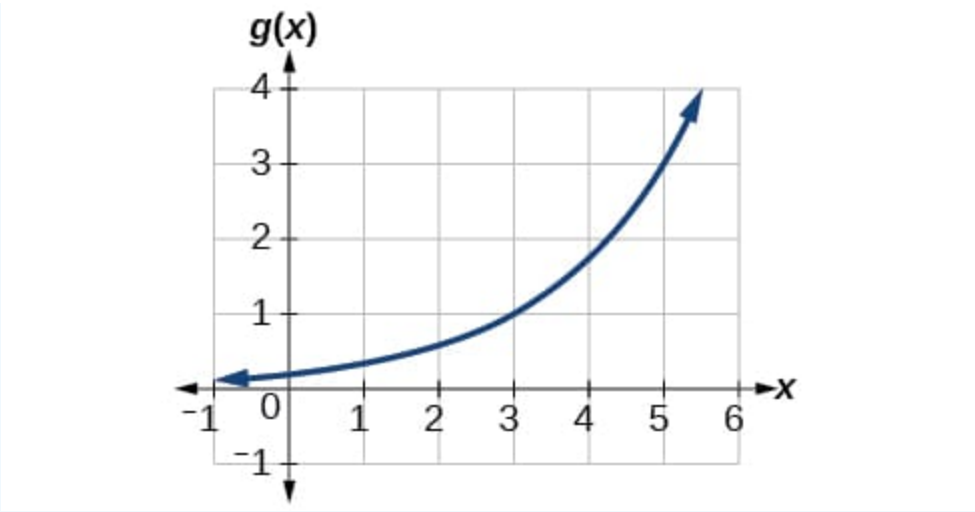



Figure 3.6.5: Graph of $g(x)$

**Solution**

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of $x$ for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 3.6.6.

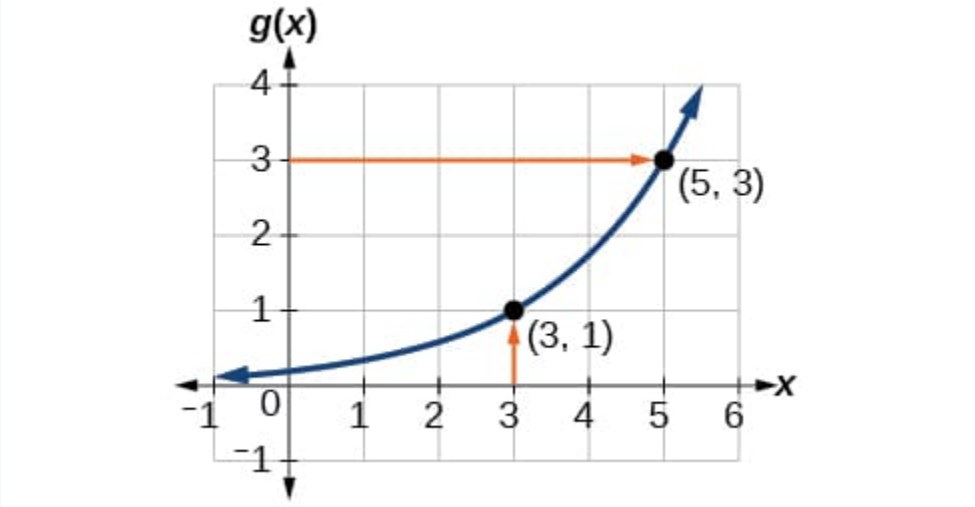



Figure 3.6.6: Graph of $g(x)$.

**Exercise 3.6.6**

Using the graph in Figure 3.6.6, (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

*Answer a available at the end of this section.*

## Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula— for example, $y$ as a function of $x$— we can often find the inverse function by solving to obtain $x$ as a function of $y$.

### How To: Given a function represented by a formula, find the inverse.

1. Make sure $f$ is a one-to-one function.
2. Solve for $x$
3. Interchange $x$ and $y$.

### Example 3.6.7: Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C = \frac{5}{9}(F - 32) \tag{3.6.19}$$

**Solution**

$$C = \frac{5}{9}(F - 32) \tag{3.6.20}$$

$$C \cdot \frac{9}{5} = F - 32 \tag{3.6.21}$$

$$F = \frac{9}{5}C + 32 \tag{3.6.22}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32) \tag{3.6.23}$$

,

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32. \tag{3.6.24}$$

In this case, we introduced a function $h$ to represent the conversion because the input and output variables are descriptive, and writing $C^{-1}$ could get confusing.

### Exercise 3.6.7

Solve for $x$ in terms of $y$ given $y = \frac{1}{3}(x - 5)$

*Answer available at the end of this section.*

### Example 3.6.8: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

**Solution**

$$y = \frac{2}{x - 3 + 4} \qquad \text{Set up an equation.} \tag{3.6.25}$$

$$y - 4 = \frac{2}{x - 3} \qquad \text{Subtract 4 from both sides.} \tag{3.6.26}$$

$$x - 3 = \frac{2}{y - 4} \qquad \text{Multiply both sides by x−3 and divide by y−4.} \tag{3.6.27}$$

$$x = \frac{2}{y - 4} + 3 \qquad \text{Add 3 to both sides.} \tag{3.6.28}$$

So $f^{-1}(y) = \frac{2}{y-4} + 3$ or $f^{-1}(x) = \frac{2}{x-4} + 3$.

**Analysis**

The domain and range of $f$ exclude the values 3 and 4, respectively. $f$ and $f^{-1}$ are equal at two points but are not the same function, as we can see by creating Table $3.6.5$.

*Table* $3.6.5$

| $x$ | 1 | 2 | 5 | $f^{-1}(y)$ |
|---|---|---|---|---|
| $f(x)$ | 3 | 2 | 5 | $y$ |

## Example $3.6.9$: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x-4}$.

**Solution**

$$y = 2 + \sqrt{x-4} \tag{3.6.29}$$
$$(y-2)^2 = x - 4 \tag{3.6.30}$$
$$x = (y-2)^2 + 4 \tag{3.6.31}$$

So $f^{-1}(x) = (x-2)^2 + 4$.

The domain of $f$ is $[4, \infty)$. Notice that the range of $f$ is $[2, \infty)$, so this means that the domain of the inverse function $f^{-1}$ is also $[2, \infty)$

**Analysis**

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real $x$. However, $f^{-1}$ itself must have an inverse (namely, $f$ ) so we have to restrict the domain of $f^{-1}$ to $[2, \infty)$ in order to make $f^{-1}$ a one-to-one function. This domain of $f^{-1}$ is exactly the range of $f$.

## Exercise $3.6.8$

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

*Answer available at the end of this section.*

# Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in Figure $3.6.7$.

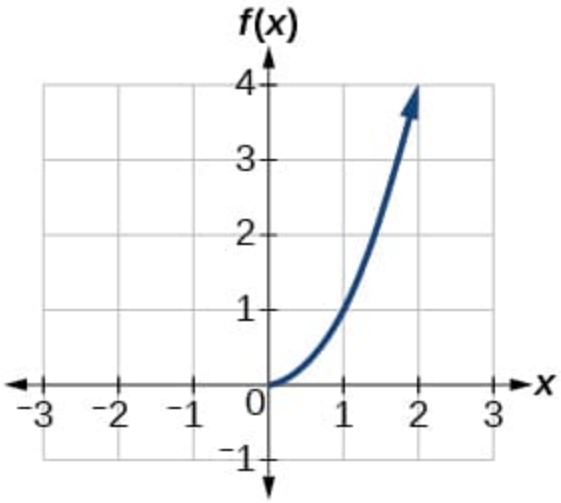



***Figure*** $3.6.7$***:*** *Quadratic function with domain restricted to* $[0, \infty)$.

**Restricting the domain** to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both $f$ and $f^{-1}$ on the same set of axes, using the x-axis for the input to both $f$ and $f^{-1}$?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure $3.6.8$**.**

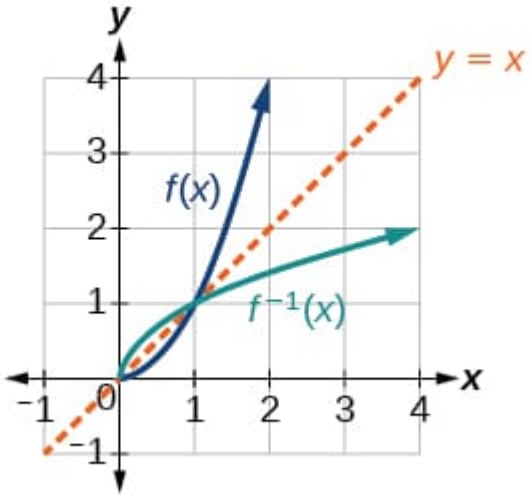



***Figure*** $3.6.8$***:*** *Square and square-root functions on the non-negative domain*

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

## Example $3.6.10$: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure $3.6.9$, sketch a graph of $f^{-1}(x)$.

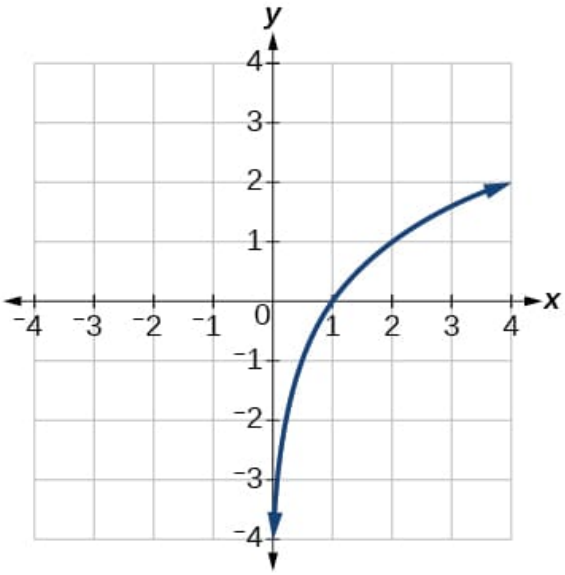



Figure 3.6.9: Graph of $f^{(}-1)(x)$.

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 3.6.10.

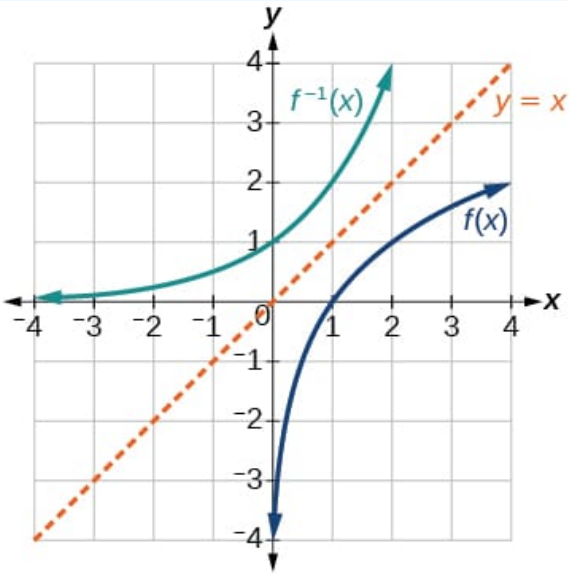



Figure 3.6.10: The function and its inverse, showing reflection about the identity line

## Exercise 3.6.1

Draw graphs of the functions $f$ and $f^{-1}$ from Example 3.6.8.

*Answer available at the end of this section.*

**Is there any function that is equal to its own inverse?**

*Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because*

$$\frac{1}{\frac{1}{x}} = x \tag{3.6.32}$$

*Any function $f(x) = c - x$, where $c$ is a constant, is also equal to its own inverse.*

# Key Concepts

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x)) = f(g(x)) = x$.
- Each of the toolkit functions has an inverse.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse.
- The inverse of a function can be determined at specific points on its graph.
- To find the inverse of a formula, solve the equation $y = f(x)$ for $x$ as a function of $y$. Then exchange the labels $x$ and $y$.
- The graph of an inverse function is the reflection of the graph of the original function across the line $y = x$.

## Answers

**Exercise** $3.6.11A$

a. yes, because each bank account has a single balance at any given time;

b. no, because several bank account numbers may have the same balance;

c. no, because the same output may correspond to more than one input.

**Exercise** $3.6.11B$

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

**Exercise** $3.6.13$

No, because it does not pass the horizontal line test.

**Exercise** $3.6.1$

$f(2) = 6$

**Exercise** $3.6.2$

Yes

**Exercise** $3.6.3$

Yes

**Exercise** $3.6.5$

$f(60) = 50$. In 60 minutes, 50 miles are traveled.
$f^{-1}(60) = 70$. To travel 60 miles, it will take 70 minutes.

**Exercise** $3.6.6$

3

5.6

**Exercise** $3.6.7$

$x = 3y + 5$

**Exercise** $3.6.8$

$f^{-1}(x) = (2 - x)^2$; domain of $f$: $[0, \infty)$; domain of $f^{-1}$: $(-\infty, 2]$

**Exercise** $3.6.1$

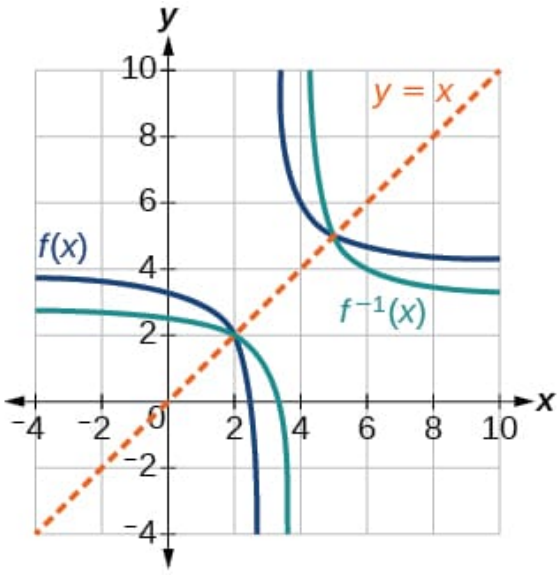



Figure 3.6.11: Graph of $f(x)$ and $f^(-1)(x)$.

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# 4: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.


4.1: Exponential Functions and Graphs

4.2: Logarithmic Functions and Graphs

4.3: Properties of Logarithms

4.4: Exponential and Logarithmic Equations

4.5: Applications of Exponential Functions


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## Contributors

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# 4.1: Exponential Functions and Graphs

**Learning Objectives**

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Graph exponential functions.
- Use compound interest formulas.
- Evaluate exponential functions with base $e$.

The population of India surpassed China's population in the year 2021. India is now the most populous country in the world with a population of about $1.45$ billion people in 2024. The population has been growing at a rate of about $0.89\%$ per year. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.

**Identifying Exponential Functions**

When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x) = 3x + 4$, the slope tells us the output increases by $3$ each time the input increases by $1$. The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

## Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, $2.5\%$ of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continues, vegans will make up $10\%$ of the U.S. population in 2015, $40\%$ in 2019, and $80\%$ in 2050.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of **exponential growth**, let us contrast exponential growth with **linear growth**. We will construct two functions. The first function is exponential. We will start with an input of $0$, and increase each input by $1$. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of $0$, and increase each input by $1$. We will add $2$ to the corresponding consecutive outputs (Table $4.1.1$).

From Table $4.1.1$ we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increases by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

*Table* 4.1.1

| $x$ | $f(x) = 2^x$ | $g(x) = 2x$ |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 2 | 2 |
| 2 | 4 | 4 |
| 3 | 8 | 6 |
| 4 | 16 | 8 |
| 5 | 32 | 10 |
| 6 | 64 | 12 |

Apparently, the difference between “the same percentage” and “the same amount” is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding $2$ to the output whenever the input was increased by one.

A basic **exponential function** is $f(x) = a^x$, where $a$ is a positive real number not equal to $1$.

- If $a > 1$,the function grows at a rate proportional to its size.
- If $0 < a < 1$, the function decays at a rate proportional to its size.

Let’s look at the function $f(x) = 2^x$ from our example. We will create a table (Table $4.1.2$) to determine the corresponding outputs over an interval in the domain from $-3$ to $3$.

*Table* 4.1.2

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = 2^x$ | $2^{-3} = \frac{1}{8}$ | $2^{-2} = \frac{1}{4}$ | $2^{-1} = \frac{1}{2}$ | $2^0 = 1$ | $2^1 = 2$ | $2^2 = 4$ | $2^3 = 8$ |

Let us examine the graph of $f$ by plotting the ordered pairs from Table $4.1.2$ and then make a few observations $4.1.1$.

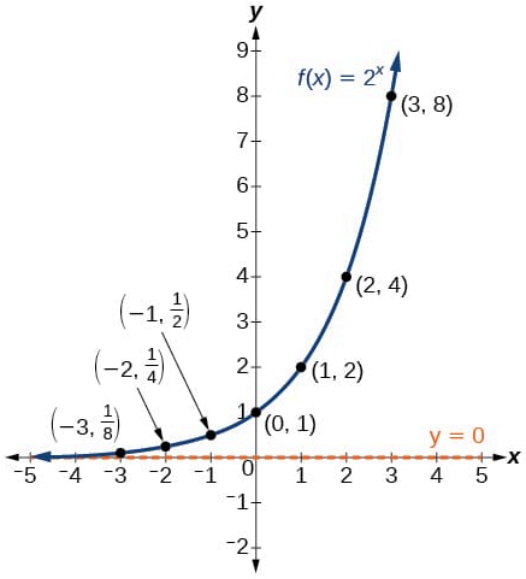



Figure 4.1.1

Let’s define the behavior of the graph of the exponential function $f(x) = 2^x$ and highlight some its key characteristics.

- the domain is $(-\infty, \infty)$,

- the range is $(0, \infty)$,
- as $x \to \infty$, $f(x) \to \infty$,
- as $x \to -\infty$, $f(x) \to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the *x*-axis because base two raised to any exponent never has the result of zero.
- $y = 0$ is the horizontal asymptote.
- the *y*-intercept is $1$.

## Definition: Exponential Functions (General Form)

For any real number $x$, an **exponential function** in general form is a function with the form

$$f(x) = Ca^x \tag{4.1.1}$$

where

- $C$ is a non-zero real number called the initial value and
- $a$ is any positive real number such that $a \neq 1$.
- The domain of $f$ is all real numbers.
- The range of $f$ is all positive real numbers if $C > 0$.
- The range of $f$ is all negative real numbers if $C < 0$.
- The *y*-intercept is $(0, C)$,and the horizontal asymptote is $y = 0$.

*Note: Since the value $a$ is called the "base", we sometimes use the letter $b$ instead of $a$.*

## Example 4.1.1: Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

**Solution**

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base $b$ of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, $-2$, is less than $0$.

## Exercise 4.1.1

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

*Answer available at the end of this section.*

# Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than $1$.Why do we limit the base bb to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

- Let $b=-9$ and $x=\frac{1}{2}$. Then $f(x)=f\left(\frac{1}{2}\right)=(-9)^{\frac{1}{2}}=\sqrt{-9}$,which is not a real number.

Why do we limit the base to positive values other than $1$? Because base $1$ results in the constant function. Observe what happens if the base is $1$:

- Let $b=1$. Then $f(x)=1^x=1$ for any value of $x$.

To evaluate an exponential function with the form $f(x)=b^x$,we simply substitute $x$ with the given value, and calculate the resulting power. For example:

Let $f(x)=2^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 2^x \\ f(3) &= 2^3 && \text{Substitute } x=3 \\ &= 8 && \text{Evaluate the power} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x)=30(2)^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 30(2)^x \\ f(3) &= 30(2)^3 && \text{Substitute } x=3 \\ &= 30(8) && \text{Simplify the power first} \\ &= 240 && \text{Multiply} \end{aligned}$$

Note that if the order of operations were not followed, the result would be incorrect:

$$f(3)=30(2)^3 \neq 60^3=216,000$$

## Example 4.1.2: Evaluating Exponential Functions

Let $f(x)=5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

**Solution**

Follow the order of operations. Be sure to pay attention to the parentheses.

$$\begin{aligned} f(x) &= 5(3)^{x+1} \\ f(2) &= 5(3)^{2+1} && \text{Substitute } x=2 \\ &= 5(3)^3 && \text{Add the exponents} \\ &= 5(27) && \text{Simplify the power} \\ &= 135 && \text{Multiply} \end{aligned}$$

## Exercise 4.1.2

Let $f(x)=8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

*Answer available at the end of this section.*

# Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term “exponential growth” is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

## Definition: Exponential Growth

A function that models exponential growth by a rate proportional to the amount present. For any real number $x$ and any positive real numbers $C$ and $b$ such that $b > 1$,an exponential growth function has the form

$$f(x) = Cb^x \tag{4.1.2}$$

where

- $C$ is the initial or starting value of the function.
- $b$ is the growth factor or growth multiplier per unit $x$.

In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent. To differentiate between linear and exponential functions, let’s consider two companies, A and B. Company A has $100$ stores and expands by opening $50$ new stores a year, so its growth can be represented by the function $A(x) = 100 + 50x$. Company B has $100$ stores and expands by increasing the number of stores by $50\%$ each year, so its growth can be represented by the function $B(x) = 100(1 + 0.5)^x$.

A few years of growth for these companies are illustrated in Table $4.1.3$.

*Table* $4.1.3$

| Year, $x$ | Stores, Company A | Stores, Company B |
|---|---|---|
| $0$ | $100 + 50(0) = 100$ | $100(1 + 0.5)^0 = 100$ |
| $1$ | $100 + 50(1) = 150$ | $100(1 + 0.5)^1 = 150$ |
| $2$ | $100 + 50(2) = 200$ | $100(1 + 0.5)^2 = 225$ |
| $3$ | $100 + 50(3) = 250$ | $100(1 + 0.5)^3 = 337.5$ |
| $x$ | $A(x) = 100 + 50x$ | $B(x) = 100(1 + 0.5)^x$ |

The graphs comparing the number of stores for each company over a five-year period are shown in Figure $4.1.2$. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.

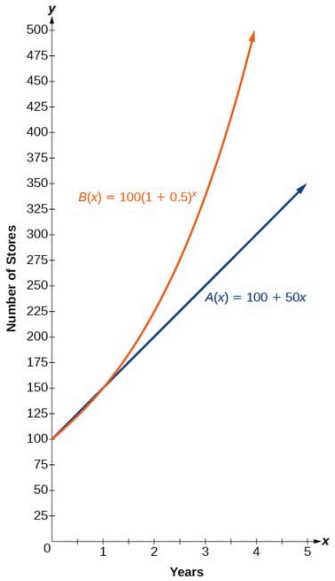



Figure 4.1.2: The graph shows the numbers of stores Companies A and B opened over a five-year period.

Notice that the domain for both functions is $[0, \infty)$,and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Now we will turn our attention to the function representing the number of stores for Company $B$, $B(x) = 100(1 + 0.5)^x$. In this exponential function, $100$ represents the initial number of stores, $0.50$ represents the growth rate, and $1 + 0.5 = 1.5$ represents the growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$,where $100$ is the initial value, $1.5$ is called the *base*, and $x$ is called the *exponent*.

## Example 4.1.3: Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about $1.25$ billion in the year 2013, with an annual growth rate of about $1.2\%$. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where $t$ is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

**Solution**

To estimate the population in 2031, we evaluate the models for $t = 18$, because 2031 is $18$ years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549$$

There will be about $1.549$ billion people in India in the year 2031.

## Exercise 4.1.3

The population of China was about $1.39$ billion in the year 2013, with an annual growth rate of about $0.6\%$. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where $t$ is the number of years since 2013.To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in Example 4.1.3?

*Answer available at the end of this section.*

# Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants $a, a$ and $b, b$,and evaluate the function.

## How to: Given two data points, write an exponential model

1. If one of the data points has the form $(0, C)$, then $C$ is the initial value. Using $C$, substitute the second point into the equation $f(x) = C(a)^x$, and solve for $a$.
2. If neither of the data points have the form $(0, C)$, substitute both points into two equations with the form $f(x) = C(a)^x$. Solve the resulting system of two equations in two unknowns to find $C$ and $a$.
3. Using the $C$ and $a$ found in the steps above, write the exponential function in the form $f(x) = C(a)^x$.

## Example 4.1.4: Writing an Exponential Model When the Initial Value Is Known

In 2006, $80$ deer were introduced into a wildlife refuge. By 2012, the population had grown to $180$ deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population $(N)$ of deer over time $t$.

**Solution**

We let our independent variable $t$ be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $C = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find $b$:

$$\begin{aligned} N(t) &= 80b^t \\ 180 &= 80b^6 \qquad \text{Substitute using point } (6, 180) \\ \frac{9}{4} &= b^6 \qquad \text{Divide and write in lowest terms} \\ b &= \left(\frac{9}{4}\right)^{\frac{1}{6}} \qquad \text{Isolate b using properties of exponents} \\ b &\approx 1.1447 \qquad \text{Round to 4 decimal places} \end{aligned}$$

> Unless otherwise stated, do not round any intermediate calculations. Then round the final answer to four places for the remainder of this section.

The exponential model for the population of deer is $N(t) = 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in Figure $4.1.3$ passes through the initial points given in the problem, $(0, 80)$ and $(6, 180)$. We can also see that the domain for the function is $[0, \infty)$,and the range for the function is $[80, \infty)$.

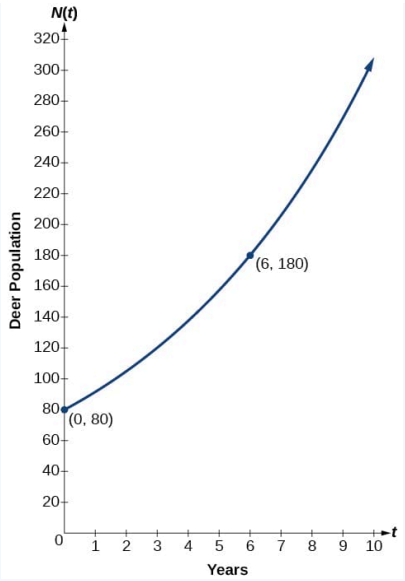



Figure 4.1.3: Graph showing the population of deer over time, $N(t) = 80(1.1447)^t$, $t$ years after 2006

## Exercise 4.1.4

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population $N$ of wolves over time $t$.

*Answer available at the end of this section.*

## Example 4.1.5: Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

**Solution**

Because we don't have the initial value, we substitute both points into an equation of the form $f(x) = ab^x$, and then solve the system for $a$ and $b$.

- Substituting $(-2, 6)$ gives $6 = Ca^{-2}$
- Substituting $(2, 1)$ gives $1 = Ca^2$

Use the first equation to solve for $a$ in terms of $b$:

$$\begin{aligned} 6 &= Ca^{-2} \\ \frac{6}{a^{-2}} &= C \quad \text{Divide} \\ C &= 6a^2 \quad \text{Use properties of exponents to rewrite the denominator} \end{aligned}$$

Substitute a in the second equation, and solve for $b$:

$$\begin{aligned} 1 &= Ca^2 \\ 1 &= 6a^2 \\ &= 6a^4 \quad \text{Substitute C} \\ a &= \left(\frac{1}{6}\right)^{\frac{1}{4}} \quad \text{Round 4 decimal places rewrite the denominator} \\ a &\approx 0.6389 \end{aligned}$$

Use the value of $a$ in the first equation to solve for the value of $C$:

$$\begin{aligned} C &= 6a^2 \\ &\approx 6(0.6389)^2 \\ &\approx 2.4492 \end{aligned}$$

Thus, the equation is $f(x) = 2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in Figure $4.1.4$ passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

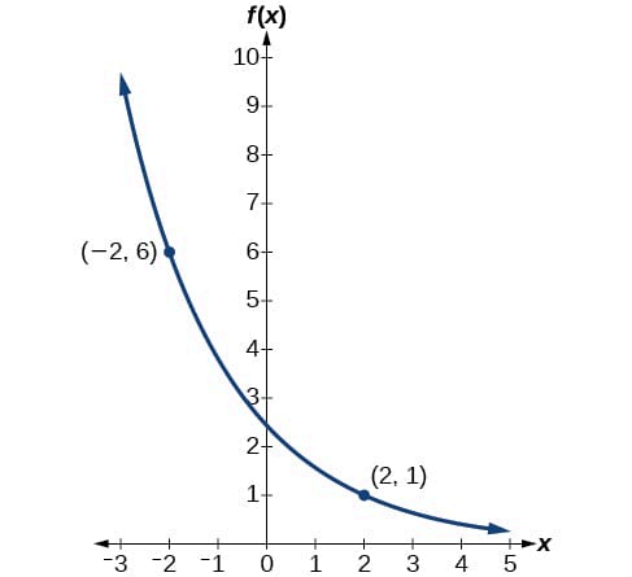



Figure $4.1.4$: The graph of $f(x) = 2.4492(0.6389)^x$ models exponential decay.

### Definition: Exponential Decay

A function that models exponential decay by a rate proportional to the amount present. For any real number $x$ and any positive real numbers $C$ and $b$ such that $0 < a < 1$, an exponential decay function has the form

$$f(x) = Ca^x \tag{4.1.3}$$

where

- $C$ is the initial or starting value of the function.
- $a$ is the decay factor per unit $x$

### Exercise $4.1.5$

Given the two points $(1, 3)$ and $(2, 4.5)$,find the equation of the exponential function that passes through these two points.

*Answer available at the end of this section.*

## Q&A: Do two points always determine a unique exponential function?

*Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in $x$, which in many real world cases involves time.*

## How to: Given the graph of an exponential function, write its equation

1. First, identify two points on the graph. Choose the $y$-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the $y$-intercept $(0, C)$, then $C$ is the initial value. Using $C$, substitute the second point into the equation $f(x) = C(a)^x$, and solve for $a$
3. If neither of the data points have the form $(0, C)$, substitute both points into two equations with the form $f(x) = C(a)^x$. Solve the resulting system of two equations in two unknowns to find $C$ and $a$.
4. Write the exponential function, $f(x) = C(a)^x$.

## Example 4.1.6: Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in Figure 4.1.5.

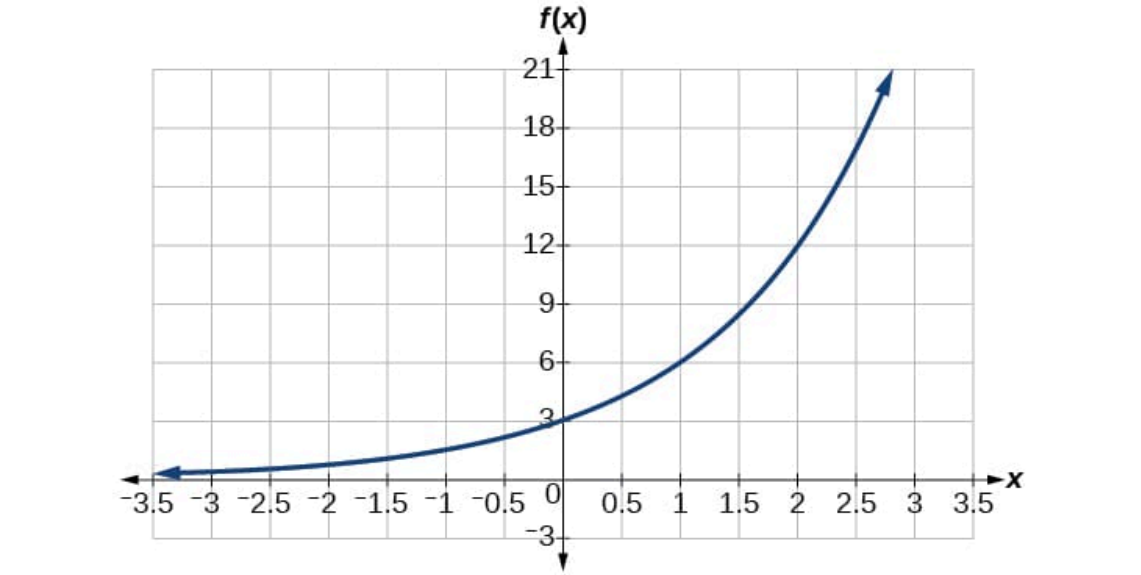



Figure 4.1.5

**Solution**

We can choose the $y$-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

$$
\begin{aligned}
y &= Ca^x && \text{Write the general form of an exponential equation} \\
y &= 3a^x && \text{Substitute the initial value 3 for } a \\
12 &= 3a^2 && \text{Substitute in 12 for } y \text{ and 2 for } x \\
4 &= a^2 && \text{Divide by 3} \\
b &= \pm 2 && \text{Take the square root}
\end{aligned}
$$

Because we restrict ourselves to positive values of $a$, we will use $a = 2$. Substitute $C$ and $a$ into the standard form to yield the equation $f(x) = 3(2)^x$.

**Exercise 4.1.6**

Find an equation for the exponential function graphed in Figure 4.1.6.

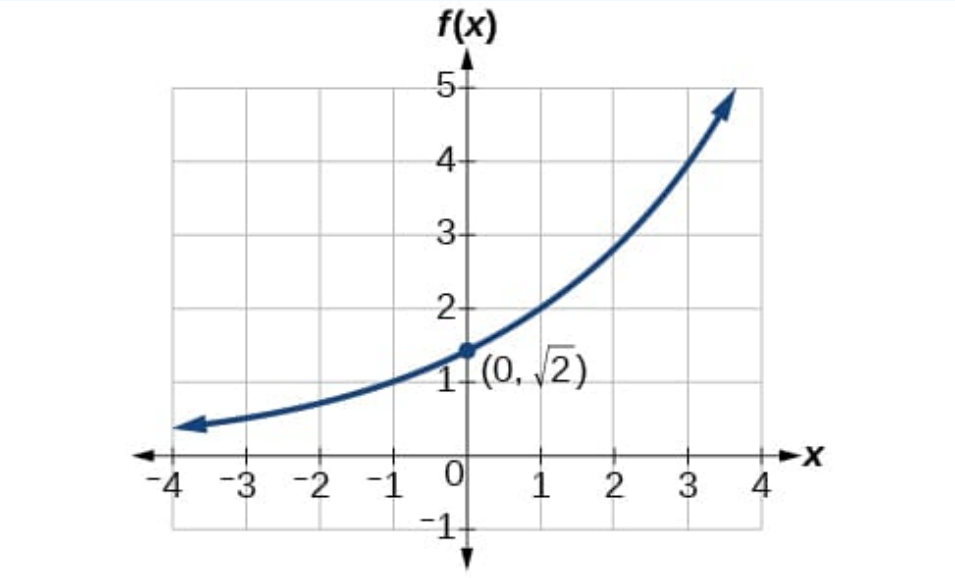



Figure 4.1.6

*Answer available at the end of this section.*

## Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table 4.1.1 change as the input increases by 1.

*Table* 4.1.1

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = 2^x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $2$ | $4$ | $8$ |

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f(x) = b^x$, $b$ is the constant ratio of the function. This means that as the input increases by 1, the output value will increase by a multiple of the base.

Notice from the table that

- the output values are positive for all values of $x$;
- as $x$ increases, the output values increase without bound; and
- as $x$ decreases, the output values grow smaller, approaching zero.

Figure 4.1.1 shows the exponential growth function $f(x) = 2^x$.

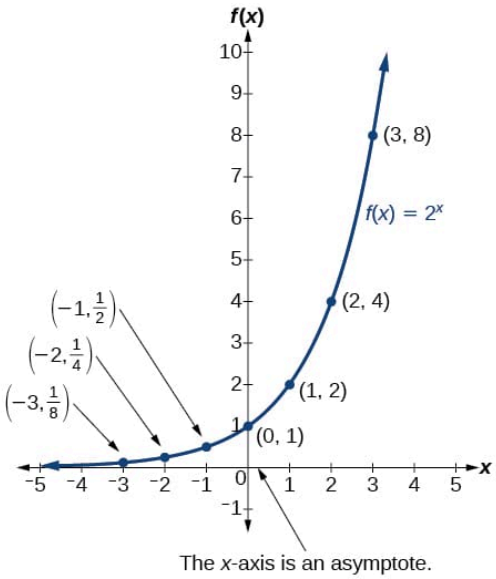



Figure $4.1.1$: Notice that the graph gets close to the x-axis, but never touches it.

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table $4.1.2$ change as the input increases by $1$.

*Table* $4.1.2$

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $g(x) = \left(\frac{1}{2}\right)^x$ | $8$ | $4$ | $2$ | $1$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |

Again, because the input is increasing by $1$, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of $x$;
- as $x$ increases, the output values grow smaller, approaching zero; and
- as $x$ decreases, the output values grow without bound.

Figure $4.1.2$ shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

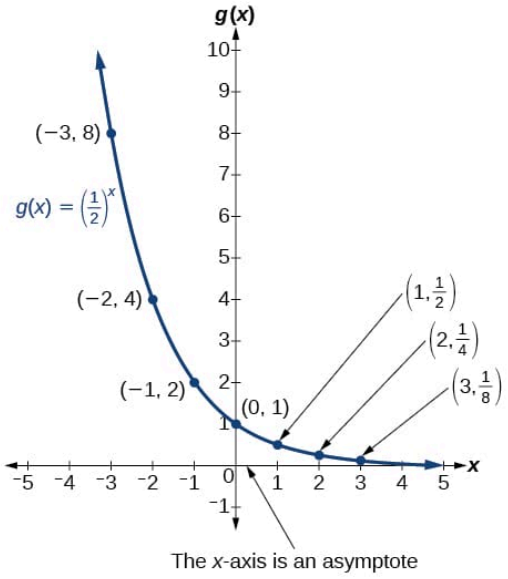



Figure $4.1.2$

The domain of $g(x) = (\frac{1}{2})^{x}$ is all real numbers, the range is $(0, \infty)$,and the horizontal asymptote is $y = 0$.

## CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION $f(X) = b^x$

An exponential function with the form $f(x) = b^x, b > 0, b \neq 1$,has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- *x*-intercept: none
- *y*-intercept: $(0, 1)$
- increasing if $b > 1$
- decreasing if $b < 1$

Figure $4.1.3$ compares the graphs of exponential growth and decay functions.

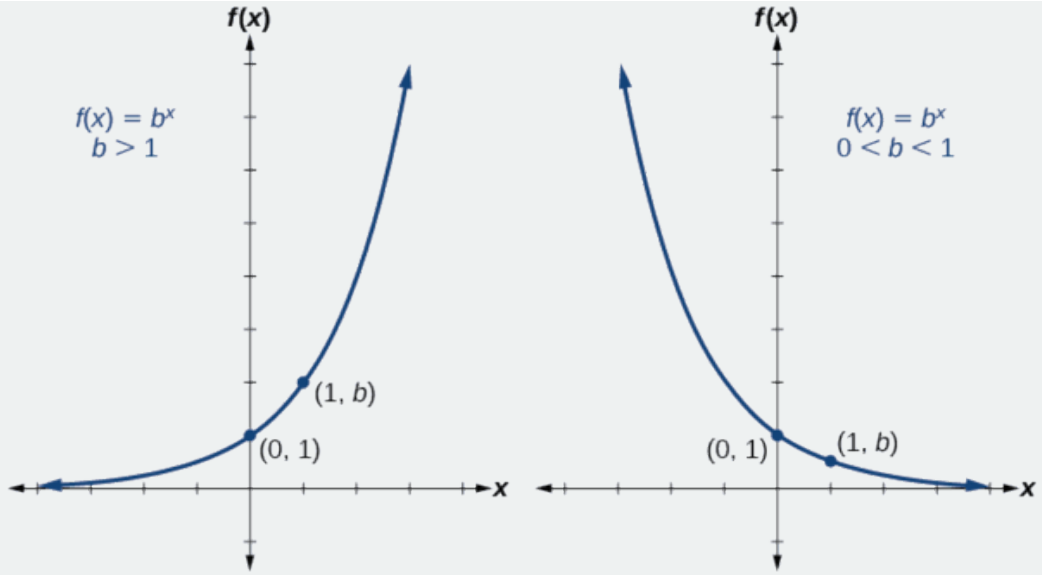



Figure 4.1.3

## How to: Given an exponential function of the form $f(x) = b^x$,graph the function

1. Create a table of points.
2. Plot at least $3$ point from the table, including the *y*-intercept $(0, 1)$.
3. Draw a smooth curve through the points.
4. State the domain, $(-\infty, \infty)$,the range, $(0, \infty)$,and the horizontal asymptote, $y = 0$.

## Example 4.1.7: Sketching the Graph of an Exponential Function of the Form $f(x) = b^x$

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

**Solution**

Before graphing, identify the behavior and create a table of points for the graph.

*Table* 4.1.3

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = 0.25^x$ | $64$ | $16$ | $4$ | $1$ | $0.25$ | \(0.0625\0 | $0.015625$ |

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in Table 4.1.3.
- Plot the *y*-intercept, $(0, 1)$,along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in Figure 4.1.4.

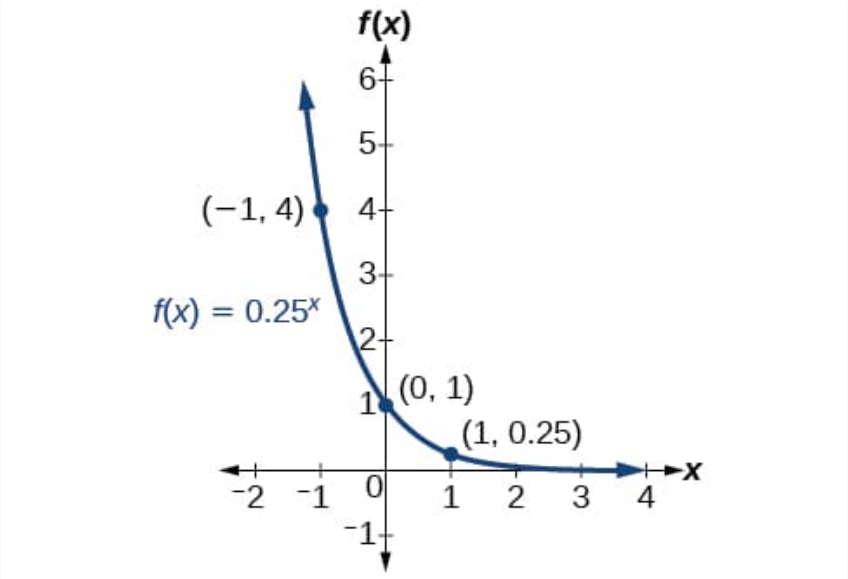



Figure 4.1.4

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

**Exercise 4.1.7**

Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

### Graphing a Vertical Shift

The first transformation occurs when we add a constant $d$ to the parent function $f(x) = b^x$, giving us a vertical shiftd d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in Figure 4.1.5.

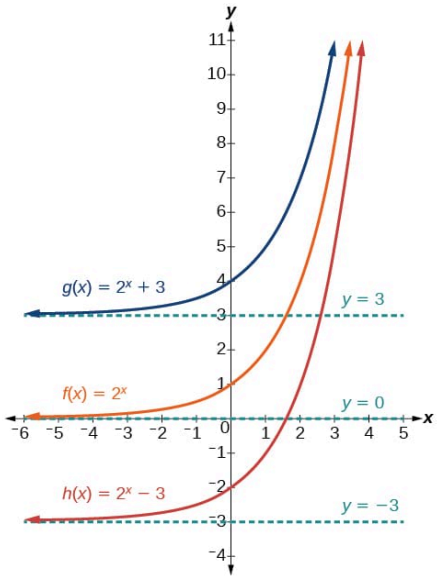



Figure 4.1.5

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up $3$ units to $g(x) = 2^x + 3$:
  - The *y*-intercept shifts up $3$ units to $(0, 4)$.
  - The asymptote shifts up $3$ units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down $3$ units to $h(x) = 2^x - 3$:
  - The *y*-intercept shifts down $3$ units to $(0, -2)$.
  - The asymptote also shifts down $3$ units to $y = -3$.
  - The range becomes $(-3, \infty)$.

## Graphing a Horizontal Shift

The next transformation occurs when we add a constant $c$ to the input of the parent function $f(x) = b^x$, giving us a horizontal shift $c$ units in the *opposite* direction of the sign. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. $h(x) = 2^{x-3}$. Both horizontal shifts are shown in Figure 4.1.6.

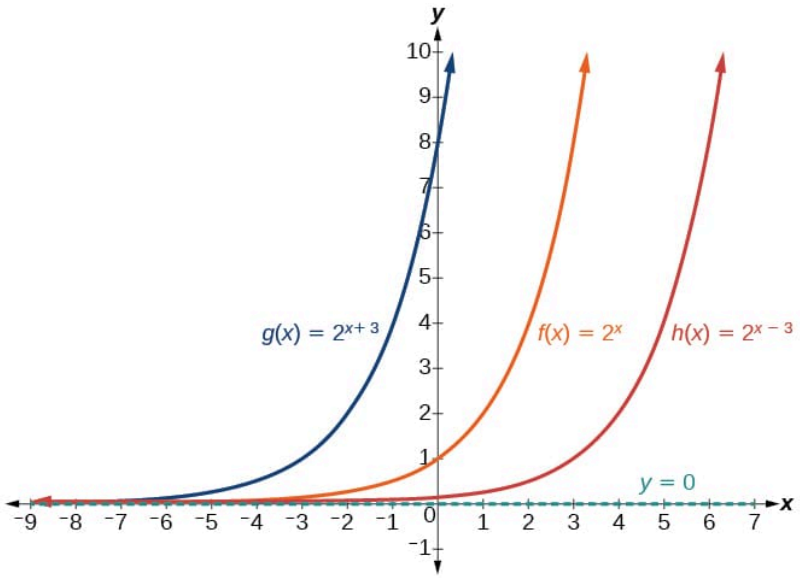



Figure 4.1.6

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$,remains unchanged.
- The asymptote, $y = 0$,remains unchanged.
- The *y*-intercept shifts such that:
  - When the function is shifted left $3$ units to $g(x) = 2^{x+3}$,the *y*-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)2^x$,so the initial value of the function is $8$.
  - When the function is shifted right $3$ units to $h(x) = 2^{x-3}$,the *y*-intercept becomes $(0, \frac{1}{8})$. Again, see that $2^{x-3} = (\frac{1}{8})2^x$,so the initial value of the function is $\frac{1}{8}$.

## SHIFTS OF THE PARENT FUNCTION $f(X) = b^x$

For any constants $h$ and $k$,the function $f(x) = b^{x+h} + k$ shifts the parent function $f(x) = b^x$

- vertically $k$ units, in the *same* direction of the sign of $k$.
- horizontally $h$ units, in the *opposite* direction of the sign of $h$.
- The *y*-intercept becomes $(0, b^h + k)$.
- The horizontal asymptote becomes $y = k$.
- The range becomes $(k, \infty)$.
- The domain, $(-\infty, \infty)$,remains unchanged.

## How to: Given an exponential function with the form $f(x) = b^{x+c} + d$, graph the translation

1. Draw the horizontal asymptote $y = k$.
2. Identify the shift as $(-h, k)$. Shift the graph of $f(x) = b^x$ left $h$ units if $h$ is positive, and right $h$ units if $h$ is negative.
3. Shift the graph of $f(x) = b^x$ up $k$ units if $k$ is positive, and down $k$ units if $k$ is negative.
4. State the domain, $(-\infty, \infty)$,the range, $(k, \infty)$,and the horizontal asymptote $y = k$.

## Example 4.1.8: Graphing a Shift of an Exponential Function

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

**Solution**

We have an exponential equation of the form $f(x) = b^{x+h} + k$, with $b = 2$, $h = 1$, and $k = -3$.

Draw the horizontal asymptote $y = k$, so draw $y = -3$.

Identify the shift as $(-h, k)$, so the shift is $(-1, -3)$.

Shift the graph of $f(x) = b^x$ left $1$ units and down $3$ units.

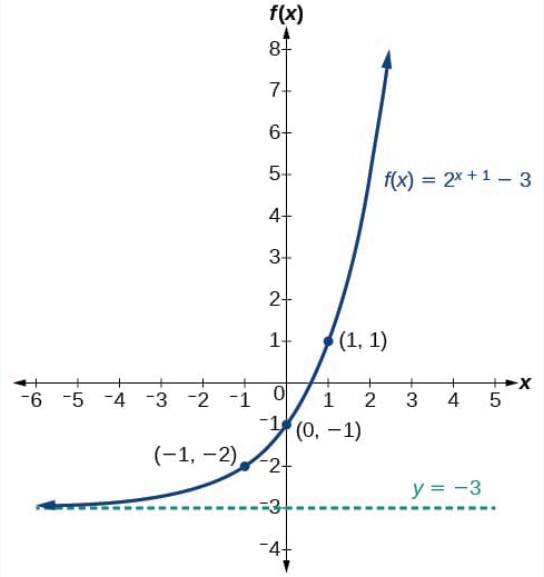



Figure $4.1.7$

The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

## Exercise $4.1.8$

Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

*Answer available at the end of this section.*

## Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a **stretch** or **compression** occurs when we multiply the parent function $f(x) = b^x$ by a constant $|a| > 0$. For example, if we begin by graphing the parent function $f(x) = 2^x$,we can then graph the stretch, using $a = 3$,to get $g(x) = 3(2)^x$ as shown on the left in Figure $4.1.8$, and the compression, using $a = \frac{1}{3}$,to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure $4.1.8$.

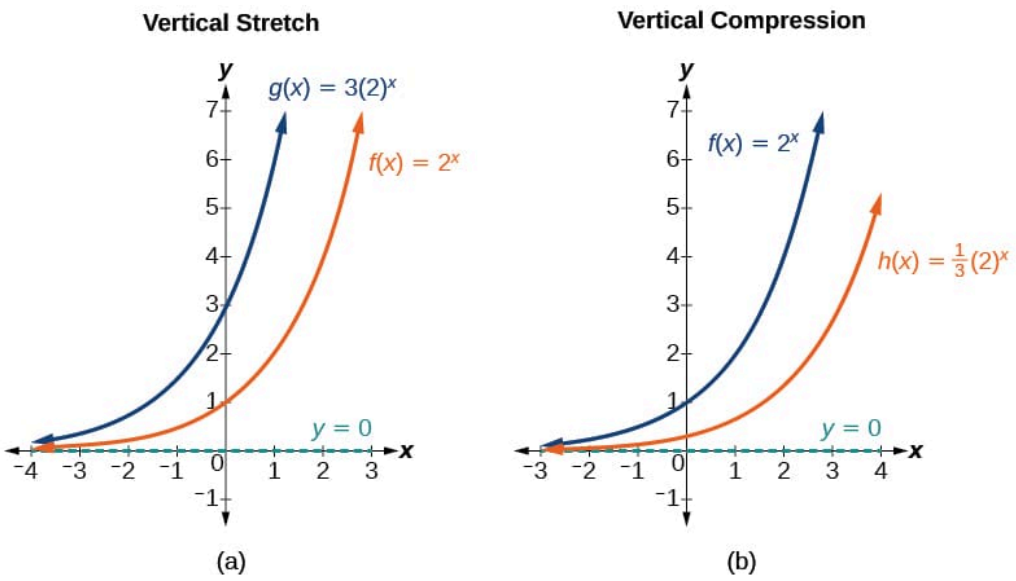



Figure 4.1.8: (a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

## STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $f(x) = b^x$

For any factor $C > 0$,the function $f(x) = C(b)^x$

- is stretched vertically by a factor of $C$ if $|C| > 1$.
- is compressed vertically by a factor of $C$ if $|C| < 1$.
- has a *y*-intercept of $(0, C)$.
- has a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$,which are unchanged from the parent function.

## Example 4.1.9: Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

**Solution**

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as $x$ decreases, and the right tail will approach the *x*-axis as $x$ increases.
- Since $a = 4$,the graph of $f(x) = (\frac{1}{2})^x$ will be stretched by a factor of 4.
- Create a table of points as shown in Table 4.1.4.

*Table* 4.1.4

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x) = 4(\frac{1}{2})^x$ | 32 | 16 | 8 | 4 | 2 | 1 | 0.5 |

- Plot the *y*-intercept, $(0, 4)$,along with two other points. We can use $(-1, 8)$ and $(1, 2)$.

Draw a smooth curve connecting the points, as shown in Figure 4.1.9.

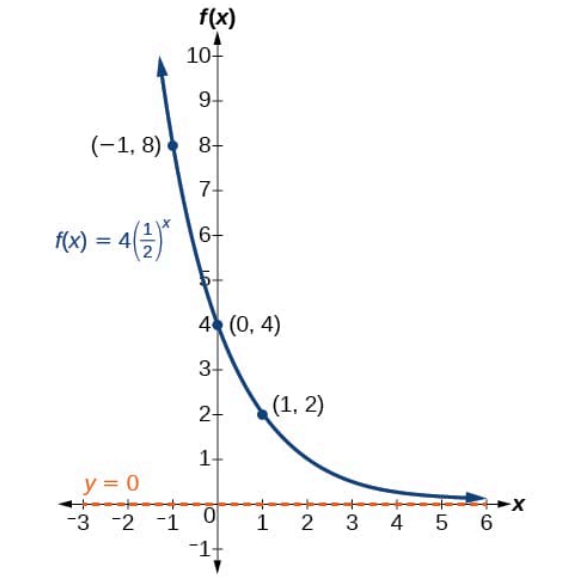



Figure $4.1.9$

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

**Exercise $4.1.9$**

Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the *x*-axis or the *y*-axis. When we multiply the parent function $f(x) = b^x$ by $-1$,we get a reflection about the *x*-axis. When we multiply the input by $-1$,we get a reflection about the *y*-axis. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection about the *x*-axis, $g(x) = -2^x$,is shown on the left side of Figure $4.1.10$, and the reflection about the y-axis $h(x) = 2^{-x}$ , is shown on the right side of Figure $4.1.10$.

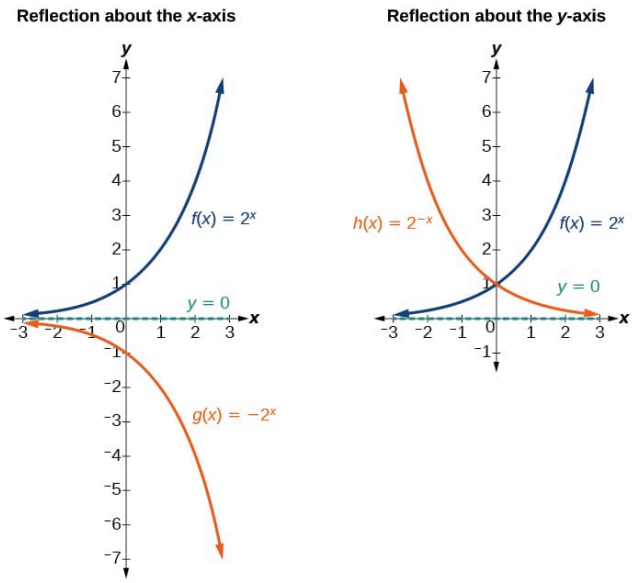



Figure 4.1.10: (a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the y-axis.

## REFLECTIONS OF THE PARENT FUNCTION $f(X) = b^x$

The function $f(x) = -b^x$

- reflects the parent function $f(x) = b^x$ about the *x*-axis.
- has a *y*-intercept of $(0, -1)$.
- has a range of $(-\infty, 0)$
- has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$,which are unchanged from the parent function.

The function $f(x) = b^{-x}$

- reflects the parent function $f(x) = b^x$ about the *y*-axis.
- has a *y*-intercept of $(0, 1)$, a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

## Example 4.1.10: Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$,that reflects $f(x) = (\frac{1}{4})^x$ about the *x*-axis. State its domain, range, and asymptote.

**Solution**

Since we want to reflect the parent function $f(x) = (\frac{1}{4})^x$ about the *x*-axis, we multiply $f(x)$ by $-1$ to get, $g(x) = -(\frac{1}{4})^x$. Next we create a table of points as in Table 4.1.5.

*Table* 4.1.5

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $g(x) = -(\frac{1}{4})^x$ | $-64$ | $-16$ | $-4$ | $-1$ | $-0.25$ | $-0.0625$ | $-0.0156$ |

Plot the *y*-intercept, $(0, -1)$,along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

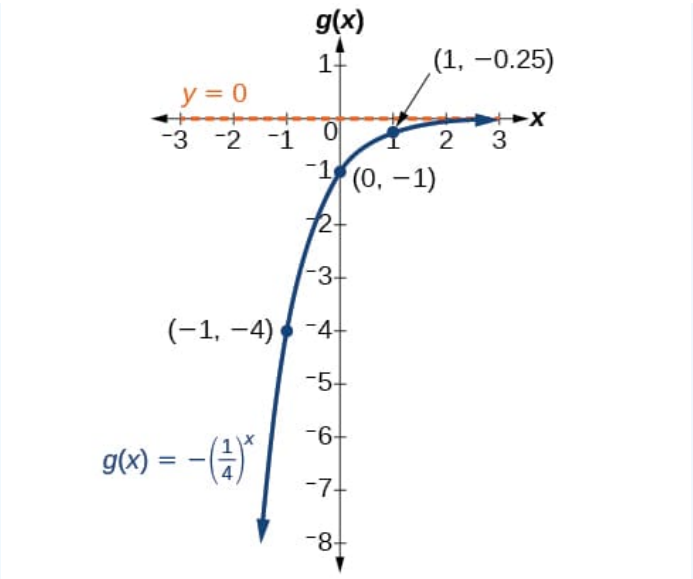



Figure 4.1.11

The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

**Exercise 4.1.10**

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the *y*-axis. State its domain, range, and asymptote.

*Answer available at the end of this section.*

## Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in **Table** 4.1.6 to arrive at the general equation for translating exponential functions.

*Table* 4.1.6*: Translations of the Parent Function* $f(x) = b^x$

| Translation | Form |
|---|---|
| Shift<br>• Horizontally $c$ units to the left<br>• Vertically $d$ units up | $f(x) = b^{x+h} + k$ |
| Stretch and Compress<br>• Stretch if $\|a\| > 1$<br>• Compression if $0 < \|a\| < 1$ | $f(x) = Cb^x$ |
| Reflect about the x-axis | $f(x) = -b^x$ |
| Reflect about the y-axis | $f(x) = b^{-x} = (\frac{1}{b})^x$ |
| General equation for all translations | $f(x) = Cb^{x+h} + k$ |

### TRANSFORMATIONS OF EXPONENTIAL FUNCTIONS

A transformation of an exponential function has the form

$$f(x) = Cb^{x+h} + k$$

Where the parent function, $y = b^x, b \neq 1$,is

- shifted horizontally $h$ units (right if $h < 0$ and left if $h > 0$)
- stretched vertically by a factor of $|C|$ if $|C| > 1$.
- compressed vertically by a factor of $|C|$ if $0 < |C| < 1$.
- shifted vertically $k$ units (down if $k < 0$ and up if $k > 0$)
- reflected about the *x*-axis when $C < 0$.

Note the order of the shifts, transformations, and reflections follow the order of operations.

### Example 4.1.11: Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is vertically stretched by a factor of $2$ , reflected across the *y*-axis, and then shifted up $4$ units.

**Solution**

We want to find an equation of the general form $f(x) = Cb^(x + h) + k$. We use the description provided to find $C$, $b$, (\h\), and $k$.

- We are given the parent function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of 2, so $C = 2$.
- The function is reflected about the *y*-axis. We replace $x$ with $-x$ to get: $e^{-x}$.
- The graph is shifted vertically 4 units, so $k = 4$.

Substituting in the general form we get,

$$f(x) = Cb^{x+h} + k$$

$$= 2e^{-x+0} + 4$$

$$= 2e^{-x} + 4$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

### Exercise 4.1.11

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the *x*-axis and then shifted down $2$ units.

*Answer available at the end of this section.*

## The Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the

account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time $t$, principal $P$, $APR$ $r$, and number of compounding periods in a year $n$:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe Table $4.1.4$, which shows the result of investing $\$1,000$ at $10\%$ for one year. Notice how the value of the account increases as the compounding frequency increases.

*Table* $4.1.4$

| Frequency | Value after $1$ year |
|---|---|
| Annually | $\$1100$ |
| Semiannually | $\$1102.50$ |
| Quarterly | $\$1103.81$ |
| Monthly | $\$1104.71$ |
| Daily | $\$1105.16$ |

## Definition: Compound Interest

*Compound interest* can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \tag{4.1.4}$$

where

- $A(t)$ is the account value,
- $t$ is measured in years,
- $P$ is the starting amount of the account, often called the principal, or more generally present value,
- $r$ is the annual percentage rate (APR) expressed as a decimal, and
- $n$ is the number of compounding periods in one year.

## Example $4.1.12$: Calculating Compound Interest

If we invest $\$3,000$ in an investment account paying $3\%$ interest compounded quarterly, how much will the account be worth in $10$ years?

**Solution**

Because we are starting with $\$3,000$, $P = 3000$. Our interest rate is $3\%$, so $r = 0.03$. Because we are compounding quarterly, we are compounding $4$ times per year, so $n = 4$. We want to know the value of the account in $10$ years, so we are looking for $A(10)$,the value when $t = 10$.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Use the compound interest formula}$$
$$A(10) = 3000\left(1 + \frac{0.03}{4}\right)^{(4)\cdot(10)} \quad \text{Substitute using given values}$$
$$\approx \$4045.05 \quad \text{Round to two decimal places}$$

The account will be worth about $\$4,045.05$ in $10$ years.

### Exercise 4.1.12

An initial investment of $\$100,000$ at $12\%$ interest is compounded weekly (use $52$ weeks in a year). What will the investment be worth in $30$ years?

*Answer available at the end of this section.*

### Example 4.1.13: Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $\$40,000$ over $18$ years. She believes the account will earn $6\%$ compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

**Solution**

The nominal interest rate is $6\%$, so $r = 0.06$. Interest is compounded twice a year, so $k = 2$.

We want to find the initial investment, $P$, needed so that the value of the account will be worth $\$40,000$ in $18$ years. Substitute the given values into the compound interest formula, and solve for $P$.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Use the compound interest formula}$$
$$40,000 = P\left(1 + \frac{0.06}{2}\right)^{2(18)} \quad \text{Substitute using given values } A, r, n, t$$
$$40,000 = P(1.03)^{36} \quad \text{Simplify}$$
$$\frac{40,000}{(1.03)^{36}} = P \quad \text{Isolate } P$$
$$P \approx \$13,801 \quad \text{Divide and round to the nearest dollar}$$

Lily will need to invest $\$13,801$ to have $\$40,000$ in $18$ years.

### Exercise 4.1.13

Refer to Example $4.1.9$. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

*Answer available at the end of this section.*

## Evaluating Functions with Base $e$

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. Table $4.1.5$ shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of $\$1$ invested at $100\%$ interest for $1$ year, compounded at various frequencies, listed in Table $4.1.5$.

*Table* 4.1.5

| Frequency | $A(t) = \left(1 + \frac{1}{n}\right)^n$ | Value |
|---|---|---|
| Annually | $\left(1 + \frac{1}{1}\right)^1$ | $2 |
| Semiannually | $\left(1 + \frac{1}{2}\right)^2$ | $2.25 |
| Quarterly | $\left(1 + \frac{1}{4}\right)^4$ | $2.441406 |
| Monthly | $\left(1 + \frac{1}{12}\right)^{12}$ | $2.613035 |
| Daily | $\left(1 + \frac{1}{365}\right)^{365}$ | $2.714567 |
| Hourly | $\left(1 + \frac{1}{8760}\right)^{8760}$ | $2.718127 |
| Once per minute | $\left(1 + \frac{1}{525600}\right)^{525600}$ | $2.718279 |
| Once per second | $\left(1 + \frac{1}{31536000}\right)^{31536000}$ | $2.718282 |

These values appear to be approaching a limit as $n$ increases without bound. In fact, as $n$ gets larger and larger, the expression $\left(1 + \frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter $e$. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

### Definition: The Number e

The letter $e$ represents the irrational number

$$\left(1 + \frac{1}{n}\right)^n \tag{4.1.5}$$

as $n$ increases without bound

The letter $e$ is used as a base for many real-world exponential models. To work with base $e$, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

### Example 4.1.14: Using a Calculator to Find Powers of $e$

Calculate $e^2$. Round to four decimal places.

**Solution**

On a calculator, press the button labeled $[e^x]$. The window shows $[e^(]$. Type $2$ and then close parenthesis, $[)]$. Press [ENTER]. Rounding to $4$ decimal places, $e^2 \approx 7.3891$.

### Exercise 4.1.14

Use a calculator to find $e^{-0.5}$. Round to four decimal places.

*Answer available at the end of this section.*

## Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, $e$ is used as the base for exponential functions. Exponential models that use $e$ as the base are called continuous growth or decay models. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

### Definition: The Continuous Growth/decay Formula

For all real numbers $t$,and all positive numbers $C$ and $k$, continuous growth or decay is represented by the formula

$$f(t) = Ce^{kt} \tag{4.1.6}$$

where

- $C$ is the initial value,
- $k$ is the continuous growth rate per unit time,
- $t$ is the elapsed time.

If $k > 0$, then the formula represents continuous growth. If $k < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t) = Pe^{rt} \tag{4.1.7}$$

where

- $P$ is the principal or the initial invested,
- $r$ is the growth or interest rate per unit time,
- $t$ is the period or term of the investment.

### How to: Given the initial value, rate of growth or decay, and time $t$, solve a continuous growth or decay function

1. Use the information in the problem to determine $a$, the initial value of the function.
2. Use the information in the problem to determine the growth rate $r$.
   - If the problem refers to continuous growth, then $r > 0$.
   - If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time $t$.
4. Substitute the given information into the continuous growth formula and solve for $A(t)$.

### Example 4.1.15: Calculating Continuous Growth

A person invested $\$1,000$ in an account earning a nominal $10\%$ per year compounded continuously. How much was in the account at the end of one year?

**Solution**

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was $\$1,000$, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Use the continuous compounding formula} \\ &= 1000(e)^{0.1} && \text{Substitute known values for } P, r, t \\ &\approx 1105.17 && \text{Use a calculator to approximate} \end{aligned}$$

The account is worth $\$1,105.17$ after one year.

### Exercise 4.1.15

A person invests $\$100,000$ at a nominal $12\%$ interest per year compounded continuously. What will be the value of the investment in $30$ years?

*Answer available at the end of this section.*

### Example 4.1.16: Calculating Continuous Decay

$Radon-222$ decays at a continuous rate of $17.3\%$ per day. How much will $100mg$ of $Radon-222$ decay to in $3$ days?

**Solution**

Since the substance is decaying, the rate, $17.3\%$, is negative. So, $r=-0.173$. The initial amount of $Radon-222$ was $100$ mg, so $a=100$. We use the continuous decay formula to find the value after $t=3$ days:

$$\begin{aligned} A(t) &= ae^{rt} && \text{Use the continuous growth formula} \\ &= 100e6{-}0.173(3) && \text{Substitute known values for } a, r, t \\ &\approx 59.5115 && \text{Use a calculator to approximate} \end{aligned}$$

So $59.5115$ mg of $Radon-222$ will remain.

### Exercise 4.1.16

Using the data in Example $4.1.12$, how much $Radon-222$ will remain after one year?

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with exponential functions.

- Exponential Growth Function
- Compound Interest

## Key Equations

| | |
|---|---|
| definition of the exponential function | $f(x)=b^x$, where $b>0$, $b\neq 1$ |
| definition of exponential growth | $f(x)=ab^x$, where $a>0$, $b>0$, $b\neq 1$ |

| | |
|---|---|
| compound interest formula | $A(t) = P(1 + \frac{r}{n})^{nt}$,<br>where $A(t)$ is the account value at time $t$<br>$t$ is the number of years<br>$P$ is the initial investment, often called the principal<br>$r$ is the annual percentage rate (APR), or nominal rate<br>$n$ is the number of compounding periods in one year |
| continuous growth formula | $A(t) = ae^{rt}$, where $t$ is the number of unit time periods of growth $a$ is the starting amount (in the continuous compounding formula a is replaced with $P$, the principal) $e$ is the mathematical constant, $e \approx 2.718282$ |

## Key Concepts

- An exponential function is defined as a function with a positive constant other than $1$ raised to a variable exponent. See Example.
- A function is evaluated by solving at a specific value. See Example and Example.
- An exponential model can be found when the growth rate and initial value are known. See Example.
- An exponential model can be found when the two data points from the model are known. See Example.
- An exponential model can be found using two data points from the graph of the model. See Example.
- An exponential model can be found using two data points from the graph and a calculator. See Example.
- The value of an account at any time $t$ can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known. See Example.
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known. See Example.
- The number $e$ is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e \approx 2.718282$.
- Scientific and graphing calculators have the key $[ex]$ or $[exp(x)]$ for calculating powers of $e$. See Example.
- Continuous growth or decay models are exponential models that use $e$ as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known. See Example and Example.

### Answers

**Exercise** $4.1.1$

$g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions.

**Exercise** $4.1.2$

$5.5556$

**Exercise** $4.1.3$

About $1.548$ billion people; by the year 2031, India's population will exceed China's by about $0.001$ billion, or $1$ million people.

**Exercise** $4.1.4$

$(0, 129)$ and $(2, 236)$; $N(t) = 129(1.3526)^t$

**Exercise** $4.1.5$

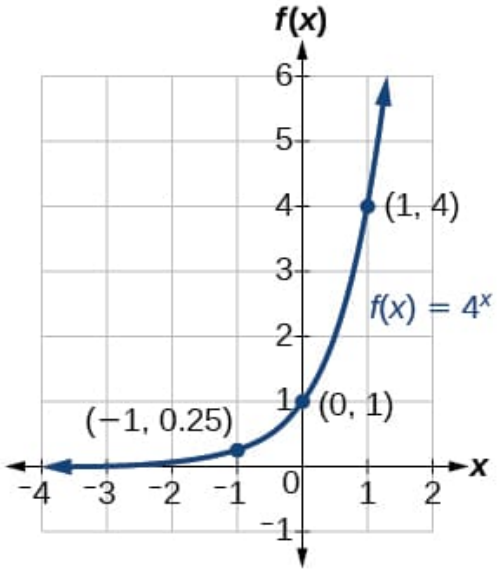
$f(x) = 2(1.5)^x$

**Exercise** $4.1.6$

$f(x) = \sqrt{2}(\sqrt{2})^x$. Answers may vary due to round-off error. The answer should be very close to $1.4142(1.4142)^x$.

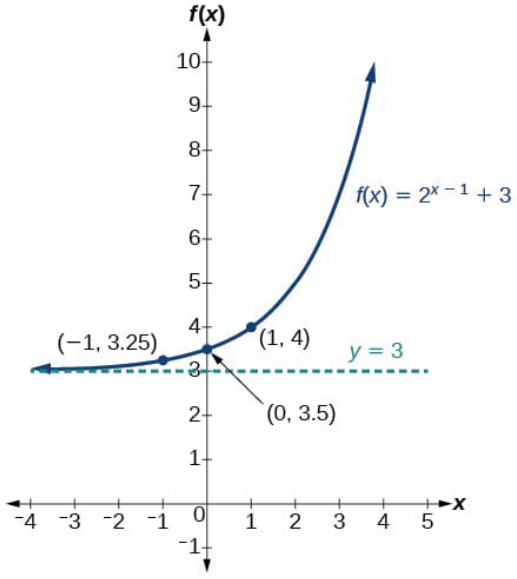
**Exercise** $4.1.7$

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



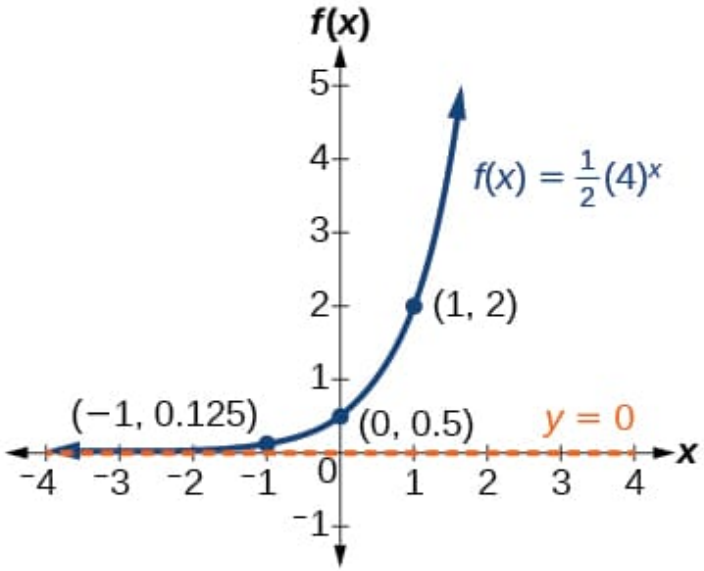

**Exercise** 4.1.8

The domain is $(-\infty, \infty)$; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$.



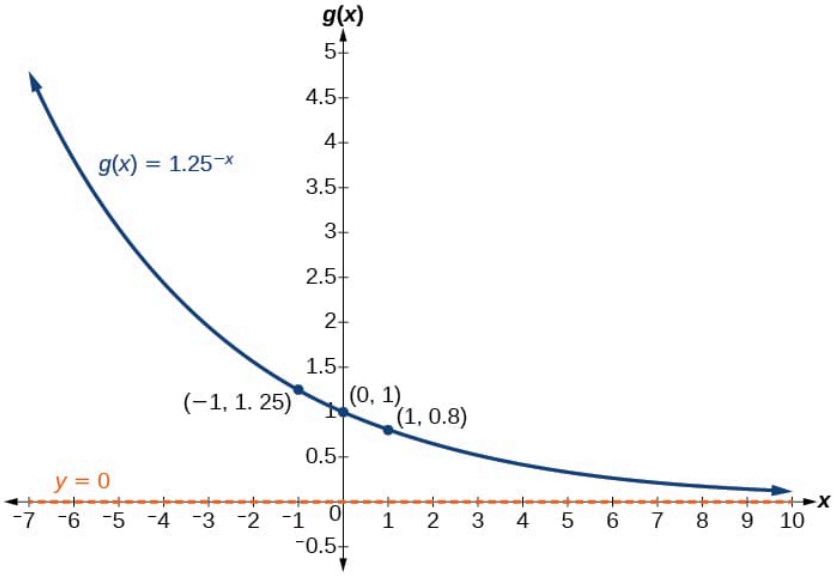

**Exercise** 4.1.9

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.




**Exercise** 4.1.10

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.




**Exercise** 4.1.11

$f(x) = -\frac{1}{3}e^x - 2$; the domain is $(-\infty, \infty)$; the range is $(-\infty, 2)$; the horizontal asymptote is $y = 2$.

**Exercise** 4.1.12

about $\$3,644,675.88$

**Exercise** 4.1.13

$\$13,693$

**Exercise** 4.1.14

$e^{-0.5} \approx 0.6065$

**Exercise** 4.1.15

$\$3,659,823.44$

**Exercise** 4.1.16

$3.77E-26$ (This is calculator notation for the number written as $3.77 \times 10^{-26}$ in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)

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# 4.2: Logarithmic Functions and Graphs

**Learning Objectives**

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.
- Find the domain of a log function.
- Graph log functions.

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings, like those shown in Figure $4.2.1$. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale whereas the Japanese earthquake registered a 9.0.



***Figure*** $4.2.1$***:*** *Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce).*

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude $8$ is not twice as great as an earthquake of magnitude $4$. It is

$$10^{8-4} = 10^4 = 10,000$$

times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

## Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where $x$ represents the difference in magnitudes on the Richter Scale. How

would we solve for $x$?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that $x$ must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in Figure $4.2.1$, to better estimate the solution.

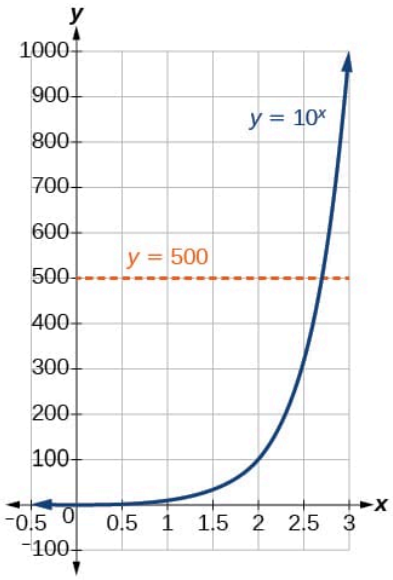



***Figure*** $4.2.2$

Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure $4.2.2$ passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange $x$ and $y$ and solve for $y$ to find the inverse function. To represent $y$ as a function of $x$, we use a logarithmic function of the form $y = \log_b(x)$. The base $b$ **logarithm** of a number is the exponent by which we must raise $b$ to get that number.

We read a logarithmic expression as, "The logarithm with base $b$ of $x$ is equal to $y$," or, simplified, "log base $b$ of $x$ is $y$." We can also say, "$b$ raised to the power of $y$ is $x$," because logs are exponents. For example, the base $2$ logarithm of $32$ is $5$, because $5$ is the exponent we must apply to $2$ to get $32$. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base $2$ of $32$ is $5$."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1 \tag{4.2.1}$$

Note that the base $b$ is always positive.




Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the $x$.

We can illustrate the notation of logarithms as follows:

CNX_Precalc_Figure_04_03_003.jpg

Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y$

$= \log^b(x)$ and $y = b^x$ are inverse functions.

## Definition: Logarithmic Function

A logarithm base $b$ of a positive number $x$ satisfies the following definition.

For $x > 0$, $b > 0$, $b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x \tag{4.2.2}$$

where,

- we read $\log_b(x)$ as, "the logarithm with base $b$ of $x$" or the "log base $b$ of $x$."
- the logarithm $y$ is the exponent to which $b$ must be raised to get $x$.

Also, since the logarithmic and exponential functions switch the $x$ and $y$ values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base $b$ is $(0, \infty)$.
- the range of the logarithm function with base $b$ is $(-\infty, \infty)$.

## Q&A: Can we take the logarithm of a negative number?

*No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.*

## How to: Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form

1. Examine the equation $y = \log_b x$ and identify $b$, $y$,and $x$.
2. Rewrite $\log_b x = y$ as $b^y = x$.

## Example 4.2.1: Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$
b. $\log_3(9) = 2$

**Solution**

First, identify the values of $b$, $y$,and $x$. Then, write the equation in the form $b^y = x$.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$

Here, $b = 6$, $y = \frac{1}{2}$,and $x = \sqrt{6}$. Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to

$$6^{\frac{1}{2}} = \sqrt{6}$$

b. $\log_3(9) = 2$

Here, $b = 3$, $y = 2$,and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to

$$3^2 = 9$$

### Exercise 4.2.1

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1,000,000) = 6$
b. $\log_5(25) = 2$

*Answer a available at the end of this section.*

*Answer b available at the end of this section.*

## Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base $b$,exponent $x$,and output $y$. Then we write $x = \log_b(y)$.

### Example 4.2.2: Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \dfrac{1}{10,000}$

**Solution**

First, identify the values of $b$, $y$,and $x$. Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$

Here, $b = 2$, $x = 3$,and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $5^2 = 25$

Here, $b = 5$, $x = 2$,and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. $10^{-4} = \dfrac{1}{10,000}$

Here, $b = 10$, $x = -4$,and $y = \dfrac{1}{10,000}$. Therefore, the equation $10^{-4} = \dfrac{1}{10,000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10,000}\right) = -4$.

### Exercise 4.2.2

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \dfrac{1}{2}$

*Answer a available at the end of this section.*

*Answer b available at the end of this section.*

*Answer c available at the end of this section.*

## Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, "To what exponent must $2$ be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, "To what exponent must $7$ be raised in order to get $49$?" We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, "To what exponent must $3$ be raised in order to get $27$?" We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$? " We know $2^2 = 4$ and $3^2 = 9$, so

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

Therefore,

$$\log_{\frac{2}{3}} \left(\frac{4}{9}\right) = 2.$$

### How to: Given a logarithm of the form $y = \log_b(x)$,evaluate it mentally

1. Rewrite the argument $x$ as a power of $b$: $b^y = x$.
2. Use previous knowledge of powers of $b$ identify $y$ by asking, "To what exponent should $b$ be raised in order to get $x$?"

### Example 4.2.3: Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

**Solution**

First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must $4$ be raised in order to get $64$?"

We know

$$4^3 = 64$$

Therefore,

$$\log_4(64) = 3$$

### Exercise 4.2.3

Solve $y = \log_{121}(11)$ without using a calculator.

*Answer available at the end of this section.*

### Example 4.2.4: Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

**Solution**

First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must $3$ be raised in order to get $\frac{1}{27}$?"

We know $3^3 = 27$,but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

### Exercise 4.2.4

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

*Answer available at the end of this section.*

## Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is $10$. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base $-10$ logarithm a *common logarithm*. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

### Definition: Common Log

A common logarithm is a logarithm with base $10$. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number $x$ satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x \tag{4.2.3}$$

We read $\log(x)$ as, "the logarithm with base $10$ of $x$ " or "log base $10$ of $x$."

The logarithm $y$ is the exponent to which $10$ must be raised to get $x$.

### How to: Given a common logarithm of the form $y = \log(x)$, evaluate it mentally

1. Rewrite the argument $x$ as a power of $10$: $10^y = x$.
2. Use previous knowledge of powers of $10$ to identify $y$ by asking, "To what exponent must $10$ be raised in order to get $x$?"

### Example 4.2.5: Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

**Solution**

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must $10$ be raised in order to get $1000$?" We know

$$10^3 = 1000$$

Therefore, $\log(1000) = 3$.

### Exercise 4.2.5

Evaluate $y = \log(1,000,000)$.

*Answer available at the end of this section.*

### How to: Given a common logarithm with the form $y = \log(x)$,evaluate it using a calculator

1. Press **[LOG]**.
2. Enter the value given for $x$,followed by **[ ) ]**.
3. Press **[ENTER]**.

### Example 4.2.6: Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

**Solution**

- Press **[LOG]**.
- Enter 321, followed by **[ ) ]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

**Analysis**

Note that $10^2 = 100$ and that $10^3 = 1000$. Since $321$ is between $100$ and $1000$, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$. This gives us the following:

$$100 < 321 < 1000$$

$$2 < 2.5065 < 3$$

### Exercise 4.2.6

Evaluate $y = \log(123)$ to four decimal places using a calculator.

*Answer available at the end of this section.*

### Example 4.2.7: Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was $500$ times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where $x$ is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

**Solution**

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500$$

$$\log(500) = x \text{ Use the definition of the common log.}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter $500$,followed by **[ ) ]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$.

The difference in magnitudes was about $2.699$.

### Exercise 4.2.7

The amount of energy released from one earthquake was $8,500$ times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where $x$ is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

*Answer available at the end of this section.*

## Using Natural Logarithms

The most frequently used base for logarithms is $e$. Base $e$ logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base $e$ logarithm, $\log_e(x)$, has its own notation,$\ln(x)$. Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of $1$ is always $0$ in any base, $\ln 1 = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of $e$ using the inverse property of logarithms.

### Definition: Natural Log

A natural logarithm is a logarithm with base $e$. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number $x$ satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base $e$ of $x$" or "the natural logarithm of $x$."

The logarithm $y$ is the exponent to which $e$ must be raised to get $x$.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all $x$ and $e^{\ln(x)} = x$ for $x > 0$.

### How to: Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator

1. Press **[LN]**.
2. Enter the value given for $x$, followed by **[ ) ]**.
3. Press **[ENTER]**.

### Example 4.2.8: Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

**Solution**

- Press **[LN]**.
- Enter $500$,followed by **[ ) ]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

### Exercise 4.2.8

Evaluate $\ln(-500)$.

*Answer available at the end of this section.*

## Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number $x$ and constant $b > 0$, $b \neq 1$, where

- The domain of $y$ is $(-\infty, \infty)$.
- The range of $y$ is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

In Graphs of Exponential Functions we saw that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of $x$ such that the argument, in this case $2x - 3$ ,is greater than zero. To find the domain, we set up an inequality and solve for $x$:

$$\begin{aligned} 2x - 3 &> 0 && \text{Show the argument greater than zero} \\ 2x &> 3 && \text{Add 3} \\ x &> 1.5 && \text{Divide by 2} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

### Given a logarithmic function, identify the domain

1. Set up an inequality showing the argument greater than zero.
2. Solve for $x$.
3. Write the domain in interval notation.

### Example 4.2.9: Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x) = \log_2(x + 3)$?

**Solution**

The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

$$\begin{aligned} x + 3 &> 0 && \text{The input must be positive} \\ x &> -3 && \text{Subtract 3} \end{aligned}$$

The domain of $f(x) = \log_2(x + 3)$ is $(-3, \infty)$.

### Exercise 4.2.9

What is the domain of $f(x) = \log_5(x - 2) + 1$?

*Answer available at the end of this section.*

### Example 4.2.10: Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x) = \log(5 - 2x)$?

**Solution**

The logarithmic function is defined only when the input is positive, so this function is defined when $5–2x > 0$. Solving this inequality,

$$\begin{aligned} 5 - 2x &> 0 && \text{The input must be positive} \\ -2x &> -5 && \text{Subtract 5} \\ x &< \frac{5}{2} && \text{Divide by -2 and switch the inequality} \end{aligned}$$

The domain of $f(x) = \log(5 - 2x)$ is $\left(-\infty, \frac{5}{2}\right)$.

**Exercise 4.2.10**

What is the domain of $f(x) = \log(x - 5) + 2$?

*Answer available at the end of this section.*

## Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the parent function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in Table 4.2.1.

*Table* 4.2.1

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $2^x = y$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $2$ | $4$ | $8$ |
| $\log_2(y) = x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |

Using the inputs and outputs from Table 4.2.1, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See Table 4.2.2.

*Table* 4.2.2

| $f(x) = 2^x$ | $\left(-3, \frac{1}{8}\right)$ | $\left(-2, \frac{1}{4}\right)$ | $\left(-1, \frac{1}{2}\right)$ | $(0, 1)$ | $(1, 2)$ | $(2, 4)$ | $(3, 8)$ |
|---|---|---|---|---|---|---|---|
| $g(x) = \log_2(x)$ | $\left(\frac{1}{8}, -3\right)$ | $\left(\frac{1}{4}, -2\right)$ | $\left(\frac{1}{2}, -1\right)$ | $(1, 0)$ | $(2, 1)$ | $(4, 2)$ | $(8, 3)$ |

As we'd expect, the $x$- and $y$-coordinates are reversed for the inverse functions. Figure 4.2.2 shows the graph of $f$ and $g$.

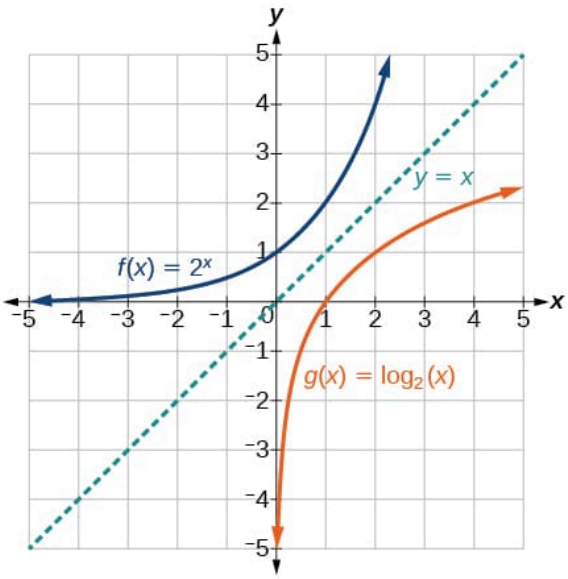



Figure 4.2.2: Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections about the line $y = x$.

