MTH 95: ALGEBRA 3



Math Department
Clackamas Community College

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Math 95 - Algebra 3

OER Textbook

First Edition - 2025

Welcome to MATH 95 - Algebra 3!

This book is here to support you every step of the way as you build a strong foundation in algebra and discover how it connects to the world around you. Algebra isn't just a series of rules; it's a tool for solving real problems, spotting patterns, and making confident decisions. You'll find clear explanations, guided practice, and plenty of opportunities to check your understanding so you can learn with ease and grow your math skills with confidence.

Here's how this book was designed to help you succeed:

- **Concept Introductions**: New concepts and terminology in each section are introduced through definitions and visual demonstrations.
- **"Example" Problems**: To help you solidify your understanding of new ideas after each class, several fully worked-out examples of applying the concepts are included, which show you not only how to find the answer, but *why* it works.
- **Practice Sets:** After each section is a set of carefully sequenced problems that build in complexity and offer varied opportunities to apply new concepts. Working through these problems will help you deepen your understanding and grow your confidence.
- **Answer Key**: All answers to practice problems are collected in the back of the book, so you can quickly confirm you're on the right track.
- **Self-Check**: At the end of each practice set, use the "self-check" questions to pinpoint any areas you'd like to revisit or get extra help with.

As you progress through the course, you'll take an in-depth look at linear, quadratic, and rational relations, each with a focus on graphical properties, algebraic techniques, and real-world applications.. Along the way, you'll sharpen your number sense, deepen your critical thinking, and build the persistence to face new challenges head-on. Remember: math is a way of seeing the world, and your entire Clackamas Community College community—including instructors, tutors, advisors, and peers—is here to help you view it with clarity.

Let's get started on this exciting journey together!

Acknowledgements

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PARCE 2024–2025 Team Members

- **Kaiwen Amrein**
- **Forrest Carter**
- **Berri Haiao**
- **Erin Kelly**
- **Kelly Mercer**
- **Misay Partnof**
- **Scot Pruyn**

TABLE OF CONTENTS

Preface

- Title Page
- Welcome
- Acknowledgements
- Table of Contents
- Copyright and Licensing

Unit 1: Linear Relationships, Equations, and Inequalities

- 1.1: Solving Equations - Essential Skills
  - 1.1P: Practice
- 1.2: Linear Models and Rates of Change
  - 1.2P: Practice
- 1.3: Linear Equations and Systems
  - 1.3P: Practice
- 1.4: Linear Inequalities
  - 1.4P: Practice
- 1.5: Two Variable Linear Inequalities and Systems
  - 1.5P: Practice

Unit 2: Nonlinear Functions- Absolute Value, Piecewise, and Parabolas

- 2.1: Absolute Value
  - 2.1P: Practice
- 2.2: Absolute Value Equations
  - 2.2P: Practice
- 2.3: Piecewise-Defined Functions
  - 2.3P: Practice
- 2.4: The Parabola
  - 2.4P: Practice
- 2.5: Optimization
  - 2.5P: Practice
- 2.6: Zeros of the Quadratic
  - 2.6P: Practice
- 2.7: The Quadratic Formula
  - 2.7P: Practice

Unit 3: Ratios, Rates, and Rational Functions

- 3.1: Direct and Inverse Variation
  - 3.1P: Practice
- 3.2: Graphs of Rational Functions

- 3.2P: Practice
- 3.3: Reducing Rational Expressions
  - 3.3P: Practice
- 3.4: Sums and Differences of Rational Functions
  - 3.4P: Practice
- 3.5: Solving Rational Equations
  - 3.5P: Practice
- 3.6: Rational Applications
  - 3.6P: Practice

Unit 4: Practice Problem Answers

- 4.1: Unit 1 Practice Answers
- 4.2: Unit 2 Practice Answers
- 4.3: Unit 3 Practice Answers

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CHAPTER OVERVIEW

Preface

In this preface, you'll find a welcome message to set the stage for your learning, guidance on how to use the book effectively, acknowledgments of those who contributed to its creation, and a table of contents to help you navigate the material. This section will help you get oriented and ready to make the most of the resources ahead.

Title Page
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Table of Contents
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TABLE OF CONTENTS

Preface

- Title Page
- Welcome
- Acknowledgements
- Table of Contents
- Copyright and Licensing

Unit 1: Linear Relationships, Equations, and Inequalities

- 1.1: Solving Equations - Essential Skills
  - 1.1P: Practice
- 1.2: Linear Models and Rates of Change
  - 1.2P: Practice
- 1.3: Linear Equations and Systems
  - 1.3P: Practice
- 1.4: Linear Inequalities
  - 1.4P: Practice
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Unit 2: Nonlinear Functions- Absolute Value, Piecewise, and Parabolas

- 2.1: Absolute Value
  - 2.1P: Practice
- 2.2: Absolute Value Equations
  - 2.2P: Practice
- 2.3: Piecewise-Defined Functions
  - 2.3P: Practice
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  - 2.5P: Practice
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- 2.7: The Quadratic Formula
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- 3.1: Direct and Inverse Variation
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- 3.2: Graphs of Rational Functions

- 3.2P: Practice
- 3.3: Reducing Rational Expressions
  - 3.3P: Practice
- 3.4: Sums and Differences of Rational Functions
  - 3.4P: Practice
- 3.5: Solving Rational Equations
  - 3.5P: Practice
- 3.6: Rational Applications
  - 3.6P: Practice

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CHAPTER OVERVIEW

Unit 1: Linear Relationships, Equations, and Inequalities

This unit develops your linear toolkit: we'll simplify expressions, solve one-variable equations, and connect tables, graphs, and formulas to write and interpret lines. We'll model real situations and solve systems using graphing, substitution, and elimination. Finally, we'll use inequalities in one and two variables to represent constraints, sketch solution sets, and interpret what the feasible region means in context.

1.1: Solving Equations - Essential Skills

1.1P: Practice

1.2: Linear Models and Rates of Change

1.2P: Practice

1.3: Linear Equations and Systems

1.3P: Practice

1.4: Linear Inequalities

1.4P: Practice

1.5: Two Variable Linear Inequalities and Systems

1.5P: Practice

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1.1: Solving Equations - Essential Skills

Learning Objectives

- Explain what it means to solve an equation for a variable
- Solve a linear equation for the variable
- Solve a linear equation with a variable on both sides of the equation

Solving for x

When asked to solve an equation for x, the goal is to manipulate the equation into the final form

$$x = \text{'Stuff'}$$

where "Stuff" is a valid mathematical expression that may contain other variables, mathematical symbols, etc., but must not contain any occurrence of the variable x.

> When asked to **solve an equation for** x, a common strategy is to isolate all terms containing the variable x on one side of the equation and move all terms not containing the variable x to the other side.

Example 1

Solve for x : $3x + 5 = 14$

Solution

On the left, the order of operations dictates that we first multiply x by 3 and then add 5. To solve this equation for x, we must "undo" each of these operations in inverse order. Thus, we will first subtract 5 from both sides of the equation, then divide both sides by 3.

$$3x + 5 = 14$$

$$3x + 5 - 5 = 14 - 5$$

$$3x = 9$$

$$\frac{3x}{3} = \frac{9}{3}$$

$$x = 3$$

Check: To check the solution, substitute 3 for x in the original equation and simplify.

$$3x + 5 = 14$$

$$3(3) + 5 \stackrel{?}{=} 14$$

$$9 + 5 \stackrel{?}{=} 14$$

$$14 \stackrel{\checkmark}{=} 14$$

Because the last line of the check is a true statement, this guarantees that 3 is a solution of the original equation.

Let's try an equation with fractions.

Example 2

Solve for x : $\frac{x}{5} - \frac{2}{3} = \frac{1}{2}$

Solution

On the left, order of operations demands that we first divide x by 5, then subtract $2/3$. To solve this equation for x, we must "undo" each of these operations in inverse order. Thus, we will first add $2/3$ to both sides of the equation, then multiply both sides of the resulting equation by 5.

$$\frac{x}{5} - \frac{2}{3} = \frac{1}{2}$$

$$\frac{x}{5} - \frac{2}{3} + \frac{2}{3} = \frac{1}{2} + \frac{2}{3}$$

$$\frac{x}{5} = \frac{3}{6} + \frac{4}{6}$$

$$\frac{x}{5} = \frac{7}{6}$$

$$5 \cdot x = 5 \cdot \frac{7}{6}$$

$$x = \frac{35}{6}$$

Check: To check the solution, substitute $\frac{35}{6}$ for x in the original equation and simplify.

$$\frac{35}{6} \div 5 - \frac{2}{3} \stackrel{?}{=} \frac{1}{2}$$

$$\frac{35}{6} \cdot \frac{1}{5} - \frac{2}{3} \stackrel{?}{=} \frac{1}{2}$$

$$\frac{7}{6} - \frac{2}{3} \stackrel{?}{=} \frac{1}{2}$$

$$\frac{7}{6} - \frac{4}{6} = \frac{3}{6} \stackrel{\checkmark}{=} \frac{1}{2}$$

Because the last line of the check is a true statement, this guarantees that $\frac{35}{6}$ is a solution of the original equation.

Let's try an equation with decimals.

Example 3

Solve for x : $5.2x + 2.3 = -3.94$

Solution

On the left, order of operations demands that we first multiply x by 5.2, then add 2.3. To solve this equation for x, we must "undo" each of these operations in inverse order. Thus, we will first subtract 2.3 from both sides of the equation, then divide both sides by 5.2.

$$5.2x + 2.3 = -3.94$$

$$5.2x + 2.3 - 2.3 = -3.94 - 2.3$$

$$5.2x = -6.24$$

$$\frac{5.2x}{5.2} = \frac{-6.24}{5.2}$$

$$x = -1.2$$

Check: To check the solution, substitute -1.2 for x in the original equation and simplify.

$$5.2(-1.2) + 2.3 \stackrel{?}{=} -3.94$$

$$-6.24 + 2.3 \stackrel{?}{=} -3.94$$

$$-3.94 \stackrel{\checkmark}{=} -3.94$$

Because the last line of the check is a true statement, this guarantees that -1.2 is a solution of the original equation.

Variable on Both Sides of the Equation

It is not uncommon that the variable you are solving for appears in terms on both sides of the equation. Consider, for example, the equation $2x + 3 = 5 - 7x$. In cases like this, it is helpful to understand what it means to "solve for x."

Example 4

Solve $3 - 2x = 5x + 9$ for x

Solution

We must isolate all terms containing x on one side of the equation. We can eliminate $5x$ from the right-hand side of $3 - 2x = 5x + 9$ by subtracting $5x$ from both sides of the equation.

$$3 - 2x = 5x + 9$$
$$3 - 2x - 5x = 5x + 9 - 5x$$
$$3 - 7x = 9$$
$$3 - 7x - 3 = 9 - 3$$
$$-7x = 6$$
$$\frac{-7x}{-7} = \frac{6}{-7}$$
$$x = -\frac{6}{7}$$

Check: To check the solution, substitute $-6/7$ for x in the original equation.

$$3 - 2\left(-\frac{6}{7}\right) \stackrel{?}{=} 5\left(-\frac{6}{7}\right) + 9$$
$$3 + \frac{12}{7} \stackrel{?}{=} -\frac{30}{7} + 9$$
$$\frac{21}{7} + \frac{12}{7} \stackrel{?}{=} -\frac{30}{7} + \frac{63}{7}$$
$$\frac{33}{7} \stackrel{\checkmark}{=} \frac{33}{7}$$

Because the last line of the check is a true statement, this guarantees that $-6/7$ is a solution of the original equation.

Simplifying Expressions when Solving Equations

Sometimes, we must simplify expressions before isolating terms that contain x.

Example 5

Solve for x : $2(3x + 1) - 3(4 - 2x) = -34$

Solution

We'll first simplify the expression on the left-hand side of the equation using the *Order of Operations*.

$$2(3x+1)-3(4-2x)=-34$$

$$6x+2-12+6x=-34$$

$$12x-10=-34$$

$$12x-10+10=-34+10$$

$$12x=-24$$

$$\frac{12x}{12}=\frac{-24}{12}$$

$$x=-2$$

Check: To check the solution, substitute -2 for x in the original equation.

$$2(3(-2)+1)-3(4-2(-2)) \stackrel{?}{=} -34$$

$$2(-6+1)-3(4+4) \stackrel{?}{=} -34$$

$$2(-5)-3(8) \stackrel{?}{=} -34$$

$$-10-24 \stackrel{\checkmark}{=} -34$$

Because the last line of the check is a true statement, this guarantees that -2 is a solution of the original equation.

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1.1P: Practice

Practice Makes Progress

Solve the given equation for x:

Exercise 1

$-11+3x=-44$

Exercise 2

$5-13x=70$

Exercise 3

$\frac{x}{2}+\frac{2}{3}=\frac{4}{7}$

Exercise 4

$\frac{x}{5}-\frac{9}{2}=-\frac{5}{3}$

Exercise 5

$11x+10=19x+20$

Exercise 6

$7x+11=16-18x$

Exercise 7

$2(6-2x)-(4x-9)=9$

Exercise 8

$2x-2(4-9x)=8(6x+2)$

Self-Check

In this section, you practiced:

a. Applying order of operations to simplify expressions
b. Combining like terms
c. Solving linear equations of a single variable with integer coefficients
d. Solving linear equations of a single variable with fraction coefficients

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It’s okay to not be fully confident with every skill yet! But it’s essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

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1.2: Linear Models and Rates of Change

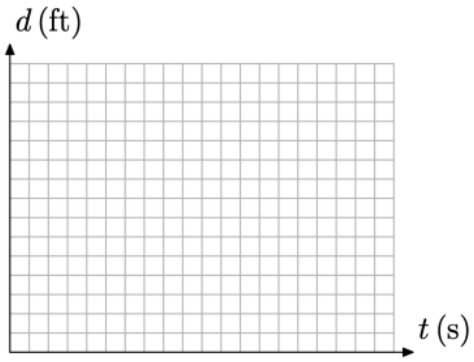
Learning Objectives

- Explain what a linear relation is, and how it differs from a linear equation of one variable
- Represent a linear relation graphically on a coordinate plane, or numerically in a table
- Identify the slope of a linear relation
- Interpret the slope as a constant rate of change in a real-world scenario

Sebastian waves good-bye to his brother, who is talking to a group of his friends approximately 20 feet away. Sebastian then begins to walk away from his brother at a constant rate of 4 feet per second. Let's model the distance separating the two brothers as a function of time.

Our first approach will be graphical. We will let the variable d represent the distance (in feet) between the brothers and the variable t represent the amount of time (in seconds) that has passed since Sebastian waved good-bye to his brother. Because the distance separating the brothers depends on the amount of time that has passed, we will say that the distance d is the dependent variable and the time t is the independent variable.

It is somewhat traditional in the modeling process to place the independent variable on the horizontal axis and the dependent variable on the vertical axis. This is not a hard-and-fast rule, but rather a matter of personal taste. Nonetheless, we will follow this tradition in our example. Thus, we will place distance on the vertical axis and time on the horizontal axis, as shown in the figure below. Notice that we've labeled each axis with its variable representation and included the units, which is an important practice.



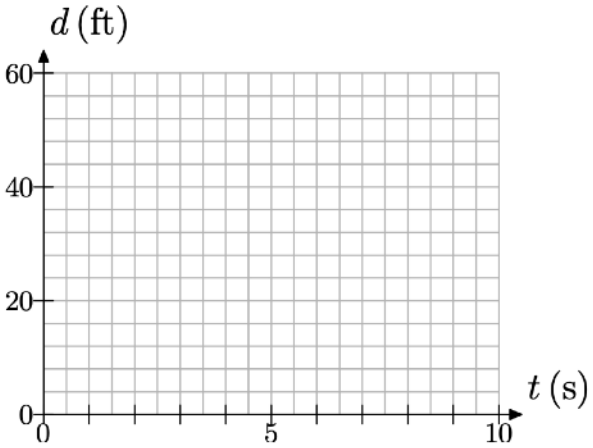
Distance depends on time.

Guidelines for Scaling Axes

Here is some good advice to follow when scaling the dependent and independent axes.

1. We want to avoid graphs that are postage-stamp-sized. A large graph is easier to interpret than one that is cramped in a small corner of our graph paper.
2. Having the same scale on each axis is unnecessary, but once a scale is chosen, consistency must be maintained.
3. We want to choose a scale that correlates easily with the given rate

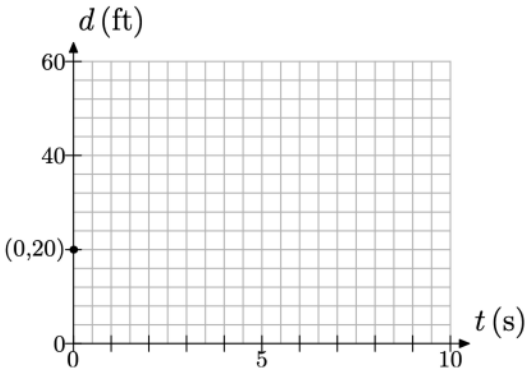
Sebastian walks away from his brother at a constant rate of 4 feet per second. Let's let each box on the vertical axis represent 4 feet and every two boxes on the horizontal axis represent 1 second, as shown here.



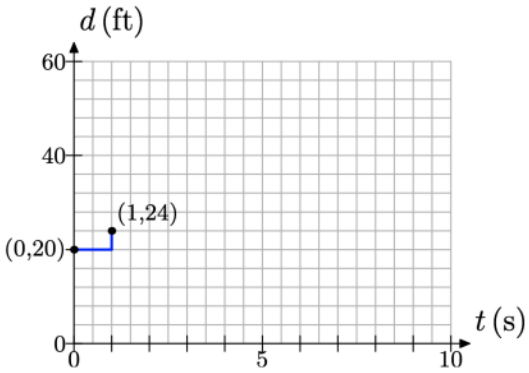
Scaling each axis to accommodate the rate.

At time $t = 0$, Sebastian is separated from his brother by a distance of $d = 20$ feet. This corresponds to the point $(t, d) = (0, 20)$.

Next, Sebastian walks away from his brother at a constant rate of 4 feet per second. This means that for every second of time that elapses, the distance between the brothers increases by 4 feet. Starting at the point (0, 20), move 1 second (two boxes) to the right and 4 feet (1 box) upward to the point (1, 24).



(a) Initially 20 feet from his brother.



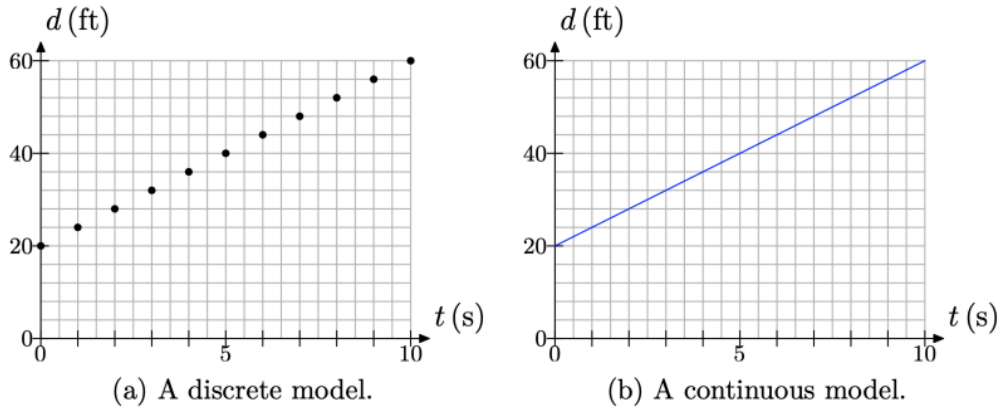
(b) Walking away at 4 ft/s.

The rate of separation is a constant 4 feet per second. So, continue indefinitely, moving 1 second (2 boxes) to the right, then 4 feet (1 box) upward. This will produce the linear relationship between distance and time, as shown in the figure on the right.

If we assume that the distance is a continuous function of time, a legitimate assumption because the distance is increasing continuously at a constant rate of 4 feet per second, then we can replace the discrete set of data points with the line shown.

The line represents a continuous model. It can be drawn with a simple stroke of the pencil, without the tip ever leaving contact with our graph paper. On the other hand, the set of points represents a discrete model. After plotting a point, our pencil must break contact with our graph paper before plotting the next point. This is the essential difference between a discrete model and a continuous model.

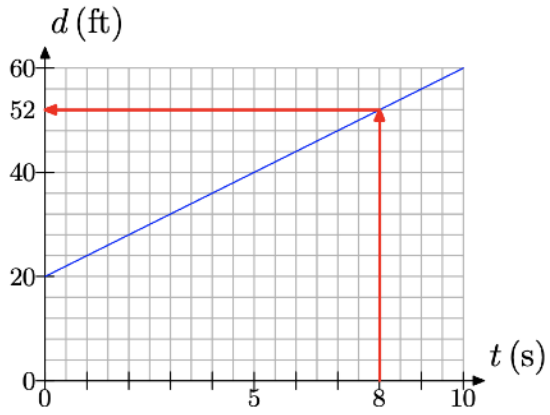
In this case, the continuous model is a more accurate representation of the distance between the brothers. We say this because the distance between them is increasing at a constant rate of 4 feet per second, or 2 feet every half second, or 1 foot every quarter second, etc. They are still occupying space even at partial seconds and for partial feet. Soon, we will provide an example where this type of continuous model is unreasonable.



