MGF 1131: MATHEMATICS IN CONTEXT



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Mathematics in Context

MGF 1131

Florida Southwestern State College

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1: NUMBER REPRESENTATION IN DIFFERENT BASES AND CRYPTOGRAPHY

1.1: Hindu-Arabic Positional System

1.2: Early Numeration Systems

1.3: Converting to Different Base Systems

1.4: Addition and Subtraction in Base Systems

1.5: Cryptography

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1.1: Hindu-Arabic Positional System

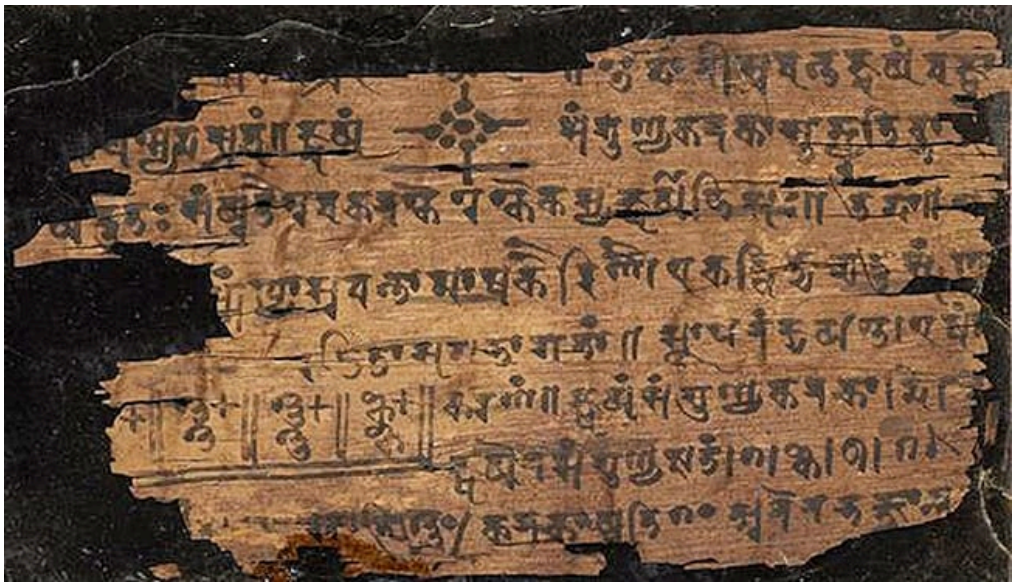


Figure 1.1.1 : This manuscript is an early example of Hindu numerals. (credit: modification of work “Bakshali manuscript”, Bodleian Libraries/ University of Oxford, public domain)

Learning Objectives

After completing this section, you should be able to:

- Evaluate an exponential expression.
- Convert a Hindu-Arabic numeral to expanded form.
- Convert a number in expanded form to a Hindu-Arabic numeral.

The modern system of counting and computing isn’t necessarily natural. The fact that different symbols are used to indicate different quantities or amounts is a relatively new invention. Simple marking by scratches or dots, one for each item being counted, was the norm long into human history. The modern system doesn’t use repeated symbols to indicate more than one thing. It uses the place of a digit in a numeral to determine what that digit represents. A numeral is a symbol used to represent a number. A number is an abstract idea that represents quantity or amount.

Being clear about the difference between a numeral and a number is important. Just like a person can be called by various names, such as brother, father, husband, uncle, they all represent the same person, John Smith. The person John Smith is the number, and the names brother, father, husband, and uncle are the numerals.

Who Knew?: Hindu-Arabic Numerals

The numerals we currently use are referred to as Hindu-Arabic numerals, although they have changed as time has passed. Early forms of the numerals for $0, 1, 2, 3, 4, 5, 6, 7, 8,$ and 9 began in India, and passed through Persia to the Middle East. Place value was also employed in India's early systems. Once this system was in North Africa and the Middle East, it spread to Europe, eventually replacing Roman numerals. Over time, the original symbols transformed into our modern ones.

Place Value in the Hindu-Arabic System

Place values in the Hindu-Arabic System rely on powers of 10. They are

$$\ldots, 10^6 \quad 10^5 \quad 10^4 \quad 10^3 \quad 10^2 \quad 10^1 \quad 10^0$$

Or

$$\ldots, 1{,}000{,}000 \quad 100{,}000 \quad 10{,}000 \quad 1{,}000 \quad 100 \quad 10 \quad 1$$

$1,000,000$	$100,000$	$10,000$	$1,000$	100	10	1
millions	hundred thousands	ten thousands	thousands	hundreds	tens	ones

Your Turn 1.1.1: Place Values and Digits used in Hindu-Arabic System

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The system we use for counting and computing uses place values based on powers of 10. In this section, we review exponents and our positional system.

Evaluating Exponential Expressions

Most modern numerical systems depend on place values, where the quantity represented depends not only on the digit, but also on where the digit is in the number. The place value is a power of some specific number, which means most numbering systems are actually exponential expressions. An exponential expression is any mathematical expression that includes exponents. So, evaluating such an expression means performing the calculation.

In this chapter, we will be using exponents that are positive integer values. Before we do so, let's remind ourselves about exponents and what they represent. Suppose you want to multiply a number. Let's label that number a, by itself some number of times. Let's label the number of times b. We denote that as a^b. We say a, or the **base**, raised to the bth power, or the **exponent**. For example, if we are multiplying 13 by itself eight times, we write 13^8 and say 13 to the eighth power.

When computing exponential expressions, we should be careful to remember the order of operations. Using the order of operation rules, calculations inside the parentheses are done first, then exponents are calculated, then multiplication and division calculations are performed, and then addition and subtraction.

Example 1.1.2: Evaluating an Exponential Expression

1. $4 \times 5^2 + 2 \times 6^3$
2. $6 \times 8^2 + 3 \times 8^1 + 4 \times 8^0$
3. $3 \times 10^2 + 0 \times 10^1 + 6 \times 10^0$

Answer 1 available at the end of this section.

Your Turn 1.1.2: Evaluate

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Converting Hindu-Arabic Numerals to Expanded Form

When you see the number 738, and you speak the number out loud, what do you say? You probably said "seven hundred thirty-eight" while wondering what point could possibly be made by asking this. What you didn't say was "seven, and three, and eight." A pre-K student might say that. Which should make you wonder, why?

The reason is that you've been taught place values, or the positions of digits in a number that determine the values of those digits. You know that in a three-digit number, the first digit is hundreds, the second digit is tens, and the last digit is ones. These place values rely on powers of 10, making this system a base 10 system.

This sense of place value makes our system of numbers so useful. You've also been taught the Hindu-Arabic numeration system. This system, which uses the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$, and 9, and also employs place value based on powers of 10, is in use today.

Example 1.1.3: Identify the Place Value

Write the place value of each digit below in the number $75,140,382$.

Answer available at the end of this section.

Your Turn 1.1.3: Find Place Values

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People in Mathematics: Aryabhata of Kusumapura and Brahmagupta

The Hindu-Arabic numeral system developed in India, and Aryabhata of Kusumapura is credited with the place value notation in the 5th century. However, the system wasn't as complete as it could be until. Roughly a century later, when Brahmagupta introduced the symbol for 0. The 0 is necessary to indicate that a given place value has been skipped, as in $4,098$. In $4,098$, the 10^2 power is skipped. Without such a symbol, $4,098$ and 498 look similar. The value of both the place value notation and the introduction of the symbol 0 cannot be overstated, for math and the sciences.

Example 1.1.4: Writing a Number in Expanded Form

Write the following in expanded form.

1. 563
2. $4,821$
3. $903,786$

Answer available at the end of this section.

Your Turn 1.1.4: Write in Expanded Form

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Converting Numbers in Expanded Form to Hindu-Arabic Numerals

Converting from an expanded form back into a Hindu-Arabic numeral is the reverse process of expanding a number and is equivalent to evaluating the exponential expression.

Example 1.1.5: Converting a Number from Expanded Form to a Hindu-Arabic Numeral

Convert the following into Hindu-Arabic numerals.

1. $3 \times 10^2 + 4 \times 10^1 + 8 \times 10^0$
2. $5 \times 10^3 + 0 \times 10^2 + 9 \times 10^1 + 9 \times 10^0$
3. $6 \times 10^6 + 2 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 1 \times 10^1 + 7 \times 10^0$

Answer available at the end of this section.

Your Turn 1.1.5: Write in Standard Form

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Place Value to the Right of the Decimal Point

Place values after the decimal point represent **fractions of one** and follow a specific order, decreasing in size from left to right. Here's how they go: Tenths ($\frac{1}{10}$), Hundredths ($\frac{1}{100}$), Thousandths ($\frac{1}{1000}$), Ten-thousandths ($\frac{1}{10000}$), etc. and those place values are summarized in table.

$\frac{1}{10,000,000}$	$\frac{1}{1,000,000}$	$\frac{1}{100,000}$	$\frac{1}{10,000}$	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$
Ten Millionths	Millionths	Hundred Thousandths	Ten Thousandths	Thousandths	Hundredths	tenths

Example 1.1.6: Identify the Place Value

Which number in 1642.895 represents the hundredths place value?

Answer available at the end of this section.

Your Turn 1.1.6: Find Place Value

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Videos for Section 1.1

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Answers

Example 1.1.2**: Evaluating an Exponential Expression**

To evaluate or calculate this expression, we use the order of operations, which means the exponents are done first, then multiplications, and then addition.

$$4 \times 5^2 + 2 \times 6^3 = 4 \times 25 + 2 \times 216 = 100 + 432 = 532$$

To evaluate the expression, we use the order of operations, which means the exponents are done first, then the multiplications, then the additions. Remember that any base raised to the exponent 0 is 1.

$$6 \times 8^2 + 3 \times 8^1 + 4 \times 8^0 = 6 \times 64 + 3 \times 8 + 4 \times 1 = 384 + 24 + 4 = 412$$

To evaluate the expression, we use the order of operations, which means the exponents are done first, then the multiplications, and then the additions. Remember that any base raised to the power 0 is 1.

$$3 \times 10^2 + 0 \times 10^1 + 6 \times 10^0 = 3 \times 100 + 0 \times 10 + 6 \times 1 = 306$$

Example 1.1.3**: Identify the Place Value**

The place values of each digit are shown in the table.

Millions			Thousands			Ones		
Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
	7	5	1	4	0	3	8	2

Figure 1.1.2: Place value of $75,140,382$.

Table displaying data with columns labeled: "Activity," "Thousands," "Crime," including various numerical entries with plave values of 75140382.

Example 1.1.4**: Writing a Number in Expanded Form**

Multiply each digit by its respective place value and add them. From right to left, the place value of 3 is 10^0, the place value of 6 is 10^1, and the place value of 5 is 10^2. So expanded form of 563 is

$$5 \times 10^2 + 6 \times 10^1 + 3 \times 10^0 \tag{1.1.1}$$

Multiply each digit by its respective place value and add them. From right to left, the place value of 1 is 10^0, the place value of 2 is 10^1, the place value of 8 is 10^2, and the place value of 4 is 10^3. So expanded form of $4,821$ is

$$4 \times 10^3 + 8 \times 10^2 + 2 \times 10^1 + 1 \times 10^0 \tag{1.1.2}$$

Similarly, we can expand $903,786$.

$$9 \times 10^5 + 0 \times 10^4 + 3 \times 10^3 + 7 \times 10^2 + 8 \times 10^1 + 6 \times 10^0 \tag{1.1.3}$$

Example 1.1.5**: Converting a Number from Expanded Form to a Hindu-Arabic Numeral**

Evaluating the expression results in:

$$3 \times 10^2 + 4 \times 10^1 + 8 \times 10^0 = 300 + 40 + 8 = 348$$

Evaluating the expression results in:

$$5 \times 10^3 + 0 \times 10^2 + 9 \times 10^1 + 9 \times 10^0 = 5000 + 0 + 90 + 9 = 5,099$$

Evaluating the expression results in:

$$6 \times 10^6 + 2 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 1 \times 10^1 + 7 \times 10^0 = 6,209,117$$

Example 1.1.6**: Identify the Place Value**

9 is in the hundredths place.

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1.2: Early Numeration Systems

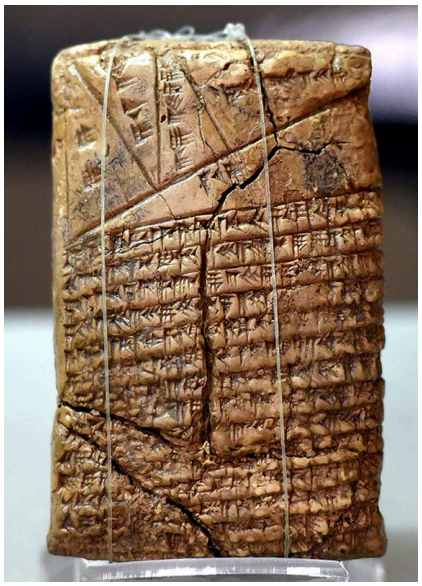


Figure 1.2.1 : Babylonians used clay tablets for writing and record keeping. (credit: modification of work by Osama Shukir Muhammed Amin frcp(Glasg), CC BY 4.0 International)

Learning Objectives

1. Understand and convert Babylonian numerals to Hindu-Arabic numerals.
2. Understand and convert Mayan numerals to Hindu-Arabic numerals.
3. Understand and convert between Roman numerals and Hindu-Arabic numerals.

Each culture throughout history had to develop its own method of counting and recording quantity. The system used in Australia would necessarily differ from the system developed in Babylon, which would, in turn, differ from the system developed in sub-Saharan Africa. These differences arose due to cultural differences. In nearly all societies, knowing the difference between one and two would be useful. But it might not be useful to know the difference between 145 and 167, as those quantities never had a practical use. For example, a shepherd likely didn't manage more than 100 sheep, so quantities larger than 100 might never have been encountered. This can even be seen in our use of the term few, which is an inexact quantity that most would agree means more than two. However, as societies became more complex, as commerce arose, as military bodies developed, so did the need for a system to handle large numbers. No matter the system, the issues of representing multiple values and how many symbols to use had to be addressed. In this section, we explore how the Babylonians, Mayans, and Romans addressed these issues.

Understand and Convert Babylonian Numerals to Hindu-Arabic Numerals

Babylonian numerals are part of one of the earliest known number systems, developed by the ancient Babylonians around 2000 BCE. The time period for the Babylonian civilization is roughly from 1894 **BCE to** 539 **BCE.** Unlike our modern decimal (base - 10) system, Babylonian numerals were **sexagesimal**, meaning they were based on 60. This is why we still divide an hour into 60 minutes or a circle into 360 degrees.

The Babylonians used a mix of an additive system of numbers and a positional system of numbers. An additive system is a number system where the value of repeated instances of a symbol is added the number of times the symbol appears. A positional system is a system of numbers that multiplies a “digit” by a number raised to a power, based on the position of the “digit.”

The Babylonian place values didn’t use powers of 10, but instead powers of 60. They didn’t use 60 different symbols, though. For the value 1, they used the following symbol:



For values up to 9, that symbol would be repeated, so three would be written as



To represent the quantity 10, they used

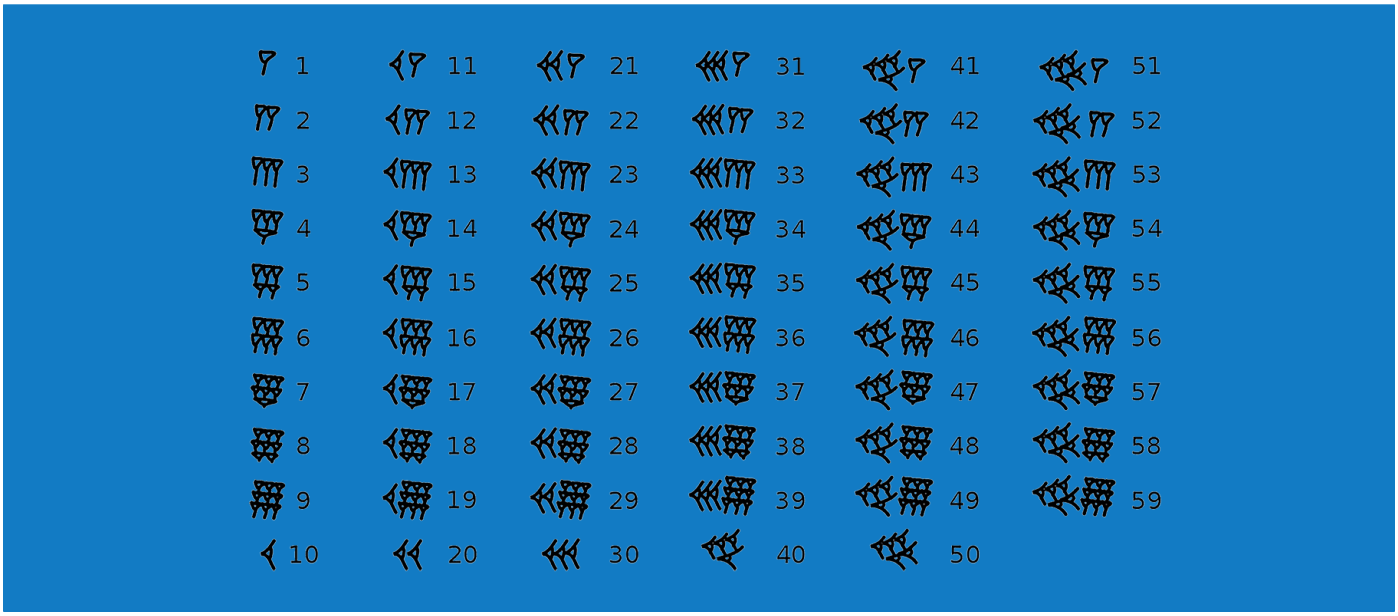


For $20, 30, 40,$ and 50,they repeated the symbol for 10 however many times it was needed, so 40 would be written.



When they reached 60, they moved to the next place value. The complete list of the Babylonian numerals up to 59 is given in the following table.

Babylonain Numerals



You can see how the Babylonians repeated the symbols to indicate multiples of a value. The number 6 is 6 of the symbol for 1 grouped together. The symbol for 30 is three of the symbols for 10 grouped together. However, their system doesn't go past 59. To go past 59, they used place values. As opposed to the Hindu-Arabic system, which was based on powers of 10, the Babylonian positional system was based on powers of 60.

You should also notice there is no symbol for 0, which has some impact on the number system. Since the Babylonian number system lacked a 0, they didn't have a placeholder when a power of 60 was absent. Without a 0, 101, 110, and 11 all look the same. However, there is some evidence that the Babylonians left a small space between "digits" where we would use a 0, allowing them to represent the absence of that place value.

To summarize, the **Babylonian system of numbers** used repeating a symbol to indicate more than one, used place values, and lacked a 0.

The place values in the Babylonian numeration system use a power of 60. The place values are

$$\ldots, 60^3, 60^2, 60^1, 60^0$$

We can expand the above exponential notation as follows.

$$\ldots, 216000, 3600, 60, 1$$

Who Knew?: Invention of 0

The idea of 0 is not a natural one. Most cultures failed to recognize the need for a 0. If someone asked a farmer in 300 B.C.E. how many cows they had, but they had none, they would not answer "zero." They'd say "I don't have any" and be done with it. It wasn't until roughly 3 B.C.E. that 0 appeared in Mesopotamia. It was independently discovered (or invented!) in the Mayan culture around 4 C.E. It made its appearance in India in the 400 s C.E., and began to spread at that point. It wasn't developed earlier, mostly because positional systems were not yet fully developed. Once positional systems arose, the need to represent a missing power had to be addressed.

The place values in the Babylonian system are based on powers of 60. If you have n digits in the Babylonian number, you multiply the first "digit" by 10 raised to one less than the number of "digits." You then continue through the "digits," multiplying each by 60 raised to a power that is one smaller. However, be careful of spaces, since they represent a zero in that place.

Step to Convert Babylonian Numbers to Hindu-Arabic Numbers

1. Identify how many digits a Babylonian number has.
2. Identify the Hindu-Arabic value and place value of each Babylonian digit.
3. Multiply each digit by its place value.
4. **Add them up** to get the Hindu-Arabic number.

Note: The rightmost Babylonian digit has place value 1, the next one has place value 60^1, and so on...

Example 1.2.1: Converting Two-Digit Babylonian Number to Hindu-Arabic Number

Convert the Babylonian number to the Hindu-Arabic Number.



Answer available at the end of this section.

Example 1.2.2: Converting Three-Digit Babylonian Number to Hindu-Arabic Number

Convert the Babylonian number into a Hindu-Arabic number.



Answer available at the end of this section.

Example 1.2.3: Converting Four-Digit Babylonian Number to Hindu-Arabic Number

Convert the Babylonian number into a Hindu-Arabic number.



Answer available at the end of this section.

Your Turn 1.2.3: Babylonian to Hindu Arabic

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Who Knew?: The Legacy of Babylonian System

The Babylonian system is still visible today. An hour is 60 minutes, and a minute is 60 seconds. Additionally, when measuring angles in degrees, each degree can be split into 60 minutes $(\frac{1}{60})^{\text{th}}$ of a degree) and 60 seconds $(\frac{1}{60})^{\text{th}}$ of minute.

Understand and Convert Mayan Numerals to Hindu-Arabic Numerals

The Mayans employed a positional system just as we do, and the Babylonians did, but they based their position values on powers of 20, and they had a dedicated symbol for zero. Like the Babylonians, the Mayans repeated symbols to indicate specific values. A single dot was a 1, two dots were a 2, up to four dots. Then, a five was a horizontal bar. The horizontal bars could be used three times since the fourth horizontal bar would make a 20, which was a new position in the number. The 0 was a unique picture, which appears like a turtle lying on its back. The shell would then be "empty," so maybe that's why the symbol was 0. The complete list is provided in Table $1.2.1$. Another feature of Mayan numbers was that they were written vertically. The powers of 20 increased from bottom to top.Adapt $1.2.1$

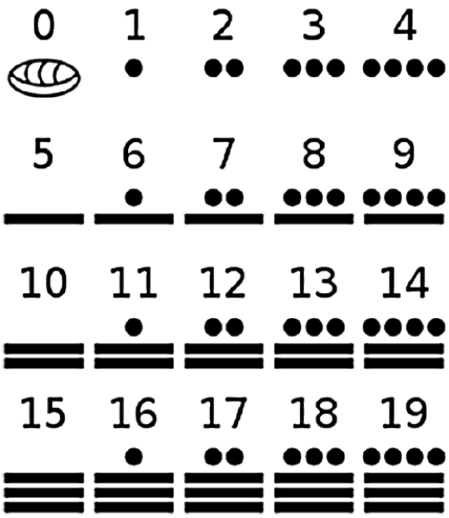


Table 1.2.1: Mayan Numerals. (Copyright; author via source)

To summarize, the **Mayan system of numbers** used a repeating symbol to indicate more than one, used place values, and employed a 0. So, how do we convert from Mayan numbers to Hindu-Arabic numbers? To do so, we need to use the symbols from Table $1.2.1$ and then place values based on powers of 20. If you have n digits in the Mayan number, you multiply the first "digit" by 20 raised to one less than the number of "digits." You then continue through the "digits," multiplying each by 20 raised to a power that is one smaller than the previous power. Fortunately, there is an explicit 0, so there is no ambiguity about numbers like $110, 101,$ and 11.

Mayan numerals are written vertically, with the first digit (place 1) positioned at the bottom.

The place values in the Mayan numeration systems are

$$\ldots, 18 \times 20^3, 18 \times 20^2, 18 \times 20^1, 20^1, 20^0$$

We can expand the above exponential notation as follows.

$$\ldots, 144000, 7200, 360, 20, 1$$

It is believed that the Mayans used 18×20 so that their numeration system would conform to their calendar of 18 months, 20 days each, plus 5 "ghost days" to complete the 365 day year.

Steps to convert Mayan Numerals to Hindu-Arabic numerals?

1. The number in the bottom row is to be multiplied by 1.
2. The number in the second row from the bottom is to be multiplied by 20.
3. The number in the third row will be multiplied by or 360.
4. The number in the fourth row will be multiplied by 7200 and so on, and then add all the products.

Example 1.2.4: Converting Three-Digit Mayan Numbers to Hindu-Arabic Numbers

Convert the following Mayan number into a Hindu-Arabic number.







Answer available at the end of this section.

Example 1.2.5: Converting Four-Digit Mayan Numbers to Hindu-Arabic Numbers

Convert the following Mayan number into a Hindu-Arabic number.









Answer available at the end of this section.

Your Turn 1.2.5: Mayan to Hindu Arabic

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Who Knew?: The Mayan Calendar

The Mayans used this base 20 system for everyday situations. But their culturally important, and extremely accurate, calendar system used a slightly different system. For their calendars, they used a system where the place values were $1, 20$, then 20×18, then $20 \times 20 \times 18$. The reason for this is 20×18 is 360, which is closer to the number of days in a year. Had they used a purely base 20 system for their calendar, they'd be very far off with 400 days in a year.

Three hundred sixty days still left the Mayans a bit short, as there are 365 days in a year (ignoring leap years). The Mayan calendar also included 5 days, called Wayeb days, which brings their calendar to 365 days. As it happens, Wayeb is the Mayan god of misfortune, so these 5 days were considered the bad luck days.

Understand and Convert Between Roman Numerals to Hindu-Arabic Numerals

The Mayan and Babylonian systems shared two features, one of which we are familiar with (place value) and one that we don't use

(repeated symbols). The **Roman system of numbers** used repeated symbols but did not employ a place value. It also lacks a 0. The Roman system is built on the following symbols in Table $1.2.2$.

Roman Numeral	I	V	X	L	C	D	M
Hindu-Arabic Numeral	1	5	10	50	100	500	1000

(Red Value: Only Can Subtract from next two greater Numeral.)

Table $1.2.1$: Roman Numeral

As in the Mayan and Babylonian systems, a symbol may be repeated to indicate a larger value. However, at 4, they did not use IIII. They instead used IV. Since I came before the V, the number stands for "one before five." A similar process was used for 9 , which was written IX, or "one before ten." The value 40 was written XL, or "ten before fifty," while 49 was written XLIX, or "forty plus nine."

Roman Numerals: Remember Those Rules.

1. Up to three symbols may be grouped together; for example, III for 3, or XXX for 30, or CC for 200. Only I, X, and C can be repeated. You can not have XXXX or VV.
2. I can be placed before V and X. So $IV = 4$ and $IX = 9$. These are the only ways I is used as a subtraction.
3. X can be placed before L and C. So $XL = 40$ and $XC = 90$. These are the only ways X is used as a subtraction.
4. C can be placed before D and M. So $CD = 400$ and $CM = 900$. These are the only ways C is used as a subtraction.
5. If there is a smaller value in front of a larger value, those two values are grouped, and then we use the subtraction property in the Roman numeral system. For example, XXIX for 29, because $IX = 9$ and you add $XX = 20$ and 9 to find the value as a whole.
6. Larger value followed by a smaller value indicated addition; for example, VII for 7, XIII for 13, LV for 55, and MCC for 1200.

Be careful: If the values of the symbols decrease from left to right, add them to find the total value of the Roman numeral. If the values increase from left to right, subtract them instead.

So, look for any smaller symbol in front of a large symbol for subtraction?

Your Turn $1.2.6$: Roman to Hindu Arabic

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Who Knew?: Legacy of Roman Numerals

The Roman numbering system is still used today in some situations. Many cornerstones of buildings have the year written in Roman numerals. Movie titles often represent the year the movie was produced as Roman numerals. The most recognizable might be that the Super Bowl is numbered using Roman numerals.

Example 1.2.7: Converting Roman Numerals to Hindu-Arabic Numbers

Convert the following Roman numerals into Hindu-Arabic numerals.

1. XXVII
2. XXXIV
3. $MMCMXLVIII$

Answer available at the end of this section.

Example 1.2.8: Converting Hindu-Arabic Numbers to Roman Numerals

Convert the following Hindu-Arabic numerals into Roman numerals.

1. 38
2. 94
3. 846
4. $2,987$

Answer available at the end of this section.

Your Turn 1.2.8: Roman to Hindu Arabic and Vice Versa

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Vinculum

Roman numerals with a **bar over them** are a cool little feature! The bar—called a **vinculum**—is used to **multiply the numeral by 1,000**. For example $\overline{LV}III = 55 \times 1000(\overline{LV}) + 3$ (III) = $55,003$

Converting Roman Numerals to Hindu-Arabic Numbers

Example 1.2.9: Converting Roman Numerals to Hindu-Arabic Numbers

Convert the following Roman numeral to a Hindu-Arabic numeral.

1. $\overline{X}$
2. $\overline{CM}$
3. $\overline{X}IV$

Answer available at the end of this section.

Your Turn 1.2.9: Roman number with Vinculum

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Answers

Example 1.2.1: Converting Two-Digit Babylonian Number to Hindu-Arabic Number

This number has two digits (symbols). Note that the space between digits separates place values in a numeral from one another. Now convert each Babylonian numeral to an Arabic numeral, multiply them by their place value, and add those products.

Place value is Babylonian systems are $\ldots, 216000, 3600, 60, 1$

The left digit is 4, and it place value is 60^1 and the right digit is 27, and its place value is $1 = 60^0$.

$$4 \times 60^1 + 27 \times 60^0 = 240 + 27 = 267$$

So, the above Babylonian number equals 267 in the Hindu-Arabic number system.

Example 1.2.2: Converting Three-Digit Babylonian Number to Hindu-Arabic Number

This number has three digits (symbols). Note that the space between digits separates place values in a numeral from one another.

Now, convert each Babylonian numeral to a Hindu-Arabic numeral, multiply it by its place value, and add those products.

Place value is Babylonian systems are $\ldots, 216000, 3600, 60, 1$

The left digit is 13, and its place value is 3600. The middle digit is 8, and its pace value is 60. The right digit is 54, and its place value is 1.

Multiplying and adding them yields

$$13 \times 3,600 + 8 \times 60 + 54 \times 1 = 46,800 + 480 + 54 = 47,334$$

So, the above Babylonian number equals $47,334$ in the Hindu-Arabic number system.

Example 1.2.3: Converting Four-Digit Babylonian Number to Hindu-Arabic Number

This number has three digits (symbols). Note that the space between digits separates place values in a numeral from one another.

Now convert each Babylonian numeral to an Arabic numeral, multiply them by their place value, and add those products. And place value in Babylonian systems is $\ldots, 216000, 3600, 60, 1$

$$\begin{aligned} &11 \times 216,000 + 40 \times 3,600 + 3 \times 60 + 3 \times 1 \\ &= 2,376,000 + 144,000 + 180 + 3 \\ &= 2,520,183 \end{aligned}$$

So, the above Babylonian number equals $2,520,183$ in the Hindu-Arabic number system.

Example 1.2.4: Converting Three-Digit Mayan Numbers to Hindu-Arabic Numbers

The first symbol (bottom) represents 4 in the Mayan system, and this is multiplied by 1. The second row from the bottom represents 8, so this is multiplied by 20, and the last digit (third from the bottom) represents 6, multiplied by 18×20.

$$6 \times 18 \times 20 + 8 \times 20 + 4 \times 1 = 2160 + 160 + 4 = 2324$$

So, the above Mayan number equals $2,234$ in the Hindu-Arabic number system.

Example 1.2.5: Converting Four-Digit Mayan Numbers to Hindu-Arabic Numbers

As we did in the previous examples,

$$8 \times 18 \times 20^2 + 0 \times 18 \times 20 + 16 \times 20 + 5 \times 1 = 57,600 + 0 + 320 + 5 = 57,925$$

So, the above Mayan number equals $57,925$ in the Hindu-Arabic number system.

Example 1.2.7: Converting Roman Numerals to Hindu-Arabic Numbers

1. Is there a smaller value in front of the large value? No. So, add the value of each Roman numeral symbol to get the value of the whole. Therefore

$$\begin{aligned} XXVII &= 10 + 10 + 5 + 1 + 1 \\ &= 27 \end{aligned} \tag{1.2.1}$$

2. Is there a smaller value in front of the large value? Yes. The symbol I (smaller) is in front of V (bigger). So, we group IV, whose value is $5 - 1 = 4$. Therefore

$$\begin{aligned} XXXIV &= 10 + 10 + 10 + (5 - 1) \\ &= 34 \end{aligned} \tag{1.2.2}$$

3. Is there a smaller value in front of the large value? Yes. The symbol X (smaller) is in front of L (bigger). So we group XL, whose value is $50 - 10 = 40$, and also the symbol C (smaller) is in front of M (bigger). So, we group CM whose value is $1000 - 100 = 900$. Therefore

$$\begin{aligned} MMCMXLVIII &= 1000 + 1000 + 900 + 40 + 5 + 1 + 1 + 1 \\ &= 2,948 \end{aligned} \tag{1.2.3}$$

Example 1.2.8: Converting Hindu-Arabic Numbers to Roman Numerals

1. Thirty is represented as three $X's$, and the 8 is represented with VIII, so 38 in Roman numerals is XXXVIII. Or You can write 38 as 30 plus 8, 30 = XXX and 8= VIII. So 38 = XXXVIII.
2. 94 is 90 plus 4. 90 is represented by XC, and 4 is represented by IV, so 94 in Roman numerals is $XCIV$.

3. The number is less than 900 and more than 500, so the first symbol to be used is D, which is 500. To get to 800, we need 300 more, which is represented by three C's. 40 is represented by XL, and 6 is represented by VI. The Roman numerals are $DCCCXLVI$. Or you can write 846 is 500 plus 300 plus 40 plus 6. So 846 = $DCCCXLVI$
4. The 2000 is represented by two $M's$. The 900 is represented by CM. The 80 is represented by LXXX $(50 + 30)$. Finally, the 7 is represented by VII. We have that $2,987$ in Roman numerals is $MMCMLXXXVII$.

Example $1.2.9$: Converting Roman Numerals to Hindu-Arabic Numbers

1. $\overline{X} = 10 \times 1,000 = 10,000$
2. $\overline{CM} = (1000 - 100) \times 1,000 = 900,000$
3. $\overline{X}IV = 10 \times 1,000 + (5 - 1) = 10,004$

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1.3: Converting to Different Base Systems

Learning Objectives

1. Convert base ten to different bases.
2. Convert different bases to base ten.

Introduction and Basic

During the previous discussions, we have been referring to positional base systems. In this chapter, we will explore exactly what a base system is and what it means if a system is “positional.” We will do so by first examining our familiar base-ten system and then expanding our exploration by considering other possible base systems.

A base system is a structure within which we count. The easiest way to describe a base system is to think about our own base-ten system. The base-ten system, which we call the “decimal” system, requires a total of ten different symbols/digits to write any number. They are, of course, $0, 1, 2, \ldots ..9$.

The decimal system is also an example of a *positional* base system, which means that the position of a digit gives its place value. Not all civilizations had a positional system, even though they did have a base with which they worked.

In our base-ten system, a number like $5,783,216$ has meaning to us because we are familiar with the system and its places. As we know, there are six ones since there is a 6 in the ones' place. Likewise, there are seven hundred thousand since the 7 reside in that place. Each digit has a value explicitly determined by its position within the number. We distinguish between a digit, a symbol such as 5, and a number, which comprises one or more digits. We can take this number and assign each of its digits a value. One way to do this is with a table, which follows:



The third column in the table shows that each place is simply a multiple of ten. Of course, this makes sense, given that our base is ten. The digits that are multiplying each place tell us how many of that place we have. We are restricted to having at most 9 in any one place before we have to “carry” over to the next place. We cannot, for example, have 11 in the hundreds place. Instead, we would carry 1 to the thousands place and retain 1 in the hundreds place. This comes as no surprise to us since we readily see that 11 hundreds is the same as one thousand, one hundred. Carrying is a pretty typical occurrence in a base system.

However, base ten is not the only option we have. Practically any positive integer greater than or equal to 2 can be used as a base for a number system. Such systems can work just like the decimal system, except the number of symbols will differ, and each position will depend on the base itself.

Other Bases

For example, let’s suppose we adopt a base-five system. The only modern digits we would need for this system are $0, 1, 2, 3,$ and

4. What are the place values in such a system? To answer that, we start with the ones place, as most base systems do. However, if we were to count in this system, we could only get to four (4) before we had to jump up to the next place. Our base is 5, after all! What is the next place that we would jump to? It would not be tens since we are no longer in base ten. We're in a different numerical world. As the base-ten system progresses from 10^0 to 10^1, so the base-five system moves from 5^0 to $5^1 = 5$. Thus, we move from the ones to the fives.

After the fives, we would move to the 5^2 place, or the twenty-fives. Note that in base-ten, we would have gone from the tens to the hundreds, which is, of course, 10^2.

Let's take an example and build a table. Consider the number 30412 in base five. We will write this as 30412_5, where the subscript 5 is not part of the number but indicates the base we're using. First off, note that this is NOT the number "thirty thousand, four hundred twelve." We must be careful not to impose the base-ten system on this number.

How to read a number in other bases?

You must read 30412_5 as "three-zero-four-one-two" in base 5". Do not read as thirty thousand four hundred and twelve.

Place Values and Digits Used in Other Bases

Place Value and Digits Used in Other Bases

The digits used in any base, b, can not reach b. For example, the digits used in the base b are $0, 1, 2, ..., b-1$. And the place values in base b are the power of b.. Note that $b^0 = 1$

$$..., b^4, b^3, b^2, b^1, 1$$

Place values in different bases

Base	Digit symbols	Place Value
2	$0, 1$	$..., 2^3, 2^2, 2^1, 1$
3	$0, 1, 2$	$..., 3^3, 3^2, 3^1, 1$
4	$0, 1, 2, 3$	$..., 4^3, 4^2, 4^1, 1$
5	$0, 1, 2, 3, 4$	$..., 5^3, 5^2, 5^1, 1$
6	$0, 1, 2, 3, 4, 5$	$..., 6^3, 6^2, 6^1, 1$
7	$0, 1, 2, 3, 4, 5, 6$	$..., 7^3, 7^2, 7^1, 1$
8	$0, 1, 2, 3, 4, 5, 6, 7$	$..., 8^3, 8^2, 8^1, 1$
9	$0, 1, 2, 3, 4, 5, 6, 7, 8$	$..., 9^3, 9^2, 9^1, 1$
10	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$	$..., 10^3, 10^2, 10^1, 1$

Be Careful!

It's important to use only the allowed digits for a given base. For example, in a base 6 system, we can only use the symbols $0, 1, 2, 3, 4$, and 5. Therefore, we cannot write 23570_6, because the digit 7 is not valid in base 6.

Also, you do not need to specify the base for the base 10 number. For example, $21048_{10} = 21048$.

Example 1.3.1: Identify the place value of a digit in other bases

What is the place value of the digit 2 in each number below?

1. 17526_8
2. 203_5
3. 2110_9
4. 3012_4

Answer available at the end of this section.

Your Turn 1.3.1: Identify Bases

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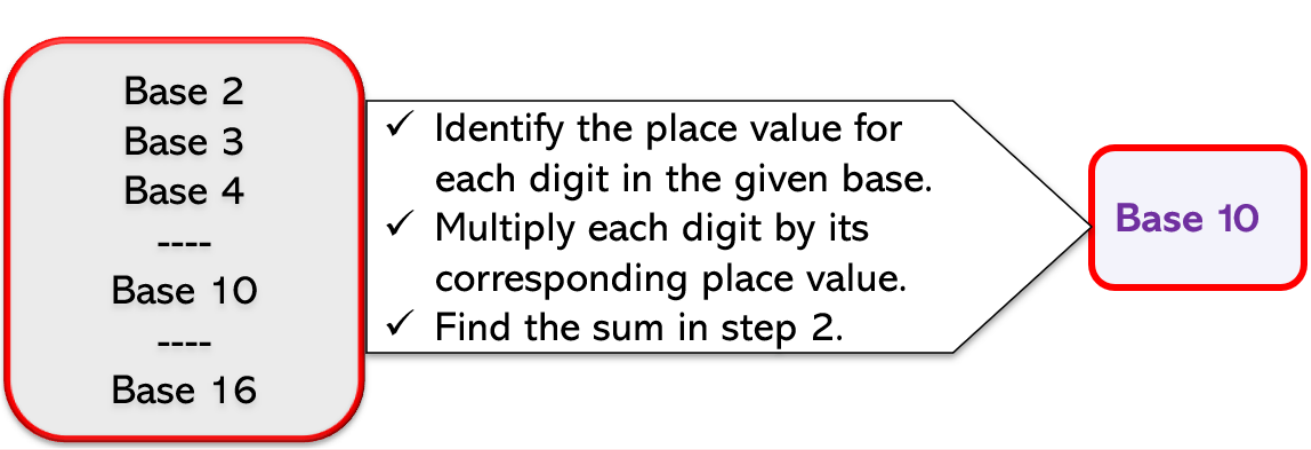
Converting Other Bases to Base 10

To convert a number from any base to base 10, you need to understand how numbers are represented in different bases. In base b each digit has a place value as a power of b.. The step to convert to base 10 from other bases is summarized below.

Steps to Convert Other Bases to Base 10

1. Identify the place value of each digit in the given base.
2. Multiply each digit by its corresponding place value.
3. Add (sum) the results from Step 2 to obtain the number in base 10.

The following diagram illustrates the given diagram.



Example 1.3.2: Convert Other Bases to Base Ten

1. Convert 6234_7 to a base 10 number.
2. Convert the base-seven number 6234_9 to base 10.

Answer 1 available at the end of this section.

Example 1.3.3: Convert Other Bases to Base Ten

1. Convert Convert 110011_2 to a base 10 number.
2. Convert 30412_5 to base 10.

Answer 1 available at the end of this section.

Your Turn 1.3.3: Covert Other Bases to Base Ten

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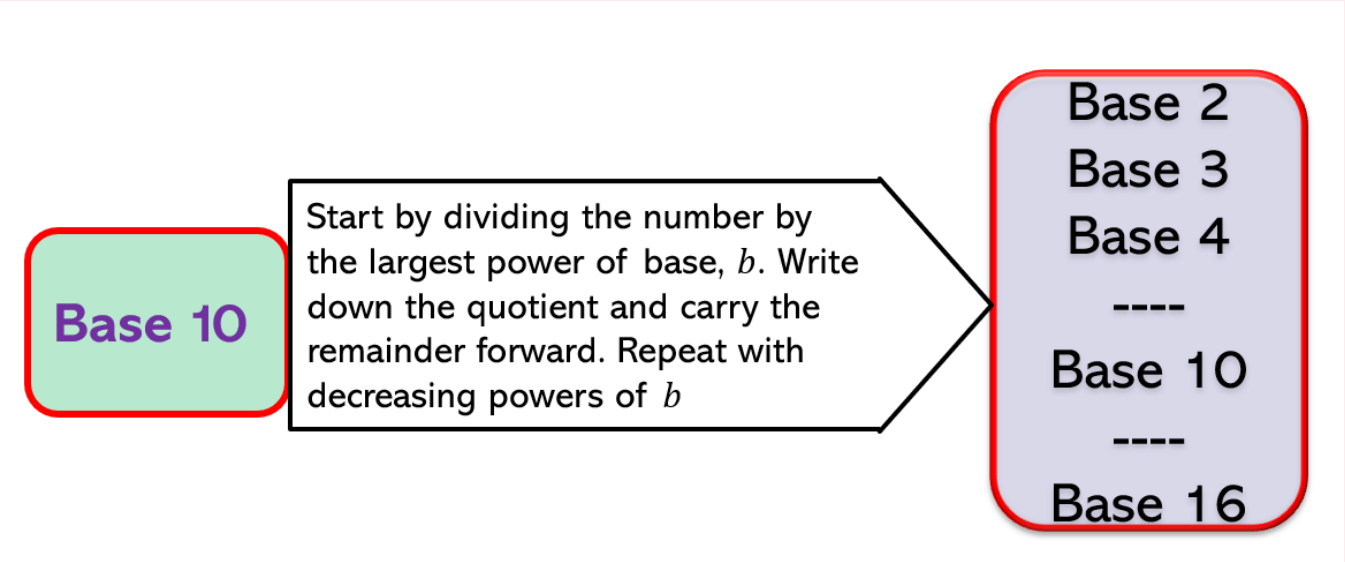
Converting Base Ten to Other Bases

Converting from an unfamiliar base to a familiar decimal system is not that difficult once you get the hang of it. It's only a matter of identifying each place and multiplying each digit by the appropriate power. However, going the other direction can be a little trickier. Suppose you have a base-ten number and want to convert it to base-five. Let's start with some simple examples before we get to a more complicated one.

Steps to Convert Base 10 to Other Bases

1. Find the highest power of the base b that will divide into the given number at least once, and then divide.
2. Write down the whole number part, then use the remainder from the division in the next step.
3. Repeat step two, dividing by the next highest power of the base b, writing down the whole number part (including 0), and using the remainder in the next step.
4. Continue until the remainder is smaller than the base. This last remainder will be in the "ones" place.
5. Collect all your whole-number parts and the last remainder to get your number in base b notation.

Note: You do not need to use place value 1 in the division process. The above step is illustrated in the following diagram.



In the given chart, we see the base-5 representation of the base-10 number.

For example: $5_{10} = 10_5$, $6_{10} = 11_5$, $15_{10} = 30_5$, $19_{10} = 34_5$, and so on.

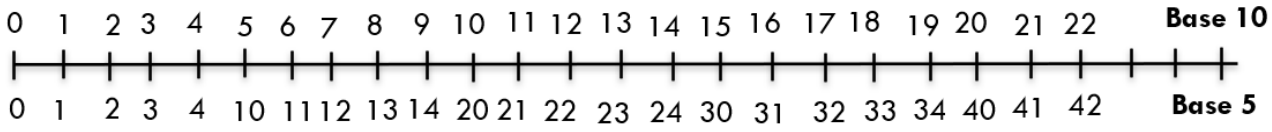


Figure $1.3.1$: Base 5 number chart

Example $1.3.4$: Convert Base 10 to Other Bases

1. Convert the base-ten number 16 to a base-five number.
2. Convert the base-ten number 2 to a base-two number.
3. Convert the base-ten number 25 to a base-nine number.

Answer 1 available at the end of this section.

Your Turn $1.3.4$: Convert Base 10 to Other Bases

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Example $1.3.5$: Convert Base 10 to Other Bases

1. Convert base 10 number 23 to a base four number.
2. Convert base 10 number 2763 to a base five number.

Answer 1 available at the end of this section.

Your Turn $1.3.5$: Convert Base 10 to Other Bases

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Working on the Bases Higher Than Ten.

Using Base 12

As mentioned in the text, working in base 10 is mathematically awkward. Ten has only two natural number divisors: 2 and 5. This means dividing into groups is not easy. However, 12, or a dozen, has more divisors: 2, 3, 4, and 6. The Dozenal Society recognizes this more mathematically pleasant detail. It advocates for a switch to using base 12 for numbers. Their argument is based on the divisibility of the number 12. But has there ever been a society that used such a system? The answer is yes. A dialect of the Gwandara language in Nigeria uses the base 12 system. It is unlikely, though, that the Dozenal Society will achieve their goal, as the base 10 system is so entrenched in our society.

A number base higher than ten and less than or equal to sixteen includes systems like base-9 through base-16, which use additional symbols beyond the standard digits 0 to 9. To represent values greater than 9, these systems introduce letters from the alphabet, starting with **A** for 10, **B** for 11, **C** for 12, **D** for 13, **E** for 14 and **F** for 15 in base-16 (hexadecimal). These bases are especially useful in fields like computer science and digital electronics. For example, hexadecimal (base-16) is commonly used to simplify binary code since each hex digit represents four binary digits (bits). These number systems allow for more compact and human-readable representations of data, making them practical for tasks such as memory addressing, color codes in web design, and machine-level programming.

Place values and digits used in bases higher than 10 up to base 16

Bases	Digits Used	Place Values
11	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}$	$\ldots, 11^3, 11^2, 11^1, 1$
12	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}, \mathrm{B}$	$\ldots, 12^3, 12^2, 12^1, 1$
13	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}$	$\ldots, 13^3, 13^2, 13^1, 1$
14	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$	$\ldots, 14^3, 14^2, 14^1, 1$
15	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}$	$\ldots, 15^3, 15^2, 15^1, 1$
16	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}$	$\ldots, 16^3, 16^2, 16^1, 1$

Example 1.3.6: Convert Base Sixteen to Base Ten

Convert $FC52E_{16}$ into base 10.

Answer available at the end of this section.

Your Turn 1.3.6: Convert Base Sixteen to Base Ten

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Answers

Example 1.3.1: Identify the place value of a digit in other bases

1. The digit 2 is in the $8^1 = 8$'s place
2. The digit 2 is in the $5^2 = 25$'s place
3. The digit 2 is in the $9^3 = 729$'s place

4. The digit 2 is in the $4^0 = 1$'s place

Example $1.3.2$: Convert Other Bases to Base Ten

We first note that we are given a base-7 number that we are to convert. Thus, our places will start at the ones (7^0, and then move up to the $7's, 49's\ (7^2)$, etc. Here's the breakdown:



Thus $6234_7 = 2181_{10}$.

The following is a slightly different way to do the conversion. The place values in base 9 are the powers of 9

Place value of 4 is $9^0 = 1$, so multiply 4 by 1.

Place value of 3 is $9^1 = 9$, so multiply 3 by 9.

Place value of 2 is $9^2 = 81$, so multiply 2 by 81.

Place value of 6 is $9^3 = 729$, so multiply 6 by 729.

Add them together as shown below.

$$6234_9 = (6 \times 729) + (2 \times 81) + (3 \times 9) + (4 \times 1) = 4567_{10}$$

Thus $6234_9 = 4567_{10}$.

Example $1.3.3$: Convert Other Bases to Base Ten

You can also make a table similar to what we did in example $1.3.2$. The following is a slightly different way to do the conversion. Here, we identify the place value of each digit in base 2, multiply those digits by their place value, and then ADD the product.

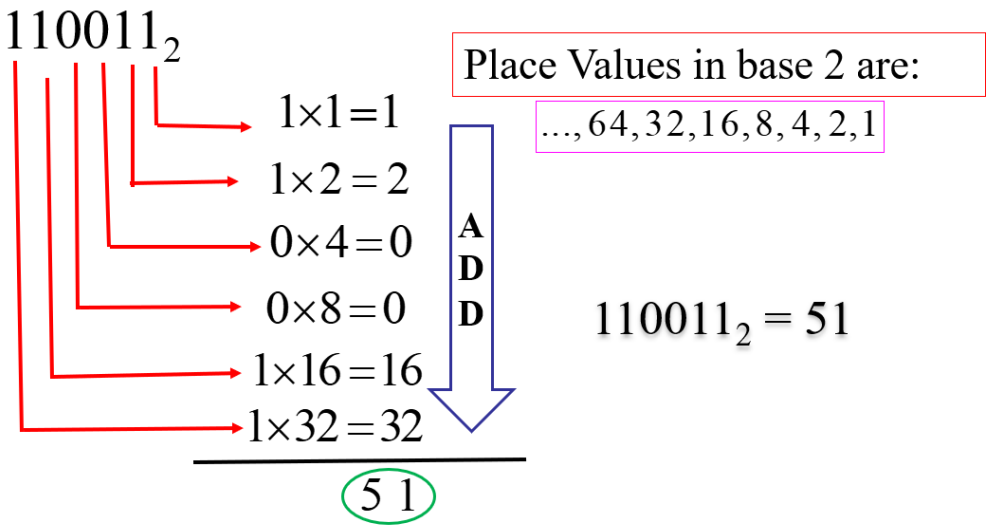


Figure $1.3.1$: Convert Base 2 to base 10.

Let's make a table to do the conversion.



As you can see, the number 30412_5 is equivalent to $1,982$ in base-ten. We will say $30412_5 = 1982_{10}$.

Example $1.3.4$: Convert Base 10 to Other Bases

Place values in base five are $..., 25, 5, 1$

The largest place value smaller than 16 that divides the given number 16 is 5.

$16 \div 5 = \quad 3 \quad \mathrm{R} \quad 1$

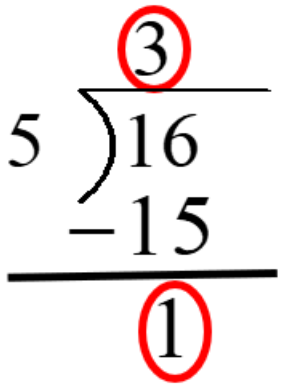
We'll say that $16_{10} = 31_5$.

The above step is illustrated in the following diagram.

Convert 16 to base 5

Place Values in base 5 are:

$\ldots, 25, 5, 1$



Place values in base two are $.., 4, 2, 1$

The largest place value smaller than (or equal to) 2 that divides the given number 2 is 2.

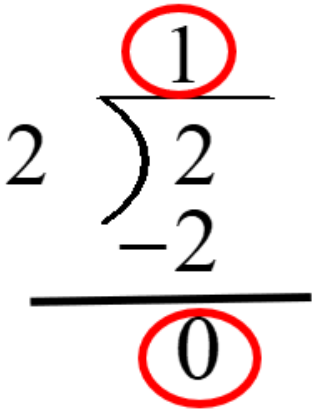
$2 \div 2 = \quad 1 \quad \mathrm{R} \quad 0$

We'll say that $2_{10} = 10_2$.

Convert 2 to base 2

Place Values in base 2 are:

$\ldots, 8, 4, 2, 1$

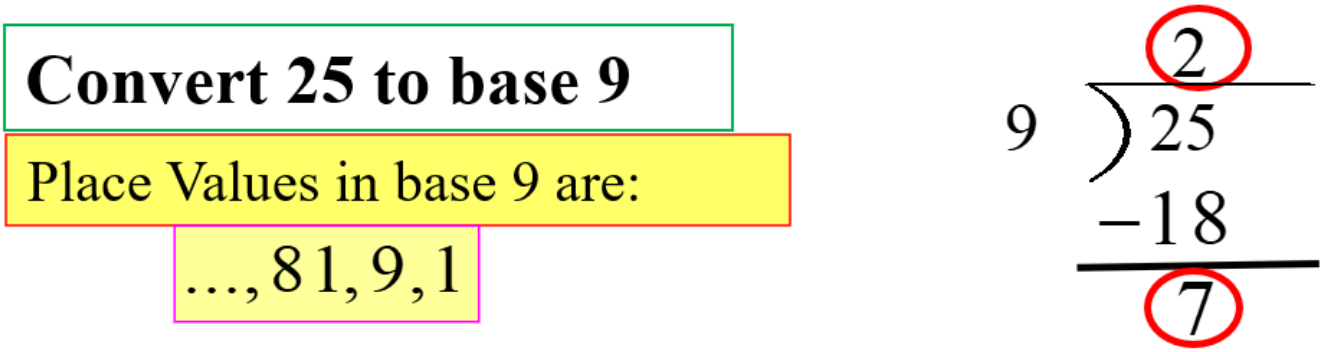


Place values in base nine are $..., 81, 9, 1$

The largest place value smaller than 25 that divides the given number 25 is 9.

$25 \div 9 = \quad 2 \quad \mathrm{R} \quad 7$

We'll say that $25_{10} = 27_9$.



Example $1.3.5$**: Convert Base 10 to Other Bases**

Place values in base four are $..., 16, 4, 1$

The largest place value smaller than 23 that divides the given number 23 is 16.

$23 \div 16 = \quad 1 \quad \text{R} \quad 7$

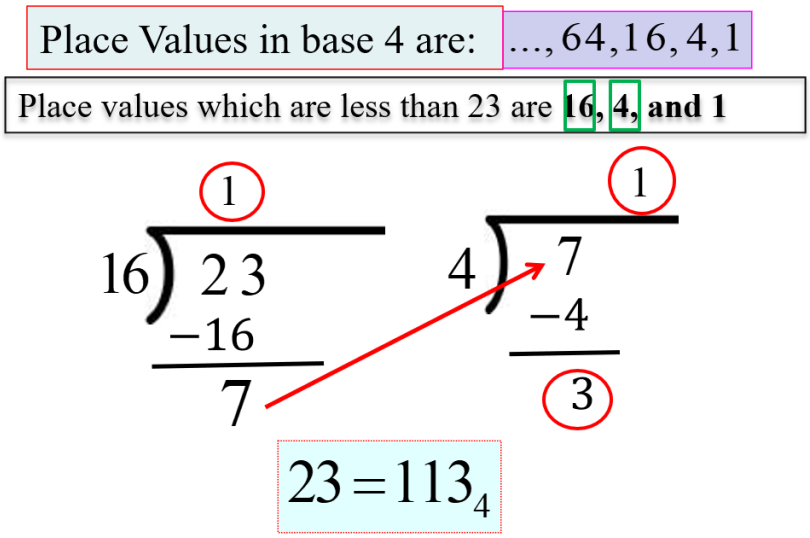
$7 \div 4 = \quad 1 \quad \text{R} \quad 3$

We'll say that $23_{10} = 113_4$

We can check our work by converting 113_4 to base ten.

$$(1 \times 16) + (1 \times 4) + (3 \times 1) = 23$$

Here, we have 1 sixteen, 1 four, and 3 ones. Hence, we have 113 in base four Thus, $23_{10} = 113_4$.. The above step is illustrated in the following diagram.



Note: In general, when converting from base-ten to some other base, it is often helpful to determine the highest power of the base that will divide into the given number at least once. In the last example, $4^2 = 16$ is the largest power of five that is present in 23, so that was our starting point. If we had moved to $4^3 = 64$, then $64)$ would not divide into 23 at least once.

Place values in base five are $..., 3125, 625, 125, 25, 5, 1$

The largest place value smaller than 2763 that divides the given number 2763 is 625.

$2763 \div 625 = \quad 4 \quad \text{R} \quad 263$

$263 \div 125 = \quad 2 \quad \text{R} \quad 13$

$13 \div 25 = \quad 0 \quad \text{R} \quad 13$

$13 \div 5 = \quad 2 \quad \text{R} \quad 3$

Putting all of this together means that $2763_{10} = 42023_5$. The above step is illustrated in the following diagram.

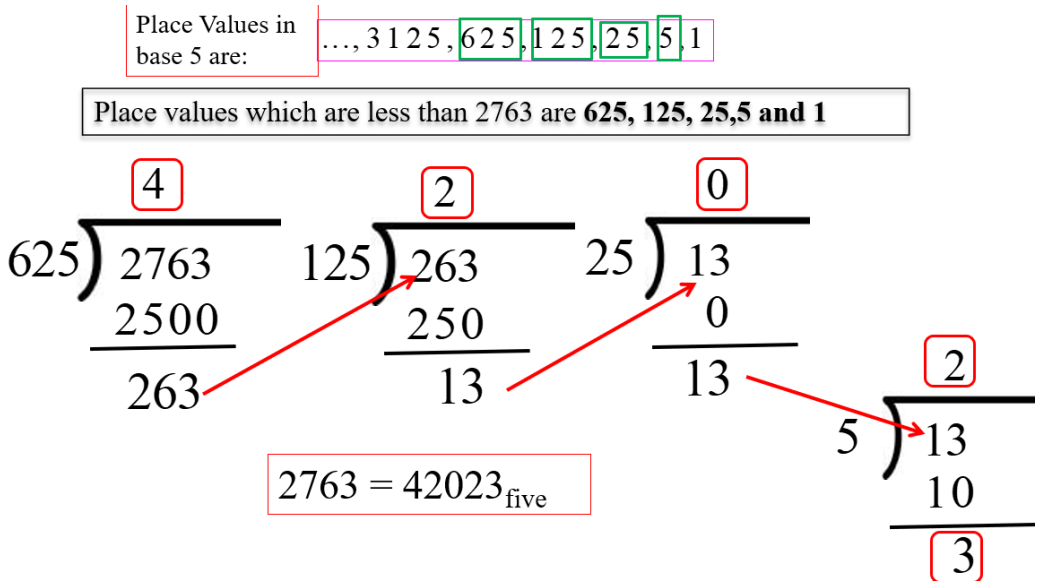


Figure 1.3.2: Convert 2763 to base 5.

This diagram shows the long division step to converst base ten number 2763 to base five number.

Example 1.3.6: Convert Base Sixteen to Base Ten

The place values in ase 16 are powers of 16 :

$$16^0 = 1$$
$$16^1 = 16$$
$$16^2 = 256$$
$$16^3 = 4096$$
$$16^4 = 65,536$$

Etc...

$$FC52E_{16} = (F(15) \times 65,536) + (12 \times 4096) + (5 \times 256) + (2 \times 16) + (E(14) \times 1) = 1,033,518_{10}$$

Thus $FC52E_{16} = 1,033,518_{10}$.

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1.4: Addition and Subtraction in Base Systems

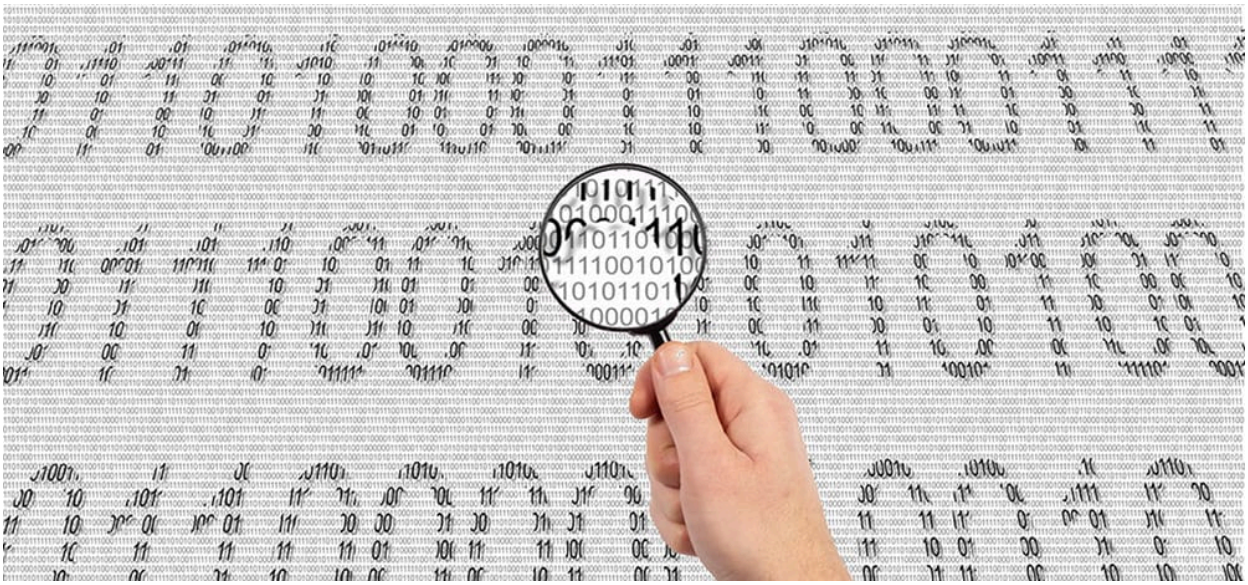


Figure 1.4.1: All information in computers is represented by 0's and 1's, including quantity, which means computers use Base 2 for arithmetic. (credit: modification of work "Magnifying glass and binary code" by Marco Verch Professional Photographer/Flickr, CC BY 2.0)

Learning Objectives

Add and subtract in bases 2 to base 16.

Once we decide on a system for counting, we need to establish rules for combining the numbers we're using. This begins with the rules for addition and subtraction. We are familiar with base 10 arithmetic, such as $2+5=7$ or $3\times 5=15$. How does that change if we use a different base instead? A larger base? A smaller one? In particular, computers use base 2 for all number representations. When your calculator adds or subtracts, multiplies or divides, it uses base 2. This is because the circuitry recognizes only two things, high current and low current, which means the system is uses only has two symbols. Which is what base 2 is.

In this section, we use addition and subtraction in bases other than 10 by referencing the processes of base 10, but applied to a new base system.

Addition in Bases Other Than Base 10

Now that we understand what it means for numbers to be expressed in a base other than 10, we can look at arithmetic using other bases, starting with addition. When you think back to when you first learned addition, it is very likely you learned the addition table. Once you knew the addition table, you moved on to the addition of numbers with more than one digit. The same process holds for addition in other bases. We begin with an addition table and then move on to adding numbers with two or more digits.

Let's create the base 6 addition table.

Look in the row that starts with "5" and select the number in the column that starts with "1". At the intersection of the selected row and column, you will find the number 10.
Is $5_6+1_6=6_6$? Since 6 is not one of the digits in base 6. So, we need to convert 6 to base 6.

$6\div 6=\quad 1\quad \mathrm{R}\quad 0$

We'll say that $6_{10} = 10_6$

Is $5_6 + 2_6 = 7_6$? Since 7 is not one of the digits in base 6. So, we need to convert 7 to base 6.

$7 \div 6 = \quad 1 \quad \mathrm{R} \quad 1$

We'll say that $7_{10} = 11_6$

Similarly, $5_6 + 3_6 = 12_6$

And so it goes. Using that process, stepping one more along the list, we can create the addition table for base 6.

Careful

When adding two single-digit numbers in the given base, if the sum is greater than or equal to the base, we need to convert the sum to the given base by dividing it by the base.

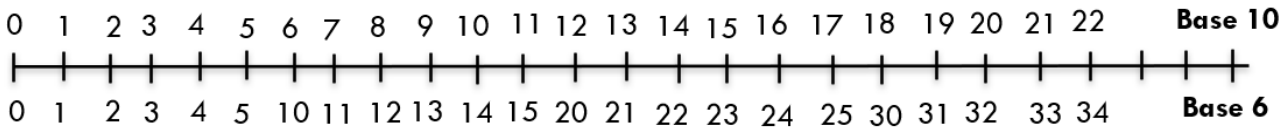


Figure $1.4.1$: Base 6 chart

Your Turn $1.4.1$: Adding Single Digit Number in Different Bases

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Example $1.4.2$: Adding Base Six Numbers

Calculate $251_6 + 133_6$

Answer available at the end of this section.

In base 7, the number that follows 6 is 10 (since we've run out of symbols!). So, $67 + 17 = 107$. In another word $6_7 + 1_7 = 7$, but 7 is not base 7 number so we need to convert 7 to base 7 dividing by 7 (See previous section).Once that is established,$67 + 27$ will be two numbers past 6, which is (11\) in base 7 (That is 11_7). Continuing, we can fill in the rows as we would in base 10, but be aware that we are working in base 7.

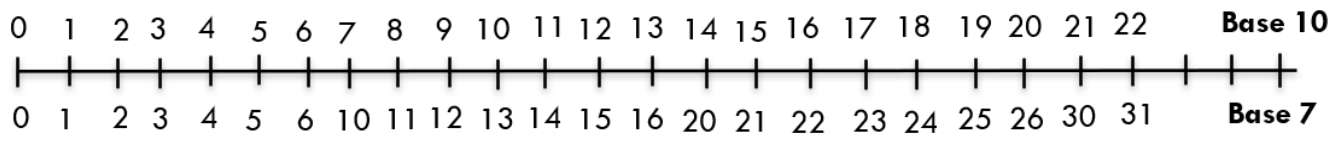


Figure 1.4.2: Base 7 chart

Your Turn 1.4.2: Base 7 Addition Table

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Example 1.4.3: Adding Base Seven Numbers

Create the addition table for base 7. and add $536_7 + 433_7$.

Answer available at the end of this section.

Your Turn 1.4.3: Add in Different Bases

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As seen previously, when performing addition in another base, set up the problem exactly as you would for addition in base 10. At each step, check the addition table for the base. As in base 10 addition, move right to the left, adding down the columns using the rules in the addition table. When necessary and just as in base 10, be sure to carry the 1.

Adding and Subtracting in a Base Higher Than Ten

As we know from the previous section, when the **base is higher than** 10, we usually use **letters** to represent digits beyond 9. For example:

Base 11 : Digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}$, where $10 = \mathrm{A}$

Base 12 : Digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}, \mathrm{B}$, where $10 = \mathrm{A}, 11 = \mathrm{B}$

Base 13 : Digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}$, where $10 = \mathrm{A}, 11 = \mathrm{B}, 12 = \mathrm{C}$

Base 14 : Digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$, where $10 = \mathrm{A}, 11 = \mathrm{B}, 12 = \mathrm{C}, 13 = \mathrm{D}$

Base 15 : Digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}$, where $10 = \mathrm{A}, 11 = \mathrm{B}, 12 = \mathrm{C}, 13 = \mathrm{D}, 14 = \mathrm{E}$

Base 16 : Digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathrm{A}, \mathrm{B}, ..., \mathrm{F}$, where $10 = \mathrm{A}, 11 = \mathrm{B}, 12 = \mathrm{C}, 13 = \mathrm{D}, 14 = \mathrm{E}, 15 = \mathrm{F}$

Let's make an **addition table for base** 12, just as an example:

Example $1.4.4$: Adding Base Twelve Numbers

Create the addition table for base 12 and calculate $3A7_{12} + 9BA_{12}$.

Answer available at the end of this section.

Subtraction in Bases Other Than Base 10

Subtraction in bases other than base 10 follows the same processes as base 10 subtraction, but, as with addition, using the addition table for the base.

Example $1.4.5$: Subtracting in Base Lower than Ten

1. Calculate $52_6 - 34_6$.
2. Calculate $563_7 - 164_7$.

Answer 1 available at the end of this section.

Your Turn $1.4.5$: Subtraction in Different Bases

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Example $1.4.6$: Subtracting in Base 12

Calculate $A17_{12} - 4B3_{12}$

Answer available at the end of this section.

Your Turn 1.4.6: Subtraction in Base Twelve

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Errors When Adding and Subtracting in Bases Other Than Base 10

When computing in bases other than 10, errors often involve applying base 10 rules or symbols to an arithmetic problem in a base other than base 10. The first type of error is using a symbol that is not in the symbol set for the base. For instance, if a 9 shows up when working in base 7, you know an error has happened because 9 is not a legal symbol in base 7.

Example 1.4.7: Identifying an Illegal Symbol in Arithmetic in a Base Other Than Base 10

Explain the error in the following calculation:

$$15_6 + 34_6 = 49_6$$

Answer available at the end of this section.

Remember: The second type of error is using a base 10 rule when the numbers are not in base 10. For instance, if you are working in base 13, then $9_{13} + 9_{13}$ is not 18_{13}, even though 18 is the correct answer in base 10.

Your Trun 1.4.8: Identifying an Arithmetic Error in a Base Other Than Base 10

Explain the error in the following calculation and correct the error:

$$\begin{array}{r} 89_{12} \\ +\ 76_{12} \\ \hline 165_{12} \end{array} \tag{1.4.1}$$

Answer available at the end of this section.

Answers

Example 1.4.2: Adding Base Six Numbers

$$\begin{array}{r} 251_{\text{six}} \\ +\ 133_{\text{six}} \\ \hline 424_{\text{six}} \end{array} \tag{1.4.2}$$

Let's do for the place of one's first. According to the base 6 addition table see above, $1_6 + 3_6 = 4_6$.
Now, we do the "tens" place (it's the sixes place). According to the base 6 addition table, we have $5_6 + 3_6 = 12_6$.
So, like in base 10, we use the 2 and carry the 1. Now the "hundreds" place (really, thirty-sixes place). There, we have $1(\text{carried}) + 2_6 + 1_6 = 4_6$.
Therefore

$$251_6 + 133_6 = 424_6$$

Note: When you add sixes places (middle column), do not write 8_6. Be Careful! We are adding in base 6.

Example 1.4.3: Adding Base Seven Numbers

$$\begin{array}{r} 536_{\text{seven}} \\ +\ 433_{\text{seven}} \\ \hline 1302_{\text{seven}} \end{array} \tag{1.4.3}$$

Let's do for the place of one's first. According to the base 7 addition table, $6_7 + 3_7 = 12_7$. We will carry the 1. Now, we do the "tens" place (it's the sevens place). According to the base 7 addition table, we have $1(\text{carried}) + 3_7 + 3_7 = 7 = 10_7$. We use the 0 and carry the 1. Now the "hundreds" place (really, forty-nines place). There, we have $1(\text{carried}) + 5_7 + 4_7 = 10 = 13_7$.

So,

$$536_7 + 433_7 = 1302_7$$

Example $1.4.4$**: Adding Base Twelve Numbers**

When the sum is up to 9, we are good. If sum is 10, we use A for 10 as a digit in base 12. When sum is 11 we use B for 11 as digit in base 12. What if sum is 12, for example, $6 + 6 = 12$, we do not have digits that represent 12 in base 10. (We only have up to 11, which is B)

In other words, since 12 is not one of the digits in base 12. So we need to convert 12 to base 12 by using division.

$12 \div 12 = \quad 1 \quad \text{R} \quad 0$

We'll say that $12 = 10_{12}$

Again, what is $9 + 7$ in base 12? It is 16, not one of the digits in base 12. So we need to convert 16 to base 12 by division.

$16 \div 12 = \quad 1 \quad \text{R} \quad 4$

We'll say that $16 = 14_{12}$.

Finally, what is $B + B$? $B(11) + B(11) = 22$, and 22 is more than base 12. So we need to convert 22 to base 12 by division.

$22 \div 12 = \quad 1 \quad \text{R} \quad 10(A)$

We'll say that $22 = 1A_{12}$

Base 12 Addition Table

+	0	1	2	3	4	5	6	7	8	9	A (10)	B (11)
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	10
2	2	3	4	5	6	7	8	9	A	B	10	11
3	3	4	5	6	7	8	9	A	B	10	11	12
4	4	5	6	7	8	9	A	B	10	11	12	13
5	5	6	7	8	9	A	B	10	11	12	13	14
6	6	7	8	9	A	B	10	11	12	13	14	15
7	7	8	9	A	B	10	11	12	13	14	15	16
8	8	9	A	B	10	11	12	13	14	15	16	17
9	9	A	B	10	11	12	13	14	15	16	17	18
A (10)	A	B	10	11	12	13	14	15	16	17	18	19
B (11)	B	10	11	12	13	14	15	16	17	18	19	1A

$$\begin{array}{r} 3A7_{12} \\ +\ 9BA_{12} \\ \hline 11A5_{12} \end{array}$$

Let's do the for the place of ones According to the base 12 addition table, $7 + A = 15_{12}$. We will carry the 1. Now, we do the "tens" place (it's really a twelves place). According to the base 12 addition table, we have $1 + A + B = 22 = 1A_{12}$. So, like in base 10, we use the A and carry the 1. Now the "hundreds" place (really, 144 s place). There, we have $1 + 3 + 9 = 13 = 11_{12}$.

So,

$$3A7_{12} + 9BA_{12} = 11A5_{12}$$

Example $1.4.5$**: Subtracting in Base Lower than Ten**

For the place of ones (right column), when we borrow a 1 from the 5, we borrow one group of 6. After borrowing, 2 becomes $2 + 6 = 8$, so $8 - 4 = 4$. Now top 5 on the left column becomes new 4 and $4 - 3 = 2$.

$$\begin{array}{r} 52_6 \\ -\ 34_6 \\ \hline 14_6 \end{array}$$

For the place of ones (right column), when we borrow a 1 from the 6, we borrow one group of 7. After borrowing, 3 becomes $3 + 7 = 10$, so $10 - 4 = 6$. In the middle column, top 6 becomes new 5. Again, we borrow 1, from 5 (the left column's top number), we borrow one group of 7, new 5 becomes $5 + 7 = 12$, so $12 - 6 = 6$. In

the left column (third column from the right), 5 becomes new 4, and $4-1=3$.

$$\begin{array}{r} 563_7 \\ -\ 164_7 \\ \hline 366_7 \end{array}$$

Example $1.4.6$**: Subtracting in Base** 12

$$\begin{array}{r} A17_{12} \\ -\ 4B3_{12} \\ \hline 524_{12} \end{array}$$

For one place, $7-3=2$, all good. For middle column, we borrow 1, from $A=10$, we borrow one group of 12, after borrowing 1 becomes $1+12=13$, so $13-B(11)=2$. Now in third column from the right, $A=10$, becomes new 9, and $9-4=5$.

Example $1.4.7$**: Identifying an Illegal Symbol in Arithmetic in a Base Other Than Base** 10

Since the problem is in base 6, the symbol set available is $0,1,2,3,4$, and 5. The 9 in the answer is clearly not a legal symbol for base 6. Returning to the base 6 addition table, we see that $5_6+4_6=13_6$. Correcting the error, we know the sum is $15_6+34_6=53_6$.

Your Trun $1.4.8$**: Identifying an Arithmetic Error in a Base Other Than Base** 10

This would be the correct answer if this problem were a base 10 problem. However, in base 12, $9+6$ is not 15, but is instead 13. To correct this error, use the addition table for base 12. If properly used, the correct answer would be 143_{12}, as seen below:

$$\begin{array}{r} 89_{12} \\ +\ 76_{12} \\ \hline 143_{12} \end{array} \tag{1.4.4}$$

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1.5: Cryptography

Learning Objectives

1. Encrypt and decrypt a message using a substitution cipher method.
2. Encrypt and decrypt a message using a transposition cipher method.
3. Encrypt and decrypt a message using a shared symmetric key cipher method.
4. Encrypt and decrypt a message using a Bifid cipher method.

When people need to secretly store or communicate messages, they turn to cryptography. Cryptography involves using techniques to obscure a message so outsiders cannot read it. It is typically split into two steps: encryption, in which the message is obscured, and decryption, in which the original message is recovered from the obscured form.

Definitions

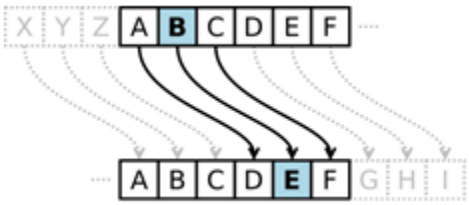
- **Plaintext** – the original, readable data (e.g., "Hello").
- **Key** – a secret value (like a password) used by the algorithm to encrypt and decrypt.
- **Ciphertext** – the scrambled, unreadable version of the data (e.g., "X9&@1z").
- **Encryption** is a process used to protect information by converting it into a format that can only be read or accessed by someone with the correct decryption key or password.
- **Decryption** is the process of converting encrypted data (**ciphertext**) back into its original, readable form (**plaintext**), using a **key** and an algorithm.

  In other words, **decryption reverses the process of encryption**.

Substitution Cipher

How does the Substitution Cipher work?

A substitution cipher replaces each letter in the message with a different letter, following some established mapping.



A simple example of a substitution cipher is called the **Caesar cipher**, sometimes called a shift cipher. In this approach, each letter is replaced with a letter some fixed number of positions later in the alphabet. For example, if we use a shift of 3, then the letter A would be replaced with D, the letter 3 positions later in the alphabet. The entire mapping would look as shown in the figure. [1]

Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ

Maps to: DEFGHIJKLMNOPQRSTUVWXYZABC

Example 1.5.1: Encryption

Use the Caesar cipher with a shift of 3 to encrypt the message: “We ride at noon.”

Answer available at the end of this section.

Your Turn 1.5.1: Encrypt Using Caesar Shift

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Example 1.5.2: Decryption

Decrypt the message GZD KNK YDX MFW JXA if it was encrypted using a shift cipher with a shift of 5.

Answer available at the end of this section.

Your Turn 1.5.2: Decrypt Using Caesar Shift

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Notice that in both the ciphers above, the extra part of the alphabet wraps around to the beginning. Because of this, a handy version of the shift cipher is a cipher disc, such as the Alberti cipher disk shown here [2] from the 1400 s. In a cipher disc, the inner wheel could be turned to change the cipher shift. This same approach is used for "secret decoder rings."

The security of a cryptographic method is very important to the person relying on their message being kept secret. The security depends on two factors:

1. The security of the method being used
2. The security of the encryption key used

In the case of a shift cipher, the method is "a shift cipher is used." The encryption key is the specific amount of shift used.

Suppose an army is using a shift cipher to send their messages and one of their officers is captured by their enemy. It is likely the method and encryption key could become compromised. It is relatively hard to change encryption methods, but relatively easy to change encryption keys.

During World War II, the Germans' Enigma encryption machines were captured, but having details on the encryption method only slightly helped the Allies, since the encryption keys were still unknown and hard to discover. Ultimately, the security of a message cannot rely on the method being kept secret; it needs to rely on the key being kept secret.

Encryption Security

The security of any encryption method should depend only on the encryption key being difficult to discover. It is not safe to rely on the encryption method (algorithm) being kept secret.

With that in mind, let's analyze the security of the Caesar cipher.

Example 1.5.3

Suppose you intercept a message and know the sender is using a Caesar cipher, but do not know the shift being used. The message begins EQZP. How hard would it be to decrypt this message?

Answer available at the end of this section.

Brute Force Attack

A brute force attack is a method for breaking encryption by trying all possible encryption keys.

To make a brute force attack harder, we could make a more complex substitution cipher by using something other than a shift of the alphabet. By choosing a random mapping, we could get a more secure cipher, with the tradeoff that the encryption key is harder to describe; the key would now be the entire mapping, rather than just the shift amount.

Example 1.5.4: Encryption

Use the substitution mapping below to encrypt the message "March 12 0300"

Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789

Maps to: 2BQF5WRTD8IJ6HLCOSUVK3A0X9YZN1G4ME7P

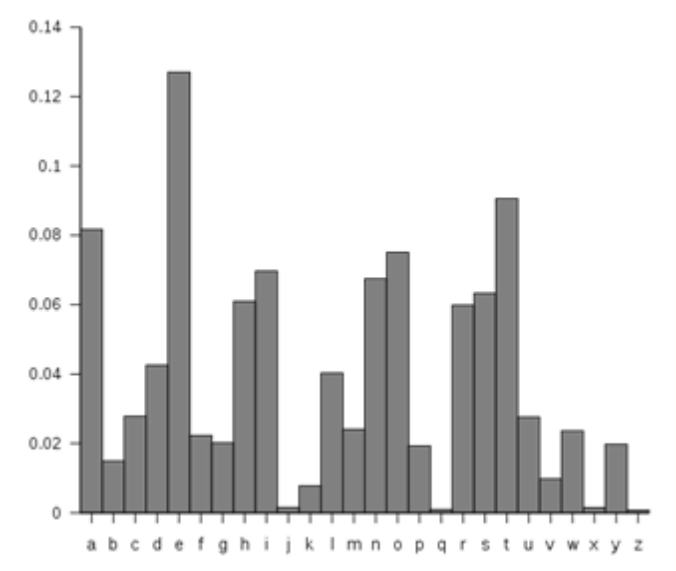
Answer available at the end of this section.

Your Turn 1.5.4: Decrypt Using Mapping

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While there were only 25 possible shift cipher keys (35 if we had included numbers), there are about 10^{40} possible substitution ciphers[3]. That's much more than a trillion trillions. It would be essentially impossible, even with supercomputers, to try every

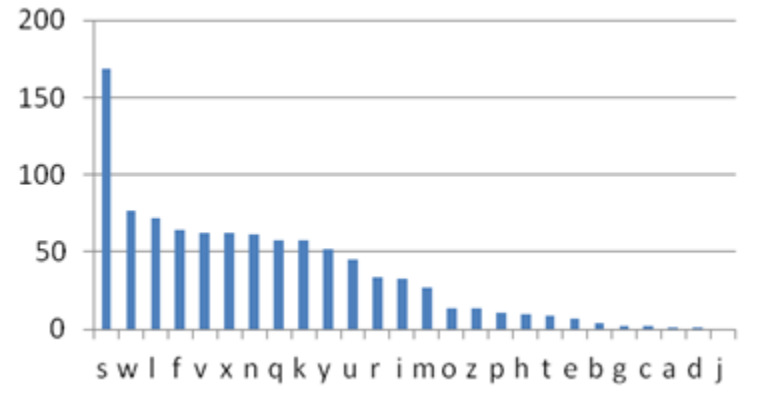


A frequency chart showing each letter of the alphabet on the horizontal, and frequency of that letter on the vertical.

possible combination. Having a huge number of possible encryption keys is one important part of key security.

Unfortunately, this cipher is still not secure because of a technique called frequency analysis, discovered by Arab mathematician Al-Kindi in the 9th century. English and other languages have certain letters that show up more often in writing than others.[4] For example, the letter E shows up the most frequently in English. The chart to the right shows the typical distribution of characters.

Example 1.5.5



A bar chart showing letters of the alphabet on the horizontal in random order, and frequency on the vertical. The first letter with highest frequency is S.

The chart to the right shows the frequency of different characters in some encrypted text. What can you deduce about the mapping?

Answer available at the end of this section.

In addition to looking at individual letters, certain pairs of letters show up more frequently, such as the pair "th." The substitution mapping can be deduced by analyzing how often different letters and letter pairs show up in an encrypted message [5].

Transposition Cipher

A transposition cipher is one in which the order of characters is scrambled to obscure the message. The characters themselves are not changed.



An early version of a transposition cipher was a Scytale[6], in which paper was wrapped around a stick and the message was written. Once unwrapped, the message would be unreadable until it was wrapped around a stick of the same size again.

One modern transposition cipher is written in rows, then forms the encrypted message from the text in the columns.

Example 1.5.6: Encryption

Encrypt the message "Meet at First and Pine at midnight" using rows 8 characters long.

Answer available at the end of this section.

Your Turn 1.5.6: Encrypt Using Transposition

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Example 1.5.7: Decryption

Decrypt the message CEE IAI MNL NOG LTR VMH NW using the method above with a table with rows of 5 characters.

Answer available at the end of this section.

Your Turn 1.5.7: Decrypt Using Transposition

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More complex versions of this rows-and-column-based transposition cipher can be created by specifying an order in which the columns should be recorded. For example, the method could specify that after writing the message out in rows that you should record the third column, then the fourth, then the first, then the fifth, then the second. This adds additional complexity that would make it harder to make a brute-force attack.

To make the encryption key easier to remember, a word could be used. For example, if the keyword was "MONEY", it would specify that rows should have 5 characters each. The order of the letters in the alphabet would dictate which order to read the columns in. Since E, the 4th letter in the word, is the earliest letter in the alphabet from the word MONEY, the 4th column would be used first, followed by the 1st column (M), the 3rd column (N), the 2nd column (O), and the 5th column (Y).

Example 1.5.8: Encryption

Encrypt the message BUY SOME MILK AND EGGS using a transposition cipher with the keyword MONEY.

Answer available at the end of this section.

Your Turn 1.5.8: Encrypt Using Keyword

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To decrypt a keyword-based transposition cipher, we'd reverse the process. In the example above, the keyword MONEY tells us to begin with the 4th column, so we'd start by writing SIDP down the 4th column, then continue to the 1st column, 3rd column, etc.

Example 1.5.9: Decryption

Decrypt the message RHA VTN USR EDE AIE RIK ATS OQR using a row-and-column transposition cipher with keyword PRIZED.

Answer available at the end of this section.

Your Turn 1.5.9: Decrypt Using Keyword

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Unfortunately, since the transposition cipher does not change the frequency of individual letters, it is still susceptible to frequency analysis, though the transposition does eliminate information from letter pairs.

Shared Symmetric-Key Methods

Both the substitution and transposition methods discussed so far are shared **symmetric-key** methods, meaning that both sender and receiver would have to have agreed upon the same secret encryption key before any messages could be sent.

All of the methods so far have been susceptible to frequency analysis since each letter is always mapped to the same encrypted character. More advanced methods get around this weakness. For example, the Enigma machines used in World War II had wheels that rotated. Each wheel was a substitution cipher, but the rotation would cause the substitution used to shift after each character.

For a simplified example, in the initial setup, the wheel might provide the mapping

- Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789
- Maps to: 2BQF5WRTD8IJ6HLCOSUVK3A0X9YZN1G4ME7P

After the first character is encrypted, the wheel rotates, shifting the mapping one space, resulting in a new shifted mapping:

- Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789
- Maps to: P2BQF5WRTD8IJ6HLCOSUVK3A0X9YZN1G4ME7

Using this approach, no letter gets encrypted as the same character over and over.

Example 1.5.10: Encryption

Encrypt the message “See me”. Use a basic Caesar cipher with a shift of 3 as the initial substitution, but shift the substitution one place after each character.

Answer available at the end of this section.

Your Turn 1.5.11: Encrypt With Keyword and Caesar Shift

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The actual Enigma machines used in WWII were more complex. Each wheel consisted of a complex substitution cipher, and multiple wheels were used in a chain[7]. The specific wheels used, the order of the wheels, and the starting position of the wheels formed the encryption key. While captured Enigma devices provided the Allied forces details on the encryption method, the keys still had to be broken to decrypt messages.

These code-breaking efforts led to the development of some of the first electronic computers by Alan Turing at Bletchley Park in the United Kingdom. This is generally considered the beginning of modern computing[8].

In the 1970s, the U.S. government had a competition and ultimately approved an algorithm, deemed DES (Data Encryption Standard), to be used for encrypting government data. It became the standard encryption algorithm used. This method used a combination of multiple substitution and transposition steps, along with other steps in which the encryption key is mixed with the message. This method uses an encryption key with a length of 56 bits, meaning there are 256 possible keys.

The number of keys makes a brute force attack extremely difficult and costly, but not impossible. In 1998 a team was able to find the decryption key for a message in 2 days, using about $250,000 worth of hardware. However, the price and time will go down as computer power increases.

From 1997 to 2001, the government held another competition, ultimately adopting a new method, deemed AES (Advanced Encryption Standard). This method uses encryption keys with 128, 192, or 256 bits, providing up to 2^{256} possible keys, making brute force attacks essentially impossible.

Bifid Cipher

The **Bifid cipher** is a classical encryption technique that combines the **Polybius square** with **transposition**, using fractionation to break up plaintext letters and spread their information across the ciphertext. This method was invented by Félix Delastelle (early 20 th century). This method combines substitution (via Polybius square) and transposition (reordering digits)

The **Polybius square** is a simple substitution cipher used to convert letters into pairs of numbers based on a 5×5 grid. Since the alphabet has 26 letters and a 5×5 grid only allows for 25, **"I" and "J" are usually combined.**

	1	**2**	**3**	**4**	**5**
1	A	B	C	D	E
2	F	G	H	I / J	K
3	L	M	N	O	P
4	Q	R	S	T	U
5	V	W	X	Y	Z

Where each pair or number represents a letter. For example

A $\rightarrow 11$ **B** $\rightarrow 12$ **H** $\rightarrow 23$ **J** or **I** $\rightarrow 24$ **Z** $\rightarrow 55$ and so on.

The word **"HELLO"** becomes: $23 \quad 15 \quad 31 \quad 31 \quad 34$

You can customize the Polybius square using a **keyword** (e.g., "SECRET") to fill the square before the rest of the alphabet.Query $1.5.1$

The following is Polybius square with the keyword "SECRET":

Start with the keyword: Remove duplicate letters from **SECRETE** $\rightarrow$ **SECRT. Complete the square** with the remaining unused letters of the alphabet (I and J are usually combined in a 5×5 square)

Polybius Square with Keyword SECRETE

	1	2	3	4	5
1	S	E	C	R	T
2	A	B	D	F	G
3	H	I/J	K	L	M
4	N	O	P	Q	U
5	V	W	X	Y	Z

Decryption: Bifid Cipher

- Convert ciphertext letters back to coordinates.
- Split the sequence into two equal parts (first half = rows, second half = columns).

- Reconstruct the original letter coordinates.
- Use the Polybius square to map back to letters.

Example 1.5.11: Decryption

Decode the following word using the bifid cipher with the given key.

Word to decode: **KONOUPYVVY**
Key: **UNDER**

Answer available at the end of this section.

Your Turn 1.5.11: Decrypt Using Bifid Cipher

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Example 1.5.12: Encryption

Encode the following word using the Bifid cipher with the given key.

Word to encode: **TELEVISION**
Key: **TOTAL**

Answer available at the end of this section.