Observe the following from the graph:

- $f(x) = 2^x$ has a $y$-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an $x$- intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

## CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION, $f(X) = log_b(x)$

For any real number $x$ and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- $x$-intercept: $(1, 0)$ and key point $(b, 1)$
- $y$-intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$

See Figure 4.2.3.

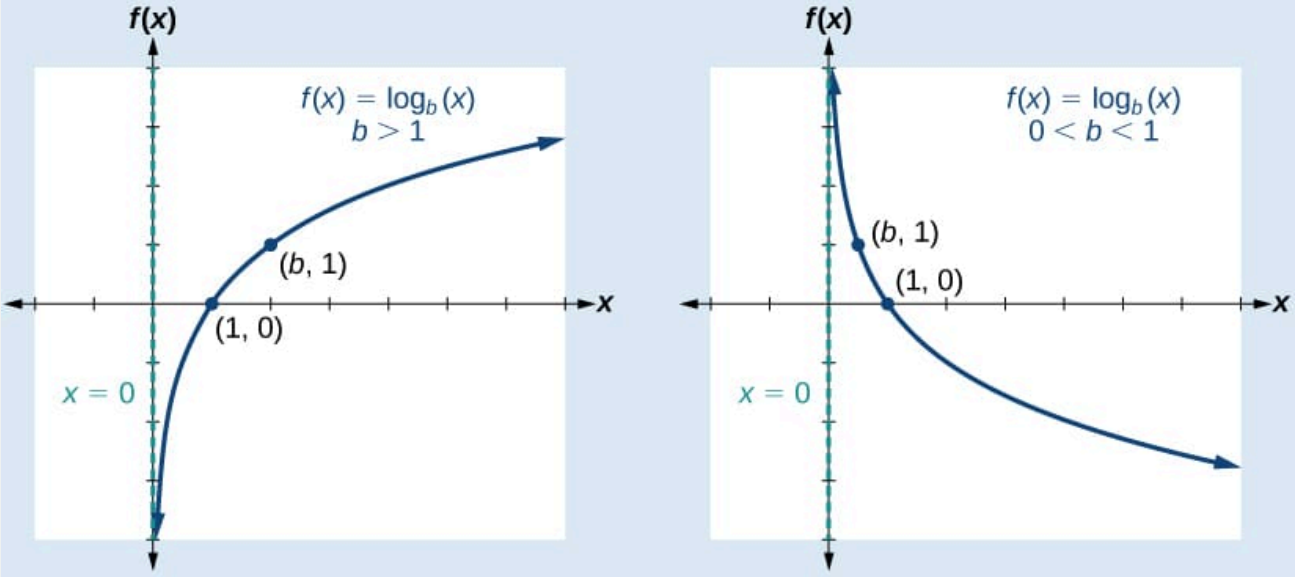



Figure 4.2.3

Figure 4.2.4 shows how changing the base $b$ in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (Note: recall that the function $\ln(x)$ has base $e \approx 2.718$.)

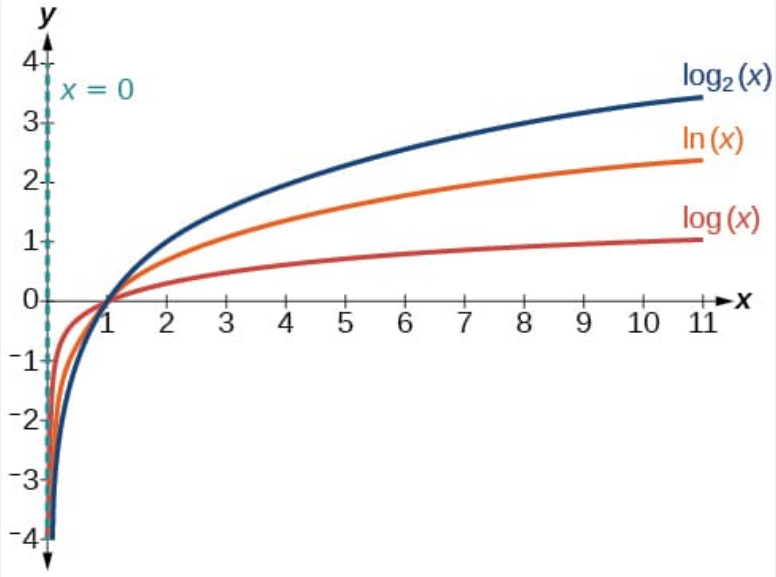



Figure 4.2.4: The graphs of three logarithmic functions with different bases, all greater than 1.

## Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.
3. Plot the key point $(b, 1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$,the range, $(-\infty, \infty)$, and the vertical asymptote, $x = 0$.

## Example 4.2.11: Graphing a Logarithmic Function with the Form $f(x) = log_b(x)$

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

**Solution**

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.
- The $x$-intercept is $(1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see Figure 4.2.5 ).

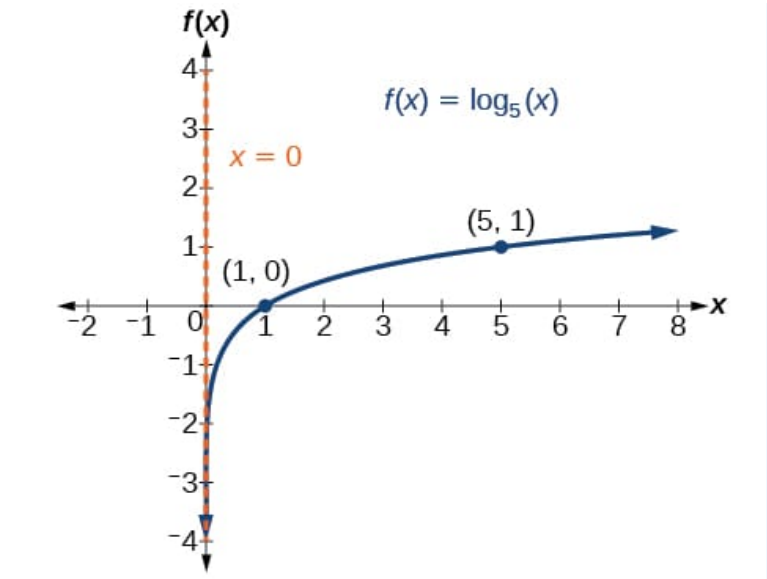



Figure 4.2.5

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

**Exercise 4.2.11**

Graph $f(x) = \log_{\frac{1}{5}}(x)$. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y = \log_b(x)$ without loss of shape.

### Graphing a Horizontal Shift of $f(x) = log_b(x)$

When a constant $c$ is added to the input of the parent function $f(x) = \log_b(x)$, the result is a horizontal shift $c$ units in the *opposite* direction of the sign on $c$. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 4.2.7.

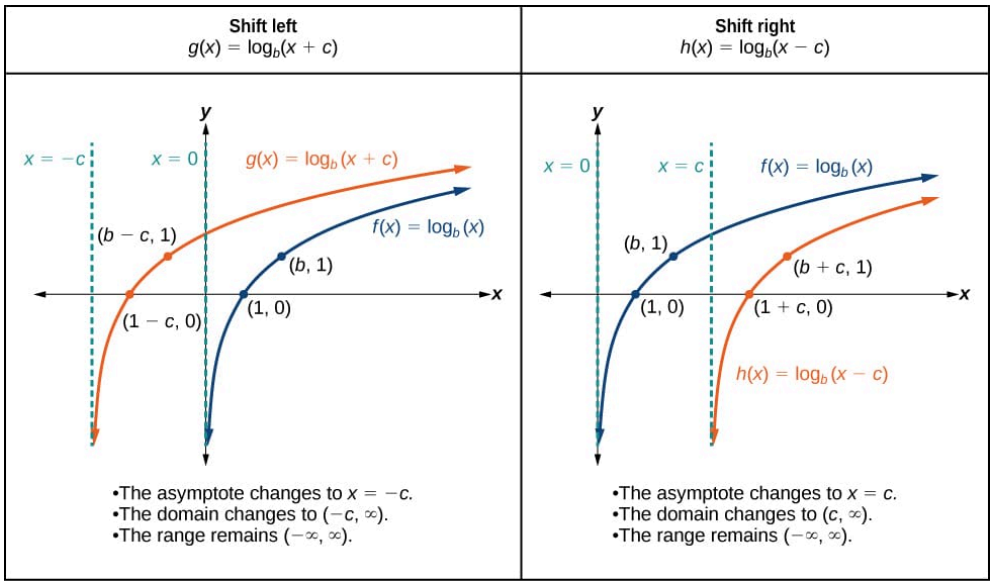



Figure 4.2.7

## HORIZONTAL SHIFTS OF THE PARENT FUNCTION $y = log_b(x)$

For any constant $c$,the function $f(x) = \log_b(x + c)$

- shifts the parent function $y = \log_b(x)$ left $c$ units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right $c$ units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

## Given a logarithmic function with the form $f(x) = \log_b(x + c)$, graph the translation.

1. Identify the horizontal shift:
   - If $c > 0$,shift the graph of $f(x) = \log_b(x)$ left $c$ units.
   - If $c < 0$,shift the graph of $f(x) = \log_b(x)$ right $c$ units.
2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting $c$ from the $x$ coordinate.
4. Label the three points.
5. The Domain is $(-c, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -c$.

## Example 4.2.12: Graphing a Horizontal Shift of the Parent Function $y = log_b(x)$

Sketch the horizontal shift $f(x) = \log_3(x - 2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

**Solution**

Since the function is $f(x) = \log_3(x - 2)$, we notice $x + (-2) = x$–2.

Thus $h = -2$, so $h < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$,and $(3, 1)$.

The new coordinates are found by adding $2$ to the $x$ coordinates.

Label the points $\left(\frac{7}{3}, -1\right)$, $(3, 0)$,and $(5, 1)$.

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.

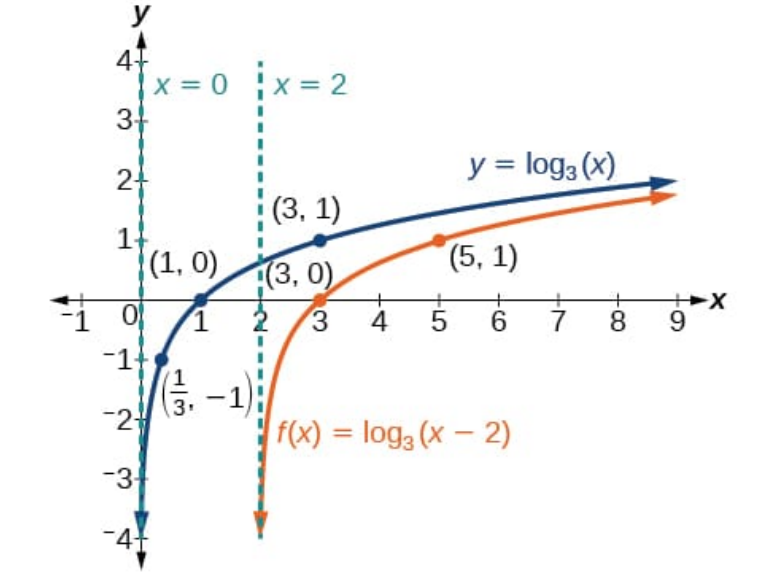



Figure $4.2.8$

**Exercise $4.2.12$**

Sketch a graph of $f(x) = \log_3(x + 4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Graphing a Vertical Shift of $y = log_b(x)$

When a constant $d$ is added to the parent function $f(x) = \log_b(x)$,the result is a vertical shift $d$ units in the direction of the sign on $d$. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$.See Figure $4.2.10$.

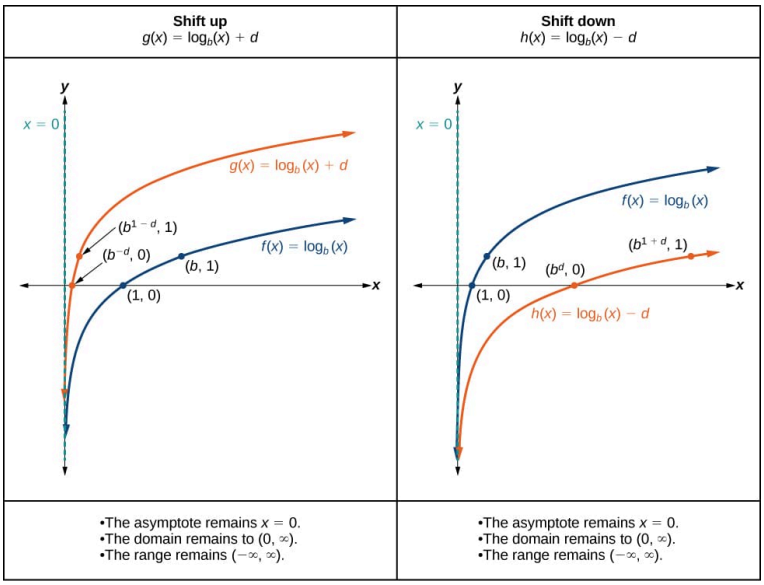



Figure 4.2.10

## VERTICAL SHIFTS OF THE PARENT FUNCTION $Y = LOG_B(X)$

For any constant $d$,the function $f(x) = \log_b(x) + d$

- shifts the parent function $y = \log_b(x)$ up $d$ units if $d > 0$.
- shifts the parent function $y = \log_b(x)$ down $d$ units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

## Given a logarithmic function with the form $f(x) = \log_b(x) + d$, graph the translation.

1. Identify the vertical shift:
   - If $d > 0$, shift the graph of $f(x) = \log_b(x)$ up $d$ units.
   - If $d < 0$, shift the graph of $f(x) = \log_b(x)$ down $d$ units.
2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding $d$ to the $y$ coordinate.
4. Label the three points.
5. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

## Example 4.2.13: Graphing a Vertical Shift of the Parent Function $y = log_b(x)$

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

**Solution**

Since the function is $f(x) = \log_3(x) - 2$,we will notice $d = -2$. Thus $d < 0$.

This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$,and $(3, 1)$.

The new coordinates are found by subtracting $2$ from the *y* coordinates.

Label the points $\left(\frac{1}{3}, -3\right)$, $(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

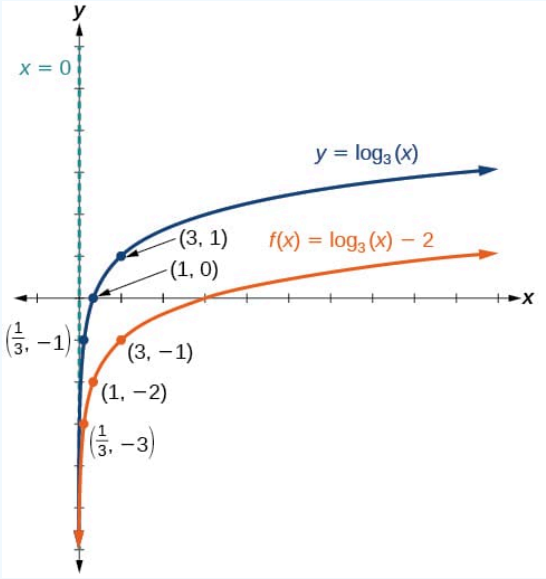



Figure $4.2.11$

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

**Exercise $4.2.13$**

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Graphing Stretches and Compressions of $y = log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a > 1$ and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$.See Figure $4.2.13$.

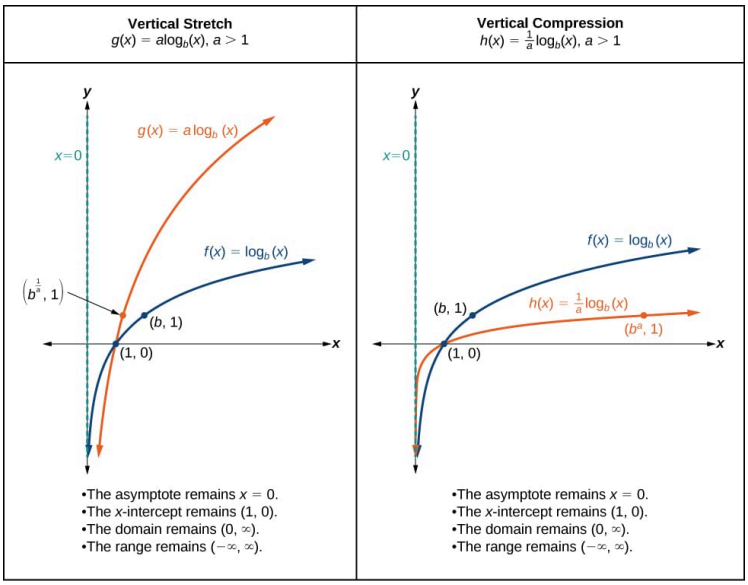



Figure 4.2.13

## VERTICAL STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $Y = LOG_B(X)$

For any constant $a > 1$,the function $f(x) = a\log_b(x)$

- stretches the parent function $y = \log_b(x)$ vertically by a factor of $a$ if $a > 1$.
- compresses the parent function $y = \log_b(x)$ vertically by a factor of $a$ if $0 < a < 1$.
- has the vertical asymptote $x = 0$.
- has the $x$-intercept $(1, 0)$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

## Given a logarithmic function with the form $f(x) = a\log_b(x)$, $a > 0$,graph the translation.

1. Identify the vertical stretch or compressions:
   - If $|a| > 1$, the graph of $f(x) = \log_b(x)$ is stretched by a factor of $a$ units.
   - If $|a| < 1$, the graph of $f(x) = \log_b(x)$ is compressed by a factor of $a$ units.
2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying they y coordinates by $a$.
4. Label the three points.
5. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

## Example 4.2.14: Graphing a Stretch or Compression of the Parent Function $y = log_b(x)$

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

**Solution**

Since the function is $f(x) = 2\log_4(x)$,we will notice $a = 2$.

This means we will stretch the function $f(x) = \log_4(x)$ by a factor of $2$.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{4}, -1\right)$, $(1, 0)$, and $(4, 1)$.

The new coordinates are found by multiplying the $y$ coordinates by $2$.

Label the points $\left(\frac{1}{4}, -2\right)$, $(1, 0)$, and $(4, 2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$. See Figure $4.2.14$.

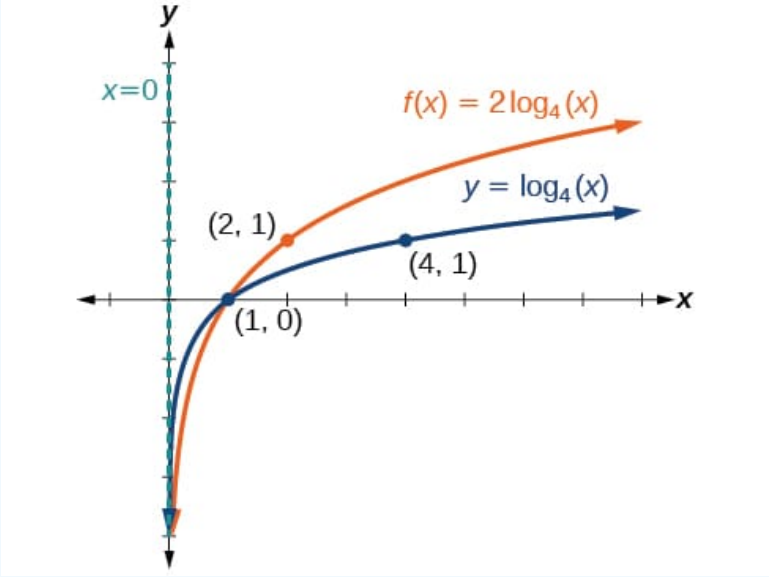



Figure $4.2.14$

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

## Exercise $4.2.14$

Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Example $4.2.15$: Combining a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x + 2)$. State the domain, range, and asymptote.

**Solution**

Remember: what happens inside parentheses happens first. First, we move the graph left $2$ units, then stretch the function vertically by a factor of $5$, as in Figure $4.2.16$. The vertical asymptote will be shifted to $x = -2$. The $x$-intercept will be $(-1, 0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5)$. We chose $x = 8$ as the $x$-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.

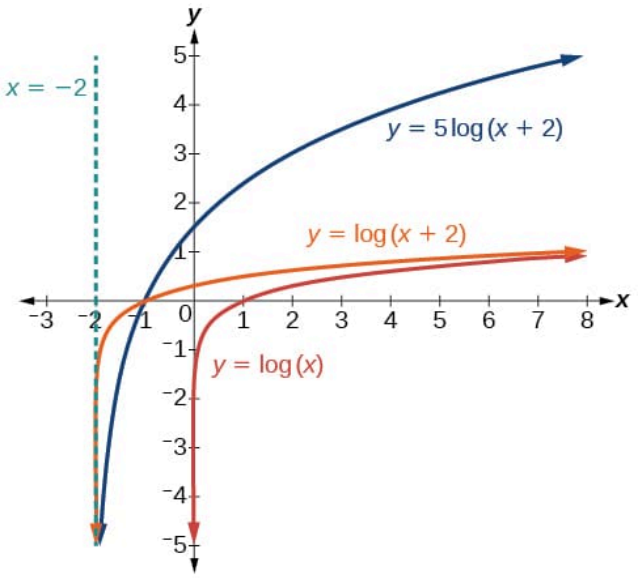



Figure 4.2.16

The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = -2$.

## Exercise 4.2.15

Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Graphing Reflections of $f(x) = log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by $-1$,the result is a reflection about the $x$-axis. When the *input* is multiplied by $-1$,the result is a reflection about the $y$-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the $x$-axis, $g(x) = -\log_b(x)$ and the reflection about the $y$-axis, $h(x) = \log_b(-x)$.

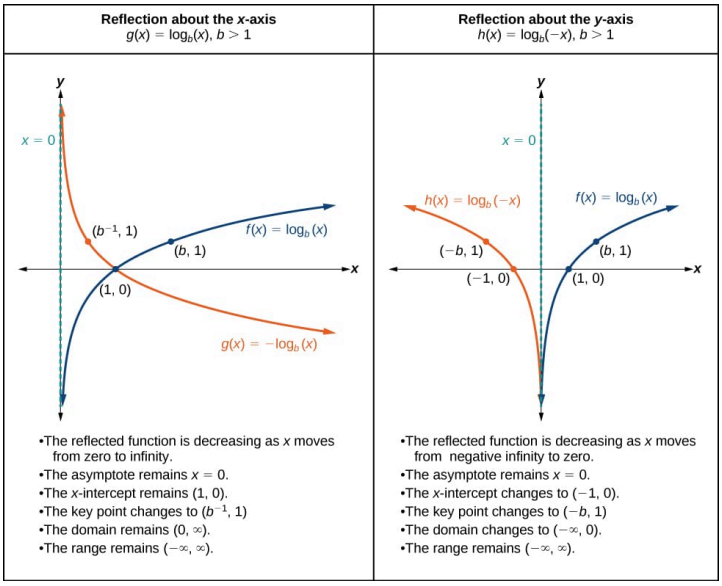



Figure 4.2.18

## REFLECTIONS OF THE PARENT FUNCTION $Y = LOG_B(X)$

The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the $x$-axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$

- reflects the parent function $y = \log_b(x)$ about the $y$-axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

## Given a logarithmic function with the parent function $f(x) = \log_b(x)$, graph a translation.

*Table* 4.2.3

| **If** $f(x) = -\log_b(x)$ | **If** $f(x) = \log_b(-x)$ |
|---|---|
| 1. Draw the vertical asymptote, $x = 0$. | 1. Draw the vertical asymptote, $x = 0$. |
| 2. Plot the *x*-intercept, $(1, 0)$. | 2. Plot the *x*-intercept, $(1, 0)$. |
| 3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *x*-axis. | 3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the *y*-axis. |
| 4. Draw a smooth curve through the points. | 4. Draw a smooth curve through the points. |
| 5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$. | 5. State the domain, $(-\infty, 0)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$. |

## Example 4.2.16: Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

**Solution**

Before graphing $f(x) = log(-x)$, $f(x) = log(-x)$,identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by $-1$, $f(x)$ is a reflection of the parent graph about the $y$-axis. Thus, $f(x) = \log(-x)$ will be decreasing as $x$ moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The $x$-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

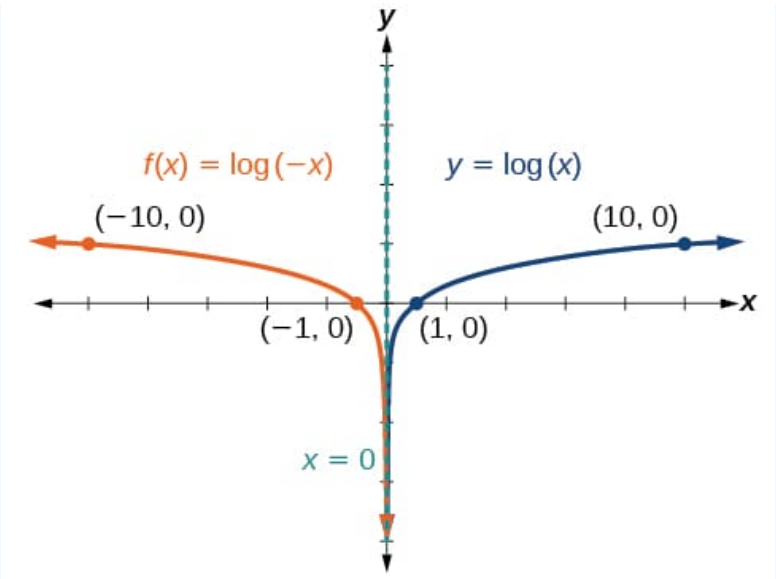



Figure 4.2.19

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

## Exercise 4.2.16

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

*Answer available at the end of this section.*

## Given a logarithmic equation, use a graphing calculator to approximate solutions.

1. Press **[Y=]**. Enter the given logarithm equation or equations as **$Y_1$=** and, if needed, **$Y_2$=**.
2. Press **[GRAPH]** to observe the graphs of the curves and use **[WINDOW]** to find an appropriate view of the graphs, including their point(s) of intersection.
3. To find the value of $x$, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of $x$,for the point(s) of intersection.

## Example 4.2.17: Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x) + 1 = -2\ln(x - 1)$ graphically. Round to the nearest thousandth.

**Solution**

Press **[Y=]** and enter $4\ln(x) + 1$ next to **$Y_1$=**. Then enter $-2\ln(x - 1)$ next to **$Y_2$=**. For a window, use the values $0$ to $5$ for $x\backslash 0 \; and \backslash (-10$ to $10$ for $y$. Press **[GRAPH]**. The graphs should intersect somewhere a little to right of $x = 1$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as $1.3385297$. (Your answer may be different if you use a different window or use a different value for **Guess?**) So, to the nearest thousandth, $x \approx 1.339$.

**Exercise 4.2.17**

Solve $5\log(x+2) = 4 - \log(x)$ graphically. Round to the nearest thousandth.

*Answer available at the end of this section.*

## Summarizing Translations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in Table 4.2.4 to arrive at the general equation for translating exponential functions.

***Table*** 4.2.4

| Translations of the Parent Function $y = \log_b(x)$ | |
|---|---|
| **Translation** | **Form** |
| Shift<br>Horizontally $c$ units to the left<br>Vertically $d$ units up | $y = \log_b(x+c) + d$ |
| Stretch and Compress<br>Stretch if $\lvert a\rvert > 1$<br>Compression if $\lvert a\rvert < 1$ | $y = a\log_b(x)$ |
| Reflect about the x-axis | $y = -\log_b(x)$ |
| Reflect about the y-axis | $y = \log_b(-x)$ |
| General equation for all translations | $y = a\log_b(x+c) + d$ |

### TRANSLATIONS OF LOGARITHMIC FUNCTIONS

All translations of the parent logarithmic function, $y = \log_b(x)$, have the form

$$f(x) = a\log_b(x+c) + d$$

where the parent function, $y = \log_b(x)$, $b > 1$,is

- shifted vertically up $d$ units.
- shifted horizontally to the left $c$ units.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- reflected about the *x*-axis when $a < 0$.

For $f(x) = \log(-x)$, the graph of the parent function is reflected about the *y*-axis.

## Example 4.2.18: Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2\log_3(x+4) + 5$?

**Solution**

The vertical asymptote is at $x = -4$.

**Analysis**

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve $4$ units to the left shifts the vertical asymptote to$x = -4$.

## Exercise 4.2.18

What is the vertical asymptote of $f(x) = 3 + \ln(x-1)$?

*Answer available at the end of this section.*

## Example 4.2.19: Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in Figure 4.2.21.

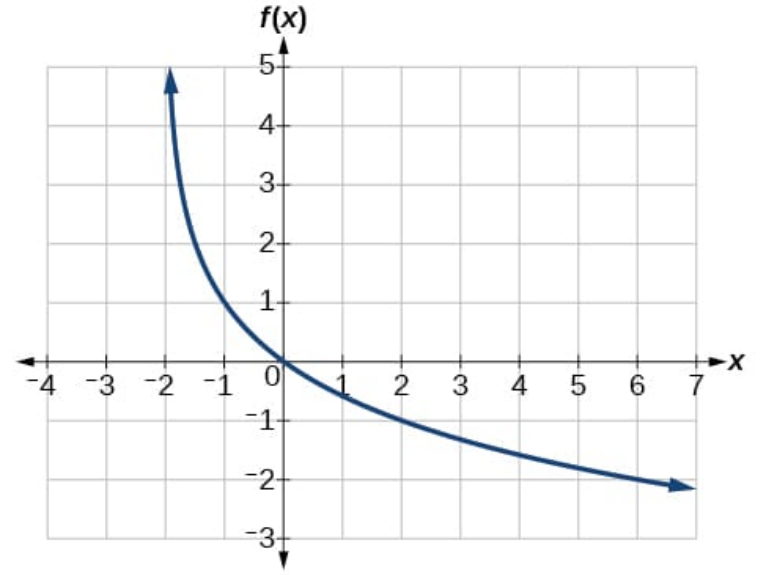



Figure 4.2.21

**Solution**

This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

$$f(x) = -a\log(x+2) + k$$

It appears the graph passes through the points $(-1, 1)$ and $(2, -1)$. Substituting $(-1, 1)$,

$$
\begin{aligned}
1 &= -a\log(-1+2)+k \qquad \text{Substitute}(-1,1)\\
1 &= -a\log(1)+k \qquad \text{Arithmetic}\\
1 &= k\log(1)\\
&= 0
\end{aligned}
$$

Next, substituting in $(2,-1)$,

$$
\begin{aligned}
-1 &= -a\log(2+2)+1 \qquad \text{Substitute}(2,-1)\\
-2 &= -a\log(4) \qquad \text{Arithmetic}\\
a &= \frac{2}{\log(4)} \qquad \text{Solve for a}
\end{aligned}
$$

This gives us the equation $f(x) = -\frac{2}{\log(4)}\log(x+2)+1$.

**Analysis**

We can verify this answer by comparing the function values in Table 4.2.5 with the points on the graph in Figure 4.2.21.

*Table* 4.2.5

| $x$ | –1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 0 | –0.58496 | –1 | –1.3219 |
| $x$ | 4 | 5 | 6 | 7 | 8 |
| $f(x)$ | –1.5850 | –1.8074 | –2 | –2.1699 | –2.3219 |

## Exercise 4.2.19

Give the equation of the natural logarithm graphed in Figure 4.2.22.

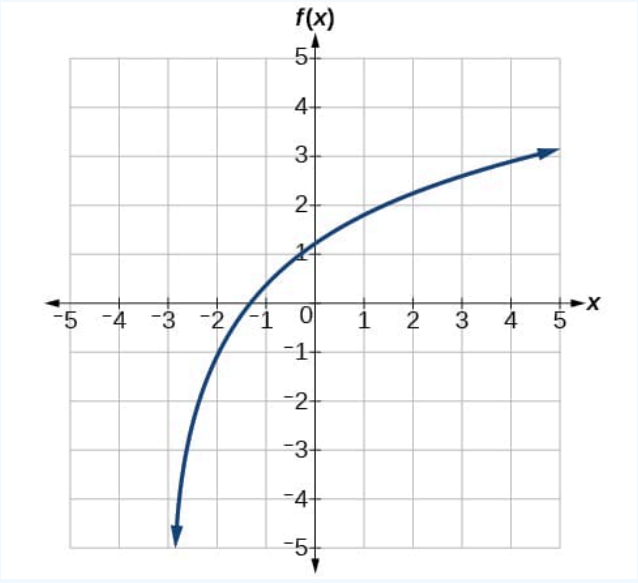



Figure 4.2.22

*Answer available at the end of this section.*

## Key Equations

| | |
|---|---|
| Definition of the logarithmic function | For $x > 0$, $b > 0$, $b \neq 1$, $y = \log_b(x)$ if and only if $b^y = x$. |
| Definition of the common logarithm | For $x > 0$, $y = \log(x)$ if and only if $10^y = x$. |
| Definition of the natural logarithm | For $x > 0$, $y = \ln(x)$ if and only if $e^y = x$. |

## Key Concepts

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See Example $4.2.1$.
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See Example $4.2.2$.
- Logarithmic functions with base $b$ can be evaluated mentally using previous knowledge of powers of $b$. See Example $4.2.3$ and Example $4.2.4$.
- Common logarithms can be evaluated mentally using previous knowledge of powers of $10$. See Example $4.2.5$.
- When common logarithms cannot be evaluated mentally, a calculator can be used. See Example $4.2.6$.
- Real-world exponential problems with base $10$ can be rewritten as a common logarithm and then evaluated using a calculator. See Example $4.2.7$.
- Natural logarithms can be evaluated using a calculator Example $4.2.8$.

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### Answers

**Exercise** $4.2.1$

$\log_{10}(1,000,000) = 6$ is equivalent to $10^6 = 1,000,000$

**Exercise** $4.2.1$

$\log_5(25) = 2$ is equivalent to $5^2 = 25$

**Exercise** $4.2.2$

$3^2 = 9$ is equivalent to $\log_3(9) = 2$

**Exercise** $4.2.2$

$5^3 = 125$ is equivalent to $\log_5(125) = 3$

**Exercise** $4.2.2$

$2^{-1} = \frac{1}{2}$ is equivalent to $\log_2\left(\frac{1}{2}\right) = -1$

**Exercise** $4.2.3$

$\log_{121}(11) = \frac{1}{2}$ (recalling that $\sqrt{121} = (121)^{\frac{1}{2}} = 11$)

**Exercise** $4.2.4$

$\log_2\left(\frac{1}{32}\right) = -5$

**Exercise** $4.2.5$

$\log(1,000,000) = 6$

**Exercise** $4.2.6$

$\log(123) \approx 2.0899$

**Exercise** $4.2.7$

The difference in magnitudes was about $3.929$.

**Exercise** $4.2.8$

It is not possible to take the logarithm of a negative number in the set of real numbers.

**Exercise** 4.2.9

$(2, \infty)$

**Exercise** 4.2.10

$(5, \infty)$

**Exercise** 4.2.11

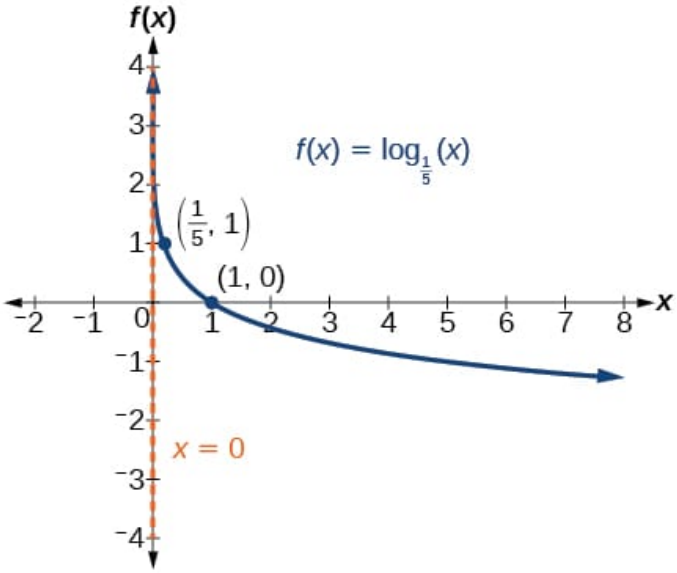



Figure 4.2.6

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

**Exercise** 4.2.12

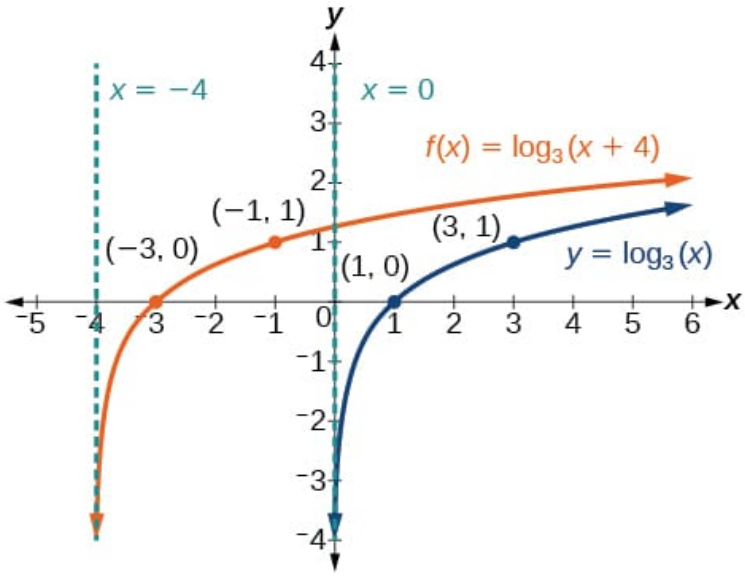



Figure 4.2.9

The domain is $(-4, \infty)$,the range $(-\infty, \infty)$,and the asymptote $x = -4$.

**Exercise** 4.2.13

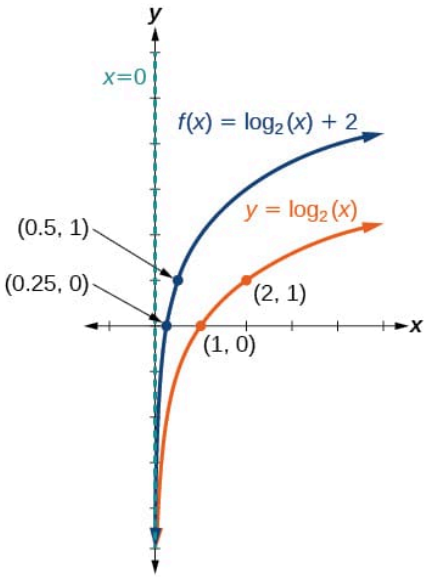



Figure 4.2.12

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

**Exercise** 4.2.14

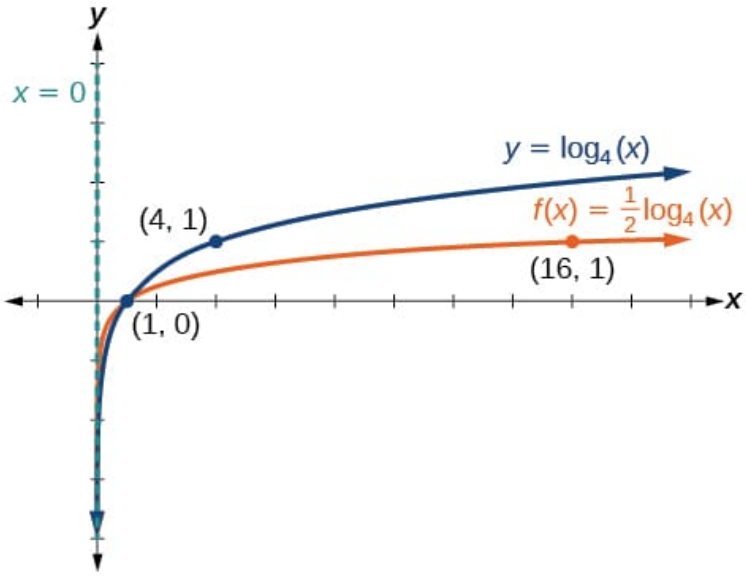



Figure 4.2.15

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

**Exercise** 4.2.15

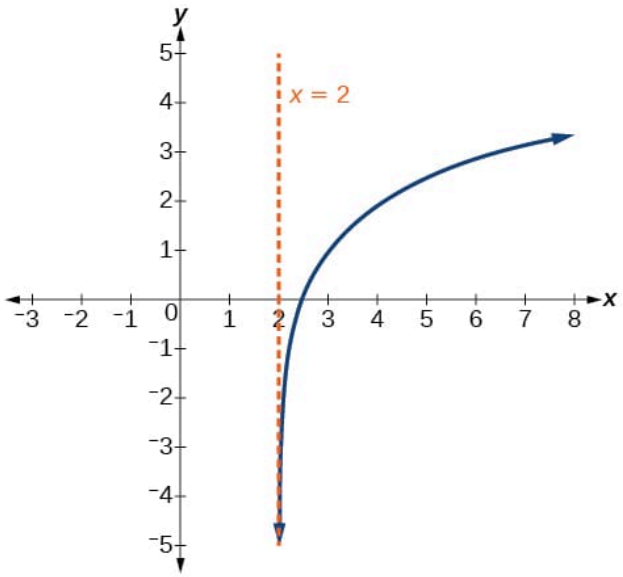



Figure 4.2.17

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.

**Exercise** 4.2.16

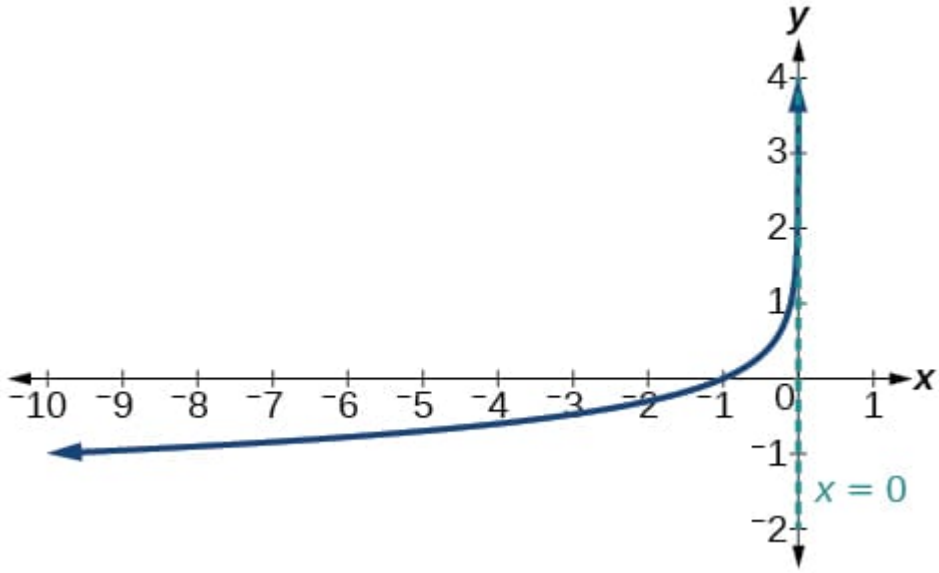



Figure 4.2.20

The domain is $(-\infty, 0)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

**Exercise** 4.2.17

$x \approx 3.049$

**Exercise** 4.2.18

$x = 1$

**Exercise** 4.2.19

$f(x) = 2\ln(x + 3) - 1$

---

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# 4.3: Properties of Logarithms

## Learning Objectives

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

In chemistry, the pH scale is used as a measure of the acidity or alkalinity of a substance. Substances with a pH less than $7$ are considered acidic, and substances with a pH greater than $7$ are said to be alkaline. Our bodies, for instance, must maintain a pH close to $7.35$ in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: $0.8$
- Stomach acid: $2.7$
- Orange juice: $3.3$
- Pure water: $7$ at $25^{\circ}C$
- Human blood: $7.35$
- Fresh coconut: $7.8$
- Sodium hydroxide (lye): $14$

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where $[\mathrm{H}^+]$ is the concentration of hydrogen ions in the solution

$$pH = -\log([\mathrm{H}^+]) \tag{4.3.1}$$

$$= \log\left(\frac{1}{[\mathrm{H}^+]}\right) \tag{4.3.2}$$

The equivalence of Equations 4.3.1 and 4.3.2 is one of the logarithm properties we will examine in this section.



Figure 4.3.1: The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan).

# Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

$$\log_b 1 = 0$$
$$\log_b b = 1$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

$$\log_b(b^x) = x$$
$$b^{\log_b x} = x,\ x > 0$$

For example, to evaluate $\log(100)$, we can rewrite the logarithm as $\log_{10}(10^2)$, and then apply the inverse property $\log_b(b^x) = x$ to get $\log_{10}(10^2) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the inverse property $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the one-to-one property.

$$\log_b M = \log_b N \text{ if and only if } M = N \tag{4.3.3}$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x + 5)$ for $x$. Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for $x$:

$3x = 2x + 5$ Set the arguments equal.

$x = 5$ Subtract $2x$.

But what about the equation $\log_3(3x) + \log_3(2x + 5) = 2$? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number $x$ and positive real numbers $M$, $N$, and $b$, where $b \neq 1$, we will show

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned}
\log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for M and N} \\
&= \log_b(b^{m+n}) && \text{Apply the product rule for exponents} \\
&= m + n && \text{Apply the inverse property of logs} \\
&= \log_b(M) + \log_b(N) && \text{Substitute for m and n}
\end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

### The Product Rule for Logarithms

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0 \tag{4.3.4}$$

### How to: Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

### Example 4.3.1: Using the Product Rule for Logarithms

Expand $\log_3(30x(3x+4))$.

**Solution**

We begin by factoring the argument completely, expressing $30$ as a product of primes.

$$\log_3(30x(3x+4)) = \log_3(2 \cdot 3 \cdot 5 \cdot x \cdot (3x+4))$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30x(3x+4)) = \log_3(2) + \log_3(3) + \log_3(5) + \log_3(x) + \log_3(3x+4)$$

### Exercise 4.3.1

Expand $\log_b(8k)$.

*Answer available at the end of this section.*

## Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $x^{\frac{a}{b}} = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms.

### The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N \tag{4.3.5}$$

Just as with the product rule, we can use the inverse property to derive the quotient rule.

**Proof**

Given any real number $x$ and positive real numbers $M$, $N$, and b, b, where $b \neq 1$, we will show

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned}
\log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{Substitute for M and N} \\
&= \log_b(b^{m-n}) && \text{Apply the quotient rule for exponents} \\
&= m - n && \text{Apply the inverse property of logs} \\
&= \log_b(M) - \log_b(N) && \text{Substitute for m and n}
\end{aligned}$$

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{aligned}
\log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{Factor the numerator and denominator} \\
&= \log\left(\frac{2x}{3}\right) && \text{Cancel the common factors}
\end{aligned}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.

$$\begin{aligned}
\log\left(\frac{2x}{3}\right) &= \log(2x) - \log(3) \\
&= \log(2) + \log(x) - \log(3)
\end{aligned}$$

**How to: Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms**

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

**Example 4.3.2: Using the Quotient Rule for Logarithms**

Expand $log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

**Solution**

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2(\frac{15x(x-1)}{(3x+4)(2-x)}) = \log_2(15x(x-1)) - \log_2((3x+4)(2-x))$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$$\begin{aligned}
\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1)] - [\log_2(3x+4) + \log_2(2-x)] \\
&= \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1) - \log_2(3x+4) - \log_2(2-x)
\end{aligned}$$

**Analysis**

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

**Exercise 4.3.2**

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

*Answer available at the end of this section.*

## Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as $x^2$? One method is as follows:

$$\begin{aligned}\log_b(x^2) &= \log_b(x \cdot x)\\ &= \log_b x + \log_b x\\ &= 2\log_b x\end{aligned}$$

Notice that we used the **product rule for logarithms** to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \ \sqrt{3} = 3^{\frac{1}{2}} \ \frac{1}{e} = e^{-1}$$

**The Power Rule for Logarithms**

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n\log_b M \tag{4.3.6}$$

**How to: Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm**

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

**Example 4.3.3: Expanding a Logarithm with Powers**

Expand $\log_2 x^5$.

**Solution**

The argument is already written as a power, so we identify the exponent, 5, and the base, $x$, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2(x^5) = 5\log_2 x$$

### Exercise 4.3.3

Expand $\ln x^2$.

*Answer available at the end of this section.*

### Example 4.3.4: Rewriting an Expression as a Power before Using the Power Rule

Expand $\log_3(25)$ using the power rule for logs.

**Solution**

Expressing the argument as a power, we get $\log_3(25) = \log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(52) = 2\log_3(5)$$

### Exercise 4.3.4

Expand $\ln\left(\frac{1}{x^2}\right)$.

*Answer available at the end of this section.*

### Example 4.3.5: Using the Power Rule in Reverse

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

**Solution**

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, $x$, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

### Exercise 4.3.5

Rewrite $2\log_3 4$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

*Answer available at the end of this section.*

## Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called “laws of logs.” Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\log_b\left(\frac{6x}{y}\right) = \log_b(6x) - \log_b y$$
$$= \log_b 6 + \log_b x - \log_b y$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\log_b\left(\frac{A}{C}\right) = \log_b(AC^{-1})$$
$$= \log_b(A) + \log_b(C^{-1})$$
$$= \log_b A + (-1)\log_b C$$
$$= \log_b A - \log_b C$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

## Example 4.3.6: Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite $ln\left(\frac{x^4y}{7}\right)$ as a sum or difference of logs.

**Solution**

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4y}{7}\right) = \ln(x^4y) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln(x^4y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

## Exercise 4.3.6

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

*Answer available at the end of this section.*

## Example 4.3.7: Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log(x)$.

**Solution**

$$\log(\sqrt{x}) = \log x^{\left(\frac{1}{2}\right)}$$
$$= \frac{1}{2}\log x$$

### Exercise 4.3.7

Expand $\ln(\sqrt[3]{x^2})$.

*Answer available at the end of this section.*

### Q&A: Can we expand $\ln(x^2 + y^2)$?

*No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.*

### Example 4.3.8: Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

**Solution**

We can expand by applying the Product and Quotient Rules.

$$\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) = \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) \quad \text{Apply the Quotient Rule}$$
$$= \log_6 26 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) \quad \text{Simplify by writing 64 as } 2^6$$
$$= 6\log_6 2 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1) \quad \text{Apply the Power Rule}$$

### Exercise 4.3.8

Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$.

*Answer available at the end of this section.*

## Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

### How to: Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

## Example 4.3.9: Using the Product and Quotient Rules to Combine Logarithms

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

**Solution**

Using the product and quotient rules

$$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$$

This reduces our original expression to

$$\log_3(40) - \log_3(2)$$

Then, using the quotient rule

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

## Exercise 4.3.9

Condense $\log_3 - \log_4 + \log_5 - \log_6$.

*Answer available at the end of this section.*

## Example 4.3.10: Condensing Complex Logarithmic Expressions

Condense $\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2)$.

**Solution**

We apply the power rule first:

$$\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2) = \log_2(x^2) + \log_2(\sqrt{x-1}) - \log_2((x+3)^6)$$

Next we apply the product rule to the sum:

$$\log_2(x^2) + \log_2(\sqrt{x-1}) - \log_2((x+3)^6) = \log_2(x^2\sqrt{x-1}) - \log_2((x+3)^6)$$

Finally, we apply the quotient rule to the difference:

$$\log_2(x^2\sqrt{x-1}) - \log_2((x+3)^6) = \log_2\frac{x^2\sqrt{x-1}}{(x+3)^6}$$

## Example 4.3.11: Rewriting as a Single Logarithm

Rewrite $2\log x - 4\log(x+5) + \frac{1}{x}\log(3x+5)$ as a single logarithm.

**Solution**

We apply the power rule first:

$$2\log x - 4\log(x+5) + \frac{1}{x}\log(3x+5) = \log(x^2) - \log(x+5)^4 + \log((3x+5)^{x^{-1}})$$

Next we rearrange and apply the product rule to the sum:

$$\begin{aligned}\log(x^2) - \log(x+5)^4 + \log((3x+5)^{x^{-1}}) &= \log(x^2) + \log((3x+5)^{x^{-1}} - \log(x+5)^4 \\ &= \log(x^2(3x+5)^{x^{-1}}) - \log(x+5)^4 \\ &= \log\frac{x^2(3x+5)^{x^{-1}}}{(x+5)^4} \qquad \text{Apply the quotient rule to the difference}\end{aligned}$$

## Exercise 4.3.10

Rewrite $\log(5) + 0.5\log(x) - \log(7x-1) + 3\log(x-1)$ as a single logarithm.

*Answer available at the end of this section.*

## Exercise 4.3.11

Condense $4(3\log(x) + \log(x+5) - \log(2x+3))$.

*Answer available at the end of this section.*

## Example 4.3.12: Applying of the Laws of Logs

Recall that, in chemistry, $pH = -\log[H+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

**Solution**

Suppose $C$ is the original concentration of hydrogen ions, and $P$ is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH = -\log(2C)$$

Using the product rule of logs

$$pH = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$,the new pH is

$$pH = P - \log(2) \approx P - 0.301$$

## Exercise 4.3.12

When the concentration of hydrogen ions is doubled, the pH decreases by about $0.301$.

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

*Answer available at the end of this section.*

# Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than $10$ ore, e,we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the **one-to-one** property and **power rule for logarithms**.

Given any positive real numbers $M$, $b$, and $n$, where $n \neq 1$ and $b \neq 1$,we show

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M$. By taking the log base $n$ of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\begin{aligned} \log_n(b^y) &= \log_n M && \text{Apply the one-to-one property} \\ y\log_n b &= \log_n M && \text{Apply the power rule for logarithms} \\ y &= \frac{\log_n M}{\log_n b} && \text{Isolate y} \\ \log_b M &= \frac{\log_n M}{\log_n b} && \text{Substitute for y} \end{aligned}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\begin{aligned} \log_5 36 &= \frac{\log(36)}{\log(5)} && \text{Apply the change of base formula using base 10} \\ &\approx 2.2266 && \text{Use a calculator to evaluate to 4 decimal places} \end{aligned}$$

## THE CHANGE-OF-BASE FORMULA

The change-of-base formula can be used to evaluate a logarithm with any base.

For any positive real numbers $M$, $b$, and $n$, where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b} \tag{4.3.7}$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b} \tag{4.3.8}$$

and

$$\log_b M = \frac{\log M}{\log b} \tag{4.3.9}$$

## How to: Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base $n$, where $n \neq 1$

1. Determine the new base $n$, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$,has base $e$.
2. Rewrite the log as a quotient using the change-of-base formula
   - The numerator of the quotient will be a logarithm with base $n$ and argument $M$.

- The denominator of the quotient will be a logarithm with base $n$ and argument $b$.

### Example 4.3.13: Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms.

**Solution**

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument $3$. The denominator of the quotient will be the natural log with argument 5.

$$\log_b M = \frac{\ln M}{\ln b}$$

$$\log_5 3 = \frac{\ln 3}{\ln 5}$$

### Exercise 4.3.13

Change $\log 0.58$ to a quotient of natural logarithms.

*Answer available at the end of this section.*

### Q&A: Can we change common logarithms to natural logarithms?

*Yes. Remember that* $\log 9$ *means*l $\log_{10} 9$. *So,* $\log 9 = \frac{\ln 9}{\ln 10}$.

### Example 4.3.14: Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

**Solution**

According to the change-of-base formula, we can rewrite the log base $2$ as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base $e$.

$$\begin{aligned} \log_2 10 &= \frac{\ln 10}{\ln 2} && \text{Apply the change of base formula using base } e \\ &\approx 3.3219 && \text{Use a calculator to evaluate to 4 decimal places} \end{aligned}$$

### Exercise 4.3.14

Evaluate $\log_5(100)$ using the change-of-base formula.

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with laws of logarithms.

- The Properties of Logarithms
- Expand Logarithmic Expressions
- Evaluate a Natural Logarithmic Expression

## Key Equations

| | |
|---|---|
| The Product Rule for Logarithms | $\log_b(MN) = \log_b(M) + \log_b(N)$ |
| The Quotient Rule for Logarithms | $\log_b(\frac{M}{N}) = \log_b M - \log_b N$ |
| The Power Rule for Logarithms | $\log_b(M^n) = n\log_b M$ |
| The Change-of-Base Formula | $\log_b M = \frac{\log_n M}{\log_n b}$ $n > 0$, $n \neq 1$, $b \neq 1$ |

## Key Concepts

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See Example $4.3.1$.
- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See Example $4.3.2$.
- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base. See Example $4.3.3$, Example $4.3.4$, and Example $4.3.5$.
- We can use the product rule, the quotient rule, and the power rule together to combine or expand a logarithm with a complex input. See Example $4.3.6$, Example $4.3.7$, and Example $4.3.8$.
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See Example $4.3.9$, Example $4.3.10$, Example $4.3.11$, and Example $4.3.12$.
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See Example $4.3.13$.
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 and $e$ as the quotient of natural or common logs. That way a calculator can be used to evaluate. See Example $4.3.14$.

### Answers

**Exercise** $4.3.1$

$\log_b 2 + \log_b 2 + \log_b 2 + \log_b k = 3\log_b 2 + \log_b k$

**Exercise** $4.3.2$

$\log_3(x+3) - \log_3(x-1) - \log_3(x-2)$

**Exercise** $4.3.3$

$2\ln x$

**Exercise** $4.3.4$

$-2\ln(x)$

**Exercise** $4.3.5$

$\log_3 16$

**Exercise** $4.3.6$

$2\log x + 3\log y - 4\log z$

**Exercise** $4.3.7$

$\frac{2}{3}\ln x$

**Exercise** $4.3.8$

$\frac{1}{2}\ln(x-1) + \ln(2x+1) - \ln(x+3) - \ln(x-3)$

**Exercise** $4.3.9$

$\log\left(\frac{3\cdot 5}{4\cdot 6}\right)$; can also be written $\log\left(\frac{5}{8}\right)$ by reducing the fraction to lowest terms.

**Exercise** $4.3.10$

$\log\frac{5(x-1)^3\sqrt{x}}{(7x-1)}$

**Exercise** $4.3.11$

$\log \dfrac{x^{12}(x+5)^4}{(2x+3)^4}$; this answer could also be written $\log\left(\dfrac{x^3(x+5)}{(2x+3)}\right)^4$

**Exercise** $4.3.12$

The pH increases by about $0.301$.

**Exercise** $4.3.13$

$\dfrac{\ln 8}{\ln 0.5}$

**Exercise** $4.3.14$

$\dfrac{\ln 100}{\ln 5} \approx \dfrac{4.6051}{1.6094} = 2.861$

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# 4.4: Exponential and Logarithmic Equations

**Learning Objectives**

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

In 1859, an Australian landowner named Thomas Austin released $24$ rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.



Figure $4.4.1$: Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the "rabbit plague." (credit: Richard Taylor, Flickr)

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

## Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers $b$, $S$, and $T$, where $b > 0$, $b \neq 1$, $b^S = b^T$ if and only if $S = T$.

In other words, when an **exponential equation** has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7} = \dfrac{3^{2x}}{3}$. To solve for $x$, we use the division property of exponents to rewrite the right side so that both sides have the common base, $3$. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for $x$:

$$
\begin{aligned}
3^{4x-7} &= \frac{3^{2x}}{3} \\
3^{4x-7} &= \frac{3^{2x}}{3^1} && \text{Rewrite 3 as } 3^1 \\
3^{4x-7} &= 3^{2x-1} && \text{Use the division property of exponents} \\
4x-7 &= 2x-1 && \text{Apply the one-to-one property of exponents} \\
2x &= 6 && \text{Subtract 2x and add 7 to both sides} \\
x &= 3 && \text{Divide by 3}
\end{aligned}
$$

### USING THE ONE-TO-ONE PROPERTY OF EXPONENTIAL FUNCTIONS TO SOLVE EXPONENTIAL EQUATIONS

For any algebraic expressions $S$ and $T$, and any positive real number $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T \tag{4.4.1}$$

### How to: Given an exponential equation with the form $b^S = b^T$, where $S$ and $T$ are algebraic expressions with an unknown, solve for the unknown.

1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S = T$, for the unknown.

### Example 4.4.1: Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

**Solution**

$$
\begin{aligned}
2^{x-1} &= 2^{2x-4} && \text{The common base is 2} \\
x-1 &= 2x-4 && \text{By the one-to-one property the exponents must be equal} \\
x &= 3 && \text{Solve for x}
\end{aligned}
$$

### Exercise 4.4.1

Solve $5^{2x} = 5^{3x+2}$.

*Answer available at the end of this section.*

## Rewrite Equations So All Powers Have the Same Base

Sometimes the *common base* for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for $x$:

$$
\begin{aligned}
256 &= 4^{x-5} && \\
2^8 &= (2^2)^{x-5} && \text{Rewrite each side as a power with base 2} \\
2^8 &= 2^{2x-10} && \text{Use the one-to-one property of exponents} \\
8 &= 2x-10 && \text{Apply the one-to-one property of exponents} \\
18 &= 2x && \text{Add 10 to both sides} \\
x &= 9 && \text{Divide by 2}
\end{aligned}
$$

### How to: Given an exponential equation with unlike bases, use the one-to-one property to solve it.

1. Rewrite each side in the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
3. Use the one-to-one property to set the exponents equal.
4. Solve the resulting equation, $S = T$, for the unknown.

### Example 4.4.2: Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2} = 16^{x+1}$.

**Solution**

$$
\begin{aligned}
8^{x+2} &= 16^{x+1} && \\
(2^3)^{x+2} &= (2^4)^{x+1} && \text{Write 8 and 16 as powers of 2} \\
2^{3x+6} &= 2^{4x+4} && \text{To take a power of a power, multiply exponents} \\
3x+6 &= 4x+4 && \text{Use the one-to-one property to set the exponents equal} \\
x &= 2 && \text{Solve for x}
\end{aligned}
$$

### Exercise 4.4.2

Solve $5^{2x} = 25^{3x+2}$.

*Answer available at the end of this section.*

### Example 4.4.3: Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve $2^{5x} = \sqrt{2}$.

**Solution**

$$
\begin{aligned}
2^{5x} &= 2^{\frac{1}{2}} && \text{Write the square root of 2 as a power of 2} \\
5x &= \frac{1}{2} && \text{Use the one-to-one property} \\
x &= \frac{1}{10} && \text{Solve for x}
\end{aligned}
$$

### Exercise 4.4.3

Solve $5^x = \sqrt{5}$.

*Answer available at the end of this section.*

**Q&A: Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?**

*No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.*

**Example 4.4.4: Solving an Equation with Positive and Negative Powers**

Solve $3^{x+1} = -2$.

**Solution**

This equation has no solution. There is no real value of $x$ that will make the equation a true statement because any power of a positive number is positive.

Analysis

Figure 4.4.2 shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

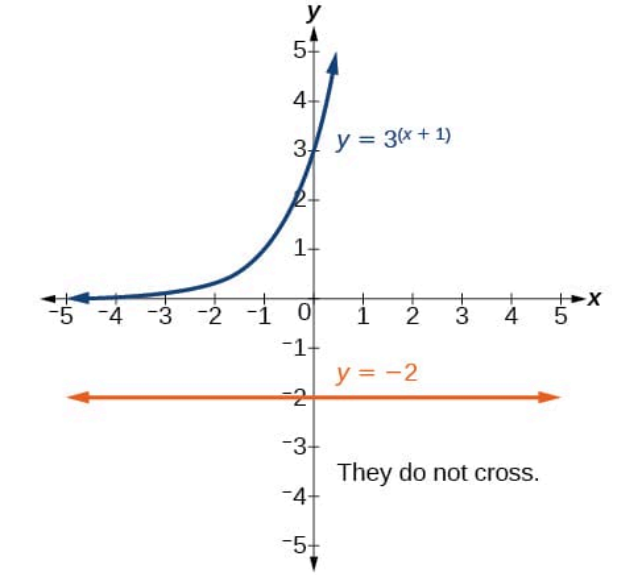



Figure 4.4.2

**Exercise 4.4.4**

Solve $2^x = -100$.

*Answer available at the end of this section.*

## Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

**How to: Given an exponential equation in which a common base cannot be found, solve for the unknown**

1. Apply the logarithm of both sides of the equation.
   - If one of the terms in the equation has base 10, use the common logarithm.
   - If none of the terms in the equation has base 10, use the natural logarithm.
2. Use the rules of logarithms to solve for the unknown.

**Example 4.4.5: Solving an Equation Containing Powers of Different Bases**

Solve $5^{x+2} = 4^x$.

**Solution**

| | |
|---|---|
| $5^{x+2} = 4^x$ | There is no easy way to get the powers to have the same base |
| $\ln 5^{x+2} = \ln 4^x$ | Take ln of both sides |
| $(x+2)\ln 5 = x \ln 4$ | Use laws of logs |
| $x \ln 5 + 2 \ln 5 = x \ln 4$ | Use the distributive law |
| $x \ln 5 - x \ln 4 = -2 \ln 5$ | Get terms containing x on one side, terms without x on the other |
| $x(\ln 5 - \ln 4) = -2 \ln 5$ | On the left hand side, factor out an x |
| $x \ln\left(\frac{5}{4}\right) = \ln\left(\frac{1}{25}\right)$ | Use the laws of logs |
| $x = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}$ | Divide by the coefficient of x |

**Exercise 4.4.5**

Solve $2^x = 3^{x+1}$.

*Answer available at the end of this section.*

**Q&A: Is there any way to solve $2^x = 3^x$?**

*Yes. The solution is* $0$.

## Equations Containing $e$

One common type of exponential equations are those with base $e$. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base $e$ on either side, we can use the natural logarithm to solve it.

**How to: Given an equation of the form $y = Ae^{kt}$, solve for $t$.**

1. Divide both sides of the equation by $A$.
2. Apply the natural logarithm of both sides of the equation.
3. Divide both sides of the equation by $k$.

### Example 4.4.6: Solve an Equation of the Form $y = Ae^{kt}$

Solve $100 = 20e^{2t}$.

**Solution**

$$\begin{aligned} 100 &= 20e^{2t} \\ 5 &= e^{2t} && \text{Divide by the coefficient of the power} \\ \ln 5 &= 2t && \text{Take ln of both sides. Use the fact that } ln(x) \text{ and } e^x \text{ are inverse functions} \\ t &= \frac{\ln 5}{2} && \text{Divide by the coefficient of t} \end{aligned}$$

**Analysis**

Using laws of logs, we can also write this answer in the form $t = \ln \sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

### Exercise 4.4.6

Solve $3e^{0.5t} = 11$.

*Answer available at the end of this section.*

### Q&A: Does every equation of the form $y = Ae^{kt}$ have a solution?

No. There is a solution when $k \neq 0$,and when $y$ and $A$ are either both 0 or neither 0, and they have the same sign. An example of an equation with this form that has no solution is $2 = -3e^t$.

### Example 4.4.7: Solving an Equation That Can Be Simplified to the Form $y = Ae^{kt}$

Solve $4e^{2x} + 5 = 12$.

**Solution**

$$\begin{aligned} 4e^{2x} + 5 &= 12 \\ 4e^{2x} &= 7 && \text{Combine like terms} \\ e^{2x} &= \frac{7}{4} && \text{Divide by the coefficient of the power} \\ 2x &= \ln\left(\frac{7}{4}\right) && \text{Take ln of both sides} \\ x &= \frac{1}{2}\ln\left(\frac{7}{4}\right) && \text{Solve for x} \end{aligned}$$

### Exercise 4.4.7

Solve $3 + e^{2t} = 7e^{2t}$.

*Answer available at the end of this section.*

## Extraneous Solutions

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct

algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

### Example 4.4.8: Solving Exponential Functions in Quadratic Form

Solve $e^{2x} - e^x = 56$.

**Solution**

$$
\begin{aligned}
e^{2x} - e^x &= 56 && \\
e^{2x} - e^x - 56 &= 0 && \text{Get one side of the equation equal to zero} \\
(e^x + 7)(e^x - 8) &= 0 && \text{Factor by the FOIL method} \\
e^x + 7 &= 0 && \text{or} \\
e^x - 8 &= 0 && \text{If a product is zero, then one factor must be zero} \\
e^x &= -7 && \text{or} \\
e^x &= 8 && \text{Isolate the exponentials} \\
e^x &= 8 && \text{Reject the equation in which the power equals a negative number} \\
x &= \ln 8 && \text{Solve the equation in which the power equals a positive number}
\end{aligned}
$$

**Analysis**

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation $e^x = -7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

### Exercise 4.4.8

Solve $e^{2x} = e^x + 2$.

*Answer available at the end of this section.*

### Q&A: Does every logarithmic equation have a solution?

*No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.*

## Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for $x$:

$$
\begin{aligned}
\log_2(2) + \log_2(3x - 5) &= 3 && \\
\log_2(2(3x - 5)) &= 3 && \text{Apply the product rule of logarithms} \\
\log_2(6x - 10) &= 3 && \text{Distribute} \\
2^3 &= 6x - 10 && \text{Apply the definition of a logarithm} \\
8 &= 6x - 10 && \text{Calculate } 2^3 \\
18 &= 6x && \text{Add 10 to both sides} \\
x &= 3 && \text{Divide by 6}
\end{aligned}
$$

## USING THE DEFINITION OF A LOGARITHM TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expression $S$ and real numbers $b$ and $c$,where $b > 0$, $b \neq 1$,

$$\log_b(S) = c \text{ if and only if } b^c = S \tag{4.4.2}$$

### Example 4.4.9: Using Algebra to Solve a Logarithmic Equation

Solve $2\ln x + 3 = 7$.

**Solution**

$$\begin{aligned} 2\ln x + 3 &= 7 \\ 2\ln x &= 4 \quad \text{Subtract 3} \\ \ln x &= 2 \quad \text{Divide by 2} \\ x &= e^2 \quad \text{Rewrite in exponential form} \end{aligned}$$

### Exercise 4.4.9

Solve $6 + \ln x = 10$.

*Answer available at the end of this section.*

### Example 4.4.10: Using Algebra Before and After Using the Definition of the Natural Logarithm

Solve $2\ln(6x) = 7$.

**Solution**

$$\begin{aligned} 2\ln(6x) &= 7 \\ \ln(6x) &= \frac{7}{2} \quad \text{Divide by 2} \\ 6x &= e^{\left(\frac{7}{2}\right)} \quad \text{Use the definition of ln} \\ x &= \frac{1}{6}e^{\left(\frac{7}{2}\right)} \quad \text{Divide by 6} \end{aligned}$$

### Exercise 4.4.10

Solve $2\ln(x + 1) = 10$.

*Answer available at the end of this section.*

### Example 4.4.11: Using a Graph to Understand the Solution to a Logarithmic Equation

Solve $\ln x = 3$.

**Solution**

$$\begin{aligned} \ln x &= 3 \\ x &= e^3 \quad \text{Use the definition of the natural logarithm} \end{aligned}$$

Figure $4.4.3$ represents the graph of the equation. On the graph, the *x*-coordinate of the point at which the two graphs intersect is close to $20$. In other words $e^3 \approx 20$. A calculator gives a better approximation: $e^3 \approx 20.0855$.

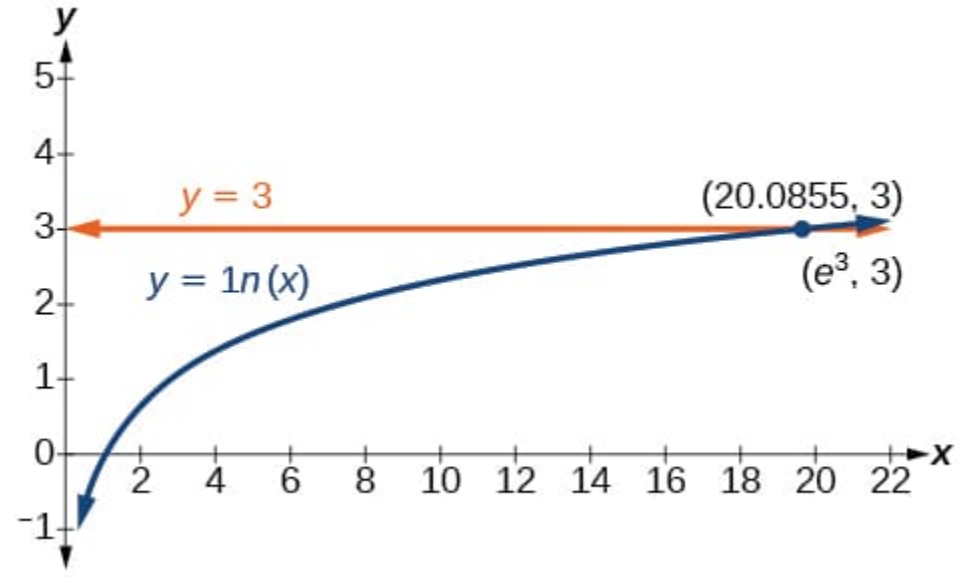



Figure $4.4.3$: The graphs of $y = \ln x$ and $y = 3$ cross at the point $(e^3, 3)$, which is approximately $(20.0855, 3)$.

**Exercise $4.4.11$**

Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to $2$ decimal places.

*Answer available at the end of this section.*

## Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number $b$, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{ then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for $x$,and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9 - 1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x - 2) - \log(2) = \log(x + 4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for $x$:

$$\begin{aligned}
\log(3x-2)-\log(2) &= \log(x+4)\\
\log\left(\frac{3x-2}{2}\right) &= \log(x+4) \quad \text{Apply the quotient rule of logarithms}\\
\frac{3x-2}{2} &= x+4 \quad \text{Apply the one to one property of a logarithm}\\
3x-2 &= 2x+8 \quad \text{Multiply both sides of the equation by 2}\\
x &= 10 \quad \text{Subtract 2x and add 2}
\end{aligned}$$

To check the result, substitute $x = 10$ into $\log(3x-2)-\log(2) = \log(x+4)$.

$$\begin{aligned}
\log(3(10)-2)-\log(2) &= \log((10)+4)\\
\log(28)-\log(2) &= \log(14)\\
\log\left(\frac{28}{2}\right) &= \log(14) \quad \text{The solution checks}
\end{aligned}$$

## USING THE ONE-TO-ONE PROPERTY OF LOGARITHMS TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expressions $S$ and $T$ and any positive real number $b$, where $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T \tag{4.4.3}$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

## How to: Given an equation containing logarithms, solve it using the one-to-one property

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log_b S = \log_b T$ .
2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S = T$, for the unknown.

## Example 4.4.12: Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2) = \ln(2x+3)$.

**Solution**

$$\begin{aligned}
\ln(x^2) &= \ln(2x+3)\\
x^2 &= 2x+3 \quad \text{Use the one-to-one property of the logarithm}\\
x^2-2x-3 &= 0 \quad \text{Get zero on one side before factoring}\\
(x-3)(x+1) &= 0 \quad \text{Factor using FOIL}\\
x-3 &= 0 \quad \text{or } x+1=0 \text{ If a product is zero, one of the factors must be zero}\\
x &= 3 \quad \text{or}\\
x &= -11 \quad \text{Solve for x}
\end{aligned}$$

**Analysis**

There are two solutions: $3$ or $-1$. The solution $-1$ is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

## Exercise 4.4.12

Solve $\ln(x^2) = \ln 1$.

*Answer available at the end of this section.*

## Solving Applied Problems Using Exponential and Logarithmic Equations

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its **half-life**. Table $4.4.1$ lists the half-life for several of the more common radioactive substances.

*Table* $4.4.1$

| Substance | Use | Half-life |
|---|---|---|
| *gallium-67* | *nuclear medicine* | *80 hours* |
| *cobalt-60* | *manufacturing* | *5.3 years* |
| *technetium-99m* | *nuclear medicine* | *6 hours* |
| *americium-241* | *construction* | *432 years* |
| *carbon-14* | *archeological dating* | *5,715 years* |
| *uranium-235* | *atomic power* | *703,800,000 years* |

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:

$$A(t) = A_0 e^{\frac{\ln(0.5)}{T}t} \tag{4.4.4}$$

$$A(t) = A_0 e^{\frac{\ln(0.5)t}{T}} \tag{4.4.5}$$

$$A(t) = A_0 (e^{\ln(0.5)})^{\frac{t}{T}} \tag{4.4.6}$$

$$A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{T}} \tag{4.4.7}$$

where

- $A_0$ is the amount initially present
- $T$ is the half-life of the substance
- $t$ is the time period over which the substance is studied
- $y$ is the amount of the substance present after time $t$

### Example $4.4.13$: Using the Formula for Radioactive Decay to Find the Quantity of a Substance

How long will it take for ten percent of a $1000$-gram sample of uranium-235 to decay?

**Solution**

$$
\begin{aligned}
y &= 1000e^{\frac{\ln(0.5)}{703{,}800{,}000}t} \\
900 &= 1000e^{\frac{\ln(0.5)}{703{,}800{,}000}t} && \text{After 10\% decays, 900 grams are left} \\
0.9 &= e^{\frac{\ln(0.5)}{703{,}800{,}000}t} && \text{Divide by 1000} \\
\ln(0.9) &= \ln\left(e^{\frac{\ln(0.5)}{703{,}800{,}000}t}\right) && \text{Take ln of both sides} \\
\ln(0.9) &= \frac{\ln(0.5)}{703,800,000}t && \ln(e^M) = M \\
t &= 703,800,000 \times \frac{\ln(0.9)}{\ln(0.5)} && \text{years Solve for t} \\
t &\approx 106,979,777 && \text{years}
\end{aligned}
$$

**Analysis**

Ten percent of $1000$ grams is $100$ grams. If $100$ grams decay, the amount of uranium-235 remaining is $900$ grams.

### Exercise 4.4.13

How long will it take before twenty percent of our 1000-gram sample of uranium-235 has decayed?

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Exponential Equations with Logarithms

## Key Equations

| | |
|---|---|
| One-to-one property for exponential functions | For any algebraic expressions $S$ and $T$ and any positive real number $b$, where $b^S = b^T$ if and only if $S = T$. |
| Definition of a logarithm | For any algebraic expression S and positive real numbers $b$ and $c$, where $b \neq 1$, $\log_b(S) = c$ if and only if $b^c = S$. |
| One-to-one property for logarithmic functions | For any algebraic expressions $S$ and $T$ and any positive real number $b$, where $b \neq 1$, $\log_b S = \log_b T$ if and only if $S = T$. |

## Key Concepts

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown. See Example $4.4.1$.
- When we are given an exponential equation where the bases are *not* explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown. See Example $4.4.2$, Example $4.4.3$, and Example $4.4.4$.
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side. See Example $4.4.5$.

- We can solve exponential equations with base $e$,by applying the natural logarithm of both sides because exponential and logarithmic functions are inverses of each other. See Example $4.4.6$ and Example $4.4.7$.
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions. See Example $4.4.8$.
- When given an equation of the form $\log_b(S) = c$, where $S$ is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown. See Example $4.4.9$ and Example $4.4.10$.
- We can also use graphing to solve equations with the form $\log_b(S) = c$. We graph both equations $y = \log_b(S)$ and $y = c$ on the same coordinate plane and identify the solution as the *x*-value of the intersecting point. See Example $4.4.11$.
- When given an equation of the form $\log_b S = \log_b T$, where $S$ and $T$ are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown. See Example $4.4.12$.
- Combining the skills learned in this and previous sections, we can solve equations that model real world situations, whether the unknown is in an exponent or in the argument of a logarithm. See Example $4.4.13$.

## Answers

**Exercise** $4.4.1$

$x = -2$

**Exercise** $4.4.2$

$x = -1$

**Exercise** $4.4.3$

$x = \frac{1}{2}$

**Exercise** $4.4.4$

The equation has no solution.

**Exercise** $4.4.5$

$x = \frac{\ln 3}{\ln\left(\frac{2}{3}\right)}$

**Exercise** $4.4.6$

$t = 2\ln\left(\frac{11}{3}\right)$ or $\ln\left(\frac{11}{3}\right)^2$

**Exercise** $4.4.7$

$t = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln(2)$

**Exercise** $4.4.8$

$x = \ln 2$

**Exercise** $4.4.9$

$x = e^4$

**Exercise** $4.4.10$

$x = e^5 - 1$

**Exercise** $4.4.11$

$x \approx 9.97$

**Exercise** $4.4.12$

$x = 1$ or $x = -1$

**Exercise** $4.4.13$

$t = 703,800,000 \times \frac{\ln(0.8)}{\ln(0.5)}$years ≈ 226,572,993 years.

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# 4.5: Applications of Exponential Functions

**Learning Objectives**

- Build an exponential growth model
- Build an exponential decay model
- Use a model to solve application problems

In previous sections of this chapter, we were either given a function explicitly to graph or evaluate. In this section, we will use as set of data to "model" the situation with an exponential function. Then, we can use that function to answer questions about that particular problem.

Do not be confused by the word *model*. In mathematics, we often use the terms *function*, *equation*, and *model* interchangeably, even though they each have their own formal definition. The term *model* is typically used to indicate that the equation or function approximates a real-world situation.

We will concentrate on two types of models in this section: exponential growth and exponential decay. Having already worked with each of these functions gives us an advantage. Knowing their formal definitions, the behavior of their graphs, and some of their real-world applications gives us the opportunity to deepen our understanding. As each regression model is presented, key features and definitions of its associated function are included for review. Take a moment to rethink each of these functions, reflect on the work we've done so far, and then explore the ways regression is used to model real-world phenomena.

## Building an Exponential Model from Data

As we've learned, there are a multitude of situations that can be modeled by exponential functions, such as investment growth, radioactive decay, atmospheric pressure changes, and temperatures of a cooling object. What do these phenomena have in common? For one thing, all the models either increase or decrease as time moves forward. But that's not the whole story. It's the *way* data increase or decrease that helps us determine whether it is best modeled by an exponential equation. Knowing the behavior of exponential functions in general allows us to recognize when to use exponential regression, so let's review exponential growth and decay.

Recall that exponential functions have the form $y = Ca^x$ or $y = Ce^{kx}$. When performing regression analysis, we use the form most commonly used on graphing utilities, $y = Ca^x$. Take a moment to reflect on the characteristics we've already learned about the exponential function $y = Ca^x$ (assume $a > 0$):

- $a$ must be greater than zero and not equal to one.
- The initial value of the model is $y = C$.
  - If $a > 1$, the function models **exponential growth**. As $x$ increases, the outputs of the model increase slowly at first, but then increase more and more rapidly, without bound.
  - If $0 < a < 1$, the function models **exponential decay**. As $x$ increases, the outputs for the model decrease rapidly at first and then level off to become asymptotic to the *x*-axis. In other words, the outputs never become equal to or less than zero.

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function:

$$y = Ce^{kt} \tag{4.5.1}$$

where $C$ is equal to the value at time zero, $e$ is Euler's constant, and $k$ is a positive constant that determines the rate (percentage) of growth. We may use the exponential growth function in applications involving **doubling time**, the time it takes for a quantity to double. Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit

growth based on a doubling time. In some applications, however, as we will see when we discuss the logistic equation, the logistic model sometimes fits the data better than the exponential model.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the **exponential decay** model. Again, we have the form $y = Ce^{kt}$ where $C$ is the starting value, and $e$ is Euler's constant. Now $k$ is a negative constant that determines the rate of decay. We may use the exponential decay model when we are calculating half-life, or the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape, as we can see in Figure $4.5.2$ and Figure $4.5.3$. It is important to remember that, although parts of each of the two graphs seem to lie on the $x$-axis, they are really a tiny distance above the $x$-axis.

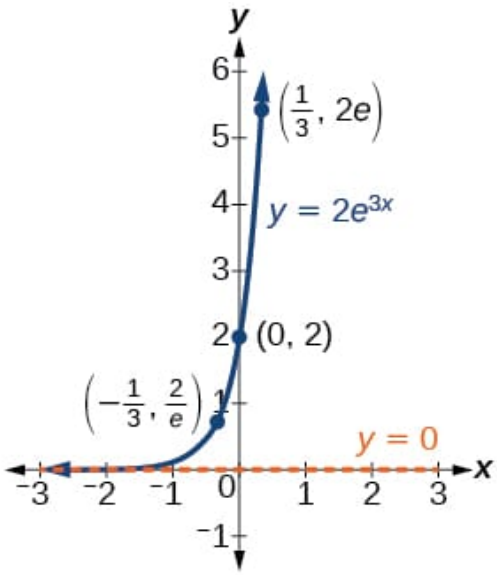



Figure $4.5.2$: A graph showing exponential growth. The equation is $y = 2e^{3x}$.

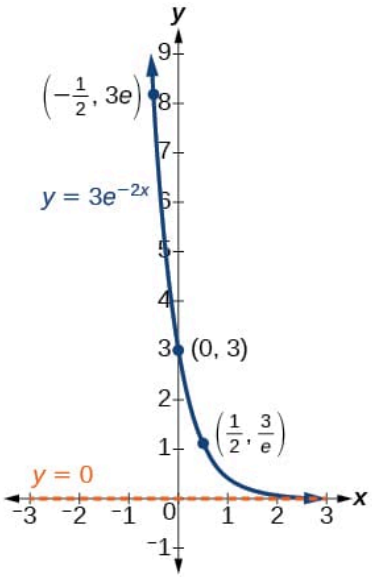



Figure $4.5.3$: A graph showing exponential decay. The equation is $y = 3e^{-2x}$.

Exponential growth and decay often involve very large or very small numbers. To describe these numbers, we often use orders of magnitude. The **order of magnitude** is the power of ten, when the number is expressed in scientific notation, with one digit to the left of the decimal. For example, the distance to the nearest star, **Proxima Centauri**, measured in kilometers, is

$40,113,497,200,000$ kilometers. Expressed in scientific notation, this is $4.01134972 \times 1013$. So, we could describe this number as having order of magnitude $1013$.

### $y = Ce^{kt}$

An exponential function with the form $y = Ce^{kt}$ has the following characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- $x$ intercept: none
- $y$-intercept: $(0, C)$
- increasing if $k > 0$
- decreasing if $k < 0$

## Example 4.5.1: Graphing Exponential Growth

A population of bacteria doubles every hour. If the culture started with $10$ bacteria, graph the population as a function of time.

**Solution**

When an amount grows at a fixed percent per unit time, the growth is exponential. To find $C$ we use the fact that $C$ is the amount at time zero, so $C = 10$. To find $k$, use the fact that after one hour $(t = 1)$ the population doubles from $10$ to $20$.The formula is derived as follows

$$\begin{aligned} 20 &= 10e^{k \cdot 1} \\ 2 &= e^k \qquad \text{Divide by } 10 \\ \ln 2 &= k \qquad \text{Take the natural logarithm} \end{aligned}$$

so $k = \ln(2)$. Thus the equation we want to graph is $y = 10e^{(\ln 2)t} = 10(e^{\ln 2})^t = 10 \cdot 2^t$. The graph is shown in Figure $4.5.5$.