(a) A discrete model. (b) A continuous model.

Constant rate yields a linear relationship.

Now that we've modeled the distance between the brothers with a graph, we can use the graph to make predictions. For example, to determine the distance between the brothers after 8 seconds, locate 8 seconds on the time axis, draw a vertical arrow to the line, then a horizontal arrow to the distance axis, as shown here.



Predicting the distance between the brothers after 8 seconds.

However, suppose that we want to determine the distance between the brothers after 2 minutes. As the graph above only models the distance over the first 10 seconds, we would have to redraw the graph over the first 2 minutes (120 seconds) to determine the answer. We did not plan ahead for this contingency, so perhaps we can model the distance between the brothers in another way, one that will more easily predict the distance between the brothers after an arbitrary amount of time t.

To this end, we search for a pattern that describes the distance d between the brothers as a function of time t. Because the distance between the brothers is increasing at a rate of 4 feet per second, we note that:

- At $t = 0$ seconds, the distance between the brothers is $d = 20$ feet.
- At $t = 1$ second, the distance between the brothers is $d = 24$ feet.
- At $t = 2$ seconds, the distance between the brothers is $d = 28$ feet.
- At $t = 3$ seconds, the distance between the brothers is $d = 32$ feet.

We summarize these results in the table below.

t	d
0	20
1	24
2	28
3	32

However, you don't want to simplify the distances as we have in a table, because this hides the pattern or relationship between the distance d and the time t. It is more efficient to seek a relationship between distance and time in the following manner. After $t = 1$ second, the distance increases by 1 increment of 4 feet, so $d = 20 + 4(1)$. After $t = 2$ seconds, the distance increases by 2 increments of 4 feet, so $d = 20 + 4(2)$. Continuing in this manner, we have:

• At $t = 3$ seconds, the distance between the brothers is $d = 20 + 4(3)$ feet.

• At $t = 4$ seconds, the distance between the brothers is $d = 20 + 4(4)$ feet.

These results are summarized here.

t	d
0	20
1	20 + 4(1)
2	20 + 4(2)
3	20 + 4(3)

Unlike the first table, this table reveals a relationship between distance d and time t that the equation can describe

$$d = 20 + 4t$$

The careful reader will check that this equation reveals the correct distances for $t = 0, 1, 2$, and 3 seconds. Two important observations can be made about the equation.

1. The 20 in $d = 20 + 4t$ is the initial distance between the brothers and corresponds to the point (0, 20).
2. The 4 in $d = 20 + 4t$ is the rate at which the distance between the brothers is increasing (4 feet per second).

Moreover, this equation can be used to predict the distance between the brothers at 2 minutes. First, convert $t = 2$ minutes to $t = 120$ seconds, and then substitute this value into our equation.

$$d = 20 + 4(120) = 500$$

Thus, the distance between the brothers after 2 minutes is $d = 500$ feet.

If the dependent variable changed at a constant rate with respect to the independent variable, then the graph would be a line. If the rate was positive, then as we swept our eyes from left to right, the line rose upward, indicating that the dependent variable increased with increasing changes in the independent variable. If the rate was negative, then the graph fell downward, indicating that the dependent variable decreased with increasing changes in the independent variable. You may have also learned that higher rates led to steeper lines (lines that rose more quickly) and lower rates led to lines that were less steep.

Now, we will connect the intuitive concept of rate developed in the previous section with a formal definition of the slope of a line. To start, let's state what the slope of a line means. If slope is a number that is directly connected to the "steepness" of a line, then we should have certain expectations.

Expectations

1. Lines with positive slope should slant uphill (as our eyes sweep from left to right).
2. Lines with negative slope should slant downhill (as our eyes sweep from left to right).
3. Because any horizontal line neither slants uphill nor downhill, we expect that it should have a slope equal to zero.
4. Lines with a larger positive slope should rise more quickly than lines with a smaller positive slope.
5. If two lines have negative slopes, then the line with the slope with the larger absolute value should fall more quickly than the other line.

We would also like the slope definition to conform to the concept of rate developed in the previous section. Therefore, we define the following.

Slope

Slope is a number that tells us how quickly a line rises or falls. The **slope** of a line is the rate at which the dependent variable changes with respect to the independent variable.

Note how the word "change" is used in the Definition. It is essential to understand that the change in some quantity can be positive, negative, or zero. For example, if the temperature outside is $40°\text{F}$ when I leave my home at 6 AM, and at noon the temperature is 65◦ F, then the change in temperature is a positive $25°\text{F}$. On the other hand, if the temperature outside is $65°\text{F}$ at noon, and the temperature is $50°\text{F}$ when I return home in the evening, then the change in temperature is a negative 15 degrees Fahrenheit.

Follow this rule when calculating the change in a quantity.

Change in Quantity

Change in Quantity = Latter Measurement − Former Measurement.

Thus, if T represents the temperature and ΔT represents the change in the temperature, then in our first case (taking the temperature in the morning then later at noon), the change in temperature is

$$\Delta T = \text{Latter} - \text{Former} = 65°\text{F} - 40°\text{F} = 25°\text{F}$$

This positive result represents a temperature increase of $25°\text{F}$.

In the second case (taking the temperature at noon, then later in the evening), the change in temperature is

$$\begin{aligned}\Delta T &= \text{Latter} - \text{Former} \\ &= 50°\text{F} - 65°\text{F} \\ &= -15°\text{F}\end{aligned}$$

This negative result represents a decrease in the temperature of $15°\text{F}$

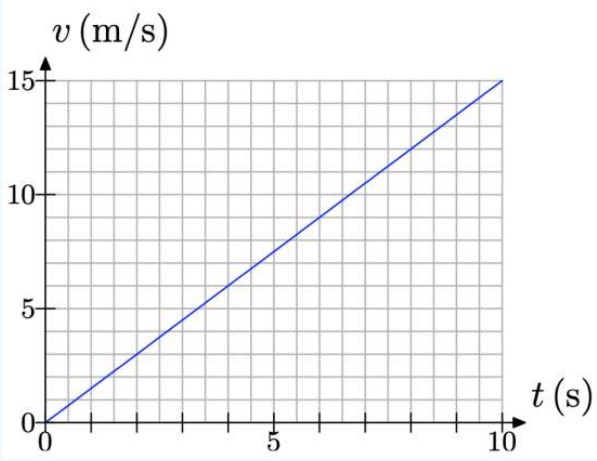
Example 1

A ball is perched at rest at the top of a long ramp. It's given a little tap, and it begins to roll down the ramp. The speed v of the ball (in meters per second) is plotted versus the time t (in seconds).

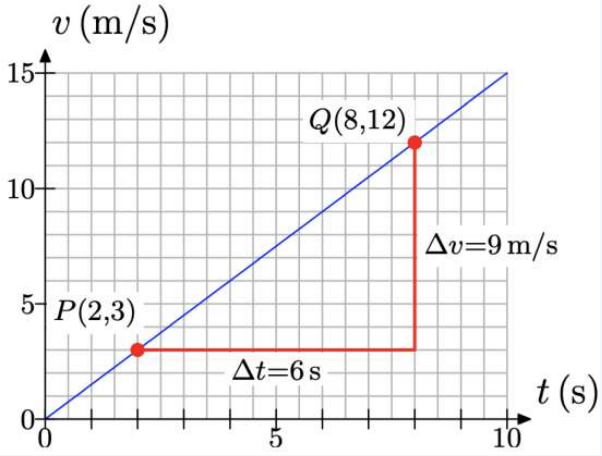
Determine the slope of the line.

Solution

We've defined the slope as the rate at which the dependent variable is changing with respect to the independent variable. In this case, the speed v of the ball "depends" upon the amount of time t that has elapsed. Consequently, v is the dependent variable and has been placed on the vertical axis. On the other hand, t is the independent variable and has been assigned the horizontal axis.



To determine the rate at which v is changing with respect to t (the slope of the line), we first select two points P(2, 3) and Q(8, 12) on the line. As we sweep our eyes from left to right (a convention we will always follow when dealing with slope), the point P occurs before the point Q. Hence, we consider P the "former" measurement and point Q the "latter" measurement.



At point P, the time is t = 2 seconds, then at point Q the time is t = 8 seconds. The change in t is found by subtracting the former measurement from the latter measurement.

$$\Delta t = 8s - 2s = 6s$$

At point P, the speed is $v = 3$ meters per second, then at point Q the speed is $v = 12$ meters per second. Hence, the change in v is

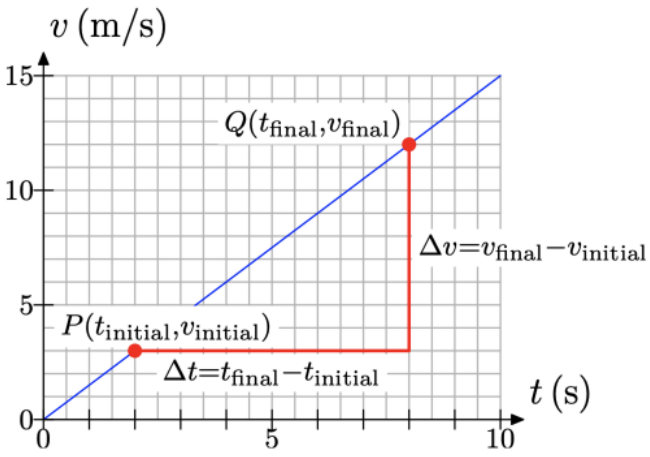
$$\Delta v = 12\text{m/s} - 3\text{m/s} = 9\text{m/s}$$

Finally, the slope of the line is defined as the rate at which the dependent variable v changes with respect to the independent variable t. That is,

$$\text{Slope} = \frac{\Delta v}{\Delta t} = \frac{9\text{m/s}}{6\text{s}} = \frac{1.5\text{m/s}}{\text{s}}$$

Scientists prefer to write this as 1.5 m/s^2, but this might not be as intuitive as writing 1.5 (m/s)/s, which indicates that the speed is increasing at a rate of 1.5 m/s every second. This makes good sense, as a ball rolling down a ramp will pick up speed over time. The slope provides an exact numerical description of how the speed increases with respect to time.

Suppose that we had labeled our points $P\,(t_{\text{initial}}, v_{\text{initial}})$ and $Q\,(t_{\text{final}}, v_{\text{final}})$ as shown here.



Now the change in speed v would be

$$\Delta v = v_{\text{final}} - v_{\text{initial}}$$

and the change in time t would be

$$\Delta t = t_{\text{final}} - t_{\text{initial}}$$

Therefore, the slope of the line is computed using the following formula.

$$\text{Slope} = \frac{\Delta v}{\Delta t} = \frac{v_{\text{final}} - v_{\text{initial}}}{t_{\text{final}} - t_{\text{initial}}}$$

With $P(t_{\text{initial}}, v_{\text{initial}}) = (2\text{s}, 3\text{m/s})$ and $Q(t_{\text{final}}, v_{\text{final}}) = (8\text{s}, 12\text{m/s})$, this becomes

$$\text{Slope} = \frac{12\text{m/s} - 3\text{m/s}}{8\text{s} - 2\text{s}} = \frac{9\text{m/s}}{6\text{s}} = 1.5\text{m/s}^2$$

Forumla: Slope

The slope of the line that passes through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the formula

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Let's look at some more examples.

Example 2

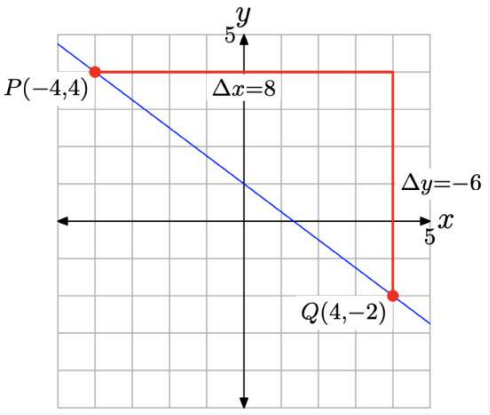
Find the slope of the line passing through the points P(−4, 4) and Q(4, −2).

Solution

We can use the slope formula to determine the slope. With $(x_1, y_1) = P(-4, 4)$ and $(x_2, y_2) = Q(4, -2)$,

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 4}{4 - (-4)} = \frac{-6}{8} = -\frac{3}{4}$$

We can also get the slope of the line from the graph. Starting at the point P(−4, 4), move 8 units to the right, then 6 units downward.



Thus, the slope of the line is

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{-6}{8} = -\frac{3}{4}$$

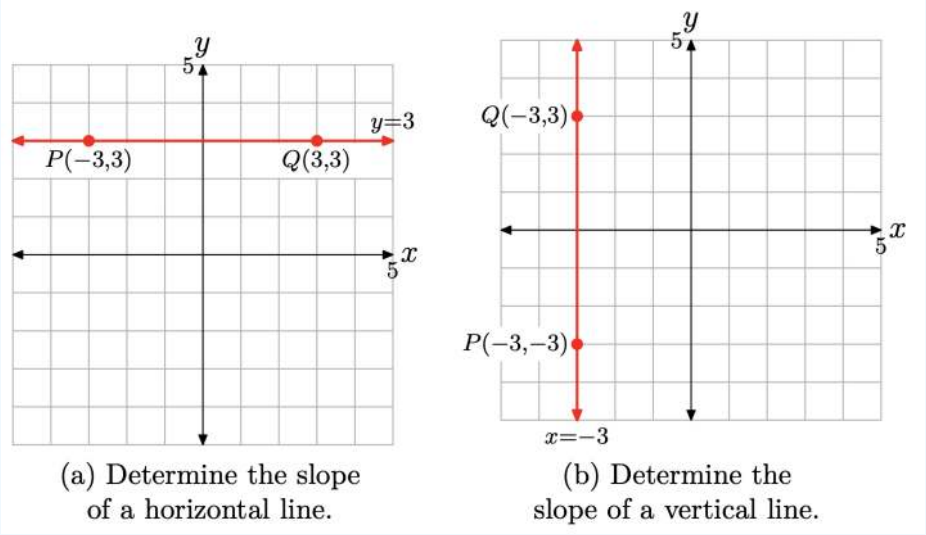
Again, one of our earlier expectations regarding the slope of a line is met in this example. The slope is −3/4, which is a negative number, and the line slants downhill (as we sweep our eyes from left to right).

Example 3

What is the slope of any horizontal line? What is the slope of any vertical line?

Solution

One would expect that our definition would verify that the slope of any horizontal line is zero. Select, for example, the horizontal line shown below. Select the points (−3, 3) and (3, 3) on this line.



With $(x_1, y_1) = (-3, 3)$ and $(x_2, y_2) = (3, 3)$

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{3-(-3)} = \frac{0}{6} = 0$$

Thus, the horizontal line $y = 3$ has slope equal to zero, exactly as expected. Further, all horizontal lines are parallel to this horizontal line and have the same slope. Therefore, all horizontal lines have a slope of zero.

For the vertical line $x = -3$, we've selected the points P(−3, −3) and Q(−3, 3) on the vertical line. With $(x_1, y_1) = P(-3, -3)$ and $(x_2, y_2) = Q(-3, -3)$,

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-3)}{-3-(-3)} = \frac{6}{0}, \text{ which is undefined.}$$

The slope of the vertical line $x = -3$ is undefined because division by zero is meaningless. Further, all vertical lines are parallel to this vertical line and have an undefined slope.

Parallel Lines

Because slope controls the "steepness" of a line, it is a simple matter to see that parallel lines must have the same slope.

Let L_1 be a line having slope m_1. Let L_2 be a line having slope m_2. If L_1 and L_2 are parallel, then

$$m_1 = m_2$$

That is, any two parallel lines have the same slope.

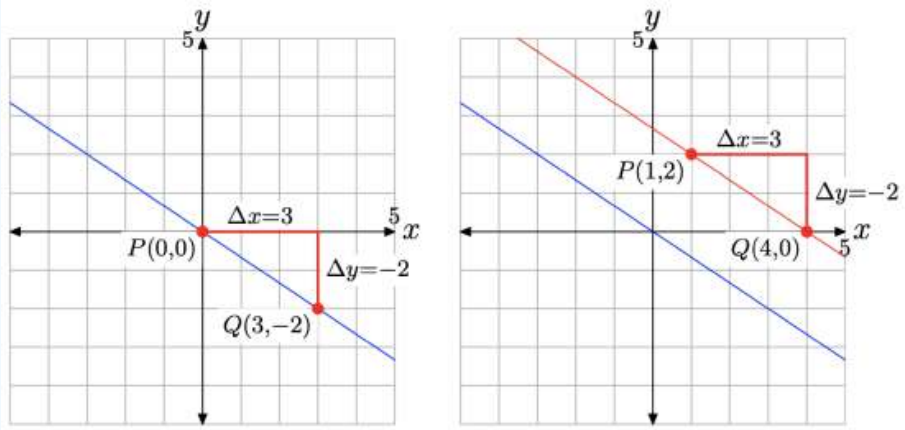
Example 4

Draw a line through the point P(1, 2) that is parallel to the line passing through the origin with slope −2/3.

Solution

We will first draw a line through the origin with slope −2/3. Plot the point P(0, 0), then move 3 units to the right and 2 units downward to the point Q(3, −2) and then draw a line through the points P and Q, as shown in the left figure.

Next, plot the point P(1, 2) as shown in right figure. To draw a line through this point that is parallel to the line through the origin, this second line must have the same slope as the first line. Therefore, start at the point P(1, 2), then move 3 units to the right and 2 units downward to the point Q(4, 0), and finally draw a line through the points P and Q as shown. Note that this second line is parallel to the first.



(a) A line through the origin with slope −2/3.

(b) A line through $P(1,2)$ that is parallel to the line through the origin.

Perpendicular Lines

The relationship between the slopes of two perpendicular lines is not as straightforward as the relationship between the slopes of two parallel lines. Let's begin by stating the pertinent property.

Let L_1 be a line having slope m_1. Let L_2 be a line having slope m_2. If L_1 and L_2 are perpendicular, then

$$m_1 m_2 = -1$$

That is, the product of the slopes of two perpendicular lines is −1.

We can solve this equation for m_1 in terms of m_2.

$$m_1 = -\frac{1}{m_2}$$

This tells us that the slope of the first line is the negative reciprocal of the slope of the second line.

For example, suppose that L_1 and L_2 are perpendicular lines with slopes m_1 and m_2, respectively

- If $m_2 = 2$, then $m_1 = -\frac{1}{2}$
- If $m_2 = \frac{3}{5}$, then $m_1 = -\frac{5}{3}$
- If $m_2 = -\frac{2}{3}$, then $m_1 = \frac{3}{2}$

Note that in each bulleted item, the product of the slopes is −1.

We won't provide proof of this equation, but we will provide some motivating evidence in the form of a graph.

Example 5

Sketch the graphs of the lines passing through the origin having slopes 2 and −1/2.

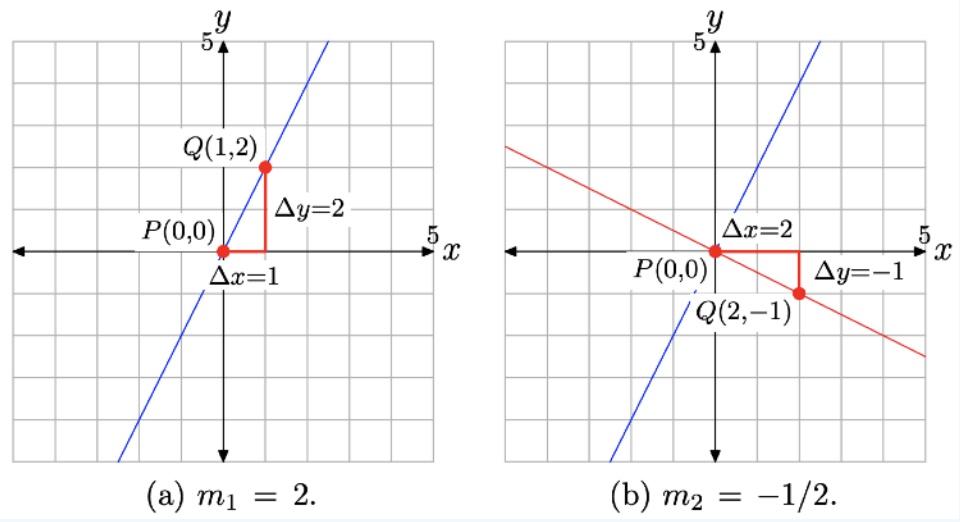
Solution

In the figure below on the left, we've plotted the point P(0, 0) at the origin, then moved 1 unit to the right and 2 units upward to the point Q(1, 2). The resulting line passes through the origin and has slope $m_1 = 2$ (alternatively, $m_1 = 2/1$).

In the figure below on the right, we've again plotted the point P(0, 0) at the origin, then moved 2 units to the right and 1 unit downward to the point Q(2, -1). The resulting line passes through the origin and has slope $m_2 = -1/2$.

The product of the two slopes is

$$m_1 m_2 = 2 \cdot \left(-\frac{1}{2}\right) = -1$$



(a) $m_1 = 2$. (b) $m_2 = -1/2$.