Your Turn 1.5.12: Encrypt Using Bifid Cipher

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Your Turn 1.5.13: Understanding Definition

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[1] en.Wikipedia.org/w/index.php?title=File:Caesar3.svg&page=1. PD

[2] en.Wikipedia.org/wiki/File:Alberti_cipher_disk.JPG

[3] There are 35 choices for what *A* maps to, then 34 choices for what *B* maps to, and so on, so the total number of possibilities is 35*34*33*...*2*1 = 35! = about 10^{40}

[4] en.Wikipedia.org/w/index.php?title=File:English_letter_frequency_(alphabetic).svg&page=1 PD

[5] For an example of how this is done, see en.Wikipedia.org/wiki/Frequency_analysis

[6] en.Wikipedia.org/wiki/File:Skytala%26EmptyStrip-Shaded.png

[7] Wikipedia(opens in new window) [en.Wikipedia.org]

[8] For a good overview, see Youtube Video(opens in new window) [www.youtube.com]

Answers

Example 1.5.1**: Encryption**

We use the mapping above to replace each letter. W gets replaced with Z, and so forth, giving the encrypted message:

ZH ULGH DW QRRQ.

Notice that the length of the words could give an important clue to the cipher shift used. If we saw a single letter in the encrypted message, we would assume it must be an encrypted A or I, since they are the only single letters that form valid English words.

To obscure the message, the letters are often rearranged into equally sized blocks. The message ZH ULGH DW QRRQ could be written in blocks of three characters as

ZHU LGH DWQ RRQ.

Example 1.5.2**: Decryption**

We start by writing out the character mapping by shifting the alphabet, with A mapping to F, five characters later in the alphabet.

Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ

Maps to: FGHIJKLMNOPQRSTUVWXYZABCDE

We now work backwards to decrypt the message. The first letter G is mapped to by B, so B is the first character of the original message. Continuing, our decrypted message is

BUY FIF TYS HAR ESA.

Removing spaces, we get BUYFIFTYSHARESA. In this case, an extra character was added to the end to make the groups of three come out even, and the original message was “Buy fifty shares.”

Example 1.5.3

Since there are only 25 possible shifts, we would only have to try 25 different possibilities to see which one produces results that make sense. While that would be tedious, one person could easily do this by hand in a few minutes. A modern computer could try all possibilities in under a second.



In this case, a shift of 12 (A mapping to M) decrypts EQZP to SEND. Because of the ease of trying all possible encryption keys, the Caesar cipher is not a very secure encryption method.

Example 1.5.4**: Encryption**

Using the mapping, the message would be encrypted to 62SQT ZN Y1YY

Example 1.5.5

Because of the high frequency of the letter S in the encrypted text, it is very likely that the substitution maps E to S. Since W is the second most frequent character, it is likely that T or A maps to W. Because C, A, D, and J show up rarely in the encrypted text, it is likely they are mapped to from J, Q, X, and Z.

Example 1.5.6**: Encryption**

We write the message in rows of 8 characters each. **We pad the message with A** to the end to complete the last row.

MEETATFI

RSTANDPI

NEATMIDN

IGHTAAAA

We could then encode the message by recording down the columns. The first column, reading down, would be MRNI. Altogether, the encoded message would be MRNI ESEG ETAH TATT ANMA TDIA FPDA IINA. The spaces would be removed or repositioned to hide the size of the table used, since that is the encryption key in this message. So the final encrypted message is

MRNIESEGETAHTATTANMATDIAFPDAIINA

Example 1.5.7**: Decryption**

Since there are total of 20 characters and each row should have 5 characters, then there will be $20/5 = 4$ rows.

We start writing, putting the first 4 letters, CEEI, down the first column.

CALLM

EINTH

EMORN

INGVW

We can now read the message: CALL ME IN THE MORNING VW. The VW is likely nonsense characters used to fill out the message.

Example 1.5.8**: Encryption**

Writing out the message in rows of 5 characters:

BUYSO
MEMIL
KANDE
GGSPK

We now record the columns in order $4 \quad 1 \quad 3 \quad 2 \quad 5$:

SIDP BMKG YMNS UEAG OLEK

As before, we'd then remove or reposition the spaces to conceal evidence of the encryption key.

Example 1.5.9**: Decryption**

The keyword PRIZED tells us to use rows with 6 characters. Since D comes first in the alphabet, we start with 6th column. Since E is next in the alphabet, we'd follow with the 5th column. Continuing, the word PRIZED tells us the message was recorded with the columns in order $4 \quad 5 \quad 3 \quad 6 \quad 2 \quad 1$.

For the decryption, we set up a table with 6 characters in each row. Since the beginning of the encrypted message came from the last column, we start writing the encrypted message down the last column.



The 5th column was the second one the encrypted message was read from, so is the next one we write to.



Continuing, we can fill out the rest of the message.



Reading across the rows gives our decrypted message: AIR STRIKE ON HEADQUARTERS.

Example 1.5.10**: Encryption**

The initial mapping is

- Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ
- Maps to: DEFGHIJKLMNOPQRSTUVWXYZABC

This would map the first letter, S to V. We would then shift the mapping by one.

- Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ
- Now maps to: EFGHIJKLMNOPQRSTUVWXYZABCD

Now the next letter, E, will map to I. Again we shift the cipher

- Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ
- Now maps to: FGHIJKLMNOPQRSTUVWXYZABCDE

The next letter, E, now maps to J. Continuing this process, the final message would be VIJSL.

Notice that frequency analysis is much less useful now, since the character E has been mapped to three different characters due to the shifting of the substitution mapping.

Example 1.5.11**: Decryption**

Polybius square using a **keyword** ("UNDER") to fill the square before the rest of the alphabet.

Polybius Square with Key word UNDER

	1	2	3	4	5
1	U	N	D	E	R
2	A	B	C	F	G
3	H	I/J	K	L	M
4	O	P	Q	S	T
5	V	W	X	Y	Z

Ciphertext: **KONOUPYVVY**

We locate each letter in the Polybius square and note its pairs:

Pair of numbers that represent a letter in ciphertext

K	O	N	O	U	P	Y	V	V	Y
33	41	12	41	11	42	54	51	51	54

Break the above pairs in half as follows. Space those numbers out so that instead of writing 10 pairs horizontally, we really write 10 pairs vertically.

33 41 12 41 11 : 3 3 4 1 1 2 4 1 1 1

42 54 51 51 54 : 4 2 5 4 5 1 5 1 5 4

Convert back to Plaintext Letters, looking at the Polybius square.

34 → L 32 → I 45 → T 14 → E 15 → R 21 → A 45 → T 11 → U 15 → R 14 → E

LITERATURE is the final decoded message.

Example 1.5.12**: Encryption**

Polybius square using a **keyword** ("TOTAL") to fill the square before the rest of the alphabet.

Polybius square using keyword TOTAL

	1	2	3	4	5
1	T	O	A	L	B
2	C	D	E	F	G
3	H	I/J	K	M	N
4	P	Q	R	S	U
5	V	W	X	Y	Z

Word to encode: **TELEVISION**

We locate each letter in the Polybius square and note its pairs and place them vertically.

Pair of numbers that represent the each letter in plaintext

T	E	L	E	V	I	S	I	O	N
1	2	1	2	5	3	4	3	1	3
1	3	4	3	1	2	4	2	2	5

Separate the above into two rows of numbers as follows. We want to move the second row next to the first row, the fourth row next to the third row, and so on.

12 12 53 43 13

13 43 12 42 25

Now we put above two rows above together

12 12 53 43 13 13 43 12 42 25

Convert back to Plaintext Letters, looking at the Polybius square.

12 → O 12 → O 53 → X 43 → R 13 → A 13 → A 43 → R 12 → O 42 → Q 25 → G

OOXRAAROQG is the final decoded message.

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1.6: Modular Arithmetic



Figure 1.6.1 : If a credit card number is entered incorrectly, error checking algorithms will often catch the mistake. (credit: modification of work “Senior couple at home checking finance on credit card from above” by Nenad Stojkovic/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Add, subtract using modulo arithmetic.
2. Apply modular arithmetic to calculate real-world applications.

Working with Mod

Online shopping requires you to enter your credit card number, which is then sent electronically to the vendor. Using an ATM involves sliding your bank card into a reader, which then reads, sends, and verifies your card. Swiping or tapping for a purchase in a brick–and-mortar store is how your card sends its information to the machine, which is then communicated to the store’s computer and your credit card company. This information is read, recorded, and transferred many times. Each instance provides one more opportunity for error to creep into the process, a misrecorded digit, transposed digits, or missing digits. Fortunately, these card numbers have a built-in error-checking system that relies on modular arithmetic, which is often referred to as clock arithmetic. In this section, we explore clock, or modular, arithmetic.

We want to create a new system of arithmetic based on remainders, always keeping in mind the number we are dividing by, known as the modulus. Modular arithmetic is a system of arithmetic for integers where numbers "wrap around" upon reaching a certain value—the modulus. This concept is often referred to as "clock arithmetic" because it resembles how we tell time on a 12-hour clock.

For example, in a "days of the week" scenario, the modulus is 7. The possible remainders we can get are $0, 1, 2, 3, 4, 5,$ and $6,$ with each remainder representing a different day of the week. Similarly, in a "months of the year" problem, the modulus is 12.

We typically assign numbers to days like the following.

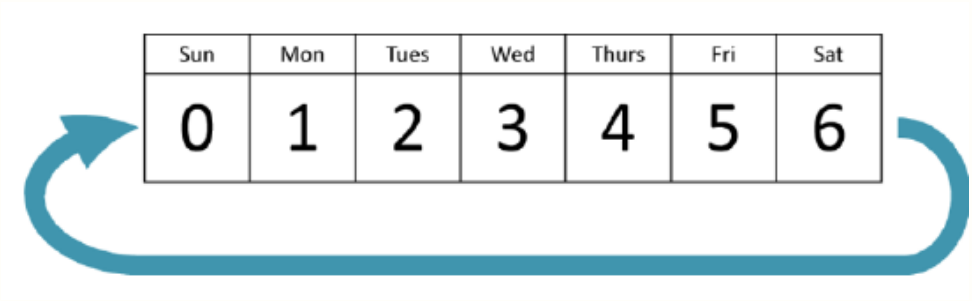


Figure 1.6.2 : Assigning numbers to days

What is Mod? Example of Mod

Let a be a positive integer. Then a modulo b, written a mod b, is the remainder when a is divided by b.

$17 \bmod 5$?

To find $17 \bmod 5$ we divide 17 by 5 and find the remainder.

$17 \div 5 = 3 \text{ remainder of } 2$

Because $5 \times 3 = 15$ and $17 - 15 = 2$. So $17 \bmod 5 = 2$

How to Find the Mod of a Negative Number?

$-a$ mod b is same as $[b - (a \bmod b)]$
For example, $-12 \bmod 5 = 5 - (12 \bmod 5) = 5 - 2 = 3$

When we do arithmetic, numbers can become larger and larger. But when we work with time, specifically with clocks, the numbers cycle back on themselves. It will never be 49 o'clock. Once 12 o'clock is reached, we go back to 1 and repeat the numbers. If it's 11 AM and someone says, "See you in four hours," you know that 11AM plus 4 hours is 3 PM, not 15 AM (ignoring military time for now). Math worked on the clock, where numbers restart after passing 12, is called clock arithmetic.

Clock arithmetic hinges on the number 12. Each cycle of 12 hours returns to the original time Figure $1.6.3$. Imagine going around the clock one full time. Twelve hours pass, but the time is the same. So, if it is $3:00$, 14 hours later and two hours later, both read the same on the clock, $5:00$. Adding 14 hours and adding 2 hours are identical. As is adding 26 hours. And adding 38 hours.

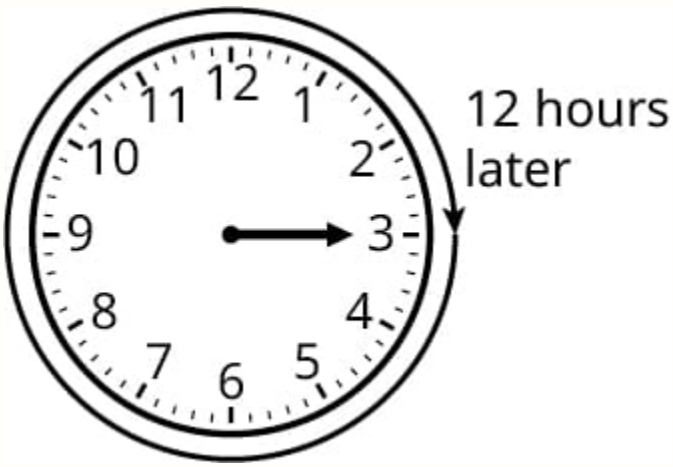


Figure $1.6.3$: lock showing $3:00$ o'clock with arrow going around the clock one full time, or 12 hours

What do $2, 14, 26,$ and 38 have in common in relation to 12? When they are divided by 12, they each have a remainder of 2. That's the key. When you add a number of hours to a specific time on the clock, first divide the number of hours being added by 12 and determine the remainder. Add that remainder to the time on the clock to know what time it will be.

A good visualization is to wrap a number line around the clock, with the 0 at the starting time. Then, each time 12 on the number line passes, the number line passes the starting spot on the clock. This is referred to as modulo 12 arithmetic. Even though the process says to divide the number being added by 12 first, perform the addition; the result will be the same if you add the numbers first, and then divide by 12 and determine the remainder

Example 1.6.1

Find the value of the following numbers modulo.

1. $88 \mod 7$
2. $539 \mod 12$
3. $-40 \mod 7$

Answer a available at the end of this section.

Your Turn 1.6.1: Find Mod

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Working with Clock Using Modular Arithmetic

Clock arithmetic is modulo 12 arithmetic, but applied to time. As time is divided into 12 hours that repeat a cycle, we use modulo 12 for clock arithmetic.

Example 1.6.2: Find Future Time or Past Time

1. If it is $3:00$ o'clock now, what time will it be in 89 hours?
2. If it is $4:00$ o'clock now, what time was it 67 hours ago?

Answer available at the end of this section.

Your Turn 1.6.2: Find the Time in Past and Future

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24 Hours Clock

Clock arithmetic in modulo 24 (also called mod 24 arithmetic) is used to represent time on a 24-hour clock. It is similar to clock 12 arithmetic.

Modulo 24 means we only care about the remainder after dividing by 24. For example:

1. $25 \mod 24 = 1$
2. $48 \mod 24 = 0$

Example 1.6.3 Applying Clock Arithmetic

1. Suppose it is $3:00$, and you decide to check your email every 5 hours. What time will it be when you check your email the ninth time?
2. Your family has a cat, and no one wants to empty the litter box. However, it has to be done daily. The six of you agree to take turns, so everyone has to empty the litter box every 6 days. You empty the box on a Thursday. What day will you empty the box for the 10th time?

Answer available at the end of this section.

Your Turn 1.6.3

1. You have agreed to text your friend every 3 hours while driving across the country. You began your trip at 8 AM. What time will it be when you text your friend the 15th time?

2. Your family shares the cooking duties in the home. You've agreed to prepare the meal every 5 days. The last time you prepared dinner was on a Tuesday. What day of the week will it be after you've prepared the meals 20 more times?

Working with Days Using Modular Arithmetic

Clock arithmetic processes can be applied to days of the week. Every 7 days the day of the week repeats, much like every 12 hours the time on the clock repeats. The only difference will be that we work with remainders after dividing by 7. In technical terms, this is referred to as modulo 7. More generally, let n be a positive integer. Then n modulo 7, written n mod 7, is the remainder when n is divided by 7.

If today is Tuesday, what day will it be in 10 days?

In Figure $1.6.2$, we know Tuesday = 2

$(2+10) \mod 7 = 12 \mod 7 = 5$. From the table in the Figure $1.6.1$, 5 corresponds to Friday.

Example $1.6.4$: Find Future Day or Past Days

1. If today is Thursday, what day is it 1000 days from today?

2. If today is Saturday, what day was it 100 days ago?

Answer available at the end of this section.

Your Turn $1.6.4$: Find the Future and Past Day

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Working with months using Modular Arithmetic

When working with months of the year, we use modulo 12 arithmetic, since there are 12 months in a year.

Note: Assigning Numbers to Months

We can number the months with our possible remainders, 0 through 11. We could start with 0 for January, 1 for February, etc., but this wouldn't match up with our standard numbering. Instead, we'll start with 1 for January, 2 for February, etc., up to 11 for November. We can't use 12 for December because 12 isn't a valid remainder. We'll use 0 for December.

JAN	FEB	MAR	APR	MAY	JUNE	JULY	AUG	SEP	OCT	NOV	DEC
1	2	3	4	5	6	7	8	9	10	11	0

Figure $1.6.5$: Assigning numbers to months

Example $1.6.5$: Find the Future and Past Month

1. If this month is April, what month will it be 100 months from this month?

2. If the month is April, what month was it 27 months ago?

Answer available at the end of this section.

Your Turn 1.6.5: Find the Future and Past Month

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Answers

Example 1.6.1

To find $88 \bmod 7$, we divide 88 by 7 and find the remainder.

$$88 \div 7 = 12\backslash) \text{ remainder of } \backslash(4\backslash).So\backslash[88 \bmod 7 = 4$$

To find $539 \bmod 12$ we divide 539 by 12 and find the remainder.

$$539 \div 12 = 44 \text{ remainder of } 11 \tag{1.6.1}$$

So

$$539 \bmod 12 = 11$$

Note that $-a$ mod b is same as $b - (a \bmod b)$ So $-40 \bmod 7 = (7 - (40 \bmod 7) = 7 - 5 = 2$

Example 1.6.2**: Find Future Time or Past Time**

1. To find that future time, we may determine the value of $89 \mod 12$

$89 \div 12 = 4 \text{ remainder of } 5$. So $89 \mod 12 = 5$. Counting 5 hours from to $3:00$ o'clcok results in $8:00$.

2. To find that past time, we may determine the value of $67 \mod 12$.

$67 \div 12 = 5 \text{ remainder of } 7$. So $67 \mod 12 = 7$. Counting 7 hours backward from to $4:00$ o'clcok results in $9:00$.We see this in the Figure 1.6.4.

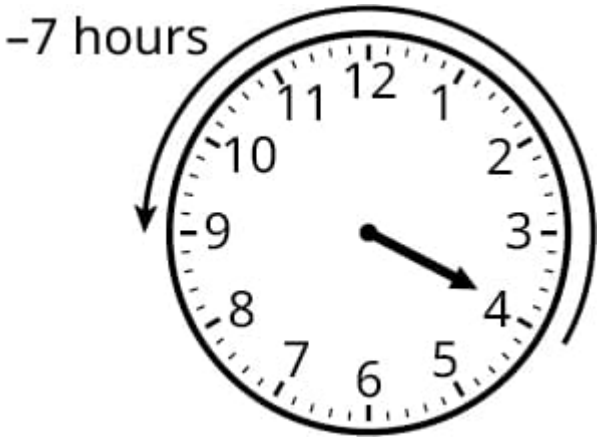


Figure 1.6.4 : Clock showing 7 hours subtracted from $4:00$

Example 1.6.3 **Applying Clock Arithmetic**

1. If you check your email every 5 hours nine times, that ninth check will occur 45 hours after $3:00$, which is an addition of 45 hours to $3:00$. So, we find 45 modulo 12, which is 9. Nine hours after $3:00$ is $12:00$. It will be $12:00$ when you check your email the ninth time.

2. The first time you emptied the litter box was on a Thursday. So, the 10 th time you empty the litter box will be 9 times later (you've already had your first turn, so 9 turns left!). This will happen $54(9 \times 6)$ days later. Finding the value of 54 modulo 7, that is $54 \mod 7$ gives the answer 5. Five days after a Thursday is Tuesday.

Example 1.6.4**: Find Future Day or Past Days**

1. Since there are 7 days in a week, every 7 days we will return to Thursday, if today is Thursday. If we divide 1000 by 7, we get a remainder of 6. This is the same as finding $1000 \mod 7$, which is 6. So we count 6 days forward starting from Thursday.

1 day after Thursday is → Friday

2 days after Thursday is → Saturday

..

..

6 days after Thursday is → **Wednesday**

There is another way to solve this problem. Thursday is 4, see the above table. Now $4 + 1000 = 1004$. $1004 \mod 7 \text{ is } 3$ (This is the same as saying when we divide 1004 by 7, we get a remainder of 3). From table in the Figure $1.6.2$, 3 corresponds to Wednesday.

2. We divide 100 by 7, and we get a remainder of 2. This is the same as finding $100 \mod 7 = 2$. So we count 2 days backward starting from Saturday.

1 day before Saturday → Friday

2 days before Saturday → **Thursday**

There is another way to solve this problem. Thursday is 6, see the above table. Now $6 - 100 = -94$.

Since $-a$ mod b is same as $b - (a \mod b)$

$-94 \mod 7 = 7 - (94 \mod 7) = 7 - 3 = 4$

From table in the Figure $1.6.2$, the number 4 corresponds to Thursday.

Example $1.6.5$: Find the Future and Past Month

1. To find what month it will be 100 months from April, we can use modular arithmetic. There are 12 months in a year, so:

$100 \mod 12 = 4$. This means 100 months from now is 4 **months ahead** of April.

Starting from April and counting 4 months forward:

1 month after April → May

2 months after April is → June

3 months after April is → July

4 months after April is → **August**

There is another way to solve this problem. April is 4, see the above table in Figure $1.6.5$. Now $4 + 100 = 104$. $104 \mod 12 \text{ is } 8$ (This is the same as saying when we divide 104 by 12, we get a remainder of 8). From the same table, the number 8 corresponds to August.

2. To find the month that is 27 months ago from **April**, we use modular arithmetic again. There are 12 months in a year:

$27 \mod 12 = 3$. This means 27 months ago is 3 **months before April.**

So, 27 months ago is 3 **months before April**.

Starting from April and counting 3 months backward:

1 month before April → March

2 months before April is → February

3 months before April is → **January**

There is another way to solve this problem. April is 4. Now $4 - 27 = -23$.

Since $-a$ mod b is same as $b - (a \mod b)$

$-23 \mod 12 = 12 - (23 \mod 12) = 12 - 11 = 1$

From table in the Figure $1.6.5$, the number 1 corresponds to January.

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2: MODELING IN MATHEMATICS

2.1: Linear Growth

2.2: Exponential, Natural, and Logistic Models

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2.1: Linear Growth

Learning Objectives

1. Apply a model to the linear relationship between two variables.

Introduction

Marco is a collector of antique soda bottles. His collection currently contains 437 bottles. Every year, he budgets enough money to buy 32 new bottles. Can we determine how many bottles he will have in 5 years and how long it will take for his collection to reach 1000 bottles?

While both of these questions you could probably solve without an equation or formal mathematics, we are going to formalize our approach to this problem to provide a means to answer more complicated questions.

Suppose that P_n represents the number, or population, of bottles Marco has after n years. So P_0 would represent the number of bottles now, P_1 would represent the number of bottles after 1 year, P_2 would represent the number of bottles after 2 years, and so on. We could describe how Marco's bottle collection is changing using:

$$\begin{aligned} P_0 &= 437 \\ P_n &= P_{n-1} + 32 \end{aligned}$$

This is called a **recursive relationship**. A recursive relationship is a formula that relates the next value in a sequence to the previous values. Here, the number of bottles in year n can be found by adding 32 to the number of bottles in the previous year, P_{n-1}. Using this relationship, we could calculate:

$$\begin{aligned} P_1 &= P_0 + 32 = 437 + 32 = 469 \\ P_2 &= P_1 + 32 = 469 + 32 = 501 \\ P_3 &= P_2 + 32 = 501 + 32 = 533 \\ P_4 &= P_3 + 32 = 533 + 32 = 565 \\ P_5 &= P_4 + 32 = 565 + 32 = 597 \end{aligned}$$

We have answered the question of how many bottles Marco will have in 5 years. However, solving how long it will take for his collection to reach 1000 bottles would require a lot more calculations.

While recursive relationships are excellent for describing simply and cleanly *how* a quantity is changing, they are not convenient for making predictions or solving problems that stretch far into the future. For that, a closed or explicit form of the relationship is preferred.

An **explicit equation** allows us to calculate P_n directly without needing to know P_{n-1}. While you may already be able to guess the explicit equation, let us derive it from the recursive formula. We can do so by selectively not simplifying as we go:

$$\begin{aligned} P_1 &= 437 + 32 &&= 437 + 1(32) \\ P_2 &= P_1 + 32 = 437 + 32 + 32 &&= 437 + 2(32) \\ P_3 &= P_2 + 32 = (437 + 2(32)) + 32 &&= 437 + 3(32) \\ P_4 &= P_3 + 32 = (437 + 3(32)) + 32 &&= 437 + 4(32) \end{aligned} \tag{2.1.1}$$

You can probably see the pattern now and generalize that

$$P_n = 437 + n(32) = 437 + 32n$$

Using this equation, we can calculate how many bottles he'll have after 5 years:

$$P_5 = 437 + 32(5) = 437 + 160 = 597$$

We can now also solve for when the collection reaches 1000 bottles by substituting in 1000 for P_n and solving for n.

$$\begin{aligned} 1000 &= 437 + 32n \\ 563 &= 32n \\ n &= \frac{563}{32} \\ n &= 17.59 \end{aligned}$$

So Marco will reach 1000 bottles in 18 years.

In the previous example, Marco's collection grew by the *same number* of bottles every year. This constant change is the defining characteristic of linear growth. Plotting the values we calculated for Marco's collection, we can see they form a straight line, indicating linear growth.

Linear Growth or Depreciation and Connection with Linear Equation with Slope Intercept Form

Linear Growth and Depreciation:

If a quantity starts at size P_0 and grows (increases) by m every time period, then the quantity after n time periods can be determined using the following model (relation). Where P_n is the quantity after n steps.

$$P_n = P_0 + nm$$

This equation represents the common difference m, which is the constant change in each time period. If m is negative, the model represents linear depreciation

$$P_n = P_0 - mn$$

Note: P_n is the same as $P(n)$, amount after time n.

This equation should look familiar – it is the same linear equation you learned in algebra, probably stated as

$$f(x) = y = mx + b \qquad \text{Linear Equation in Slope Intercept Form}$$

Recall that m was the slope and b is an initial amount.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } x}{\text{change in } y}$$

Linear Model Equation in Word

The linear model in the form of a word is

$$\boxed{\text{Amount after time } t = \text{Initial Amount} + \text{Rate of change (slope)} \times t}$$

Common Example of Linear Model

Common Examples of Linear Model

Scenario	Constant Rate of Change	
Hourly Wages	Earning $\$15$ for every hour worked.	

Scenario	Constant Rate of Change
Water Filling a Tank	A hose is pouring 5 gallons into a pool every minute.
Uniform Motion	A car traveling at a steady 60 mph on a highway.
Subscription Services	A company gains 100 new users every month.
Car Depreciation	A car decrease in value $\$2500$ per year

Function Model Notations: Do not get Confused!

Rather than using y or $f(x)$ to represent a model, we can use a variety of other notations. For example, a model may be written as

$G(w)$, $C(x)$, $V(x)$, $P(n)$, or $S(t)$ etc.

Sometimes the model is written simply as G, C, V, P, or S etc, depending on the context.

Your Turn $2.1.1$: Find Linear Function

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Example $2.1.1$: Interpret Slope

A tour guide is planning a ski trip. The equation $C(x) = 23x + 450$ models the relation between the cost in dollars, $C(x)$, of the ski trip and the number of skiers, x. The slope tells us.

1. For every additional skier, the ski trip costs an additional $\$450$.
2. If no skiers show up, the ski trip costs $\$23$.
3. For every additional skier, the ski trip costs an additional $\$23$.
4. If no skiers show up, the ski trip costs $\$450$.

Answer available at the end of this section.

Your Turn 2.1.1: Find Linear Function

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Example 2.1.2: Streetlight in the City

A city currently has 130 streetlights. As part of an urban renewal program, the city council has decided to install 2 additional streetlights at the end of each week for the next year. Write an equation that models the number of streetlights, $P(t)$, after t weeks. How many streetlights will the city have at the end of 35 weeks? When will there be 230 streetlights?

Answer available at the end of this section.

Your Turn 2.1.2: Find the Linear Function

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Example 2.1.3: City Population

A city currently has 1.89 thousand residents. Each year, the city's population grows by around 220 persons.

1. Write a function to model this situation where t is the number of years and $P(t))$ the population after t years.
2. After 3 years, what will the approximate population of the city be?

Answer available at the end of this section.

Your Turn 2.1.3: Gift Card

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Find the Linear Model that has Given Two Points

If the line passes through two points (x_1, y_1) and (x_2, y_2), the slope of the line is given by the following formula. To find the slope, we need to identify two points and use the following formula:

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Now we have m. The y-intercept, b, is the initial amount (or value). Use $y = mx + b$ to find the model.

Example 2.1.4: Linear Function Using Two Points

The population of elk in a national forest was measured to be $12,000$ in 2003 and was measured again to be $15,000$ in 2007. If the population continues to grow linearly at this rate. Find the function that models the elk population in time t. What will the elk population be in 2027?

Answer available at the end of this section.

Your Turn 2.1.4: Linear Function

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Example 2.1.5: Car Depreciation

Assume a car depreciates by the same amount each year. Max purchased a car in 2020 for $\$26,800$. In 2024, it was worth $\$22,000$. Find the linear model that expresses the value $V(t)$ as a function of how many years old is, t. Predict how much the car will be worth in 2028.

Answer available at the end of this section.

Your Turn 2.1.5: Value of Bulldozer

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When Good Models go Bad?

[1] When using mathematical models to predict future behavior, it is essential to consider that very few trends will persist indefinitely. Consider the following example.

Example: Suppose a four-year-old boy is currently 39 inches tall, and you are told to expect him to grow 2.5 inches a year. How tall will he be when he is 6 years old? 50 years old?

We can set up a growth model, with $t = 0$ corresponding to 4 years old

$$P(t) = P_t = P_0 + dt = 39 + 2.5t$$

So at 6 years old, $t = 2$, and we expect him to be:

$$P(2) = 39 + 2.5(2) = 44 \text{ inches}$$

Any mathematical model will eventually break down. Certainly, we shouldn't expect this boy to continue growing at the same rate throughout his life. If he did, at age 50 he would be inches tall 154 inches or 12.8 feet tall as we see below.

So at 50 years old, $t = 46$, and we expect him to be:

$$P(46) = 39 + 2.5(46) = 154 \text{ inches}$$

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Answers

Example $2.1.1$**: Interpret Slope**

The slope is the ratio of the rise (change in cost in this case) and the run (change in the number of skiers). Since the slope is 23, then we know:

$$23 = \frac{\text{rise}}{\text{run}}$$
$$\frac{23}{1} = \frac{\text{change in cost}}{\text{change in number of skier}}$$

So we might read this as follows: If the number of skiers increases by 1, then the cost will increase by 23.
Alternatively, this says that for every additional skier, the ski trip will cost an additional 23 dollars.

Example $2.1.2$**: Streetlight in the City**

The initial streetlights are $P_0 = 130$, and the common difference is $d = 2$, considered a slope. The linear growth model for this problem is:

$$P(t) = P_t = P_0 + dt = 130 + 2t$$

The above equation models the number of streetlights after t weeks.

The next question asks how many streetlights are in the city at the end of 35 weeks, $t = 35$.

$$P(35) = 130 + 2(35) = 130 + 70 = 200$$

Finally, we are asked to find the time when there is 230 straight light. So, $P(t) = 230$, we will solve for *t*.

$$230 = 130 + 2t$$
$$230 - 130 = 2t$$
$$50 = t$$

There will be 230 streetlights after 50 weeks. We can also see the linear relationship in the following graph.

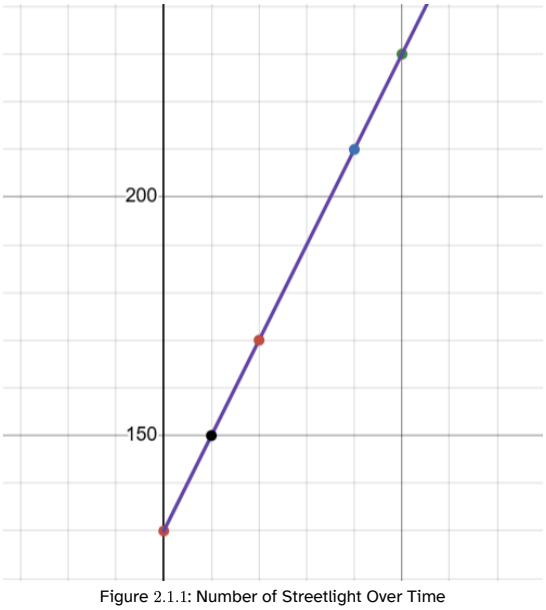


Figure $2.1.1$: Number of Streetlight Over Time

Example $2.1.3$**: City Population**

(a) The initial population is $P_0 = 1.89 \times 1,000 = 1890$ and slope (rate), m, is 220 person grow per year. Note that the slope is positive here. Using a linear model that models the number of people after t years.

$$P(t) = P_t = P_0 + mt = 1890 + 220t$$

(b) To find the approximate population of the city after 3 years, put $t = 3$.

$$P(3) = 1,890 + 3(220) = 1,890 + 660 = 2,550$$

So the city population after 3 years is $2,550$ or 2.55 thousand.

Example $2.1.4$: Linear Function Using Two Points

To begin, we need to define how we're going to measure t. Remember that $P.0$ is the population when $t = 0$, so we probably don't want to literally use the year 0. Since we already know the population in 2003, let us define $t = 0$ to be the year 2003. Then

$P.0 = P(0) = 12,000$,. This is the y-intercept of the linear equation.

Next, we need to find d. Remember, d is the growth per time period, in this case, growth per year (slope, m). Between the two measurements, the population grew by $15,000 - 12,000 = 3,000$, but it took $2007 - 2003 = 4$ years to grow that much.

To find the annual growth, we can divide 3000 elk by 4 years, giving 750 elk in 1 year.

Alternatively, you can use the slope formula from algebra to determine the common difference or slope. Here $(x_1, y_1) = (2003, 12000)$ and $(x_2, y_2) = (2007, 15000)$

$$\begin{aligned}\text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{15,000 - 12,000}{2007 - 2003} \\ &= \frac{3000}{4} \\ &= 750\end{aligned}$$

We can now write an equation for the growth model of the elk population as follows

$$P_t = P(t) = 12,000 + 750t$$

To answer the question, we need to first note that the year 2027 will be $t = 24$, since 2027 is 24 years after 2003. Use a linear model to find $P(24)$

$$\begin{aligned}P_{24} &= 12,000 + 750(24) \\ &= 30,000\end{aligned}$$

The elk population in 2027 will be $30,000$

Example $2.1.5$: Car Depreciation

Letting $n = 0$ correspond with 2020 would give $b = \$26,800$, initial value (y-intercept) or initial amount.

Next, we need to find m. Remember, m is the rate of change per year (slope, m).

We can find the slope using the two-point formula.

Here $(x_1, y_1) = (2020, \$26,800)$ and $(x_2, y_2) = (2024, \$22,000)$

$$\begin{aligned}\text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$22,000 - \$26,800}{2024 - 2020} \\ &= \frac{-\$4,800}{4} \\ &= -\$1,200\end{aligned}$$

That means the car value is decreasing by $-\$1,200$ per year. So the model equation in the form of $y = mx + b$ is

$$V(t) = 26,800 - 1,200t$$

Now, to find out how much the car will be worth in 2028, we need to know how many years that is from the purchase year. Since it is eight years later. So put $t = 8$ in the model equation.

$$\begin{aligned}V(t) &= 26,800 - 1,200t \qquad \text{Equation of the model} \\ &= 26,800 - 1,200(8) \\ &= 26,800 - 9,600 \\ &= 17,200\end{aligned}$$

So the value of car in 2028 (after 8 years) will be $\$17,200$

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2.2: Exponential, Natural, and Logistic Models

Learning Objectives

1. Apply a formula to model the exponential relationship between two variables.
2. Apply the natural growth formula to solve the problem.
3. Apply the logistic growth formula to solve the problem.

Exponential Growth

Suppose that every year, only 10% of the fish in a lake have surviving offspring. Assuming that none of the adult fish dies, if there were 100 fish in the lake last year, there would now be 110 fish. If there were 1000 fish in the lake last year, there would now be 1100 fish. Absent any inhibiting factors, populations of people and animals tend to grow by a percentage of the existing population each year. Whenever a quantity grows by a fixed percentage rather than a fixed amount, the growth is exponential, like compound interest.

Suppose our lake began with 1000 fish, and 10% of the fish have surviving offspring each year. Since we start with 1000 fish, $P_0 = 1000$. How do we calculate P_1? The new population will be the old population plus an additional 10%. Symbolically:

$$P_1 = P_0 + 0.10P_0$$

Notice this could be condensed to a shorter form by factoring:

$$P_1 = P_0 + 0.10P_0 = (1 + 0.10)P_0 = 1.10P_0$$

While 10% is the **growth rate**, 1.10 is the **growth multiplier**. Notice that 1.10 can be thought of as "the original 100% plus an additional 10%", which is 110% (of the original population).

For our fish population,

$$P_0 = 1.10(1000) = 1100$$

We could then calculate the population in later years:

$$P_2 = 1.10P_1 = 1.10(1100) = 1210$$

$$P_3 = 1.10P_2 = 1.10(1210) = 1331$$

Notice that in the first year, the population grew by 100 fish; in the second year, the population grew by 110 fish; and in the third year, the population grew by 121 fish. While there is a constant *percentage* growth, the actual increase in number of fish is increasing each year.

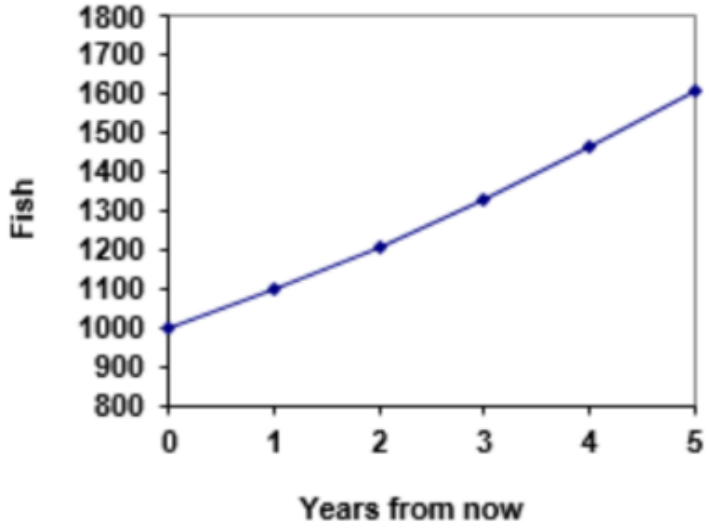


Figure 2.2.1: Fish growth graph

Graphing these values reveals that this growth doesn't appear to be linear.

To gain a clearer understanding of how this percentage-based growth impacts things, we need an explicit formula to quickly calculate values further into the future.

Like we did for the linear model, we will start building from the recursive equation:

$$\begin{aligned} P_1 &= 1.10P_0 = 1.10(1000) \\ P_2 &= 1.10P_1 = 1.10(1.10(1000)) = 1.10^2(1000) \\ P_3 &= 1.10P_2 = 1.10\left(1.10^2(1000)\right) = 1.00^3(1000) \\ P_4 &= 1.10P_3 = 1.10\left(1.10^3(1000)\right) = 1.10^4(1000) \end{aligned}$$

Observing a pattern, we can generalize the explicit form to be:

$P_n = 1.10^n(1000)$ or equivalently, $P_n = 1000\left(1.10^n\right)$. This is exactly like the compound interest formula with $r = 10\%$ annually.

From this, we can quickly calculate the number of fish in $10, 20,$ or 30 years:

$$\begin{aligned} P_{10} &= 1.10^{10}(1000) = 2594 \\ P_{20} &= 1.10^{20}(1000) = 6727 \\ P_{30} &= 1.10^{30}(1000) = 17,449 \end{aligned}$$

Adding these values to our graph reveals a shape that is definitely not linear. If our fish population had been growing linearly, by 100 fish each year, the population would have only reached 4000 in 30 years, compared to almost $18,000$ with this percent-based growth, called **exponential growth**.

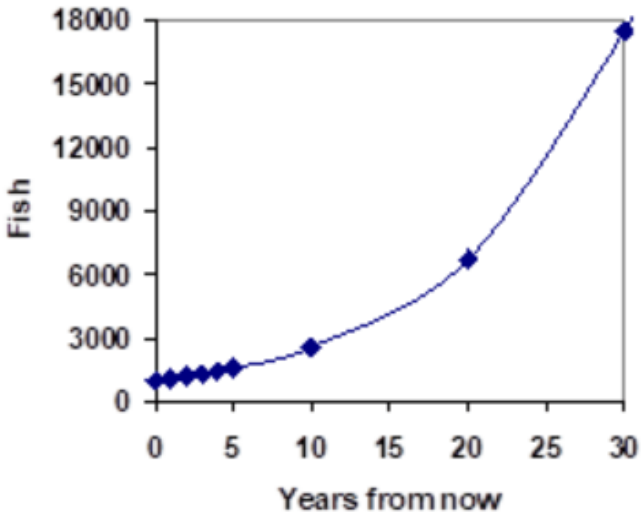


Figure 2.2.2: Fish population exponential growth graph

In exponential growth, the population grows proportionally to its own size, so as the population increases, the same percentage growth will yield a larger numeric growth.

Exponential Growth Formula: Growth or Decay Model

If a quantity starts at size P_0 and grows (or decays) by $r\%$ (written as a decimal) every time period, then the quantity after n time periods can be determined by following the model (function).

Exponential Growth and Exponential Decay

Exponential Growth
$P(n) = P_0(1+r)^n$

Where $P(n) = P_n$, which is the population (or amount) after n time.

$P(0) = P_0$, which is the population (or amount) in the beginning or the initial amount.

We call r the growth (decay) rate. Usually, it is expressed as a percentage, but you need to convert it to a decimal when you put it into the equation.
The term $(1+r)$ is called the growth (decay) multiplier, or growth (decay) factor, or common ratio.

Formula for Growth Factor if $P(0)$ and $P(1)$ are given

$$\text{Growth Factor} = \frac{P_1}{P_0}$$

If we know the growth factor, we can find the growth rate using the following formula

$$\text{Growth Rate} = \text{Growth Factor} - 1$$

Additionally, we can express the explicit form slightly differently, as shown below, with n time periods denoted as t.

$$f(t) = ab^t$$

Where $f(t)$ is similar to $P(t)$, $P(0)$ similar to a, and $1+r$ is similar to b. The two formulas above can be combined into a single formula in word.

$$\boxed{\text{Amount after time } t = \text{Initial Amount} \times (\text{Growth Factor})^t}$$

If growth factor b is >1, the model $f(x)$ is called an exponential growth model.

If growth factor b is <1, the model $f(x)$ is called an exponential decay model.

Interpretation of Exponential Growth Equation

A friend is using the equation $P_n = 4600(1.072)^n$ to predict the annual tuition at a local college. She says the formula is based on years after 2010. What does this equation tell us?

In the equation, $P_0 = 4600$ is the tuition's starting value when $n = 0$. This tells us that the tuition in 2010 was $\$4,600$.

The growth multiplier or growth factor is 1.072.

$$\begin{aligned}\text{Growth Rate} &= \text{Growth Factor} - 1\\ &= 1.072 - 1\\ &= 0.072\\ &= 7.2\%\end{aligned}$$

This tells us that the tuition is expected to grow by 7.2% yearly.

Putting this together, we could say that the tuition in 2010 was $\$4,600$ and is expected to grow by 7.2% each year.

Your Turn 2.2.1: Exponential Model

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Example 2.2.2: Use Exponential Model Function

Between 2007 and 2008, Olympia, WA, grew by almost 3% to a population of 245 thousand people. If this growth rate were to continue, what would the population of Olympia be in 2014?

Answer available at the end of this section.

Your Turn 2.2.2: Use Exponential Model Function

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Here's a simple table showing **exponential growth** over time.

Population over the time

Time, t	0	1	2	3	4	5	6	7	8	9	10
Value, $P(t)$	100	150	225	337.5	506.25	759.38	1139.06	1708.59	2562.89	3844.34	5766.51

What is an exponential model equation representing the above table?

Here, $P_0 = 100$, $P_1 = 150$, $P_2 = 225$ and so on.

Since the growth factor is a common ratio. To find it, we have to divide the one value by the value that follows it.

$$\text{Growth factor } = \frac{P_1}{P_0} = \frac{150}{100} = 1.5$$

and

$$\begin{aligned}\text{Growth Rate} &= \text{Growth Factor} - 1\\ &= 1.50 - 1\\ &= 0.50\\ &= 50\%\end{aligned}$$

So, the exponential model is

$$P_t = P_0(1+r)^t = 100(1.5)^t$$

The graph of the above model is shown below.

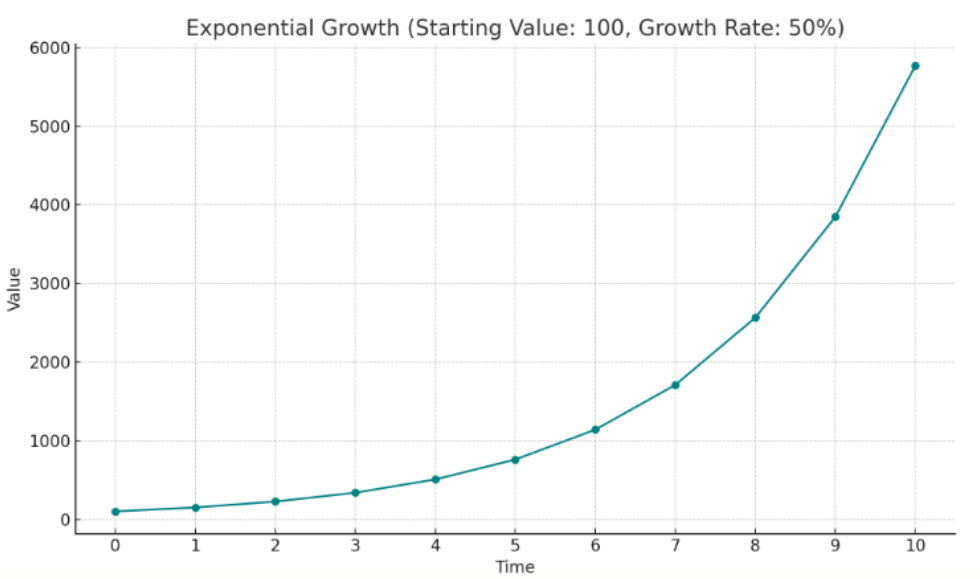


Figure 2.2.3: Exponential Growth, Growth rate 50%

What is the predicted value when $t = 25$. To find it, we put in the model equation

$$P_{25} = P_0(1+r)^t = 100(1.5)^{25} = 2,525,116.829$$

Your Turn 2.2.3: Exponential Model

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Example 2.2.4 : Find Exponential Model

Year	1998	1999	2000	2001	2002	2003
Balance ($)	8,000.00	8,440.00	8,904.20	9,393.93	9,910.60	10,455.68

Figure 2.2.4: Balance over Time

The table above shows a person's bank account balance for 5 years.

1. Find the growth factor and growth rate.
2. Find an exponential model for this data, with y representing the balance t years after 1998 and $t = 0$ corresponding to 1998.
3. Find this person's balance in 2010 if the bank account continues to grow at the same rate.

Answer available at the end of this section.

Your Turn 2.2.4: Use Exponential Formula

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Example 2.2.5 : Use Exponential Model

A population grows according to an exponential growth model. The initial population is $P_0 = 19$, and the growth rate is $r = 15\%$.

1. Find the exponential model that predicts the population after n period.
2. Find P_1 and P_2.
3. Find P_{10}.

Answer available at the end of this section.

Your Turn 2.2.5: Use Exponential Model

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Natural Growth

In this section, we will examine natural and logistic growth. Most naturally occurring phenomena **grow continuously**. That is called natural growth. Population growth is usually modeled using natural growth models, which are also exponential. However, they use the mathematical constant ***e as the base***. The growth rate per given period, *k*, is a part of the exponent, which is multiplied by the number of periods.

Natural growth is **a form of exponential growth** involving the mathematical constant **e, approximately equal to ≈ 2.718.**

Explicit Formula: The Natural Growth Model

The Natural Growth Model is

$$P(t) = P_0 e^{kt}$$

where P_0 is the initial population, k is the growth rate per unit of time, and t is the number of time periods.
Note: $P(t)$ is same as P_t.

The Natural Decay Model is

$$P(t) = P_0 e^{-kt}$$

where P_0 is the initial population, k is the decay rate per unit of time, and t is the number of time periods.
Note: $P(t)$ is same as P_t.

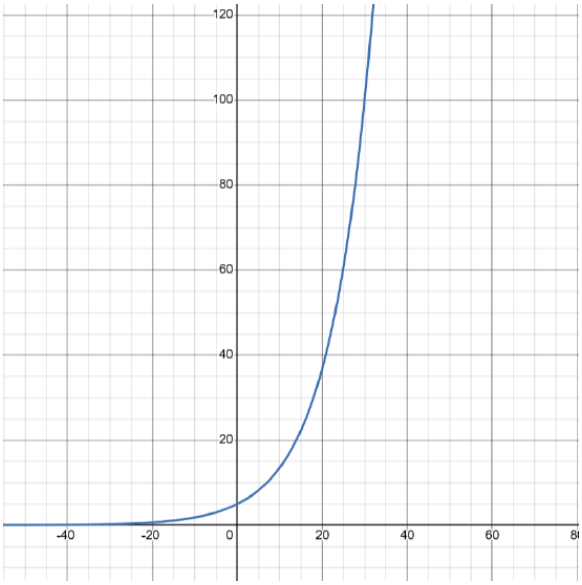


Figure 2.2.1: Natural Growth,$P(t) = P_0 e^{kt}, k > 0$

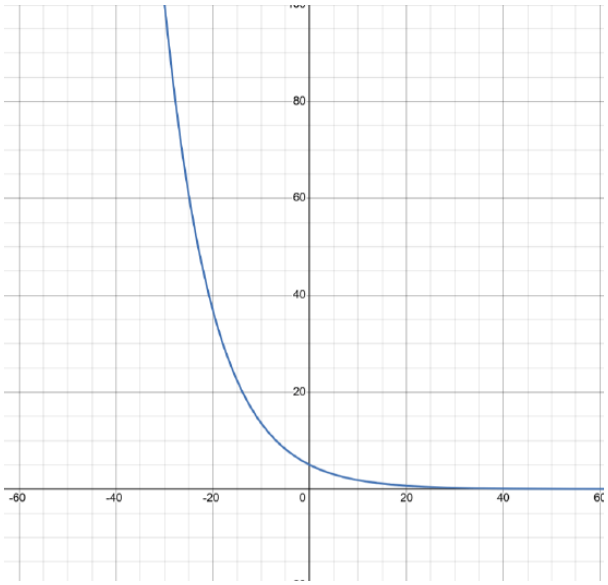


Figure 2.2.2: Natural Decay, $P(t) = P_0 e^{-kt}, k > 0$

Example 2.2.6: Natural Decay

When a certain drug is administered to a patient, the number of milligrams remaining in the bloodstream after *t* hours is given by the model

$$P(t) = 40e^{-0.25t}$$

How many milligrams are in the blood after two hours?

Answer available at the end of this section.

Your Turn 2.2.6: Natural Growth

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Example 2.2.7: Natural Growth

Bob has an ant problem. On the first day of May, Bob discovers he has a small red ant hill in his backyard, with a population of about 100 ants. If conditions are just right, red ant colonies will have a growth rate of 240% per year during the first four years. If Bob does nothing, how many ants will he have next May? How many in five years?

Answer available at the end of this section.

Your Turn 2.2.7: Natural Decay

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The problem with exponential growth is that the population grows without bound, and at some point, the model will no longer predict what is actually happening because the amount of resources available is limited. Populations cannot continue to grow on a purely physical level; eventually, death occurs, and a limiting population is reached.

Logistic Growth Model

Another growth model for living organisms is the logistic growth model. In the real world, "unlimited" doesn't last. Space runs out, food becomes scarce, and waste builds up. This introduces the concept of **Carrying Capacity** C, the maximum population an environment can sustain. The logistic growth model has a maximum population, known as the carrying capacity. As the population grows, it reaches the carrying capacity and remains there. This is the maximum population the environment can sustain. **Logistic growth** is used to model situations where growth starts out exponentially but then **slows down** as it approaches a **maximum limit** (called the **carrying capacity**).

For our fish, the carrying capacity is the largest population that the resources in the lake can sustain. If the population in the lake is far below the carrying capacity, then we would expect the population to grow essentially exponentially. However, as the population approaches the carrying capacity, there will be a scarcity of food and space available, and the growth rate will decrease. If the population exceeds the carrying capacity, there won't be enough resources to sustain all the fish and there will be a negative growth rate, causing the population to decrease back to the carrying capacity.

Carrying Capacity

The carrying capacity, or maximum sustainable population, is the largest population that an environment can support.

Explicit Formula: Logistic Growth

$$P_t = P(t) = \frac{C}{1 + \left(\frac{C}{P_0} - 1\right) e^{-rt}}$$

where $P(t)$ = Population over the time t

C = Carrying capacity

r = growth rate

t = time

$P(0) = P_0$ = Initial Population
Note: $P(t)$ is same as P_t.

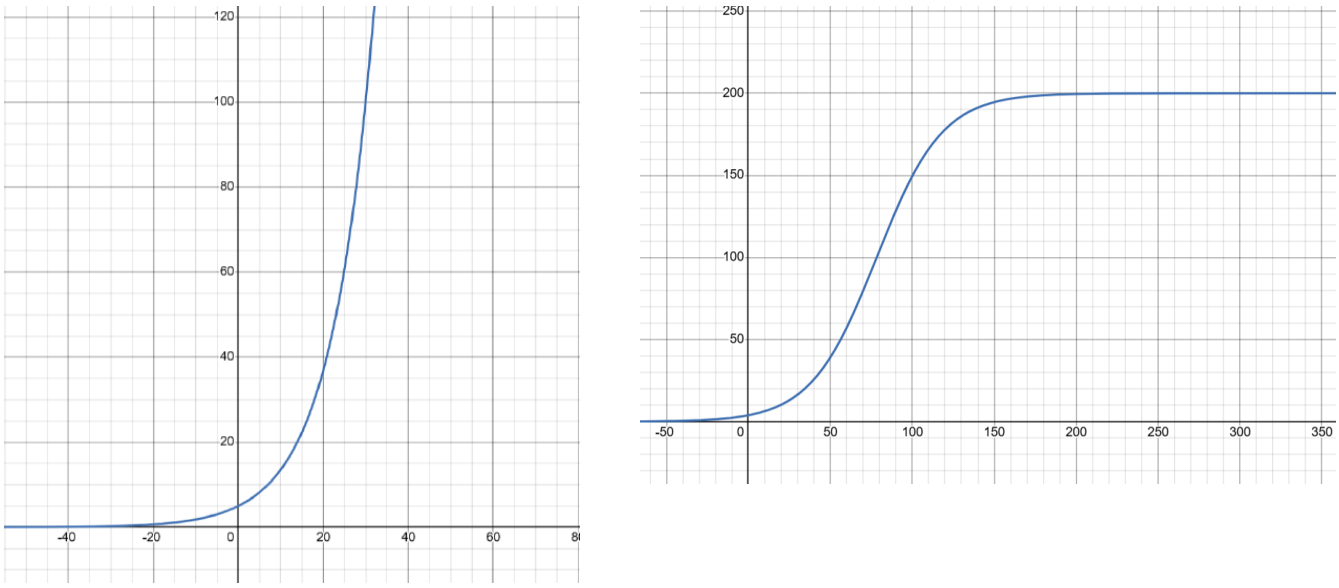


Figure 2.2.4: Logistic (Right) and Exponential (Left) Model Graph

The horizontal line on this graph illustrates the carrying capacity.

(Logistic Growth Image 1, n.d.)

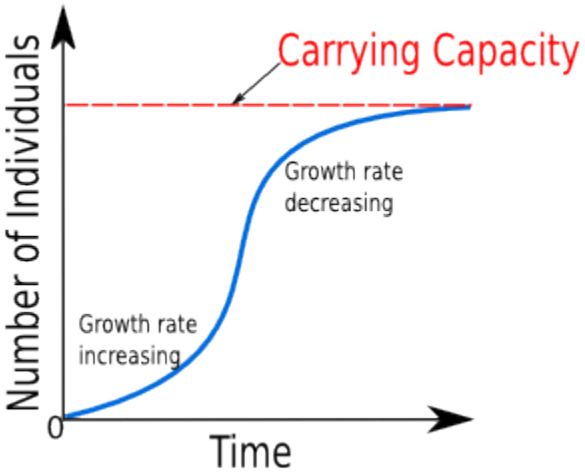


Figure 2.2.5: Logistic Growth Model

(Logistic Growth Image 2, n.d.)

The graph for logistic growth starts with a small population. When the population is small, the growth is fast because there is more elbow room in the environment. As the population approaches the carrying capacity, the growth slows.

Example 2.2.8: Bird Population

The population of an endangered bird species on an island grows according to the logistic growth model.

$$P(t) = \frac{3640}{1 + 25e^{-0.04t}}$$

Identify the initial population. What will the bird population be in five years? What will the population be like in 150 years? What will the population be like in 500 years?

Answer available at the end of this section.

Your Turn 2.2.8: Logistic Growth

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Example 2.2.9: Logistic Model

Suppose 16 blackberry plants started growing in a yard. Absent constraint, the number of blackberry plants will increase continuously at a monthly rate of 80%. If the yard can only sustain 150 plants, use a logistic growth model to estimate the number of plants after 4 months.

Answer available at the end of this section.

NEED CALCULATOR?

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Your Turn 2.2.9: Logistic Growth

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Answers

Example 2.2.2: Use Exponential Model Function

As we did before, we first need to define which year corresponds to $n = 0$. Since we know the population in 2008, it would make sense to have 2008 correspond to $n = 0$, so $P_0 = 245,000$.The year 2014 would then be $2014 - 2008 = 6$.

We know the growth rate is 3%, giving $r = 0.03$.

Using the model function:

$$\begin{aligned} P(6) = P_6 &= 245,000(1 + 0.03)^6 \\ &= 245,000 \times 1.19405 \\ &= 292,542.25 \end{aligned}$$

The model predicts that in 2014, Olympia would have a population of about 293 thousand people.

Example 2.2.4 : Find Exponential Model

1. Here, $P_0 = 8000$, $P_1 = 8440$, $P_2 = 8904.20$ and so on.

$$\begin{aligned}\text{Growth Factor: } (1+r) &= \frac{P_1}{P_0}\\ &= \frac{8440}{8000}\\ &= 1.055\end{aligned}$$

In the above exponential model, the growth rate is 5.5%. See the calculation below

$$\begin{aligned}\text{Growth Rate} &= \text{Growth Factor} - 1\\ &= 1.055 - 1\\ &= 0.055\\ &= 5.5\%\end{aligned}$$

2. The exponential model in is

$$y = P_t = P(t) = P_0(1+r)^t = 8000(1.055)^t$$

3. To find the predicted balance in 2010, which is $t = 2010 - 1998 = 12$ years after 1998, we put $t = 12$ into the equation

$$\begin{aligned}P(t) &= P_0(1+r)^t\\ P(12) &= 8000(1.055)^{12}\\ &= 15209.65989\end{aligned}$$

So this person's balance in 2010 will be $\$15209.66$. Round to two decimal places.

Example $2.2.5$ **: Use Exponential Model**

1. The exponential model is

$$P_n = 19(1+0.15)^n = 19(1.15)^n$$

2. To find $P(1)$, put $n = 1$ in the model

$$\begin{aligned}P(1) &= P_0(1+r)^n\\ &= 19(1.15)^1\\ &= 21.85\end{aligned}$$

To find $P(2)$, put $n = 2$ in the model

$$\begin{aligned}P(1) &= P_0(1+r)^n\\ &= 19(1.15)^2\\ &= 25.1275\end{aligned}$$

3. To find P_{10}. we put $n = 10$ into above model

$$\begin{aligned}P_{10} &= 19 \times (1.15)^{10}\\ &= 76.87\end{aligned}$$

Example $2.2.6$**: Natural Decay**

To solve this problem, we use the given equation with $t = 2$. Here growth rate is 0.25, which is same as 25%.

$$\begin{aligned}P(2) &= 40e^{-0.25(2)}\\ &= 24.26\end{aligned} \tag{2.2.1}$$

There are approximately 24.6 milligrams of the drug in the patient's bloodstream after two hours.

Example $2.2.7$**: Natural Growth**

We solve this problem using the natural growth model. Here $k = 2.40$.

$$P(t) = 100e^{2.4t}$$

In one year, $t = 1$, we have

$$P(1) = 100e^{2.4(1)} = 1102 \text{ ants}$$

In one year, $t = 5$, we have

$$P(5) = 100e^{2.4(5)} = 16,275,479 \text{ ants}$$

That is a lot of ants! Bob will not let this happen in his backyard!

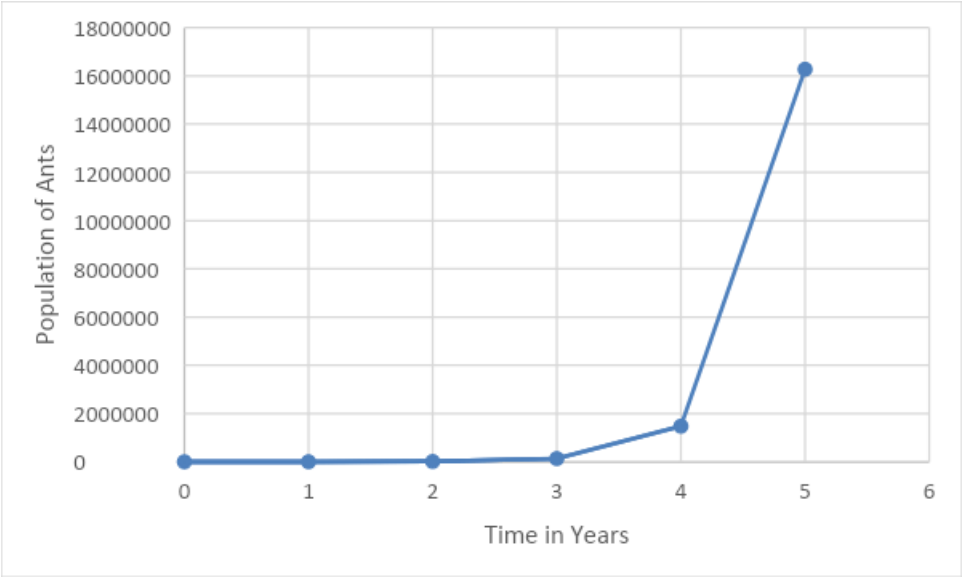


Figure 2.2.3 : Graph of Ant Population Growth in Bob's Yard.

Note: The population of ants in Bob's backyard follows an exponential (or natural) growth model.

Example 2.2.8**: Bird Population**

We know the initial population, P_0, occurs when $t = 0$.

$$P_0 = P(0) = \frac{3640}{1 + 25e^{-0.04(0)}} = \frac{3640}{1 + 25 \times 1} = 140$$

Note: $e^{-0.04(0)} = e^0 = 1$

When $t = 5$, calculate the population in five years.

$$P(5) = \frac{3640}{1 + 25e^{-0.04(5)}} = 169.6$$

The island will be home to approximately 170 birds in five years

Calculate the population in 150 years, when $t = 150$.

$$P(150) = \frac{3640}{1 + 25e^{-0.04(150)}} = 3427.6$$

The island will be home to approximately 3428 birds in 150 years.

Calculate the population in 500 years when $t = 500$.

$$P(500) = \frac{3640}{1 + 25e^{-0.04(500)}} = 3640.0$$

The island will be home to approximately 3640 birds in 500 years.

This example shows that the population grows quickly between five years and 150 years, with an overall increase of over 3000 birds, but slows dramatically between 150 years and 500 years (a longer span of time) with an increase of just over 200 birds.

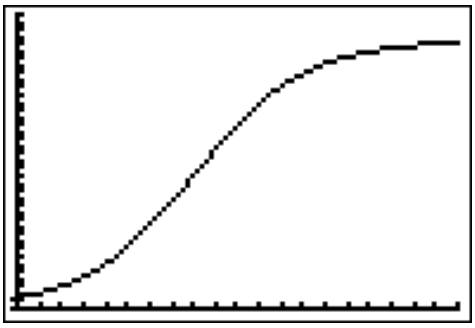


Figure 2.2.6 : Bird Population over a 200-Year Span

Example 2.2.9**: Logistic Model**

The following formula gives the logistic growth model.

$$P(t) = \frac{C}{1 + \left(\frac{C}{P_0} - 1\right) e^{-rt}}$$

Where

$C = 150$

$P_0 = 16$

$r = 80\% = 0.8$

Put those values in the given model.

$$\begin{aligned} P(t) &= \frac{C}{1+\left(\frac{C}{P_0}-1\right)e^{-rt}} \\ P(4) &= \frac{150}{1+\left(\frac{150}{16}-1\right)e^{-0.80t}} \\ &= \frac{150}{1+8.375e^{-0.80\times 4}} \\ &= 111.82484 \end{aligned}$$

After 4 month, there will be about 112 blackberry plants.

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3: COUNTING AND PROBABILITY

3.1: The Multiplication Rule for Counting
3.2: Permutations and Combination
3.3: Fundamental Concepts of Probability
3.4: The Addition and Complement Rule for Probability
3.5: Odd and Expected Value

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3.1: The Multiplication Rule for Counting

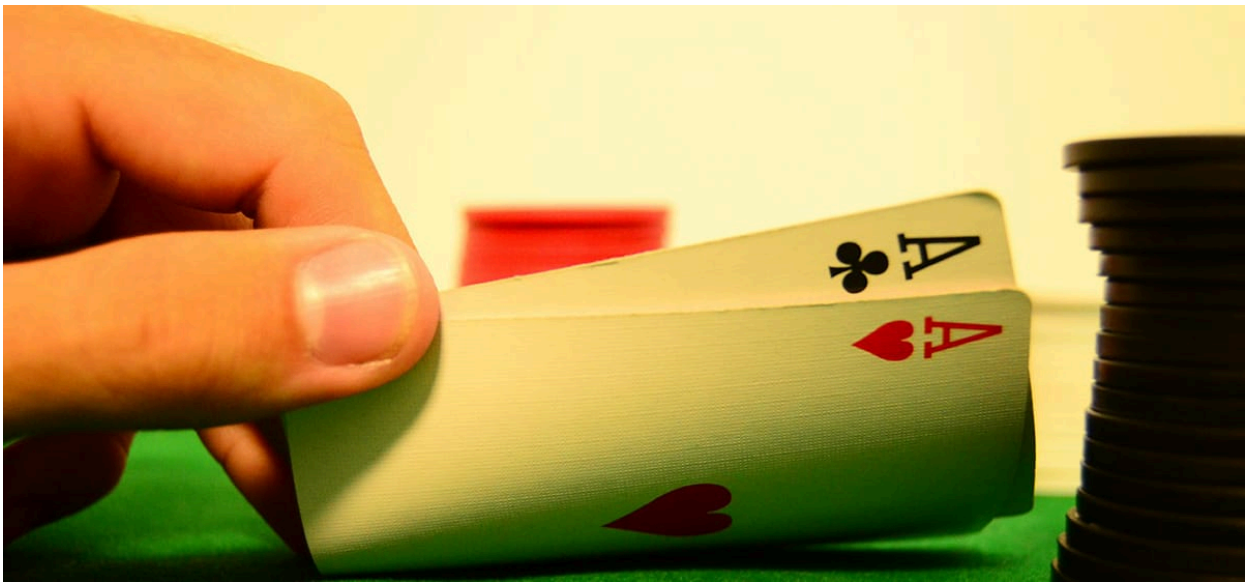


Figure 3.1.1 : The Multiplication Rule for Counting allows us to compute more complicated probabilities, like drawing two aces from a deck. (credit: "Pair of Aces – Poker" by Poker Photos/Flickr, CC BY 2.0)

Learning Objectives

Apply the Multiplication Rule for Counting to solve problems.

One of the first bits of mathematical knowledge children learn is how to count objects by pointing to them in turn and saying: "one, two, three, ..." That's a useful skill. Still, when the number of things that we need to count grows large, that method becomes onerous (or, for *very* large numbers, impossible for humans to accomplish in a typical human lifespan). So, mathematicians have developed shortcuts to counting big numbers. These techniques fall under the mathematical discipline of combinatorics, which is devoted to counting.

Multiplication as a Combinatorial Shortcut

One of the first combinatorial shortcuts to counting students learn in school has to do with areas of rectangles. If we have a set of objects to be counted that can be physically arranged into a rectangular shape, then we can use multiplication to do the counting for us. Consider this set of objects (Figure 3.1.2):



Figure 3.1.2

Indeed, we can count them by pointing and running through the numbers, but it's more efficient to group them (Figure 3.1.3.)

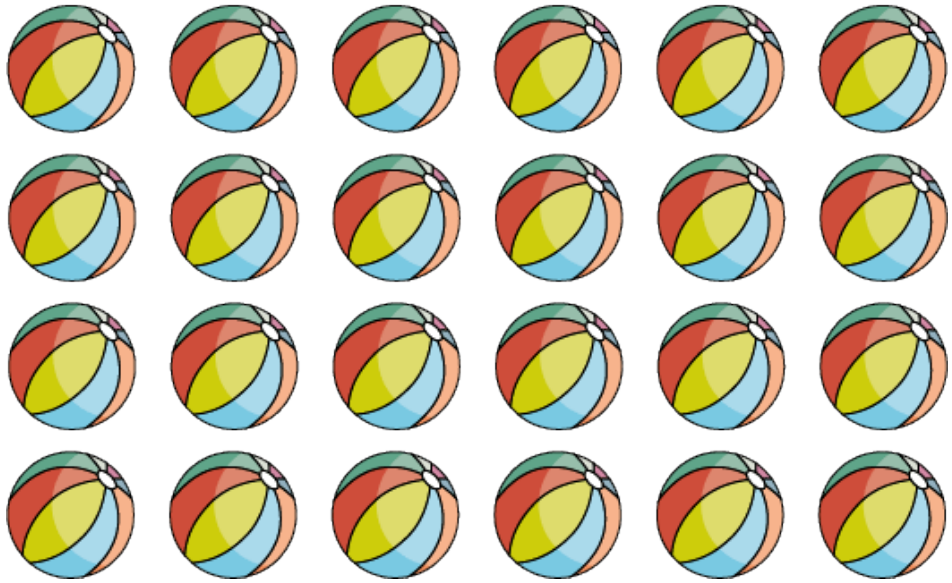


Figure 3.1.3

If we group the balls by 4s, we see that we have 6 groups (or, we can see this arrangement as 4 groups of 6 balls). Since multiplication is repeated addition (i.e., $6 \times 4 = 4 + 4 + 4 + 4 + 4 + 4$, we can use this grouping to quickly see that there are 24 balls.

Let's generalize this idea a little bit. Let's say that we're visiting a bakery that offers customized cupcakes. For the cake, we have three choices: vanilla, chocolate, and strawberry. Each cupcake can be topped with one of four types of frosting: vanilla, chocolate, lemon, and strawberry. How many different cupcake combinations are possible? We can think of laying out all the possibilities in a grid, with cake choices defining the rows and frosting choices defining the columns (Figure 3.1.4.)

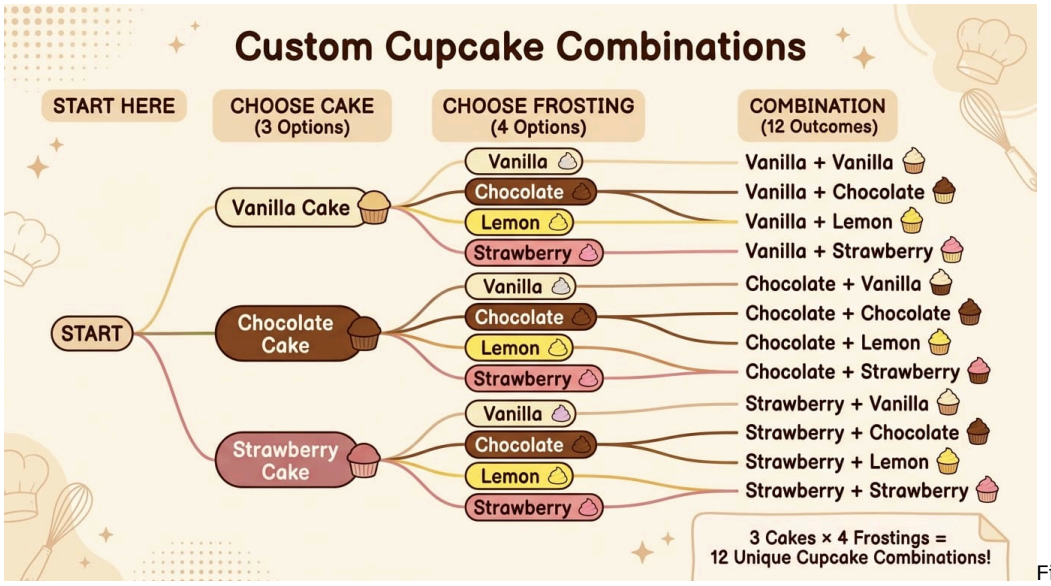


Figure 3.1.4: 12 Possible Combination

Since there are 3 rows (cakes) and 4 columns (frostings), we have $3 \times 4 = 12$ possible combinations. This is the reasoning behind the Multiplication Rule for Counting, which is also known as the Fundamental Counting Principle. This rule says that if there are n ways to accomplish one task and m ways to accomplish a second task, then there are $n \times m$ ways to accomplish both tasks. We can tackle additional tasks by multiplying the number of ways to accomplish those tasks using our previous product.

Another example:

A boy owns 3 pairs of pants, 2 shirts, 3 jackets. How many different outfits can he wear to school if he must wear one of each item? Since there are 3 pairs of pants, 2 shirts, 3 jackets, we have $3 \times 2 \times 3 = 18$ possible combinations.

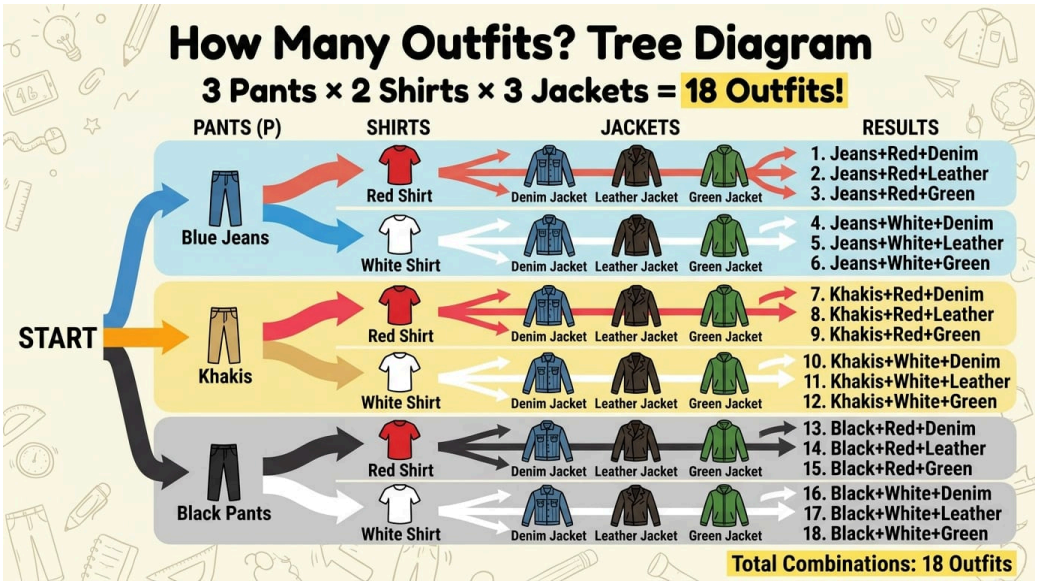


Figure $3.1.5$: 18 possible combination for outfit

Multiplication Rule for Counting

Let's ask to choose one item from each of two separate categories where there are m items in the first category and n items in the second category. The total number of available choices is $m \times n$.

Multiplication Rule: Key Point

To use the Multiplication rule for counting, you need to look for a couple of things.

1. How many groups are you choosing items from? Are those the same group or a different group? Can you repeat our selection?
2. How many ways can you choose each item from that group?

For example, in the above figure $3.1.4$, there are two different groups: "cake group" and "frosting group". In the cake group, we can choose any one item at a time from 3, and in the frosting group, we can choose any one item at a time from 4.

Example $3.1.1$ Using the Multiplication Rule for Counting for 3 Groups

At I Love Food restaurant, you can choose from 5 appetizers, 13 entrees, and 3 desserts. How many three-course meals can you order?

Answer available at the end of this section.

Your Turn 3.1.1: Multiplication Rule

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Example 3.1.2 Using the Multiplication Rule for Counting for 4 Groups

The University Combinatorics Club has 31 members: 8 seniors, 7 juniors, 5 sophomores, and 11 first-years. How many possible 4-person committees can be formed by selecting one member from each class?

Answer available at the end of this section.

Your Turn 3.1.2: Multiplication Rule

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Example 3.1.3: Using the Multiplication Rule for Counting for More Groups

1. The standard license plates for vehicles in a certain state are six characters: 3 letters followed by 3 digits. There are 26 letters in the alphabet and 10 digits (0 through 9) to choose from. How many license plates can be made using this format?
2. 10 people are to be seated in a row of 10 chairs. How many different seating arrangements are there?

Answer available at the end of this section.

Your Turn 3.1.3: Multiplication Rule

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Example 3.1.4: Using the Multiplication Rule for Counting for More Groups

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

Answer available at the end of this section.

Your Turn 3.1.4: Multiplication Rule

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Example 3.1.5

How many license plates can one create using the DDLLLL format, in which D stands for any odd digit and L stands for any capital letter A-N?

1. If the repetition of digits and letters is allowed.
2. If the repetition of digits is allowed, but the repetition of letters is not allowed.
3. If the repetition of digits is not allowed, but the repetition of letters is allowed.
4. If the repetition of digits and letters is not allowed.

Answer available at the end of this section.

Your Turn 3.1.5: Multiplication Rule

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Answers

Example 3.1.1 **Using the Multiplication Rule for Counting for 3 Groups**

Since there are three groups "appetizers" (5 items), "entrees" (13) items, and "desserts (3) items.

Appetizers can be chosen in 5 different ways.

Entrees can be chosen in 13 different ways.

Desserts can be chosen in 3 different ways.

So, you can order a three-course meal $5 \times 13 \times 3 = 195$ in different ways.

Example 3.1.2 **Using the Multiplication Rule for Counting for 4 Groups**

Since we have four groups: seniors, juniors, sophomores, and first-year students. And number of choices: 8 choices for the senior, 7 choices for the junior, 8 for the sophomore, and 11 for the first year, there are

$$8 \times 7 \times 5 \times 11 = 3,080$$

There are $3,080$ different ways to fill out the committee members.

Example 3.1.3**: Using the Multiplication Rule for Counting for More Groups**

1. Since we are choosing 6 characters (items): There are 6 group: First three group are all 26 alphabets and last 3 group are 10 number. Since there are 26 different letters and 10 different digits. Each of the first three characters can be selected in 26 different ways, and the last three can be chosen in 10 different ways. (Note that there is **no restriction**. Digits and letters can be repeated. The total number of possible license plates

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$$

2. Here we have to choose 10 items (person) for each chair. So we consider there are ten groups. In the first chair, there are 10 choices. Since one person (the first person that is selected) can sit in that chair, there are now 9 persons left, and there are 9 choices for the second chair (in other words, we have 9 items in this group), and so on. The total number of setting arrangements will be

$$10 \times 9 \times 8 \times 7 \times 5 \times 6 \times 4 \times 3 \times 2 \times 1 = 3,628,800$$

Example 3.1.4**: Using the Multiplication Rule for Counting for More Groups**

There are 3 questions. Each question has 2 possible answers (true or false). There are two ways to answer each of the three questions. So, the quiz can be answered in $2 \times 2 \times 2 = 8$ different ways.

Example $3.1.5$

There are a total of 6 symbols on the license plate. The first two are odd digits, and the last 4 are alphabet from A to N.

1. Note that repetition is allowed. Since there are 5 odd digits, the number of choices for the first two D is 5, and since there are 14 capital letters A-N, the number of choices for the last four L is 14, thus:
$$5 \times 5 \times 14 \times 14 \times 14 \times 14 = 960,400 \tag{3.1.1}$$
2. If the repetition of digits is allowed, but the repetition of letters is not allowed.
$$5 \times 5 \times 14 \times 13 \times 12 \times 11 = 600,600 \tag{3.1.2}$$
3. If the repetition of digits is not allowed, but the repetition of letters is allowed.
$$5 \times 4 \times 14 \times 14 \times 14 \times 14 = 768,320 \tag{3.1.3}$$
4. If the repetition of digits and letters is not allowed.
$$5 \times 4 \times 14 \times 13 \times 12 \times 11 = 480,480 \tag{3.1.4}$$

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3.2: Permutations and Combination

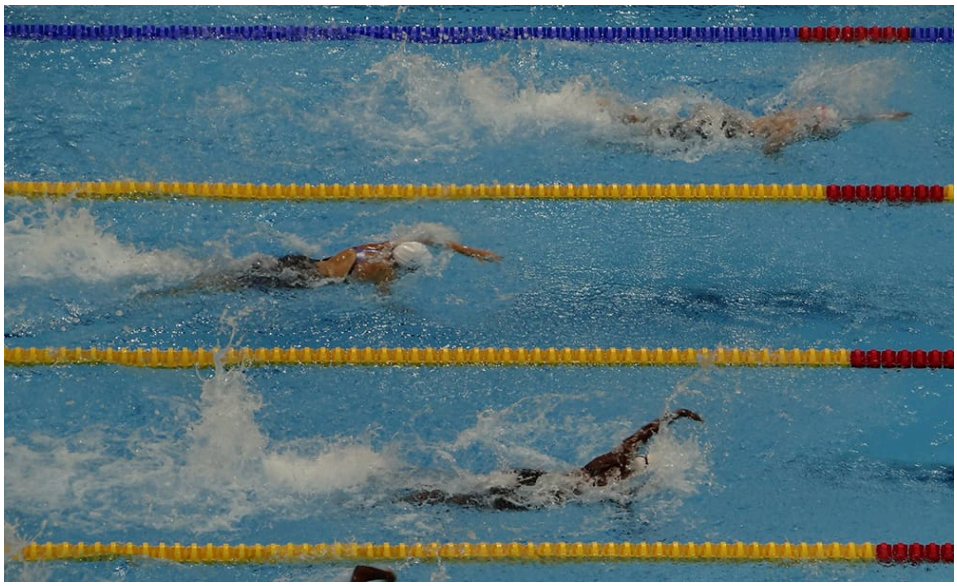


Figure 3.2.1 : We can use permutations to calculate the number of different orders of finish in an Olympic swimming heat. (credit: “London 2012 Olympics Park Stratford London” by Gary Bembridge/Flickr, CC BY 2.0)

Learning Objectives

1. Use the Multiplication Rule for Counting to determine the number of permutations.
2. Compute expressions containing factorials.
3. Compute permutations.
4. Apply permutations to solve problems.
5. Distinguish between permutation and combination uses.
6. Compute combinations.
7. Apply combinations to solve applications.

Swimming events are some of the most popular events at the Summer Olympic Games. In the finals of each event, 8 swimmers compete at the same time, making for some exciting finishes. How many different orders of finish are possible in these events? In this section, we'll extend the Multiplication Rule for Counting to help answer questions like this one related to permutations. A permutation is an ordered list of objects taken from a given population. The length of the list is given, and the list cannot contain any repeated items.

Applying the Multiplication Rule for Counting to Permutations

In the case of the swimming finals, one possible permutation of length 3 would be the list of medal winners (first, second, and third place finishers). A permutation of length 8 would be the full order of finish (first place through eighth place). Let's use the Multiplication Rule for Counting to figure out how many of each of these permutations there are.

Example 3.2.1: Using the Multiplication Rule for Counting to Find the Number of Permutations

The final heat of Olympic swimming events features eight swimmers (or teams of swimmers).

1. How many podium placements (first, second, and third place) are possible?
2. How many complete orders of finish (first place through eighth place) are possible?

Answer 1 available at the end of this section.

Answer 2 available at the end of this section.

Your Turn 3.2.1: Find Permutation

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Factorials

A **factorial** is a mathematical operation denoted by an exclamation point (!), and it's used to multiply a number by all the whole numbers below it.

Definition: Factorial

For any positive whole number n, we define the factorial of n (denoted $n!$ read " n factorial") to be the product of every whole number less than or equal to n.

$$n! = 1 \times 2 \times \ldots \times n$$

And

$$0! = 1$$

We will use factorials in a couple of different contexts, so let's get some practice doing computations with them.

How to Split Factorial?

Note that we can split (!) differently according to our needs. For example

$$100! = 99! \times 100$$
$$100! = 98! \times 99 \times 100$$
$$100! = 97! \times 98 \times 99 \times 100$$
$$100! = 90! \times 91 \times 92 \times \ldots \times 100$$

Your Turn 3.2.2: Split Factorial

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Example 3.2.3: Computing Factorials

Compute the following:

1. $4!$
2. $\frac{8!}{6!}$
3. $\frac{9!}{3!4!}$

Answer 1 available at the end of this section.

Your Turn 3.2.3: Evaluate Factorial

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Permutations and Combinations

As we've seen, factorials can pop up when we're computing permutations. There is a formula that we can use to make that connection explicit. Let's define some notation first. If we have a collection of n objects and we wish to create an ordered list of r of the objects (where $1 \leq r \leq n$), we'll call the number of these **arrangements** a permutation. The order of an arrangement matters in a permutation. **The order of arrangement matters in permutation.**

The symbol ${}_nP_r$ can be read as "the number of permutations of n objects taken r at a time". We formalize the formula we'll use to compute permutations below.

FORMULA: Permutation (Arrangement)

The number of permutations of n objects taken r at a time is

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Sometimes ${}_nP_r$ is also denoted by $P(n, r)$.

In Permutations, we studied permutations, which we use to count the number of ways to generate an ordered list of a given length from a group of objects. An important property of permutations is that the order of the list matters: The results of a race and the selection of club officers are examples of lists where the order is important. In other situations, the order is not important. For example, in most card games where a player receives a hand of cards, the order in which the cards are received is irrelevant; in fact, players often rearrange the cards in a way that helps them keep the cards organized.

When the order of a list of objects doesn't matter, the lists are no longer permutations. Instead, we call them combinations. A **combination** refers to the selection of objects without regard to the order. Unlike permutations, the order of selection does not matter in combinations.

FORMULA: Combination (Selection)

The number of combinations when r objects are chosen from n objects is

$$ {}_nC_r = \frac{n!}{r!(n-r)!} $$

Sometime ${}_nC_r$ is also denoted by $C(n,r)$.

Note: Permutation and Combination: How are they Different?

If you have the letters A, B, and C, the possible ***permutations*** of 2 letters are: 2 objects are chosen out of 3.
AB, BA, AC, CA, BC, CB.
Here, the order of the letters makes a difference. There are **six** permutations.

If you have the letters A, B, and C, the possible ***combinations*** of 2 letters are: 2 objects are chosen out of 3.
AB, AC, BC.
In this case, AB and BA are considered the same combination because the order doesn't matter. Similarly, AC and CA are the same, BC and CB are the same.
There are 3 combination.

These examples show the difference clearly: **permutations** are about **arrangements** (order matters), while **combinations** are about **selections** (order doesn't matter)

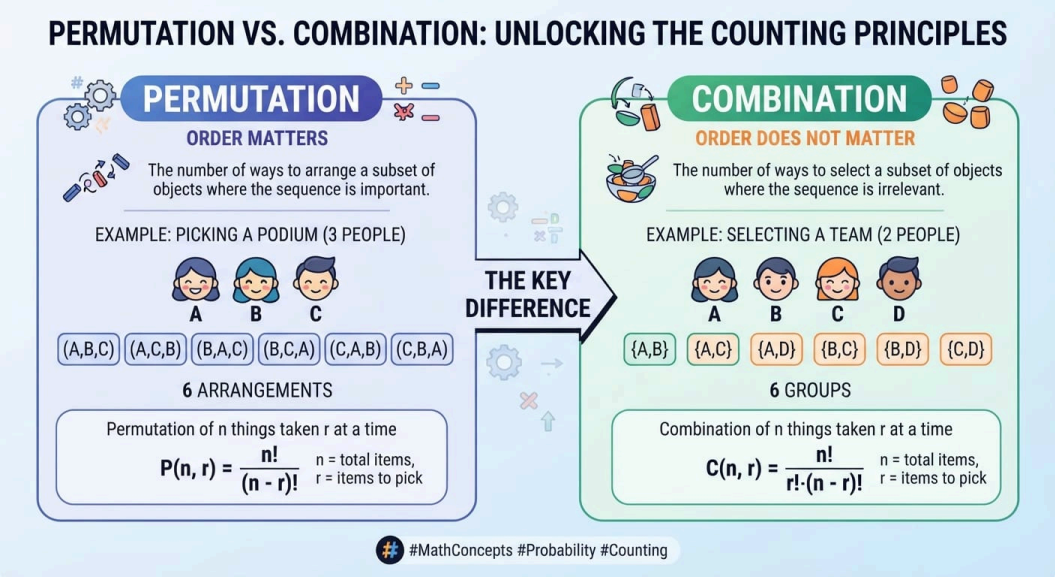


Figure 3.2.2: Permutation Vs Combination Key Differences

Example 3.2.4: Computing Permutations

Find the following numbers.

1. The number of permutations of 12 objects taken 3 at a time.
2. The number of permutations of 8 objects taken 5 at a time.
3. The number of permutations of 32 objects taken 2 at a time.

Answer 1 available at the end of this section.

Your Turn 3.2.4: Evaluate Permutation

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Example 3.2.5: Applying Permutations

1. A high school graduating class has 312 students. The top student is declared valedictorian, and the second-best is named salutatorian. How many possible outcomes are there for the valedictorian and salutatorian?
2. In the card game blackjack, the dealer's hand of 2 cards is dealt with one face up and one facedown. If the game is being played with a single deck of (52) cards, how many possible hands could the dealer get?
3. The University Combinatorics Club has three officers: president, vice president, and treasurer. If there are 18 club members, how many ways are there to fill the officer positions?

Answer 1 available at the end of this section.

Your Turn 3.2.5: Problem Involving Permutation

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Who Knew?: Very Big Permutations

Permutations involving relatively small sets of objects can get very big very quickly. A standard deck contains 52 cards. So, the number of different ways to shuffle the cards—in other words, the number of permutations of 52 objects taken 52 at a time—is $52! \approx 8 \times 10^{67}$ (written out, that's an 8 followed by 67 zeroes). The estimated age of the universe is *only* about 4×10^{17} seconds. So, if a very bored, all-powerful being started shuffling cards at the instant the universe began, it would have to have averaged at least $\frac{8 \times 10^{67}}{4 \times 10^{17}} \approx 2 \times 10^{50}$ shuffles *per second since the beginning of time* to have covered every possible arrangement of a deck of cards. That means the next time you pick up a deck of cards and give it a good shuffle, it's almost certain that the particular arrangement you created has never been created before and likely never will be created again.

Permutation With Duplicate Items

In some cases, the available choices are identical. Suppose someone was given the chance to choose five gift cards out of 10 available gift cards, and they noticed that 3 were from their favorite coffee shop. They might choose the first 3 and then decide on the other two. The number of choices changes when duplicate items are available.