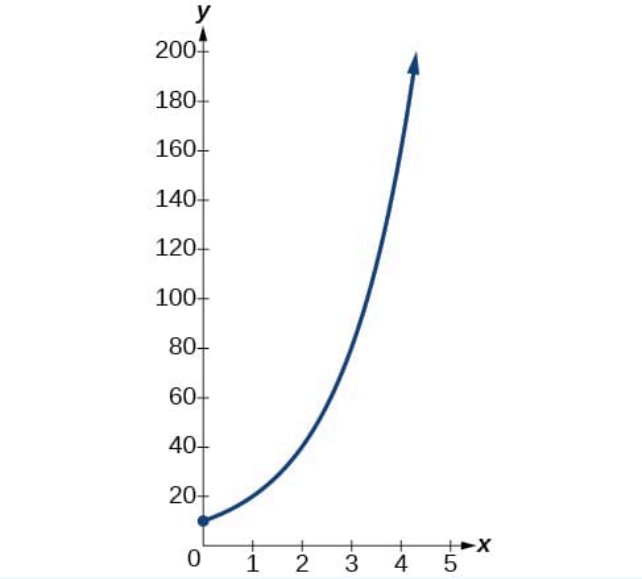



Figure $4.5.5$: The graph of $y = 10e^{(\ln 2)t}$

**Analysis**

The population of bacteria after ten hours is $10,240$. We could describe this amount is being of the order of magnitude $10^4$. The population of bacteria after twenty hours is $10,485,760$ which is of the order of magnitude $10^7$, so we could say that the population has increased by three orders of magnitude in ten hours.

## Calculating Doubling Time

For decaying quantities, we determined how long it took for half of a substance to decay. For growing quantities, we might want to find out how long it takes for a quantity to double. As we mentioned above, the time it takes for a quantity to double is called the **doubling time**.

Given the basic **exponential growth** equation $y = Ce^{kt}$, doubling time can be found by solving for when the original quantity has doubled, that is, by solving $2C = Ce^{kt}$.

The formula is derived as follows:

$$\begin{aligned} 2A_0 &= Ce^{kt} \\ 2 &= e^{kt} && \text{Divide by } C \\ \ln 2 &= kt && \text{Take the natural logarithm} \\ t &= \frac{\ln 2}{k} && \text{Divide by the coefficient of t} \end{aligned}$$

Thus the doubling time is

$$t = \frac{\ln 2}{k} \tag{4.5.2}$$

### Example 4.5.4: Finding a Function That Describes Exponential Growth

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

**Solution**

The formula is derived as follows:

$$\begin{aligned} t &= \frac{\ln 2}{k} && \text{The doubling time formula} \\ 2 &= \frac{\ln 2}{k} && \text{Use a doubling time of two years} \\ k &= \frac{\ln 2}{2} && \text{Multiply by k and divide by 2} \\ y &= Ce^{\frac{\ln 2}{2}t} && \text{Substitute k into the continuous growth formula} \end{aligned}$$

The function is $y = Ce^{(\frac{\ln 2}{2})t}$.

### Exercise 4.5.3

Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.

*Answer available at the end of this section.*

# Half-Life

We now turn to **exponential decay**. One of the common terms associated with exponential decay, as stated above, is **half-life**, the length of time it takes an exponentially decaying quantity to decrease to half its original amount. Every radioactive isotope has a half-life, and the process describing the exponential decay of an isotope is called radioactive decay.

To find the half-life of a function describing exponential decay, solve the following equation:

$$\frac{1}{2}C = Ce^{kt}$$

We find that the half-life depends only on the constant $k$ and not on the starting quantity $C$.

The formula is derived as follows

$$\begin{aligned} \frac{1}{2}A_0 &= Ce^{kt} \\ \frac{1}{2} &= e^{kt} && \text{Divide by } C \\ \ln\left(\frac{1}{2}\right) &= ktv && \text{Take the natural log} \\ -\ln(2) &= kt && \text{Apply laws of logarithms} \\ -\ln(2)k &= t && \text{Divide by k} \end{aligned}$$

Since $t$, the time, is positive, $k$ must, as expected, be negative. This gives us the half-life formula

$$t = -\frac{\ln(2)}{k} \tag{4.5.3}$$

## Given the half-life, find the decay rate.

1. Write $y = Ce^{kt}$.
2. Replace $y$ by $\frac{1}{2}C$ and replace $t$ by the given half-life.
3. Solve to find $k$. Express $k$ as an exact value (do not round).

Note: It is also possible to find the decay rate using $k = -\ln(2)t$.

## Example 4.5.2: Finding the Function that Describes Radioactive Decay

The half-life of carbon-14 is $5,730$ years. Express the amount of carbon-14 remaining as a function of time, $t$.

**Solution**

This formula is derived as follows.

$$\begin{aligned} y &= Ce^{kt} && \text{The continuous growth formula} \\ 0.5C &= Ce^{k\cdot 5730} && \text{Substitute the half-life for t and } 0.5A_0 \text{ for } f(t) \\ 0.5 &= e^{5730k} && \text{Divide by } C \\ \ln(0.5) &= 5730k && \text{Take the natural log of both sides} \\ k &= \frac{\ln(0.5)}{5730} && \text{Divide by the coefficient of k} \\ y &= Ce^{\left(\frac{\ln(0.5)}{5730}\right)t} && \text{Substitute for r in the continuous growth formula} \end{aligned}$$

The function that describes this continuous decay is $f(t) = Ce^{\left(\frac{\ln(0.5)}{5730}\right)t}$. We observe that the coefficient of $t$, $\frac{\ln(0.5)}{5730} \approx -1.2097 \times 10^{-4}$ is negative, as expected in the case of exponential decay.

**Exercise 4.5.1**

The half-life of plutonium-244 is $80,000,000$ years. Find function gives the amount of carbon-14 remaining as a function of time, measured in years.

*Answer available at the end of this section.*

## Radiocarbon Dating

The formula for radioactive decay is important in **radiocarbon dating**, which is used to calculate the approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby, who won a Nobel Prize for his discovery. It compares the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about $1\%$ error for plants or animals that died within the last $60,000$ years.

Carbon-14 is a radioactive isotope of carbon that has a half-life of $5,730$ years. It occurs in small quantities in the carbon dioxide in the air we breathe. Most of the carbon on Earth is carbon-12, which has an atomic weight of $12$ and is not radioactive. Scientists have determined the ratio of carbon-14 to carbon-12 in the air for the last $60,000$ years, using tree rings and other organic samples of known dates—although the ratio has changed slightly over the centuries.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Since the half-life of carbon-14 is $5,730$ years, the formula for the amount of carbon-14 remaining after $t$ years is

$$y \approx Ce^{\left(\frac{\ln(0.5)}{5730}\right)t}$$

where

- $y$ is the amount of carbon-14 remaining
- $C$ is the amount of carbon-14 when the plant or animal began decaying.

This formula is derived as follows:

$$\begin{aligned}
y &= Ce^{kt} && \text{The continuous growth formula} \\
0.5C &= Ce^{k\cdot 5730} && \text{Substitute the half-life for t and } 0.5C \text{ for f(t)} \\
0.5 &= e^{5730k} && \text{Divide by } C \\
\ln(0.5) &= 5730k && \text{Take the natural log of both sides} \\
k &= \frac{\ln(0.5)}{5730} && \text{Divide by the coefficient of k} \\
y &= Ce^{\left(\frac{\ln(0.5)}{5730}\right)t} && \text{Substitute for r in the continuous growth formula}
\end{aligned}$$

To find the age of an object, we solve this equation for $t$:

$$t = \frac{\ln\left(\frac{y}{C}\right)}{-0.000121}$$

Out of necessity, we neglect here the many details that a scientist takes into consideration when doing carbon-14 dating, and we only look at the basic formula. The ratio of carbon-14 to carbon-12 in the atmosphere is approximately $0.0000000001$. Let $r$ be the ratio of carbon-14 to carbon-12 in the organic artifact or fossil to be dated, determined by a method called liquid scintillation. From the equation $y \approx Ce^{-0.000121t}$ we know the ratio of the percentage of carbon-14 in the object we are dating to the percentage of carbon-14 in the atmosphere is $r = \frac{y}{C} \approx e^{-0.000121t}$. We solve this equation for $t$, to get

$$t = \frac{\ln(r)}{-0.000121}$$

### Given the percentage of carbon-14 in an object, determine its age.

1. Express the given percentage of carbon-14 as an equivalent decimal, $k$.
2. Substitute for $k$ in the equation $t = \dfrac{\ln(r)}{-0.000121}$ and solve for the age, $t$.

### Example 4.5.3: Finding the Age of a Bone

A bone fragment is found that contains $20\%$ of its original carbon-14. To the nearest year, how old is the bone?

**Solution**

We substitute $20\% = 0.20$ for $k$ in the equation and solve for $t$:

$$\begin{aligned} t &= \frac{\ln(r)}{-0.000121} && \text{Use the general form of the equation} \\ &= \frac{\ln(0.20)}{-0.000121} && \text{Substitute for r} \\ &\approx 13301 && \text{Round to the nearest year} \end{aligned}$$

The bone fragment is about $13,301$ years old.

**Analysis**

The instruments that measure the percentage of carbon-14 are extremely sensitive and, as we mention above, a scientist will need to do much more work than we did in order to be satisfied. Even so, carbon dating is only accurate to about $1\%$, so this age should be given as $13,301$ years $\pm 1\%$ or $13,301$ years $\pm 133$ years.

### Exercise 4.5.2

Cesium-137 has a half-life of about $30$ years. If we begin with $200$ mg of cesium-137, will it take more or less than $230$ years until only $1$ milligram remains?

*Answer available at the end of this section.*

## Answers

**Exercise** $4.5.3$

$f(t) = Ce^{\frac{\ln 2}{3}t}$

**Exercise** $4.5.1$

$f(t) = Ce^{-0.0000000087t}$

**Exercise** $4.5.2$

less than $230$ years, $229.3157$ to be exact

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# 5: POLYNOMIAL AND RATIONAL FUNCTIONS


5.1: Quadratic Functions with Applications

5.2: Power Functions and Polynomial Functions

5.3: Graphs of Polynomial Functions

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# 5.1: Quadratic Functions with Applications

**Learning Objectives**

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.
- Solve applications using quadratic functions.

## Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the **minimum value** of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure $5.1.2$.

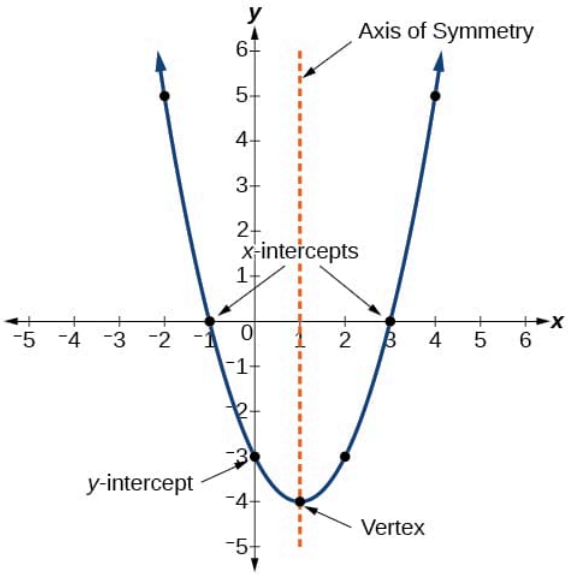



Figure $5.1.1$: Graph of a parabola showing where the $x$ and $y$ intercepts, vertex, and axis of symmetry are.

The y-intercept is the point at which the parabola crosses the $y$-axis. The x-intercepts are the points at which the parabola crosses the $x$-axis. If they exist, the x-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of $x$ at which $y = 0$.

**Example $5.1.1$: Identifying the Characteristics of a Parabola**

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure $5.1.3$.

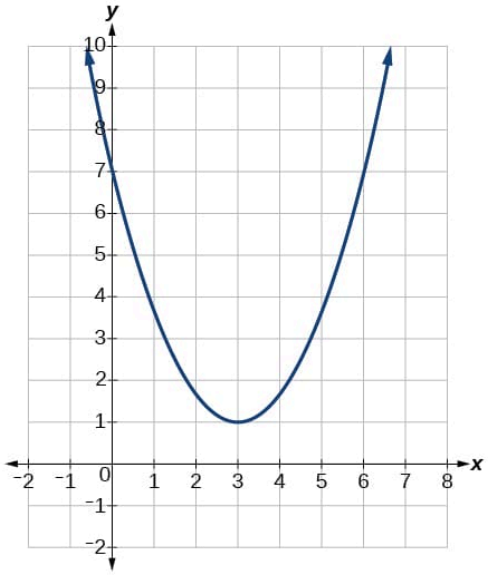



Figure 5.1.2.

**Solution**

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the $y$-axis at $(0, 7)$ so this is the y-intercept.

## Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

**Definition: Standard Form of a Quadratic Function**

The **standard (or general) form of a quadratic function** is any function of the form

$$f(x) = ax^2 + bx + c \tag{5.1.1}$$

where $a$, $b$, and $c$ are real numbers and $a \neq 0$.

If we use the quadratic formula to solve $ax^2 + bx + c = 0$, this will give the $x$ intercepts

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5.1.2}$$

And, the value that is halfway between these $x$ values can be found by

$$
\begin{aligned}
x &= \left(\frac{-b-\sqrt{b^2-4ac}}{2a} + \frac{-b+\sqrt{b^2-4ac}}{2a}\right) \div 2 \\
&= \left(\frac{-b-\sqrt{b^2-4ac}-b+\sqrt{b^2-4ac}}{2a}\right) \div 2 \\
&= \left(\frac{-b-b}{2a}\right) \div 2 \\
&= \left(\frac{-2b}{2a}\right) \div 2 \\
&= \left(\frac{-b}{a}\right) \cdot \frac{1}{2} \\
&= \frac{-b}{2a}
\end{aligned}
$$

so, we have a nice formula for the axis of symmetry $x = -\frac{b}{2a}$

Figure $5.1.4$ represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

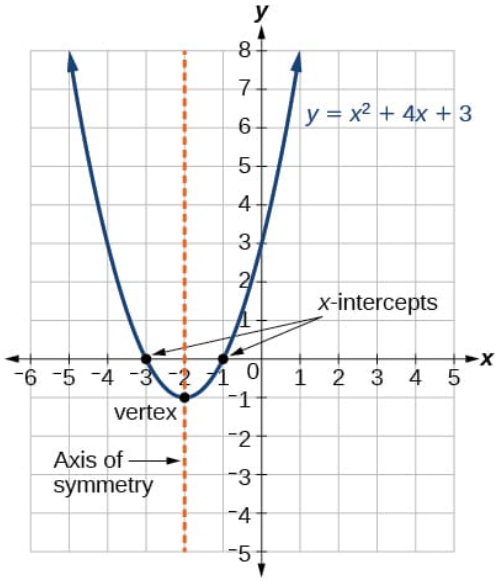



Figure $5.1.3$: Graph of a parabola showing where the $x$ and $y$ intercepts, vertex, and axis of symmetry are for the function $y = x^2 + 4x + 3$.

## Definition: Vertex Form of a Quadratic Function

- The vertex form of a quadratic function is $f(x) = a(x - h)^2 + k$
- The vertex is $(h, k)$ where $h = -\frac{b}{2a}$ and $k = f(h)$

As with the standard form form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure $5.1.5$ represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x–h = x + 2$ in this example, $h = –2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

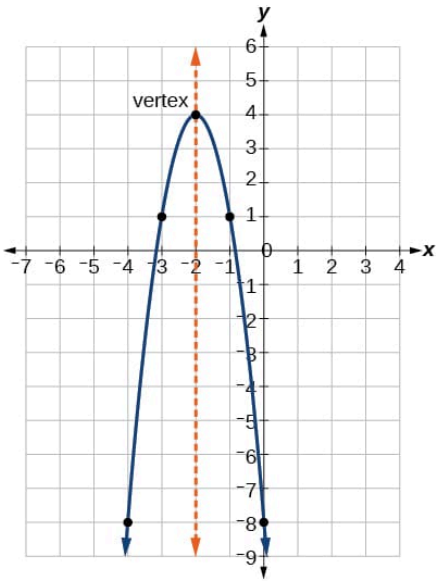



Figure $5.1.4$: Graph of a parabola showing where the $x$ and $y$ intercepts, vertex, and axis of symmetry are for the function $y = -3(x + 2)^2 + 4$.

The vertex

form is useful for determining how the graph is transformed from the graph of $y = x^2$. Figure $5.1.6$ is the graph of this basic function.

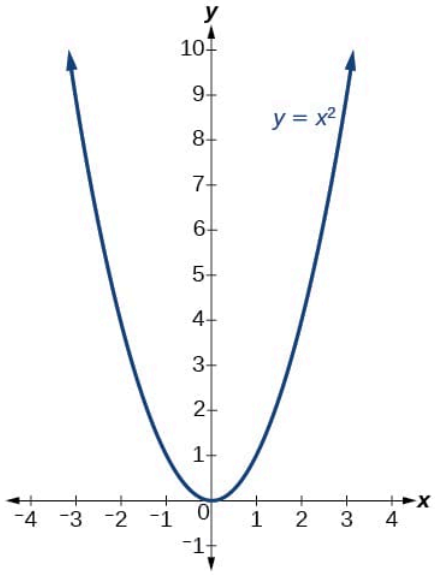



Figure $5.1.5$: Graph of $y = x^2$.

If $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In Figure $5.1.5$, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In Figure $5.1.5$, $h < 0$, so the graph is shifted 2 units to the left. The magnitude of $a$ indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In Figure $5.1.5$, $|a| > 1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$a(x - h)^2 + k = ax^2 + bx + c$$
$$ax^2 - 2ahx + (ah^2 + k) = ax^2 + bx + c$$

For the linear terms to be equal, the coefficients must be equal.

$$\begin{aligned} -2ah &= b \\ h &= -\frac{b}{2a} \end{aligned}$$

This is the **axis of symmetry** we defined earlier.

Setting the constant terms equal:

$$\begin{aligned} ah^2 + k &= c \\ k &= c - ah^2 \\ &= c - a \cdot \left(-\frac{b}{2a}\right)^2 \\ &= c - \frac{ab^2}{4a^2} \\ k &= c - \frac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that $k$ is the output value of the function when the input is $h$, so $f(h) = k$.

## Rewriting Quadratics in Vertex Form




**Given a quadratic function, find the x-intercepts by rewriting in vetrex form.**

1. Substitute a and $b$ into $h = -\frac{b}{2a}$.
2. Substitute $x = h$ into the general form of the quadratic function to find $k$.
3. Rewrite the quadratic in standard form using $h$ and $k$.
4. Solve for when the output of the function will be zero to find the x-intercepts.

### Example 5.1.2: Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

**Solution**

We begin by solving for when the output will be zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x - h)^2 + k$$

We know that $a = 2$. Then we solve for $h$ and $k$.

$$\begin{aligned} h &= -\frac{b}{2a} & k &= f(-1) \\ &= -\frac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

$$f(x) = 2(x+1)^2 - 6$$

We can now solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x+1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has x-intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$.

**Analysis**

We can check our work by graphing the given function on a graphing utility and observing the x-intercepts. See Figure $5.1.15$.

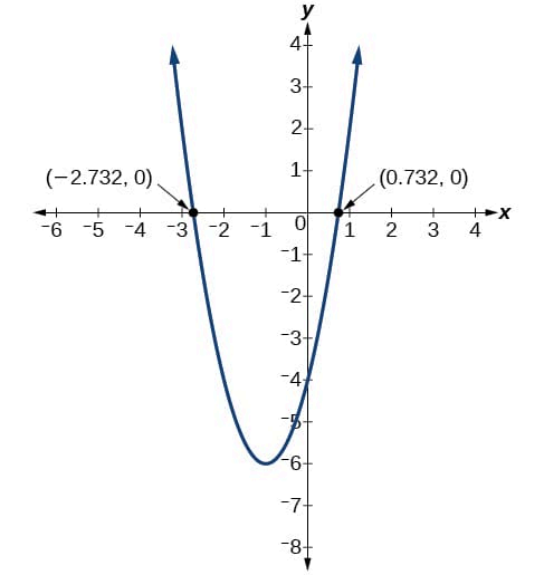



Figure $5.1.6$: Graph of a parabola which has the following x-intercepts: $(-2.732, 0)$ and $(0.732, 0)$.

## Exercise $5.1.1$

Find the vertex form for the function $g(x) = 13 + x^2 - 6x$. Then find the y- and x-intercepts (if any).

*Answer available at the end of this section.*

## Example $5.1.3$: Solving a Quadratic Equation with the Quadratic Formula

Solve $x^2 + x + 2 = 0$.

**Solution**

Let's begin by writing the quadratic formula: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

When applying the quadratic formula, we identify the coefficients $a$, $b$ and $c$. For the equation $x^2 + x + 2 = 0$, we have $a = 1$, $b = 1$, and $c = 2$. Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (2)}}{2 \cdot 1} \\ &= \frac{-1 \pm \sqrt{1-8}}{2} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $x = \frac{-1+i\sqrt{7}}{2}$ and $x = \frac{-1-i\sqrt{7}}{2}$ or $x = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $x = \frac{-1}{2} - \frac{i\sqrt{7}}{2}$.

Given a graph of a quadratic function, write the equation of the function in standard form.

1. Identify the horizontal shift of the parabola; this value is $h$. Identify the vertical shift of the parabola; this value is $k$.
2. Substitute the values of the horizontal and vertical shift for $h$ and $k$. in the function $f(x) = a(x-h)^2 + k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for $x$ and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. If the parabola opens up, $a > 0$. If the parabola opens down, $a < 0$ since this means the graph was reflected about the x-axis.
6. Expand and simplify to write in standard form.

## Example 5.1.4: Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function $g$ in Figure 5.1.7 as a transformation of $f(x) = x^2$, and then expand the formula, and simplify terms to write the equation in general form.

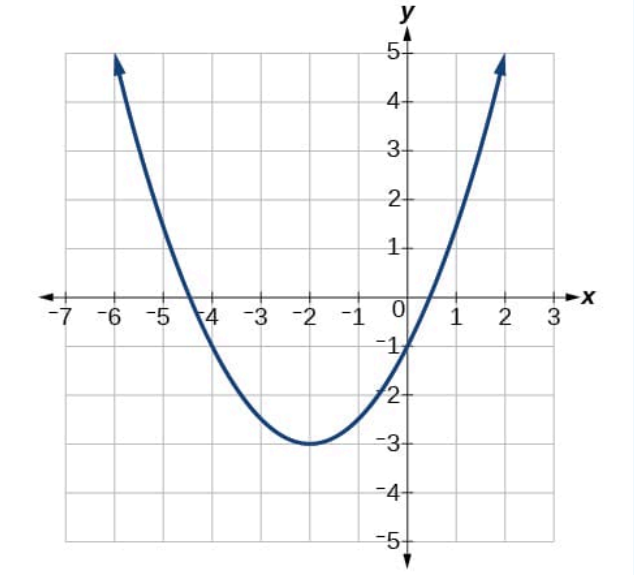



Figure 5.1.7: Graph of a parabola with its vertex at $(-2, -3)$.

**Solution**

We can see the graph of $g$ is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2–3$.

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$-1 = a(0+2)^2 - 3 \tag{5.1.3}$$
$$2 = 4a \tag{5.1.4}$$
$$a = \frac{1}{2} \tag{5.1.5}$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x+2)^2–3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$g(x) = \frac{1}{2}(x+2)^2 - 3 \tag{5.1.6}$$
$$= \frac{1}{2}(x+2)(x+2) - 3 \tag{5.1.7}$$
$$= \frac{1}{2}(x^2+4x+4) - 3 \tag{5.1.8}$$
$$= \frac{1}{2}x^2 + 2x + 2 - 3 \tag{5.1.9}$$
$$= \frac{1}{2}x^2 + 2x - 1 \tag{5.1.10}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

**Analysis**

We can check our work using a table of values.

*Table* 5.1.1

| $x$ | -6 | -4 | -2 | 0 | 2 |
|---|---|---|---|---|---|
| $y$ | -5 | -1 | -3 | -1 | 5 |

The ordered pairs in the table correspond to points on the graph.




**Given a quadratic function in standard form, find the vertex of the parabola.**

1. Identify $a$, $b$, and $c$.
2. Find $h$, the x-coordinate of the vertex, by substituting $a$ and $b$ into $h = -\frac{b}{2a}$.
3. Find $k$, the y-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

## Example 5.1.5: Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2–6x + 7$. Rewrite the quadratic in vertex form.

**Solution**

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} \tag{5.1.11}$$
$$= -\frac{-6}{2(2)} \tag{5.1.12}$$
$$= \frac{6}{4} \tag{5.1.13}$$
$$= \frac{3}{2} \tag{5.1.14}$$

The vertical coordinate of the vertex will be at

$$k = f(h) \tag{5.1.15}$$
$$= f\left(\frac{3}{2}\right) \tag{5.1.16}$$
$$= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \tag{5.1.17}$$
$$= \frac{5}{2} \tag{5.1.18}$$

Rewriting into standard form, the stretch factor will be the same as the $a$ in the original quadratic.

$$f(x) = ax^2 + bx + c \tag{5.1.19}$$
$$f(x) = 2x^2 - 6x + 7$$

Using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2} \tag{5.1.20}$$

**Analysis**

One reason we may want to identify the vertex of the parabola is that this point will inform us what the maximum or minimum value of the function is, $(k)$,and where it occurs, $(h)$.

**Exercise 5.1.2**

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in standard form and then in vertex form.

*Answer available at the end of this section.*

## Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure $5.1.13$ that the number of x-intercepts can vary depending upon the location of the graph.

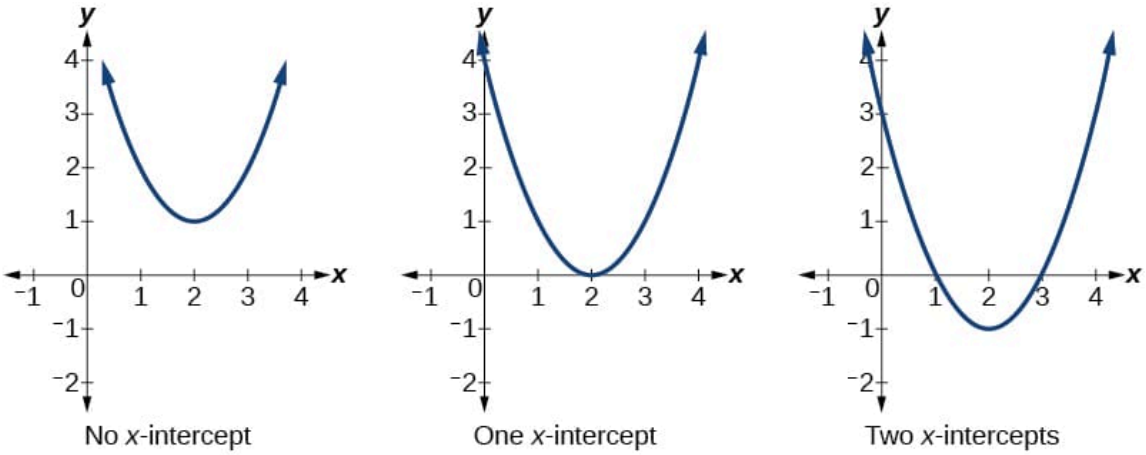



Figure 5.1.8: Number of x-intercepts of a parabola.

**How To: Given a quadratic function $f(x)$, find the y- and x-intercepts.**

1. Evaluate $f(0)$ to find the y-intercept.
2. Solve the quadratic equation $f(x) = 0$ to find the x-intercepts.

## Example 5.1.6: Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

**Solution**

We find the y-intercept by evaluating $f(0)$.

$$f(0) = 3(0)^2 + 5(0) - 2 \tag{5.1.21}$$
$$= -2 \tag{5.1.22}$$

So the y-intercept is at $(0, -2)$.

For the x-intercepts, we find all solutions of $f(x) = 0$.

$$0 = 3x^2 + 5x - 2 \tag{5.1.23}$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = (3x - 1)(x + 2) \tag{5.1.24}$$

$$0 = 3x - 1 \qquad 0 = x + 2 \tag{5.1.25}$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -2 \tag{5.1.26}$$

So the x-intercepts are at $(\frac{1}{3}, 0)$ and $(-2, 0)$.

**Analysis**

By graphing the function, we can confirm that the graph crosses the $y$-axis at $(0, -2)$. We can also confirm that the graph crosses the x-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$. See Figure 5.1.14.

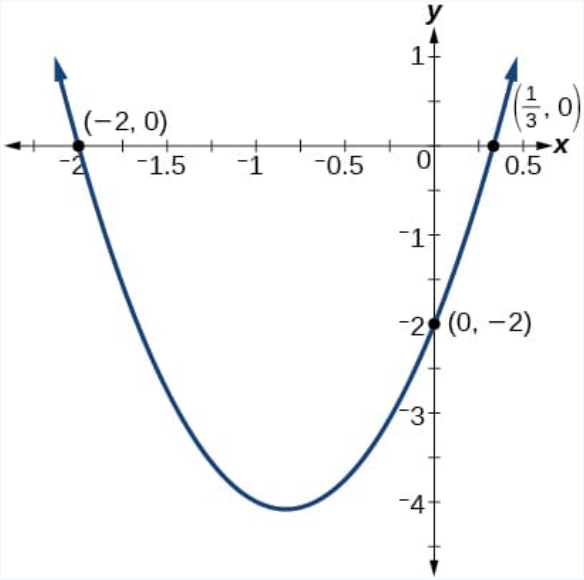



Figure 5.1.9: Graph of a parabola.

## Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

### Definition: Domain and Range of a Quadratic Function

The domain of any **quadratic function** is all real numbers.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with $a > 0$ is $[-\frac{b}{2a})$, or $[f(-\frac{b}{2a}), \infty)$; the range of a quadratic function written in general form with $a < 0$ is $(-\infty, f(-\frac{b}{2a})]$.

The range of a quadratic function written in standard form $f(x) = a(x - h)^2 + k$ with $a > 0$ is $[k, oo)$ the range of a quadratic function written in standard form with $a < 0$ is $(-oo, k]$.

**How To:**

**Given a quadratic function, find the domain and range**

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether $a$ is positive or negative. If $a$ is positive, the parabola has a minimum. If $a$ is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, $k$.
4. If the parabola has a minimum, the range is $[k, \infty)$. If the parabola has a maximum, the range is $(-\infty, k]$.

### Example 5.1.4: Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

**Solution**

As with any quadratic function, the domain is all real numbers.

Because $a$ is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$h = -\frac{b}{2a} \tag{5.1.27}$$
$$= -\frac{9}{2(-5)} \tag{5.1.28}$$
$$= \frac{9}{10} \tag{5.1.29}$$

The maximum value is given by $f(h)$.

$$f(\frac{9}{10}) = 5(\frac{9}{10})^2 + 9(\frac{9}{10}) - 1 \tag{5.1.30}$$
$$= \frac{61}{20} \tag{5.1.31}$$

The range is $f(x) \leq \frac{61}{20}$, or $(-\infty, \frac{61}{20}]$.

### Exercise 5.1.4

Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

*Answer available at the end of this section.*

## Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the **parabola**. We can see the maximum and minimum values in Figure 5.1.9.

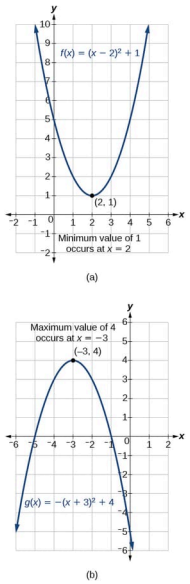



Figure 5.1.9: Minimum and maximum of two quadratic functions.

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

## Example 5.1.5: Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length $L$.
b. What dimensions should she make her garden to maximize the enclosed area?

**Solution**

Let's use a diagram such as Figure 5.1.10 to record the given information. It is also helpful to introduce a temporary variable, $W$, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

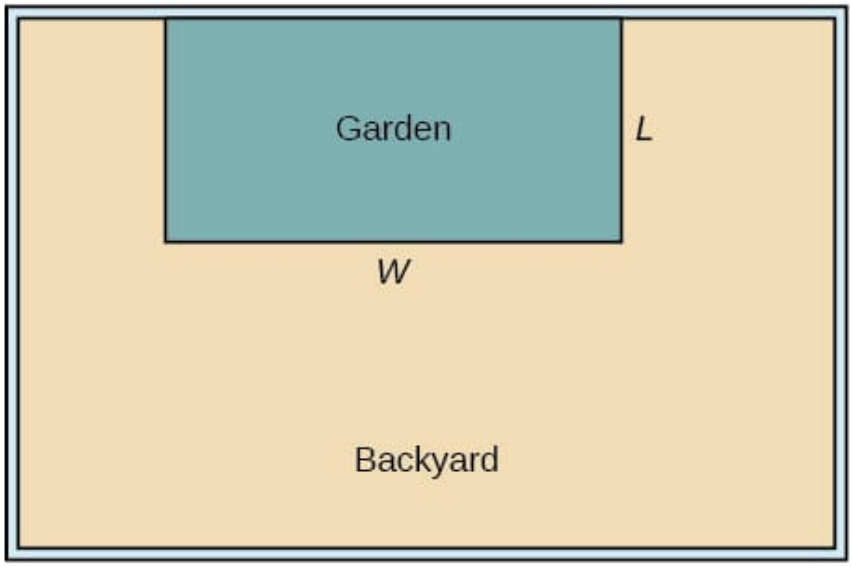



Figure 5.1.10: Diagram of the garden and the backyard.

a. We know we have only 80 feet of fence available, and $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, $W$, in terms of $L$.

$$W = 80 - 2L \tag{5.1.32}$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80 - 2L) \tag{5.1.33}$$
$$A(L) = 80L - 2L^2 \tag{5.1.34}$$

This formula represents the area of the fence in terms of the variable length $L$. The function, written in general form, is

$$A(L) = -2L^2 + 80L \tag{5.1.35}$$

.

The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since $a$ is the coefficient of the squared term, $a = -2$, $b = 80$, and $c = 0$.

To find the vertex:

$$h = -\frac{80}{2(-2)} \qquad k = A(20) \tag{5.1.36}$$
$$= 20 \qquad \text{and} \qquad = 80(20) - 2(20)^2 \tag{5.1.37}$$
$$= 800 \tag{5.1.38}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

**Analysis**

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 5.1.11.

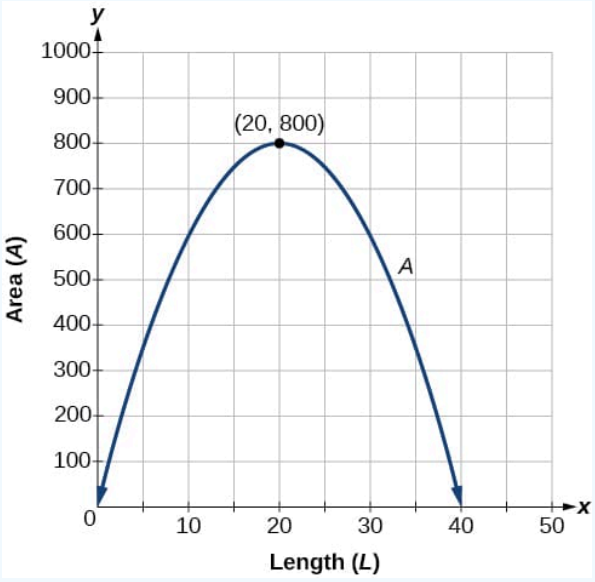



Figure 5.1.11: Graph of the parabolic function $A(L) = -2L^2 + 80L$

**Given an application involving revenue, use a quadratic equation to find the maximum.**

1. Write a quadratic equation for revenue.
2. Find the vertex of the quadratic equation.
3. Determine the y-value of the vertex.

## Example 5.1.6: Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

**Solution**

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, $p$ for price per subscription and $Q$ for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30} \tag{5.1.39}$$
$$= -\frac{5,000}{2} \tag{5.1.40}$$
$$= -2,500 \tag{5.1.41}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$Q = -2500p + b \qquad \text{Substitute in the point } Q = 84,000 \text{ and } p = 30 \tag{5.1.42}$$
$$84,000 = -2500(30) + b \qquad \text{Solve for } b \tag{5.1.43}$$
$$b = 159,000 \tag{5.1.44}$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\text{Revenue} = pQ \tag{5.1.45}$$
$$\text{Revenue} = p(-2,500p + 159,000) \tag{5.1.46}$$
$$\text{Revenue} = -2,500p^2 + 159,000p \tag{5.1.47}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$h = -\frac{159,000}{2(-2,500)} \tag{5.1.48}$$
$$= 31.8 \tag{5.1.49}$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\text{maximum revenue} = -2,500(31.8)^2 + 159,000(31.8) \tag{5.1.50}$$
$$= 2,528,100 \tag{5.1.51}$$

**Analysis**

This could also be solved by graphing the quadratic as in Figure 5.1.12. We can see the maximum revenue on a graph of the quadratic function.

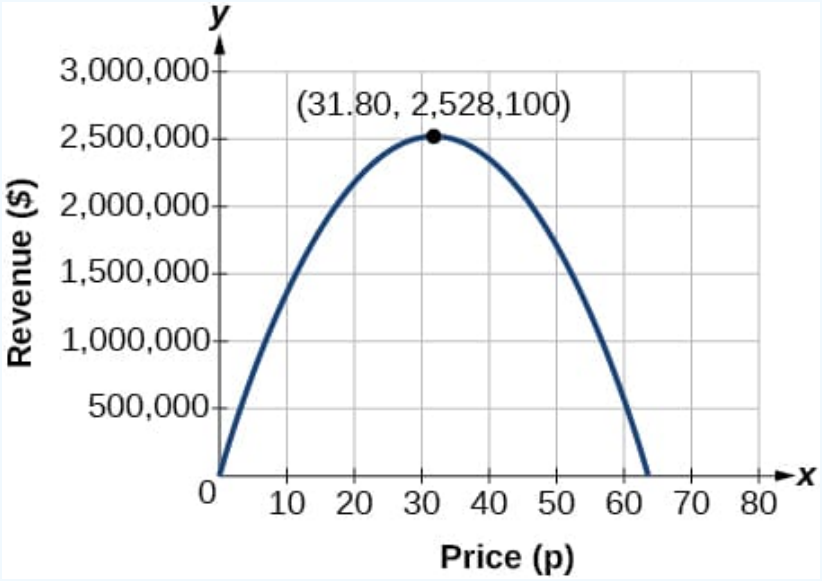



Figure 5.1.12: Graph of the parabolic function

## Example 5.1.7: Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

When does the ball reach the maximum height?
What is the maximum height of the ball?
When does the ball hit the ground?

The ball reaches the maximum height at the vertex of the parabola.

$$h = -\frac{80}{2(-16)} \tag{5.1.52}$$
$$= \frac{80}{32} \tag{5.1.53}$$
$$= \frac{5}{2} \tag{5.1.54}$$
$$= 2.5 \tag{5.1.55}$$

The ball reaches a maximum height after 2.5 seconds.

To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$k = H(-\frac{b}{2a}) \tag{5.1.56}$$
$$= H(2.5) \tag{5.1.57}$$
$$= -16(2.5)^2 + 80(2.5) + 40 \tag{5.1.58}$$
$$= 140 \tag{5.1.59}$$

The ball reaches a maximum height of 140 feet.

To find when the ball hits the ground, we need to determine when the height is zero, $H(t) = 0$.

We use the quadratic formula.

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \tag{5.1.60}$$

$$= \frac{-80 \pm \sqrt{8960}}{-32} \tag{5.1.61}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458 \tag{5.1.62}$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 5.1.16.

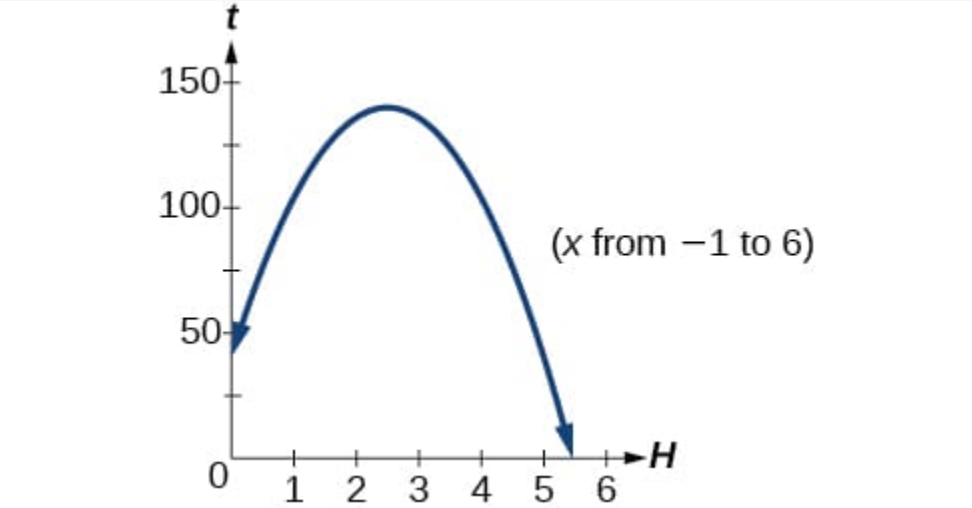



Figure \({5{.}1{.}\eatSpaces{16} \endSpaces}\

## Exercise 5.1.5

A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

**Solution**

a. 3 seconds b. 256 feet c. 7 seconds

# Key Equations

- the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- standard form of a quadratic function: $f(x) = a(x - h)^2 + k$
- vertex form of a quadratic function: $f(x) = ax^2 + bx + c$

## Key Concepts

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x-intercepts, are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the $y$-axis.
- Quadratic functions are often written in standard form.
- Vertex form is useful to easily identify the vertex of a parabola.
- Either form can be written from a graph.
- The vertex can be found from an equation representing a quadratic function. .
- The domain of a quadratic function is all real numbers.
- The range depends on the leading coefficient of the quadratic function.
- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area of a rectangle, revenue and height of a projectile.
- Quadratic equations in standard form can be solved by using the quadratic formula.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems.

## Glossary

**axis of symmetry**
a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

**general form of a quadratic function**
the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where $a, b,$ and $c$ are real numbers and a≠0.

**standard form of a quadratic function**
the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$, where $(h, k)$ is the vertex.

**vertex**
the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

**vertex form of a quadratic function**
another name for the standard form of a quadratic function

**zeros**
in a given function, the values of $x$ at which $y = 0$, also called roots

## Answers

**Exercise** $5.1.1$
y-intercept at $(0, 13)$, No x-intercepts

**Exercise** $5.1.2$
$g(x) = x^2 - 6x + 13$ in standard form; $g(x) = (x - 3)^2 + 4$ in vertex form.

**Exercise** $5.1.4$
The domain is all real numbers. The range is $f(x) \geq \frac{8}{11}$, or $[\frac{8}{11}, \infty)$.

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# 5.2: Power Functions and Polynomial Functions

**Learning Objectives**

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.

## Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function. A power function is a function with a single term that is the product of a real number, a coefficient, and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)

As an example, consider functions for area or volume. The function for the area of a circle with radius $r$ is

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius $r$ is

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, $\pi$ or $\frac{4}{3}\pi$, multiplied by a variable $r$ raised to a power.

**Definition 4.1.1: Power Function**

A *power function* is a function that can be represented in the form

$$f(x) = ax^n \tag{5.2.1}$$

where $a$ and $n$ are real numbers, and $a$ is known as the *coefficient*.

**Q&A: Is $f(x) = 2^x$ a power function?**

*No. A power function contains a variable base raised to a fixed power (Equation 5.2.1). This function has a constant base raised to a variable power. This is called an exponential function, not a power function. This function will be discussed in a later chapter.*

**Example 5.2.1: Identifying Power Functions**

Which of the following functions are power functions?

| | |
|---|---|
| $f(x) = 1$ | Constant function |
| $f(x) = x$ | Identify function |
| $f(x) = x^2$ | Quadratic function |
| $f(x) = x^3$ | Cubic function |
| $f(x) = \frac{1}{x}$ | Reciprocal function |
| $f(x) = \frac{1}{x^2}$ | Reciprocal squared function |
| $f(x) = \sqrt{x}$ | Square root function |
| $f(x) = \sqrt[3]{x}$ | Cube root function |

**Solution**

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The **reciprocal** and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and **cube root** functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

### Exercise 5.2.1

Which functions are power functions?

- $f(x) = 2x^2 \cdot 4x^3$
- $g(x) = -x^5 + 5x^3 - 4x$
- $r(x) = \frac{2x^5-1}{3x^2+4}$
- $q(x) = \frac{5}{-2x^4}$

*Answer available at the end of this section.*

## Identifying End Behavior of Power Functions

Although definition 4.1.1 allows the exponent of a power function to be negative. For the remainder of this section, we will explore only those power functions for which the exponent is positive. The other case shall be discussed later in this chapter and shall be known as reciprocal or rational functions.

Figure 5.2.2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

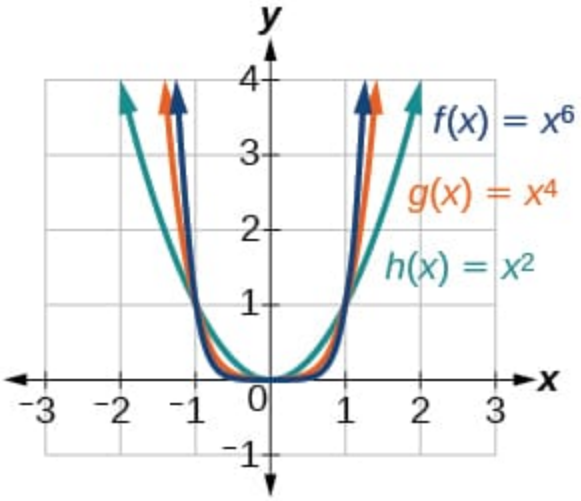



Figure 5.2.2: Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol $\infty$ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that $x$ is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as $x$ approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty,\ f(x) \to \infty$$

Figure 5.2.3 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

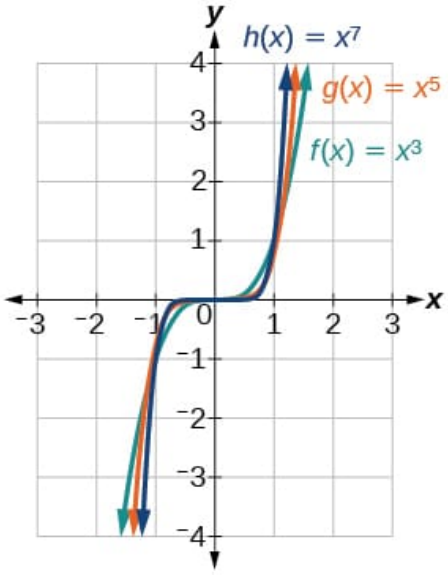



Figure 5.2.3: Odd-power function

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure 5.2.2 we see that even functions of the form $f(x) = x^n$, $n$ even, are symmetric about the $y$-axis. In Figure 5.2.3 we see that odd functions of the form $f(x) = x^n$, $n$ odd, are symmetric about the origin.

For these odd power functions, as $x$ approaches negative infinity, $f(x)$ decreases without bound. As $x$ approaches positive infinity,

$f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$
$$\text{as } x \to \infty,\ f(x) \to \infty$$

The behavior of the graph of a function as the input values get very small $(x \to -\infty)$ and get very large $x \to \infty$ is referred to as the *end behavior* of the function. We can use words or symbols to describe end behavior.

Figure $5.2.4$ shows the end behavior of power functions in the form $f(x) = kx^n$ where $n$ is a non-negative integer depending on the power and the constant.

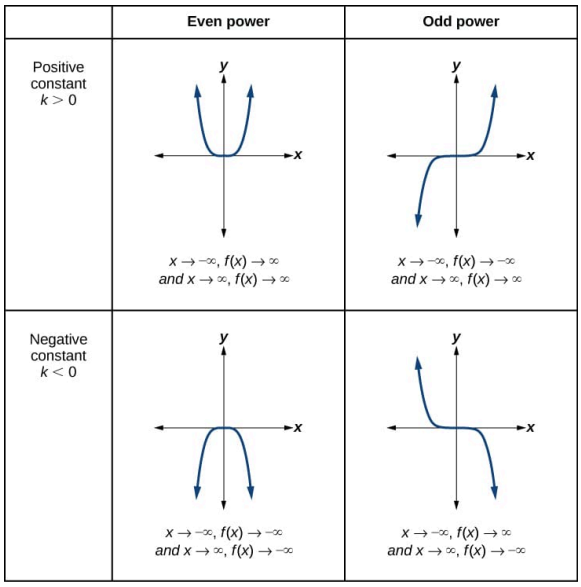



Figure $5.2.4$

**How To: Given a power function $f(x) = kx^n$ where $n$ is a non-negative integer, identify the end behavior.**

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use Figure $5.2.4$ to identify the end behavior.

**Example $5.2.2$: Identifying the End Behavior of a Power Function**

Describe the end behavior of the graph of $f(x) = x^8$.

**Solution**

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As $x$ approaches infinity, the output (value of $f(x)$ ) increases without bound. We write as $x \to \infty,\ f(x) \to \infty$. As $x$ approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty,\ f(x) \to \infty$. We can graphically represent the function as shown in Figure $5.2.5$.

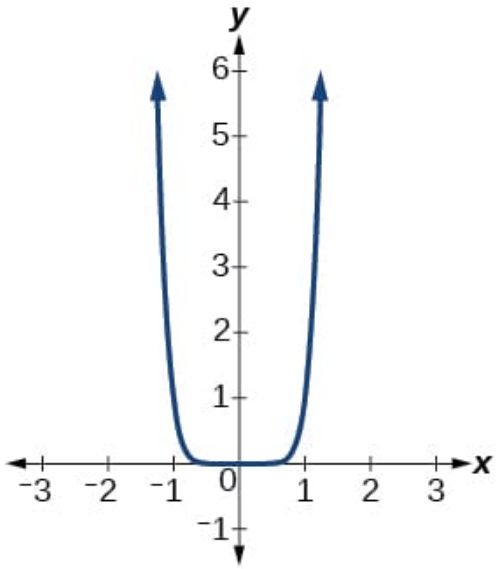



Figure 5.2.5: Graph of $f(x) = x^8$.

## Example 5.2.3: Identifying the End Behavior of a Power Function.

Describe the end behavior of the graph of $f(x) = -x^9$.

**Solution**

The exponent of the power function is 9 (an odd number). Because the coefficient is –1 (negative), the graph is the reflection about the $x$-axis of the graph of $f(x) = x^9$. Figure 5.2.6 shows that as $x$ approaches infinity, the output decreases without bound. As $x$ approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty,\ f(x) \to \infty$$
$$\text{as } x \to \infty,\ f(x) \to -\infty$$

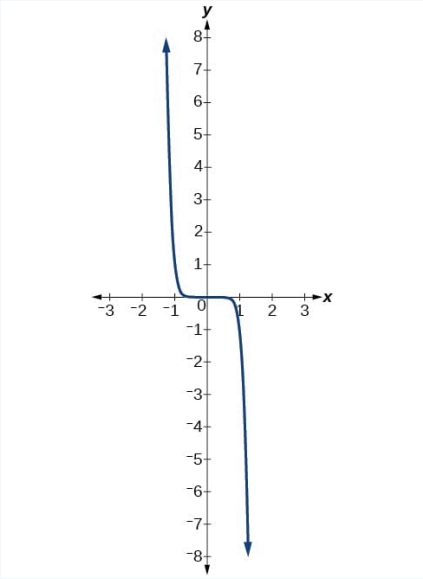



Figure 5.2.6: Graph of $f(x) = -x^9$.

**Analysis**

We can check our work by using the table feature on a graphing utility.

*Table* 5.2.2

| $x$ | $f(x)$ |
|---|---|
| -10 | 1,000,000,000 |
| -5 | 1,953,125 |
| 0 | 0 |
| 5 | -1,953,125 |
| 10 | -1,000,000,000 |

We can see from Table 5.2.2 that, when we substitute very small values for $x$, the output is very large, and when we substitute very large values for $x$, the output is very small (meaning that it is a very large negative value).

**Exercise 5.2.2**

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

*Answer available at the end of this section.*

## Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius $r$ of the spill depends on the number of weeks $w$ that have passed. This relationship is linear.

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area A of a circle.

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24 + 8w) \\ &= \pi(24 + 8w)^2 \end{aligned}$$

Multiplying gives the formula.

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a polynomial function. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

**Definition: Polynomial Function**

Let $n$ be a non-negative integer. A *polynomial function* is a function that can be written in the form

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \tag{5.2.2}$$

This is called the general form of a polynomial function. Each $a_i$ is a coefficient and can be any real number. Each product $a_i x^i$ is a *term of a polynomial function*.

### Example 5.2.4: Identifying Polynomial Functions

Which of the following are polynomial functions?

- $f(x) = 2x^3 \cdot 3x + 4$
- $g(x) = -x(x^2 - 4)$
- $h(x) = 5\sqrt{x} + 2$

**Solution**

The first two functions are examples of polynomial functions because they can be written in the form of Equation 5.2.2, where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

## Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The degree of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The leading term is the term containing the highest power of the variable, or the term with the highest degree. The leading coefficient is the coefficient of the leading term.

### Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.

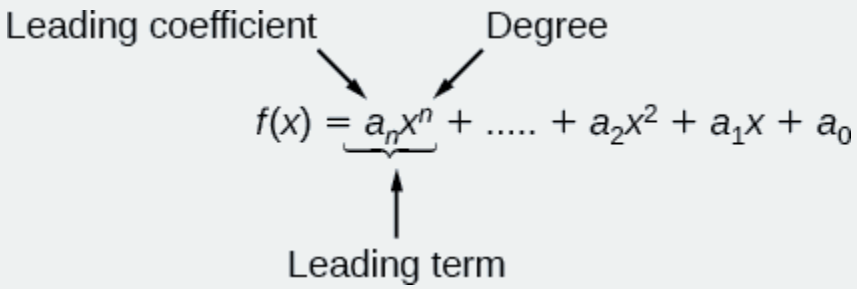



Figure 5.2.7

When a polynomial is written in this way, we say that it is in *general form*.

### How To: Given a polynomial function, identify the degree and leading coefficient

1. Find the highest power of $x$ to determine the degree function.
2. Identify the term containing the highest power of $x$ to find the leading term.
3. Identify the coefficient of the leading term.

### Example 5.2.5: Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$f(x) = 3 + 2x^2 - 4x^3$$

$$g(t) = 5t^5 - 2t^3 + 7t$$

$$h(p) = 6p - p^3 - 2$$

**Solution**

For the function $f(x)$, the highest power of $x$ is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, –4.

For the function $g(t)$, the highest power of $t$ is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of $p$ is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, –1.

### Exercise 5.2.3

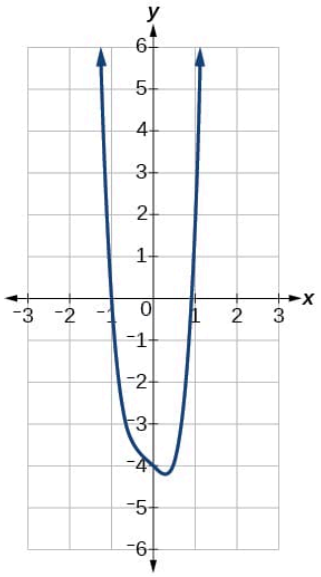
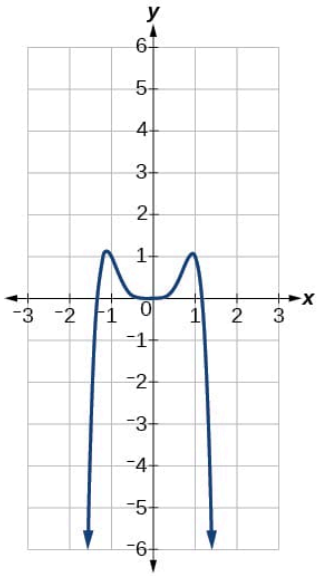
Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 6$.

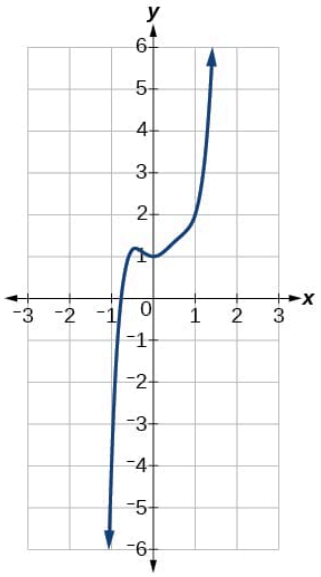
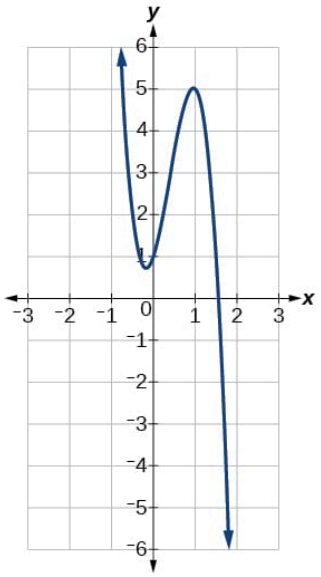
*Answer available at the end of this section.*

## Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as $x$ gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree (Table 5.2.3).

*Table* 5.2.3

| Polynomial Function | Leading Term | Graph of Polynomial Function |
|---|---|---|
| $f(x) = 5x^4 + 2x^3 - x - 4$ | $5x^4$ |  |
| $f(x) = -2x^6 - x^5 + 3x^4 + x^3$ | $-2x^6$ |  |

| Polynomial Function | Leading Term | Graph of Polynomial Function |
|---|---|---|
| $f(x) = 3x^5 - 4x^4 + 2x^2 + 1$ | $3x^5$ |  |
| $f(x) = -6x^3 + 7x^2 + 3x + 1$ | $-6x^3$ |  |

## Example 5.2.6: Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure 5.2.8.

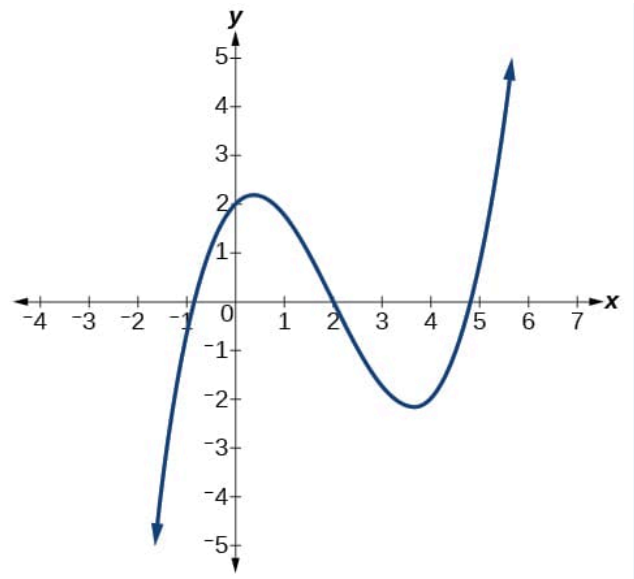



Figure 5.2.8.

**Solution**

As the input values $x$ get very large, the output values $f(x)$ increase without bound. As the input values $x$ get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to \infty,\ f(x) \to \infty$$

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

In words, we could say that as $x$ values approach infinity, the function values approach infinity, and as $x$ values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

## Exercise 5.2.1

Describe the end behavior, and determine a possible degree of the polynomial function in Figure 5.2.9.

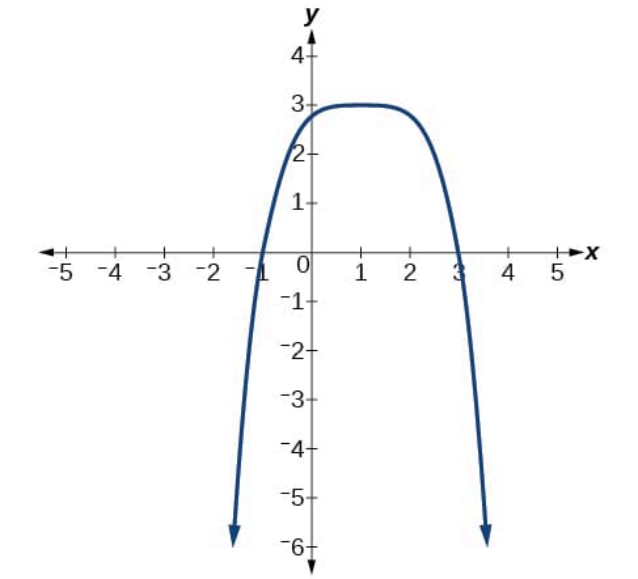



Figure 5.2.9

*Answer available at the end of this section.*

### Example 5.2.7: Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

**Solution**

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2+3x-4) \\ &= -3x^4 - 9x^3 + 12x^2 \end{aligned}$$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

$$\text{as } x \to \infty,\ f(x) \to -\infty$$

### Exercise 5.2.7

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

*Answer available at the end of this section.*

## Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the $y$-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one $y$-intercept $(0, a_0)$. The $x$-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one $x$-intercept. See Figure $5.2.10$.

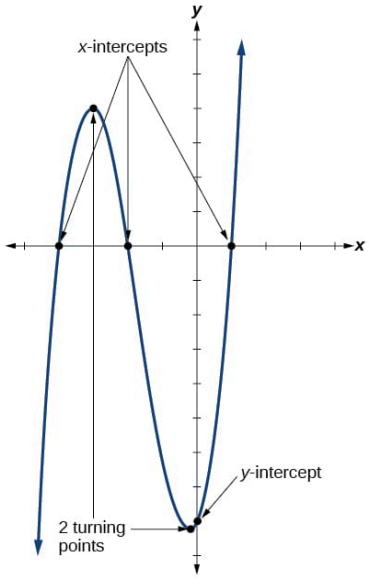



Figure $5.2.10$

## Defintion: Intercepts and Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The $y$-intercept is the point at which the function has an input value of zero. The $x$-intercepts are the points at which the output value is zero.

**How To:** **Given a polynomial function, determine the intercepts.**

1. Determine the $y$-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the $x$-intercepts by solving for the input values that yield an output value of zero.

## Example $5.2.8$: Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x - 2)(x + 1)(x - 4)$, written in factored form for your convenience, determine the $y$- and $x$-intercepts.

**Solution**

The $y$-intercept occurs when the input is zero, so substitute $0$ for $x$.

$$\begin{aligned} f(0) &= (0 - 2)(0 + 1)(0 - 4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The $y$-intercept is $(0, 8)$.

The $x$-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$\begin{array}{ccccccc} x-2=0 & \text{or} & x+1=0 & \text{or} & x-4=0 \\ x=2 & \text{or} & x=-1 & \text{or} & x=4 \end{array}$$

The $x$-intercepts are $(2,0)$,$(-1,0)$, and $(4,0)$.

We can see these intercepts on the graph of the function shown in Figure 5.2.11.

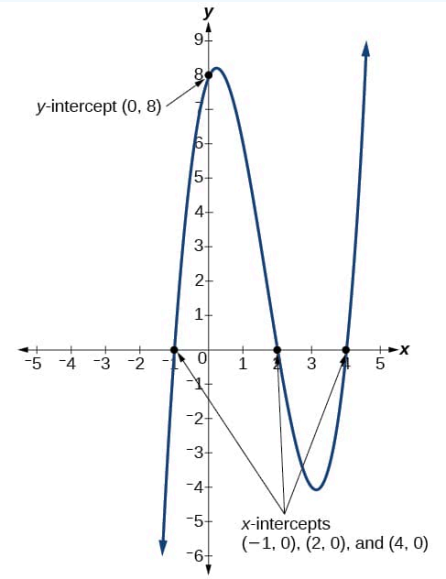



Figure 5.2.11: Graph of $f(x) = (x-2)(x+1)(x-4)$.

## Example 5.2.9: Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the $y$- and $x$-intercepts.

**Solution**

The $y$-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^2 - 45 \\ &= -45 \end{aligned}$$

The $y$-intercept is $(0,-45)$.

The $x$-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$\begin{aligned} f(x) &= x^4 - 4x^2 - 45 \\ &= (x^2-9)(x^2+5) \\ &= (x-3)(x+3)(x^2+5) \end{aligned}$$

$$0 = (x-3)(x+3)(x^2+5)$$

$$\begin{array}{ccccccc} x-3=0 & \text{or} & x+3=0 & \text{or} & x^2+5=0 \\ x=3 & \text{or} & x=-3 & \text{or} & (\text{no real solution}) \end{array}$$

The $x$-intercepts are $(3,0)$ and $(-3,0)$.

We can see these intercepts on the graph of the function shown in Figure 5.2.12. We can see that the function is even because $f(x) = f(-x)$.

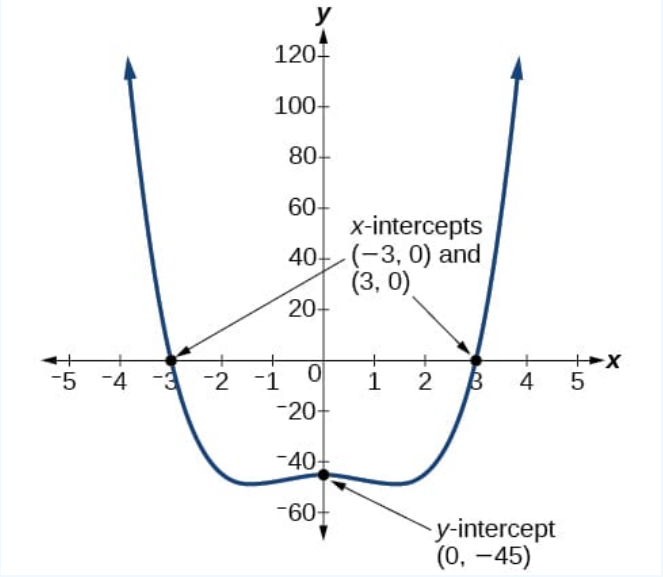



Figure 5.2.12: Graph of $f(x) = x^4 - 4x^2 - 45$.

### Exercise 5.2.5

5.2.5**:** Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the $y$- and $x$-intercepts.

*Solution available at the end of this section.*

## Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of $x$-intercepts and the number of turning points. A polynomial function of $n^{\text{th}}$ degree is the product of $n$ factors, so it will have at most $n$ roots or zeros, or $x$-intercepts. The graph of the polynomial function of degree $n$ must have at most $n–1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A smooth curve is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

### Intercepts and Turning Points of Polynomials

A polynomial of degree $n$ will have, at most, $n$ $x$-intercepts and $n - 1$ turning points.

### Example 5.2.10: Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of $x$-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

**Solution**

The polynomial has a degree of **10**, so there are at most $10$ $x$-intercepts and at most $9$ turning points.

## Exercise 5.2.6

Without graphing the function, determine the maximum number of $x$-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$

*Answer available at the end of this section.*

## Example 5.2.11: Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure 5.2.12 based on its intercepts and turning points?

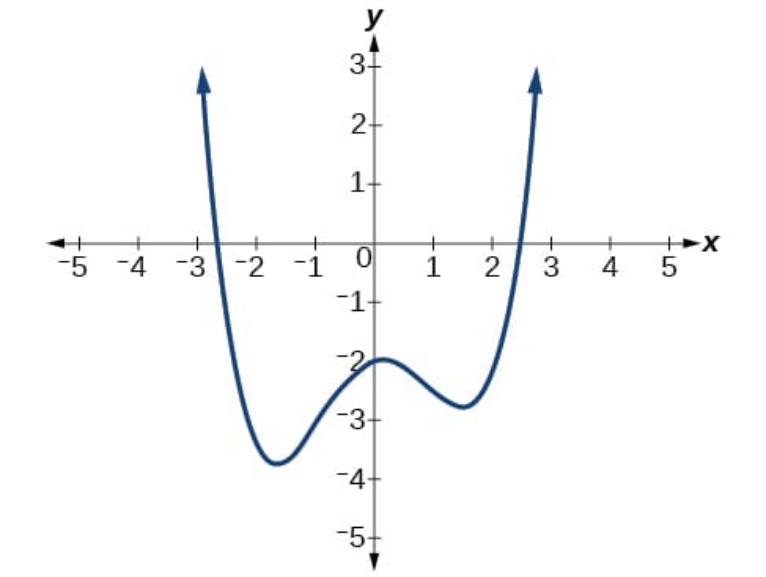



Figure 5.2.13.

**Solution**

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See Figure 5.2.14.

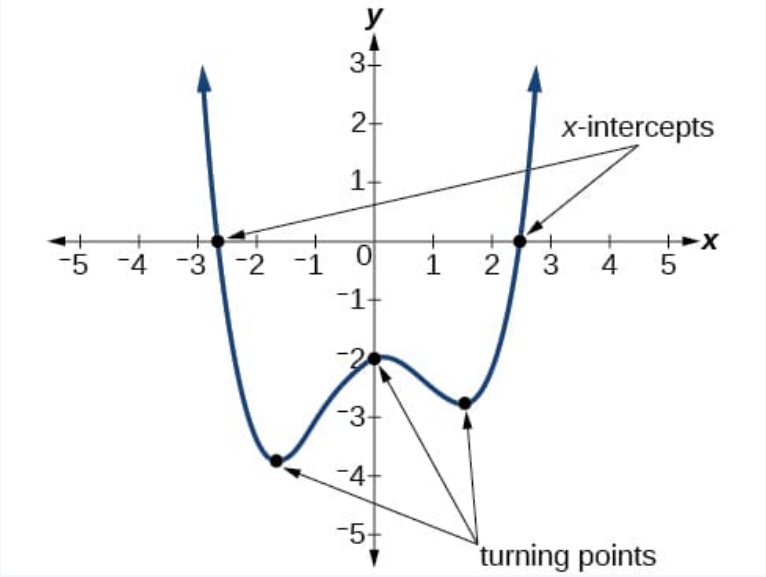



Figure 5.2.14: Graph of an even-degree polynomial.

The graph has 2 $x$-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

## Exercise 5.2.7

What can we conclude about the polynomial represented by the graph shown in Figure 5.2.15 based on its intercepts and turning points?

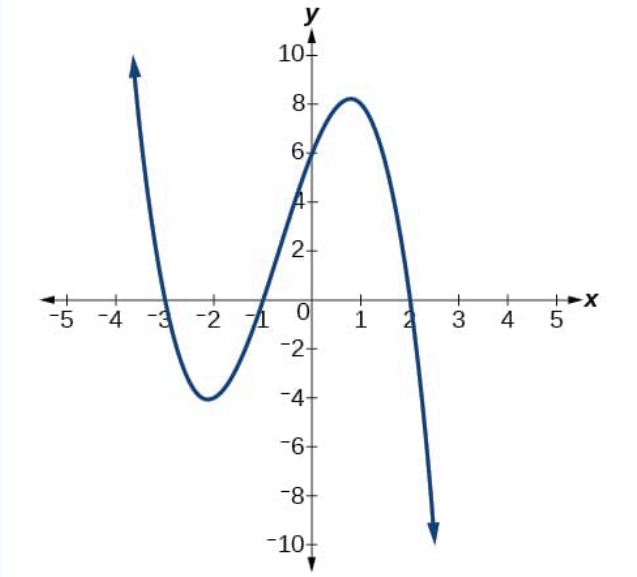



**Figure 5.2.15.**

*Answer available at the end of this section.*

**Solution**

The end behavior indicates an odd-degree polynomial function; there are 3 $x$-intercepts and 2 turning points, so the degree is

odd and at least 3. Because of the end behavior, we know that the lead coefficient must be negative.

### Example 5.2.12: Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

**Solution**

The $y$-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The $y$-intercept is $(0, 0)$.

The $x$-intercepts are found by determining the zeros of the function.

$$\begin{aligned} 0 &= -4x(x+3)(x-4) \\ x &= 0 \quad \text{or} \quad x+3=0 \quad \text{or} \quad x-4=0 \\ x &= 0 \quad \text{or} \quad x=-3 \quad \text{or} \quad x=4 \end{aligned}$$

The $x$-intercepts are $(0, 0)$,$(-3, 0)$, and $(4, 0)$.

The degree is 3 so the graph has at most 2 turning points.

### Exercise 5.2.8

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

*Answer available at the end of this section.*

## Key Equations

- general form of a polynomial function: $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

## Key Concepts

- A power function is a variable base raised to a number power.
- The behavior of a graph as the input decreases without bound and increases without bound is called the end behavior.
- The end behavior depends on whether the power is even or odd.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function.
- A polynomial of degree $n$ will have at most $n$ $x$-intercepts and at most $n-1$ turning points.

## Glossary

**coefficient**

a nonzero real number that is multiplied by a variable raised to an exponent (only the number factor is the coefficient)

**continuous function**

a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

**degree**

the highest power of the variable that occurs in a polynomial

**end behavior**

the behavior of the graph of a function as the input decreases without bound and increases without bound

**leading coefficient**

the coefficient of the leading term

**leading term**

the term containing the highest power of the variable

**polynomial function**

a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

**power function**

a function that can be represented in the form $f(x) = kx^p$ where $k$ is a constant, the base is a variable, and the exponent, $p$, is a constant

**smooth curve**

a graph with no sharp corners

**term of a polynomial function**

any $a_i x^i$ of a polynomial function in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} ... + a_2 x^2 + a_1 x + a_0$

**turning point**

the location at which the graph of a function changes direction

## Answers

**Exercise** $5.2.1$

$f(x)$ is a power function because it can be written as $f(x) = 8x^5$ and $q(x)$ is a power function because it can be written as $q(x) = -\frac{5}{2}x^{-4}$.

$g(x)$ and $r(x)$ are not power functions.

**Exercise** $5.2.2$

As $x$ approaches positive or negative infinity, $f(x)$ decreases without bound: as $x \to \pm\infty$, $f(x) \to -\infty$ because of the negative coefficient.

**Exercise** $5.2.3$

The degree is $6$. The leading term is $-x^6$. The leading coefficient is $-1$.

**Exercise** $5.2.1$

As $x \to \infty$, $f(x) \to -\infty$; as $x \to -\infty$, $f(x) \to -\infty$. It has the shape of an even degree power function with a negative coefficient.

**Exercise** $5.2.7$

The leading term is $0.2x^3$, so it is a degree 3 polynomial. As $x$ approaches positive infinity, $f(x)$ increases without bound; as $x$ approaches negative infinity, $f(x)$ decreases without bound.

**Exercise** $5.2.5$

$y$-intercept $(0, 0)$; $x$-intercepts $(0, 0)$,$(-2, 0)$, and $(5, 0)$

**Exercise** $5.2.6$

There are at most 12 $x$-intercepts and at most 11 turning points.

**Exercise** $5.2.7$

The end behavior indicates that this is an odd degree polynomial. Since there are 3 $x$ intercepts and 2 turning points, we can conclude that the degree is odd and at least 3.

**Exercise** $5.2.8$

The $x$-intercepts are $(2,0)$, $(-1,0)$, and $(5,0)$, the $y$-intercept is $(0,2)$, and the graph has at most 2 turning points.

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# 5.3: Graphs of Polynomial Functions

**Learning Objectives**

- Recognize characteristics of graphs of polynomial functions.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Determine end behavior.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.
- Use the Intermediate Value Theorem.



**François Viète (1540 - 1603)**, Viète introduced the first consistent, fully symbolic notation for algebra in works like *In Artem Analyticem Isagoge* (1591). He used vowels for variables and consonants for given constants, which allowed for the general consideration of polynomials as abstract entities rather than specific numerical problems. This move from a rhetorical (word-based) to a symbolic approach was fundamental to the development of modern algebra.

## Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. Figure $5.3.1$ shows a graph that represents a **polynomial function** and a graph that represents a function that is not a polynomial.

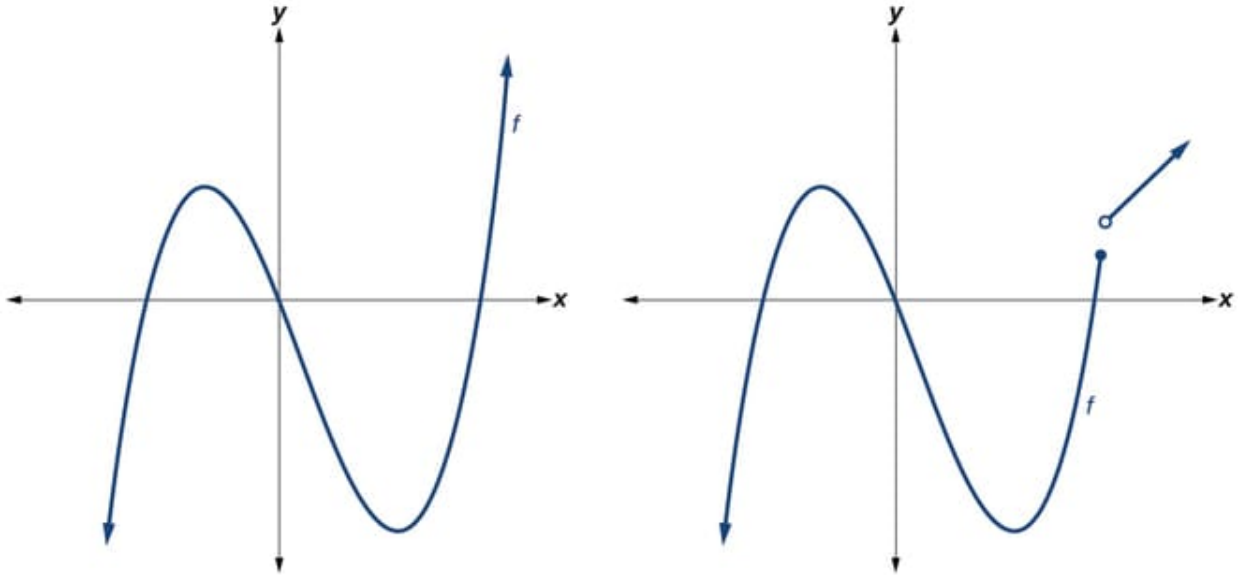



Figure 5.3.1: Graph of $f(x) = x^3 - 0.01x$.

## Example 5.3.1: Recognizing Polynomial Functions

Which of the graphs in Figure 5.3.2 represents a polynomial function?

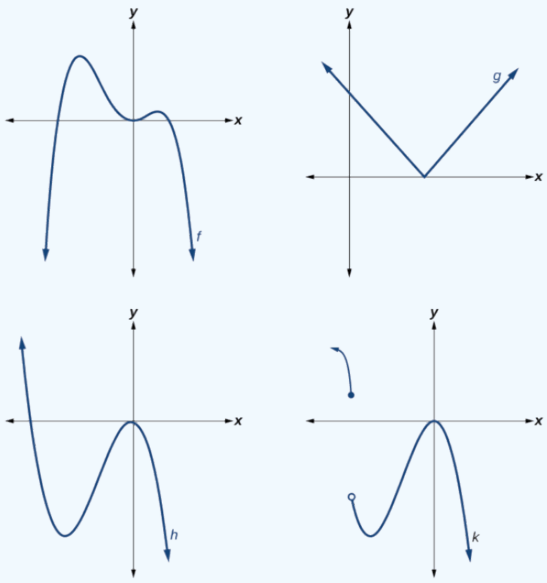



***Figure*** 5.3.2

**Solution**

- The graphs of $f$ and $h$ are graphs of polynomial functions. They are smooth and **continuous**.
- The graphs of $g$ and $k$ are graphs of functions that are not polynomials. The graph of function $g$ has a sharp corner. The graph of function $k$ is not continuous.

## Q&A

Do all polynomial functions have as their domain all real numbers?

- Yes. Any real number is a valid input for a polynomial function.

## Using Factoring to Find Zeros of Polynomial Functions

Recall that if $f$ is a polynomial function, the values of $x$ for which $f(x) = 0$ are called **zeros** of $f$. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
The polynomial is given in factored form.
Technology is used to determine the intercepts.

### How To: Given a polynomial function $f$, find the x-intercepts by factoring

1. Set $f(x) = 0$.
2. If the polynomial function is not given in factored form:
   a. Factor out any common monomial factors.
   b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x-intercepts.

### Example 5.3.2: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

**Solution**

We can attempt to factor this polynomial to find solutions for $f(x) = 0$.

$$\begin{aligned} x^6 - 3x^4 + 2x^2 &= 0 && \text{Factor out the greatest common factor.} \\ x^2(x^4 - 3x^2 + 2) &= 0 && \text{Factor the trinomial.} \\ x^2(x^2 - 1)(x^2 - 2) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x^2 &= 0 \\ x^2 &= 0 \\ x &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} (x^2 - 1) &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned} \quad \text{or} \quad \begin{aligned} (x^2 - 2) &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

.

This gives us five x-intercepts: $(0, 0)$, $(1, 0)$, $(-1, 0)$, $(\sqrt{2}, 0)$,and $(-\sqrt{2}, 0)$ (Figure 5.3.3). We can see that this is an even function.

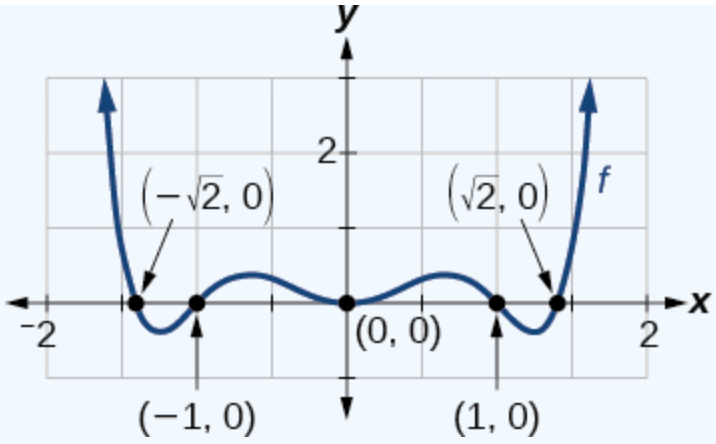



Figure 5.3.3.

## Example 5.3.3: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

**Solution**

Find solutions for $f(x) = 0$ by factoring.

$$\begin{aligned} x^3 - 5x^2 - x + 5 &= 0 && \text{Factor by grouping.} \\ x^2(x-5) - (x-5) &= 0 && \text{Factor out the common factor.} \\ (x^2-1)(x-5) &= 0 && \text{Factor the difference of squares.} \\ (x+1)(x-1)(x-5) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x+1 &= 0 \quad \text{or} \quad x-1 = 0 \quad \text{or} \quad x-5 = 0 \\ x &= -1 \qquad\quad x = 1 \qquad\quad x = 5 \end{aligned}$$

There are three x-intercepts: $(-1, 0)$, $(1, 0)$, and $(5, 0)$ (Figure 5.3.4).

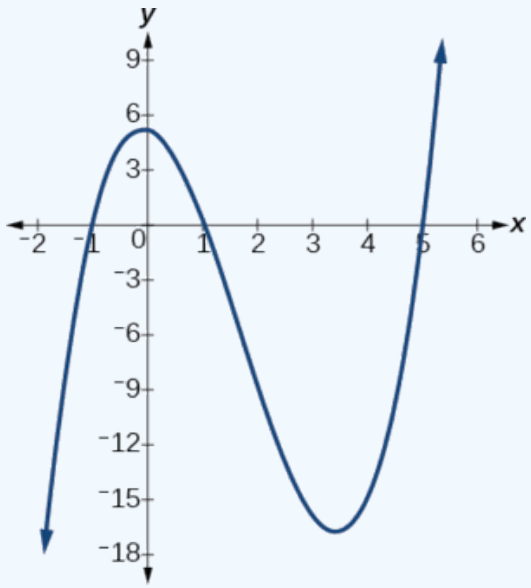



Figure 5.3.4: Graph of $f(x)$.

## Example 5.3.4: Finding the y- and x-Intercepts of a Polynomial in Factored Form

Find the y- and x-intercepts of $g(x) = (x-2)^2(2x+3)$.

**Solution**

The y-intercept can be found by evaluating $g(0)$.

$$\begin{aligned} g(0) &= (0-2)^2(2(0)+3) \\ &= 12 \end{aligned}$$

So the y-intercept is $(0, 12)$.

The x-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3) = 0 \tag{5.3.1}$$

$$\begin{aligned} (x-2)^2 &= 0 \qquad\qquad (2x+3) = 0 \\ x-2 &= 0 \quad \text{or} \quad x = -\frac{3}{2} \\ x &= 2 \end{aligned}$$

So the x-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

**Analysis**

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial as shown in Figure 5.3.5.

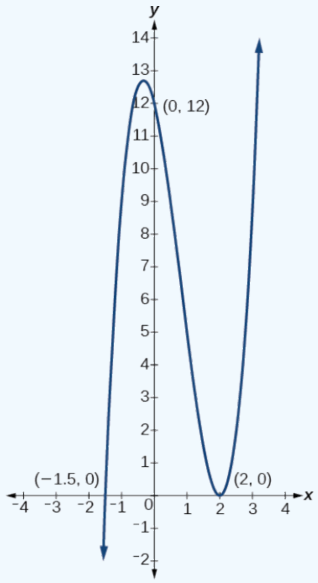



Figure 5.3.5: Graph of $g(x)$.

## Example 5.3.5: Finding the x-Intercepts of a Polynomial Function Using a Graph

Find the x-intercepts of $h(x) = x^3 + 4x^2 + x - 6$.

**Solution**

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in Figure 5.3.6, it appears that there are x-intercepts at $x = -3, -2, \text{ and } 1$.

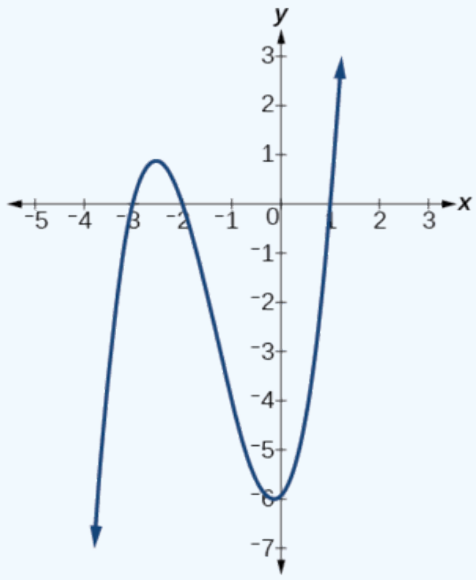



Figure 5.3.6: Graph of $h(x)$.

We can check whether these are correct by substituting these values for $x$ and verifying that

$$h(-3) = h(-2) = h(1) = 0.$$

Since $h(x) = x^3 + 4x^2 + x - 6$, we have:

$$\begin{aligned} h(-3) &= (-3)^3 + 4(-3)^2 + (-3) - 6 = -27 + 36 - 3 - 6 = 0 \\ h(-2) &= (-2)^3 + 4(-2)^2 + (-2) - 6 = -8 + 16 - 2 - 6 = 0 \\ h(1) &= (1)^3 + 4(1)^2 + (1) - 6 = 1 + 4 + 1 - 6 = 0 \end{aligned}$$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$\begin{aligned} h(x) &= x^3 + 4x^2 + x - 6 \\ &= (x+3)(x+2)(x-1) \end{aligned}$$

**Exercise 5.3.1**

Find the y-and x-intercepts of the function $f(x) = x^4 - 19x^2 + 30x$.