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1.2P: Practice

Practice Makes Progress

Exercise 1



Earl the squirrel has only ten more days until hibernation. He needs to save 50 more acorns. He is tired of collecting acorns and so he is only able to gather 8 acorns every 2 days.

a) Let t represent time in days (independent variable). Let N represent the number of acorns collected (dependent variable). Set up an appropriately scaled coordinate system on graph paper.

b) At time $t = 0$, Earl has collected zero of the acorns he needs. To what point does this correspond? Plot this point.

c) After two days ($t = 2$), Earl has collected 8 acorns. Move 2 days to the right and 8 acorns up. Plot this point. What are its coordinates?

d) Continue plotting every 2 days, adding 8 acorns, until you reach 14 days.

e) Treating the situation as continuous, draw a line through your data points.

f) Use the graph to determine how many acorns Earl will have collected after 10 days. Will it be enough?

g) Convert the rate of 8 acorns every 2 days to a unit rate of acorns per day.

h) Fill in the missing entries in the table below.

t	N
0	0
1	0 + 4(1)=4
2	
3	
	0 + 4(4)=16
	24
8	
10	
12	
14	

i) Write a linear equation that represents N, the number of acorns collected after t days.

j) Use your equation to predict the number of acorns after 10 days. Does this agree with your graph in part (f)?

Exercise 2

Set up a coordinate system on a sheet of graph paper, plotting the points $P(-1,3)$ and $Q(5,-3)$ and drawing the line through them.

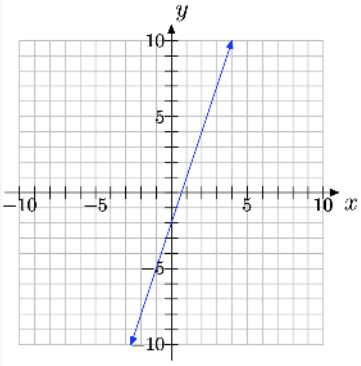
a) What can you say about the slope of the line?

b) Find the slope of the line.

c) Find the equation of this line.

Exercise 3

Find the equation of the line graphed below.



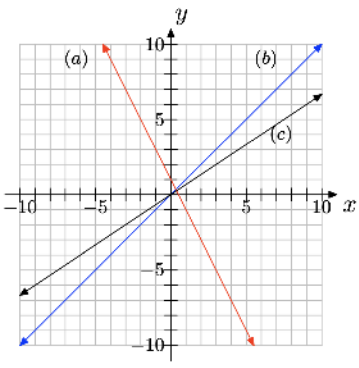
Exercise 4

The following coordinate system shows the graphs of three lines. Match each slope:

slope $= 1$

slope $= \frac{2}{3}$

slope $= -2$



Exercise 5

Draw a line through the point $P(1,3)$ that is parallel to the line through the origin and the point $(4,-1)$. Then find the equation of both lines.

https://math.libretexts.org/@go/page/189683

Exercise 6

Draw a coordinate system from -10 to 10 on both axes.

a) Draw a line through $(-1, -2)$ with slope $\frac{3}{4}$

b) Draw another through $(0, 1)$ with slope $\frac{4}{3}$

c) Are they parallel, perpendicular, or neither?

Exercise 7



When construction crews talk about the *grade* of a road, it is equivalent to slope but written as a percentage. For example a slope of 1/4 could be written as a grade of 25%.

a) Express 8% grade as slope

b) Hill drops 16 ft over 180 ft — what is the grade?

c) What does 0% grade mean?

Self-Check

In this section, you practiced:

a. Finding the slope of a line through 2 points
b. Finding the equation of a line through 2 points
c. Finding the equation of a line given its slope and a point
d. Finding the equation of a line given its graph
e. Representing a linear relation numerically using a table
f. Representing a linear relation graphically in a coordinate plane
g. Representing a linear relation symbolically as a formula
h. Using linear equations to model real-world scenarios

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

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1.3: Linear Equations and Systems

Learning Objectives

- Represent a linear relation symbolically as a formula in standard form, slope-intercept form, and point-slope form
- Translate between graphical and symbolic representations of a line
- Identify key features of a line, including intercepts and slope
- Solve a system of linear equations graphically

In this section, we will develop the slope-intercept form of a line. When you have completed the work in this section, you should be able to look at a line's graph and determine its equation in slope-intercept form.

The Slope-Intercept Form of a Line

If a line intercepts the y-axis at the point $(0, b)$ and has slope m, then the equation of the line is

$$y = mx + b$$

This form of the line equation is known as the slope-intercept form. A **linear function** is defined by the equation

$$f(x) = mx + b$$

It is important to note two key facts about the slope-intercept form $y = mx + b$.

- The coefficient of x (the m in $y = mx + b$) is the slope of the line.
- The constant term (the b in $y = mx + b$) is the y-coordinate of the y-intercept $(0, b)$.

Let's look at some examples of use of this all-important formula.

Example 1

What is the equation of the line having slope $\frac{-2}{3}$ and y-intercept at $(0, 3)$?

Solution

We're given that the slope is $\frac{-2}{3}$. Hence, $m = \frac{-2}{3}$. Secondly, we're given that the line intercepts the y-axis at the point $(0, 3)$. Hence, $b = 3$. Substituting in these values, we obtain the equation

$$y = -\frac{2}{3}x + 3$$

The Standard Form of a Line

We now know that if our equation has the form $y = mx + b$ (or can be manipulated into this form), the graph will be a line. Let's take a moment to demonstrate that the graph of the equation $Ax + By = C$, where A, B, and C are constants, is a line.

If we can place the form $Ax + By = C$ into slope-intercept form $y = mx + b$, then that will demonstrate that the graph of $Ax + By = C$ is a line. So, start with $Ax + By = C$ and subtract Ax from both sides of the equation.

$$By = -Ax + C$$

Divide both sides of this last equation by B. Note that there is an assumption here that $B \neq 0$. At the end of this section, we will handle the case when $B = 0$ separately.

$$\frac{By}{B} = \frac{-Ax + C}{B}$$

$$y = -\frac{A}{B}x + \frac{C}{B}$$

When we compare $y = -(A/B)x + (C/B)$ with $y = mx + b$, we note that the slope is $m = \frac{-A}{B}$ and the y-coordinate of the y-intercept is $b = \frac{C}{B}$. Because we were successful in placing the equation $Ax + By = C$ into slope-intercept form, we now know that the graph of $Ax + By = C$ is a line (we'll need this result in later work).

The Standard Form of a Line

For constants $A, B,$ and C, the equation $Ax + By = C$ gives the standard form of a line.

Let's look at an example.

Example 2

The equation $3x + 4y = 12$ is in standard form. Place this equation in slope-intercept form and determine the slope and y-intercept.

Solution

First, solve the equation $3x + 4y = 12$ for y.

$$3x + 4y = 12$$

$$4y = -3x + 12$$

$$y = -\frac{3}{4}x + 3$$

Note in the last step how the distributive property is applied. When we divided $-3x + 12$ by 4, we divided each term by 4, getting $-\frac{3}{4}x + 3$.

When we compare $y = -\frac{3}{4}x + 3$ with the general slope-intercept form $y = mx + b$, we determine that the slope is $m = -3/4$ and the y-coordinate of the y-intercept is $b = 3$.

How to Find Intercepts

- To find an x-intercept, let $y = 0$ in the equation and solve for x.
- To find a y-intercept, let $x = 0$ in the equation and solve for y.

Let's put what we've learned into practice.

Example 3

Sketch the graph of $3x + 4y = 12$.

Solution

We will first determine the x- and y-intercepts, plot them, and then draw a line through these intercepts.

To find the x-intercept, let $y = 0$ in $3x + 4y = 12$ and solve for x.

$$3x + 4y = 12$$

$$3x + 4(0) = 12$$

$$3x = 12$$

$$x = 4$$

Hence, the x-intercept is the point Q(4, 0). To find the y-intercept, let $x = 0$ in $3x + 4y = 12$ and solve for y.

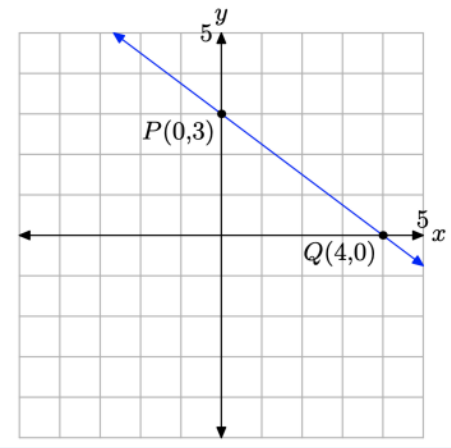
$$3x + 4y = 12$$

$$3(0) + 4y = 12$$

$$4y = 12$$

$$y = 3$$

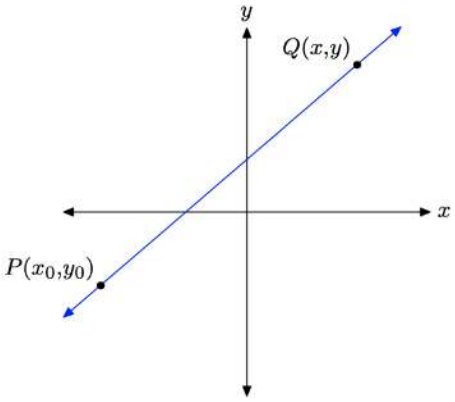
Hence, the y-intercept is the point P(0, 3). We've plotted both x- and y-intercepts and drawn a line through them.



We recommend that whenever the line is given in Standard Form $Ax + By = C$, find the x- and y-intercepts, plot them, then draw a line through them. This technique is quite efficient because working with the number zero greatly simplifies the calculations.

Now that we have developed the slope-intercept form of a line $y = mx + b$. The slope-intercept form of a line is applicable when you're given the slope and y-intercept of the line. However, there will be times when the y-intercept is unknown.

Suppose, for example, that you are asked to find the equation of a line that passes through a particular point $P = (x_0, y_0)$ with slope $= m$.



Let the point Q(x, y) be an arbitrary point on the line. We can determine the equation of the line by using the slope formula with points P and Q. Hence,

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0}$$

Because the slope equals m, we can set Slope $= m$ in this last result to obtain

$$m = \frac{y - y_0}{x - x_0}$$

If we multiply both sides of this last equation by $x - x_0$, we get

$$m(x - x_0) = y - y_0$$

or exchanging sides of this last equation,

$$y - y_0 = m(x - x_0)$$

This last result is the equation of the line.

The Point-Slope Form of a Line

If a line passes through the point (x_0, y_0) and has slope m, then the equation of the line is

$$y - y_0 = m(x - x_0)$$

This form of the equation of a line is called the point-slope form.

To use the point-slope form of a line, follow these steps.

Procedure for Using the Point-Slope Form of a Line

When given the slope of a line and a point on the line, use the point-slope form as follows:

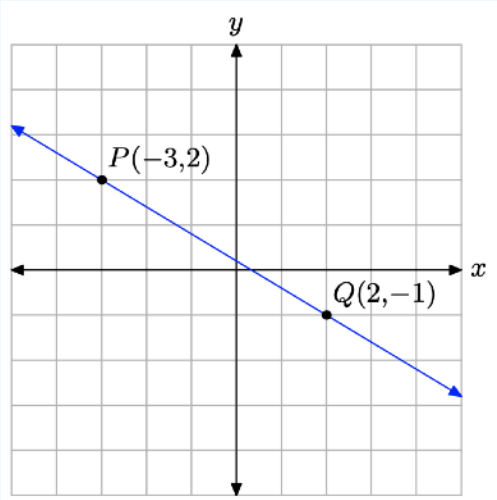
1. Substitute the given slope for m in the formula $y - y_0 = m(x - x_0)$.
2. Substitute the coordinates of the given point for x_0 and y_0 in the formula $y - y_0 = m(x - x_0)$.

Example 4

Find the equation of the line passing through the points P(-3, 2) and Q(2, -1). Place your final answer in standard form.

Solution

Again, to help keep our focus, we draw the line passing through the points P(-3, 2) and Q(2, -1).



Use the slope formula to determine the slope of the line through the points P(-3, 2) and Q(2, -1).

$$m = \frac{\Delta y}{\Delta x} = \frac{-1 - 2}{2 - (-3)} = -\frac{3}{5}$$

We'll use the point-slope form of the line

$$y - y_0 = m(x - x_0)$$

Let's use point P(-3, 2) as the given point (x_0, y_0). That is, $(x_0, y_0) = (-3, 2)$. Substitute $m = -3/5, x_0 = -3,$ and $y_0 = 2$, in the point-slope form, simplifying and rearranging, obtaining the standard form $Ax + By = C$.

$$y - 2 = -\frac{3}{5}(x - (-3))$$

$$y - 2 = -\frac{3}{5}x - \frac{9}{5}$$

$$5(y - 2) = 5\left(-\frac{3}{5}x - \frac{9}{5}\right)$$

$$5y - 10 = -3x - 9$$

$$3x + 5y = 1$$

Alternatively, we could also use the point Q(2, -1) as the given point (x_0, y_0). That is, $(x_0, y_0) = (2, -1)$. Substitute $m = -3/5, x_0 = 2,$ and $y_0 = -1$ in the point-slope form, simplifying and rearranging, obtaining the standard form $Ax + By = C$.

$$y - (-1) = -\frac{3}{5}(x - 2)$$

$$y + 1 = -\frac{3}{5}x + \frac{6}{5}$$

$$5(y + 1) = 5\left(-\frac{3}{5}x + \frac{6}{5}\right)$$

$$5y + 5 = -3x + 6$$

$$3x + 5y = 1$$

Note that the equations are identical. Thus, it doesn't matter which point you use in the point-slope form. Both lead to the same result.

Parallel Lines

Recall that slope controls the "steepness" of a line. Consequently, if two lines are parallel, they must have the same "steepness" or slope. Let's look at an example of parallel lines.

✔ Example 5

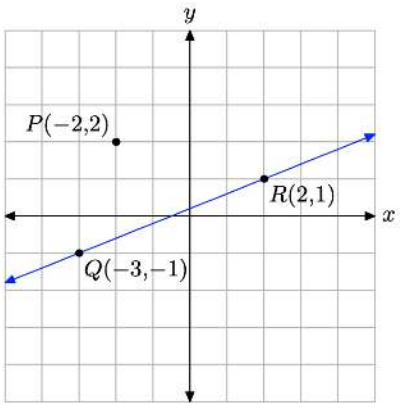
Find the equation of the line that passes through the point P(-2, 2) that is parallel to the line passing through the points Q(-3, -1) and R(2, 1).

Solution

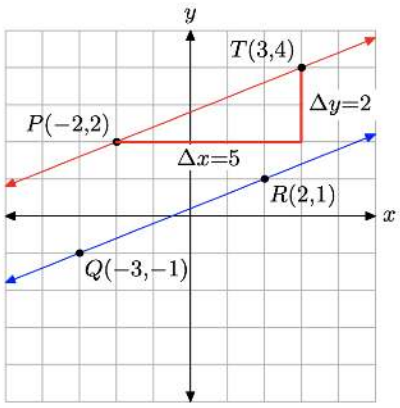
First, to help us stay focused, we draw the line through the points Q(-3, -1) and R(2, 1), then plot the point P(-2, 2).

We can use the slope formula to calculate the slope of the line passing through the points Q(-3, -1) and R(2, 1).

$$m = \frac{\Delta y}{\Delta x} = \frac{1 - (-1)}{2 - (-3)} = \frac{2}{5}$$



(a) The line through $Q(-3,-1)$ and $R(2,1)$.



(b) The line through $P(-2,2)$ that is parallel to the line through Q and R.

We now draw a line through the point P(-2, 2) that is parallel to the line through the points Q and R. Parallel lines must have the same slope, so we start at the point P(-2, 2), "run" 5 units to the right, then "rise" 2 units up to the point T(3, 4).

We seek the equation of the line through the points P and T. We'll use the point-slope form of the line

$$y - y_0 = m\,(x - x_0)$$

We'll use the point P(-2, 2) as the given point (x_0, y_0). That is, $(x_0, y_0) = (-2, 2)$. The line through P has slope 2/5. Substitute $m = 2/5, x_0 = -2,$ and $y_0 = 2$, to obtain

$$y - 2 = \frac{2}{5}(x - (-2))$$

$$y - 2 = \frac{2}{5}x + \frac{4}{5}$$

$$5(y - 2) = 5\left(\frac{2}{5}x + \frac{4}{5}\right)$$

$$5y - 10 = 2x + 4$$

$$-14 = 2x - 5y$$

$$2x - 5y = -14$$

This is the standard form of the equation of the line passing through the point P and parallel to the line passing through the points Q and R.

Perpendicular Lines

Suppose that two lines L_1 and L_2 have slopes m_1 and m_2, respectively. If L_1 and L_2 are perpendicular, then the product of their slopes is $m_1 m_2 = -1$. Alternatively, the slope of the first line is the negative reciprocal of the second line, and vice-versa; i.e., $m_1 = -1/m_2$ and $m_2 = -1/m_1$. Let's look at an example of perpendicular lines.

✔ Example 6

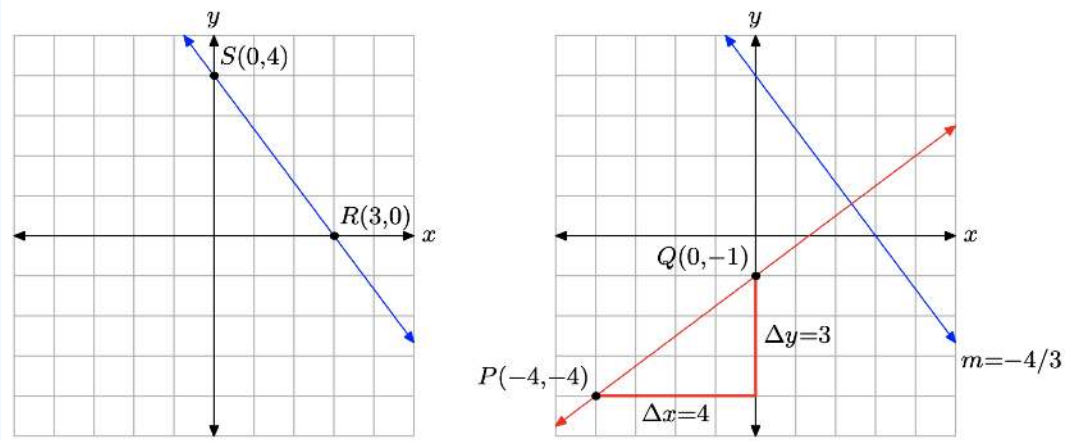
Find the equation of the line passing through the point P(-4, -4) that is perpendicular to the line $4x + 3y = 12$.

Solution

It will help our focus if we draw the given line $4x + 3y = 12$. The easiest way to plot a line in standard form $Ax + By = C$ is to find the x- and y-intercepts.

$$\begin{aligned} 4x+3y &= 12 \quad & 4x+3y &= 12 \\ 4x+3(0) &= 12 \quad & 4(0)+3y &= 12 \\ 4x &= 12 \quad & 3y &= 12 \\ x &= 3 \quad & y &= 4 \end{aligned}$$

Plot the x- and y-intercepts R(3, 0) and S(0, 4). The line through points R and S is the graph of the equation $4x+3y=12$.



(a) Plot the x and y-intercepts of $4x+3y=12$.

(b) A line through $P(-4,-4)$ that is perpendicular to $4x+3y=12$.

Next, determine the slope of the line $4x+3y=12$ by placing this equation in slope-intercept form.

$$4x+3y=12$$

$$3y=-4x+12$$

$$y=-\frac{4}{3}x+4$$

If two lines are perpendicular, then their slopes are negative reciprocals of one another. Therefore, the slope of the line that is perpendicular to the line $4x+3y=12$ (which has slope -4/3) is $m=\frac{3}{4}$. Our second line must pass through the point P(-4, -4). To draw this second line, first plot the point P(-4, -4), then move 4 units to the right and 3 units upward to the point Q(0, -1). The line through the points P and Q is perpendicular to the line 4x + 3y = 12.

To determine the equation of the line through the points P and Q, we will use the point-slope form of the line, namely

$$y-y_0=m\,(x-x_0)$$

The slope of the line through points P and Q is m = 3/4. If we use the point P(-4, -4), then $(x_0,y_0)=(-4,-4)$. Set $m=3/4, x_0=-4$, and $y_0=-4$, obtaining

$$y-(-4)=\frac{3}{4}(x-(-4))$$

or equivalently,

$$y+4=\frac{3}{4}(x+4)$$

Alternatively, we could use the slope-intercept form of the line. We know that the line through points P and Q crosses the y-axis at Q(0, -1). So, with slope m = 3/4 and y-coordinate of the y-intercept b = -1, the slope-intercept form y = mx + b becomes

$$y=\frac{3}{4}x-1$$

We should get the same results if we solve for y,

$$y+4=\frac{3}{4}(x+4)$$

$$y+4=\frac{3}{4}x+3$$

$$y=\frac{3}{4}x-1$$

Note that this is identical to the result found using the slope-intercept form above.

It is comforting to note that the two forms (point-slope and slope-intercept) give the same result, but how do we determine the most efficient form to use for a particular problem? Here's a good hint.

Determining the Form of the Line to Use

Here is some sound advice when deciding whether to use a line's slope-intercept form or point-slope form.