FORMULA

The number of distinct permutations of n items when p times are identical, q items are identical, r items are identical, and so on... is given by,

$$\text{Number of choices} = \frac{n!}{p!q!r!\ldots} \tag{3.2.1}$$

Note: Here, the number of choices is also the number of arrangements.

Example 3.2.6: Computing Permutations for Duplicate Items

1. How many distinct ways could you arrange the letters in the word MISSISSIPPI?
2. Sarah bought 12 plants to arrange along the border of her garden. How many distinguishable arrangements can she make if the plants comprise three tulips, six roses, and three daisies?

Answer 1 available at the end of this section.

Your Turn 3.2.6: Permutation of Duplicate Items

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Example 3.2.7: Distinguishing Between Permutations and Combinations

For each of the following situations, decide whether the chosen subset is a permutation or a combination.

1. A social club selects 3 members to form a committee. Each of the members has an equal share of responsibility.
2. You are prompted to reset your email password; select a password consisting of 10 characters without repeats.
3. At a dog show, the judge must choose first, second, and third-place finishers from 16 dogs.
4. At a restaurant, the day's special comes with the customer's choice of 3 sides taken from a list of 6 possibilities.
5. A club has 5 members. How many ways can you choose a president and a vice president?

Answer available at the end of this section.

Your Turn 3.2.7: Permutation or Combination

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Example 3.2.8: Using the Combination Formula

Compute the following:

1. ${}_8C_3$
2. ${}_{12}C_5$
3. ${}_{15}C_9$

Answer available at the end of this section.

Your Turn 3.2.8: Evaluate Combination

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Example 3.2.9: Applying the Combination Formula

1. In the card game Texas Hold'em (a variation of poker), players are dealt 2 cards from a standard deck to form their hands. How many different hands are possible?
2. The board game *Clue* uses a deck of 21 cards. If 3 people are playing, each person gets 6 cards for their hand. How many different 6 -card *Clue* hands are possible?
3. Palmetto Cash 5 is a game offered by the South Carolina Education Lottery. Players choose 5 numbers from the whole numbers between 1 and 38 (inclusive); the player wins the jackpot of $\$100,000$ if the randomizer selects those numbers in any order. How many different sets of winning numbers are possible?

Answer available at the end of this section.

Your Turn 3.2.9: Use Combination

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Checkpoint

The notation and nomenclature used for the number of combinations is not standard across all sources. You'll sometimes see $\binom{n}{r}$ instead of ${}_nC_r$. Sometimes you'll hear that expression read as "n choose r" as shorthand for "the number of combinations of n objects taken r at a time.

People in Mathematics: Early Eastern Mathematicians

Although combinations weren't studied in Europe until around the 13th century, mathematicians of the Middle and Far East had already been working on them for hundreds of years. The Indian mathematician known as Pingala had described them by the second century BCE; Varāhamihira (fl. sixth century) and Halayudha (fl. 10th century) extended Pingala's work. In the ninth century, an Indian mathematician named Mahāvīra gave the formula for combinations that we use today.

In 10 10th-century Baghdad, a mathematician named Al-Karaji also knew formulas for combinations. However, his work is now lost, it was known to (and repeated by) Persian mathematician Omar Khayyam, whose work survives. Khayyam is probably best remembered as a poet, with his *Rubaiyat* being his most famous work.

Meanwhile, in 11th-century China, Jia Xian was also working with combinations, as was his 13th-century successor, Yang Hui.

It is not known whether the discoveries of any of these men were known in other regions, or whether the Indians, Persians, and Chinese arrived at their discoveries independently. We know that mathematical knowledge and texts were sometimes passed along trade routes, so it can't be ruled out.

Example 3.2.10: Combining Combinations with the Multiplication Rule for Counting

The student government at a university consists of 10 seniors, 8 juniors, 6 sophomores, and 4 freshmen.

1. How many ways are there to choose a committee of 8 people from this group?
2. How many ways are there to choose a committee of 8 people if the committee must consist of 2 people from each class?

Answer available at the end of this section.

Your Turn 3.2.10: Combining Combination

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Answers

Example 3.2.1: Using the Multiplication Rule for Counting to Find the Number of Permutations

Let's start with the first-place finisher. How many options are there? Since eight swimmers compete, there are 8 possibilities. Once that first swimmer completes the race, there are 7 swimmers left competing for second place. After the second finisher is decided, there are 6 swimmers remaining who could possibly finish in third place. Thus, there are 8 possibilities for first place, 7 for second place, and 6 for third place.

The Multiplication Rule for Counting then tells us there are

$$8 \times 6 \times 7 = 336$$

There are 336 different ways the winners' podium can be filled out.

Example 3.2.1: Using the Multiplication Rule for Counting to Find the Number of Permutations

To look at the complete order of finish, we can continue the pattern we can see in part 1 of this example: There are 5 possibilities for fourth place, 4 for fifth place, 3 for sixth place, 2 for seventh place, and then just 1 swimmer is left to finish in eighth place.

Using the Multiplication Rule for Counting, we see that there are

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320$$

There are $40,320$ possible orders of finish

Example 3.2.3: Computing Factorials

$4! = 4 \times 3 \times 2 \times 1 = 24$

There are two ways to approach this calculation. The first way is to compute the factorials first, then divide.

One way

$$\begin{aligned} \frac{8!}{6!} &= \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}{1 \times 2 \times 3 \times 4 \times 5 \times 6} \\ &= 54 \end{aligned}$$

Let's approach this one using our canceling technique by splitting factorial. When we see two factorials in either the numerator or denominator, we should focus on the larger one first. So

$$\begin{aligned} \frac{9!}{3!4!} &= \frac{4! \times 5 \times 6 \times 7 \times 8 \times 9}{1 \times 2 \times 3 \times 4!} \\ &= \frac{15,120}{6} \\ &= 2,520 \end{aligned}$$

With that in mind, we can proceed this way by canceling out the $6!$. That's much easier!

$$\begin{aligned} \frac{9!}{3!4!} &= \frac{4! \times 5 \times 6 \times 7 \times 8 \times 9}{1 \times 2 \times 3 \times 4!} \\ &= \frac{15,120}{6} \\ &= 2,520 \end{aligned}$$

Example 3.2.4: Computing Permutations

Here $n = 12$ and $r = 3$

$$
\begin{aligned}
{}_{12}P_3 = \frac{12!}{(12-3)!} &= \frac{12!}{9!} \\
&= \frac{9!(10)(11)(12)}{9!} \\
&= 1,320
\end{aligned}
$$

Here $n = 8$ and $r = 5$

$$
\begin{aligned}
{}_8P_5 = \frac{8!}{(8-5)!} &= \frac{8!}{3!} \\
&= \frac{3!(4)(5)(6)(7)(8)}{3!} \\
&= 6,720
\end{aligned}
$$

Here $n = 32$ and $r = 2$

$$
\begin{aligned}
{}_{32}P_2 = \frac{32!}{(32-2)!} &= \frac{32!}{30!} \\
&= \frac{30!(31)(32)}{30!} \\
&= 992
\end{aligned}
$$

Example 3.2.5: Applying Permutations

This is the number of permutations of 312 students taken 2 at a time, and the chosen students' order matters. So

$$
\begin{aligned}
P(312,2) = {}_{312}P_2 = \frac{312!}{(312-2)!} &= \frac{312!}{310!} \\
&= \frac{310!(311)(312)}{310!} = 97,032
\end{aligned}
$$

We want the number of permutations of cards taken two at a time, and the order of the cards matters. So

$$
\begin{aligned}
P(52,2) = {}_{52}P_2 = \frac{52!}{(52-2)!} &= \frac{52!}{50!} \\
&= \frac{50! \times 51 \times 52}{50!} = 2,652
\end{aligned}
$$

Here, we're looking for the number of permutations of 18 members taken three at a time, and the officers' order matters. So

$$
\begin{aligned}
P(18,3) = {}_{18}P_3 = \frac{18!}{(18-3)!} &= \frac{18!}{15!} \\
&= \frac{15! \times 16 \times 17 \times 18}{15!} = 4,896
\end{aligned}
$$

Example 3.2.6: Computing Permutations for Duplicate Items

Consider the total number of items first. There are 11 letters in the word MISSISSIPPI. This means $n = 11$. Next, consider the duplicate items.

There are four of the letter I, four of the letter S, and two of the letter P.

$$
\text{Choices} = \frac{n!}{p!q!r!\ldots} = \frac{11!}{4!4!2!} = 34,650
$$

In comparison, if we were asked to arrange the letters in an 11 -letter word with no duplicate letters, the answer would be

$$
{}_{11}P_{11} = \frac{11!}{(11-11)!} = 11! = 39,916,800
$$

There are 12 plants. This means $n = 12$

Next, consider the duplicate items. There are three tulips, six roses, and three daisies. The number of distinguishable arrangements will be

$$
\text{Choices} = \frac{n!}{p!q!r!\ldots} = \frac{12!}{3!6!3!} = 18,480
$$

Example 3.2.7: Distinguishing Between Permutations and Combinations

1. Since there is no distinction among the responsibilities of the 3 committee members, the order isn't necessary. So, this is a combination.
2. The order of the characters in a password matters, so this is a permutation.
3. The order of finish matters in a dog show, so this is a permutation.
4. A plate with mashed potatoes, peas, and broccoli is functionally equivalent to a plate with peas, broccoli, and mashed potatoes, so this is a combination.
5. **Order matters** because the roles of **president** and **vice president** are different, so this is a permutation.

Example 3.2.8: Using the Combination Formula

1. Here $n = 8$ and $r = 3$

   Unknown environment ' align* '

2. Here $n = 12$ and $r = 5$

   Unknown environment ' align* '

3. Here $n = 15$ and $r = 9$

   Unknown environment ' align* '

Example 3.2.9: Applying the Combination Formula

1. A standard deck has 52 cards, and a hand has 2 cards. Since the order doesn't matter, we use the formula for counting combinations:

$$\begin{aligned} {}_{52}C_2 &= \frac{52!}{2!(52-2)!} \\ &= \frac{52!}{2! \times 50!} \\ &= \frac{50! \times 51 \times 52}{1 \times 2 \times 50!} \\ &= 1,326 \end{aligned} \tag{3.2.2}$$

2. Again, the order doesn't matter, so the number of combinations is:

$$\begin{aligned} {}_{21}C_6 &= \frac{21!}{6!(21-6)!} \\ &= \frac{21!}{6! \times 15!} \\ &= \frac{15! \times 16 \times \ldots \times 21}{1 \times \ldots \times 6 \times 15!} \\ &= 54,264 \end{aligned}$$

3. There are 38 numbers to choose from, and we must pick 5. Since order doesn't matter, the number of combinations is:

$$\begin{aligned} {}_{38}C_5 &= \frac{38!}{5!(38-5)!} \\ &= \frac{38!}{5! \times 33!} \\ &= \frac{33! \times 34 \times \ldots \times 38}{1 \times \ldots \times 5 \times 33!} \\ &= 501,942 \end{aligned}$$

Example $3.2.10$: Combining Combinations with the Multiplication Rule for Counting

1. There are 28 people to choose from, and we need 8. So, the number of possible committees is

$${}_{28}C_8 = \frac{28!}{8!(28-8)!} = \frac{28!}{8! \times 20!} = 3,108,105$$

2. Break the selection of committee members into a 4-step process: choose the seniors, then the juniors, then the sophomores, and finally the freshmen. The number of ways to choose 2 people from 10 seniors

$${}_{10}C_2 = \frac{10!}{2!(10-2)!} = 45$$

The number of ways to choose 2 people from 8 juniors

$${}_{8}C_2 = \frac{8!}{2!(8-2)!} = 28$$

The number of ways to choose 2 people from 6 sophomores

$${}_{6}C_2 = \frac{6!}{2!(6-2)!} = 15$$

The number of ways to choose 2 people from 4 freshmen

$${}_{4}C_2 = \frac{4!}{2!(4-2)!} = 6$$

The Multiplication Rule for Counting tells us that we can get the total number of ways to complete this task by multiplying the number of ways to do each of the four subtasks. So, there are

$$45 \times 28 \times 15 \times 6 = 113,400$$

There are $113,400$ possible committees with these restrictions.

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3.3: Fundamental Concepts of Probability

Learning Objectives

1. Determine outcomes, sample spaces, and event probabilities in probability experiments.
2. Calculate basic theoretical probabilities.
3. Calculate basic empirical probabilities.

Introducing Probability

It all comes down to this. The game *Monopoly,* which started hours ago, is in the home stretch. Your sister has the dice, and if she rolls a 4, 5, or 7, she'll land on one of your best spaces, and the game will be over. How likely is it that the game will end on the next turn? Is it more likely than not? How can we measure that likelihood? This section addresses this question by introducing a way to measure uncertainty.

If you roll a die, pick a card from the deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology

Uncertainty is, almost by definition, a nebulous concept. In order to put enough constraints on it that we can mathematically study it, we will focus on uncertainty strictly in the context of experiments. Recall that experiments are processes whose outcomes are unknown; the sample space for the experiment is the collection of all those possible outcomes. When we want to talk about the likelihood of particular outcomes, we sometimes group outcomes together; for example, in the *Monopoly* example at the beginning of this section, we were interested in the roll of 4 dice that might fall as a 4, 5, or 7. A grouping of outcomes that we're interested in is called an event. In other words, an event is a subset of the sample space of an experiment; it often consists of the outcomes of interest to the experimenter.

When we are talking about combinatorics or probability, the word "experiment" has a slightly different meaning than it does in the sciences. Experiments can range from very simple ("flip a coin") to very complex ("count the number of uranium atoms that undergo nuclear fission in a sample of a given size over the course of an hour"). Experiments have unknown outcomes that generally rely on chance, so if the experiment is repeated (or replicated), the outcome might be different. No matter the experiment, analysis typically begins by identifying its sample space.

Probability

Probability is the measure of the chance of an event occurring. Probability takes a numerical value that represents how likely an event is to occur. It is always a value between 0 (impossible) and 1 (certain).

Experiment

In probability, an experiment is any process or action that can be repeated, but where the exact outcome isn't known in advance.

Example of a probability experiment:

1. Tossing a coin.
2. Roll a die
3. Pick one card from a standard deck of 52.
4. Testing a light bulb from a factory line to see if it is "defective" or "non-defective.
5. Observing whether it rains or stays dry on a specific day.

Outcome

The possible results of an experiment are called **outcomes**. For example, in the experiment of tossing a coin, possible outcomes are head and tail.

Sample Space

The **sample space** of an experiment is the set of all possible outcomes.

It's often expressed as a set (i.e., as a list bound by braces)

If the experiment is *a roll, a single die is rolled once,* the sample space would be written $\{1, 2, 3, 4, 5, 6\}$.

The sample space depends on what you observed

The sample space isn't a fixed property of the physical object alone; it is defined by **the question you are asking**. Changing what you decide to measure changes what counts as a distinct outcome.

Experiment: Rolling 6-sided die

Imagine you roll a standard die. What your sample space looks like depends entirely on what you decide to record:

- **Recording the Exact Number Upward:** You care about the precise face value.

$$S = \{1, 2, 3, 4, 5, 6\}$$

- **Recording Even or Odd:** You group the numbers into two categories based on parity.

$$S = \{\text{Even}, \text{Odd}\}$$

Tossing Two Coins

Experiment: Imagine you flip a coin twice (or flip two coins at the same time). Depending on what you measure, the size and nature of your sample space shift dramatically:

- **Recording the Sequence:** You track the exact result of each flip in order.

$$S = \{\text{H H}, \text{HT}, \text{TH}, \text{TT}\} \quad (\text{Total: } 4 \text{ outcomes})$$

- **Recording the Total Number of Heads:** You ignore the order and just count how many heads appeared.

$$S = \{0, 1, 2\} \quad (\text{Total: } 3 \text{ outcomes})$$

Event

An **event** is any particular outcome or group of outcomes. The event is a subset of the sample space.

If the experiment is *a roll, a single die is rolled once, the event could be "roll an even number"* or $\{2, 4, 6\}$.

Example 3.3.1: Finding the Sample Space (Dice and Coin)

For each of the following experiments, identify the sample space.

1. Flip a coin (which has 2 faces, typically called “heads” and “tails”).
2. Roll a 6-sided die.
3. Roll two 6-sided dice.

Answer 1 available at the end of this section.

Your Turn $3.3.1$: Find Sample Space, Event and Experiment

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A standard deck of cards includes:

4 suits: Hearts, Diamonds, Clubs, Spades

13 cards in each suit. So total $13 \times 4 = 52$ cards (excluding Jokers).

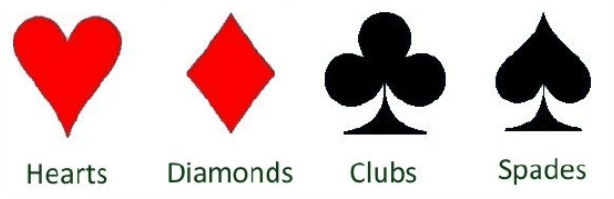


Figure $3.3.4A$: Suit of Cards (4 Suit) (TOP), Rank of cards (13 rank (BOTTOM)

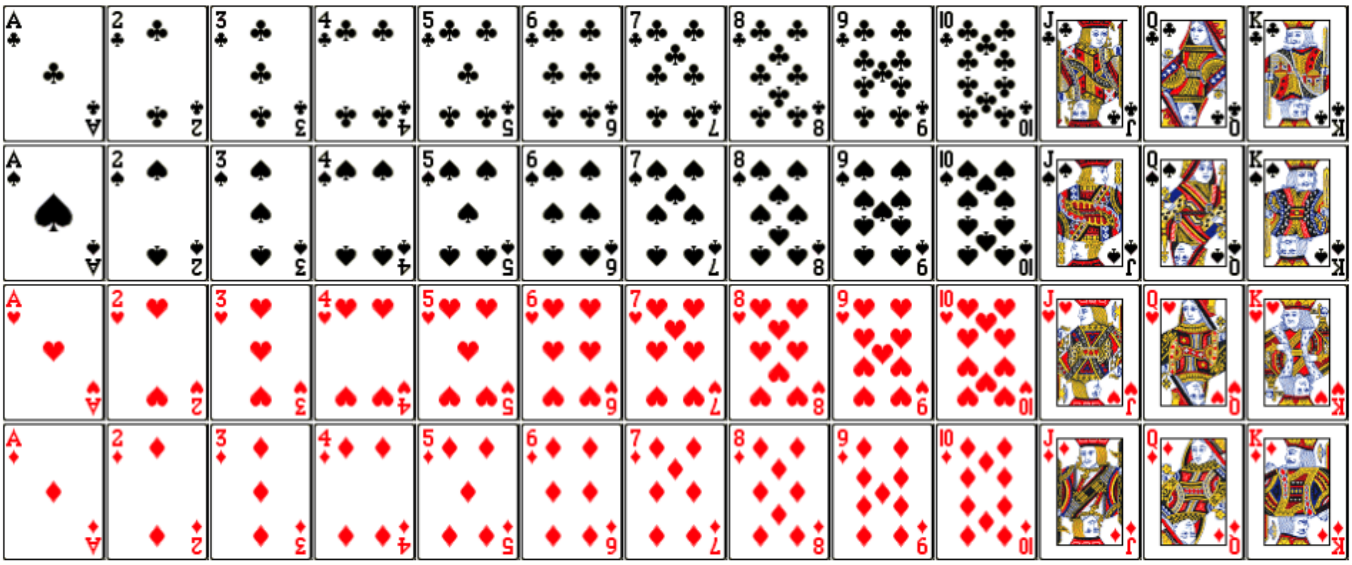


Figure $3.3.4B$: Deck of 52 cards

Depending on the experiment's definition, the sample space changes:

- **Recording Suit Only:** 4 outcomes $\{\text{Hearts}, \text{Diamonds}, \text{Clubs}, \text{Spades}\}$

- **Recording Rank Only:** 13 outcomes $\{2, 3, 4, 5, 6, 7, 8, 9, 10, \text{Jack}, \text{Queen}, \text{King}, \text{Ace}\}$
- **Recording the Full Card:** 52 outcomes (4 suits $\times$ 13 ranks)

Example 3.3.2: Finding the Event (Die and Deck of Cards)

Identify the event in the following experiment.

1. Roll one 6-sided die. Find the event rolling an even number.
2. Roll two 6-sided dice. Find the event rolling two of the same numbers.
3. Draw a card from a deck of 52 cards. Find the event drawing king.

Answer available at the end of this section.

Using Tree Diagrams to Identify Sample Spaces

In experiments with two or more stages, or when the stages are dependent, a tree diagram is a helpful tool for systematically identifying the sample space. Tree diagrams are built by first drawing a single point (or *node*), then, from that node, drawing one branch (a short line segment) for each outcome of the first stage. Each branch gets its own node at the other end (which we typically label with the corresponding outcome for that branch); from each of these, we draw another branch for each outcome of the second stage, assuming that the outcome of the first stage matches the branch we were on. If there are other stages, we can continue from there by adding branches and nodes. This sounds really complicated, but it's easier to understand through an example.

The following tree diagram shows the sample space for flipping 2 and 3 coins.

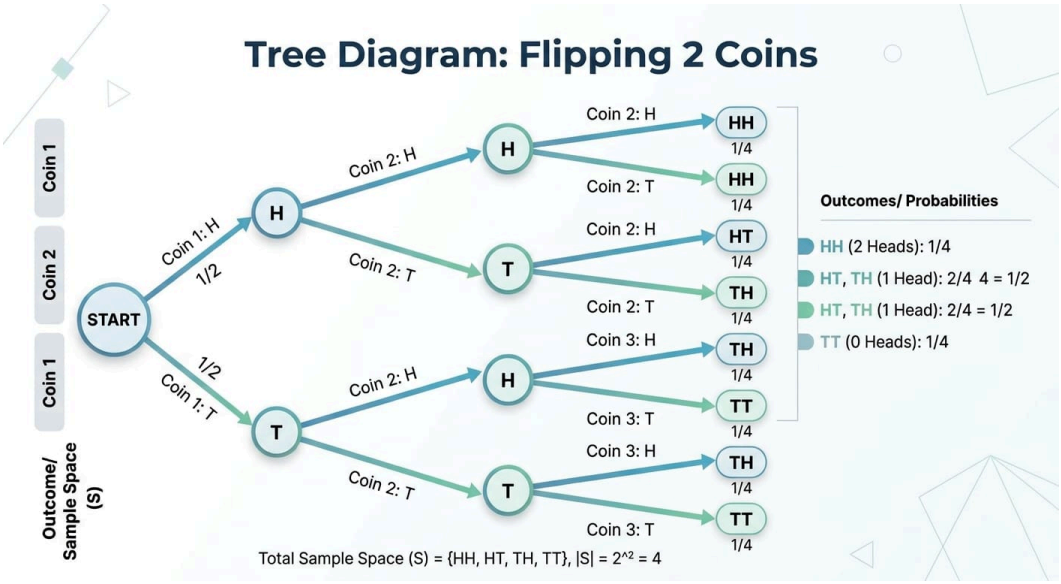


Figure 3.3.5: Flip 2 coins

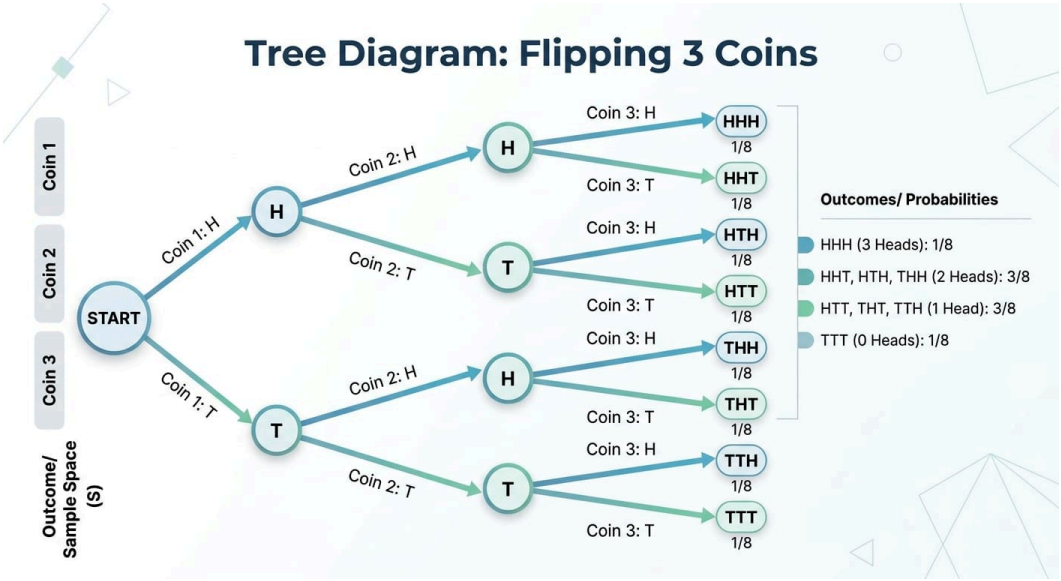


Figure 3.3.6: Flip 3 coins

Example 3.3.3: Finding the Sample Space Using Tree Diagram

Use a tree diagram to find the sample spaces of each of the following experiments:

You flip a coin. If the result is a head, you roll a 4 -sided die. If it's a tail, you roll a 6-sided die.

Answer available at the end of this section.

Your Turn 3.3.3: Identify Sample Space

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Theoretical Probability

The theoretical method gives the most reliable results, but cannot always be used. Suppose the sample space of an experiment consists of equally likely outcomes. In that case, the theoretical probability of an event is defined as the ratio of the number of outcomes in the event to the number of outcomes in the sample space.

Probability Notation: Possible Values of Probability

For any event A, the notation $P(A)$ is for the probability of event A occurring. The smallest possible probability is 0, if no outcomes correspond with the event. The largest possible probability is 1, if all possible outcomes correspond with the event.

The probability of any event must be between zero and one, inclusive. That is

$$0 \leq P(A) \leq 1$$

Note: You can express **probability as a percentage** by simply converting the decimal value to a percentage. For example, the probability of tossing heads when we flip one coin is 0.5. We can also say that when we flip a coin, there is a 50% chance that the coin lands as a head.

In the course of this chapter, if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.

An **impossible event** has a probability of 0. In another word, if $P(A) = 0$, the event A is impossible.

A **certain event** has a probability of 1 or 100%. In another word, if $P(A) = 1$, the event A is certain.

Probabilities are essentially fractions and can be reduced to lower terms, like fractions, or expressed as decimals.

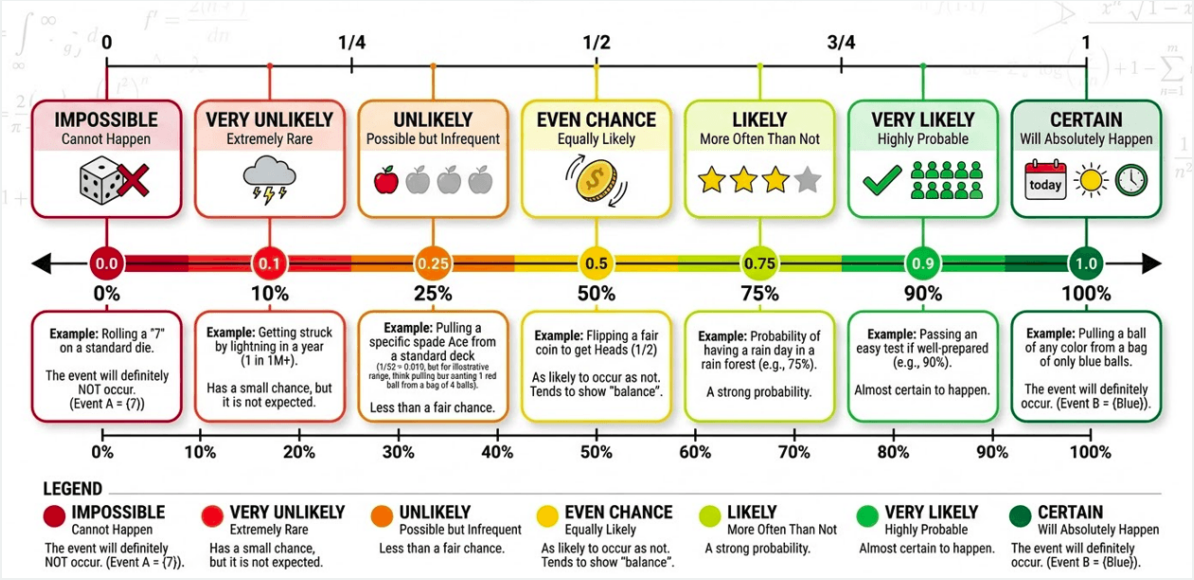


Figure 3.3.8: Probability Scale

Sum of the Probabilities Rule

The sum of the probabilities of all possible mutually exclusive outcomes in a sample space S must equal 1 (or 100%).

A fair coin flip has two outcomes:

$P(\text{Heads}) + P(\text{Tails}) = 0.5 + 0.5 = 1.0.$

Note: Mutually exclusive (or disjoint) if they cannot happen at the same time, or they do not have common outcomes.

Your Turn 3.3.4: Possible Values of Probability

FORMULA: Theoretical Probability

For an experiment whose sample space S consists of equally likely outcomes, the **theoretical probability** of the event E is the ratio

$$P(\text{Event}) = \frac{\text{number of outcomes in the event}}{\text{number of outcomes in sample space}}$$
$$= \frac{\text{How many outcomes in the event?}}{\text{How many outcomes in the sample space?}}$$

Example 3.3.5: Compute Theoretical Probability (Die)

If we roll a 6-sided die, calculate

1. P(rolling a 1).
2. P(rolling a number bigger than 4).

Answer available at the end of this section.

Your Turn 3.3.5: Roll a Die

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Example 3.3.6: Determining Certain and Impossible Events

A. Consider an experiment that consists of rolling a single standard 6-sided die (with faces numbered $1-6$). Decide if these probabilities are equal to zero, equal to one, or somewhere in between.

1. P(roll 4).
2. P(roll 7).
3. P(roll a positive number).
4. P(roll $\frac{1}{3}$).
5. P(roll an even number).

B. Consider an experiment of drawing a card from a deck of cards. Decide if these probabilities are equal to zero, equal to one, or somewhere in between.

1. A card is a king.
2. A card is a king or queen.
3. A card is red and black.
4. A card is red or black.

Answer available at the end of this section.

Your Turn 3.3.6: Multi Sided Die

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Your Turn 3.3.7: Deck of Cards

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Example 3.3.8: Spinner

Suppose the spinner shown below is used for a game. Find the following probability.



Figure 3.3.9: Spinning Wheel with Six number 1 to 6

1. P(spin 5).
2. P(spin multiple of 2).
3. P(spin a number greater than 4).
4. P(spin 2 and 3).

Answer available at the end of this section.

Your Turn 3.3.8: Spinner

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The theoretical method gives the most reliable results, but it cannot always be used. If the sample space of an experiment consists of equally likely outcomes, then the theoretical probability of an event is defined to be the ratio of the number of outcomes in the event to the number of outcomes in the sample space.

Checkpoint: Equally Likely for Theoretical Probability

It is critical that you make sure that every outcome in a sample space is equally likely before you compute theoretical probabilities!

Example 3.3.9: Find Theoretical Probabilities (Roll of Two Dice)

In the Basic Concepts of Probability, we were considering a *Monopoly* game where, if your sister rolled a sum of 4, 5, or 7 with 2 standard dice, you would win the game. What is the probability of this event? Use tables to determine your answer.

Answer available at the end of this section.

Your Turn 3.3.9: Roll Two Dice

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Example 3.3.10: Tree Diagrams to Compute Theoretical Probability (Flip Coins 3 times)

If you flip a fair coin 3 times, what is the probability of each event? Use the tree diagram in this book to determine your answer

1. You flip exactly 2 heads.
2. You flip 2 consecutive heads at some point in the three flips.
3. All 3 flips show the same result.

Answer available at the end of this section.

Your Turn 3.3.10: Flip 3 or 4 Coins

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People in Mathematics: Gerolamo Cardano

The first known text that provided a systematic approach to probabilities was written in 1564 by Gerolamo Cardano (1501–1576). Cardano was a physician whose illegitimate birth closed many doors that would have otherwise been open to someone with a medical degree in 16th-century Italy. As a result, Cardano often turned to gambling to help ends meet. He was a remarkable mathematician, and he used his knowledge to gain an edge when playing cards or dice. His 1564work, titled *Liber de ludo aleae* (which translates as *Book on Games of Chance*), summarized everything he knew about probability. Of course, if that book fell into the hands of those he played against, his advantage would disappear. That's why he never allowed it to be published in his lifetime (it was eventually published in 1663). Cardano made other contributions to mathematics; he was the first person to publish the third-degree analogue of the Quadratic Formula (though he didn't discover it himself), and he popularized the use of negative numbers.

Empirical Probability

Theoretical probabilities are precise but may not be available in every situation. If the outcomes in the sample space are not

equally likely, then we're out of luck. Suppose you're watching a baseball game, and your favorite player is about to step up to the plate. What is the probability that he will get a hit?

In this case, the sample space is {hit, not a hit}. That doesn't mean that the probability of a hit is $\frac{1}{2}$, since those outcomes aren't equally likely. The theoretical method simply can't be used in this situation. Instead, we might look at the player's statistics up to this point in the season, and see that he has 122 hits in 531 opportunities. So, we might think that the probability of a hit in the next plate appearance would be about $\frac{122}{531} \simeq 0.23$. When we use the outcomes of previous replications of an experiment to assign a probability to the next replication, we're defining an empirical probability. Empirical probability is assigned using the outcomes of previous replications of an experiment by finding the ratio of the number of times the event occurred in the previous replications to the total number of previous replications.

Empirical probabilities aren't exact, but when the number of previous replications is large, we expect them to be close. Also, if the previous runs of the experiment were not conducted under the exact set of circumstances as the one we're interested in, the empirical probability is less reliable. For instance, in the case of our favorite baseball player, we might try to get a better estimate of the probability of a hit by looking only at his history against left- or right-handed pitchers (depending on the handedness of the pitcher he is about to face).

Theoretical Probability vs Empirical Probability

Side-by-side comparison between **Theoretical Probability** and **Empirical Probability.**

Theoretical Probability vs Empirical Probability

Feature	Theoretical Probability
Basis	Calculated using **mathematical rules and logic** based on all possible equally likely outcomes.
Formula	$$P(E) = \frac{\text{Number of outcomes in } E}{\text{Total number of possible outcomes}}$$
Constancy	Always stays the same for a fair, defined event.
Requirement	Requires knowing the full structure of the sample space beforehand.
Example (Coin Flip)	$$P(\text{Heads}) = \frac{1}{2} = 0.50(50\%)$$

Law of Large Number

As the total number of trials performed in an experiment grows significantly (approaching infinity), the **empirical probability** will get closer and closer to the **theoretical probability**.

FORMULA: EMPIRICAL PROBABILITY

Empirical probability is the probability of an event occurring based on actual experiments or historical data rather than theoretical calculations. It is calculated using the formula:

$$P(\text{Event}) = \frac{\text{number of times event occurs}}{\text{total number of trials}}$$

If you flip a coin 100 times and it lands on heads 48 times, the empirical probability of getting heads is:

$$P(\text{Heads}) = \frac{48}{100} = 0.48$$

Empirical probability differs from theoretical probability because it is based on observed outcomes rather than assumed equal likelihoods. It is useful in real-world scenarios where theoretical probabilities cannot be easily determined.

Example 3.3.11: Finding Empirical Probabilities

Find the empirical probability of the following events:

1. Jose is on the basketball court practicing his shots from the free-throw line. He made 47 out of his last 80 attempts. What is the probability he will make his next shot?
2. Amy is about to begin her morning commute. Over her last 60 commutes, she arrived at work 12 times in under half an hour. What is the probability that she arrives at work in 30 minutes or less?
3. Felix is playing *Yahtzee* with his sister. Felix won 14 of the last 20 games he played against her. How likely is he to win this game?

Answer available at the end of this section.

Your Turn 3.3.11: Empirical Probability

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Example 3.3.12: Empirical Probability

The number of M&Ms of each color found in a case containing 1600 M&Ms is given in the table below.

M&M Color and its Frequency

Color	Freq
blue	238
brown	338
green	419
orange	166
red	323
yellow	116

1. Find the percentage of yellow M&M's in the case.
2. Find the probability that a randomly selected M&M will be in the category "yellow or green".

Answer available at the end of this section.

Your Turn 3.3.12: Probability from Two Way Table

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Answers

Example 3.3.1: Finding the Sample Space (Dice and Coin)

If we use "H" to denote "heads is facing up" and "T" to denote "tails is facing up", then

$$\text{Sample space: } \{H, T\}$$

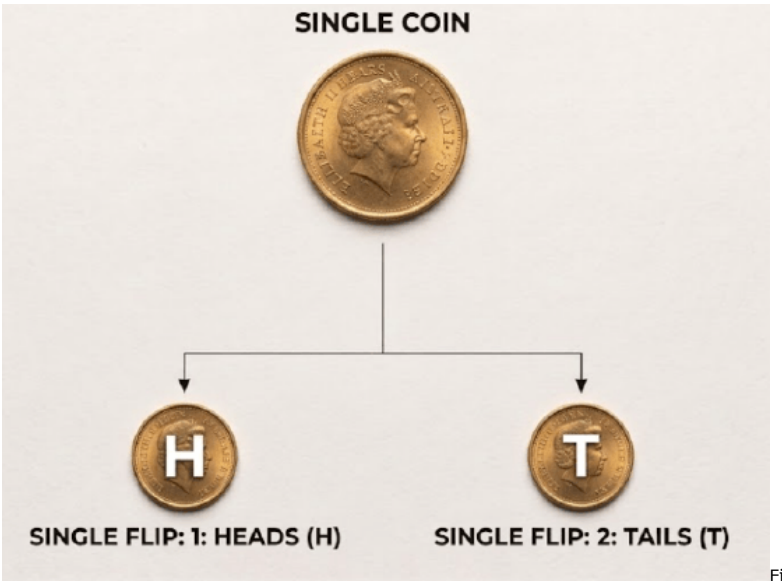


Figure 3.3.1: Possible Outcomes of Flipping One Coin

There are 6 numbers on the die: $1, 2, 3, 4, 5,$ and 6, then

$$\text{Sample space: } \{1, 2, 3, 4, 5, 6\}$$

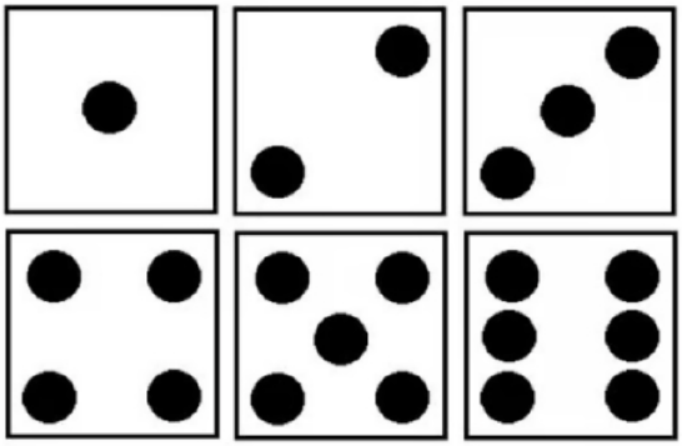


Figure 3.3.2: Possible Outcome of Rolling Single Die

Note that each ordered pair below shows the result of the first die followed by the second die, and the collection of all 36 ordered pairs is the sample space.

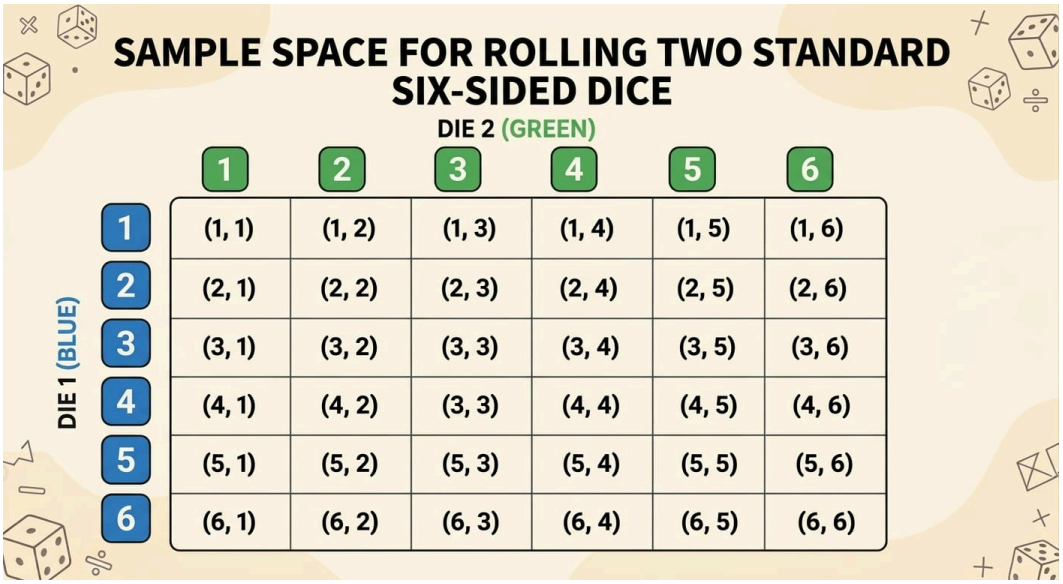


Figure 3.3.3: possible outcomes of rolling two dice

Example 3.3.2: Finding the Event (Die and Deck of Cards)

1. The sample space is the set of all possible outcomes: $\{1, 2, 3, 4, 5, 6\}$. There are 6 outcomes in the sample space.

$$\text{Event: } \{2, 4, 6\}$$

2. All 36 possible outcomes while rolling two dice are given in the table in the example 3.3.1.
   So, for example, $(3, 2)$ represents the outcome where one 6-sided roll results in a 3, and another 6-sided roll gives us a 2.

$$\text{Event: } \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

3. Recall that a standard deck of cards consists of 52 unique cards which are labeled with a rank (the whole numbers from 2 to 10, plus J, Q, K, and A) and a suit (♣ , ♢, ♡, or ♠). A standard deck is thoroughly shuffled, and you draw one card at random (so every card has an equal chance of being drawn). There are 52 possible outcomes while drawing a card from a deck of cards. (See the above figure $3.3.2C$)

$$\text{Event: } \{\clubsuit K, \diamondsuit K, \heartsuit K, \spadesuit K\}$$

Example 3.3.3: Finding the Sample Space Using Tree Diagram

If the result is H, you roll either $1, 2, 3, \text{and } 4$. If result is T, you roll either $1, 2, 3, 4, 5, \text{and} 6$. See figure 3.3.7.

The sample space is {H1, H2, H3, H4, T1, T2, T3, T4, T5, T6} shown in the given tree diagram.

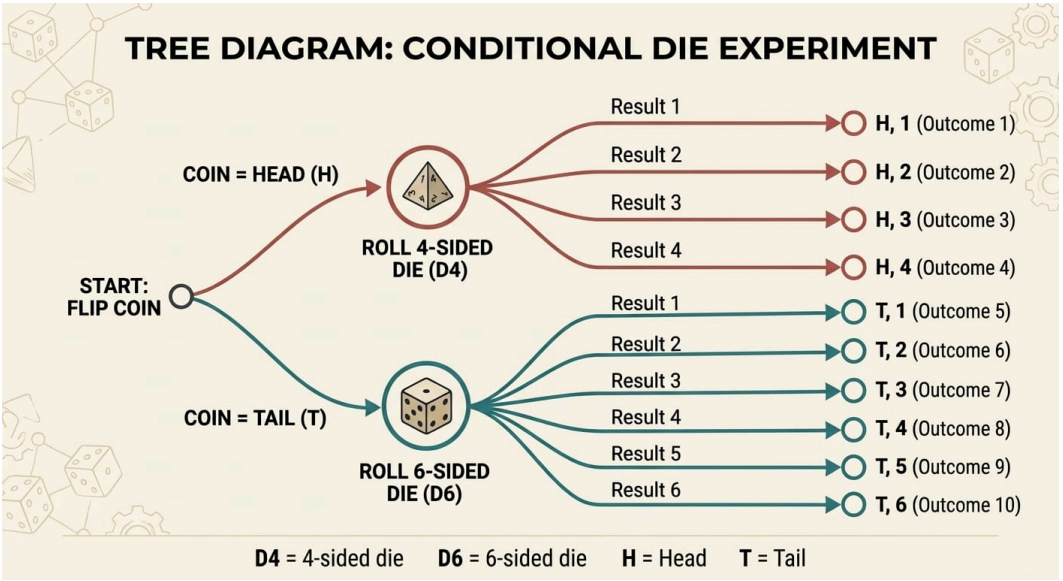


Figure 3.3.7: Sample space or flipping a coin and rolling a die

Example 3.3.5: Compute Theoretical Probability (Die)

Recall that the sample space when rolling a die is $\{1, 2, 3, 4, 5, 6\}$.

1. There is one outcome corresponding to "rolling a 1", so

$$P(\text{ rolling a } 1) = \frac{1}{6}$$

2. There are two outcomes bigger than a 4, so

$$\mathrm{P}(\text{rolling a number bigger than } 4) = \frac{2}{6} = \frac{1}{3}$$

Example $3.3.6$: Determining Certain and Impossible Events

A. Let's start by identifying the sample space. For one roll of this die, the possible outcomes are $\{1, 2, 3, 4, 5, 6\}$. We can use that to assess these probabilities:

1. We see that 4 is in the sample space, so it's possible that it will be the outcome. It's not certain to be the outcome, though. P(roll 4) is somewhere in between.
2. Notice that 7 is not in the sample space. So, $P(\text{roll a } 7) = 0$.
3. Every outcome in the sample space is a positive number, so this event is certain. P(roll a positive number) is certain.
4. Since $\frac{1}{3}$ is not in the sample space, P(roll $\frac{1}{3}$) = 0, so this is impossible event.
5. Some outcomes in the sample space are even numbers $2, 4$, and 6, but the others aren't. So, $0 < P(\text{roll an even number}) < 1$

B. There are 52 cards in the deck, so the sample space for each of these experiments has 52 elements. That will be the denominator for each of our probabilities.

1. There are 4 king cards in total in the deck, so $P(\text{king}) = \frac{4}{52} = \frac{1}{13}$, so this event probability is in between zero and one.
2. There are 8 king or queen cards in the deck, so $P(\text{king or queen}) = \frac{8}{52} = \frac{2}{13}$, so this event probability is in between zero and one.
3. There are 0 red and black cards in the deck, so $P(\text{red and black}) = \frac{0}{52} = 0$, so this event is impossible.
4. There are 52 red or black cards in the deck, so $P(\text{red or black}) = \frac{52}{52} = 1$, so this event is certain.

Example $3.3.8$: Spinner

There are 6 possible outcomes $\{1, 2, 3, 4, 5, 6\}$.

1. There is only one outcome 5, so

$$P(\text{ spin } 5) = \frac{1}{6}$$

2. There is only three outcomes as multiple of 2, they are $2, 4, 6$, so

$$P(\text{ spin multiple of } 2) = \frac{3}{6} = \frac{1}{2}$$

3. There is only two outcomes as greater than 4, they are 5 and 6, so

$$\mathrm{P}(\text{spin a number greater than } 4) = \frac{2}{6} = \frac{1}{3}$$

4. Ther is not number that has both values as 2 and 3, so

$$\mathrm{P}(\text{spin 2 and 3}) = \frac{0}{6} = 0$$

Note: In Q4, if there is "**or**" instead of "**and**", the probability will be different

$$\mathrm{P}(\text{spin 2 or 3}) = \frac{2}{6} = \frac{1}{3}$$

Example $3.3.9$: Find Theoretical Probabilities (Roll of Two Dice)

We should think of this experiment as occurring in two stages: (1) one die roll, then (2) another die roll. Even though these two stages will usually occur simultaneously in practice, since they're independent, it's okay to treat them separately.

Now, each of the 36 ordered pairs in the table represents an equally likely outcome.

Sample space for the sum of rolling two dice is given in the figure $3.3.9$.

Sample Space for Sum of Rolling Two Six-Sided Dice

Each cell shows: (Die 1, Die 2) → Sum

DIE 1 \ DIE 2	1	2	3	4	5	6
1	(1, 1) → 2	(1, 2) → 3	(1, 3) → 4	(1, 4) → 5	(1, 5) → 6	(1, 6) → 7
2	(2, 1) → 3	(2, 2) → 4	(2, 3) → 5	(2, 4) → 6	(2, 5) → 7	(2, 6) → 8
3	(3, 1) → 4	(3, 2) → 5	(3, 3) → 6	(3, 4) → 7	(3, 5) → 8	(3, 6) → 9
4	(4, 1) → 5	(4, 2) → 6	(4, 3) → 7	(4, 4) → 8	(4, 5) → 9	(4, 6) → 10
5	(5, 1) → 6	(5, 2) → 7	(5, 3) → 8	(5, 4) → 9	(5, 5) → 10	(5, 6) → 11
6	(6, 1) → 7	(6, 2) → 8	(6, 3) → 9	(6, 4) → 10	(6, 5) → 11	(6, 6) → 12

Possible Sums and Number of Outcomes

Sum	2	3	4	5	6	7	8	9	10	11	12
Number of Outcomes	1	2	3	4	5	6	5	4	3	2	1

Total number of outcomes = 36 (6 × 6)

Figure $3.3.10$: Possible Sum When Rolling Two Dice

Our event has $13 = 3 + 4 + 6$ outcomes, because the sum 4 occurs 3 times, sum 5 occurs 4 times, and sum 7 occurs 6 times.

The probability that you win a game is $\frac{13}{36}$.

Example $3.3.10$: Tree Diagrams to Compute Theoretical Probability (Flip Coins 3 times)

As we see in figure $3.3.5$, the sample space is {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}, which has 8 outcomes.

1. Flipping exactly 2 heads occurs three times (HHT, HTH, THH), so the probability is

$$P(\text{ Exaclty two heads}) = \frac{3}{8}$$

2. Flipping 2 consecutive heads at some point in the experiment happens 3 times: HHH, HHT, THH. So, the probability is

$$P(\text{ Consecutive two heads}) = \frac{3}{8}$$

3. There are 2 outcomes that all show the same result: HHH and TTT. So, the probability is

$$P(\text{ All same result}) = \frac{2}{8} = \frac{1}{4}$$

Example $3.3.11$**: Finding Empirical Probabilities**

1. Since Jose made 47 out of his last 80 attempts, assign this event an empirical probability of

$$P(\text{Make}) = \frac{47}{80} \simeq 59\%$$

2. Amy completed the commute in under 30 minutes in 12 of the last 60 commutes, so we can estimate her probability of making it in under 30 minutes is

$$P(\text{Making under 30 minutes}) = \frac{12}{60} \simeq 20\%$$

3. Since Felix has won 14 of the last 20 games, assign a probability for a win this time of

$$P(\text{Win}) = \frac{14}{20} \simeq 70\%$$

Example $3.3.12$**: Empirical Probability**

1. There are $1,600$ M&M. Number of yellow M&M 116. So the probability of drawing yellow is

$$P(\text{Yellow}) = \frac{116}{1,600} = 0.0725 = 7.25\%$$

2. There are $116 + 419 = 535$ "yellow or green" M&M in a case.

$$P(\text{Yellow or Green}) = \frac{535}{1,600} = 0.3344$$

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3.4: The Addition and Complement Rule for Probability

Learning Objectives

1. Find the probability of the complement of an event.
2. Determine if two events are mutually exclusive.
3. Use the Addition Rule to find the probability of "or" events.

Complementary Events

Now, let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a 6-sided die and we compute the probability of of an event $A = \{2, 3, 4, 5\}$: The answer is $P(A) = \frac{4}{6}$.

Now consider the probability that we not A: there are 2 outcomes that are not in A, so the answer is $P(\text{not } A) = \frac{2}{6}$. Notice that

$$P(A) + P(\text{not } A) = \frac{4}{6} + \frac{2}{6} = \frac{6}{6} = 1$$

So we have a rule

$$P(\text{not } A) = 1 - P(A)$$

Let's summarize the above discussion.

Complement of an Event

The **complement** of event A is the event "A does not happen (not A)." It is the set of outcomes in the sample space S that are not in event A.

- The notation A' is used for the complement of event A.
- The probability of the complement of event A is $P(A') = 1 - P(A)$.
- Notice also that $P(A) = 1 - P(A')$ since event A and its complement make up the entire sample space, $P(A) + P(A') = 1$.
- Note sometime A' is also written as A^c

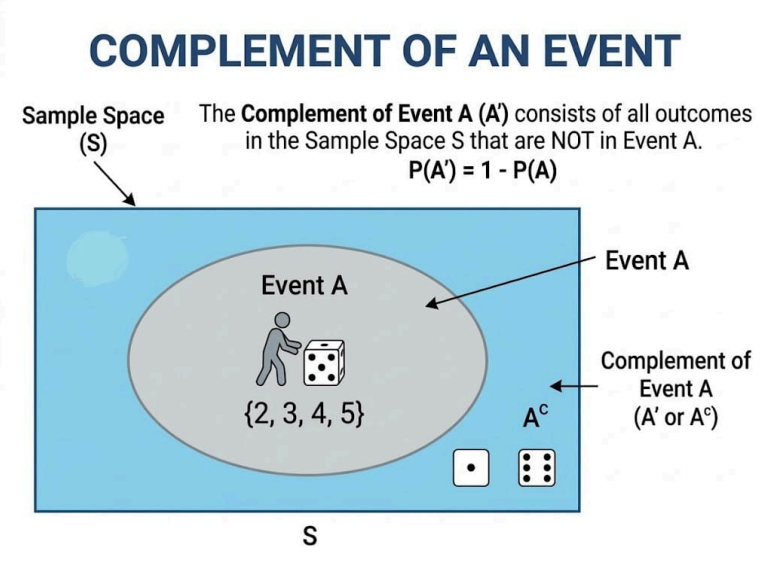


Figure 3.4.1: Showing an Event and Its Complement

Note that the complement of an event in probability is essentially the same as that of a set since an event is a subset of the sample space.

Example 3.4.1: Complement of an Event

1. If you pick a random card from a deck of playing cards, what is the probability it is not a heart?
2. ABC Airlines reports that 90.6% of its flights arrive on time. What percent of its flights do not arrive on time?

Answer available at the end of this section.

Your Turn 3.4.1: Find the Probability of Complement

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Addition Rule for "Or" Probabilities

Many probabilities in real life involve more than one event. If we draw a single card from a deck we might want to know the probability that it is either red or a jack. If we look at a group of students, we might want to know the probability that a single student has brown hair and blue eyes. When we combine two events, we make a single event called a **compound event**. To create a compound event, we can use the word "and" or the word "or" to combine events. It is very important in probability to pay attention to the words "and" and "or" if they appear in a problem. The word "and" restricts the field of possible outcomes to only those outcomes that simultaneously describe all events. The word "or" broadens the field of possible outcomes to those that describe one or more events. This is essentially the same as unions and intersections in sets.

Note that the non-exclusive use of the word "or" will double-count certain events. Say we look at the event of drawing a single card from a deck and the outcome of drawing a "red card or a jack". If you count the number of options for red cards and the number of options for a jack, you are counting the red jacks twice. This double counting needs to be accounted for.

Suppose a teacher wants to know the probability that a student in her class of 30 is taking either Art or English. She asks the class to raise their hands if they are taking Art and counts 13 hands. Then she asks the class to raise their hands if they are taking English and counts 21 hands. The teacher then calculates (the incorrect answer)

$$P(\text{Art or English}) = \frac{13}{30} + \frac{21}{30} = \frac{13+21}{30} = \frac{34}{30} > 1$$

The teacher knows this is wrong because probabilities must be inclusive and between 0 and 1. After thinking about it, she remembered nine students taking both Art and English. These students raised their hands each time she counted, so the teacher counted them twice. When we calculate probabilities, we have to be careful to count each outcome only once.

The correct answer would be:

$$P(\text{Art or English}) = \frac{13+21-9}{30} = \frac{25}{30} = \frac{5}{6}$$

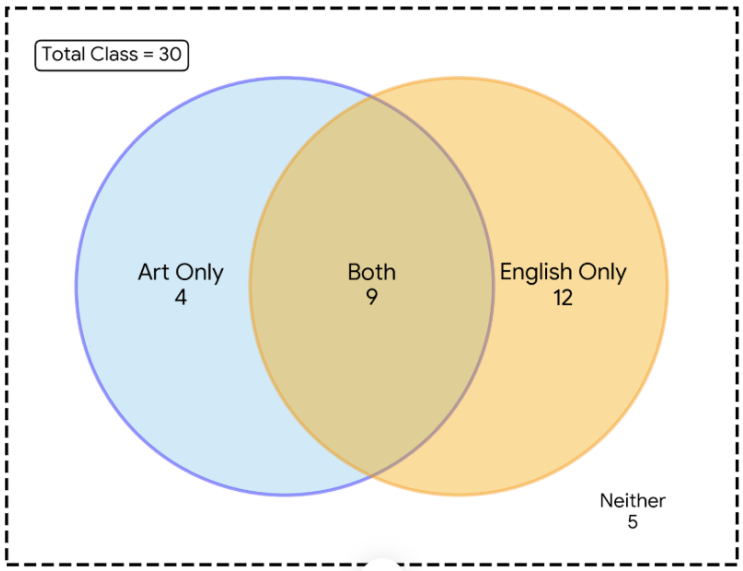


Figure 3.4.2: Venn Diagram of Showing Taking Arts and English

We will work out how to get the correct answer in general below.

An experiment involves drawing one card from a well-shuffled deck of 52 cards. Consider the events E = the card is red, F = the card is a 5, and G = the card is a spade. A card can be both red and a 5 simultaneously, but it is not possible for a card to be both red and a spade simultaneously. It would be easy to accidentally count a red 5 twice by mistake. It is not possible to count a red spade twice.

If two events have events in common or can happen simultaneously, the overlap is called the **intersection** of the events. The intersection of events is denoted as A "and" B, and is the same as the intersection of two sets A and B.

Definition: Mutually Exclusive (Disjoints)

Two events are **mutually exclusive** if they have no outcomes in common. In other words, **mutually exclusive events** are those that **cannot occur simultaneously**.

The probability of two mutually exclusive events A and B is $P(A \text{ and } B) = 0$.

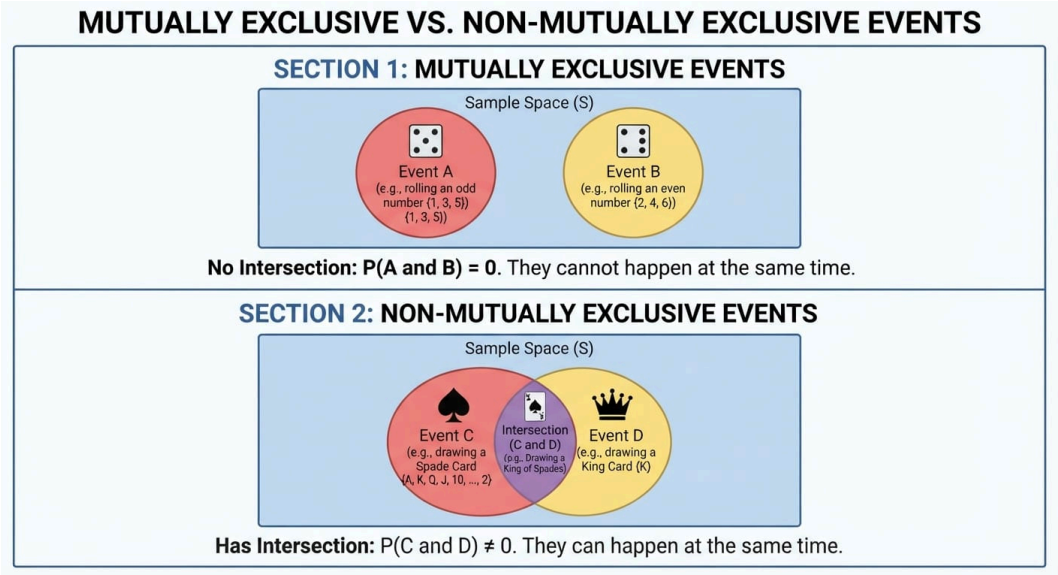


Figure 3.4.3: Showing Mutually Exclusive and Not Mutually Exclusive Events

Examples of Mutually Exclusive and Not Mutually Exclusive Events

Example of mutually exclusive events:

1. Drawing one card from a deck, A: Drawing a red card, and B: Drawing a black card
2. Rolling a die, A: Rolling a 2 and B: Rolling a 3
3. A student is chosen randomly: A: Student passes the exam, and B: Student fails the exam

Examples of not mutually exclusive events:

1. Rolling a die, A: Rolling an even number, and B: Rolling a number greater than 3
2. Drawing one card from a deck, A: Drawing a red card (hearts or diamonds), and B: Drawing a king
3. A student is chosen randomly. A: Student is in the drama club, and B: Student is in the soccer team

Example 3.4.2: Identify if the Events are Mutually Exclusive or not?

Two fair dice are tossed, and different events are recorded. Let the events A, B and C be as follows:

$A = \text{The sum is } 5 = \{(1,4),(2,3),(3,2),(4,1)\}$

$B = \text{Both numbers are even} = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$

$C = \text{Both numbers are less than } 5 = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$

a. Are events A and B mutually exclusive?
b. Are events A and C mutually exclusive?
c. Are events B and C mutually exclusive?

Answer available at the end of this section.

Answer available at the end of this section.

Your Turn 3.4.2: Disjoint or Not

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Do Not Count the Same Outcome Twice.

The addition rule for probabilities is used when the word " or " connects the events ." Remember, the teacher in her class wanted to know the probability that her students took either Art or English. Her problem was that she counted some students twice. She needed to add the number of students taking Art to the number of students taking English, and then subtract the number of students she counted twice. After dividing the result by the total number of students, she will find the desired probability. The calculation is as follows:

$$\begin{aligned} P(\text{Art or English}) &= \frac{\#\text{ taking Art} + \#\text{ taking English - }\#\text{ taking both}}{\text{total number of students}} \\ &= \frac{13+21-9}{30} \\ &= \frac{25}{30} \approx 0.833 \end{aligned}$$

The probability that a student takes Art or English is 0.833 or $\$83.3\%$.

When calculating the probability for compound events connected by the word “or,” we must be careful not to count the same thing twice.

If we want the probability of drawing a red card or a 5 we cannot count the red 5s twice. If we want to know the probability of a person being blonde-haired or blue-eyed, we cannot count the blue-eyed blondes twice. The addition rule for probabilities adds the number of blonde-haired people to the number of blue-eyed people and then subtracts the number of people we counted twice.

Formula: Addition Rule for “Or” Probabilities

If A and B are any events, then the probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

If A and B are mutually exclusive events then $P(A \text{ and } B) = 0$, so then

$$P(A \text{ or } B) = P(A) + P(B)$$

In a slightly different way, we can see the above formula

$$P(\text{A or B}) = \frac{\text{n(A)} + \text{n(B)} - \text{n(A and B)}}{\text{total number outcomes}}$$

Where n(A) = Number of outcomes in A.

Example 3.4.3: Deck of Card Mutually Exclusive

A single card is drawn from a well-shuffled deck of 52 cards. Find the probability that the card is a club or a face card.

Answer available at the end of this section.

Your Turn 3.4.3: Find Or Probability

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Example 3.4.4: Deck of Cards (Mutually Exclusive and Not Mutually Exclusive)

1. Suppose we draw one card from a standard deck. What is the probability that we get a queen or a king?
2. Suppose we draw one card from a standard deck. What is the probability that we will get a red card or a king?

Answer 1 available at the end of this section.

Your Turn 3.4.4: Find Or Probability (Two Dice Rolled)

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Find the Probability Using a Two-way Table

A **two-way table** (also called a **contingency table**) is a way to organize data that shows the relationship between **two categorical variables**. It helps you see how different categories intersect. It's like a grid with one variable across the **top (columns)** and another variable down the **side (rows)**, and each **cell** shows the **frequency/count** of observations for that pair of categories.

KEEP IN MIND

When you find a probability using a **two-way table**, you **must always use the total number of observations** (the **grand total**) in the **denominator**, unless the problem specifically tells you to find a *conditional probability*.

Example 3.4.5: Calculate Probability Using Two Way Table

A survey was conducted among 250 people who had recently purchased a car, and the results are summarized in the following table.

Satisfaction of Car Buyers

	Satisfied	Not Satisfied	Total
New Car	92	28	120
Used Car	83	47	130
Total	175	75	250

Find the probability that a person bought a new car or was not satisfied.

Answer available at the end of this section.

Your Turn 3.4.5: Probability Using Two Way Table

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Example 3.4.6: Calculate Probability Using Two Way Table

The table below shows the number of survey subjects who have received and have not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

1. Has a red car *and* got a speeding ticket
2. Has a red car *or* got a speeding ticket.



Answer available at the end of this section.

Your Turn 3.4.6: Calculate Probability Using Venn Diagram

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Answers

Example 3.4.1: Complement of an Event

1. There are 13 hearts in the deck, so

$$P(\text{heart}) = \frac{13}{52} = \frac{1}{4}$$

The probability of *not* drawing a heart is the complement of drawing a heart:

$$P(\text{not heart}) = 1 - P(\text{heart}) = 1 - \frac{1}{4} = \frac{3}{4}$$

2. Let A = arrive on time. The probability of A is $P(A) = 0.906$. The probability of not arriving on time is is

$$P(A') = 1 - P(A) = 1 - 0.906 = 0.094 = 9.4\%$$

Example 3.4.2: Identify if the Events are Mutually Exclusive or not?

Example 3.4.2: Identify if the Events are Mutually Exclusive or not?

a. Yes. A and B are mutually exclusive because they have no outcomes in common. It is not possible to add 2 even numbers to get a sum of 5.
b. No. A and C are not mutually exclusive because they have some outcomes in common. The pairs $(1,4), (2,3), (3,2)$, and $(4,1)$ all have sums of 5, and both numbers are less than 5.
c. No. B and C are not mutually exclusive because they have some outcomes in common. The pairs $(2,2), (2,4), (4,2)$, and $(4,4)$ all have two even numbers that are less than 5.

Example 3.4.3: Deck of Card Mutually Exclusive

There are 3 cards that are clubs, 12 face cards (J, Q, K in each suit), and 3 face cards that are clubs.

$$\begin{aligned} P(\text{club or face card}) &= P(\text{club}) + P(\text{face card}) - P(\text{club and face card}) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} = \frac{11}{26} \approx 0.423 \end{aligned}$$

The probability that the card is a club or a face card is approximately 0.423 or 42.3%.

OR you can do the following directly

$$\begin{aligned} P(\text{club card or face card}) &= \frac{\#\text{ of club card } + \#\text{of face card} - \#\text{ of both club and face card}}{\text{total number cards}} \\ &= \frac{13 + 12 - 3}{52} \\ &= \frac{22}{52} \approx 0.423 \end{aligned}$$

A simple way to check this answer is to take the 52 card deck and count the number of physical cards that are either clubs or face cards. If you were to set aside all of the clubs and face cards in the deck, you would end up with the following:

OR you can also do the following

{2 Clubs, 3 Clubs, 4 Clubs, 5 Clubs, 6 Clubs, 7 Clubs, 8 Clubs, 9 Clubs, 10 Clubs, J Clubs, Q Clubs, K Clubs, A Clubs, J Hearts, Q Hearts, K Hearts, J Spades, Q Spades, K Spades, J Diamonds, Q Diamonds, K Diamonds}

That is 22 cards out of the 52 card deck, which gives us a probably of:

$$\frac{22}{52} = \frac{11}{26} \approx 0.423$$

This confirms our earlier answer using the formal Addition Rule.

Example 3.4.4: Deck of Cards (Mutually Exclusive and Not Mutually Exclusive)

The event of drawing a king card and drawing a queen is mutually exclusive. There are 4 queens and 4 kings in the deck, hence 8 outcomes corresponding to a queen or king out of 52 possible outcomes. Thus, the probability of drawing a queen or a king is:

$$P(\text{ King or Queen}) = \frac{8}{52} = \frac{2}{13}$$

Note that in this case, there are no cards that are both a queen and a king, so $P(\text{King and Queen}) = 0$. Using the addition rule, we could have said:

$$P(\text{ King or Queen}) = P(\text{ King }) + P(\text{ Queen}) - P(\text{ King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52} = \frac{2}{13}$$

Note that you should not simplify the individual fractions before adding/subtracting them, since they naturally have a common denominator. Simplify after the addition.

OR you can do the following directly

$$\begin{aligned} P(\text{King or Queen}) &= \frac{\#\text{ of Kings } + \#\text{of Queen } - \#\text{ of King and Queen}}{\text{total number of outcomes in sample space}} \\ &= \frac{4 + 4 - 0}{52} \\ &= \frac{8}{52} = \frac{2}{13} \end{aligned}$$

The two events, drawing a red card and drawing a king, are not mutually exclusive. Half the cards are red, so

$$P(\text{red}) = \frac{26}{52} \tag{3.4.1}$$

There are 4 kings, so

$$P(\text{King}) = \frac{4}{52} \tag{3.4.2}$$

There are 2 red kings, so

$$P(\text{red and King}) = \frac{2}{52} \tag{3.4.3}$$

We can then calculate

$$\begin{aligned} P(\text{red or King}) &= P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) \\ &= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} \\ &= \frac{28}{52} = \frac{7}{13} \end{aligned} \tag{3.4.4}$$

Note that you should not simplify the individual fractions before adding/subtracting them, since they naturally have a common denominator. Simplify after the addition.

OR you can do the following directly

$$\begin{aligned} P(\text{King or Red}) &= \frac{\#\text{ of King} + \#\text{of Red} - \#\text{ of King and Red}}{\text{total number of outcomes in sample space}} \\ &= \frac{4+26-2}{52} \\ &= \frac{28}{52} = \frac{7}{13} \end{aligned}$$

Example 3.4.5**: Calculate Probability Using Two Way Table**

$$\begin{aligned} P(\text{new car or not satisfied}) &= P(\text{new car}) + P(\text{not satisfied}) - P(\text{new car and not satisfied}) \\ &= \frac{120}{250} + \frac{75}{250} - \frac{28}{250} = \frac{167}{250} \approx 0.668 \end{aligned}$$

The probability that a person bought a new car or was not satisfied is approximately 0.668 or 66.8%.

OR you can find the probability directly as shown below

$$\begin{aligned} P(\text{new car or not satisfied}) &= \frac{\#\text{ of new car} + \#\text{of not satisfied} - \#\text{ of new are and not satisfied}}{\text{total number people}} \\ &= \frac{120+75-28}{250} \\ &= \frac{167}{250} \approx 0.668 \end{aligned}$$

Example 3.4.6**: Calculate Probability Using Two Way Table**

1. We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is

$$P(\text{red car and ticket}) = \frac{15}{665} \approx 0.0226 \tag{3.4.5}$$

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of the probabilities of each one occurring.

2. Of the 665 surveyed, we can see that 150 people had red cars, 60 people had speeding tickets, and 15 people had red cars and speeding tickets. So

$$\begin{aligned} P(\text{red car or ticket}) &= \frac{\#\text{ of red car} + \#\text{of ticket} - \#\text{ of red car and got ticket}}{\text{total number people surveyed}} \\ &= \frac{150+60-15}{665} \\ &= \frac{195}{665} = \frac{39}{133} = 0.2932 \end{aligned}$$

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3.5: Odd and Expected Value

Learning Objectives

After completing this section, you should be able to:

1. Determine odds from probabilities.
2. Determine probabilities from odds.
3. Calculate the expected value of an experiment.
4. Interpret the expected value of an experiment.
5. Use expected value to analyze applications.



Figure 3.5.1: Scratch-off lottery tickets, as well as many other games, represent the likelihood of winning using odds. (credit: "My Scratch-off Winnings" by Shoshanah/Flickr, CC BY 2.0)

A particular lottery instant-win game has 2 million tickets available. Of those, $500,000$ win a prize. If there are $500,000$ winners, then it follows that there are $1,500,000$ losing tickets. When we evaluate the risk associated with a game like this, it can be useful to compare the number of ways to win the game to the number of ways to lose. In the case of this game, we would compare the $500,000$ wins to the $1,500,000$ losses. In other words, there are three losing tickets for every winning ticket. Comparisons of this type are the focus of this section.

Computing Odds

The ratio of the number of equally likely outcomes in an event, E, to the number of equally likely outcomes *not* in the event, E', odds for (or odds in favor of) the event. The opposite ratio (the number of outcomes not in the event to the number in the event to the number in the event is called the odds against the event. The formula is given below

Let's say, based on past data, in a certain city, it rains 60 days in a year, and it doesn't rain 305 days a year. Find the **odds in favor** of a rainy day and interpret your answer**.**

Now, suppose you're trying to find the **odds in favor** of **a rainy day**:

Favorable outcomes (rainy days) = 60

Unfavorable outcomes (non-rainy days) = 305

$$\text{Odds in favor of rain} = 60{:}305 = \frac{60}{305} = \frac{12}{61} = 12{:}61$$

Interpretation: The odds in favor of rain on a randomly chosen day are $12:61$, meaning for every 12 rainy day, there are 61 days without rain.

FORMULA: Odd in Favor and Odd Against

For an event E

$$\begin{aligned} \text{P(not E)} &= 1 - P(E) \\ \text{Odd in favor of E} &= \frac{P(E)}{P(\text{not } E)} \\ \text{Odd against E} &= \frac{P(\text{not } E)}{P(E)} \end{aligned}$$

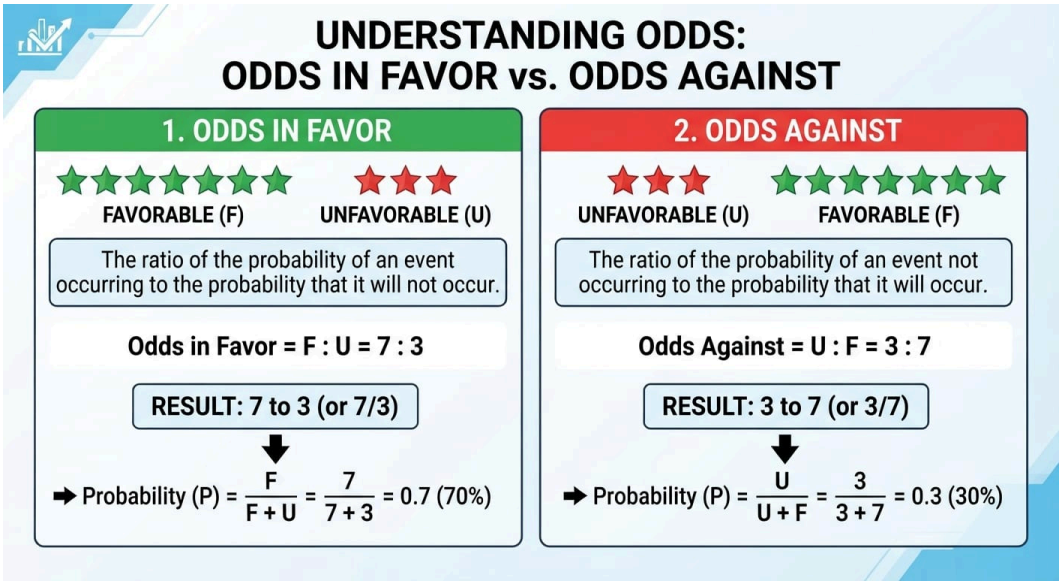


Figure 3.5.2: Understanding Odd in Favor and Odd Against

If we know the numbers of favorable and unfavorable outcomes, we can use the following to calculate the odds against and in favor.

Odd in Favor and Odd Against: Interpretation

Odd in Favor and Odd Against

Type	Formula
Odds in Favor	$\text{Odd in Favor} = \dfrac{\text{Number of favorable outcomes}}{\text{Number of unfavorable outcomes}}$

Type	Formula
Odds Against	$\text{Odd Against} = \dfrac{\text{Number of unfavorable outcomes}}{\text{Number of favorable outcomes}}$

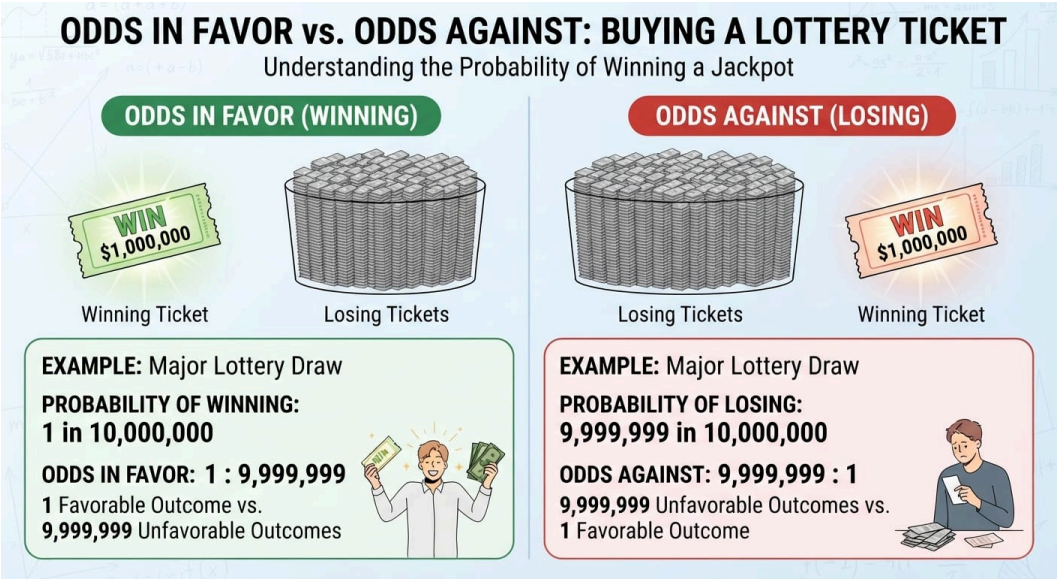


Figure 3.5.3: Odd in Facor vs Odd Aganist

Checkpoint

Both odds and probabilities are calculated as ratios. To avoid confusion, we will always use fractions, decimals, or percents for probabilities and colons to indicate odds. The rules for simplifying fractions also apply to odds. Thus, the odds for winning a prize in the game described in the section opener are $500,000:1, 500,000 = 1:3$ and the odds against winning this prize are $3:1$. These would often be expressed in words as "the odds of winning are one to three in favor" or "the odds of winning are three to one against.

Example 3.5.1: Computing Odds

If you draw a card at random from a standard deck, what are the odds of drawing a face card?

Answer available at the end of this section.

Your Turn 3.5.2: Computing Odds

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Example 3.5.2: Converting Probabilities to Odds

Given the following probabilities of an event, find the corresponding odds for and against that event.

$\mathrm{P(E\ )} = 17\%$

Answer available at the end of this section.

Your Turn 3.5.2: Computing Odds

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Odds as a Ratio of Probabilities

We can use these formulas to convert between probabilities and odds.

We can also think of odds as a ratio of probabilities. Consider the instant-win game from the section opener again, with $500,000$ winning tickets out of $2,000,000$ total tickets.

Number of favorable tickets is $500,000$.

Number of unfavorable tickets is $2,000,000 - 500,000 = 1,500,000$.

If a player buys one ticket, the probability of winning is

$$\frac{5000,000}{2,000,000} = \frac{1}{4}$$

And the probability of losing is $1 - \frac{1}{4} = \frac{3}{4}$. Notice that the ratio of the probability of winning to the probability of losing is $\frac{1}{4} \div \frac{3}{4} = \frac{1}{3} = 1:3$, which matches the odds in favor of winning.

FORMULA: Converting Odd to Probability

Here, "favorable" means the number of favorable outcomes.
If the odds in favor of E are $\mathrm{favorable:unfavorable}$, then the probability of E is

$$\mathrm{Total} = \mathrm{favorable} + \mathrm{unfavorable}$$

$$\mathrm{P(E)} = \frac{\mathrm{favorable}}{\mathrm{Total}}$$

If the odds against of E are $\mathrm{uavorable:favorable}$, then the probability of E is

$$\mathrm{P(E)} = \frac{\mathrm{unfvorable}}{\mathrm{Total}}$$

Example 3.5.3: Converting Odds to Probability

Find $P(E)$ if the odds in favor of E are $4:9$.

Answer available at the end of this section.

Your Turn 3.5.3: Converting Odd to Probability

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Checkpoint

Some places, particularly state lottery websites, will use the words "odds" and "probability" interchangeably. Never assume that the word "odds" is being used correctly! Compute one of the odds/probabilities yourself to make sure you know how the *word.* Expected Value*d is being used!*

Expected Value



Figure 3.5.4: The concept of expected value allows us to analyze games that involve randomness, like Roulette. (credit: "Roulette Table and Roulette Wheel in a Casino with People betting on numbers" by Marco Verch/Flickr, CC BY 2.0)

The casino game roulette has dozens of different bets that can be made. These bets have different probabilities of winning but also have different payouts. In general, the lower the probability of winning a bet, the more money a player wins for that bet. With so many options, is there one bet that's "smarter" than the rest? What's the best play to make at a roulette table? This section will develop the tools we need to answer these questions.

Many experiments have numbers associated with their outcomes. Some are easy to define; if you roll 2 dice, the sum of the numbers showing is a good example. In some card games, cards have different point values associated with them; for example, in some forms of the game rummy, aces are worth 15 points; 10 s, jacks, queens, and kings are worth 10; and all other cards are worth 6. The outcomes of casino and lottery games are all associated with an amount of money won or lost. These outcome values

are used to find the expected value of an experiment: the mean of the values associated with the outcomes that we would observe over a large number of repetitions of the experiment.

That definition is a little vague; How many is “a large number?” In practice, it depends on the experiment; the number must be large enough that every outcome is expected to appear at least a few times. For example, if we’re rolling a standard 6-sided die and noting the number that shows, a few dozen replications should be sufficient to ensure the mean is representative. Since the probability of each outcome is $\frac{1}{6}$, we would expect to see each outcome about 8 times over the course of 48 replications. However, if we’re talking about the Powerball lottery, where the probability of winning the jackpot is about $\frac{1}{292{,}000{,}000}$, we would need several *billion* replications to ensure that every outcome appears a few times. Luckily, we can find the theoretical expected value before we even run the experiment for the first time.

FORMULA: Expected Value

If X represents the value of that outcome, $P(X)$ represents the probability of the outcome. The expected value of the experiment is:

$$\Sigma(X) \times P(X))$$

where Σ is the “sum,” meaning we add up the results of the formula that follows over all possible outcomes.

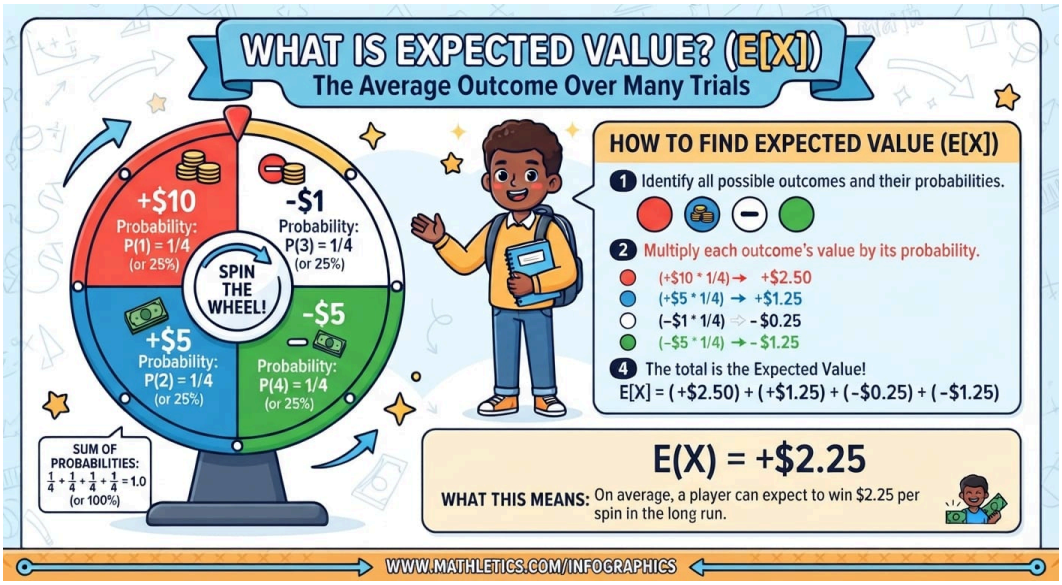


Figure 3.5.5: Expected Value: What is it?

We can find the probability distribution table for the above diagram.

Outcome and Value of the Outcome (X)	Probability $P(X)$
Land on Red: $\$5$	$\frac{1}{4}$
Land on Blue: $\$5$	$\frac{1}{4}$
Land on Green: $-\$5$	$\frac{1}{4}$
Land on White:$-\$1$	$\frac{1}{4}$

Key Points to Remember for Finding Expected Value

1. Identify all possible outcomes and their values.

These values can be positive or negative depending on the situation.

2. Determine the probability of each outcome.
3. Set up a probability distribution table:
   - First column (or row) → outcome probabilities
   - Second column (or row) → corresponding outcome numerical value
4. **Multiply each outcome value by its probability**, then **add all the products together** to find the **expected value**.

Here are the most common areas where people use expected value:

Gambling and Games: Used to determine the average gain or loss in activities like lottery, cards, dice, and slot machines, and to decide if a game is fair or profitable.

Insurance: Insurance companies use expected value to set premiums by estimating the expected cost of accidents, health claims, or property damage. Finance and Investments: Used to evaluate risky investments and predict the average profit or loss over time.

Sports: Helps predict average points, wins, or scoring outcomes in a game.

Weather and Natural Events: Used to estimate expected rainfall, average temperature, or the likelihood of natural disasters.

Example 3.5.4: Finding Expected Values

Find the expected values of the following experiments.

An **unfair** six-sided die has sides numbered $1, 2, 3, 4, 5, 6$ with probabilities given in the table below. Find the expected value of one roll of the die.

Outcomes for rolling a six-sided die and their probabilities

Value of Outcome	1	2	3	4	5	6
Probability	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

Answer available at the end of this section.

Let's look at another example. Where the probability distribution table must be created.

Example 3.5.5: Finding Expected Values

Find the expected values of the following experiments.

Draw a card from a well-shuffled standard deck of cards and note its rummy value (15 for aces; 10 for tens, jacks, queens, and kings; 5 for everything else).

Answer available at the end of this section.

Your Turn 3.5.5: Calculated Expected Value

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As we noted, the expected value of an experiment is the mean of the values we would observe if we repeated the experiment a large number of times. (This interpretation is due to an important theorem in the theory of probability called the Law of Large Numbers.)

Example 3.5.6: Finding Expected Value (Sum of Two Dice)

A bag contains 3 gold marbles, 9 silver marbles, and 26 black marbles. Someone offers to play this game: You randomly select one marble from the bag. If it is gold, you win $\$4.00$. If it is silver, you win $\$3.00$. If it is black, you lose $\$1.00$. What is your expected value if you play this game?

Answer available at the end of this section.

To find the expected value, you need to determine the outcomes and their probabilities. Those are listed in the following table.

Outcomes and their probabilities

Outcome	Value	Probability	Value × probability
Gold	$\$4$	$\frac{3}{38}$	$\frac{12}{38}$
Silver	$\$3$	$\frac{9}{38}$	$\frac{27}{38}$
Black	$-\$1$	$\frac{26}{38}$	$\frac{-26}{38}$

$$\begin{aligned} \text{Expected value} &= \$\frac{12}{38} + \$\frac{27}{38} + \$\frac{-26}{38} \\ &= \frac{-4}{38} \\ &= -\$0.3421 \end{aligned} \tag{3.5.1}$$

Your Turn 3.5.6: Find Expected Value a Game

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Example 3.5.7: Risk of Dying

A 40-year-old man in the U.S. has a 0.24% risk of dying during the next year. An insurance company charges $\$300$ annually for a life insurance policy that pays a $\$100,000$ death benefit. What is the expected value for the person buying the insurance? Round your answer to the nearest dollar.

Answer available at the end of this section.

Your Turn 3.5.7: Risk of Dying

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Example 3.5.8: Every Game Has a Negative Expected Value to the player

Suppose someone offers to play a game with you. If you roll a die and get a 6, you get $\$10$. However, getting a five or below means you lose $\$1$. Is this a game you'd want to play?

Answer available at the end of this section.

Your Turn 3.5.8: Expected Value of Raffle Ticket

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Who Knew?: Expected Values in Football

In the 21st century, data analytics tools have revolutionized the way sports are coached and played. One tool in particular is used in football at crucial moments in the game. When a team faces a fourth down (the last possession in a series of four possessions, a fairly common occurrence), the coach faces a decision: Run one play to try to gain a certain number of yards, or kick the ball away to the other team. Here's the interesting part of the decision: If the team "goes for it" and runs the play and they are successful, then they keep possession of the ball and can continue in their quest to score more points. If they are unsuccessful, then they lose possession of the ball, giving the other team an opportunity to score points. If, instead, the team punts, or kicks the ball away, then the other team gets possession of the ball, but in a worse position for them than if the original team goes for it and fails. To analyze this situation, data analysts have generated empirical probabilities for every fourth-down situation and computed the expected value (in terms of points) for each decision. Coaches frequently use those calculations when they decide which option to take!

People in Mathematics: Pierre de Fermat and Blaise Pascal

In 1654, a French writer and amateur mathematician named Antoine Gombaud (who called himself the Chevalier du Mére) reached out to his gambling buddy Blaise Pascal to answer a question that he'd read about called the "problem of points." The question goes like this: Suppose you're playing a game that is scored using points, and the first person to earn 5 points is the winner. The game is interrupted with the score 4 points to 2. If the winner stood to win $\$100$, how should the prize money be divided between the players? Certainly, the person who is 1 point away from victory should get more, but *how much* more?

We have developed tools in this section to answer this question. At its heart, it's a question about conditional probabilities and expected value. Pascal first started thinking about it at the time, though those ideas hadn't yet been invented. Pascal reached out to a colleague named Pierre de Fermat, and over the course of a couple of months, their correspondence with each other would eventually solve the problem. In the process, they first described conditional probabilities and expected values!

Pascal is remembered for the "arithmetical triangle" that is named after him (though he wasn't the first person to discover it; see the section on the binomial distribution for more), as well as his work in geometry. In physics, Pascal worked on hydrodynamics and air pressure (the SI unit for pressure is named after him), and in philosophy, Pascal advocated a mathematical approach to philosophical problems.

Example 3.5.9: Using Expected Values

In the casino game keno, a machine randomly chooses 20 numbers between 1 and 80 (inclusive) without replacement. Players try to predict which numbers will be selected. Players don't try to guess all 20; generally, they'll try to predict between 1 and 10 of the chosen numbers. The amount won depends on the number of guesses they made and the number of guesses that were correct. At one casino, a player can try to guess just one number. If that number is among the 20 selected, the player wins $\$2$; otherwise, the player loses $\$1$. What is the expected value?

Answer available at the end of this section.

Your Turn 3.5.9: Keno Game

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WORK IT OUT: Make Your Own Lottery

By yourself or with a partner, devise your own lottery scheme. Assume you would have access to one or more machines that choose numbers randomly. What will a lottery draw look like? How many numbers are players choosing from? How many will be drawn? Will they be drawn with replacement or without replacement? What conditions must be met for a player to win the first or second (or more!) prize? Once you've decided on that, determine the payoff structure for winners and how much the game will cost. Try to make the game enticing enough that people will want to play it but with enough negative expected value that the lottery will make money. Aim for the expected value to be about -0.25 times the cost of playing the game.