*Answer available at the end of this section.*

## Identifying Zeros and Their Multiplicities

**Definition: Turning Point**

A **turning point** is a point of a polynomial is where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). These points are also called *turns*.

Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and turn around. Suppose, for example, we graph the function

$$f(x) = (x+3)(x-2)^2(x+1)^3. \tag{5.3.2}$$

Notice in Figure 5.3.7 that the behavior of the function at each of the x-intercepts is different.

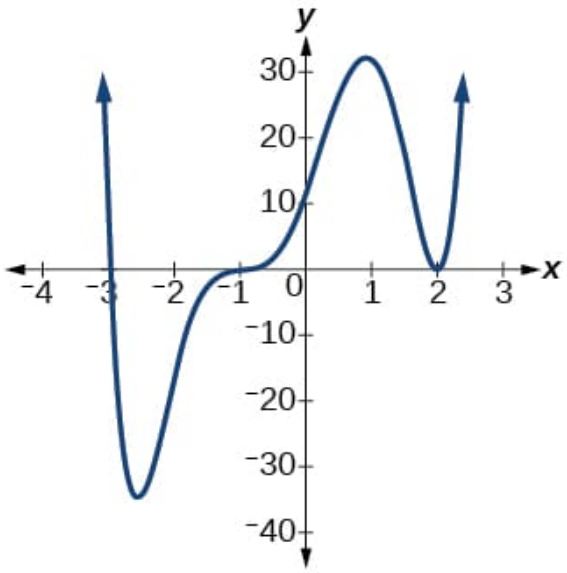



Figure 5.3.7: Identifying the behavior of the graph at an x-intercept by examining the multiplicity of the zero.

The x-intercept –3 is the solution of equation $(x+3) = 0$. The graph passes directly through thex-intercept at $x = -3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

The x-intercept 2 is the repeated solution of equation $(x-2)^2 = 0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.

$$(x-2)^2 = (x-2)(x-2) \tag{5.3.3}$$

The factor is repeated, that is, the factor $(x-2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x = 2$, has multiplicity 2 because the factor $(x-2)$ occurs twice.

The x-intercept –1 is the repeated solution of factor $(x+1)^3 = 0$.The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near the intercept as the toolkit function $f(x) = x^3$. We call this a triple zero, or a zero with multiplicity 3.

For **zeros** with even multiplicities, the graphs touch or are tangent to the x-axis. For zeros with odd multiplicities, the graphs cross or intersect the x-axis. See Figure 5.3.8 for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.

- $p = 1$ appears *linear* as it crosses the x axis.
- $p = 2$ *turning* point on the x axis.
- $p = 3$ *inflection* point on the x axis.

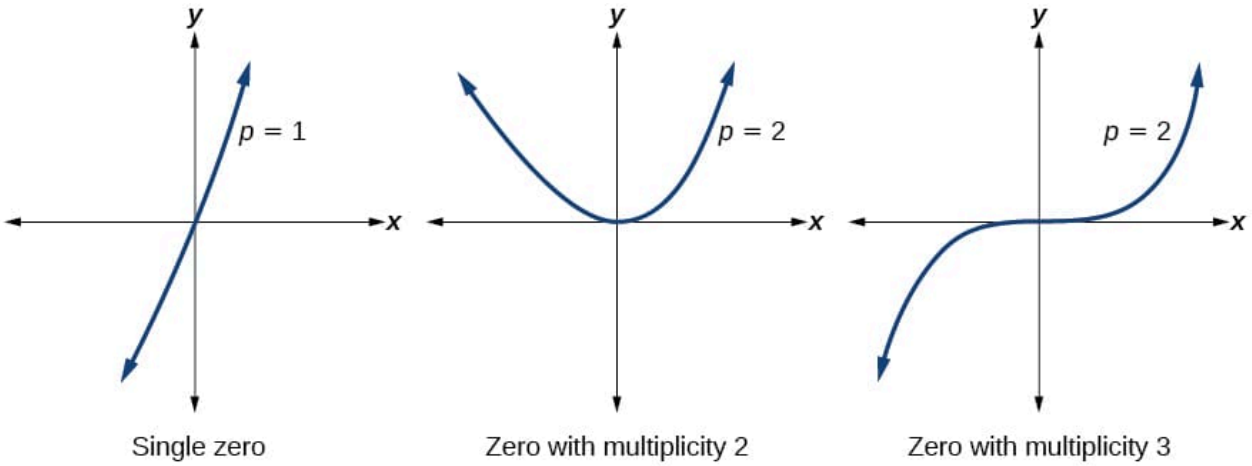



Figure 5.3.8: Three graphs showing three different polynomial functions with multiplicity 1, 2, and 3.

For higher even powers, such as 4, 6, and 8, the graph will touch the horizontal axis and turn around but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis.

For higher odd powers, such as 5, 7, and 9, the graph will have an inflection point on the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

### Graphical Behavior of Polynomials at x-Intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercept is determined by the power $p$. We say that $x=h$ is a zero of **multiplicity** $p$.

- The graph will cross the x-axis without inflection at zeros with multiplicity equal to one.
- The graph will have an inflection point the x-axis at zeros with odd multiplicities greater than one.
- The graph of a polynomial function will have a turn on the x-axis at zeros with even multiplicities.
- The sum of the multiplicities of all zeros is equal to the degree of the polynomial.

### HOW TO: Given a graph of a polynomial function of degree $n$, identify the zeros and their multiplicities

1. If the graph crosses the x-axis and appears linear at the intercept, then it is a zero of multiplicity ONE.
2. If the graph crosses the x-axis at an inflection point, then it is a zero with odd multiplicity greater than ONE.
3. If the graph touches the x-axis at a turning point, then it is a zero with even multiplicity.
4. The sum of the multiplicities of ALL zeros is equal to $n$.

### Example 5.3.6: Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in Figure 5.3.9 to identify the zeros of the function and their possible multiplicities.

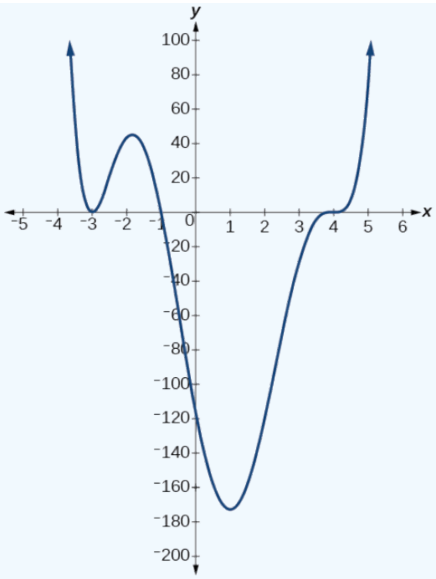



Figure 5.3.9: Graph of a polynomial function with degree 6.

**Solution**

The polynomial function is of degree $n$. The sum of the multiplicities must be $n$.

Starting from the left, the first zero occurs at $x = -3$. The graph has a turning point on the x-axis, so the multiplicity of that zero must be even. The zero at $x = -3$ is of multiplicity 2 or 4, since the degree of the polynomial is 6 and there are two additional zeros.

The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is a zero of multiplicity 1.

The last zero occurs at $x = 4$.The graph has an inflection point on the x-axis, so the multiplicity of the zero must be odd and greater than 1. We know that the multiplicity must be 3 since there are two other zeros, a multiplicity of 5 would be too large since the degree of the polynomial is 6.

Now, we can conclude that since the sum of the multiplicities is equal to 6, we have only valid possibility 2 + 1 + 3 = 6

Hence, we can summarize the results:

The at $x = -3$ must be of multiplicity 2.

The at $x = -1$ must be of multiplicity 1.

The at $x = 4$ must be of multiplicity 3.

## Exercise 5.3.2

Use the graph of the function of degree 5 in Figure 5.3.10 to identify the zeros of the function and their multiplicities.

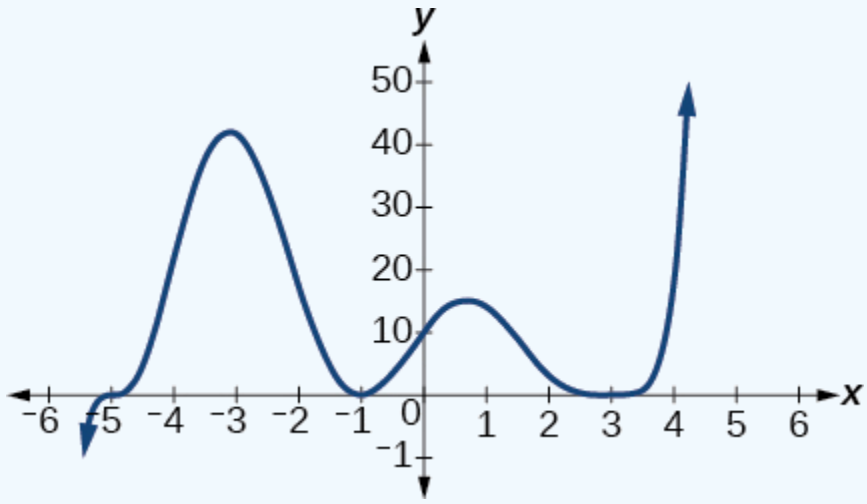



***Figure*** $5.3.10$***:*** *Graph of a polynomial function with degree 5.*

*Answer available at the end of this section.*

## Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \tag{5.3.4}$$

will either ultimately rise or fall as $x$ increases without bound and will either rise or fall as $x$ decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the end behavior of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as $x$ increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as $x$ decreases without bound, $f(x)$ also decreases without bound; as $x$ increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure $5.3.11$ summarizes all four cases.

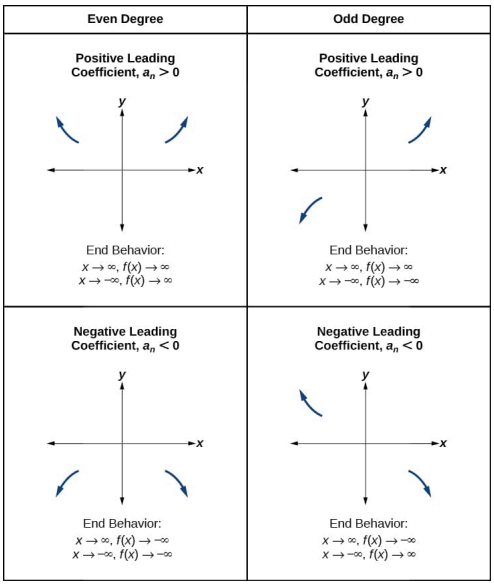



Figure 5.3.11.

## Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure 5.3.12. The graph has three turning points.

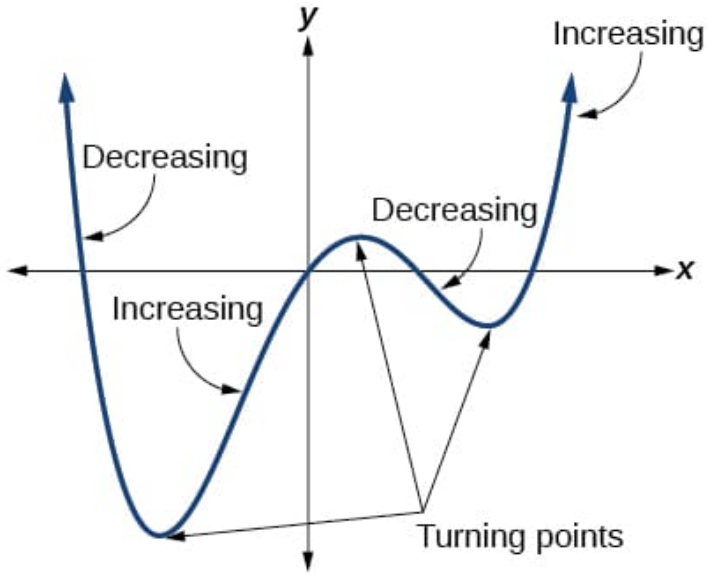



Figure 5.3.12: Graph of $f(x) = x^4 - x^3 - 4x^2 + 4x$

This function $f$ is a 4th degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

**Theorem 5.3.1 Turns Theorem**

A polynomial of degree $n$ will have *at most* $n - 1$ turning points.

If the graph of a polynomial has $n$ turns, then the degree is *at least* $n + 1$

### Example 5.3.7: Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$
b. $f(x) = -(x-1)^2(1+2x^2)$

**Solution**

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$

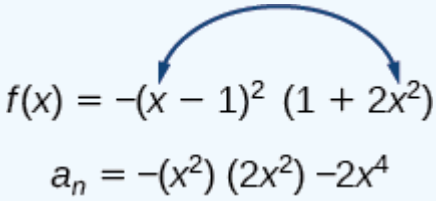
First, rewrite the polynomial function in descending order: $f(x) = 4x^5 - x^3 - 3x^2 + 1$

Identify the degree of the polynomial function. This polynomial function is of degree 5.

The maximum number of turning points is $5 - 1 = 4$.

b. $f(x) = -(x-1)^2(1+2x^2)$

First, identify the leading term of the polynomial function if the function were expanded.




Then, identify the degree of the polynomial function. This polynomial function is of degree 4.

The maximum number of turning points is $4 - 1 = 3$.

## Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

### How to: Given a polynomial function, sketch the graph

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

### Example 5.3.8: Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

**Solution**

This graph has two x-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x = 5$,the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -2(0+3)^2(0-5) \\ &= -2 \cdot 9 \cdot (-5) \\ &= 90 \end{aligned}$$

The y-intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See Figure $5.3.13$.

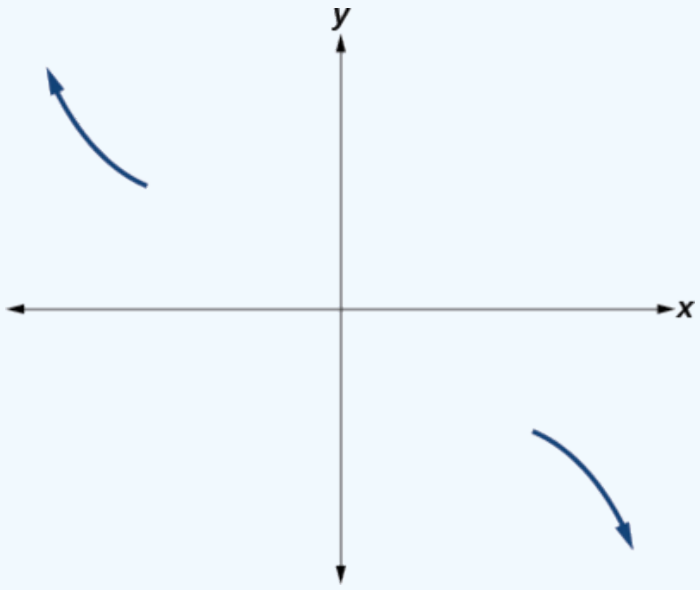



Figure $5.3.13$: Showing the distribution for the leading term.

To sketch this, we consider that:

- As $x \to -\infty$ the function $f(x) \to \infty$,so we know the graph starts in the second quadrant and is decreasing toward the x-axis.
- Since $f(-x) = -2(-x+3)^2(-x–5)$ is not equal to $f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of thex-axis, so the function must start increasing.
- At $(0, 90)$, the graph crosses the y-axis at the y-intercept. See Figure $5.3.14$.

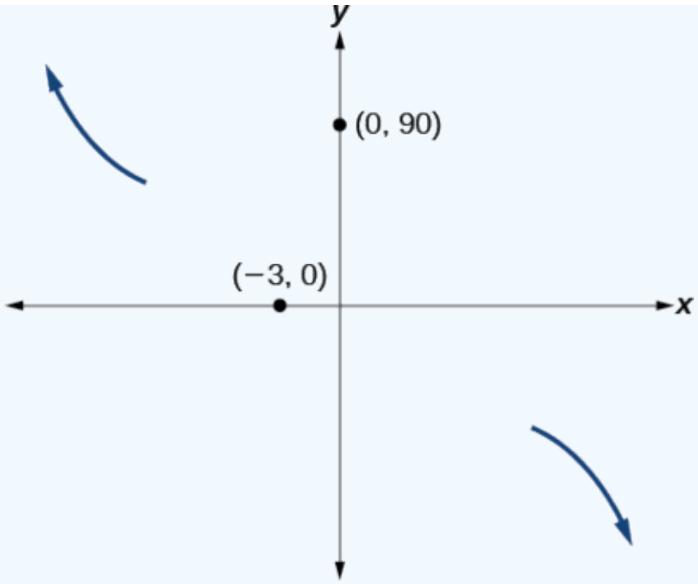



Figure $5.3.14$: Graph of the end behavior and intercepts, $(-3, 0)$ and $(0, 90)$, for the function $f(x) = -2(x + 3)^2(x - 5)$.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5, 0)$. See Figure $5.3.15$.

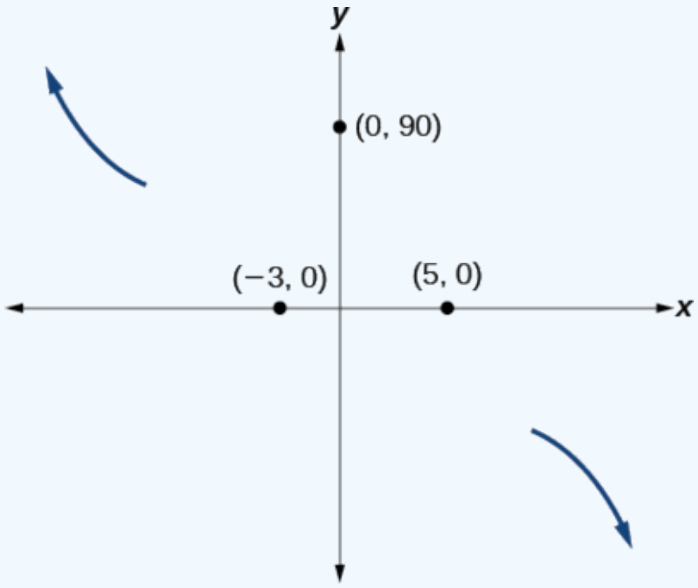



Figure $5.3.15$: Graph of the end behavior and intercepts, $(-3, 0)$, $(0, 90)$ and $(5, 0)$, for the function $f(x) = -2(x + 3)^2(x - 5)$.

As $x \to \infty$ the function $f(x) \to -\infty$,

so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure $5.3.16$, and verify that the resulting graph looks like our sketch in Figure $5.3.15$.

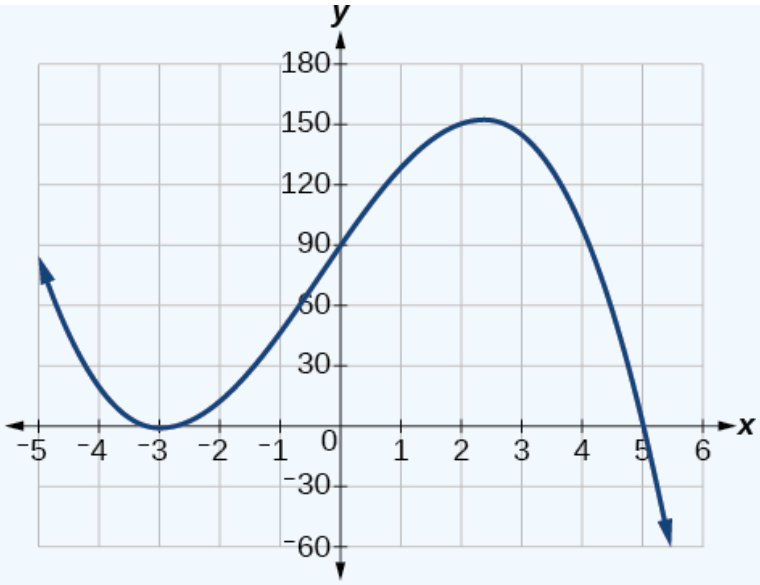



***Figure*** 5.3.16***:*** *The complete graph of the polynomial function* $f(x) = -2(x+3)^2(x-5)$.

**Exercise** 5.3.8

Sketch a graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$.

*Answer available at the end of this section.*

## Using the Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the x-axis, we can confirm that there is a zero between them. Consider a polynomial function $f$ whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers $a$ and $b$ in the domain of $f$,if $a < b$ and $f(a) \neq f(b)$,then the function $f$ takes on every value between $f(a)$ and $f(b)$. We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function $f$ at $x = a$ lies above the x-axis and another point at $x = b$ lies below thex-axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x-axis. Call this point $(c, f(c))$.This means that we are assured there is a solution $c$ where $f(c) = 0$.

In other words, the **Intermediate Value Theorem** tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis. Figure 5.3.18 shows that there is a zero between $a$ and $b$.

**Theorem** 5.3.1**Value Theorem**

Let $f$ be a polynomial function. If $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value $c$ between $a$ and $b$ for which $f(c) = 0$.

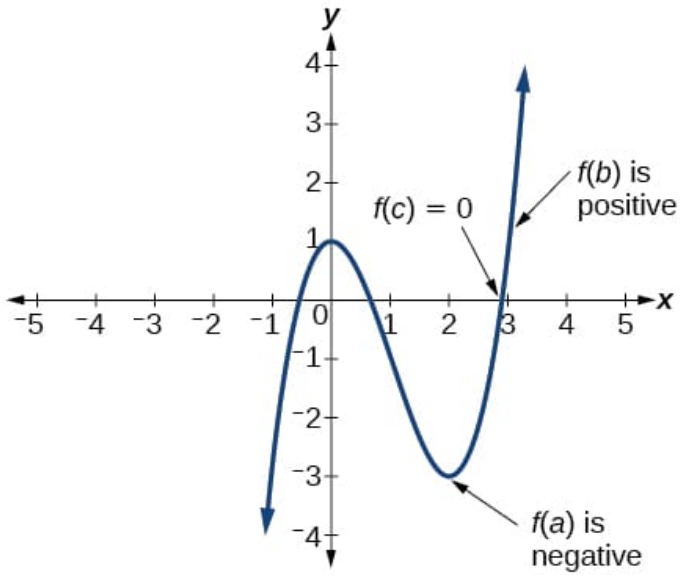



Figure 5.3.18: Using the Intermediate Value Theorem to show there exists a zero.

## Example 5.3.9: Using the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

**Solution**

As a start, evaluate $f(x)$ at the integer values $x = 1,\ 2,\ 3,\ \text{and } 4$ (Table 5.3.2).

*Table* 5.3.2

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 5 | 0 | -3 | 2 |

We see that one zero occurs at $x = 2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4.

We have shown that there are at least two real zeros between $x = 1$ and $x = 4$.

**Analysis**

We can also see on the graph of the function in Figure 5.3.19 that there are two real zeros between $x = 1$ and $x = 4$.

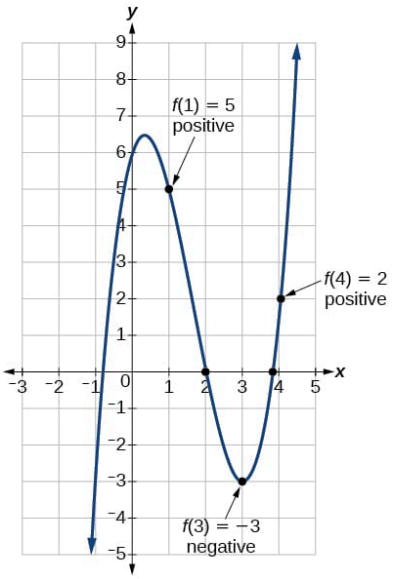



Figure 5.3.19.

**Exercise 5.3.4**

Show that the function $f(x) = 7x^5 - 9x^4 - x^2$ has at least one real zero between $x = 1$ and $x = 2$.

*Answer available at the end of this section.*

## Writing Formulas for Polynomial Functions

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a **polynomial function** written in factored form will have an x-intercept where each factor is equal to zero, we can form a function that will pass through a set of x-intercepts by introducing a corresponding set of factors.

**Note: Factored Form of Polynomials**

If a polynomial of lowest degree $p$ has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, then the polynomial can be written in the factored form: $f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$ where the powers $p_i$ on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor $a$ can be determined given a value of the function other than the x-intercept.

**Given a graph of a polynomial function, write a formula for the function.**

1. Identify the x-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the x-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the y-intercept may be easiest) to determine the stretch factor.

## Example 5.3.10: Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in Figure 5.3.20.

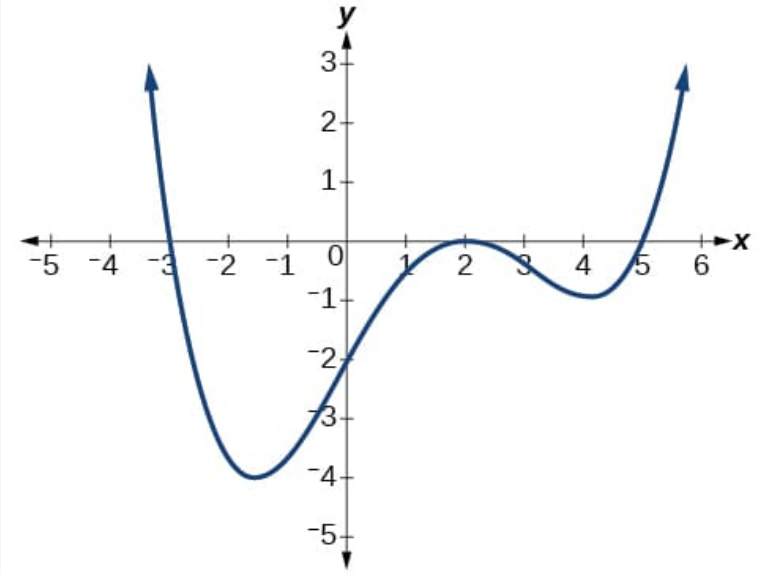



Figure 5.3.20.

**Solution**

This graph has three x-intercepts: $x = -3,\ 2,\ \text{and}\ 5$. The y-intercept is located at $(0, 2)$.At $x = -3$ and $x = 5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

$$f(x) = a(x + 3)(x - 2)^2(x - 5) \tag{5.3.5}$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, -2)$, to solve for $a$.

$$\begin{aligned} f(0) &= a(0 + 3)(0 - 2)^2(0 - 5) \\ -2 &= a(0 + 3)(0 - 2)^2(0 - 5) \\ -2 &= -60a \\ a &= \frac{1}{30} \end{aligned}$$

The graphed polynomial appears to represent the function $f(x) = \frac{1}{30}(x + 3)(x - 2)^2(x - 5)$.

## Exercise 5.3.5

Given the graph shown in Figure 5.3.21, write a formula for the function shown.

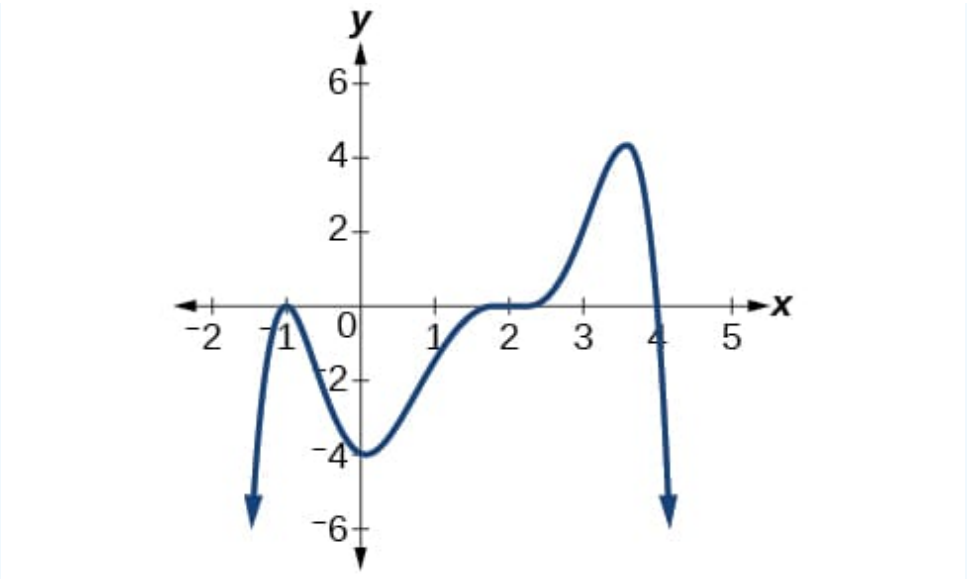



Figure 5.3.21.

*Answer available at the end of this section.*

## Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a **global maximum** or a **global minimum**. These are also referred to as the absolute maximum and absolute minimum values of the function.

### Note: Local and Global Extrema

A **local maximum** or **local minimum** at $x = a$ (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around $x = a$.If a function has a local maximum at $a$, then $f(a) \geq f(x)$for all $x$ in an open interval around $x = a$. If a function has a local minimum at $a$, then $f(a) \leq f(x)$for all $x$ in an open interval around $x = a$.

A **global maximum** or **global minimum** is the output at the highest or lowest point of the function. If a function has a global maximum at $a$, then $f(a) \geq f(x)$ for all $x$. If a function has a global minimum at $a$, then $f(a) \leq f(x)$ for all $x$.

We can see the difference between local and global extrema in Figure 5.3.22.

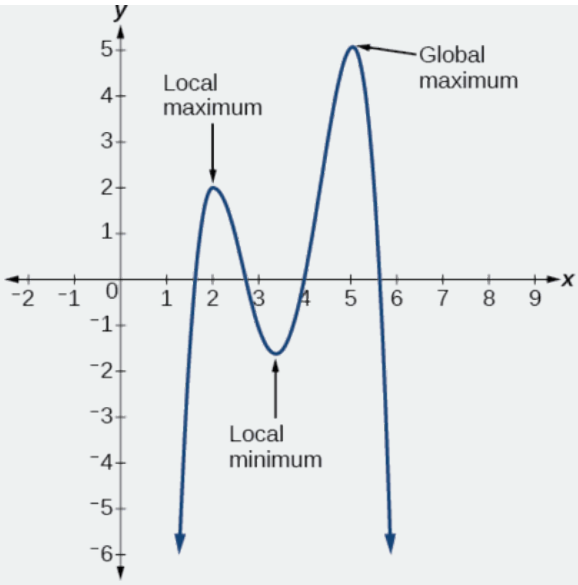



Figure 5.3.22: Graph of an even-degree polynomial that denotes the local maximum and minimum and the global maximum.

**Do all polynomial functions have a global minimum or maximum?**

*No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ has neither a global maximum nor a global minimum.*

## Example 5.3.11: Using Local Extrema to Solve Applications

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

**Solution**

We will start this problem by drawing a picture like that in Figure 5.3.23, labeling the width of the cut-out squares with a variable, $w$.

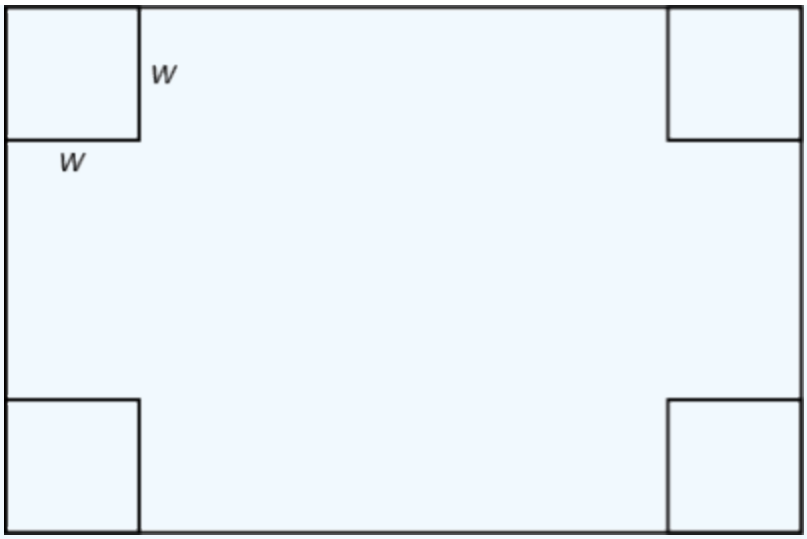



Figure 5.3.23: Diagram of a rectangle with four squares at the corners.

Notice that after a square is cut out from each end, it leaves $a(14-2w)$ cm by $(20-2w)$ cm rectangle for the base of the box, and the box will be $w$ cm tall. This gives the volume

$$\begin{aligned} V(w) &= (20-2w)(14-2w)w \\ &= 280w - 68w^2 + 4w^3 \end{aligned}$$

Notice, since the factors are $20–2w$, $14–2w$, and $w$, the three zeros are $10,\ 7$, and $0,)respectively.Becauseaheightof\(0$ cm is not reasonable, we consider the only the zeros $10$ and $7$. The shortest side is $14$ and we are cutting off two squares, so values $w$ may take on are greater than zero or less than $7$. This means we will restrict the domain of this function to $0 < w < 7$.Using technology to sketch the graph of $V(w)$ on this reasonable domain, we get a graph like that in Figure $5.3.24$. We can use this graph to estimate the maximum value for the volume, restricted to values for $w$ that are reasonable for this problem—values from 0 to 7.

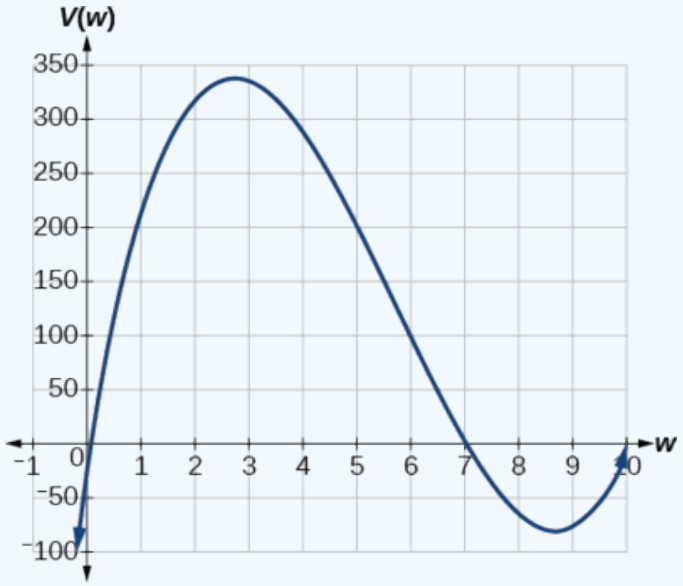



Figure 5.3.24: Graph of $V(w) = (20-2w)(14-2w)w$

From this graph, we turn our focus to only the portion on the reasonable domain, $[0, 7]$. We can estimate the maximum value to be around 340 cubic cm, which occurs when the squares are about 2.75 cm on each side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce Figure $5.3.25$.

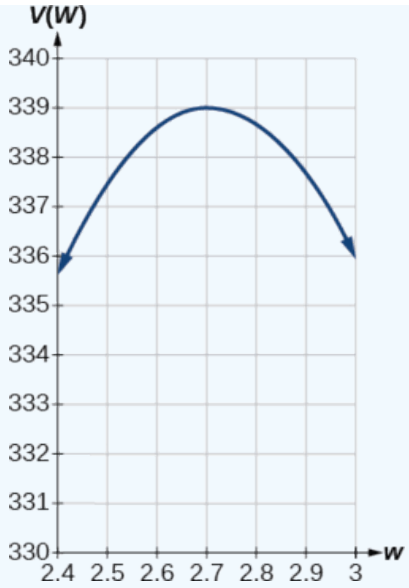


Figure 5.3.25: Graph of $V(w) = (20 - 2w)(14 - 2w)w$.

From this zoomed-in view, we can refine our estimate for the maximum volume to about 339 cubic cm, when the squares measure approximately 2.7 cm on each side.

**Exercise 5.3.1**

Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function $f(x) = -0.2(x - 2)^3(x + 1)^2(x - 4)$.

*Answer available at the end of this section.*

## Key Concepts

- Polynomial functions of degree 2 or more are smooth, continuous functions.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero.
- Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x-axis.
- The multiplicity of a zero determines how the graph behaves at the x-intercepts.
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree $n$ has at most $n - 1$ turning points.
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n - 1$ turning points.
- Graphing a polynomial function helps to estimate local and global extremas.
- The Intermediate Value Theorem tells us that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value $c$ between $a$ and $b$ for which $f(c) = 0$.

## Glossary

**global maximum**
highest turning point on a graph; $f(a)$ where $f(a) \geq f(x)$ for all $x$.

**global minimum**

lowest turning point on a graph; $f(a)$ where $f(a) \leq f(x)$ for all $x$.

**Intermediate Value Theorem**
for two numbers $a$ and $b$ in the domain of $f$, if $a < b$ and $f(a) \neq f(b)$, then the functionf takes on every value between $f(a)$ and $f(b)$ ; specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis

**multiplicity**
the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x-h)^p$, $x=h$ is a zero of multiplicity $p$.

## Answers

**Exercise** 5.3.1

- y-intercept $(0,0)$;
- x-intercepts $(0,0)$, $(-5,0)$, $(2,0)$, and $(3,0)$

**Exercise** 5.3.2

The graph has a zero of –5 with multiplicity 1, a zero of –1 with multiplicity 2, and a zero of 3 with even multiplicity.

**Exercise** 5.3.8

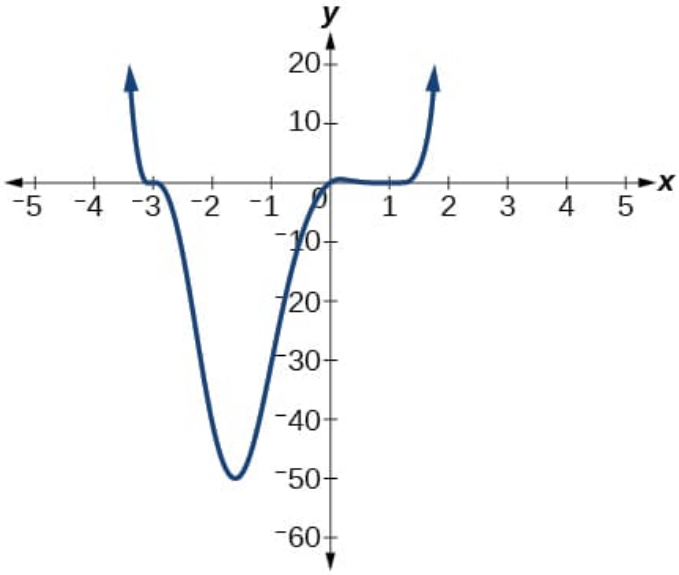



Figure 5.3.17: Graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$

**Exercise** 5.3.4

Because $f$ is a polynomial function and since $f(1)$ is negative and $f(2)$ is positive, there is at least one real zero between $x=1$ and $x=2$.

**Exercise** 5.3.5

$f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

**Exercise** 5.3.1

The minimum occurs at approximately the point $(0,-6.5)$,
and the maximum occurs at approximately the point $(3.5,7)$.

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# 5.4: Dividing Polynomials

**Learning Objectives**

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

## Using Long Division to Divide Polynomials

We are familiar with the **long division** algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

**Long Division**

$$\begin{array}{r} 59 \\ 3\overline{)178} \\ \underline{-15}\downarrow \\ 28 \\ \underline{-27} \\ 1 \end{array}$$

Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$
Step 2: Bring down the 8
Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$

Answer: $59\,R\,1$ or $59\frac{1}{3}$

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \text{dividend} &= (\text{divisor}\cdot\text{quotient}) + \text{remainder} \\ 178 &= (3\cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm, it would look like this:

$$\begin{array}{rl} 2x^2 - 7x + 18 & \text{Step 1. Divide: } \frac{2x^3}{x} \quad \text{Step 4. Divide: } \frac{-7x^2}{x} = -7x \quad \text{Step 7. Divide: } \frac{18x}{x} = 18 \\ x + 2\,\overline{)2x^3 - 3x^2 + 4x + 5} & \text{Original problem} \\ -(2x^3 + 4x^2) & \text{Step 2. Multiply: } 2x^2(x+2) \\ -7x^2 + 4x & \text{Step 3. Subtract and Bring Down} \\ -(-7x^2 - 14x) & \text{Step 5. Multiply: } -7x(x+2) \\ 18x + 5 & \text{Step 6. Subtract and Bring Down} \\ -(18x + 36) & \text{Step 8. Multiply: } 18(x+2) \\ -31 & \text{Step 9. Subtract. This is the remainder} \end{array}$$

The result is: $\dfrac{2x^3 - 3x^2 + 4x + 5}{x + 2} = 2x^2 - 7x + 18 - \dfrac{31}{x + 2}$

or $2x^3 - 3x^2 + 4x + 5 = (x + 2)(2x^2 - 7x + 18) - 31$

We can identify the dividend, the **divisor**, the **quotient**, and the **remainder**.

Writing the result in this manner illustrates the Division Algorithm.

## The Division Algorithm

The Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$. If $r(x) = 0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

## How to: Given a polynomial and a binomial, use long division to divide the polynomial by the binomial

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom **binomial** from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until the degree of the remainder is less than the degree of the divisor.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

## Example 5.4.1: Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$.

**Solution**

$$\begin{array}{rl} 5x - 2 & \text{Step 1. Divide: } \frac{5x}{x} \qquad \text{Step 4. Divide: } \frac{-2}{x} = -7x \\ x + 1 \,/\overline{5x^2 + 3x - 2} & \text{Original problem} \\ \underline{-(5x^2 + 5x)} & \text{Step 2. Multiply: } 5x(x + 1) \\ -2x - 2 & \text{Step 3. Subtract and Bring Down} \\ \underline{-(-2x - 2)} & \text{Step 5. Multiply: } -2(x + 1) \\ 0 & \text{Step 6. Subtract. There is no remainder} \end{array}$$

The quotient is $5x - 2$. The remainder is $0$. The result is $\dfrac{5x^2 + 3x - 2}{x + 1} = 5x - 2$

or $5x^2 + 3x - 2 = (x + 1)(5x - 2)$

**Analysis**

This division problem had a remainder of $0$. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

## 5.4.2: Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$.

**Solution**

$$
\begin{array}{r}
2x^2 + 5x - 7 \\
3x - 2\overline{)\,6x^3 + 11x^2 - 31x + 1} \\
\underline{-(6x^3 - 4x^2)} \\
15x^2 - 31x \\
\underline{-(15x^2 + 10x)} \\
21x + 15 \\
\underline{-(-21x + 14)} \\
1
\end{array}
$$

$6x^3$ divided by $3x$ is $2x^2$.

Multiply $x + 2$ by $2x^2$.
Subtract. Bring down the next term. $15x^2$ divided by $3x$ is $5x$.
Multiply $3x - 2$ by $5x$.
Subtract. Bring down the next term. $-21x$ divided by $3x$ is $-7$.
Multiply $3x - 2$ by $-7$.
Subtract. The remainder is 1.

There is a remainder of 1. We can express the result as: $\frac{6x^3 + 11x^2 - 31x + 15}{3x - 2} = 2x^2 + 5x - 7 + \frac{1}{3x - 2}$

**Analysis**

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x - 2)(2x^2 + 5x - 7) + 1 = 6x^3 + 11x^2 - 31x + 15$$

Notice, as we write our result,

- the dividend is $6x^3 + 11x^2 - 31x + 15$
- the divisor is $3x - 2$
- the quotient is $2x^2 + 5x - 7$
- the remainder is $1$

### Try It 5.4.2

Divide $16x^3 - 12x^2 + 20x - 3$ by $4x + 5$.

*Solution available at the end of this section.*

The divisor can also be a higher degree polynomial.

### Example 5.4.3

Divide.

$$x^2 + 3x + 1\overline{)\,2x^4 + 7x^3 + 4x^2 - 2x - 1}$$

**Solution**

$$
\begin{array}{rl}
 & 2x^2 + x - 1 \qquad \text{Step 1. Divide: } \frac{2x^4}{x^2} \quad \text{Step 4. Divide: } \frac{x^3}{x^2} \quad \text{Step 7. Divide: } \frac{-x^2}{x^2} \\
x^2+3x+1 \,\overline{)\, 2x^4+7x^3+4x^2-2x-1} & \text{Original problem} \\
-(2x^4+6x^3+2x^2) & \text{Step 2. Multiply: } 2x^2(x^2+3x+1) \\
x^3+2x^2-2x & \text{Step 3. Subtract and Bring Down} \\
-(x^3+3x^2+x) & \text{Step 5. Multiply: } x(x^2+3x+1) \\
-x^2-3x-1 & \text{Step 6. Subtract and Bring Down} \\
-(-x^2-3x-1) & \text{Step 8. Multiply: } -1(x^2+3x+1) \\
0 & \text{Step 9. Subtract. There is no remainder}
\end{array}
$$

Because $x^2+3x+1$ divides evenly into $2x^4+7x^3+4x^2-2x-1$ we have a zero remainder.

The final answer is $\dfrac{2x^4+7x^3+4x^2-2x-1}{x^2+3x+1} = 2x^2+x-1$

**Example 5.4.4**

Divide. $\dfrac{3x^4-8x^3+19x^2-15x+10}{x^2-x+4}$

**Solution**

$$
\begin{array}{rl}
 & 3x^2 - 5x + 2 \qquad \text{Step 1. Divide: } \frac{3x^4}{x^2} \quad \text{Step 4. Divide: } \frac{-5x^3}{x^2} \quad \text{Step 7. Divide: } \frac{2x^2}{x^2} \\
x^2-x+4 \,\overline{)\, 3x^4-8x^3+19x^2-15x+10} & \text{Original problem} \\
-(3x^4-3x^3+12x^2) & \text{Step 2. Multiply: } x^2(x^2-x+4) \\
-5x^3+7x^2-15x & \text{Step 3. Subtract and Bring Down} \\
-(-5x^3+5x^2-20x) & \text{Step 5. Multiply: } x(x^2-x+4) \\
2x^2+5x+10 & \text{Step 6. Subtract and Bring Down} \\
-(2x^2-2x+8) & \text{Step 8. Multiply: } 2(x^2-x+4) \\
7x+2 & \text{Step 9. Subtract. This is the remainder}
\end{array}
$$

Thus $3x^4-8x^3+19x^2-15x+10 = (x^2-x+4)(3x^2-5x+2)+(7x+2)$

The final Answer is: $\dfrac{3x^4-8x^3+19x^2-15x+10}{x^2-x+4} = 3x^2-5x+2+\dfrac{7x+2}{x^2-x+4}$

## Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3-3x^2+4x+5$ by $x+2$ using the long division algorithm.

The final form of the process looked like this:

$$
\begin{array}{r}
2x^2 + x + 18 \\
x+2\overline{)\,2x^3 - 3x^2 + 4x + 5} \\
\underline{-(2x^3 + 4x^2)\phantom{+ 5}} \\
-7x^2 + 4x \\
\underline{-(-7x^2 - 14x)} \\
18x + 5 \\
\underline{-(18x + 36)} \\
-31
\end{array}
$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$
\begin{array}{r|rrrr}
2 & 2 & -3 & 4 & 5 \\
 & -2 & -4 & & \\
\hline
 & & -7 & 14 & \\
 & & & 18 & -36 \\
\hline
 & & & & -31
\end{array}
$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$
\begin{array}{r|rrrr}
-2 & 2 & -3 & 4 & 5 \\
 & & -4 & 14 & -36 \\
\hline
 & 2 & -7 & 18 & -31
\end{array}
$$

We then multiply it by the "divisor" and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2–7x + 18$ and the remainder is –31.The process will be made more clear in the next example.

## Synthetic Division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x - k$. In *synthetic division*, only the coefficients are used in the division process.

## How to: Given two polynomials, use synthetic division to divide

1. Write $k$ for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by $k$. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by $k$. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

## Example 5.4.5: Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

**Solution**

Begin by setting up the synthetic division. Write $k$ and the coefficients.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ \hline & & & \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by $k$.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by $k$. Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The quotient is $5x + 12$. The remainder is 0. Therefore, $x - 3$ is a factor of $5x^2 - 3x - 36$.

**Analysis**

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x - 3)(5x + 12) + 0 = 5x^2 - 3x - 36 \tag{5.4.1}$$

## Example 5.4.6: Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

**Solution**

The binomial divisor is $x + 2$ so $k = -2$. Add each column, multiply the result by –2, and repeat until the last column is reached.

$$\begin{array}{r|cccc} -2 & 4 & 10 & -6 & -20 \\ \hline & 4 & & & \end{array}$$

$$\begin{array}{r|cccc} -2 & 4 & 10 & -6 & -20 \\ & & -8 & & \\ \hline & 4 & & & \end{array}$$

$$\begin{array}{r|cccc} -2 & 4 & 10 & -6 & -20 \\ & & -8 & & \\ \hline & 4 & 2 & & \end{array}$$

$$\begin{array}{r|cccc} -2 & 4 & 10 & -6 & -20 \\ & & -8 & -4 & \\ \hline & 4 & 2 & & \end{array}$$

$$\begin{array}{r|cccc} -2 & 4 & 10 & -6 & -20 \\ & & -8 & -4 & \\ \hline & 4 & 2 & -10 & \end{array}$$

$$\begin{array}{r|cccc} -2 & 4 & 10 & -6 & -20 \\ & & -8 & -4 & 20 \\ \hline & 4 & 2 & -10 & \end{array}$$

$$\begin{array}{r|cccc} -2 & 4 & 10 & -6 & -20 \\ & & -8 & -4 & 20 \\ \hline & 4 & 2 & -10 & 0 \end{array}$$

The quotient is $4x^2 + 2x - 10$.

The remainder is 0. Thus, $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

## Example 5.4.7: Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

**Solution**

Notice there is no x-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The quotient is $-9x^3 + x^2 + 8x + 8$

The remainder is $2$

The result is $-9x^3 + x^2 + 8x + 8 + \frac{2}{x-1}$.

Try It 5.4.7

Use synthetic division to divide $3x^4 + 18x^3 - 3x + 40$ by $x + 7$.

*Solution available at the end of this section.*

## Key Equations

Division Algorithm $f(x) = d(x)q(x) + r(x)$ where $q(x) \neq 0$

## Key Concepts

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form x–k.
- Polynomial division can be used to solve application problems, including area and volume.

## Glossary

**Division Algorithm**

given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

**synthetic division**

a shortcut method that can be used to divide a polynomial by a binomial of the form $x - k$

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### Answers



**Try It** 5.4.2

$4x^2 - 8x + 15 - \dfrac{78}{4x + 5}$



5.4.7

$3x^3 - 3x^2 + 21x - 150 + \frac{1,090}{x+7}$

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# 5.5: Zeros of Polynomial Functions

**Learning Objectives**

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.
- Use the Rational Zero Theorem to find rational zeros.
- Find zeros of a polynomial function.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes' Rule of Signs.

## Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by $x–c$, the remainder may be found quickly by evaluating the polynomial function at $x = c$, that is, $f(c)$. Let's walk through the proof of the theorem.

Recall that the Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$,there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

If the divisor, $d(x)$, is $x - c$, this takes the form

$$f(x) = (x - c)q(x) + r$$

Since the divisor $x - c$ is linear, the remainder will be a constant, $r$.

And, if we evaluate this for $x = k$, we have

$$\begin{aligned} f(c) &= (c - c)q(c) + r \\ &= 0 \cdot q(k) + r \\ &= r \end{aligned}$$

In other words, $f(c)$ is the remainder obtained by dividing $f(x)$by $x - c$.

**Theorem 5.5.1: The Remainder Theorem**

If a polynomial $f(x)$ is divided by $x - c$, then the remainder is the value $f(c)$.




Given a polynomial function $f$, evaluate $f(x)$ at $x = c$ using the Remainder Theorem.

1. Use synthetic division to divide the polynomial by $x - c$.
2. The remainder is the value $f(c)$.

### Example 5.5.1: Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate $f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7$ at $x = 2$.

**Solution**

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x - 2$.

$$\begin{array}{r|rrrrr} 2 & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array} \tag{5.5.1}$$

The remainder is 25. Therefore, $f(2) = 25$.

**Analysis**

We can check our answer by evaluating $f(2)$.

$$\begin{aligned} f(x) &= 6x^4 - x^3 - 15x^2 + 2x - 7 \\ f(2) &= 6(2)^4 - (2)^3 - 15(2)^2 + 2(2) - 7 \\ &= 25 \end{aligned}$$

### Exercise 5.5.1

se the Remainder Theorem to evaluate $f(x) = 2x^5 - 3x^4 - 9x^3 + 8x^2 + 2$ at $x = -3$.

*Answer available at the end of this section.*

## Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm.

$$f(x) = (x - c)q(x) + r \tag{5.5.2}$$

If $x = c$ is a zero, then the remainder $r$ is $f(c) = 0$ and $f(x) = (x - c)q(x) + 0$ or $f(x) = (x - c)q(x)$.

Notice, written in this form, $x - c$ is a factor of $f(x)$. We can conclude if $x = c$ is a zero of $f(x)$, then $x - c$ is a factor of $f(x)$.

Similarly, if $x - c$ is a factor of $f(x)$, then the remainder of the Division Algorithm $f(x) = (x - c)q(x) + r$ is $0$. This tells us that $c$ is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree $n$ in the complex number system will have $n$ zeros. We can use the Factor Theorem to completely factor a polynomial into the product of $n$ factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

### Theorem 5.5.2 Factor Theorem

According to the Factor Theorem, $c$ is a zero of $f(x)$ if and only if $(x - c)$ is a factor of $f(x)$.

**How to: Given a factor and a third-degree polynomial, use the Factor Theorem to factor the polynomial**

1. Use synthetic division to divide the polynomial by $(x-k)$.
2. Confirm that the remainder is $0$.
3. Write the polynomial as the product of $(x-k)$ and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

**Example 5.5.2: Using the Factor Theorem to Solve a Polynomial Equation**

Show that $(x+2)$ is a factor of $x^3-6x^2-x+30$. Find the remaining factors. Use the factors to determine the zeros of the **polynomial**.

**Solution**

We can use synthetic division to show that $(x+2)$ is a factor of the polynomial.

$$\begin{array}{r|rrrr} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array} \tag{5.5.3}$$

The remainder is zero, so $(x+2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$$(x+2)(x^2-8x+15) \tag{5.5.4}$$

We can factor the quadratic factor to write the polynomial as

$$(x+2)(x-3)(x-5) \tag{5.5.5}$$

By the Factor Theorem, the zeros of $x^3-6x^2-x+30$ are –2, 3, and 5.

**Exercise 5.5.2**

Use the Factor Theorem to find the zeros of $f(x)=x^3+4x^2-4x-16$ given that $(x-2)$ is a factor of the polynomial.

*Answer available at the end of this section.*

## Using the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading **coefficient** of the polynomial

Consider a quadratic function with two zeros, $x=\frac{2}{5}$ and $x=\frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0, and then construct the original quadratic function absent its stretching factor.

| | |
|---|---|
| $x - \frac{2}{5} = 0$ or $x - \frac{3}{4} = 0$ | Set each factor equal to 0. |
| $5x - 2 = 0$ or $4x - 3 = 0$ | Multiply both sides of the equation to eliminate fractions. |
| $f(x) = (5x - 2)(4x - 3)$ | Create the quadratic function, multiplying the factors. |
| $f(x) = 20x^2 - 23x + 6$ | Expand the polynomial. |
| $f(x) = (5 \cdot 4)x^2 - 23x + (2 \cdot 3)$ | |

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

### Theorem 5.5.3 The Rational Zeros Theorem

The Rational Zero Theorem states that, if the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ has integer coefficients, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where $p$ is a factor of the constant term $a_0$ and $q$ is a factor of the leading coefficient $a_n$.

When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

### How to: Given a polynomial function $f(x)$, use the Rational Zero Theorem to find rational zeros.

1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of $\frac{p}{q}$, where $p$ is a factor of the constant term and $q$ is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of $f(\frac{p}{q})$.

### Example 5.5.3: Listing All Possible Rational Zeros

List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$.

**Solution**

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, −4, and the factors of the leading coefficient, 2.

The constant term is −4; the factors of −4 are $p = \pm 1, \pm 2, \pm 4$.

The leading coefficient is 2; the factors of 2 are $q = \pm 1, \pm 2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of −4 divided by one of the factors of 2.

$$\frac{p}{q} = \pm\frac{1}{1}, \pm\frac{1}{2} \qquad \frac{p}{q} = \pm\frac{2}{1}, \pm\frac{2}{2} \qquad \frac{p}{q} = \pm\frac{4}{1}, \pm\frac{4}{2}$$

Note that $\frac{2}{2} = 1$ and $\frac{4}{2} = 2$, which have already been listed. So we can shorten our list.

$$\frac{p}{q} = \frac{\text{Factors of the last}}{\text{Factors of the first}} = \pm 1, \pm 2, \pm 4, \pm\frac{1}{2}$$

### Example 5.5.4: Using the Rational Zero Theorem to Find Rational Zeros

Use the Rational Zero Theorem to find the rational zeros of $f(x) = 2x^3 + x^2 - 4x + 1$.

**Solution**

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then $p$ is a factor of 1 and $q$ is a factor of 2.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$
$$= \frac{factor\ of\ 1}{factor\ of\ 2}$$

The factors of 1 are ±1 and the factors of 2 are ±1 and ±2. The possible values for $\frac{p}{q}$ are ±1 and $\pm\frac{1}{2}$. These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by substituting these values for $x$ in $f(x)$.

$$f(-1) = 2(-1)^3 + (-1)^2 - 4(-1) + 1 = 4 \tag{5.5.6}$$

$$f(1) = 1(1)^3 + (1)^2 - 4(1) + 1 = 0 \tag{5.5.7}$$

$$f(-\frac{1}{2}) = 2(-\frac{1}{2})^3 + (-\frac{1}{2})^2 - 4(-\frac{1}{2}) + 1 = 3 \tag{5.5.8}$$

$$f(\frac{1}{2}) = 2(\frac{1}{2})^3 + (\frac{1}{2})^2 - 4(\frac{1}{2}) + 1 = -\frac{1}{2} \tag{5.5.9}$$

Of those, $-1$,$-\frac{1}{2}$, and $\frac{1}{2}$ are not zeros of $f(x)$. 1 is the only rational zero of $f(x)$.

### Exercise 5.5.3

Use the Rational Zero Theorem to find the rational zeros of $f(x) = x^3 - 5x^2 + 2x + 1$.

*Answer available at the end of this section.*

## Finding the Zeros of Polynomial Functions

The Rational Zero Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use **synthetic division** repeatedly to determine all of the **zeros** of a polynomial function.

### How to: Given a polynomial function $f$, use synthetic division to find its zeros.

1. Use the Rational Zero Theorem to list all possible rational zeros of the function.
2. Use synthetic division to evaluate a given possible zero by synthetically dividing the candidate into the polynomial. If the remainder is 0, the candidate is a zero. If the remainder is not zero, discard the candidate.
3. Repeat step two using the quotient found with synthetic division. If possible, continue until the quotient is a quadratic.
4. Find the zeros of the quadratic function. Two possible methods for solving quadratics are factoring and using the quadratic formula.

### Example 5.5.5: Finding the Zeros of a Polynomial Function with Repeated Real Zeros

Find the zeros of $f(x) = 4x^3 - 3x - 1$.

**Solution**

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then $p$ is a factor of –1 and $q$ is a factor of 4.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$
$$= \frac{factor\ of\ -1}{factor\ of\ 4}$$

The factors of –1 are ±1 and the factors of 4 are ±1,±2, and ±4. The possible values for $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}$, and $\pm\frac{1}{4}$. These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 1.

Dividing by $(x-1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

$$(x-1)(4x^2+4x+1)$$

The quadratic is a perfect square. $f(x)$ can be written as

$$(x-1)(2x+1)^2$$

We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

$$2x+1=0$$
$$x=-\frac{1}{2}$$

The zeros of the function are 1 and $-\frac{1}{2}$ with multiplicity 2.

**Analysis**

Look at the graph of the function $f$ in Figure 5.5.1. Notice, at $x=-0.5$, the graph bounces off the x-axis, indicating the even multiplicity (2,4,6...) for the zero –0.5. At $x=1$, the graph crosses the x-axis, indicating the odd multiplicity (1,3,5...) for the zero $x=1$.

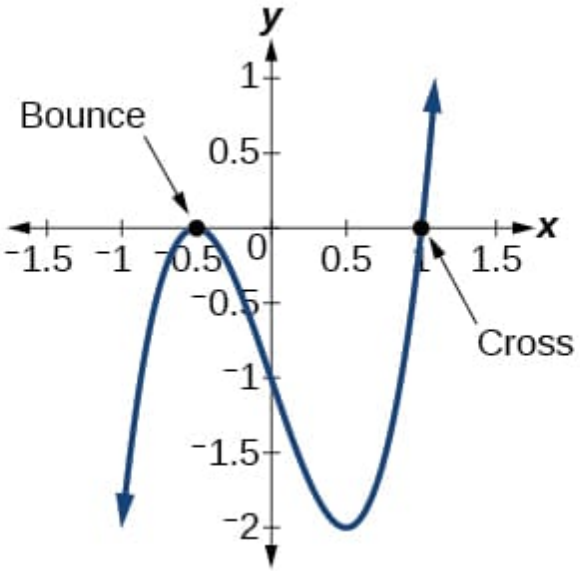



Figure 5.5.1.

# Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose $f$ is a polynomial function of degree four, and $f(x) = 0$. The Fundamental Theorem of Algebra states that there is at least one complex solution, call it $c_1$. By the Factor Theorem, we can write $f(x)$ as a product of $x - c_1$ and a polynomial quotient. Since $x - c_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it $c_2$. So we can write the polynomial quotient as a product of $x - c_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of $f(x)$.

## Theorem 5.5.3 The Fundamental Theorem of Algebra

The *Fundamental Theorem of Algebra* states that, if $f(x)$ is a polynomial of degree $n > 0$, then $f(x)$ has at least one complex zero.

We can use this theorem to argue that, if $f(x)$ is a polynomial of degree $n > 0$, and a is a non-zero real number, then $f(x)$ has exactly $n$ linear factors

$$f(x) = a(x - c_1)(x - c_2)...(x - c_n) \tag{5.5.10}$$

where $c_1, c_2,...,c_n$ are complex numbers. Therefore, $f(x)$ has $n$ roots if we allow for multiplicities.

## Q&A: Does every polynomial have at least one imaginary zero?

No. Real numbers are a subset of complex numbers, but not the other way around. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

## Example 5.5.6: Finding the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of $f(x) = 3x^3 + 9x^2 + x + 3$.

**Solution**

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then $p$ is a factor of 3 and $q$ is a factor of 3.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient}$$
$$= \frac{factor\ of\ 3}{factor\ of\ 3}$$

The factors of 3 are ±1 and ±3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are ±3,±1, and $\pm\frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with –3.

Dividing by $(x + 3)$ gives a remainder of 0, so –3 is a zero of the function. The polynomial can be written as

$$(x + 3)(3x^2 + 1)$$

We can then set the quadratic equal to 0 and solve to find the other zeros of the function.

$$
\begin{aligned}
3x^2 + 1 &= 0 \\
x^2 &= -\frac{1}{3} \\
x &= \pm\sqrt{\frac{-1}{3}} \\
&= \pm\frac{i}{\sqrt{3}} \\
&= \pm\frac{i\sqrt{3}}{3}
\end{aligned}
$$

The zeros of $f(x)$ are $-3$ and $\pm\frac{i\sqrt{3}}{3}$.

**Analysis**

Look at the graph of the function $f$ in Figure $5.5.2$. Notice that, at $x = -3$, the graph crosses the x-axis, indicating an odd multiplicity (1) for the zero $x = -3$. Also note the presence of the two turning points. This means that, since there is a $3^{rd}$ degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the x-intercepts for the function are shown. So either the multiplicity of $x = -3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x = -3$ is three. Either way, our result is correct.

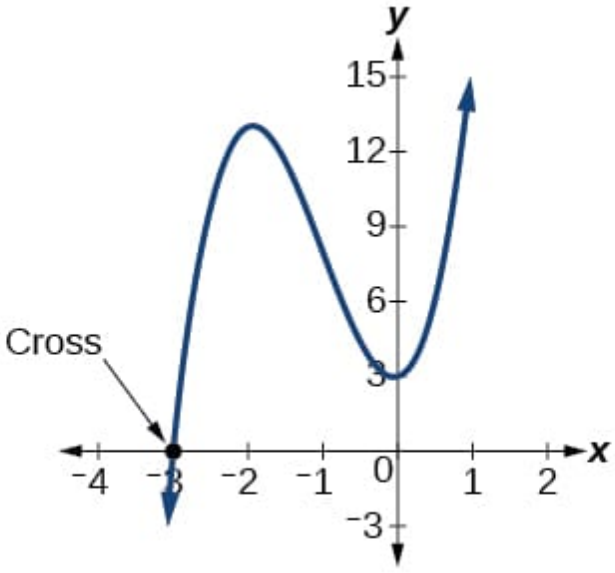



Figure $5.5.2$.

**Example $5.5.4$**

Find the zeros of $f(x) = 2x^3 + 5x^2 - 11x + 4$.

**Solution**

The factors of 4 are ±1, ±2, and ±4 and the factors of 2 are ±1 and ±2. The possible values for $\frac{p}{q}$ are ±1, 2, 4 and $\pm\frac{1}{2}$.

Use synthetic division to find one rational zero. Let's start with $x = 1$

$$
\begin{array}{r|cccc}
1 & 2 & 5 & -11 & 4 \\
 & & 2 & 7 & -4 \\
\hline
 & 2 & 7 & -4 & 0
\end{array} \tag{5.5.11}
$$

Since the remainder is zero, we have that $x = 1$ is a zero and $x - 1$ is a factor of $f(x)$

The quotient is $2x^2 + 7x - 4$

This can be factored as $(2x - 1)(x + 4)$

Setting this equal to zero gives us

$2x - 1 = 0$

hence,

$x = \frac{1}{2}$

And

$x + 4 = 0$

so,

$x = -4$

Therefore, the zeros are $-4$, $\frac{1}{2}$, and $1$.

## Using the Linear Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the **Fundamental Theorem of Algebra**, as we stated above, is that a polynomial function of degree n will have $n$ zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into $n$ factors. The **Linear Factorization Theorem** tells us that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x - c)$, where c is a complex number.

Let $f$ be a polynomial function with real coefficients, and suppose $a + bi$, $b \neq 0$, is a zero of $f(x)$. Then, by the Factor Theorem, $x - (a + bi)$ is a factor of $f(x)$. For $f$ to have real coefficients, $x - (a - bi)$ must also be a factor of $f(x)$. This is true because any factor other than $x - (a - bi)$, when multiplied by $x - (a + bi)$, will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function $f$ with real coefficients has a complex zero $a + bi$, then the complex conjugate $a - bi$ must also be a zero of $f(x)$. This is called the **Complex Conjugate Theorem**.

### Theorem 5.5.4 Complex Conjugate Zeros Theorem

According to the Linear Factorization Theorem, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x - c)$, where $c$ is a complex number.

If the polynomial function $f$ has real coefficients and a complex zero in the form $a + bi$, then the complex conjugate of the zero, $a - bi$, is also a zero.

### How to

**Given the zeros of a polynomial function $f$ and a point $(c, f(c))$ on the graph of $f$, use the Linear Factorization Theorem to find the polynomial function.**

1. Use the zeros to construct the linear factors of the polynomial.
2. Multiply the linear factors to expand the polynomial.

3. Substitute $(c, f(c))$ into the function to determine the leading coefficient.
4. Simplify.

### Example 5.5.7: Using the Linear Factorization Theorem to Find a Polynomial with Given Zeros

Find a fourth degree polynomial with real coefficients that has zeros of –3, 2, $i$, such that $f(-2) = 100$.

**Solution**

Because $x = i$ is a zero, by the Complex Conjugate Theorem $x = -i$ is also a zero. The polynomial must have factors of $(x + 3), (x - 2), (x - i)$, and $(x + i)$. Since we are looking for a degree 4 polynomial, and now have four zeros, we have all four factors. Let's begin by multiplying these factors.

$$f(x) = a(x + 3)(x - 2)(x - i)(x + i) \tag{5.5.12}$$
$$f(x) = a(x^2 + x - 6)(x^2 + 1) \tag{5.5.13}$$
$$f(x) = a(x^4 + x^3 - 5x^2 + x - 6) \tag{5.5.14}$$

We need to find $a$ to ensure $f(-2) = 100$. Substitute $x = -2$ and $f(-2) = 100$ into $f(x)$.

$$100 = a((-2)^4 + (-2)^3 - 5(-2)^2 + (-2) - 6) \tag{5.5.15}$$
$$100 = a(-20) \tag{5.5.16}$$
$$-5 = a \tag{5.5.17}$$

So the polynomial function is

$$f(x) = -5(x^4 + x^3 - 5x^2 + x - 6) \tag{5.5.18}$$

or

$$f(x) = -5x^4 - 5x^3 + 25x^2 - 5x + 30 \tag{5.5.19}$$

Analysis

We found that both $i$ and $-i$ were zeros, but only one of these zeros needed to be given. If $i$ is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of $i$.

### Q&A

**If $2 + 3i$ were given as a zero of a polynomial with real coefficients, would $2 - 3i$ also need to be a zero?**

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

### 5.5.5

Find a third degree polynomial with real coefficients that has zeros of $5$ and $-2i$ such that $f(1) = 10$.

**Solution**

$f(x) = -\frac{1}{2}x^3 + \frac{5}{2}x^2 - 2x + 10$

## Using Descartes' Rule of Signs

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign

changes in $f(x)$ and the number of positive real zeros. For example, the polynomial function below has one sign change.

$$f(x) = x^4 + x^3 + x^2 + x - 1$$

This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in $f(-x)$ and the number of negative real zeros.

$$f(-x) = (-x)^4 + (-x)^3 + (-x)^2 + (-x) - 1$$
$$f(-x) = +x^4 - x^3 + x^2 - x - 1$$

In this case, $f(-x)$ has 3 sign changes. This tells us that $f(x)$ could have 3 or 1 negative real zeros.

### Theorem 5.5.5 Descartes' Rules of Signs

According to *Descartes' Rule of Signs*, if we let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial function with real coefficients:

- The number of positive real zeros is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.Example 5.5.8: Using Descartes' Rule of Signs

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for $f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$.

**Solution**

Begin by determining the number of sign changes.

There are two sign changes, so there are either 2 or 0 positive real roots. Next, we examine $f(-x)$ to determine the number of negative real roots.

$$f(-x) = -(-x)^4 - 3(-x)^3 + 6(-x)^2 - 4(-x) - 12 \quad (5.5.20)$$
$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12 \quad (5.5.21)$$

$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12$$

Again, there are two sign changes, so there are either 2 or 0 negative real roots.

There are four possibilities, as we can see in Table 5.5.1.

*Table* 5.5.1

| Positive Real Zeros | Negative Real Zeros | Complex Zeros | Total Zeros |
|---|---|---|---|
| 2 | 2 | 0 | 4 |
| 2 | 0 | 2 | 4 |
| 0 | 2 | 2 | 4 |

| Positive Real Zeros | Negative Real Zeros | Complex Zeros | Total Zeros |
|---|---|---|---|
| 0 | 0 | 4 | 4 |

**Analysis**

We can confirm the numbers of positive and negative real roots by examining a graph of the function. See Figure 5.5.3. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.

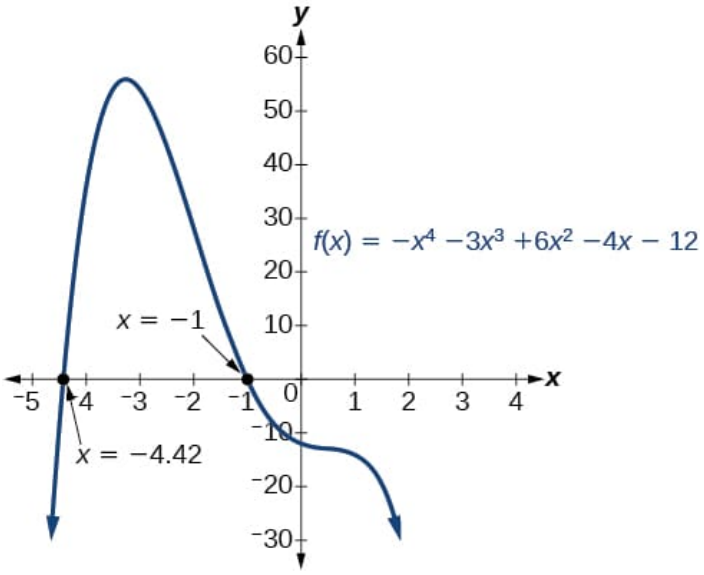



Figure 5.5.3.

5.5.6

Use Descartes' Rule of Signs to determine the maximum possible numbers of positive and negative real zeros for $f(x) = 2x^4 - 10x^3 + 11x^2 - 15x + 12$. Use a graph to verify the numbers of positive and negative real zeros for the function.

**Solution**

There must be 4, 2, or 0 positive real roots and 0 negative real roots. The graph shows that there are 2 positive real zeros and 0 negative real zeros.

## Key Concepts

- To find $f(k)$, determine the remainder of the polynomial $f(x)$ when it is divided by $x - k$. This is known as the Remainder Theorem. See Example 5.5.1.
- According to the Factor Theorem, $k$ is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$. See Example5.5.2.
- According to the Rational Zero Theorem, each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient. See Example 5.5.3 and Example 5.5.4.
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function. See Example 5.5.5.
- According to the Fundamental Theorem, every polynomial function with degree greater than 0 has at least one complex zero. See Example 5.5.6.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form $(x - c)$, where $c$ is a complex number. See Example 5.5.7.
- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of $f(-x)$ or less than the

number of sign changes by an even integer. See Example 5.5.8.
- Polynomial equations model many real-world scenarios. Solving the equations is easiest done by synthetic division. See Example 5.5.9.

# Glossary of Theorems

**Descartes' Rule of Signs**

a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $f(x)$ and $f(-x)$

**Factor Theorem**

$k$ is a zero of polynomial function $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$

**Fundamental Theorem of Algebra**

a polynomial function with degree greater than 0 has at least one complex zero

**Linear Factorization Theorem**

allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where $c$ is a complex number

**Rational Zero Theorem**

the possible rational zeros of a polynomial function have the form $\frac{p}{q}$ where $p$ is a factor of the constant term and $q$ is a factor of the leading coefficient.

**Remainder Theorem**

if a polynomial $f(x)$ is divided by $x-k$,then the remainder is equal to the value $f(k)$

## Answers

**Exercise** 5.5.1

$f(-3)=-412$

**Exercise** 5.5.2

The zeros are 2, –2, and –4.

**Exercise** 5.5.3

There are no rational zeros.

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# 5.6: Rational Functions

**Learning Objectives**

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.

Suppose we know that the cost of making a product is dependent on the number of items, $x$, produced. This is given by the equation $C(x) = 15,000x - 0.1x^2 + 1000$. If we want to know the average cost for producing $x$ items, we would divide the cost function by the number of items, $x$. The average cost function, which yields the average cost per item for $x$ items produced, is

$$f(x) = \frac{15,000x - 0.1x^2 + 1000}{x}$$

Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

## Using Arrow Notation

We have seen the graphs of the basic **reciprocal function** and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure $5.6.1$, and notice some of their features.

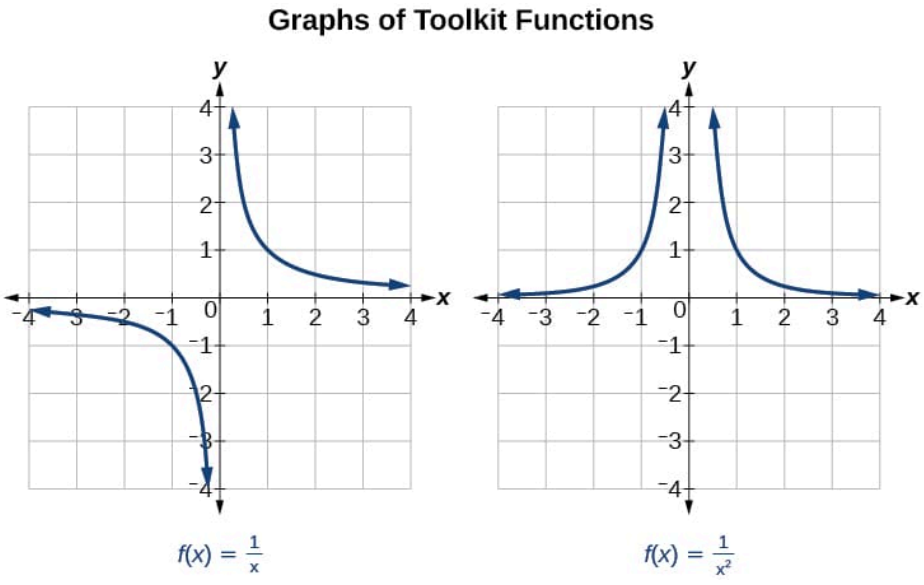



Figure $5.6.1$

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

- On the left branch of the graph, the curve approaches the $x$-axis $(y = 0)$ as $x \to -\infty$.
- As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.

- Finally, on the right branch of the graph, the curves approaches the $x$-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that $x$ or $f(x)$ is approaching a particular value (Table 5.6.1).

*Table* 5.6.1

| Symbol | Meaning |
|---|---|
| $x \to a^-$ | $x$ approaches a from the left ($x < a$ but close to $a$ ) |
| $x \to a^+$ | $x$ approaches a from the right ($x > a$ but close to $a$ ) |
| $x \to \infty$ | $x$ approaches infinity ($x$ increases without bound) |
| $x \to -\infty$ | $x$ approaches negative infinity ($x$ decreases without bound) |
| $f(x) \to \infty$ | the output approaches infinity (the output increases without bound) |
| $f(x) \to -\infty$ | the output approaches negative infinity (the output decreases without bound) |
| $f(x) \to a$ | the output approaches $a$ |

## Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table 5.6.2.

*Table* 5.6.2

| $x$ | -0.1 | -0.01 | -0.001 | -0.0001 |
|---|---|---|---|---|
| $f(x) = \frac{1}{x}$ | -10 | -100 | -1000 | -10,000 |

We write in arrow notation

$$\text{as } x \to 0^-, f(x) \to -\infty$$

As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in Table 5.6.3.

*Table* 5.6.3

| $x$ | 0.1 | 0.01 | 0.001 | 0.0001 |
|---|---|---|---|---|
| $f(x) = \frac{1}{x}$ | 10 | 100 | 1000 | 10,000 |

We write in arrow notation

$$\text{As } x \to 0^+, f(x) \to \infty.$$

See Figure 5.6.2.

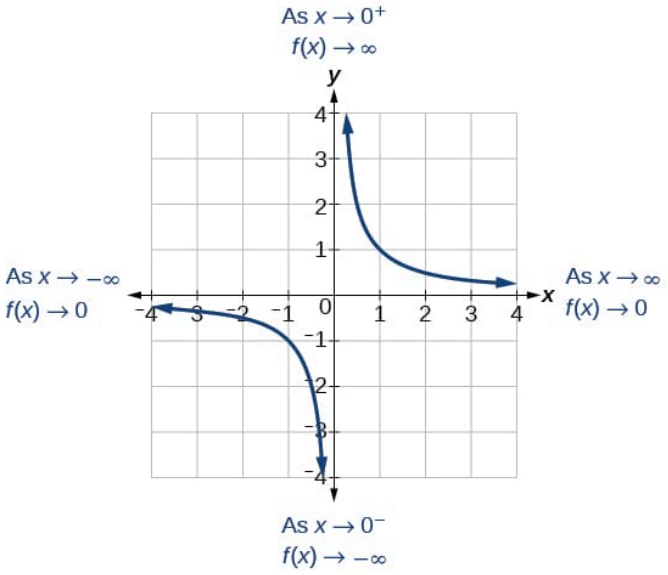



Figure 5.6.2.

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero (Figure 5.6.3).

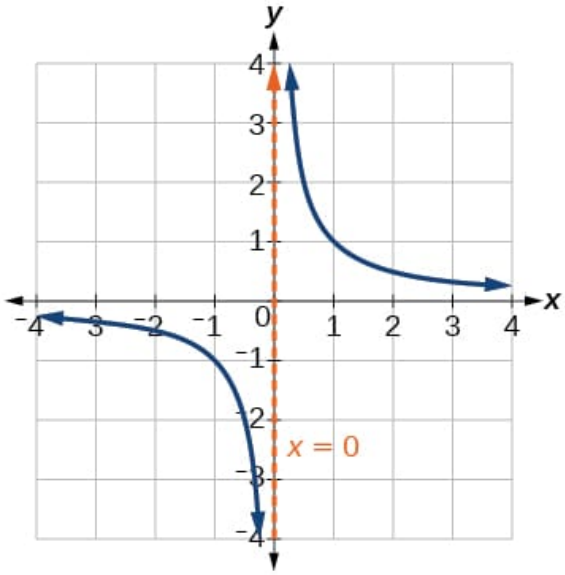



Figure 5.6.3.

**Definition: VERTICAL ASYMPTOTE**

A vertical asymptote of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach $a$. We write

As $x \to a$, $f(x) \to \infty$, or as $x \to a$, $f(x) \to -\infty$.

## End Behavior of $f(x) = \frac{1}{x}$

As the values of $x$ approach infinity, the function values approach $0$. As the values of $x$ approach negative infinity, the function values approach $0$ (Figure 5.6.4). Symbolically, using arrow notation

As $x \to \infty$, $f(x) \to 0$,and as $x \to -\infty$, $f(x) \to 0$.

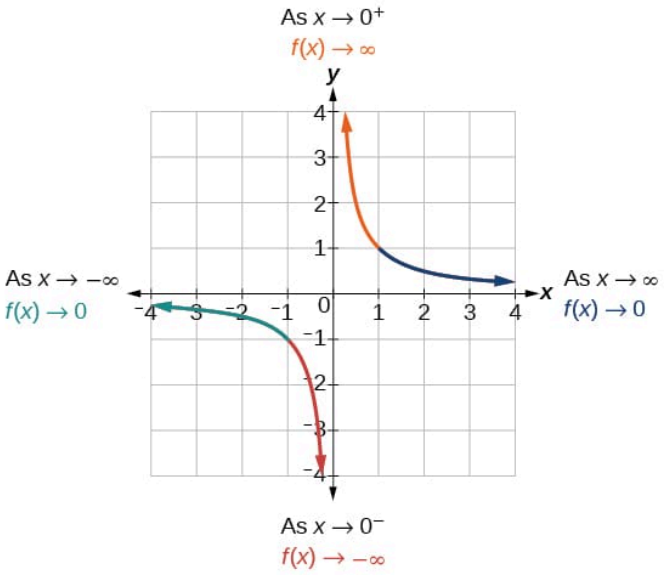



Figure 5.6.4.

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure 5.6.5.

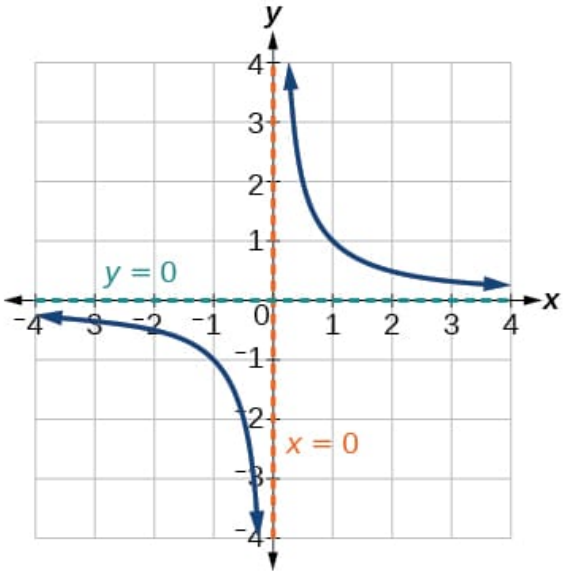



Figure 5.6.5.

## Definition: HORIZONTAL ASYMPTOTE

A *horizontal asymptote* of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$$\text{As } x \to \infty \text{ or } x \to -\infty, f(x) \to b.$$

## Example 5.6.1: Using Arrow Notation.

Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure 5.6.6.

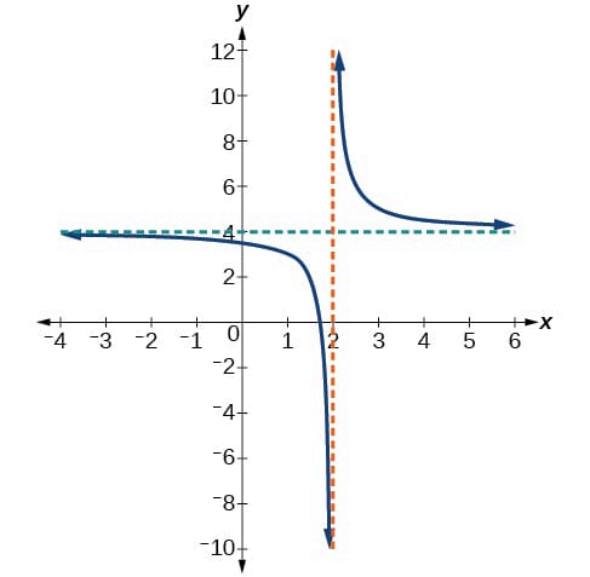



Figure 5.6.6.

**Solution**

Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$.

$$\text{As } x \to 2^-,\ f(x) \to -\infty, \text{ and as } x \to 2^+,\ f(x) \to \infty.$$

And as the inputs decrease without bound, the graph appears to be leveling off at output values of $4$, indicating a horizontal asymptote at $y = 4$. As the inputs increase without bound, the graph levels off at $4$.

$$\text{As } x \to \infty,\ f(x) \to 4 \text{ and as } x \to -\infty,\ f(x) \to 4.$$

## Exercises 5.6.1

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

*Answer available at the end of this section.*

## Example 5.6.2: Using Transformations to Graph a Rational Function.

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

**Solution**

Shifting the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3 \tag{5.6.1}$$

or equivalently, by giving the terms a common denominator,

$$f(x) = \frac{3x+7}{x+2} \tag{5.6.2}$$

The graph of the shifted function is displayed in Figure 5.6.7.