- If you are given the slope and the y-intercept, use the slope-intercept form $y=mx+b$.
- If you are given a point (other than the y-intercept) and the slope, use the point-slope form $y-y_0=m(x-x_0)$.

Applications of Linear Functions

In this section, we will examine some applications of linear functions. We begin by developing a function relating Fahrenheit and Celsius temperatures.

Example 7

Water freezes at $32°F$ and $0°C$. Water boils at $212°F$ and $100°C$. F and C are abbreviations for Fahrenheit and Celsius temperature scales, respectively. Assuming a linear relationship, develop a model relating Fahrenheit and Celsius temperatures.

Solution

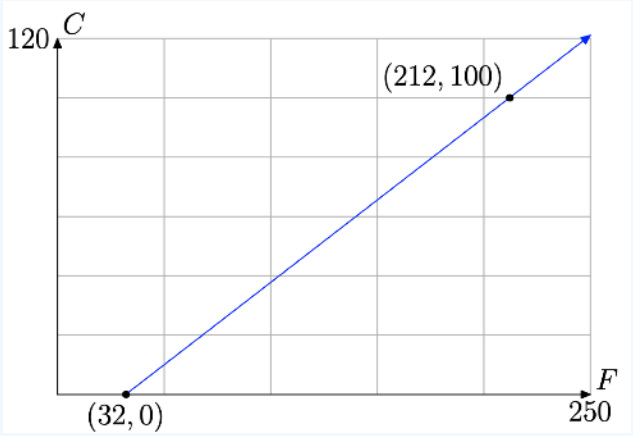
First, to help us stay focused, we set up a coordinate system on a sheet of graph paper. We've decided to make the Celsius temperature the dependent variable and assigned it to the vertical axis. Similarly, we've declared the Fahrenheit temperature the independent variable and assigned it to the horizontal axis.

Interpret the given data:

- Water freezes at $32°F$ and $0°C$. This gives us the point (F, C) = (32, 0), which we plot.
- Water boils at $212°F$ and $100°C$. This gives us the point (F, C) = (212, 100), which we plot.

Now we are on familiar ground. We want to find the equation of the line through these two points, which is the same type of problem we tackled in Example 6. First, use the points (32, 0) and (212, 100) to determine the slope of the line.

$$m=\frac{\Delta C}{\Delta F}=\frac{100-0}{212-32}=\frac{100}{180}=\frac{5}{9}$$



We will now use the point-slope form of the line, $y - y_0 = m(x - x_0)$ with m = 5/9 and $(x_0, y_0) = (32, 0)$. Substitute $m = 5/9, x_0 = 32$, and $y_0 = 0$ in $y - y_0 = m(x - x_0)$ to obtain

$$y - 0 = \frac{5}{9}(x - 32)$$

However, our dependent axis is labeled C, not y, and our independent axis is labeled F, not x. So, we must replace y and x in the equation with C and F, respectively, obtaining

$$C = \frac{5}{9}(F - 32)$$

Suppose that we know that the Fahrenheit temperature outside is $80°\text{F}$ and we wish to express this using the Celsius scale. To do so, we simply evaluate for F=80, as in

$$C = \frac{5}{9}(80 - 32) \approx 26.6$$

Hence, the Celsius temperature is approximately $26.6°\text{C}$.

On the other hand, suppose that we know the Celsius temperature on a metal roof is $80°\text{C}$ and we wish to find the Fahrenheit temperature. To do so, we need to solve for F, or equivalently,

$$\frac{5}{9}(F - 32) = 80$$

$$5(F - 32) = 720$$

$$F - 32 = 144$$

$$F = 176°\text{F}$$

Wow, that's hot!

Solving Systems

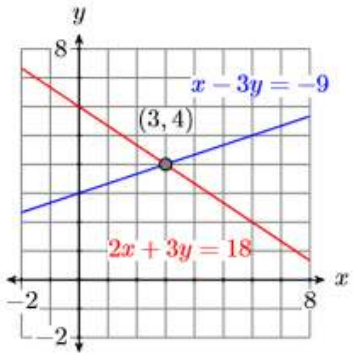
Now, we introduce a graphical technique for solving systems of two linear equations in two unknowns. If a point satisfies an equation, then that point lies on the graph of the equation. If we are looking for a point that satisfies two equations, then we are looking for a point that lies on the graphs of both equations; that is, we are looking for a point of intersection.

For example, consider the two equations:

$$x - 3y = -9$$

$$2x + 3y = 18$$

which is called a *system of linear equations*. The equations are linear equations because their graphs are lines. Note that the two lines in the figure below intersect at the point $(3, 4)$. Therefore, the point $(3, 4)$ should satisfy both equations. Let's check.



Substitute 3 for x and 4 for y.

$$x - 3y = -9$$

$$3 - 3(4) = -9$$

$$3 - 12 = -9$$
$$-9 = -9$$

Substitute 3 for x and 4 for y.

$$2x + 3y = 18$$

$$2(3) + 3(4) = 18$$

$$6 + 12 = 18$$

$$18 = 18$$

Hence, the point $(3, 4)$ satisfies both equations and is called a solution of the system.

Solution of a linear system

A point (x, y) is called a solution of a system of two linear equations if and only if it satisfies both equations. Furthermore, because a point satisfies an equation if and only if it lies on the graph of the equation, to solve a system of linear equations graphically, we need to determine the point of intersection of the two lines having the given equations.

Let's try an example.

Example 8

Solve the following system of equations:

$$3x + 2y = 12$$

$$y = x + 1$$

Solution

We are looking for the point (x, y) that satisfies both equations; that is, we are looking for the point that lies on the graph of both equations. Therefore, the logical approach is to plot the graphs of both lines, then identify the point of intersection.

First, let's determine the x- and y-intercepts of $3x + 2y = 12$.

To find the x-intercept, let $y = 0$.

$$3x + 2y = 12$$

$$3x + 2(0) = 12$$

$$3x = 12$$

$$x = 4$$

To find the y-intercept, let $x = 0$.

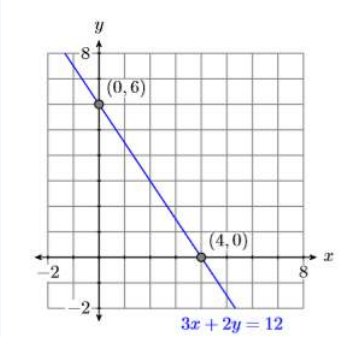
$$3x + 2y = 12$$

$$3(0) + 2y = 12$$

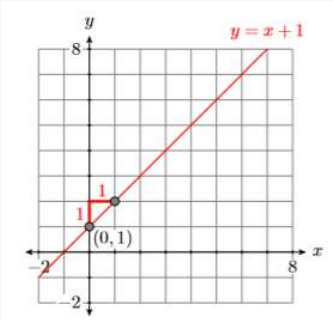
$$2y = 12$$

$$y = 6$$

Hence, the x-intercept is $(4, 0)$ and the y-intercept is $(0, 6)$. These intercepts are plotted, and the line $3x + 2y = 12$ is drawn through them.

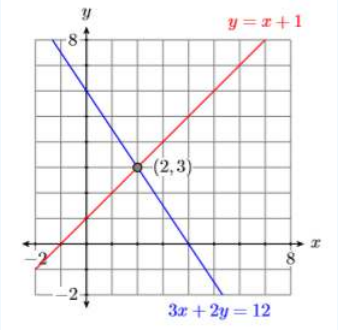


Comparing the second equation $y = x + 1$ with the slope-intercept form $y = mx + b$, we see that the slope is $m = 1$ and the y-intercept is $(0, 1)$. Plot the intercept $(0, 1)$, then go up 1 unit and right 1 unit, then draw the line.



We are trying to find the point that lies on both lines, so we plot both lines on the same coordinate system, labeling each with its equation. It appears that the lines intersect at the point $(2, 3)$, making $(x, y) = (2, 3)$ the solution of the system.

Check: To show that $(x, y) = (2, 3)$ is a solution, we must show that we get true statements when we substitute 2 for x and 3 for y in both equations of the system.



Substituting 2 for x and 3 for y in $3x + 2y = 12$, we get:

$$3x + 2y = 12$$

$$3(2) + 2(3) = 12$$

$$6 + 6 = 12$$

$$12 = 12$$

Hence, $(2, 3)$ satisfies the equation $3x + 2y = 12$.

Substituting 2 for x and 3 for y in $y = x + 1$, we get:

$$y = x + 1$$

$$3 = 2 + 1$$

$$3 = 3$$

Hence, $(2, 3)$ satisfies the equation $y = x + 1$.

Because $(2, 3)$ satisfies both equations, this makes $(2, 3)$ a solution of the system.

Additional Cases

Most of the time, given the graphs of two lines, they will intersect in exactly one point. But there are two exceptions to this.

Example 9

Solve the following system of equations:

$$\begin{aligned} 2x + 3y &= 6 \\ 2x + 3y &= -6 \end{aligned}$$

Solution

Let's place each equation in slope-intercept form by solving each equation for y.

Solve $2x + 3y = 6$ for y:

$$2x + 3y = 6$$

$$2x + 3y - 2x = 6 - 2x$$

$$3y = 6 - 2x$$

$$\frac{3y}{3} = \frac{6 - 2x}{3}$$

$$y = -\frac{2}{3}x + 2$$

Solve $2x + 3y = -6$ for y:

$$2x + 3y = -6$$

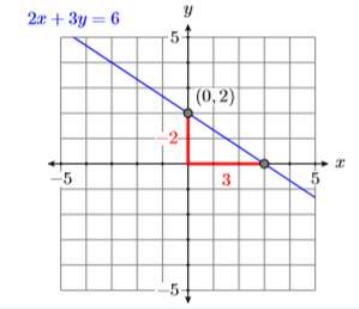
$$2x + 3y - 2x = -6 - 2x$$

$$3y = -6 - 2x$$

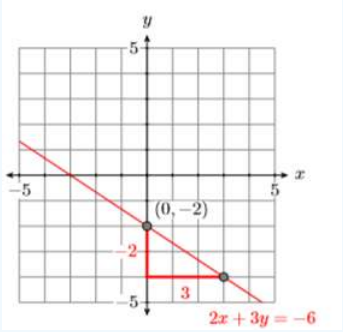
$$\frac{3y}{3} = \frac{-6 - 2x}{3}$$

$$y = -\frac{2}{3}x - 2$$

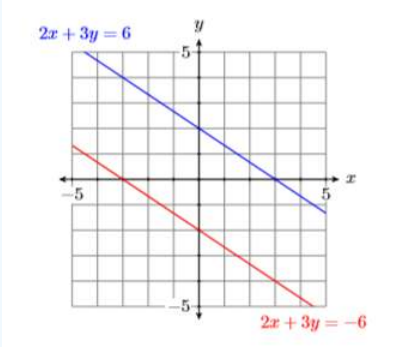
Comparing $y = (-2/3)x + 2$ with the slope-intercept form $y = mx + b$ tells us that the slope is $m = -2/3$ and the y-intercept is $(0, 2)$. Plot the intercept $(0, 2)$, then go down 2 units and right 3 units and draw the line.



Comparing $y = (-2/3)x - 2$ with the slope-intercept form $y = mx + b$ tells us that the slope is $m = -2/3$ and the y-intercept is $(0, -2)$. Plot the intercept $(0, -2)$, then go down 2 units and right 3 units and draw the line.



To find the solution of the system, draw both lines on the same coordinate system. Note how the lines appear to be parallel (they don't intersect). The fact that both lines have the same slope $-2/3$ confirms our suspicion that the lines are parallel. However, note that the lines have different y-intercepts. Hence, we are looking at two parallel but distinct lines that do not intersect. Hence, the system has no solution.



Example 10

Solve the following system of equations:

$$x - y = 3$$

$$-2x + 2y = -6$$

Solution

Let's solve both equations for y.

Solve $x - y = 3$ for y:

$$x - y = 3$$

$$x - y - x = 3 - x$$

$$-y = -x + 3$$

$$-1(-y) = -1(-x + 3)$$

$$y = x - 3$$

Solve $-2x + 2y = -6$ for y:

$$-2x + 2y = -6$$

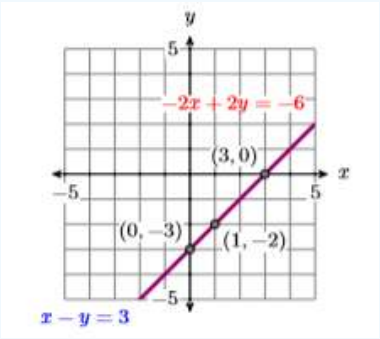
$$-2x + 2y + 2x = -6 + 2x$$

$$2y = 2x - 6$$

$$\frac{2y}{2} = \frac{2x - 6}{2}$$

$$y = x - 3$$

Both lines have slope $m = 1$, and both have the same y-intercept $(0, -3)$. Hence, the two lines are identical. Hence, the system has an infinite number of points of intersection. Any point on either line is a solution of the system. Examples of points of intersection (solutions satisfying both equations) are $(0, -3)$, $(1, -2)$, and $(3, 0)$.



Alternate solution:

A much easier approach is to note that if we divide both sides of the second equation $-2x + 2y = -6$ by -2, we get:

$$-2x + 2y = -6$$

$$\frac{-2x + 2y}{-2} = \frac{-6}{-2}$$

$$\frac{-2x}{-2} + \frac{2y}{-2} = \frac{-6}{-2}$$

$$x - y = 3$$

Hence, the second equation in the system is identical to the first. Thus, there are an infinite number of solutions. Any point on either line is a solution.

These examples lead us to the following conclusion.

Number of solutions of a linear system

When dealing with a system of two linear equations in two unknowns, there are only three possibilities:

1. There is exactly one solution.
2. There are no solutions.
3. There are an infinite number of solutions.

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1.3P: Practice

Practice Makes Progress

Exercise 1

$y = -\frac{3}{4}x + 3$

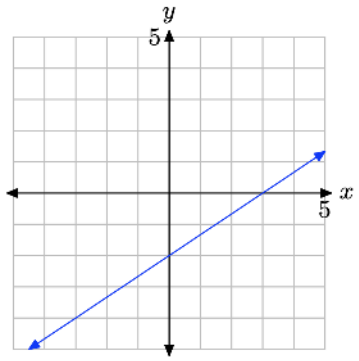
a. Identify the slope and y-intercept of the graph of the given line.

b. Determine the x-intercept of the line.

c. Sketch the graph on graph paper. Label the intercepts and at least one additional point.

Exercise 2

Use the graph below to find the y-intercept and the equation of the line in slope-intercept form.



Exercise 3

Enrique had $1,000 saved in an account, and saves an additional $25 each month.

a) Let t represent time (months), and S represent savings (dollars). Which is the independent variable?

b) Identify the S-intercept and slope.

c) Write a linear equation for Enrique's savings.

d) Predict Enrique's savings after one year.

e) When will Enrique's account reach $2,000?

f) Graph your equation.

g) Anne-Marie saves the same each month but starts with $1,200. Graph her savings on the same graph.

h) How do the graphs compare? Discuss the slopes.

i) Write a linear function for Anne-Marie's savings.

j) Explain how you know the graphs are parallel using formulas and context.

k) Will Anne-Marie reach $2,000 sooner or later than Enrique?

l) Predict how long it will take Anne-Marie to reach $2,000. Does this match part (k)?

? Exercise 4

Consider the line $2x - 3y = 15$

a. Find the x- and y- intercepts
b. Find the slope of the line
c. Sketch a graph of the line, labeling at least two points

? Exercise 5

Consider the line $4x + 5y = 15$

a. Put the equation into slope-intercept form
b. Sketch a graph of the line, labeling at least two points

? Exercise 6

Sketch the vertical line passing through $(2, -1)$. Label the equation.

? Exercise 7

Sketch the horizontal line passing through $(2, -1)$. Label the equation.

? Exercise 8

Draw the line with slope $-\frac{3}{4}$ through $(-2, 3)$. Use point-slope form, then convert to slope-intercept form.

? Exercise 9

Find the equation of the line through $(1, 9)$ with slope $-\frac{2}{3}$.

? Exercise 10

Find the equation of the line through $(-4, 6)$ and $(2, -4)$.

? Exercise 11

An automobile is traveling down the highway, and the driver applies the brakes. After 2 seconds, the speed is 60 km/h. After 4 seconds, the speed is 50 km/h.

a) On graph paper, sketch the graph of velocity v versus time t, assuming it is a ***linear*** relationship. Assume velocity is the dependent variable.

b) Determine the slope, including units. Explain its meaning.

c) Write an equation modeling the velocity as a function of time.

d) Determine when the car comes to a stop.

e) What is the meaning of the v-intercept of your graph?

For each system of equations below,
1. Graph both lines on graph paper, and determine their intersection point on the graph.
2. Put both lines into slope-intercept form, then form an equation to find their intersection algebraically.
3. Verify your graphical and algebraic solutions match.

Exercise 12

$$\begin{aligned} 3x-4y&=24 \\ y&=-\frac{1}{2}x-1 \end{aligned} \tag{1.3P.1}$$

Exercise 13

$$\begin{aligned} 2x+y&=6 \\ y&=x+3 \end{aligned} \tag{1.3P.2}$$

Exercise 14

$$\begin{aligned} 6x+3y&=12 \\ -2x-y&=4 \end{aligned} \tag{1.3P.3}$$

Exercise 15

$$\begin{aligned} -6x+2y&=-12 \\ -12x+3y&=-36 \end{aligned} \tag{1.3P.4}$$

Self-Check

In this section, you practiced:

a. Converting between point-slope, slope-intercept, and standard forms of a linear equation
b. Finding the slope and intercepts of a line given its formula
c. Finding the slope, intercepts, and equation of a line given its graph
d. Graphing a linear equation given its formula or a description
e. Using linear equations to model and analyze real-world scenarios
f. Solving a system of linear equations graphically
g. Solving a system of linear equations algebraically

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

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1.4: Linear Inequalities

Learning Objectives

- Express an interval of real numbers using set-builder notation, interval notation, and a number line
- Solve a linear inequality in one variable
- Determine whether a linear inequality has a solution or not
- Use the notation and properties of linear inequalities to investigate real-world scenarios

The Real Numbers

If we combine all rational and irrational numbers into one collection, we have a set of numbers called the set of real numbers. The set of real numbers is denoted by the symbol $\mathbb{R}$.

Every point on the number line is associated with a unique real number. Conversely, every real number is associated with a unique position on the number line. In lieu of this correspondence, the number line is commonly referred to as the real line.

Ordering the Real Numbers

The real numbers are ordered on the real line in a manner identical to how we order integers on a number line.

Order on the Real Line

Suppose that a and b are real numbers positioned on the real line as shown below.



- Because a lies to the "left of" b, we say that a is "less than" b, or in mathematical symbols, $a < b$. The inequality symbol $<$ is read "less than."
- Alternately, b lies to the "right of" a, so we can also say that b is "greater than" a, or in mathematical symbols, $b > a$. The inequality symbol $>$ is read "greater than."

Here are two more inequality symbols that we will use in this section.

Less than or equal to

If we want to say that a lies to the "left of" b, or shares the same position as b, then we say that a is "less than or equal to" b and write $a \leq b$. The inequality symbol $\leq$ is read "less than or equal to."

Greater than or equal to

If we want to say that b lies to the "right of" a, or shares the same position as a, then we say that b is "greater than or equal to" a and write $b \geq a$. The inequality symbol $\geq$ is read "greater than or equal to."

Set-Builder Notation

Mathematicians use a construct called set-builder notation to describe sets or collections of numbers. The general form of set-builder notation looks as follows:

$$\{x : \text{ some statement about } x\}$$

For example, suppose that we want to describe the set of "all real numbers that are less than 2." We could use the following notation:

$$A = \{x : x < 2\}$$

This is read aloud as follows: "A equals the set of all x such that x is less than 2." Some prefer to use a vertical bar instead of a colon.

$$A = \{x \mid x < 2\}$$

In this text we use the colon in set-builder notation, but feel free to use the vertical bar instead. They mean the same thing. One might still object that the notation

$$\{x : x < 2\}$$

is a bit vague. One objection could be "What type of numbers x are you referring to? Do you want the integers that are less than two or do you want the real numbers that are less than two?" As you can see, this is a valid objection. One way of addressing this objection is to write:

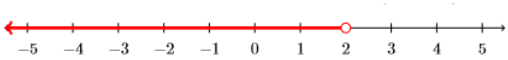
$$A = \{x \in \mathbb{R} : x < 2\} \quad \text{or} \quad A = \{x \in \mathbb{N} : x < 2\}$$

The first is read "A is the set of all x in the Real Numbers ($\mathbb{R}$) that are less than two," while the second is read "A is the set of all x in the Natural Numbers ($\mathbb{N}$) that are less than two."

Set-builder Assumption

In this text, unless there is a specific reference to the set of numbers desired, we will assume that the notation $\{x : x < 2\}$ is asking for the set of all real numbers less than 2.

Here, we've shaded the set of real numbers $\{x : x < 2\}$.



"less than" is the same as saying "left of," we've shaded (in red) all points on the real line that lie to the left of the number two. Note that there is an "empty circle" at the number two. The point representing the number two is not shaded because we were only asked to shade the numbers that are strictly less than two.

While the shading in the above is perfectly valid, much of the information provided is unnecessary (and perhaps distracting). We only need to label the endpoint and shade the real numbers to the left of two, as we've done in here.