Answers

Example 3.5.1: Computing Odds

One Method:

There are 12 of those outcomes in the event, E, "drawing a face card," so E ={J♥, Q♥, K♥,J♥, Q♦, K♦, J♣, Q♣, K♣ J♠, Q♠, K♠}. Note that a deck has 52 cards.

Number of favorable outcomes (number of face cards) = 12

Number of unfavorable outcomes (number of not face cards) = $52 - 12 = 40$

$$\text{Odd in Favor} = \frac{\text{Number of favorable outcomes}}{\text{Number of unfavorable outcomes}} = \frac{12}{40} = \frac{3}{10}$$

$$\text{Odd against} = \frac{\text{Number of unfavorable outcomes}}{\text{Number of favorable outcomes}} = \frac{40}{12} = \frac{10}{3}$$

Another Method:

We can find odds another way as shown in the following

$$\begin{aligned} \mathrm{P(E\ )} &= \frac{12}{52} \\ &= \frac{3}{13} \\ \mathrm{P(not\ E\ )} &= 1 - \frac{3}{13} = \frac{10}{13} \end{aligned} \tag{3.5.2}$$

$$\text{Odd in favor of E } = \frac{P(E)}{P(\text{not}E)} = \frac{\frac{3}{13}}{\frac{10}{13}} = \frac{3}{13} \times \frac{13}{10} = \frac{3}{10} = 3{:}10 \tag{3.5.3}$$

Example 3.5.2**: Converting Probabilities to Odds**

One Method:

Note that $\mathrm{P(E\ )} = 17\%$ means, out of 100, 17 times that outcome occurs. In other words,

Number of favorable outcomes for event E = 17

Number of unfavorable outcomes for event E = $100 - 17 = 83$

$$\text{Odd in Favor} = \frac{\text{Number of favorable outcomes}}{\text{Number of unfavorable outcomes}} = \frac{17}{83}$$

$$\text{Odd against} = \frac{\text{Number of unfavorable outcomes}}{\text{Number of favorable outcomes}} = \frac{83}{17}$$

Another Method

We'll use another formula:

$$\begin{aligned} \mathrm{P(E\ )} &= 0.17 = \frac{17}{100} \\ \mathrm{P(not\ E\ )} &= 1 - 0.17 = 0.83 = \frac{83}{100} \\ \text{Odd in favor of E } &= \frac{\frac{17}{100}}{\frac{83}{100}} \\ &= \frac{17}{100} \times \frac{100}{83} \\ &= \frac{17}{83} \\ &= 17{:}83 \end{aligned}$$

Example 3.5.3**: Converting Odds to Probability**

The odds in favor of E are $4:9$. That means, the number of favorable outcomes is 4, and the number of unfavorable outcomes is 9, and the number ot total outcomes is $4 + 9 = 13$.

$$\mathrm{P(E)} = \frac{\text{favorable}}{\text{Total}} = \frac{4}{13}$$

Example 3.5.4**: Finding Expected Values**

We multiply each value by its probability and sum the results.

$$\begin{aligned} \text{Expected value} &= 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8} + 4 \times \frac{1}{4} + 5 \times \frac{1}{4} + 6 \times \frac{1}{8} \\ &= \frac{1}{8} + \frac{2}{8} + \frac{3}{8} + \frac{4}{4} + \frac{5}{4} + \frac{6}{8} \\ &= \frac{30}{8} = 3.75 \end{aligned}$$

So, the expected value of one roll of the dice is 3.750.

Example 3.5.5**: Finding Expected Values**

Let's find the value of the outcome and its corresponding probabilities. We have three outcomes that we are looking for:

1. aces
2. tens, jacks, queens, and kings
3. Everything else.

Possible outcomes and probabilities for the given experiment

Outcome	Probability	Value of Outcome
$\{A\}$	$\frac{4}{52}$	15
$\{10, J, K, Q\}$	$\frac{16}{52}$	10
$\{2, 3, 4, 5, 6, 7, 8, 9\}$	$\frac{32}{52}$	5

We multiply each value by its probability and add them.

$$\begin{aligned} \text{Expected value} &= 15 \times \frac{4}{52} + 10 \times \frac{16}{52} + 5 \times \frac{32}{52} \\ &= \frac{60}{52} + \frac{160}{52} + \frac{160}{52} \\ &= \frac{380}{52} = 7.3 \end{aligned}$$

Example 3.5.6: Finding Expected Value (Sum of Two Dice)

Example 3.5.7: Risk of Dying

To find the expected value, we need to determine the value of each outcome and its corresponding probability. Those are listed in the following table.

Outcomes and their Probabilities

Outcomes (Events)	Values $n(O)$	Probability $P(O)$
Survive	$-\$300$	$1 - 0.0024 = 0.9976$
Die	$\$100,000 - \$300 = \$99,700$	0.0024

$$\text{Expected value} = (-\$300)(0.9976) + (\$99,700)(0.0024) = -\$60$$

Note: The negative sign in front of $\$300$ is because we are looking for the insured person's expected value; if that person survives, she/he will lose $\$300$. You also see that we subtract $\$300$ to find the value of the outcome die. It is because the insured person gets $\$300$ less than the policy value because the insured person has already paid $\$300$ for the policy.

Example 3.5.8: Every Game Has a Negative Expected Value to the player

Let's look at the expected value: The probability of winning is $\frac{1}{6}$, and the probability of losing is$\frac{5}{6}$, so the expected value is

$$\$10 \times \frac{1}{6} + (-\$1)\frac{5}{6} = -\$0.83$$

That means, on average, you'll come out ahead by about 83 cents every time you play this game.

It's a great deal! On the other hand, if the winnings for rolling a 6 drop to $\$3$, the expected value becomes

$$\$3 \times \frac{1}{6} + (-\$1)\frac{5}{6} = -\$0.33$$

This means you should expect to lose about 33 cents on average for every time you play.

Playing that game is not a good idea! This is how casinos and lottery corporations make money:

Example 3.5.9: Using Expected Values

There are 20 winning numbers out of 80, so if we try to guess one of them, the probability of guessing correctly is $\frac{20}{80} = \frac{1}{4}$. If the player guesses correctly, the player wins $\$2$. The probability of losing is then $\frac{3}{4}$. If the player guesses incorrectly, the player loses $\$1$.

Outcomes and their corresponding Probabilities

Outcomes (Events)	Values $n(O)$	Probability $P(O)$
Winner	$\$2$	$\frac{1}{4}$
Die	$-\$1$	$\frac{3}{4}$

$$\begin{aligned}\text{Expected value} &= \$2 \times \frac{1}{4} + (-\$1) \times \frac{3}{4} \\ &= -\$0.25\end{aligned}$$

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4: FINANCIAL MATHEMATICS

4.1: Understanding Percent

4.2: Discounts, Markups, and Sales Tax

4.3: Simple and Compound Interest

4.4: Methods of Savings (Annuities, Stock, and Bonds)

4.5: Basics of Loans (Mortgage, Cars, and Credit Cards)

4.6: Income Tax and FICA Taxes

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4.1: Understanding Percent

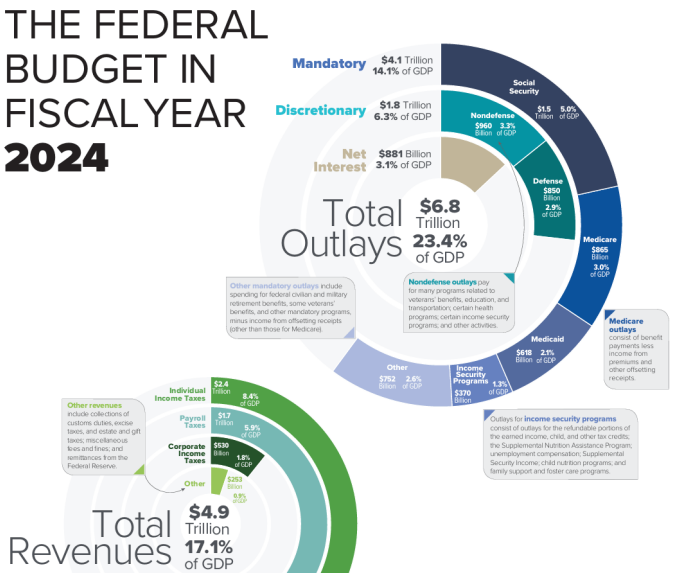


Figure $4.1.1$: "The federal budget describes how money is spent (Total Outlays) and how money is earned (Total Revenues)".. (credit: [1])

Learning Objectives

1. Convert between percent, decimal, and fractional values.
2. Calculate the total, percent, or part.
3. Solve application problems involving percent.
4. Calculate the percent increase and percent decrease.

In 2024, the U.S. federal government budgeted $3.8 billion for the National Park Service(opens in new window), which appears to be a very large number (and is!) and a large portion of the total federal budget. However, the total outlays from the U.S. federal government in 2024 were $6.8 trillion(opens in new window). So, the amount budgeted for the National Park Service was less than one-tenth of 1 percent, or $\frac{1}{10}\%$, of the total outlays. This **percentage** describes a specific number. Understanding that ratio puts the $\$3.8$ billion budgeted to the National Park Service in perspective.

This chapter 4 focuses on percent as a primary tool for understanding money management. The interest paid on debt, the interest earned through investments, and even taxes are entirely determined using percentages. This section introduces the basics of working with this invaluable tool.

The word percent comes from the Latin phrase per centum, which means "by the hundred." So any percent is a number divided by 100. Changing a percent to a fraction means writing the percent as a fraction. To write $n\%$ in its fractional form is to write the percent as the fraction $\frac{n}{100}$.

Convert Between Percent, Decimal, and Fractional Values

Formula: Percent to Decimal

The fraction form of $n\%$ is found by dividing $\frac{n}{100}$ and calculating the decimal value of the fraction.

Or you move the decimal to the left and remove the percent sign.

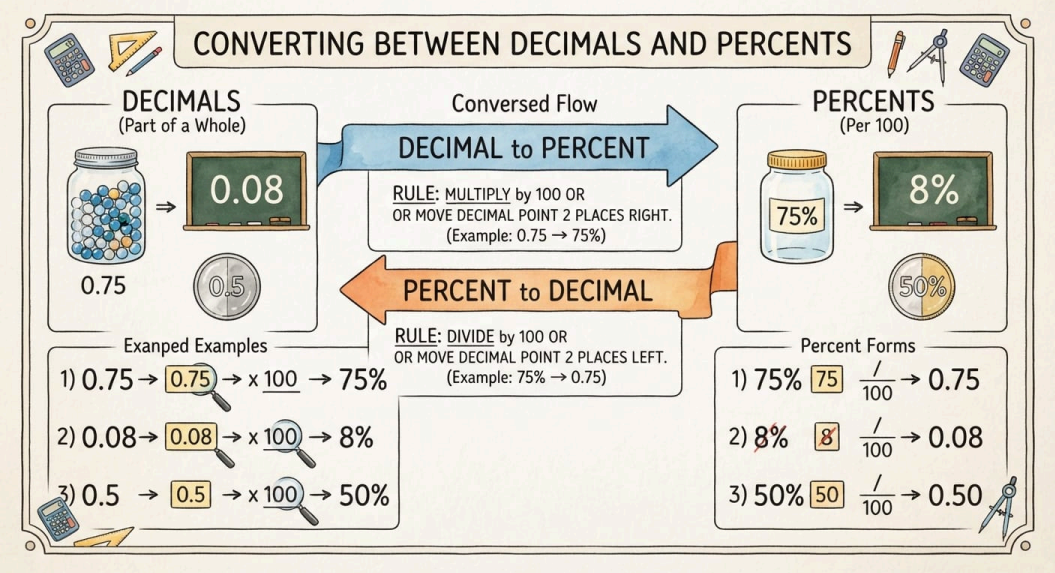


Figure 4.1.1: Percent to Decimal (or Fraction) and Decimal (Fraction) to Percent

Formula: Decimal to Percent

To convert the number x from decimal form (or fraction form) to percent, multiply x by 100 and place a percent sign, $\%$, after the number.

Or to convert from decimal form to percent form, you can simply move the decimal two places to the right without performing the multiplication.

Do You Know?

A percent need not be an integer and does not have to be less than 100.

Example 4.1.1: Rewriting a Percent as a Fraction or Decimal Form

Rewrite the following as fractions and decimal form:

1. 8%
2. 38.7%
3. $\frac{1}{4}\%$
4. 213%

Answer available at the end of this section.

When any calculation with a percent is to be performed, the form of the percent must be changed to its fractional form or its **decimal form**. We can convert a percent to a decimal by dividing the percent by 100 and representing the result as a decimal.

Example 4.1.2: Converting the Decimal Form or Fraction to Percent

Convert each of the following to a percent:

1. 0.34
2. 4.15

3. 0.0391
4. $\frac{3}{8}$

Answer available at the end of this section.

Your Turn $4.1.2$: Percent as Decimals and Vice Versa

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Calculate the Total (Whole), Percent, or Part

The word "of" indicates multiplication using fractions, as in "one-fourth of 56." To find "one-fourth of 56" we would multiply 56 by one-fourth. We can think of percents as fractions with a specific denominator - 100.

So, to calculate "25% of 52," we multiply 52 by 25%. But first, we need to convert the percent to decimal form, 0.25. Using the decimal form of 25%, we have 0.25×52, which equals 13.

In this problem, 52 is the **total** or **base**, 25 is the **percentage**, and 13 is **the percentage of** 52 or the **part (amount)** of 52.

FORMULA: Percent, Total and Part

If we have a **part** that is some percent of a **whole (Total)**, then

$$\begin{aligned} \text{percent} &= \frac{\text{part}}{\text{total}} \\ \text{part} &= \text{percent} \cdot \text{total} \\ \text{total} &= \frac{\text{part}}{\text{percent}} \end{aligned}$$

We can find any missing component if the other two are known by using the above formula.

To perform the calculations, we convert the percent to a decimal. Recall, to rewrite the percent as a decimal, we move the decimal over two places to the left, or, equivalently, divide the percent number by 100.

Checkpoint

1. *In all calculations, the percent is expressed as a decimal.*
2. In Calculation, the word **"of"** implies multiplication.
3. Any number after the word **"of"** is total or whole.

Knowing any two of the values in our formula allows us to calculate the third value. In the following example, we know the total and the percent, and are asked to find the percentage of the total.

Example $4.1.3$: Finding the Percent of a Total

Determine 156% of 720.

Answer available at the end of this section.

Your Turn 4.1.3: Find Part

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In the previous example, we knew the total and the percent and found the part using our formula. We may instead know the percent and the part, but not the total. We can use our formula again to solve for the total.

Example 4.1.4: Finding the Total from the Percent and the Part

What is the total if 10% of the total is $4,000$?

Answer available at the end of this section.

Your Turn 4.1.4: Find the Total

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Similarly, the percent can be found if the total and the percent of the total (the part) are known. This will result in the decimal form of the percent, so it must be converted to percent form.

Example 4.1.5: Finding the Percent from the Total and the Part

What percent of 500 is 175?

Answer available at the end of this section.

Your Turn 4.1.5: Find Percentage

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Solve Application Problems Involving Percent

Percentages are frequently used in finance, research, science experiments, and casual conversations. Understanding these values helps when consuming media or discussing finances, for instance. Effectively working with and interpreting numbers and percentages will help you become an informed consumer of this information.

In most cases, working through what is presented requires you to identify that you are indeed working with a question of percent, which two of the three values that are related through percent are known, and which of the three values you need to find.

Example 4.1.6: Retention Rate at College

Justine applied to a medium-sized university outside her hometown and found out that the retention rate (percent of students who return for their sophomore year) for the 2021 academic year at the university was 84%. During a visit to the registrar's office, she found out that 1,350 people had enrolled in the academic year 2021. How many students from the academic year 2021 are returning for the 2022 academic year?

Answer available at the end of this section.

Your Turn 4.1.6: Find the Parts

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Example 4.1.7: Percent of Chemistry Majors

Cameron enrolls in a calculus class. In this class of 45 students, there are 18 chemistry majors. What percent of the class are chemistry majors?

Answer available at the end of this section.

Your Turn 4.1.7: Find Percentage

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Example 4.1.8: Total Sales and Commission

Mariel makes a 20% commission on every sale she makes. One week, her commission check is for $153.00. What were her total sales that week?

Answer available at the end of this section.

Your Turn 4.1.8: Find Total

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Who Knew?: LED Lightbulbs

According to the U.S. government's energy website, LED lightbulbs use at least 75% less energy than incandescent bulbs. They also last up to 25 times as long as an incandescent bulb. If lighting accounts for a significant percentage of your electricity use, replacing all incandescent bulbs with LED bulbs can significantly reduce your electricity bill.

Definition: Percent Increase or Decrease

Percent increase and percent decrease are ways to express a change in a quantity as a percentage of its original value. To calculate the percent decrease, you use the following formula:

$$\text{Change in Value} = \text{New Value - Old Value}$$

The value of the Increase (Decrease) is also called the **absolute change**. If the value decreases, the absolute change will be negative.

$$\text{Percent Increase (Decrease)} = \frac{\text{Change in Value}}{\text{Old Value}} \times 100\%$$

The percent Increase (or Decrease) is also called **a relative change.**

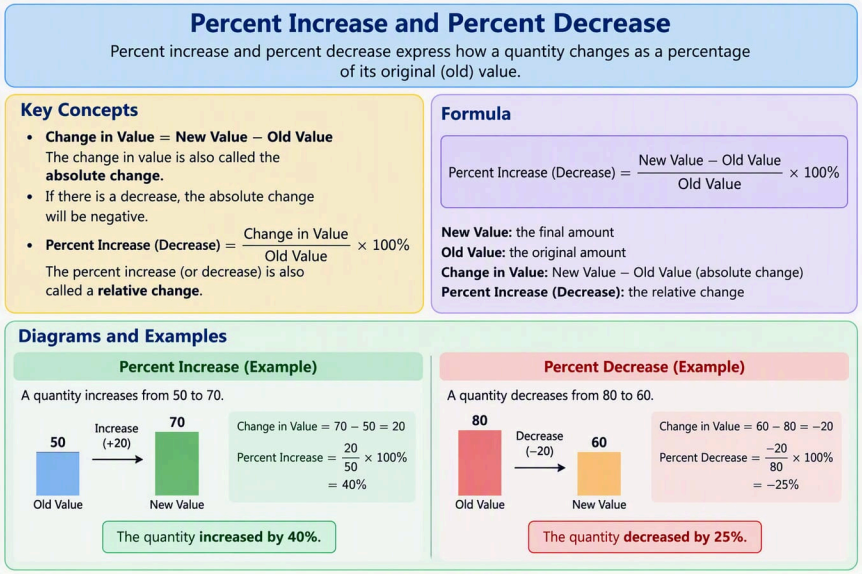


Figure 4.1.1: Copy and Paste Caption here. (Copyright; author via source)

Example 4.1.9: Percent Increase or Decrease

1. If the price of a shirt increases from $\$50$ to $\$70$, what is the percent increase?
2. If the price of a shirt decreases from $\$80$ to $\$60$, what is the percent decrease?

Answer available at the end of this section.

Your Turn 4.1.9: Find Percent Increase or Percentage Decrease

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[1] THE FEDERAL BUDGET IN FISCALYEAR(opens in new window) [www.cbo.gov]

Answers

Example 4.1.1: Rewriting a Percent as a Fraction or Decimal Form

1. Using the definition, $n = 8$. Decimal form = 0.08 and fractional form
$$\frac{8}{100} = \frac{2}{25}$$
2. Using the definition, $n = 38.7$. Decimal form = 0.387 and fractional form
$$\frac{387}{100}$$
3. Using the definition, $n = \frac{1}{4} = 0.25$. Decimal form = 0.0025 and fractional form
$$\frac{0.25}{100} = \frac{\frac{1}{4}}{100} = \frac{1}{400}$$
4. Using the definition, $n = 213$. Decimal form = 2.13 and fractional form
$$\frac{213}{100}$$

Example 4.1.2: Converting the Decimal Form or Fraction to Percent

1. Using the formula, $x = 0.34$, we calculate, 0.34×100, which gives us 34%.
2. Using the formula, $x = 4.15$, we calculate, 4.15×100, which gives us 415%.
3. Using the formula, $x = 0.0391$, we calculate, 0.0391×100, which gives us 3.91%.
4. Using the formula, $x = \frac{3}{8}$, we calculate, $(\frac{3}{8} \times 100\%)$, which gives us 37.5%.

Example 4.1.3: Finding the Percent of a Total

The total is 720, and the percent is 156%. The decimal form of 156% is 1.56. To find the part, or percent of the total, substitute those values into the formula and calculate.

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ &= 1.56 \times 720 \\ &= 1,123.2 \end{aligned}$$

From this, we say that 156% of 720 is $1,123.20$.

Example 4.1.4: Finding the Total from the Percent and the Part

The percent is 10, which in decimal form is 0.10. We were given that 10% of the total is $4,000$, so the part is $4,000$. Substituting into the formula, we have

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 4,000 &= 0.10 \times \text{total} \end{aligned}$$

To find the total, we solve the equation for the total.

$$\begin{aligned} 4,000 &= 0.10 \times \text{ total} \\ \frac{4,000}{0.10} &= \frac{\cancel{0.10} \times \text{ total}}{\cancel{0.10}} \\ 40,000 &= \text{ total} \\ &= 40,000 \end{aligned}$$

From this we see that $40,000$ is the total, or that 10% of $40,000$ is $4,000$.

Or you can use the formula directly

$$\begin{aligned} \text{total} &= \frac{\text{part}}{\text{percent}} \\ &= \frac{4,000}{0.10} \\ &= 40,000 \end{aligned}$$

Example $4.1.5$**: Finding the Percent from the Total and the Part**

The total is 500, the percent of the total is 175. Substituting into the formula, we have

$$\begin{aligned} \text{part} &= \text{ percent } \times \text{ total} \\ 175 &= \text{ percent } \times 500 \end{aligned}$$

To find the percent, we solve the equation for the percent.

$$\begin{aligned} 175 &= \text{ percent } \times 500 \\ \frac{175}{500} &= \frac{\text{ percent } \times \cancel{500}}{\cancel{500}} \\ 0.35 &= \text{ percent} \end{aligned}$$

We see the percent in decimal form is 0.35. Converting from the decimal form yields 35%. We say that 175 is 35% of 500.

Or we can use a direct formula:

$$\begin{aligned} \text{percent} &= \frac{\text{part}}{\text{total}} \\ &= \frac{175}{500} \\ &= 0.35 \\ &= 35\% \end{aligned}$$

Example $4.1.6$**: Retention Rate at College**

The percentage of students who will return for the 2022 academic year (the retention rate) is 84%. The total number of students enrolled in the 2021 academic year was $1,350$. This means the percent is known, and the total is known. From this, we can determine the number of students who will return (percent of the total) for the 2022 academic year using the formula.

$$\begin{aligned} \text{part} &= \text{ percent } \times \text{total} \\ \text{part} &= 0.84 \times 1,350 \\ \text{part} &= 1,134 \end{aligned}$$

So, $1,134$ students will return for the 2022 academic year.

Example $4.1.7$**: Percent of Chemistry Majors**

In this situation, the percentage is to be determined. We know the total number of students is 45, and the part of the students who are chemistry majors is 18. Using that information and the formula.

$$\begin{aligned} \text{percent} &= \frac{\text{part}}{\text{total}} \\ &= \frac{18}{45} \\ &= 0.4 \\ &= 40\% \end{aligned}$$

Converting the 0.4 from decimal form, we find that 40% of the students in the calculus class are chemistry majors.

Example $4.1.8$**: Total Sales and Commission**

In this problem, Mariel's total sales are to be determined. We know the percent she earns is 20%. We also know that her sales commission was $\$153.00$, which is a part of the total. Using this information and the formula

$$\begin{aligned} \text{total} &= \frac{\text{part}}{\text{percent}} \\ &= \frac{153}{0.2} \\ &= 765 \end{aligned}$$

Mariel's total sales were $\$765.00$.

Example $4.1.9$**: Percent Increase or Decrease**

Old value = $\$50$ AND New value = $\$70$

$$\begin{aligned} \text{Change in Value} &= \$70 - \$50 = \$20 \\ \text{Percent Increase} &= \frac{\text{Change in Value}}{\text{Old Value}} \times 100\% \\ &= \frac{\$20}{\$50} \times 100\% = 40\% \end{aligned} \tag{4.1.1}$$

Old value = $\$80$ AND New value = $\$60$

$$\begin{aligned} \text{Change in Value} &= \$60 - \$80 = -\$20 \\ \text{Percent Decrease} &= \frac{\text{Change in Value}}{\text{Old Value}} \times 100\% \\ &= \frac{-\$20}{\$80} \times 100\% = -25\% \end{aligned} \tag{4.1.2}$$

So, the percent decrease is 25%. Negative signs indicate a decrease.

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4.2: Discounts, Markups, and Sales Tax



Figure 4.2.1: Sale prices are often described as percent discounts. (credit: "Close-up of a discount sign" by Ivan Radic/Flickr, CC BY 2.0)

Learning Objectives

1. Compute discounts, markups, and sales tax.
2. Solve application problems involving discounts, markups, and sales tax.

Many people first encounter percentages during a retail transaction such as a percent discount (SALE! 25% off!!), or through sales tax ("Wait, I thought this was $\$1.99$?"), a report that something has increased by some percentage of the previous value (NOW! 20 more!!). These are examples of percent decreases and percent increases. In this section, we discuss the decrease, increase, and then the case of sales tax.

Calculating Discounts

Retailers frequently hold sales to help move merchandise. The sale price is almost always expressed as some amount off the original price. These are discounts, a reduction in the price of something. The price after the discount is sometimes referred to as the reduced price or the sale price.

When a reduction is a percent discount, it is an application of percent, which was introduced in Understanding Percent. The formula used was "part=percentage×total." In the formula, it would be helpful to remind that the present is expressed as a decimal. For 25% discount, use 0.25 as percent discount.In a discount application, the discount plays the role of the part, the percent discount is the percentage, and the original price plays the role of the total.

FORMULA: Discount, Mark Up, and Sale Tax

If the final value (or amount) is $p\%$ more than the initial

$$\text{Final} = (100+p)\% \times \text{Initial}$$
$$\text{Initial} = \frac{\text{final}}{(100+p)\%}$$

If the final value (or amount) is $p\%$ less than the initial

$$\text{Final} = (100 - p)\% \times \text{Initial}$$
$$\text{Initial} = \frac{\text{final}}{(100 - p)\%}$$

Note: p in the above formula could be the sales tax rate, markup percent, or discount percent.

If we have a **part** that is some percent of a **whole (Total)**, then

$$\text{percent} = \frac{\text{part}}{\text{total}}$$
$$\text{part} = \text{percent} \cdot \text{total}$$
$$\text{total} = \frac{\text{part}}{\text{percent}}$$

If we solve the sales tax and markup problem, the final price is always less than the initial price.

Note: When you solve a percent problem, you always change percent to decimal either by dividing by 100 or by moving the decimal two places to the left.

Example 4.2.1: Calculating Discount for a Percent Discount

Calculate the discount for the given price and discount percentage. Then calculate the sale price after calculating the discount.

original price = $\$75.80$; percent discount is 25%.

Answer available at the end of this section.

Your Turn 4.2.1: Find Sale Price

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Example 4.2.2: Calculating the Percent Discount from the Original and Sale Prices

Determine the percent discount based on the given original and sale prices.

original price = $\$1,200.00$; sale price = $\$900.00$

Answer available at the end of this section.

Your Turn 4.2.2: Find Percent Discount

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Example 4.2.3: Calculating the Original Price from the Percent Discount and Sale Price

Determine the original price based on the percent discount and sale price.

percent discount 10%; sale price = $\$450.00$

Answer available at the end of this section.

Your Turn 4.2.3: Find Original Price Before Discount

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WORK IT OUT

There are cases where retailers allow multiple discounts to be applied. However, it is rare that the discount percentages are added together. For example, if you have a 15% coupon and qualify for a 20% price reduction, the retailer typically does not add those two percentages together to determine the new price. The retailer instead applies one discount and then applies the second discount to the price obtained after the first discount was deducted.

Research the original prices of two different laptops offered by one retail outlet. Assume you will receive a student discount of 12% and your outlet of choice is having a 15% off sale on all laptops.

For each laptop:

1. List the original price and calculate the price after applying the student discount 12% only.
2. Then find the price after applying the sale discount (15% off) to the price found in Step 1.
3. Determine the total saved on the laptop and what percent discount the total savings represent.
4. Now, apply the discounts in reverse order (first the sale discount, then the student discount).
5. Note anything interesting about your findings.

Calculate Markups

When retailers purchase goods to sell, they pay a certain price, called the **cost**. The retailer then charges more than that amount for the goods. This increase is called the **markup**. This selling price, or **retail price**, is what the retailer charges the consumer to cover their own costs and make a profit. **Markup, then, is very similar to discount, except we add the markup rather than subtract the discount.**

Mark Up

Markup is also similar to a discount, but instead of subtracting the percentage, we add it. In markup, the final price is greater than the initial price.
The selling price is the retail price, and the cost is the original price before the markup is added. Sometimes it is also called wholesale price. The markup amount is the part that is added to the cost. When the markup is added to the cost, the result is the retail (selling) price.

Example 4.2.4: Determining the Retail Price Based on the Cost and the Percent Markup

For the given cost and markup percentage, calculate the retail price.

cost = $\$62.00$; percent markup is 15%.

Answer available at the end of this section.

Your Turn 4.2.4: Price after Mark Up

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Example 4.2.5: Calculating the Percent Markup from the Cost and Retail Price

Determine the percent markup based on the given cost and retail price. Round percentages to two decimal places.

cost = $\$5.20$; retail price = $\$9.90$

Answer available at the end of this section.

Your Turn 4.2.5: Mark Up Percentage

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Example 4.2.6: Calculating the Cost from the Percent Markup and Retail Price

Determine the cost based on the percent markup and retail price.

percent markup 225%; retail price =$\$26.55$

Answer available at the end of this section.

Your Turn 4.2.6: Price Before Mark Up

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Compute Sales Tax

Sales tax is applied to the sale or lease of some goods and services in the United States but is not determined by the federal government. It is most often set, collected, and spent by individual states, counties, parishes, and municipalities. None of these sales tax revenues go to the federal government.

For example, North Carolina has a state sales tax of 4.75% while New Mexico has a state sales tax of 5%. Additionally, many counties in North Carolina charge an additional 2% sales tax, bringing the total sales tax for most 72 of the 100 counties in North Carolinians to 6.75%. However, in Durham, the county sales tax is 2.25% plus an additional 0.5% tax used to fund public transportation, bringing Durham County's sales tax to 7%. To find the sales tax in a particular place, add other locality sales taxes to the base state sales tax rate.

How much we pay in sales tax depends on where we are and what we are buying.

To determine the amount of sales tax on a taxable purchase, we need to find the purchase price and add the sales tax for that locality

FORMULA: Sales Tax

The formula for sales tax is similar to markup because both involve adding a percentage to the original amount. In sales tax, the amount before tax is the total or whole (original price), and the tax is the part added to it. To find the final amount, you add the sales tax to the original price. The final amount is always greater than the initial amount because the tax increases the total cost.

Checkpoint

Round to the nearest penny or round to the nearest cent means round to two decimal places.

You should notice that this is the same as markup, except using a different term. Sales tax plays the role of markup, the purchase price plays the role of cost, and the tax rate plays the role of percent markup. This means all the strategies developed for markups apply to this situation, with the changes indicated.

Example 4.2.7: Calculating the Sales Tax from the Purchase Price

The sales tax in Kankakee, Illinois, is 8.25%. Find the sales tax and final price of items based on the purchase price listed.
Purchase price = $\$428.99$

Answer available at the end of this section.

Your Turn 4.2.7: Price After Sales Tax

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Be aware that almost all sales tax rates are structured as full percentages, half percentages, one-quarter percentages, or three-quarter percentages. This means the decimal value of the sales tax rate, written as a percent, will be either 0, as in 5.0%, 5 as in 7.5%, 25 as in 3.25%, or 75 as in 4.75%. When rounding for the sales tax percentage, be sure to use this guideline.

Example 4.2.9: Calculating the Purchase Price from the Sales Tax and Total Price

Determine the purchase price for the specified sales tax rate and total cost.

sales tax rate = 5.75%; price after tax = $\$36.56$

Answer available at the end of this section.

Your Turn 4.2.9: Price Before Tax

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Who Knew?

West Virginia was the first state to impose a sales tax. This happened on May 3, 1921.

Look up your locality on the website that lists standard state-level sales tax rates(opens in new window) and compare the sales tax structure in your state to two nearby states.

State Sales Tax Rates(opens in new window) [www.salestaxinstitute.com]

Answers

Example 4.2.1: Calculating Discount for a Percent Discount

Here discount is part of the original price (whole or total).

$$\begin{aligned}\text{discount (part)} &= \text{percent} \times \text{total} \\ &= 0.25 \times 75.80 \\ &= 18.95\end{aligned}$$

The sale price of the item is then

$$\text{Sale price (final price)} = \$75.80 - \$18.95 = \$56.85$$

The sale price is $\$56.85$.

Or you can find the sale price (final price) directly using the following formula. Note that the Final price is 25% less than the original price.

$$\begin{aligned}\text{Final} &= (100 - p)\% \times \text{Initial} \\ &= (100 - 25)\% \times 75.80 \\ &= 75\% \times 75.80 \\ &= 0.75 \times 75.80 \\ &= 56.85\end{aligned}$$

Example 4.2.2: Calculating the Percent Discount from the Original and Sale Prices

Find the discount, which is part of the $\$1,200.00$.

discount = original price – sale price, and it would be $\$300.00$.

To find the discount percent, substitute the part $\$300.00$ and the original price (total)$\$1,200.00$ into the formula

$$\begin{aligned}\text{percent} &= \frac{\text{part}}{\text{total}} \\ &= \frac{300}{1200} \\ &= 0.25 = 25\%\end{aligned}$$

25% discount of $\$1,200$ is $\$300$.

Example 4.2.3: Calculating the Original Price from the Percent Discount and Sale Price

Using the percent discount and the final price (sale price), we can find the original price (initial price) with the formula

$$\begin{aligned}\text{Initial (origianl price)} &= \frac{\text{Final (sale price)}}{\text{(100-p)\%}} \\ &= \frac{450}{(100 - 10)\%} \\ &= \frac{450}{90\%} \\ &= \frac{450}{0.90} \\ &= 500\end{aligned}$$

Original Price is $\$500$. And 10% discount of it is $\$50.00$ and you pay $\$450$.

Example 4.2.4: Determining the Retail Price Based on the Cost and the Percent Markup

Here, markup is part of the cost (whole or total).

$$\begin{aligned}\text{Mark up (part)} &= \text{percent} \times \text{total} \\ &= 0.15 \times 62 \\ &= 9.3\end{aligned}$$

The sale price of the item is then

$$\text{Sale price (final price)} = \$62 + \$9.30 = \$71.30$$

So retail price (after markup added) is $\$71.30$

Or you can find the retail price (final price) directly using the following formula. Note that the final price is 15%more than the Initial price.

$$\begin{aligned}\text{Final (retail)} &= \text{Initial} \times (100 + p)\% \\ &= 62 \times (100 + 15)\% \\ &= 62 \times (1.15) \\ &= 71.30\end{aligned}$$

So retail price of item is $\$71.30$.

Example 4.2.5: Calculating the Percent Markup from the Cost and Retail Price

Using the cost (total) and the retail price, we can find the markup is $\$9.90 - \$5.20$, which is $\$4.70$, which is part.
After substituting the part and whole into the formula,

$$\begin{aligned}\text{Percent} &= \frac{\text{part}}{\text{total (whole)}} \\ &= \frac{4.70}{5.20} \\ &= 0.9038 \\ &= 90.38\%\end{aligned}$$

Converting to percent form, the percent markup is 90.38%.
So the percent markup is 90.38%.

Example 4.2.6**: Calculating the Cost from the Percent Markup and Retail Price**

Note that the retail price is 225%; more than the initial price (cost).

$$\begin{aligned}\text{Initial (origianl price)} &= \frac{\text{Final (retail price)}}{(100+\text{p})\%} \\ &= \frac{26.55}{(100+225)\%} \\ &= \frac{26.55}{325\%} \\ &= \frac{26.55}{3.25} \\ &= 8.17\end{aligned}$$

So cost is $\$8.17$. And 225% markup in this price will be $\$26.55$.

Example 4.2.7**: Calculating the Sales Tax from the Purchase Price**

Here, sales tax is part of the original price (whole or total).

$$\begin{aligned}\text{Sale tax (part)} &= \text{percent} \times \text{total} \\ &= 0.0825 \times 428.99 \\ &= 35.39\end{aligned}$$

The final price of the item is then

$$\text{Final price} = \$428.99 + \$35.39 = \$464.38$$

The final price including tax is $\$464.38$.

Or you can find the final price directly using the following formula. Note that the final price is 25% less than the original price (purchased price).

$$\begin{aligned}\text{Fina price} &= \text{Initial price} \times (100+p)\% \\ &= 428.99 \times (100+8.25)\% \\ &= 428.99 \times (1.0825) \\ &= 464.38\end{aligned}$$

Example 4.2.9**: Calculating the Purchase Price from the Sales Tax and Total Price**

$$\begin{aligned}\text{Initial} &= \frac{\text{Final}}{(100+p)\%} \\ &= \frac{36.56}{(100+5.75)\%} \\ &= \frac{36.56}{105.75\%} \\ &= \frac{36.56}{1.0575} \\ &= 34.57\end{aligned}$$

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4.3: Simple and Compound Interest

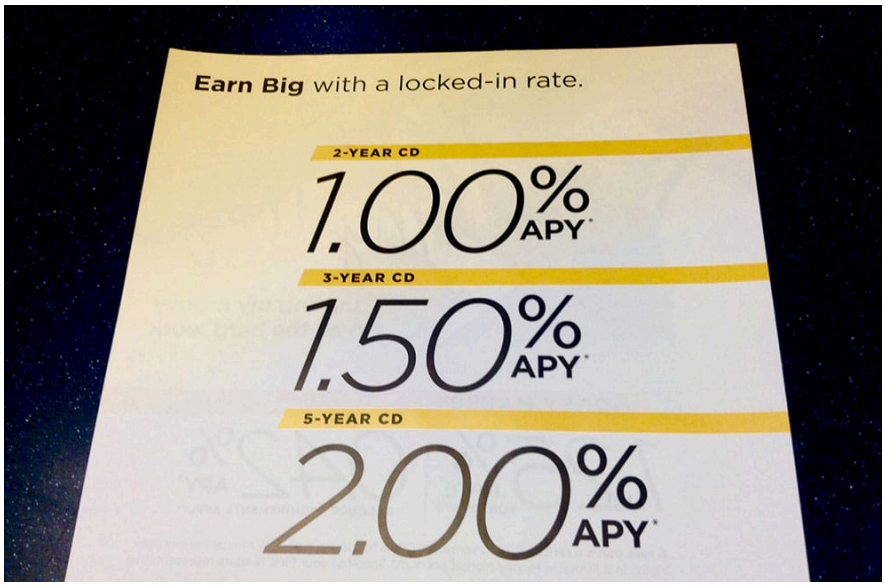


Figure 4.3.1: Interest is how savings earn money. (Credit: "Interest Rates" by Mike Mozart/Flickr, CC BY 2.0)

Learning Objectives

1. Compute simple interest.
2. Understand and compute future value.
3. Compute simple interest loans with partial payments.
4. Understand and compute present value.
5. Understand and compute future value.
6. Compute present value.
7. Compute and interpret effective annual yield.

There is truth in the phrase "You need to have money to make money." In essence, if you have money to lend, you can lend it at a cost to a borrower and make money on that transaction.

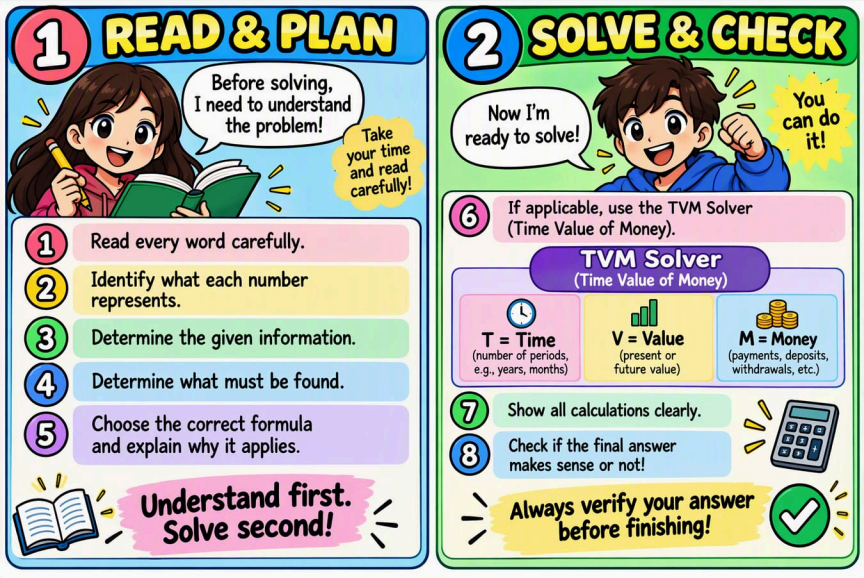
When money is borrowed, the person borrowing the money (the borrower) typically has to pay the person or entity that lent the money (the lender) more than the amount borrowed. This extra money is the interest that is to be paid. Interest is sometimes referred to as the cost of borrowing, the cost of the loan, or the finance charge.

This idea also applies when someone deposits money into a bank account or invests in another form of investment. That person is essentially lending the money to the bank or company. The money earned by the depositor is also referred to as interest. The interest is typically based on the amount borrowed, or the principal.

The pairing of borrower and lender can take various forms. The borrower may be a consumer using a credit card or taking out a loan from a bank, the lender. Companies also borrow from lending banks. Someone who invests in a company's stock is the lender in this case; the company is essentially the borrower.

In this section, we examine the basic building block of interest paid on loans and borrowed credit, as well as the returns on investments such as bank accounts and simple interest.

Problem Solving Process



Simple Interest and Future Value

Let's get some terminology understood. The interest to be paid by a borrower is often expressed as an annual percentage rate, which is the percentage of the principal paid as interest each year the money is borrowed. This means that the more that is borrowed, the more that must be paid back. Sometimes, the interest to be repaid is simple interest, meaning it is calculated only on the amount borrowed.

The length of time until the loan must be repaid is the **loan term**. The date when the loan must be paid off is the due **date**. The day that the loan is issued is the origination date. We'll put this terminology to use in the following examples. Note that in this section, we will use letters, called variables, to represent the different parts of the formulas we'll be using. This will help keep our formulas and calculations manageable.

Calculating simple interest is similar to the percent calculations we made in Understanding Percent and Discounts, Markups, and Sales Tax, but it must be multiplied by the loan term (in years when using an annual percentage rate).

FORMULA: Simple Interest

The simple interest, I, to be paid on a loan with annual interest rate r, with principal P, is found using

$$I = P \times r \times t$$

where the decimal form of the interest rate, r, is used and time, t, is expressed in years.

The amount that is loaned is called the present value, PV, or just P.

FORMULA: Future Value

The future value, FV, of an investment that yields simple interest is

$$FV = P + I$$
$$Or$$
$$FV = P(1 + rt)$$

Also, the interest that is earned by investing P can be calculated as

P is the principal (amount invested at the start), r is the annual interest rate in decimal form, and t is the length of time the money is invested (in years).

$$I = FV - P$$

Note: The future value of the amount is also denoted by just A.

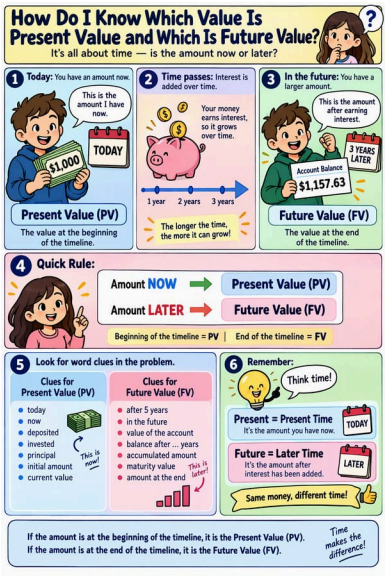


Figure 4.3.2: Future and Present Value

The Money can also be invested for a certain period, earning simple interest during that time. The terms and formulas we use are the same as before.
The investor wants to know how much the investment will be worth after interest is added. This is called the **future value**.

As we see above, **future value** is the amount of money you end up with after the principal (the amount originally invested) and all the interest earned during that time are added together.

Example 4.3.1: Simple Interest on Loans with Whole Number Year Terms

1. Calculate the simple interest to be paid on a loan with the given principal, annual percentage rate, and number of years. Then, calculate the loan payoff amount. (Round your answers to the nearest cent.): Principal P = $\$4,000,$ annual interest rate r = 5.5%, and number of years t = 4.
2. Riley runs an auto repair shop and needs to purchase a new brake lathe, which costs $\$11,995.$ She takes out a two-year, simple interest loan at an annual interest rate of $14.9\%.$ How much interest will she pay, and how much will she repay on the loan? (Round your answers to the nearest cent)

Answer 1 available at the end of this section.

Your Turn 4.3.1: Use Simple Interest

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Simple Interest Loans with Other Lengths of Terms

In the previous example and the Your Turn exercise, the loans were repaid in a single payment after an integer number of years. However, there are also loans lasting a length of time not equal to an integer number of years (like 1, 2, or 3 years or more), but in a number of months (like 4 months, 18 months, and so on). What model would apply to these situations?

When the loan is repaid after a term that is not a year but rather a number of months, days, or weeks, t must be expressed in years. Here are some examples.

1. For a 5-month loan, the time in years is $t = \frac{5}{12}$.
2. For a 18-month loan, the time in years is $t = \frac{18}{12}$.
3. For a 100-day term, the term in years is $t = \frac{100}{365}$.
4. For a 30-week loan, the time in years is $t = \frac{30}{52}$.

Caution

Do not forget to convert time to years before you put it in the formula.

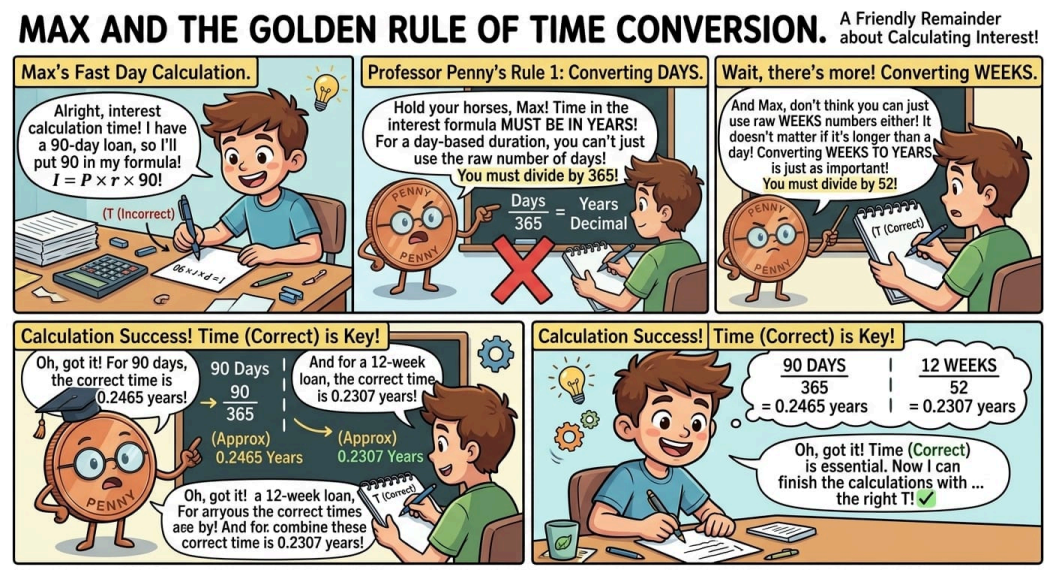


Figure 4.3.3: Do not forget to change time to years

Example 4.3.2: Loan to Purchase Equipment

1. Abeje needs a loan to purchase equipment for the gym she is going to open. She visits the bank and secures a 4-month loan of $\$20,000$. Her annual percentage rate is 6.75%. How much interest will Abeje pay, and what is her loan payoff amount?
2. David plans to move his family from Raleigh, North Carolina, to Tempe, Arizona. His company will reimburse (pay after the move) David for the move. David does research and determines that movers will cost $\$5,600$ to move his family's belongings to Tempe. He takes out a simple interest, 45-day loan at 11.75% interest to pay this cost. How much interest will be paid on this 45-day loan, and what is David's loan payoff amount? (Round your answers to the nearest cent)

Answer 1 available at the end of this section.

Your Turn 4.3.2: Use Simple Interest

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Who Knew?

It seems reasonable to use 365 as the number of days in the year since there are 365 days in most years. However, sometimes, banks have used (and continue to use) 360 as the number of days in a year. They may also treat all months as if they have 30 days. These differences lead to (sometimes small) differences in how much interest is paid. Since the number of days is in the denominator, a smaller denominator (360) will result in larger numbers (interest); therefore, 365 is used for the denominator. See this page from ACRE(opens in new window) for a comparison.

Example 4.3.3: Simple Interest on a Investment

In the following, determine how much interest was earned on the investment and the future value of the investment, if the investment yields simple interest. Round your answer to the nearest cent.

1. Principal is $\$1,000$, annual interest rate is 2.01%, and time is 5 years.
2. Principal is $\$10,000$, annual interest rate is 1.25%, and time is 18 months.
3. Principal is $\$7,000$, annual interest rate is 3.26%, and time is 100 days.

Answer 1 available at the end of this section.

Answer 2 available at the end of this section.

Answer 3 available at the end of this section.

Your Turn 4.3.3: Use Simple Interest for Investment

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Understand and Compute Present Value for Simple Interest Investments

When finding the future value of an investment, we know how much is deposited, but we have no idea how much that money will be worth in the future. If we set a future goal, it would be useful to know how much to deposit now for the account to reach it. The amount that needs to be deposited now to reach a goal in the future is called the **present value, PV, or simply P,** which is a notation used for a loan.

FORMULA: Present Value

The present value, PV of money deposited at an annual, simple interest rate of r (in decimal form) for time t (in years) with a specified future value of FV is calculated with the formula

$$\mathrm{PV} = \frac{FV}{(1+rt)}$$

Note: Present value, in this calculation, is always **rounded UP**. Otherwise, future value may fall short of the target future value. Sometimes, present value may be denoted by just P.

Understanding what this tells you is important. When you find the present value, that is how much you need to invest now to reach the goal, under the conditions (time and rate) at which the money will be invested.

Example 4.3.4: Find Present Value

Compute the present value of the investment described. Interpret the result.

Beatriz will invest some money in a CD that yields 3.99% simple interest when invested for 30 years. How much must Beatriz invest so that after those 30 years, her CD is worth $\$300,000$?

Answer available at the end of this section.

Your Turn 4.3.4: Find Present Value

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Determining Simple Interest Rate and Time

Using the same formula, we can also calculate the simple interest rate r and the time t.

Rate and Time Formula: Simple Interest

$$\text{Interest rate: } r = \frac{FV - P}{P \times t}$$

$$\text{Time: } t = \frac{FV - P}{P \times r}$$

Example 4.3.5: Find r and t

1. A simple interest loan for $\$20,000$ is taken at a 12.6% annual rate when the loan's future value is $\$25,000$.
2. A simple interest loan for $\$1500$ is taken, and you pay it back to double in 4 years. What simple interest rate do you pay?

Answer 1 available at the end of this section.

Your Turn 4.3.5: Find Simple Interest Rate

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Caution!

CAREFUL: When solving problems in the Simple Interest section, make sure to read each question carefully. Look for the word **"simple"** so you know to use the **simple interest formula**. Always ensure you understand **which formula** is required to solve the problem correctly.

Compound Interest

For a very long time in certain parts of the world, interest was not charged due to religious dictates. Once this restriction was relaxed, interest-bearing loans became possible. Initially, such loans had short terms, so only simple interest was applied. However, when loans began to stretch out for years, it was natural to add interest at the end of each year to the loan principal. After another year, the interest was calculated on the initial principal plus the interest from year 1, or the interest earned. Each year, more interest was added to the money owed, and that interest continued to earn interest.

Since the amount in the account grows each year, more money earns interest, increasing the account faster. This growth follows a geometric series (Geometric Sequences). It is this feature that gives compound interest its power. This module covers the mathematics of compound interest.

As we saw in Simple Interest, an account that pays simple interest only pays based on the original principal and the term of the loan. Accounts offering compound interest pay interest at regular intervals. After each interval, the interest is added to the original principal. Later, interest is calculated on the original principal plus the interest previously added.

After each period, the interest on the account is calculated and then added to the account balance. Then, after the next period,

interest is computed based on the original principal AND the interest added in previous periods. The period of time between two interest payments is called the compounding period.

What is Compound Interest?

Compound interest is interest calculated on both the **principal** (the original amount of money) and any interest that has already been earned. In other words, interest is added to the balance, allowing you to earn interest on previously earned interest.

With compound interest, your money grows faster because the interest earned is added to the balance. In each subsequent period, interest is calculated on the new, larger balance.

The comic in the following figure illustrates the concept of compound interest and helps you understand how it works.



Figure 4.3.4: How Compound Interest Work?

Compound Interest: Formula to Find the Future Value Using Compound Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

A: The balance (future value) in the account after t years.

P: The starting balance of the account (also called initial deposit, or principal)

r: The annual interest rate in decimal form.

n: The number of compounding periods in one year.

Additionally, we can calculate the compound interest earned by subtracting the principal from the future value.

$$I = A - P$$

Note: Sometimes, future value A can also be represented as FV, and P can also be represented as PV.

What is a Compounding Period?

In finance and mathematics, **a compounding period** is the time interval between interest calculations.

When you invest money or take out a loan, interest is usually not just added once at the end—it's added many times along the way. Each time interest is added (or "compounded"), that's a **compounding period**. We use n to denote the compounding period.



Figure 4.3.6: Compounding Periods

Example 4.3.6: Computing Future Value for Compound Interest

In the following, compute the future value of the investment with the given conditions.

Cody invests $\$18,500$, in an account that earns 6.25%, compounded quarterly, for 17 years. Determine the value of Cody's investment after 17 years. How much interest is included in Cody's investment? (Round your final answer to the nearest cent.)

Answer available at the end of this section.

Caution!

CAREFUL: When solving problems in the **Compound Interest** section, make sure to read each question carefully. Look for the word **"compounded,"** so you know to use the *compound interest formula*. Always be sure you understand **which formula** is required to solve the problem correctly.

If you are using the **TVM Solver**:

- Do **not** use commas and dollar signs in the input boxes.
- Do **not** convert the interest rate to a decimal.
- If your answer comes out **negative**, enter the final answer as a **positive** value.
- Choose the value of n compounding periods based on the words monthly, quarterly, etc. (See Figure 4.3.6)

- Remember that $N = n \times t$.

What is a Certificate of Deposit (CD)

A **Certificate of Deposit (CD)** is a type of savings account offered by banks and credit unions that holds a fixed amount of money for a fixed period of time (such as 6 months, 1 year, or 5 years).

In exchange for keeping your money locked away for the agreed term, the financial institution pays you a fixed interest rate—typically higher than what standard savings accounts offer.

Almost all major traditional banks, online banks, credit unions, and brokerage firms offer Certificates of Deposit (CDs). See the figure below for a list of some financial institutions that offer CDs.



Figure $4.3.6$: Copy and Paste Caption here. (Copyright; author via source)

Example $4.3.7$: Computing Future Value for Compound Interest

In the following, compute the future value of the investment with the given conditions.

You invest $\$5,000,$ into a CD at Allay Bank, which gives you a fixed annual interest rate of 3.80%, compounded monthly, for 5 years. How much will you have in the account in 5 years? How much interest is included in the future value? (Round your final answer to the nearest cent)

Answer available at the end of this section.

Calculators

When you do not use a Formula	When you use a Formula
TVM SOLVER	**DESMOS SCIENTIFIC**

Your Turn 4.3.7: Future Value and Interest from Compound Interest

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth see figure 4.3.7.

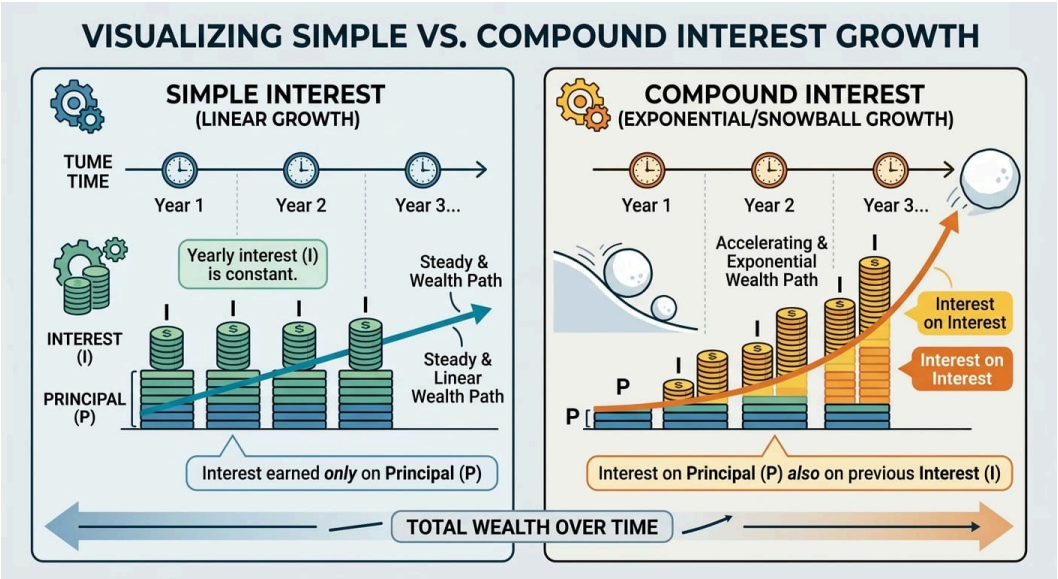


Figure 4.3.7: Visualizing Simple and Compound Interest

Understand and Compute Present Value from Compound Interest

When investing, there is often a goal to reach, such as "after 20 years, I'd like the account to be worth $\$100,000$." The question to be answered in this case is "How much money must be invested now to reach the goal?" As with simple interest, this is referred to as the present value.

FORMULA: Compute Present Value from Compound Interest

The money invested in an account bearing an annual interest rate of r (in decimal form), compounded n times per year for t years, is called the present value, PV (or just P), of the account (or of the money) and found using the formula

$$P = \frac{A}{(1+\frac{r}{n})^{nt}} \tag{4.3.1}$$

where A is the value of the account at the investment's end. Always round this value **UP** to the nearest cent unless mentioned otherwise.

Example 4.3.8: Computing Present Value

Pilar plans for retirement early, believing she will need $1,500,000$ to live comfortably after age 67. How much will she need to deposit at age 23 in an account bearing 6.35% annual interest compounded monthly? How much of $\$1,500,000$ comes from interest?

Answer available at the end of this section.

Your Turn 4.3.8: Find the Present Value

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Compute and Interpret Effective Annual Yield

As we've seen, quarterly compounding pays interest 4 times a year or every 3 months; monthly compounding pays 12 times a year; daily compounding pays interest every day, and so on. Effective annual yield allows direct comparisons between simple interest and compound interest by converting compound interest to its equivalent simple interest rate. We can even directly compare different compound interest situations. This provides information to identify the best investment from a yield perspective.

Using a formula, we can express compound interest in terms of simple interest. The effective annual yield formula stems from the compound interest formula and is based on an investment of $\$1$ for one year.

Formula: Effective Annual Yield

$$Y = \left(1+\frac{r}{n}\right)^{n} - 1$$

where Y = effective annual yield, r = interest rate in decimal form, and n= number of times the interest is compounded in a year.

Y is interpreted as the equivalent annual simple interest rate.

Note: Effective annual yield is also called Annual Percentage Yield (APY).

Example 4.3.9: Determine and Interpret Effective Annual Yield for 6% Compounded Quarterly

Suppose you have an investment paying a rate of 6% compounded quarterly. Determine and interpret the effective annual yield of the investment.

Answer available at the end of this section.

Your Turn 4.3.9: Effective Annual Yield

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Answers

Example 4.3.1: Simple Interest on Loans with Whole Number Year Terms

Substitute the principal P = $\$4,000$, the decimal form of the annual interest rate r = 0.055, and number of years $t = 4$ into the formula for simple interest, and calculate.

$$I = 4,000 \times 0.055 \times 4 = 880$$

The simple interest, or cost of the loan, to be paid on the loan is $\$880$, which is I. The loan payoff amount, or the total to be repaid, is

$$A = P + I = \$4,000 + 880 = \$4,880$$

So $\$4,880.00$ is the loan payoff amount.

Determine the variables or parts of the formula. The principal P = $\$11,995$. The interest rate Riley pays is 14.9%, or $r = 0.149$ in decimal form. The length of the loan is two years, so $t = 2$. We are first asked to find I, the interest Riley will pay. Substitute the known variables into the formula for simple interest $I = P \times r \times t$ and solve for I.

$$I = 11,995 \times 0.149 \times 2 = 3,574.51$$

This tells us that the simple interest, or cost to borrow, to be paid on the loan is $\$3,574.51$.

Use the formula $A = P + I$ to determine the total amount Riley will repay.

$$A = P + I = \$11,995 + \$3,574.51 = \$15,569.51$$

The total to be repaid is $\$15,569.51$.

Example 4.3.2: Loan to Purchase Equipment

Abeje's loan is for $\$20,000$, so her principal is $P = \$20,000$. The interest rate Abeje will pay is 6.75%, or $r = 0.0675$ in decimal form. The length of the loan is 4 months, so $t = \frac{4}{12}$. Substituting these in the formula for simple interest, we find her interest to be

$$I = 20,000 \times 0.0675 \times \frac{4}{12} = 450$$

The simple interest, or cost to borrow, to be paid on the loan is $\$450.00$. The payoff is $\$20,000 + \$450 = \$20,450$.

The principal for the loan is the moving cost, or $P = \$5,600$. The annual interest rate that David will pay is 11.75%, which in decimal is $r = 0.1175$. The length of time for the loan is 45 days, so $t = \frac{45}{365}$ year. Substituting these values into the formula and calculating, we find that the interest to be paid is

$$I = 5600 \times 0.1175 \times \frac{45}{365} = 81.12$$

$\$81.12$ is the interest for 45 days.

For payoff

$$A = P + I = \$5,600 + \$81.12 = \$5,681.12$$

The payoff for the loan is $\$5,681.12$.

Example 4.3.3: Simple Interest on a Investment

The principal is $P = \$1,000$, the annual interest rate, in decimal form, is $r = 0.0201$, and the term is 5 years, or $t = 5$.

$$FV = P(1 + rt) = 1000(1 + 0.0201 \times 5) = 1100.50$$

The interest earned on the investment is

$$I = FV - P = 1100.50 - 1000 = 100.50$$

The future value of the investment at the end of 5 years is $\$1,100.50$, where $\$100.50$ comes from interest and $\$1000$ comes from principal..

Example $4.3.3$**: Simple Interest on a Investment**

The principal is $P = \$10,000$, the annual interest rate, in decimal form, is 0.0125, and the term is 18 months. Since the term is in months, we have to write the months in terms of years. For 18 months, we have used $\frac{18}{12}$ as t.

$$\begin{aligned} FV &= P(1+rt) \\ &= 10,000\left(1+0.0125\times\frac{18}{12}\right) \\ &= 10,187.50 \end{aligned}$$

The interest earned on the investment is

$$I = FV - P = 10,187.50 - 10,000 = 187.50$$

The future value of the investment at the end of 18 months is $\$10,187.50$, where $\$187.50$ comes from interest and $\$10,000$ comes from principal..

Example $4.3.3$**: Simple Interest on a Investment**

The principal is $P = \$7,000$, the annual interest rate, in decimal form, is $r = 0.0326$, and the term is 100 days. Since the term is in days, we have to write the time in years, or $t = \frac{100}{365}$.

$$\begin{aligned} FV &= P(1+rt) \\ &= 7,000\left(1+0.0326\times\frac{100}{365}\right) \\ &= 7,062.52 \end{aligned}$$

The interest earned on the investment is

$$I = FV - P = \$7,062.52 - \$7,000 = \$62.52$$

The future value of the investment at the end of 100 days is $\$7,062.52$, where $\$62.52$ comes from interest and $\$7,000$ comes from principal.

Example $4.3.4$**: Find Present Value**

Beatriz needs to know how much to deposit now so that her CD is worth $\$300,000$ after 30 years. This means she needs to know the present value of that $\$300,000$. The time is 30 years and the annual simple interest rate, in decimal form, is $r = 0.0399$. Using that information and the formula for present value, we calculate the present value of that $\$300,000$.

$$\begin{aligned} PV &= \frac{FV}{(1+rt)} \\ &= \frac{300,000}{(1+0.0399\times 30)} \\ &= \frac{300,000}{2.197} \\ &= 136,459.8407 \end{aligned}$$

Rounding up, Beatriz needs to invest $\$136,549.85$ so that she has $\$300,000$ in 30 years.

Example $4.3.5$**: Find** r **and** t

Here we have, $PV = \$20,000,\ \ FV = \$25,000$, $r = 0.126$, and t = Unknown

Use the formula for t

$$\begin{aligned} \text{Time: } t &= \frac{FV-P}{P\times r} \\ &= \frac{25,000-20,000}{20,000\times 0.126} \\ &= \frac{5,000}{2520} \\ &= 1.98 \end{aligned}$$

So A simple interest loan for $\$20,000$ will be $\$25,000$ in almost 2 years with a 12.6% interest rate.

Here we have, $PV = \$1500$, $FV = \$3000$, $t = 4$, and r = Unknown

Use the formula for r

$$\begin{aligned} \text{Time: } r &= \frac{FV-P}{P\times t} \\ &= \frac{3,000-1,500}{1,500\times 4} \\ &= \frac{1,500}{6000} \\ &= 0.25 \end{aligned}$$

So $\$1500$ loan will be doubled in 4 year with 25% simple interest rate.

Example $4.3.6$**: Computing Future Value for Compound Interest**

The principal is $P = \$18,500$, interest rate, in decimal form, $r = 0.0625$, compounded quarterly so $n = 4$, and for $t = 17$ years. Substituting these values into the formula, we

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 18,500\left(1+\frac{0.0625}{4}\right)^{4\times 17} \\ &= 18,500\left(1+0.015625\right)^{68} \\ &= 18,500\left(1.015625\right)^{68} \\ &= 18,500\times 2.86992152 \\ &= 53,093.54812 \end{aligned}$$

The future value of the investment is $\$53093.55$.

Interest earned

$$I = A - P = \$53,093.55 - \$18,500 = \$34,593.55 \tag{4.3.2}$$

You can also find future value using the following TVM (Time Value of Money) Calculator. **But you should learn how to do it manually using a formula.** For example, input the following in the TVM solver.

N: The number of payments = P/Y $\times$ # of years = $4 \times 17 = 68$

I%: The interest rate, as a percentage. Put I =6.25

PV: Present value. Put PV = $18,500$

PMT: Periodic payment (We don't have this information, so we entered **0**.)

FV: Future value (We need to solve for it, so click it after you put all the values)

P/Y: Payments per year. Put P/Y = 4 (quarterly)

C/Y: Times per year that interest is compounded. Put C/Y = 4 (quarterly

Example $4.3.7$**: Computing Future Value for Compound Interest**

The principal is $P = \$5,000,$ interest rate, in decimal form, $0.038,$ compounded monthly so =$12,$ and for =5 years. Substituting these values into the formula, we find

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 5000\left(1+\frac{0.038}{12}\right)^{12\times 5} \\ &= 5000\left(1+0.00316666\right)^{60} \\ &= 5000\left(1.00316666\right)^{60} \\ &= 5000 \times 1.20888615 \\ &= 6044.4307 \end{aligned}$$

The future value of the investment is $\$6,044.43.$

Interest earned

$$I = A - P = \$6,044.43 - \$5,000 = \$1,044.43 \tag{4.3.3}$$

You can also find future value using the following TVM (Time Value of Money) Calculator. **But you should learn how to do it manually using a formula. Input the following in TVM solver.**

N: The number of payments = P/Y $\times$ # of years = $5 \times 12 = 60$

I%: The interest rate, as a percentage. Put I =3.80

PV: Present value. Put PV = $5,000$

PMT: Periodic payment (We don't have this information, so we entered **0**.)

FV: Future value (We need to solve for it, so click it after you put all the values)

P/Y: Payments per year. Put P/Y = 12 (quarterly)

C/Y: Times per year that interest is compounded. Put C/Y = 12 (Monthly)

Example $4.3.8$**: Computing Present Value**

She's 23 now and will leave the money in the account until the age of 67. She leaves the money for $67 - 23 = 44$ years, making $t = 44$. $r = 6.35\% = 0.0635$, and $n = 12$ because it is compounded monthly. Using this information, substitute into the formula for present value given above.

$$\begin{aligned} P &= \frac{A}{\left(1+\frac{r}{n}\right)^{nt}} \\ &= \frac{1,150,000}{\left(1+\frac{0.0635}{12}\right)^{12\times 44}} \\ &= \frac{1,150,000}{\left(1.005291667\right)^{528}} \\ &= 70,872.5737 \end{aligned}$$

Rounding up to the nearest cent, we get $P = \$70,872.58.$

So, for this account to reach $\$1,150,000$ after 44 years, $\$70,872.58$ needs to be invested at the given interest rate, compounded monthly. Interest earned during that period is:

$$I = A - P = \$1,150,000 - \$70,872.58 = \$1,079,127.42$$

Also, you can find the present value using the TVM solver

N: The number of payments = $12 \times 44 = 528$

I%: The interest rate, as a percentage. Put I = 6.35

PV: Present value. We need to solve it, so click it after you put all the values in.

PMT: Periodic payment (We don't have this information, so we have entered zero for it.)

FV: Future value. Put FV =$1,500,000$

P/Y: Payments per year. Put P/Y = 12 (monthly)

C/Y: Times per year that interest is compounded. Put C/Y = 12 (monthly)

Example $4.3.9$**: Determine and Interpret Effective Annual Yield for 6% Compounded Quarterly**

Here, $n = 4$ (quarterly) and $r = 0.06$ (decimal form). Substituting into the formula, we find that the effective annual yield is

$$\begin{aligned} Y &= \left(1+\frac{r}{n}\right)^n - 1 \\ &= \left(1+\frac{0.06}{4}\right)^n - 1 \\ &= (1.015)^4 - 1 \\ &= 0.06136 \end{aligned}$$

Therefore, a rate of 6% compounded quarterly is equivalent to a simple interest rate of 6.14%.

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4.4: Methods of Savings (Annuities, Stock, and Bonds)



Figure 4.4.1 : Money wisely invested grows over time. (credit: “Stack of Cash” by Janak Raja/Flickr, Public Domain Mark 1.0)

Learning Objectives

1. Distinguish various basic forms of savings plans.
2. Compute return on investment for basic forms of savings plans.
3. Compute the payment to reach a financial goal.
4. Distinguish between basic forms of investments, including stocks, bonds, and mutual funds.
5. Read and derive information from a stock table
6. Compute return on investment for basic forms of investments.
7. Identify and distinguish between retirement savings accounts.

The stock market crash of 1929 led to the Great Depression, a decade-long global downturn in productivity and employment. A state of shock swept through the United States; the damage to people’s lives was immeasurable. Americans no longer trusted established financial institutions. By October 1931, the banking industry’s biggest challenge was restoring confidence to the American public. In the next 10 years, the federal government would impose strict regulations and guidelines on the financial industry. The Emergency Banking Act of 1933 created the Federal Deposit Insurance Corporation (FDIC), which insures bank deposits. The new federal guidelines helped ease suspicions among the general public about the banking industry. Gradually, things returned to normal, and today we have more investment instruments, many insured through the FDIC, than ever before.

In this section, we will first look at the different types of savings accounts and proceed to discuss the various types of investments. There is some overlap, but we will try to differentiate among these financial instruments. Saving money should be a goal of every adult, but it can also be a difficult goal to attain.

There are at least three types of savings accounts. Traditional savings accounts, certificates of deposit (CDs), and money market accounts are the primary options for saving.

Savings Account

There are at least three types of savings accounts. Traditional savings accounts, certificates of deposit (CDs), and money market accounts are the primary options for saving.

- Traditional savings accounts: Allow you to deposit money, earn interest, and generally withdraw money when needed.

- Certificates of deposit (CDs): Typically pay interest in exchange for keeping your money deposited for a specified period.
- Money market accounts: Typically earn interest and may offer check-writing or debit card access, depending on the bank.

A savings account is probably the most well-known type of investment, and for many people, it is their first experience with a bank. A savings account is a deposit account held at a bank or other financial institution that bears interest on the deposited money. Savings accounts are intended as a place to save money for emergencies or to achieve short-term goals. They typically pay a low interest rate, but there is virtually no risk involved, and they are insured by the FDIC for up to $\$250,000$

Savings accounts have some strengths. They are highly flexible. Generally, there are no limitations on the number of withdrawals allowed and no limit on how much you can deposit. It is not unusual, however, for a savings account to have a minimum balance in order for the bank to cover maintenance costs. If your account balance drops below the minimum, fees may be applied.

Who Knew?

Many banks are covered by FDIC insurance. The FDIC, or Federal Deposit Insurance Corporation, is an independent agency created by the U.S. Congress. One of its purposes is to provide insurance for deposits in banks, including savings accounts and other deposit types. Be aware, not all banks are FDIC insured. The FDIC insures up to $\$250,000$ for a savings account, so you do not want your balance to exceed that federally insured limit.

Having your savings account at the same bank as your checking account does offer a real advantage. For example, if your checking account is approaching its lower limit, you can transfer funds from your savings account and avoid any bank fees. Similarly, if you have an excess of funds in your checking account, you can transfer funds to your savings account and earn some interest. Checking accounts rarely pay interest.

50-30-20 Rule

The $50-30-20$ **rule** is a **simple personal budgeting method** designed to help individuals manage their finances by dividing **after-tax income** into three main spending categories:

50% — **Needs** (**Essential expenses)**

- Utilities (electricity, water, gas)
- Groceries
- Transportation (car payment, gas, public transit)
- Insurance (health, auto)
- Minimum loan payments

30% — **Wants (Non-essential expenses)**

- Entertainment (movies, streaming, events)
- Shopping (clothes, gadgets)
- Vacations
- Hobbies and subscriptions

20% — **Savings & Debt Repayment**

- Emergency fund
- Retirement accounts (IRA, 401(k))
- Investments
- Extra payments toward credit cards or loans
- Saving for a home, education, etc

Look at an example

David gathers his pay stubs and bills from the past 6 months. His income, after taxes, is $\$3,450$ per month. His rent, utilities included, is $\$925$. His car payments are $\$178.54$ per month, his car insurance is $\$129.49$ per month, his credit cards cost him $\$117.00$ per month, he spends $\$195$ per month on gas, and his food costs are $\$290$ per month. He also spends $\$21.99$ on Amazon Prime, $\$49.99$ on his internet bill, and $\$400$ per month going out. Create David's monthly budget, including totals, based on that information.

1. Using David's Budget, how much income does he have per month after accounting for his expenses?
2. Apply the $50-30-20$ budget philosophy to David's budget.
3. Evaluate David's budget with respect to the $50-30-20$ budget philosophy.

50-30-20 Rule

Amount ($)	Expense	
3450.00	Rent, with utilities	
	Car payments	
	Car Insurance	
	Credit card debt	
	Gas	
	Food	
	Amazon prime	
	Internet	
	Going out	
3450.00	Total	

David saved in one month

$$\$3450.00–\$2307.01 = \$1142.99$$

Necessities 50% :

$$0.50 \times \$3450 = \$1725$$

Wants 30% :

$$0.30 \times \$3450 = \$1035$$

Savings and Debt Repyment 20% :

$$0.20 \times \$3450 = \$690$$

Necessities: $\$925 + \$178.54 + \$129.49 + \$117 + \$195 + \$290 = \$1835.03,$ This is very close to the 50%.

Wants: $\$21.99 + \$49.99 + \$400 = \$471.98,$ This is much lower than the 30%.

Saving and Debt Repayment: $\$1142.99,$ This is much greater than the 20%.

People in Mathematics: J.P. Morgan

J.P. Morgan was a wealthy banker around the turn of the 20 th century. His business interests included railroads and the steel industry. However, it was in 1907 that a financial crisis, caused by poor banking decisions and followed by such great distrust in the banking system that a frenzy of withdrawals from banks occurred, that J.P. Morgan and other wealthy bankers lent from their own funds to help stabilize and save the system.

There are some weaknesses to savings accounts. Primarily, it is because savings accounts earn very low interest rates. This means they are not the best way to grow your money. Experts, though, recommend keeping a savings account balance to cover 3 to 6 months of living expenses in case you should lose your job, have a sudden medical expense, or other emergency.

What is 1099 form?

Around tax time, you will receive a 1099-INT form stating the amount of interest earned on your savings, which is the amount that must be reported when you file your tax return. A 1099 form is a tax form that reports earnings that do not come from your employer, including interest earned on savings accounts. These 1099 forms have the suffix INT to indicate that the income is interest income.

Savings accounts earn interest, and those earnings can be found using the interest formulas from previous sections. The final value of these accounts is sometimes referred to as the future value of the account.

Who Knew? Does the bank offer interest on the Saving Account?

Banks have not always offered interest on savings accounts. An 1836 publication from Indiana noted that banks in other states allow small interest on deposits. It specifically says that in these other states, these deposits are what business transactions are based upon. And that giving interest would encourage deposits, and thus increase the business that banks can do.

Journal of the House of Representatives of the State of Indiana(opens in new window)

Certificates of Deposit, or CDs, and Money Market Accounts

We discussed **certificates of deposit** (CDs) in earlier sections. CDs differ from savings accounts in a few ways. First, the investment lasts for a fixed period of time, agreed to when the money is invested in the CD. These time periods often range from 6 months to 5 years. Money from the CD cannot be withdrawn (without penalty) until the end of the investment period. Also, money cannot be added to an existing CD.

Certificates of deposit share features similar to those of savings accounts. They are insured by the FDIC. They are entirely safe. They do, though, offer a better interest rate. The trade-off is that once the money is invested in a CD, that money is unavailable

until the investment period ends.

A money market account is similar to a savings account, except the number of transactions (withdrawals and transfers) is generally limited to six each month. Money market accounts typically require a minimum balance to be maintained. If the balance in the account falls below the minimum, a penalty may be incurred. Money market accounts offer the flexibility of checks and ATM cards. Ultimately, the interest rate on a money market account is generally higher than that of a savings account.

FORMULA: Return on Investment

The return on investment, often denoted ROI, is the percent difference between the initial investment, P, and the final value of the investment, FV.

$$\text{ROI} = \frac{FV - P}{P} \times 100\%$$

Note that *the length of time of the investment is not considered in ROI.*

Example 4.4.1: 5-Year CD ROI

Silvio deposits $\$10,000$ in a CD that yields 2.17% compounded semiannually for 5 years. How much is the CD worth after 5 years? How much did she earn in interest?

Answer available at the end of this section.

Your Turn 4.4.1: Find the Future Value for Annuity

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Annuities as Savings

In Compound Interest, we talked about the future value of a single deposit. In reality, people often open accounts that allow them to add deposits or payments to the account at regular intervals. When a deposit is made at the end of each compounding period, such a savings account is called an ordinary annuity. The $50 - 30 - 20$ is a simple budgeting guideline that helps individuals manage their money effectively. It divides your **after-tax income** into three broad categories:

Formula: Future Value of Annuity and Interest Earned

$$FV = PMT\left[\frac{(1+\frac{r}{n})^{nt}-1}{(\frac{r}{n})}\right]$$

where

FV: The future value of the annuity.

PMT: The payment or periodic deposit.

r: The annual interest rate (in decimal form).

n: The number of compounding periods per year.

t: The number of years.

$$\text{Total deposit} = PMT \times n \times t$$
$$\text{Interest earned from annuity} = FV - \text{Total deposit}$$

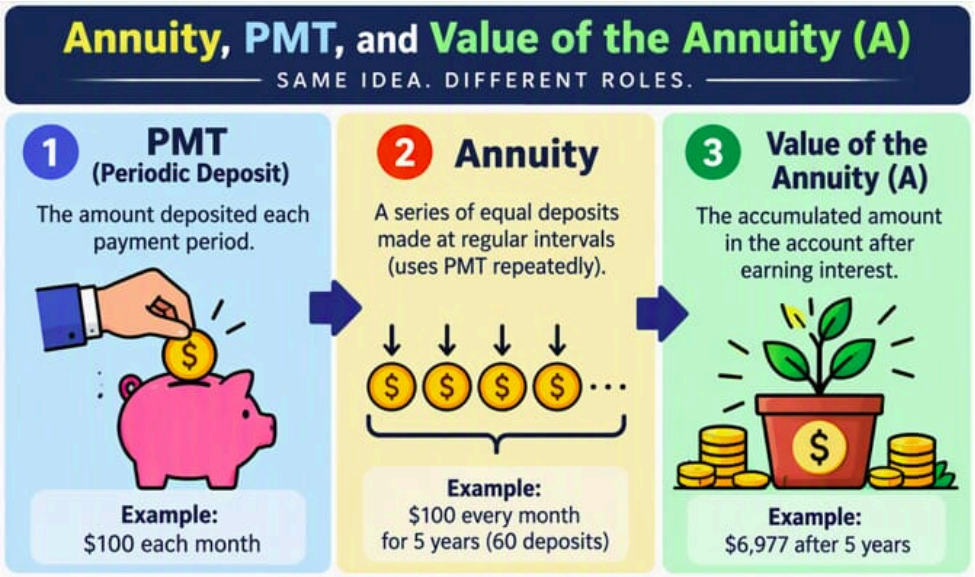


Figure 4.4.1: Annuity and Value of Annuity

Checkpoint: Another form of Annuity Due

Another form of annuity is the annuity due, which has deposits at the start of each compounding period. This other annuity type has different formulas and is not addressed in this text.

Example 4.4.2: Future Value of an Ordinary Annuity

Jill has an account that bears 5% interest compounded quarterly. She decides to deposit $\$500$ quarterly into this account.

1. What is the future value of this account after 35 years?
2. How much money will Jill have put into the account in 35 years?
3. How much interest will Jill have earned?

Answer 1 available at the end of this section.

Your Turn $4.4.2$: Find the Future Value from Annuity

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Who Knew?: Setting Savings Account Interest Rates

There are a number of factors that contribute to the interest rates a bank gives for savings accounts. The interest rate reflects how much the bank values deposits. It also reflects the money that the bank will earn when it lends out money. Finally, interest rates are impacted by the Federal Reserve Bank. When the Fed raises interest rates, so do banks.

People in Mathematics: The Federal Reserve Chairperson

The Federal Reserve Board monitors the risks in the financial system to help ensure a healthy economy for individuals, companies, and communities. The Board oversees the 12 regional reserve banks. The Chairperson of the Federal Reserve Board testifies to Congress twice per year, meets with the Secretary of the Treasury, chairs the Federal Open Market Committee, and is the face of federal monetary policy. Mr. Jerome Powell is the current Chairman of the Federal Reserve Board. Jerome Powell(opens in new window) was sworn in as chairman on February $5, 2018$. He had been first nominated to the position by President Donald Trump(opens in new window) on November $2, 2017,$ and confirmed by the Senate. He was nominated to a second term by President Joe Biden(opens in new window), confirmed by the Senate, and sworn in on May $23, 2022$.

The formula used to calculate the future value of an ordinary annuity is useful for determining the final amount in the account. However, that isn't how planning works. To plan, we need to determine how much to invest in the ordinary annuity each compounding period in order to reach our goal. Fortunately, that formula exists.

Periodic Payment or Periodic Deposits (Or Annuity)

The formula for the amount that needs to be deposited per period.

$$PMT = FV\left[\frac{\left(\frac{r}{n}\right)}{(1+\frac{r}{n})^{nt}-1}\right]$$

where

FV: The future value of the annuity,

PMT: The periodic payment or deposits

r: The annual interest rate (in decimal form)

n: The number of compounding periods per year

t: The number of years

Example 4.4.3: Saving for a Car

Yaroslava wants to save in order to buy a car in 3 years, without taking out a loan. She determines that she'll need $\$35,500$ for the purchase. If she deposits money into an ordinary annuity that yields 4.25% interest compounded monthly.

1. How much will she need to deposit each month?
2. How much money will she have put into the account?
3. How much interest will she have earned?

Answer 1 available at the end of this section.

Your Turn4.4.3: Find Periodic Deposit

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Bonds

You can save your money in a safe or a vault (or worse, under the mattress!), but that money does not grow. It would be hard to save enough for retirement that way. What can be done to increase the value of the money you already have?

The answer is to invest it. Use the money that you have to earn more money back. For instance, as we saw in Methods of Savings, you can save it in a bank. Or, to reach loftier goals, invest in something more likely to grow, such as stocks.

A great example of this is Apple stock. Anyone who bought stock in Apple in 1997 and held onto the shares earned a lot of money. To be more specific, $\$100$ worth of Apple shares bought in 1980, when it was first sold to the public, was valued at $\$67,564$ in 2019, or 676 times more! Perhaps you have heard a story like that, of an investment opportunity taken that paid off, or the story of an investment opportunity missed. But such stories are the exceptions.

In this section, we'll investigate bonds, stocks, and mutual funds and their comparative strengths and weaknesses. We close the section with a discussion of retirement savings accounts.

Bonds, stocks, and mutual funds tend to offer higher returns, but to varying degrees, come with higher risks. Stocks and mutual funds also vary in the amount of earnings they generate. Their predicted rates of return on investment are not guaranteed, but are educated guesses based on market trends and historical performance.

Bonds are issued by big companies and by governments. Selling bonds is an alternative to an institution borrowing from a bank. Funds from bond sales are often used for large projects, such as constructing a new highway or hospital.

Bonds are considered a conservative investment. They are bought for what is known as the issue price. The interest is fixed (does not change) at the time of purchase and is based on the issue price of the bond. The interest rate is often referred to as the coupon rate; the interest paid is often called the coupon yield. The interest paid is often higher than that of savings accounts, and the risk is exceptionally low. The bond is for a fixed term. The end of this time is the maturity date of the bond.

A **bond** is like a loan you give to a company or government. The **face value** (or par value) is the amount of money the bond will pay you back when it matures, usually $\$1,000$. The **coupon rate** is the interest rate the bond pays based on its face value. For example, if a bond has a face value of $\$1,000$ and a 5% coupon rate, it will pay $\$50$ in interest each year. The coupon rate tells you how much income the bond will generate, while the face value tells you how much you will receive at the end of the bond's

term.

There are several types of bonds and their return level

- **Treasury bonds** are issued by the federal government: safer, lower returns
- **Municipal bonds** are issued by state and local governments: tax-free, moderate returns
- **Corporate bonds** are issued by major corporations: riskier, higher returns

The **maturity date of a bond** is the **specific date in the future when the bond issuer must repay the full face value (principal) to the investor**

Short-term: 1–3 years

Medium-term: 4–10 years

Long-term: 10–30 years or more

If you buy a 10-year bond on January $1, 2025$. The **maturity date** is **January** $1, 2035$. You will receive coupon payments each year until 2035. On Jan $1, 2035$, you get the **full** $\$1,000$ **face value back**

Who Knew?: Trading Bonds

Bonds are often part of larger investment portfolios. These bonds may be traded. However, the interest paid is based on the price when the bond was bought (the issue price). These bonds can be bought and sold for more or less money than the issue price. If the bond is bought for more than the issue price, the interest is still paid on the issue price, not on the purchase price at the time of the trade. This means the actual return on the bond decreases. If the bond is purchased at a price below the issue price, the bond's return increases.

FORMULA: Annual Return (AR)

As you see in the ROI formula above, it does not account for the duration of the investment. A good way to do that is to equate the ROI to an account bearing interest that is compounded annually. The annual return is the **average annual rate**, or the annual percentage yield (APY) that would result in the same amount **if** the interest **were** paid once a year.

The formula for the annual return of the investment

$$\text{AR} = \left[\left(\frac{FV}{P}\right)^{\frac{1}{t}} - 1\right] \times 100\%$$

where

t = the number of years

FV = new value

P = starting principal

AR = Annual Return

Also, if you know the annual return (AR), we can find the value of investment (FV) after t years by the following formula

$$\text{Future Value: FV} = P\,(1 + AR)^t$$

Example 4.4.4: Bond Investment

Muriel purchases a $\$3,000$ bond with a maturity of 4 years at a fixed coupon rate of 5.5% paid annually.

1. How much is Muriel paid each year, and what is the total amount earned with the bond?
2. What is Mauriel's return on investment?
3. What was Muriel's annual return on investment? Interpret this as compound interest.

Answer 1 available at the end of this section.

Your Turn 4.4.4: Return on Bond

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Stocks

Stocks are part ownership in a company. They come in units called shares. The performance and earnings of stocks is not guaranteed, which makes them riskier than any other investment discussed earlier. However, they can offer a higher return on investment than other investments. Their value grows in two ways. They offer dividends, which is a portion of the profit made by the company. And the price per share can increase based on how others see the value of the company changing. If the value of the company drops or the company folds, the money invested in the stock also drops.

Most stock transactions are executed through a broker. Brokers' commissions can be a percentage of the value of the trades made or a flat fee. There are full-service brokers who charge higher commission rates, but they also offer financial advice and perform the research that you may not have the time or the expertise to do on your own. A discount broker only executes the stock transactions, buying or selling, so they charge lower rates than full-service brokers. There are also brokers that offer commission-free trading.

An important thing to remember is that stocks may provide a substantial return on investment, but the trade-off is the risk associated with owning them.

Who Knew?: Chapter 11 Bankruptcy and Stocks

In the fall of 2022, the parent company of Regal Theaters, named Cineworld, filed for Chapter 11 bankruptcy. According to news articles, the bankruptcy was necessitated due to its heavy debt load. Generally, a company can file for Chapter 11 bankruptcy to allow them time to reorganize and restructure debts. When this happens, the company, after the Chapter 11 process is over, offers new stock. This makes the previous stock worthless. However, the company may allow an exchange of old stock for a discounted amount of the new stock. This, in effect, reduces (maybe vastly) the wealth held by those who owned the original stock.

Who Knew?: Risk and Volkswagen

The question of risk hovers over every investment. How risky can it get? Volkswagen seems to be a rather safe investment. But in 2015, Volkswagen's stock tumbled 30% over a few days when it was revealed that the company had installed software that altered the emission performance of some of its diesel engines. Volkswagen's hope was that lower emissions would bolster US sales of some of their diesel models. This was a drastic drop, and many investors lost a lot of money. However, the stock has since recovered. This was mild compared to the 65% drop in the Martha Stewart Living Omnimedia stocks.

People in Mathematics: Warren Buffett

Warren Buffett is an investment legend. He began his career as an investment salesman in the $1950s$. He formed Buffett Associates in 1956. In 1965, he was in control of Berkshire Hathaway, which began as a merger between two textile companies. In his role there, he began to invest in a variety of companies. It is now a conglomerate holding company, and fully owns GEICO, Duracell, Diary Queen, and other large companies.