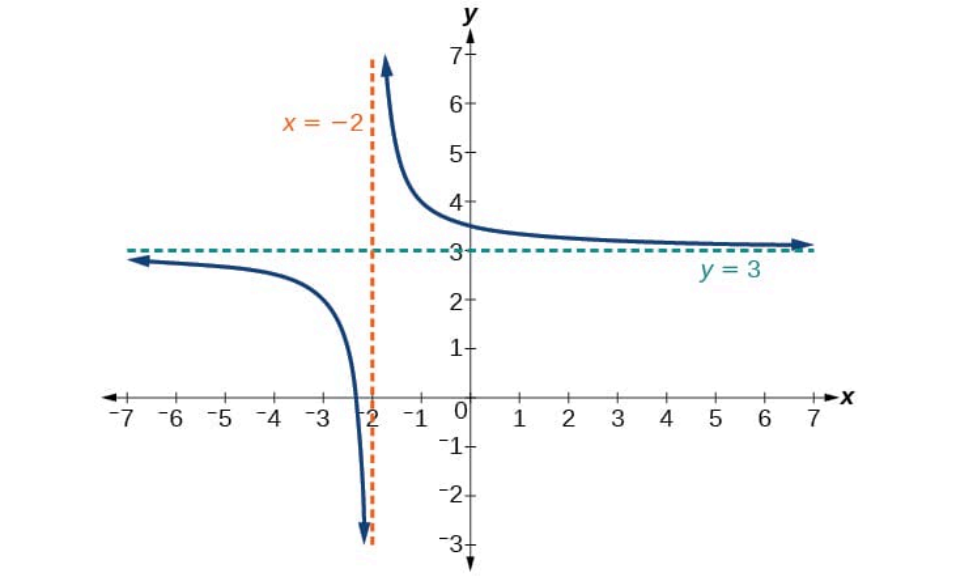



Figure 5.6.7.

Notice that this function is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

$$\text{As } x \to -2^-,\ f(x) \to -\infty, \text{ and as } x \to -2^+,\ f(x) \to \infty.$$

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

$$\text{As } x \to \pm\infty,\ f(x) \to 3.$$

**Analysis**

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

## Exercise 5.6.2

Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

**Solution**

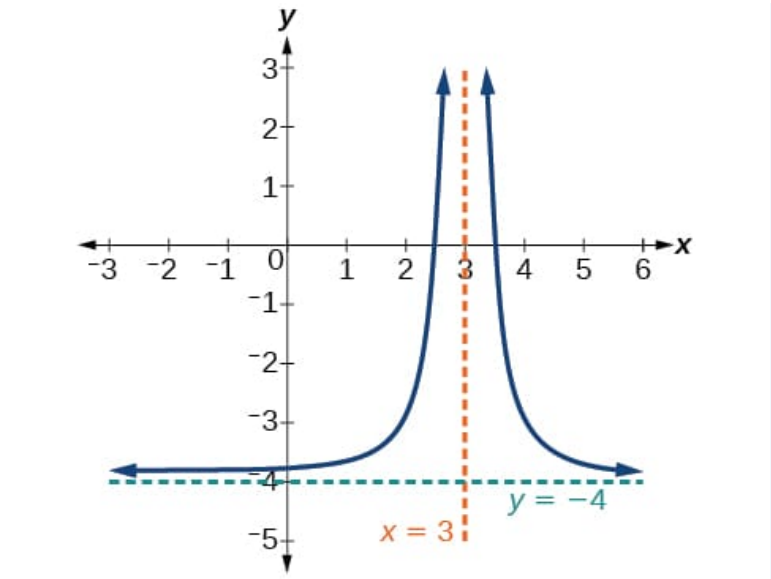



Figure 5.6.8.

The function and the asymptotes are shifted 3 units right and 4 units down. As $x \to 3$, $f(x) \to \infty$, and as $x \to \pm\infty$, $f(x) \to -4$.

The function is $f(x) = \frac{1}{(x-3)^2} - 4$.

## Solving Applied Problems Involving Rational Functions

In Example 5.6.2, we shifted a toolkit function in a way that resulted in the function $f(x) = \frac{3x+7}{x+2}$. This is an example of a rational function. A *rational function* is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

### Definition: RATIONAL FUNCTION

A ***rational function*** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0},\ Q(x) \neq 0 \tag{5.6.3}$$

### Example 5.6.3: Solving an Applied Problem Involving a Rational Function

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

**Solution**

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

water: $W(t) = 100 + 10t$ in gallons
sugar: $S(t) = 5 + 1t$ in pounds

The concentration, $C$, will be the ratio of pounds of sugar to gallons of water

$$C(t) = \frac{5+t}{100+10t} \tag{5.6.4}$$

The concentration after 12 minutes is given by evaluating $C(t)$ at $t = 12$.

$$C(12) = \frac{5+12}{100+10(12)} \tag{5.6.5}$$

$$= \frac{17}{220} \tag{5.6.6}$$

This means the concentration is 17 pounds of sugar to 220 gallons of water.

At the beginning, the concentration is

$$C(0) = \frac{5+0}{100+10(0)} \tag{5.6.7}$$

$$= \frac{1}{20} \tag{5.6.8}$$

Since $\frac{17}{220} \approx 0.08 > \frac{1}{20} = 0.05$, the concentration is greater after 12 minutes than at the beginning.

**Analysis**

To find the horizontal asymptote, divide the leading coefficient in the numerator by the leading coefficient in the denominator:

$$\frac{1}{10} = 0.1 \tag{5.6.9}$$

Notice the horizontal asymptote is $y = 0.1$. This means the concentration, $C$, the ratio of pounds of sugar to gallons of water, will approach 0.1 in the long term.

### Exercise 5.6.3

There are 1,200 freshmen and 1,500 sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

*Answer available at the end of this section.*

## Finding the Domains of Rational Functions

A *vertical asymptote* represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

### Definition: DOMAIN OF A RATIONAL FUNCTION

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

### How To: Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

### Example 5.6.4: Finding the Domain of a Rational Function

Find the domain of $f(x) = \dfrac{x+3}{x^2-9}$.

**Solution**

Begin by setting the denominator equal to zero and solving.

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$.

**Analysis**

A graph of this function, as shown in Figure 5.6.9, confirms that the function is not defined when $x = \pm 3$.

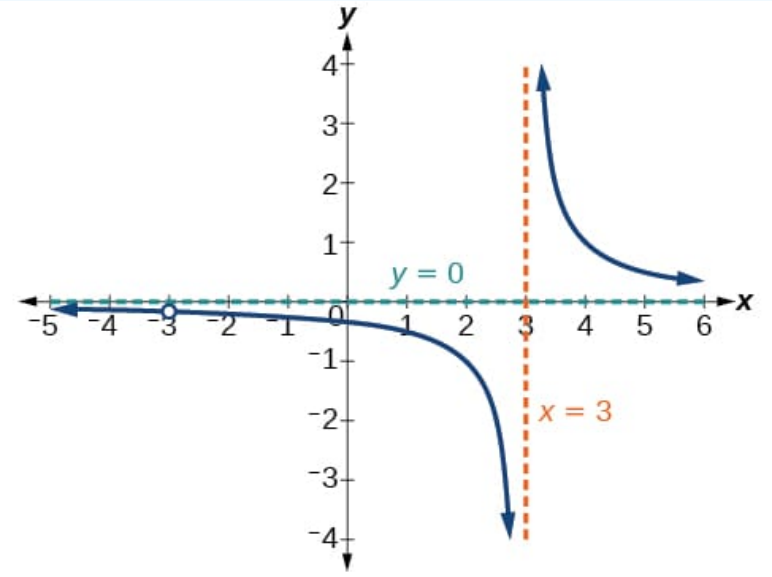



Figure 5.6.9.

There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

### Exercise 5.6.4

Find the domain of $f(x) = \dfrac{4x}{5(x-1)(x-5)}$.

*Answer available at the end of this section.*

## Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

# Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

## How To: Given a rational function, identify any vertical asymptotes of its graph

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or “holes.”

## Example 5.6.5: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \dfrac{5+2x^2}{2-x-x^2}$.

**Solution**

First, factor the numerator and denominator.

$$k(x) = \frac{5+2x^2}{2-x-x^2}$$

$$= \frac{5+2x^2}{(2+x)(1-x)}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2+x)(1-x) = 0$$

$$x = -2,\ x = 1$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 5.6.10 confirms the location of the two vertical asymptotes.

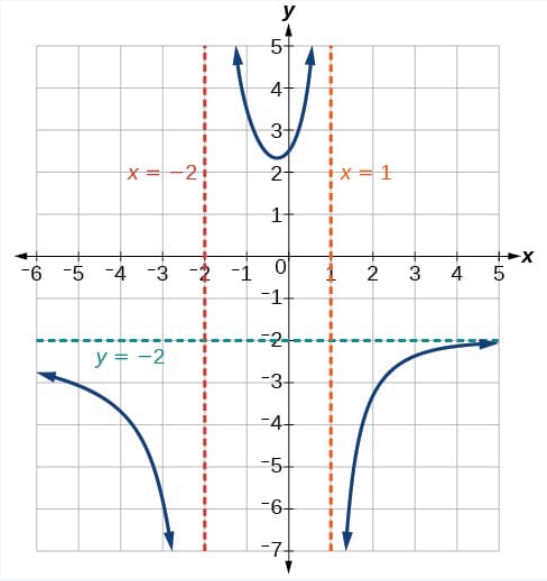


***Figure*** 5.6.10.

## Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a *removable discontinuity*. For example, the function $f(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x + 1$ is a common factor to the numerator and the denominator. The zero of this factor, $x = -1$, is the location of the removable discontinuity. Notice also that $(x–3)$ is not a factor in both the numerator and denominator. The zero of this factor, $x = 3$, is the vertical asymptote. See Figure 5.6.11. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

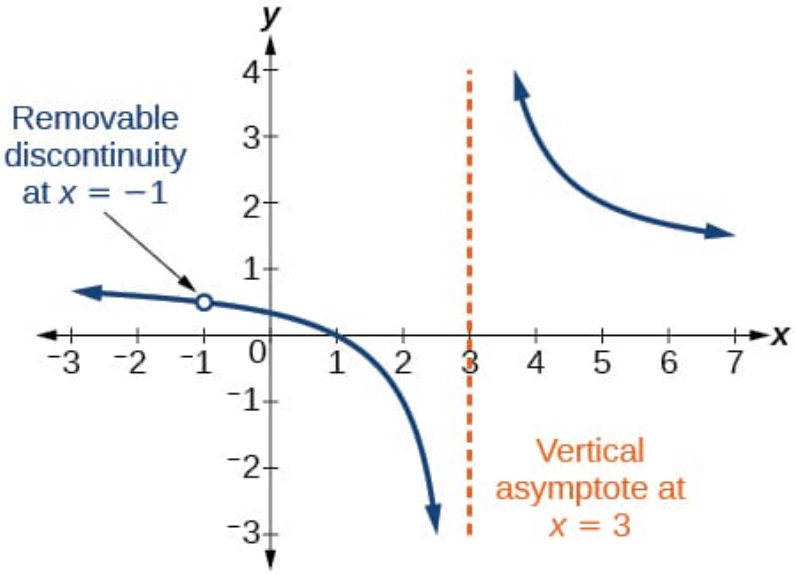



Figure 5.6.11.

### REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x = a$ if $a$ is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

### Example 5.6.6: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \dfrac{x - 2}{x^2 - 4}$.

**Solution**

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x$–2. The zero for this factor is $x = 2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x + 2$. The zero for this factor is $x = -2$. The vertical asymptote is $x = -2$. See Figure 5.6.12.

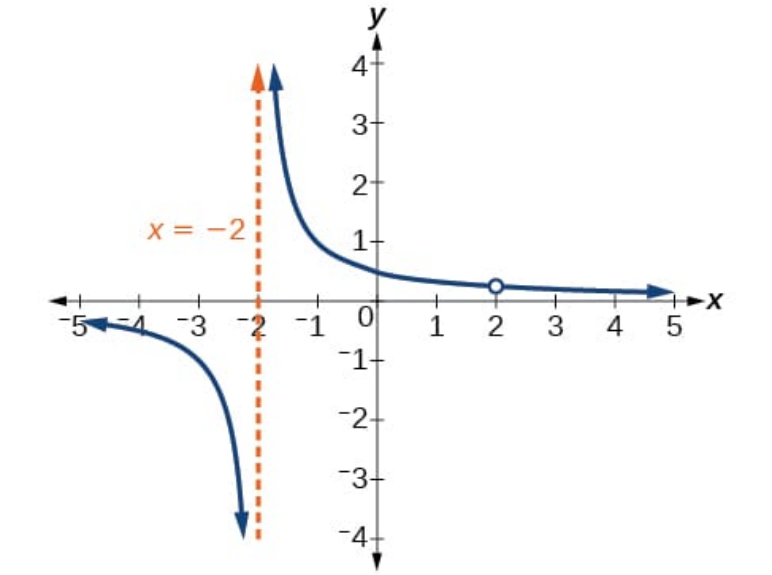



Figure 5.6.12.

The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole.

**Exercise 5.6.5**

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \frac{x^2 - 25}{x^3 - 6x^2 + 5x}$.

*Answer available at the end of this section.*

## Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are three distinct outcomes when checking for horizontal asymptotes:

**Case 1**: If the degree of the denominator > degree of the numerator, there is a **horizontal asymptote** at $y = 0$.

**Example**: $f(x) = \frac{4x + 2}{x^2 + 4x - 5}$

In this case, the end behavior is $f(x) \approx \frac{4x}{x^2} = \frac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = \frac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y = 0$. See **Figure** 5.6.13. Note that this graph crosses the horizontal asymptote.

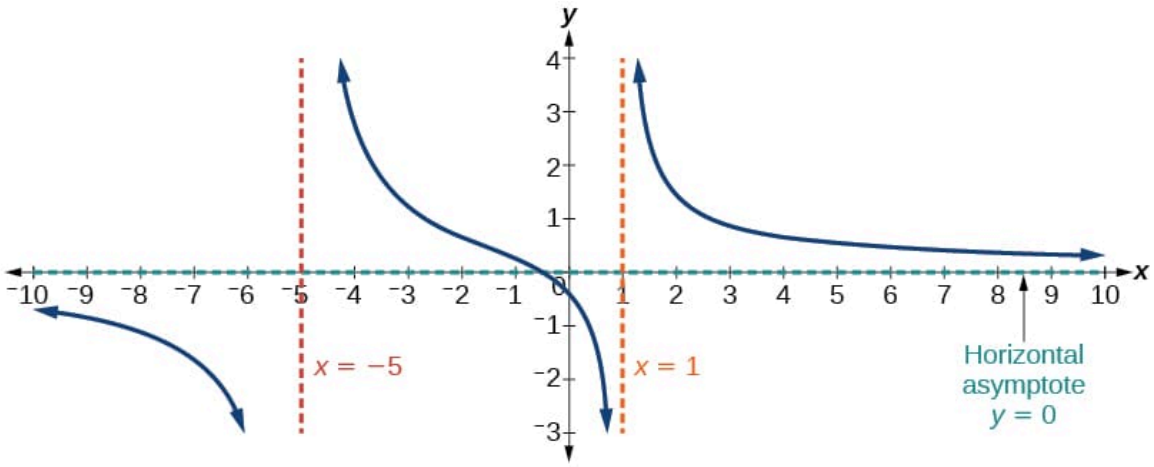



Figure 5.6.13: Horizontal asymptote $y = 0$ when $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of $p$ < degree of $q$.

**Case 2**: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

**Example**: $f(x) = \dfrac{3x^2 - 2x + 1}{x - 1}$

In this case, the end behavior is $f(x) \approx \dfrac{3x^2}{x} = 3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x) = 3x$ looks like a diagonal line, and since $f$ will behave similarly to $g$, it will approach a line close to $y = 3x$. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\dfrac{3x^2 - 2x + 1}{x - 1}$. The quotient is $3x + 1$, and the remainder is 2. The slant asymptote is the graph of the line $g(x) = 3x + 1$. See Figure 5.6.14.

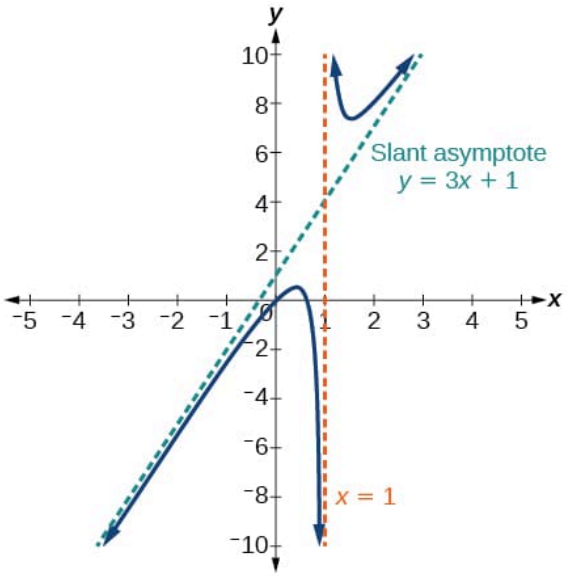



Figure 5.6.14: Slant asymptote when $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of $p$ >degree of $q$ by 1.

**Case 3**: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y = \frac{a_n}{b_n}$, where $a_n$ and $b_n$ are respectively the leading coefficients of $p(x)$ and $q(x)$ for $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$.

**Example**: $f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $y = 3$. See Figure 5.6.15. Note that this graph crosses the horizontal asymptote.

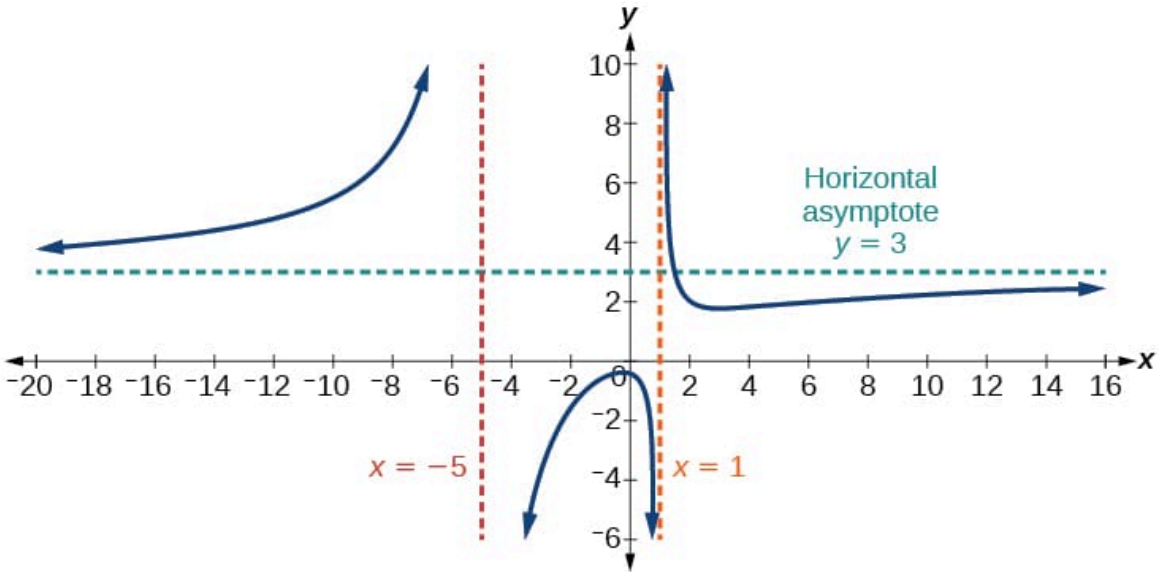



Figure 5.6.15: Horizontal asymptote when $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of $p$ = degree of $q$.

Notice that, while the graph of a rational function will never cross a **vertical asymptote**, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the **end behavior** of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x + 3}$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \to \pm\infty, f(x) \to \infty$$

## HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator is less than degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

## Example 5.6.7: Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$
2. $h(x) = \dfrac{x^2 - 4x + 1}{x + 2}$
3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$

**Solution**

For these solutions, we will use $f(x) = \dfrac{p(x)}{q(x)},\ q(x) \neq 0$.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$: The degree of $p =$ degree of $q = 3$, so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.
2. $h(x) = \dfrac{x^2 - 4x + 1}{x + 2}$: The degree of $p = 2$ and degree of $q = 1$. Since $p > q$ by 1, there is a slant asymptote found at $\dfrac{x^2 - 4x + 1}{x + 2}$.

$$\begin{array}{r|rr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$ : The degree of $p = 2$ < degree of $q = 3$, so there is a horizontal asymptote $y = 0$.

## Example 5.6.8 Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \dfrac{5 + t}{100 + 10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

**Solution**

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is $t$, with coefficient 1. In the denominator, the leading term is 10t, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty,\ C(t) \to \tfrac{1}{10}$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of $t$ increase, the values of $C$ will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

## Example 5.6.9: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

**Solution**

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1, -2,$and $5$, indicating vertical asymptotes at these values.

The numerator has degree $2$, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure 5.6.16.

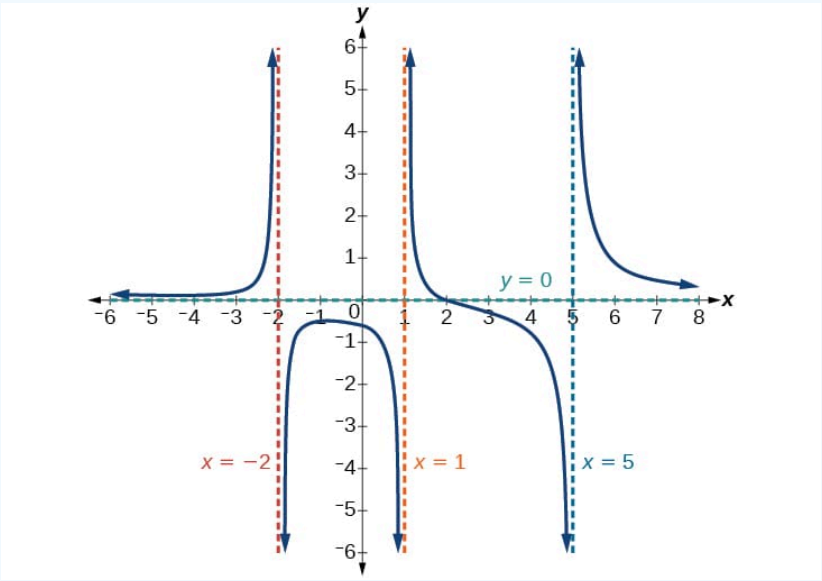



Figure 5.6.16.

## Exercise 5.6.6

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

*Answer available at the end of this section.*

## INTERCEPTS OF RATIONAL FUNCTIONS

A *rational function* will have a $y$-intercept at $f(0)$, if the function is defined at zero. A rational function will not have a $y$ -intercept if the function is not defined at zero.

Likewise, a rational function will have $x$-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

## Example 5.6.10: Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

**Solution**

We can find the y-intercept by evaluating the function at zero

$$\begin{aligned} f(0) &= \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} \\ &= -\frac{6}{10} \\ &= -\frac{3}{5} \\ &= -0.6 \end{aligned}$$

The x-intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \quad \text{This is zero when the numerator is zero.}$$

$$0 = (x-2)(x+3)$$

$$x = 2, x = -3$$

The y-intercept is $(0, -0.6)$, the x-intercepts are $(2, 0)$ and $(-3, 0)$. See Figure 5.6.17.

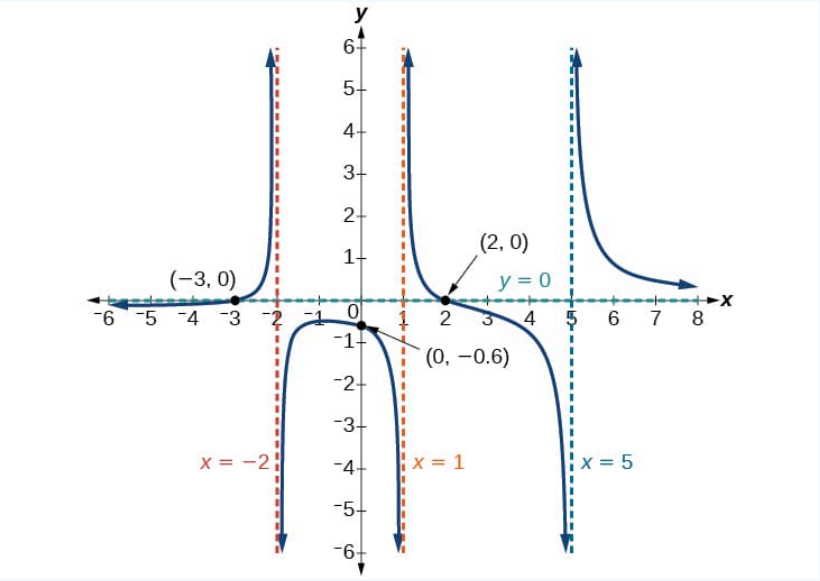



Figure 5.6.17.

## Exercise 5.6.7

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

*Answer available at the end of this section.*

## Graphing Rational Functions

In Example 5.6.10, we see that the numerator of a rational function reveals the *x*-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure 5.6.18.

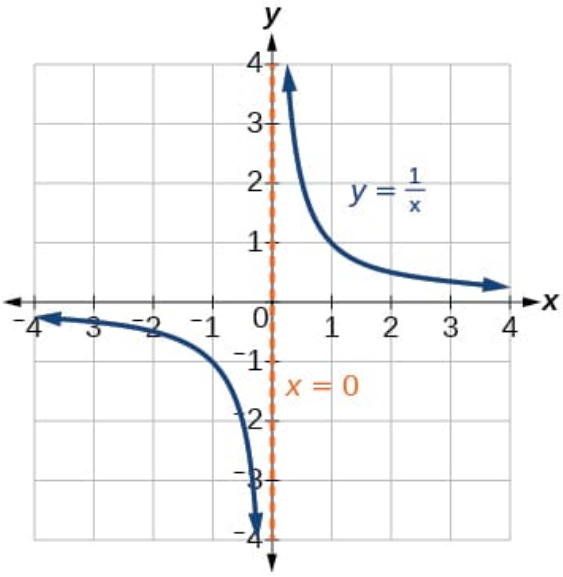



Figure 5.6.18.

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure 5.6.19.

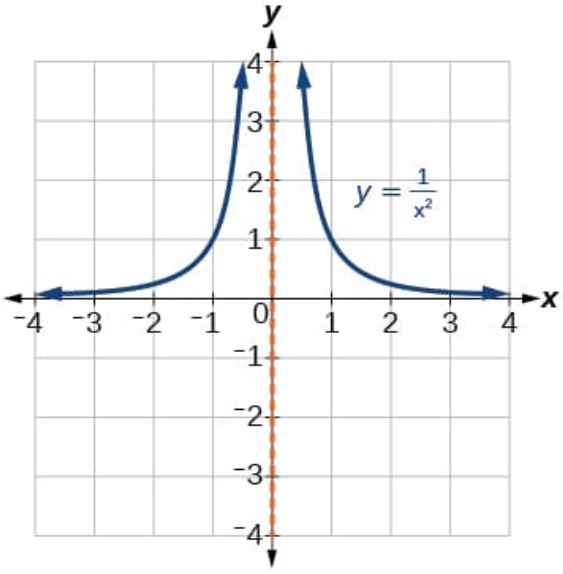



Figure 5.6.19.

For example, the graph of $f(x) = \dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure 5.6.20.

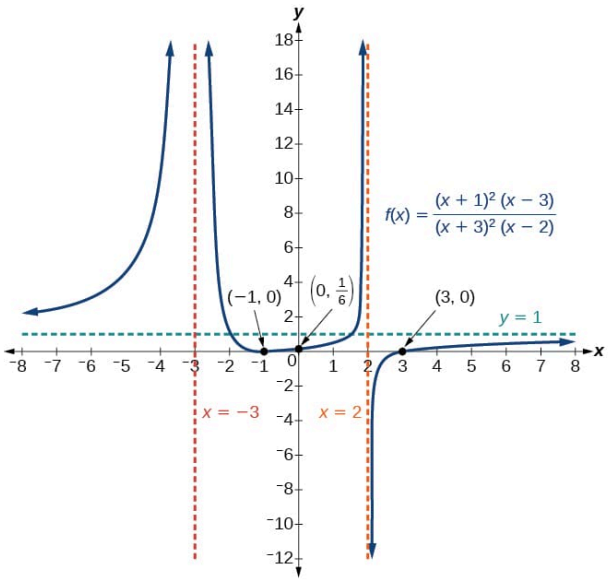



Figure 5.6.20.

- At the x-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.
- At the x-intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \dfrac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function $f(x) = \dfrac{1}{x}$.

### Howto: Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

### Example 5.6.11: Graphing a Rational Function

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

**Solution**

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the y-intercept:

$$f(0) = \frac{(0+2)(0-3)}{(0+1)^2(0-2)} = 3$$

To find the x-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find x-intercepts at $x = -2$ and $x = 3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a y-intercept at $(0, 3)$ and x-intercepts at $(-2, 0)$ and $(3, 0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x + 1 = 0$ and when $x - 2 = 0$, giving us vertical asymptotes at $x = -1$ and $x = 2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no x-intercepts between the vertical asymptotes, and the y-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure 5.6.21.

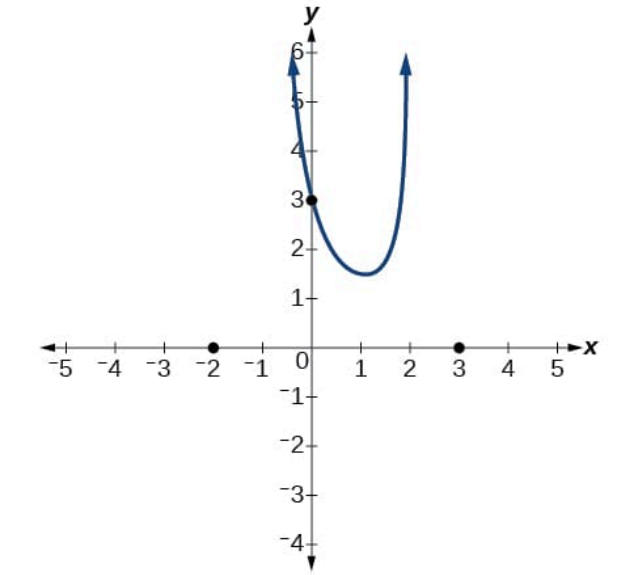



Figure 5.6.21.

The factor associated with the vertical asymptote at $x = -1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x = 2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure 5.6.22. After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

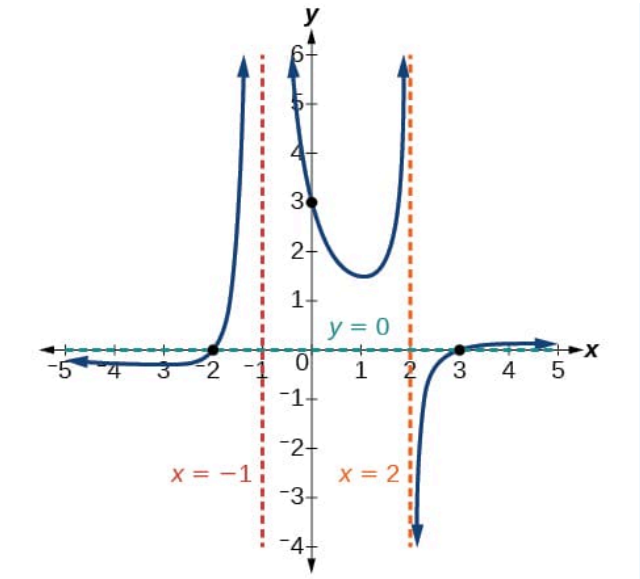



Figure 5.6.22.

**Exercise 5.6.8**

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

*Answer available at the end of this section.*

## Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an *x*-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of *x*-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

**WRITING RATIONAL FUNCTIONS FROM INTERCEPTS AND ASYMPTOTES**

If a **rational function** has x-intercepts at $x = x_1, x_2, ..., x_n$, vertical asymptotes at $x = v_1, v_2, ..., v_m$, and no $x_i =$ any $v_j$, then the function can be written in the form:

$$f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}}$$

where the powers $p_i$ or $q_i$ on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor $a$ can be determined given a value of the function other than the x-intercept or by the horizontal asymptote if it is nonzero.

## Given a graph of a rational function, write the function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the *x*-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the "simplest" function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

## Example 5.6.12: Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in Figure 5.6.24.

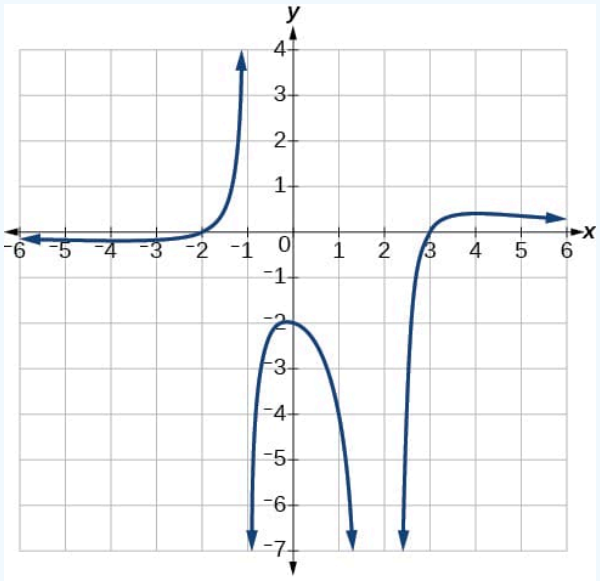



Figure 5.6.24.

**Solution**

The graph appears to have x-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See Figure 5.6.25.

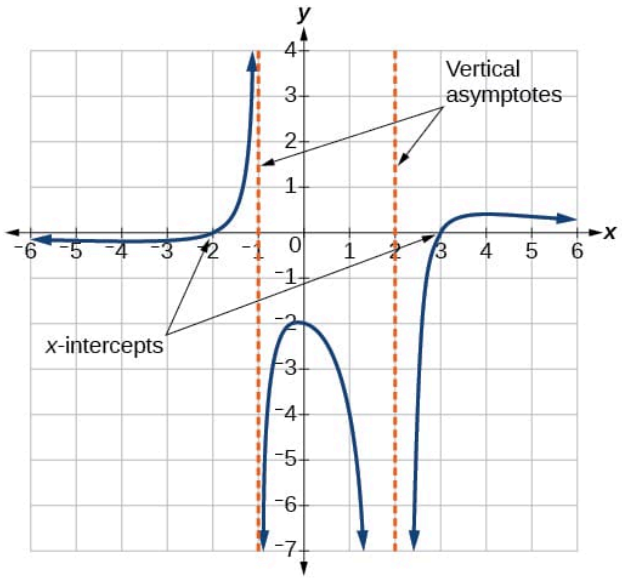



Figure 5.6.25.

We can use this information to write a function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $(0,-2)$.

$$\begin{aligned} -2 &= a\frac{(0+2)(0-3)}{(0+1)(0-2)^2} \\ -2 &= a\frac{-6}{4} \\ a &= \frac{-8}{-6} = \frac{4}{3} \end{aligned}$$

This gives us a final function of $f(x) = \dfrac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

## Media

Access these online resources for additional instruction and practice with rational functions.

- Graphing Rational Functions
- Find the Equation of a Rational Function
- Determining Vertical and Horizontal Asymptotes
- Find the Intercepts, Asymptotes, and Hole of a Rational Function

# Key Equations

| | |
|---|---|
| Rational Function | $f(x) = \dfrac{P(x)}{Q(x)} = \dfrac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0},\ Q(x) \neq 0$ |

# Key Concepts

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$. See Example $5.6.1$.
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See Example.
- Application problems involving rates and concentrations often involve rational functions. See Example.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See Example.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See Example.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See Example.
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See Example, Example, Example, and Example.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See Example.
- If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any $v_j$, then the function can be written in the form

## Answers

**Exercises** $5.6.1$

End behavior: as $x \to \pm\infty$, $f(x) \to 0$;

Local behavior: as $x \to 0$, $f(x) \to \infty$ (there are no x- or y-intercepts)

**Exercise** $5.6.3$

$\frac{12}{11}$

**Exercise** $5.6.4$

The domain is all real numbers except $x = 1$ and $x = 5$.

**Exercise** $5.6.5$

Removable discontinuity at $x = 5$.

Vertical asymptotes: $x = 0$, $x = 1$.

**Exercise** $5.6.6$

Vertical asymptotes at $x = 2$ and $x = -3$

horizontal asymptote at $y = 4$.

**Exercise** $5.6.7$

For the transformed reciprocal squared function, we find the rational form.

$$f(x) = \frac{1}{(x-3)^2} - 4 = \frac{1-4(x-3)^2}{(x-3)^2} = \frac{1-4(x^2-6x+9)}{(x-3)(x-3)} = \frac{-4x^2+24x-35}{x^2-6x+9}$$

Because the numerator is the same degree as the denominator we know that as $x \to \pm\infty$, $f(x) \to -4$; so $y = -4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x = 3$, because as $x \to 3$, $f(x) \to \infty$. We then set the numerator equal to $0$ and find the x-intercepts are at $(2.5, 0)$ and $(3.5, 0)$. Finally, we evaluate the function at $0$ and find the y-intercept to be at $(0, -\frac{35}{9})$.

**Exercise** $5.6.8$

Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1$ and $x = 3$. y-intercept at $(0, \frac{4}{3})$.

x-intercepts at $(2, 0)$ and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity $2$, and the graph bounces off the x-axis at this point. $(2, 0)$ is a single zero and the graph crosses the axis at this point.

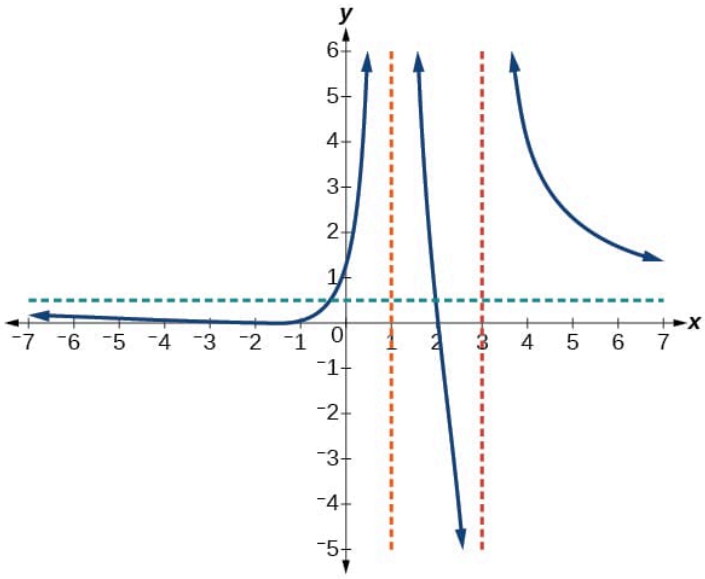



Figure 5.6.23.

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# 6: ANALYTIC GEOMETRY WITH CONIC SECTIONS

6.1: The Pythagorean Theorem, Distance and Midpoint
6.2: The Circle
6.3: The Ellipse
6.4: The Parabola
6.5: The Hyperbola

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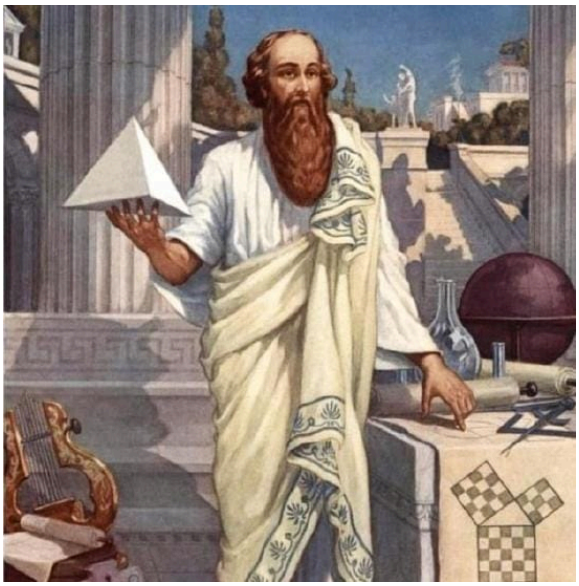
# 6.1: The Pythagorean Theorem, Distance and Midpoint

**Learning Objectives**

- Using the Pythagorean Theorem
- Use the distance formula.
- Use the midpoint formula.

## The Pythagorean Theorem

We have learned how the measures of the angles of a triangle relate to each other. Now, we will learn how the lengths of the sides relate to each other. An important property that describes the relationship among the lengths of the three sides of a right triangle is called the **Pythagorean Theorem**. This theorem has been used around the world since ancient times. It is named after the Ionian-Greek philosopher and mathematician, Pythagoras of Samos, who lived between 570 - 495 BC.



Before we state the Pythagorean Theorem, we need to introduce some terms for the sides of a triangle. Remember that a right triangle has a 90° angle, marked with a small square in the corner. The side of the triangle opposite the 90°90° angle is called the *hypotenuse* and each of the other sides are called *legs*.

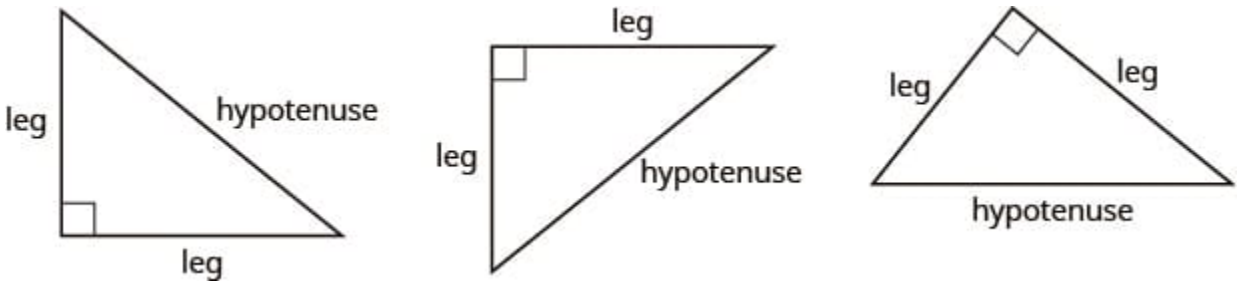


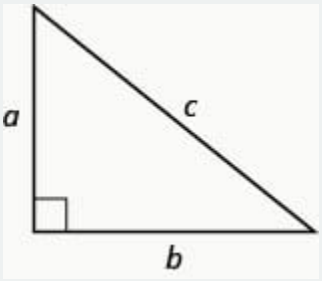

Figure 6.1.4

The Pythagorean Theorem tells how the lengths of the three sides of a right triangle relate to each other. It states that in any right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse. In symbols we say: in any right triangle, $a^2 + b^2 = c^2$, where a and b are the lengths of the legs and cc is the length of the hypotenuse.

Writing the formula in every exercise and saying it aloud as you write it, may help you remember the Pythagorean Theorem.

## THE PYTHAGOREAN THEOREM

In any right triangle, where $a$ and $b$ are the lengths of the legs, $c$ is the length of the hypotenuse.




Then

$$a^2 + b^2 = c^2 \tag{6.1.1}$$

To solve exercises that use the Pythagorean Theorem (Equation 6.1.1), we will need to find square roots. We have used the notation $\sqrt{m}$ and the definition:

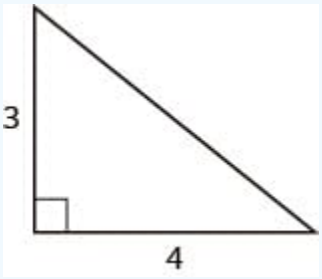
If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

For example, we found that $\sqrt{25}$ is 5 because $25 = 5^2$.

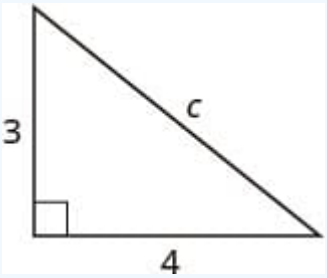
Because the Pythagorean Theorem contains variables that are squared, to solve for the length of a side in a right triangle, we will have to use square roots.

## Example 6.1.1

Use the Pythagorean Theorem to find the length of the hypotenuse shown below.

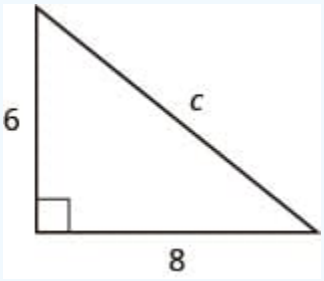



**Solution**

| | |
|---|---|
| Label side $c$ on the figure. | Let $c$ = the length of the hypotenuse.<br>  |
| Write the appropriate formula. | $a^2 + b^2 = c^2$ |
| Substitute. | $3^2 + 4^2 = c^2$ |
| Solve the equation. | $9 + 16 = c^2$ |
| Simplify. | $25 = c^2$ |
| Use the definition of square root. | $\sqrt{25} = c$ |
| Simplify. | $5 = c$ |
| Check<br>$3^2 + 4^2 \stackrel{?}{=} 5^2$<br>$9 + 16 \stackrel{?}{=} 25$<br>$25 = 25$ ✓ | |
| **Answer** | The length of the hypotenuse is 5. |

## Try It 6.1.1

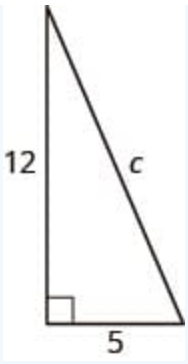
Use the Pythagorean Theorem to find the length of the hypotenuse in the triangle shown below.




*Answer available at the end of this section.*

## Exercise 6.1.1

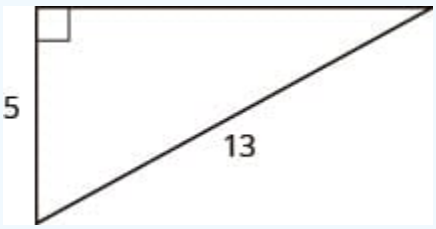
Use the Pythagorean Theorem to find the length of the hypotenuse in the triangle shown below.



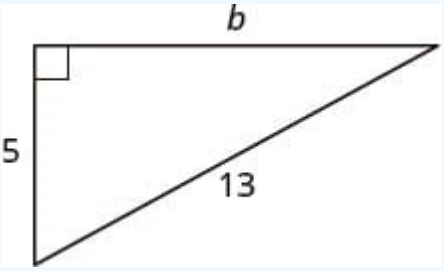

*Answer available at the end of this section.*

## Example 6.1.2

Use the Pythagorean Theorem to find the length of the leg shown below.



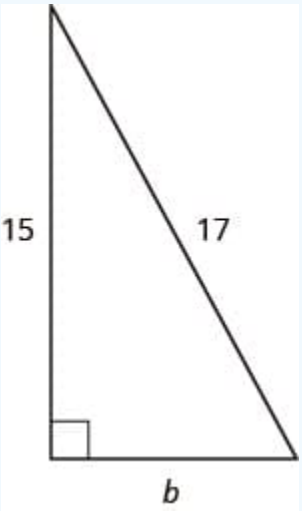

**Solution**

| | |
|---|---|
| Label side $b$. |   |
| Write the appropriate formula. | $a^2 + b^2 = c^2$ |
| Substitute. | $5^2 + b^2 = 13^2$ |
| Solve the equation. | $25 + b^2 = 169$ |
| Isolate the variable term. | $b^2 = 144$ |
| Use the definition of square root. | $b = \sqrt{144}$ |
| Simplify. | $b = 12$ |

| | |
|---|---|
| **Check.** $5^2 + 12^2 \stackrel{?}{=} 13^2$ <br> $25 + 144 \stackrel{?}{=} 169$ <br> $169 = 169$ ✓ | |
| **Answer.** | The length of the leg is 12. |

## Try It 6.1.2

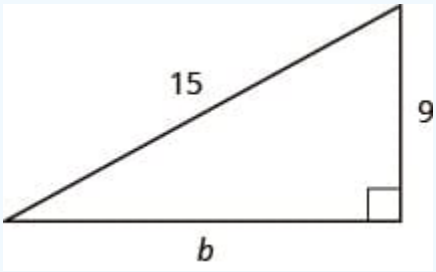
Use the Pythagorean Theorem to find the length of the leg in the triangle shown below.




*Answer available at the end of this section.*

## Exercise 6.1.2

Use the Pythagorean Theorem to find the length of the leg in the triangle shown below.




*Answer available at the end of this section.*

## Example 6.1.3: Finding the Length of the Missing Side of a Right Triangle

Find the length of the missing side of the right triangle in Figure 6.1.5.

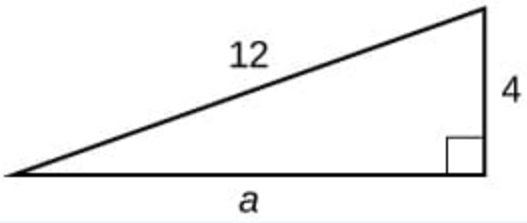



Figure 6.1.5

**Solution**

As we have measurements for side $b$ and the hypotenuse, the missing side is $a$.

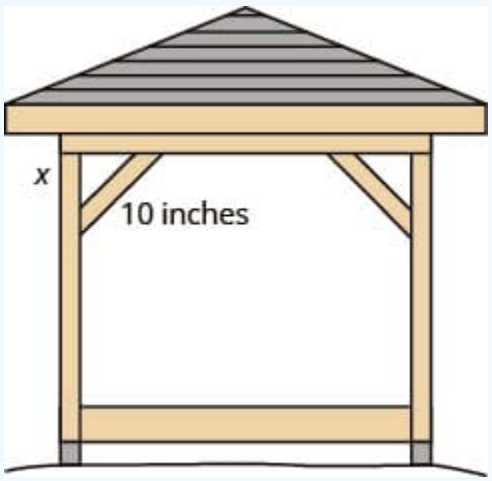
$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + (4)^2 &= (12)^2 \\ a^2 + 16 &= 144 \\ a^2 &= 128 \\ a &= \sqrt{128} \\ &= 8\sqrt{2} \end{aligned}$$

## Exercise 6.1.3

Use the Pythagorean Theorem to solve the right triangle problem: Leg a measures 4 units, leg b measures 3 units. Find the length of the hypotenuse.

*Answer available at the end of this section.*

## Example 6.1.4




Kelvin is building a gazebo and wants to brace each corner by placing a 10″ piece of wood diagonally as shown above.

If he fastens the wood so that the ends of the brace are the same distance from the corner, what is the length of the legs of the right triangle formed? Approximate to the nearest tenth of an inch.

**Solution**

**Step 1.** Read the problem.

| | |
|---|---|
| **Step 2.** Identify what we are looking for. | the distance from the corner that the bracket should be attached |
| **Step 3.** Name. Choose a variable to represent it. | Let x = distance from the corner. |
| **Step 4.**Translate | |
| Write the appropriate formula and substitute. | $a^2 + b^2 = c^2$<br>$x^2 + x^2 = 10^2$ |
| **Step 5. Solve the equation.** | |
| | $2x^2 = 100$ |
| Isolate the variable. | $x^2 = 50$ |
| Simplify. Approximate to the nearest tenth. | $x \approx 7.1$ |
| **Step 6.** Check.<br>$a^2 + b^2 = c^2$<br>$(7.1)^2 + (7.1)^2 \approx 10^2$ Yes. | |
| **Step 7. Answer the question.** | Kelven should fasten each piece of wood approximately 7.1" from the corner. |

## Try It 6.1.4

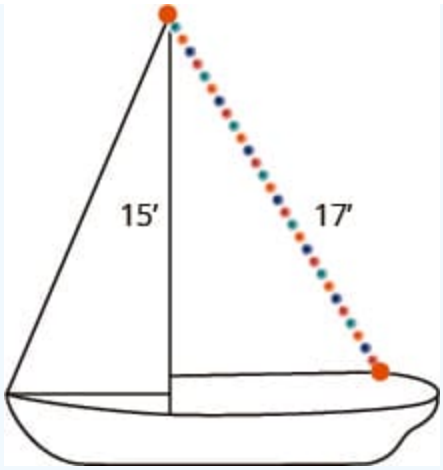
John puts the base of a 13-foot ladder five feet from the wall of his house as shown below. How far up the wall does the ladder reach?




*Answer available at the end of this section.*

## Exercise 6.1.4

Randy wants to attach a 17 foot string of lights to the top of the 15 foot mast of his sailboat, as shown below. How far from the base of the mast should he attach the end of the light string?




*Answer available at the end of this section.*

## The Distance Formula

Derived from the **Pythagorean Theorem**, the **distance formula** is used to find the distance between two points in the plane. The Pythagorean Theorem, $a^2 + b^2 = c^2$, is based on a right triangle where $a$ and $b$ are the lengths of the legs adjacent to the right angle, and $c$ is the length of the hypotenuse. See Figure $6.1.15$.

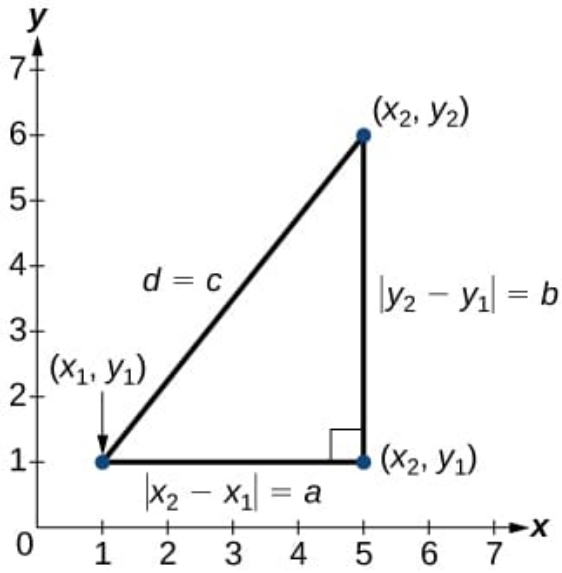



Figure $6.1.15$

The relationship of sides $|x_2 - x_1|$ and $|y_2 - y_1|$ to side $d$ is the same as that of sides $a$ and $b$ to side $c$. We use the absolute value symbol to indicate that the length is a positive number because the absolute value of any number is positive. (For example, $|-3| = 3$. ) The symbols $|x_2 - x_1|$ and $|y_2 - y_1|$ indicate that the lengths of the sides of the triangle are positive. To find the length $c$, take the square root of both sides of the Pythagorean Theorem.

$$c^2 = a^2 + b^2 \rightarrow c = \sqrt{a^2 + b^2} \tag{6.1.2}$$

It follows that the distance formula is given as

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \rightarrow d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{6.1.3}$$

We do not have to use the absolute value symbols in this definition because any number squared is positive.

## The distance formula

Given endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$, the distance between $A$ and $B$ is given by

$$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{6.1.4}$$

*note:* \(d(A,B)\) *is also called the* ***length*** *of the line segment* $AB$

## Example 6.1.5: Finding the Distance between Two Points

Find the distance between the points $(-3, -1)$ and $(2, 3)$.

**Solution**

Let us first look at the graph of the two points. Connect the points to form a right triangle as in Figure 6.1.16

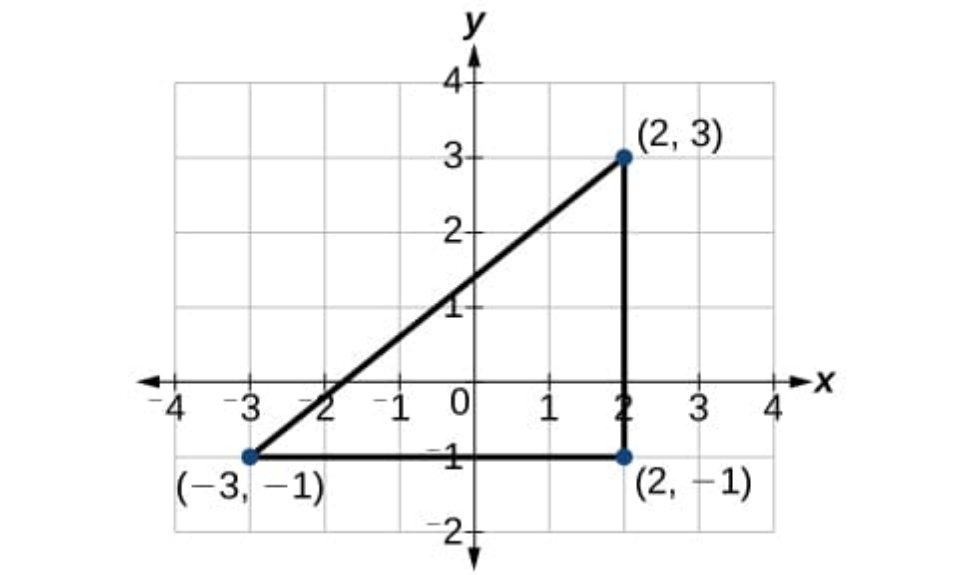



Figure 6.1.16

Then, calculate the length of $d$ using the distance formula.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2 - (-3))^2 + (3 - (-1))^2} \\ &= \sqrt{(5)^2 + (4)^2} \\ &= \sqrt{25 + 16} \\ &= \sqrt{41} \end{aligned}$$

## Exercise 6.1.5

Find the distance between two points: $(1, 4)$ and $(11, 9)$.

*Answer available at the end of this section.*

# The Midpoint Formula

When the endpoints of a line segment are known, we can find the point midway between them. This point is known as the midpoint and the formula is known as the **midpoint formula**. Given the endpoints of a line segment, $(x_1, y_1)$ and $(x_2, y_2)$, the midpoint formula states how to find the coordinates of the midpoint M.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \tag{6.1.5}$$

A graphical view of a midpoint is shown in Figure 6.1.18. Notice that the line segments on either side of the midpoint are congruent.

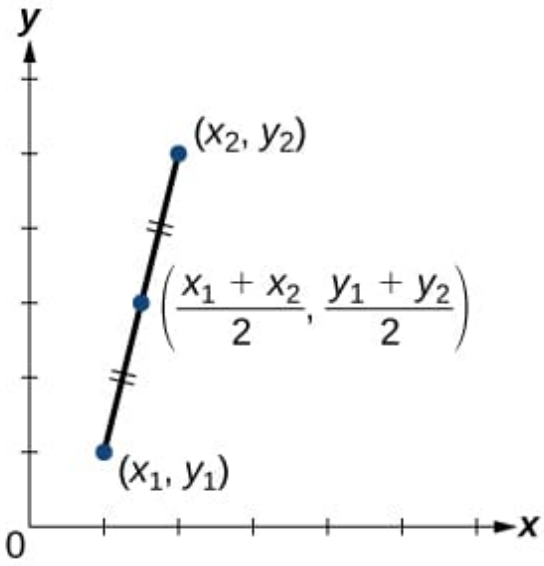



Figure 6.1.18

## Example 6.1.6: Finding the Midpoint of the Line Segment

Find the midpoint of the line segment with the endpoints $(7, -2)$ and $(9, 5)$.

**Solution**

Use the formula to find the midpoint of the line segment.

$$\begin{aligned}\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) &= \left(\frac{7+9}{2}, \frac{-2+5}{2}\right) \\ &= \left(8, \frac{3}{2}\right)\end{aligned}$$

## Exercise 6.1.6

Find the midpoint of the line segment with endpoints $(-2, -1)$ and $(-8, 6)$.

*Answer available at the end of this section.*

## Example 6.1.7: Finding the Center of a Circle

The diameter of a circle has endpoints $(-1, -4)$ and $(5, -4)$. Find the center of the circle.

**Solution**

The center of a circle is the center, or midpoint, of its diameter. Thus, the midpoint formula will yield the center point.

$$\begin{aligned}\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right) &= \left(\frac{-1+5}{2},\frac{-4-4}{2})\right)\\ &= \left(\frac{4}{2},-\frac{8}{2}\right)\\ &= (2,4)\end{aligned}$$

# Key Concepts

- The Pythagorean Theorem, among the most famous theorems in history, is used to solve right-triangle problems and has applications in numerous fields. Solving for the length of one side of a right triangle requires solving a quadratic equation. See Example.We can locate, or plot, points in the Cartesian coordinate system using ordered pairs, which are defined as displacement from the $x$-axis and displacement from the $y$-axis. See Example.
- The distance formula is derived from the Pythagorean Theorem and is used to find the length of a line segment. See Example and Example.
- The midpoint formula provides a method of finding the coordinates of the midpoint dividing the sum of the $x$-coordinates and the sum of the $y$-coordinates of the endpoints by $2$. See Example and Example.

## Answers

**Try It** $6.1.1$
c=10

**Exercise** $6.1.1$
c=13

**Try It** $6.1.2$
8

**Exercise** $6.1.2$
12

**Exercise** $6.1.3$
$5$ units

**Try It** $6.1.4$
12 feet

**Exercise** $6.1.4$
8 feet

**Exercise** $6.1.5$
$\sqrt{125}=5\sqrt{5}$

**Exercise** $6.1.6$
$\left(-5,\frac{5}{2}\right)$

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# 6.2: The Circle

## The Conic Sections

A conic section, or conic, is a shape resulting from intersecting a right circular cone with a plane. The angle at which the plane intersects the cone determines the shape, as shown in Figure $6.2.1$

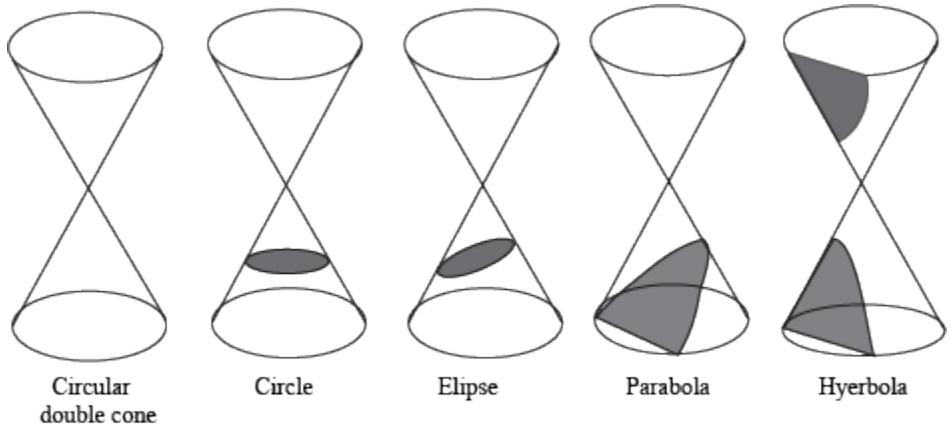



Figure $6.2.1$

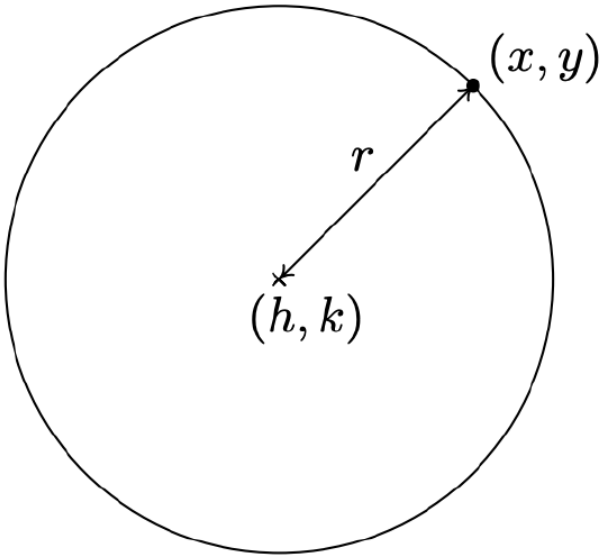
Conic sections can also be described by a set of points in the coordinate plane. Later in this chapter, we will see that the graph of any quadratic equation in two variables is a conic section. The signs of the equations and the coefficients of the variable terms determine the shape.

## Defining Circles

This section focuses on the four variations of the standard form of the equation for a circle. A circle is a "conic section" that can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

**Definition 2.2.1**

A **circle** with center $(h, k)$ and radius $r > 0$ is the set of all points $(x, y)$ in the plane whose distance to $(h, k)$ is $r$.




From the picture, we see that a point $(x, y)$ is on the circle if and only if its distance to $(h, k)$ is $r$. We express this relationship algebraically using the Distance Formula, Equation 1.1, as

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

**Definition 2.2.2 The Standard Equation of a Circle**

The Standard Equation of a Circle: The equation of a circle with center $(h, k)$ and radius $r > 0$ is $(x-h)^2 + (y-k)^2 = r^2$.

**Example 2.2.1**

Write the standard equation of the circle with center at the origin and radius $3$.

**Solution**

Here, $(h, k) = (0, 0)$ and $r = 3$, so we get

$$\begin{aligned} (x-0)^2 + (y-0)^2 &= (3)^2 \\ x^2 + y^2 &= 9 \end{aligned}$$

**Example 2.2.2**

Write the standard equation of the circle with center at $-2, 3)$ and radius $5$.

**Solution**

Here, $(h, k) = (-2, 3)$ and $r = 5$, so we get

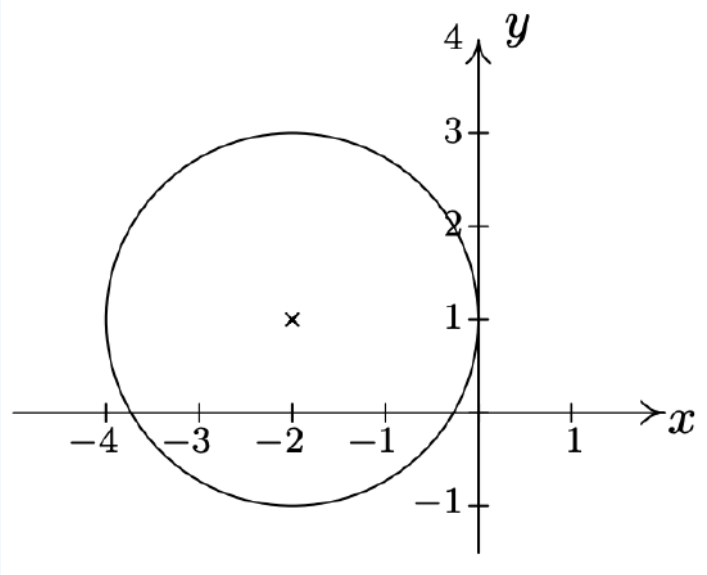
$$\begin{aligned} (x-(-2))^2 + (y-3)^2 &= (5)^2 \\ (x+2)^2 + (y-3)^2 &= 25 \end{aligned}$$

## Example 2.2.3

Graph $(x+2)^2+(y-1)^2=4$. Find the center and radius.

**Solution**

From the standard form of a circle, Equation 7.1, we have that $x+2$ is $x-h$, so $h=-2$ and $y-1$ is $y-k$ so $k=1$. This tells us that our center is $(-2,1)$. Furthermore, $r^2=4$, so $r=2$. Thus we have a circle centered at $(-2,1)$ with a radius of $2$. Graphing gives us




If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x+2)^2+(y-1)^2=4$, we'd be contending with $x^2+4x+y^2-2y+1=0$. If we're given such an equation, we can complete the square in each of the variables to see if it fits the form given in Equation 7.1 by following the steps given below.

**To Write the Equation of a Circle in Standard Form**

1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square on both variables as needed.
3. Divide both sides by the coefficient of the squares. (For circles, they will be the same.)

## Example 2.2.4

Complete the square to find the center and radius of $3x^2-6x+3y^2+4y-4=0$.

**Solution.**

$$
\begin{aligned}
3x^2 - 6x + 3y^2 + 4y - 4 &= 0 \\
3x^2 - 6x + 3y^2 + 4y &= 4 && \text{add 4 to both sides} \\
3\left(x^2 - 2x\right) + 3\left(y^2 + \frac{4}{3}y\right) &= 4 && \text{factor out leading coefficients} \\
3\left(x^2 - 2x + \underline{1}\right) + 3\left(y^2 + \frac{4}{3}y + \underline{\underline{\frac{4}{9}}}\right) &= 4 + 3\underline{(1)} + 3\underline{\underline{\left(\frac{4}{9}\right)}} && \text{complete the square in } x,\ y \\
3(x-1)^2 + 3\left(y + \frac{2}{3}\right)^2 &= \frac{25}{3} && \text{factor} \\
(x-1)^2 + \left(y + \frac{2}{3}\right)^2 &= \frac{25}{9} && \text{divide both sides by 3}
\end{aligned}
$$

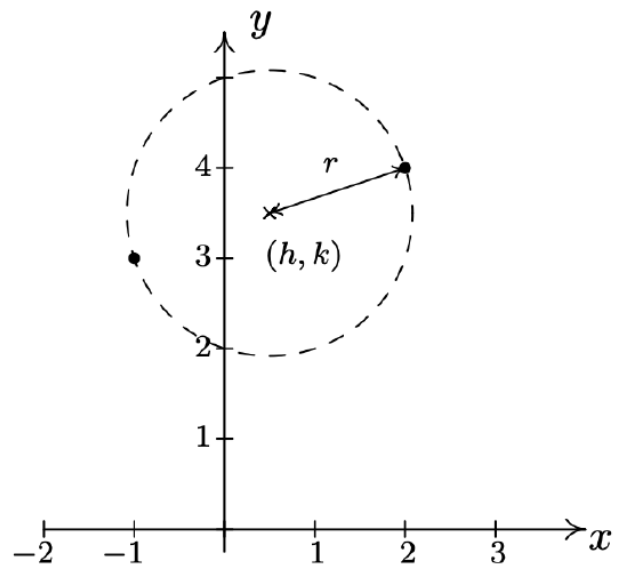
Hence, the center is $(h,k) = \left(1, -\frac{2}{3}\right)$. Furthermore, we see that $r^2 = \frac{25}{9}$ so the radius is $r = \frac{5}{3}$.

It is possible to obtain equations like $(x-3)^2 + (y+1)^2 = 0$ or $(x-3)^2 + (y+1)^2 = -1$, neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations. The next example uses the Midpoint Formula, Equation 1.2, in conjunction with the ideas presented so far in this section.

## Example 2.2.5

Write the standard equation of the circle which has $(-1,3)$ and $(2,4)$ as the endpoints of a diameter.

We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields




Since the given points are endpoints of a diameter, we know their midpoint $(h,k)$ is the center of the circle. Equation 1.2 gives us

$$
\begin{aligned}
(h,k) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\
&= \left(\frac{-1+2}{2}, \frac{3+4}{2}\right) \\
&= \left(\frac{1}{2}, \frac{7}{2}\right)
\end{aligned}
$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned} r &= \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2} \\ &= \frac{1}{2}\sqrt{3^2+1^2} \\ &= \frac{\sqrt{10}}{2} \end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2 = \frac{10}{4}$, our answer becomes $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2 = \frac{10}{4}$

## The Unit Circle

We close this section with perhaps the most important circle in all of mathematics. The equation of this circle is quite simple. However, its importance should not be underestimated.

**Definition 2.2.3**

The **Unit Circle** is the circle centered at $(0,0)$ with a radius of $1$. The standard equation of the Unit Circle is $x^2+y^2=1$.

**Example 2.2.6**

Find the points on the unit circle with $y$-coordinate $\frac{\sqrt{3}}{2}$.

**Solution**

We replace $y$ with $\frac{\sqrt{3}}{2}$ in the equation $x^2+y^2=1$ to get

$$\begin{aligned} x^2+y^2 &= 1 \\ x^2+\left(\frac{\sqrt{3}}{2}\right)^2 &= 1 \\ \frac{3}{4}+x^2 &= 1 \\ x^2 &= \frac{1}{4} \\ x &= \pm\sqrt{\frac{1}{4}} \\ x &= \pm\frac{1}{2} \end{aligned}$$

Our final answers are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.

---

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# 6.3: The Ellipse

**Learning Objectives**

- Write equations of ellipses in standard form.
- Graph ellipses centered at the origin.
- Graph ellipses not centered at the origin.
- Solve applied problems involving ellipses.

Can you imagine standing at one end of a large room and still being able to hear a whisper from a person standing at the other end? The National Statuary Hall in Washington, D.C., shown in Figure 6.3.1, is such a room. It is an oval-shaped room called a *whispering chamber* because the shape makes it possible for sound to travel along the walls. In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.



Figure 6.3.1: The National Statuary Hall in Washington, D.C. (credit: Greg Palmer, Flickr)

## Writing Equations of Ellipses in Standard Form

We can draw an ellipse using a piece of cardboard, two thumbtacks, a pencil, and string. Place the thumbtacks in the cardboard to form the foci of the ellipse. Cut a piece of string longer than the distance between the two thumbtacks (the length of the string represents the constant in the definition). Tack each end of the string to the cardboard, and trace a curve with a pencil held taut against the string. The result is an ellipse. See Figure 6.3.3.

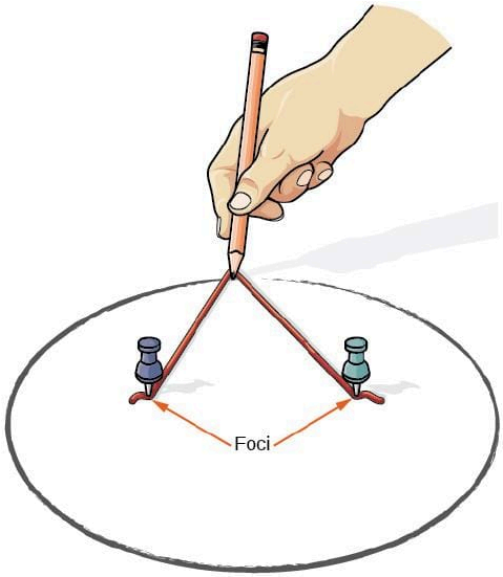



Figure 6.3.3

Every ellipse has two axes of symmetry. The longer axis is called the major axis, and the shorter axis is called the minor axis. Each endpoint of the major axis is the vertex of the ellipse (plural: vertices), and each endpoint of the minor axis is a co-vertex of the ellipse. The center of an ellipse is the midpoint of both the major and minor axes. The axes are perpendicular at the center. The foci always lie on the major axis, and the sum of the distances from the foci to any point on the ellipse (the constant sum) is greater than the distance between the foci (Figure 6.3.4).

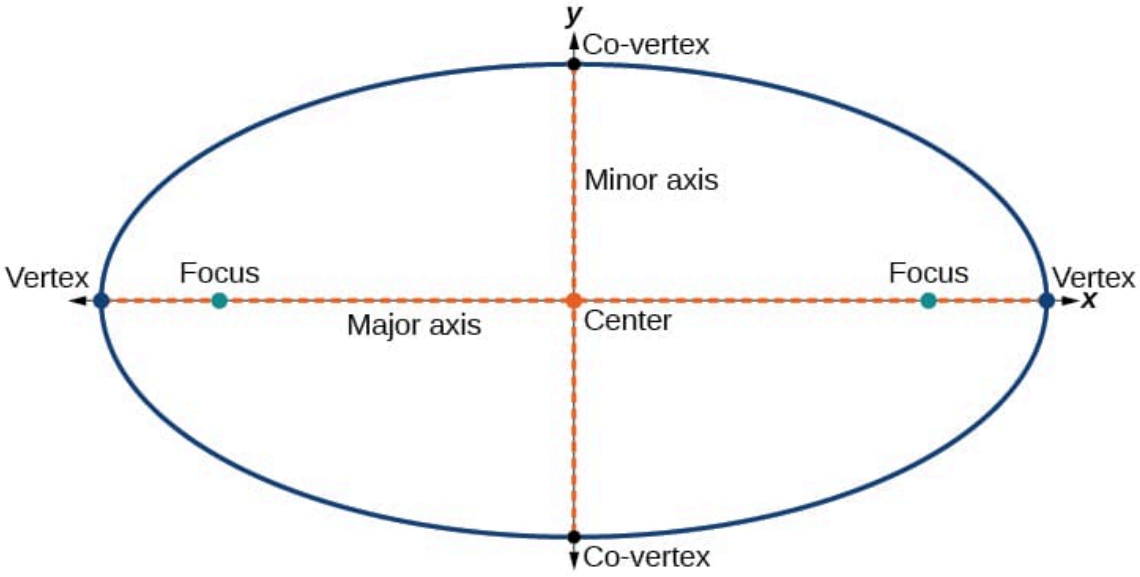



Figure 6.3.4

In this section, we restrict ellipses to those that are positioned vertically or horizontally in the coordinate plane. That is, the axes will either lie on or be parallel to the $x$- and $y$-axes. Later in the chapter, we will see ellipses that are rotated in the coordinate plane.

To work with horizontal and vertical ellipses in the coordinate plane, we consider two cases: those that are centered at the origin and those that are centered at a point other than the origin. First we will learn to derive the equations of ellipses, and then we will learn how to write the equations of ellipses in standard form. Later we will use what we learn to draw the graphs.

# Deriving the Equation of an Ellipse Centered at the Origin

To derive the equation of an **ellipse** centered at the origin, we begin with the foci $(-c, 0)$ and $(c, 0)$. The ellipse is the set of all points $(x, y)$ such that the sum of the distances from $(x, y)$ to the foci is constant, as shown in Figure $6.3.5$.

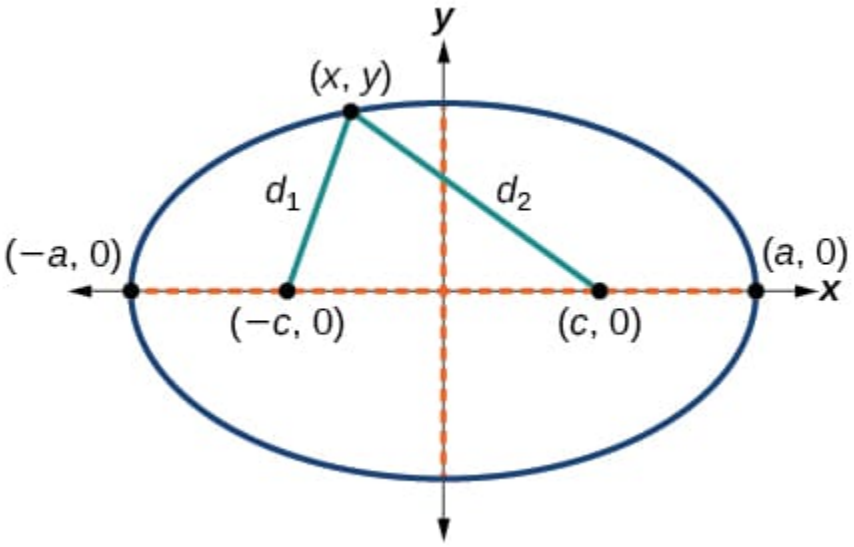



Figure $6.3.5$

If $(a, 0)$ is a vertex of the ellipse, the distance from $(-c, 0)$ to $(a, 0)$ is $a - (-c) = a + c$. The distance from $(c, 0)$ to $(a, 0)$ is $a - c$. The sum of the distances from the foci to the vertex is

$$(a + c) + (a - c) = 2a$$

If $(x, y)$ is a point on the ellipse, then we can define the following variables:

- $d_1 =$ the distance from $(-c, 0)$ to $(x, y)$
- $d_2 =$ the distance from $(c, 0)$ to $(x, y)$

By the definition of an ellipse, $d_1 + d_2$ is constant for any point $(x, y)$ on the ellipse. We know that the sum of these distances is $2a$ for the vertex $(a, 0)$. It follows that $d_1 + d_2 = 2a$ for any point on the ellipse. We will begin the derivation by applying the distance formula. The rest of the derivation is algebraic.

$$
\begin{aligned}
d_1 + d_2 &= 2a \\
\sqrt{(x-(-c))^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} &= 2a \quad \text{Distance formula} \\
\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} &= 2a \quad \text{Simplify expressions.} \\
\sqrt{(x+c)^2 + y^2} &= 2a - \sqrt{(x-c)^2 + y^2} \quad \text{Move radical to opposite side.} \\
(x+c)^2 + y^2 &= \left[2a - \sqrt{(x-c)^2 + y^2}\right]^2 \quad \text{Square both sides.} \\
x^2 + 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 \quad \text{Expand the squares.} \\
x^2 + 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2 \quad \text{Expand remaining squares.} \\
2cx &= 4a^2 - 4a\sqrt{(x-c)^2 + y^2} - 2cx \quad \text{Combine like terms.} \\
4cx - 4a^2 &= -4a\sqrt{(x-c)^2 + y^2} \quad \text{Isolate the radical.} \\
cx - a^2 &= -a\sqrt{(x-c)^2 + y^2} \quad \text{Divide by 4.} \\
\left[cx - a^2\right]^2 &= a^2\left[\sqrt{(x-c)^2 + y^2}\right]^2 \quad \text{Square both sides.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2(x^2 - 2cx + c^2 + y^2) \quad \text{Expand the squares.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 \quad \text{Distribute } a^2 \\
a^2x^2 - c^2x^2 + a^2y^2 &= a^4 - a^2c^2 \quad \text{Rewrite.} \\
x^2(a^2 - c^2) + a^2y^2 &= a^2(a^2 - c^2) \quad \text{Factor common terms.} \\
x^2b^2 + a^2y^2 &= a^2b^2 \quad \text{Set } b^2 = a^2 - c^2 \\
\frac{x^2b^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} &= \frac{a^2b^2}{a^2b^2} \quad \text{Divide both sides by } a^2b^2 \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \quad \text{Simplify}
\end{aligned}
$$

Thus, the standard equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.This equation defines an ellipse centered at the origin. If $a > b$,the ellipse is stretched further in the horizontal direction, and if $b > a$, the ellipse is stretched further in the vertical direction.

## Writing Equations of Ellipses Centered at the Origin in Standard Form

Standard forms of equations tell us about key features of graphs. Take a moment to recall some of the standard forms of equations we've worked with in the past: linear, quadratic, cubic, exponential, logarithmic, and so on. By learning to interpret standard forms of equations, we are bridging the relationship between algebraic and geometric representations of mathematical phenomena.

The key features of the **ellipse** are its center, **vertices**, **co-vertices**, **foci**, and lengths and positions of the **major and minor axes**. Just as with other equations, we can identify all of these features just by looking at the standard form of the equation. There are four variations of the standard form of the ellipse. These variations are categorized first by the location of the center (the origin or not the origin), and then by the position (horizontal or vertical). Each is presented along with a description of how the parts of the equation relate to the graph. Interpreting these parts allows us to form a mental picture of the ellipse.

### STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER $(0, 0)$

The standard form of the equation of an ellipse with center $(0, 0)$ and major axis on the $x$-*axis* is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{6.3.1}$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(\pm a, 0)$

- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the coordinates of the foci are $(\pm c, 0)$, where $c^2 = a^2 - b^2$. See Figure $6.3.6a$.

The standard form of the equation of an ellipse with center $(0, 0)$ and major axis on the $y$-*axis* is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \tag{6.3.2}$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(0, \pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(\pm b, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = a^2 - b^2$. See Figure $6.3.6b$.

Note that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 - b^2$. When we are given the coordinates of the foci and vertices of an ellipse, we can use this relationship to find the equation of the ellipse in standard form.

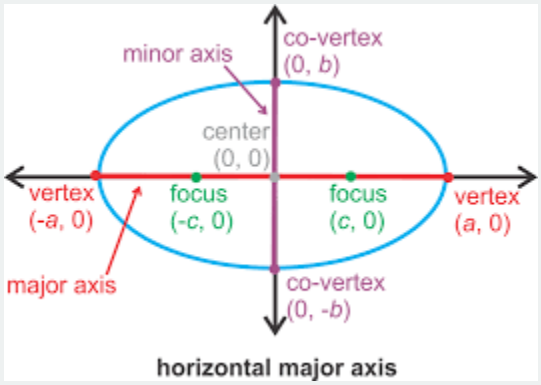



Figure $6.3.6$: Horizontal ellipse with center $(0, 0)$

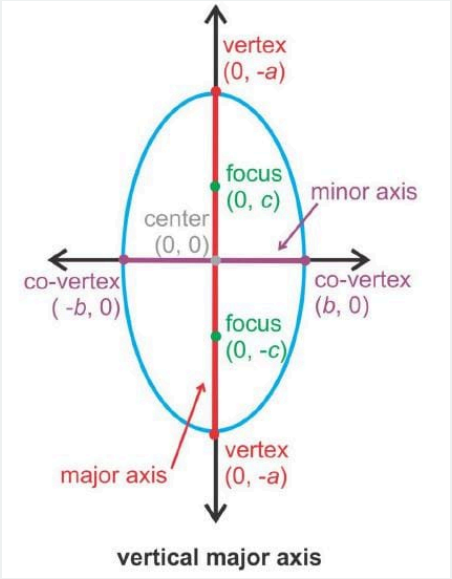



Figure $6.3.7$:Vertical ellipse with center $(0, 0)$

### How to: Given the vertices and foci of an ellipse centered at the origin, write its equation in standard form

1. Determine whether the major axis lies on the *x*- or *y*-axis.
   - If the given coordinates of the vertices and foci have the form $(\pm a, 0)$ and $(\pm c, 0)$ respectively, then the major axis is the *x*-axis. Use the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
   - If the given coordinates of the vertices and foci have the form $(0, \pm a)$ and $(\pm c, 0)$,respectively, then the major axis is the *y*-axis. Use the standard form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$
2. Use the equation $c^2 = a^2 - b^2$, along with the given coordinates of the vertices and foci, to solve for $b^2$.
3. Substitute the values for $a^2$ and $b^2$ into the standard form of the equation determined in Step 1.

### Example 6.3.1: Writing the Equation of an Ellipse Centered at the Origin in Standard Form

What is the standard form equation of the ellipse that has vertices $(\pm 8, 0)$ and foci $(\pm 5, 0)$?

**Solution**

The foci are on the $x$-axis, so the major axis is the $x$-axis. Thus, the equation will have the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The vertices are $(\pm 8, 0)$,so $a = 8$ and $a^2 = 64$.

The foci are $(\pm 5, 0)$,so $c = 5$ and $c^2 = 25$.

We know that the vertices and foci are related by the equation $c^2 = a^2 - b^2$. Solving for $b^2$, we have:

$$\begin{aligned} c^2 &= a^2 - b^2 \\ 25 &= 64 - b^2 \qquad \text{Substitute for } c^2 \text{ and } a^2 \\ b^2 &= 39 \qquad \text{Solve for } b^2 \end{aligned}$$

Now we need only substitute $a^2 = 64$ and $b^2 = 39$ into the standard form of the equation. The equation of the ellipse is $\frac{x^2}{64} + \frac{y^2}{39} = 1$.

### Exercise 6.3.1

What is the standard form equation of the ellipse that has vertices $(0, \pm 4)$ and foci $(0, \pm\sqrt{15})$?

*Answer available at the end of this section.*

### Q&A

**Can we write the equation of an ellipse centered at the origin given coordinates of just one focus and vertex?**

*Yes. Ellipses are symmetrical, so the coordinates of the vertices of an ellipse centered around the origin will always have the form $(\pm a, 0)$ or $(0, \pm a)$. Similarly, the coordinates of the foci will always have the form $(\pm c, 0)$ or $(0, \pm c)$. Knowing this, we can use $a$ and $c$ from the given points, along with the equation $c^2 = a^2 - b^2$, to find $b^2$.*

## Writing Equations of Ellipses Not Centered at the Origin

Like the graphs of other equations, the graph of an **ellipse** can be translated. If an ellipse is translated $h$ units horizontally and

$k$ units vertically, the center of the ellipse will be $(h, k)$. This **translation** results in the standard form of the equation we saw previously, with $x$ replaced by $(x - h)$ and $y$ replaced by $(y - k)$.

## STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER $(H, K)$

The standard form of the equation of an ellipse with center $(h, k)$ and major axis parallel to the $x$-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{6.3.3}$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the coordinates of the foci are $(h \pm c, k)$,where $c^2 = a^2 - b^2$. See Figure $6.3.7a$.

The standard form of the equation of an ellipse with center $(h, k)$ and major axis parallel to the $y$-axis is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \tag{6.3.4}$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^2 = a^2 - b^2$. See Figure $6.3.7b$.

Just as with ellipses centered at the origin, ellipses that are centered at a point $(h, k)$ have vertices, co-vertices, and foci that are related by the equation $c^2 = a^2 - b^2$. We can use this relationship along with the midpoint and distance formulas to find the equation of the ellipse in standard form when the vertices and foci are given.

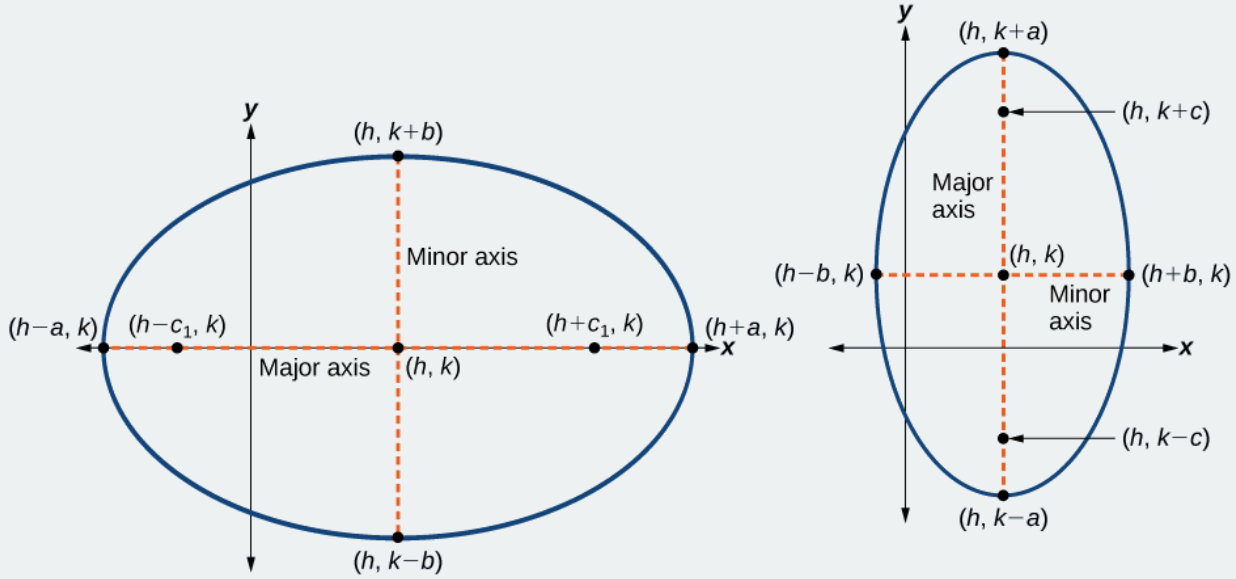



Figure $6.3.8$: (a) Horizontal ellipse with center $(h, k)$ (b) Vertical ellipse with center $(h, k)$

**How to: Given the vertices and foci of an ellipse not centered at the origin, write its equation in standard form**

1. Determine whether the major axis is parallel to the $x$- or $y$-axis.
   - If the *y*-coordinates of the given vertices and foci are the same, then the major axis is parallel to the $x$-axis. Use the standard form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$
   - If the *x*-coordinates of the given vertices and foci are the same, then the major axis is parallel to the *y*-axis. Use the standard form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$
2. Identify the center of the ellipse $(h,k)$ using the midpoint formula and the given coordinates for the vertices.
3. Find $a^2$ by solving for the length of the major axis, $2a$, which is the distance between the given vertices.
4. Find $c^2$ using $h$ and $k$, found in Step 2, along with the given coordinates for the foci.
5. Solve for $b^2$ using the equation $c^2=a^2-b^2$.
6. Substitute the values for $h$, $k$, $a^2$, and $b^2$ into the standard form of the equation determined in Step 1.

**Example 6.3.2: Writing the Equation of an Ellipse Centered at a Point Other Than the Origin**

What is the standard form equation of the ellipse that has vertices $(-2,-8)$ and $(-2,2)$ and foci $(-2,-7)$ and $(-2,1)$?

## Solution

The $x$-coordinates of the vertices and foci are the same, so the major axis is parallel to the $y$-axis. Thus, the equation of the ellipse will have the form

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

First, we identify the center, $(h,k)$. The center is halfway between the vertices, $(-2,-8)$ and $(-2,2)$. Applying the midpoint formula, we have:

$$\begin{aligned}(h,k)&=\left(\frac{-2+(-2)}{2},\frac{-8+2}{2}\right)\\&=(-2,-3)\end{aligned}$$

Next, we find $a^2$. The length of the major axis, $2a$, is bounded by the vertices. We solve for $a$ by finding the distance between the *y*-coordinates of the vertices.

$$\begin{aligned}2a&=2-(-8)\\2a&=10\\a&=5\end{aligned}$$

So $a^2=25$.

Now we find $c^2$. The foci are given by $(h,k\pm c)$. So, $(h,k-c)=(-2,-7)$ and $(h,k+c)=(-2,1)$. We substitute $k=-3$ using either of these points to solve for $c$.

$$\begin{aligned}k+c&=1\\-3+c&=1\\c&=4\end{aligned}$$

So $c^2=16$.

Next, we solve for $b^2$ using the equation $c^2=a^2-b^2$.

$$c^2 = a^2 - b^2$$
$$16 = 25 - b^2$$
$$b^2 = 9$$

Finally, we substitute the values found for $h$, $k$, $a^2$, and $b^2$ into the standard form equation for an ellipse:

$$\frac{(x+2)^2}{9} + \frac{(y+3)^2}{25} = 1$$

**Exercise 6.3.2**

What is the standard form equation of the ellipse that has vertices $(-3, 3)$ and $(5, 3)$ and foci $(1 - 2\sqrt{3}, 3)$ and $(1 + 2\sqrt{3}, 3)$?

*Answer available at the end of this section.*

## Graphing Ellipses Centered at the Origin

Just as we can write the equation for an ellipse given its graph, we can graph an ellipse given its equation. To graph ellipses centered at the origin, we use the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b \text{ for horizontal ellipses}$$

and

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b \text{ for vertical ellipses}$$

**How to: Given the standard form of an equation for an ellipse centered at $(0, 0)$, sketch the graph.**

1. Use the standard forms of the equations of an ellipse to determine the major axis, vertices, co-vertices, and foci.
   - If the equation is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, then
     - the major axis is the $x$-axis
     - the coordinates of the vertices are $(\pm a, 0)$
     - the coordinates of the co-vertices are $(0, \pm b)$
     - the coordinates of the foci are $(\pm c, 0)$
   - If the equation is in the form $x^2b^2 + y^2a^2 = 1$,where $a > b$, then
     - the major axis is the $y$-axis
     - the coordinates of the vertices are $(0, \pm a)$
     - the coordinates of the co-vertices are $(\pm b, 0)$
     - the coordinates of the foci are $(0, \pm c)$
2. Solve for $c$ using the equation $c^2 = a^2 - b^2$.
3. Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.

**Example 6.3.3: Graphing an Ellipse Centered at the Origin**

Graph the ellipse given by the equation, $\frac{x^2}{9} + \frac{y^2}{25} = 1$. Identify and label the center, vertices, co-vertices, and foci.

**Solution**

First, we determine the position of the major axis. Because $25 > 9$,the major axis is on the $y$-axis. Therefore, the equation is in the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$,where $b^2 = 9$ and $a^2 = 25$. It follows that:

- the center of the ellipse is $(0, 0)$
- the coordinates of the vertices are $(0, \pm a) = (0, \pm\sqrt{25}) = (0, \pm 5)$
- the coordinates of the co-vertices are $(\pm b, 0) = (\pm 9, 0) = (\pm 3, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = a^2 - b^2$ Solving for $c$, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{25 - 9} \\ &= \pm\sqrt{16} \\ &= \pm 4 \end{aligned}$$

Therefore, the coordinates of the foci are $(0, \pm 4)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse. See Figure $6.3.8$.

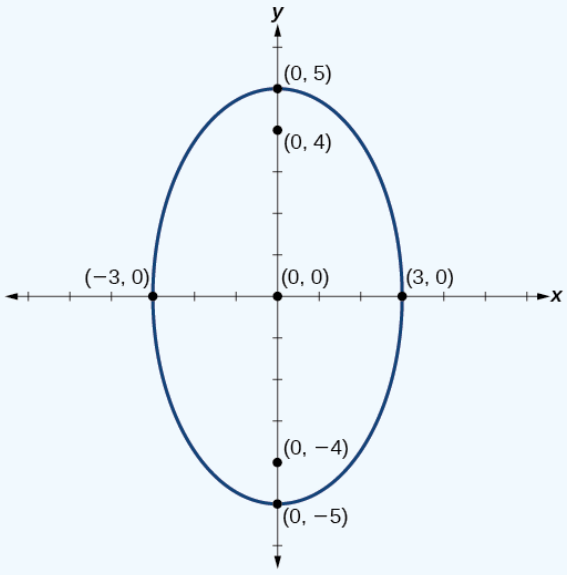



Figure $6.3.9$

## Exercise $6.3.3$

Graph the ellipse given by the equation $\frac{x^2}{36} + \frac{y^2}{4} = 1$. Identify and label the center, vertices, co-vertices, and foci.

*Answer available at the end of this section.*

## Example $6.3.4$: Graphing an Ellipse Centered at the Origin from an Equation Not in Standard Form

Graph the ellipse given by the equation $4x^2 + 25y^2 = 100$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

**Solution**

First, use algebra to rewrite the equation in standard form.

$$4x^2 + 25y^2 = 100$$
$$\frac{4x^2}{100} + \frac{25y^2}{100} = \frac{100}{100}$$
$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

Next, we determine the position of the major axis. Because $25 > 4$, the major axis is on the $x$-axis. Therefore, the equation is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 = 25$ and $b^2 = 4$. It follows that:

- the center of the ellipse is $(0, 0)$
- the coordinates of the vertices are $(\pm a, 0) = (\pm\sqrt{25}, 0) = (\pm 5, 0)$
- the coordinates of the co-vertices are $(0, \pm b) = (0, \pm\sqrt{4}) = (0, \pm 2)$
- the coordinates of the foci are $(\pm c, 0)$, where $c^2 = a^2 - b^2$. Solving for $c$, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{25 - 4} \\ &= \pm\sqrt{21} \end{aligned}$$

Therefore the coordinates of the foci are $(\pm\sqrt{21}, 0)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

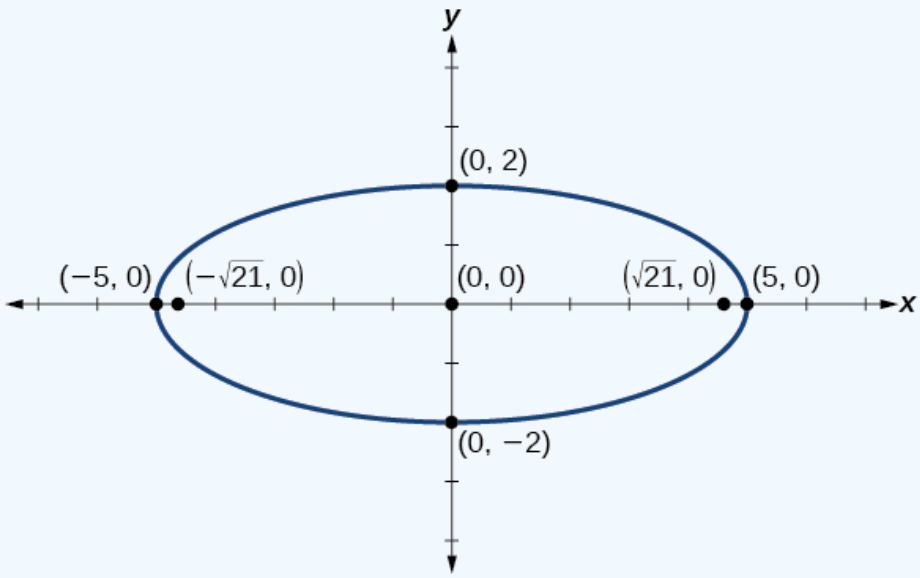



Figure 6.3.11

**Exercise 6.3.4**

Graph the ellipse given by the equation $49x^2 + 16y^2 = 784$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

*Answer available at the end of this section.*

## Graphing Ellipses Not Centered at the Origin

When an **ellipse** is not centered at the origin, we can still use the standard forms to find the key features of the graph. When the ellipse is centered at some point, $(h, k)$,we use the standard forms $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $a > b$ for horizontal ellipses and

$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, $a>b$ for vertical ellipses. From these standard equations, we can easily determine the center, vertices, co-vertices, foci, and positions of the major and minor axes.

### How to: Given the standard form of an equation for an ellipse centered at $(h,k)$, sketch the graph.

1. Use the standard forms of the equations of an ellipse to determine the center, position of the major axis, vertices, co-vertices, and foci.
   - If the equation is in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a>b$, then
     - the center is $(h,k)$
     - the major axis is parallel to the $x$-axis
     - the coordinates of the vertices are $(h\pm a,k)$
     - the coordinates of the co-vertices are $(h,k\pm b)$
     - the coordinates of the foci are $(h\pm c,k)$
   - If the equation is in the form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, where $a>b$, then
     - the center is $(h,k)$
     - the major axis is parallel to the $y$-axis
     - the coordinates of the vertices are $(h,k\pm a)$
     - the coordinates of the co-vertices are $(h\pm b,k)$
     - the coordinates of the foci are $(h,k\pm c)$
2. Solve for $c$ using the equation $c^2=a^2-b^2$.
3. Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.

### Example 6.3.5: Graphing an Ellipse Centered at $(h,k)$

Graph the ellipse given by the equation, $\frac{(x+2)^2}{4}+\frac{(y-5)^2}{9}=1$. Identify and label the center, vertices, co-vertices, and foci.

**Solution**

First, we determine the position of the major axis. Because $9>4$, the major axis is parallel to the $y$-axis. Therefore, the equation is in the form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, where $b^2=4$ and $a^2=9$. It follows that:

- the center of the ellipse is $(h,k)=(-2,5)$
- the coordinates of the vertices are $(h,k\pm a)=(-2,5\pm\sqrt{9})=(-2,5\pm3)$, or $(-2,2)$ and $(-2,8)$
- the coordinates of the co-vertices are $(h\pm b,k)=(-2\pm\sqrt{4},5)=(-2\pm2,5)$, or $(-4,5)$ and $(0,5)$
- the coordinates of the foci are $(h,k\pm c)$, where $c^2=a^2-b^2$. Solving for $c$,we have:

$$\begin{aligned} c&=\pm\sqrt{a^2-b^2}\\ &=\pm\sqrt{9-4}\\ &=\pm\sqrt{5}\end{aligned}$$

Therefore, the coordinates of the foci are $(-2,5-\sqrt{5})$ and $(-2,5+\sqrt{5})$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

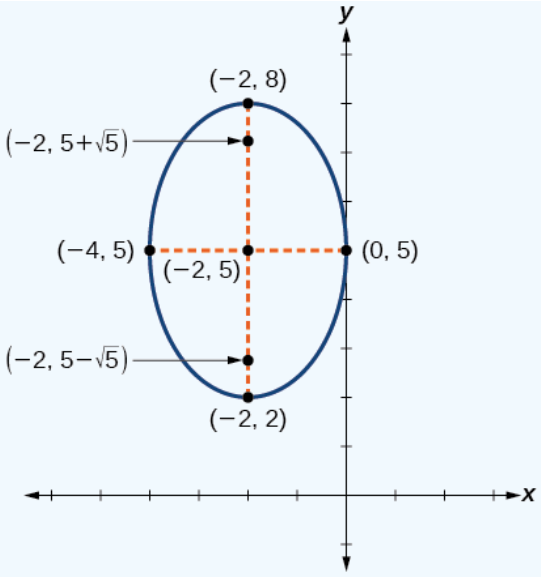



Figure 6.3.13

## Exercise 6.3.5

Graph the ellipse given by the equation $\frac{(x-4)^2}{36}+\frac{(y-2)^2}{20}=1$. Identify and label the center, vertices, co-vertices, and foci.

*Answer available at the end of this section.*

## How to: Given the general form of an equation for an ellipse centered at $(h,k)$, express the equation in standard form.

1. Recognize that an ellipse described by an equation in the form $ax^2+by^2+cx+dy+e=0$ is in general form.
2. Rearrange the equation by grouping terms that contain the same variable. Move the constant term to the opposite side of the equation.
3. Factor out the coefficients of the $x^2$ and $y^2$ terms in preparation for completing the square.
4. Complete the square for each variable to rewrite the equation in the form of the sum of multiples of two binomials squared set equal to a constant, $m_1(x-h)^2+m_2(y-k)^2=m_3$, where $m_1$, $m_2$,and $m_3$ are constants.
5. Divide both sides of the equation by the constant term to express the equation in standard form.

## Example 6.3.6: Graphing an Ellipse Centered at $(h,k)$ by First Writing It in Standard Form

Graph the ellipse given by the equation $4x^2+9y^2-40x+36y+100=0$. Identify and label the center, vertices, co-vertices, and foci.

**Solution**

We must begin by rewriting the equation in standard form.

$$4x^2+9y^2-40x+36y+100=0$$

Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$(4x^2-40x)+(9y^2+36y)=-100$$

Factor out the coefficients of the squared terms.

$$4(x^2 - 10x) + 9(y^2 + 4y) = -100$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$4(x^2 - 10x + 25) + 9(y^2 + 4y + 4) = -100 + 100 + 36$$

Rewrite as perfect squares.

$$4(x - 5)^2 + 9(y + 2)^2 = 36$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x - 5)^2}{9} + \frac{(y + 2)^2}{4} = 1$$

Now that the equation is in standard form, we can determine the position of the major axis. Because $9 > 4$, the major axis is parallel to the $x$-axis. Therefore, the equation is in the form $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$, where $a^2 = 9$ and $b^2 = 4$. It follows that:

- the center of the ellipse is $(h, k) = (5, -2)$
- the coordinates of the vertices are $(h \pm a, k) = (5 \pm \sqrt{9}, -2) = (5 \pm 3, -2)$, or $(2, -2)$ and $(8, -2)$
- the coordinates of the co-vertices are $(h, k \pm b) = (5, -2 \pm \sqrt{4}) = (5, -2 \pm 2)$, or $(5, -4)$ and $(5, 0)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$. Solving for $c$, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{9 - 4} \\ &= \pm\sqrt{5} \end{aligned}$$

Therefore, the coordinates of the foci are $(5 - \sqrt{5}, -2)$ and $(5 + \sqrt{5}, -2)$.

Next we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse as shown in Figure 6.3.14.

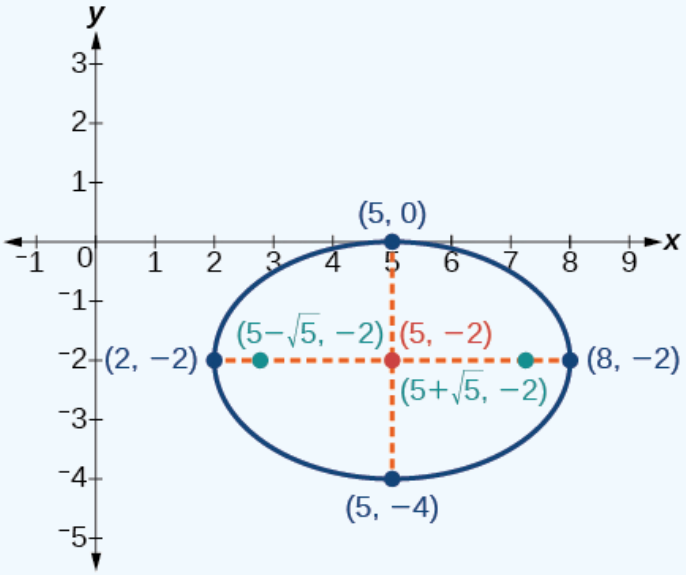



Figure 6.3.15

## Exercise 6.3.6

Express the equation of the ellipse given in standard form. Identify the center, vertices, co-vertices, and foci of the ellipse.

$$4x^2 + y^2 - 24x + 2y + 21 = 0$$

*Answer available at the end of this section.*

## Solving Applied Problems Involving Ellipses

Many real-world situations can be represented by ellipses, including orbits of planets, satellites, moons and comets, and shapes of boat keels, rudders, and some airplane wings. A medical device called a lithotripter uses elliptical reflectors to break up kidney stones by generating sound waves. Some buildings, called whispering chambers, are designed with elliptical domes so that a person whispering at one focus can easily be heard by someone standing at the other focus. This occurs because of the acoustic properties of an ellipse. When a sound wave originates at one focus of a whispering chamber, the sound wave will be reflected off the elliptical dome and back to the other focus (Figure $6.3.15$). In the whisper chamber at the Museum of Science and Industry in Chicago, two people standing at the foci—about $43$ feet apart—can hear each other whisper.

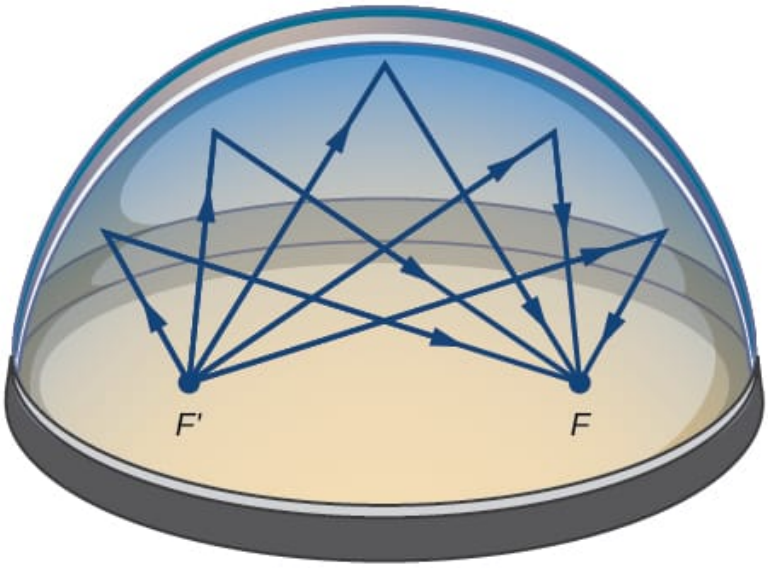



Figure $6.3.16$: Sound waves are reflected between foci in an elliptical room, called a whispering chamber.

### Example $6.3.7$: Locating the Foci of a Whispering Chamber

The Statuary Hall in the Capitol Building in Washington, D.C. is a whispering chamber. Its dimensions are $46$ feet wide by $96$ feet long as shown in Figure $6.3.16$.

a. What is the standard form of the equation of the ellipse representing the outline of the room? ***Hint:*** assume a horizontal ellipse, and let the center of the room be the point $(0, 0)$.
b. If two senators standing at the foci of this room can hear each other whisper, how far apart are the senators? Round to the nearest foot.

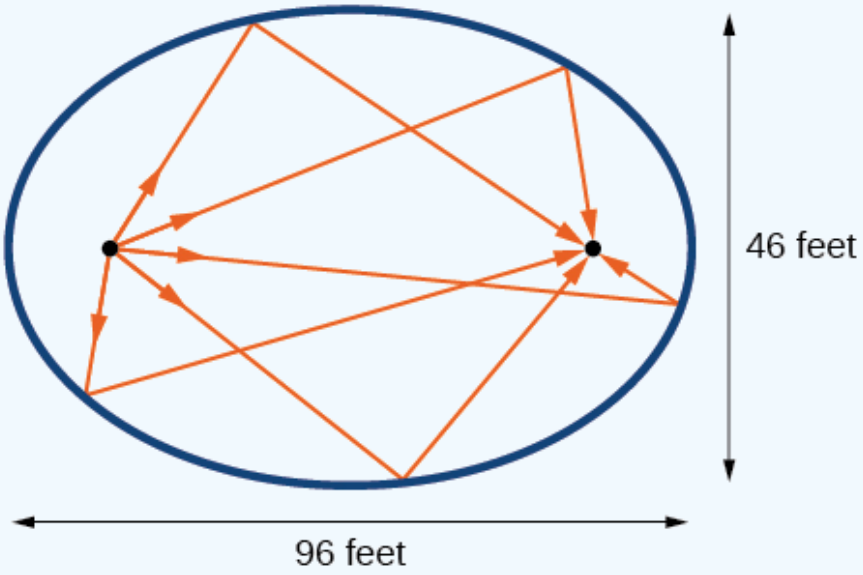



Figure 6.3.17

**Solution**

a. We are assuming a horizontal ellipse with center $(0,0)$, so we need to find an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b$. We know that the length of the major axis, $2a$, is longer than the length of the minor axis, $2b$. So the length of the room, 96, is represented by the major axis, and the width of the room, 46, is represented by the minor axis.

   Therefore, the equation of the ellipse is \[(dfrac{x^2}{2304}+\dfrac{y^2}{529}=1\)

   - Solving for $a$, we have $2a=96$, so $a=48$, and $a^2=2304$.
   - Solving for $b$, we have $2b=46$, so $b=23$, and $b^2=529$.

b. To find the distance between the senators, we must find the distance between the foci, $(\pm c,0)$, where $c^2=a^2-b^2$. Solving for $c$,we have:

$$\begin{aligned} c^2 &= a^2-b^2 \\ c^2 &= 2304-529 \quad \text{Substitute using the values found in part } (a) \\ c &= \pm\sqrt{2304-529} \quad \text{Take the square root of both sides.} \\ c &= \pm\sqrt{1775} \quad \text{Subtract.} \\ c &\approx \pm 42 \quad \text{Round to the nearest foot.} \end{aligned}$$

The points $(\pm 42,0)$ represent the foci. Thus, the distance between the senators is $2(42)=84$ feet.

## Exercise 6.3.7

Suppose a whispering chamber is $480$ feet long and $320$ feet wide.

a. What is the standard form of the equation of the ellipse representing the room? Hint: assume a horizontal ellipse, and let the center of the room be the point $(0,0)$.
b. If two people are standing at the foci of this room and can hear each other whisper, how far apart are the people? Round to the nearest foot.

*Answer a available at the end of this section.*

*Answer b available at the end of this section.*

## Media

Access these online resources for additional instruction and practice with ellipses.

- Conic Sections: The Ellipse
- Graph an Ellipse with Center at the Origin
- Graph an Ellipse with Center Not at the Origin

# Key Equations

| | |
|---|---|
| Horizontal ellipse, center at origin | $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$ |
| Vertical ellipse, center at origin | $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,\ a>b$ |
| Horizontal ellipse, center $(h,k)$ | $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,\ a>b$ |
| Vertical ellipse, center $(h,k)$ | $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,\ a>b$ |

# Key Concepts

- An ellipse is the set of all points $(x,y)$ in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a focus (plural: foci).
- When given the coordinates of the foci and vertices of an ellipse, we can write the equation of the ellipse in standard form. See Example $6.3.1$ and Example $6.3.2$.
- When given an equation for an ellipse centered at the origin in standard form, we can identify its vertices, co-vertices, foci, and the lengths and positions of the major and minor axes in order to graph the ellipse. See Example $6.3.3$ and Example $6.3.4$.
- When given the equation for an ellipse centered at some point other than the origin, we can identify its key features and graph the ellipse. See Example $6.3.5$ and Example $6.3.6$.
- Real-world situations can be modeled using the standard equations of ellipses and then evaluated to find key features, such as lengths of axes and distance between foci. See Example $6.3.7$.

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## Answers

**Exercise** $6.3.1$

$x^2+\frac{y^2}{16}=1$

**Exercise** $6.3.2$

$\frac{(x-1)^2}{16}+\frac{(y-3)^2}{4}=1$

**Exercise** $6.3.3$

center: $(0,0)$; vertices: $(\pm 6,0)$; co-vertices: $(0,\pm 2)$; foci: $(\pm 4\sqrt{2},0)$

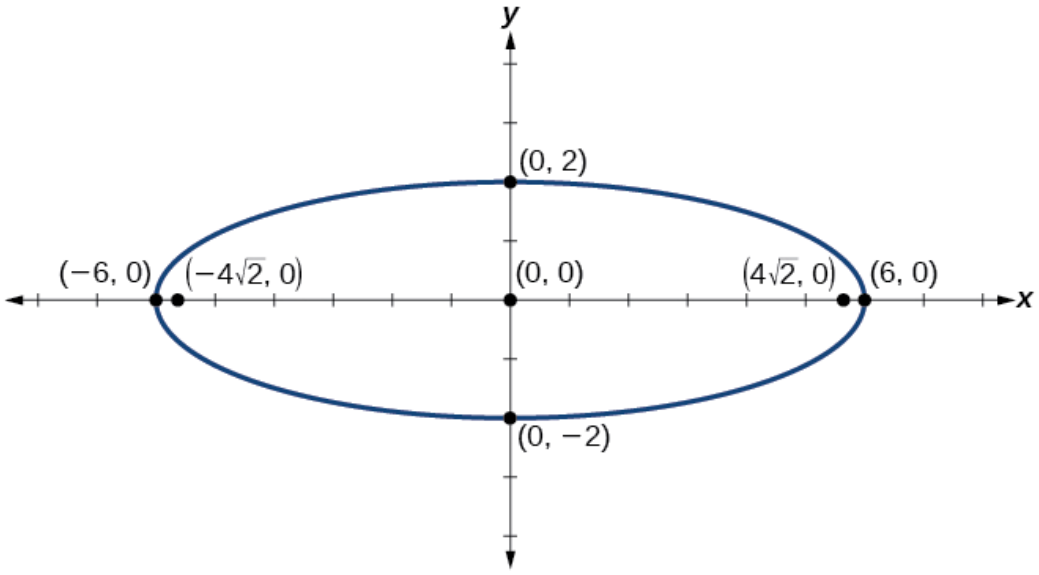



Figure 6.3.10

**Exercise** 6.3.4

Standard form: $\frac{x^2}{16}+\frac{y^2}{49}=1$; center: $(0,0)$; vertices: $(0,\pm 7)$; co-vertices: $(\pm 4,0)$; foci: $(0,\pm\sqrt{33})$

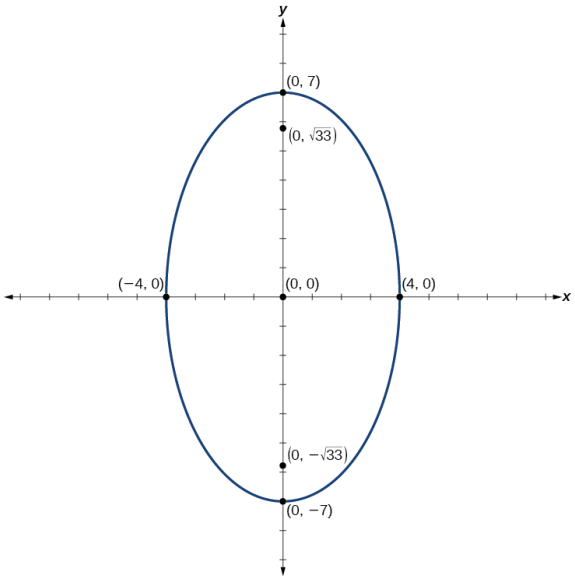



Figure 6.3.12

**Exercise** 6.3.5

Center: $(4,2)$; vertices: $(-2,2)$ and $(10,2)$; co-vertices: $(4,2-2\sqrt{5})$ and $(4,2+2\sqrt{5})$; foci: $(0,2)$ and $(8,2)$

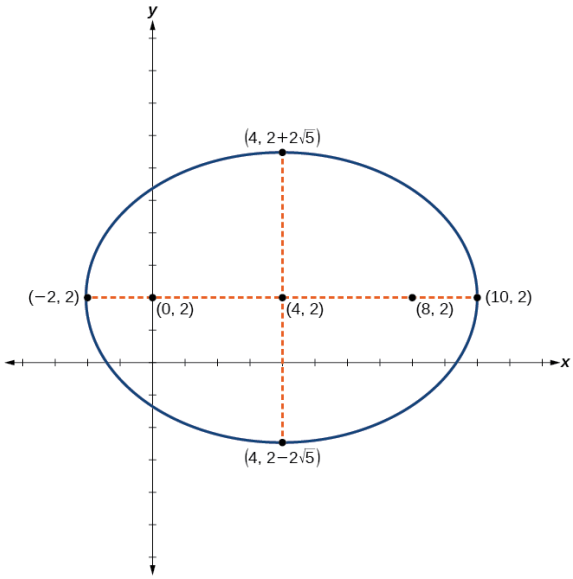



Figure 6.3.14

**Exercise** 6.3.6

$\frac{(x-3)^2}{4}+\frac{(y+1)^2}{16}=1$; center: $(3,-1)$; vertices: $(3,-5)$ and $(3,3)$; co-vertices: $(1,-1)$ and $(5,-1)$; foci: $(3,-1-2\sqrt{3})$ and $(3,-1+2\sqrt{3})$

**Exercise** 6.3.7

$\frac{x^2}{57,600}+\frac{y^2}{25,600}=1$

**Exercise** 6.3.7

The people are standing $358$ feet apart.

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# 6.4: The Parabola

**Learning Objectives**

- Graph parabolas with vertices at the origin.
- Write equations of parabolas in standard form.
- Graph parabolas with vertices not at the origin.
- Solve applied problems involving parabolas.

Did you know that the Olympic torch is lit several months before the start of the games? The ceremonial method for lighting the flame is the same as in ancient times. The ceremony takes place at the Temple of Hera in Olympia, Greece, and is rooted in Greek mythology, paying tribute to Prometheus, who stole fire from Zeus to give to all humans. One of eleven acting priestesses places the torch at the focus of a parabolic mirror (Figure 6.4.1), which focuses light rays from the sun to ignite the flame.



Figure 6.4.1: The Olympic torch concludes its journey around the world when it is used to light the Olympic cauldron during the opening ceremony. (credit: Ken Hackman, U.S. Air Force)

Parabolic mirrors (or reflectors) are able to capture energy and focus it to a single point. The advantages of this property are evidenced by the vast list of parabolic objects we use every day: satellite dishes, suspension bridges, telescopes, microphones, spotlights, and car headlights, to name a few. Parabolic reflectors are also used in alternative energy devices, such as solar cookers and water heaters, because they are inexpensive to manufacture and need little maintenance. In this section we will explore the parabola and its uses, including low-cost, energy-efficient solar designs.

## Graphing Parabolas with Vertices at the Origin

Previously, we saw that an ellipse is formed when a plane cuts through a right circular cone. If the plane is parallel to the edge of the cone, an unbounded curve is formed. This curve is a *parabola* (Figure 6.4.2).

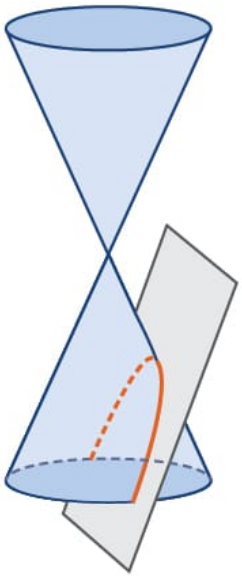


Figure 6.4.2: Parabola

Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane. A parabola is the set of all points $(x, y)$ in a plane that are the same distance from a fixed line, called the *directrix*, and a fixed point (the *focus*) not on the directrix.

Previously, we learned about a parabola's vertex and axis of symmetry. Now we extend the discussion to include other key features of the parabola (Figure 6.4.3). Notice that the axis of symmetry passes through the focus and vertex and is perpendicular to the directrix. The vertex is the midpoint between the directrix and the focus. The line segment that passes through the focus and is parallel to the directrix is called the *latus rectum*. The endpoints of the latus rectum lie on the curve. By definition, the distanced d from the focus to any point $P$ on the parabola is equal to the distance from $P$ to the directrix.

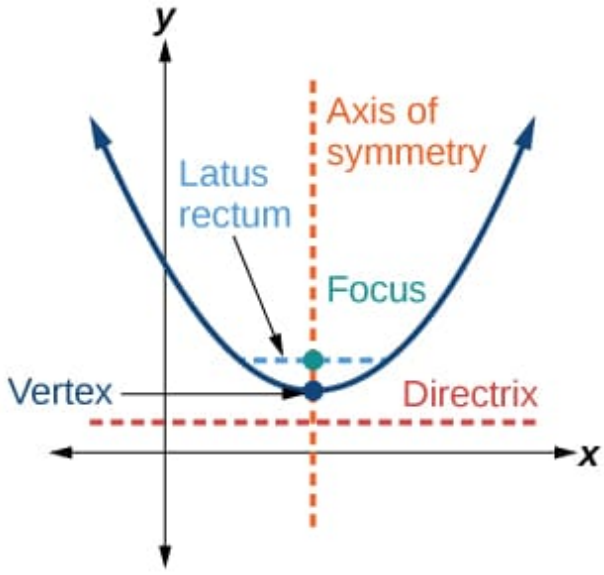



Figure 6.4.3: Key features of the parabola

To work with parabolas in the *coordinate plane*, we consider two cases: those with a vertex at the origin and those with a *vertex* at a point other than the origin. We begin with the former.

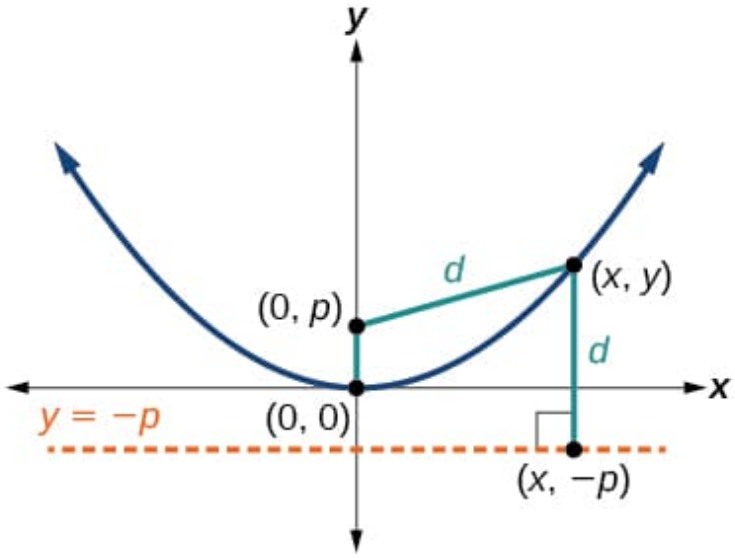



Figure 6.4.4

Let $(x, y)$ be a point on the parabola with vertex $(0, 0)$, focus $(0, p)$,and directrix $y = -p$ as shown in Figure 6.4.4. The distanced d from point $(x, y)$ to point $(x, -p)$ on the directrix is the difference of the *y*-values: $d = y + p$. The distance from the focus $(0, p)$ to the point $(x, y)$ is also equal to $d$ and can be expressed using the distance formula.

$$\begin{aligned} d &= \sqrt{(x-0)^2 + (y-p)^2} \\ &= \sqrt{x^2 + (y-p)^2} \end{aligned}$$

Set the two expressions for $d$ equal to each other and solve for $y$ to derive the equation of the parabola. We do this because the distance from $(x, y)$ to $(0, p)$ equals the distance from $(x, y)$ to $(x, -p)$.

$$\sqrt{x^2 + (y-p)^2} = y + p \tag{6.4.1}$$

We then square both sides of the equation, expand the squared terms, and simplify by combining like terms.

$$\begin{aligned} x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\ x^2 - 2py &= 2py \\ x^2 &= 4py \end{aligned}$$

The equations of parabolas with vertex $(0, 0)$ are $y^2 = 4px$ when the *x*-axis is the axis of symmetry and $x^2 = 4py$ when the *y*-axis is the axis of symmetry. These standard forms are given below, along with their general graphs and key features.

## STANDARD FORMS OF PARABOLAS WITH VERTEX $(0, 0)$

Table 6.4.1 and Figure 6.4.5 summarize the standard features of parabolas with a vertex at the origin.

*Table* 6.4.1

| Axis of Symmetry | Equation | Focus | Directrix | Endpoints of Latus Rectum |
|---|---|---|---|---|
| *x*-axis | $y^2 = 4px$ | $(p, 0)$ | $x = -p$ | $(p, \pm 2p)$ |
| *y*-axis | $x^2 = 4py$ | $(0, p)$ | $y = -p$ | $(\pm 2p, p)$ |

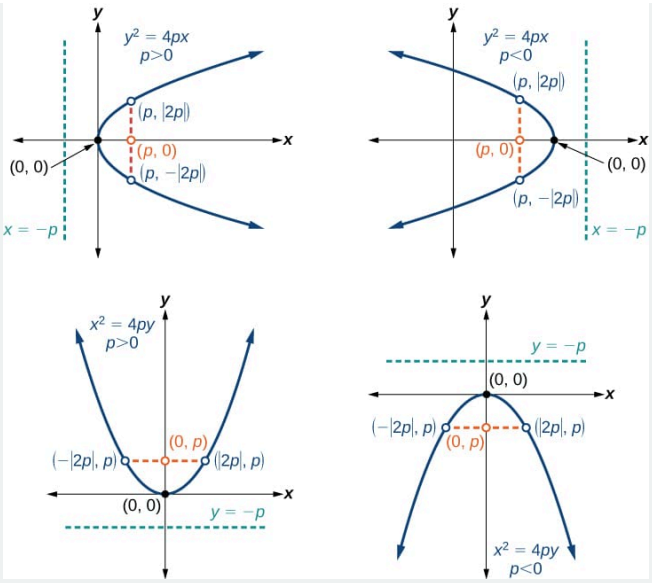



Figure $6.4.5$: (a) When $p > 0$ and the axis of symmetry is the x-axis, the parabola opens right. (b) When $p < 0$ and the axis of symmetry is the x-axis, the parabola opens left. (c) When $p < 0$ and the axis of symmetry is the y-axis, the parabola opens up. (d) When $p < 0$ and the axis of symmetry is the y-axis, the parabola opens down.

The key features of a parabola are its vertex, axis of symmetry, focus, directrix, and *latus rectum* (Figure $6.4.5$). When given a standard equation for a parabola centered at the origin, we can easily identify the key features to graph the parabola. A line is said to be tangent to a curve if it intersects the curve at exactly one point. If we sketch lines tangent to the parabola at the endpoints of the *latus rectum*, these lines intersect on the axis of symmetry, as shown in Figure $6.4.6$.

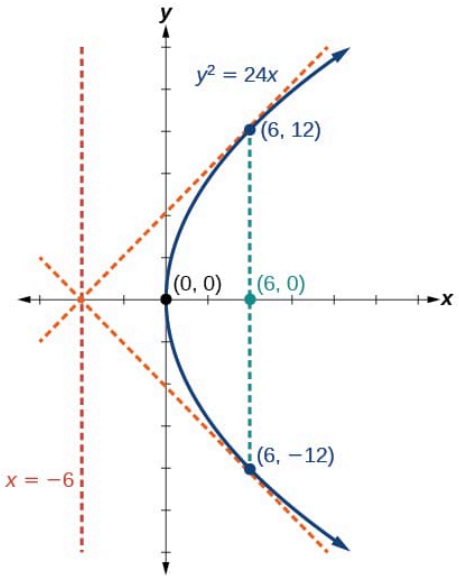



Figure $6.4.6$

## How to: Given a standard form equation for a parabola centered at $(0, 0)$, sketch the graph

1. Determine which of the standard forms applies to the given equation: $y^2 = 4px$ or $x^2 = 4py$.
2. Use the standard form identified in Step 1 to determine the axis of symmetry, focus, equation of the directrix, and endpoints of the latus rectum.
   - If the equation is in the form $y^2 = 4px$, then
     - the axis of symmetry is the $x$-axis, $y = 0$

- set $4p$ equal to the coefficient of $x$ in the given equation to solve for $p$. If $p > 0$, the parabola opens right. If $p < 0$ , the parabola opens left.
- use $p$ to find the coordinates of the focus, $(p, 0)$
- use $p$ to find the equation of the directrix, $x = -p$
- use $p$ to find the endpoints of the latus rectum, $(p, \pm 2p)$. Alternately, substitute $x = p$ into the original equation.

◦ If the equation is in the form $x^2 = 4py$,then

- the axis of symmetry is the $y$-axis, $x = 0$
- set $4p$ equal to the coefficient of $y$ in the given equation to solve for $p$. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down.
- use $p$ to find the coordinates of the focus, $(0, p)$
- use $p$ to find equation of the directrix, $y = -p$
- use $p$ to find the endpoints of the latus rectum, $(\pm 2p, p)$

3. Plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

## -axis as the Axis of Symmetry

Graph $y^2 = 24x$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

**Solution**

The standard form that applies to the given equation is $y^2 = 4px$. Thus, the axis of symmetry is the *x*-axis. It follows that:

- $24 = 4p$, so $p = 6$. Since $p > 0$, the parabola opens right
- the coordinates of the focus are $(p, 0) = (6, 0)$
- the equation of the directrix is $x = -p = -6$
- the endpoints of the latus rectum have the same *x*-coordinate at the focus. To find the endpoints, substitute $x = 6$ into the original equation: $(6, \pm 12)$

Next we plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola (Figure 6.4.7).

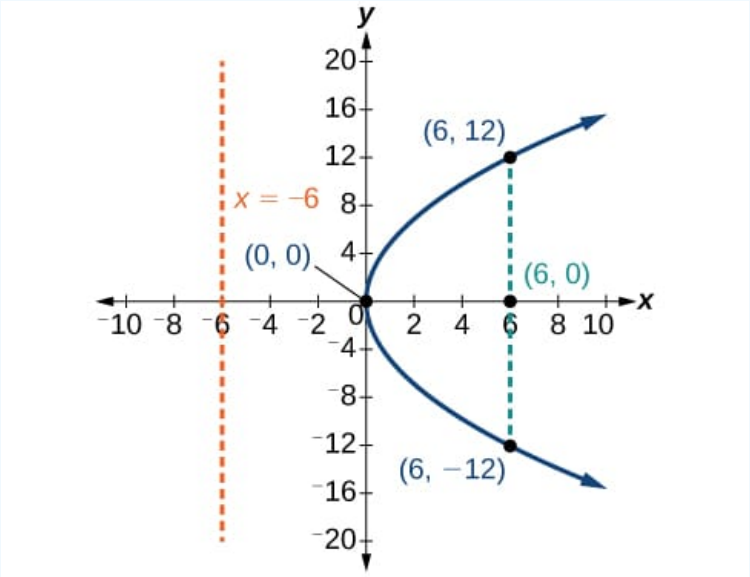



Figure 6.4.7

## Exercise 6.4.1

Graph $y^2 = -16x$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

*Answer available at the end of this section.*

## -axis as the Axis of Symmetry

Graph $x^2 = -6y$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

**Solution**

The standard form that applies to the given equation is $x^2 = 4py$. Thus, the axis of symmetry is the $y$-axis. It follows that:

- $-6 = 4p$,so $p = -\frac{3}{2}$. Since $p < 0$, the parabola opens down.
- the coordinates of the focus are $(0, p) = (0, -\frac{3}{2})$
- the equation of the directrix is $y = -p = \frac{3}{2}$
- the endpoints of the latus rectum can be found by substituting $y = \frac{3}{2}$ into the original equation, $(\pm 3, -\frac{3}{2})$

Next we plot the focus, directrix, and *latus rectum*, and draw a smooth curve to form the parabola.

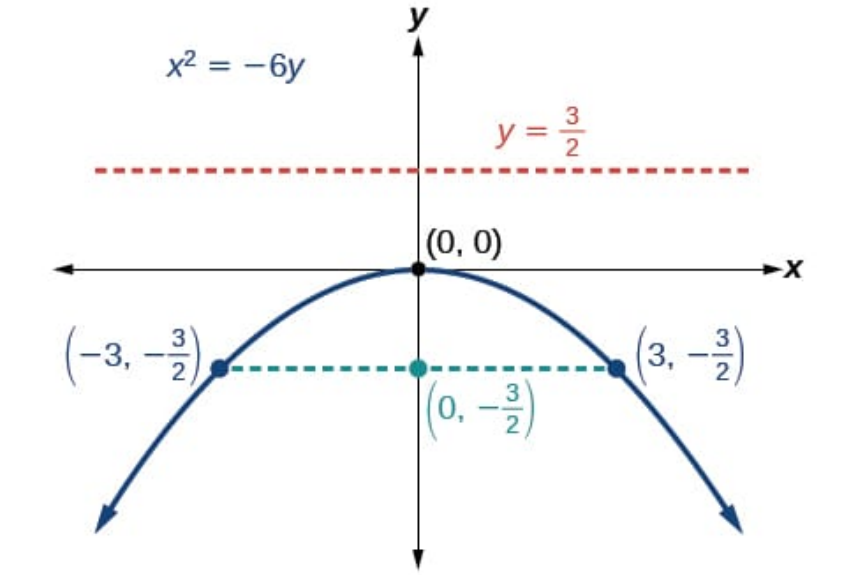



Figure 6.4.9

## Exercise 6.4.2

Graph $x^2 = 8y$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

*Answer available at the end of this section.*

# Writing Equations of Parabolas in Standard Form

In the previous examples, we used the standard form equation of a parabola to calculate the locations of its key features. We can also use the calculations in reverse to write an equation for a parabola when given its key features.

### How to: Given its focus and directrix, write the equation for a parabola in standard form

1. Determine whether the axis of symmetry is the $x$- or $y$-axis.
   i. If the given coordinates of the focus have the form $(p, 0)$, then the axis of symmetry is the $x$-axis. Use the standard form $y^2 = 4px$.

ii. If the given coordinates of the focus have the form $(0, p)$, then the axis of symmetry is the $y$-axis. Use the standard form $x^2 = 4py$.
2. Multiply $4p$.
3. Substitute the value from Step 2 into the equation determined in Step 1.

### Example 6.4.3: Writing the Equation of a Parabola in Standard Form Given its Focus and Directrix

What is the equation for the parabola with focus $(-\frac{1}{2}, 0)$ and directrix $x = \frac{1}{2}$?

**Solution**

The focus has the form $(p, 0)$, so the equation will have the form $y^2 = 4px$.

- Multiplying $4p$, we have $4p = 4(-\frac{1}{2}) = -2$.
- Substituting for $4p$, we have $y^2 = 4px = -2x$.=

Therefore, the equation for the parabola is $y^2 = -2x$.

### Exercise 6.4.3

What is the equation for the parabola with focus $\left(0, \frac{7}{2}\right)$ and directrix $y = -\frac{7}{2}$?

*Answer available at the end of this section.*

## Graphing Parabolas with Vertices Not at the Origin

Like other graphs we've worked with, the graph of a parabola can be translated. If a parabola is translated $h$ units horizontally and $k$ units vertically, the vertex will be $(h, k)$. This translation results in the standard form of the equation we saw previously with $x$ replaced by $(x - h)$ and $y$ replaced by $(y - k)$.

To graph parabolas with a vertex $(h, k)$ other than the origin, we use the standard form $(y - k)^2 = 4p(x - h)$ for parabolas that have an axis of symmetry parallel to the $x$-axis, and $(x - h)^2 = 4p(y - k)$ for parabolas that have an axis of symmetry parallel to the $y$ -axis. These standard forms are given below, along with their general graphs and key features.

### STANDARD FORMS OF PARABOLAS WITH VERTEX $(H, K)$

Table 6.4.2 and Figure 6.4.11 summarize the standard features of parabolas with a vertex at a point $(h, k)$.

*Table* 6.4.2

| Axis of Symmetry | Equation | Focus | Directrix | Endpoints of Latus Rectum |
|---|---|---|---|---|
| $y = k$ | $(y - k)^2 = 4p(x - h)$ | $(h + p, k)$ | $x = h - p$ | $(h + p, k \pm 2p)$ |
| $x = h$ | $(x - h)^2 = 4p(y - k)$ | $(h, k + p)$ | $y = k - p$ | $(h \pm 2p, k + p)$ |

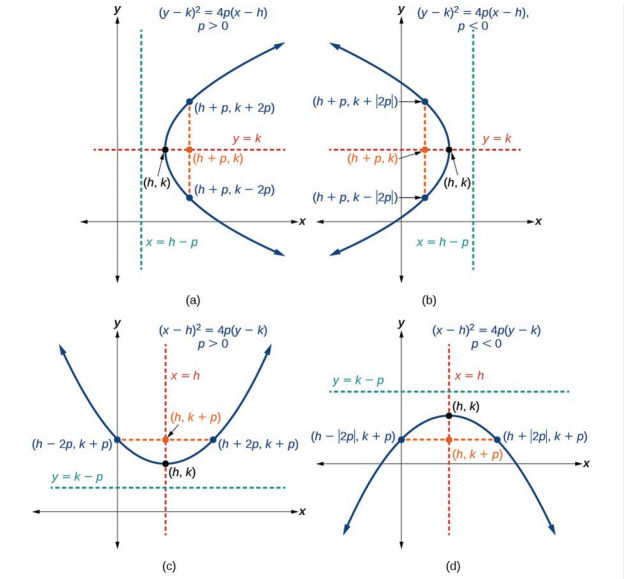



Figure 6.4.11: (a) When $p > 0$, the parabola opens right. (b) When $p < 0$, the parabola opens left. (c) When $p > 0$,the parabola opens up. (d) When $p < 0$, the parabola opens down.

## How to: Given a standard form equation for a parabola centered at $(h, k)$, sketch the graph

1. Determine which of the standard forms applies to the given equation: $(y-k)^2 = 4p(x-h)$ or $(x-h)^2 = 4p(y-k)$.
2. Use the standard form identified in Step 1 to determine the vertex, axis of symmetry, focus, equation of the directrix, and endpoints of the latus rectum.
   - If the equation is in the form $(y-k)^2 = 4p(x-h)$,then:
     - use the given equation to identify $h$ and $k$ for the vertex, $(h, k)$
     - use the value of $k$ to determine the axis of symmetry, $y = k$
     - set $4p$ equal to the coefficient of $(x-h)$ in the given equation to solve for $p$. If $p > 0$,the parabola opens right. If $p < 0$, the parabola opens left.
     - use $h$, $k$, and $p$ to find the coordinates of the focus, $(h+p, k)$
     - use $h$ andp p to find the equation of the directrix, $x = h - p$
     - use $h$, $k$, and $p$ to find the endpoints of the latus rectum, $(h+p, k \pm 2p)$
   - If the equation is in the form $(x-h)^2 = 4p(y-k)$,then:
     - use the given equation to identify $h$ and $k$ for the vertex, $(h, k)$
     - use the value of $h$ to determine the axis of symmetry, $x = h$
     - set $4p$ equal to the coefficient of $(y-k)$ in the given equation to solve for $p$. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down.
     - use $h$, $k$, and $p$ to find the coordinates of the focus, $(h, k+p)$
     - use $k$ and $p$ to find the equation of the directrix, $y = k - p$
     - use $h$, $k$, and $p$ to find the endpoints of the latus rectum, $(h \pm 2p, k+p)$
3. Plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

## Example 6.4.4: Graphing a Parabola with Vertex $(h, k)$ and Axis of Symmetry Parallel to the $x$-axis

Graph $(y-1)^2 = -16(x+3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

**Solution**

The standard form that applies to the given equation is $(y-k)^2 = 4p(x-h)$. Thus, the axis of symmetry is parallel to the $x$ -axis. It follows that:

- the vertex is $(h, k) = (-3, 1)$
- the axis of symmetry is $y = k = 1$

- $-16 = 4p$,so $p = -4$. Since $p < 0$, the parabola opens left.
- the coordinates of the focus are $(h + p, k) = (-3 + (-4), 1) = (-7, 1)$
- the equation of the directrix is $x = h - p = -3 - (-4) = 1$
- the endpoints of the latus rectum are $(h + p, k \pm 2p) = (-3 + (-4), 1 \pm 2(-4))$, or $(-7, -7)$ and $(-7, 9)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola (Figure 6.4.10).

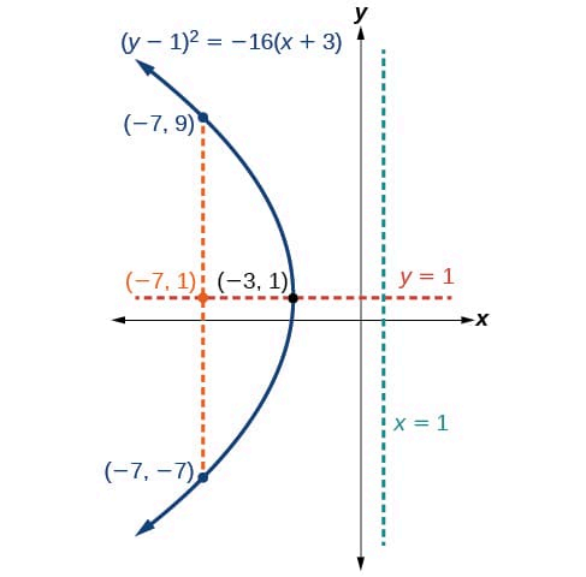



Figure 6.4.12

## Exercise 6.4.4

Graph $(y + 1)^2 = 4(x - 8)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the *latus rectum*.

*Answer available at the end of this section.*

## Example 6.4.5: Graphing a Parabola from an Equation Given in General Form

Graph $x^2 - 8x - 28y - 208 = 0$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the *latus rectum*.

**Solution**

Start by writing the equation of the parabola in standard form. The standard form that applies to the given equation is $(x - h)^2 = 4p(y - k)$. Thus, the axis of symmetry is parallel to the $y$-axis. To express the equation of the parabola in this form, we begin by isolating the terms that contain the variable $x$ in order to complete the square.

$$\begin{aligned} x^2 - 8x - 28y - 208 &= 0 \\ x^2 - 8x &= 28y + 208 \\ x^2 - 8x + 16 &= 28y + 208 + 16 \\ (x - 4)^2 &= 28y + 224 \\ (x - 4)^2 &= 28(y + 8) \\ (x - 4)^2 &= 4 \cdot 7 \cdot (y + 8) \end{aligned}$$

It follows that:

- the vertex is $(h, k) = (4, -8)$
- the axis of symmetry is $x = h = 4$
- since $p = 7$, $p > 0$ and so the parabola opens up
- the coordinates of the focus are $(h, k + p) = (4, -8 + 7) = (4, -1)$
- the equation of the directrix is $y = k - p = -8 - 7 = -15$
- the endpoints of the latus rectum are $(h \pm 2p, k + p) = (4 \pm 2(7), -8 + 7)$, or $(-10, -1)$ and $(18, -1)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola (Figure 6.4.14).

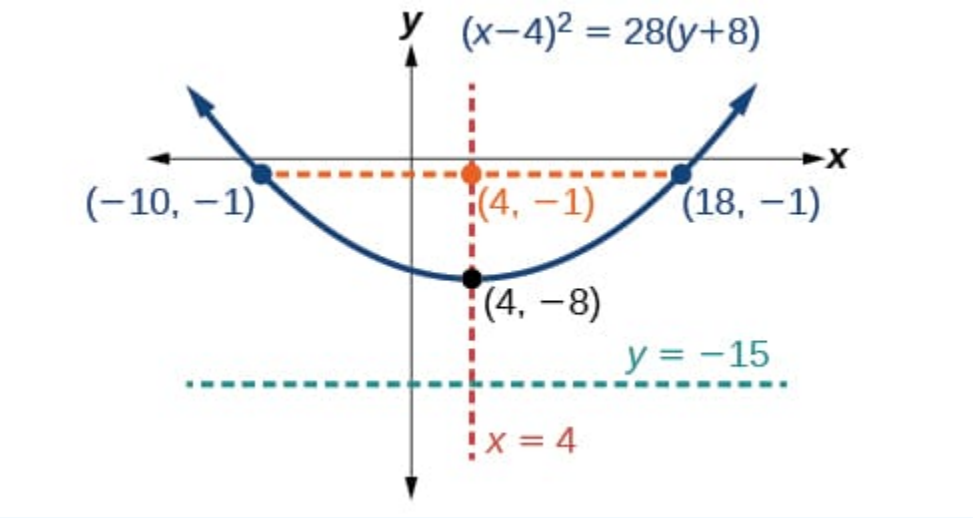



Figure 6.4.14

**Exercise 6.4.5**

Graph $(x + 2)^2 = -20(y - 3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the *latus rectum*.

*Answer available at the end of this section.*

## Solving Applied Problems Involving Parabolas

As we mentioned at the beginning of the section, parabolas are used to design many objects we use every day, such as telescopes, suspension bridges, microphones, and radar equipment. Parabolic mirrors, such as the one used to light the Olympic torch, have a very unique reflecting property. When rays of light parallel to the parabola's axis of symmetry are directed toward any surface of the mirror, the light is reflected directly to the focus (Figure 6.4.16). This is why the Olympic torch is ignited when it is held at the focus of the parabolic mirror.

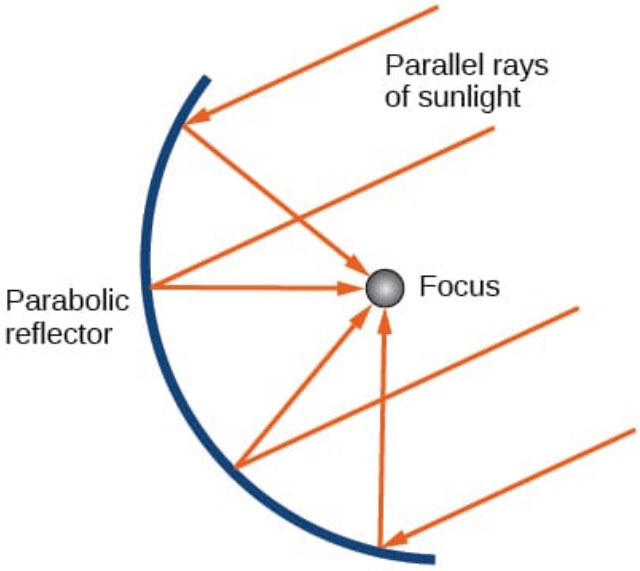



Figure 6.4.16: Reflecting property of parabolas

Parabolic mirrors have the ability to focus the sun's energy to a single point, raising the temperature hundreds of degrees in a matter of seconds. Thus, parabolic mirrors are featured in many low-cost, energy efficient solar products, such as solar cookers, solar heaters, and even travel-sized fire starters.

## Example 6.4.6: Solving Applied Problems Involving Parabolas

A cross-section of a design for a travel-sized solar fire starter is shown in Figure 6.4.17. The sun's rays reflect off the parabolic mirror toward an object attached to the igniter. Because the igniter is located at the focus of the parabola, the reflected rays cause the object to burn in just seconds.

a. Find the equation of the parabola that models the fire starter. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane.
b. Use the equation found in part (a) to find the depth of the fire starter.

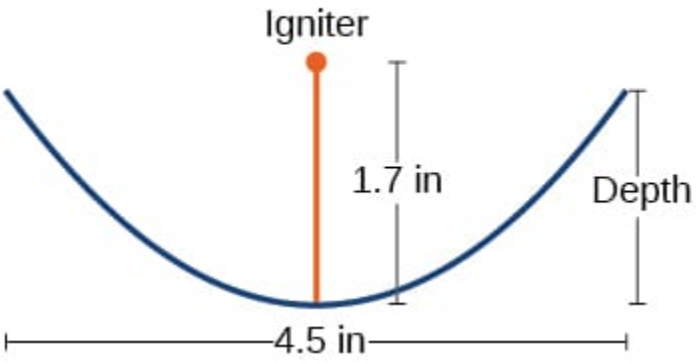



Figure 6.4.17 Cross-section of a travel-sized solar fire starter

**Solution**

a. The vertex of the dish is the origin of the coordinate plane, so the parabola will take the standard form $x^2 = 4py$,where $p > 0$. The igniter, which is the focus, is $1.7$ inches above the vertex of the dish. Thus we have $p = 1.7$.

$$
\begin{aligned}
x^2 &= 4py && \text{Standard form of upward-facing parabola with vertex } (0,0)\\
x^2 &= 4(1.7)y && \text{Substitute } 1.7 \text{ for } p\\
x^2 &= 6.8y && \text{Multiply.}
\end{aligned}
$$

b. The dish extends $\frac{4.5}{2} = 2.25$ inches on either side of the origin. We can substitute $2.25$ for $x$ in the equation from part (a) to find the depth of the dish.

$$
\begin{aligned}
x^2 &= 6.8y && \text{Equation found in part } (a)\\
(2.25)^2 &= 6.8y && \text{Substitute } 2.25 \text{ for } x\\
y &\approx 0.74 && \text{Solve for } y
\end{aligned}
$$

The dish is about $0.74$ inches deep.

### Exercise 6.4.6

Balcony-sized solar cookers have been designed for families living in India. The top of a dish has a diameter of $1600$ mm. The sun's rays reflect off the parabolic mirror toward the "cooker," which is placed $320$ mm from the base.

a. Find an equation that models a cross-section of the solar cooker. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane, and that the parabola opens to the right (i.e., has the *x*-axis as its axis of symmetry).
b. Use the equation found in part (a) to find the depth of the cooker.

*Answer a available at the end of this section.*

## Key Equations

| | |
|---|---|
| Parabola, vertex at origin, axis of symmetry on *x*-axis | $y^2 = 4px$ |
| Parabola, vertex at origin, axis of symmetry on *y*-axis | $x^2 = 4py$ |
| Parabola, vertex at $(h,k)$,axis of symmetry on *x*-axis | $(y-k)^2 = 4p(x-h)$ |
| Parabola, vertex at $(h,k)$,axis of symmetry on *y*-axis | $(x-h)^2 = 4p(y-k)$ |

## Key Concepts

- A parabola is the set of all points $(x,y)$ in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the directrix.
- The standard form of a parabola with vertex $(0,0)$ and the *x*-axis as its axis of symmetry can be used to graph the parabola. If $p>0$, the parabola opens right. If $p<0$, the parabola opens left. See Example $6.4.1$.
- The standard form of a parabola with vertex $(0,0)$ and the *y*-axis as its axis of symmetry can be used to graph the parabola. If $p>0$, the parabola opens up. If $p<0$, the parabola opens down. See Example $6.4.2$.
- When given the focus and directrix of a parabola, we can write its equation in standard form. See Example $6.4.3$.
- The standard form of a parabola with vertex $(h,k)$ and axis of symmetry parallel to the $x$-axis can be used to graph the parabola. If $p>0$, the parabola opens right. If $p<0$, the parabola opens left. See Example $6.4.4$.
- The standard form of a parabola with vertex $(h,k)$ and axis of symmetry parallel to the $y$-axis can be used to graph the parabola. If $p>0$, the parabola opens up. If $p<0$, the parabola opens down. See Example $6.4.5$.
- Real-world situations can be modeled using the standard equations of parabolas. For instance, given the diameter and focus of a cross-section of a parabolic reflector, we can find an equation that models its sides. See Example $6.4.6$.

### Answers

**Exercise** $6.4.1$

- Focus: $(-4,0)$
- Directrix: $x=4$

- Endpoints of the latus rectum: $(-4, \pm 8)$

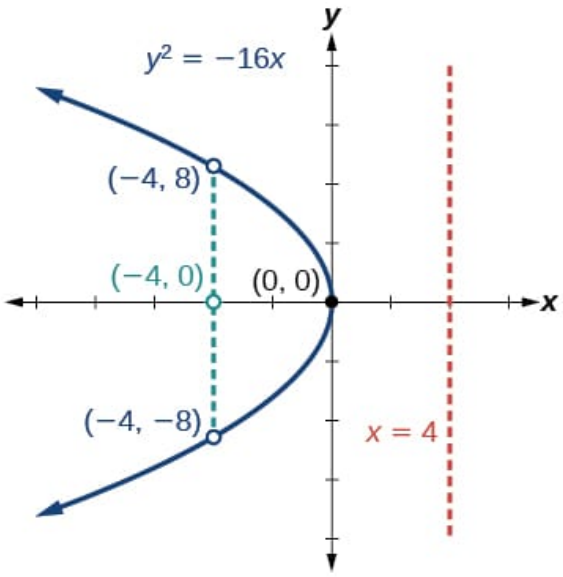



Figure 6.4.8

**Exercise** 6.4.2

- Focus: $(0, 2)$
- Directrix: $y = -2$
- Endpoints of the latus rectum: $(\pm 4, 2)$.

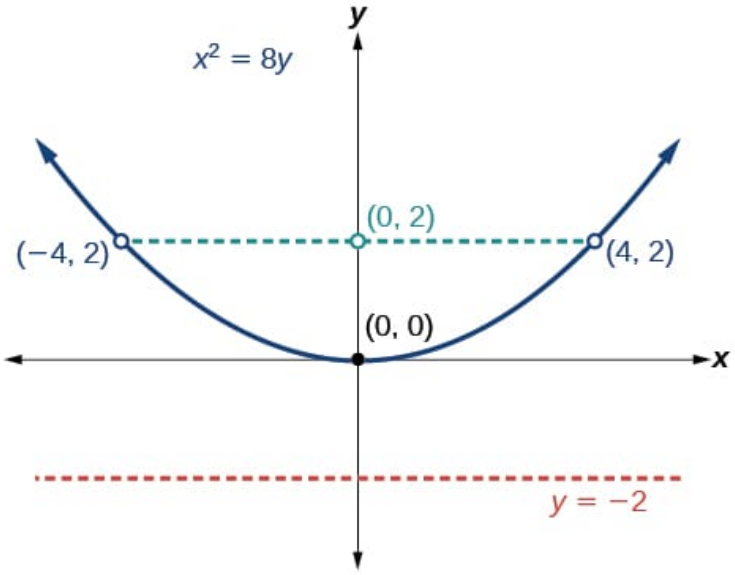



Figure 6.4.10

**Exercise** 6.4.3

$x^2 = 14y$.

**Exercise** 6.4.4

- Vertex: $(8, -1)$
- Axis of symmetry: $y = -1$
- Focus: $(9, -1)$
- Directrix: $x = 7$
- Endpoints of the *latus rectum*: $(9, -3)$ and $(9, 1)$.

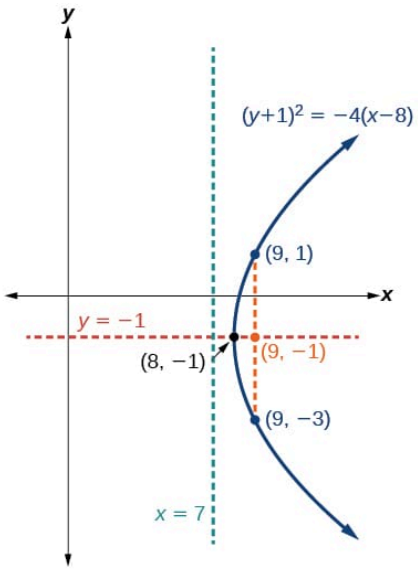



Figure 6.4.13

**Exercise** 6.4.5

- Vertex: $(-2, 3)$
- Axis of symmetry: $x = -2$
- Focus: $(-2, -2)$
- Directrix: $y = 8$
- Endpoints of the *latus rectum*: $(-12, -2)$ and $(8, -2)$.

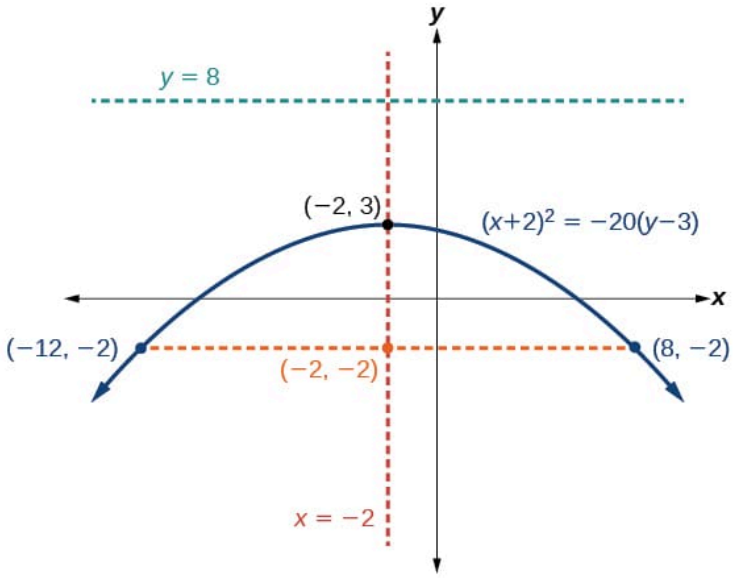



Figure 6.4.15

**Exercise** 6.4.6

$y^2 = 1280x$

The depth of the cooker is $500$ mm

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# 6.5: The Hyperbola

**Learning Objectives**

- Locate a hyperbola's vertices and foci.
- Write equations of hyperbolas in standard form.
- Graph hyperbolas centered at the origin.
- Graph hyperbolas not centered at the origin.
- Solve applied problems involving hyperbolas.

What do paths of comets, supersonic booms, ancient Grecian pillars, and natural draft cooling towers have in common? They can all be modeled by the same type of **conic**. For instance, when something moves faster than the speed of sound, a shock wave in the form of a cone is created. A portion of a conic is formed when the wave intersects the ground, resulting in a sonic boom (Figure 6.5.1).

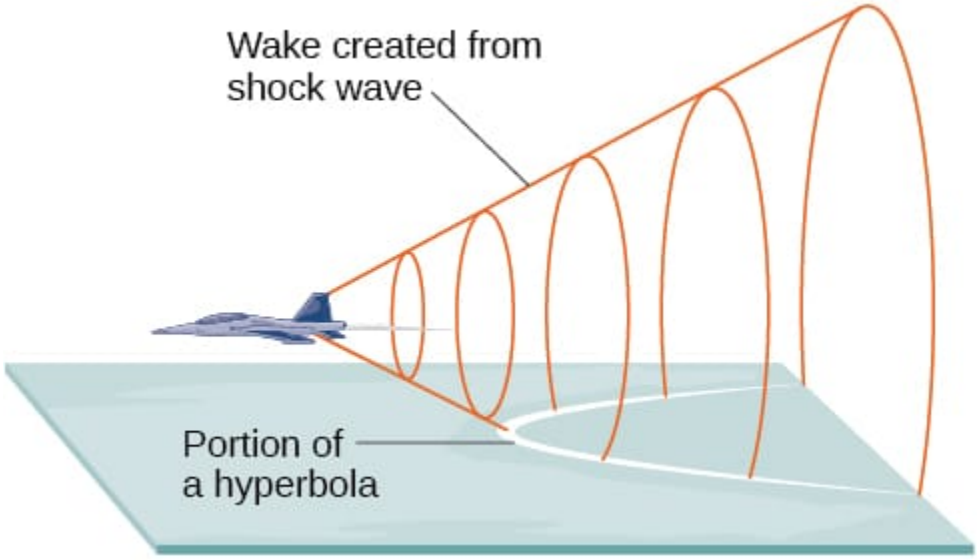



Figure 6.5.1: A shock wave intersecting the ground forms a portion of a conic and results in a sonic boom.

Most people are familiar with the sonic boom created by supersonic aircraft, but humans were breaking the sound barrier long before the first supersonic flight. The crack of a whip occurs because the tip is exceeding the speed of sound. The bullets shot from many firearms also break the sound barrier, although the bang of the gun usually supersedes the sound of the sonic boom.

## Locating the Vertices and Foci of a Hyperbola

In analytic geometry, a **hyperbola** is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other (Figure 6.5.2).

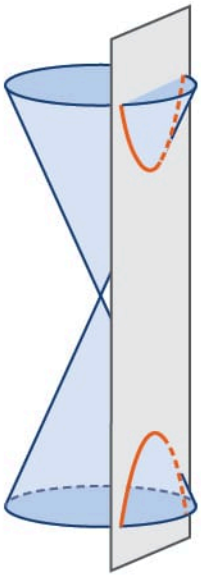


Figure 6.5.2: A hyperbola

Like the ellipse, the hyperbola can also be defined as a set of points in the coordinate plane. A hyperbola is the set of all points $(x, y)$ in a plane such that the difference of the distances between $(x, y)$ and the foci is a positive constant.

Notice that the definition of a hyperbola is very similar to that of an ellipse. The distinction is that the hyperbola is defined in terms of the *difference* of two distances, whereas the ellipse is defined in terms of the *sum* of two distances.

As with the ellipse, every hyperbola has two **axes of symmetry**. The **transverse axis** is a line segment that passes through the center of the hyperbola and has vertices as its endpoints. The foci lie on the line that contains the transverse axis. The **conjugate axis** is perpendicular to the transverse axis and has the co-vertices as its endpoints. The **center of a hyperbola** is the midpoint of both the transverse and conjugate axes, where they intersect. Every hyperbola also has two **asymptotes** that pass through its center. As a hyperbola recedes from the center, its branches approach these asymptotes. The **central rectangle** of the hyperbola is centered at the origin with sides that pass through each vertex and co-vertex; it is a useful tool for graphing the hyperbola and its asymptotes. To sketch the asymptotes of the hyperbola, simply sketch and extend the diagonals of the central rectangle (Figure 6.5.3).

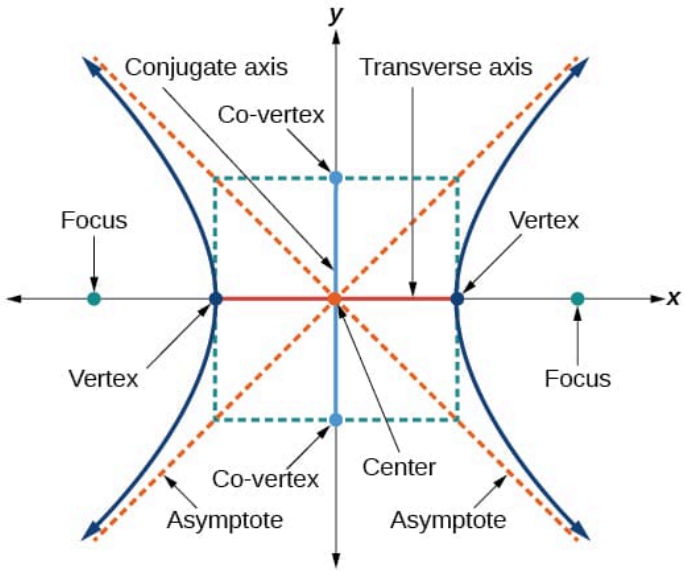



Figure 6.5.3: Key features of the hyperbola

In this section, we will limit our discussion to hyperbolas that are positioned vertically or horizontally in the coordinate plane; the axes will either lie on or be parallel to the $x$- and $y$-axes. We will consider two cases: those that are centered at the origin, and

those that are centered at a point other than the origin.

## Deriving the Equation of an Ellipse Centered at the Origin

Let $(-c, 0)$ and $(c, 0)$ be the foci of a hyperbola centered at the origin. The hyperbola is the set of all points $(x, y)$ such that the difference of the distances from $(x, y)$ to the foci is constant. See Figure $6.5.4$.

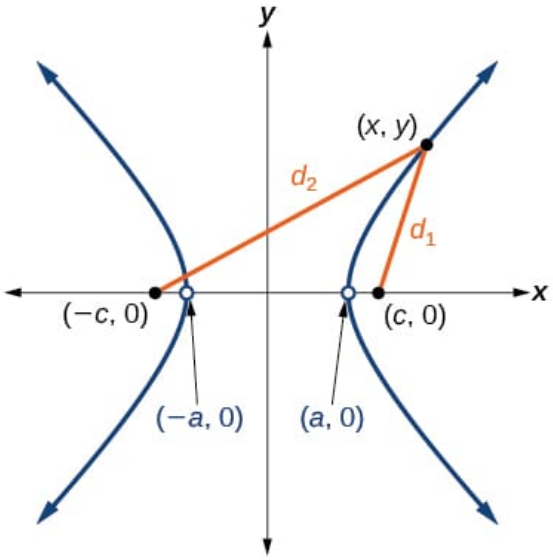



Figure $6.5.4$

If $(a, 0)$ is a vertex of the hyperbola, the distance from $(-c, 0)$ to $(a, 0)$ is $a - (-c) = a + c$. The distance from $(c, 0)$ to $(a, 0)$ is $c - a$. The sum of the distances from the foci to the vertex is

$$(a + c) - (c - a) = 2a$$

If $(x, y)$ is a point on the hyperbola, we can define the following variables:

$$d_2 = \text{the distance from } (-c, 0) \text{ to } (x, y)$$

$$d_1 = \text{the distance from } (c, 0) \text{ to } (x, y)$$

By definition of a hyperbola, $d_2 - d_1$ is constant for any point $(x, y)$ on the hyperbola. We know that the difference of these distances is $2a$ for the vertex $(a, 0)$. It follows that $d_2 - d_1 = 2a$ for any point on the hyperbola. As with the derivation of the equation of an ellipse, we will begin by applying the **distance formula**. The rest of the derivation is algebraic. Compare this derivation with the one from the previous section for ellipses.

$$
\begin{aligned}
d_2 - d_1 &= 2a \\
\sqrt{(x-(-c))^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} &= 2a \qquad \text{Distance Formula} \\
\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} &= 2a \qquad \text{Simplify expressions.} \\
\sqrt{(x+c)^2+y^2} &= 2a + \sqrt{(x-c)^2+y^2} \qquad \text{Move radical to opposite side.} \\
(x+c)^2+y^2 &= \left(2a + \sqrt{(x-c)^2+y^2}\right)^2 \qquad \text{Square both sides.} \\
x^2+2cx+c^2+y^2 &= 4a^2 + 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2 \qquad \text{Expand the squares.} \\
x^2+2cx+c^2+y^2 &= 4a^2 + 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2 \qquad \text{Expand remaining square.} \\
2cx &= 4a^2 + 4a\sqrt{(x-c)^2+y^2} - 2cx \qquad \text{Combine like terms.} \\
4cx - 4a^2 &= 4a\sqrt{(x-c)^2+y^2} \qquad \text{Isolate the radical.} \\
cx - a^2 &= a\sqrt{(x-c)^2+y^2} \qquad \text{Divide by 4.} \\
(cx-a^2)^2 &= a^2\left[\sqrt{(x-c)^2+y^2}\right]^2 \qquad \text{Square both sides.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2(x^2 - 2cx + c^2 + y^2) \qquad \text{Expand the squares.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 \qquad \text{Distribute } a^2 \\
a^4 + c^2x^2 &= a^2x^2 + a^2c^2 + a^2y^2 \qquad \text{Combine like terms.} \\
c^2x^2 - a^2x^2 - a^2y^2 &= a^2c^2 - a^4 \qquad \text{Rearrange terms.} \\
x^2(c^2-a^2) - a^2y^2 &= a^2(c^2-a^2) \qquad \text{Factor common terms.} \\
x^2b^2 - a^2y^2 &= a^2b^2 \qquad \text{Set } b^2 = c^2 - a^2 \\
\frac{x^2b^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} &= \frac{a^2b^2}{a^2b^2} \qquad \text{Divide both sides by } a^2b^2 \\
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1
\end{aligned}
$$

This equation defines a hyperbola centered at the origin with vertices $(\pm a, 0)$ and co-vertices $(0, \pm b)$.

## STANDARD FORMS OF THE EQUATION OF A HYPERBOLA WITH CENTER $(0,0)$

The standard form of the equation of a hyperbola with center $(0,0)$ and transverse axis on the $x$-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(\pm a, 0)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(\pm c, 0)$
- the equations of the asymptotes are $y = \pm \frac{b}{a}x$

See Figure $6.5.5a$.

The standard form of the equation of a hyperbola with center $(0,0)$ and transverse axis on the $y$-axis is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(0, \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(\pm b, 0)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(0, \pm c)$
- the equations of the asymptotes are $y = \pm \frac{a}{b}x$

See Figure $6.5.5b$.

Note that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 + b^2$. When we are given the equation of a hyperbola, we can use this relationship to identify its vertices and foci.

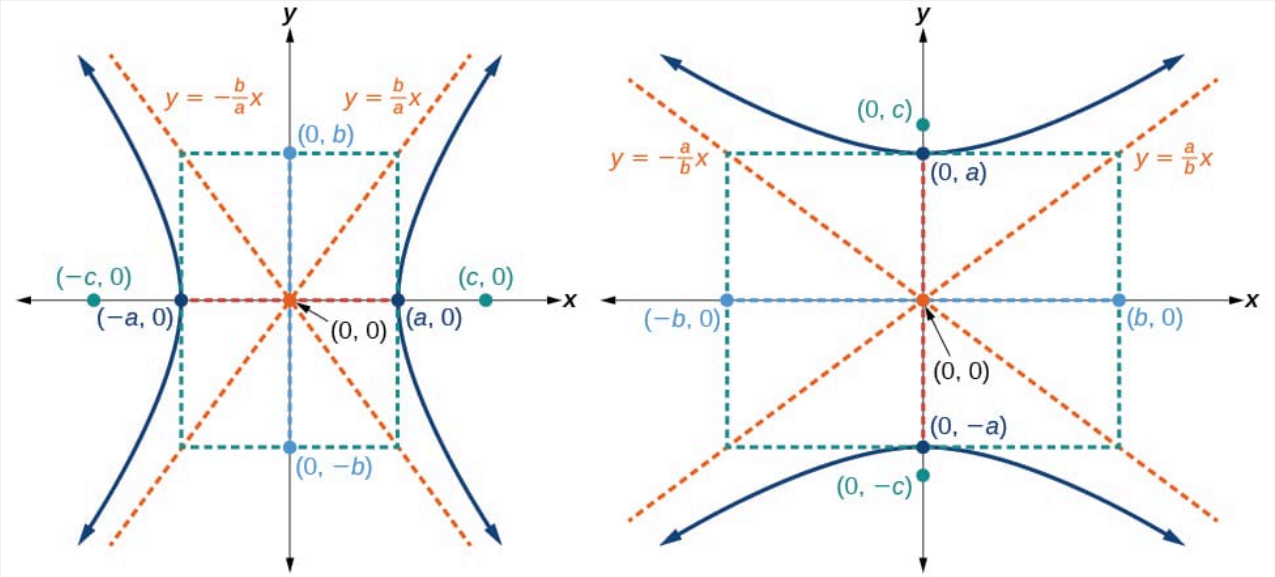



Figure $6.5.5$: (a) Horizontal hyperbola with center $(0, 0)$ (b) Vertical hyperbola with center $(0, 0)$

## How to: Given the equation of a hyperbola in standard form, locate its vertices and foci

1. Determine whether the transverse axis lies on the $x$- or $y$-axis. Notice that $a^2$ is always under the variable with the positive coefficient. So, if you set the other variable equal to zero, you can easily find the intercepts. In the case where the hyperbola is centered at the origin, the intercepts coincide with the vertices.
   - If the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the transverse axis lies on the $x$-axis. The vertices are located at $(\pm a, 0)$, and the foci are located at $(\pm c, 0)$.
   - If the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, then the transverse axis lies on the $y$-axis. The vertices are located at $(0, \pm a)$, and the foci are located at $(0, \pm c)$.
2. Solve for $a$ using the equation $a = \sqrt{a^2}$.
3. Solve for $c$ using the equation $c = \sqrt{a^2 + b^2}$.