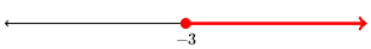
For contrast, suppose instead that we're asked to shade the set of real numbers $\{x : x \leq 2\}$. This means that we must shade all the real numbers that are "less than or equal to 2" or "left of and including 2."



Note the difference between the two figures. In the first, we're shading the set $\{x : x < 2\}$, so the number 2 is left unshaded (an empty dot). In the second, we're shading the set $\{x : x \leq 2\}$, so the number 2 is shaded (a filled-in dot).

Interval Notation

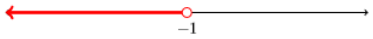
There is another mathematical symbolism, called interval notation, that can be used to describe sets of real numbers. Consider the first set of numbers $\{x : x \geq -3\}$.



Sweeping our eyes "from left to right", we use $[-3, \infty)$ to describe this set of real numbers. Some comments are in order:

1. The bracket at the left end means that -3 is included in the set.
2. As you move toward the right end of the real line, the numbers grow without bound. Hence, the ∞ symbol (positive infinity) is used to indicate that we are including all real numbers to the right of -3. However, ∞ is not really a number, so we use parentheses to indicate we are "not including" this fictional point.

The set of numbers $\{x : x < -1\}$ has the number line:



Sweeping our eyes “from left to right”, this set of real numbers is described with $(-\infty, -1)$. Again, comments are in order:

1. The number -1 is not included in this set. To indicate that it is not included, we use a parenthesis.
2. As you move toward the left end of the real line, the numbers decrease without bound. Hence, the $-\infty$ symbol (negative infinity) is used to indicate that we are including all real numbers to the left of -1. Again, $-\infty$ is not an actual number, so we use a parenthesis to indicate that we are not including this “fictional” point.

Sweep your eyes from “left to right”

If you would like to ensure that you correctly use interval notation, place the numbers in your interval notation in the same order as they are encountered as you sweep your eyes from “left to right” on the real line.

A nice summary of set-builder and interval notation is presented in a table at the end of the section.

Equivalent Inequalities

Like equations, two inequalities are equivalent if they have the same solution sets.

Adding or Subtracting the Same Quantity from Both Sides of an Inequality

Let a and b be real numbers with

$$a < b$$

If c is any real number, then

$$a + c < b + c$$

and

$$a - c < b - c$$

That is, adding or subtracting the same amount from both sides of an inequality produces an equivalent inequality (does not change the solution).

Example 1

Solve for x: $x - 2 \leq 7$. Sketch the solution on the real line, then use set-builder and interval notation to describe your solution.

Solution

To “undo” subtracting 2, we add 2 to both sides of the inequality.

$$\begin{aligned} x - 2 &\leq 7 && \text{Original inequality.} \\ x - 2 + 2 &\leq 7 + 2 && \text{Add 2 to both sides.} \\ x &\leq 9 && \text{Simplify both sides.} \end{aligned}$$

To shade the real numbers less than or equal to 9, we shade the number 9 and all real numbers to the left of 9.



Using set-builder notation, the solution is $\{x : x \leq 9\}$. Using interval notation, the solution is $(-\infty, 9]$.

If we multiply or divide both sides of an inequality by a positive number, we have an equivalent inequality.

Multiplying or Dividing by a Positive Number

Let a and b be real numbers with $a < b$. If c is a real positive number, then

$$ac < bc$$

and

$$\frac{a}{c} < \frac{b}{c}$$

Example 2

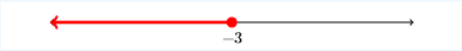
Solve for x : $3x \leq -9$ Sketch the solution on the real line, then use set-builder and interval notation to describe your solution.

Solution

To "undo" multiplying by 3, divide both sides of the inequality by 3. Because we are dividing both sides by a positive number, we do not reverse the inequality sign.

$$\begin{aligned} 3x &\leq -9 && \text{Original inequality.} \\ \frac{3x}{3} &\leq \frac{-9}{3} && \text{Divide both sides by 3.} \\ x &\leq -3 && \text{Simplify both sides.} \end{aligned}$$

Shade the real numbers less than or equal to -3.



Using set-builder notation, the solution is $\{x : x \leq -3\}$. Using interval notation, the solution is $(-\infty, -3]$.

Reversing the Inequality Sign

Up to this point, the technique for solving inequalities appears to be nearly identical to the technique used to solve equations. However, in this section, we will encounter one exception.

Suppose we start with the valid inequality $-2 < 5$, then we multiply both sides by 2, 3, and 4.

$$\begin{array}{rrr} -2 < 5 & -2 < 5 & -2 < 5 \\ 2(-2) < 2(5) & 3(-2) < 3(5) & 4(-2) < 4(5) \\ -4 < 10 & -6 < 15 & -8 < 20 \end{array}$$

Note that in each case, the resulting inequality is still valid.

Caution! We're about to make an error!

Start again with $-2 < 5$, but this time multiply both sides by -2, -3, and -4.

$$\begin{array}{rrr} -2 < 5 & -2 < 5 & -2 < 5 \\ -2(-2) < -2(5) & -3(-2) < -3(5) & -4(-2) < -4(5) \\ 4 < -10 & 6 < -15 & 8 < -20 \end{array}$$

Caution

In each of the resulting inequalities, the inequality symbol is pointing the wrong way!

When you multiply both sides of an inequality by a negative number, you must reverse the inequality sign. Starting with $-2 < 5$, multiply both sides by -2, -3, and -4, but reverse the inequality symbol.

Some readers might prefer a more formal explanation for why we reverse the inequality when we multiply both sides by a negative number. Suppose that $a < b$. Then, subtracting b from both sides gives the result $a - b < 0$. This means that $a - b$ is a negative number. Now, if c is a negative number, then the product $(a - b)c$ is positive. Then:

$$\begin{aligned}(a-b)c &> 0\\ ac-bc &> 0\\ ac-bc+bc &> 0+bc\\ ac &> bc\end{aligned}$$

Thus, if you start with $a < b$ and $c < 0$, then $ac > bc$.

Multiplying or Dividing by a Negative Number

Let a and b be real numbers with $a < b$. If c is a real negative number, then

$$ac > bc$$

and

$$\frac{a}{c} > \frac{b}{c}$$

That is, when multiplying or dividing both sides of an inequality by a negative number, you must **reverse** the inequality sign.

Example 3

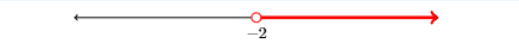
Solve for x: $-2x < 4$. Sketch the solution on the real line, then use set-builder and interval notation to describe your solution.

Solution

To "undo" multiplying by -2, divide both sides by -2. Because we are dividing both sides by a negative number, we reverse the inequality sign.

$$\begin{aligned}-2x &< 4 && \text{Original inequality.}\\ \frac{-2x}{-2} &> \frac{4}{-2} && \text{Divide both sides by } -2\\ x &> -2 && \text{Reverse the inequality sign.}\\ x &> -2 && \text{Simplify both sides.}\end{aligned}$$

Shade the real numbers greater than -2.



Using set-builder notation, the solution is $\{x : x > -2\}$. Using interval notation, the solution is $(-2, \infty)$.

Multiple Steps

Sometimes you need to perform a sequence of steps to arrive at the solution.

Example 4

Solve for x: $3 - 5x \leq 2x + 17$. Sketch the solution on the real line, then use set-builder and interval notation to describe your solution.

Solution

We need to isolate terms containing x on one side of the inequality. Start by subtracting $2x$ from both sides of the inequality.

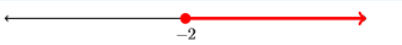
$$\begin{aligned}3 - 5x &\leq 2x + 17 && \text{Original inequality.}\\ 3 - 5x - 2x &\leq 2x + 17 - 2x && \text{Subtract } 2x \text{ from both sides.}\\ 3 - 7x &\leq 17 && \text{Simplify both sides.}\end{aligned}$$

We continue to isolate terms containing x on one side of the inequality. Subtract 3 from both sides.

$$\begin{aligned}3 - 7x - 3 &\leq 17 - 3 && \text{Subtract 3 from both sides.}\\ -7x &\leq 14 && \text{Simplify both sides.}\end{aligned}$$

To "undo" multiplying by -7, divide both sides by -7. Because we are dividing both sides by a negative number, we reverse the inequality sign.

$$\begin{aligned} \frac{-7x}{-7} &\geq \frac{14}{-7} && \text{Multiply both sides by } -7\text{. Reverse the inequality sign.} \\ x &\geq -2 && \text{Simplify both sides.} \end{aligned}$$



Using set-builder notation, the solution is $\{x : x \geq -2\}$. Using interval notation, the solution is $[-2, \infty)$.

We can also clear fractions from an inequality by multiplying both sides by the least common denominator.

Example 5

Solve for x: $\quad \frac{3}{4} - \frac{x}{12} > \frac{1}{3}$.

Solution

First, clear the fractions from the inequality by multiplying both sides by the least common denominator, which in this case is 12.

$$\begin{aligned} \frac{3}{4} - \frac{x}{12} &> \frac{1}{3} && \text{Original inequality.} \\ 12\left[\frac{3}{4} - \frac{x}{12}\right] &> \left[\frac{1}{3}\right]12 && \text{Multiply both sides by 12.} \\ 12\left[\frac{3}{4}\right] - 12\left[\frac{x}{12}\right] &> \left[\frac{1}{3}\right]12 && \text{Distribute the 12.} \\ 9 - x &> 4 && \text{Cancel and Multiply.} \end{aligned}$$

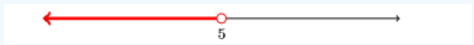
To "undo" adding 9, subtract 9 from both sides.

$$\begin{aligned} 9 - x - 9 &> 4 - 9 && \text{Subtract 9 from both sides.} \\ -x &> -5 && \text{Simplify both sides.} \end{aligned}$$

We could divide both sides by -1, but multiplying both sides by -1 will also do the job. Because we are multiplying both sides by a negative number, we reverse the inequality sign.

$$\begin{aligned} (-1)(-x) &< (-5)(-1) && \text{Multiply both sides by } -1\text{. Reverse the inequality sign.} \\ x &< 5 && \text{Simplify both sides.} \end{aligned}$$

Shade the real numbers less than 5.



Using set-builder notation, the solution is $\{x : x < 5\}$. Using interval notation, the solution is $(-\infty, 5)$.

We clear decimals from an inequality in the usual manner, by multiplying both sides by the appropriate power of ten.

Example 6

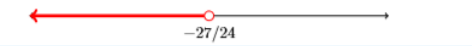
Solve for x: $\quad 3.25 - 1.2x > 4.6$.

Solution

First, clear the decimals from the inequality by multiplying both sides by 100, which moves each decimal point two places to the right.

$$
\begin{aligned}
3.25 - 1.2x &> 4.6 && \text{Original inequality.} \\
325 - 120x &> 460 && \text{Multiply both sides by 100.} \\
325 - 120x - 325 &> 460 - 325 && \text{Subtract 325 from both sides.} \\
-120x &> 135 && \text{Simplify both sides.} \\
\frac{-120x}{-120} &< \frac{135}{-120} && \text{Divide both sides by } -120\text{. Reverse the inequality sign.} \\
x &< -\frac{27}{24} && \text{Reduce to lowest terms.}
\end{aligned}
$$

Shade the real numbers less than $-27/24$.



Using set-builder notation, the solution is $\{x : x < -27/24\}$. Using interval notation, the solution is $(-\infty, -27/24)$.

Just like some equations are identities with infinite solutions and some are contradictions with no solutions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

Example 7

Solve the inequality $8x - 2(5 - x) < 4(x + 9) + 6x$, graph the solution on the number line, and write the solution in interval notation.

Solution

Simplify each side as much as possible.	8x−2(5−x)<4(x+9)+6x
Distribute.	8x−10+2x<4x+36+6x
Combine like terms.	10x−10<10x+36
Subtract 10x from both sides to collect the variables on the left.	10x−10−10x<10x+36−10x
Simplify.	−10<36
The x's are gone, and we have a true statement.	The inequality is an identity. The solution is all real numbers.
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, \infty)$

Example 8

Solve the inequality $\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$, graph the solution on the number line, and write the solution in interval notation.

Solution

Multiply both sides by the LCD, 24, to clear the fractions.	$24\left(\frac{1}{3}a-\frac{1}{8}a\right)>24\left(\frac{5}{24}a+\frac{3}{4}\right)$
Simplify.	$8a-3a>5a+18$
Combine like terms.	$5a>5a+18$
Subtract 5a from both sides to collect the variables on the left.	$5a-5a>5a-5a+18$
Simplify.	$0>18$
The statement is false!	The inequality is a contradiction.
	There is no solution.
Graph the solution on the number line.	
Write the solution in interval notation.	There is no solution.

Translate to an Inequality and Solve

To translate English sentences into inequalities, we must recognize the phrases indicating inequality. Some words are easy, like 'more than' and 'less than', but others are not as obvious.

Think about the phrase 'at least' – what does it mean to be 'at least 21 years old'? It means 21 or more. The phrase 'at least' is the same as 'greater than or equal to'.

This table shows some common phrases that indicate inequalities.

>	≥	<	≤
is greater than	is greater than or equal to	is less than	is less than or equal to
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

Solve Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. The method we will use to solve applications involving linear inequalities is very similar to the one we used when solving applications with equations.

We will read the problem and make sure all the words are understood. Next, we will identify what we are looking for and assign a variable to represent it. We will restate the problem in a single sentence to facilitate its translation into an inequality. Then, we will solve the inequality.

Sometimes an application requires the solution to be a whole number, but the algebraic solution to the inequality is not a whole number. In that case, we must round the algebraic solution to a whole number. The context of the application will determine whether we round up or down.

Example 9

Dawn won a mini-grant of $4,000 to buy tablet computers for her classroom. The tablets she would like to buy cost $254.12 each, including tax and delivery. What is the maximum number of tablets Dawn can buy?

Solution

Step 1. Read the problem.	

Step 2. Identify what you are looking for.	the maximum number of tablets Dawn can buy
Step 3. Name what you are looking for. Choose a variable to represent that quantity.	Let n = the number of tablets.
Step 4. Translate Write a sentence that gives the information to find it.	$254.12 times the number of tablets is no more than $4,000.
Translate into an inequality.	$254.12n \leq 4000$
Step 5. Solve the inequality.	$n \leq 15.74$ but n must be a whole number of tablets, so round down to 15 since 16 would cost too much. $n \leq 15$ tablets
Step 6. Check the answer in the problem and make sure it makes sense.	Rounding down the price to $250, 15 tablets would cost $3,750, while 16 tablets would be $4,000. So a maximum of 15 tablets at $254.12 seems reasonable.
Step 7. Write a sentence that answers the question.	Dawn can buy a maximum of 15 tablets.

Example 10

Felicity has a calligraphy business. She charges $2.50 per wedding invitation. Her monthly expenses are $650. How many invitations must she write to earn a profit of at least $2,800 per month?

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the number of invitations Felicity needs to write
Step 3. Name what you are looking for. Choose a variable to represent that quantity.	Let j = the number of invitations.
Step 4. Translate Write a sentence that gives the information to find it.	$2.50 times the number of invitations minus $650 is at least $2,800.
Translate into an inequality.	$2.50j - 650 \geq 2,800$
Step 5. Solve the inequality.	$2.5j \geq 3,450$ $j \geq 1,380$ invitations
Step 6. Check the answer in the problem and make sure it makes sense.	If Felicity wrote 1400 invitations, her profit would be $2.50(1400) - 650$, or $2,850. This is more than $2800.
Step 7. Write a sentence that answers the question.	Felicity must write at least 1,380 invitations.

In many situations, several quantities contribute to the total expense. When we solve problems like this, we must account for all the individual expenses.

Example 11

Malik is planning a six-day summer vacation trip. He has $840 in savings, and he earns $45 per hour for tutoring. The trip will cost him $525 for airfare, $780 for food and sightseeing, and $95 per night for the hotel. How many hours must he tutor to have enough money to pay for the trip?

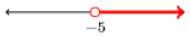
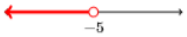
Solution

Step 1. Read the problem.	

Step 2. Identify what you are looking for.	the number of hours Malik must tutor
Step 3. Name what you are looking for. Choose a variable to represent that quantity.	Let h = the number of hours.
Step 4. Translate Write a sentence that gives the information to find it.	The expenses must be less than or equal to the income. The cost of airfare plus the cost of food and sightseeing and the hotel bill must be less than the savings plus the amount earned tutoring.
Translate into an inequality.	$525 + 780 + 95(6) \leq 840 + 45h$
Step 5. Solve the inequality.	$1,875 \leq 840 + 45h$ $1,035 \leq 45h$ $h \geq 23$ hours
Step 6. Check the answer in the problem and make sure it makes sense.	We substitute 23 into the inequality. $1,875 \leq 840 + 45h$ $1,875 \leq 840 + 45(23)$ $1,875 \leq 1875$
Step 7. Write a sentence that answers the question.	Malik must tutor at least 23 hours.

Summary Table of Set-Builder and Interval Notation

Here is a summary table of the set-builder and interval notation.

Shading on the real line	Set-builder	Interval
	$\{x : x > -5\}$	$(-5, \infty)$
	$\{x : x \geq -5\}$	$[-5, \infty)$
	$\{x : x < -5\}$	$(-\infty, -5)$
	$\{x : x \leq -5\}$	$(-\infty, -5]$

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1.4P: Practice

Practice Makes Progress

Exercise 1

Draw a number line, then plot the numbers $4, 3, -4, \frac{7}{8}, -\frac{8}{3}$ on your number line. Label each point with its value. Finally, list the numbers in order, from smallest to largest.

Solve each inequality algebraically, and write your solution in both interval notation and set-builder notation.

Exercise 2

$x + 10 < 19$

Exercise 3

$4x < 8$

Exercise 4

$-2y \leq -2$

Exercise 5

$-5n - 6 \geq 4 - 9n$

Exercise 6

$15x + 5 < 6x + 2$

Exercise 7

$\frac{3}{2}x > \frac{9}{8}$

Exercise 8

$x + \frac{3}{2} < \frac{9}{5}$

Exercise 9

$\frac{4}{7} - \frac{1}{6}x \leq \frac{4}{3}x - \frac{1}{2}$

Exercise 10

$\frac{9}{7}y + \frac{9}{2} > \frac{1}{7}y + \frac{7}{2}$

Exercise 11

$6n - 12(3 - n) \leq 9(n - 4) + 9n$

Exercise 12

$\frac{4}{5}h - \frac{2}{3}(h-9) > \frac{1}{15}(2h+90)$

For each application problem,
1. Define variables and write an inequality that represents the scenario
2. Solve the inequality algebraically
3. State the solution in the context of the problem.

Exercise 13

Kimuyen needs to earn $4,150 per month to cover all her expenses. Her job pays her $3,475 per month plus 4% of her total sales. What is the minimum Kimuyen's total sales must be for her to pay all her expenses?

Exercise 14

Nataly is considering two job offers. The first job would pay her $83,000 per year. The second would pay her $66,500 plus 15% of her total sales. What would her total sales need to be for her salary on the second offer to be higher than the first?

Exercise 15

Kellen wants to rent a banquet room in a restaurant for her cousin's baby shower. The restaurant charges $350 for the banquet room plus $32.50 per person for lunch. How many people can Kellen have at the shower if she wants the maximum cost to be $1,500?

Self-Check

In this section, you practiced:

a. Writing inequalities in interval notation
b. Writing inequalities in set-builder notation
c. Sketching inequalities on a number line
d. Solving linear inequalities
e. Writing an inequality to model a scenario
f. Interpreting the solution to an inequality in context

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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1.5: Two Variable Linear Inequalities and Systems

Learning Objectives

- Determine whether an ordered pair is a solution to a linear inequality in two variables
- Sketch the set of all solutions to a linear inequality in two variables, including correct shading and boundary line
- Sketch the set of all solutions ot a system of linear inequalities, including correct shading and boundary lines
- Explain the similarities and differences between a linear inequality in one variable, a linear inequality in two variables, and a system of linear inequalities
- Use the notation and techniques of systems of linear inequalities to find solutions to real-world problems

Previously, we graphed inequalities in one variable, but now we learn to graph inequalities in two variables. Although this section may seem similar to linear equations in two variables, linear inequalities in two variables have many applications. For example, business owners want to know when revenue is greater than cost so that their business makes a profit, e.g., revenue $>$ cost.

Verifying Solutions

Solution to a Linear Inequality in Two Variables

An ordered pair (x, y) is a solution to a linear inequality in two variables, $ax + by < c$, if the ordered pair (x, y) makes the inequality true, where the the same is for $>$, $\leq$, $\geq$ and a, $b \neq 0$.

Example 1

Verify whether each ordered pair is a solution to the inequality $y > x + 4$.

a. $(0, 0)$
b. $(1, 6)$

Solution

We substitute the ordered pairs into the inequality and determine if the results are true.

a. Let's substitute $(0, 0)$ into the inequality and determine if the left side is greater than the right side.

$$\begin{aligned} y &\overset{?}{>} x + 4 && \text{Substitute } x = 0 \text{ and } y = 0 \\ 0 &\overset{?}{>} 0 + 4 && \text{Simplify} \\ 0 &\not> 4 && \text{False} \end{aligned}$$

Hence, $(0, 0)$ is not a solution to the inequality $y > x + 4$.

b. Let's substitute $(1, 6)$ into the inequality and determine if the left side is greater than the right side.

$$\begin{aligned} y &\overset{?}{>} x + 4 && \text{Substitute } x = 1 \text{ and } y = 6 \\ 6 &\overset{?}{>} 1 + 4 && \text{Simplify} \\ 6 &> 5 && \checkmark \text{ True} \end{aligned}$$

Hence, $(1, 6)$ is a solution to the inequality $y > x + 4$.