His investment philosophy involves finding stocks and bonds from companies that have high intrinsic worth compared to their stock or bond prices. This means he focuses not on the supply and demand side of stock investing, but instead on the company's worth in total. Using this philosophy, he has become one of the world's most successful investors.

Trading Platform

A stock trading platform is a digital tool—typically a website or mobile app—that enables investors to buy and sell stocks, mutual funds, bonds, and other securities. These platforms connect users to financial markets and provide the necessary tools to manage their investments. Some popular stock trading platforms include Robinhood, E*TRADE, Charles Schwab, Fidelity, TD Ameritrade, Vanguard, and Webull.

On the other hand, a crypto trading platform is an online exchange—also available as a website or mobile app—where you can buy, sell, trade, and store cryptocurrencies such as Bitcoin, Ethereum, Solana, and many others. Some well-known crypto trading platforms include Coinbase, Binance, Kraken, and Crypto.com.

Reading Stock Tables

Information about particular stocks is contained in **stock tables**. This information includes how much the stock is selling for, and its high and low values from the past year 52 weeks).

Teminology: Reading Stock Chart/Table

52-Week High Low	The highest and the lowest share prices during the past 52 weeks.
SYM	The company name and the ticker symbol are used to identify the company.
DIV	The current annual dividend per share.
Yld %	$\frac{\text{annual divided}}{\text{share price}} \times 100\%$
Vol	The number of shares that have been traded today.
Open/High/Low	Opening, highest, and lowest prices so far today
Close	The price at which the stock traded at the close of the in prior trading day.
Net Chg	The difference between the prior trading period's closing price and the current trading period's closing price
Market Cap	Total stock value of the company.

Share Outstanding	The total number of shares that exist for the company to trade.
P/E	The share price divided by the earnings per share over the past year (dd indicates loss.)

The formulas for yield and price to earnings is a good way to measure how much the stock returns per share. Their values are calculated in the stock table, but deserve attention here.

FORMULA: P/E Ratio and EPS

$$\begin{aligned} \text{P/E} &= \frac{\text{Share price}}{\text{Dividend}} \\ \text{Yld} &= \frac{\text{Annual Dividend}}{\text{Share price}} \times 100\% \\ \text{Earning per share} &= \frac{\text{Share price in opening day}}{\text{P/E ratio}} \end{aligned} \tag{4.4.1}$$

It should be noted that the price of a stock increases and decreases every moment, and so these values change as the share price changes.

Example 4.4.5: Computing Percent Yield

Find the Annual percent yield for a stock with a price of $\$30.69$ and quarterly dividends of $\$1.48$ per share.

Answer available at the end of this section.

Your Turn 4.4.5: Annual Percent Yield

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Example 4.4.6: Reading an Online Stock Table

Consider the stock table Figure 4.4.1, and answer the questions based on the table.

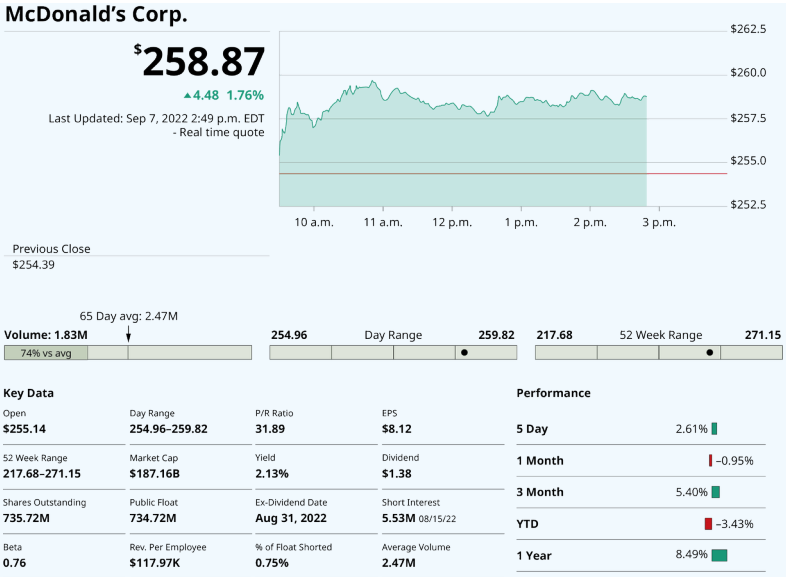


Figure 4.4.2: Key data for McDonald's stock 9-7-2022 (data source: marketwatch.com)

1. What is the current price of McDonald's Corp on this date?
2. What is the price of the stock (share) at the start of the day?
3. What is the 52-week high? 52-week low?
4. When is the dividend expected?
5. What is its yield?
6. Sujan owns 500 shares of McDonald's Corp stock. How much will Sujan earn in dividends in the quarter?
7. What are the earnings per share (EPS)?
8. What is the total value of the company?

Answer available at the end of this section.

Your Turn 4.4.7: Read Stock Chart

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Example 4.4.7: Stock Price Increases

Vincent buys 100 stocks in the REM company for $21.87 per share. One year later, he sells those 100 shares for $29.15 per share.

1. How much money did Vincent make?
2. What was his return on investment for that one year?

Answer available at the end of this section.

Your Turn 4.4.7: Find ROI

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Mutual Funds

A **mutual fund** is a collection of investments that are combined into a single portfolio. When you buy shares of a mutual fund, your money is pooled with the assets of other investors. This pooled money is invested in stocks, bonds, money market instruments, and other assets. Mutual funds are typically managed by professional money managers who allocate the fund's assets and strive to generate capital gains or income for the fund's investors. Some mutual funds are

Examples of some mutual funds

1. Vanguard 500 Index Fund (VFIAX)
2. Fidelity 500 Index Fund (FXAIX)
3. Schwab S&P 500 Index Fund (SWPPX)

An example of a mutual fund, VFIAX, is provided online below, with some parts labeled.

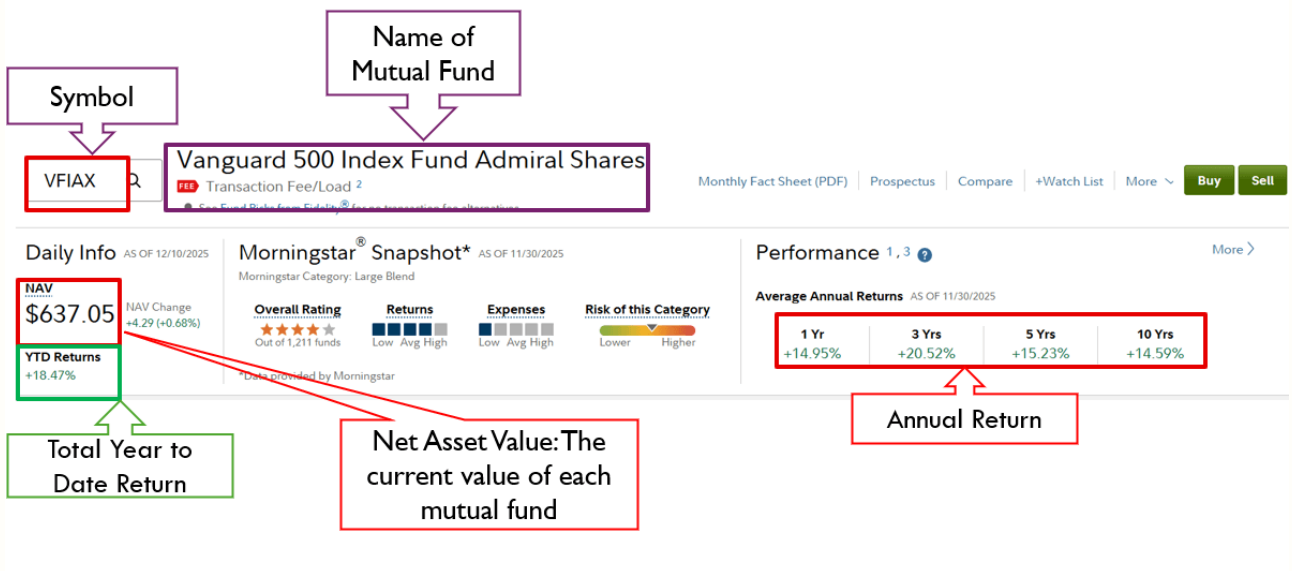


Figure 4.4.3: VFIAX Mutual Fund

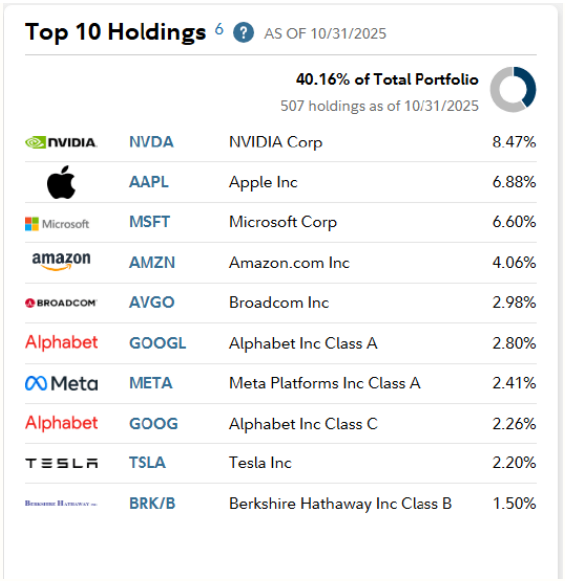


Figure 4.4.4:Top 10 holding

As we mentioned, a **mutual fund** is a collection of investments. Figure 4.4.4 shows the top 10 collection of stocks in the VFIAX mutual fund. Those collection is called holdings in a mutual fund.

A key benefit of mutual funds is that they allow small or individual investors to invest in professionally managed portfolios of equities, bonds, and other securities. This means that each shareholder participates proportionally in the fund's gains or losses. The performance of a mutual fund is usually stated as how much the mutual fund's total value has increased or decreased. Since there are many different investments within the mutual fund, the risk is significantly reduced compared to direct ownership of stocks. Even so, mutual funds historically perform well and can earn more than 10% annually.

The investments that comprise a mutual fund are structured and managed to align with the stated investment objectives, as specified in its **prospectus**. A prospectus is a document that provides information about a mutual fund. Before buying shares of a mutual fund, consult its prospectus to consider its goals and strategies and determine if they align with your objectives and values. Additionally, research any associated fees.

Example 4.4.8: Reading Mutual Fund Table

Answer the following questions based on an actual **Fidelity Select Communication Service mutual** fund.

$a.$ If you invest $\$5,000$ in this fund today, how many shares will you be able to buy?
$b.$ If you had invested $\$5,000$ in this fund three years ago, how much would it be worth today?
$c.$ If you had invested $\$5,000$ in this fund ten years ago, how much would it be worth today?

Answer available at the end of this section.

Your Turn 4.4.8: Find Future Value of Mutual Fund Investment

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Retirement Savings Plans

We close this section by investigating the three main forms of retirement savings accounts: traditional individual retirement accounts (IRAs), Roth IRAs, and 401(k) accounts. Each has distinct characteristics that are suited to the needs of different investors.

Individual Retirement Accounts

A **traditional IRA** lets you contribute up to an amount set by the government, which may change from year to year. For 2026, if you're 49 or younger, you can contribute up to $\$7,500$, and if you're 50 or older, you can contribute up to $\$8,600$.

Anyone is eligible to contribute to a traditional IRA, regardless of their income level. Your money grows tax-deferred(You pay taxes later when you withdraw the money in retirement), but withdrawals after age $59½$ are taxed at current rates. Traditional IRAs also allow you to use the contribution itself as a deduction on a current-year tax return.

Roth IRAs allow contributions at the same levels as traditional IRAs. For 2026, if you're 49 or younger, you can contribute up to $\$7,500$, and if you're 50 or older, you can contribute up to $\$8,600$. However, to be eligible to make contributions, your earned income must be below a certain level. A Roth IRA allows after-tax contributions. In other words, the contribution itself is not tax-deductible, unlike the traditional IRA. However, your money grows tax-free. If you make no withdrawals until you are age $59½$, there are no penalties. IRAs pay a modest interest rate.

In either case, IRA deposits have to be from earned income, which in effect means if your earned income is over $\$6,000$ ($\$7,000$), then you can deposit the maximum.

401(k) Accounts

Your employer may offer a retirement account to you. These are often in the form of a **401(k)** account. There are traditional and Roth 401(k) accounts, which differ in how they are taxed, much as with other IRAs. In the traditional 401(k) plans, the money is deposited before tax is assessed, which means you do not pay taxes on this money. However, that means when money is withdrawn, it is taxed. These accounts are similar to mutual funds in that the money is invested in a wide range of assets, spreading the risk.

One of the perks some employers offer is to match some amount of your contributions to the 401(k) plan. For instance, they may match your deposits up to 5% of your income. This is an instant 100% return on the matched funds.

401(k) plans with matching funds provide great value, as their rates of return are high compared to savings accounts, and are less risky than stocks since such funds invest across many investment vehicles. The next example demonstrates the power of consistent deposits into a 401(k) plan with an employer match.

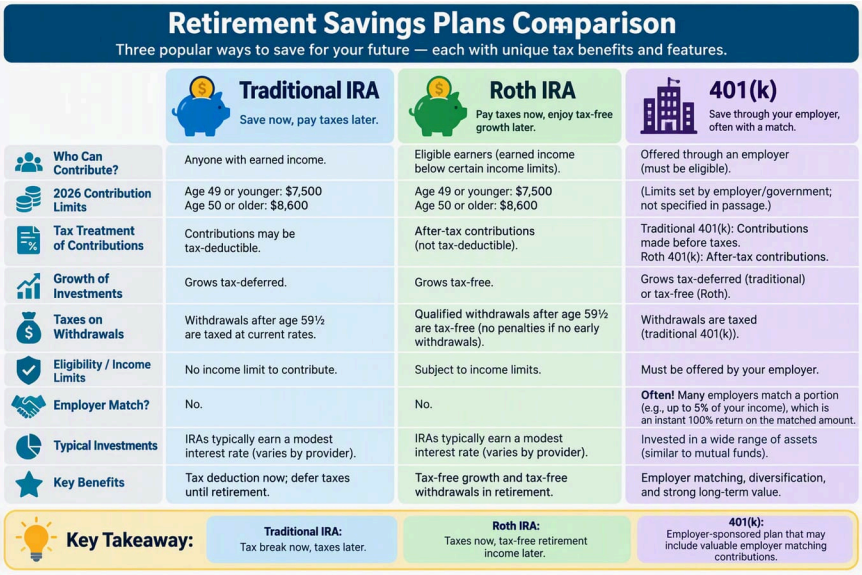


Figure \({4{.}4{.}\eatSpaces{6\): Comparison Between IRA and 401K

Example Extra close brace or missing open brace: Contribution in 401K

Alice signs up for her employer-based 401(k). The employer matches any 401(k) contribution up to 6% of the employee's salary. Alice's annual salary is $\$51,600$.

1. What is the maximum amount that Alice can deposit, which will be fully matched by the company?
2. How much total will be deposited into Alice's account if she deposits the full 6%?

Answer available at the end of this section.

Your Turn $4.4.9$: Contribution to 401 K

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Note: Which Investment is Riskier?

Which of the types of investments listed would be considered to have the **highest** level of risk?

Bonds are considered the lowest-risk investment, while stocks are considered the highest-risk investment.

Answers

Example $4.4.1$: 5-Year CD ROI

This also uses the compound interest formula from the Compound Interest section.
Substituting the values $P = \$10,000$, $r = 0.0217$, $n = 2$ (semiannually means twice per year), and $t = 5$. We find the account will be worth

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 10,000\left(1+\frac{0.0217}{2}\right)^{2\times 5} \\ &= 10,000\,(1.01085)^{10} \\ &= 10,000\times 1.113953736 \\ &= 11,139.53736 \end{aligned} \tag{4.4.2}$$

The CD will be worth $\$11,139.54$ after 5 years.
Interest earned during that period is:

$$I = A - P = \$11,139.54 - \$10,000 = \$1,139.54$$

The initial deposit in the CD was $\$10,000$, so P = $P = \$10,000$. The value at the end of 5 years was $\$11,139.54$. and \(FV = $11,139.54.\) Substituting and computing, we find the return on investment.

$$\begin{aligned} \text{ROI} &= \frac{FV-P}{P}\times 100\% \\ &= \frac{\$11,139.54-\$10,000}{\$10,000}\times 100\% \\ &= \frac{\$1,139.54}{\$10,000}\times 100\% \\ &= 11.3954\% \end{aligned} \tag{4.4.3}$$

The ROI is 11.40%.

Note: As we did in the compound interest section, you can find the future value using the TVM solver. Put the following values in the given variables

$$\begin{aligned} \text{N} &= n\times t = 2\times 5 = 10 \\ \text{I} &= 2.17 \text{ (Do not put percent)} \\ \text{PV} &= 10,000 \text{ (Deposited amount or present value)} \\ \text{FV} &= \text{We need to find it.} \\ \text{PMT} &= 0 \\ \text{P/Y} &= 2 \\ \text{C/Y} &= 2 \end{aligned}$$

And click solve for FV.

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Example $4.4.2$**: Future Value of an Ordinary Annuity**

Here
FV is the future value of the annuity (We do not know)
$PMT = \$500$ is the payment (Periodic deposit)
$r = 0.05$ is the annual interest rate (in decimal form)
$n = 4$ is the number of compounding periods per year
$t = 35$ is the number of years

$$\begin{aligned} FV &= PMT\left[\frac{(1+\frac{r}{n})^{nt}-1}{(\frac{r}{n})}\right] \\ &= 500\left[\frac{(1+\frac{0.05}{4})^{4\times 35}-1}{(\frac{0.05}{4})}\right] \\ &= 500\left[\frac{(1.0125)^{140}-1}{0.0125}\right] \\ &= 500\left[\frac{4.692518676}{0.0125}\right] \\ &= 500\times 375.4014941 \\ &= 187,700.747 \end{aligned} \tag{4.4.4}$$

So the future value of this account, after 35 years, will be $\$187,700.75$.

$$\begin{aligned} \text{Total deposit} &= PMT\times n\times t \\ &= \$500\times 4\times 35 \\ &= \$70,000 \end{aligned}$$

The interest Jill will have earned will be

$$\begin{aligned} \text{Interest earned} &= FV - \text{Total deposit} \\ &= \$187,700.75 - \$70,000 \\ &= \$117,700.75 \end{aligned}$$

Note: As we did before, you can check your answer using the TVM solver. Put the following values in the given variables to find the future value of the annuity.

$$\begin{aligned} \text{N} &= n \times t = 4 \times 35 = 140 \\ \text{I} &= 5 \ \text{ (Do not put percent)} \\ \text{PV} &= 0 \ \text{ (We don't have it, so put it as zero)} \\ \text{FV} &= \text{We need to find it.} \\ \text{PMT} &= 500 \\ \text{P/Y} &= 4 \\ \text{C/Y} &= 4 \end{aligned}$$

And solve for FV.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Example 4.4.3**: Saving for a Car**

Yaroslava has a goal and needs to know the payments to make to reach the goal. Her goal is $FV = \$35,500,$ with an interest rate $r = 0.0425,$ compounded per month so $n = 12,$ and for 3 years, making $t = 3$. Substituting into the formula, Yaroslava finds the necessary payment.

$$\begin{aligned} PMT &= FV\left[\frac{\left(\frac{r}{n}\right)}{(1+\frac{r}{n})^{nt}-1}\right] \\ &= 35,500\left[\frac{\left(\frac{0.0425}{12}\right)}{(1+\frac{0.0425}{12})^{12\times 3}-1}\right] \\ &= 35,500\left[\frac{0.003541667}{(1.003541667)^{36}-1}\right] \\ &= \frac{125.72916}{0.13572907} \\ &= 926.3248 \end{aligned} \tag{4.4.5}$$

That means if Yaroslava deposits $\$926.33$ every months, after 3 years, her account will grow to $\$35,500$ with 4.25% interest rate.

Yaroslava puts $\$926.33$ monthly. So here the total deposit will be

$$\begin{aligned} \text{Total deposit} &= PMT \times n \times t \\ &= \$926.33 \times 12 \times 3 \\ &= \$33,347.88 \end{aligned}$$

The interest Yaroslava will have earned will be

$$\begin{aligned} \text{Interest earned} &= FV - \text{Total deposit} \\ &= \$35,500 - \$33,347.88 \\ &= \$2,152.12 \end{aligned}$$

You can check your work by using the TVM solver. Put the following

$$\begin{aligned} \text{N} &= n \times t = 12 \times 3 = 36 \\ \text{I} &= 4.25 \ \text{ (Do not put percent)} \\ \text{PV} &= 0 \ \text{(We don't have it, so put it as a zero.)} \\ \text{FV} &= 35500 \\ \text{PMT} &= \text{We need to find it.} \\ \text{P/Y} &= 12 \\ \text{C/Y} &= 12 \end{aligned}$$

And click solve for PMT.

You will find PMT = $\$926.33$

To reach her goal, Yaroslava would need to deposit $\$926.33$ (ROUND UP) in her account each month.

Example 4.4.4**: Bond Investment**

The coupon rate is $5.5\%.$ per year.

Money paid in one year is

$$0.055 \times \$3,000 = \$165$$

Each year Muriel receives $\$165.$ Total amount earned in the bond is $\$165 \times 4 = \$660.$

Each year, Muriel received $\$165.$ She received this money four times, so she earned a total of $\$660.$ Muriel's bond face value is $\$3,000 + \$660 = \$3,660$ and investment on the bond is $3,000,$ which is $P.$

$$\begin{aligned} \text{ROI} &= \frac{\text{FV-P}}{\text{P}} \times 100\% \\ \text{ROI} &= \frac{660}{3000} \times 100\% \\ \text{ROI} &= 22\% \end{aligned}$$

So 22% is Muriel's ROI.

Muriel purchases a $\$3,000$ bond so her investment $P = \$3,000$. Muriel earned a total of $\$660$. $FV = \$3,000 + \$660 = \$3,660$. Using that, we find that the annual return is

$$\begin{aligned} \text{Annual return} &= \left[\left(\frac{FV}{P}\right)^{\frac{1}{t}} - 1\right] \times 100\% \\ &= \left[\left(\frac{3,660}{3000}\right)^{\frac{1}{4}} - 1\right] \times 100\% \\ &= (1.22^{\frac{1}{4}} - 1) \times 100\% \\ &== 5.0969\% \end{aligned}$$

5.1% is Muriel's annual return.

The 5.5% bond earned the equivalent of 5.10% compounded annually.

Example $4.4.5$**: Computing Percent Yield**

Substituting the values for price, $\$30.69$, and annual dividend, $\$4 \times 1.48$,, becuase quarterly dividends of $\$1.48$ per share. We find the Annual percent yield for the stock to be

$$\begin{aligned} \text{Yld} &= \frac{4 \times \$1.48}{\$30.69} \times 100\% \\ &= 19.28\% \end{aligned} \tag{4.4.6}$$

So the percent yield for a stock is 19.28%

Example $4.4.6$**: Reading an Online Stock Table**

1. Looking at the table, the current price of a share is $\$258.87$.
2. The high was $\$271.15$, and the low was $\$217.68$.
3. $\$255.14$
4. August 31, 2022
5. 2.13
6. $\$500 \times 1.38 = \$690$
7. The EPS value is $\$8.12$.
8. $\$187.16$ billion

Example $4.4.7$**: Stock Price Increases**

1. Vincent spent $\$21.87$ per share to buy the stock. The total he spent on the stock was $\$21.87 \times 100 = \$2,187$. When he sold the stock, the price was $\$29.15$, so he receive $\$29.15 \times 100 = \$2,915$.
2. He made $\$2,915 - 2,187 = \$728$. His return on investment was

$$\begin{aligned} \text{ROI} &= \frac{\text{earning}}{\text{original price}} \\ &= \frac{\$728}{\$2,187} \\ &= 0.3329 \\ &= 33.29\% \end{aligned} \tag{4.4.7}$$

Example $4.4.8$**: Reading Mutual Fund Table**

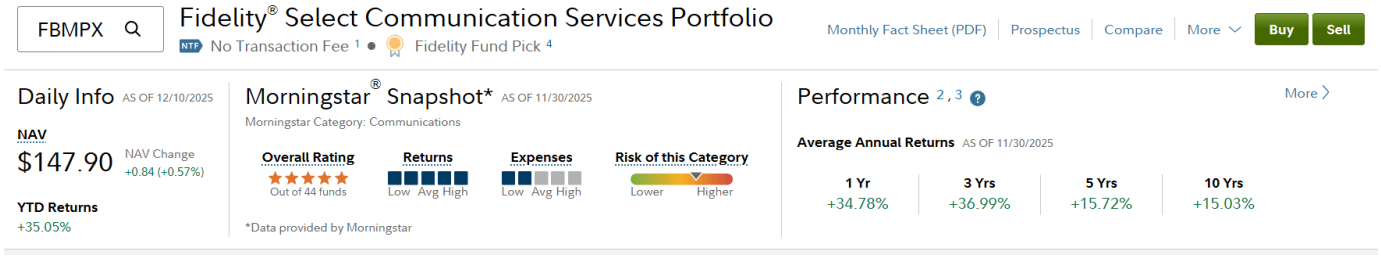


Figure $4.4.5$: Copy and Paste Caption here. (Copyright; author via source)

$a.$ Since each shares cost $\$147.90$, the umber of share we can buy with $\$5000$ is: $\frac{5000}{147.90} = 33.8066$

$b.$ Here we need to find future value of $\$5,000$ investment after 3 years.

$$\begin{aligned} \text{Future Value: FV} &= P(1+r)^t \\ &= 5,000(1+0.3699)^3 \\ &= 12853.94986 \end{aligned}$$

$c.$ Here we need to find future value of $\$5,000$ investment after 10 years.

$$\begin{aligned} \text{Future Value: FV} &= P(1+r)^t \\ &= 5,000(1+0.1503)^{10} \\ &= 20280.6188 \end{aligned}$$

Example $4.4.9$**: Contribution in** 401**K**

1. The employer will match up to 6% of any employee's salary. The 6% of Alice's salary is

$$0.06 \times \$51,600 = \$3,096$$

So Alice can deposit up to $\$3,096$ and receive that amount in matching funds in her account.

2. Alice's contribution plus the company's contribution is

$$\$3,096 + \$3,096 = \$6,192$$

which is the total that is deposited into Alice's account.

3. She earns a 100% return on the day she deposits her $\$3,096$.

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4.5: Basics of Loans (Mortgage, Cars, and Credit Cards)



Figure 4.5.1: Loans are contracts that allow people to buy now but require them to pay more. (credit: "Closing" by Tim Pierce/Flickr, CC BY 2.0)

Learning Objectives

1. Understand how credit scoring works.
2. Calculate the payment necessary to pay off a loan.
3. Read an amortization table.
4. Calculate the monthly payment for a mortgage and related interest costs.
5. Solve application problems involving the affordability of a mortgage.
6. Read and understand the basic parts of a credit card statement.
7. Compute interest, balance due, and minimum payment due for a credit card.

New car envy is real. Some people look at a new car and feel that they, too, should have a new car. The search begins. They find the model they want, in the color they want, with the features they want, and then they look at the price. That's often the point where the new car fever breaks, and the reality of borrowing money to purchase the car enters the picture. This borrowing takes the form of a loan.

In this section, we look at the basics of loans, including terminology, credit scores, payments, and the cost of borrowing money.

Reasons for Loans

Even if you want a new car because you need one, or if you need a new computer since your current one no longer runs as fast or smoothly as you would like, or you need a new chimney because the one on your house is crumbling, it's likely you do not have that cost in cash. Those are very large purchases. How do you buy that if you don 't have the cash? You borrow the money. And to help you with your purchase, the company or bank charges you interest.

Loans are taken out to pay for goods or services when a person does not have the cash to pay for the goods or services. We are most familiar with loans for the big purchases in our lives, such as cars, homes, and a college education. Loans are also taken out to pay for repairs, smaller purchases, and home goods like furniture and computers.

Loans can come from a bank, or from the company selling the goods or providing the service. The borrower agrees to pay back more than the amount borrowed. So there is a cost to borrowing that should be considered when deciding on a purchase bought with credit or borrowed money. Even using a credit card is a form of a loan.

Essentially, a loan can be obtained for just about any purchase, large or small, that has a cost beyond a person's cash on hand.

The Terminology of Loans

There are many words and acronyms that are used in relation to loans. A few are below.

APR is the annual percentage rate. It is the annual interest paid on the money that was borrowed. The principal is the total amount of the loan that has been financed. A fixed-interest rate loan has an interest rate that remains constant throughout the life of the loan. A variable interest rate loan has an interest rate that may change during the life of the loan. The **term** of the loan refers to the duration for which the borrower is required to repay the loan. An installment loan is a loan with a fixed period, and the borrower pays a fixed amount per period until the loan is paid off. The periods are almost uniformly monthly. Loan amortization is the process used to calculate how much of each payment will be applied to the principal and how much is applied to interest. Revolving credit, also known as open-end credit, is how most credit cards work, but is also a kind of loan account. (We will learn about credit cards in Credit Cards) You can use up to some specified value, called the limit, any way you want, and as long as you pay the issuer of the credit according to their terms, you can keep borrowing from this account.

These and other terminologies can be researched further at Forbes.

Who Knew?: Credit Scores

Not everyone pays the same interest for the same loan. One person might get an APR of 2.9% while another pays 6.9%. These rates are based on your credit score.

Data about you and your credit is collected by three credit bureaus—Experian, Equifax, and TransUnion. They calculate your score using one of two main models: FICO and Vantage Score. The score they develop is based on the following categories:

- **Payment History:** Making your payments on time and not missing payments is by far the most important factor. All three credit types—revolving, installment, and open—contribute to this factor.
- **Credit Utilization or Amount Owed:** How much do you owe on your credit card accounts? This category is concerned with the ratio of how much you owe on revolving credit accounts relative to your available credit, also known as your credit utilization ratio.
  This is the only category that depends solely on your revolving credit accounts.
- **Length of Credit History:** This is the average age of your credit history, including the age of the oldest and newest accounts. All three types of credit accounts play a role in this category.
- **Credit Mix:** This number represents the different types of credit accounts you have, such as credit cards, car loans, mortgages, and whether you are successful managing both revolving and installment accounts.
- **New Credit:** Have you recently opened a new account or applied for new credit? Lenders want to know how much new credit you are taking on. If you plan to buy a car and make another large purchase with a credit card, consider spacing these purchases out.

If you have done well in these categories, your credit score will be high, and you will qualify for lower interest rates because you are not perceived as being a risky investment. However, if you do poorly in these categories, your score will be low, and you will pay higher interest rates since you present a greater risk.

Calculating Loan Payments

Loan payments are made up of two components. One component is the interest that accrued during the payment period. The other component is part of the principal. This should remind you of partial payments from Simple Interest.

Over the course of the loan, the amount of principal remaining to be paid decreases. The interest you pay in a month is based on the remaining principal, just as in the partial payments of Simple Interest.

The payment of the loan has to be such that the principal of the loan is paid off with the last payment. In any period, the amount of interest is defined by the formula above, but changes from period to period since the principal is decreasing with each payment. The trick is knowing how much principal should be paid each payment so that the loan is paid off at the stated time. Fortunately, that is found using the following formula.

FORMULA: Monthly Payment for Loans

$$PMT = \frac{P\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}$$

The payment, PMT, per period to pay down a loan with beginning principal, P is where r is the annual interest rate in decimal form, t is the number of years of the loan, and n is the number of payments per year (typically, loans are paid monthly, making $n = 12$.) Note, payment to lenders is always **rounded UP** to the next penny unless otherwise mentioned.

Formula: Interest Paid and Total Paid

The total paid on a t years of loan with monthly payments PMT is

$$\text{Total Paid} = PMT \times 12 \times t$$

The cost of financing (Total Interest Paid),

$$\text{Interest Paid} = \text{Total Paid} - P$$

where P is the starting principal or loan amount.

Example 4.5.1: Calculating the Payment for a Loan

In the following, calculate the payment necessary to pay off the loan with the given details. The payments are monthly.

1. A car loan was taken out for $\$28,500$ at an annual interest rate of 3.99% for 5 years. What is the interest paid in this case?

2. A home loan was taken out for $\$136,700$ with an annual interest rate of 5.75% for 15 years.

Answer available at the end of this section.

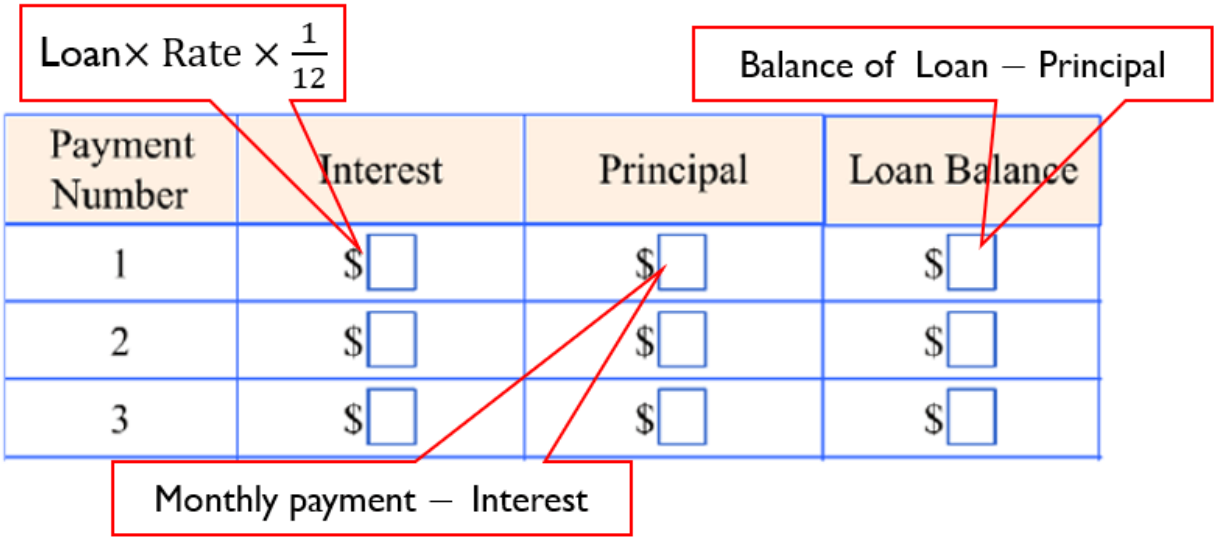
Your Turn 4.5.1: Find the Monthly Payment and Interest of the loan

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Reading Amortization Tables

An **amortization table** or amortization schedule is a table that provides the details of the periodic payments for a loan, where the payments are applied to both the principal and the interest. The principal of the loan is paid down over the life of the loan. Typically, the payments each period are equal. Importantly, the second column will show how much of each payment is used for interest, the third column will show how much is applied to the outstanding principal, and the fourth column will show the remaining principal or balance. As you see in the following table.



Formula to Find the Amortization Table

Interest Payment = $P \times r \times \frac{1}{12}$: For one month.

Principal Payment = Monthly PMT – Interest Payment

Balance of loan = Principal BAL (Loan) - Principal payment

Where P is the loan amount, r is the annual interest rate, and t is one month

We will find the first two rows of example $4.5.3$ using the above formula.

For the **FIRST** month

$$\begin{aligned}\text{Interest} &= P \times r \times \frac{1}{12} \\ &= \$10,000 \times 0.0475 \times \frac{1}{12} \\ &= \$39.58\end{aligned}$$

$$\begin{aligned}\text{Principal Payment} &= \text{Monthly PMT–Interest Payment} \\ &= \$64.62 - \$39.58 \\ &= \$25.04\end{aligned}$$

$$\begin{aligned}\text{Balance of loan} &= \text{Principal BAL (Loan)} - \text{Principal payment} \\ &= \$10,000 - \$25.04 \\ &= \$9,974.96\end{aligned}$$

For the **SECOND** month

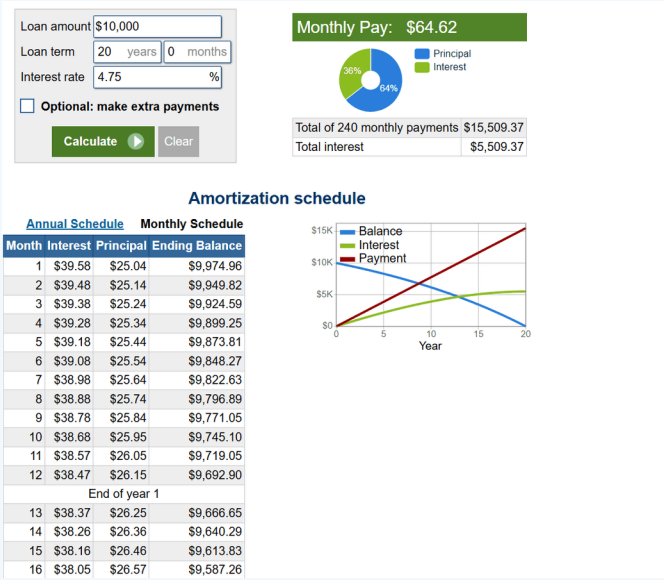
$$\begin{aligned}\text{Interest} &= P \times r \times \frac{1}{12} \\ &= \$9,974.96 \times 0.0475 \times \frac{1}{12} \\ &= \$39.48\end{aligned}$$

$$\begin{aligned}\text{Principal Payment} &= \text{Monthly PMT–Interest Payment} \\ &= \$64.62 - \$39.48 \\ &= \$25.14\end{aligned}$$

$$\begin{aligned}\text{Balance of loan} &= \text{Principal BAL (Loan)} - \text{Principal payment} \\ &= \$9,974.96 - \$25.14 \\ &= \$9,949.82\end{aligned}$$

Example 4.5.2: Reading from an Amortization Table

Using the partial amortization table 4.5.2, answer the following questions.



Month	Interest	Principal	Ending Balance
1	$39.58	$25.04	$9,974.96
2	$39.48	$25.14	$9,949.82
3	$39.38	$25.24	$9,924.59
4	$39.28	$25.34	$9,899.25
5	$39.18	$25.44	$9,873.81
6	$39.08	$25.54	$9,848.27
7	$38.98	$25.64	$9,822.63
8	$38.88	$25.74	$9,796.89
9	$38.78	$25.84	$9,771.05
10	$38.68	$25.95	$9,745.10
11	$38.57	$26.05	$9,719.05
12	$38.47	$26.15	$9,692.90
End of year 1			
13	$38.37	$26.25	$9,666.65
14	$38.26	$26.36	$9,640.29
15	$38.16	$26.46	$9,613.83
16	$38.05	$26.57	$9,587.26

Figure 4.5.2 : Amortization table

1. What is the loan amount (principal), the interest rate, and the term of the loan?
2. How much is the monthly payment?
3. How much remaining balance is there after the payment in month 15?
4. How much was the interest on payment 10?
5. Looking at the table or graph, what do you conclude about the interest payment and the principal payment in each month?
6. How much interest is paid when the loan is paid of after 20 years?

Answer available at the end of this section.

Your Turn 4.5.2: Amortization Table

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Buying House

After renting an apartment for 10 years, you realize that it may be time to purchase a home. Your job is stable, and you could use more space. It is time to investigate becoming a homeowner. What are the things that you must consider, and what is the financial benefit of owning as opposed to renting? This section is about the advantages, disadvantages, and costs of homeownership as opposed to renting.

Advantages and Disadvantages of Renting

When renting, you will likely sign a lease, which is a contract between a renter and a landlord. A landlord is the person or company that owns property that is rented. The lease will detail your responsibilities, restrictions on activities, deposits, fees, maintenance, repairs, and rent during the term of the lease. It also defines what your landlord can and cannot do with the property while you occupy it.

There are advantages to renting, but also some disadvantages.

Advantages	Disadvantages
• Lower cost. • Short-term commitment. • Little to no maintenance cost. The landlord pays for or performs most maintenance. • You need not stay at the end of the lease. Once the lease term is over (the lease is up), you are not obligated to stay. • If renting in an apartment complex, there may be a pool, gym, or community room for renters to use.	• No tax incentives. • Housing cost is not fixed. When the lease is up, the rent can change. • No equity. When you are done living in a rental, you have built no value. • Restrictions on occupants. There may be a limit on how many can live in the apartment. • Restrictions on decorating. The property is not yours, so any decorating or improvements need the landlord's permission. • Limits on pets. Permission for pets and their number and type will be set forth in the lease. • May not be able to remain when the lease term is over. The landlord can, at the end of your lease, invite you to leave. • The building may be sold, and the new landlord may institute changes to the lease when the previous lease expires.

Renting has fees to be paid at the start of the lease. Typically, when you rent, you will pay the first month's rent, last month's rent, and a security deposit. A security deposit is a sum of money that the landlord holds until the renter leaves the rental property. The deposit will cover repairs for any damage to the apartment during the renter's stay, but it may be returned if the apartment is in good condition upon departure. If your landlord runs a credit check on you, the landlord may charge you for

that.

Advantages and Disadvantages of Buying a Home

The advantages to buying a home mirror the disadvantages of renting, and the disadvantages of home ownership mirror the advantages of renting.

Advantages	Disadvantages
• There are tax incentives. The interest you pay for your mortgage (more on that later) is deductible on your federal income tax. • There are no restrictions on pets or occupants unless laws in your area specify limits for homes. • You can redecorate any way you wish, limited only by the laws in your area. • Once your mortgage is set with a fixed interest rate, your housing cost is fixed. • Your home grows equity; that is, the difference between what you owe and what the house is worth grows. You can use the equity to secure loans, and you recover the equity (and more if you're fortunate) when you sell the house. • As long as you pay your mortgage and maintain the home to the standards of your community, you can stay as long as you wish.	• The cost is higher than renting. Mortgages and associated costs are typically higher than rent for a similar living space. • The owner is responsible for upkeep, maintenance, and repairs. These can be extremely costly. • The owner cannot walk away from the property. It can be sold, but simply leaving the property, especially if not paid off yet, has serious consequences.

The big question of affordability looms large over the decision to rent or buy. Renting, strictly from an affordability viewpoint, comes with much less initial outlay and smaller commitment. If you do not have sufficient income to regularly save for possibly expensive repairs, or your credit isn't quite as good as it needs to be, then renting may be the best choice. Of course, even if you can afford to buy a home, you may choose to rent based on the comparative advantages.

Buying a home really involves two buyers. You and the mortgage company. The mortgage company has an interest in the home, as they are providing the funds for the home. They want to protect their investment, and many fees are about the bank as much as the buyer. They fund a mortgage based on the value they assign the property. Not you. This means they will want some certainty that the home is sound and that you are a good investment.

Who Knew?: Closing Costs

When a home is bought, there are many costs that need to be paid at the time of purchase, which are lumped under the term closing costs. At the start of 2022, the average closing costs for a single-family home exceeded $\$6,800$. These costs include:

- The appraisal fee is the amount paid to a professional to determine the home's value. The bank's value of the home may differ from what the home is listed for or what an app says it's worth. It may run approximately $\$350$.
- The home inspection fee. The inspection should reveal any problems with the house that will need to be fixed either before or after you obtain the home.
- The title search. There is a records search to ensure there are no issues with who actually owns the property. It can cost about 0.5% to 1% of the amount you are financing.
- Prepaid taxes. You will need to pay about 6 months of taxes at the time of purchase.
- The credit report fee. This is a fee for checking your credit. You might pay $\$25$ or more for this.
- The origination fee. This is the price the mortgage company charges you to cover the costs of creating the mortgage. This could be 0.5% to 1% (or more) of the amount you are borrowing.
- The application fee. This is just a processing fee and could come to several hundred dollars.
- The underwriting fee. This covers the cost of verifying your financial qualifications. It could be a flat fee or some small percentage of the amount financed. Such as 0.5% or 1%.

- Attorney fees. If you use an attorney, you will have to pay the attorney.
- State of local fees. This may include a filing fee charged by the county or municipality in which you reside.

That's a long list, and it is not even complete. When buying, be prepared to see these costs. It can be surprising. But in the end, you will have equity in the home, which means when you sell your home, you will get some of your money back.

In the end, you must weigh your options and carefully consider your priorities in choosing to rent or buy a home.

Mortgages

Some people will purchase a home or condo with cash, but the majority of people will apply for a mortgage. A mortgage is a long-term loan, and the property itself is the security. The bank decides the minimum down payment (with your input), the payment schedule, the duration of the loan, whether the loan can be assumed by another party, and the penalty for late payments. The title of the home belongs to the bank.

Since a mortgage is a loan, everything about loans from The Basics of Loans holds, including the formula for the payments. The terms that are related to mortgages are given below.

Down Payment	This is the **initial amount of money** you pay upfront when purchasing something such as a house or a car.
Mortgage	This is a **loan used to buy a home or property**. You borrow money from a bank or lender, and in return, you agree to pay it back over time with interest. $$\text{Morgage} = \text{Sale Price} - \text{down payment}$$
Monthly Payment	This is the amount you pay each month, which includes principal plus interest for the loan you took.
Closing points	Generally paid by the buyer at the time of closing the home. It is part of the loan. 1 point = 1% of the loan amount. 2 point = 2% of the loan amount.... so on.
Total paid	This is the total amount of your payments over the life of the loan
Interest paid (Cost of financing)	This is the difference between the principal of the mortgage and the total paid over the life of the mortgage.
Amortization table	A table that shows how a loan is paid off over time through regular payments. It breaks each payment into three parts: principal payment, Interest payment, and remaining balance.
Escrow account	An account is a separate account held by a neutral third party (usually your mortgage lender) to manage and pay certain property-related expenses on your behalf, mainly property taxes and homeowners' insurance.

Monthly Mortgage Payments

The formula to calculate your monthly payments of principal and interest uses APR as the annual interest rate.

FORMULA: Monthly Mortgage Payment

The payment, PMT, per month to pay down a mortgage with a beginning principal P (Loan or mortgage)is

$$PMT = \frac{P\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}$$

where r is the annual interest rate in decimal form, and t is the number of years of the payment. Here, $n = 12$. Note that payments to lenders are always rounded up to the next penny, unless otherwise stated.



Figure 4.5.1: Home Loan Details

Example 4.5.3: 30-years Mortgage

Evan buys a house that is listed as $\$300,00$. He put 20% down payment. His 30-year mortgage comes to $\$240,650$. The interest rate is 6.8%. He is paying 2 points during closing. Find Evan's monthly payments.

Answer available at the end of this section.

Using the mortgage principal and the total amount paid over the life of the mortgage, the cost of financing can be found by subtracting the principal from the total paid.

Example 4.5.4: Total Paid and Interest

In an example $4.5.3$,

1. How much is Evan's total payment over the life of the mortgage?

2. What is the cost of financing a mortgage, or how much interest Evan pays over the life of the mortgage?

Answer available at the end of this section.

Your Turn 4.5.4: Mortgage

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With the principal of the mortgage and how much total is paid over the life of the mortgage, the cost of financing can be found by subtracting the principal of the mortgage from the total paid over the life of the mortgage.

Who Knew?: Private Mortgage Insurance (PMI)

When you purchase a home, you will have to pay a down payment. This means you have money tied to the property, which lenders believe makes you less likely to default on the loan. The amount of the down payment will be decided between you and the mortgage company. However, if your down payment is less than 20% of the property value, you will be required to pay private mortgage insurance (PMI). This is insurance you pay for so that the mortgage company is protected if *you* default on the loan. It often comes to between 0.5% and 2.25% of the original loan amount. It increases your monthly payment. Once you reach 20% of the loan value, you can request that the PMI be dropped. Even if you do not request cancelling the PMI, it will eventually and automatically be dropped.

For more information, see this article on ways to eliminate PMI.

Credit Cards

It can be difficult to get along these days without at least one credit card. Most hotels and rental car agencies require that a credit card is used. There are even a number of retailers and restaurants that no longer accept cash. They make online purchasing easier. And nothing contributes more to a good credit rating than a solid history of making credit card payments on time.

Being granted a credit card is a privilege. Used unwisely, that privilege can become a curse, and the privilege may be withdrawn. In this section, we will talk about the different types of credit cards and their advantages and disadvantages. The more knowledge a cardholder has about the credit card industry, the better able credit accounts can be managed, and that knowledge may cause major adjustments to a cardholder's lifestyle.

All credit cards are not equal, but they all represent consumers borrowing money, usually from a bank, to pay for needs and "wants." As such, they are a type of loan, and your repayment may include interest. (You might want to review Section 6.8, which discusses loans and repayment plans.)

There are many institutions and credit cards to choose from. Use caution as you shop around for a credit card that suits you. Your top concern is likely the interest rates on purchases and cash advances. But be careful to also read the small print regarding charges for late payments, and other fees such as an **annual fee,** where the credit card charges you (the cardholder) a fee each year for the privilege of using the cards. Many cards charge no such fee, but there are many that charge modest to heavy fees. Make sure to understand rules for **reward programs**, where the credit card issuer grants benefits based on one's spending. Finally, once one applies for and is granted a credit card, pay attention to the credit limit the bank offers. Once a company owes that much money, use of the card for purchases should be curtailed until some of the debt is paid off.

Checkpoint

The interest rate will not matter if the balance is paid every month. When the balance is paid every month, there is NO INTEREST charged.

Types of Credit Cards

There are basically three types of credit cards: bank-issued credit cards, store-issued credit cards, and travel/entertainment credit cards. We will examine all three and discuss the strengths and weaknesses of each.

Bank-Issued Credit Cards

Perhaps the most widely used credit card type is the **bank-issued credit card**, like Visa or MasterCard (and even American Express and Discover cards). These types of cards are an example of revolving credit, meaning that additional credit is extended before the previous balance is paid—but only up to the assigned credit limit. Bank-issued cards are considered the most convenient, as they can be used to purchase anything, including apparel, furniture, groceries, fuel for automobiles, meals, hotel bills, and so on, just as if paying with cash. The interest rates on bank-issued credit cards are usually lower than those for other credit cards we'll discuss, and the credit limits are generally higher. Currently, bank-issued cards have an average 20.09% APR.

Store-Issued Credit Cards

Store-issued credit cards are issued by retailers. One can hardly walk into a store these days without being offered a discount on purchases if one applies for the store credit card. These cards can only be used in that store or family of stores that issues the card. However, if a store credit card is associated with Visa, MasterCard, or American Express, then the card might be used the same way that the bank-issued cards are used. This is called cobranding. The logo of the bank-issued card will be present on the store card. Many stores offer both types. Like other credit cards, they may come with an annual fee.

Store credit cards usually charge higher interest rates than bank-issued cards. Currently, store credit cards have an APR (annual percentage rate) of 24.15%. Any rewards offered by store credit cards are usually limited to purchases made in their own store, and it typically takes longer to accumulate enough rewards or points to redeem them, whereas cobranded credit cards offer opportunities to earn rewards on all purchases, regardless of whether purchases are made in the issuing store or not.

Store credit cards usually offer lower credit limits, at least in the beginning. After being proved to be a responsible credit card owner, credit limits can be raised. Nevertheless, store credit cards are a good choice for those new to the credit card industry. If on-time payments are consistently made, it is an excellent way to get started building a credit history.

Travel/Entertainment Cards, or Charge Cards

This is the third type of credit card. The **travel and entertainment cards,** also known as **charge cards,** first and foremost offer very high limits or unlimited credit, but they must be paid in full every month. They generally charge high annual fees and impose expensive penalties should a payment be late. On the other hand, they typically have longer grace periods and offer many and various kinds of rewards.

Checkpoint

An interest rate will greatly depend on a credit score. Responsible use of credit cards will increase a credit score. See the WHO KNEW? from The Basics of Loans.

Credit Card Statements

Cardholders usually receive monthly statements and have 21 days to pay the minimum amount due. The statements itemize and summarize activity on the credit card for that statement's billing period. The billing period for a credit card is generally a month long, but typically does not start and end on the first and last days of the month. The statement will include the current balance, interest rate, the minimum payment due, and the due date. Be aware, different companies produce statements that are laid out

differently. The information will be clearly labeled, though.

The due date is a top concern. Missing a due date is one of the worst things a cardholder can do financially, and this is by far the biggest downfall of owning a credit card. Not only is the cardholder subject to late fees, but when a payment is late more than once, there is a high probability that the cardholder will be negatively reported to the credit bureaus, which can quickly erode a credit score. Figure $4.5.4$ shows an excerpt from an actual statement from a Chase Bank Visa card, based on the current $\$668.25$ balance.

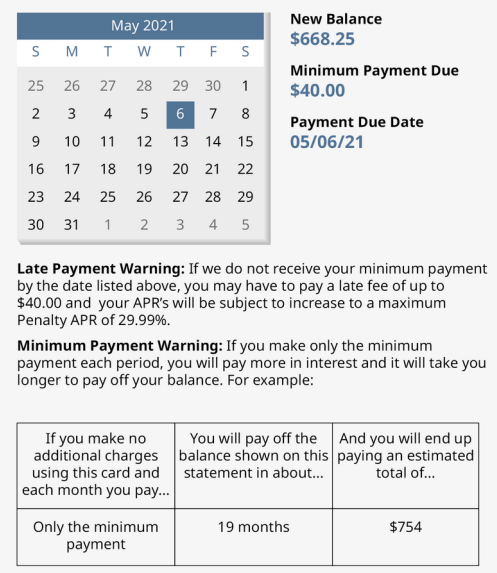


Figure $4.5.4$: Credit card statement

Specifically, pay attention to the late payment penalty and minimum payment warning statements. stating that if no other purchases are made and you continue making only the minimum payment, it will take 19 months to pay off the balance and you will pay $\$754.00$. You can't say you were not warned.

It is critical that you examine your statement every month because it is always a possibility that your account may have been compromised. If you should notice fraudulent charges on your statement, notifying the credit card company is often enough to have those charges researched by the company and removed. The card with the fraudulent charges will be canceled, and a new card with a new account number will be sent to you.

Example $4.5.5$: Reading a Credit Card Statement

On the credit card statement Figure $4.5.5$,

1. What is the balance due?

2. What is the minimum required payment?

3. What is the length of time it takes to pay off the balance by paying the minimum payments and without charging more to the card?

4. What is the interest rate for purchases?

5. What is the maximum amount you can spend on the credit card?



Figure 4.5.5 : Credit card account statement

Answer available at the end of this section.

Your Turn 4.5.5: Understanding Credit Card Statement

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Compute Interest, Balance Due, and Minimum Payment Due for a Credit Card

Computing all of these values depends on understanding and computing the average daily balance on a credit card. Once that is known, the interest, balance due, and minimum payment can be found.

Above all else, if you pay off the entire balance each month, interest is not charged.

Averge Daily Balance

Most credit card companies compute interest using the **average daily balance** method.

To find the average daily balance on your credit card, determine the balance on the card each day of the billing period (often that month), and take the average. One process to find that average daily balance follows these steps:

1. Start with a list of transactions with their dates and amounts.
2. For each day that had transactions, find the total of the transactions for the day. Expenditures are treated as positive values, payments are treated as negative values.
3. Create a table containing each day with a different balance. The balance is the previous balance plus the day's total transactions.

4. Add a column for the number of days those balances until the balance changed.
5. Add a column that contains the balances multiplied by the number of days until the balance changed.
6. Find the sum of that last column.
7. Divide the sum by the number of days in the billing period (often the number of days in the month). This is the average daily balance.

Interest and Balance on Credit Card

The interest charge, I , for a credit card during a billing cycle is

$$\text{Interest Charge } = \text{average daily balance} \times \text{interest rate} \times \text{time} \tag{4.5.1}$$

As before, interest is rounded **up** to the next penny; otherwise, it is stated.

Balance on Credit Card

The **balance,** or sometimes balance due, on a credit card is the previous balance, plus all expenses, minus all payments and credits, plus the interest on the card. As stated earlier, if the card is paid off, no interest is due.

$$\text{Balance Due} = \text{balance in the end of the month} + \text{interest charge} \tag{4.5.2}$$

Minimum Payment Due

The **minimum payment** due is the smallest required amount to be paid on a credit card to avoid late fees and penalties, such as an increased interest rate. The calculations for this may differ from card to card. They also depend on the balance of the credit card. General guidelines for minimum payment due are:

- For larger balances (usually over $\$1,000$), the minimum payment will be some percentage of the balance due.
- For moderate balances (between $\$25$ and $\$1,000$), the minimum would be a specified dollar amount. $\$25$ seems to be a common value.
- If the balance is small (under $\$25$ for instance), then the minimum payment is the balance.

Those are just guidelines. Individual cards may vary in these values.

Example 4.5.6: Computing Average Daily Balance

Assume the annual interest rate is 13.7% of the average daily balance and the billing period is from July 1 - July 31. Complete the following, rounded to the nearest cent.

1. Calculate the average daily balance.
2. Calculate the interest to be paid at the beginning of the next month.
3. Calculate the balance due at the beginning of the next month.
4. This credit card requires a $\$20$ minimum payment or $\frac{1}{36}$ of the amount due, whichever is higher. What is the minimum monthly payment due for this month?

Transaction	**Amount**
Previous balance: $\$669.69$	
July 1: Billing date	
July 4: Carnival tickets	$\$71$ charge
July 15: Payment	$\$139$ credit
July 19: Bookstore purchase	$\$109.62$ charge

July 25: Gym membership	$59 charge
July 31: End of billing cycle	

Answer available at the end of this section.

Your Turn 4.5.6: Average Daily Balance

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Who Knew?: Credit Cards Charge Stores Fees

The interest you pay is not the only way a credit card company generates revenue. It also charges fees to the retailers, online stores, and service providers that allow you, the consumer, to use your credit card to pay them. These are called processing fees. Currently they typically range from 2.87% to 4.35% of each transaction. That means if you use your credit card at a store and spend $100, the store will have to pay the credit card company somewhere between $2.87 and $4.35.

The Basics of Car Purchasing

There are people who don't need a car and won't purchase one. However, for many people, the question of whether to have a car is not a matter of choice. Having a car is a basic necessity for these people.

Obtaining a car can be daunting. The models, features, additional costs, and securing funding are all steps that need to be taken. One of the big decisions is whether to buy the car or to lease the car. This section will address some of the issues associated with each option.

The biggest questions you will answer before purchasing a car are, what do you want and what do you need?

Does it have to be new? Does it have to be a make and model you are familiar with? Does it require assisted driving? What other details are important to you? For a new vehicle, every feature beyond standard features comes with additional cost, which leads to the question that constrains all of your decisions about a car. How much can you afford to spend on a car?

What you can afford must include insurance costs (discussed later in this section), as well as maintenance and upkeep. Once you have this in mind, you can search for a car that matches, as closely as possible, what you want and can afford. Most, if not all, dealers have websites that you can search through to identify the car you want. If new cars are not affordable, used cars are often less expensive but come with the wear and tear of use.

The sticker price of the car, called the manufacturer's suggested retail price (MSRP), or the negotiated price you arrive at, isn't the end of the cost to buying a car.

Fees accociate with the Purchase of the Car

There are several fees associated with purchasing the car, including sales tax. These include, but aren't necessarily limited to, the following:

- The title and registration fee include registering your car with the state, getting the license plate, and assigning the title of the car to the lender. This cannot be avoided.
- a destination fee, which covers the cost of delivering the vehicle to the dealer
- a documentation fee, sometimes referred to a processing fee of handling fee, is the cost of all the paperwork the dealer did to get you the car
- a dealer preparation fee, which is for washing the car and other preparation of that sort. You should try to negotiate that out of the cost of the dealer tries to charge for those.
- extended warranties and maintenance plans, which help cover some of the costs of caring for the car.
- Sales tax.

You could pay for these immediately, but they are often added to the financing of the car, meaning they become part of the principal of your loan.

One way to bring down payments on a car is to provide a down payment or a trade-in. This is money applied to the purchase price before financing happens. Be warned, the sales tax applies to the full purchase price! If you reduce the amount financed, the payments will go down. This often becomes part of the negotiating process.

Example $4.5.7$: Total Cost to Purchase a Car with Down Payment

Sophia negotiates a $\$19,800$ price for her new car. The sales tax is 9.5% in her area, and the dealership charges her $\$300$ in documentation fees. Her title, plates, and registration come to $\$321.50$. The dealership adds to this a destination fee of $\$1,100$. If she places a down payment of $\$5,000$ on the car, what is the total she will finance for the car?

Answer available at the end of this section.

Your Turn $4.5.7$: Cost on Purchasing Car

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When purchasing a car, the total cost to obtain the car is not the only factor in your monthly price. You will also pay an interest rate for the loan you obtain. The interest rate you will get is dependent on your credit score (see The Basics of Loans). But you can choose from different lenders. The dealership will likely offer to finance your car loan. Frequently, dealerships offer special financing with very low rates. This is to help move inventory, and may indicate their desire to make sales. This might make negotiating easier. Even if the dealership offers financing, check with your bank or credit union to determine the interest rates they are offering. To reduce your payments, choose the lowest rate you can find.

Purchase Payments and Interest

Whether or not you buy a new car or a used car, if you finance the purchase, you are taking out a loan. The interest rates available for used cards are frequently higher than those for new cards. These loan payments work exactly the same way as other loans do as far as payments are concerned. The payment function comes from The Basics of Loans. The difference between financing a new car and a used car is that financing a new car typically comes with a lower interest rate and a longer term than financing a used car.

FORMULA: Monty Payment For Car Loan

The payment, PMT, per month to pay for a car loan with a beginning principal P is

$$PMT = \frac{P\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]} \tag{4.5.3}$$

where r the annual interest rate in decimal form and t is the number of years of the payment.
Please note that payments to lenders are always rounded up to the next penny, unless otherwise specified.

Example 4.5.8: New Car Payments

You decide to finance a $\$18,000$ car at a 6% interest rate for 5 years.

1. How much will your monthly payment?

2. By the end of the loan, what will be the total of all your payments?

3. How much interest will you pay over the life of the loan?

Answer available at the end of this section.

Your Turn 4.5.8: Car Payment

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The Basics of Leasing a Car

Leasing a car is an alternative to purchasing a car. It is still a loan and acts like one in many respects. They typically last either 24 months or 36 months, though other terms are available. Leases also come with mileage limits, frequently $10,000$, $12,000$, or $15,000$ miles per year. When the lease is over, the car is returned to the dealer. At that time, there may be fees that have to be paid, such as for damage to the car or for extra miles driven over the limit.

There are two components to lease costs. One is the monthly payment for the lease. The other is the fees for leasing, These often are paid before the lease is complete. These include:

- a down payment, which is your initial payment that is applied to the price of the car. It reduces the amount you finance, much the same as when you purchase a car. It is recommended that this be negotiated away.
- The acquisition fee, sometimes called the bank fee. This is the money charged for the company to set up the lease. It is essentially a paperwork fee. It is not likely that this can be negotiated.
- a security deposit, which might be required. It is about the same as 1 month's payment for the lease. The deposit is returned to you if the car is in good shape at the end. This can be negotiated away.
- disposition fees, which cover the cost the company will incur when they take your car back and are typically between $\$200$ and $\$450$.
- the title, registration, and license fees, just as with the purchase of a car.
- sales tax, which will likely be applied. The sales tax only covers the depreciated portion of the car (more on depreciation later) in many states. Since this depends on the state in which the car is leased, you should determine the sales tax rules for

where you lease the car.

As you can imagine, this can come to a fairly high dollar amount.

You have some obligations when you lease a car. You must keep the car in good condition, cleaned, maintained, and free of anything more than minor damage. If the car is in poor condition when the car is returned, you will be responsible for the cost to bring the car to an acceptable condition. You are also expected to keep the mileage under its limit. If you go over, you will pay 10 to 25 cents per mile over.

Lease Payments

Lease payments are similar to regular loan payments, but have some other details. Calculating a lease payment involves knowing the following values:

- **The price of the car.** This is the cost you would pay for the car after applying all discounts, incentives, and negotiations.
- **Residual Value.** This is the manufacturer's estimate of the car's value after a set period of time. The residual value is expressed as a percentage of the manufacturer's suggested retail price (MSRP).
- **Months.** This is the length of the lease. Most leases are either 24- or 36-month leases, but other terms are available.
- **Monthly Depreciation**. The monthly depreciation is the difference between the price of the car and the residual value, divided by the number of months of the lease, and represents the monthly loss of value of the car while it's being used

Comparing Purchasing and Leasing

When deciding to buy or lease a car, the differences between the two options should be carefully evaluated. The following is a list of points of comparison between the two.

- The payments for a lease are likely less than the payments for purchasing.
- When leasing, you get a new car after the lease term is over, typically 24 or 36 months. Buying the car means the same car is driven until it is resold, and a new one is bought. Essentially, leasing a car is equivalent to renting a car.
- The leased car is new, so all warranties are in force, and you drive the car during its best years. When the car is purchased, it may be kept past its warranty and may be driven until it is quite old.
- Each time you lease a new car, all the fees and taxes must be paid again. When buying a car, these fees are only paid once.
- Leasing contracts carry restrictions on the mileage you can drive per year, and going over incurs more cost at the end of the lease. Buying the car means no mileage limits.
- When leasing, you are obligated to keep the vehicle in good condition and maintain it according to the dealer's schedule. Some dealerships will even pay for oil changes over the life of the lease. When the car is purchased, the upkeep schedule is the choice of the owner.
- When a car is purchased and kept for long enough, the warranty expires, and the owner is responsible for all maintenance items and repairs. The warranty for a car won't expire during the lease term.
- When a new car is purchased and the loan is paid off, the car is still owned by the buyer and may be traded in when a new car is purchased. When leasing, the car is returned to the dealer when the lease term is over.

When deciding between the two, you are choosing between these features. If you aren't willing to drive an older car or deal with the upkeep that accompanies an older car, you may want to lease. This means you will need to pay those beginning costs each time the lease is up. If you want to own the car after the payments are over, then you may want to buy a car. This means you are paying for all the upkeep after the warranties expire, but you have no limits on mileage and own the car at the end. It really depends on your preferences.s.

Car Insurance

Car insurance is meant to cover costs associated with accidents involving cars. Most states (all except New Hampshire and Virginia) require some insurance. Without insurance, the state may not let you get a license for your car or register your car. Your state's requirements can be hard to follow. Fortunately, insurance companies and brokers will make sure your insurance is

sufficient for your state and will warn you if you try to not meet the requirements. Of course, they may offer more than what is sufficient, so it is your responsibility to determine how much coverage you want, as long as the minimum insurance requirements are met. The cost of insurance should be accounted for when evaluating the affordability of buying or leasing a car.

Whether your car is leased or owned, you do need insurance. This contributes to the cost of having the vehicle. Leasing or owning makes no difference to the insurance company you choose, because they are insuring you based on what you are driving, your driving record, and other information about you, including where you live and your age. These insurance policies have many components that address different costs that can come from auto accidents. This may make details confusing, and you may not realize what you are paying for until you must use it. Here is a brief outline of the different components of auto insurance, many of which are required by the state that issues your driver's license.

- **Liability insurance** is mandatory coverage in most states. Liability insurance covers property damage and injuries to others should you be found legally responsible for an accident. You are required to have the minimum amount of coverage, as determined by your state, in both areas.
- **Collision insurance** is insurance that covers the damage caused to your car if it is involved in an accident with another vehicle.
- **Comprehensive insurance** is an extra level of coverage if involved in an accident with another vehicle and covers other things like theft, vandalism, fire, or weather events as outlined in your policy. There is a deductible assigned to each type of insurance, an amount that you pay out of pocket before your comprehensive coverage takes effect. Comprehensive insurance is often required if you lease or finance the purchase of a vehicle.
- **Uninsured or underinsured motorist insurance:** If you are hit by an uninsured or underinsured motorist, this insurance will help pay medical bills and damage to your car.
- **Medical payments insurance** is mandatory in some states and helps cover the costs of medical expenses associated with an accident, regardless of who is at fault.
- **Personal injury protection insurance** is coverage for certain medical bills and other expenses due to a car accident. Other covered expenses may include loss of income or childcare, depending on the specific terms of your policy.
- **Gap insurance** is designed to cover the gap between what is owed on the car and what the car is worth in the event your car is a total loss.
- **Rental reimbursement insurance** provides coverage for a rental car while your vehicle is under repair due to an accident.

You can also purchase other special insurance policies, such as classic car insurance, new car replacement insurance, and sound system replacement insurance, to name a few. It is essential to determine exactly what you need, as insurance policies can be expensive and vary based on your age, driving history, and location.

Equity

Equity is the difference between the market value of a property (car, house, etc) and the amount still owed on the loan.

Example $4.5.9$: Equity in House

You bought a house 15 years ago, taking out a $\$179,000$ mortgage at a 4% interest rate for 30 years. Your monthly payments are $\$854.58$.

1. How much will still be owed after making payments for 15 years?

2. If the house's value is now $\$299,000$, then how much equity do you have after making payments for 15 years?

Answer available at the end of this section.

Your Turn 4.5.9: Find Equity

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Answers

Example 4.5.1: Calculating the Payment for a Loan

1. The loan is for $\$28,500$, which is the principal. The rate is 3.99%, so $r = 0.0399$. The term of the loan is 5 years, so $t = 5$. Monthly payments mean $n = 12$.
Substituting these values into the formula

$$\begin{aligned} PMT &= \frac{P\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]} \\ &= \frac{28,500\left(\frac{0.0399}{12}\right)}{\left[1-\left(1+\frac{0.0399}{12}\right)^{-12\times 5}\right]} \\ &= \frac{28,500\,(0.003325)}{\left[1-(1+0.003325)^{-60}\right]} \\ &= \frac{94.7625}{\left[1-(1.003325)^{-60}\right]} \\ &= \frac{94.7625}{1-0.8194113482} \\ &= \frac{94.7625}{0.1805886518} \\ &= 524.7422751 \end{aligned}$$

The monthly payment needed is $\$524.74$.
Total paid is given by

$$\begin{aligned} \text{Total Paid} &= PMT \times 12 \times t \\ &= 524.75 \times 12 \times 5 \\ &= 31,484.40 \end{aligned}$$

We found that the interest paid by subtracting the loan from the total paid.

$$\begin{aligned} \text{Intrest Paid} &= \text{Total Paid} - P \\ &= \$31,484.40 - \$28,500 \\ &= \$2,984.40 \end{aligned}$$

We can also find PMT using the TVM calculator: Put the following values for the variable

$$\begin{aligned} \text{N} &= n \times t = 12 \times 5 = 60 \\ \text{I} &= 3.99 \ \text{ (Do not put percent)} \\ \text{PV} &= 28,500 \ \text{(LOAN)} \\ \text{FV} &= 0 \\ \text{PMT} &= \ \text{We need to find it.} \\ \text{P/Y} &= 12 \\ \text{C/Y} &= 12 \end{aligned}$$

And click Solve for PMT.

2. As we did above, we use the same formula to find the monthly payment, and we can verify your answer using the TVM calculator.

Put the following values for the variable

$$\begin{aligned} \text{N} &= n \times t = 12 \times 15 = 180 \\ \text{I} &= 5.75 \ \text{ (Do not put percent)} \\ \text{PV} &= 136,700 \ \text{(LOAN)} \\ \text{FV} &= 0 \\ \text{PMT} &= \ \text{We need to find it.} \\ \text{P/Y} &= 12 \\ \text{C/Y} &= 12 \end{aligned}$$

And click "Solve for PMT". The monthly payment needed is $\$1,135.18$.

Example 4.5.2: Reading from an Amortization Table

1. Reading the values at the top of the table, we see the principal is $\$10,000$, the interest rate is 4.75% and the term is 20 years.
2. The monthly payment is listed below the term of the loan, and is $\$64.62$.
3. $\$9,613.83$
4. $\$38.68$
5. Interest payment decreases each month, while the principal payment increases each month
6. $\$5509.37$

Example 4.5.3: 30-years Mortgage

Using the information above, $P = \$240,000$, $r = 0.065$ and $t = 30$. Substituting those values into the formula

$$\begin{aligned} PMT &= \frac{P\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]} \\ &= \frac{240,000\left(\frac{0.065}{12}\right)}{\left[1-\left(1+\frac{0.065}{12}\right)^{-12\times 30}\right]} \\ &\approx 1516.96 \end{aligned} \tag{4.5.4}$$

You input the above data into the scientific calculator, and you will find the answer.

His mortgage payment is $\$1516.96$.

We can check our answer using TVM calculator:

$$\begin{aligned} N &= n \times t = 12 \times 30 = 360 \\ \mathrm{I} &= 6.50 \ \ (\text{Do not put percent}) \\ PV &= 240,000 \ (\text{LOAN}) \\ FV &= 0 \\ \mathrm{PMT} &= (\text{We need to find it}) \\ \mathrm{P/Y} &= 12 \\ \mathrm{C/Y} &= 12 \end{aligned}$$

And click Solve for PMT.

Example 4.5.4: Total Paid and Interest

1. Using the mortgage payment of $\$1,516.96$ and t = 30 years in the formula

$$\begin{aligned} \text{Total Paid} &= PMT \times 12 \times t \\ &= 1,516.96 \times 12 \times 30 \\ &= 546,105.60 \end{aligned} \tag{4.5.5}$$

Evan will pay for the mortgage, which is $\$546,105.60$. This is his total payment.

2. We found that the total Evan will pay for the $\$240,000$ and mortgage (Principal) is $\$546,105.60$.

$$\begin{aligned} \text{Interest Paid} &= \text{Total Paid} - P \\ &= \$546,105.60 - \$240,000 \\ &= \$306,105.20 \end{aligned} \tag{4.5.6}$$

So Evan's interest will be $\$306,105.20$ over the lifetime of the mortgage.

Example 4.5.5: Reading a Credit Card Statement

1. The balance due is under the payment information heading and is $\$3,663.23$.
2. The minimum payment due is also under the payment information heading, and is $\$36.63$.
3. The time to pay off the balance using only minimum payments is below the payment information, and says it takes 2 years and 4 months to pay off the balance.
4. The interest rate for purchases is toward the bottom of the statement. It is 19.99%.
5. The maximum amount you can spend on the credit card is $\$9,000$. This is the credit limit.

Example 4.5.6: Computing Average Daily Balance

1. To find the average daily balance, we use the following steps.

Date	Balance	Days Until Balance Changes	Balance Times Days
July 1	$\$669.69$	$4-1=3$	$\$669.69 \times 3 = \$2,009.07$
July 4	$\$669.69 + \$71 = \$740.69$	$15-4=11$	$\$740.69 \times 11 = \$8,147.59$
July 15	$\$740.69 - \$139 = \$601.69$	$19-15=4$	$\$601.69 \times 4 = \$2,406.76$
July 19	$\$601.69 + \$109.62 = \$711.31$	$25-19=6$	$\$711.31 \times 6 = \$4,267.86$
July 25	$\$711.31 + \$59 = \$770.31$	$31-25=6$ and $6+1=7$	$\$770.31 \times 7 = \$5,392.17$

Find the sum of that last column. Adding that last column, we have a sum of $\$22,223.45$. There are 31 days in July, so divide the sum by 31, which gives an average of $\$716.89$, which is the average daily balance.

$$\text{average daily balance} = \frac{\$22,223.45}{31} = \$716.89 \tag{4.5.7}$$

2. We use $\$716.89$, as a principal and time will be 1 month with is same as $\frac{1}{12}$ year.

$$\begin{aligned}\text{Interst} &= \text{average daily balance} \times \text{interest rate} \times \text{time} \\ &= \$716.89 \times 0.137 \times \frac{1}{12} \\ &= \$8.18\end{aligned} \tag{4.5.8}$$

Or we can find monthly interest rate $\frac{13.7\%}{12} = 1.1416666$

$$\begin{aligned}\text{Interst} &= \text{average daily balance} \times \text{interest rate} \times \textit{time} \\ &= \$716.89 \times 0.01416666 \times 1 \\ &= \$8.18\end{aligned} \tag{4.5.9}$$

Interest to pay is $\$8.18$.

3. The balance due at the beginning of the next month is the sum of the unpaid balance in the given month, which is $\$770.31$ plus interest, $\$8.18$.

$$\begin{aligned}\text{Balance due} &= \$770.31 + \$8.18 \\ &= \$778.49\end{aligned} \tag{4.5.10}$$

4. $\frac{1}{36}$ of $\$778.49 = \$21.62$, which is greater than $\$20$. So minimum payment due is $\$21.62$.

Example 4.5.7**: Total Cost to Purchase a Car with Down Payment**

The price was $\$19,800$. The sales tax of 9.5% is based on this number. The sales tax comes to

$$\$19,800 \times 0.095 = \$1,881$$

Adding all the fees to the price and the sales tax brings the total cost of the car to

$$19,800 + \$1,88 + \$300 + \$321.50 + \$1,100 = \$23,402.50$$

Her down payment is applied to this number, so the $\$5,000$ is subtracted from $\$23,402.50$.
The subtraction yields the amount to be financed, which is $\$18,402.50$.

Example 4.5.8**: New Car Payments**

1. The loan is for $\$18,000$, which is the principal. The rate is 6%, so $r = 0.06$. The term of the loan is 5 years, Monthly payments mean $n = 12$. Substituting these values into the formula as we did in the previous example.

$$PMT = \frac{P\left(\frac{r}{n}\right)}{\left[1 - \left(1 + \frac{r}{n}\right)^{-nt}\right]} \tag{4.5.11}$$

Your monthly payment needed is $\$348$.

We can check our answer using the TVM solver.

$$\begin{aligned}\text{N} &= n \times t = 12 \times 5 = 60 \\ \text{I} &= 6 \ \text{(Do not put percent)} \\ \text{PV} &= 18,000 \ \text{(LOAN)} \\ \text{FV} &= 0 \\ \text{PMT} &= \text{We need to find it.} \\ \text{P/Y} &= 12 \\ \text{C/Y} &= 12\end{aligned}$$

And click Solve for PMT.

2. Using the mortgage payment of $\$348$ and $t = 5$ years in the formula

$$\begin{aligned}\text{Total Paid} &= PMT \times 12 \times t \\ &= 348 \times 12 \times 5 \\ &= 20,880.00\end{aligned} \tag{4.5.12}$$

You will pay for a total of $\$20,880$.

3. We found that the total you will pay for the $\$18,000$ loan (Principal) is $\$20,880$.

$$\begin{aligned}\text{Intrest Paid} &= \text{Total Paid} - P \\ &= 20,880 - 18,000 \\ &= 2,880\end{aligned} \tag{4.5.13}$$

So your interest will be $\$2,880$ over the life of the loan.

Example 4.5.9**: Equity in House**

1. We use TVM calculator to find the amount that you still owe.

$$\begin{aligned}N &= n \times t = 12 \times 15 = 180 \\ \text{I} &= 4 \ \text{(Do not put percent)} \\ PV &= 179,000 \ \text{(LOAN)} \\ \text{PMT} &= -854.58 \ \text{(Do not forget to put negative symbol)} \\ \text{P/Y} &= 12 \\ \text{C/Y} &= 12\end{aligned}$$

Click Solve for FV. The face value (FV) will be $\$115,529.98$, and this is the value you still owe after 15 years.

2. The house equity is $\$299,000 - \$115,529.98 = \$183,470.02$

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4.6: Income Tax and FICA Taxes

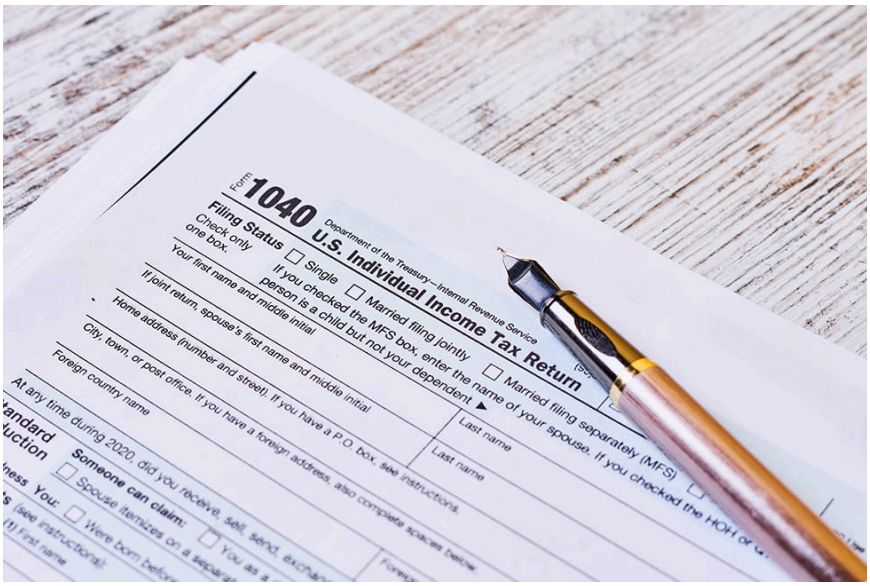


Figure 4.6.1: Federal income tax is a concern for most US citizens. (credit: "1040 US tax form" by Marco Verch Professional Photographer/ Flickr, CC BY 2.0)

Learning Objectives

1. Determine gross, adjusted gross, and taxable income.
2. Apply deductions and credits to basic income tax calculations.
3. Compute FICA tax and income tax.

Before the start of the American Civil War in 1861, most of the country's revenue came from tariffs on trade and excise taxes. However, this fell far short of the high cost of the war. Because of this, the federal government enacted the nation's first income tax with the Revenue Act of 1861, which created the Internal Revenue Service as we know it today.

No one likes paying income tax, but it is a reality of life. In this section, we will learn about Form 1040, the U.S. Individual Income Tax Return, and ways to prepare for tax time.

The U.S. tax code may change from year to year. Because of this, this section includes examples of how taxes, deductions, and exemptions might be computed. The types of income, deductions, and exemptions that are used in the examples are used in the current tax code.

Income Tax Terminology

Your income drives how much you pay in taxes. The more you earn, the more you are likely to pay. But your income alone is not the full story. When you add all the money you earn from your job, freelance work, interest from savings, and other sources, you have your gross income. If you are an employee, your income from your job will be reported on a W-2, which is sent to you by your employer. Income from freelance work will be reported on a 1099-MISC form, and is sent by the company that paid you. Income from interest is reported on a 1099 -INT form and comes from the entity that paid the interest.

Before you determine how much you owe in taxes, you will make certain adjustments to that gross income. You will deduct or subtract some of the income from the gross income. That's your adjusted gross income or AGI. That is still not what you are taxed on. Next, you need to apply exemptions to your income. These are pieces of income that the government does not tax. After that is done, you reach your taxable income. We will look at each of these parts of the taxable income.

Tax Withholding: W-2 Form

"Tax withholding" refers to the portion of an employee's wages that an employer deducts and sends directly to the government as partial payment of the employee's income tax. It serves as a way to pay income taxes gradually over the course of the year rather than all at once when you file your return. You can find **federal income tax withholding** on your **Form W-2.**

Who Knew? Gifts and Winning

Money given as a gift may be taxed if the gift amount is high enough. If you win $\$50,000$ in the lottery, that money is taxed as income. If you give a family member a large cash gift, that gift will be subject to tax, provided that the gift exceeds the federally set limits.

You will notice that your paycheck already has taxes taken out of it. Your employer will withhold some of your income, sending it directly to the federal, state, and local governments. It is an estimate of how much you will owe in income tax. Ultimately, it reduces the amount you will pay when your taxes are due. If they withhold too much income, you will receive the extra they withheld in the form of a refund.

States with No State Tax

Here are the U.S. states that **do not have a state income tax** as of 2025: **Alaska, Florida, Nevada, South Dakota, Texas, Washington, and Wyoming**

Gross Income (Total Income)

Gross income (GI) refers to the total earnings an individual or business receives before any deductions or taxes are taken out. Gross income includes wages, salaries, bonuses, rental income, tips, investment income, unemployment compensation, profit from business, and game-show winnings.

Adjusted Gross Income

Adjusted Gross Income (AGI) is your **gross income minus certain allowable deductions**, known as **adjustments**. Some of the adjustments include student loan interest, contributions to a traditional IRA, contributions to a health savings account (HSA), alimony paid, educator expenses, tuition, and fees.

$$\text{Adjusted Gross Income (AGI)} = \text{Gross Income} - \text{adjustment}$$

What is Deduction? Which Deduction Do We Use?

A **deduction** is an amount you can subtract from your **adjusted gross income (AGI)** to reduce the amount of income that's subject to tax. Deductions help lower your **taxable income**, which can decrease the amount of tax you owe. There are two types of deductions:

Standard Deduction: A fixed amount set by the IRS based on your **filing status** (e.g., single, married, filing jointly).

Itemize Deduction: If your deductible expenses are higher than the standard deduction, you can itemize. Common itemized deductions include: Mortgage interest that you paid, charitable contributions if you made any, medical bills over a threshold, medical insurance under certain circumstances, and property taxes. If you add all these up, and they are all legal deductions, the sum is subtracted from your gross adjusted gross income.

During calculation, you have to use only one deduction, whichever is higher, **NOT BOTH.**

Your Turn 4.6.1: Which Deduction? Find Taxable Income

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How to Find Taxable Income?

Taxable income (TI) is the portion of your income that is subject to income tax after all **deductions** (standard or itemized) and **adjustments** (like those used to calculate AGI) have been subtracted from your gross income.

$$\text{Taxable Income} = \text{AGI} - \text{Deduction}$$

Remember that your AGI is not your **taxable income**.

Example 4.6.2: Find GI, AGI, and TI

Puspa and Sami were calculating their taxes for 2022. Their income for that year was $\$176,815$. They contributed $\$7,900$ to tax-deferred retirement plans. They also have a rental income of $\$19,800$ for a home they rent and $\$18,200$ for profits from a small business. They have the following deduction: mortgage interest payments of $\$15,600$. State and local taxes deduction of $\$7,400$. And charitable contribution amount of $\$10,200$. The standard deduction for a married couple filing jointly is $\$25,900$.

1. What is their gross income?
2. What is their adjusted gross income?
3. What is their taxable income?

Answer available at the end of this section.

Your Turn 4.6.2: Find Taxable Income

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Tax Credits

Another piece of the tax puzzle is tax credits. This is money subtracted from the tax you owe.

Tax credits are very different from deductions or exemptions. Deductions and exemptions are taken away from your gross income before the tax you owe is calculated. A tax credit is subtracted, dollar for dollar, from your tax bill. Once the tax you

owe is calculated, subtract any tax credits from that calculated tax.

Some of the tax credits are refundable. This means that if subtracting them from your tax results in a negative number, you receive a tax refund. For more details, see this article about tax credits(opens in new window).

Refundable and Non-Refundable Tax Credit

Refundable Tax Credit	Non-Refundable Tax Credit
1. Can reduce your tax bill **below zero**. 2. If the credit is larger than the tax you owe, the **extra amount is refunded to you**	1. Can reduce your tax bill **to zero**, but **not below zero**. 2. If the credit exceeds the tax you owe, the excess is **lost**.

The federal government has imposed income limits and restrictions on those eligible to receive tax credits because the value of these credits is so high. Here is a partial list of tax credits that you might qualify for:

- **Earned income credit** is a refundable tax credit for low- to moderate-income workers and ranges from $\$632$ to $\$7,830$ depending on dependents and income. This is **refundable**.
- **The American Opportunity Credit** is a tax credit available to parents with children enrolled at least half-time in college and pursuing a degree. This credit is worth $\$2,500$ per student for the first 4 years of undergraduate school, subject to income limits. This is a **refundable** tax credit.
- **The lifetime learning credit** is a credit equal to 20% of educational expenses, up to $\$2,000$ per year, subject to income limits. There is no cap on how many years you can apply this credit to. This is a **nonrefundable** tax credit.
- **The child tax credit** is worth $\$2,000$ per child under the age of 17 if that child lives at home for at least half the year, subject to income limits. This is a refundable tax credit.
- **The Child and Dependent Care Tax Credit** was designed to help pay for child care while the parent is working. The amount of the credit is dependent on your income. However, the maximum amount that can be received is, in 2026, $\$3,000$ for one eligible person, or $\$6,000$ for two or more qualifying people. A dependent qualifies if they are a child under 13 years old, a spouse who is unable to care for themselves, or some other qualifying person. This is a **refundable** tax credit.
- **The premium tax credit** was created by the Affordable Care Act and is received by many people throughout the year. In essence, it is a health insurance premium subsidy. The amount of the credit is based on your income and the price of health insurance in your area. This is a **refundable** tax credit.

Example $4.6.3$: Apply a Tax Credit

1. Kaitlyn has calculated the tax she owes to be $\$5,200$. However, she receives an earned income tax credit of $\$1,715$. How much does Kaitlyn owe after applying the earned income tax credit?

2. Chanajah calculated their tax owed, which came to $\$4,300$. They have an earned income tax credit of $\$2,190$, a child tax credit of $\$2,000$, and a child and dependent care tax credit of $\$4,000$. How much tax does Chanajah owe, or how much will they get in a refund?

Answer available at the end of this section.

Your Turn $4.6.3$: Applying Tax Credit

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Deduction vs Credits

- **Deduction**: Lowers how much of your income is *taxable*.
  Example: If you make $\$60,000$ and have a $\$5,000$ deduction, you now only pay taxes on $\$55,000$.

- **Tax credit**: Lowers your *actual tax bill*, dollar for dollar.
  Example: If you owe $\$5,000$ in taxes and you have a $\$1,000$ credit, you only owe $\$4,000$.

Big difference:

- A **deduction** reduces the *amount of income* taxed.
- A **credit** directly reduces *how much tax* you owe.

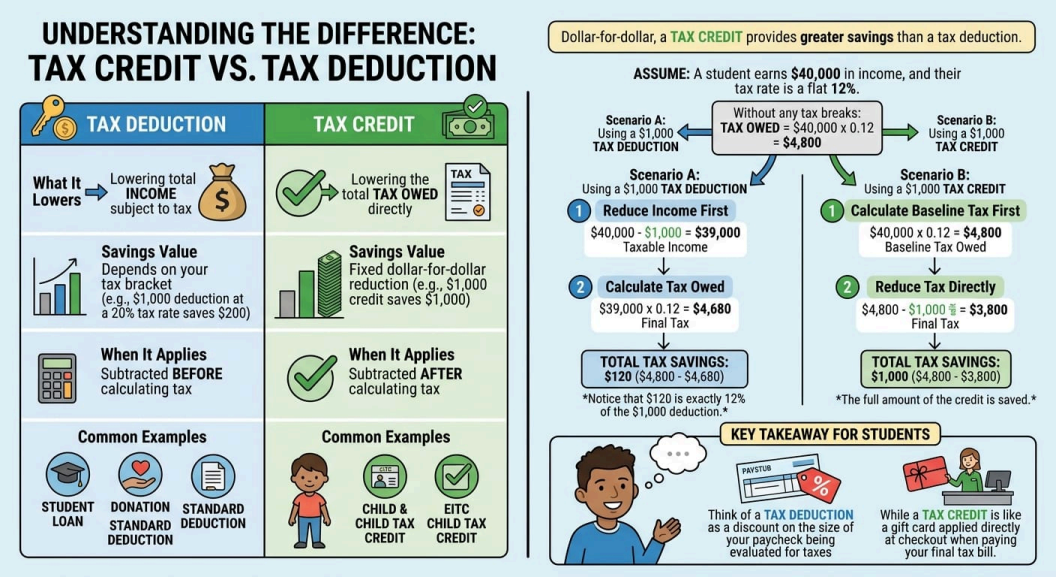


Figure 4.6.2: Tax Credit Vs Tax Deduction

Example 4.6.4: Tax Credit or Deduction

Assume a student earns $\$40,000$ in income for the year, and their tax rate is a flat 12.

1. How much will his tax bill be reduced if he makes a $\$2000$ contribution to charity?
2. How much will his tax bill be reduced if he qualifies for a $\$2,000$ tax credit?

Answer available at the end of this section.