## Example $6.5.1$: Locating a Hyperbola's Vertices and Foci

Identify the vertices and foci of the hyperbola with equation $\frac{y^2}{49} - \frac{x^2}{32} = 1$.

**Solution**

The equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, so the transverse axis lies on the $y$-axis. The hyperbola is centered at the origin, so the vertices serve as the $y$-intercepts of the graph. To find the vertices, set $x = 0$, and solve for $y$.

$$\begin{aligned} 1 &= \frac{y^2}{49} - \frac{x^2}{32} \\ 1 &= \frac{y^2}{49} - \frac{0^2}{32} \\ 1 &= \frac{y^2}{49} \\ y^2 &= 49 \\ y &= \pm\sqrt{49} \\ &= \pm 7 \end{aligned}$$

The foci are located at $(0, \pm c)$. Solving for $c$,

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{49 + 32} \\ &= \sqrt{81} \\ &= 9 \end{aligned}$$

Therefore, the vertices are located at $(0, \pm 7)$, and the foci are located at $(0, 9)$.

### Exercise 6.5.1

Identify the vertices and foci of the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{25} = 1$.

*Answer available at the end of this section.*

## Writing Equations of Hyperbolas in Standard Form

Just as with ellipses, writing the equation for a hyperbola in standard form allows us to calculate the key features: its center, vertices, co-vertices, foci, asymptotes, and the lengths and positions of the transverse and conjugate axes. Conversely, an equation for a hyperbola can be found given its key features. We begin by finding standard equations for hyperbolas centered at the origin. Then we will turn our attention to finding standard equations for hyperbolas centered at some point other than the origin.

## Hyperbolas Centered at the Origin

Reviewing the standard forms given for hyperbolas centered at $(0, 0)$,we see that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 + b^2$. Note that this equation can also be rewritten as $b^2 = c^2 - a^2$. This relationship is used to write the equation for a hyperbola when given the coordinates of its foci and vertices.

### How to: Given the vertices and foci of a hyperbola centered at $(0, 0)$, write its equation in standard form

1. Determine whether the transverse axis lies on the $x$- or $y$-axis.
   - If the given coordinates of the vertices and foci have the form $(\pm a, 0)$ and $(\pm c, 0)$, respectively, then the transverse axis is the $x$-axis. Use the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
   - If the given coordinates of the vertices and foci have the form $(0, \pm a)$ and $(0, \pm c)$, respectively, then the transverse axis is the $y$-axis. Use the standard form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

2. Find $b^2$ using the equation $b^2 = c^2 - a^2$.
3. Substitute the values for $a^2$ and $b^2$ into the standard form of the equation determined in Step 1.

## Example 6.5.2: Finding the Equation of a Hyperbola Centered at $(0, 0)$ Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices $(\pm 6, 0)$ and foci $(\pm 2\sqrt{10}, 0)$?

**Solution**

The vertices and foci are on the $x$-axis. Thus, the equation for the hyperbola will have the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The vertices are $(\pm 6, 0)$, so $a = 6$ and $a^2 = 36$.

The foci are $(\pm 2\sqrt{10}, 0)$, so $c = 2\sqrt{10}$ and $c^2 = 40$.

Solving for $b^2$, we have

$$\begin{aligned} b^2 &= c^2 - a^2 \\ b^2 &= 40 - 36 \qquad \text{Substitute for } c^2 \text{ and } a^2 \\ b^2 &= 4 \qquad \text{Subtract.} \end{aligned}$$

Finally, we substitute $a^2 = 36$ and $b^2 = 4$ into the standard form of the equation, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The equation of the hyperbola is $\frac{x^2}{36} - \frac{y^2}{4} = 1$, as shown in Figure 6.5.6.

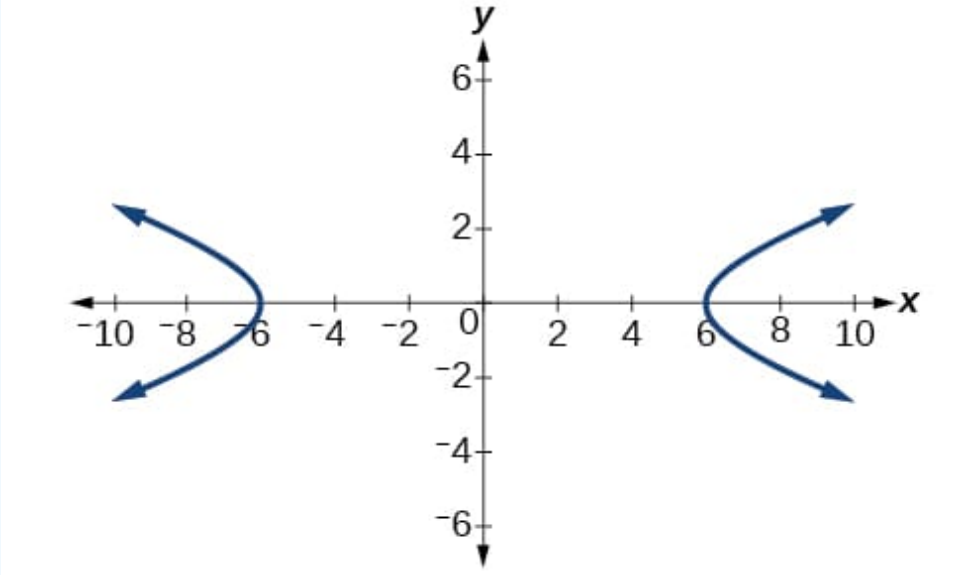



Figure 6.5.6

## Exercise 6.5.2

What is the standard form equation of the hyperbola that has vertices $(0, \pm 2)$ and foci $(0, \pm 2\sqrt{5})$?

*Answer available at the end of this section.*

## Hyperbolas Not Centered at the Origin

Like the graphs for other equations, the graph of a hyperbola can be translated. If a hyperbola is translated $h$ units horizontally and $k$ units vertically, the center of the hyperbola will be $(h, k)$. This translation results in the standard form of the equation we saw previously, with $x$ replaced by $(x - h)$ and $y$ replaced by $(y - k)$.

### STANDARD FORMS OF THE EQUATION OF A HYPERBOLA WITH CENTER $(H, K)$

The standard form of the equation of a hyperbola with center $(h, k)$ and transverse axis parallel to the $x$-axis is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{6.5.1}$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(h \pm c, k)$

The asymptotes of the hyperbola coincide with the diagonals of the central rectangle. The length of the rectangle is $2a$ and its width is $2b$. The slopes of the diagonals are $\pm\frac{b}{a}$,and each diagonal passes through the center $(h, k)$. Using the point-slope formula, it is simple to show that the equations of the asymptotes are $y = \pm\frac{b}{a}(x - h) + k$. See Figure $6.5.7a$.

The standard form of the equation of a hyperbola with center $(h, k)$ and transverse axis parallel to the $y$-axis is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \tag{6.5.2}$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(h, k \pm c)$

Using the reasoning above, the equations of the asymptotes are $y = \pm\frac{a}{b}(x - h) + k$. See Figure $6.5.7b$.

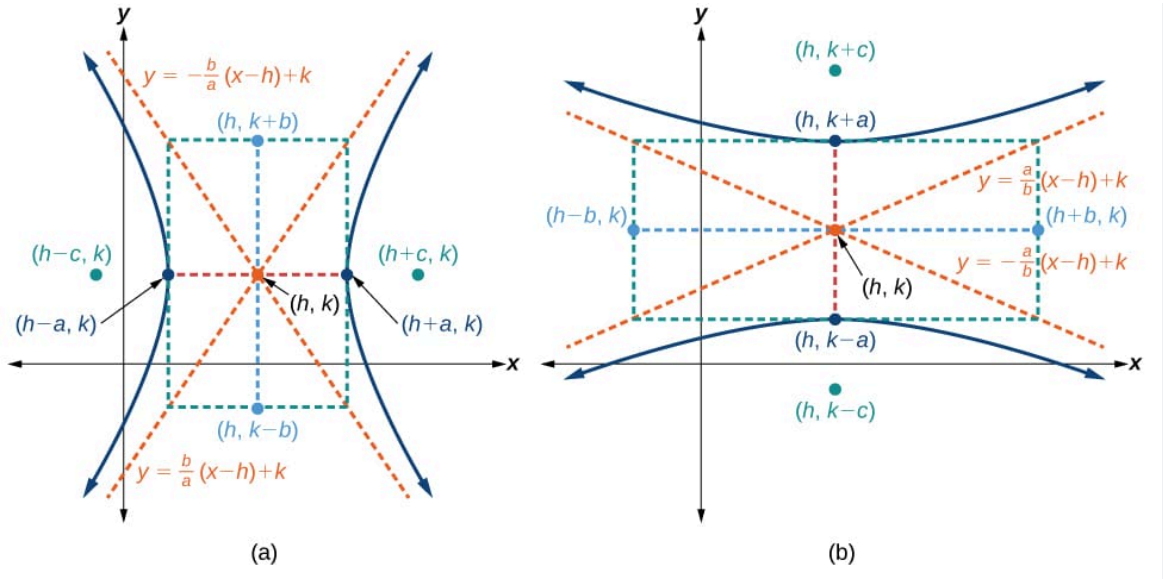



Figure 6.5.7: (a) Horizontal hyperbola with center $(h, k)$ (b) Vertical hyperbola with center $(h, k)$

Like hyperbolas centered at the origin, hyperbolas centered at a point $(h, k)$ have vertices, co-vertices, and foci that are related by the equation $c^2 = a^2 + b^2$. We can use this relationship along with the midpoint and distance formulas to find the standard equation of a hyperbola when the vertices and foci are given.

### How to: Given the vertices and foci of a hyperbola centered at $(h, k)$,write its equation in standard form

1. Determine whether the transverse axis is parallel to the $x$- or $y$-axis.
   - If the $y$-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the $x$-axis. Use the standard form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.
   - If the $x$-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the $y$-axis. Use the standard form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.
2. Identify the center of the hyperbola, $(h, k)$,using the midpoint formula and the given coordinates for the vertices.
3. Find $a^2$ by solving for the length of the transverse axis, $2a$, which is the distance between the given vertices.
4. Find $c^2$ using $h$ and $k$ found in Step 2 along with the given coordinates for the foci.
5. Solve for $b^2$ using the equation $b^2 = c^2 - a^2$.
6. Substitute the values for $h$, $k$, $a^2$, and $b^2$ into the standard form of the equation determined in Step 1.

### Example 6.5.3: Finding the Equation of a Hyperbola Centered at $(h, k)$ Given its Foci and Vertices

What is the standard form equation of the **hyperbola** that has vertices at $(0, -2)$ and $(6, -2)$ and foci at $(-2, -2)$ and $(8, -2)$?

**Solution**

The $y$-coordinates of the vertices and foci are the same, so the transverse axis is parallel to the $x$-axis. Thus, the equation of the hyperbola will have the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

First, we identify the center, $(h, k)$. The center is halfway between the vertices $(0, -2)$ and $(6, -2)$. Applying the midpoint formula, we have

$$(h,k)=(\frac{0+6}{2},\frac{-2+(-2)}{2})=(3,-2)$$

Next, we find $a^2$. The length of the transverse axis, $2a$,is bounded by the vertices. So, we can find $a^2$ by finding the distance between the $x$-coordinates of the vertices.

$$\begin{aligned}2a&=|0-6|\\2a&=6\\a&=3\\a^2&=9\end{aligned}$$

Now we need to find $c^2$. The coordinates of the foci are $(h\pm c,k)$. So $(h-c,k)=(-2,-2)$ and $(h+c,k)=(8,-2)$. We can use the $x$-coordinate from either of these points to solve for $c$. Using the point $(8,-2)$, and substituting $h=3$,

$$\begin{aligned}h+c&=8\\3+c&=8\\c&=5\\c^2&=25\end{aligned}$$

Next, solve for $b^2$ using the equation $b^2=c^2-a^2$:

$$\begin{aligned}b^2&=c^2-a^2\\&=25-9\\&=16\end{aligned}$$

Finally, substitute the values found for $h$, $k$, $a^2$,and $b^2$ into the standard form of the equation.

$$\frac{(x-3)^2}{9}-\frac{(y+2)^2}{16}=1$$

**Exercise 6.5.3**

What is the standard form equation of the hyperbola that has vertices $(1,-2)$ and $(1,8)$ and foci $(1,-10)$ and $(1,16)$?

*Answer available at the end of this section.*

## Graphing Hyperbolas Centered at the Origin

When we have an equation in standard form for a hyperbola centered at the origin, we can interpret its parts to identify the key features of its graph: the center, vertices, co-vertices, asymptotes, foci, and lengths and positions of the transverse and conjugate axes. To graph hyperbolas centered at the origin, we use the standard form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for horizontal hyperbolas and the standard form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ for vertical hyperbolas.

**How to: Given a standard form equation for a hyperbola centered at $(0,0)$, sketch the graph**

1. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the vertices, co-vertices, and foci; and the equations for the asymptotes.
   - If the equation is in the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then
     - the transverse axis is on the $x$-axis
     - the coordinates of the vertices are \((\pm a,0)\0
     - the coordinates of the co-vertices are $(0,\pm b)$
     - the coordinates of the foci are $(\pm c,0)$

- the equations of the asymptotes are $y = \pm\frac{b}{a}x$

◦ If the equation is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, then

- the transverse axis is on the $y$-axis
- the coordinates of the vertices are $(0, \pm a)$
- the coordinates of the co-vertices are $(\pm b, 0)$
- the coordinates of the foci are $(0, \pm c)$
- the equations of the asymptotes are $y = \pm\frac{a}{b}x$

3. Solve for the coordinates of the foci using the equation $c = \pm\sqrt{a^2 + b^2}$.
4. Plot the vertices, co-vertices, foci, and asymptotes in the coordinate plane, and draw a smooth curve to form the hyperbola.

### Example 6.5.4: Graphing a Hyperbola Centered at $(0, 0)$ Given an Equation in Standard Form

Graph the hyperbola given by the equation $\frac{y^2}{64} - \frac{x^2}{36} = 1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

**Solution**

The standard form that applies to the given equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Thus, the transverse axis is on the $y$-axis

The coordinates of the vertices are $(0, \pm a) = (0, \pm\sqrt{64}) = (0, \pm 8)$

The coordinates of the co-vertices are $(\pm b, 0) = (\pm\sqrt{36}, 0) = (\pm 6, 0)$

The coordinates of the foci are $(0, \pm c)$, where $c = \pm\sqrt{a^2 + b^2}$. Solving for $c$, we have

$$c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{64 + 36} = \pm\sqrt{100} = \pm 10$$

Therefore, the coordinates of the foci are $(0, \pm 10)$

The equations of the asymptotes are $y = \pm\frac{a}{b}x = \pm\frac{8}{6}x = \pm\frac{4}{3}x$

Plot and label the vertices and co-vertices, and then sketch the central rectangle. Sides of the rectangle are parallel to the axes and pass through the vertices and co-vertices. Sketch and extend the diagonals of the central rectangle to show the asymptotes. The central rectangle and asymptotes provide the framework needed to sketch an accurate graph of the hyperbola. Label the foci and asymptotes, and draw a smooth curve to form the hyperbola, as shown in Figure 6.5.8.

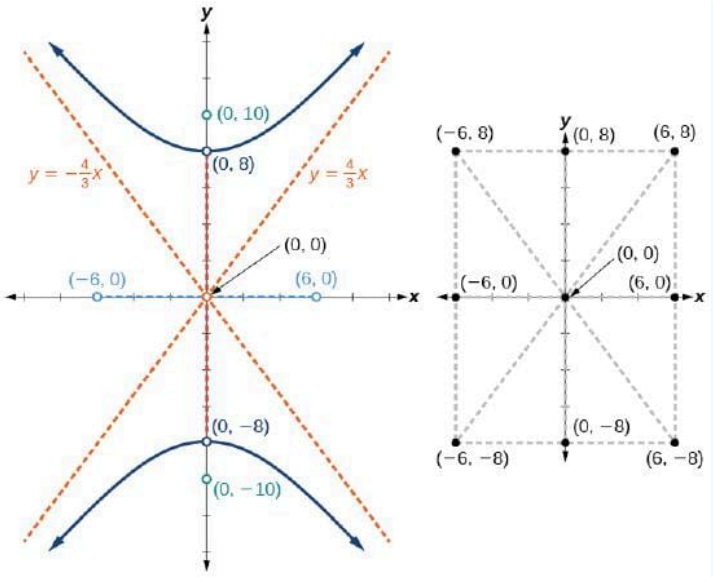



Figure 6.5.8

**Exercise 6.5.4**

Graph the hyperbola given by the equation $\frac{x^2}{144} - \frac{y^2}{81} = 1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

*Answer available at the end of this section.*

## Graphing Hyperbolas Not Centered at the Origin

Graphing hyperbolas centered at a point $(h, k)$ other than the origin is similar to graphing ellipses centered at a point other than the origin. We use the standard forms $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ for horizontal hyperbolas, and $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ for vertical hyperbolas. From these standard form equations we can easily calculate and plot key features of the graph: the coordinates of its center, vertices, co-vertices, and foci; the equations of its asymptotes; and the positions of the transverse and conjugate axes.

**How to: Given a general form for a hyperbola centered at $(h, k)$, sketch the graph**

1. Convert the general form to that standard form. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the center, vertices, co-vertices, foci; and equations for the asymptotes.
    1. If the equation is in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, then
        - the transverse axis is parallel to the $x$-axis
        - the center is $(h, k)$
        - the coordinates of the vertices are $(h \pm a, k)$
        - the coordinates of the co-vertices are $(h, k \pm b)$
        - the coordinates of the foci are $(h \pm c, k)$
        - the equations of the asymptotes are $y = \pm\frac{b}{a}(x-h) + k$
    2. If the equation is in the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, then
        - the transverse axis is parallel to the $y$-axis
        - the center is $(h, k)$
        - the coordinates of the vertices are $(h, k \pm a)$

- the coordinates of the co-vertices are $(h \pm b, k)$
- the coordinates of the foci are $(h, k \pm c)$
- the equations of the asymptotes are $y = \pm\frac{a}{b}(x-h)+k$

3. Solve for the coordinates of the foci using the equation $c = \pm\sqrt{a^2+b^2}$.
4. Plot the center, vertices, co-vertices, foci, and asymptotes in the coordinate plane and draw a smooth curve to form the hyperbola.

## Example 6.5.5: Graphing a Hyperbola Centered at $(h, k)$ Given an Equation in General Form

Graph the hyperbola given by the equation $9x^2 - 4y^2 - 36x - 40y - 388 = 0$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

**Solution**

Start by expressing the equation in standard form. Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$(9x^2 - 36x) - (4y^2 + 40y) = 388$$

Factor the leading coefficient of each expression.

$$9(x^2 - 4x) - 4(y^2 + 10y) = 388$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$9(x^2 - 4x + 4) - 4(y^2 + 10y + 25) = 388 + 36 - 100$$

Rewrite as perfect squares.

$$9(x-2)^2 - 4(y+5)^2 = 324$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x-2)^2}{36} - \frac{(y+5)^2}{81} = 1$$

The standard form that applies to the given equation is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where $a^2 = 36$ and $b^2 = 81$,or $a = 6$ and $b = 9$. Thus, the transverse axis is parallel to the $x$-axis. It follows that:

the center of the ellipse is $(h, k) = (2, -5)$

the coordinates of the vertices are $(h \pm a, k) = (2 \pm 6, -5)$, or $(-4, -5)$ and $(8, -5)$

the coordinates of the co-vertices are $(h, k \pm b) = (2, -5 \pm 9)$, or $(2, -14)$ and $(2, 4)$

the coordinates of the foci are $(h \pm c, k)$, where $c = \pm\sqrt{a^2+b^2}$. Solving for $c$,we have

$$c = \pm\sqrt{36+81} = \pm\sqrt{117} = \pm 3\sqrt{13}$$

Therefore, the coordinates of the foci are $(2 - 3\sqrt{13}, -5)$ and $(2 + 3\sqrt{13}, -5)$.

The equations of the asymptotes are $y = \pm\frac{b}{a}(x-h)+k = \pm\frac{3}{2}(x-2)-5$.

Next, we plot and label the center, vertices, co-vertices, foci, and asymptotes and draw smooth curves to form the hyperbola, as shown in Figure 6.5.10.

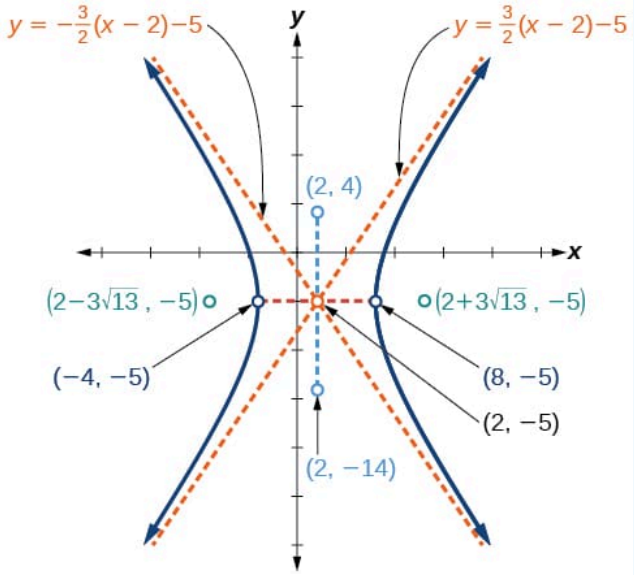



Figure 6.5.10

**Exercise 6.5.5**

Graph the hyperbola given by the standard form of an equation $\frac{(y+4)^2}{100} - \frac{(x-3)^2}{64} = 1$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

*Answer available at the end of this section.*

## Solving Applied Problems Involving Hyperbolas

As we discussed at the beginning of this section, hyperbolas have real-world applications in many fields, such as astronomy, physics, engineering, and architecture. The design efficiency of hyperbolic cooling towers is particularly interesting. Cooling towers are used to transfer waste heat to the atmosphere and are often touted for their ability to generate power efficiently. Because of their hyperbolic form, these structures are able to withstand extreme winds while requiring less material than any other forms of their size and strength (Figure 6.5.12). For example, a 500-foot tower can be made of a reinforced concrete shell only 6 or 8 inches wide!



Figure 6.5.12: Cooling towers at the Drax power station in North Yorkshire, United Kingdom (credit: Les Haines, Flickr)

The first hyperbolic towers were designed in 1914 and were 35 meters high. Today, the tallest cooling towers are in France, standing a remarkable 170 meters tall. In Example 6.5.6 we will use the design layout of a cooling tower to find a hyperbolic equation that models its sides.

## Example 6.5.6: Solving Applied Problems Involving Hyperbolas

The design layout of a cooling tower is shown in Figure 6.5.13. The tower stands 179.6 meters tall. The diameter of the top is 72 meters. At their closest, the sides of the tower are 60 meters apart.

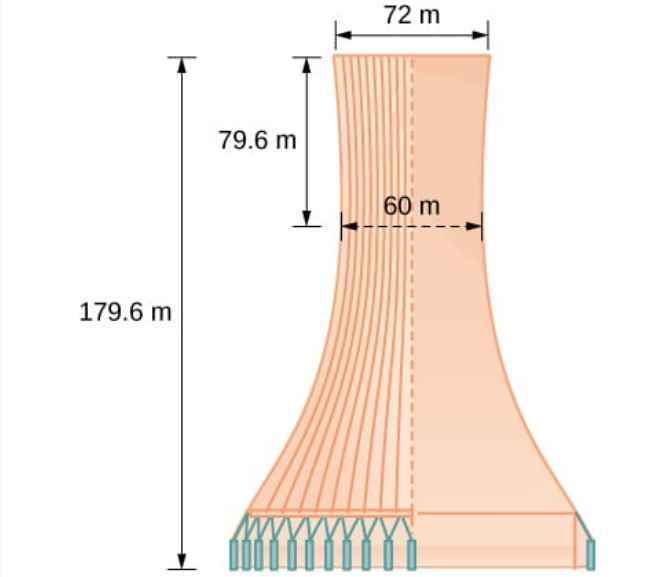



Figure 6.5.13: Project design for a natural draft cooling tower

Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

**Solution**

We are assuming the center of the tower is at the origin, so we can use the standard form of a horizontal hyperbola centered at the origin: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where the branches of the hyperbola form the sides of the cooling tower. We must find the values of $a^2$ and $b^2$ to complete the model.

First, we find $a^2$. Recall that the length of the transverse axis of a hyperbola is $2a$. This length is represented by the distance where the sides are closest, which is given as $65.3$ meters. So, $2a = 60$. Therefore, $a = 30$ and $a^2 = 900$.

To solve for $b^2$,we need to substitute for $x$ and $y$ in our equation using a known point. To do this, we can use the dimensions of the tower to find some point $(x, y)$ that lies on the hyperbola. We will use the top right corner of the tower to represent that point. Since the $y$-axis bisects the tower, our $x$-value can be represented by the radius of the top, or $36$ meters. The *y*-value is represented by the distance from the origin to the top, which is given as $79.6$ meters. Therefore,

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 && \text{Standard form of horizontal hyperbola.} \\ b^2 &= \frac{y^2}{\frac{x^2}{a^2} - 1} && \text{Isolate } b^2 \\ &= \frac{(79.6)^2}{\frac{(36)^2}{900} - 1} && \text{Substitute for } a^2,\ x,\ \text{and } y \\ &\approx 14400.3636 && \text{Round to four decimal places} \end{aligned}$$

The sides of the tower can be modeled by the hyperbolic equation

$$\frac{x^2}{900} - \frac{y^2}{14400.3636} = 1, \text{or } \frac{x^2}{30^2} - \frac{y^2}{120.0015^2} = 1$$

## Exercise 6.5.6

A design for a cooling tower project is shown in Figure $6.5.14$. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

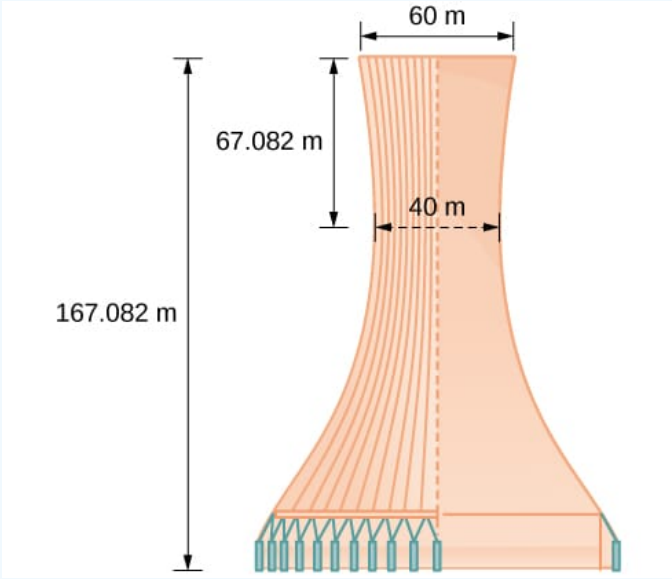



Figure $6.5.14$

*Answer available at the end of this section.*

## Media

Access these online resources for additional instruction and practice with hyperbolas.

- Conic Sections: The Hyperbola Part 1 of 2
- Conic Sections: The Hyperbola Part 2 of 2
- Graph a Hyperbola with Center at Origin
- Graph a Hyperbola with Center not at Origin

# Key Equations

| | |
|---|---|
| Hyperbola, center at origin, transverse axis on *x*-axis | $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ |
| Hyperbola, center at origin, transverse axis on *y*-axis | $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ |
| Hyperbola, center at $(h,k)$,transverse axis parallel to *x*-axis | $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ |
| Hyperbola, center at $(h,k)$,transverse axis parallel to *y*-axis | $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ |

# Key Concepts

- A hyperbola is the set of all points $(x,y)$ in a plane such that the difference of the distances between $(x,y)$ and the foci is a positive constant.
- The standard form of a hyperbola can be used to locate its vertices and foci. See Example $6.5.1$.
- When given the coordinates of the foci and vertices of a hyperbola, we can write the equation of the hyperbola in standard form. See Example $6.5.2$ and Example $6.5.3$.
- When given an equation for a hyperbola, we can identify its vertices, co-vertices, foci, asymptotes, and lengths and positions of the transverse and conjugate axes in order to graph the hyperbola. See Example $6.5.4$ and Example $6.5.5$.
- Real-world situations can be modeled using the standard equations of hyperbolas. For instance, given the dimensions of a natural draft cooling tower, we can find a hyperbolic equation that models its sides. See Example $6.5.6$.

## Answers

**Exercise** $6.5.1$

Vertices: $(\pm 3,0)$; Foci: $(\pm\sqrt{34},0)$

**Exercise** $6.5.2$

$\frac{y^2}{4}-\frac{x^2}{16}=1$

**Exercise** $6.5.3$

$\frac{(y-3)^2}{25}+\frac{(x-1)^2}{144}=1$

**Exercise** $6.5.4$

vertices: $(\pm 12,0)$; co-vertices: $(0,\pm 9)$; foci: $(\pm 15,0)$; asymptotes: $y=\pm\frac{3}{4}x$;

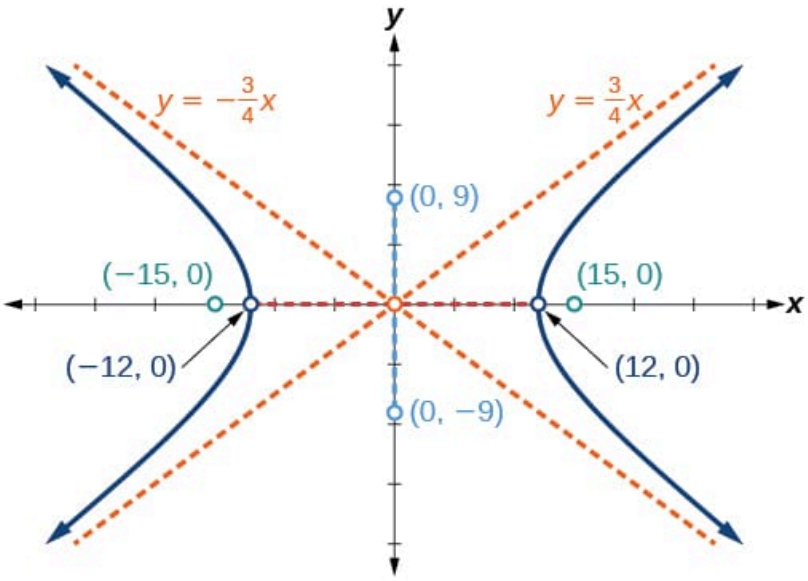



Figure 6.5.9

**Exercise** 6.5.5

center: $(3, -4)$; vertices: $(3, -14)$ and $(3, 6)$; co-vertices: $(-5, -4)$; and $(11, -4)$; foci: $(3, -4 - 2\sqrt{41})$ and $(3, -4 + 2\sqrt{41})$; asymptotes: $y = \pm\frac{5}{4}(x - 3) - 4$

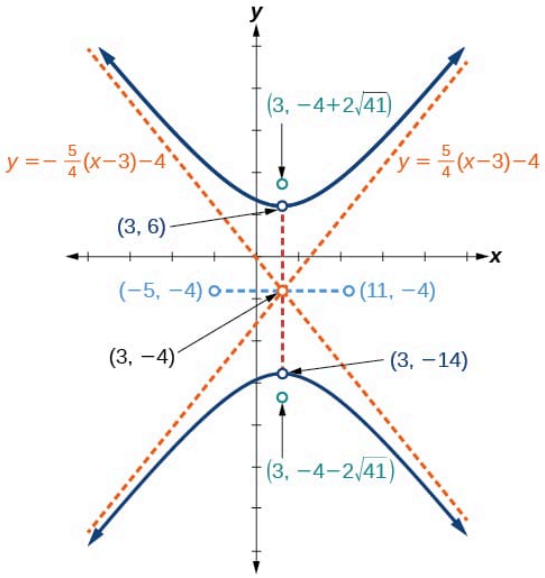



Figure 6.5.11

**Exercise** 6.5.6

The sides of the tower can be modeled by the hyperbolic equation. $\frac{x^2}{400} - \frac{y^2}{3600} = 1$ or $\frac{x^2}{20^2} - \frac{y^2}{60^2} = 1$.

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# 7: SEQUENCES, SERIES, AND COUNTING

In this chapter, we will explore the mathematics behind situations involving probabilities and counting. We will take an in-depth look at annuities. We will also look at the branch of mathematics that would allow us to calculate the number of ways to choose lottery numbers and the probability of winning.


7.1: Introduction to Sequences

7.2: Arithmetic Sequences

7.3: Geometric Sequences

7.4: Summations and Series

7.5: The Principles of Counting

7.6: The Binomial Theorem


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# 7.1: Introduction to Sequences

**Learning Objectives**

- Write the terms of a sequence defined by an explicit formula.
- Write the terms of a sequence defined by a recursive formula.
- Use factorial notation.

## Sequences

A **sequence**[1] is a function whose domain is a set of consecutive natural numbers beginning with $1$.

For example, the following function with domain $\{1, 2, 3, ...\}$ defines a **sequence**[2]:

$$a(n) = 5n - 3$$

The elements in the range of this function are called **terms** of the sequence. It is common to define the $n$th term, or the **general term of a sequence**[3], using the subscripted notation $a_n$, which reads "$a$ sub $n$." Terms can be found using substitution as follows:

$$\begin{aligned} First\ term (n = 1): &\quad a_1 = 5(1) - 3 = 2 \\ Second\ term (n = 2): &\quad a_2 = 5(2) - 3 = 7 \\ Third\ term (n = 3): &\quad a_3 = 5(3) - 3 = 12 \\ Fourth\ term (n = 4): &\quad a_4 = 5(4) - 3 = 17 \\ Fifth\ term (n = 5): &\quad a_5 = 5(5) - 3 = 22 \\ &\quad \vdots \\ General\ term: &\quad a_n = 5n - 3 \end{aligned}$$

This produces an ordered list,

$$2, 7, 12, 17, 22, ...$$

Unlike a set, order matters. If the domain of a sequence consists of natural numbers that end, such as $\{1, 2, 3, ..., k\}$, then it is called a **finite sequence**[4].

A sequence that continues indefinitely is called an **infinite sequence**. The ellipsis $(...)$ indicates that this sequence continues forever. The domain of an infinite sequence is the set of natural numbers.

Because the domain is usually implied, we are typically only presented with the range of the function. As described above, we call these values the "**terms**" of the sequence.

## Writing the Terms of a Sequence Defined by an Explicit Formula

A sequence that you probably learned as a child is known as the **even natural numbers.**

$$\{2, 4, 6, 8, 10, ...\}$$

The first five terms of this sequence are $2$, $4$, $6$, $8$, and $10$.

Another way to represent a sequence is by using a table. The first five terms of the sequence and the nth term of the sequence are shown in Table $7.1.1$.

*Table* 7.1.1

| $n$ | 1 | 2 | 3 | 4 | 5 | $n$ |
|---|---|---|---|---|---|---|
| $n^{th}$ **term of the sequence,** $a_n$ | 2 | 4 | 6 | 8 | 10 | $2n$ |

Similarly, the **odd natural numbers** can be described by the sequence.

$$\{1, 3, 5, 7, 9, ...\}$$

The first five terms of the sequence and the nth term of the sequence are shown in Table 7.1.1.

*Table* 7.1.1

| $n$ | 1 | 2 | 3 | 4 | 5 | $n$ |
|---|---|---|---|---|---|---|
| $n^{th}$ **term of the sequence,** $a_n$ | 1 | 3 | 5 | 7 | 1 | $2n-1$ |

Next, we explore more elaborate examples of sequences.

## Example 7.1.1

A video game company launches an exciting new advertising campaign. They predict the number of online visits to their website, or hits, will double each day. The model they are using shows 2 hits the first day, 4 hits the second day, 8 hits the third day, and so on (Table 7.1.2).

*Table* 7.1.2

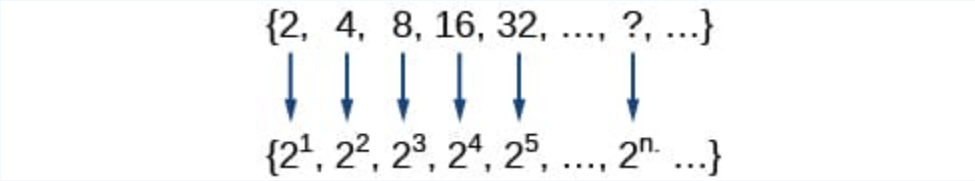
| **Day** | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|
| **Hits** | 2 | 4 | 8 | 16 | 32 | ... |

If their model continues, how many hits will there be at the end of the month?

**Solution**

Listing all of the terms for a sequence can be cumbersome. For example, finding the number of hits on the website at the end of the month would require listing out as many as 31 terms. A more efficient way to determine a specific term is by writing a formula to define the sequence.

One type of formula is an *explicit formula*, which defines the terms of a sequence using their position in the sequence. Explicit formulas are helpful if we want to find a specific term of a sequence without finding all of the previous terms. We can use the formula to find the nth *term of the sequence*, where nn is any positive number. In our example, each number in the sequence is double the previous number, so we can use powers of 2 to write a formula for the nth term.




The first term of the sequence is $2^1 = 2$, the second term is $2^2 = 4$, the third term is $2^3 = 8$, and so on. The nth term of the sequence can be found by raising 2 to the nth power. An explicit formula for a sequence is named by a lower case letter $a$, $b$, $c$... with the subscript $n$. The explicit formula for this sequence is

$$a_n = 2^n$$

Now that we have a formula for the nth term of the sequence, we can answer the question posed at the beginning of this section. We were asked to find the number of hits at the end of the month, which we will take to be 31 days. To find the number of hits on the last day of the month, we need to find the 31st term of the sequence. We will substitute 31 for $n$ in the formula.

$$a_{31} = 2^{31}$$

$$= 2,147,483,648$$

If the doubling trend continues, the company will get $2,147,483,648$ hits on the last day of the month. That is over $2.1$ billion hits! The huge number is probably a little unrealistic because it does not take consumer interest and competition into account. It does, however, give the company a starting point from which to consider business decisions.

Lastly, we can write this particular sequence as

$$\{2, 4, 8, 16, 32, \ldots, 2^n, \ldots\}$$

Graphing provides a visual representation of the sequence as a set of distinct points. We can see from the graph in Figure $7.1.2$ that the number of hits is rising at an exponential rate. This particular sequence forms an exponential function.

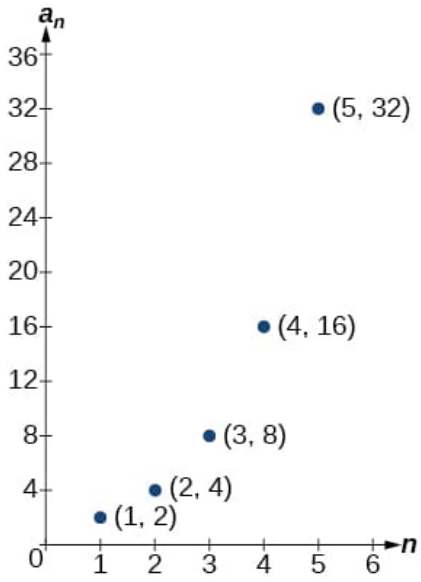



Figure $7.1.2$

If we consider only the first $10$ terms of the sequence, we could write

$$\{2, 4, 8, 16, 32, \ldots, 2^n, \ldots, 1024\}$$

This sequence is called a **finite sequence** because it does not continue indefinitely.

## Definition $6.1.1$ : Sequence

A *sequence* is a function whose domain is the set of positive integers (also known as "natural" numbers).

A *finite sequence* is a sequence whose domain consists of only the first $k$ positive integers, where $k$ is a fixed value. The numbers in a sequence are called *terms*. The variable $a$ with a number subscript is used to represent the terms in a sequence and to indicate the position of the term in the sequence.

$$a_1, a_2, a_3, \ldots, a_n, \ldots a_k \tag{7.1.1}$$

We call $a_1$ the first term of the sequence, $a_2$ the second term of the sequence, $a_3$ the third term of the sequence, and so on. The term $a_n$ is called the nth term of the sequence, or the general term of the sequence. An explicit formula defines the nth term of a sequence using the position of the term.

An *infinite sequence* is a sequence whose domain consists of ALL positive integers.

$$a_1, a_2, a_3, \ldots, a_n, \ldots \tag{7.1.2}$$

The *ellipsis* (...) indicates that the sequence continues indefinitely. In other words, this sequence never ends.

### Q&A: Does a sequence always have to begin with $a_1$?

*No. In certain problems, it may be useful to define the initial term as $a_0$ instead of $a_1$. In these problems, the domain of the function includes $0$.*

### How to: Given an explicit formula, write the first $n$ terms of a sequence

1. Substitute each value of $n$ into the formula. Begin with $n = 1$ to find the first term, $a_1$.
2. To find the second term, $a_2$, use $n = 2$.
3. Continue in the same manner until you have identified all nn terms.

### Example 7.1.2: Writing the Terms of a Sequence Defined by an Explicit Formula

Write the first five terms of the sequence defined by the explicit formula $a_n = -3n + 8$.

**Solution**

Substitute $n = 1$ into the formula. Repeat with values $2$ through $5$ for $n$.

$$\begin{aligned} n = 1 \quad & a_1 = -3(1) + 8 = 5 \\ n = 2 \quad & a_2 = -3(2) + 8 = 2 \\ n = 3 \quad & a_3 = -3(3) + 8 = -1 \\ n = 4 \quad & a_4 = -3(4) + 8 = -4 \\ n = 5 \quad & a_5 = -3(5) + 8 = -7 \end{aligned}$$

The first five terms are $\{5, 2, -1, -4, -7\}$.

**Analysis**

The sequence values can be listed in a table.

*Table* 7.1.3

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $a_n$ | 5 | 2 | -1 | -4 | -7 |

A graph can be made from this table of values. From the graph in Figure $7.1.3$, we can see that this sequence represents a linear function, but notice the graph is not continuous because the domain is over the positive integers only.

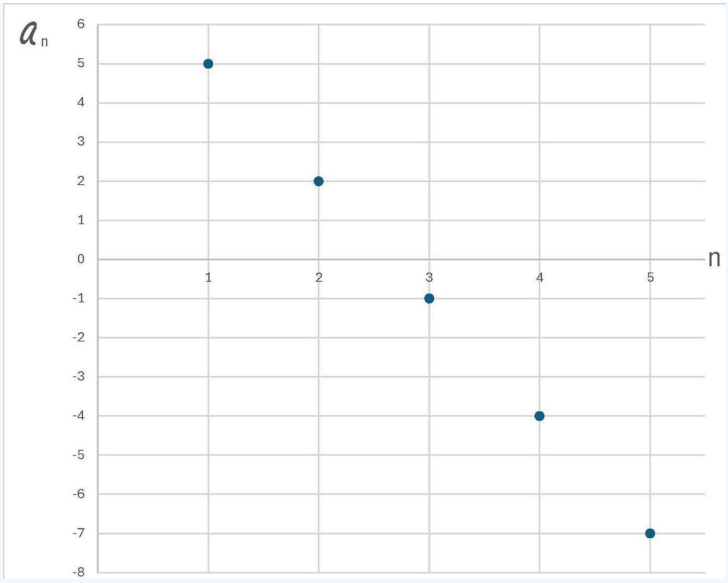



Figure 7.1.3

**Exercise 7.1.2**

Write the first five terms of the sequence defined by the *explicit formula* $a_n = 5n - 4$.

*Answer available at the end of this section.*

## Investigating Alternating Sequences

Sometimes sequences have terms that are alternate. In fact, the terms may actually alternate in sign. The steps to finding terms of the sequence are the same as if the signs did not alternate. However, the resulting terms will not show increase or decrease as $n$ increases. Let's take a look at the following sequence.

$$\{2, -4, 6, -8\}$$

Notice the first term is greater than the second term, the second term is less than the third term, and the third term is greater than the fourth term. This trend continues forever. Do not rearrange the terms in numerical order to interpret the sequence.

**How to: Given an explicit formula with alternating terms, write the first nn terms of a sequence**

1. Substitute each value of nn into the formula. Begin with $n = 1$ to find the first term, $a_1$. The sign of the term is given by the $(-1)^n$ in the explicit formula.
2. To find the second term, $a_2$, use $n = 2$.
3. Continue in the same manner until you have identified all $n$ terms.

**Example 7.1.3: Writing the Terms of an Alternating Sequence Defined by an Explicit Formula**

Write the first five terms of the sequence.

$$a_n = \frac{(-1)^n n^2}{n+1}$$

**Solution**

Substitute $n = 1$, $n = 2$, and so on in the formula.

$$n = 1 \quad a_1 = \frac{(-1)^1 2^2}{1+1} = -\frac{1}{2}$$
$$n = 2 \quad a_2 = \frac{(-1)^2 2^2}{2+1} = \frac{4}{3}$$
$$n = 3 \quad a_3 = \frac{(-1)^3 3^2}{3+1} = -\frac{9}{4}$$
$$n = 4 \quad a_4 = \frac{(-1)^4 4^2}{4+1} = \frac{16}{5}$$
$$n = 5 \quad a_5 = \frac{(-1)^5 5^2}{5+1} = -\frac{25}{6}$$

The first five terms are $\{-12, 43, -94, 165, -256\}$.

**Analysis**

The graph of this function, shown in Figure 7.1.3, looks different from the ones we have seen previously in this section because the terms of the sequence alternate between positive and negative values.

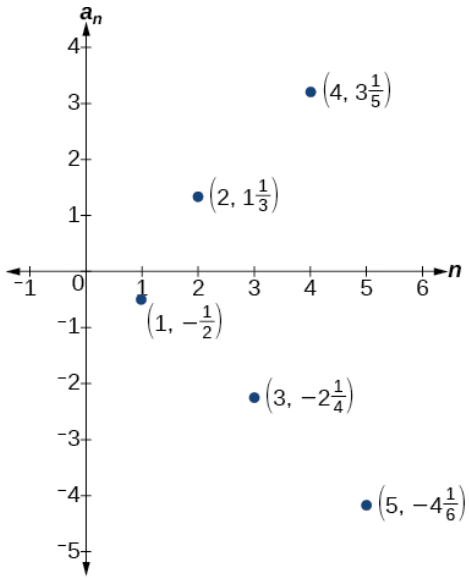



Figure 7.1.4

**Q&A: In the example above, does the $(-1)$ to the power of $n$ account for the oscillations of signs?**

*Yes, the power might be $n$, $n+1$, or $n-1$, but all cases, the odd powers will result in a negative term, and the even powers will always result in a positive term.*

**Exercise 7.1.4**

Write the first five terms of the sequence:

$$a_n = \frac{4n}{(-2)^n}$$

*Answer available at the end of this section.*

# Investigating Piecewise Explicit Formulas

We've learned that sequences are functions whose domain is over the positive integers. This is true for other types of functions, including some piecewise functions. Recall that a *piecewise function* is a function defined by multiple subsections. A different formula might represent each individual subsection.

### How to: Given an explicit formula for a piecewise function, write the first $n$ terms of a sequence

1. Identify the formula to which $n = 1$ applies.
2. To find the first term, $a_1$, use $n = 1$ in the appropriate formula.
3. Identify the formula to which $n = 2$ applies.
4. To find the second term, $a_2$, use $n = 2$ in the appropriate formula.
5. Continue in the same manner until you have identified all $n$ terms.

### Example 7.1.4: Writing the Terms of a Sequence Defined by a Piecewise Explicit Formula

Write the first six terms of the sequence.

$$a_n = \begin{cases} n^2 \text{ if n is not divisible by 3} \\ \frac{n}{3} \text{if n is divisible by 3} \end{cases}$$

## Solution

Substitute $n = 1$, $n = 2$, and so on in the appropriate formula. Use $n^2$ when $n$ is not a multiple of 3. Use $n^3$ when $n$ is a multiple of 3.

$$\begin{aligned} a_1 &= 1^2 = 1 && 1 \text{ is not a multiple of 3. Use } n^2. \\ a_2 &= 2^2 = 4 && 2 \text{ is not a multiple of 3 . Use } n^2. \\ a_3 &= \frac{3}{3} = 1 && 3 \text{ is a multiple of 3. Use } \frac{n}{3}. \\ a_4 &= 4^2 = 16 && 4 \text{ is not a multiple of 3. Use } n^2. \\ a_5 &= 5^2 = 25 && 5 \text{ is not a multiple of 3.Use } n^2. \\ a_6 &= \frac{6}{3} = 2 && 6 \text{ is a multiple of 3.Use } \frac{n}{3} \end{aligned}$$

The first six terms are $\{1, 4, 1, 16, 25, 2\}$.

### Analysis

Every third point on the graph shown in Figure 7.1.4 stands out from the two nearby points. This occurs because the sequence was defined by a piecewise function.

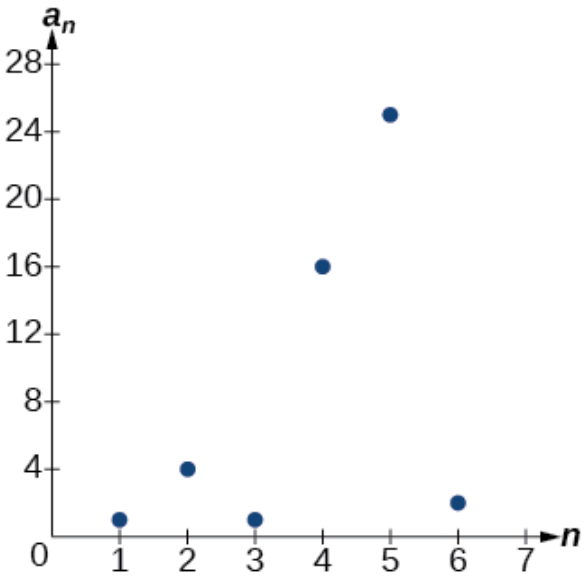



Figure 7.1.4

**Exercise 7.1.5**

Write the first six terms of the sequence.

$$a_n = \begin{cases} 2n^3 \text{ if } n \text{ is odd} \\ \dfrac{5n}{2} \text{ if } n \text{ is even} \end{cases}$$

*Answer available at the end of this section.*

## Finding an Explicit Formula

Thus far, we have been given the explicit formula and asked to find a number of terms of the sequence. Sometimes, the explicit formula for the nth term of a sequence is not given. Instead, we are given several terms from the sequence. When this happens, we can work in reverse to find an explicit formula from the first few terms of a sequence. The key to finding an explicit formula is to look for a pattern in the terms. Keep in mind that the pattern may involve alternating terms, formulas for numerators, formulas for denominators, exponents, or bases.

**How to: Given the first few terms of a sequence, find an explicit formula for the sequence**

1. Look for a pattern among the terms.
2. If the terms are fractions, look for a separate pattern among the numerators and denominators.
3. Look for a pattern among the signs of the terms.
4. Write a formula for $a_n$ in terms of $n$. Test your formula for $n = 1$, $n = 2$, and $n = 3$.

**Example 7.1.5: Writing an Explicit Formula for the *n*th Term of a Sequence**

Write an explicit formula for the nth term of each sequence.

a. $\{-11, 13, -15, 17, -19, ...\}$
b. $\{-\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, -\frac{2}{3,125}, -\frac{2}{15,625}, ...\}$

c. $\{81, 243, 729, ...\}$

**Solution**

Look for the pattern in each sequence.

a. The terms alternate between positive and negative. We can use $(-1)^n$ to make the terms alternate. The values are all odd integers that can be represented by $2n + 9$.

$$a_n = (-1)^n(2n + 9)$$

b.

$$\left\{-\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, -\frac{2}{3,125}, -\frac{2}{15,625}, \cdots\right\}$$

The numerator is 2.

$$\left\{-\frac{2}{5^2}, -\frac{2}{5^3}, -\frac{2}{5^4}, -\frac{2}{5^6}, -\frac{2}{5^7}, \cdots, -\frac{2}{5^n}\right\}$$

The denominators are increasing powers of 5.

The terms are all negative. So we know that the fraction is negative, the numerator is $2$, and the denominator can be represented by $5n + 1$.

$$a_n = -\frac{2}{5^{n+1}}$$

c. The terms are ALL powers of $3$. For $n = 1$, the first term is $3^4$ so the exponent must be $4$ when $n = 1$, hence the general power is $n + 3$.

$$a_n = 3^{n+3}$$

## Exercise 7.1.6

Write an explicit formula for the nth term of the sequence.

$$\{-4, -8, -12, -16, -20, ...\}$$

*Answer available at the end of this section.*

## Exercise 7.1.7

Write an explicit formula for the nth term of the sequence.

$$\{9, -81, 729, -6561, 59049, ...\}$$

*Answer available at the end of this section.*

## Exercise 7.1.8

Write an explicit formula for the nth term of the sequence.

$$\left\{\frac{1}{4}, \frac{1}{2}, 1, 2, 4, ...\right\}$$

*Answer available at the end of this section.*

# Writing the Terms of a Sequence Defined by a Recursive Formula

One interesting example is the Fibonacci sequence. The first two numbers in the Fibonacci sequence are $1$, and each successive term is the sum of the previous two. Therefore, the general term is expressed in terms of the previous two as follows:

$$F_n = F_{n-2} + F_{n-1}$$

Here $F_1 = 1, F_2 = 1$, and $n > 2$. A formula that describes a sequence in terms of its previous terms is called a **recursive formula** or a **recurrence relation**.

Each term of the Fibonacci sequence depends on the terms that come before it.

## Example 7.1.7

Find the first $7$ Fibonacci numbers.

**Solution**

Given that $F_1 = 1$ and $F_2 = 1$, use the recurrence relation $F_n = F_{n-2} + F_{n-1}$ where $n$ is an integer starting with $n = 3$ to find the next $5$ terms:

$$\begin{aligned}
F_3 &= F_{3-2} + F_{3-1} = F_1 + F_2 = 1 + 1 = 2 \\
F_4 &= F_{4-2} + F_{4-1} = F_2 + F_3 = 1 + 2 = 3 \\
F_5 &= F_{5-2} + F_{5-1} = F_3 + F_4 = 2 + 3 = 5 \\
F_6 &= F_{6-2} + F_{6-1} = F_4 + F_5 = 3 + 5 = 8 \\
F_7 &= F_{7-2} + F_{7-1} = F_5 + F_6 = 5 + 8 = 13
\end{aligned}$$

**Answer**:

$1, 1, 2, 3, 5, 8, 13$



***Figure*** 7.1.7 *: Leonardo Fibonacci (1170–1250)*

Fibonacci numbers appear in applications ranging from art to computer science and biology. The beauty of this sequence can be visualized by constructing a Fibonacci spiral. Consider a tiling of squares where each side has a length that matches each Fibonacci number:

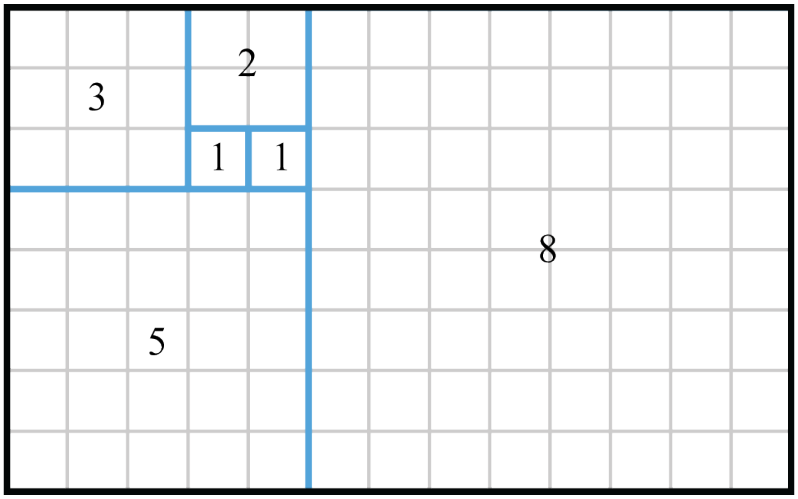



***Figure*** 7.1.8

Connecting the opposite corners of the squares with an arc produces a special spiral shape.

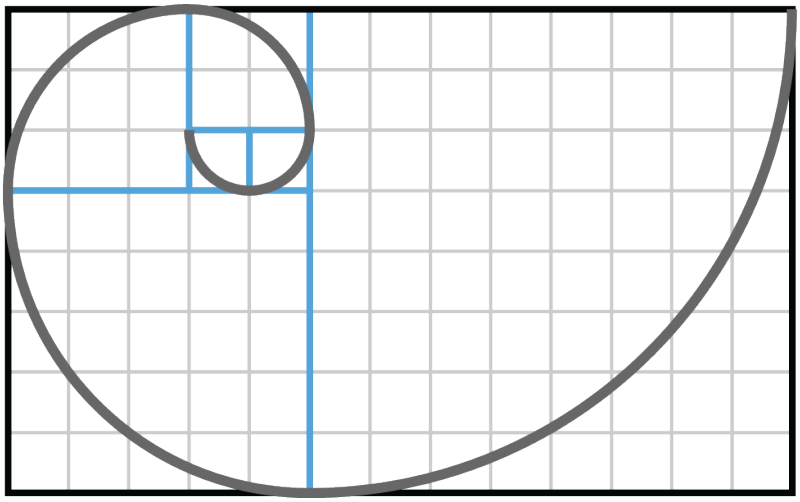


***Figure*** 7.1.9

This shape is called the Fibonacci spiral and approximates many spiral shapes that occur naturally in the growth patterns of nautilus shells, pinecones, tree branches, and many other natural structures. We may see the sequence in the leaf or branch arrangement, the number of petals of a flower, or the pattern of the chambers in a nautilus shell. Their growth follows the Fibonacci sequence, a famous sequence in which each term can be found by adding the preceding two terms. The numbers in the sequence are $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. Other examples from the natural world that exhibit the Fibonacci sequence are the Calla Lily, which has just one petal, the Black-Eyed Susan with $13$ petals, and different varieties of daisies that may have $21$ or $34$ petals.

## Using Recursive Formulas

A recursive formula always has two parts: the value of an initial term (or terms), and an equation defining $a_n$ in terms of preceding terms. For example, suppose we know the following:

$$\begin{aligned} a_1 &= 3 \\ a_n &= 2a_{n-1} - 1, \text{ for } n \geq 2 \end{aligned}$$

We can find the subsequent terms of the sequence using the first term.

$$
\begin{aligned}
a_1 &= 3 \\
a_2 &= 2a_1 - 1 = 2(3) - 1 = 5 \\
a_3 &= 2a_2 - 1 = 2(5) - 1 = 9 \\
a_4 &= 2a_3 - 1 = 2(9) - 1 = 17
\end{aligned}
$$

So the first four terms of the sequence are $\{3, 5, 9, 17\}$.

The recursive formula for the Fibonacci sequence states the first two terms and defines each successive term as the sum of the preceding two terms.

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= 1 \\
a_n &= a_{n-1} + a_{n-2}, \text{ for } n \geq 3
\end{aligned}
$$

To find the tenth term of the sequence, for example, we would need to add the eighth and ninth terms. We were told previously that the eighth and ninth terms are $21$ and $34$, so

$$a_{10} = a_9 + a_8 = 34 + 21 = 55$$

### Definition 6.1.2 : Recursive Formula

A recursive formula is a formula that defines each term of a sequence using preceding term(s). Recursive formulas must always state the initial term, or terms, of the sequence.

### Q&A: Must the first two terms always be given in a recursive formula?

*No. The Fibonacci sequence defines each term using the two preceding terms, but many recursive formulas define each term using only one preceding term. These sequences need only the first term to be defined.*

### How to: Given a recursive formula with only the first term provided, write the first n terms of a sequence

1. Identify the initial term, $a_1$, which is given as part of the formula. This is the first term.
2. To find the second term, $a_2$, substitute the initial term into the formula for $a_{n-1}$. Solve.
3. To find the third term, $a_3$, substitute the second term into the formula. Solve.
4. Repeat until you have solved for the nth term.

### Example 7.1.6: Writing the Terms of a Sequence Defined by a Recursive Formula

Write the first five terms of the sequence defined by the recursive formula.

$$
\begin{aligned}
a_1 &= 9 \\
a_n &= 3a_{n-1} - 20, \text{ for } n \geq 2
\end{aligned}
$$

**Solution**

The first term is given in the formula. For each subsequent term, we replace $a_n - 1$ with the value of the preceding term.

$$
\begin{aligned}
n = 1 \quad & a_1 = 9 \\
n = 2 \quad & a_2 = 3a_1 - 20 = 3(9) - 20 = 27 - 20 = 7 \\
n = 3 \quad & a_3 = 3a_2 - 20 = 3(7) - 20 = 21 - 20 = 1 \\
n = 4 \quad & a_4 = 3a_3 - 20 = 3(1) - 20 = 3 - 20 = -17 \\
n = 5 \quad & a_5 = 3a_4 - 20 = 3(-17) - 20 = -51 - 20 = -71
\end{aligned}
$$

The first five terms are $\{9, 7, 1, -17, -71\}$. See Figure 7.1.5.

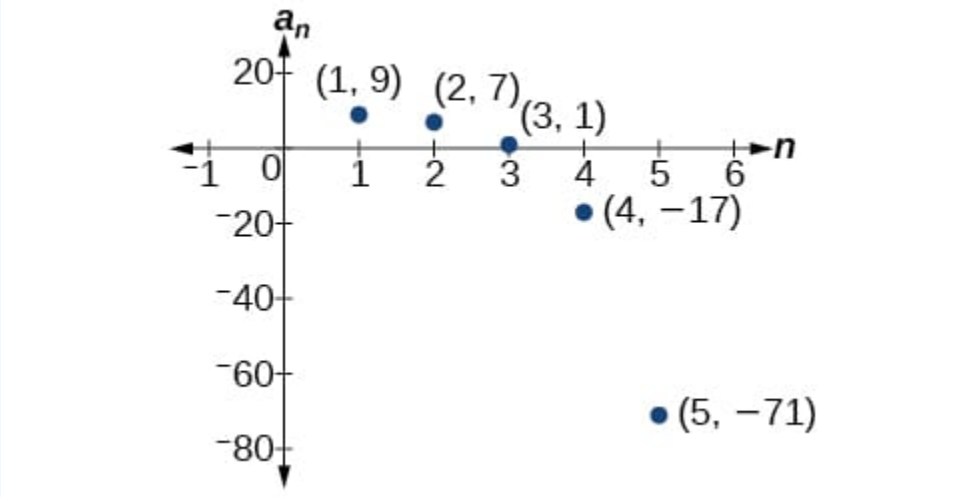



Figure 7.1.5

## Exercise 7.1.9

Write the first five terms of the sequence defined by the recursive formula.

$$a_1 = 2$$
$$a_n = 2a_{n-1} + 1, \text{ for } n \geq 2$$

*Answer available at the end of this section.*

## How to: Given a recursive formula with two initial terms, write the first $n$ terms of a sequence

1. Identify the initial term, $a_1$, which is given as part of the formula.
2. Identify the second term, $a_2$, which is given as part of the formula.
3. To find the third term, substitute the initial term and the second term into the formula. Evaluate.
4. Repeat until you have evaluated the nth term.

## Example 7.1.7: Writing the Terms of a Sequence Defined by a Recursive Formula

Write the first six terms of the sequence defined by the recursive formula.

$$a_1 = 1$$
$$a_2 = 2$$
$$a_n = 3a_{n-1} + 4a_{n-2}, \text{ for } n \geq 3$$

**Solution**

The first two terms are given. For each subsequent term, we replace $a_{n-1}$ and $a_{n-2}$ with the values of the two preceding terms.

$$\begin{aligned} n = 3 \quad & a_3 = 3a_2 + 4a_1 = 3(2) + 4(1) = 10 \\ n = 4 \quad & a_4 = 3a_3 + 4a_2 = 3(10) + 4(2) = 38 \\ n = 5 \quad & a_5 = 3a_4 + 4a_3 = 3(38) + 4(10) = 154 \\ n = 6 \quad & a_6 = 3a_5 + 4a_4 = 3(154) + 4(38) = 614 \end{aligned}$$

The first six terms are $\{1, 2, 10, 38, 154, 614\}$. See Figure 7.1.6.

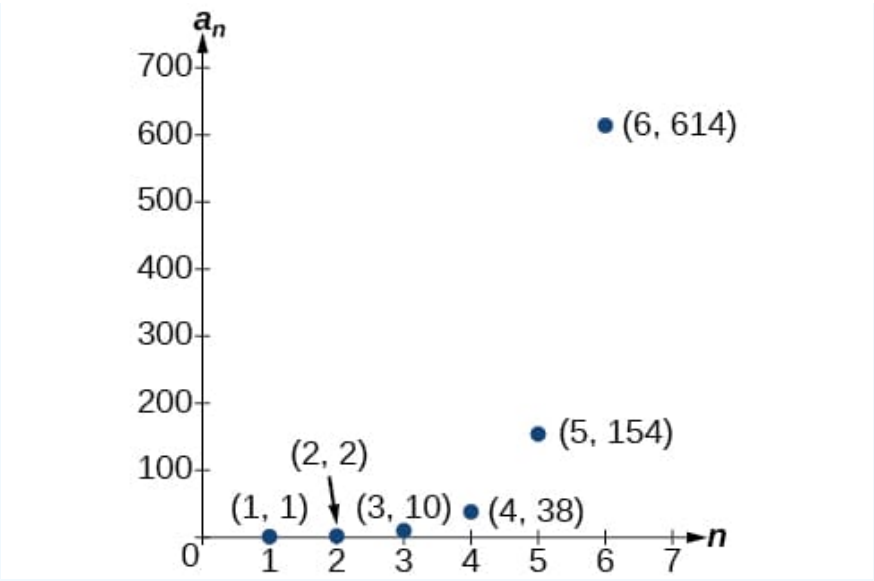



Figure 7.1.6

**Exercise 7.1.10**

Write the first $8$ terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 1 \\ a_3 &= 1 \\ a_n &= \frac{a_{n-1}}{a_{n-2}} + a_{n-3}, \text{ for } n \geq 4 \end{aligned}$$

*Answer available at the end of this section.*

## Using Factorial Notation

The formulas for some sequences include products of consecutive positive integers. $n$ **factorial**, written as $n!$, is the product of the positive integers from $1$ to $n$. For example,

$$\begin{aligned} 4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \end{aligned}$$

An example of formula containing a **factorial** is $a_n = (n+1)!$. The sixth term of the sequence can be found by substituting $6$ for $n$ .

$$a_6 = (6+1)! = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

The factorial of any whole number $n$ is $n(n-1)!$ We can therefore also think of $5!$ as $5 \cdot 4!$.

**Definition 6.1.3 : Factorial**

$n$ factorial is a mathematical operation that can be defined using a recursive formula. The factorial of $n$, denoted $n!$, is defined for a positive integer $n$ as:

$$0! = 1 \tag{7.1.3}$$
$$1! = 1 \tag{7.1.4}$$
$$n! = n(n-1)(n-2)\cdots(2)(1), \text{ for } n \geq 2 \tag{7.1.5}$$

The special case $0!$ is defined as $0! = 1$.

### Q&A: Can factorials always be found using a calculator?

*No. Factorials get large very quickly—faster than even exponential functions! When the output gets too large for the calculator, it will not be able to calculate the factorial.*

### Example 7.1.8: Writing the Terms of a Sequence Using Factorials

Write the first five terms of the sequence defined by the explicit formula $a_n = \dfrac{5n}{(n+2)!}$.

**Solution**

Substitute $n = 1$, $n = 2$, and so on in the formula.

$$\begin{aligned}
n = 1 \quad a_1 &= \frac{5(1)}{(1+2)!} = \frac{5}{3!} = \frac{5}{3 \cdot 2 \cdot 1} = \frac{5}{6} \\
n = 2 \quad a_2 &= \frac{5(2)}{(2+2)!} = \frac{10}{4!} = \frac{10}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{5}{12} \\
n = 3 \quad a_3 &= \frac{5(3)}{(3+2)!} = \frac{15}{5!} = \frac{15}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{8} \\
n = 4 \quad a_4 &= \frac{5(4)}{(4+2)!} = \frac{20}{6!} = \frac{20}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{36} \\
n = 5 \quad a_5 &= \frac{5(5)}{(5+2)!} = \frac{25}{7!} = \frac{25}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{5}{1,008}
\end{aligned}$$

The first five terms are $\left\{\frac{5}{6}, \frac{5}{12}, \frac{1}{8}, \frac{1}{36}, \frac{5}{1,008}\right\}$.

**Analysis**

Figure 7.1.7 shows the graph of the sequence. Notice that, since factorials grow very quickly, the presence of the factorial term in the denominator results in the denominator becoming much larger than the numerator as nn increases. This means the quotient gets smaller and, as the plot of the terms shows, the terms are decreasing and nearing zero.

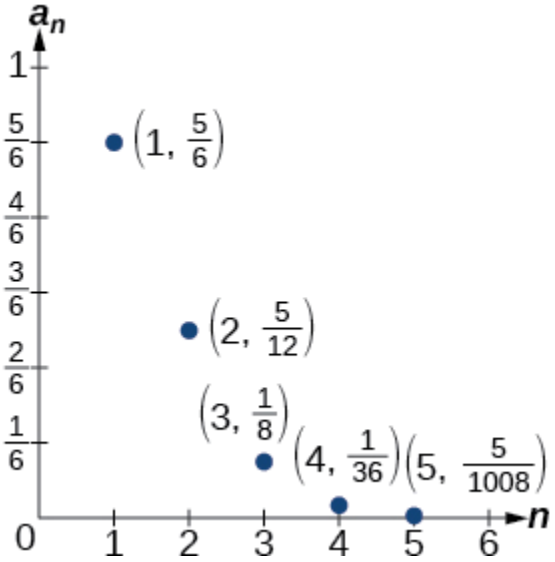



Figure 7.1.7

**Exercise 7.1.11**

Write the first five terms of the sequence defined by the explicit formula $a_n = \frac{(n+1)!}{2n}$.

*Answer available at the end of this section.*

**Media**

Access this online resource for additional instruction and practice with sequences.

- Finding Terms in a Sequence

## Key Equations

| | |
|---|---|
| Formula for a factorial | $0! = 1$<br>$1! = 1$<br>$n! = n(n-1)(n-2)\cdots(2)(1)$, for $n \geq 2$ |

## Key Concepts

- A sequence is a list of numbers, called terms, written in a specific order.
- Explicit formulas define each term of a sequence using the position of the term. See Example 7.1.1, Example 7.1.2, and Example 7.1.3.
- An explicit formula for thenth nth term of a sequence can be written by analyzing the pattern of several terms. See Example 7.1.4.
- Recursive formulas define each term of a sequence using previous terms.
- Recursive formulas must state the initial term, or terms, of a sequence.
- A set of terms can be written by using a recursive formula. See Example 7.1.5 and Example 7.1.6.
- A factorial is a mathematical operation that can be defined recursively.
- The factorial of $n$ is the product of all integers from $1$ to $n$. See Example 7.1.7.

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### Answers

**Exercise** 7.1.2

The first five terms are $\{1, 6, 11, 16, 21\}$.

**Exercise** 7.1.4

The first five terms are $\{-2, 2, -32, 1, -58\}$.

**Exercise** 7.1.5

The first six terms are $\{2, 5, 54, 10, 250, 15\}$.

**Exercise** 7.1.6

$a_n = -4n$

**Exercise** 7.1.6

$a_n = (-1)^{n+1} 9^n$

**Exercise** $7.1.8$

$a_n = 2^{n-3}$

**Exercise** $7.1.9$

$\{2, 5, 11, 23, 47\}$

**Exercise** $7.1.10$

$\{0, 1, 1, 1, 2, 3, 52, 176\}$.

**Exercise** $7.1.11$

The first five terms are $\left\{1, \frac{3}{2}, 4, 15, 72\right\}$.

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# 7.2: Arithmetic Sequences

**Learning Objectives**

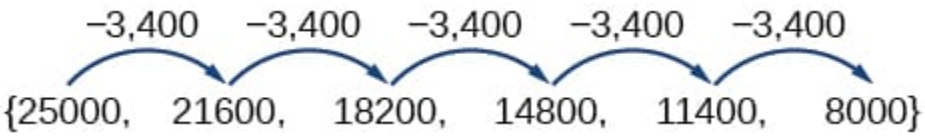
- Find the common difference for an arithmetic sequence.
- Write terms of an arithmetic sequence.
- Use a recursive formula for an arithmetic sequence.
- Use an explicit formula for an arithmetic sequence.

Companies often make large purchases, such as computers and vehicles, for business use. The book-value of these supplies decreases each year for tax purposes. This decrease in value is called depreciation. One method of calculating depreciation is straight-line depreciation, in which the value of the asset decreases by the same amount each year.

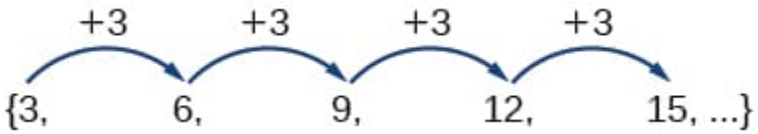
As an example, consider a woman who starts a small contracting business. She purchases a new truck for $\$25,000$. After five years, she estimates that she will be able to sell the truck for $\$8,000$. The loss in value of the truck will therefore be $\$17,000$, which is $\$3,400$ per year for five years. The truck will be worth $\$21,600$ after the first year; $\$18,200$ after two years; $\$14,800$ after three years; $\$11,400$ after four years; and $\$8,000$ at the end of five years. In this section, we will consider specific kinds of sequences that will allow us to calculate depreciation, such as the truck's value.

## Finding Common Differences

The values of the truck in the example are said to form an **arithmetic sequence** because they change by a constant amount each year. Each term increases or decreases by the same constant value called the **common difference** of the sequence. For this sequence, the common difference is $-3,400$.




The sequence below is another example of an arithmetic sequence. In this case, the constant difference is $3$. You can choose any term of the sequence, and add $3$ to find the subsequent term.




**Definition 6.2.1: Arithmetic Sequence**

An *arithmetic sequence* is a sequence in which the difference between any two consecutive terms is a constant. This constant is called the *common difference*. If $a_1$ is the first term of an arithmetic sequence and $d$ is the common difference, the sequence will be:

$$\{a_n\} = \{a_1, a_1 + d, a_1 + 2d, a_1 + 3d, ...\} \tag{7.2.1}$$

**Example 7.2.1: Finding Common Differences**

Is each sequence arithmetic? If so, find the common difference.

a. $\{1, 2, 4, 8, 16, ...\}$
b. $\{-3, 1, 5, 9, 13, ...\}$

**Solution**

Subtract each term from the subsequent term to determine whether a common difference exists.

a. The sequence is not arithmetic because there is no common difference.

$$2 - 1 = 1 \qquad 4 - 2 = 2 \qquad 8 - 4 = 4 \qquad 16 - 8 = 8$$

b. The sequence is arithmetic because there is a common difference. The common difference is $4$.

$$1 - (-3) = 4 \qquad 5 - 1 = 4 \qquad 9 - 5 = 4 \qquad 13 - 9 = 4$$

**Analysis**

The graph of each of these sequences is shown in Figure $7.2.1$. We can see from the graphs that, although both sequences show growth, (a) is not linear whereas (b) is linear. Arithmetic sequences have a constant rate of change so their graphs will always be points on a line.

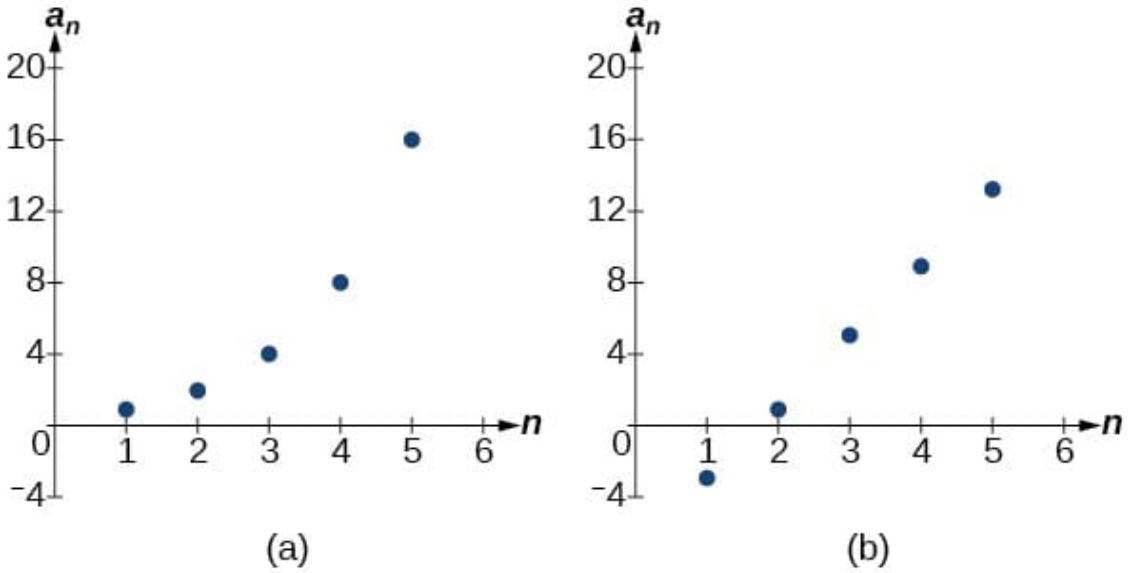



Figure $7.2.1$

**Q&A**

If we are told that a sequence is arithmetic, do we have to subtract every term from the following term to find the common difference?

*No. If we know that the sequence is arithmetic, we can choose any one term in the sequence, and subtract it from the subsequent term to find the common difference.*

**Exercise $7.2.1A$**

Is the given sequence arithmetic? If so, find the common difference.

$$\{18, 16, 14, 12, 10, ...\}$$

*Answer available at the end of this section.*

### Exercise $7.2.1B$

Is the given sequence arithmetic? If so, find the common difference.

$$\{1, 3, 6, 10, 15, ...\}$$

*Answer available at the end of this section.*

## Writing Terms of Arithmetic Sequences

Now that we can recognize an arithmetic sequence, we will find the terms if we are given the first term and the common difference. The terms can be found by beginning with the first term and adding the common difference repeatedly. In addition, any term can also be found by plugging in the values of $n$ and $d$ into formula below.

$$a_n = a_1 + (n-1)d \tag{7.2.2}$$

### How to: Given the first term and the common difference of an arithmetic sequence, find the first several terms.

1. Add the common difference to the first term to find the second term.
2. Add the common difference to the second term to find the third term.
3. Continue until all of the desired terms are identified.
4. Write the terms separated by commas within brackets.

### Example $7.2.2$: Writing Terms of Arithmetic Sequences

Write the first five terms of the arithmetic sequence with $a_1 = 17$ and $d = -3$.

**Solution**

Adding $-3$ is the same as subtracting $3$. Beginning with the first term, subtract $3$ from each term to find the next term.

The first five terms are $\{17, 14, 11, 8, 5\}$

**Analysis**

As expected, the graph of the sequence consists of points on a line as shown in Figure $7.2.2$.

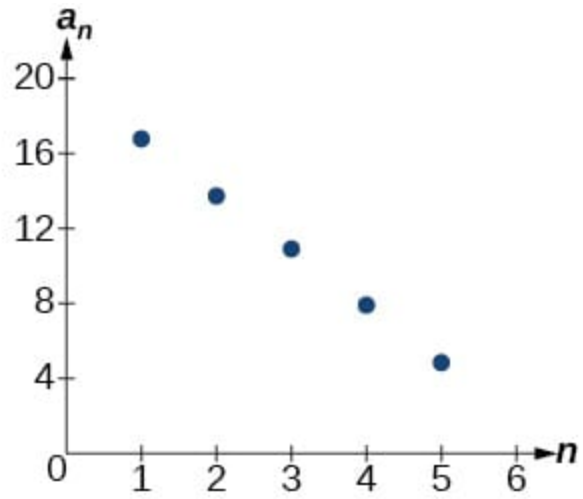



Figure 7.2.2

## Exercise 7.2.2

List the first five terms of the arithmetic sequence with $a_1 = 1$ and $d = 5$.

*Answer available at the end of this section.*

## How to: Given any the first term and any other term in an arithmetic sequence, find a given term.

1. Substitute the values given for $a_1$, $a_n$, $n$ into the formula $a_n = a_1 + (n-1)d$ to solve for $d$.
2. Find a given term by substituting the appropriate values for $a_1$, $n$, and $d$ into the formula $a_n = a_1 + (n-1)d$.

## Example 7.2.3: Writing Terms of Arithmetic Sequences

Given $a_1 = 8$ and $a_4 = 14$, find $a_5$.

**Solution**

The sequence can be written in terms of the initial term $8$ and the common difference $d$.

$$\{8, 8+d, 8+2d, 8+3d\}$$

We know the fourth term equals $14$; we know the fourth term has the form $a_1 + 3d = 8 + 3d$.

We can find the common difference $d$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ a_4 &= a_1 + 3d \\ a_4 &= 8 + 3d \quad \text{Write the fourth term of the sequence in terms of } a_1 \text{ and } d. \\ 14 &= 8 + 3d \quad \text{Substitute 14 for } a_4. \\ d &= 2 \quad \text{Solve for the common difference.} \end{aligned}$$

Find the fifth term by adding the common difference to the fourth term.

$$a_5 = a_4 + 2 = 16$$

**Analysis**

Notice that the common difference is added to the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by adding the common difference to the first term nine times or by using the equation $a_n = a_1 + (n-1)d$.

**Exercise 7.2.3**

Given $a_3 = 7$ and $a_5 = 17$, find $a_2$.

*Answer available at the end of this section.*

## Using Recursive Formulas for Arithmetic Sequences

Some arithmetic sequences are defined in terms of the previous term using a *recursive formula*. The formula provides an algebraic rule for determining the terms of the sequence. A recursive formula allows us to find any term of an arithmetic sequence using a function of the preceding term. Each term is the sum of the previous term and the common difference. For example, if the common difference is $5$, then each term is the previous term plus $5$. As with any recursive formula, the first term must be given.

When the first term of an arithmetic sequence is given. The recursive formula for each successive term with common difference $d$ is given by:

$$a_n = a_{n-1} + d \tag{7.2.3}$$

for $n \geq 2$

**How to: Given an arithmetic sequence, write its recursive formula.**

1. Subtract any term from the subsequent term to find the common difference.
2. State the initial term and substitute the common difference into the recursive formula for arithmetic sequences.

**Example 7.2.4: Writing a Recursive Formula for an Arithmetic Sequence**

Write a **recursive formula** for the **arithmetic sequence**.

$$\{-18, -7, 4, 15, 26, ...\}$$

**Solution**

The first term is given as $-18$. The common difference can be found by subtracting the first term from the second term.

$$d = -7 - (-18) = 11$$

Substitute the initial term and the common difference into the recursive formula for arithmetic sequences.

$$a_1 = -18$$

$$a_n = a_{n-1} + 11$$

for $n \geq 2$

**Analysis**

We see that the common difference is the slope of the line formed when we graph the terms of the sequence, as shown in Figure

7.2.3. The growth pattern of the sequence shows the constant difference of 11 units.

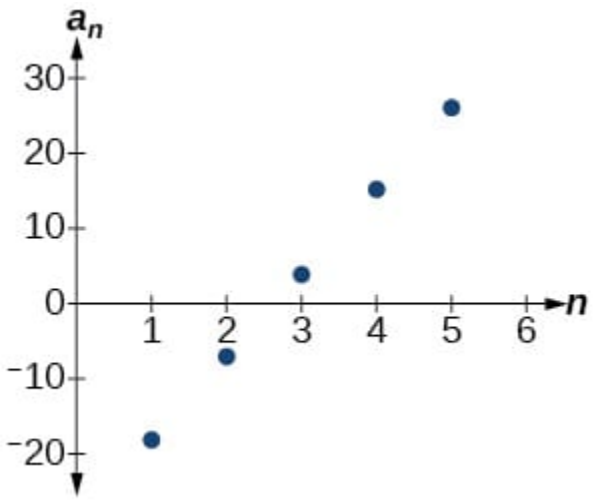



Figure 7.2.3

Q&A

**Do we have to subtract the first term from the second term to find the common difference?**

*No. We can subtract any term in the sequence from the subsequent term. It is, however, most common to subtract the first term from the second term because it is often the easiest method of finding the common difference.*

Exercise 7.2.4

Write a recursive formula for the arithmetic sequence.

$$\{25, 37, 49, 61, ...\}$$

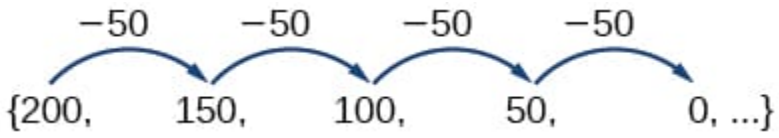
*Answer available at the end of this section.*

## Using Explicit Formulas for Arithmetic Sequences

We can think of an arithmetic sequence as a function on the domain of the natural numbers; it is a linear function because it has a constant rate of change. The common difference is the constant rate of change, or the slope of the function. We can construct the linear function if we know the slope and the vertical intercept.

$$a_n = a_1 + d(n-1)$$

To find the $y$-intercept of the function, we can subtract the common difference from the first term of the sequence. Consider the following sequence.




The common difference is $-50$, so the sequence represents a linear function with a slope of $-50$. To find the $y$-intercept, we subtract $-50$ from $200$: $200-(-50)=200+50=250$. You can also find the $y$-intercept by graphing the function and determining where a line that connects the points would intersect the vertical axis. The graph is shown in Figure 7.2.4.

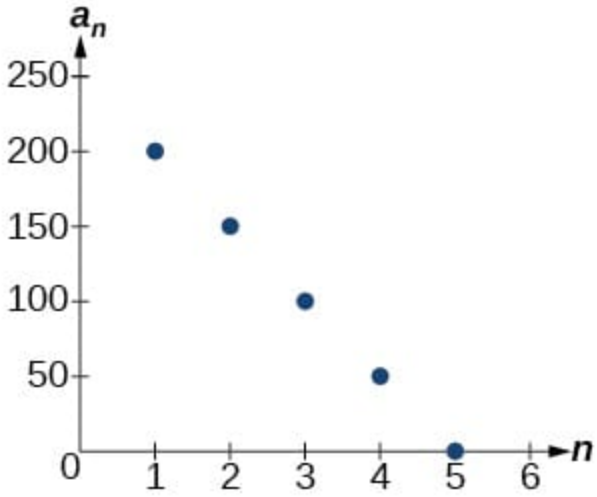



Figure 7.2.4

Recall the slope-intercept form of a line is $y = mx + b$. When dealing with sequences, we use $a_n$ in place of $y$ and $n$ in place of $x$. If we know the slope and vertical intercept of the function, we can substitute them for $m$ and $b$ in the slope-intercept form of a line. Substituting $-50$ for the slope and $250$ for the vertical intercept, we get the following equation:

$$a_n = -50n + 250$$

We do not need to find the vertical intercept to write an explicit formula for an arithmetic sequence. Another explicit formula for this sequence is $a_n = 200 - 50(n-1)$, which simplifies to $a_n = -50n + 250$.