Boundary Lines

If we are given a linear inequality, $ax + by < c$, we can see that not all ordered pairs are a solution, only some. Why? Well, notice that $(0, 0)$ is below the line $y = x + 4$ and $(1, 6)$ is above the line $y = x + 4$. This implies that ordered pairs in certain regions are solutions to the inequality $y > x + 4$. Hence, the line $y = x + 4$ is critical when finding solutions to the inequality. We call the line $y = x + 4$ a ***boundary line***, a line that separates the ordered pairs that are solutions and the ordered pairs that are not solutions of the linear inequality in two variables $y > x + 4$.

Definition: Boundary Line

A linear equation in two variables $ax + by = c$ is called the ***boundary line***, the line that separates the region where $ax + by > c$ from the region where $ax + by < c$.

Note

Since there are four inequality symbols: $>$, $<$, $\geq$, $\leq$, then we have linear inequalities in two variables that include the boundary, e.g., inequalities with $\leq$ and $\geq$, and linear inequalities in two variables that exclude the boundary, e.g., inequalities with $<$ and $>$.

We can use the table below to help identify the boundary line, determine whether to include it, and visualize its graphical representation.

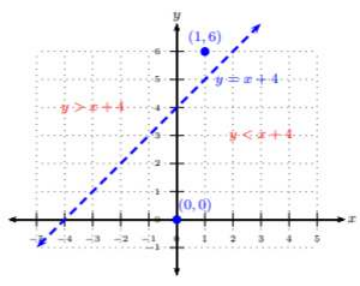
Case 1.	Case 2.
$ax + by < c$	$ax + by \leq c$
$ax + by > c$	$ax + by \geq c$
Boundary line: $ax + by = c$	Boundary line: $ax + by = c$
Boundary line is excluded in solution	Boundary line is included in solution
Boundary line is dashed	Boundary line is solid

Example 2

Let's revisit the first example and graph the boundary line, as well as the points $(0, 0)$ and $(1, 6)$.

Solution

Since we have $y > x + 4$, we can see from the table above that we have Case 1, and the boundary line is excluded. We represent this by graphing the line $y = x + 4$ as a dashed line.



We can see from the graph that the point $(0, 0)$ lies below the boundary line $y = x + 4$ and point $(1, 6)$ lies above the boundary line. Recall, point $(1, 6)$ was verified as a solution of $y > x + 4$. Furthermore, any ordered pair that lies above $y = x + 4$ will verify as a solution, i.e., making the inequality true. We typically represent this area by *shading the region* where the set of ordered pairs satisfies the inequality.

Graphing Linear Inequalities

Steps for Graphing Linear Inequalities in Two Variables

Given a linear inequality in two variables, $ax + by < c$, we can use the steps below to graph $ax + by < c$, where the the same process is applied for $>$, $\leq$, $\geq$ and $a,\ b \neq 0$.

Step 1. Rewrite the inequality in slope-intercept form, i.e., $y = mx + b$.

Step 2. Graph the boundary line according to the two cases:

Case 1. If the inequality is $<$ or $>$, then the boundary line is dashed.

Case 2. If the inequality is $\geq$ or $\leq$, then the boundary line is solid.

Step 3. Select a test point that is *not on the boundary line*. **Ask:** *Does this ordered pair make the inequality true?*

Step 4. If the ordered pair is

- a solution to the inequality, i.e., makes the inequality true, then shade the side that includes the ordered pair.
- not a solution, then shade the opposite side of the boundary line.

Note

If we choose a test point on the boundary line, we will obtain an identity, where both sides of the inequality symbol are the same number. Hence, it is crucial to select a point that is **not** on the boundary line.

Example 3

Graph the inequality:

$$2x - y > 3.$$

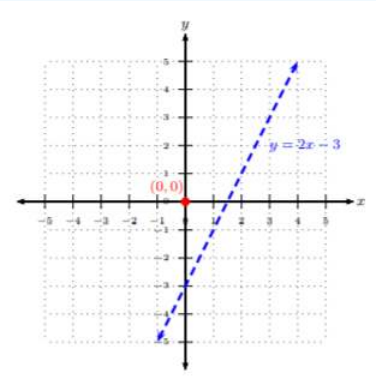
Solution

Let's follow the steps given above to graph the inequality.

Step 1. Rewrite the inequality in slope-intercept form, i.e., $y = mx + b$.

$$\begin{aligned} 2x - y &> 3 \\ -y &> -2x + 3 \\ y &< 2x - 3 \end{aligned} \tag{1.5.1}$$

Step 2. Graph the boundary line according to the two cases. Since the given inequality is $<$, then we have Case 1.



Step 3. Select a test point that is *not on the boundary line*. **Ask:** *Does this ordered pair make the inequality true?*
Let's pick the test point $(0, 0)$ as it is a great choice!

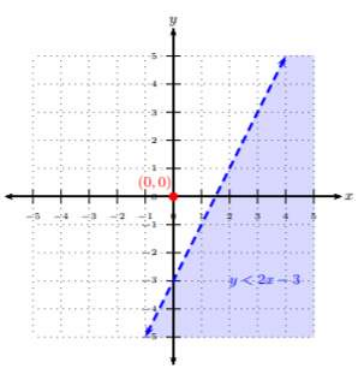
$$\begin{aligned} y &\overset{?}{<} 2x - 3 \\ 0 &\overset{?}{<} 2(0) - 3 \\ 0 &\not< -3 \end{aligned} \tag{1.5.2}$$

Hence, $(0, 0)$ doesn't make the inequality true.

Step 4. If the ordered pair is

- a solution to the inequality, i.e., makes the inequality true, then shade the side that includes the ordered pair.
- not a solution, then shade the opposite side of the boundary line.

Since the ordered pair $(0,0)$ is not a solution to the inequality, we shade on the opposite side of the boundary line from the location of the ordered pair.



Note

Another way of graphing linear inequalities in two variables is to complete **Step 1.** and **Step 2.**, but instead of taking a test point in **Step 3.**, we can observe the inequality symbols. If the inequality has $y <$ or $y \leq$, then we shade below the boundary line, i.e., below the y-intercept. Similarly, if the inequality has $y >$ or $y \geq$, then we shade above the boundary line, i.e., above the y-intercept.

Example 4

Graph the inequality:

$$3x + 2y \geq -6.$$

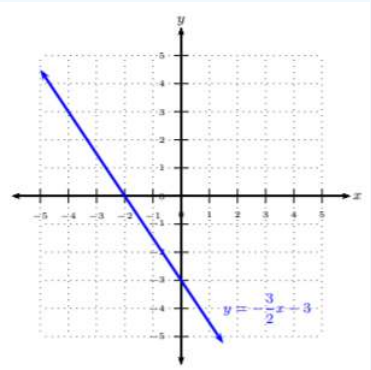
Solution

Let's follow the steps given above to graph the inequality, but try skipping **Step 3.** and **Step 4.**

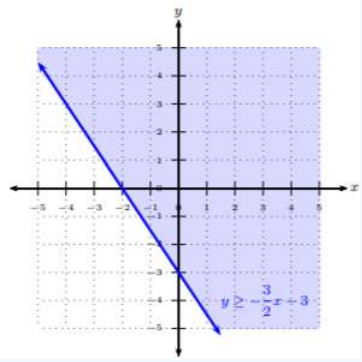
Step 1. Rewrite the inequality in slope-intercept form, i.e., $y = mx + b$.

$$\begin{aligned} 3x + 2y &\geq -6 \\ 2y &\geq -3x - 6 \\ y &\geq -\frac{3}{2}x - 3 \end{aligned} \tag{1.5.3}$$

Step 2. Graph the boundary line according to the two cases. Since the given inequality is $\geq$, then the boundary line is solid.



Since this inequality is $y \geq$, where all ordered pairs above the boundary line are solutions to the inequality, we can easily shade above the y-intercept:



Determine whether an ordered pair is a solution of a system of linear inequalities

The definition of a **system of linear inequalities** is similar to that of a system of linear equations.

Definition: System of Linear Inequalities

Two or more linear inequalities grouped together form a system of linear inequalities.

A system of linear inequalities resembles a system of linear equations, but it contains inequalities instead of equations. A system of two linear inequalities is shown here.

$$\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs (x, y) that make both inequalities true.

Solutions of a System of Linear Inequalities

Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true.

The solution of a system of linear inequalities is represented as a shaded region in the xy-coordinate system, encompassing all points whose ordered pairs satisfy the inequalities.

To determine if an ordered pair is a solution to a system of two inequalities, we substitute the values of the variables into each inequality. If the ordered pair satisfies both inequalities, it is a solution to the system.

Example 5

Determine whether the ordered pair is a solution to the system $\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$

a. $(-2, 4)$ b. $(3, 1)$

Solution:

a. Is the ordered pair $(-2, 4)$ a solution?

We substitute $x = -2$ and $y = 4$ into both inequalities.

$$\begin{array}{ll} x + 4y \geq 10 & 3x - 2y < 12 \\ -2 + 4(4) \stackrel{?}{\geq} 10 & 3(-2) - 2(4) \stackrel{?}{<} 12 \\ 14 \geq 10 \text{ true} & -14 < 12 \text{ true} \end{array}$$

The ordered pair $(-2, 4)$ made both inequalities true. Therefore $(-2, 4)$ is a solution to this system.

b. Is the ordered pair $(3, 1)$ a solution?

We substitute $x = 3$ and $y = 1$ into both inequalities.

$x + 4y \geq 10$	$3x - 2y < 12$
$3 + 4(1) \stackrel{?}{\geq} 10$	$3(3) - 2(1) \stackrel{?}{<} 12$
$7 \geq 10$ false	$7 < 12$ true

The ordered pair $(3, 1)$ made one inequality true, but the other one false. Therefore $(3, 1)$ is not a solution to this system.

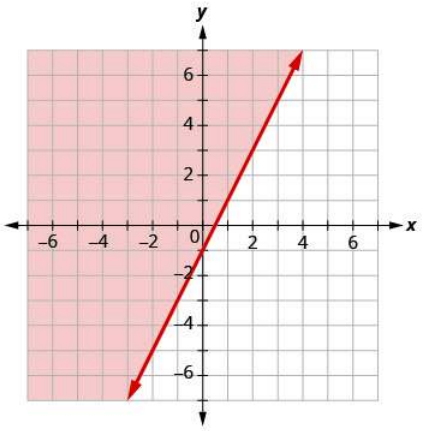
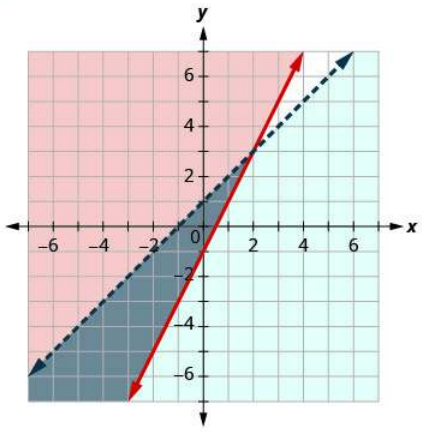
Solve a System of Linear Inequalities by Graphing

The solution to a single linear inequality is the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this region, we will graph each inequality separately and then identify the region where both inequalities are true. The solution is always shown as a graph.

Example 6

Solve the system by graphing: $\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$

Solution:

Step 1. Graph the first inequality. • Graph the boundary line. • Shade in the side of the boundary line where the inequality is true.	We will graph $y \geq 2x - 1$. We graph the line $y = 2x - 1$. It is a solid line because the inequality sign is $\geq$. We choose (0, 0) as a test point. It is a solution to $y \geq 2x - 1$, so we shade in above the boundary line.	$\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$ 
Step 2. On the same grid, graph the second inequality. • Graph the boundary line. • Shade in the side of that boundary line where the inequality is true.	We will graph $y < x + 1$ on the same grid. We graph the line $y = x + 1$. It is a dashed line because the inequality sign is $<$. Again, we use (0, 0) as a test point. It is a solution so we shade in that side of the line $y = x + 1$.	

Step 3. The solution is the region where the shading overlaps.	The point where the boundary lines intersect is not a solution because it is not a solution to $y < x + 1$.	The solution is all points in the area shaded twice—which appears as the darkest shaded region.
Step 4. Check by choosing a test point.	We'll use (–1, –1) as a test point.	Is (–1, –1) a solution to $y \geq 2x - 1$? $-1 \stackrel{?}{\geq} 2(-1) - 1$ $-1 \geq -3$ true Is (–1, –1) a solution to $y < x + 1$? $-1 \stackrel{?}{<} -1 + 1$ $-1 < 0$ true The region containing (–1, –1) is the solution to this system.

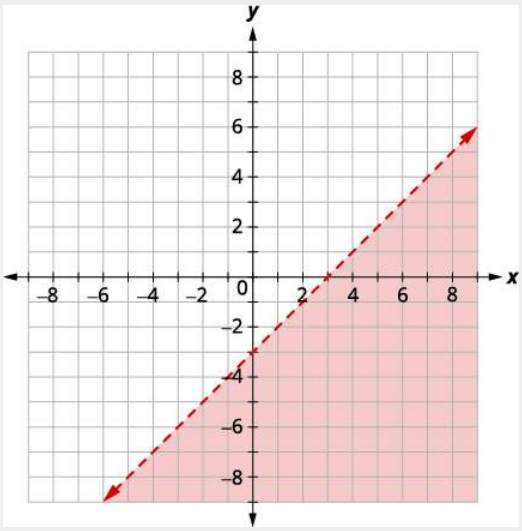
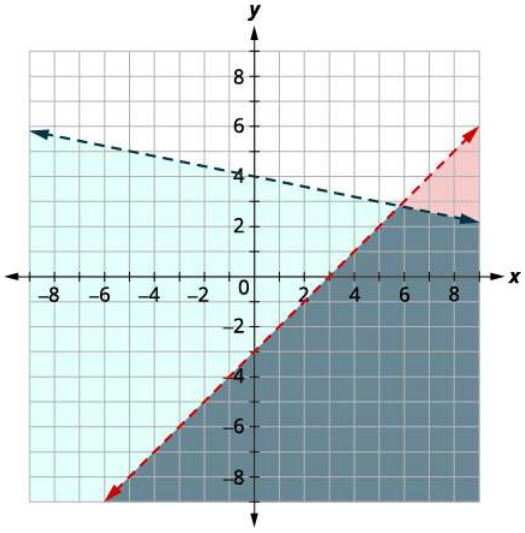
Solve a System of Linear Inequalities by Graphing

1. Graph the first inequality.
   - Graph the boundary line.
   - Shade the side of the boundary line where the inequality is true.
2. On the same grid, graph the second inequality.
   - Graph the boundary line.
   - Shade in the side of that boundary line where the inequality is true.
3. The solution is the region where the shading overlaps.
4. Check by choosing a test point.

Example 7

Solve the system by graphing: $\begin{cases} x - y > 3 \\ y < -\frac{1}{5}x + 4 \end{cases}$

Solution:

	$\begin{cases} x-y>3 \\ y<-\frac{1}{5}x+4 \end{cases}$
Graph $x-y>3$, by graphing $x-y=3$ and testing a point. The intercepts are $x=3$ and $y=-3$ and the boundary line will be dashed. Test $(0,0)$ which makes the inequality false so shade (red) the side that does not contain $(0,0)$.	
Graph $y<-\frac{1}{5}x+4$ by graphing $y=-\frac{1}{5}x+4$ using the slope $m=-\frac{1}{5}$ and y-intercept $b=4$. The boundary line will be dashed Test $(0,0)$ which makes the inequality true, so shade (blue) the side that contains $(0,0)$.	

The point of intersection of the two lines is not included, as both boundary lines were dashed. The solution is the area shaded twice, which appears as the darkest shaded region.

Example 8

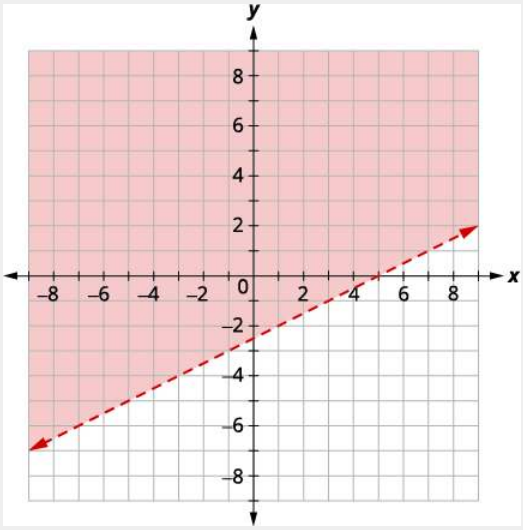
Solve the system by graphing: $\begin{cases} x-2y<5 \\ y>-4 \end{cases}$

Solution:

$\begin{cases} x-2y<5 \\ y>-4 \end{cases}$

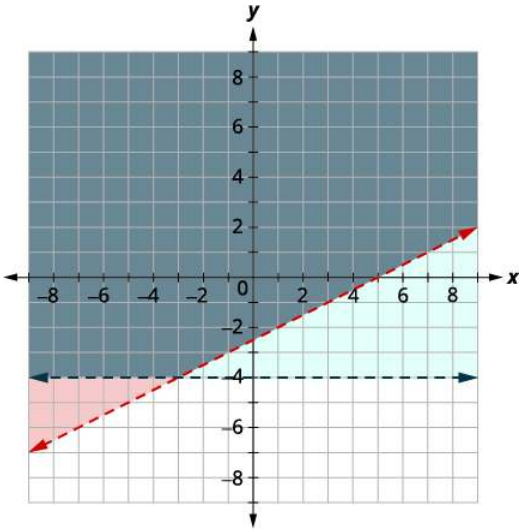
Graph $x-2y<5$, by graphing $x-2y=5$ and testing a point. The intercepts are $x=5$ and $y=-2.5$ and the boundary line will be dashed.

Test $(0,0)$ which makes the inequality true, so shade (red) the side that contains $(0,0)$.



Graph $y>-4$, by graphing $y=-4$ and recognizing that it is a horizontal line through $y=-4$. The boundary line will be dashed.

Test $(0,0)$ which makes the inequality true so shade (blue) the side that contains $(0,0)$.



The point $(0,0)$ is in the solution, and we have already found it to be a solution of each inequality. The point of intersection of the two lines is not included, as both boundary lines were dashed.

The solution is the area shaded twice, which appears as the darkest shaded region.

Systems of linear inequalities where the boundary lines are parallel might have no solution. We'll see this in the next example.

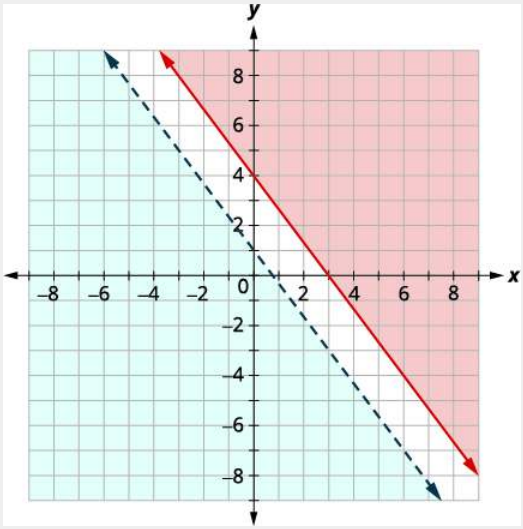
Example 9

Solve the system by graphing: $\begin{cases} 4x+3y\geq 12 \\ y<-\frac{4}{3}x+1 \end{cases}$

Solution:

$\begin{cases} 4x+3y \geq 12 \\ y < -\frac{4}{3}x+1 \end{cases}$

1. Graph $4x+3y \geq 12$, by graphing $4x+3y=12$ and testing a point. The intercepts are $x=3$ and $y=4$, and the boundary line will be solid. Test $(0,0)$ which makes the inequality false, so shade (red) the side that does not contain $(0,0)$.
2. Graph $y < -\frac{4}{3}x+1$ by graphing $y=-\frac{4}{3}x+1$ using the slope $m=-\frac{4}{3}$ and y-intercept $b=1$. The boundary line will be dashed. Test $(0,0)$ which makes the inequality true, so shade (blue) the side that contains $(0,0)$.



There is no point of intersection between the two shaded regions, so the system has no solution.

Some systems of linear inequalities with parallel boundary lines will have a solution. We'll see this in the next example.