Your Turn 4.6.4: Deduction or Tax Credit

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Computing FICA Taxes

FICA stands for the **Federal Insurance Contributions Act** of 1935. FICA taxes are used solely to fund Social Security and Medicare and are separate from federal income tax.

FICA taxes are the taxes taken out of your paycheck to fund **Social Security** and **Medicare**.

Social Security helps people when they retire, become disabled, or die (their family can sometimes get benefits).

Medicare helps mainly older people $(65+)$ with health insurance.

2022 FICA Taxes Rate

Employee's Rates	Self-Employed Rates1
• SSI: 6.20% on the first $\$147,000$ of wages • Medicare: 1.45% on all wages • Tota FICA tax: 7.65% on upto $\$147,000$ plus medicare tax above $\$147,000$	• 12.4% on the first $\$147,000$ of wages • Medicare: 2.90% on all wages • Tota FICA tax:15.3% on upto $\$147,000$ plus medicare tax above $\$147,000$

FICA taxes include Social Security and Medicare. The Social Security tax applies to the first $\$147,000$ of income, while the Medicare tax applies to all income.

It amounts to 7.65% of your gross pay, which is withheld from your paycheck automatically. Your employer is required to match the 7.65% amount. Of the 7.65%, 6.2% goes to Social Security (SSI), and 1.45% goes to Medicare.

As of 2022, SSI tax only applies to the first $\$147,000$ of earnings. Any gross income above that is not taxed for Social Security. This limit changes every year.

Medicare tax, on the other hand, applies to the entirety of your gross income.

Example $4.6.5$: Computed FICA Taxes

1. McKenzi earned $\$2,700$ in gross income, before taxes, in a given 2-week period. How much does she owe in FICA taxes, and how much of that is for SSI?

2. Suppose you earn $\$50,000$ from your own business in 2022. How much do you owe in FICA taxes, and how much of that is for SSI?

3. Renard earned $\$195,000$ in wages for the year as an employee. How much in SSI taxes does Renard owe for the year? What are the total FICA taxes?

Answer available at the end of this section.

Answer available at the end of this section.

Your Turn 4.6.5: Calculate FICA Taxes

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Calculating Your Income Tax

Your income tax bill and your income tax rate are based on your taxable income. The tax system in the United States is progressive, meaning that the tax rates are marginal, so the higher your taxable income, the higher the tax rate you will pay.

Taxable income is broken into brackets, or ranges of income. Each bracket has a different tax rate. The tax brackets and rates for single filers as of 2022 are given below:

Marginal Tax Rate

Marginal Tax Rate	For Single Filers	For Married Individuals Filing Joint Returns	For Heads of Households
10%	$0 – $10,275	$0 – $20,550	$0 – $14,650
12%	$10,275 – $41,775	$20,550 – $83,550	$14,650 – $55,900
22%	$41,775 – $89,075	$83,550 – $178,150	$55,900 – $89,050
24%	$89,075 – $170,050	$178,150 – $340,100	$89,050 – $170,050
32%	$170,050 – $215,950	$340,100 – $431,900	$170,050 – $215,950
35%	$215,950 – $539,900	$431,900 – $647,850	$215,950 – $539,900
37%	$539,900 or more	$647,850 or more	$539,900 or more
Standard Deduction	**$12,950**	**$25,900**	**$19,400**

Question: So if your taxable income is $76,500 and you file as a single filer, your tax bill will be 22% of that $76,500, right?

Wrong. Your income is split among those brackets, and the money in each bracket is taxed at that bracket's marginal tax rate. Seems confusing. Here is a list of steps to follow to find the tax owed.

Why is the person in 22% tax bracket?

Since that person's taxable income $76,500 falls in the single filer column in a 22% marginal rate row. We call the person whose taxable income falls in the 22% tax bracket. So that person needs to pay 10%, 12%, and 22% of his portion of the taxable income as shown in the following table.

The following calculation shows how to calculate the tax bill of the above single filer. That person's taxable income is in 22% tax bracket,

Marginal Tax Rate for Taxpayer

Marginal Tax Rate	Taxable income that applies to the given marginal tax rate	Tax calculation

10%	$\$10,275 - \$0 = \$10,275$	$0.10 \times \$10,275 = \$1,027.50$
12%	$\$41,775 - \$10,275 = \$31,500$	$0.12 \times \$31,500 = \$3,780$
22%	$\$76,500 - \$41,775 = \$34,725$	$0.22 \times \$34,275 = \$7,639.50$

So the total tax bill will be

$$\$1,027.50 + \$3,780 + \$7,639.50 = \$12,447$$

We use the following steps to calculate a tax bill. It is important to understand how to calculate the marginal tax amount for each tax rate.

Step to Find the Tax Bill: 2021 Tax Rate

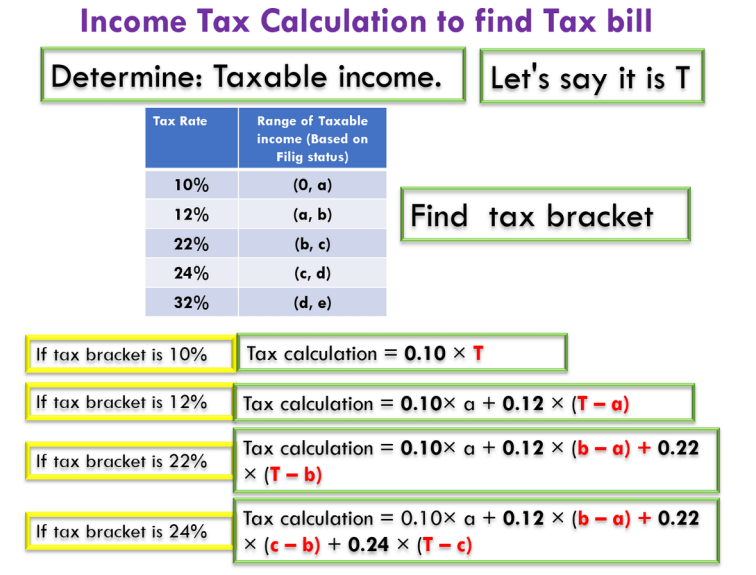


Figure 4.6.1: Tax Calculation 2021 Tax Rate

There are various tax brackets, and the rates may change from one year to the next. The income limits may also change. For all examples going forward, we will use the single filer tax brackets, even if those brackets are not appropriate (e.g., married or head of household filers).

Example 4.6.6: Income Tax on Taxable Income

Use the 2022 Marginal Tax Rate Table to find the federal tax bill. Faith has a taxable income of $\$103,650$, and she is single.

Answer available at the end of this section.

Your Turn 4.6.6: Determine Marginal Tax

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Tax Due, Refund, or Owe Taxes

Remember that your employer will estimate how much tax you will owe and withhold it from your paychecks. This means you may have already paid some, if not all, and more of the income tax you owe.

If the total taxes owed after tax credit (if there is one) is more than the tax withholding (in W-2), the taxpayer is due a refund, and if the total taxes owed after tax credit (if there is one) is less than the tax withholding (in W-2), the taxpayer owes taxes.

Example 4.6.7: Finding Income Tax Owed

Emmanuel is preparing his taxes for the year 2022. His filing status is married jointly. His and his wife's income that is in W-2 for the year was $\$95,250$ He also has income from some freelance artwork he did, amounting to $\$7,500$, which is shown in the form 1099. Emmanuel also deposited $\$4,500$ into a tax-deferred retirement plan. He paid $\$7,920$ in mortgage interest for the year and $\$3,740$ in property taxes. He also qualified for $\$4,000$ in child tax credits. Based on this information, how much tax does Emmanuel owe, or how much does he get in a refund?

Answer available at the end of this section.

Your Turn 4.6.7: Calculate Income Tax

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Example 4.6.88: Refund or Tax Owe?

The Halls are a married couple with two children. Their total adjusted gross income is $\$181,770$. They have following itemize deductions:

$$\begin{aligned} &\$9050 \text{ in property taxes} \\ &\$9850 \text{ in charitable contributions} \\ &\$910 \text{ in childcare expenses} \end{aligned} \tag{4.6.1}$$

In addition, they are eligible for a $\$2000$ tax credit for each child. (For this problem, treat this credit as non-refundable.) Round all answers in this problem to the nearest cent, if necessary. To complete this problem, you will need to refer to the 2020 tax table.

2020 Marginal Tax Rates

Rate	Single	Married Filing Jointly
10%	$0 — $9,875	$0 — $19,750
12%	$9,875 — $40,125	$19,750 — $80,250
22%	$40,125 — $85,525	$80,250 — $171,050
24%	$85,525 — $163,300	$171,050 — $326,600
32%	$163,300 — $207,350	$326,600 — $414,700
35%	$207,350 — $518,400	$414,700 — $622,050
37%	$518,400 or more	$622,050 or more

Standard Deduction

Single	Married Filing Jointly
$12,400	$24,800

1. If they itemize their deductions, how much will the Hall' total deduction be?

2. Should they itemize their deductions or use the standard deduction?

3. What is the Hall' taxable income for 2020?

4. Calculate the Hall' total taxes owed, *before* tax credits.

5. Calculate the Hall' total taxes owed, *after* tax credits.

6. They have prepaid $22,253 over the course of the year through withholding. Do the Hall owe taxes, or are they due a refund? How much do they owe, or what refund are they due?

Answer available at the end of this section.

Your Turn 4.6.8: Tax Owe or Refund?

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Your Turn 4.6.9: Check Your Understanding

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A **tax deduction** reduces your **taxable income**, while a **tax credit** directly reduces your **final tax bill**.

Dollar for dollar, a **tax credit provides greater savings** than a tax deduction.

Core Comparison

Feature	Tax Deduction	Tax Credit
What it lowers	Lowering total **income** subject to tax	Lowering the total **tax owed** directly
Savings value	Depends on your tax bracket (e.g., $1,000 deduction at a 20% tax rate saves $200)	Fixed dollar-for-dollar reduction (e.g., $1,000 credit saves $1,000)
When it applies	Subtracted **before** calculating tax	Subtracted **after** calculating tax
Common examples	Student loan interest, charitable donations, standard deduction	Child Tax Credit, Earned Income Tax Credit (EITC)

Step-by-Step Example

Assume a student earns **$40,000** in income for the year, and their tax rate is a flat **12%**.

Without any tax breaks:

* **Tax Owed:** $$40,000 \times 0.12 = \mathbf{$4,800}$

Scenario A: Using a $1,000 Tax Deduction

1. **Reduce Income First:** Subtract the $1,000 deduction from total earnings.

$$\text{Taxable Income} = \$40,000 - \$1,000 = \$39,000 \tag{4.6.2}$$

2. **Calculate Tax Owed:** Apply the 12% tax rate to the new taxable income.

$$\text{Final Tax} = \$39,000 \times 0.12 = \mathbf{\$4,680} \tag{4.6.3}$$

* **Total Tax Savings:** **$120** ($$4,800 - $4,680$). *Notice that $120 is exactly 12% of the $1,000 deduction.*

Scenario B: Using a $1,000 Tax Credit

1. **Calculate Baseline Tax:** Calculate tax on full earnings first.

$$\text{Tax Owed} = \$40,000 \times 0.12 = \$4,800 \tag{4.6.4}$$

2. **Reduce Tax Directly:** Subtract the $1,000 credit directly from the tax owed.

$$\text{Final Tax} = \$4,800 - \$1,000 = \mathbf{\$3,800} \tag{4.6.5}$$

* **Total Tax Savings:** **$1,000** ($$4,800 - $3,800$). *The full amount of the credit is saved.*

Key Takeaway for Students

Think of a **tax deduction** as a discount on the size of your paycheck being evaluated for taxes, while a **tax credit** is like a gift card applied directly at checkout when paying your final tax bill.

Answers

Example 4.6.2: Find GI, AGI, and TI

1. $\text{Adjusted Gross Income (AGI)} = \$176,815 + \$19,800 + \$18,200 = \$214,815$
2. Here, $\$7,900$ to a tax-deferred retirement account is the adjustment. So

$$\begin{aligned}\text{Adjusted Gross Income (AGI)} &= \text{Gross Income} - \text{adjustment} \\ &= \$214,815 - \$7,900 \\ &= \$206,915\end{aligned}$$

3. Since itemized deduction = $\$15,600 + \$7,400 + \$10,200 = \$33,200$ is more than the standard deduction, we must subtract the itemized deduction to find taxable income.

$$\begin{aligned}\text{Taxable Income} &= AGI - \text{deduction} \\ &= \$206,915 - \$33,200 \\ &= \$173,715\end{aligned}$$

Note: For deductions, always find both options (itemize and standard) and use the higher one. If you subtract the higher deduction, your taxable income will be less. That's what all taxpayers want.

Example 4.6.3: Apply a Tax Credit

1. The amount of taxes Katilyn owes is her calculated tax of $\$5,200$ minus the credit she receives. The amount she owes in taxes is

$$\$5,200 - \$1,715 = \$3,485$$

2. Adding Chanajah's tax credits together, we find their total to be $\$8,190$. That is more than the tax they owe, which was $\$4,300$. Each of those credits is refundable, which means they will receive a refund. Subtracting the credits from the tax owed yields

$$\$4,300 - \$8,190 = -\$3,890$$

This is negative, so it represents a refund of $\$3,890$.

Example 4.6.4: Tax Credit or Deduction

If you itemize your deduction, the $\$2000$ contribution to the charity reduces your taxable income by $\$2000$. Since you are in 12% tax bracket, your tax bill will be reduced by $\$2000 \times 0.12 = \$240.$, but if you use the standard deduction (when your itemized deduction is less than the standard deduction), your tax bill is reduced by nothing..

Whether you use itemize or standard deduction, your tax bill is reduced by $\$2000$.

Example 4.6.5: Computed FICA Taxes

Example 4.6.5: Computed FICA Taxes

1. The FICA tax is 7.65% of her gross earnings. 7.65% of her $\$2,700$ is $\$206.55$. Also, the SSI is 6.2% of her income, which is $0.062 \times \$2700 = \$167.40$
2. The FICA tax is $15.3\% \times \$50,000 = \$7,650$. SSI is $12.4\% \times \$50,000 = \$6,200$.
3. Since Renard earned more than the taxable limit of $\$147,000$, he only pays the 6.2% SSI tax on $\$147,000$. This comes to $\$9,114 = 0.062 \times \$147,000$.

$$\begin{aligned}\text{FICA Tax} &= 7.65\% \text{ of } \$147,000 + 1.45\% \text{ of } (\$195,000 - \$147,000) \\ &= 0.0765 \times \$147,000 + 0.0145 \times \$48,000 \\ &= \$11,941.50\end{aligned}$$

Example 4.6.6: Income Tax on Taxable Income

$$0.10 \times \$10,275 + 0.12 \times (\$51,775 - \$10,275) + 0.22 \times (\$89,075 - \$41,775 + 0.24 \times (\$103,650 - \$89,075) = \$18,711.50 \tag{4.6.6}$$

Faith's taxable income falls in the marginal tax bracket of 24%. So she needs to pay 10%, 12%, 22%, and 24% of her portion of the taxable income as shown in the following table.

Marginal Tax Rate for Taxpayer

Marginal Tax Rate	Taxable income that applies to the given marginal tax rate	Tax calculation
10%	$\$10,275 - \$0 = \$10,275$	$0.10 \times \$10,275 = \$1,027.50$
12%	$\$41,775 - \$10,275 = \$31,500$	$0.12 \times \$31,500 = \$3,780$
22%	$\$89,075 - \$41,775 = \$47,300$	$0.22 \times \$47,300 = \$10,406$
24%	$\$103,650 - \$89,075 = \$14,575$	$0.24 \times 14,575 = \$3,498$

So Faith tax bill is

$$\$1,027.50 + \$3,780 + \$10,406 + \$3,498 = \$18,711.50$$

Example 4.6.7**: Finding Income Tax Owed**

We need to know Emmanuel's taxable income, based on gross income, deductions, and exemptions.

His gross income is the amount from his W-2 and his 1099-MISC. Adding these gives a

$$\text{Gross income} = 95,250 + \$7,500 = \$102,750$$

$\$7,500$ is his adjustment because it is deposited in the tax-deferred retirement plan. So

$$\begin{aligned} \text{Adjusted Gross Income (AGI)} &= \text{Gross Income} - \text{adjustment} \\ &= \$102,750 - \$4,500 \\ &= \$98,250 \end{aligned}$$

The amount $\$7,920$ for mortgage interest, $\$3,740$ for property taxes are itemized deduction.

$$\begin{aligned} \text{Itemize Deduction} &= \$7,920 + \$3,740 \\ &= \$11,660 \end{aligned}$$

From the 2022 tax table, his standard deduction (for a married couple) is $\$25,900$, which is more than the itemized deduction. So we need to use the standard deduction to find the taxable income.

$$\begin{aligned} \text{Taxable Income} &= \$98,250 - \text{deduction} \\ &= \$98,250 - \$25,900 \\ &= \$72,350 \end{aligned}$$

His taxable income belongs to the 12% marginal tax bracket.

Marginal Tax Rate

Marginal Tax Rate	Taxable income that applies to the given marginal tax rate	Tax calculation
10%	$\$20,550 - \$0 = \$20,550$	$0.10 \times \$20,550 = \$2,055$
12%	$\$72,350 - \$20,550 = \$51,800$	$0.12 \times \$51,800 = \$6,216$

Total tax computation $\$2,055 + \$6,216 = \$8,271$

Once his taxes are computed, he subtracts his tax credits. His only tax credit is $\$4,000$. The total he owes in taxes is

$$\text{Total tax owe} = \$8,271 - \$4,000 = \$4,271$$

So Emmanuel owe $\$4,271$ tax.

Example 4.6.88**: Refund or Tax Owe?**

1. Itemize deduction: $\$9,050 + \$9,850 + \$9,100 = \$28,000$
2. They should use itemize deduction of $\$28,000$ since it is more then standard deduction $\$24,800$.
3. Taxable income is

$$\begin{aligned} \text{Taxable income} &= \text{AGI} - \text{deduction} \\ &= \$181,770 - \$28,000 \\ &= \$153,770 \end{aligned}$$

4. The Hall's taxable income belongs to the tax bracket of 22%.

Marginal Tax calculation

Marginal Tax Rate	Taxable income that applies to the given marginal tax rate	Tax calculation
10%	$\$19,750 - \$0 = \$19,750$	$0.10 \times \$19,750 = \$1,975$
12%	$\$80,250 - \$19,750 = \$60,500$	$0.12 \times \$60,500 = \$7,260$
22%	$\$153,770 - \$80,250 = \$73,520$	$0.22 \times \$73,520 = \$16,174.40$

Calculate the Hall' total taxes owed, *before* tax credits, is $\$1,975 + \$7,260 + \$16,174.40 = \$25,409.40$

5. Once his taxes are computed, he subtracts his tax credits. His only tax credit is $\$4,000$. Because $2 \times \$2000$, since for each for each kid child tax credit is $\$2,000$.

$$\begin{aligned}\text{Total tax after tax credit} &= \$25,409.40 - \$4,000 \\ &= \$21,409.40\end{aligned}$$

6. They have prepaid $\$22,253$ over the course of the year through withholding. That means they are already more than they are liable to. So they will get a refund of $\$22,253 - 21,409.40 = \$843.60$

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5: VOTING THEORY AND FAIRNESS CRITERIA

5.1: Voting Methods
5.2: Fairness Criteria

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5.1: Voting Methods

Learning Objectives

1. Use the Plurality method to determine the winner of an election.
2. Use the Borda count method to determine the winner of an election.
3. Use the Instant Runoff Voting method to determine the winner of an election.
4. Use the Copland method to determine the winner of an election.

To begin, we will want more information than a traditional ballot typically provides. A traditional ballot usually asks you to pick your favorite from a list of choices. This ballot fails to provide any information on how a voter would rank the alternatives if their first choice were unsuccessful.

Preference ballot

A **preference ballot** is a type of voting system where voters rank candidates or options in order of preference, rather than selecting just one.

Example 5.1.1: Understanding Perference Ballot

A vacation club is trying to decide which destination to visit this year: Hawaii (H), Orlando (O), or Anaheim (A). Their votes are shown below:



Answer available at the end of this section.

Your Turn 5.1.1: Understanding Preference Table

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The Plurality Method.

The voting method we're most familiar with in the United States is the **plurality method**.

How to find Winner by Plurality Method?

This method declares the choice with the most first-place votes as the winner. Ties are possible and must be settled through some sort of run-off vote.

Answer available at the end of this section.

This method is sometimes mistakenly referred to as the majority method or “majority rules,” but it is not necessary for a choice to have gained a majority of votes to win.

A majority is over 50% of the total votes; a winner can have a **plurality** without having a majority.

Example $5.1.2$: Using Plurality Method

We had the preference table in our election, for example $5.1.1$, find the winner by the plurality method. Does that candidate have the majority of the first-place votes?



Answer available at the end of this section.

Your Turn $5.1.2$: Plurality Method

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Borda Count Method

Borda Count is another voting method, named for Jean-Charles de Borda, who developed the system in 1770.

How to Find Winner by Borda Count Method?

This method assigns points to candidates based on their ranking: 1 points for the last choice, 2 points for the second-to-last choice, and so on. The point totals for all ballots are summed, and the candidate with the highest total is the winner.

Example 5.1.3:The Borda Count Method



A group of mathematicians is getting together for a conference. The members are coming from four cities: Seattle, Tacoma, Puyallup, and Olympia. Their approximate locations on a map are shown to the right.

The votes for where to hold the conference were:

[redacted]

Answer available at the end of this section.

Your Turn 5.1.3: Borda Count Method

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The Instant Runoff Voting (Plurality with Elimination) Method

Instant Runoff Voting (IRV), also called Plurality with Elimination, is a modification of the plurality method that attempts to address the issue of insincere voting.

Insincere Voting

Situations like the one in example 6.1.4 in the previous section (see the table below), when there is more than one candidate that shares somewhat similar points of view, can lead to insincere voting. Insincere voting is when a person casts a ballot contrary to their actual preference for strategic purposes.



In that case, the democratic leadership might realize that Don and Key will split the vote, and encourage voters to vote for Key by officially endorsing him. Not wanting to see their party lose the election, as happened in the scenario above, Don's supporters might insincerely vote for Key, effectively voting against Elle.

How to find the Winner by Instant Runoff Voting (IRV) Method?

In IRV (Plurality with Elimination), if the candidate or choice gets the majority of the first-place votes, that choice would be the winner. If not, the choice with the *least* first-place votes is then eliminated from the election, and any votes for that candidate are redistributed to the voter's next choice. This continues until a choice has a majority (over 50%).

This is similar to the idea of holding runoff elections, but since every voter's order of preference is recorded on the ballot, the runoff can be computed without requiring a second costly election.

This voting method is used in several political elections around the world, including the election of members of the Australian House of Representatives, and was used for county positions in Pierce County, Washington until it was eliminated by voters in 2009. A version of IRV is used by the International Olympic Committee to select host nations.

Example $5.1.4$: Find the Winner Using IRV

Consider the preference schedule below, in which a company's advertising team is voting on five different advertising slogans: A, B, C, D, and E. The election result is summarized in the given preference table. Find the winner using IRV.



Answer available at the end of this section.

Your Turn $5.1.4$: IVR Method

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Copland Method

So far none of our voting methods have satisfied the Condorcet Criterion. The Copeland Method specifically attempts to satisfy the Condorcet Criterion by looking at pairwise (one-to-one) comparisons.

How to Find the Winner by Copeland's (Pairwise Comparison Method) Method?

In this method, each pair of candidates is compared, using all preferences to determine which of the two is more preferred. The more preferred candidate is awarded 1 point. If there is a tie, each candidate is awarded $\frac{1}{2}$ point. After all pairwise comparisons are made, the candidate with the most points, and hence the most pairwise wins, is declared the winner.

Variations of Copeland's Method are used in many professional organizations, including the election of the Board of Trustees for the Wikimedia Foundation that runs Wikipedia.

Formula: Number of Comparisons

number of comparison for n candidate = $\frac{n(n-1)}{2}$

Example 5.1.5: Copland Method

Consider our vacation group example from the beginning of the chapter. Determine the winner using Copeland's Method.



Answer available at the end of this section.

Example 5.1.6: Copland Method

Consider the advertising group's vote we explored earlier. Determine the winner using Copeland's method.



Answer available at the end of this section.

Your Turn 5.1.6: Copland Method

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Attribution

[2] This data is loosely based on the 2008 County Executive election in Pierce County, Washington. See

Answers

Example $5.1.1$**: Understanding Perference Ballot**

These individual ballots are typically combined into one **preference schedule (or preference table)**, which shows the number of voters in the top row that voted for each option:



Notice that by totaling the vote counts across the top of the preference schedule, we can recover the total number of votes cast: $1+3+3+3=10$ total votes.

How to find Winner by Plurality Method?

Example $5.1.2$**: Using Plularity Method**

For the plurality method, we only care about the first-choice options. Totaling them up:

Anaheim: $1+3=4$ first-choice votes

Orlando: 3 first-choice votes

Hawaii: 3 first-choice votes

Anaheim is the winner using the plurality voting method.

Notice that Anaheim won with 4 out of 10 votes, 40% of the votes, which is a plurality of the votes but not a majority.

Example $5.1.3$**:The Borda Count Method**

In each of the 51 ballots ranking Seattle first, Puyallup will be given 1 point, Olympia 2 points, Tacoma 3 points, and Seattle 4 points. Multiplying the points per vote times the number of votes allows us to calculate the points awarded:



Adding up the points:

Seattle: $204+25+10+14=253$ points

Tacoma: $153+100+30+42=325$ points

Puyallup: $51+75+40+28=194$ points

Olympia: $102+50+20+56=228$ points

Under the Borda Count method, Tacoma is the winner of this vote.

Example $5.1.4$**: Find the Winner Using IRV**

If this were a plurality election, note that B would be the winner with 9 first-choice votes, compared to 6 for D, 4 for C, and 1 for E.

There are a total of $3+4+4+6+2+1=20$ votes. A majority would be 11 votes. No one yet has a majority, so we proceed to elimination rounds.

Round 1: We make our first elimination. Choice A has the fewest first-place votes (0 vote) , so we remove that choice



We then shift everyone's choices up to fill the gaps. There is still no choice with a majority, so we eliminate again.



Round 2: We make our second elimination. Choice E has the fewest first-place votes (1 vote) , so we remove that choice, shifting everyone's options to fill the gaps.



Now B has 9 first-choice votes, C has 4 votes, and D has 7 votes. Still no majority, so we eliminate again.

Round 3: We make our third elimination. C has the fewest votes (4 votes).



D has gained a majority (11 votes) and is declared the winner under IRV.

Example 5.1.5: Copland Method

We need to look at each choice and see which would win in a one-to-one comparison.

number of comparison for 3 candidate = $\frac{3(3-1)}{2} = 3$.

They are Hawaii vs Orlando, Hawaii vs Anaheim, and Orlando vs Anaheim.

You may recall we did this earlier when determining the Condorcet Winner. For example, comparing Hawaii vs Orlando, we see that 6 voters, those shaded below in the first table below, would prefer Hawaii to Orlando. Note that Hawaii doesn't have to be the voter's first choice – we're imagining that Anaheim wasn't an option. If it helps, you can imagine removing Anaheim, as in the second table below.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

	1	3	3	3
1st choice			O	H
2nd choice	O	H	H	
3rd choice	H	O		O

Based on this, in comparing Hawaii vs. Orlando, Hawaii wins and receives 1 point. Similarly, we compare Hawaii vs Anaheim, and Orlando vs Anaheim.

Hawaii vs Orlando: $6(H)$ votes to $4(O)$ votes: Hawaii gets 1 point
Anaheim vs Orlando: $7(A)$ votes to $3(O)$ votes: Anaheim gets 1 point
Hawaii vs Anaheim: $6(H)$ votes to $4(A)$ votes: Hawaii gets 1 point

Total points for Hawaii is 2, Anaheim is 1, and Orlando is 0.

Hawaii is the winner under Copeland's Method, having earned the most points.

Notice this process is consistent with our determination of a Condorcet Winner.

Example 5.1.6: Copland Method

With 5 candidates, there are 10 comparisons to make:

number of comparison for 5 candidate = $\frac{5(5-1)}{2} = 10$.

A vs B: 11 votes to 9 votes — A gets 1 point
A vs C: 3 votes to 17 votes — C gets 1 point
A vs D: 10 votes to 10 votes — A gets $\frac{1}{2}$ point, D gets $\frac{1}{2}$ point
A vs E: 17 votes to 3 votes — A gets 1 point
B vs C: 10 votes to 10 votes — B gets $\frac{1}{2}$ point, C gets $\frac{1}{2}$ point
B vs D: 9 votes to 11 votes — D gets 1 point
B vs E: 13 votes to 7 votes — B gets 1 point
C vs D: 9 votes to 11 votes — D gets 1 point
C vs E: 17 votes to 3 votes — C gets 1 point
D vs E: 17 votes to 3 votes — D gets 1 point

Totaling these up:

A gets $2\frac{1}{2}$ points
B gets $1\frac{1}{2}$ points
C gets $2\frac{1}{2}$ points
D gets $3\frac{1}{2}$ points
E gets 0 points

Using Copeland's Method, we declare D as the winner.

Notice that in this case, D is not a Condorcet Winner. While Copeland's method will also select a Condorcet Candidate as the winner, the method still works in cases where there is no Condorcet Winner.

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5.2: Fairness Criteria

Learning Objectives

1. Use the Condorcet criterion to determine the fairness of the voting system.
2. Use the Majority criterion to determine the fairness of the voting system.
3. Use the Monotonicity criterion to determine the fairness of the voting system.
4. Use the Independence of Irrelevant Alternatives (IIA) criterion to determine the fairness of the voting system.

Voting Fairness Criteria

Voting fairness criteria are rules used to judge whether a voting method produces fair and reasonable election results. In other words, they are standards used to decide whether an election method gives fair and logical outcomes. Sometimes, a voting method does not satisfy a fairness criterion. When this happens, we say the voting method **violates** that criterion, meaning the method fails to meet one of the rules that define a fair election outcome.

In other words, the election result produced by that voting method **contradicts what the fairness criterion states should happen**.

There are four fairness criteria.

1. Condorcet criterion (head-to-head)
2. Majority criterion
3. Monotonicity criterion
4. Independence of Irrelevant Alternatives (IIA) Criterion

What's Wrong With the Plurality Method?

The election from example $5.1.1$ in the previous section may seem clean, but there is a problem lurking that arises whenever there are three or more choices. Looking back at our preference table, how would our members vote if they only had two choices? Let's compare the winner, Anaheim, to Orlando and Hawaii.

Anaheim vs Orlando: 7 out of the 10 would prefer Anaheim over Orlando

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Anaheim vs Hawaii: 6 out of 10 would prefer Hawaii over Anaheim

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

This doesn't seem right. Anaheim just won the election, yet 6 out of 10 voters, 60%, would have preferred Hawaii! That hardly seems fair. Marquis de Condorcet, a French philosopher, mathematician, and political scientist, wrote about how this could happen in 1785, and for him, we name our first **fairness criterion**.

Condorcet Criterion

If a choice is preferred in every one-to-one comparison with the other options, that choice should be the winner. We call this winner the **Condorcet Winner** or Condorcet Candidate. The Condorcet criterion is also called the **head-to-head** criterion.

Note: One-to-one comparison $\equiv$ Head-to-head comparison.

Example $5.2.1$: Find Condercet Winner

What choice is the Condorcet Winner in the following election?

[redacted]

Answer available at the end of this section.

When Does the Plurality Method Violate the Condorcet Criterion?

The plurality voting method meets (or satisfies) the Condorcet criterion if the winner according to this method is the same as the Condorcet winner. Otherwise, the voting method (Plurality) violates the criterion (Condorcet).

Example $5.2.2$: The Plularity Method Violates the Condorcet Criterion

Consider a city council election in a district that is historically 60% Democratic voters and 40% Republican voters. Even though the city council is technically a nonpartisan office, people generally know the affiliations of the candidates. In this election, there are three candidates: Don and Key, both Democrats, and Elle, a Republican. A preference schedule for the votes looks as follows:

[redacted]

1. Who is the Condorcet candidate?
2. Under the Plurality method, who is the winner?
3. Does the plurality method violate the head-to-head criterion?

Answer available at the end of this section.

Your Turn 5.2.2: Plurality with Elimination and Head to Head

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What's Wrong with Board Count?

You might have already noticed one potential flaw of the Borda Count from the previous example. In that example, Seattle had a majority of first-choice votes, yet lost the election! This seems odd and prompts our next fairness criterion:

Majority Criterion

If a choice has a majority of first-place votes, that choice should be the winner.

The election from example 5.2.1 using the Borda Count violates the Majority Criterion. Because the total number of voters is 100 and Seattle got 51, the majority of the first-place votes. But under the Borda Count method, Tacoma is the winner.

Notice also that this automatically means that the Condorcet Criterion will also be violated, as Seattle would have been preferred by 51% of voters in any head-to-head comparison.

Borda count is sometimes described as a consensus-based voting system, since it can sometimes choose a more broadly acceptable option over the one with majority support. In the example above, Tacoma is probably the best compromise location. This is a different approach from plurality and instant runoff voting, which focus on first-choice votes; Borda Count considers every voter's entire ranking to determine the outcome.

Because of this consensus behavior, Borda Count, or some variation, is commonly used in awarding sports awards. Variations are used to determine the Most Valuable Player in baseball, to rank teams in NCAA sports, and to award the Heisman trophy.

When Does the Borda Count Method Violate the Majority Criterion?

The Borda count voting method meets (or satisfies) the Majority criterion if the winner according to this method is the same as a candidate who got the majority of the first-place votes. Otherwise, the voting method (Borda count) violates the criterion (Majority).

Example 5.2.3: The Borda Count Violate Majority Criterion

Use the following preference table to answer the given questions.

Number of voters	16	20	11	13	7
1**st choice**	B	B	A	A	D
2**and choice**	A	A	C	D	C
3**rd choice**	C	D	D	C	A

Number of voters	16	20	11	13	7
4th choice	D	C	B	B	B

Answer the following question for the above preference table.

1. How many votes were cast?
2. How many votes are needed for a majority?
3. Does any candidate have a majority of first-place votes? If so, who is that candidate?
4. Find the winner under the Borda Count. Use the count that assigns 1 point to the last place.
5. Is the majority criterion violated by the Borda Count in this election?

Answer available at the end of this section.

Your Turn 5.2.3: Borda Count and Majority

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What's Wrong with Plurality with the Elimination Method?

Let's return to our City Council Election



In this election, Don has the smallest number of first-place votes, so Don is eliminated in the first round. The 214 people who voted for Don have their votes transferred to their second choice, Key.



So Key is the winner using the IRV method.

We can immediately notice that IRV violates the Condorcet Criterion in this election since we determined earlier that Don was the Condorcet winner.

On the other hand, the temptation has been removed for Don's supporters to vote for Key; they now know their vote will be transferred to Key, not simply discarded.

Monotonicity Criterion

If voters change their votes to increase their preference for a candidate, it should not harm that candidate's chances of winning.

When Does the IRV Method Satisfy the Monotonicity Criterion?

The IRV method meets (or satisfies) the monotonicity criterion if the winner, according to this method, is the same even after people increase their preference for the candidate who is more likely to win. Otherwise, the voting method (IRV) violates the criterion (Monotonicity).

Example 5.2.4: The IRV Satisfy the Monotonicity Criterion

[1]A class is voting to choose a new class mascot, and the three nominations are Kudu (K), Ferret (F), and Gopher (G). A vote of the entire class was held, and the results are summarized in the preference table below.

Number of Votes	37	40	34	36	32	44
First Choice	F	K	G	F	K	G
Second Choice	K	G	F	G	F	K
Third Choice	G	F	K	K	G	F

1. Using the plurality-with-elimination method, determine the winner of the team mascot election.
2. In the second election, suppose that the 36 students who initially voted for F, G, K (in that order) decided to change their votes to G, K, F (in that order). Using the plurality-with-elimination method and updated voting preferences, determine the winner of the team mascot election.
3. Determine if the monotonicity criterion is satisfied in this case. Explain why or why not.

Answer 1 available at the end of this section.

In example 5.2.2 above, the IRV method satisfies the monotonicity criterion, but that would not always be true. The example below shows that the IRV method could violate the monotonicity criterion.

Your Turn 5.2.4: Monotonicity and Plularity with Elimination

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What's Wrong with Copland's Method?

As already noted, Copeland's Method does satisfy the Condorcet Criterion. It also satisfies the Majority Criterion and the Monotonicity Criterion. So is this the perfect method? Well, in a word, NO.

Example 5.2.5: Example that Leads to IIA Criterion

A committee is trying to award a scholarship to one of four students, Anna (A), Brian (B), Carlos (C), and Dimitry (D). The votes are shown below:



Answer available at the end of this section.

The Independence of Irrelevant Alternatives (IIA) Criterion

If a non-winning choice is removed from the ballot, it should not change the election's winner.

If choice A is preferred over choice B, introducing or removing choice C should not cause B to be preferred over A.

This anecdote illustrating the IIA issue is attributed to Sidney Morgenbesser: After dinner, Sidney Morgenbesser decides to order dessert. The waitress tells him he has two choices: apple pie and blueberry pie. Sidney orders the apple pie. After a few minutes, the waitress returns and says that they also have cherry pie, at which point Morgenbesser says, "In that case, I'll have the blueberry pie."

Another disadvantage of Copeland's Method is that it is fairly easy for the election to end in a tie. For this reason, Copeland's method is usually the first part of a more advanced method that uses more sophisticated methods for breaking ties and determining the winner when there is no Condorcet Candidate.

When Does the Copland Method Violate the IIA Criterion?

The Copland voting method meets (or satisfies) the IIA criterion if the winner according to this method is the same as the winner after the non-winning choice is removed. Otherwise, the voting method (Copland) violates the criterion (IIA).

Your Turn 5.2.5: Copland and IAA

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Where is the Fair Method?

At this point, you're probably asking why we keep looking at method after method just to point out that they are not fully fair. We must be holding out on the perfect method.

Unfortunately, no. A mathematical economist, Kenneth Arrow, was able to prove in 1949 that there is no voting method that will satisfy all the fairness criteria we have discussed.

Arrow's Impossibility Theorem

Arrow's Impossibility Theorem states, roughly, that it is not possible for a voting method to satisfy every fairness criterion that we've discussed.

To see a very simple example of how difficult voting can be, consider the election below:



Notice that in this election:

10 people prefer A to B

10 people prefer B to C

10 people prefer C to A

No matter whom we choose as the winner, $\frac{2}{3}$ of voters would prefer someone else! This scenario is dubbed **Condorcet's Voting Paradox**, and demonstrates how voting preferences are not transitive (just because A is preferred over B, and B over C, does not mean A is preferred over C). In this election, there is no fair resolution.

It is because of this impossibility of a totally fair method that Plurality, IRV, Borda Count, Copeland's Method, and dozens of variants are all still used. Usually, the decision of which method to use is based on what seems most fair for the situation in which it is being applied.

Summary of Violation of Fairness Criteria

Summary of Violations of Fairness Criteria

	Plurality	Borda Count	Plurality with Elimination (IRA)	Pairwise Comparisons (Copland)
Majority Criterion	*	Violation Possible	*	*
Condorcet Criterion (Head to Head)	Violation Possible	Violation Possible	Violation Possible	*
Monotonicity Criterion	*	*	Violation Possible	*
IIA Criterion	Violation Possible	Violation Possible	Violation Possible	Violation Possible

** The indicated voting method does not violate the indicated criterion in any election.*

Your Turn 5.2.6: Understaing Fairness Criteria

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[2] This data is loosely based on the 2008 County Executive election in Pierce County, Washington. See http://www.co.pierce.wa.us/xml/abtus...ec/summary.pdf

Answers

Example 5.2.1: Find Condercet Winner

To find the Condorcet choice, we compare every candidate to every other candidate. This is called a head-to-head comparison.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

A vs H: 4 prefer A; 6 prefer H. H is preferred. In other words, Hawaii is the winner between Anaheim and Hawaii.

A vs O: 7 prefer A; 3 prefer O. A is preferred. In other words, Anaheim is the winner between Anaheim and Orlando.

O vs H: 4 prefer O; 6 prefer H. H is preferred. In other words, Hawaii is the winner between Orlando and Hawaii.

Hawaii is preferred (or you can say it wins) over every other candidate when we do a one-to-one comparison, so Hawaii is the Condorcet Winner.

Example 5.2.2: The Plularity Method Violates the Condorcet Criterion

(a) We can see a total of $342 + 214 + 298 = 854$ voters participated in this election. Computing the percentage of first-place votes:

Don: $214/854 = 25.1\%$

Key: $298/854 = 34.9\%$

Elle: $342/854 = 40.0\%$

So in this election, the Democratic voters split their vote over the two Democratic candidates, allowing the Republican candidate Elle to win under the plurality method with 40% of the vote.

Analyzing this election closely, we see that it violates the Condorcet Criterion. Analyzing the one-to-one comparisons:

Elle vs Don: 342 prefer Elle; 512 prefer Don: Don is preferred

Elle vs Key: 342 prefer Elle; 512 prefer Key: Key is preferred

Don vs Key: 556 prefer Don; 298 prefer Key: Don is preferred

Don is the Condorcet winner, preferred in every one-to-one comparison with the other candidates.

(b) Let's count the most first-place votes

Don: 241; Key: 298; and Elle: 342. So the winner by the plurality method is Elle.

(c) YES. Since the winner using the plurality method (Elle) is not the same as the Condorcet winner (Don).

Example 5.2.3: The Borda Count Violate Majority Criterion

1. Total number of voter are: $16 + 20 + 11 + 13 + 7 = 67$.
2. To get the majority, one should get more than $\frac{67}{2} = 33.5$. So 34 or more should be needed in the first place to get a majority.
3. YES. B gets a majority. Since candidate B gets $16 + 20 = 36$, the most first-place votes, but this is a majority since it is more than 34.
4. Assign the points to candidates based on their ranking: 1 point for the last choice, 2 points for the second-to-last choice, and so on.

Number of voters	16	20	11	13	7
1st choice (4 points)	B $16 \times 4 = 64$	B $20 \times 4 = 80$	A $11 \times 4 = 44$	A $13 \times 4 = 52$	D $7 \times 4 = 28$

Number of voters	16	20	11	13	7
2nd choice (3 points)	A $16 \times 3 = 48$	A $20 \times 3 = 60$	C $11 \times 3 = 33$	D $13 \times 3 = 39$	C $7 \times 3 = 21$
3rd choice (2 points)	C $16 \times 2 = 32$	D $20 \times 2 = 40$	D $11 \times 2 = 22$	C $13 \times 2 = 26$	A $7 \times 2 = 14$
4th choice (1 points)	D $16 \times 1 = 16$	C $20 \times 1 = 20$	B $11 \times 1 = 11$	B $13 \times 1 = 13$	B $7 \times 1 = 7$

Adding up the points:

A: $48 + 60 + 44 + 52 + 14 = 218$ points
B: $64 + 80 + 11 + 13 + 7 = 175$ points
C: $32 + 20 + 33 + 26 + 21 = 132$ points
D: $16 + 40 + 22 + 39 + 28 = 145$ points

5. The majority criterion is violated by the Borda Count in this election because the Borda Count winner (A) is not the same as the majority winner (B).

Example 5.2.4: The IRV Satisfy the Monotonicity Criterion

Total number of voters $37 + 40 + 34 + 36 + 32 + 44 = 223$. The first-place votes must be 112 or higher to get a majority and win by the IRV method. Let's count the first-place votes.

F: $37 + 36 = 73$, K: $40 + 32 = 72$, and G: $34 + 44 = 78$. No one get more than or equal to 112. We make our first elimination. Choice K has the fewest first-place votes, so we remove that choice and then shift everyone's choices up to fill the gaps.



F: $37 + 32 + 36 = 105$, and G: $40 + 34 + 44 = 118$. The candidate G got 118 and it is the majority, so G is the winner by the IRV method.

In the second election, the 36 students who initially voted for F, G, K (in that order) decided to change their votes to G, K, F (in that order). Now we create a new preference table as given below.

Number of Votes	37	40	34	36	32	44
First Choice	F	K	G	**G**	K	G
Second Choice	K	G	F	**K**	F	K
Third Choice	G	F	K	**F**	G	F

F: 37, K: $40 + 32 = 72$, and G: $34 + 36 + 44 = 114$. The candidate G got 114 and it is the majority, so G is the winner by the IRV method in the second election.

Yes, the monotonicity criterion is satisfied because the winner of the second election was the same as the winner of the first election after the ballots were changed in favor of the first winner

Example 5.2.5: Example that Leads to IIA Criterion

Making the comparisons:

A vs B: 10 votes to 10 votes A gets $\frac{1}{2}$ point, B gets $\frac{1}{2}$ point
A vs C: 14 votes to 6 votes: A gets 1 point
A vs D: 5 votes to 15 votes: D gets 1 point
B vs C: 4 votes to 16 votes: C gets 1 point
B vs D: 15 votes to 5 votes: B gets 1 point
C vs D: 11 votes to 9 votes: C gets 1 point

Totaling:

A has $1\frac{1}{2}$ points B has $1\frac{1}{2}$ points
C has 2 points D has 1 point

So Carlos is awarded the scholarship. However, the committee then discovers that Dimitry was not eligible for the scholarship (he failed his last math class). Even though this seems like it shouldn't affect the outcome, the committee decides to recount the vote, removing Dimitry from consideration. This reduces the preference schedule to:



A vs B: 10 votes to 10 votes A gets $\frac{1}{2}$ point, B gets $\frac{1}{2}$ point
A vs C: 14 votes to 6 votes A gets 1 point
B vs C: 4 votes to 16 votes C gets 1 point

Totaling:

A has $1\frac{1}{2}$ points B has $\frac{1}{2}$ point
C has 1 point

Suddenly, Anna is the winner! This leads us to another fairness criterion.

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6: APPORTIONMENT METHOD AND PARADOX

6.1: Apportionment Methods

6.2: Apportionment Paradoxes

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6.1: Apportionment Methods

Learning Objectives

1. Find the standard advisor and standard quota.
2. Apply Hamilton's method to solve the apportionment problem.
3. Apply Jefferson's method to solve the apportionment problem.
4. Apply Adam's method to solve the apportionment problem.
5. Apply Webster's method to solve the apportionment problem.

What is the Apportionment Problem?

Apportionment is the problem of dividing a fixed number of items among groups of varying sizes. In politics, this takes the form of allocating a limited number of representatives among voters. This problem, presumably, is older than the United States, but the best-known ways to solve it have their origins in the problem of assigning each state an appropriate number of representatives in the new Congress when the country was formed. States also face this apportionment problem in defining how to draw districts for state representatives. The apportionment problem also arises in various non-political contexts.

An **apportionment problem** is a mathematical method used to decide **how to fairly distribute a fixed number of items as a whole number (**you can't assign 2.4 items) (usually seats in a legislature) among groups (usually states or regions) **based on their population**.

- Assigning seats in the **U.S. House of Representatives**
- Dividing representatives in the student government
- Allocating funding to departments based on population or size
- Assigning voting power in committees

Standard Divisor and Standard Quota

The **standard divisor** is a concept used in **apportionment problems**, particularly when dividing the number of allocated items (seats or quotas) fairly among different groups based on population. The **standard divisor** is a key value used in apportionment problems to help decide how many seats each group should receive.

Formula: Standard Divisor

$$\text{Standard Divisor (SD)} = \frac{\text{Total Population}}{\text{Number of quota}}$$

Formula: Standard Quota of a Group

The standard quota for a particular group can be found by using the following formula

$$\text{Standard Quota for Particular Group} = \frac{\text{Population of that group}}{\text{Standard Divisor (SD)}}$$

Note: If we round the standard quota DOWN to the nearest whole number, we call it standard lower quota or (Just lower quota), and if we round the standard quota UP to the nearest whole number, we call it standard upper quota or (Just upper quota).

Example 6.1.1: Finding Standard Divisor and Standard Quota

Delaware has three counties: Kent, New Castle, and Sussex. The Delaware State House of Representatives has 41 members. Find the standard divisor, standard quota for each county, upper quota, and lower quota.

County	Population
Kent	162,310
New Castle	538,479
Sussex	197,145
Total	**897,934**

Answer available at the end of this section.

Your Turn 6.1.1: Standard Divisor and Standard Quota

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As we see in example 6.1.1,, rounding the standard quota to the upper and lower quotas does not give the total number of representatives in the Delaware State House. Now, we will discuss how to solve the apportionment problem using various methods. The first one is the Hamilton Method.

Hamilton's Method

Alexander Hamilton proposed the method that now bears his name. His method was approved by Congress in 1791, but was vetoed by President Washington. It was later adopted in 1852 and used through 1911. He begins by determining, to several decimal places, how many things each group should get. Since he was interested in the question of Congressional representation, we'll use the language of states and representatives, so he can determine how many representatives each state should get. He follows these steps:

Steps to Apportion by Hamilton's Method

There are five steps we follow when applying Hamilton's Method of apportionment:

Step 1: Find the standard divisor.

Step 2: Find each state's standard quota.

Step 3: Give each state the state's lower quota (with each state receiving at least one seat).

Step 4: Give each remaining seat one at a time to the states with ***the largest fractional parts*** of their standard quotas until no seats remain.

Step 5: Check the solution by confirming that the sum of the modified quotas equals the house size from step 4.

Note on rounding: Today, we have technological advantages that Hamilton (and the others) couldn't have imagined. Please take advantage of them, and keep them in several decimal places.

Example 6.1.2: Apportionment Using Hamilton's Method

Use Hamilton's method to apportion the 75 seats of Rhode Island's House of Representatives among its five counties.

County	Population
Bristol	49,875
Kent	166,158
Newport	82,888
Providence	626,667
Washington	126,979
Total	**1,052,567**

Answer available at the end of this section.

Calculator for Hamilton's Method

Your Turn 6.1.2: Hamilton Method

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Hamilton's method obeys something called the Quota Rule. The Quota Rule isn't a law of any sort, but just an idea that some people, including Hamilton, think is a good one.

Quota Rule

The Quota Rule says that the final number of representatives a state gets should be within one of that state's quota. Since we're dealing with whole numbers for our final answers, each state should either go up to the next whole number above its quota or down to the next whole number below its quota.

For example, let's say there are 100 seats to distribute and 5 states. If State A's population justifies exactly 9.4 seats, the Quota Rule says: State A should get **either** 9 **or** 10 seats. Getting **less than nine** or **more than** 10 would violate the rule. The Hamilton Method always satisfies the quota rule.

Jefferson's Method

Thomas Jefferson proposed a different method for apportionment. After Washington vetoed Hamilton's method, Jefferson's method was adopted and used in Congress from 1791 through 1842. Jefferson had political reasons for wanting his method to be used rather than Hamilton's. Primarily, his method favors larger states, and his own home state of Virginia was the largest in the country at the time. He would also argue that it's the ratio of people to representatives that is the critical thing, and apportionment methods should be based on that. However, the paradoxes we saw also provide mathematical reasons for concluding that Hamilton's method isn't so good, and while Jefferson's method might or might not be the best one to replace it, at least we should look for other possibilities.

The first steps of Jefferson's method are the same as Hamilton's method. He finds the same divisor and the same quota, and cuts off the decimal parts in the same way, giving a total number of representatives that is less than the required total. The difference is in how Jefferson resolves that difference. He says that since we ended up with a too small answer, our divisor must have been too big. He changes the divisor by making it smaller, finding new quotas with the new divisor, cutting off the decimal parts, and

looking at the new total, until he finds a divisor that produces the required total.

Steps to Apportion by Jefferson's Method (Round Down)

Step 1: Find the standard divisor.

Step 2: Choose a modified divisor slightly less than the standard divisor.

Step 3: Find the states' **or group lower quota** and their sum (round down to the nearest whole number). Add all the lower quota numbers.

Step 4: If the total lower quota (after rounding) is **more** than available, **increase the divisor** (making quotas smaller). If the total is **less**, **decrease the divisor** (making quotas larger). Repeat until the sum of the lower quotas equals the total quota.

Step 5: Each group quota from step 4 is the final apportionment (or final quota).

Which Divisor is Higher? Standard or Modified?

In Jefferson's method, a modified divisor is less than the standard divisor.

Example 6.1.3: Jefferson's Method

Use Jefferson's method to apportion the 25 seats (senators) among six states of the small country.

Table 6.1.1: Populations by State

State	A	B	C	D	E	F	Total
Population	$24,000$	$56,000$	$28,000$	$17,000$	$65,000$	$47,000$	$237,000$

Answer available at the end of this section.

Calculator for Jefferson's Method

Example 6.1.4: Jefferson's Method

Use Jefferson's method to apportion the 75 seats of Rhode Island's House of Representatives among its five counties.

County	Population
Bristol	$49,875$
Kent	$166,158$
Newport	$82,888$
Providence	$626,667$
Washington	$126,979$
Total	$\mathbf{1,052,567}$

Answer available at the end of this section.

Your Turn 6.1.4: Jefferson Method

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Notice, in comparison to Hamilton's method, that although the results were the same, they came about in a different way, and the outcome was almost different. Providence County (the largest) almost went up to 46 representatives before Kent (which is much smaller) got to 12. Although that didn't happen here, it can. Divisor-adjusting methods, such as Jefferson's, are not guaranteed to follow the quota rule.

Adam's Method:

Adams's method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did, Adams used the upper quotas. Adam's method of apportionment is very similar to Jefferson's, but with a twist in how it rounds numbers. It is named after John Quincy Adams, the 6th President of the U.S.

Steps to Apportion by Adam's Method

Step 1: Find the standard divisor.

Step 2: Choose a modified divisor slightly greater than the standard divisor.

Step 3: Find the states' **or group upper quota** and their sum (round UP to the nearest whole number). Add all the upper quota numbers.

Step 4: If the total upper quota (after rounding) is **more** than available, **increase the divisor** (making quotas smaller). If the total is **less**, **decrease the divisor** (making quotas larger). Repeat until the sum of the upper quotas equals the total quota.

Step 5: Each group quota from step 4 is the final apportionment (or final quota).

Which Divisor is Higher? Standard or Modified?

Note: In Adam's method, a modified divisor is greater than the standard divisor.

Example 6.1.5: Using Adam's Method

In Adamstown, 42 new firefighters have just completed their training. They are to be assigned to the five firehouses in town in a manner proportional to the population in each fire district. The populations are listed in the following table.

Table 6.1.1 : Populations for the Fire Districts of Adamstown

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465

Apportion the new firefighters to the firehouses using the Adams Method.

Answer available at the end of this section.

Adam's Method Calculator

Your Turn 6.1.5: Adams Method

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Webster's Method

This method is named after Daniel Webster, an American statesman in the 19th century. Webster's method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did, or the sum of the upper quotas, as Adams did, Webster used traditional rounding.

Steps to Apportion by Webster's Method (Normal Rounding)

Step 1: Find the standard divisor.

Step 2: Choose a modified divisor either slightly greater than or slightly less than the standard divisor. But you must start with a standard divisor. This could be a modified divisor.

Step 3: Find the states or group rounded quota (If the decimal is 0.5 or more, round UP. If it's less than 0.5, round DOWN). Add all rounded quota numbers.

Step 4: If the number of quotas (after rounding) is **more** than what is available, **increase the divisor** (making quotas smaller). If the total is **less**, **decrease the divisor** (making quotas larger). Repeat until the sum of the rounded quotas equals the total seats or quotas.

Step 5: Each group quota from step 4 is the final apportionment (or final quota).

Because some quotas will be rounded up and others will be rounded down, we cannot immediately determine whether the total number of seats is too large or too small. Unlike Jefferson's and Adam's methods, we do not know how to adjust the modified divisor. This forces us to use the standard divisor as the first modified divisor. So, the modified divisor in this method can be less than, greater than, or equal to the standard divisor.

Which Divisor is Higher? Standard or Modified?

The modified divisor in the Webster's Method can be less than, greater than, or equal to the standard divisor.

Example 6.1.6: Webester's Method

Use Webster's method to apportion the 25 seats among 6 states of the small country.

Table 6.1.1 : Population by States

State	A	B	C	D	E	F	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,00

Answer available at the end of this section.

Webster Method Calculator

Your Turn 6.1.6: Webester's Method

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Example 6.1.7: Comparing all 4 Apportionment Methods

The following table summarizes the results of the Hamilton, Jefferson, Adams, and Webster methods when applied to the apportionment of 616 schools among five Hawaiian counties.

Apportion Using all 4 methods

	Hawaii	Honolulu	Kalawao	Kauai	Maui
Under 19 years old	46,310	224,230	20	16,560	38,450
Hamilton	87	424	1	31	73
Jefferson	87	425	1	31	72
Adams	88	422	1	31	73
Webster	87	424	1	31	73

1. Do any of the apportionment methods result in the same apportionment? If so, which ones?
2. Which apportionment method would the citizens of the largest county likely favor most and least? Justify your answer.
3. As a group, which apportionment method would the citizens of the other four counties likely favor most and least? Justify your answer.

Answer available at the end of this section.

Your Turn 6.1.7: Definition of Apportionment Method

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Answers

Example 6.1.1: Finding Standard Divisor and Standard Quota

First, we determine the standard divisor = $\frac{897,934}{41} = 21,900.82927$

Determine each country's standard quota by dividing the country's population by the standard divisor, and find the upper and lower quotas.

County	Population	Quota	Lower Qutoa	Upper Quota
Kent	162,310	7.4111	7	8
New Castle	538,479	24.5872	24	25
Sussex	197,145	9.0017	9	10
Total	**897,934**		40	43

Example 6.1.2: Apportionment Using Hamilton's Method

1. The Standard divisor = $\frac{1,052,567}{75} = 14,034.22667$

2. Determine each county's quota by dividing its population by the divisor:

County	Population	Standard Quota
Bristol	49,875	3.5538
Kent	166,158	11.8395
Newport	82,888	5.9061
Providence	626,667	44.6528
Washington	126,979	9.0478
Total	**1,052,567**	

3. Remove the decimal part of each quota (That is, find the lower quota):

County	Population	Standard Quota	Initial (Lower Quota)
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	**1,052,567**		**72**

4. We need 75 representatives and we only have 72, so we assign the remaining three, one each, to the three counties with the largest decimal parts, which are Newport, Kent, and Providence:

County	Population	Standard Quota	Initial (Lower Quota)	Final Quota
Bristol	49,875	3.5538	3	3
Kent	166,158	11.8395	11	$11+1=12$
Newport	82,888	5.9061	5	$5+1=6$
Providence	626,667	44.6528	44	$44+1=45$
Washington	126,979	9.0478	9	9
Total	**1,052,567**		**72**	**75**

Note that even though Bristol County's decimal part is greater than 0.5, it isn't big enough to get an additional representative, because three other counties have greater decimal parts.

Example 6.1.3: Jefferson's Method

Standard divisor is $\frac{237,000}{25} = 9480$.

In Jefferson's method, the modified divisor is always smaller than the standard divisor. There is no formula to find the modified divisor, so we start by guessing.

Let's try the modified divisor, $d = 9000$.

Table 6.1.2 : Quotas for $d = 9000$

State	A	B	C	D	E	F	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Modified Quota $d = 9000$	2.67	6.22	3.11	1.89	7.22	5.22	
Lower Quota	2	6	3	1	7	5	24

The sum 24 is smaller than the given quota 25, so we must try again by making the modified divisor smaller than 9000. Let's try $d = 8000$.

Table 6.1.3 : Quotas for $d = 8000$

State	A	B	C	D	E	F	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Modified Quota $d = 8000$	3.00	7.00	3.50	2.13	8.13	5.88	

State	A	B	C	D	E	F	Total
Lower Quota	3	7	3	2	8	5	28

This time, the sum of 28 is too big. Try again, making the modified divisor larger. We know the divisor must be between $d = 8000$ and $d = 9000$ so let's try $d = 8500$.

Table 6.1.4 : *Quotas for* $d = 8500$

State	A	B	C	D	E	F	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Modified Quota $d = 8500$	2.82	6.59	3.29	2.00	7.65	5.53	
Lower Quota Final apportion	2	6	3	2	7	5	25

This time the sum is 25, so we are done. A gets two senators, B gets six senators, C gets three senators, D gets two senators, E gets seven senators, and F gets five senators.

Any modified divisor between 8,200 and 8,500 should work in this problem.

Note: You can double-check the above calculation using the following Jefferson Method Calculator.

Example 6.1.4**: Jefferson's Method**

First, find the standard divisor. The Standard divisor = $\frac{1,052,567}{75} = 14,034.22667$

We need 75 representatives and only have 75, so we need to use a smaller divisor. Let's try 13,500.

County	Population	Modified Quota	Lower quota
Bristol	49,875	3.6944	3
Kent	166,158	12.3080	12
Newport	82,888	6.1399	6
Providence	626,667	46.4198	46
Washington	126,979	9.4059	9
Total	**1,052,567**		**76**

We've gone too far. We need a divisor that's greater than 13,500 but less than 14,034.22667. Let's try 13,700.

County	Population	Modified Quota	Lower quota (Final apportion)
Bristol	49,875	3.6405	3
Kent	166,158	12.1283	12
Newport	82,888	6.0502	6
Providence	626,667	45.7421	45
Washington	126,979	9.2685	9
Total	**1,052,567**		**75**

This works.

Any modified divisor between 13,650 and 13,800 should work in this problem.

Example 6.1.5**: Using Adam's Method**

$$\text{Standard divisor} = \frac{69,465}{42} = 1653.92857$$

Adams's method always rounds up, making the sum of the upper quotas too large. First, choose **the modified divisor slightly larger than the standard divisor to see if that works.** The results are summarized below in Table 6.1.2.

Guess #1 : $d = 1700$. The total is too still too large (43) so make the modified divisor larger. (Does not work)

Guess #2 : $d = 1800$. Now the total is too small (41) so make the modified divisor smaller. (Does not work)

Guess #3 : $d = 1750$. The total is too large again (43) so make the modified divisor larger. (Does not work)

Guess #4 : $d = 1775$. The sum is 42 so we are done. (Works)

Table 6.1.2 : *Apportion by Adams's Method for Adamstown*

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465
$d = 1700$	14.712	5.153	6.818	5.309	8.871	
quota	15	6	7	6	9	43
$d = 1800$	13.894	4.86	6.439	5.014	8.378	
quota	14	5	7	6	9	41
$d = 1750$	14.291	5.006	6.623	5.157	8.617	

District	A	B	C	D	E	Total
quota	15	6	7	6	9	43
$d = 1775$	14.090	4.935	6.530	5.085	8.496	
Final Allocation	15	5	7	6	9	42

Any modified divisor between 1755 and 1785 should work in this case.

Example $6.1.6$**: Webester's Method**

First, find the standard divisor = $\frac{237,000}{25} = 9480$. and use it as a modified divisor to see if we are lucky.

Table $6.1.2$ *: Quotas for* $d = 9480$

State	A	B	C	D	E	F	Total
Population	$24,000$	$56,000$	$28,000$	$17,000$	$65,000$	$47,000$	$237,000$
$d = 9480$	2.53	5.91	2.95	1.79	6.86	4.96	
Rounded Quota	3	6	3	2	7	5	26

Since the total of 26 seats is too big, we need to make the modified divisor larger. Try $d = 11,000$.

Table $6.1.3$ *: Quotas for* $d = 11,000$

State	A	B	C	D	E	F	Total
Population	$24,000$	$56,000$	$28,000$	$17,000$	$65,000$	$47,000$	$237,000$
$d = 11,000$	2.18	5.09	2.55	1.55	5.91	4.27	
Rounded Quota	2	5	3	2	6	4	22

The total number of seats 22 is smaller than we hoped $(25.)$ That means that $d = 11,000$ is much too big. We need to pick a new modified divisor between 9480 and $11,000$. Try a divisor closer to 9480 such as $d = 10,000$.

Table $6.1.4$ *: Quotas for* $d = 10,000$

State	A	B	C	D	E	F	Total
Population	$24,000$	$56,000$	$28,000$	$17,000$	$65,000$	$47,000$	$237,000$
$d = 10,000$	2.40	5.60	2.80	1.70	6.50	4.70	
Rounded Quota	2	6	3	2	7	5	25

Any modified divisor between $9,650$ and $10,000$ works here.

Note: You can double-check the above calculation using the following Webster's Method Calculator.

Example $6.1.7$**: Comparing all** 4 **Apportionment Methods**

1. Yes, the Hamilton and Webster methods result in the same apportionment.
2. The largest county is Honolulu. The citizens would likely favor the Jefferson method of apportionment most since they received the most seats by that method. They would likely favor the Adams method of apportionment least because they received the least number of seats by that method.
3. As a group, the other four counties received 192 seats by either the Hamilton or Webster method, 193 seats by the Adams method, and 191 seats by the Jefferson method. They would likely favor the Adams method the most and favor the Jefferson method the least.

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6.2: Apportionment Paradoxes

Learning Objectives

1. Describe and illustrate the Alabama paradox.
2. Describe and illustrate the population paradox.
3. Describe and illustrate the new-states paradox.

Controversy

There are certain characteristics that they would reasonably expect from a fair apportionment.

- If the house size is increased, the state quotas should all increase or remain the same, but never decrease.
- If one state's population is growing more rapidly than another's, the faster-growing state should not lose a seat while the slower-growing state maintains or gains a seat.
- If there is a fixed number of seats, adding a new state should not cause an existing state to gain seats while others lose them.

However, apportionment methods are known to contradict these expectations. Before you decide on the right apportionment for Imaginarians, let's explore the **apportionment paradox**, a situation that occurs when an apportionment method produces results that seem to contradict reasonable expectations of fairness.

There is a lot that the founders of Imaginaria can learn from U.S. history. The constitution of the United States requires that the seats in the House of Representatives be apportioned according to the results of the census that occurs every decade, but the number of seats and the apportionment method is not stipulated. Over the years, several different apportionment methods and house sizes have been used and scrutinized for fairness. This scrutiny has led to the discovery of several of these apportionment paradoxes.

After seeing Hamilton's method, many people find that it makes sense, it's not that difficult to use (or, at least, the difficulty comes from the numbers that are involved and the amount of computation that's needed, not from the method), and they wonder why anyone would want another method. The problem is that Hamilton's method is subject to several paradoxes. Three of them happened, on separate occasions, when Hamilton's method was used to apportion the United States House of Representatives.

The Alabama Paradox

The **Alabama Paradox** is named for an incident that happened during the apportionment that took place after the 1880 census. (A similar incident happened ten years earlier involving the state of Rhode Island, but the paradox is named after Alabama.) The post-1880 apportionment had been completed, using Hamilton's method and the new population numbers from the census. Then it was decided that because of the country's growing population, the House of Representatives should be made larger. That meant that the apportionment would need to be done again, still using Hamilton's method and the same 1880 census numbers, but with more representatives. The assumption was that some states would gain another representative and others would stay with the same number they already had (since there weren't enough new representatives being added to give one more to every state). The paradox is that Alabama ended up *losing* a representative in the process, even though no populations were changed and the total number of representatives increased.

Alabama Paradox

Discovered in 1881.

	Seat 299	Seat 300
Alabama	9	7
Illinois	18	19
Texas	9	10

Figure 6.2.1: Illustration of Alabama Paradox

When Does the Alabama Paradox Occur?

The Alabama paradox occurs when an increase in the total number of items (quotas) to be apportioned results in a loss of an item for a group.

Example 6.2.1: Does Alabama Paradox Occur?

[1] Three states, A, B, and C, comprise a small country. The population of each state is given in the table below.

Country Population by States

State	A	B	C	Total
Population	$11,251$	$18,245$	$16,468$	$45,964$

1. According to the country's constitution, its congress shall have 26 seats, divided amongst the three states according to their respective populations. Use Hamilton's method to determine each state's apportionment of congressional seats.
2. Use Hamilton's method to determine each state's apportionment of congressional seats if the number of seats in parliament is increased from 26 to 27.
3. Determine if the Alabama paradox occurred when the number of seats in Congress was increased from 26 to 27.

Answer available at the end of this section.

Your Turn 6.2.1: Alabama Paradox

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The New State Paradox

The **New States Paradox** happened when Oklahoma became a state in 1907. Oklahoma had enough population to qualify for five representatives in Congress. Those five representatives would need to come from somewhere, though, so five states, presumably, would lose one representative each. That happened, but another thing also happened: Maine gained a representative (from New York).

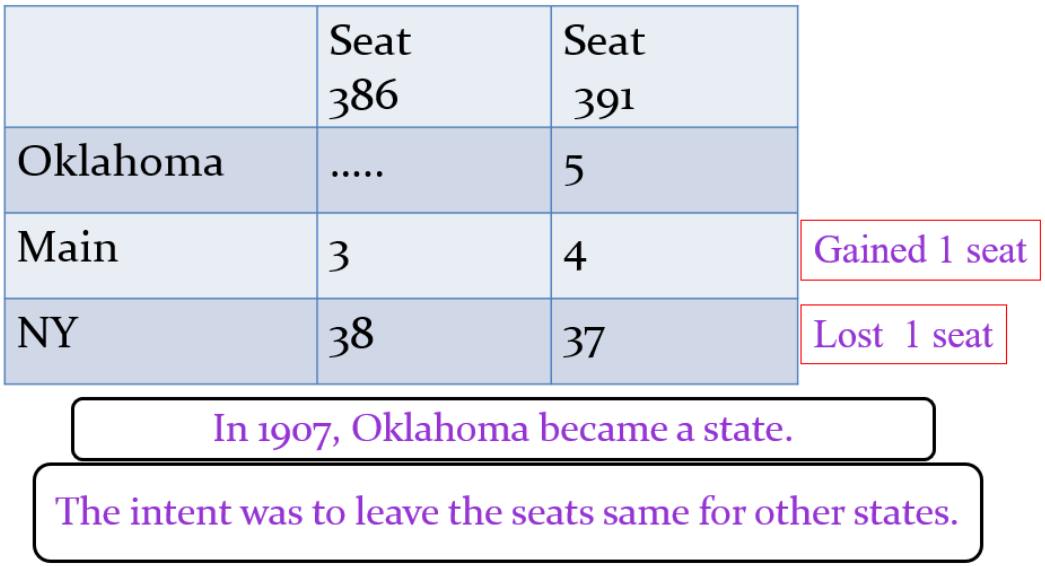


Figure $6.2.2$: Illustration of New State Paradox

When Does the New State Paradox Occur?

The new state paradox occurs when adding a new group changes the apportionment of other groups.

Example $6.2.2$: Does New State Paradox Occur?

A small city is made up of three districts and governed by a committee with 100 members. District A has a population of $5,310$. District B has a population of 1330, and District C has a population of 3308. The city annexes a small area, District D, with a population of 500. At the same time the number of committee members is increased by five. Hamilton's method was used to find the apportionment before and after the annexation.

Table $6.2.3$: *Population Before the Annexation*

State	A	B	C	Total
Population	$5,310$	1330	$3,308$	$9,948$

Table $6.2.4$: *Population After the Annexation*

State	A	B	C	D	Total
Population	$5,310$	1330	$3,308$	500	$10,448$

Answer available at the end of this section.

Your Turn 6.2.2: New State Paradox

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The Population Paradox

The **Population Paradox** happened between the apportionments after the census of 1900 and of 1910. In those ten years, Virginia's population grew at an average annual rate of 1.07%, while Maine's grew at an average annual rate of 0.67%. Virginia started with more people, grew at a faster rate, grew by more people, and ended up with more people than Maine. By itself, that doesn't mean that Virginia should gain representatives or Maine shouldn't, because there are lots of other states involved. But Virginia ended up losing a representative *to Maine*.

When Does the Population Paradox Occur?

The population paradox occurs when group A loses an item(s) to group B, even though group A's population grew faster than group B's population.

Example 6.2.3: Does Population Paradox Occur?

A mom decides to split 11 candy bars among three children based on the number of minutes they spend on chores this week. The time spent by each of her children is given in the table. Near the end of the week, Mom reminds the children of the deal, and they each do some extra work. Abby does an extra two minutes, Bobby an additional 12 minutes, and Charley an extra 86 minutes.

Table 6.2.7 : Candy Bars Before the Extra Work

	Abby	Bobby	Charley	Total
Population (Before)	54	243	703	1,000
Population (After)	56	255	789	1100

1. Use Hamilton's method to apportion the candy bars before and after the extra work.
2. Does the population paradox occur? Why or why not?

Answer available at the end of this section.

Example 6.2.4: Population Paradox

Suppose that 18 respirators are to be apportioned to three hospitals based on their capacities. The Hamilton method is used to allocate the respirators in 2020, then to reallocate based on new capacities in 2020. The results are shown in the table below. How does this demonstrate the population paradox?

Illustration of Poplulation Paradox

Hospital	Capacity in 2020	Respirators in 2020	Capacity in 2021	Respirators in 2021	Growth Rate
A	825	9	882	9	$\frac{882-825}{825} = 6.01\%$

Hospital	Capacity in 2020	Respirators in 2020	Capacity in 2021	Respirators in 2021	Growth Rate
B	613	7	626	6	$\frac{626-613}{613} = 2.12\%$
C	239	2	242	3	$\frac{242-239}{239} = 1.26\%$

Answer available at the end of this section.

Your Turn 6.2.4: Population Paradox

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Quota Rule Violation

A small college has three departments. Department A has 98 faculty, Department B has 689 faculty, and Department C has 212 faculty. The college has a faculty senate with 100 representatives. Use Jefferson's method with a modified divisor of $d = 9.83$ to apportion the 100 representatives among the departments.

Apportion using Hamilton before the extra work: Standard divisor = $\frac{999}{100} = 9.99$

Table 6.2.10 : Quota Rule Violation

State	A	B	C	Total
Population	98	689	212	999
Standard Quota	9.810	68.969	21.221	100
$d = 9.83$	9.969	70.092	21.567	
quota	9	70	21	100

Department B has a standard quota of 68.969, so it should get either its lower quota 68 or its upper quota 69 seats. Using this method, District B received 70 seats, one more than its upper quota. This is a Quota Rule violation.

Is it Possible to Find the Perfect Apportionment Method?

In 1980, Michael Balinski (State University of New York at Stony Brook) and H. Peyton Young (Johns Hopkins University) proved that all apportionment methods violate the quota rule or suffer from one of the paradoxes. This means that it is impossible to find the "perfect" apportionment method. The methods and their potential flaws are listed in the following table.

Table 6.2.11 : Methods, Quota Rule Violations, and Paradoxes

		Paradoxes		
Method	**Quota Rule**	**Alabama**	**Population**	**New-States**
Hamilton	No violations	Yes	Yes	Yes
Jefferson	Upper-quota violations	No	No	No
Adams	Lower-quota violations	No	No	No

		Paradoxes		
Webster	Lower- and upper-quota violations	No	No	No
Huntington-Hill	Lower- and upper-quota violations	No	No	No

Your Turn 6.2.5: Name Paradox

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Answers

Example 6.2.1: Does Alabama Paradox Occur?

With 26 seats: Standard divisor = $\frac{45,964}{26} = 1767.84615$

Table 6.2.1 : Apportionment with 26 seats

State	A	B	C	Total
Population	$11,251$	$18,245$	$16,468$	$45,694$
Standard Quota	6.36424	10.32047	9.31529	
Lower Quota	6	10	9	25
Apportionment	$6+1=7$	10	9	26

With 27 seats: Standard divisor = $\frac{45,964}{27} = 1702.37037$

Table 6.2.2 : Apportionment with 27 seats

State	A	B	C	Total
Population	$11,251$	$18,245$	$16,468$	$45,694$
Standard Quota	6.60902	10.71741	9.67357	
Lower Quota	6	10	9	25
Apportionment	6	$10+1=11$	$9+1=10$	27

Yes, the Alabama occurred, since state A lost a seat in Congress.

State	With 26 seats	With 27 seats
A	7 →	6
B	10	11
C	9	10

Figure 6.2.2: Illustration of Alabama Paradox

Example 6.2.2: Does New State Paradox Occur?

Before annexation: Standard divisor = $\frac{9948}{100} = 99.48$

Table 6.2.5 : Apportionment Before the Annexation

State	A	B	C	Total
Population	$5,310$	1330	$3,308$	$9,948$
Standard Quota	53.378	13.370	33.253	100.000
Lower Quota	53	13	33	99
Final apportionment	54	13	33	100

After annexation: Standard divisor = $\frac{10448}{105} = 99.505$

Table 6.2.6 : Apportionment After the Annexation

State	A	B	C	D	Total
Population	$5,310$	1330	$3,308$	500	$10,448$
Standard Quota	53.364	13.366	33.245	5.025	105.000
Lower Quota	53	13	33	5	104
Final apportionment	53	$13+1=14$	33	5	105

District D has a population of 500, so it should get five seats. When District D is added with its five seats, District A loses a seat, and District B gains a seat. This is an example of the New-States Paradox.

Example 6.2.3: Does Population Paradox Occur?

Apportion using Hamilton before the extra work: Standard divisor = $\frac{1000}{11} = 90.90909$

Table 6.2.8 : Apportion of Candy Bars Before the Extra Work

	Abby	Bobby	Charley	Total
Population	54	243	703	$1,000$
Standard Quota	0.594	2.673	7.733	
Lower Quota	0	2	7	9
Apportionment	0	$2+1=3$	$7+1=8$	11

Apportion using Hamilton after extra work: Standard divisor = $\frac{1100}{11} = 100$

Table 6.2.9 : Apportion of Candy Bars After the Extra Work

	Abby	Bobby	Charley	Total
Population	56	255	789	$1,100$
Standard Quota	0.560	2.550	7.890	
Lower Quota	0	2	7	9
Final Apportionment	$0+1=1$	2	$7+1=8$	11

Abby time increase by: $\frac{56-54}{54} = 3.7\%$

Bobby time increase by: $\frac{255-243}{243} = 4.9\%$

Charley time increase by: $\frac{789-703}{703} = 12.2\%$

Abby's time increased by 3.7% while Bobby's time increased by 4.9%, more than Abby's. However, Abby gains a candy bar while Bobby loses one. This is an example of the Population Paradox.

Example 6.2.4: Population Paradox

Hospital B lost a respirator while hospital C gained one, even though hospital B had a higher growth rate (2.12%) than hospital C $(1.26\%.)$

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7: GRAPH THEORY

7.1: Basic Graphs and Graphs Structure
7.2: Euler Circuits and Eulerization of Graph
7.3: Hamiltonian Circuits and the Traveling Salesman Problem
7.4: Trees

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7.1: Basic Graphs and Graphs Structure



Figure 7.1.1: Cell phone networks connect individuals. (credit: "Business people using their phones" by Rawpixel Ltd./Flickr, CC BY 2.0)

Learning Objectives

The following objective will be covered in this section.

Learning Objectives

1. Identify parts of a graph.
2. Model relationships with graphs.

When you hear the word, *graph*, what comes to mind? You might think of the xy-coordinate system you learned about earlier in this course, or you might think of the line graphs and bar charts that are used to display data in news reports. The graphs we discuss in this chapter are probably very different from what you think of as a graph. They look like a bunch of dots connected by short line segments. The dots represent a group of objects, and the line segments represent the connections, or relationships, between them. The objects might be bus stops, computers, Snapchat accounts, family members, or any other objects that have direct connections to each other. The connections can be physical or virtual, formal or casual, scientific or social. Regardless of the type of connections, the purpose of the graph is to help us visualize the structure of the entire network and better understand the interactions among its objects.

Parts of a Graph

In a graph, the objects are represented with dots, and their connections are represented with lines, as in Figure 7.1.2, which displays a simple graph labeled *G* and a multigraph labeled *H*. The dots are called vertices; an individual dot is a **vertex**, which is one object of a set of objects, some of which may be connected. We often label vertices with letters. For example, Graph *G* has vertices *a, b, c*, and *d*, and Multigraph *H* has vertices *e, f, g*, and *h*. Each line segment or connection between two vertices is called an **edge.** *H* is considered a **multigraph** because it has a double edge between *f* and *h* and a double edge between *h* and *g*. Another reason *H* is called a multigraph is that it has a loop connecting vertex *e* to itself; a loop is an edge that joins a vertex to itself. Loops and double edges are not allowed in a simple graph.

Graphs, Vertices, and Edges

Graph: A graph consists of a set of dots, called vertices, and a set of edges connecting pairs of vertices.

Vertex: A vertex is a dot in the graph that could represent an intersection of streets, a land mass, or a general location, like "work" or "school". Vertices are often connected by edges. Note that vertices only occur when a dot is explicitly placed, not whenever two edges cross. Imagine a freeway overpass – the freeway and side street cross, but it is not possible to change from the side street to the freeway at that point, so there is no intersection and no vertex would be placed.

Edge: Edges connect pairs of vertices. An edge can represent a physical connection between locations, like a street, or simply that a route connecting the two locations exists, like an airline flight.

To sum up, a simple graph is a collection of vertices and any edges that may connect them, such that every edge connects two vertices with no loops and no two vertices are joined by more than one edge. A **multigraph** is a graph in which loops or pairs of vertices are joined by more than one edge. In this chapter, most of our work will be with simple graphs, which we will call graphs for convenience.

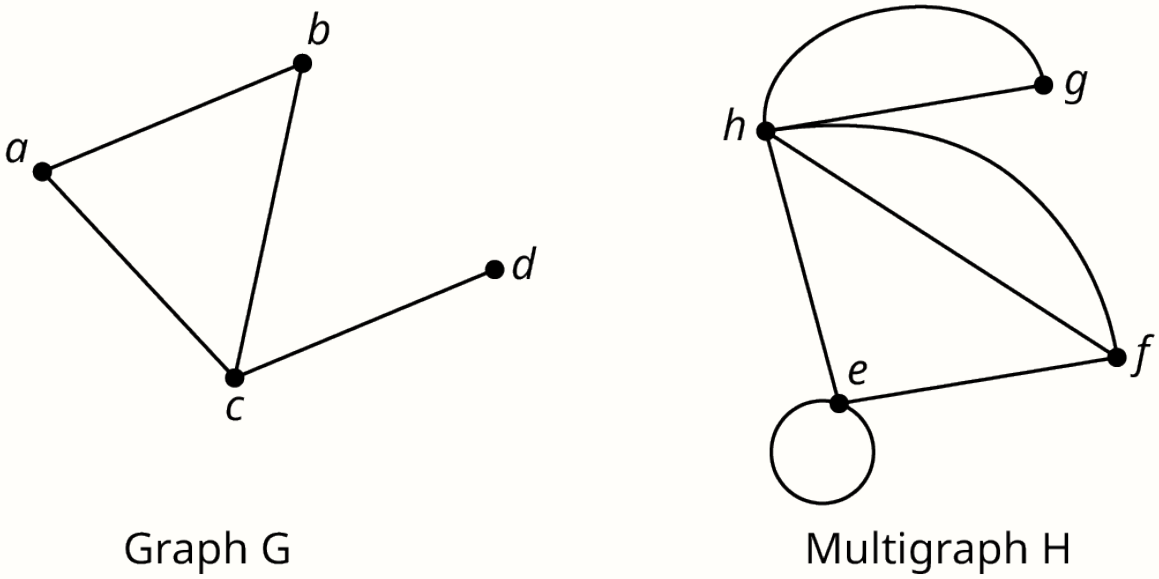


Figure 7.1.2: A Graph and a Multigraph

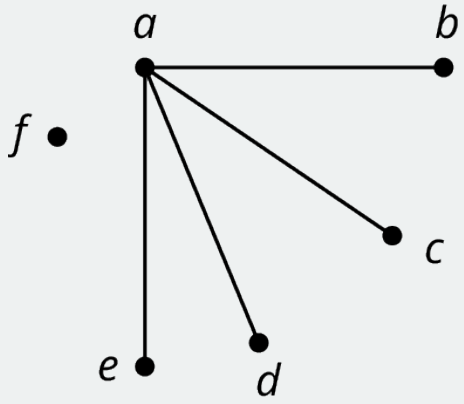
It is not necessary for the edges in a graph to be straight. In fact, you can draw an edge any way you want. Graph theory focuses on which vertices are connected, not how the connections are drawn. In a graph, each edge can be named by the two letters of the associated vertices. The four edges in Graph *X* in Figure 7.1.3 are *ab, ac, ad*, and *ae*. The order of the letters is not essential when you name the edge of a graph. For example, *ab* refers to the same edge as *ba*.

Loop of the Graph

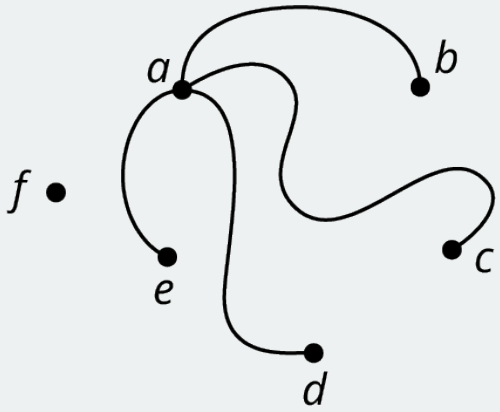
A loop is a special type of edge that connects a vertex to itself. Loops are not used much in street network graphs. In figure 7.1.2, **ee** is a loop.

Edge Could Be a Straight Line or Curve

A graph may have vertices that are not joined to other vertices by edges, such as vertex f in Graph X in Figure 7.1.3, but any edge must have a vertex at each end.



Graph X

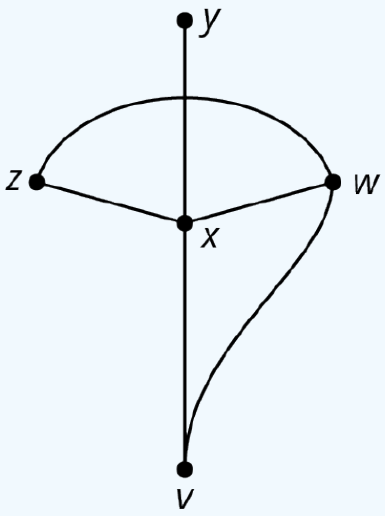


Graph X

Figure 7.1.3: Different Representations of the Same Graph

Example 7.1.1: Identifying Edges and Vertices

Name all the vertices and edges of graph *F* in Figure 7.1.4. Does this graph have a loop?



Graph F

Figure 7.1.4: Graph F

Answer available at the end of this section.

Adjacent Vertices and Adjacent Edges

Two **vertices** in a graph are called **adjacent** if they are connected directly by an **edge**.

For example, in Figure 7.1.5 : Graph *A,* vertices **s** *and* **t** are adjacent, **s** and **r** are adjacent, **s** and **q** are adjacent, but vertices **s** and **p** are not, and also **t** and **p** are not adjacent.

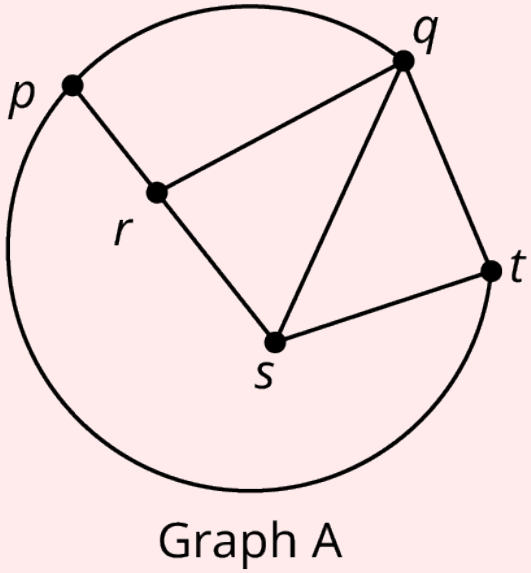


Figure 7.1.5: Graph A

Two **edges** are called **adjacent** if they **share a common vertex**.

Edges **st** and **sr** are adjacent, but **sr** and **pq** are not.

Your Turn 7.1.2: Find Adjacent Vertex

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Analyzing Geographical Maps with Graphs

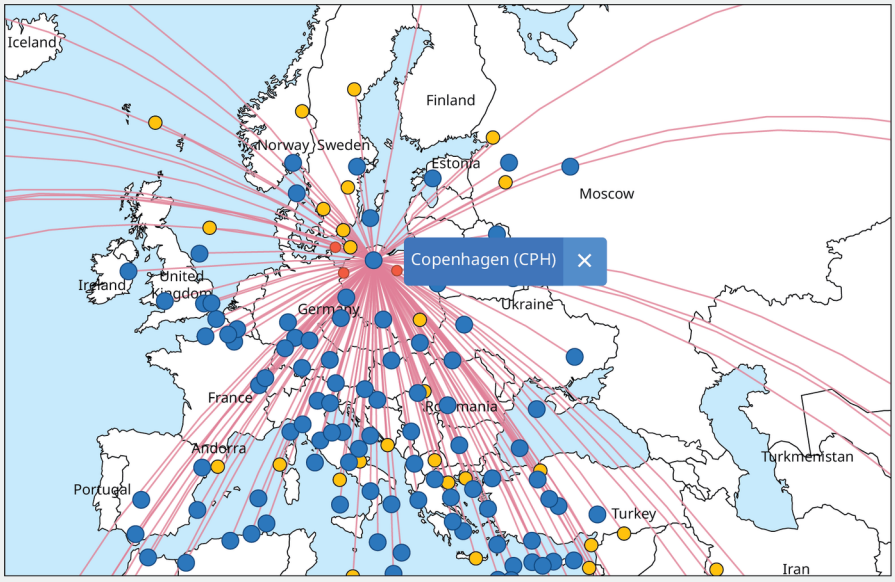


Figure 7.1.6: Commercial Airlines' Route Systems Create a Global Network.

When graphs are used to model and analyze real-world applications, the number of edges meeting at a given vertex is important. For example, a graph may represent the direct flight connections for a particular airport, as in Figure 7.1.7. Representing the connections with a graph rather than a map shifts the focus away from the relative positions and toward which airports are connected.

In Figure 7.1.7, the vertices are the airports, and the edges are the direct flight paths. The number of flight connections between a particular airport and other South Florida airports is the number of edges meeting at a particular vertex. For example, Key West has direct flights to three of the five airports on the graph. In graph theory terms, we would say that vertex FYW has degree 3. The degree of a vertex is the number of edges that connect to that vertex.

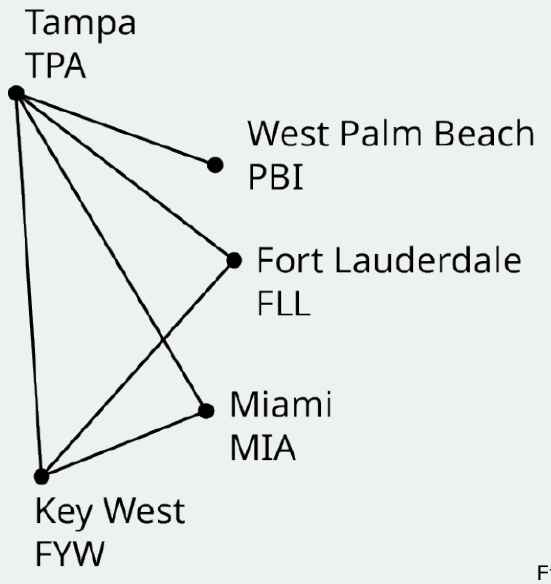


Figure 7.1.7: Direct Flights between South Florida Airport.

Degree of a Vertex

The degree of a vertex is the number of edges meeting at that vertex. It is possible for a vertex to have a degree of zero or larger. The **degree** of a vertex can be either **even** or **odd,** depending on how many edges are connected to it.