### Note: EXPLICIT FORMULA FOR AN ARITHMETIC SEQUENCE

An explicit formula for the $n^{th}$ term of an arithmetic sequence is given by

$$a_n = a_1 + d(n-1) \tag{7.2.4}$$

### How to: Given the first several terms for an arithmetic sequence, write an explicit formula.

1. Find the common difference, $a_2 - a_1$.
2. Substitute the common difference and the first term into $a_n = a_1 + d(n-1)$.

### Example 7.2.5: Writing the *n*th Term Explicit Formula for an Arithmetic Sequence

Write an explicit formula for the arithmetic sequence.

$$\{2, 12, 22, 32, 42, ...\}$$

**Solution**

The common difference can be found by subtracting the first term from the second term.

$$\begin{aligned} d &= a_2 - a_1 \\ &= 12 - 2 \\ &= 10 \end{aligned}$$

The common difference is $10$. Substitute the common difference and the first term of the sequence into the formula and simplify.

$$a_n = 2 + 10(n-1)$$
$$a_n = 10n - 8$$

**Analysis**

The graph of this sequence, represented in Figure $7.2.5$, shows a slope of $10$ and a vertical intercept of $-8$.

Graph of the arithmetic sequence. The points form a positive line.

Figure $7.2.5$

**Exercise $7.2.5$**

Write an explicit formula for the following arithmetic sequence.

$$\{50, 47, 44, 41, ...\}$$

*Answer available at the end of this section.*

## Finding the Number of Terms in a Finite Arithmetic Sequence

Explicit formulas can be used to determine the number of terms in a finite arithmetic sequence. We need to find the common difference, and then determine how many times the common difference must be added to the first term to obtain the final term of the sequence.

**How to: Given the first three terms and the last term of a finite arithmetic sequence, find the total number of terms.**

1. Find the common difference $d$.
2. Substitute the common difference and the first term into $a_n = a_1 + d(n–1)$.
3. Substitute the last term for $a_n$ and solve for $n$.

**Example $7.2.6$: Finding the Number of Terms in a Finite Arithmetic Sequence**

Find the number of terms in the finite arithmetic sequence.

$$\{8, 1, –6, ..., –41\}$$

**Solution**

The common difference can be found by subtracting the first term from the second term.

$$1 - 8 = -7$$

The common difference is $-7$. Substitute the common difference and the initial term of the sequence into the nth term formula and simplify.

$$a_n = a_1 + d(n-1)$$
$$a_n = 8 + -7(n-1)$$
$$a_n = 15 - 7n$$

Substitute $-41$ for $a_n$ and solve for $n$

$$-41 = 15 - 7n$$
$$8 = n$$

There are eight terms in the sequence.

### Exercise 7.2.6

Find the number of terms in the finite arithmetic sequence.

$$\{6, 11, 16, ..., 56\}$$

*Answer available at the end of this section.*

## Solving Application Problems with Arithmetic Sequences

In many application problems, it often makes sense to use an initial term of $a_0$ instead of $a_1$. In these problems, we alter the explicit formula slightly to account for the difference in initial terms. We use the following formula:

$$a_n = a_0 + dn \tag{7.2.5}$$

### Example 7.2.7: Solving Application Problems with Arithmetic Sequences

A five-year old child receives an allowance of $\$1$ each week. His parents promise him an annual increase of $\$2$ per week.

a. Write a formula for the child's weekly allowance in a given year.
b. What will the child's allowance be when he is $16$ years old?

**Solution**

a. The situation can be modeled by an arithmetic sequence with an initial term of $1$ and a common difference of $2$.

   Let $A$ be the amount of the allowance and $n$ be the number of years after age $5$. Using the altered explicit formula for an arithmetic sequence we get:

$$A_n = 1 + 2n$$

b. We can find the number of years since age $5$ by subtracting.

$$16 - 5 = 11$$

   We are looking for the child's allowance after $11$ years. Substitute $11$ into the formula to find the child's allowance at age $16$.

$$A_{11} = 1 + 2(11) = 23$$

   The child's allowance at age $16$ will be $\$23$ per week.

### Exercise 7.2.7

A woman decides to go for a $10$-minute run every day this week and plans to increase the time of her daily run by $4$ minutes each week. Write a formula for the time of her run after $n$ weeks. How long will her daily run be $8$ weeks from today?

*Answer available at the end of this section.*

Media

Access this online resource for additional instruction and practice with arithmetic sequences.

- Arithmetic Sequences

## Key Equations

| | |
|---|---|
| recursive formula for nth term of an arithmetic sequence | $a_n = a_{n-1} + d\ n \geq 2$ |
| explicit formula for nth term of an arithmetic sequence | $a_n = a_1 + d(n-1)$ |

## Key Concepts

- An arithmetic sequence is a sequence where the difference between any two consecutive terms is a constant.
- The constant between two consecutive terms is called the common difference.
- The common difference is the number added to any one term of an arithmetic sequence that generates the subsequent term. See Example $7.2.1$.
- The terms of an arithmetic sequence can be found by beginning with the initial term and adding the common difference repeatedly. See Example $7.2.2$ and Example $7.2.3$.
- A recursive formula for an arithmetic sequence with common difference dd is given by $a_n = a_{n-1} + d$, $n \geq 2$. See Example $7.2.4$.
- As with any recursive formula, the initial term of the sequence must be given.
- An explicit formula for an arithmetic sequence with common difference $d$ is given by $a_n = a_1 + d(n-1)$. See Example $7.2.5$.
- An explicit formula can be used to find the number of terms in a sequence. See Example $7.2.6$.
- In application problems, we sometimes alter the explicit formula slightly to $a_n = a_0 + dn$. See Example $7.2.7$.

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### Answers

**Exercise** $7.2.1A$

The sequence is arithmetic. The common difference is –2.

**Exercise** $7.2.1B$

The sequence is not arithmetic because $3-1 \neq 6-3$.

**Exercise** $7.2.2$

$\{1, 6, 11, 16, 21\}$

**Exercise** $7.2.3$

$a_2 = 2$

**Exercise** $7.2.4$

$a_1 = 25$
$a_n = a_{n-1} + 12, \text{ for } n \geq 2$

**Exercise** $7.2.5$

$a_n = 53 - 3n$

**Exercise** $7.2.6$

There are $11$ terms in the sequence.

**Exercise** $7.2.7$

The formula is $T_n = 10 + 4n$, and it will take her $42$ minutes.

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# 7.3: Geometric Sequences

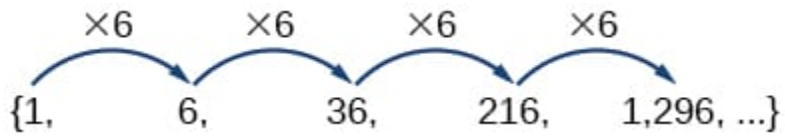
**Learning Objectives**

- Find the common ratio for a geometric sequence.
- List the terms of a geometric sequence.
- Use a recursive formula for a geometric sequence.
- Use an explicit formula for a geometric sequence.

Many jobs offer an annual cost-of-living increase to keep salaries consistent with inflation. Suppose, for example, a recent college graduate finds a position as a sales manager earning an annual salary of $\$26,000$. He is promised a $2\%$ cost of living increase each year. His annual salary in any given year can be found by multiplying his salary from the previous year by $102\%$. His salary will be $\$26,520$ after one year; $\$27,050.40$ after two years; $\$27,591.41$ after three years; and so on. When a salary increases by a constant rate each year, the salary grows by a constant factor. In this section, we will review sequences that grow in this way.

## Finding Common Ratios

The yearly salary values described form a **geometric sequence** because they change by a constant factor each year. Each term of a geometric sequence increases or decreases by a constant factor called the **common ratio**. The sequence below is an example of a geometric sequence because each term increases by a constant factor of 6. Multiplying any term of the sequence by the common ratio 6 generates the subsequent term.




**Definition 7.3.1: Geometric Sequence**

A *geometric sequence* is a sequence in which the quotient of any term and its previous term is a constant. This constant is called the *common ratio* of the sequence. The common ratio can be found by dividing any term in the sequence by the previous term. If $a_1$ is the initial term of a geometric sequence and $r$ is the common ratio, the sequence will be

$$\{a_1, a_1 r, a_1 r^2, a_1 r^3, ...\}. \tag{7.3.1}$$

**How to: Given a set of numbers, determine if they represent a geometric sequence.**

1. Divide each term by the previous term.
2. Compare the quotients. If they are the same, a common ratio exists and the sequence is geometric.

**Example 7.3.1: Finding Common Ratios**

Is the sequence geometric? If so, find the common ratio.

a. $1, 2, 4, 8, 16,...$
b. $48, 12, 4, 2,...$

**Solution**

Divide each term by the previous term to determine whether a common ratio exists.

a. $\frac{2}{1} = 2\ \frac{4}{2} = 2\ \frac{8}{4} = 2\ \frac{16}{8} = 2$

The sequence is geometric because there is a common ratio. The common ratio is $2$.

b. $\frac{12}{48} = \frac{1}{4}\ \frac{4}{12} = \frac{1}{3}\ \frac{2}{4} = \frac{1}{2}$

The sequence is not geometric because there is not a common ratio.

**Analysis**

The graph of each sequence is shown in Figure $7.3.1$. It seems from the graphs that both (a) and (b) appear have the form of the graph of an exponential function in this viewing window. However, we know that (a) is geometric and so this interpretation holds, but (b) is not.

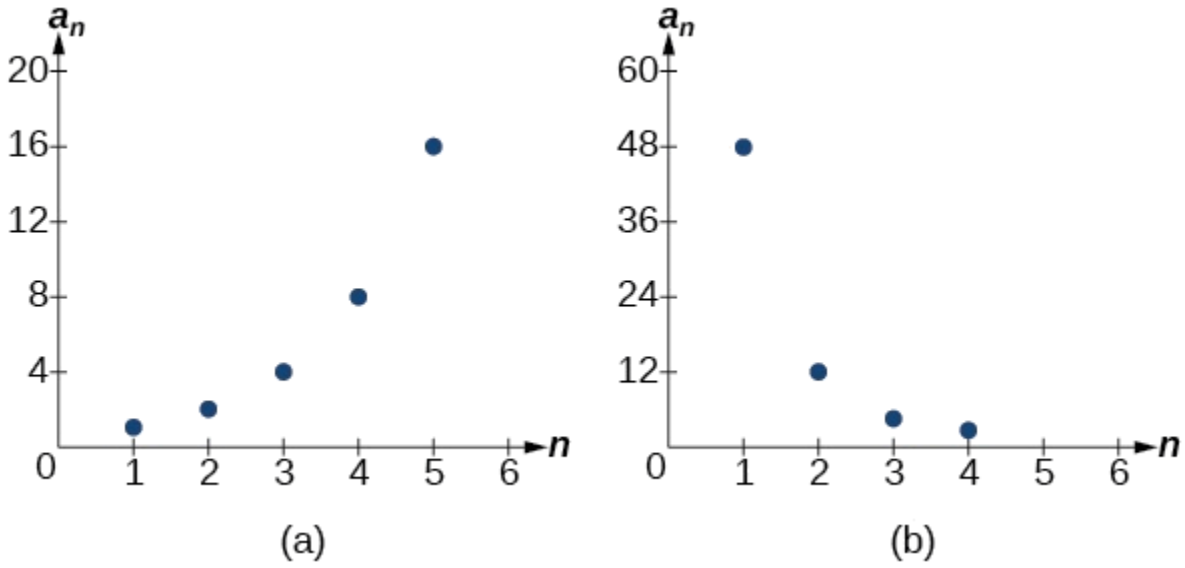



Figure $7.3.1$

Q&A

**If you are told that a sequence is geometric, do you have to divide every term by the previous term to find the common ratio?**

*No. If you know that the sequence is geometric, you can choose any one term in the sequence and divide it by the previous term to find the common ratio.*

Exercise $7.3.1A$

Is the sequence geometric? If so, find the common ratio.

$$5,\ 10,\ 15,\ 20,\ldots$$

*Answer available at the end of this section.*

**Exercise 7.3.1B**

Is the sequence geometric? If so, find the common ratio.

$$100,\ 20,\ 4,\ \frac{4}{5},\ldots$$

*Answer available at the end of this section.*

## Writing Terms of Geometric Sequences

Now that we can identify a geometric sequence, we will learn how to find the terms of a geometric sequence if we are given the first term and the common ratio. The terms of a geometric sequence can be found by beginning with the first term and multiplying by the common ratio repeatedly. For instance, if the first term of a geometric sequence is $a_1 = -2$ and the common ratio is $r = 4$, we can find subsequent terms by multiplying $-2 \cdot 4$ to get $-8$ then multiplying the result $-8 \cdot 4$ to get $-32$ and so on.

$$\begin{aligned} a_1 &= -2 \\ a_2 &= (-2 \cdot 4) = -8 \\ a_3 &= (-8 \cdot 4) = -32 \\ a_4 &= (-32 \cdot 4) = 128 \end{aligned}$$

The first four terms are $\{-2, -8, -32, -128\}$.

**How to: Given the first term and the common factor, find the first four terms of a geometric sequence.**

1. Multiply the initial term, $a_1$, by the common ratio to find the next term, $a_2$.
2. Repeat the process, using $a_n = a_2$ to find $a_3$ and then use $a_3$ to find $a_4$, until all four terms have been identified.
3. Write the terms separated by commons within brackets.

**Example 7.3.2: Writing the Terms of a Geometric Sequence**

List the first four terms of the geometric sequence with $a_1 = 5$ and $r = -2$.

**Solution**

Multiply $a_1$ by $-2$ to find $a_2$. Repeat the process, using $a_2$ to find $a_3$, and so on.

$$\begin{aligned} a_1 &= 5 \\ a_2 &= -2a_1 = -10 \\ a_3 &= -2a_2 = 20 \\ a_4 &= -2a_3 = -40 \end{aligned}$$

The first four terms are $\{5, -10, 20, -40\}$.

**Exercise 7.3.2**

List the first five terms of the geometric sequence with $a_1 = 18$ and $r = \frac{1}{3}$.

*Answer available at the end of this section.*

# Using Recursive Formulas for Geometric Sequences

A recursive formula allows us to find any term of a geometric sequence by using the previous term. Each term is the product of the common ratio and the previous term. For example, suppose the common ratio is $9$. Then each term is nine times the previous term. As with any recursive formula, the initial term must be given.

**Note: RECURSIVE FORMULA FOR A GEOMETRIC SEQUENCE**

The recursive formula for a geometric sequence with common ratio r and first term $a_1$ is

$$a_n = ra_{n-1}, \quad n \geq 2 \tag{7.3.2}$$

**How to: Given the first several terms of a geometric sequence, write its recursive formula.**

1. State the initial term.
2. Find the common ratio by dividing any term by the preceding term.
3. Substitute the common ratio into the recursive formula for a geometric sequence.

**Example 7.3.3: Using Recursive Formulas for Geometric Sequences**

Write a recursive formula for the following geometric sequence.

$$\{6, 9, 13.5, 20.25, ...\}$$

## Solution

The first term is given as $6$. The common ratio can be found by dividing the second term by the first term.

$$r = \frac{9}{6} = 1.5$$

Substitute the common ratio into the recursive formula for geometric sequences and define $a_1$.

$$\begin{aligned} a_n &= ra_{n-1} \\ a_n &= 1.5a_{n-1} \text{ for } n \geq 2 \\ a_1 &= 6 \end{aligned}$$

### Analysis

The sequence of data points follows an exponential pattern. The common ratio is also the base of an exponential function as shown in Figure $7.3.2$.

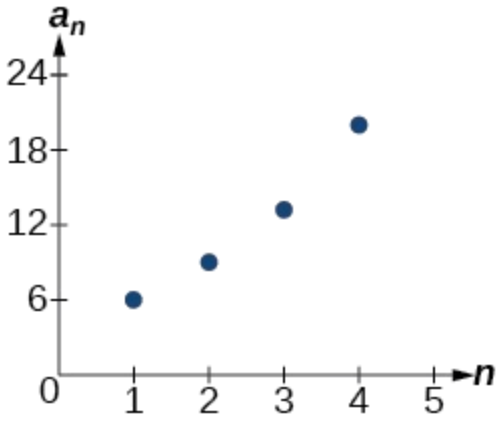



Figure 7.3.2

Q&A

**Do we have to divide the second term by the first term to find the common ratio?**

*No. We can divide any term in the sequence by the previous term. It is, however, most common to divide the second term by the first term because it is often the easiest method of finding the common ratio.*

Exercise 7.3.3

Write a recursive formula for the following geometric sequence.

$$\{2, \tfrac{4}{3}, \tfrac{8}{9}, \tfrac{16}{27}, ...\}$$

*Answer available at the end of this section.*

## Using Explicit Formulas for Geometric Sequences

Because a geometric sequence is an exponential function whose domain is the set of positive integers, and the common ratio is the base of the function, we can write explicit formulas that allow us to find particular terms.

$$a_n = a_1 r^{n-1} \tag{7.3.3}$$

Let's take a look at the sequence $\{18, 36, 72, 144, 288, ...\}$. This is a geometric sequence with a common ratio of $2$ and an exponential function with a base of $2$. An explicit formula for this sequence is

$$a_n = 18 \cdot 2^{n-1}$$

The graph of the sequence is shown in Figure 7.3.3.

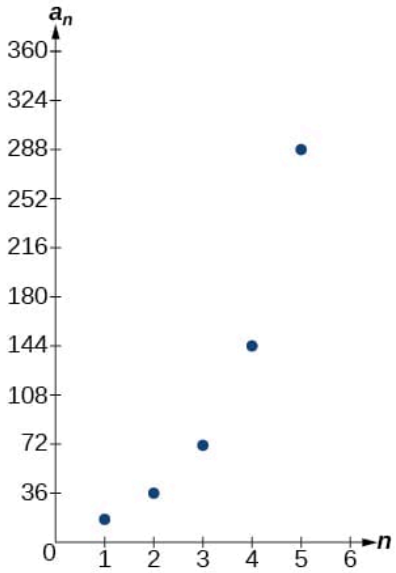



Figure 7.3.3

## Note: EXPLICIT FORMULA FOR A GEOMETRIC SEQUENCE

The $n^{th}$ term of a geometric sequence is given by the *explicit formula*:

$$a_n = a_1 r^{n-1} \tag{7.3.4}$$

## Example 7.3.4: Writing Terms of Geometric Sequences Using the Explicit Formula

Given a geometric sequence with $a_1 = 3$ and $a_4 = 24$, find $a_2$.

**Solution**

The sequence can be written in terms of the initial term and the common ratio $r$.

$$3,\ 3r,\ 3r^2,\ 3r^3,\ldots$$

Find the common ratio using the given fourth term.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_4 &= 3r^3 \quad \text{Write the fourth term of sequence in terms of } a_1 \text{ and } r \\ 24 &= 3r^3 \quad \text{Substitute 24 for } a_4 \\ 8 &= r^3 \quad \text{Divide} \\ r &= 2 \quad \text{Solve for the common ratio} \end{aligned}$$

Find the second term by multiplying the first term by the common ratio.

$$\begin{aligned} a_2 &= 2 \\ a_1 &= 2(3) \\ &= 6 \end{aligned}$$

**Analysis**

The common ratio is multiplied by the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by multiplying the first term by the common ratio nine times or by multiplying by the common ratio raised to the ninth power.

## Exercise 7.3.4

Given a geometric sequence with $a_2 = 4$ and $2a_3 = 32$, find $a_6$.

*Answer available at the end of this section.*

## Example 7.3.5: Writing an Explicit Formula for the *n*th Term of a Geometric Sequence

Write an explicit formula for the nth term of the following geometric sequence.

$$\{2, 10, 50, 250, ...\}$$

**Solution**

The first term is $2$. The common ratio can be found by dividing the second term by the first term.

$$\frac{10}{2} = 5$$

The common ratio is $5$. Substitute the common ratio and the first term of the sequence into the formula.

$$a_n = a_1 r^{(n-1)}$$
$$a_n = 2 \cdot 5^{n-1}$$

The graph of this sequence in Figure 7.3.4 shows an exponential pattern.

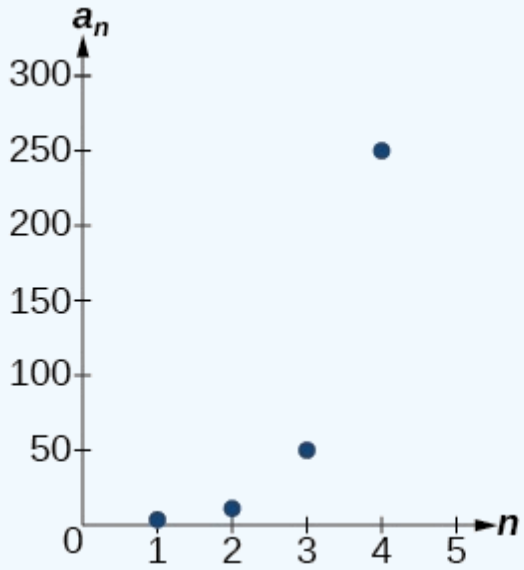



Figure 7.3.4

## Exercise 7.3.5

Write an explicit formula for the following geometric sequence.

$$\{-1, 3, -9, 27, ...\}$$

*Answer available at the end of this section.*

# Solving Application Problems with Geometric Sequences

In real-world scenarios involving arithmetic sequences, we may need to use an initial term of $a_0$ instead of $a_1$. In these problems, we can alter the explicit formula slightly by using the following formula:

$$a_n = a_0 r^n$$

## Example 7.3.6: Solving Application Problems with Geometric Sequences

In 2013, the number of students in a small school is $284$. It is estimated that the student population will increase by $4\%$ each year.

a. Write a formula for the student population.
b. Estimate the student population in 2020.

**Solution**

a. The situation can be modeled by a geometric sequence with an initial term of $284$. The student population will be $104\%$ of the prior year, so the common ratio is $1.04$.

   Let $P$ be the student population and $n$ be the number of years after 2013. Using the explicit formula for a geometric sequence we get

$$P_n = 284 \cdot 1.04^n$$

b. We can find the number of years since 2013 by subtracting.

$$2020 - 2013 = 7$$

   We are looking for the population after $7$ years. We can substitute $7$ for $n$ to estimate the population in 2020.

$$P_7 = 284 \cdot 1.04^7 \approx 374$$

   The student population will be about $374$ in 2020.

## Exercise 7.3.6

A business starts a new website. Initially the number of hits is $293$ due to the curiosity factor. The business estimates the number of hits will increase by $2.6$ per week.

a. Write a formula for the number of hits.
b. Estimate the number of hits in $5$ weeks.

*Answer a available at the end of this section.*

## Media

Access these online resources for additional instruction and practice with geometric sequences.

- Geometric Sequences
- Determine the Type of Sequence
- Find the Formula for a Sequence

## Key Equations

| | |
|---|---|
| recursive formula for nth term of a geometric sequence | $a_n = ra_{n-1}, n \geq 2$ |
| explicit formula for nth term of a geometric sequence | $a_n = a_1 r^{n-1}$ |

## Key Concepts

- A geometric sequence is a sequence in which the ratio between any two consecutive terms is a constant.
- The constant ratio between two consecutive terms is called the common ratio.
- The common ratio can be found by dividing any term in the sequence by the previous term. See Example $7.3.1$.
- The terms of a geometric sequence can be found by beginning with the first term and multiplying by the common ratio repeatedly. See Example $7.3.2$ and Example $7.3.4$.
- A recursive formula for a geometric sequence with common ratio $r$ is given by $a_n = ra_{n-1}$ for $n \geq 2$.
- As with any recursive formula, the initial term of the sequence must be given. See Example $7.3.3$.
- An explicit formula for a geometric sequence with common ratio $r$ is given by $a_n = a_1 r^{n-1}$. See Example $7.3.5$.
- In application problems, we sometimes alter the explicit formula slightly to $a_n = a_0 r^n$. See Example $7.3.6$.

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### Answers

**Exercise** $7.3.1A$

The sequence is not geometric because $\frac{10}{5} \neq \frac{15}{10}$

**Exercise** $7.3.1B$

The sequence is geometric. The common ratio is $\frac{1}{5}$

**Exercise** $7.3.2$

$\left\{18, 6, 2, \frac{2}{3}, \frac{2}{9}\right\}$

**Exercise** $7.3.3$

$a_1 = 2$

$a_n = \frac{2}{3}a_{n-1}$ for $n \geq 2$

**Exercise** $7.3.4$

$a_6 = 16,384$

**Exercise** $7.3.5$

$a_n = -(-3)^{n-1}$

**Exercise** $7.3.6$

$P_n = 293 \cdot 1.026a^n$

The number of hits will be about $333$.

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# 7.4: Summations and Series

**Learning Objectives**

- Use summation notation.
- Use the formula for the sum of the first $n$ terms of an arithmetic series.
- Use the formula for the sum of the first $n$ terms of a geometric series.
- Use the formula for the sum of an infinite geometric series.
- Solve annuity problems.

A couple decides to start a college fund for their daughter. They plan to invest $\$50$ in the fund each month. The fund pays $6\%$ annual interest, compounded monthly. How much money will they have saved when their daughter is ready to start college in $6$ years? In this section, we will learn how to answer this question. To do so, we need to consider the amount of money invested and the amount of interest earned.

## Using Summation Notation

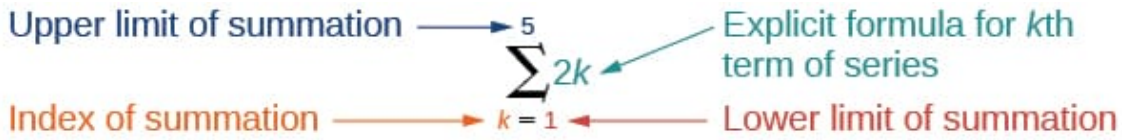
To find the total amount of money in the college fund and the sum of the amounts deposited, we need to add the amounts deposited each month and the amounts earned monthly. The sum of the terms of a sequence is called a *series*. Consider, for example, the following series.

$$3 + 7 + 11 + 15 + 19 + \ldots$$

The $n^{th}$ *partial sum* of a series is the sum of a finite number of consecutive terms beginning with the first term. The notation $S_n$ represents the partial sum.

$$\begin{aligned} S_1 &= 3 \\ S_2 &= 3 + 7 = 10 \\ S_3 &= 3 + 7 + 11 = 21 \\ S_4 &= 3 + 7 + 11 + 15 = 36 \end{aligned}$$

*Summation notation* is used to represent series. Summation notation is often known as sigma notation because it uses the Greek capital letter *sigma*, $\sum$, to represent the sum. Summation notation includes an explicit formula and specifies the first and last terms in the series. An explicit formula for each term of the series is given to the right of the sigma. A variable called the *index of summation* is written below the sigma. The index of summation is set equal to the *lower limit of summation*, which is the number used to generate the first term in the series. The number above the sigma, called the *upper limit of summation*, is the number used to generate the last term in a series.




If we interpret the given notation, we see that it asks us to find the sum of the terms in the series $a_k = 2k$ for $k = 1$ through $k = 5$. We can begin by substituting the terms for $k$ and listing out the terms of this series.

$$\begin{aligned} a_1 &= 2(1) = 2 \\ a_2 &= 2(2) = 4 \\ a_3 &= 2(3) = 6 \\ a_4 &= 2(4) = 8 \\ a_5 &= 2(5) = 10 \end{aligned}$$

We can find the sum of the series by adding the terms:

$$\sum_{k=1}^{5} 2k = 2 + 4 + 6 + 8 + 10 = 30$$

### Definition 7.4.1: Summation Notation

The sum of the first $n$ terms of a series can be expressed in *summation notation* as follows:

$$\sum_{k=1}^{n} a_k \tag{7.4.1}$$

This notation tells us to find the sum of $a_k$ from $k = 1$ to $k = n$.

$k$ is called the index of summation, $1$ is the lower limit of summation, and $n$ is the upper limit of summation.

### Q&A: Does the lower limit of summation have to be 1?

*No. The lower limit of summation can be any number, but $1$ is frequently used. We will look at examples with lower limits of summation other than $1$.*

### How to: Given summation notation for a series, evaluate the value

1. Identify the lower limit of summation.
2. Identify the upper limit of summation.
3. Substitute each value of $k$ from the lower limit to the upper limit into the formula.
4. Add to find the sum.

### Example 7.4.1: Using Summation Notation

Evaluate

$$\sum_{k=3}^{7} k^2$$

**Solution**

According to the notation, the lower limit of summation is $3$ and the upper limit is $7$. So we need to find the sum of $k^2$ from $k = 3$ to $k = 7$. We find the terms of the series by substituting $k = 3, 4, 5, 6,$ and $7$ into the function $k^2$. We add the terms to find the sum.

$$\begin{aligned}\sum_{k=3}^{7} k^2 &= 3^2 + 4^2 + 5^2 + 6^2 + 7^2 \\ &= 9 + 16 + 25 + 36 + 49 \\ &= 135\end{aligned}$$

### Exercise 7.4.1

Evaluate

$$\sum_{k=2}^{5} (3k - 1)$$

*Answer available at the end of this section.*

# Using the Formula for Arithmetic Series

Just as we studied special types of sequences, we will look at special types of series. Recall that an arithmetic sequence is a sequence in which the difference between any two consecutive terms is the *common difference*, $d$. The sum of the terms of an arithmetic sequence is called an *arithmetic series*. We can write the sum of the first $n$ terms of an arithmetic series as:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_n - d) + a_n.$$

We can also reverse the order of the terms and write the sum as

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_1 + d) + a_1.$$

If we add these two expressions for the sum of the first $n$ terms of an arithmetic series, we can derive a formula for the sum of the first $n$ terms of any arithmetic series.

$$\begin{aligned} S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_n - d) + a_n \\ +S_n &= a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_1 + d) + a_1 \\ \hline 2S_n &= (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n) \end{aligned}$$

Because there are $n$ terms in the series, we can simplify this sum to

$$2S_n = n(a_1 + a_n).$$

We divide by $2$ to find the formula for the sum of the first $n$ terms of an arithmetic series.

$$S_n = \frac{n(a_1 + a_n)}{2}$$

### FORMULA FOR THE SUM OF THE FIRST $N$ TERMS OF AN ARITHMETIC SERIES

An arithmetic series is the sum of the terms of an arithmetic sequence. The formula for the sum of the first $n$ terms of an arithmetic sequence is

$$S_n = \frac{n(a_1 + a_n)}{2} \tag{7.4.2}$$

### How to: Given terms of an arithmetic series, find the sum of the first $n$ terms

1. Identify $a_1$ and $a_n$.
2. Determine $n$.
3. Substitute values for $a_1$, $a_n$, and $n$ into the formula $S_n = \frac{n(a_1 + a_n)}{2}$.
4. Simplify to find $S_n$.

### Terms of an Arithmetic Series

Find the sum of each arithmetic series.

a. $5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32$
b. $20 + 15 + 10 + \ldots + -50$
c. $\sum_{k=1}^{12} 3k - 8$

**Solution**

a. We are given $a_1 = 5$ and $a_n = 32$.

Count the number of terms in the sequence to find $n = 10$.

Substitute values for $a_1$, $a_n$, and $n$ into the formula and simplify.

$$\begin{aligned} S_n &= \frac{n(a_1 + a_n)}{2} \\ S_{10} &= \frac{10(5 + 32)}{2} \\ &= 185 \end{aligned}$$

2. We are given $a_1 = 20$ and $a_n = -50$.

Use the formula for the general term of an arithmetic sequence to find $n$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ -50 &= 20 + (n-1)(-5) \\ -70 &= (n-1)(-5) \\ 14 &= n - 1 \\ 15 &= n \end{aligned}$$

Substitute values for $a_1$, $a_n$, $n$ into the formula and simplify.

$$\begin{aligned} S_n &= \frac{n(a_1 + a_n)}{2} \\ S_{15} &= \frac{15(20 - 50)}{2} \\ &= -225 \end{aligned}$$

c. To find $a_1$, substitute $k = 1$ into the given explicit formula.

$$\begin{aligned} a_k &= 3k - 8 \\ a_1 &= 3(1) - 8 \\ &= -5 \end{aligned}$$

We are given that $n = 12$. To find $a_12$, substitute $k = 12$ into the given explicit formula.

$$\begin{aligned} a_k &= 3k - 8 \\ a_{12} &= 3(12) - 8 \\ &= 28 \end{aligned}$$

Substitute values for $a_1$, $a_n$, and $n$ into the formula and simplify.

$$\begin{aligned} S_n &= \frac{n(a_1 + a_n)}{2} \\ S_{12} &= \frac{12(-5 + 28)}{2} \\ &= 138 \end{aligned}$$

Use the formula to find the sum of each arithmetic series.

## Exercise 7.4.2A

$1.4 + 1.6 + 1.8 + 2.0 + 2.2 + 2.4 + 2.6 + 2.8 + 3.0 + 3.2 + 3.4$

*Answer available at the end of this section.*

### Exercise $7.4.2B$

$13 + 21 + 29 + ... + 69$

*Answer available at the end of this section.*

### Exercise $7.4.2C$

$\sum_{k=1} 105 - 6k$

*Answer available at the end of this section.*

### Example $7.4.3$: Solving Application Problems with Arithmetic Series

On the Sunday after a minor surgery, a woman is able to walk a half-mile. Each Sunday, she walks an additional quarter-mile. After $8$ weeks, what will be the total number of miles she has walked?

**Solution**

This problem can be modeled by an arithmetic series with $a_1 = \frac{1}{2}$ and $d = \frac{1}{4}$. We are looking for the total number of miles walked after $8$ weeks, so we know that $n = 8$, and we are looking for $S_8$. To find $a_8$, we can use the explicit formula for an arithmetic sequence.

$$\begin{aligned} a_n &= a_1 + d(n-1) \\ a_8 &= \frac{1}{2} + \frac{1}{4}(8-1) \\ &= \frac{9}{4} \end{aligned}$$

We can now use the formula for arithmetic series.

$$\begin{aligned} S_n &= \frac{n(a_1 + a_n)}{2} \\ S_8 &= \frac{8\left(\frac{1}{2} + \frac{9}{4}\right)}{2} \\ &= 11 \end{aligned}$$

She will have walked a total of $11$ miles.

### Exercise $7.4.3$

A man earns $\$100$ in the first week of June. Each week, he earns $\$12.50$ more than the previous week. After $12$ weeks, how much has he earned?

*Answer available at the end of this section.*

## Using the Formula for Geometric Series

Just as the sum of the terms of an arithmetic sequence is called an arithmetic series, the sum of the terms in a geometric sequence is called a **geometric series**. Recall that a **geometric sequence** is a sequence in which the ratio of any two consecutive terms is the **common ratio**, $r$. We can write the sum of the first $n$ terms of a geometric series as

$$S_n = a_1 + ra_1 + r^2a_1 + ... + r^{n-1}a_1.$$

Just as with arithmetic series, we can do some algebraic manipulation to derive a formula for the sum of the first $n$ terms of a geometric series. We will begin by multiplying both sides of the equation by $r$.

$$rS_n = ra_1 + r^2a_1 + r^3a_1 + ... + r^na_1$$

Next, we subtract this equation from the original equation.

$$\begin{aligned} S_n &= a_1 + ra_1 + r^2a_1 + ... + r^{n-1}a_1 \\ -rS_n &= -(ra_1 + r^2a_1 + r^3a_1 + ... + r^na_1) \\ (1-r)S_n &= a_1 - r^na_1 \end{aligned}$$

Notice that when we subtract, all but the first term of the top equation and the last term of the bottom equation cancel out. To obtain a formula for $S_n$, divide both sides by $(1-r)$.

$$S_n = \frac{a_1(1-r^n)}{1-r} \; r \neq 1$$

## TERMS OF A GEOMETRIC SERIES

A geometric series is the sum of the terms in a geometric sequence. The formula for the sum of the first $n$ terms of a geometric sequence is represented as

$$S_n = \frac{a_1(1-r^n)}{1-r} \; r \neq 1$$

## How to: Given a geometric series, find the sum of the first $n$ terms.

1. Identify $a_1$, $r$, and $n$.
2. Substitute values for $a_1$, $r$, and $n$ into the formula $S_n = \frac{a_1(1-r^n)}{1-r}$.
3. Simplify to find $S_n$.

## Terms of a Geometric Series

Use the formula to find the indicated partial sum of each geometric series.

a. $S_{11}$ for the series $8 + -4 + 2 + ...$
b. $\sum_6^{k=1} 3 \cdot 2k$

**Solution**

a. $a_1 = 8$, and we are given that $n = 11$.

We can find $r$ by dividing the second term of the series by the first.

$$r = \frac{-4}{8} = -\frac{1}{2}$$

Substitute values for $a_1$, $r$, and $n$ into the formula and simplify.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$
$$S_{11} = \frac{8\left(1-\left(-\frac{1}{2}\right)^{11}\right)}{1-\left(-\frac{1}{2}\right)}$$
$$\approx 5.336$$

b. Find $a_1$ by substituting $k=1$ into the given explicit formula.

$$a_1 = 3 \cdot 2^1 = 6$$

We can see from the given explicit formula that $r=2$. The upper limit of summation is $6$, so $n=6$.

Substitute values for $a_1$, $r$, and $n$ into the formula, and simplify.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$
$$S_6 = \frac{6(1-2^6)}{1-2}$$
$$= 378$$

Use the formula to find the indicated partial sum of each geometric series.

## Exercise $7.4.4A$

$S_{20}$ for the series $1,000 + 500 + 250 + ...$

*Answer available at the end of this section.*

## Exercise $7.4.4B$

$\sum_{k=1}^{8} 3^k$

*Answer available at the end of this section.*

## Example $7.4.5$: Solving an Application Problem with a Geometric Series

At a new job, an employee's starting salary is $\$26,750$. He receives a $1.6\%$ annual raise. Find his total earnings at the end of $5$ years.

**Solution**

The problem can be represented by a geometric series with $a_1 = 26,750$; $n=5$; and $r=1.016$. Substitute values for $a_1$, $r$, and $n$ into the formula and simplify to find the total amount earned at the end of $5$ years.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$
$$S_5 = \frac{26,750(1-1.016^5)}{1-1.016}$$
$$\approx 138,099.03$$

He will have earned a total of $\$138,099.03$ by the end of $5$ years.

**Exercise 7.4.5**

At a new job, an employee's starting salary is $\$32,100$. She receives a $2\%$ annual raise. How much will she have earned by the end of $8$ years?

*Answer available at the end of this section.*

# Using the Formula for the Sum of an Infinite Geometric Series

Thus far, we have looked only at finite series. Sometimes, however, we are interested in the sum of the terms of an infinite sequence rather than the sum of only the first $n$ terms. An infinite series is the sum of the terms of an **infinite sequence**. An example of an infinite series is $2+4+6+8+...$

This series can also be written in summation notation as $\sum_{k=1}^{\infty} 2k$, where the upper limit of summation is infinity. Because the terms are not tending to zero, the sum of the series increases without bound as we add more terms. Therefore, the sum of this infinite series is not defined. When the sum is not a real number, we say the series **diverges**.

## Determining Whether the Sum of an Infinite Geometric Series is Defined

If the terms of an **infinite geometric series** approach $0$, the sum of an infinite geometric series can be defined. The terms in this series approach $0$:

$$1+0.2+0.04+0.008+0.0016+...$$

The common ratio $r=0.2$. As $n$ gets very large, the values of $r^n$ get very small and approach $0$. Each successive term affects the sum less than the preceding term. As each succeeding term gets closer to $0$, the sum of the terms approaches a finite value. The terms of any infinite geometric series with $-1<r<1$ approach $0$; the sum of a geometric series is defined when $-1<r<1$.

**DETERMINING WHETHER THE SUM OF AN INFINITE GEOMETRIC SERIES IS DEFINED**

The sum of an infinite series is defined if the series is geometric and $-1<r<1$.

**How to: Given the first several terms of an infinite series, determine if the sum of the series exists.**

1. Find the ratio of the second term to the first term.
2. Find the ratio of the third term to the second term.
3. Continue this process to ensure the ratio of a term to the preceding term is constant throughout. If so, the series is geometric.
4. If a common ratio, $r$, was found in step 3, check to see if $-1<r<1$. If so, the sum is defined. If not, the sum is not defined.

**Example 7.4.6: Determining Whether the Sum of an Infinite Series is Defined**

Determine whether the sum of each infinite series is defined.

a. $12+8+4+...$
b. $\frac{3}{4}+\frac{1}{2}+\frac{1}{3}+...$
c. $\sum_{k=1}^{\infty} 27\cdot\left(\frac{1}{3}\right)^k$
d. $\sum_{k=1}^{\infty} 5k$

**Solution**

a. The ratio of the second term to the first is $\frac{2}{3}$, which is not the same as the ratio of the third term to the second, $\frac{1}{2}$.The series is not geometric.
b. The ratio of the second term to the first is the same as the ratio of the third term to the second. The series is geometric with a common ratio of $\frac{2}{3}$. The sum of the infinite series is defined.
c. The given formula is exponential with a base of $\frac{1}{3}$; the series is geometric with a common ratio of $\frac{1}{3}$. The sum of the infinite series is defined.
d. The given formula is not exponential; the series is not geometric because the terms are increasing, and so cannot yield a finite sum.

Determine whether the sum of the infinite series is defined.

**Exercise 7.4.6A**

$\frac{1}{3}+\frac{1}{2}+\frac{3}{4}+\frac{9}{8}+...$

*Answer available at the end of this section.*

**Exercise 7.4.6B**

$24+(-12)+6+(-3)+...$

*Answer available at the end of this section.*

**Exercise 7.4.6C**

$\sum_{k=1}^{\infty} 15\cdot(-0.3)^k$

*Answer available at the end of this section.*

## Finding Sums of Infinite Series

When the sum of an infinite geometric series exists, we can calculate the sum. The formula for the sum of an infinite series is related to the formula for the sum of the first $n$ terms of a geometric series.

$$S_n=\frac{a_1(1-r^n)}{1-r} \tag{7.4.3}$$

We will examine an infinite series with $r=\frac{1}{2}$. What happens to $r^n$ as $n$ increases?

$$\left(\frac{1}{2}\right)^2=\frac{1}{4}$$
$$\left(\frac{1}{2}\right)^3=\frac{1}{8}$$
$$\left(\frac{1}{2}\right)^4=\frac{1}{16}$$

The value of $r^n$ decreases rapidly. What happens for greater values of $n$?

$$\left(\frac{1}{2}\right)^{10} = \frac{1}{1,024}$$
$$\left(\frac{1}{2}\right)^{20} = \frac{1}{1,048,576}$$
$$\left(\frac{1}{2}\right)^{30} = \frac{1}{1,073,741,824}$$

As $n$ gets very large, $r^n$ gets very small. We say that, as $n$ increases without bound, $r^n$ approaches 0. As $r^n$ approaches $0$, $1$, $-r^n$ approaches $1$. When this happens, the numerator approaches $a_1$. This give us a formula for the sum of an infinite geometric series.

### FORMULA FOR THE SUM OF AN INFINITE GEOMETRIC SERIES

The formula for the sum of an infinite geometric series with $-1 < r < 1$ is

$$S = \frac{a_1}{1-r} \tag{7.4.4}$$

### How to: Given an infinite geometric series, find its sum

1. Identify $a_1$ and $r$.
2. Confirm that $-1 < r < 1$.
3. Substitute values for $a_1$ and $r$ into the formula, $S = \frac{a_1}{1-r}$.
4. Simplify to find $S$.

### Example $7.4.7A$: Finding the Sum of an Infinite Geometric Series

Find the sum, if it exists, for the following:

a. $10 + 9 + 8 + 7 + \ldots$
b. $248.6 + 99.44 + 39.776 + \ldots$
c. $\sum_{k=1}^{\infty} 4,374 \cdot (-\frac{1}{3})^{k-1}$
d. $\sum_{k=1}^{\infty} \frac{1}{9} \cdot (\frac{4}{3})^{k}$

**Solution**

a. There is not a constant ratio; the series is not geometric.
b. There is a constant ratio; the series is geometric. $a_1 = 248.6$ and $r = \frac{99.44}{248.6} = 0.4$, so the sum exists. Substitute $a_1 = 248.6$ and $r = 0.4$ into the formula and simplify to find the sum:

$$\begin{aligned} S &= \frac{a_1}{1-r} \\ &= \frac{248.6}{1-0.4} \\ &= 414.\overline{3} \end{aligned}$$

c. The formula is exponential, so the series is geometric with $r = -\frac{1}{3}$. Find $a_1$ by substituting $k = 1$ into the given explicit formula:

$$a_1 = 4,374 \cdot (-\frac{1}{3})^{1-1} = 4,374$$

Substitute $a_1 = 4,374$ and $r = -\frac{1}{3}$ into the formula, and simplify to find the sum:

$$\begin{aligned} S &= \frac{a_1}{1-r} \\ &= \frac{4,374}{1-\left(-\frac{1}{3}\right)} \\ &= 3,280.5 \end{aligned}$$

d. The formula is exponential, so the series is geometric, but $r > 1$. The sum does not exist.

## Example $7.4.7B$: Finding an Equivalent Fraction for a Repeating Decimal

Find an equivalent fraction for the repeating decimal $0.\overline{3}$

**Solution**

We notice the repeating decimal $0.\overline{3} = 0.333...$ so we can rewrite the repeating decimal as a sum of terms.

$$0.\overline{3} = 0.3 + 0.03 + 0.003 + ...$$

Looking for a pattern, we rewrite the sum, noticing that we see the first term multiplied to $0.1$ in the second term, and the second term multiplied to $0.1$ in the third term.

$$0.\overline{3} = 0.3 + (0.1)\underbrace{(0.3)}_{\text{First Term}} + (0.1)\underbrace{(0.1)(0.3)}_{\text{Second Term}}$$

Notice the pattern; we multiply each consecutive term by a common ratio of $0.1$ starting with the first term of $0.3$. So, substituting into our formula for an infinite geometric sum, we have

$$S_n = \frac{a_1}{1-r} = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}.$$

Find the sum, if it exists.

## Exercise $7.4.7A$

$2 + 23 + 29 + ...$

*Answer available at the end of this section.*

## Exercise $7.4.7B$

$\sum_{k=1}^{\infty} 0.76k + 1$

*Answer available at the end of this section.*

## Exercise $7.4.7C$

$\sum_{k=1}^{\infty} \left(-\frac{3}{8}\right)^k$

*Answer available at the end of this section.*

## Solving Annuity Problems

At the beginning of the section, we looked at a problem in which a couple invested a set amount of money each month into a college fund for six years. An annuity is an investment in which the purchaser makes a sequence of periodic, equal payments. To find the amount of an annuity, we need to find the sum of all the payments and the interest earned. In the example, the couple invests $\$50$ each month. This is the value of the initial deposit. The account paid $6\%$ annual interest, compounded monthly. To find the interest rate per payment period, we need to divide the $6\%$ annual percentage interest (APR) rate by $12$. So the monthly interest rate is $0.5\%$. We can multiply the amount in the account each month by $100.5\%$ to find the value of the account after interest has been added.

We can find the value of the annuity right after the last deposit by using a geometric series with $a_1 = 50$ and $r = 100.5$. After the first deposit, the value of the annuity will be $\$50$. Let us see if we can determine the amount in the college fund and the interest earned.

We can find the value of the annuity after nn deposits using the formula for the sum of the first nn terms of a geometric series. In $6$ years, there are $72$ months, so $n = 72$. We can substitute $a_1 = 50$, $r = 1.005$, and $n = 72$ into the formula, and simplify to find the value of the annuity after 6 years.

$$S_{72} = \frac{50(1 - 1.005^{72})}{1 - 1.005} \approx 4,320.44$$

After the last deposit, the couple will have a total of $\$4,320.44$ in the account. Notice, the couple made $72$ payments of $\$50$ each for a total of $72(50) = \$3,600$. This means that because of the annuity, the couple earned $\$720.44$ interest in their college fund.

### How to: Given an initial deposit and an interest rate, find the value of an annuity.

1. Determine $a_1$, the value of the initial deposit.
2. Determine $n$, the number of deposits.
3. Determine $r$.
   - Divide the annual interest rate by the number of times per year that interest is compounded.
   - Add 1 to this amount to find $r$.
4. Substitute values for $a_1$, $r$, and $n$ into the formula for the sum of the first nn terms of a geometric series, $S_n = \frac{a_1(1-r^n)}{1-r}$.
5. Simplify to find $S_n$, the value of the annuity after $n$ deposits.

### Example 7.4.8: Solving an Annuity Problem

A deposit of $\$100$ is placed into a college fund at the beginning of every month for $10$ years. The fund earns $9\%$ annual interest, compounded monthly, and paid at the end of the month. How much is in the account right after the last deposit?

**Solution**

The value of the initial deposit is $\$100$, so $a_1 = 100$. A total of $120$ monthly deposits are made in the $10$ years, so $n = 120$. To find $r$, divide the annual interest rate by $12$ to find the monthly interest rate and add $1$ to represent the new monthly deposit.

$$r = 1 + \frac{0.09}{12} = 1.0075$$

Substitute $a_1 = 100$, $r = 1.0075$, and $n = 120$ into the formula for the sum of the first $n$ terms of a geometric series, and simplify to find the value of the annuity.

$$S_{120} = \frac{100(1 - 1.0075^{120})}{1 - 1.0075} \approx 19,351.43$$

So the account has $19,351.43 after the last deposit is made.

### Exercise 7.4.8

At the beginning of each month, $\$200$ is deposited into a retirement fund. The fund earns $6\%$ annual interest, compounded monthly, and paid into the account at the end of the month. How much is in the account if deposits are made for $10$ years?

*Answer available at the end of this section.*

### Media

Access these online resources for additional instruction and practice with series.

- Arithmetic Series
- Geometric Series
- Summation Notation

## Key Equations

| | |
|---|---|
| sum of the first $n$ terms of an arithmetic series | $S_n = \frac{n(a_1 + a_n)}{2}$ |
| sum of the first $n$ terms of a geometric series | $S_n = \frac{a_1(1 - r^n)}{1 - r} \cdot \ r \neq 1$ |
| sum of an infinite geometric series with $-1 < r < 1$ | $S_n = \frac{a_1}{1 - r} \cdot \ r \neq 1$ |

## Key Concepts

- The sum of the terms in a sequence is called a series.
- A common notation for series is called summation notation, which uses the Greek letter sigma to represent the sum. See Example $7.4.1$.
- The sum of the terms in an arithmetic sequence is called an arithmetic series.
- The sum of the first $n$ terms of an arithmetic series can be found using a formula. See Example $7.4.2$ and Example $7.4.3$.
- The sum of the terms in a geometric sequence is called a geometric series.
- The sum of the first $n$ terms of a geometric series can be found using a formula. See Example $7.4.4$ and Example $7.4.5$.
- The sum of an infinite series exists if the series is geometric with $-1 < r < 1$.
- If the sum of an infinite series exists, it can be found using a formula. See Example $7.4.6$, Example $7.4.7$, and Example $7.4.8$.
- An annuity is an account into which the investor makes a series of regularly scheduled payments. The value of an annuity can be found using geometric series. See Example $7.4.9$.

### Answers

**Exercise** $7.4.1$

$38$

**Exercise** $7.4.2A$

$26.4$

**Exercise** $7.4.2B$

$328$

**Exercise** $7.4.2C$

$-280$

**Exercise** $7.4.3$

$\$2,025$

**Exercise** $7.4.4A$

$\approx 2,000.00$

**Exercise** $7.4.4B$

$9,840$

**Exercise** $7.4.5$

$\$275,513.31$

**Exercise** $7.4.6A$

The sum is defined. It is geometric.

**Exercise** $7.4.6B$

The sum of the infinite series is defined.

**Exercise** $7.4.6C$

The sum of the infinite series is defined.

**Exercise** $7.4.7A$

$3$

**Exercise** $7.4.7B$

The series is not geometric.

**Exercise** $7.4.7C$

$-\frac{3}{11}$

**Exercise** $7.4.8$

$\$92,408.18$

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# 7.5: The Principles of Counting

**Learning Objectives**

- Solve counting problems using the Addition Principle.
- Solve counting problems using the Multiplication Principle.
- Solve counting problems using permutations involving n distinct objects.
- Solve counting problems using combinations.
- Find the number of subsets of a given set.
- Solve counting problems using permutations involving n non-distinct objects.

A new company sells customizable cases for tablets and smartphones. Each case comes in a variety of colors and can be personalized for an additional fee with images or a monogram. A customer can choose not to personalize or could choose to have one, two, or three images or a monogram. The customer can choose the order of the images and the letters in the monogram. The company is working with an agency to develop a marketing campaign with a focus on the huge number of options they offer. Counting the possibilities is challenging!

We encounter a wide variety of counting problems every day. There is a branch of mathematics devoted to the study of counting problems such as this one. Other applications of counting include secure passwords, horse racing outcomes, and college scheduling choices. We will examine this type of mathematics in this section.

## Using the Addition Principle

The company that sells customizable cases offers cases for tablets and smartphones. There are $3$ supported tablet models and $5$ supported smartphone models. The **Addition Principle** tells us that we can add the number of tablet options to the number of smartphone options to find the total number of options. By the Addition Principle, there are $8$ total options, as we can see in Figure $7.5.1$.

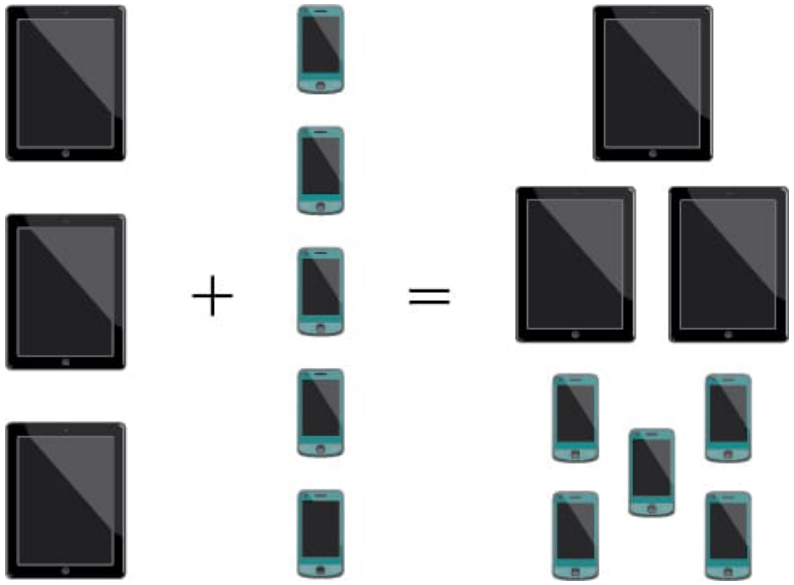


Figure $7.5.1$

THE ADDITION PRINCIPLE

According to the *Addition Principle*, if one event can occur in $m$ ways and a second event with no common outcomes can occur in $n$ ways, then the first *or* second event can occur in $m + n$ ways.

## Example 7.5.1: Using the Addition Principle

There are $2$ vegetarian entrée options and $5$ meat entrée options on a dinner menu. What is the total number of entrée options?

**Solution**

We can add the number of vegetarian options to the number of meat options to find the total number of entrée options.

The addition of the type of options for an entree.

There are $7$ total options.

## Exercise 7.5.1

A student is shopping for a new computer. He is deciding among $3$ desktop computers and $4$ laptop computers. What is the total number of computer options?

*Answer available at the end of this section.*

## Using the Multiplication Principle

The **Multiplication Principle** applies when we are making more than one selection. Suppose we are choosing an appetizer, an entrée, and a dessert. If there are $2$ appetizer options, $3$ entrée options, and $2$ dessert options on a fixed-price dinner menu, there are a total of $12$ possible choices of one each as shown in the tree diagram in Figure $7.5.2$.

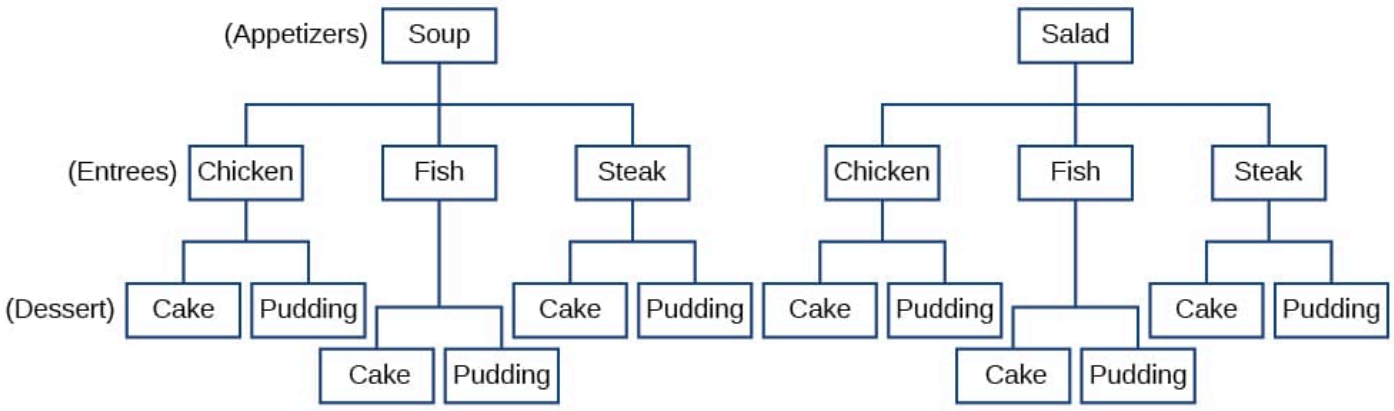



Figure $7.5.2$

The possible choices are:

1. soup, chicken, cake
2. soup, chicken, pudding
3. soup, fish, cake
4. soup, fish, pudding
5. soup, steak, cake
6. soup, steak, pudding

7. salad, chicken, cake
8. salad, chicken, pudding
9. salad, fish, cake
10. salad, fish, pudding
11. salad, steak, cake
12. salad, steak, pudding We can also find the total number of possible dinners by multiplying.

We could also conclude that there are $12$ possible dinner choices simply by applying the Multiplication Principle.

$$\#\text{of appetizer options} \times \#\text{of entree options} \times \#\text{of dessert options}$$

$$2 \times 3 \times 2 = 12$$

### THE MULTIPLICATION PRINCIPLE

According to the Multiplication Principle, if one event can occur in $m$ ways and a second event can occur in $n$ ways after the first event has occurred, then the two events can occur in $m \times n$ ways. This is also known as the Fundamental Counting Principle.

### Example $7.5.2$: Using the Multiplication Principle

Diane packed $2$ skirts, $4$ blouses, and a sweater for her business trip. She will need to choose a skirt and a blouse for each outfit and decide whether to wear the sweater. Use the Multiplication Principle to find the total number of possible outfits.

**Solution**

To find the total number of outfits, find the product of the number of skirt options, the number of blouse options, and the number of sweater options.

| # of skirt options | × | # of blouse options | × | # of sweater options | |
|---|---|---|---|---|---|
| 2 | × | 4 | × | 2 | = 16 |

There are $16$ possible outfits.

### Exercise $7.5.2$

A restaurant offers a breakfast special that includes a breakfast sandwich, a side dish, and a beverage. There are $3$ types of breakfast sandwiches, $4$ side dish options, and $5$ beverage choices. Find the total number of possible breakfast specials.

*Answer available at the end of this section.*

## Finding the Number of Permutations of $n$ Distinct Objects

The Multiplication Principle can be used to solve a variety of problem types. One type of problem involves placing objects in order. We arrange letters into words and digits into numbers, line up for photographs, decorate rooms, and more. An ordering of objects is called a permutation.

### Finding the Number of Permutations of $n$ Distinct Objects Using the Multiplication Principle

To solve permutation problems, it is often helpful to draw line segments for each option. That enables us to determine the number

of each option so we can multiply. For instance, suppose we have four paintings, and we want to find the number of ways we can hang three of the paintings in order on the wall. We can draw three lines to represent the three places on the wall.

$$\_\_\_\_\_ \times \_\_\_\_\_ \times \_\_\_\_\_$$

There are four options for the first place, so we write a $4$ on the first line.

$$4 \times \_\_\_\_\_ \times \_\_\_\_\_$$

After the first place has been filled, there are three options for the second place so we write a $3$ on the second line.

$$4 \times 3 \times \_\_\_\_\_$$

After the second place has been filled, there are two options for the third place so we write a $2$ on the third line. Finally, we find the product.

$$4 \times 3 \times 2 = 24$$

There are $24$ possible permutations of the paintings.

### How to: Given $n$ distinct options, determine how many permutations there are.

1. Determine how many options there are for the first situation.
2. Determine how many options are left for the second situation.
3. Continue until all of the spots are filled.
4. Multiply the numbers together.

### Example $7.5.3$: Finding the Number of Permutations Using the Multiplication Principle

At a swimming competition, nine swimmers compete in a race.

a. How many ways can they place first, second, and third?
b. How many ways can they place first, second, and third if a swimmer named Ariel wins first place? (Assume there is only one contestant named Ariel.)
c. How many ways can all nine swimmers line up for a photo?

**Solution**

a. Draw lines for each place.

$$\text{options for } 1^{\text{st}} \text{ place} \times \text{options for } 2^{\text{nd}} \text{ place} \times \text{options for } 3^{\text{rd}} \text{ place}$$

There are $9$ options for first place. Once someone has won first place, there are $8$ remaining options for second place. Once first and second place have been won, there are $7$ remaining options for third place.

$$9 \times 8 \times 7 = 504$$

Multiply to find that there are $504$ ways for the swimmers to place.

b. Draw lines for describing each place.

options for 1st place × options for 2nd place × options for 3rd place

We know Ariel must win first place, so there is only $1$ option for first place. There are $8$ remaining options for second place, and then $7$ remaining options for third place.

$$1 \times 8 \times 7 = 56$$

Multiply to find that there are $56$ ways for the swimmers to place if Ariel wins first.

c. Draw lines for describing each place in the photo.

_____ × _____ × _____ × _____ × _____ × _____ × _____ × _____ × _____

There are $9$ choices for the first spot, then $8$ for the second, $7$ for the third, $6$ for the fourth, and so on until only $1$ person remains for the last spot.

$$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362{,}880$$

There are $362,880$ possible permutations for the swimmers to line up.

**Analysis**

Note that in part c, we found there were $9!$ ways for $9$ people to line up. The number of permutations of $n$ distinct objects can always be found by $n!$.

## Exercise $7.5.3A$

A family of five is having portraits taken. Use the Multiplication Principle to find the following:

a. How many ways can the family line up for the portrait?
b. How many ways can the photographer line up $3$ family members?
c. How many ways can the family line up for the portrait if the parents are required to stand on each end?

*Answer a available at the end of this section.*

### Finding the Number of Permutations of *n* Distinct Objects Using a Formula

For some permutation problems, it is inconvenient to use the Multiplication Principle because there are so many numbers to multiply. Fortunately, we can solve these problems using a formula. Before we learn the formula, let's look at two common notations for permutations. If we have a set of $n$ objects and we want to choose $r$ objects from the set in order, we write $P(n,r)$. Another way to write this is ${}_nP_r$, a notation commonly seen on computers and calculators. To calculate $P(n,r)$, we begin by finding $n!$, the number of ways to line up all nn objects. We then divide by $(n-r)!$ to cancel out the $(n-r)$ items that we do not wish to line up.

Let's see how this works with a simple example. Imagine a club of six people. They need to elect a president, a vice president, and a treasurer. Six people can be elected president, any one of the five remaining people can be elected vice president, and any of the remaining four people could be elected treasurer. The number of ways this may be done is $6 \times 5 \times 4 = 120$. Using factorials, we get the same result.

$$\frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 6 \cdot 5 \cdot 4 = 120$$

There are $120$ ways to select $3$ officers in order from a club with $6$ members. We refer to this as a permutation of $6$ taken $3$ at a

time. The general formula is as follows.

$$P(n,r)=\frac{n!}{(n-r)!} \tag{7.5.1}$$

Note that the formula stills works if we are choosing all $n$ objects and placing them in order. In that case we would be dividing by $(n-n)!$ or $0!$, which we said earlier is equal to $1$. So the number of permutations of $n$ objects taken $n$ at a time is $\frac{n!}{1}$ or just $n!$.

### Note: FORMULA FOR PERMUTATIONS OF *N* DISTINCT OBJECTS

Given $n$ distinct objects, the number of ways to select $r$ objects from the set in order is

$$P(n,r)=\frac{n!}{(n-r)!} \tag{7.5.2}$$

### How to: Given a word problem, evaluate the possible permutations.

1. Identify $n$ from the given information.
2. Identify $r$ from the given information.
3. Replace $n$ and $r$ in the formula with the given values.
4. Evaluate.

### Example 7.5.4: Finding the Number of Permutations Using the Formula

A professor is creating an exam of $9$ questions from a test bank of $12$ questions. How many ways can she select and arrange the questions?

**Solution**

Substitute $n=12$ and $r=9$ into the permutation formula and simplify.

$$\begin{aligned} P(n,r) &= \frac{n!}{(n-r)!} \\ P(12,9) &= \frac{12!}{(12-9)!} \\ &= \frac{12!}{3!} \\ &= 79,833,600 \end{aligned}$$

There are $79,833,600$ possible permutations of exam questions!

**Analysis**

We can also use a calculator to find permutations. For this problem, we would enter $12$, press the $nP_r$ function, enter $9$, and then press the equal sign. The $nP_r$ function may be located under the MATH menu with probability commands.

### Q&A

**Could we have solved the example above using the Multiplication Principle?**

Yes. We could have multiplied $15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4$ to find the same answer.

### Exercise 7.5.4A

A play has a cast of 7 actors preparing to make their curtain call. Use the permutation formula to find the following.

How many ways can the 7 actors line up?

*Answer available at the end of this section.*

### Exercise 7.5.4B

How many ways can 5 of the 7 actors be chosen to line up?

*Answer available at the end of this section.*

## Find the Number of Combinations Using the Formula

So far, we have looked at problems asking us to put objects in order. There are many problems in which we want to select a few objects from a group of objects, but we do not care about the order. When we are selecting objects and the order does not matter, we are dealing with **combinations**. A selection of $r$ objects from a set of $n$ objects where the order does not matter can be written as $C(n,r)$. Just as with permutations, $C(n,r)$ can also be written as ${}_nC_r$. In this case, the general formula is as follows.

$$C(n,r)=\frac{n!}{r!(n-r)!} \tag{7.5.3}$$

An earlier problem considered choosing 3 of 4 possible paintings to hang on a wall. We found that there were 24 ways to select 3 of the 4 paintings in order. But what if we did not care about the order? We would expect a smaller number because selecting paintings $1,2,3$ would be the same as selecting paintings $2,3,1$. To find the number of ways to select 3 of the 4 paintings, disregarding the order of the paintings, divide the number of permutations by the number of ways to order 3 paintings. There are $3!=3\cdot 2\cdot 1=6$ ways to order 3 paintings. There are $\frac{24}{6}$, or 4 ways to select 3 of the 4 paintings. This number makes sense because every time we are selecting 3 paintings, we are *not* selecting 1 painting. There are 4 paintings we could choose *not* to select, so there are 4 ways to select 3 of the 4 paintings.

### Note: FORMULA FOR COMBINATIONS OF *N* DISTINCT OBJECTS

Given $n$ distinct objects, the number of ways to select $r$ objects from the set is

$$C(n,r)=\frac{n!}{r!(n-r)!} \tag{7.5.4}$$

### How to: Given a number of options, determine the possible number of combinations.

1. Identify $n$ from the given information.
2. Identify $r$ from the given information.
3. Replace $n$ and $r$ in the formula with the given values.
4. Evaluate.

### Example 7.5.5: Finding the Number of Combinations Using the Formula

A fast food restaurant offers five side dish options. Your meal comes with two side dishes.

a. How many ways can you select your side dishes?

b. How many ways can you select $3$ side dishes?

**Solution**

a. We want to choose $2$ side dishes from $5$ options.

$$C(5,2) = \frac{5!}{2!(5-2)!} = 10$$

b. We want to choose $3$ side dishes from $5$ options.

$$C(5,3) = \frac{5!}{3!(5-3)!} = 10$$

**Analysis**

We can also use a graphing calculator to find combinations. Enter 5, then press $nC_r$, enter 3, and then press the equal sign. The $nC_r$, function may be located under the MATH menu with probability commands.

Q&A

**Is it a coincidence that parts (a) and (b) in the example above have the same answers?**

No. When we choose r objects from n objects, we are **not** choosing $(n–r)$ objects. Therefore, $C(n,r) = C(n,n–r)$.

Exercise $7.5.5$

An ice cream shop offers $10$ flavors of ice cream. How many ways are there to choose $3$ flavors for a banana split?

*Answer available at the end of this section.*

## Finding the Number of Subsets of a Set

We have looked only at combination problems in which we chose exactly rr objects. In some problems, we want to consider choosing every possible number of objects. Consider, for example, a pizza restaurant that offers $5$ toppings. Any number of toppings can be ordered. How many different pizzas are possible?

To answer this question, we need to consider pizzas with any number of toppings. There is $C(5,0) = 1$ way to order a pizza with no toppings. There are $C(5,1) = 5$ ways to order a pizza with exactly one topping. If we continue this process, we get

$$C(5,0) + C(5,1) + C(5,2) + C(5,3) + C(5,4) + C(5,5) = 32$$

There are $32$ possible pizzas. This result is equal to $2^5$.

We are presented with a sequence of choices. For each of the nn objects we have two choices: include it in the subset or not. So for the whole subset we have made nn choices, each with two options. So there are a total of $2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2$ possible resulting subsets, all the way from the empty subset, which we obtain when we say “no” each time, to the original set itself, which we obtain when we say “yes” each time.

Note: FORMULA FOR THE NUMBER OF SUBSETS OF A SET

A set containing *n* distinct objects has $2^n$ subsets.

### Example 7.5.6: Finding the Number of Subsets of a Set

A restaurant offers butter, cheese, chives, and sour cream as toppings for a baked potato. How many different ways are there to order a potato?

**Solution**

We are looking for the number of subsets of a set with $4$ objects. Substitute $n = 4$ into the formula.

$$\begin{aligned} 2^n &= 2^4 \\ &= 16 \end{aligned}$$

There are $16$ possible ways to order a potato.

### Exercise 7.5.6

A sundae bar at a wedding has $6$ toppings to choose from. Any number of toppings can be chosen. How many different sundaes are possible?

*Answer available at the end of this section.*

## Finding the Number of Permutations of *n* Non-Distinct Objects

We have studied permutations where all of the objects involved were distinct. What happens if some of the objects are indistinguishable? For example, suppose there is a sheet of $12$ stickers. If all of the stickers were distinct, there would be $12!$ ways to order the stickers. However, $4$ of the stickers are identical stars, and $3$ are identical moons. Because all of the objects are not distinct, many of the $12!$ permutations we counted are duplicates. The general formula for this situation is as follows.

$$\frac{n!}{r_1!r_2!\ldots r_k!} \tag{7.5.5}$$

In this example, we need to divide by the number of ways to order the $4$ stars and the ways to order the $3$ moons to find the number of unique permutations of the stickers. There are $4!$ ways to order the stars and $3!$ ways to order the moon.

$$\frac{12!}{4!3!} = 3,326,400$$

There are $3,326,400$ ways to order the sheet of stickers.

### Note: FORMULA FOR FINDING THE NUMBER OF PERMUTATIONS OF $N$ NON-DISTINCT OBJECTS

If there are nn elements in a set and $r_1$ are alike, $r_2$ are alike, $r_3$ are alike, and so on through $r_k$, the number of permutations can be found by

$$\frac{n!}{r_1!r_2!\ldots r_k!} \tag{7.5.6}$$

### Example 7.5.7: Finding the Number of Permutations of $n$ Non-Distinct Objects

Find the number of rearrangements of the letters in the word DISTINCT.

**Solution**

There are $8$ letters. Both $I$ and $T$ are repeated $2$ times. Substitute $n = 8$, $r_1 = 2$, and $r_2 = 2$ into the formula.

$$\frac{8!}{2!2!} = 10,080$$

There are $10,080$ arrangements.

**Exercise 7.5.7**

Find the number of rearrangements of the letters in the word CARRIER.

*Answer available at the end of this section.*

**Media**

Access these online resources for additional instruction and practice with combinations and permutations.

- Combinations
- Permutations

## Key Equations

| | |
|---|---|
| number of permutations of $n$ distinct objects taken $r$ at a time | $P(n,r) = \frac{n!}{(n-r}$ |
| number of combinations of $n$ distinct objects taken $r$ at a time | $C(n,r) = \frac{n!}{r!(n-}$ |
| number of permutations of $n$ non-distinct objects | $\frac{n!}{r1!r2!\ldots rk!}$ |

## Key Concepts

- If one event can occur in $m$ ways and a second event with no common outcomes can occur in $n$ ways, then the first or second event can occur in $m + n$ ways. See Example $7.5.1$.
- If one event can occur in $m$ ways and a second event can occur in $n$ ways after the first event has occurred, then the two events can occur in $m \times n$ ways. See Example $7.5.2$.
- A permutation is an ordering of $n$ objects.
- If we have a set of $n$ objects and we want to choose $r$ objects from the set in order, we write $P(n,r)$.
- Permutation problems can be solved using the Multiplication Principle or the formula for $P(n,r)$. See Example $7.5.3$ and Example $7.5.4$.
- A selection of objects where the order does not matter is a combination.
- Given $n$ distinct objects, the number of ways to select $r$ objects from the set is $C(n,r)$ and can be found using a formula. See Example $7.5.5$.
- A set containing $n$ distinct objects has $2n$ subsets. See Example $7.5.6$.
- For counting problems involving non-distinct objects, we need to divide to avoid counting duplicate permutations. See Example $7.5.7$.

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## Answers

**Exercise** $7.5.1$

$7$

**Exercise** $7.5.2$

There are $60$ possible breakfast specials.

**Exercise** $7.5.3A$

$120$

$60$

$12$

**Exercise** $7.5.4A$

$P(7,7) = 5,040$

**Exercise** $7.5.4B$

$P(7,5) = 2,520$

**Exercise** $7.5.5$

$C(10,3) = 120$

**Exercise** $7.5.6$

$64$ sundaes

**Exercise** $7.5.7$

$840$

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# 7.6: The Binomial Theorem

**Learning Objectives**

- Apply the Binomial Theorem.

A polynomial with two terms is called a binomial. We have already learned to multiply binomials and to raise binomials to powers, but raising a binomial to a high power can be tedious and time-consuming. In this section, we will discuss a shortcut that will allow us to find $(x+y)^n$ without multiplying the binomial by itself $n$ times.

## Identifying Binomial Coefficients

In the shortcut to finding $(x+y)^n$, we will need to use combinations to find the coefficients that will appear in the expansion of the binomial. In this case, we use the notation $\binom{n}{r}$ instead of $C(n,r)$, but it can be calculated in the same way. So

$$\binom{n}{r} = C(n,r) = \frac{n!}{r!(n-r)!} \tag{7.6.1}$$

The combination $\binom{n}{r}$ is called a *binomial coefficient*. An example of a binomial coefficient is:

$$\binom{5}{2} = C(5,2) = 10$$

**Definition: 7.6.1**

If $n$ and $r$ are integers greater than or equal to $0$ with $n \geq r$, then the *binomial coefficient* is

$$\binom{n}{r} = C(n,r) = \frac{n!}{r!(n-r)!} \tag{7.6.2}$$

**Q&A: Is a binomial coefficient always a whole number?**

*Yes. Just as the number of combinations must always be a whole number, a binomial coefficient will always be a whole number.*

**Example 7.6.1: Finding Binomial Coefficients**

Find each binomial coefficient.

a. $\binom{5}{3}$
b. $\binom{9}{2}$
c. $\binom{9}{7}$

**Solution**

Use the Equation 7.6.2 to calculate each binomial coefficient. You can also use the $nC_r$ function on your calculator.

a. $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot 3!}{3!2!} = 10$
b. $\binom{9}{2} = \frac{9!}{2!(9-2)!} = \frac{9 \cdot 8 \cdot 7!}{2!7!} = 36$
c. $\binom{9}{7} = \frac{9!}{7!(9-7)!} = \frac{9 \cdot 8 \cdot 7!}{7!2!} = 36$

**Analysis**

Notice that we obtained the same result for parts (b) and (c). If you look closely at the solution for these two parts, you will see that you end up with the same two factorials in the denominator, but the order is reversed, just as with combinations.

$$\binom{n}{r} = \binom{n}{n-r}$$

**Exercise 7.6.1**

Find each binomial coefficient.

a. $\binom{7}{3}$
b. $\binom{11}{4}$

*Answer a available at the end of this section.*

## Using the Binomial Theorem

When we expand $(x+y)^n$ by multiplying, the result is called a binomial expansion, and it includes binomial coefficients. If we wanted to expand $(x+y)^{52}$, we might multiply $(x+y)$ by itself fifty-two times. This could take hours! If we examine some simple binomial expansions, we can find patterns that will lead us to a shortcut for finding more complicated binomial expansions.

$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

First, let's examine the exponents. With each successive term, the exponent for $x$ decreases and the exponent for $y$ increases. The sum of the two exponents is $n$ for each term.

Next, let's examine the coefficients. Notice that the coefficients increase and then decrease in a symmetrical pattern. The coefficients follow a pattern:

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}.$$

These patterns lead us to the **Binomial Theorem**, which can be used to expand any binomial.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$
$$= x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x y^{n-1} + y^n$$

Another way to see the coefficients is to examine the expansion of a binomial in general form, $x+y$, to successive powers $1$, $2$, $3$, and $4$.

$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

Can you guess the next expansion for the binomial $(x+y)^5$?

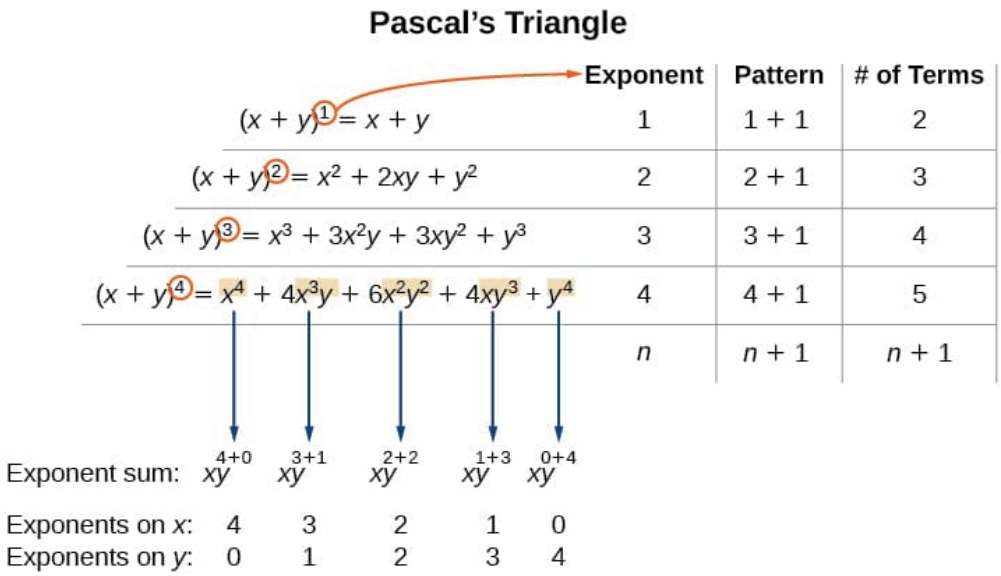



Figure 7.6.1

See Figure 7.6.1, which illustrates the following:

- There are $n+1$ terms in the expansion of $(x+y)^n$.
- The degree (or sum of the exponents) for each term is $n$.
- The powers on $x$ begin with $n$ and decrease to $0$.
- The powers on $y$ begin with $0$ and increase to $n$.
- The coefficients are symmetric.

To determine the expansion on $(x+y)^5$, we see $n=5$, thus, there will be $5+1=6$ terms. Each term has a combined degree of $5$. In descending order for powers of $x$, the pattern is as follows:

- Introduce $x^5$, and then for each successive term reduce the exponent on $x$ by $1$ until $x^0=1$ is reached.
- Introduce $y^0=1$, and then increase the exponent on yy by 1 until $y^5$ is reached.

$$x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5$$

The next expansion would be

$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

But where do those coefficients come from? The binomial coefficients are symmetric. We can see these coefficients in an array known as Pascal's Triangle, shown in Figure 7.6.2.

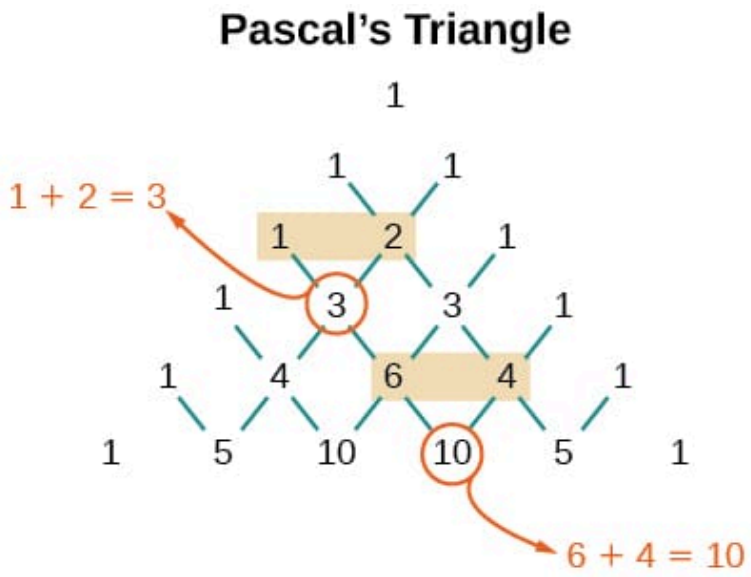



Figure 7.6.2

To generate Pascal's Triangle, we start by writing a $1$. In the row below, row 2, we write two $1's$. In the 3rd row, flank the ends of the rows with $1's$, and add $1 + 1$ to find the middle number, $2$. In the $n^{th}$ row, flank the ends of the row with $1's$. Each element in the triangle is the sum of the two elements immediately above it.

To see the connection between Pascal's Triangle and binomial coefficients, let us revisit the expansion of the binomials in general form.

$$1 \longrightarrow (x+y)^0 = 1$$

$$1 \quad 1 \longrightarrow (x+y)^1 = x + y$$

$$1 \quad 2 \quad 1 \longrightarrow (x+y)^2 = x^2 + 2xy + y^2$$

$$1 \quad 3 \quad 3 \quad 1 \longrightarrow (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1 \longrightarrow (x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \longrightarrow (x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

## THE BINOMIAL THEOREM

The *Binomial Theorem* is a formula that can be used to expand any binomial.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + y^n \tag{7.6.3}$$

## How to: Given a binomial, write it in expanded form.

1. Determine the value of $n$ according to the exponent.
2. Evaluate the $k = 0$ through $k = n$ using the Binomial Theorem formula.
3. Simplify.

### Example 7.6.2: Expanding a Binomial

Write in expanded form.

a. $(x+y)^5$
b. $(3x-y)^4$

**Solution**

a. Substitute $n=5$ into the formula. Evaluate the $k=0$ through $k=5$ terms. Simplify.

$$(x+y)^5 = \binom{5}{0}x^5y^0 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}x^1y^4 + \binom{5}{5}x^0y^5$$
$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

b. Substitute $n=4$ into the formula. Evaluate the $k=0$ through $k=4$ terms. Notice that $3x$ is in the place that was occupied by $x$ and that $-y$ is in the place that was occupied by $y$. So we substitute them. Simplify.

$$(3x-y)^4 = \binom{4}{0}(3x)^4(-y)^0 + \binom{4}{1}(3x)^3(-y)^1 + \binom{4}{2}(3x)^2(-y)^2 + \binom{4}{3}(3x)^1(-y)^3 + \binom{4}{4}(3x)^0(-y)^4$$
$$(3x-y)^4 = 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4$$

**Analysis**

Notice the alternating signs in part b. This happens because $(-y)$ raised to odd powers is negative, but $(-y)$ raised to even powers is positive. This will occur whenever the binomial contains a subtraction sign.

### Exercise 7.6.2

Write in expanded form.

a. $(x-y)^5$
b. $(2x+5y)^3$

*Answer a available at the end of this section.*

## Using the Binomial Theorem to Find a Single Term

Expanding a binomial with a high exponent such as $(x+2y)^{16}$ can be a lengthy process. Sometimes we are interested only in a certain term of a binomial expansion. We do not need to fully expand a binomial to find a single specific term.

Note the pattern of coefficients in the expansion of $(x+y)^5$.

$$(x+y)^5 = x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + y^5$$

The second term is $\binom{5}{1}x^4y$. The third term is $\binom{5}{2}x^3y^2$. We can generalize this result.

### THE $(r+1)$ th TERM OF A BINOMIAL EXPANSION

The $(r+1)$ th term of the binomial expansion of $(x+y)^n$ is:

$$\binom{n}{r}x^{n-r}y^r \tag{7.6.4}$$

**How to: Given a binomial, write a specific term without fully expanding.**

1. Determine the value of nn according to the exponent.
2. Determine $(r+1)$.
3. Determine $r$.
4. Replace $r$ in the formula for the $(r+1)$th term of the binomial expansion.

**Example 7.6.3: Writing a Given Term of a Binomial Expansion**

Find the tenth term of $(x+2y)^{16}$ without fully expanding the binomial.

**Solution**

Because we are looking for the tenth term, $r+1=10$, we will use $r=9$ in our calculations and Equation 7.6.4.

$$\binom{16}{9}x^{16-9}(2y)^9 = 5,857,280x^7y^9$$

**Exercise 7.6.3**

Find the sixth term of $(3x-y)^9$ without fully expanding the binomial.

*Answer available at the end of this section.*

**Media**

Access these online resources for additional instruction and practice with binomial expansion.

- The Binomial Theorem
- Binomial Theorem Example

## Key Equations

| | |
|---|---|
| Binomial Theorem | $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$ |
| $(r+1)$th term of a binomial expansion | $\binom{n}{r}x^{n-r}y^r$ |

## Key Concepts

- $\binom{n}{r}$ is called a binomial coefficient and is equal to $C(n,r)$. See Example 7.6.1.
- The Binomial Theorem allows us to expand binomials without multiplying. See Example 7.6.2.
- We can find a given term of a binomial expansion without fully expanding the binomial. See Example 7.6.3.

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## Answers

**Exercise** 7.6.1

35

33

**Exercise** 7.6.2

$x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

$8x^3 + 60x^2y + 150xy^2 + 125y^3$

**Exercise** 7.6.3

$-10,206x^4y^5$

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