Example 10

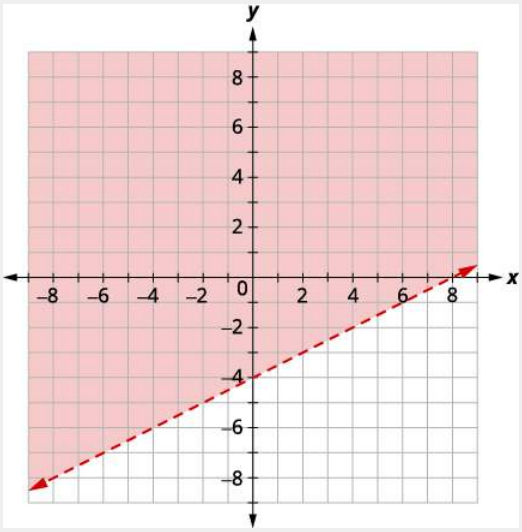
Solve the system by graphing: $\begin{cases} y > \frac{1}{2}x-4 \\ x-2y < -4 \end{cases}$

Solution:

$\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$

Graph $y > \frac{1}{2}x - 4$ by graphing $y = \frac{1}{2}x - 4$ using the slope $m = \frac{1}{2}$ and the intercept $b = -4$. The boundary line will be dashed.

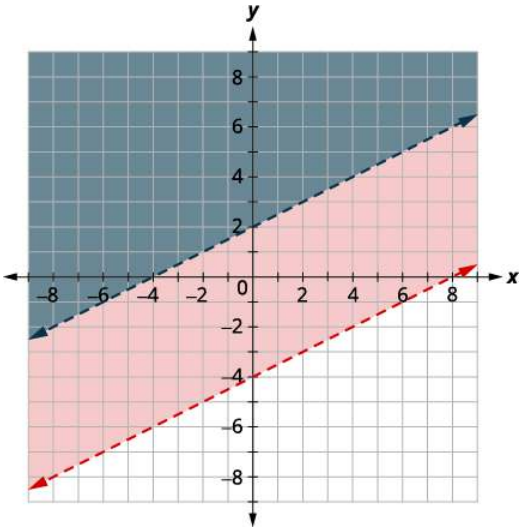
Test $(0, 0)$ which makes the inequality true, so shade (red) the side that contains $(0, 0)$.



Graph $x - 2y < -4$ by graphing $x - 2y = -4$ and testing a point. The intercepts are $x = -4$ and $y = 2$ and the boundary line will be dashed.

Choose a test point in the solution and verify that it is a solution to both inequalties.

Test $(0, 0)$ which makes the inequality false, so shade (blue) the side that does not contain $(0, 0)$.



No point on the boundary lines is included in the solution as both lines are dashed.

The solution is the region that is shaded twice, which is also the solution to $x - 2y < -4$.

Solve Applications of Systems of Inequalities

To solve applications of systems of inequalities, we translate each condition into an inequality. Then, we graph the system, as we did above, to see the region that contains the solutions. Many situations will be realistic only if both variables are positive, so we add inequalities to the system as additional requirements.

Example 11

Christy sells her photographs at a street fair. At the start of the day, she wants to have at least 25 photos to display at her booth. Each small photo she displays costs her $4, and each large photo costs her $10. She doesn't want to spend more than $200 on photos to display.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could she display 10 small and 20 large photos?
d. Could she display 20 large and 10 small photos?

Solution:

a.

Let $x =$ the number of small photos.
$y =$ the number of large photos

To find the system of equations translate the information.

She wants to have at least 25 photos.
The number of small plus the number of large should be at least 25.

$$x + y \geq 25$$

\$4 for each small and \$10 for each large must be no more than \$200

$$4x + 10y \leq 200$$

The number of small photos must be greater than or equal to 0.

$$x \geq 0$$

The number of large photos must be greater than or equal to 0.

$$y \geq 0$$

We have our system of equations.

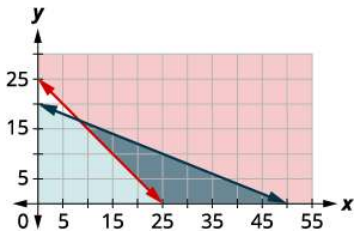
$$\begin{cases} x + y \geq 25 \\ 4x + 10y \leq 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

b.

Since $x \geq 0$ and $y \geq 0$ (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $x + y \geq 25$, graph $x + y = 25$ as a solid line. Choose $(0, 0)$ as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point $(0, 0)$.

To graph $4x + 10y \leq 200$, graph $4x + 10y = 200$ as a solid line. Choose $(0, 0)$ as a test point. Since it does make the inequality true, shade (blue) the side that include the point $(0, 0)$.



The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly-shaded section are included in the solution, as are the points on the x-axis from $(25, 0)$ to $(55, 0)$.

c. To determine if 10 small and 20 large photos would work, we look at the graph to see if the point $(10, 20)$ is in the solution region. We could also test the point to see if it is a solution of both equations.

It is not, Christy would not display 10 small and 20 large photos.

d. To determine if 20 small and 10 large photos would work, we look at the graph to see if the point $(20, 10)$ is in the solution region. We could also test the point to see if it is a solution to both equations.

It is, so Christy could choose to display 20 small and 10 large photos.

Notice that we could also test the possible solutions by substituting the values into each inequality.

When we use variables other than x and y to define an unknown quantity, we must change the names of the axes of the graph as well.

Example 12

Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers and cookies, and he doesn't want to spend more than \$5. At the convenience store near his college, each hamburger has 240 calories and costs \$1.40. Each cookie has 160 calories and costs \$0.50.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could he eat 4 cookies and 1 hamburger?
d. Could he eat 1 cookie and 4 hamburgers?

Solution:

a.

Let $c =$ the number of cookies
$h =$ the number of hamburgers.

To find the system of equations, translate the information.

The calories from hamburgers at 240 calories each, plus the calories from cookies at 160 calories each must be more that 800.

$$240h + 160c \geq 800$$

The amount spent on hamburgers at \$1.40 each, plus the amount spent on cookies at \$0.50 each must be no more than \$5.00.

$$1.40h + 0.50c \leq 5$$

The number of hamburgers must be greater than or equal to 0.

$$h \geq 0$$

The number of cookies must be greater than or equal to 0.

$$c \geq 0$$

We have our system of equations.

$$\begin{cases} 240h + 160c \geq 800 \\ 1.40h + 0.50c \leq 5 \\ h \geq 0 \\ c \geq 0 \end{cases}$$

b.

Since $h \geq 0$ and $c \geq 0$ (both are greater than or equal to), all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $240h + 160c \geq 800$, graph $240h + 160c = 800$ as a solid line.

Choose $(0, 0)$ as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point $(0, 0)$.



Graph $1.40h + 0.50c \leq 5$. The boundary line is $1.40h + 0.50c = 5$. We test $(0, 0)$ and it makes the inequality true. We shade the side of the line that includes $(0, 0)$.

The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly shaded section are included in the solution as are the points on the x-axis from $(5, 0)$ to $(10, 0)$.

c. To determine if 4 cookies and 1 hamburger would meet Omar's criteria, we see if the point $(4, 1)$ is in the solution region. It is, so Omar might choose this option.

d. To determine if 1 cookie and 4 hamburgers would meet Omar's criteria, we see if the point $(1, 4)$ is in the solution region. It is not, so Omar should not choose this option.

We could also test the possible solutions by substituting the values into each inequality.

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1.5P: Practice

Practice Makes Progress

Determine if the given points are a solution to the two-variable inequality.

Exercise 1

$x + 2y \geq -4$

a. (0, −4)
b. (1, 1)

Exercise 2

$2x - y \leq 2$

a. (1, 5)
b. (3, 1)

Graph the two-variable inequality.

Exercise 3

$2x - y \leq 2$

Exercise 4

$x > 4y - 8$

Exercise 5

$6x + 8y \leq 24$

Determine if the given points are a solution to the system of two-variable inequalities.

Exercise 6

$$\begin{cases} 4x - y < 10 \\ -2x + 2y > -8 \end{cases}$$

a. (5, −2)
b. (−1, 3)

Exercise 7

$$\begin{cases} y < \frac{3}{2}x + 3 \\ \frac{3}{4}x - 2y < 5 \end{cases}$$

a. (−4, −1)
b. (8, 3)

Graph the solution to the system of two-variable inequalities.

Exercise 8

$$\begin{cases} y < -2x + 2 \\ y \geq -x - 1 \end{cases}$$

Exercise 9

$$\begin{cases} 2x + 4y \geq 8 \\ y \leq \frac{3}{4}x \end{cases}$$

Exercise 10

$$\begin{cases} 2x + y > -6 \\ -x + 2y \geq -4 \end{cases}$$

Exercise 11

$$\begin{cases} y \leq -\frac{2}{3}x + 5 \\ x \geq 3 \end{cases}$$

Exercise 12

$$\begin{cases} 4x - 12y < -12 \\ 6y \leq 2x - 12 \end{cases}$$

Exercise 13

$$\begin{cases} y \geq \frac{1}{2}x - 1 \\ -2x + 4y \geq 4 \end{cases}$$

Solve the application

Exercise 14

Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers and cookies, and he doesn't want to spend more than $5. At the hamburger restaurant near his college, each hamburger has 240 calories and costs $1.40. Each cookie has 160 calories and costs $0.50.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could he eat 3 hamburgers and 1 cookie?
d. Could he eat 2 hamburgers and 4 cookies?

Exercise 15

Christy sells her photographs at a booth at a street fair. At the start of the day, she wants to have at least 25 photos to display at her booth. Each small photo she displays costs her $4 and each large photo costs her $10. She doesn't want to spend more than $200 on photos to display.

a. Write a system of inequalities to model this situation.
b. Graph the system.
c. Could she display 10 small and 20 large photos?
d. Could she display 20 large and 10 small photos?

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Self-Check

In this section, you practiced:

a. Checking whether an ordered pair is a solution to a two-variable inequality
b. Sketching the solution to a two-variable inequality graphically
c. Checking whether an ordered pair is a solution to a system of inequalities
d. Sketching the solution to a system of inequalities graphically
e. Writing a system of inequalities to model a real-world scenario

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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CHAPTER OVERVIEW

Unit 2: Nonlinear Functions- Absolute Value, Piecewise, and Parabolas

This unit is about seeing and using your first nonlinear models. You'll treat absolute value as distance and parabolas as "U-shapes" you can slide, stretch, and reflect—using the vertex as your landmark. You'll move smoothly between story, rule, and graph; read and write piecewise rules that describe different behaviors; and pick a solving method that fits the question. Most importantly, you'll use these models to answer practical questions—what values are allowed, where a quantity is largest or smallest, and how changes in input shift the graph—so you can explain results clearly and make confident decisions.

2.1: Absolute Value

2.1P: Practice

2.2: Absolute Value Equations

2.2P: Practice

2.3: Piecewise-Defined Functions

2.3P: Practice

2.4: The Parabola

2.4P: Practice

2.5: Optimization

2.5P: Practice

2.6: Zeros of the Quadratic

2.6P: Practice

2.7: The Quadratic Formula

2.7P: Practice

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2.1: Absolute Value

Learning Objectives

- Express a relation using function notation
- Identify input and output values of an absolute value function
- Represent an absolute value function symbolically (as a formula) and graphically
- Use an absolute value expression to represent a distance

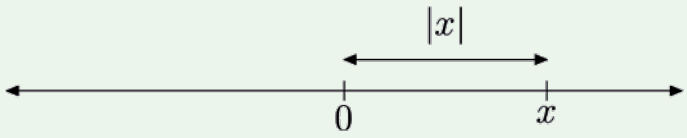
The absolute value of a number is a measure of its magnitude. That is, the "size" of the number regardless of it being positive or negative.

$$|7| = 7 \qquad \text{and} \qquad |-7| = 7$$

Here is the formal definition of the absolute value of a real number.

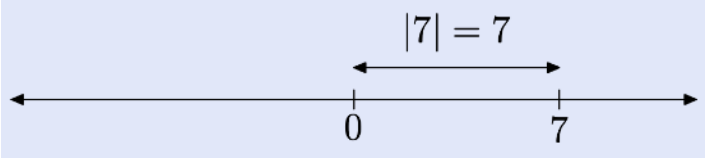
Definition: Absolute Value

To find the absolute value of any real number, first locate the number on the real line.

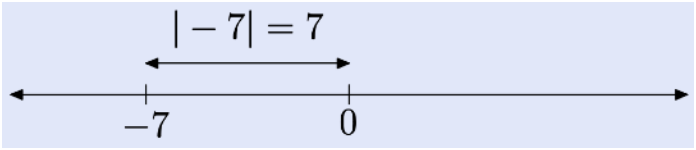


The absolute value of the number is defined as its distance from the origin.

For example, to find the absolute value of 7, locate 7 on the real line and then find its distance from the origin.



To find the absolute value of -7, locate -7 on the real line and then find its distance from the origin.



When we take the absolute value of a number, the result is always nonnegative; that is, the result is either positive or zero. In symbols,

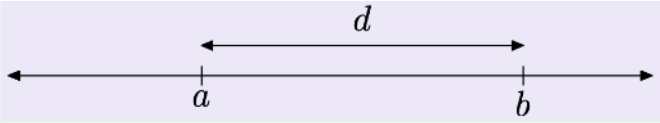
$$|x| \geq 0 \text{ for all real numbers } x$$

Distance Revisited

For any real number x, the absolute value of x is defined as the distance between the real number x and the origin on the real line. Here, we will push this distance concept a bit further.

Property 16.

Suppose that a and b are two numbers on the real line



You can determine the distance d between a and b on the real line by taking the absolute value of their difference. That is,

$$d = |b - a|$$

Of course, you could subtract in the other direction, obtaining $d = |a - b|$. This is also correct.

Now that this geometry of distance has been introduced, it is useful to pronounce the symbolism $|a - b|$ as "the distance between a and b" instead of saying "the absolute value of a minus b."

Example 1

Solve the equation

$$|x + 5| = 2$$

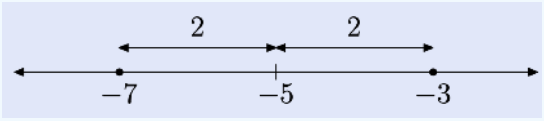
for x.

Solution

Rewrite the equation as a difference.

$$|x - (-5)| = 2$$

This is pronounced "the distance between x and -5 is 2." Locate two points on the number line that are 2 units away from -5.



Hence, the solutions of $|x + 5| = 2$ are $x = -7$ or $x = -3$.

1. If a number is negative, negating that number will make it positive. $|-5| = -(-5) = 5$, and similarly, $|-12| = -(-12) = 12$. Thus, if $x < 0$ (if x is negative), then $|x| = -x$.
2. If $x = 0$, then $|x| = 0$.
3. If a number is positive, taking the absolute value of that number does not change anything.

$$|5| = 5, \text{ and similarly, } |12| = 12$$

Thus, if $x > 0$ (if x is positive), then $|x| = x$.

Definition

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

You will notice that because |0| = 0, we can include this case with the piece |x| = x, if we adjust the condition to include zero.

Note

It is the first line in our piecewise definition that usually leaves students scratching their heads. They might say, "I thought absolute value makes a number positive (or zero), yet you have $|x| = -x$; that is, you have the absolute value of x equal to a negative x." Try as they might, this seems contradictory. Does it seem so to you?

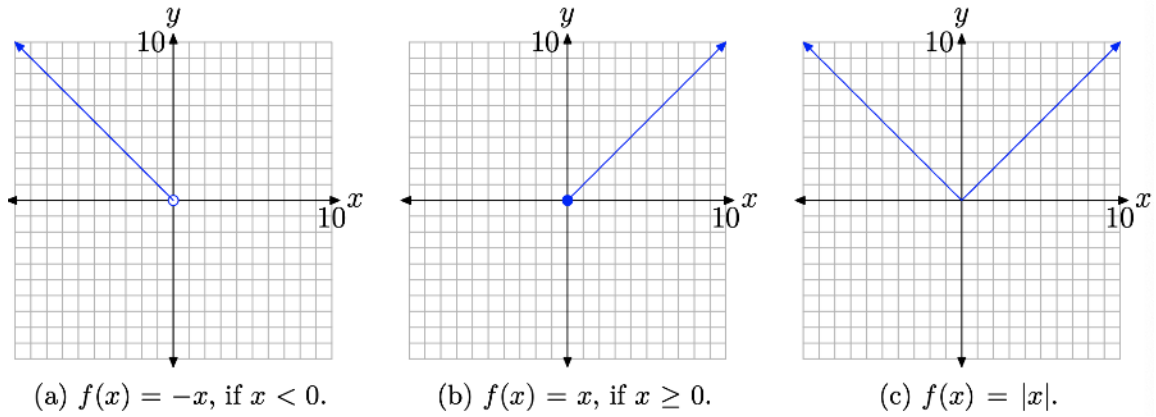
However, there is no contradiction. If x < 0, that is, if x is a negative number, then -x is a positive number, and our intuitive notion of absolute value is not dissimilar to that of our piecewise definition (4). For example, if x = -8, then -x = 8, and even

though we say “negative x,” in this case, -x is a positive number.

Graph of Absolute Value Function

Some basic observations are:

- If $x < 0$, then $f(x) = -x$. This ray starts at the origin and extends to the left with slope -1. Its graph is pictured in Figure (a).
- If $x \geq 0$, then $f(x) = x$. This ray starts at the origin and extends to the right with slope 1. Its graph is pictured in Figure (b).
- We combine the graphs in Figures (a) and (b) to produce the graph of $f(x) = |x|$ in Figure (c).



(a) $f(x) = -x$, if $x < 0$. (b) $f(x) = x$, if $x \geq 0$. (c) $f(x) = |x|$.

Things to note:

- The graph of $f(x) = |x|$ is “V-shaped.”
- The vertex of the graph is at the point $(0, 0)$.
- The left-hand branch has the equation $y = -x$ and a slope of -1.
- The right-hand branch has the equation $y = x$ and a slope of 1.
- Each branch of the graph of $f(x) = |x|$ forms a perfect 45◦ angle with the x-axis.

Now that we know how to draw the graph of $f(x) = |x|$, we can use transformations to sketch a number of simple graphs involving absolute value.

Vertical Transformations.

Vertical Transformations. Suppose we are given the graph of $y = f(x)$.

- If $a > 1$, then the graph of $y = af(x)$ is stretched vertically (away from the x-axis), both positively and negatively, by a factor of a.
- If $0 < a < 1$, then the graph of $y = af(x)$ is compressed vertically (toward the x-axis), both positively and negatively, by a factor of $\frac{1}{a}$.
- The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ across the x-axis.
- If $c > 0$, then the graph of $y = f(x) + c$ is shifted c units upward from the graph of $y = f(x)$.
- If $c > 0$, then the graph of $y = f(x) - c$ is shifted c units downward from the graph of $y = f(x)$.

Horizontal Transformations

Suppose we are given the graph of $y = f(x)$.

- If $a > 1$, the graph of $y = f(ax)$ compresses horizontally (toward the y-axis), both positively and negatively, by a factor of a.
- If $0 < a < 1$, the graph of $y = f(ax)$ stretches horizontally (away from the y-axis), both positively and negatively, by a factor of $\frac{1}{a}$.
- The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ across the y-axis.
- If $c > 0$, then the graph of $y = f(x + c)$ is shifted c units to the left of the graph of $y = f(x)$.
- If $c > 0$, then the graph of $y = f(x - c)$ is shifted c units to the right of the graph of $y = f(x)$.

Example 2

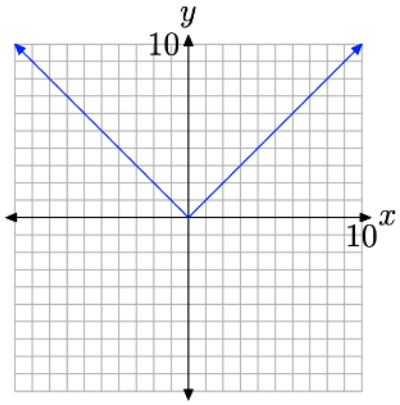
Sketch the graph of $f(x) = |x - 3|$.

Solution

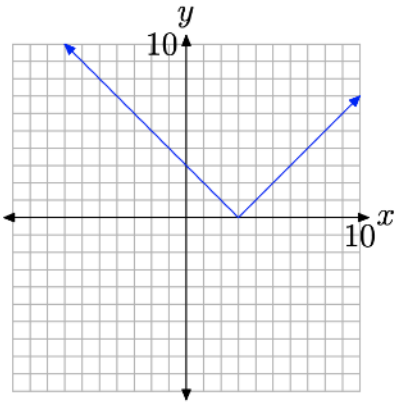
First, sketch the graph of $y = f(x) = |x|$, as shown in Figure (a). Note that if $f(x) = |x|$, then

$$y = f(x - 3) = |x - 3|$$

To sketch the graph of $y = f(x - 3) = |x - 3|$, shift the graph of $y = f(x) = |x|$ three units to the right, producing the result shown in Figure (b).



(a) $y = f(x) = |x|$.



(b) $y = f(x - 3) = |x - 3|$.

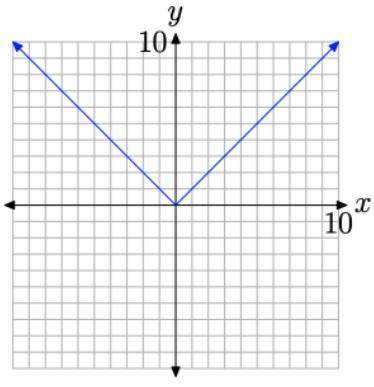
Let's look at one final example.