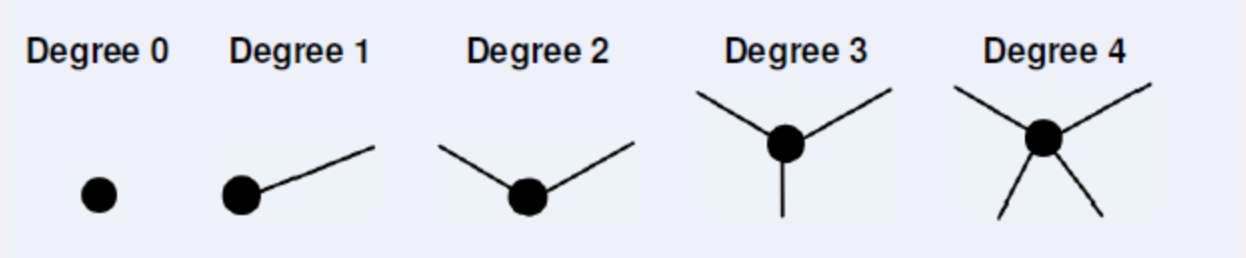


Figure 7.1.*D*: Showing degree of Vertex

If the degree of a vertex is even, it is called even; if it is odd, it is called odd.

If a graph contains a loop, that loop contributes two to the degree of the vertex.

Think about the intersection of two roads, Daniel Parkway and Simmerlin, near FSW. That intersection is a vertex, and its degree is four.

Example 7.1.3: Determining the Degree of a Vertex

Determine the degree of each vertex of Graph *J* in Figure 7.1.8. If Graph *J* represents direct flights between a set of airports. Find the degree of each vertex. Do any of the airports have direct flights to two or more of the other cities on the graph? Also, determine if the degree is even or odd.

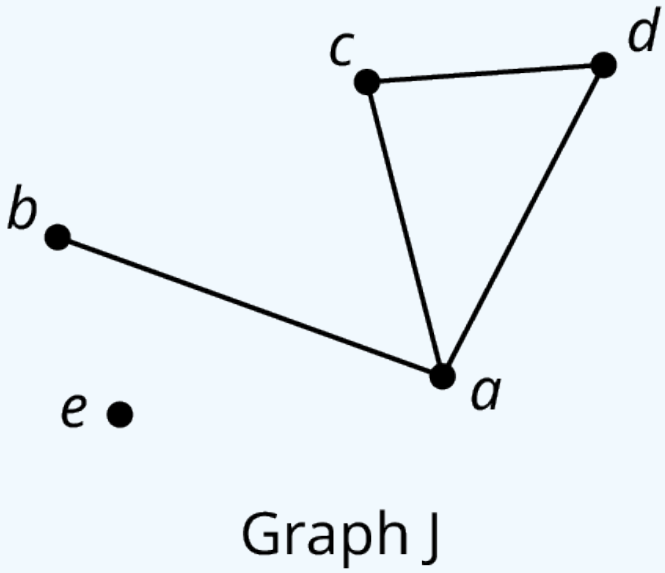


Figure 7.1.8: Graph J

Answer available at the end of this section.

Your Turn 7.1.3: Even or Odd Vertices?

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Your Turn 7.1.4: Football Schedule

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Path and Circuit

Definition: Path

A path is a sequence of vertices using the edges. Usually, we are interested in a path between two vertices. For example, a path from vertex A to vertex M is shown below. It is one of many possible paths in this graph. A path may not include an edge more than one time.

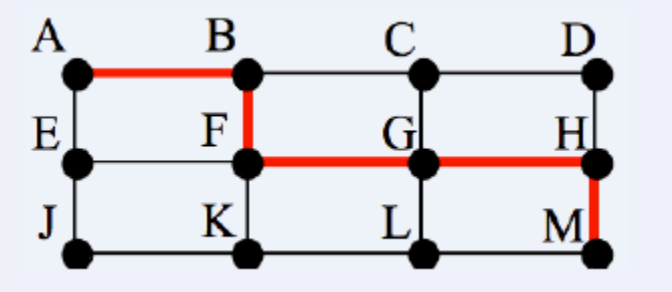


Figure 7.1.10: Graph Showing Path

One of the paths from A to H is: ABFGHM (Yes, of course, there are more)

or

$A \to B \to F \to G \to H \to M$

Definition: Circuit

A circuit is a path that begins and ends at the same vertex. A circuit starting and ending at vertex A is shown below.

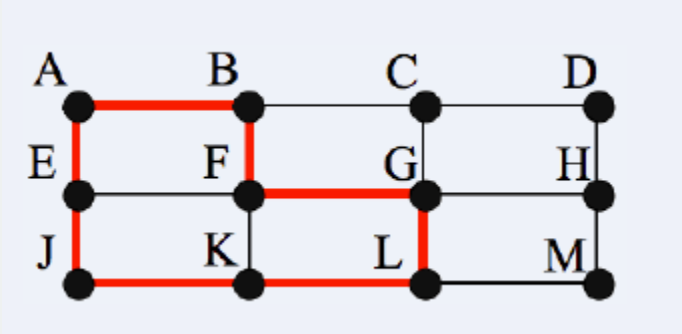


Figure 7.1.11: Graph Showing Circuit

One of the circuit from A to A is: ABFGLKJEA (Yes, of course, there are more)

or

$A \to B \to F \to G \to L \to K \to J \to E \to A$

Your Turn 7.1.5: Find the Path

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Modeling Situation Using Graph

Example 7.1.6: Modeling a Floor Plan

Figure 7.1.12 shows the floor plan of a house. Use the floor plan to answer each question.

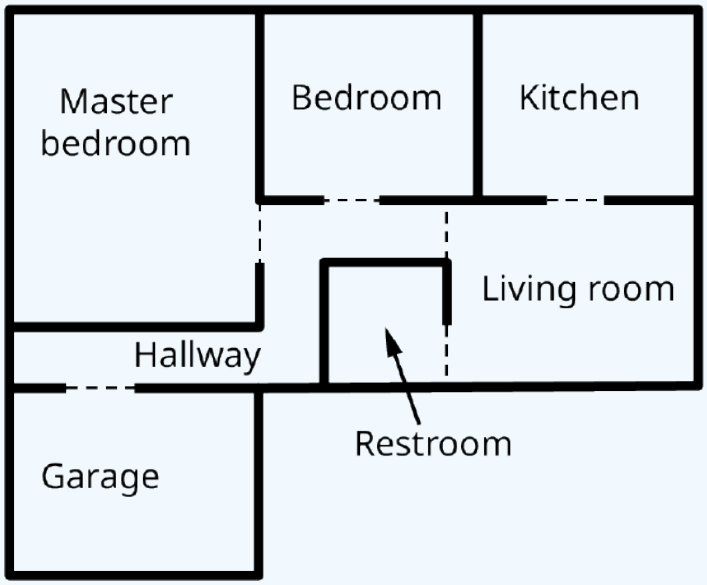


Figure 7.1.12: Floor Plan of a House

Draw a graph representing the floor plan in which each vertex represents a different room (or hallway), and edges represent doorways between rooms. Also, find one path from the Garage to the Kitchen.

Answer available at the end of this section.

Your Turn 7.1.6: Draw a Graph to Model Given Floor Plan

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Graphs are also used to analyze regional boundaries. The states of Utah, Colorado, Arizona, and New Mexico all meet at a single point known as the “Four Corners,” which is shown in the map in figure 7.1.15.

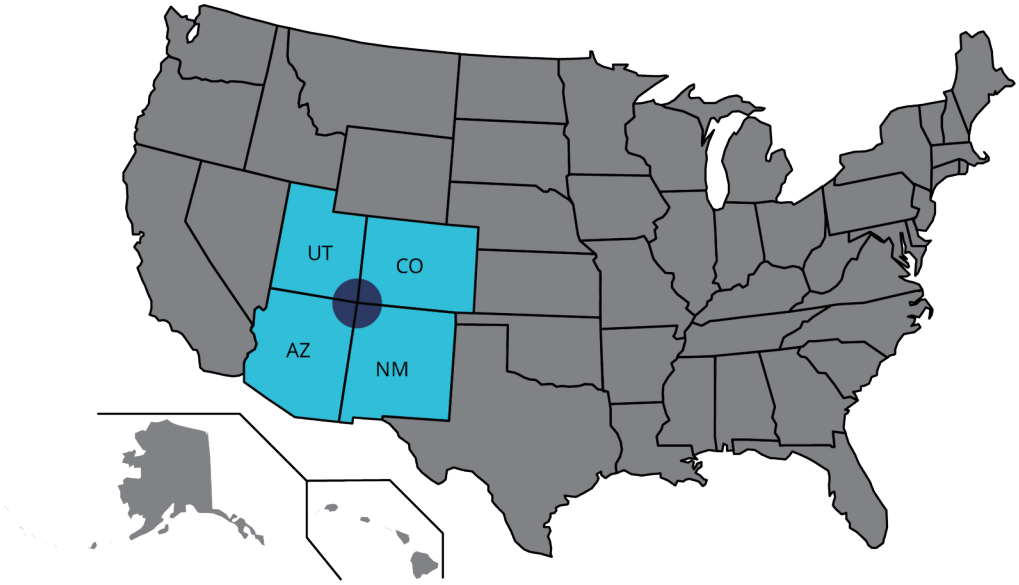


Figure 7.1.15 : Map of the Four Corners

In Figure 7.1.16, each vertex represents one of these states, and each edge represents a shared border. States like Utah and New Mexico that meet at only a single point are *not* considered to have a shared border. By representing this map as a graph, where connections are shared borders, we shift our perspective from physical attributes such as shape, size, and distance toward the relationship of shared boundaries.

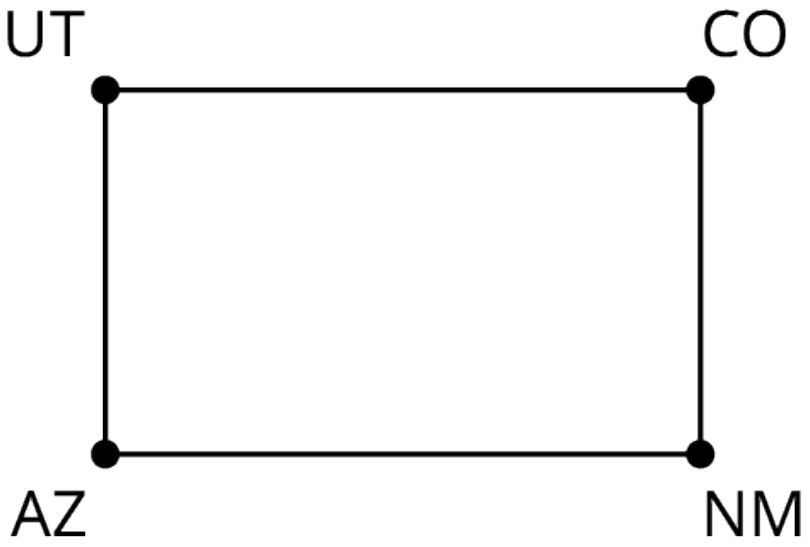


Figure 7.1.16: Graph of the Shared Boundaries of Four Corners States

Example 7.1.7: Graphing the Midwestern States

A map of the Midwest is given in Figure 7.1.17. Create a graph of the region in which each vertex represents a state and each edge represents a shared border.

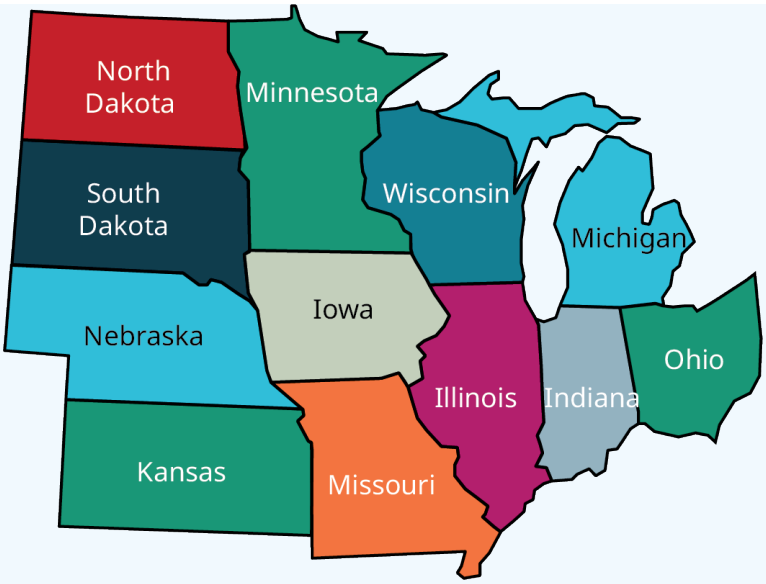


Figure 7.1.17: Map of Midwestern States

Answer available at the end of this section.

Your Turn 7.1.7: Model the Map to Graph

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Who Knew?: Using Graph Theory to Reduce Internet Fraud

Could graphs be used to reduce Internet fraud? At least one researcher thinks so. Graph theory is used every day to analyze our behavior, particularly on social network sites. Alex Buetel, a computer scientist from Carnegie Mellon University in Pittsburgh, Pennsylvania, published a research paper in 2016 that discussed the possibilities of distinguishing the normal interactions from those that might be fraudulent using graph theory. Buetel wrote, “To more effectively model and detect abnormal behavior, we model *how* fraudsters work, catching previously undetected fraud on Facebook, Twitter, and Tencent Weibo and improving classification accuracy by up to 68.” In the same paper, the researcher discusses how similar techniques can be used to model many other applications and even, “predict *why* you like a particular movie.” (Alex Beutel, "User Behavior Modeling with Large-Scale Graph Analysis," User Behaviour Modeling(opens in new window) [reports-archive.adm.cs.cmu.ed...-CS-16-105.pdf(opens in new window)], May 2016, CMU-CS-16-105, Computer Science Department, School of Computer Science, Carnegie Mellon University, Pittsburgh, PA)

Completeness and Subgraph

Suppose that there were five strangers in a room, *A*, *B*, *C*, *D*, and *E*, and each one would be introduced to each of the others. How many introductions are necessary? One way to begin answering this question is to draw a graph, with each vertex representing an individual in the room and each edge representing an introduction.

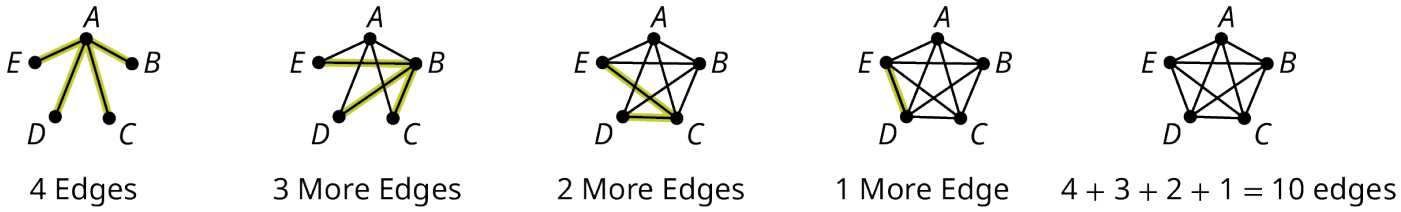


Figure 7.1.20: Model of Introductions between Five Strangers

Let's approach the problem by thinking about how many new people Person A would meet, then Person B, and so on, making sure not to repeat any introductions. The first graph in Figure 7.1.20 shows Person A meeting Persons B, C, D, and E, for a total of 4 introductions. The next graph shows that Person B still has to meet Persons C, D, and E, for a total of 3 more introductions. The next graph shows that Person C still has to meet Persons D and E, which is 2 more introductions. The next graph shows that Person D only needs to meet Person E, which is one more introduction.

The final graph has $4 + 3 + 2 + 1 = 10$ edges representing 10 introductions.

Definition: Complete Graph

A complete graph is one in which every pair of distinct vertices is connected by an edge.

The last graph in Figure 7.1.20 is an example of a complete graph because an edge joins each pair of vertices.

Another way of saying this is that the graph is complete because each vertex is adjacent to every other vertex. Figure 7.1.21 shows the complete graphs with three, four, five, and six vertices.

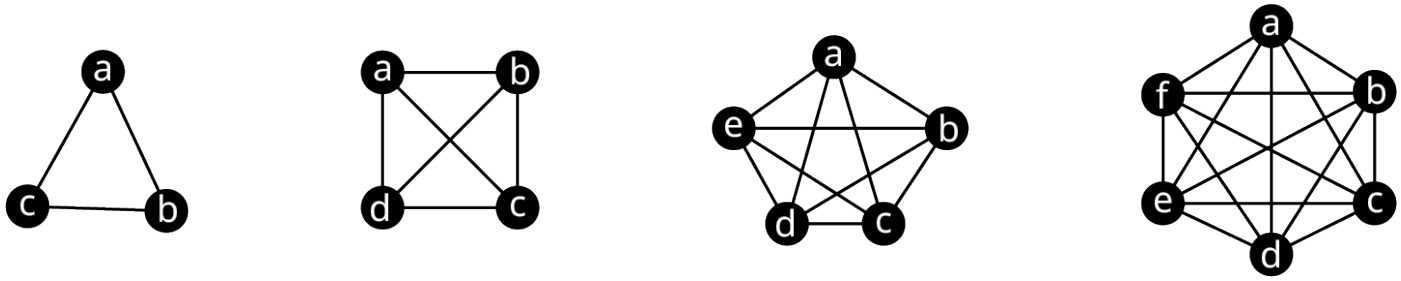


Figure 7.1.21: Complete Graphs with Up to Six Vertices

Sometimes, a graph is a part of a larger graph. This graph is called a subgraph.

Definition: Subgraph

A subgraph is a smaller graph section comprising a subset of the vertices and edges from the original graph. Every edge and vertex of the subgraph must come from the original graph.

In a global airline route map (**Original Graph**), the domestic flight paths within just one country would be a **subgraph**.

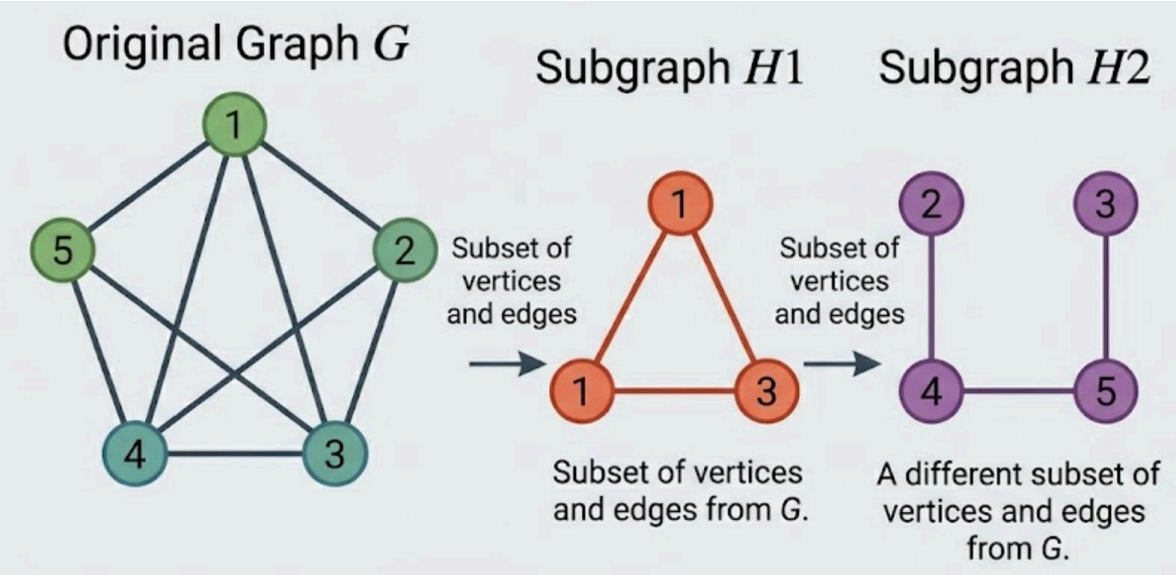


Figure 7.1.22: Two Subgraphs of a Graph

Example 7.1.8: Identifying a Subgraph

Graph G and four diagrams are given in Figure 7.1.23. Determine whether each diagram is or is not a subgraph of Graph *G* and explain why.

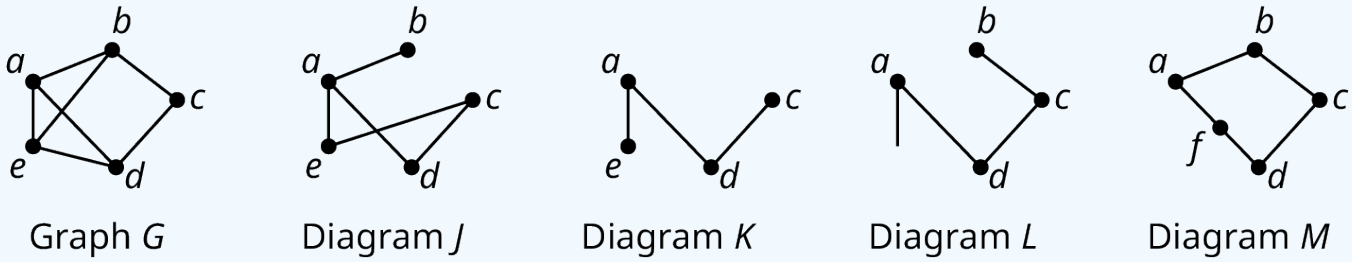


Figure 7.1.23: Graph *G* and Potential Subgraphs

Answer available at the end of this section.

Your Turn 7.1.8: Make Complete Graph

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Recognizing Isomorphic (Equivalent) Graphs

Isomorphic graphs that represent the same pattern of connections can look very different despite having the same underlying structure. The edges can be stretched and twisted. The graph can be rotated or flipped. For example, in Figure 7.1.24, each diagram represents the same pattern of connections. The isomorphic graph is also called an equivalent graph.

Equivalent Graph (Isomorphic Graph)

Two graphs are said to be **isomorphic** if they are **structurally the same**, even if they look different in a drawing.

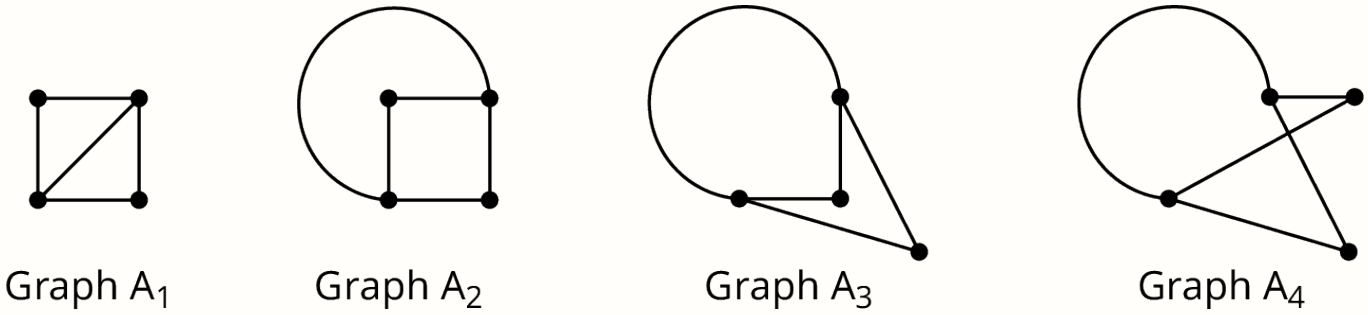


Figure 7.1.24: Four Representations of the Same Graph

Looking at Figure 7.1.24, how can we know that these graphs are isomorphic? We will start by checking for any obvious differences. Each of the graphs in Figure 7.1.24 has four vertices and five edges, so there are no differences. Next, we will focus on the degrees of the vertices, which have been labeled in Figure 7.1.25.

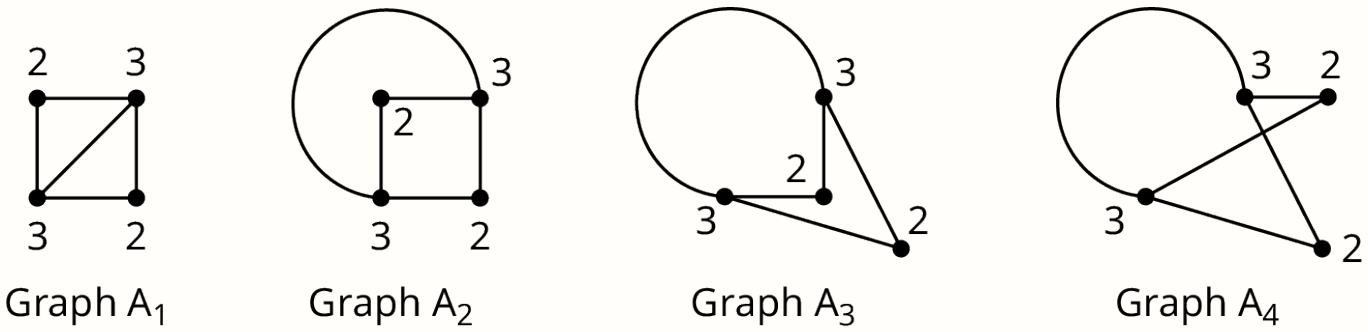


Figure 7.1.25: Graphs with Vertices of the Same Degrees

As shown in Figure 7.1.25, each graph has two vertices of degree 2 and two vertices of degree 3, so there are no differences.

Example 7.1.9: Identifying Equivalent Graph

Are the Graphs B_1 and B_2 in Figure 7.1.26 are isomorphic? Identify an isomorphism between them by listing corresponding pairs of vertices.

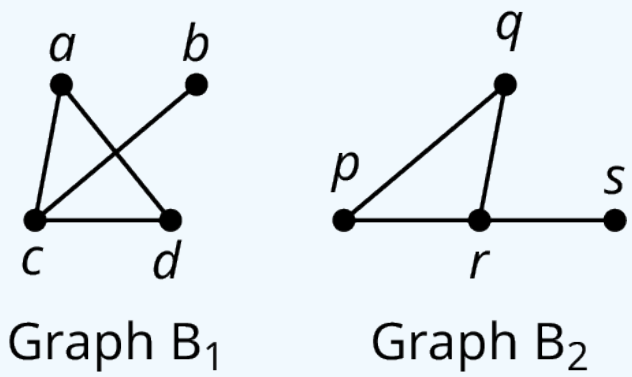


Figure 7.1.26: Graphs B1 and B2

Answer available at the end of this section.

Your Turn 7.1.9: Equivalent Graphs

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Connectedness and Bridges

Before we can discuss finding the best delivery route, we must ensure that a route exists. For example, suppose that you were tasked with visiting every airport on the graph in Figure 7.1.37 by plane. Could you accomplish that task by only taking direct flight paths between airports listed on this graph? In other words, are all the airports connected by paths? Or are some of the airports disconnected from the others?

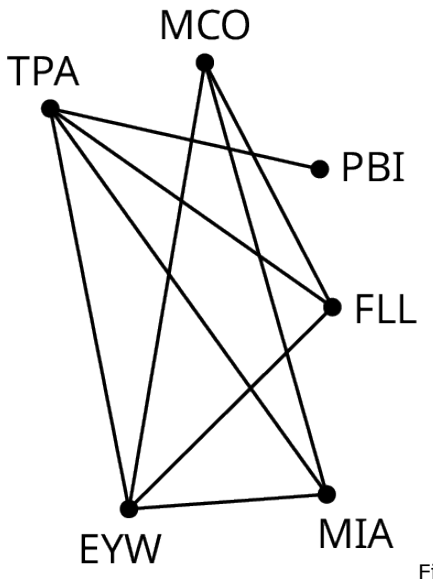


Figure 7.1.28: Major Central and South Florida Airports

In Figure 7.1.28, we can see TPA is adjacent to PBI, FLL, MIA, and EYW. Also, there is a path between TPA and MCO through FLL. This indicates there is a path between each pair of vertices. So, it is possible to travel to each of these airports only by taking direct flight paths and visiting no other airports.

Definition: Connectedness

A graph is connected if there is a path from any vertex to any other vertex. Every graph drawn so far has been connected. The graph below is disconnected; there is no path from the vertices on the left to those on the right.

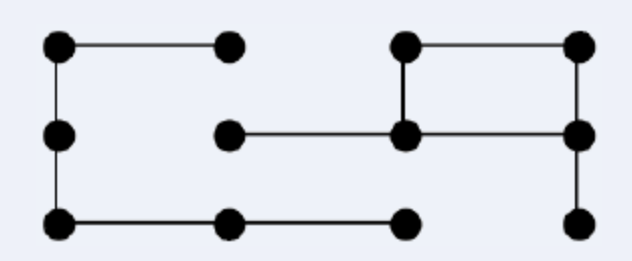


Figure 7.1.29: Disconnected Graph

Let's take a closer look at graph *X* in Figure 7.1.30. Focus on vertex *a*. There is a path between vertices *a* and *b*, but there is no path between vertex *a* and vertex *c*. So, Graph *X* is disconnected. The graph Y is connected because of vertex g.

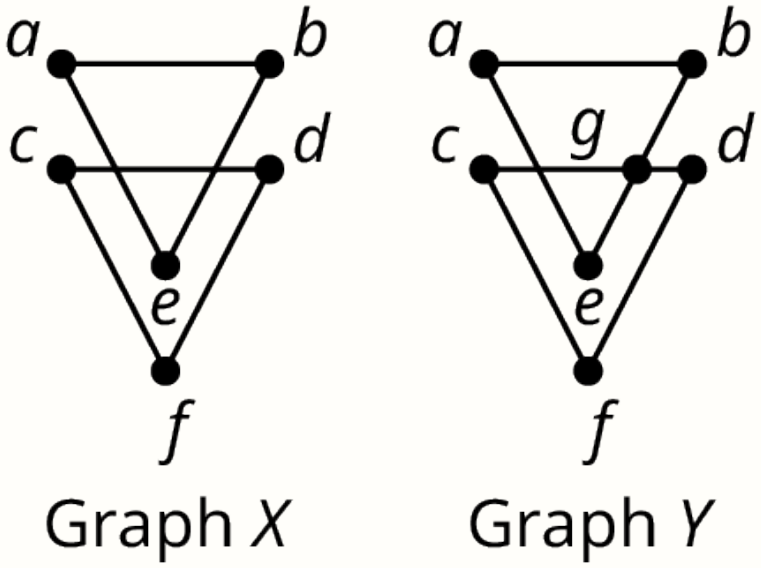


Figure 7.1.30: Connected vs. Disconnected

When you are working with a planar graph, you can also determine if a graph is connected by untangling it. If you draw it so that none of the edges are overlapping, as we did with Graph *X* in Figure 7.1.31, it is easier to see that the graph is disconnected.

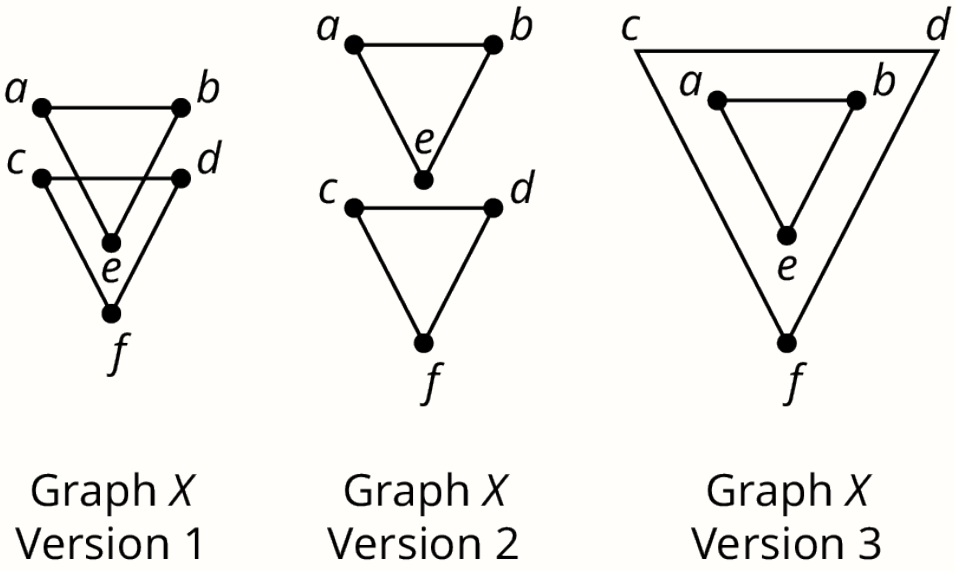


Figure 7.1.31: Untangling Graph *X*

Versions 2 and 3 of Graph *X* in Figure 7.1.31, each have the same number of vertices, number of edges, degrees of the vertices, and pairs of adjacent vertices as in version 1. In other words, each version retains the same structure as the original graph. Since versions 2 and 3 of Graph *X*, do not have overlapping edges, it is easier to identify pairs of vertices that do not have paths between them, and it is more obvious that Graph *X* is disconnected. In fact, there are two completely separate, disconnected subgraphs, one with the vertices in {*a, b, e*}, and the other with the vertices {*c, d, f*}

Example 7.1.10: Determining If a Graph Is Connected or Disconnected

Use Figure 7.1.32 to answer the following questions.

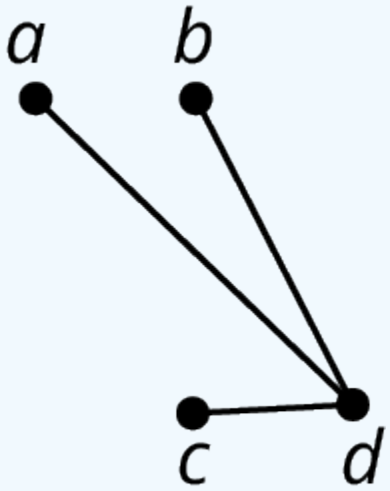


Figure 7.1.32: Graph E

1. Identify all the components of Graph *E*.
2. Determine whether the graph is connected or disconnected, and explain your reasoning.

Answer available at the end of this section.

Your Turn 7.1.10: Is Graph Connected?

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Definition: Bridge

A bridge is an edge that, if removed, would disconnect the graph, meaning there would be at least two separate parts that are no longer connected.

Example 7.1.11: Find Bridges

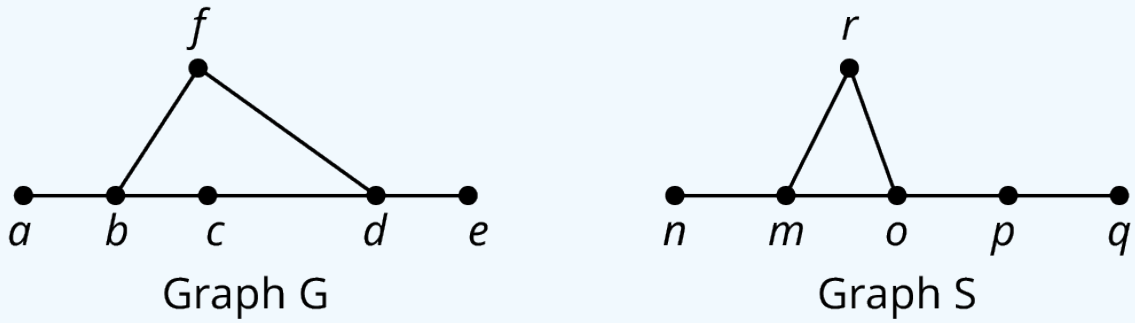


Figure 7.1.33

Find the bridges in **Graph G** and **Graph S**

Answer available at the end of this section.

Your Turn 7.1.11: Identify Bridges

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Your Turn 7.1.12: Matching Terms with Definition

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Contribution

1. Math in Society [www.opentextbookstore.com]
2. Contemporary Mathematics [openstax.org]

Answers

Example 7.1.1: Identifying Edges and Vertices

The vertices are *v*, *w*, *x*, *y*, and *z*. The edges are *vw*, *vx*, *wx*, *wz*, *xy*, and *xz*. The graph has no loop.

Example 7.1.3: Determining the Degree of a Vertex

For each vertex, count the number of edges that meet at that vertex. This value is the degree of the vertex. In Figure 7.1.9, the dashed edges indicate the edges that meet at the marked vertex.

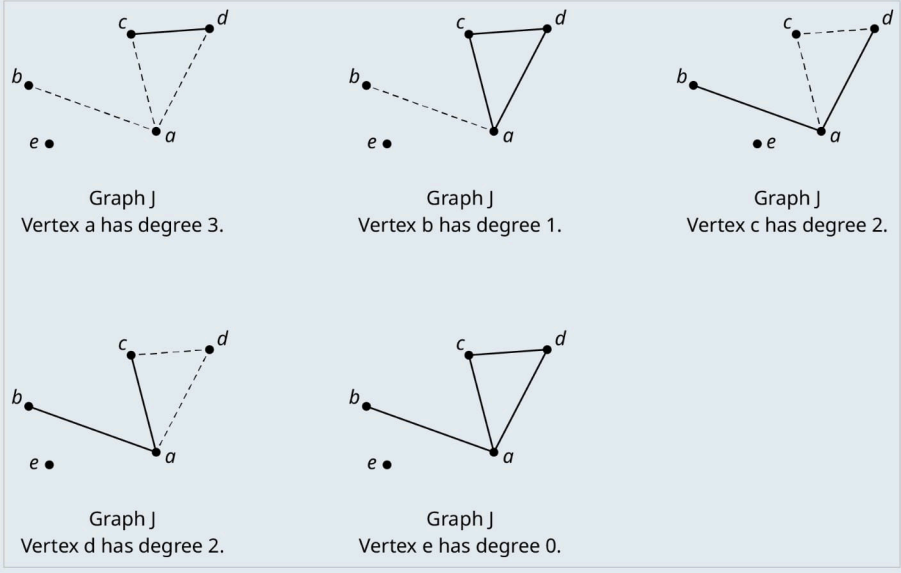


Figure 7.1.9: Degree of Vertices of Graph J

The degree of a is 3, indicating that a is ODD.
The degree of b is 1, meaning that b is also ODD.
The degree of d is 2, which classifies d as EVEN.
The degree of e is 0, so e is also considered EVEN.
Finally, the degree of c is 2, making c EVEN.

Yes , airports a, c, and d have direct flights to two or more of the other cities.

Example 7.1.6: Modeling a Floor Plan

We will need a vertex for each room, and it is convenient to label them with the rooms' names. Visualize the scenario in your head as shown in Figure 7.1.13. You don't have to write this step on your paper.

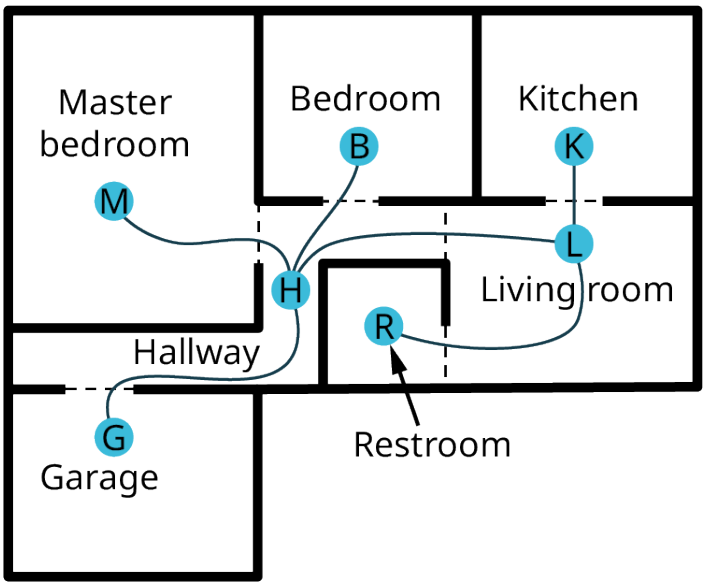


Figure 7.1.13: Assigning Vertices to Rooms

Draw a graph to represent the scenario. Start with the vertices. Then, connect those vertices that share a doorway in the floor plan, as shown in Figure 7.1.14.

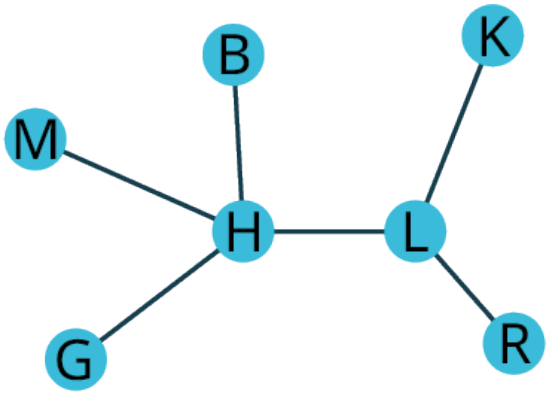


Figure 7.1.14: Graph of the Floor plan

There is only one path from Garage to Kitchen, which is GHLK, or $G \to H \to L \to K$

Example 7.1.7: Graphing the Midwestern States

For each state, draw and label a vertex. Draw edges between any two states that share a common land border as figure 7.1.18.

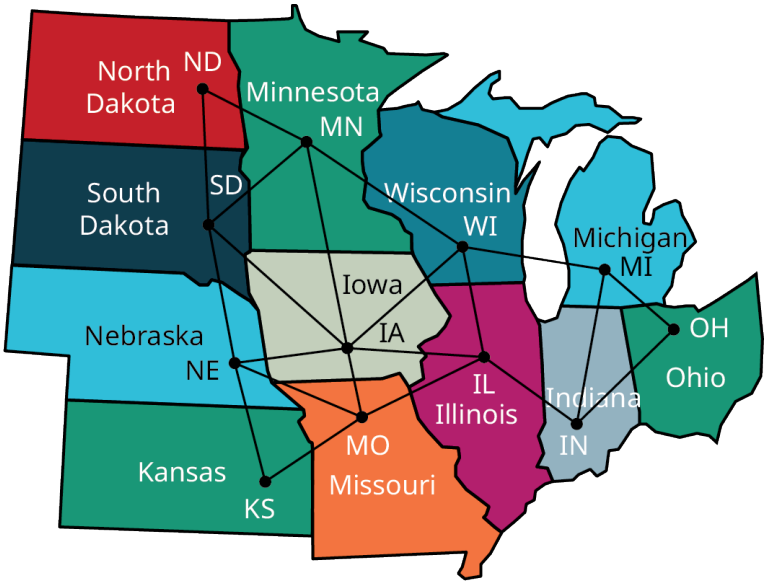


Figure 7.1.18: Edge Assigned to Each Pair of Midwestern States with Common Border

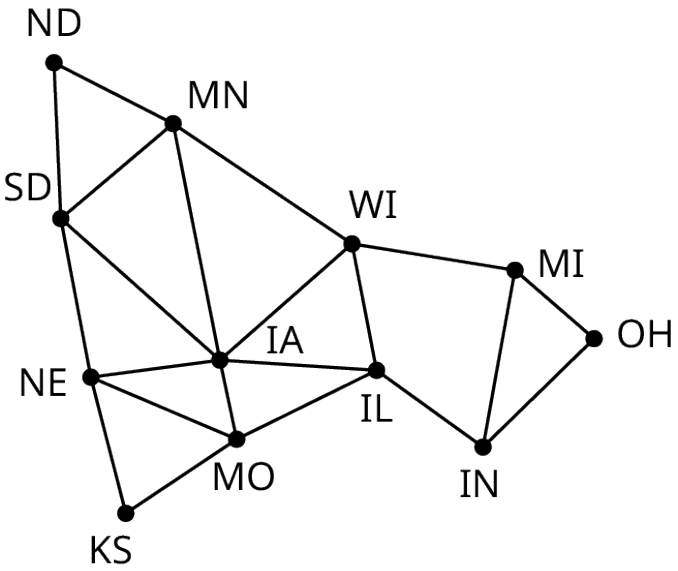


Figure 7.1.19: Final Graph Representing Common Boundaries between Midwestern States

Example 7.1.8: Identifying a Subgraph

Diagram *J* is not a subgraph of Graph *G* because edge *ec* is not in Graph *G*.

Diagram *K* is a subgraph of Graph *G* because all of its vertices and edges were part of Graph *G*.

Diagram *L* is not a graph at all because there is a line extending from vertex *a* that does not connect *a* to another vertex. So, Diagram *L* cannot be a subgraph.

Diagram *M* is not a subgraph of Graph *G* because it contains a vertex *f*, which is not part of graph *G*.

Example 7.1.9**: Identifying Equivalent Graph**

Figure 7.1.35 shows how to transform Graph B_1 to get Graph B_2.

Vertex **b** corresponds to **s**, vertex **c** corresponds to **r,** vertex **q** corresponds to **a**, and vertex **d** corresponds to **p**.

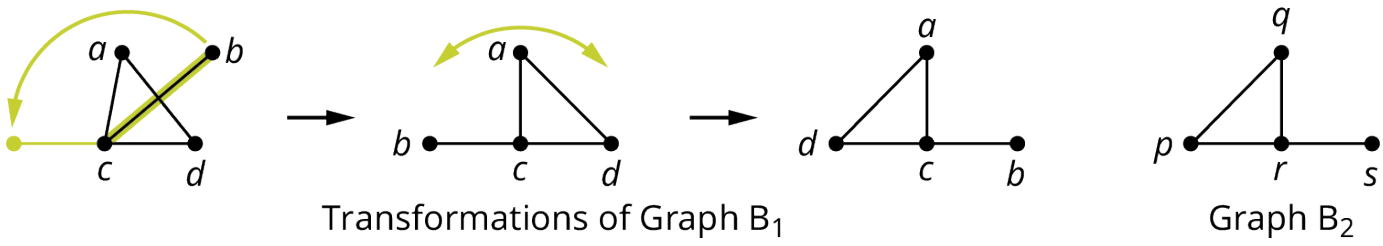


Figure 7.1.27: Transform Graph B_1 into Graph B_2

Example 7.1.10**: Determining If a Graph Is Connected or Disconnected**

1. There is only one component in Graph *E*. It has the vertices $\{a, b, c, d\}$
2. The graph is connected because there is a path between vertex a and every other vertex. We can also see that Graph E is connected because it has only one component.

Example 7.1.11**: Find Bridges**

In G, bridges are **ab** and **de.**

In S, bridges are **nm, op, and pq.**

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7.2: Euler Circuits and Eulerization of Graph



Figure 7.2.1: Delivery trucks move goods from place to place. (credit: "Mack Midliner" by Jason Lawrence/Flickr, CC BY 2.0)

Learning Objectives

1. Describe and identify Euler paths and Circuits.
2. Apply the Euler Circuit Theorem.
3. State the Chinese postman problem.
4. Evaluate Euler Circuits in real-world applications.

The delivery of goods is a huge part of our daily lives. From the factory to the distribution center to the local vendor or to your front door, nearly every product that you buy has been shipped multiple times to get to you. If the cost and time of that delivery is too great, you will not be able to afford the product. Delivery personnel have to leave from one location, deliver the goods to various places, and then return to their original location and do all of this in an organized way without losing money. How do delivery services find the most efficient delivery route? The answer lies in graph theory.

Origin of Euler Circuits and Terminology of Euler Path and Circuit

The city of Konigsberg, modern-day Kaliningrad, Russia, has waterways that divide the city. In the 1700s, the city had seven bridges over the various waterways. The map of those bridges is shown in Figure $7.2.2$. The question as to whether it was possible to find a route that crossed each bridge exactly once and return to the starting point was known as the Konigsberg Bridge Problem. In 1735, one of the most influential mathematicians of all time, Leonard Euler, solved the problem using an area of mathematics he created himself: graph theory!

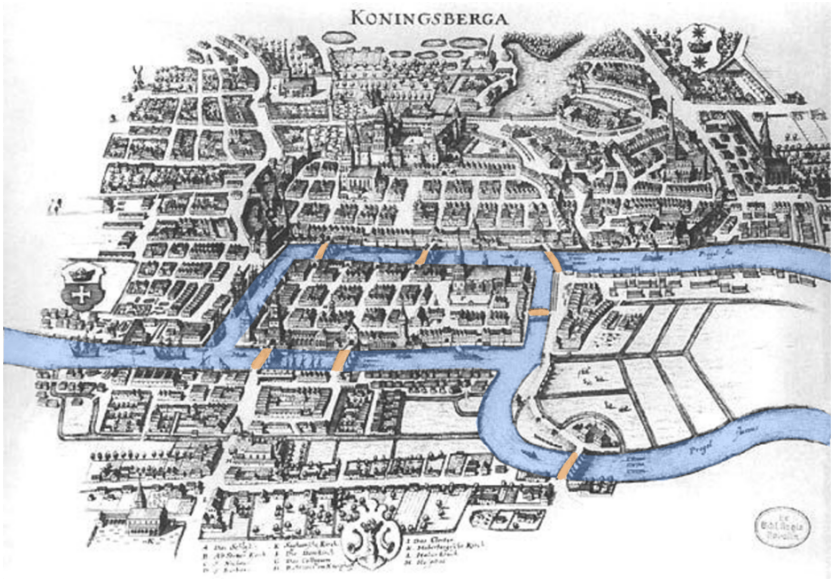


Figure 7.2.2 : Map of the Bridges of Konigsberg in 1700s (credit: “Konigsberg Bridge” by Merian Erben/Wikimedia Commons, Public Domain)

Euler drew a multigraph in which each vertex represented a land mass, and each edge represented a bridge connecting them, as shown in Figure 7.2.3. Remember from Navigating Graphs that a circuit is a trail, so it never repeats an edge, and it is closed, so it begins and ends at the same vertex. Euler pointed out that the Konigsberg Bridge Problem was the same as asking this graph theory question: Is it possible to find a circuit that crosses every edge? Since then, circuits (or closed trails) that visit every edge in a graph exactly once have come to be known as Euler circuits in honor of Leonard Euler.

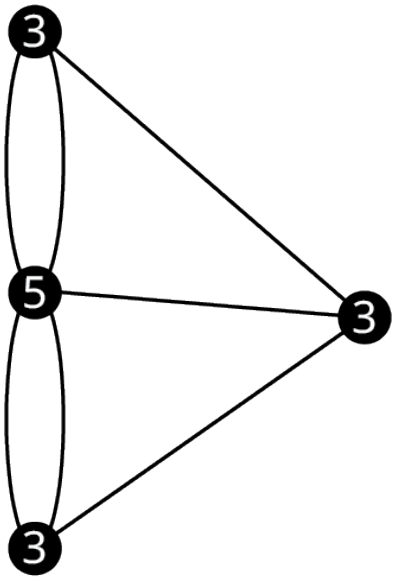


Figure 7.2.3: Graph of Map of the Bridges of Konigsberg

In the first section, we created a graph of the Königsberg bridges and asked whether it was possible to walk across every bridge once. Because Euler first studied this question, these types of paths are named after him.

Euler paths and Euler circuits are useful in many real-life situations where you need to travel through connections efficiently without repeating edges (roads, links, wires, routes, etc.). Here’s a clear and simple explanation of where they are used:

Euler Path and Euler Circuit

An **Euler path** is a path that uses every edge in a graph with no repeats. Being a path, it does not have to return to the starting vertex.

An **Euler circuit** is a circuit that uses every edge in a graph with no repeats. Being a circuit, it must start and end at the same vertex

Eulerian Graph

The graph is called Eulerian if it contains an Eulerian circuit.
Semi-Eulerian if it contains an Eulerian path but *not* an Eulerian circuit.

Your Turn 7.2.1: Euler Path Euler, Circuit or Neither?

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Euler's Path and Circuit Theorem

	Condition	Where to Start and End
Euler Circuit	All vertices are even	We can start at any even vertex and must end at the same vertex
Euler Path or Euler Trail (No Euler Circuit)	Exactly two odd vertices	We must start from one odd vertex and end at another odd vertex
No Euler Path and Euler Circuit	More than 2 odd vertices	

Your Turn 7.2.2: Euler Theorem

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Euler Path and Euler Circuit

A path that visits every edge in a graph is called an Euler path. Since trails are walks that do not repeat edges, an Euler trail visits every edge exactly once.

Example 7.2.3: Find a Euler Path

In the graph shown below, there are several Euler paths. Find one such path. What is the degree of the vertex that you start and end at to find such a path?

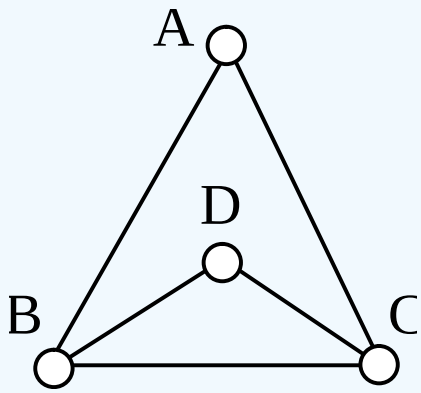


Figure 7.2.4

Answer available at the end of this section.

Your Turn 7.2.3: Euler Path

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Example 7.2.4: Find a Euler Path

Does the graph below have an Euler Circuit? If so, find one.

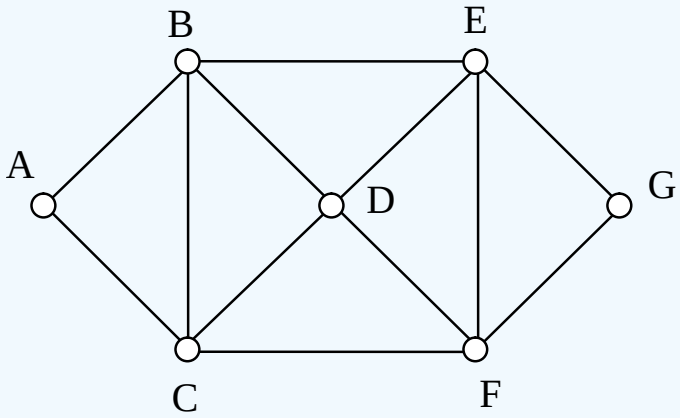


Figure 7.2.6

Answer available at the end of this section.

Your Turn 7.2.4: Euler Circuit

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Here is a portion of a housing development. As part of her job, the development's lawn inspector has to walk down every street to ensure the homeowners' landscaping conforms to the community requirements.

Naturally, she wants to minimize the amount of walking she has to do. Is it possible for her to walk down every street in this development without doing any backtracking? While you might be able to answer that question just by looking at the picture for a while, it would be ideal to be able to answer the question for any picture, regardless of its complexity job; the development's lawn inspector has to walk down every street in the development, making sure homeowners' landscaping conforms to the community requirements.



Figure 7.2.7: Portion of Housing Development

Why care if an Euler circuit exists? The lawn inspector is interested in walking as little as possible. The ideal situation would be a circuit that covers every street with no repeats. That's an Euler circuit! Luckily, Euler solved the question of whether or not an Euler path or circuit will exist.

Is there an Euler circuit on the housing development map Figure Figure 7.2.7, which is represented by the following graph in Figure 7.2.8. All the highlighted vertices have odd degrees (Figure 7.2.9.) Since there are more than two vertices with odd degrees, this graph has no Euler paths or circuits. Unfortunately, our lawn inspector will need to do some backtracking.

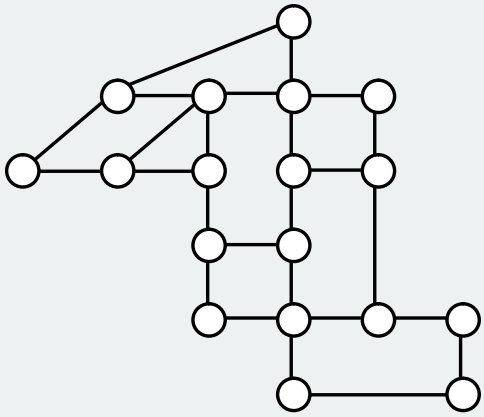


Figure 7.2.8

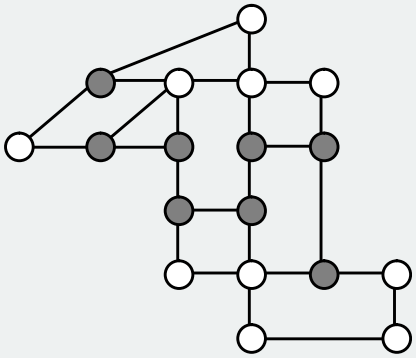


Figure 7.2.9

Now we know how to determine if a graph has an Euler circuit, but how do we find one if it does? While it is usually possible to find an Euler circuit just by pulling out your pencil and trying to find one, the more formal method is **Fleury's algorithm.**

Fleury's Algorithm: To Find the Euler Path and Euler Circuit

1. Start at any vertex if finding an Euler circuit. If finding an Euler path, start at one of the two vertices with odd degree.
2. Choose any edge leaving your current vertex, provided that deleting that edge will not separate the graph into two disconnected sets of edges.
3. Add that edge to your circuit and delete it from the graph.
4. Continue until you're done.

Example 7.2.5: Finding an Euler Circuit

Using Fleury's algorithm, starting at vertex A, let's find an Euler Circuit on this graph.

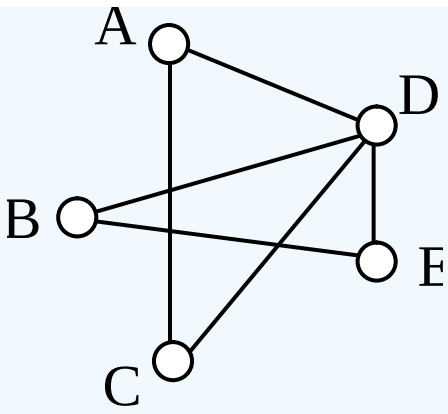


Figure 7.2.10

Answer available at the end of this section.

Your Turn 7.2.5: Find Euler Circuit

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Example 7.2.6: Finding an Euler Path using Fleury's Algorithm

Use Fleury's Algorithm to find an Euler trail for Graph *J* in Figure 7.2.13.

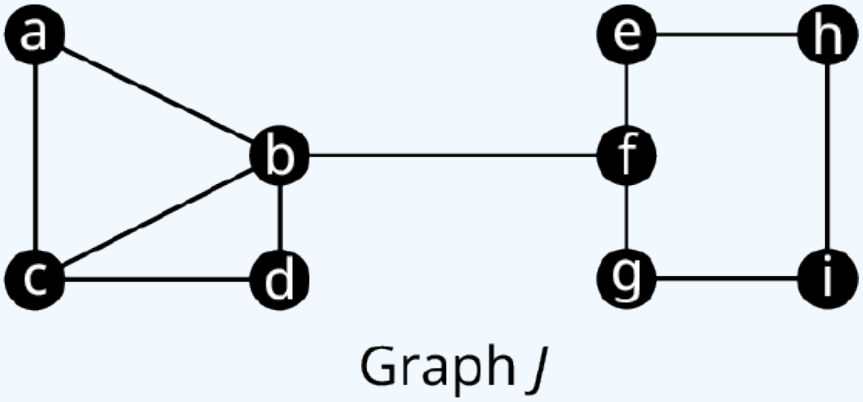


Figure 7.2.13: Graph J

Answer available at the end of this section.

TIP! To avoid errors, count the number of edges in your graph and ensure that your Euler trail represents that number of edges.

Your Turn 7.2.6: Find Euler Circuit

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Eulerization and the Chinese Postman Problem

Imagine you're a postman and must deliver mail by **walking along every street in a neighborhood at least once**, then return to your starting point. Naturally, you want to **walk the shortest possible distance**. How can you find the shortest closed path that covers every edge in a graph at least once? The name Chinese postman problem was coined in honor of the Chinese mathematician named Kwan Mei-Ko in 1960, who first studied the problem. This problem is essential in determining efficient routes for garbage trucks, school buses, parking meter checkers, and street sweepers.

Definition: Chinese Postman Problem

The task of finding the shortest circuit that visits every edge of a connected graph is often referred to as the **Chinese Postman problem**. In a given graph , the task is to find the shortest closed path that visits every edge at least once. The postman can traverse edges more than once if necessary, but each edge must be traversed at least once.

Example 7.2.7: Understanding Eulerian Graphs

A postal delivery person must deliver mail to every block on every street in a local subdivision. Figure 7.2.17 is a map of the subdivision. Use the map to answer each question.

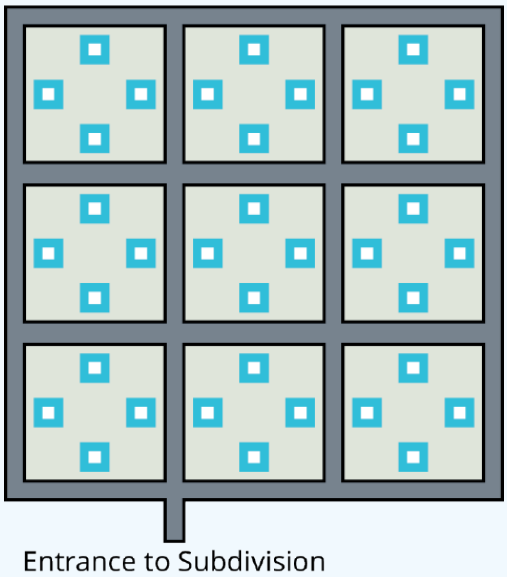


Figure 7.2.17: Map of Subdivision

Draw a graph or multigraph to represent the subdivision in which the vertices represent the intersections and the edges represent the streets.

1. Is your graph connected? Explain how you know.

2. Determine the degrees of the vertices in the graph.
3. Is your graph an Eulerian graph?
4. Is it possible for the postal delivery person to visit each block on each street exactly once? Justify your answer.

Answer available at the end of this section.

Your Turn 7.2.7: Euler Circuit

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Not every graph has an Euler path or circuit, yet our lawn inspector still needs to do her inspections. Her goal is to minimize the amount of walking she has to do. To do that, she will have to duplicate some edges in the graph until an Euler circuit exists. The problem of finding the optimal **Eulerization** is called the Chinese Postman Problem.

Eulerization

Eulerization is the process of adding edges to a graph to create an Euler circuit on a graph. To eulerize a graph, edges are duplicated to connect pairs of vertices with odd degrees. Connecting two odd-degree vertices increases the degree of each, giving them both even degrees. When two odd-degree vertices are not directly connected, we can duplicate all edges in a path connecting the two.

Note that we can only duplicate edges, not create edges where none existed. Duplicating edges would mean walking or driving down a road twice while creating an edge where there wasn't one before, which is akin to installing a new road!

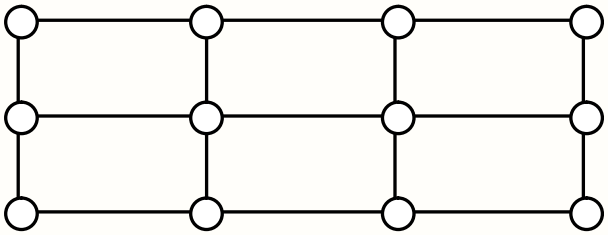


Figure 7.2.20: Non Eulerian Graph

Three possible eulerizations for graph in figure 7.2.25 are shown in figure 7.2.26. Notice that in each of these cases, the vertices that started with odd degrees have even degrees after Eulerization, allowing for an Euler circuit.

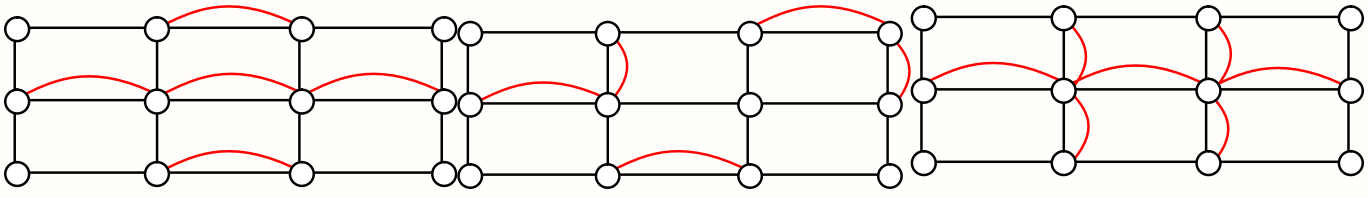


Figure 7.2.21: Three Possible Eulerization

In the example above, you'll notice that the last Eulerization in figure 7.2.21 required duplicating seven edges, while the first two only required duplicating five. If we were Eulerizing the graph to find a walking path, we would want the Eulerization with minimal duplications.

Example 7.2.8: Eulerized Map of Subdivision

Eulerized the graph of the Map of Subdivision given in figure 7.2.17.

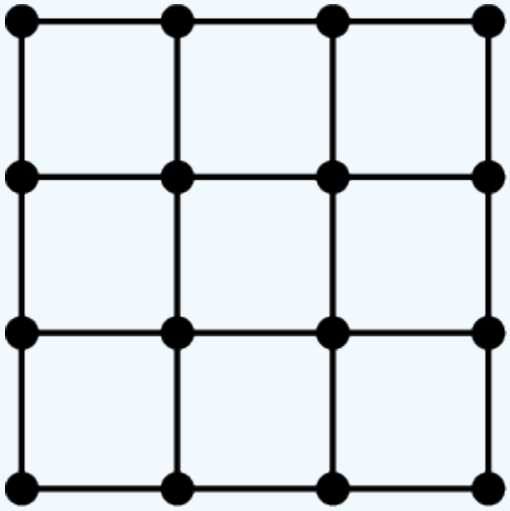


Figure 7.2.18: Graph of Subdivision

Answer available at the end of this section.

Your Turn 7.2.8: Eulerization of Graph

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Checkpoint

IMPORTANT! The duplicate edges that we add indicate places where the route will pass twice. An entirely new edge between two vertices that were not previously adjacent would indicate that our postal delivery person created a new road through someone's property! So, **we can duplicate existing edges, but we cannot create new ones.**

Your Turn 7.2.9: Understand Euler Pathh

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Answers

Example 7.2.3: Find a Euler Path

One such path is CABDCB. The path is shown in arrows to the right, with the order of edges numbered.

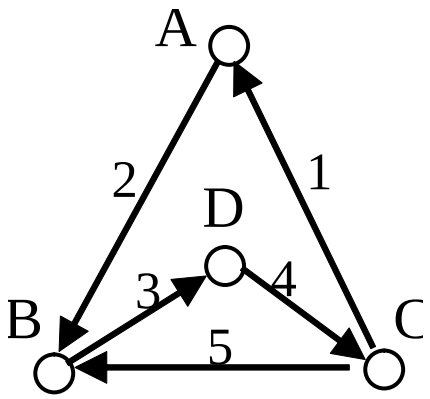


Figure 7.2.5

You can write the above path in different ways

C, A, B, D, C, B

OR

$C \to A \to B \to D \to C \to B$

The degree of the vertex that we start is 3 (C: odd degree), and the degree of the vertex that we end is 3 (B: odd degree).

Example $7.2.4$**: Find a Euler Path**

All vertices have even degrees, so this graph has an Euler Circuit (By Euler's Path and Circuit Theorem). There are several possibilities. One is ABEGFCDFEDBCA. We can write the same circuit another way, as follows

$A \to B \to E \to G \to F \to C \to D \to F \to E \to D \to B \to C \to A$

Since this graph has an Euler Circuit, we call this graph an **Eulerian graph.**

Example $7.2.5$**: Finding an Euler Circuit**

All vertices are even, so it has an Euler Circuit. In other words, it is an Eulerian graph.

Choose an edge AD.

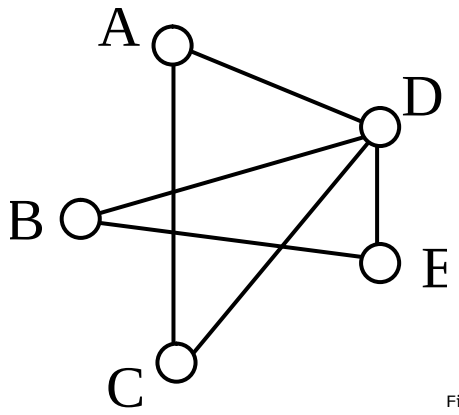


Figure 7.2.10

Delete AD: See Figure \({7{.}2{.}\eatSpaces{11} \endSpaces}. We can't choose DC since that would disconnect the graph.

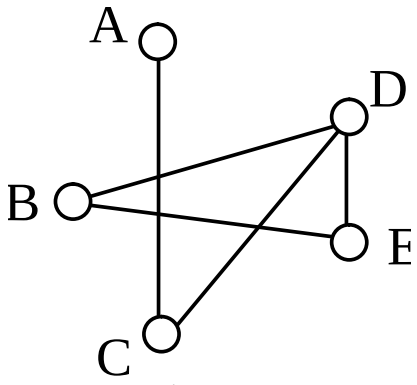


Figure 7.2.11

Choose DE. Delete DE. See figure 7.2.12. From E, there is only one option, so the rest of the circuit is determined.

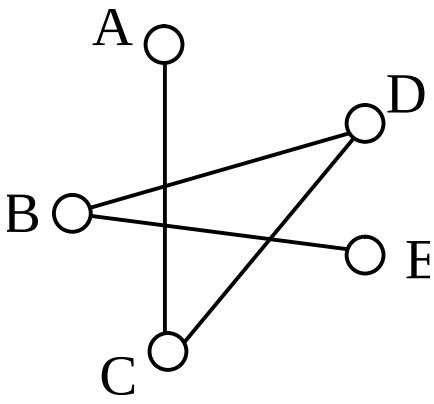


Figure 7.2.12

Circuit: ADEBDCA

or

$A \to D \to E \to B \to D \to C \to A$

Example 7.2.6: Finding an Euler Path using Fleury's Algorithm

Choose one of the two vertices of odd degree, *c* or *f*, as your starting vertex, since you must start from one odd vertex.

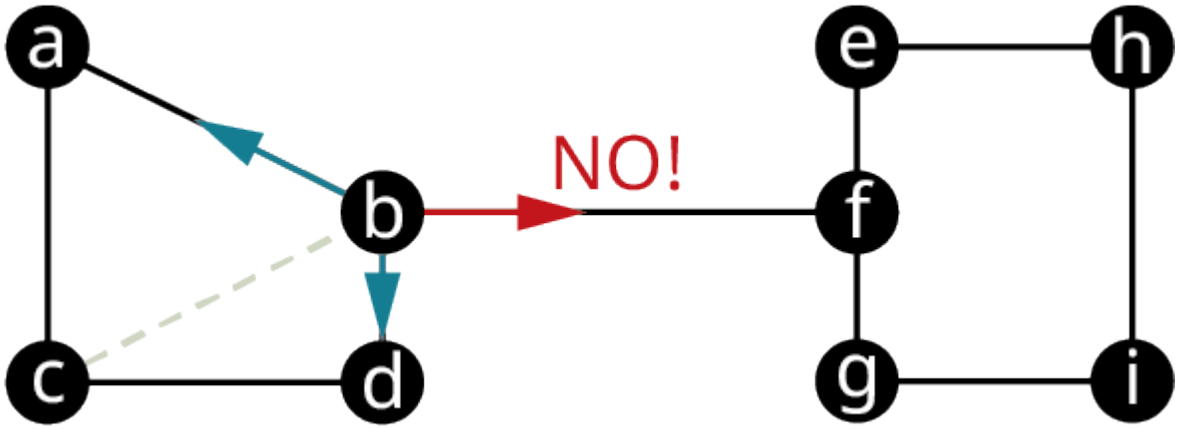


Figure 7.2.14: Graph *J* with *cb* Removed

The next choice is to remove edge *ba, bd*, or *bf,* as shown in Figure 7.2.14, but *bf* is not an option since it is a bridge. As shown in Figure 7.2.15, we will choose *ba* as the second edge removed.

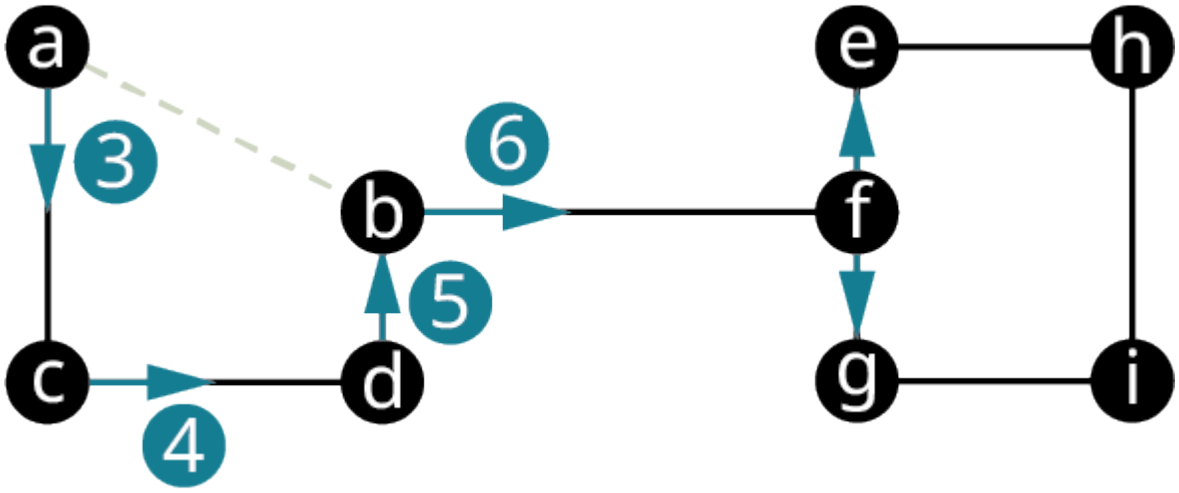


Figure 7.2.15: raph *J* with *cb* and *ba* Removed

For the third, fourth, fifth, sixth, and seventh edges. As shown in Figure 7.2.15, until we get to the seventh edge, there is only one option each time: *ac, cd, db*, and *bf* in that order. For the seventh edge, we must choose between *fe* and *fg*. Neither of these is a bridge. We choose *fe*. Figure 7.2.16 shows that *ac, cd, db, bf,* and *fe* have been removed.

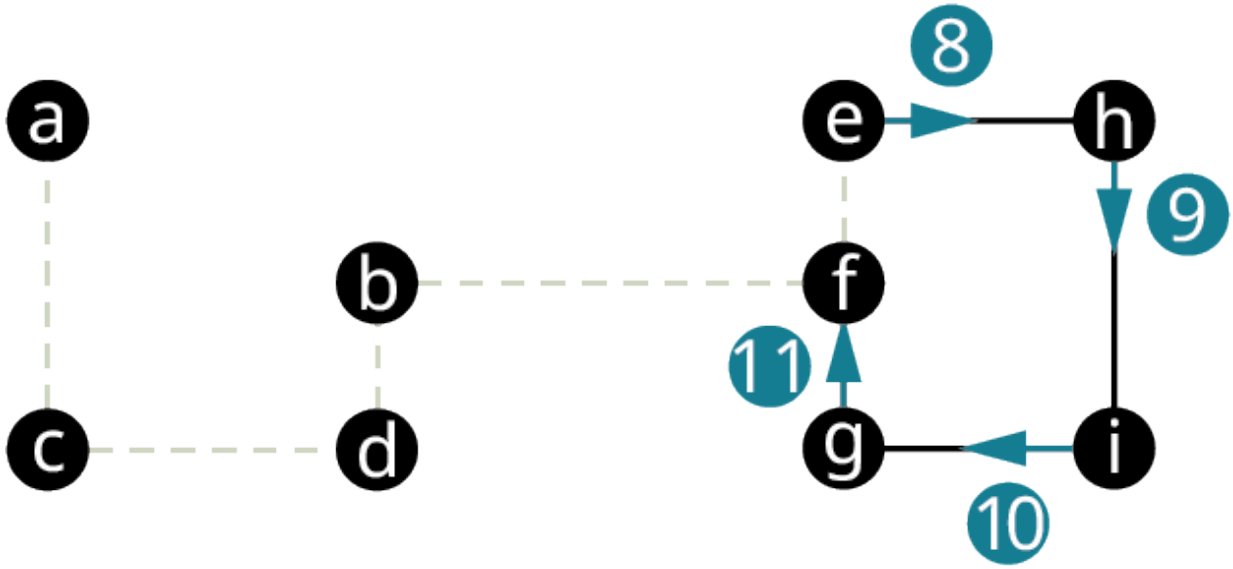


Figure 7.2.16: Graph *J* with Seven Edges Removed

Write out the Euler trail using the vertices in the sequence in which the edges were removed. We removed *cb, ba, ac, cd, db, bf, fe, eh, hi, ig*, and *gf*, in that order. The Euler trail is

$c \to b \to a \to c \to d \to b \to f \to e \to h \to i \to g \to f$

Example 7.2.7: Understanding Eulerian Graphs

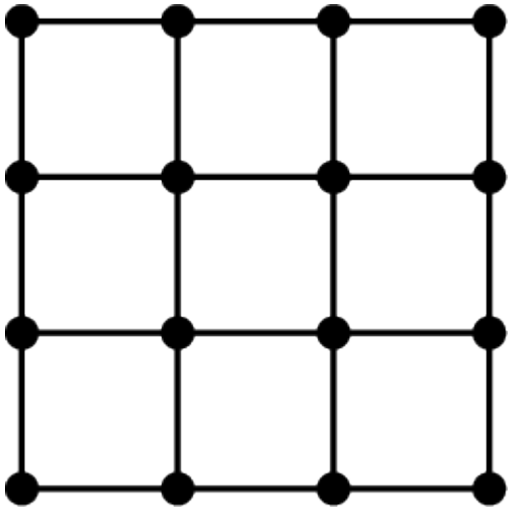


Figure 7.2.18: Graph of Subdivision

The graph is show in Figure 7.2.18.

1. The graph is connected. It has only one component and a path between each pair of vertices.
2. There are four corner vertices of degree 2, eight exterior vertices of degree 3, and four interior vertices of degree 3.
3. The graph is not Eulerian because it has vertices of odd degrees.
4. No. Since the graph is not Eulerian, there is no Euler circuit, meaning there is no route that would pass through every edge exactly once.

Example 7.2.8: Eulerized Map of Subdivision

In Figure 7.2.22, the eight vertices of odd degrees in the graph of the subdivision are circled in green. We have added duplicate edges between pairs of vertices, which changes the degrees of the vertices to even degrees, so the resulting multigraph has an Euler circuit. In other words, we have eulerized the graph shown below in figure 7.2.22.

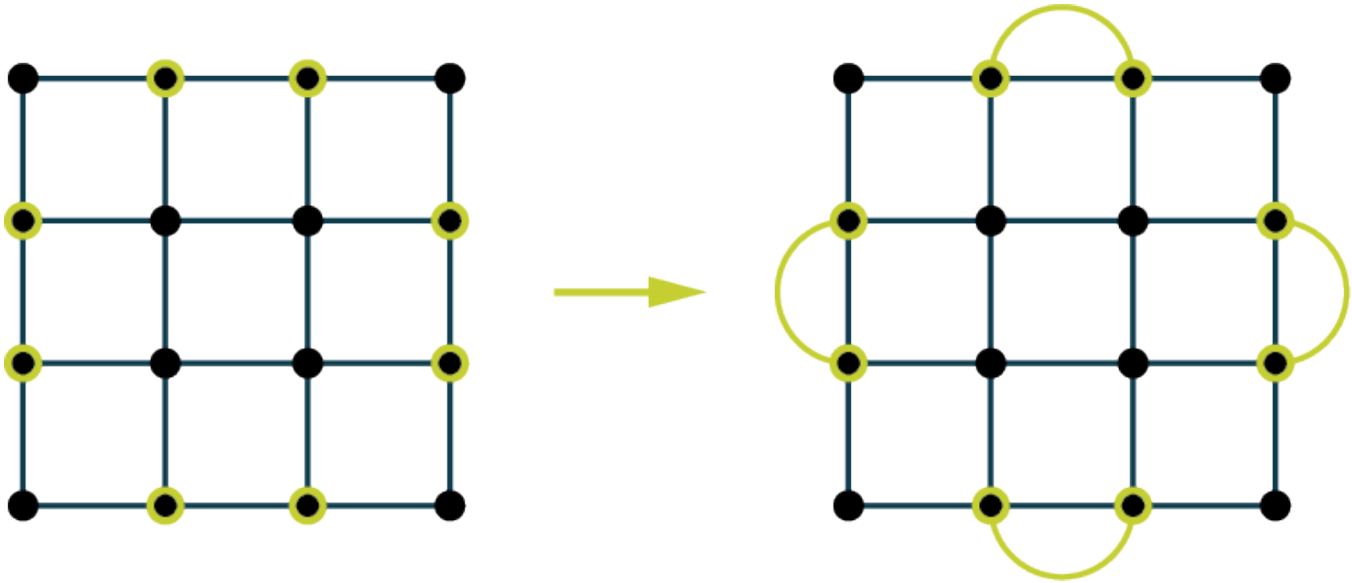


Figure 7.2.22: An Eulerized Graph

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7.3: Hamiltonian Circuits and the Traveling Salesman Problem



Figure 7.3.1 : A school bus picks up children along a planned route. (credit: “Kids at School Bus Stop” by Ty Hatch/Flickr, CC BY 2.0)

Learning Objectives

1. Describe and evaluate Hamilton paths and circuits.
2. Evaluate Hamilton paths in real-world applications.
3. Apply brute force and the nearest neighbor methods to solve traveling salesperson problems.

In the United States, school buses carry 25 million children between school and home every day. The total distance they travel is around 6 billion kilometers per year. In the city of Boston, Massachusetts, the 2016 budget for running those buses was $\$120$ million. In $2017,$ the city held a competition to find ways to cut costs and the Quantum Team from the MIT Operations Research Center came to the rescue, using a computer algorithm to identify the most efficient and least costly routes, which saved the city of Boston $\$5$ million each year and even reduced daily CO_2 emissions by $9,000$ kilograms! (*This* U.S. *city put an algorithm in charge of its school bus routes and saved $5 million*, Sean Fleming, World Economic Forum)

The problem the Quantum Team tackled involves graph theory. Imagine a graph in which the vertices are the bus depot, the school, and the bus stops along a particular route. The bus must start at the depot, visit every stop exactly once, and end at the school. The route is a special kind of path that visits every vertex exactly once. Can you guess what those paths are called?

Hamilton Paths and Hamilton Circuits

Definition: Hamiltonian Circuits and Paths

A **Hamiltonian circuit** is a circuit that visits every vertex once with no repeats. Being a circuit, it must start and end at the same vertex. A **Hamiltonian path** also visits every vertex once with no repeats, but does not have to start and end at the same vertex.

Note: "**Hamiltonian"** is the same as "**Hamilton".**

Hamiltonian circuits are named after William Rowan Hamilton, who studied them in the 1800s.

Just as circuits that visit each vertex in a graph exactly once are called Hamiltonian cycles (or Hamilton circuits), paths that visit each vertex on a graph exactly once are called Hamiltonian paths. As we explore Hamiltonian paths, you might find it helpful to refresh your memory about the relationships between walks, trails, and paths. We know that paths are walks that don't repeat vertices or edges. So, a Hamiltonian path visits every vertex without repeating any vertices or edges. Figure 7.3.2 shows a path from vertex *A* to vertex *E* and a Hamiltonian path from vertex *A* to vertex *E*.

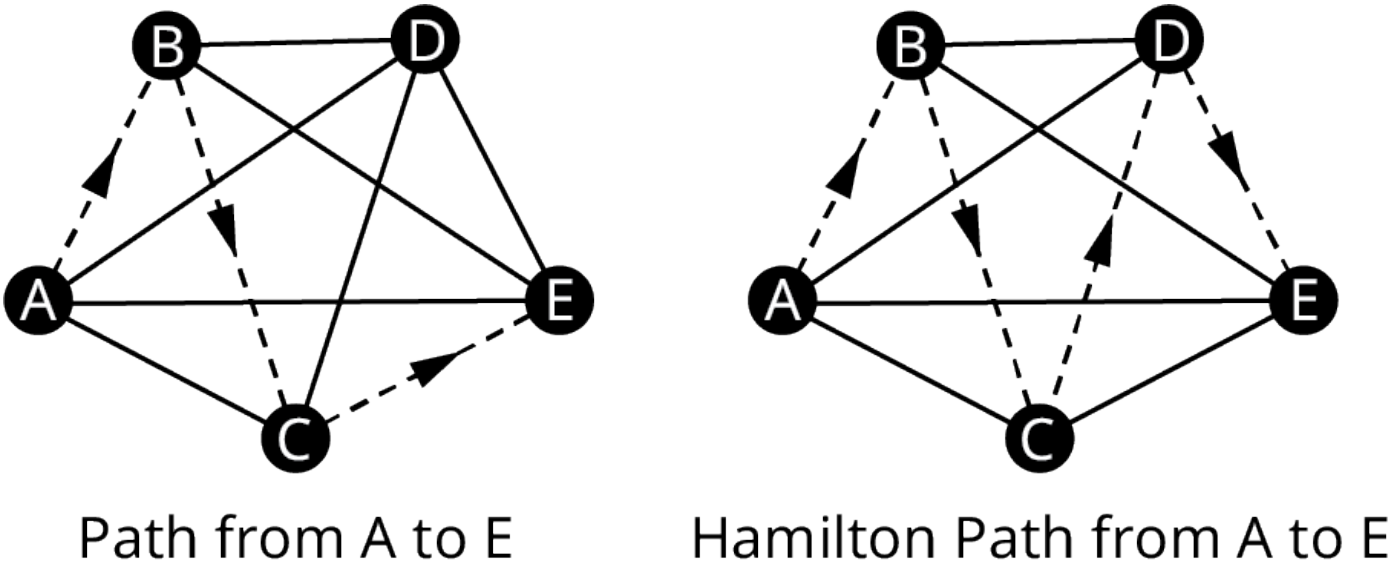


Figure 7.3.2: Path or Hamiltonian Path?

Example 7.3.1: Find the Hamiltonian Circuit and Path

Find a Hamiltonian circuit from A to A. There are several other Hamiltonian circuits possible on this graph. Can you find a Hamiltonian path from A to D?

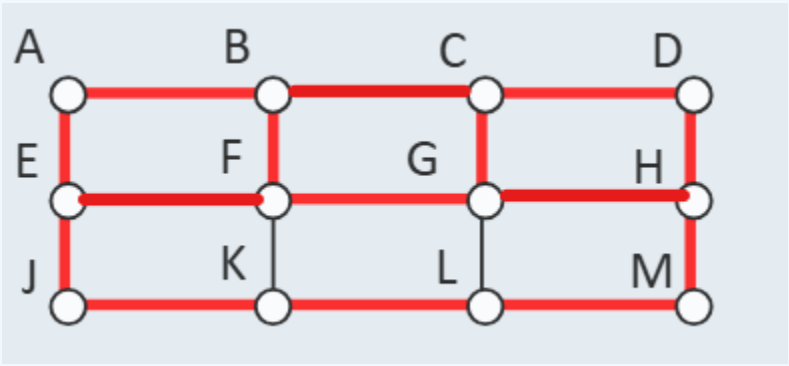


Figure 7.3.3: Graph

Answer available at the end of this section.

Your Turn 7.3.1: Hamiltonian Circuit

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Unlike with Euler circuits, no nice theorem allows us to instantly determine whether or not a Hamiltonian circuit exists for all graphs. Just as circuits that visit each vertex in a graph exactly once are called Hamilton cycles (or Hamilton circuits), paths that visit each vertex on a graph exactly once are called Hamilton paths. As we explore Hamilton paths, you might find it helpful to refresh your memory about the relationships between walks, trails, and paths. We know that paths are walks that don't repeat vertices or edges. So, a Hamilton path visits every vertex without repeating any vertices or edges.

Example 7.3.2: Identifying Hamiltonian Paths

Which of the following sequences of vertices is a Hamiltonian path for Graph Q in Figure 7.3.5?

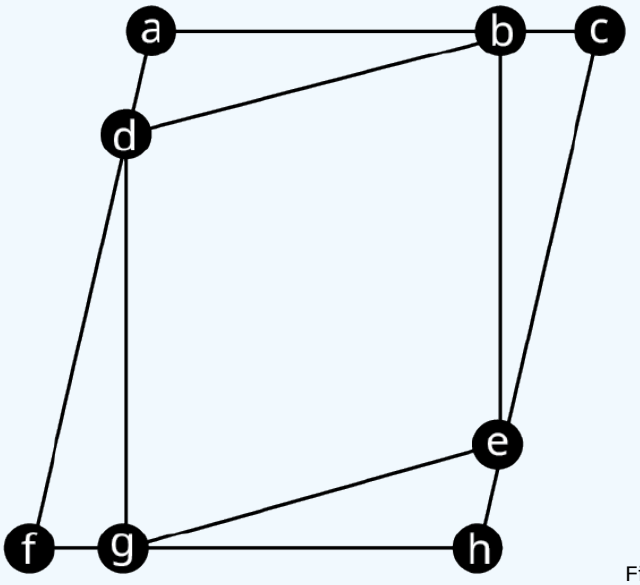


Figure 7.3.5: Graph Q

1. $a \to d \to b \to c \to e \to g \to f$
2. $c \to b \to e \to h \to g \to f \to d \to a$
3. $h \to e \to g \to d \to b \to e \to g \to f \to d \to a \to b \to c$

Answer available at the end of this section.

Checkpoint

TIP! Since a Hamiltonian path visits each vertex exactly once, it must have the same number of vertices as appear in the graph.

Your Turn 7.3.2: Hamiltonian Path

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More on Finding Hamilton Circuit and Hamilton Path

Suppose you were visiting an aquarium with some friends. The map of the aquarium is given in 7.3.5. The letters represent the exhibits.

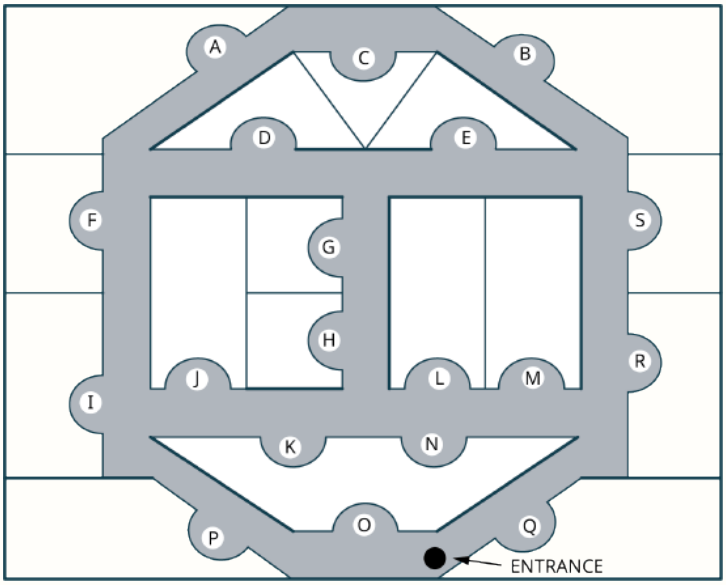


Figure 7.3.6: Map of Aquarium Exhibits

Figure 7.3.7 shows a graph of the aquarium in which each vertex represents an exhibit and each edge is a route between the pair of exhibits that doesn't bypass another exhibit. Suppose that, after exhibit *O*, we plan to visit.

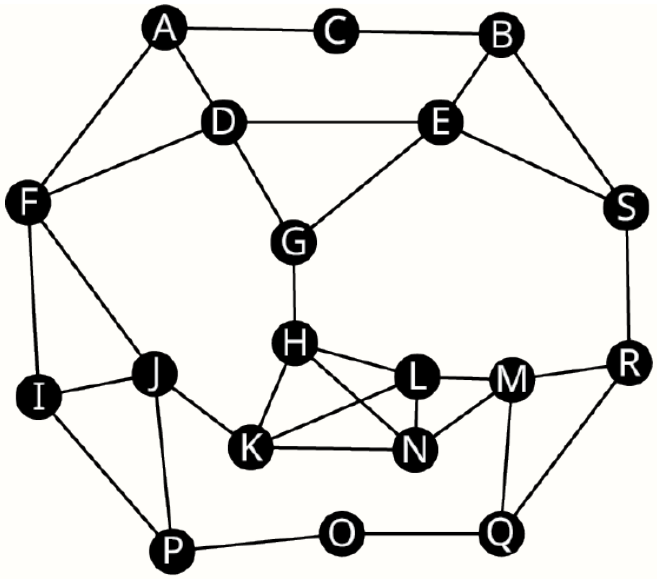


Figure 7.3.7: Graph of Aquarium

Let's see if we can plan a tour of the exhibits that visits each exhibit exactly once, beginning at exhibit *O* and ending at exhibit *C*. Suppose that, after exhibit *O*, we plan to visit exhibit *Q* and then exhibit *M*. After *M*, should we plan to visit *N*, *L*, or *R*? Take a look at Figure 7.3.8. If *R* is not chosen next, that will cause a problem later on. Do you see what it is?

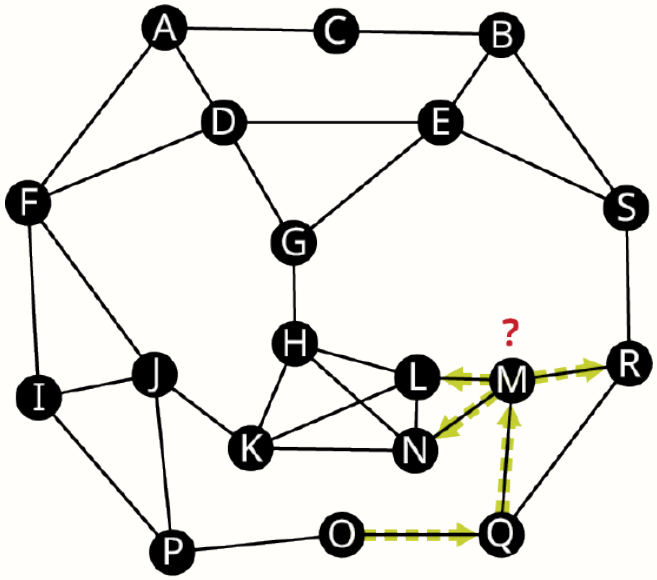


Figure 7.3.8 : Choosing Vertex *L*, *N*, or *R*

If *L* or *N* is chosen next, the only way to get to *R* later will be to go from S to *R*, and then we will not be able to continue without repeating a vertex. So, we will pick *R* next, and then the only option is S. After S we have another choice to make. As shown in Figure 7.3.9, the next choice is between *B* and *E*. Keeping in mind that the goal is to end at *C*, which would be the better choice?

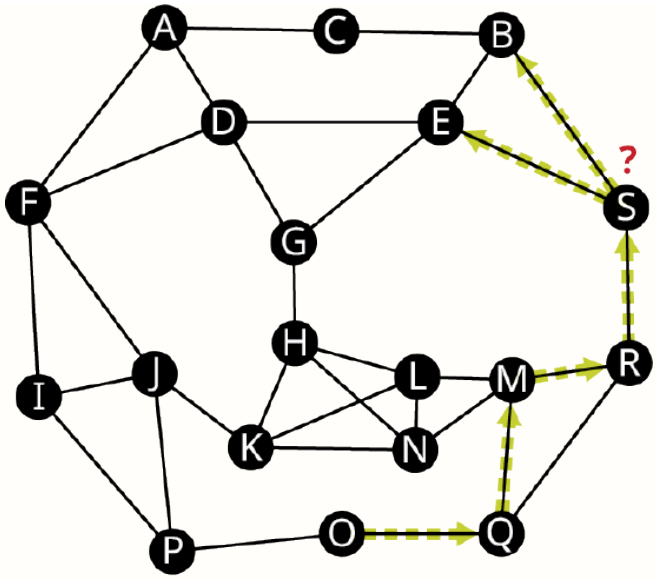


Figure 7.3.9 : Choosing Vertex *B* or *E*

If you said vertex *B*, you are right! Otherwise, we will not be able to visit *B* later. After *B*, the only option is *E*. Then we can choose either *D* or *G*. Either will work fine. Let's choose *G* as shown in Figure 7.3.10. After *G*, you must visit *H*, but should you visit *K* or *L* after that?

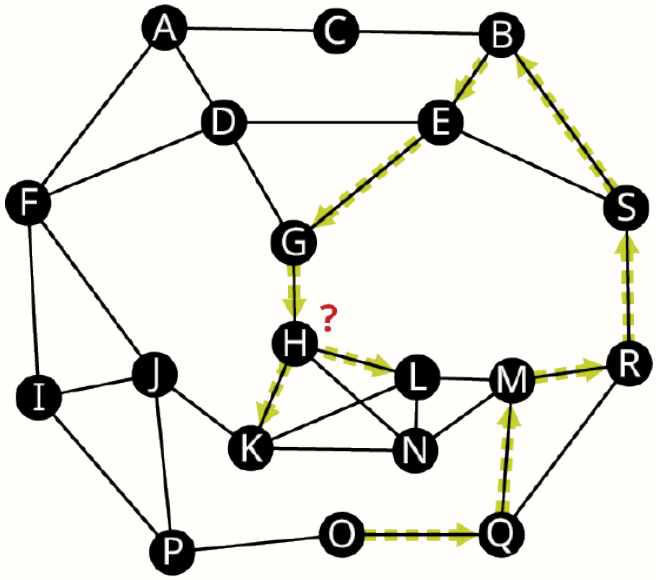


Figure 7.3.10 : Choose Vertex *L* or *K*

If you said to go to vertex *L* next, you are right! Otherwise, it will be impossible to visit *N* without repeating a vertex. So, next is *L*, then *N*, then *K*, and then at *J* you have another decision to make, we can see in Figure 7.3.11. Should you choose *F*, *I*, or *P* next?

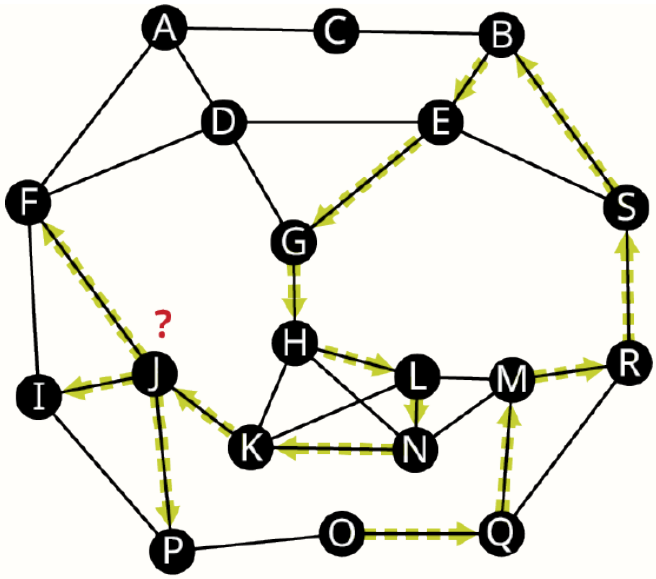
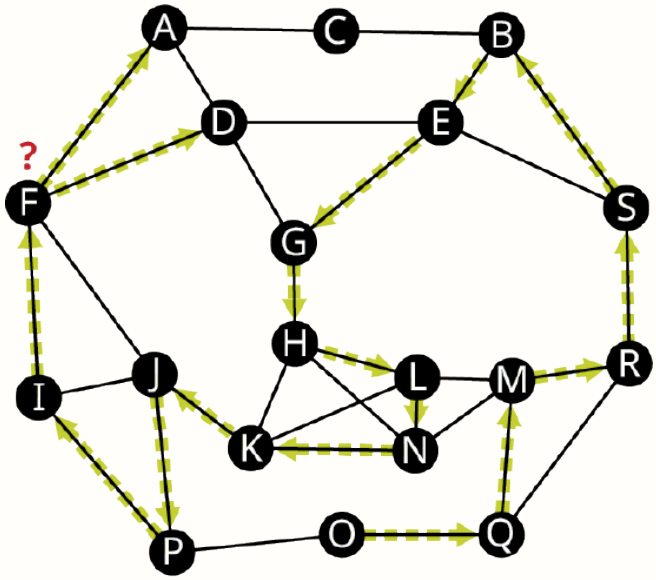


Figure 7.3.11 : Choose Vertex *F*, *I*, or *P*

If you said *P*, you are right! If you choose either of the other two vertices, you will not be able to visit *P* later without passing through another vertex twice. Once *P* is chosen, vertex *I* must be next, followed by *F*. Then you have to choose between *A* and *D* as shown in Figure 7.3.12.



Figure

:

Choose Vertex *A* or *D*

In this case, we must go to *D*, then to *A*, so that we can visit *C* without backtracking. The complete Hamilton path is shown in Figure 7.3.13.

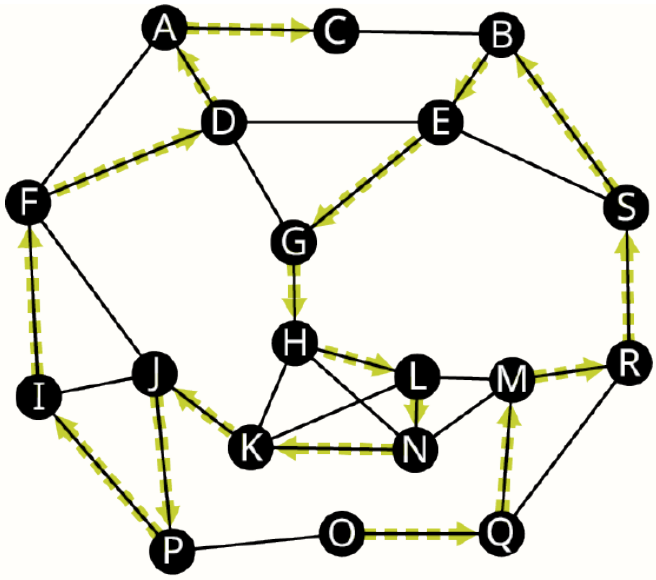


Figure 7.3.13 : Complete Hamilton Path from *O* to *C*

So, one Hamilton path that begins at *O* and ends at *C* is

$$O \to Q \to M \to R \to S \to B \to E \to G \to H \to L \to N \to K \to J \to P \to I \to F \to D \to A \to C$$

There is no set sequence of steps that can be used to find a Hamilton path if it exists, but it does help to keep in mind where we are headed and avoid choices that will make returning to a particular vertex impossible without repeating vertices. Let's practice finding Hamilton paths.

In the last section, we considered optimizing a walking route for a postal carrier. How is this different than the requirements of a package delivery driver? While the postal carrier needed to walk down every street (edge) to deliver the mail, the package delivery driver needs to visit every location in a set. Instead of looking for a circuit that covers every edge once, the package deliverer is interested in a circuit that visits every vertex once

The Traveling Salesperson Problem (TSP)

Now, let's focus on the graph-theoretic application known as the traveling salesman problem (TSP), in which we must find the shortest route to visit a set of locations and return to the starting point.

With Hamiltonian circuits, our focus will not be on existence but on the question of optimization; given a graph where the edges have weights, can we find the optimal Hamiltonian circuit, the one with the lowest total weight?

This problem is known as the **Traveling Salesman Problem** (TSP) because it can be framed as follows: Suppose a salesman needs to give sales pitches in four cities. He looks up the airfares between each city and puts the costs in a graph. In what order should he travel to visit each city once, then return home with the lowest cost?

What is TSP?

The **Traveling Salesman Problem (TSP)** is a problem in mathematics and computer science where a person (the "salesman") must visit a set of cities **exactly once**, then return to the starting city, while **minimizing the total distance traveled.**
Key Idea of TSP

- Visit every city once
- Return to the starting point

- Find the **shortest possible route**

Note: The solution that was obtained from TSP is called an ***optimal*** solution.

We will consider some possible approaches to determining the lowest-cost Hamiltonian circuit. The first option that might come to mind is to just try all the different possible circuits.

Consider the following scenario. The U.S. Air Force officer stationed at Vandenberg Air Force Base must drive to visit three other California Air Force bases before returning to Vandenberg. The officer needed to visit each base once. We looked at the weighted graph in Figure $7.3.14$ representing the four U.S. Air Force bases: Vandenberg, Edwards, Los Angeles, and Beal, and the distances between them. There are different ways officers can visit three cities and return to Vandenberg, but we are interested in finding the optimal route (at least in terms of distance).

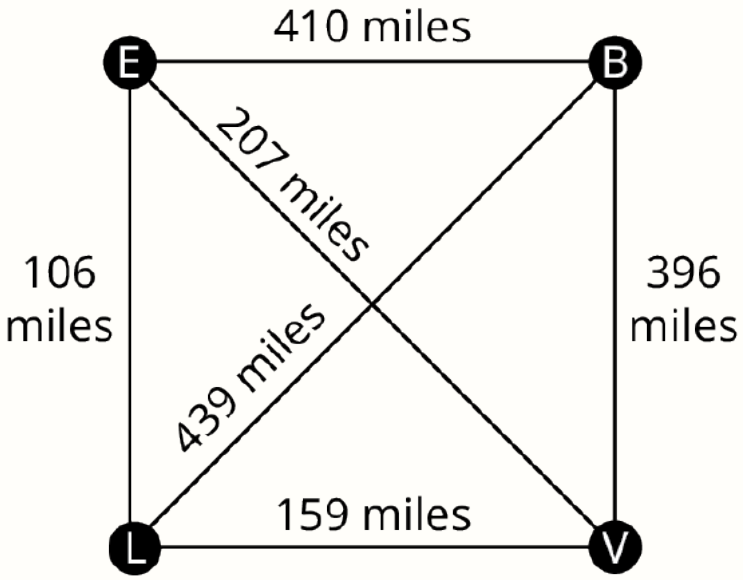


Figure $7.3.14$: Graph of Four California Air Force Bases

In the above scenario, any route that visits each base and returns to the start would be a Hamilton cycle on the graph. If the officer wants to travel the shortest distance, this corresponds to a Hamilton cycle of minimum weight. There are six distinct Hamilton circuits in a complete graph with four vertices. They are listed below.
Circuit 1: $V \to B \to E \to L \to V$
Circuit 2: $V \to B \to L \to E \to V$
Circuit 3: $V \to E \to B \to L \to V$
Circuit 4: $V \to E \to L \to B \to V$
Circuit 5: $V \to L \to B \to E \to V$
Circuit 6: $V \to L \to E \to B \to V$

Since the distance between bases is the same in either direction in Figure $7.3.15$, it does not matter if the officer travels clockwise or counterclockwise. So, there are really only three possible distances as shown in Figure $7.3.15$.

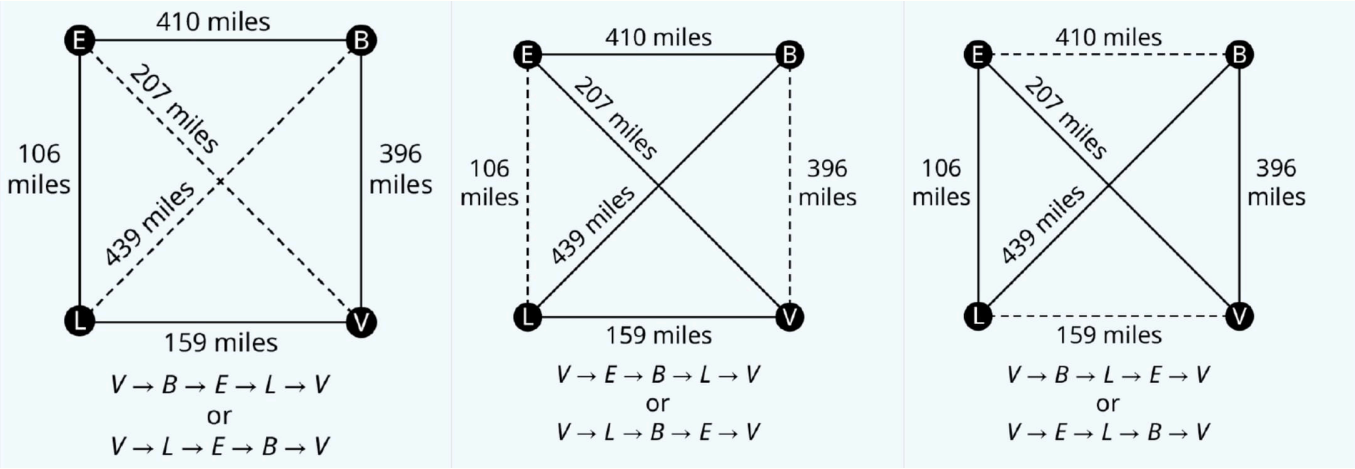


Figure 7.3.15: Three Possible Distances

The possible distances are:
$VBELV = VB + BE + EL + LV = 396 + 410 + 106 + 159 = 1071$
$VEBLV = VE + EB + BL + LV = 207 + 410 + 439 + 159 = 1215$
$VBLEV = VB + BL + LE + EV = 396 + 439 + 106 + 207 = 1148$
So, a Hamilton cycle of least weight is

$V \to B \to E \to L \to V$(or the reverse direction).

This circuit is called an optimal circuit.

The officer should travel from Vandenberg to Beal to Edwards, to Los Angeles, and back to Vandenberg.

Finding Weights of All Hamilton Cycles in Complete Graphs

Notice that we listed all of the Hamilton cycles and found their weights when we solved the TSP for the officer from Vandenberg. This is a skill you will need to practice. To make sure you don't miss any, you can calculate the number of possible Hamilton circuits in a complete graph. It is also helpful to know that half of the circuits in a complete graph are the same circuits in the reverse direction, so you only have to calculate half the number of possible weights, and the rest are duplicates.

FORMULA: Number of Possible Hamilton Circuits

For n vertices in a **complete graph**, the number of Hamilton Circuits (routes) is $n-1)!$

$$n! = 1 \times 2 \times 3 \times ... \times n$$

Of the half of these are duplicates in reverse order. The exclamation symbol, !, is pronounced: “factorial.”

For example

$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

Checkpoint

TIP! When listing all the distinct Hamilton cycles in a complete graph, you can start them all at any vertex you choose. Remember, the circuit $a \to b \to c \to a$ is the same cycle as $b \to c \to a \to b$, so there is no need to list both.

Example 7.3.3: Number of Hamilton Circuits in a Complete Graph

How many Hamilton circuits would a complete graph with eight vertices have?

Answer available at the end of this section.

Your Turn 7.3.3: Find Number of Hamilton Circuit

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Example 7.3.4: Calculating Possible Weights of Hamilton Circuits

Suppose you have a complete weighted graph with vertices *N, M, O*, and *P*. Find the following

1. The number of Hamilton circuits in the complete graph. List all of them
2. Which pairs of circuits must have the same weights? How do you know?

Answer available at the end of this section.

Checkpoint: How to Write Hamilton Circuit?

TIP! When listing the possible cycles, ignore the vertex where the cycle begins and ends, and focus on the ways to arrange the letters that represent the vertices in the middle. Using a systematic approach is best; for example, if you must arrange the letters M, O, and P, first list all those arrangements beginning with M, then beginning with O, and then beginning with P, as we did in Example 7.3.4.

Your Turn 7.3.4: Find the Weight of Circuit

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The Brute Force Method

The method we have been using to find a Hamilton cycle of the least weight in a complete graph is a brute-force algorithm; hence, it is called the brute-force method. The steps in the brute-force method are:

Steps: Brute Force Method

1. List all possible Hamiltonian circuits.
2. Find the total cost of each circuit by adding the edge weights
3. Select the circuit with minimal total weight.

Example 7.3.5: Applying the Brute Force Method

Consider the following scenario. The U.S. Air Force officer stationed at Vandenberg Air Force Base must drive to visit three other California Air Force bases before returning to Vandenberg. The officer needed to visit each base once. We looked at the weighted graph in Figure representing the four U.S. Air Force bases: Vandenberg, Edwards, Los Angeles, and Beal, and the distances between them. There are different ways officers can visit three cities and return to Vandenberg. But what is the optimal route (at least in terms of distance)**?** Use Figure 7.3.16 to find the shortest route.

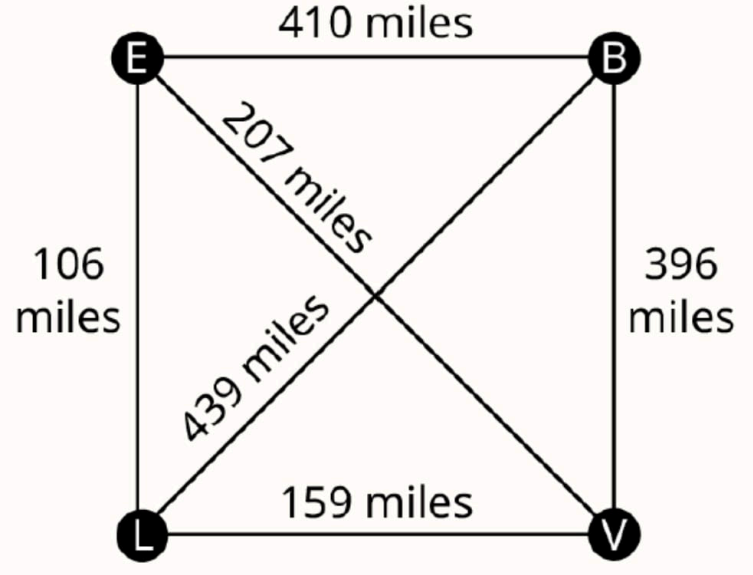


Figure 7.3.16: Graph Model the Air Force Base

Answer available at the end of this section.

Your Turn 7.3.5: Apply Brute Force Method

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Now, suppose that the officer needed a cycle that visited all 5 of the Air Force bases in the following graph.

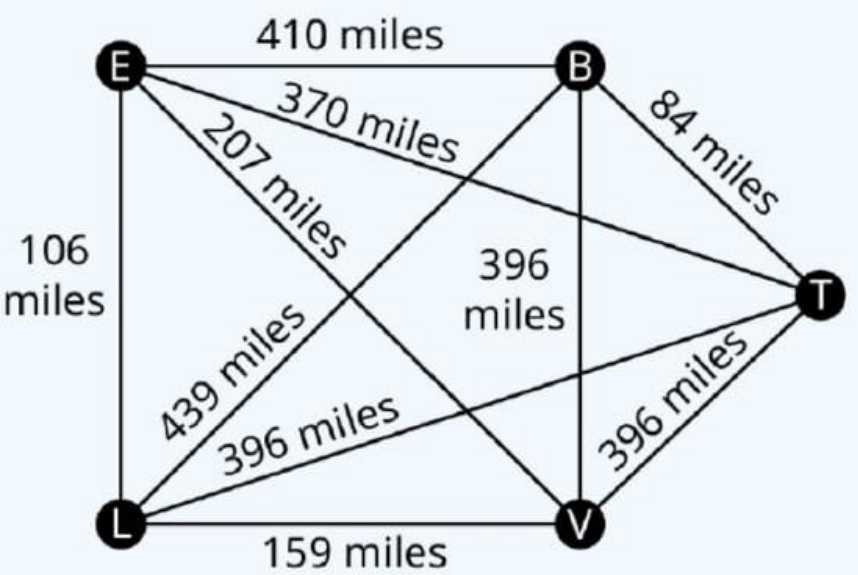


Figure 7.3.17: Five Air Force Bases

There would $(5-1)! = 4! = 4 \times 3 \times 2 \times 1 = 24$ Hamilton circuits as shown below

1. V → L → B → E → T → V
2. V → L → B → T → E → V
3. V → L → E → B → T → V
4. V → L → E → T → B → V
5. V → L → T → B → E → V
6. V → L → T → E → B → V
7. V → B → L → E → T → V
8. V → B → L → T → E → V
9. V → B → E → L → T → V
10. V → B → E → T → L → V
11. V → B → T → L → E → V
12. V → B → T → E → L → V
13. V → E → L → B → T → V
14. V → E → L → T → B → V
15. V → E → B → L → T → V
16. V → E → B → T → L → V
17. V → E → T → L → B → V
18. V → E → T → B → L → V
19. V → T → L → B → E → V
20. V → T → L → E → B → V
21. V → T → B → L → E → V
22. V → T → B → E → L → V
23. V → T → E → L → B → V
24. V → T → E → B → L → V

Half of them has same weight since they are in reverse order. That's what we need to list 12 unique Hamilton circuits and find the weight of all of those circuits.

A lot of work, right?

What if you consider 10 Air Force bases? The number of Hamilton circuits for $n = 10$ is

$$(10-1)! = 9! = 9 \times 8 \times 7 \times 6 \times \cdots \times 1 = 362,880$$

In this case, need to list $\frac{362,880}{2}$ unique Hamilton circuits and find the weight of all of those circuits.

Impossible....

Look at the table.



It is impossible for humans—and even computers- to struggle as numbers grow. **There must be another way to solve the TSP problem**

The Nearest Neighbor Method (NNA)

The brute-force algorithm is optimal; it will always produce the Hamiltonian circuit of minimum weight. Is it efficient? To answer that question, we need to consider how many Hamiltonian circuits a graph could have. For simplicity, let's look at the worst-case possibility, where every vertex is connected to every other vertex. This is called a **complete graph.**

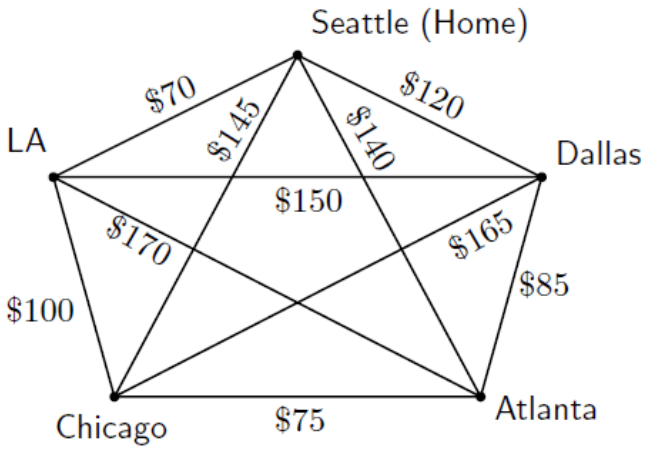


Figure 7.3.18: Air Travel Graph

Suppose we had a complete graph with five vertices, like the air travel graph above. From Seattle, we can visit four cities first. There are three choices for each of those. There are two possible cities to visit next from each of those cities. There is then only one choice for the last city before returning home.

This can be shown visually:

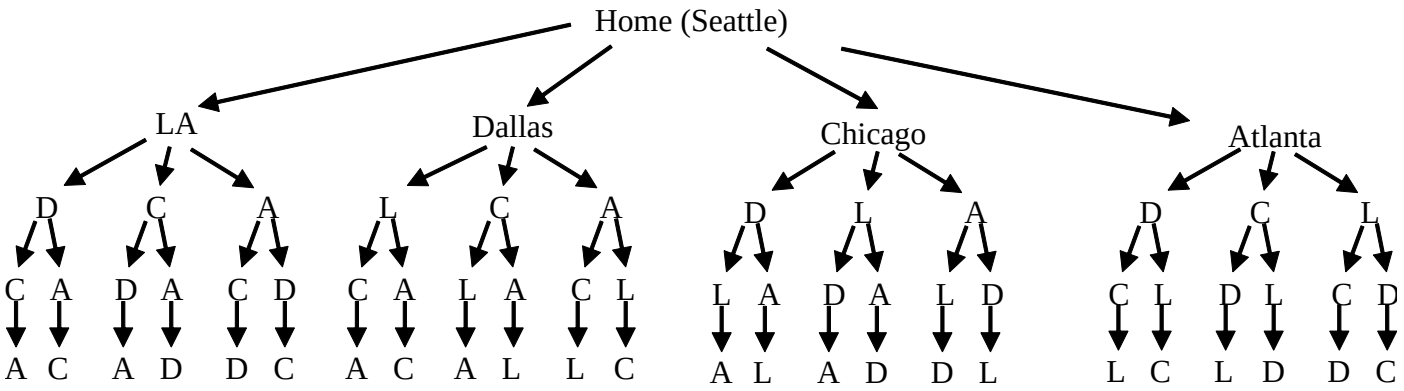


Figure 7.3.19

Counting the number of routes, we can see there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ routes. For six cities there would be $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ routes.

As you can see, the number of circuits is increasing rapidly. If a computer looked at one billion circuits a second, it would still take almost two years to examine all the possible circuits with only 20 cities! Certainly, Brute Force is **not** an efficient algorithm.

Unfortunately, no one has yet found an efficient *and* optimal algorithm to solve the TSP, and it is very unlikely anyone ever will. Since brute-force methods are impractical for solving the problem, we turn instead to **heuristic algorithms**, which provide approximate solutions. In other words, heuristic algorithms are fast, but may or may not produce the optimal circuit.

NNA is also called the greedy algorithm, meaning it only considers the immediate decision without considering future consequences.

Steps: Nearest Neighbor Algorithm

1. Select a starting point.
2. Move to the nearest unvisited vertex (the edge with the smallest weight).
3. Repeat until the circuit is complete.

In the above example, starting in Seattle, the nearest neighbor (cheapest flight) is to LA, at a cost of $\$70$. From there:

LA to Chicago: $\$100$

Chicago to Atlanta: $\$75$

Atlanta to Dallas: $\$85$

Dallas to Seattle: $\$120$

Total cost: $\$450$

In this case, the nearest neighbor did find the optimal circuit. Which is

$$Seattle \rightarrow LA \rightarrow Chicago \rightarrow Atlanta \rightarrow Dallas \rightarrow Seattle.$$

Example 7.3.6: Use the Nearest Neighbor Algorithm

Suppose that a candidate for governor wants to hold rallies around the state. They plan to leave their home in city *A*, visit cities *B, C, D, E*, and *F* each once, and return home. The airfare between cities is shown in the graph in Figure 7.3.20. Let's help the candidate keep travel costs down by applying the nearest-neighbor method to find a Hamilton cycle with a reasonably low weight.

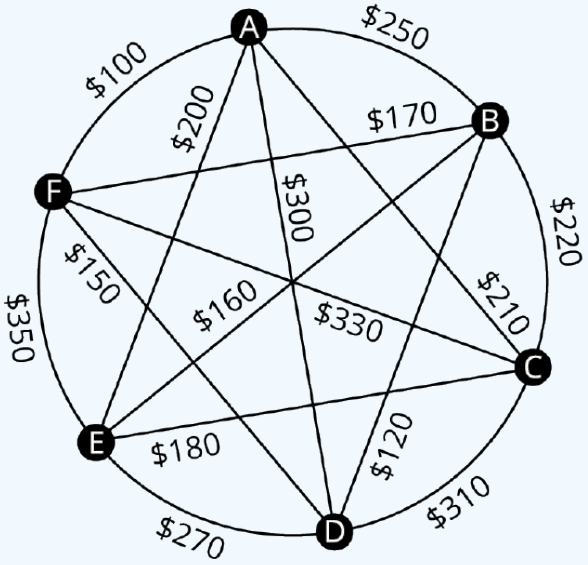


Figure 7.3.20: Airfares between Cities *A, B, C, D, E*, and *F*

Answer available at the end of this section.

Your Turn 7.3.6: Use the Nearest Neighbor Method

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Your Turn 7.3.7: Fill in the Blank

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Answers

Example 7.3.1: Find the Hamiltonian Circuit and Path

For the Hamiltonian circuit, each vertex should be visited only once, and it should start and end at the same vertex:

ABFGCDHMLKJEA is a Hamiltonian circuit that starts and ends at A.

Notice that the same circuit can be written in reverse order, or with a different starting and ending vertex.

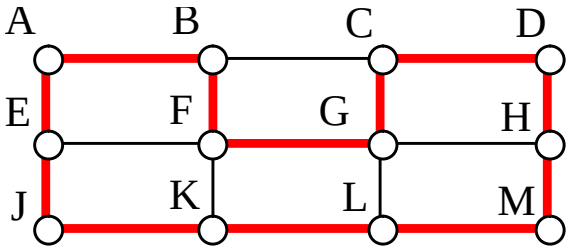


Figure 7.3.4: Hamiltonian Circuit from A to A

AEJKFBCGLMHD is a Hamiltonain path from A to D.

Example 7.3.2: Identifying Hamiltonian Paths

1. Sequence 1 is a path, because it doesn't repeat any vertices or edges, but it is not a Hamiltonian path because it skips vertex h.
2. Sequence 2 is a path that visits each vertex exactly once; so, it is a Hamiltonian path.
3. Sequence 3 is not a Hamiltonian path because it visits vertices g, d, e, b each more than once.

Example 7.3.3: Number of Hamilton Circuits in a Complete Graph

A complete graph with 8 vertices would have

$$(8-1)! = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

We have 5040 possible Hamiltonian circuits. Half of the circuits are duplicates of other circuits but in reverse order, leaving 2520 unique Hamilton circuits.

Example 7.3.4: Calculating Possible Weights of Hamilton Circuits

1. There are 4 vertices; so,$n = 4$. This means there are

$$(4-1)! = 3! = 3 \times 2 \times 1 = 6$$

There are 6 distinct Hamilton circuits beginning at any given vertex. Let's see why we choose vertex N as the starting and ending point.

Circuit 1: $N \to M \to O \to P \to N$
Circuit 2: $N \to M \to P \to O \to N$
Circuit 3: $N \to O \to M \to P \to N$
Circuit 4: $N \to O \to P \to M \to N$
Circuit 5: $N \to P \to M \to O \to N$
Circuit 6: $N \to P \to O \to M \to N$

2. Circuits 1 and 6 have the same weight, circuits2 and 4 have the same weight, and circuits 3 and 5 have the same weight, because these pairs follow the same route through the graph but in reverse.

Example 7.3.5: Applying the Brute Force Method

Step 1: Since there are 4 vertices, there $(4-1)! = 3! = 3 \times 2 \times 1 = 6$ possible Hamilton circuits.

Step 2: List all the Hamilton circuits. They are

Circuit 1: $V \to B \to E \to L \to V$
Circuit 2: $V \to B \to L \to E \to V$
Circuit 3: $V \to E \to B \to L \to V$
Circuit 4: $V \to E \to L \to B \to V$
Circuit 5: $V \to L \to B \to E \to V$
Circuit 6: $V \to L \to E \to B \to V$

The possible distances are:
$VBELV = VB + BE + EL + LV = 396 + 410 + 106 + 159 = 1071$
$VEBLV = VE + EB + BL + LV = 207 + 410 + 439 + 159 = 1215$
$VBLEV = VB + BL + LE + EV = 396 + 439 + 106 + 207 = 1148$
So, a Hamilton cycle of least weight is $V \to B \to E \to L \to V$(or the reverse direction).

The officer should travel from Vandenberg to Beal to Edwards, to Los Angeles, and back to Vandenberg.

Example 7.3.6: Use the Nearest Neighbor Algorithm

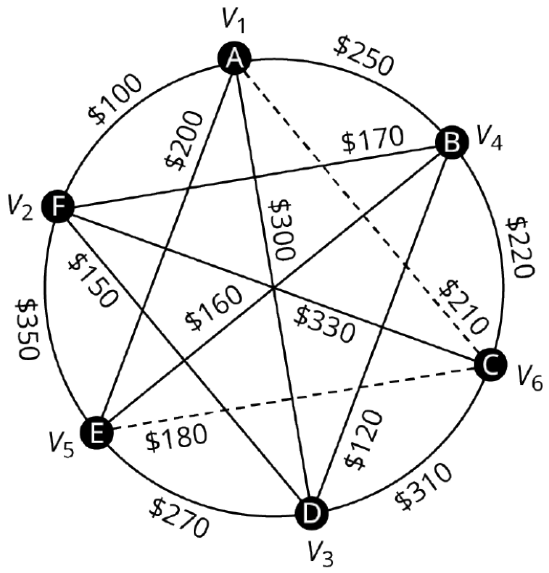


Figure 7.3.21: Copy and Paste Caption here. (Copyright; author via source)

Begin by marking starting vertex as V_1 for "visited 1st." Then to compare the weights of the edges between A and vertices adjacent to A : $\$250, \$210, \$300, \$200$, and $\$100$ as shown in Figure 7.3.20. The lowest of these is $\$100$, which is the edge between A and F.

Mark F as V_2 for "visited 2nd" then compare the weights of the edges between F and the remaining vertices adjacent to F : $\$170, \$330, \$150$ and $\$350$ as shown in Figure 7.3.21. The lowest of these is $\$150$, which is the edge between F and D.

Mark D as V_3 for "visited 3rd." Next, compare the weights of the edges between D and the remaining vertices adjacent to D : $\$120, \$310$, and $\$270$ as shown in Figure 7.3.21. The lowest of these is $\$120$, which is the edge between D and B.

So, mark B as V_4 for "visited 4th." Finally, compare the weights of the edges between B and the remaining vertices adjacent to B : $\$160$ and $\$220$ as shown in Figure 7.3.21. The lower amount is $\$160$, which is the edge between B and E.

Now you can mark E as V_5 and mark the only remaining vertex, which is C, as V_6. This is shown in Figure 7.3.21. Make a note of the weight of the edge from E to C , which is $\$180$, and from C back to A, which is $\$210$.

The Hamilton cycle we found is $A \to F \to D \to B \to E \to C \to A$. The weight of the circuit is

$$\$100 + \$150 + \$120 + \$160 + \$180 + \$210 = \$920$$

This may or may not be the route with the lowest cost, but there is a good chance it is very close since the weights are mostly of the lowest weights on the graph, and we found it in six steps instead of finding 120 different Hamilton cycles and calculating 60 weights.

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7.4: Trees



Figure 7.4.1 : In graph theory, graphs known as trees have structures in common with live trees. (credit: “Row of trees in Roslev” by AKA CJ/ Flickr, public Domain)

Learning Objectives

1. Describe and identify trees.
2. Determine a minimum spanning tree for a connected graph.
3. Solve application problems involving trees.

We saved the best for last! In this last section, we will discuss arguably the most fun kinds of graphs and trees. Have you ever researched your family tree? Family trees are a perfect example of the kind of trees we study in graph theory. One of the characteristics of a family tree graph is that it never loops back around because no one is their own grandparent!

What Is A Tree?

Whether we are talking about a family tree or a tree in a forest, none of the branches ever loops back around and rejoins the trunk. This means that a tree has no cyclic subgraphs or is acyclic. A tree also has only one component. So, a tree is a connected acyclic graph. Here are some graphs that have the same characteristics. Each of the graphs in figure 7.4.2 and figure 7.4.2 are a tree.

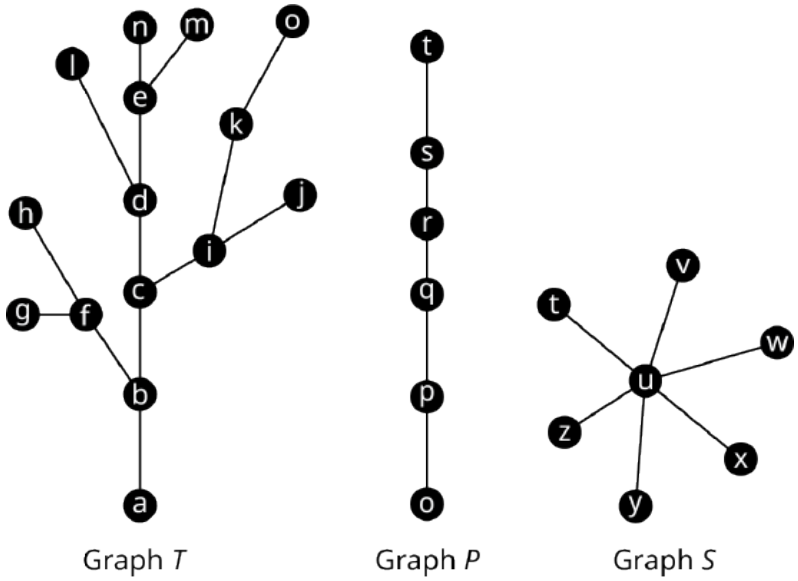


Figure 7.4.2 : Graphs *T*, *P*, and S

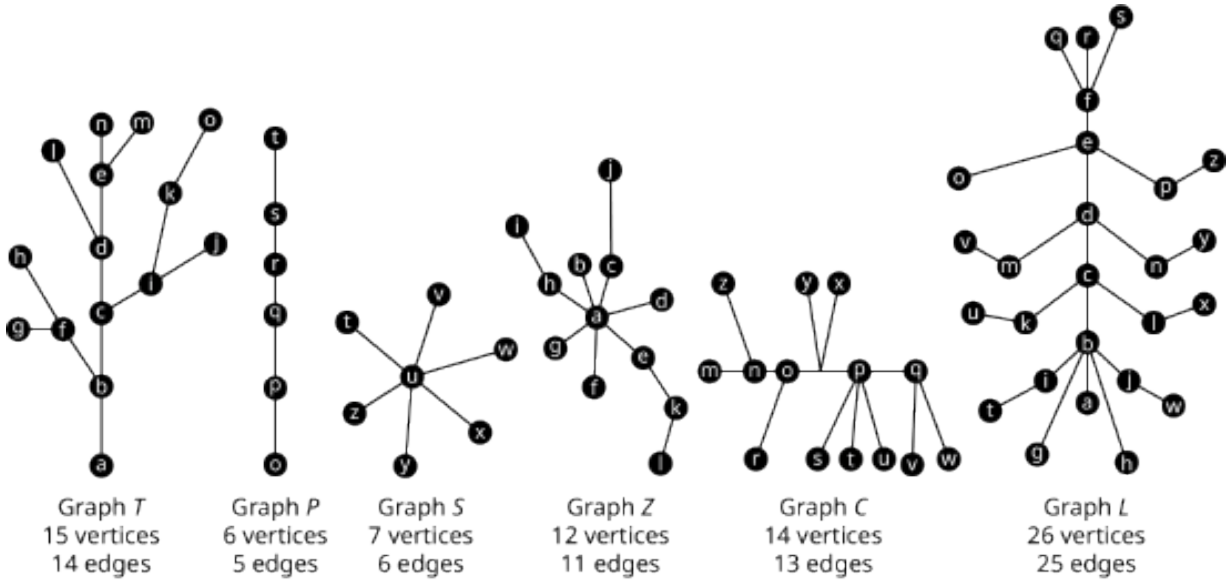


Figure 7.4.3 : Number of Vertices and Edges in Trees vs. Other Graph

Applications of Tree Graphs

Tree structures appear everywhere because they efficiently organize information. Here are the most common and important uses:

1. Folders and subfolders on a computer form a **tree**.
2. Communication networks
3. Shows relationships between generations.
4. Company Organization Chart

Definition: Characteristic of a Tree

1. There are **no loops** or closed paths.
2. Every pair of vertices is connected by **exactly one path**.
3. The number of edges is one less than the number of vertices. If tree has n vertices, It must have $n-1$ edges.
4. Every edge in a tree is a bridge.

Your Turn 7.4.1: Family Relationship

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Example 7.4.2: Identifying Trees

Identify any trees in Figure 7.4.4. If a graph is not a tree, explain how you know.

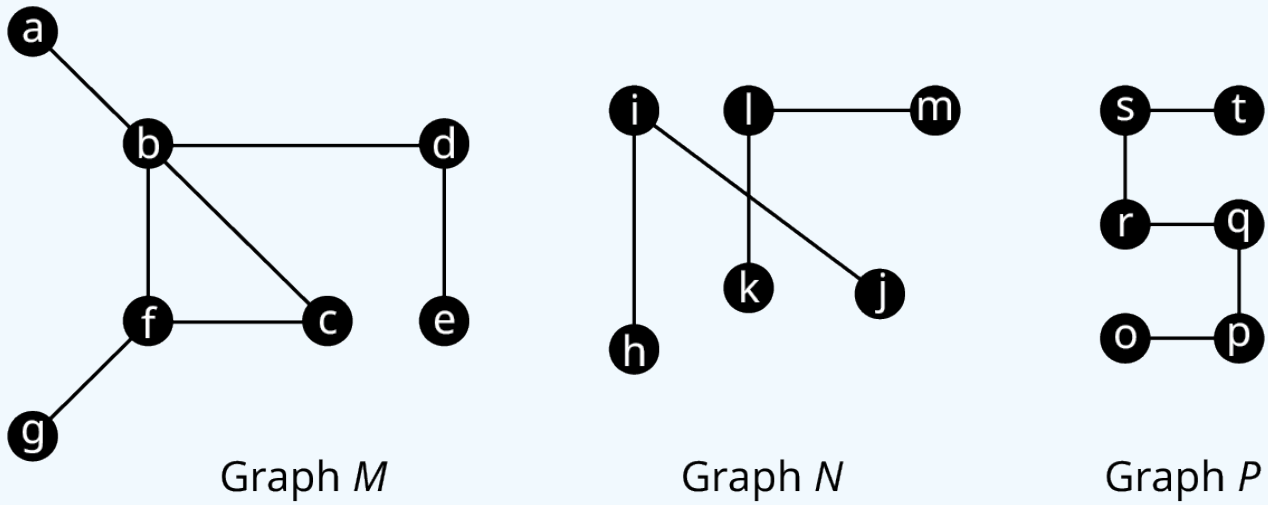


Figure 7.4.4 : Graphs *M*, *N*, and *P*

Answer available at the end of this section.

Example 7.4.3: Exploring Characteristics of Trees

Use Graphs *I* and *J* in Figure 7.4.5 to answer each question.

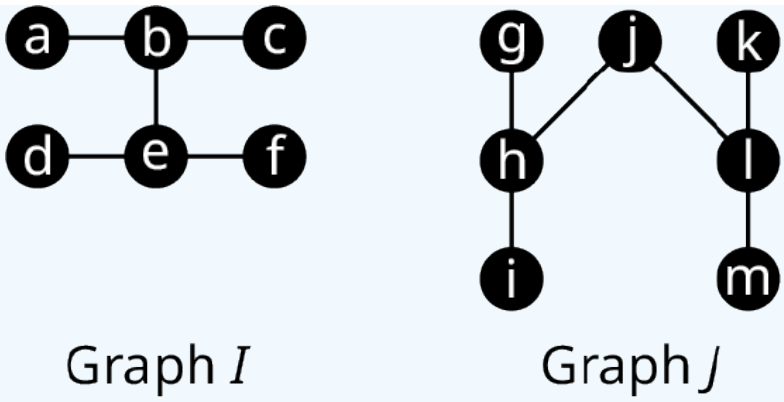


Figure 7.4.5 : Graphs I and J

1. Which vertices are in each of the components that remain when edge *be* is removed from Graph I?

2. Determine the number of edges and the number of vertices in Graph J. Explain how this confirms that Graph J is a tree.

3. What kind of cycle is created if edge *im* is added to Graph J?

Answer available at the end of this section.

Your Turn 7.4.3: Identify Tree

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Who Knew?: Graph Theory in the Movies

In the 1997 film *Good Will Hunting*, the main character, Will, played by Matt Damon, solves what is supposed to be an exceptionally difficult graph theory problem, "Draw all the homeomorphically irreducible trees of size $n = 10$." That sounds terrifying! But don't panic. Watch this great Numberphile video to see why this is actually a problem you can do at home!

Spanning Trees

Suppose that you planned to set up your own computer network with four devices. One option is to use a "mesh topology" like the one in Figure 7.4.9, in which each device is connected directly to every other device in the network.

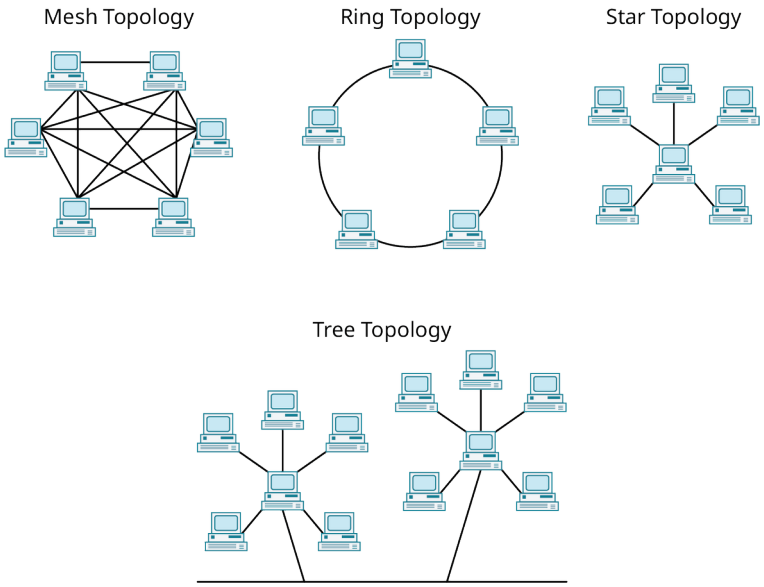


Figure 7.4.6 : Common Network Configurations

The mesh topology for four devices could be represented by the complete Graph A_1 in Figure 7.4.7 where the vertices represent the devices, and the edges represent network connections. However, the devices could be networked using fewer connections. Graphs A_2, A_3, and A_4 of Figure 7.4.7 show configurations in which three of the six edges have been removed. Each of the Graphs A_2, A_3 and A_4 in Figure 7.4.7 is a tree because it is connected and contains no cycles. Since Graphs A_2, A_3, and A_4 are also subgraphs of Graph A_1 that include every vertex of the original graph, they are also known as spanning trees.

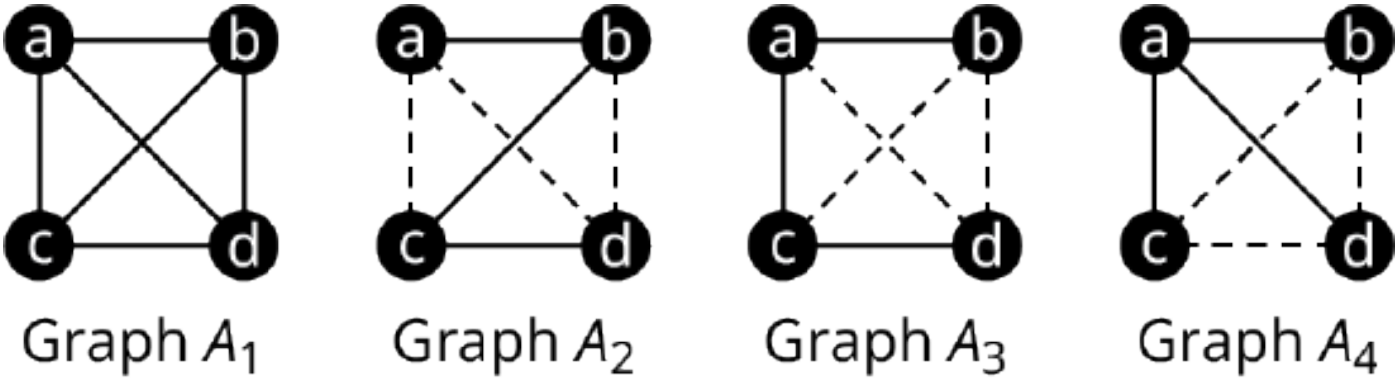


Figure 7.4.7 : Network Configurations for Four Devices

By definition, spanning trees must span the whole graph by visiting all the vertices. Since spanning trees are subgraphs, they may only have edges between vertices that were adjacent in the original graph. Since spanning trees are trees, they are connected, and they are acyclic.

So, when deciding whether a graph is a spanning tree, check the following characteristics:

- All vertices are included.
- No vertices are adjacent that were not adjacent in the original graph.
- The graph is connected.
- There are no cycles.

What is a **Spanning Tree**?

A spanning tree of a graph is a sub-graph that includes all the vertices of the original graph and has no cycles (because it is a tree).

Why use a Spanning Tree?

Because it connects all points with no redundancy and uses the least number of edges, it is used in:

1. Designing efficient communication networks
2. Reducing the cost of wiring or cabling
3. Optimizing routes in transportation

Example 7.4.4: Identifying Spanning Trees

Use Figure 7.4.8 to determine which of graphs M_1, M_2, M_3, and M_4, are spanning trees of Q.

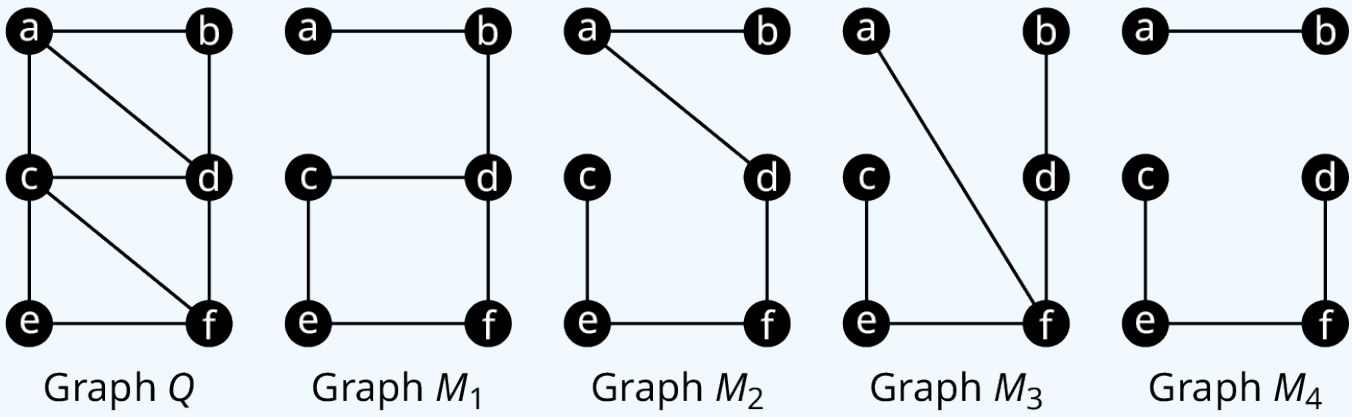


Figure 7.4.8 : Graphs Q, M_1, M_2, M_3, and M_4

Answer available at the end of this section.

Your Turn 7.4.4: Identify Spanning Tree

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Constructing a Spanning Tree Using Paths

Suppose that you wanted to find a spanning tree within a graph. One approach is to find paths within the graph. You can start at any vertex, go any direction, and create a path through the graph, stopping only when you can't continue without backtracking, as shown in Figure 7.4.9.

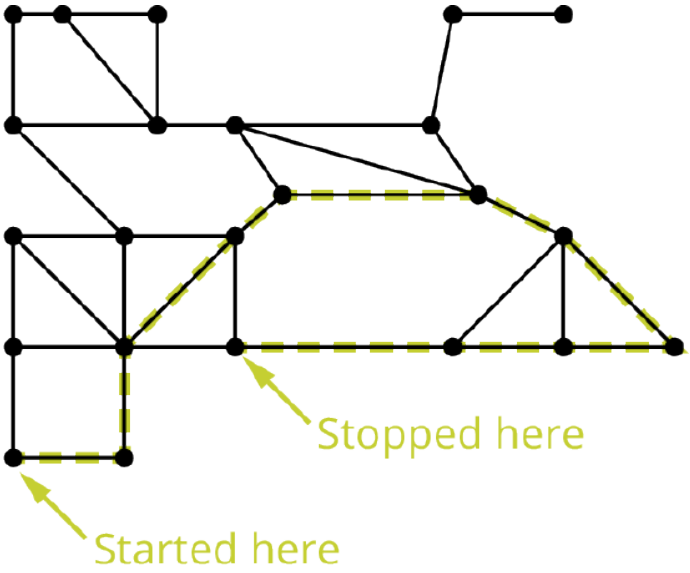


Figure 7.4.9 : First Phase to Construct a Spanning Tree

Once you have stopped, pick a vertex along the path you drew as a starting point for another path. Make sure to visit only the vertices you have not visited before, as shown in Figure 7.4.10.

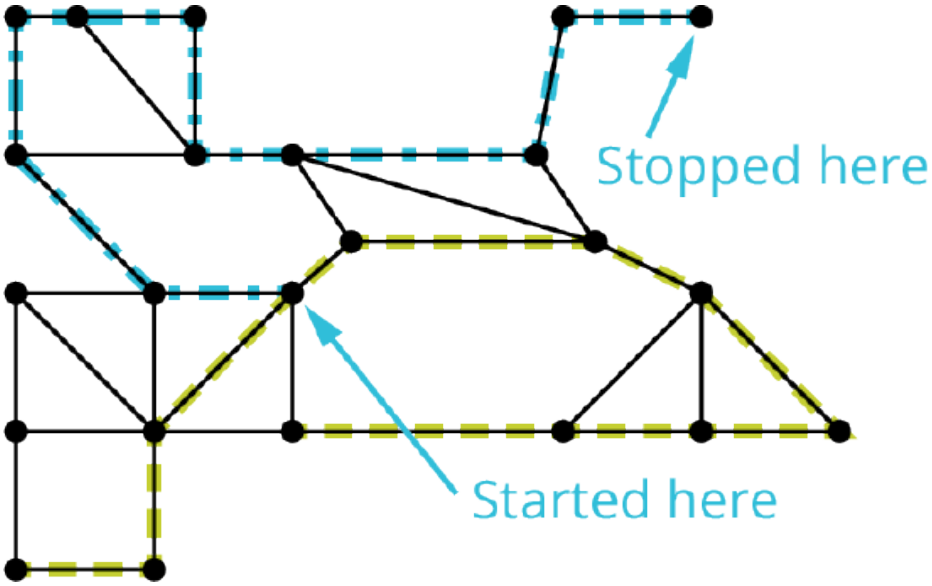


Figure 7.4.10 : Intermediate Phase to Construct a Spanning Tree

Repeat this process until all vertices have been visited as shown in Figure 7.4.11.

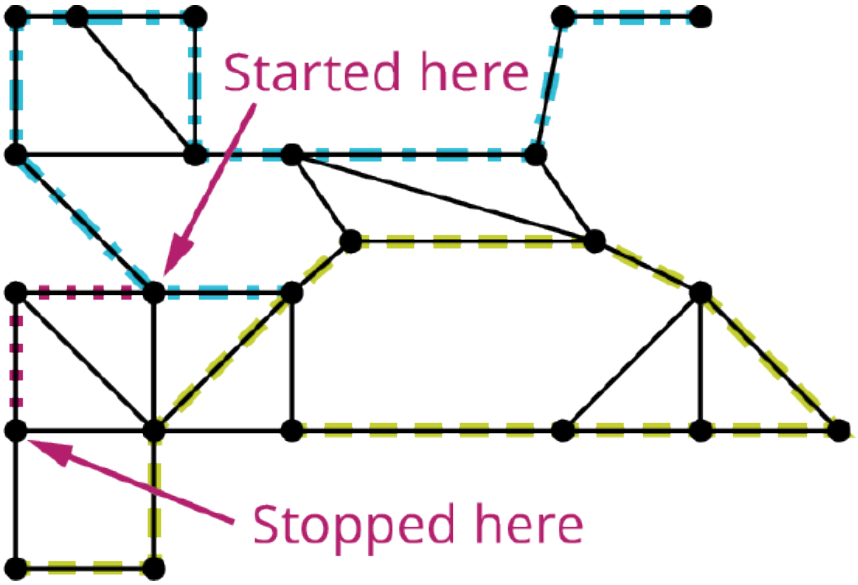


Figure 7.4.11 : Final Phase to Construct a Spanning Tree

The end result is a tree that spans the entire graph as shown in Figure 7.4.12.

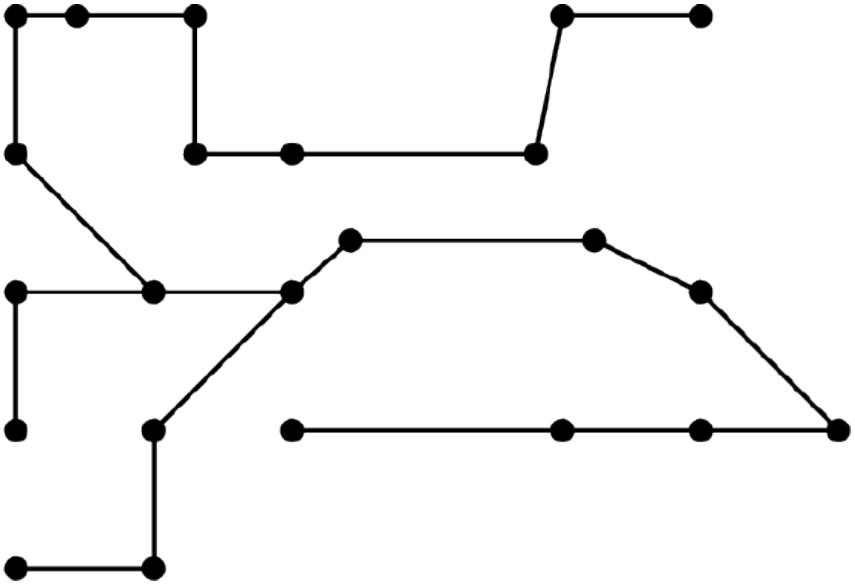


Figure 7.4.12 : The Resulting Spanning Tree

Notice that this subgraph is a tree because it is connected and acyclic. It also visits every vertex of the original graph, so it is a spanning tree. However, it is not the only spanning tree for this graph. By making different turns, we could create any number of distinct spanning trees.

Revealing Spanning Trees

Another approach to finding a spanning tree in a connected graph involves removing unwanted edges to reveal a spanning tree. Consider Graph *D* in Figure 7.4.13.

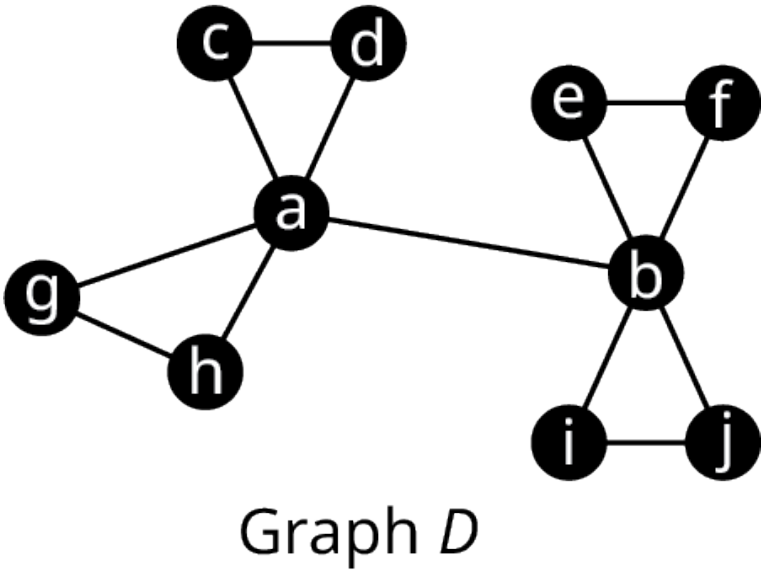


Figure $7.4.13$: Graph *D*

Graph *D* has 10 vertices. A spanning tree of Graph *D* must have 9 edges, because the number of edges is one less than the number of vertices in any tree. Graph *D* has 13 edges so 4 need to be removed. To determine which 4 edges to remove, remember that trees do not have cycles. There are four triangles in Graph *D* that we need to break up. We can accomplish this by removing 1 edge from each of the triangles. There are many ways this can be done. Two of these ways are shown in Figure \({7{.}4{.}\eatSpaces{14} \endSpaces}.\

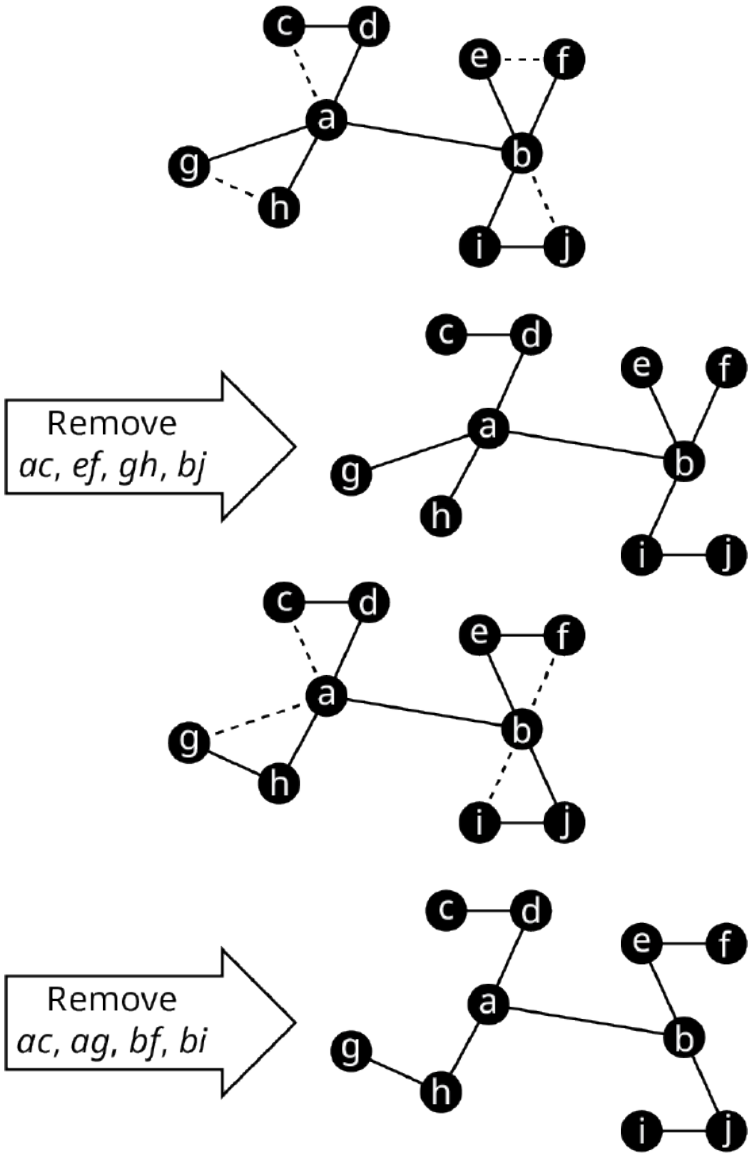


Figure 7.4.14 : Removing Four Edges from Graph *D*

Example 7.4.5: Removing Edges to Find Spanning Trees

Use the graph in Figure 7.4.15 to answer each question.

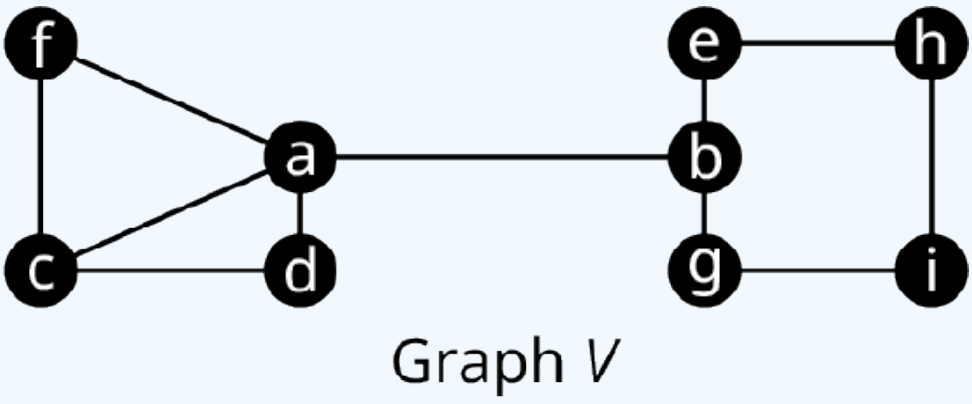


Figure 7.4.15 : Graph *V*

1. Determine the number of edges that must be removed to reveal a spanning tree.

2. Name all the undirected cycles in Graph *V*.

3. Find two distinct spanning trees of Graph *V*.

Answer available at the end of this section.

Your Turn 7.4.5: Constructing Spanning Trees

Construct two distinct spanning trees for the graph in Figure 7.4.18.

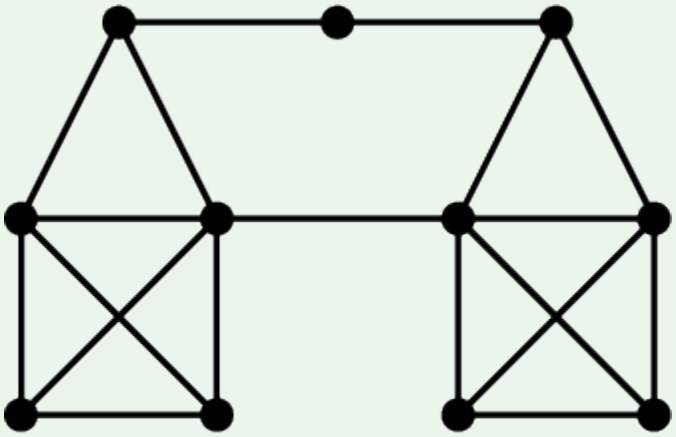


Figure 7.4.18 : Graph *L*

Answer available at the end of this section.

Minimum Spanning Tree (MST) and Kruskal's Algorithm

In many applications of spanning trees, the graphs are weighted, and we want to find the spanning tree of the least possible weight. For example, the graph might represent a computer network, and the weights might represent the cost involved in connecting two devices. So, finding a spanning tree with the lowest possible total weight, or minimum spanning tree, means saving money! The method that we will use to find a minimum spanning tree (MST) of a weighted graph is called Kruskal's algorithm. The steps for Kruskal's algorithm are:

Kruskal's Algorithm: To Find MST

Step 1: Choose any edge with the minimum weight of all edges.

Step 2: Choose another edge of minimum weight from the remaining edges. The second edge does not have to be connected to the first edge.

Step 3: Choose another edge of minimum weight from the remaining edges, but do not select any edge that creates a cycle in the subgraph you are creating.

Step 4: Repeat step 3 until all the vertices of the original graph are included and you have a spanning tree.

Where is MST used?

MSTs help design networks that connect all nodes at the minimum cost. Some examples include

1. Cable TV distribution networks, Internet wiring, fiber-optic networks, Telephone and communication networks
2. Electrical power grids
3. Cheapest road connections between cities, Railway routes, Pipeline networks (water, gas, oil)
4. Airline route optimization (connecting hubs with minimal cost)

Example 7.4.6 : Using Kruskal's Algorithm

A computer network will be set up with six devices. The vertices in the graph in Figure 7.4.21 represent the devices, and the edges represent the cost of a connection. Find the network configuration that will cost the least. What is the total cost?

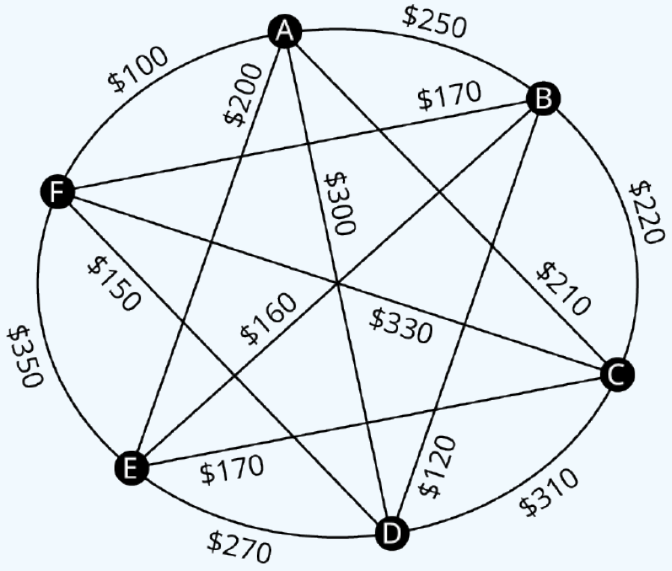


Figure 7.4.21 : Graph of Network Connection Costs

Answer available at the end of this section.

Your Turn 7.4.6: Find MST

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Your Turn 7.4.7: Definitions

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Answers

Example 7.4.2: Identifying Trees

Graph *M* is not a tree because it contains the cycle (*b*, *c*, *f*).

Graph *N* is not a tree because it is not connected. It has two components, one with vertices *h, i, j*, and another with vertices *k, l, m*.

Graph *P* is a tree. It has no cycles, and it is connected.

Example 7.4.3: Exploring Characteristics of Trees

1. When edge *be* is removed, there are two components that remain. One component includes vertices *a, b*, and *c*. The other component includes vertices *d, e*, and *f*.
2. There are seven vertices and six edges in Graph *J*. This confirms that Graph *J* is a tree because the number of edges is one less than the number of vertices.
3. The pentagon (*i*, *h*, *j*, *l*, *m*) is created when edge *im* is added to Graph *J*.

Example 7.4.4: Identifying Spanning Trees

1. Graph M_1 is not a spanning tree of Graph *Q* because it has a cycle (*c*, *d*, *f*, *e*).
2. Graph M_2 is a spanning tree of Graph *Q* because it has all the original vertices, no vertices are adjacent in M_2 that weren't adjacent in Graph *Q*, Graph M_2 is connected, and it contains no cycles.
3. Graph M_3 is not a spanning tree of Graph *Q* because vertices *a* and *f* are adjacent in Graph M_3 but not in Graph *Q*.
4. Graph M_4 is not a spanning tree of Graph *Q* because it is not connected.

So, only graph M_2 is a spanning tree of Graph *Q*.

Example 7.4.5: Removing Edges to Find Spanning Trees

1. Graph *V* has nine vertices, so a spanning tree for the graph must have 8 edges. Since Graph *V* has 11 edges, 3 edges must be removed to reveal a spanning tree.
2. (*a*, *c*, *d*), (*a*, *c*, *f*), (*a*, *d*, *c*, *f*), and (*b*, *e*, *h*, *i*, *g*)
3. To find the first spanning tree, remove edge *ac*, which will break up both of the triangles, remove edge *cf*, which will break up the quadrilateral, and remove *be*, which will break up the pentagon, to give us the spanning tree shown in Figure 7.4.16.

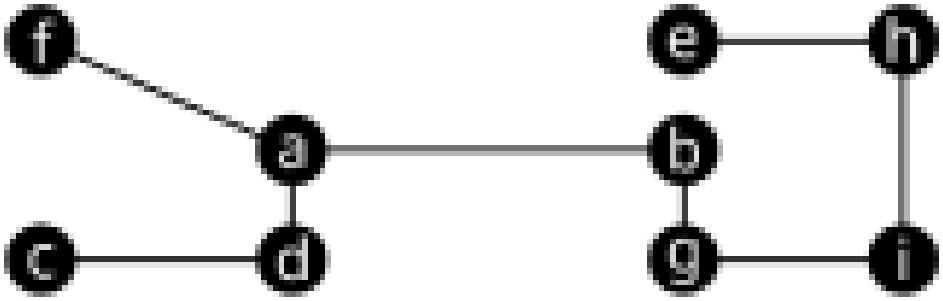


Figure 7.4.16 : Spanning Tree Formed Removing *ac, cf*, and *be*

To find another spanning tree, remove *ad*, which will break up (*a*, *c*, *d*) and (*a*, *d*, *c*, *f*), remove *af* to break up (*a*, *c*, *f*), and remove *hi* to break up (*b*, *e*, *h*, *i*, *g*). This will give us the spanning tree in Figure 7.4.17.

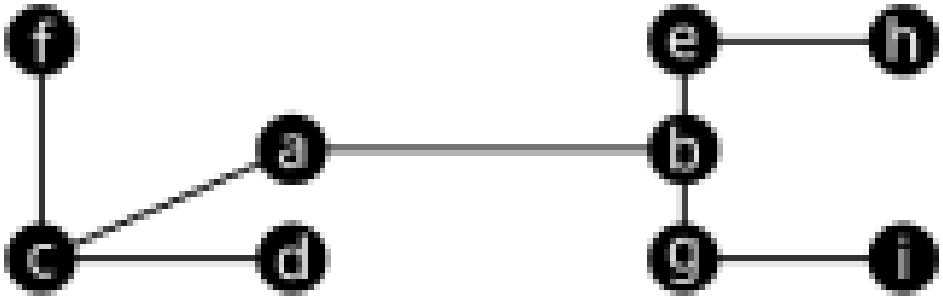


Figure 7.4.17 : Spanning Tree Formed Removing *ad*, *af*, and *hi*

Your Turn 7.4.5**: Constructing Spanning Trees**

Two possible solutions are given in Figure 7.4.19 and Figure 7.4.20.

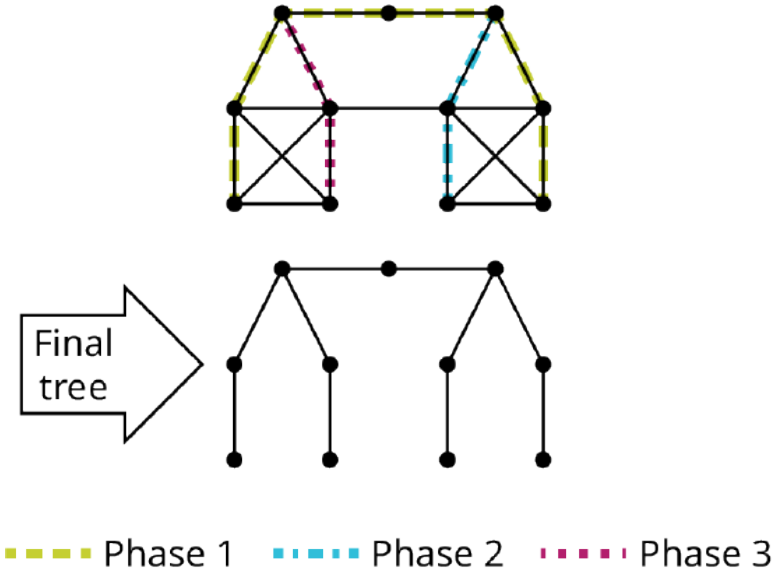


Figure 7.4.19 : First Spanning Tree for Graph *L*

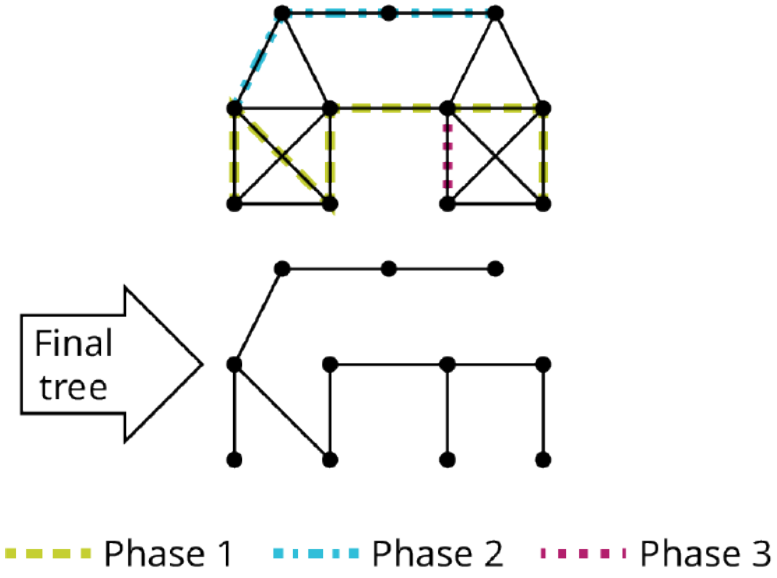
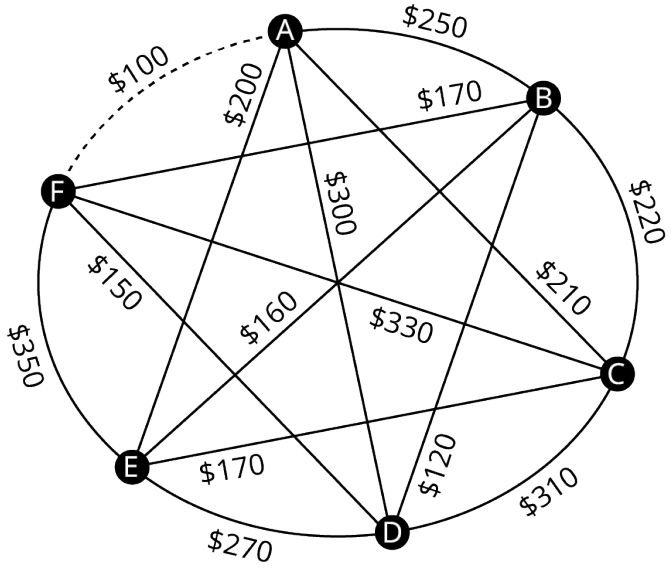


Figure 7.4.20 : Second Spanning Tree for Graph *L*

Example 7.4.6 : **Using Kruskal's Algorithm**

A minimum spanning tree will correspond to the network configuration of the least cost. We will use Kruskal's algorithm to find one. Since the graph has six vertices, the spanning tree will have six vertices and five edges.

Step 1: Choose an edge of least weight. We have sorted the weights into numerical order. The least is $100. The only edge of this weight is edge *AF* as shown in Figure 7.4.22.



Cost
~~$100~~
$120
$150
$160
$170
$170
$200
$210
$220
$250
$270
$300
$310
$330
$350

Figure 7.4.22 : Step 1 Select Edge *AF*

Step 2: Choose the edge of least weight of the remaining edges, which is *BD* with $120 Notice that the two selected edges do not need to be adjacent to each other as shown in Figure 7.4.23.

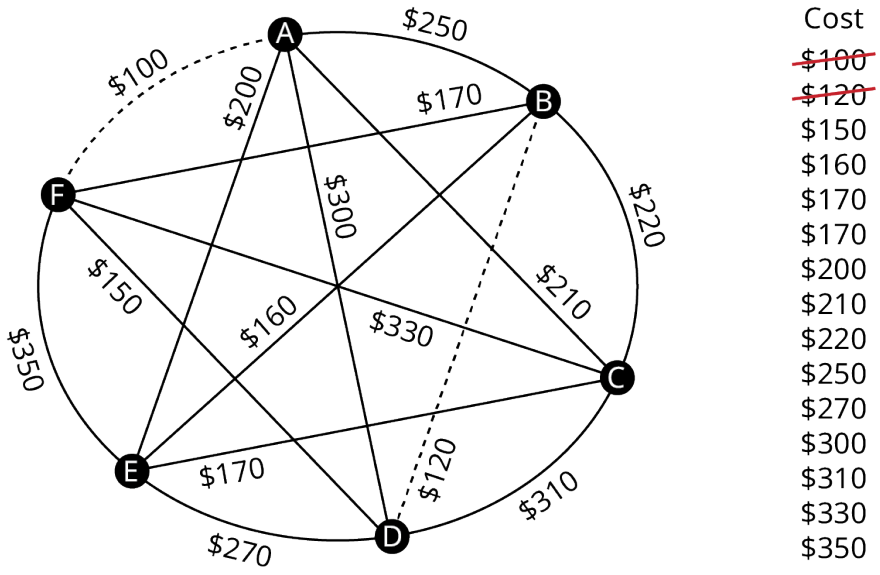


Figure 7.4.23 : Step 2 Select Edge *BD*

Step 3: Select the lowest weight edge of the remaining edges, as long as it does not result in a cycle. We select *DF* with $150 since it does not form a cycle as shown in Figure 7.4.24.

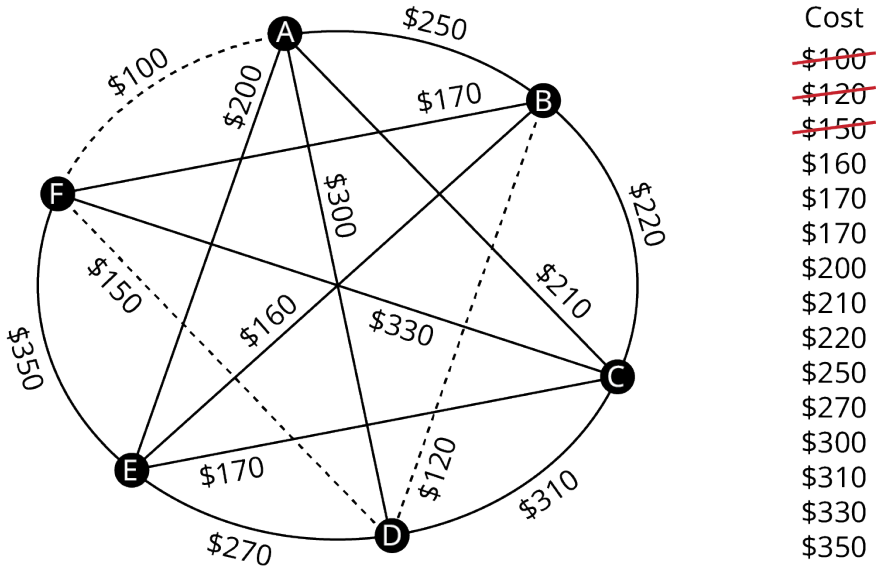


Figure 7.4.24: Step 3 Select Edge *DF*

Repeat Step 3: Select the lowest weight edge of the remaining edges, which is *BE* with $160 and it does not form a cycle as shown in Figure 7.4.25. This gives us four edges so we only need to repeat step 3 once more to get the fifth edge.

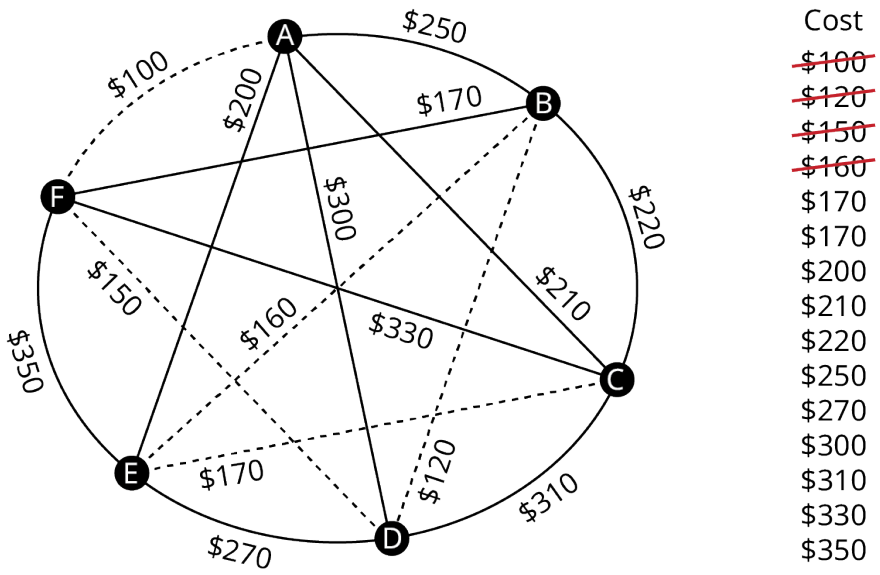


Figure 7.4.25 : Repeat Step 3 Select Edge *DF*

Repeat Step 3: The lowest weight of the remaining edges is $170. Both *BF* and *CE* have a weight of $170, but *BF* would create cycle (*b*, *d*, *f*) and there cannot be a cycle in a spanning tree as shown in Figure 7.4.26.

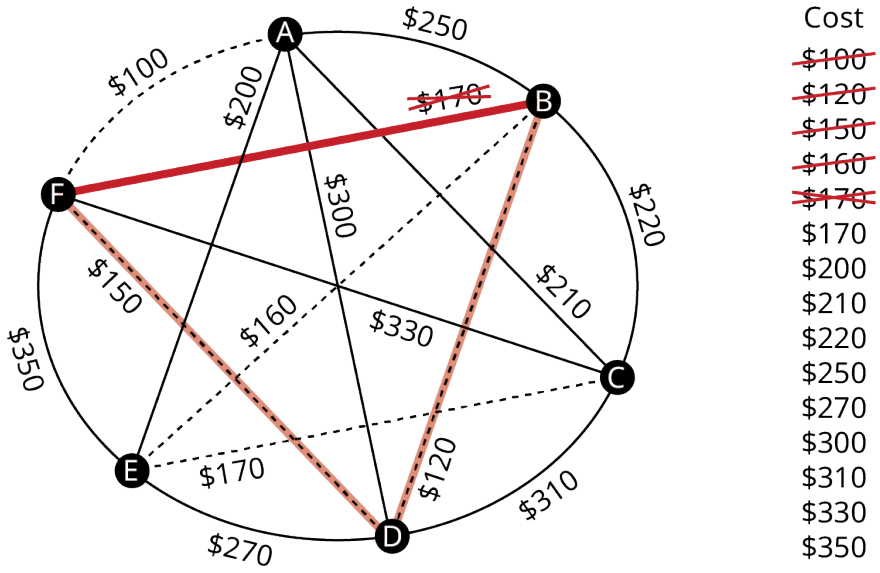


Figure 7.4.26 : Repeat Step 3 Do Not Select Edge *BF*

So, we will select *CE*, which will complete the spanning tree as shown in Figure 7.4.27.

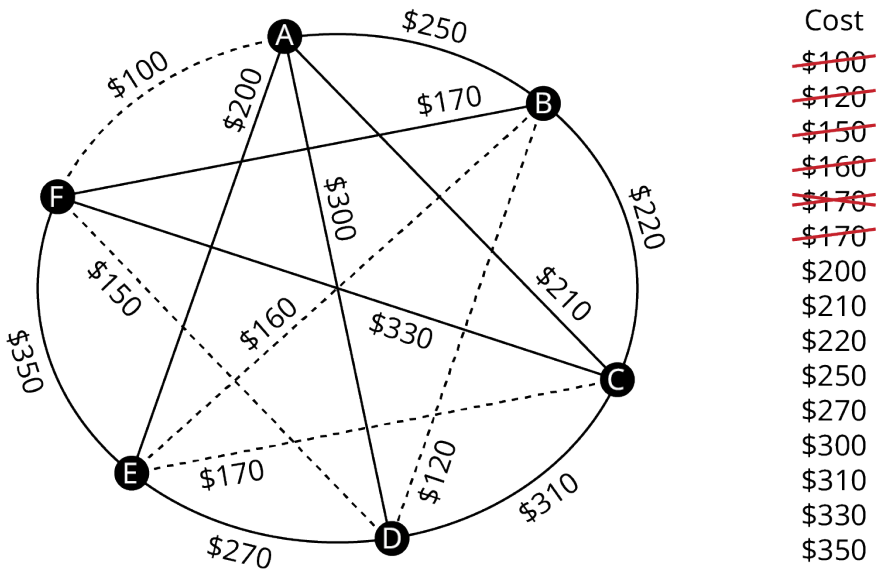


Figure 7.4.27 : Repeat Step 3 Select Edge *CE*

The minimum spanning tree is shown in Figure 7.4.28. This is the configuration of the network of least cost. The spanning tree has a total weight of $\$100 + \$120 + \$150 + \$160 + \$170 = \$700$, which is the total cost of this network configuration.

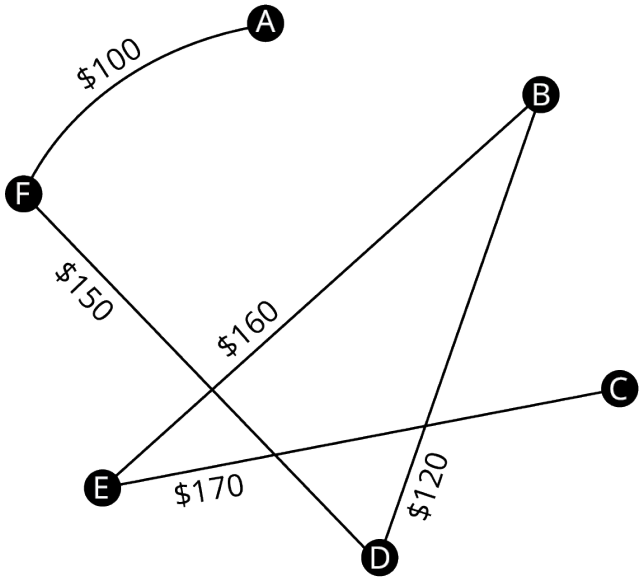


Figure 7.4.28 : Final Minimum Spanning Tree

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