Example 3

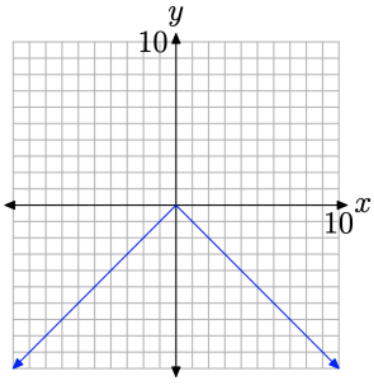
Sketch the graph of $f(x) = -|x| + 5$.

Solution

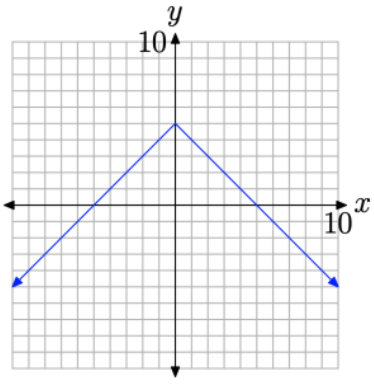
- First, sketch the graph of $y = f(x) = |x|$, as shown in Figure (a).
- Next, sketch the graph of $y = -f(x) = -|x|$, which is a reflection of the graph of $y = f(x) = |x|$ across the x-axis and is pictured in Figure (b).
- Finally, we will want to sketch the graph of $y = -f(x) + 5 = -|x| + 5$. To do this, we shift the graph of $y = -f(x) = -|x|$ in Figure (b) upward 5 units to produce the result in Figure (c).



(a) $y = f(x) = |x|$.



(b) $y = -f(x) = -|x|$.



(b) $y = -f(x) + 5 = -|x| + 5$.

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2.1P: Practice

Practice Makes Progress

Exercise 1

Set up a coordinate system on a sheet of graph paper. Label and scale each axis from -10 to 10.
Create an accurate plot of the function $y = |x|$ on your coordinate system and label this graph with its equation.

For each function below,

a. Explain in words what transformations (shifts, stretches, reflections) have been applied to $y = |x|$. Then use the transformations to sketch the graph of the given function. Clearly label the vertex.
b. Use interval notation to describe the domain and range of the given function.

Exercise 2

$f(x) = |x + 3| + 2$

Exercise 3

$f(x) = -|x - 2|$

Exercise 4

$f(x) = \frac{1}{2}|x| - 3$

Exercise 5

$f(x) = -3|x - 1| + 5$

Self-Check

In this section, you practiced:

a. Graphing absolute value expressions using transformations from the graph of $y = |x|$
b. Using function notation
c. Identifying the vertex, domain, and range of an absolute value function

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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2.2: Absolute Value Equations

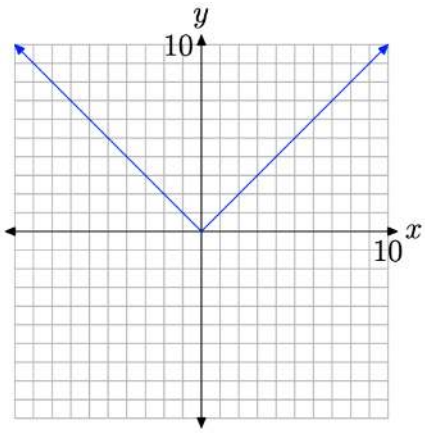
Learning Objectives

- Sketch a graph to visualize the solutions to an absolute value equation
- Solve and absolute value equation algebraically
- Use graphical and algebraic methods to show if an absolute value equation has one, two, or no solutions
- Solve absolute value inequalities, and express the solution set in set-builder and interval notation

In the previous section, we defined

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

And we saw that the graph of the absolute value function defined by $f(x) = |x|$ has the "V-shape" shown here.



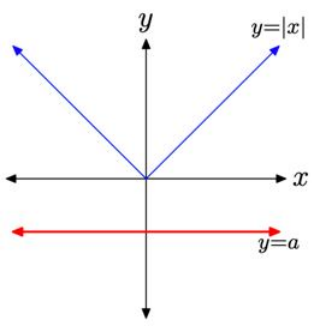
It is important to note that the equation of the left-hand branch of the "V" is $y = -x$ and the right-hand branch of the "V" has equation $y = x$.

There are three distinct cases to discuss, each of which depends upon the value and sign of the number a.

Solving $|x| = a$

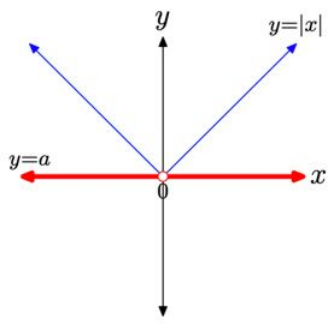
The solution of $|x| = a$ depends upon the value and sign of a.

Case I: $a < 0$



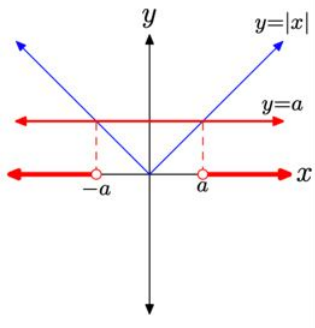
If $a < 0$, then the graph of $y = a$ is a horizontal line that lies strictly below the x-axis. In this case, the equation $|x| = a$ has no solutions because the graphs of $y = a$ and $y = |x|$ do not intersect. Therefore, the equation $|x| = a$ has no solutions. For example, $|x| = -3$ has no solutions.

Case II: $a = 0$



If $a = 0$, then the graph of $y = 0$ is a horizontal line that coincides with the x-axis. In this case, the equation $|x| = 0$ has the single solution $x = 0$, because the horizontal line $y = 0$ intersects the graph of $y = |x|$ at exactly one point, at $x = 0$.

Case III: $a > 0$



If $a > 0$, then the graph of $y = a$ is a horizontal line that lies strictly above the x-axis. In this case, the equation $|x| = a$ has two solutions, because the graphs of $y = a$ and $y = |x|$ intersect at two points. For example, the left-hand side of $|x| = 4$ is "V" portion of the graph. The right-hand side of $|x| = 4$ is the horizontal line 4 units above the x-axis. The graphs intersect at $(-4, 4)$ and $(4, 4)$. Therefore, the solutions of $|x| = 4$ are $x = -4$ or $x = 4$.

Example 1

Solve the equation $|3 - 2x| = 6$ for x.

Solution

Algebraic Approach: One can use an algebraic technique to find the two solutions of $|3 - 2x| = 6$. Much as $|k| = 6$ has solutions $k = -6$ or $k = 6$, the equation

$$|3 - 2x| = 6$$

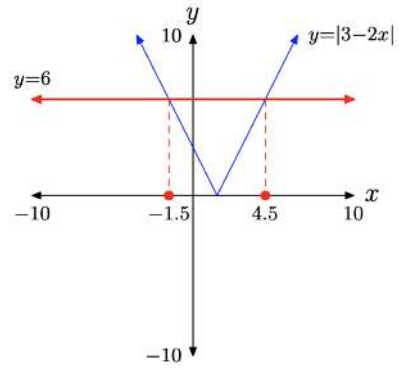
is true only if the expression inside the absolute values is either equal to -6 or 6. Therefore, write

$$3 - 2x = -6 \quad \text{or} \quad 3 - 2x = 6$$

and solve these equations independently

$$\begin{array}{rcl} 3 - 2x & = & -6 \\ -2x & = & -9 \\ x & = & \frac{9}{2} \end{array} \quad \text{or} \quad \begin{array}{rcl} 3 - 2x & = & 6 \\ -2x & = & 3 \\ x & = & -\frac{3}{2} \end{array}$$

Graphical Approach: We obtain the graph by separately graphing the left- and right-hand sides of the equation $|3 - 2x| = 6$.



The two graphs intersect when $x = -1.5$ and $x = 4.5$, which match our algebraic solutions.

Sometimes, we need to perform algebraic operations both before and after removing the absolute value bars.

Example 2

Solve the equation $|x+2|+3=8$ for x.

Solution

First, subtract 3 from both sides of the equation.

$$\begin{aligned} |x+2|+3 &= 8 \\ |x+2|+3-3 &= 8-3 \end{aligned}$$

This simplifies to

$$|x+2| = 5$$

Now, either

$$x+2=-5 \quad \text{or} \quad x+2=5$$

each of which can be solved separately.

$$\begin{aligned} x+2 &= -5 \\ x+2-2 &= -5-2 \\ x &= -7 \end{aligned} \quad \text{or} \quad \begin{aligned} x+2 &= 5 \\ x+2-2 &= 5-2 \\ x &= 3 \end{aligned}$$

Example 3

Solve the equation $\left|\frac{x}{2}-\frac{1}{3}\right| = \frac{1}{4}$ for x.

Solution

It is tempting to multiply both sides of this equation by a common denominator as follows.

$$\begin{aligned} \left|\frac{x}{2}-\frac{1}{3}\right| &= \frac{1}{4} \\ 12\left|\frac{x}{2}-\frac{1}{3}\right| &= 12\left(\frac{1}{4}\right) \end{aligned}$$

If it is permissible to move the 12 inside the absolute values, then we could proceed as follows.

$$\begin{aligned} \left|12\left(\frac{x}{2}-\frac{1}{3}\right)\right| &= 3 \\ |6x-4| &= 3 \end{aligned}$$

Assuming for the moment that this last move is allowable, either

$$6x-4=-3 \quad \text{or} \quad 6x-4=3$$

Each of these can be solved separately, first by adding 4 to both sides of the equations, then dividing by 6.

$$\begin{aligned} 6x-4 &= -3 \quad \text{or} \quad 6x-4 = 3 \\ 6x &= 1 \qquad\qquad 6x = 7 \\ x &= 1/6 \qquad\quad x = 7/6 \end{aligned}$$

As we've employed a somewhat questionable approach in obtaining these solutions, it would be wise to verify our results. First, substitute $x = 1/6$ into the original equation.

$$\begin{aligned} \left|\frac{x}{2}-\frac{1}{3}\right| &= \frac{1}{4} \\ \left|\frac{1/6}{2}-\frac{1}{3}\right| &= \frac{1}{4} \\ \left|\frac{1}{12}-\frac{1}{3}\right| &= \frac{1}{4} \end{aligned}$$

Write equivalent fractions with a common denominator and subtract.

$$\begin{aligned} \left|\frac{1}{12}-\frac{4}{12}\right| &= \frac{1}{4} \\ \left|-\frac{3}{12}\right| &= \frac{1}{4} \\ \left|-\frac{1}{4}\right| &= \frac{1}{4} \end{aligned}$$

Clearly, $x = 1/6$ checks. We'll leave the check of the second solution to our readers.

Warning

It is not permissible to multiply by a negative number and slide the negative number inside the absolute value bars. For example,

$$-2|x-3| = |-2(x-3)|$$

is clearly an error. For any x except 3, the left-hand side of this result is a negative number, but the right-hand side is a positive number. They are clearly not equal.

Absolution Value Inequalities

We now turn our attention to inequalities involving absolute value.

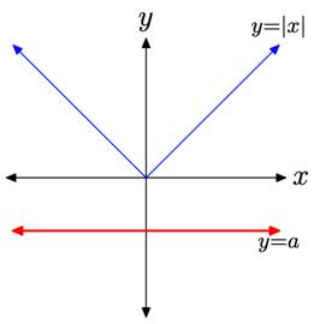
Solving $|x| < a$

The solutions of

$$|x| < a$$

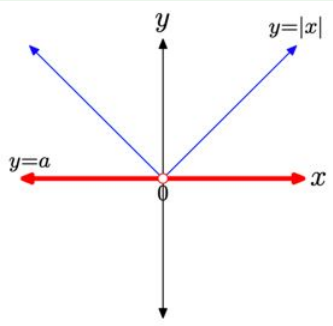
again depend upon the value and sign of the number a. To solve $|x| < a$ graphically, we must determine where the graph of the left-hand side lies below the graph of the right-hand side of the inequality $|x| < a$. There are three cases to consider.

Case I: $a < 0$



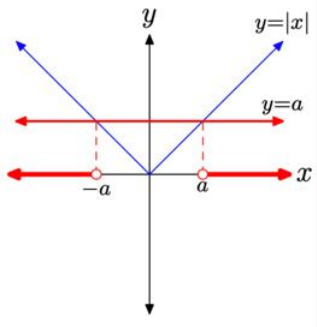
In this case, the graph of $y = a$ lies strictly below the x-axis. Said another way, the graph of $y = |x|$ never lies below the graph of $y = a$. Hence, the inequality $|x| < a$ has no solutions. For example, $|x| < -5$ has no solutions.

Case II: $a = 0$



In this case, the graph of $y = 0$ coincides with the x-axis. The graph of $y = |x|$ never lies strictly below the x-axis. Hence, the inequality $|x| < 0$ has no solutions.

Case III: $a > 0$



In this case, the graph of $y = a$ lies strictly above the x-axis. The graphs of $y = |x|$ and $y = a$ intersect at $x = -a$ and $x = a$. We also see that the graph of $y = |x|$ lies strictly below the graph of $y = a$ when x is between $-a$ and a; that is, when $-a < x < a$. For example, $|x| < 8$ has solutions $-8 < x < 8$.

Let's look at some examples.

Example 4

Solve the inequality $|5 - 2x| < 3$ for x.

Solution

Algebraic Approach: Let's now explore an algebraic solution of the inequality $|5 - 2x| < 3$. Much as $|k| < 3$ implies that $-3 < k < 3$, the inequality

$$|5 - 2x| < 3$$

requires that

$$-3 < 5 - 2x < 3$$

We can subtract 5 from all three members of this last inequality, then simplify.

$$-3 - 5 < 5 - 2x - 5 < 3 - 5$$
$$-8 < -2x < -2$$

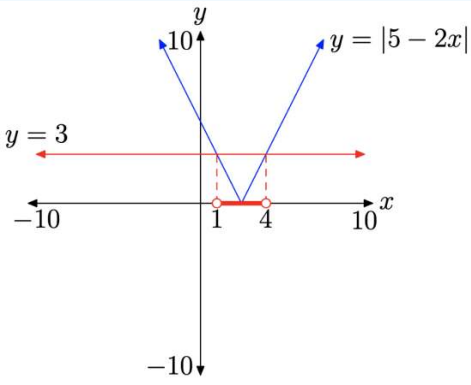
Divide all three members of this last inequality by -2, reversing the inequality symbols as you go.

$$4 > x > 1$$

We prefer that our inequalities read from "small-to-large," so we write

$$1 < x < 4$$

Graphical Approach. Separately graphing the left- and right-hand sides of the inequality $|5 - 2x| < 3$, we obtain the following graph.



The graph of $y = |5 - 2x|$ falls below the graph of $y = 3$ for all values of x between 1 and 4. Hence, the solution of the inequality $|5 - 2x| < 3$ is {x:1<x<4} in set-builder and $(1, 4)$ in interval notation.

Note

If $a > 0$, then the inequality $|x| \leq a$ is equivalent to the inequality $-a \leq x \leq a$.

Example 5

Solve the inequality $5 - 3|x - 4| \geq -4$ for x.

Solution

At first glance, the inequality

$$5 - 3|x - 4| \geq -4$$

has a form quite dissimilar from what we've done thus far. However, let's subtract 5 from both sides of the inequality.

$$-3|x - 4| \geq -9$$

Now, let's divide both sides of this last inequality by -3, reversing the inequality sign.

$$|x - 4| \leq 3$$

Aha! Familiar ground. This last inequality is equivalent to

$$-3 \leq x - 4 \leq 3$$

and when we add 4 to all three members, we have the solution.

$$1 \leq x \leq 7$$

We can sketch the solution on a number line



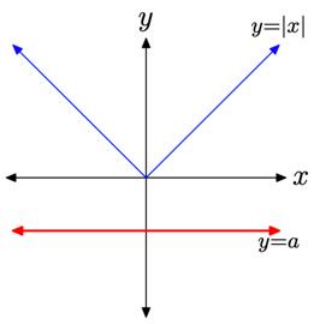
We can describe the solution using interval and set-builder notation.

$$[1, 7] = \{x : 1 \leq x \leq 7\}$$

Solving $|x| > a$

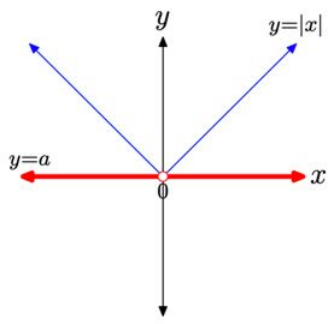
The solutions of $|x| > a$ again depend upon the value and sign of a. To solve $|x| > a$ graphically, we must determine where the graph of $y = |x|$ lies above the graph of $y = a$. Again, we consider three cases.

Case I: a < 0



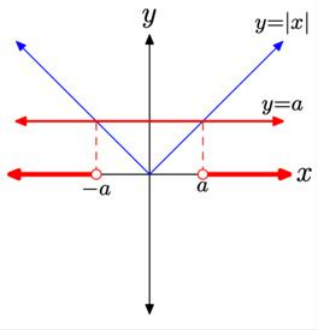
In this case, the graph of $y = a$ lies strictly below the x-axis. Therefore, the graph of $y = |x|$ always lies above the graph of $y = a$. Hence, all real numbers are solutions of the inequality $|x| > a$. For example, any value of x satisfies $|x| > -2$.

Case II: a = 0



In this case, the graph of $y = 0$ coincides with the x-axis. The graph of $y = |x|$ lies strictly above the graph of $y = 0$ for all values of x except one: x cannot equal zero. Hence, every real number except $x = 0$ is a solution of $|x| > 0$.

Case III: a > 0



In this case, the graph of $y = a$ lies strictly above the x-axis. The graph of $y = |x|$ intersects the graph of $y = a$ at $x = -a$ and $x = a$. We see that the graph of $y = |x|$ lies strictly above the graph of $y = a$ if x is less than $-a$ or greater than a. For example, $|x| > 2$ has solutions $(-\infty, -2)$ or $(2, \infty)$.

Let's look at an example.

Example 6

Solve the inequality $|4 - x| > 5$ for x.

Solution

Algebraic Approach. Let's explore an algebraic solution of $|4 - x| > 5$. In much the same manner that $|k| > 5$ leads to the conditions $k < -5$ or $k > 5$, the inequality

$$|4 - x| > 5$$

requires that

$$4 - x < -5 \quad \text{or} \quad 4 - x > 5$$

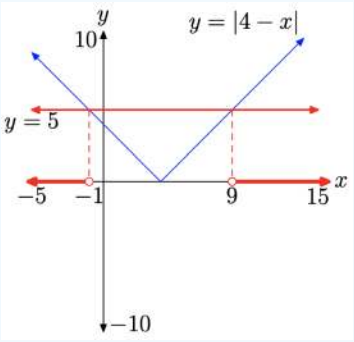
We can solve each of these independently by first subtracting 4 from both sides of the inequality, then multiplying both sides of each inequality by -1, reversing the direction of each inequality as we do so.

$$\begin{array}{rcl} 4 - x & < & -5 \\ -x & < & -9 \\ x & > & 9 \end{array} \quad \text{or} \quad \begin{array}{rcl} 4 - x & > & 5 \\ -x & > & 1 \\ x & < & -1 \end{array}$$

We prefer to write this solution in the order

$$x < -1 \quad \text{or} \quad x > 9$$

Graphical Approach. Separately graphing the left- and right-hand sides of the inequality $|4 - x| > 5$, we obtain the following graph.



The graph of $y = |4 - x|$ lies above the graph of $y = 5$ for all values of x that lie either to the left of -1 or to the right of 9. Hence, the solution of $|4 - x| > 5$ is the set {x : x < -1 or x > 9}.

Note

If $a > 0$, then the inequality $|x| \geq a$ is equivalent to the inequality $x \leq -a$ or $x \geq a$.

Example 7

Solve the inequality $3|1-x|-4 \geq |1-x|$ for x.

Solution

Again, at first glance, the inequality

$$3|1-x|-4 \geq |1-x|$$

looks unlike any inequality we've attempted to this point. However, if we subtract $|1-x|$ from both sides of the inequality, then add 4 to both sides of the inequality, we get

$$3|1-x|-|1-x| \geq 4$$

On the left, we have like terms. Note that $3|1-x|-|1-x| = 3|1-x|-1|1-x| = 2|1-x|$. Thus,

$$2|1-x| \geq 4$$

Divide both sides of the last inequality by 2.

$$|1-x| \geq 2$$

We can now write

$$1-x \leq -2 \quad \text{or} \quad 1-x \geq 2$$

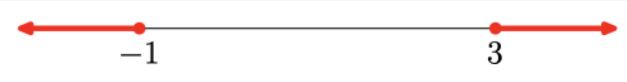
We can solve each of these inequalities independently. First, subtract 1 from both sides of each inequality, then multiply both sides of each resulting inequality by -1, reversing each inequality as you go.

$$\begin{array}{rcl} 1-x & \leq & -2 \\ -x & \leq & -3 \\ x & \geq & 3 \end{array} \quad \text{or} \quad \begin{array}{rcl} 1-x & \geq & 2 \\ -x & \geq & 1 \\ x & \leq & -1 \end{array}$$

We prefer to write this in the order

$$x \leq -1 \quad \text{or} \quad x \geq 3$$

We can sketch the solutions on a number line.



And we can describe the solutions using interval and set-builder notation.

$$(-\infty, -1] \cup [3, \infty) = \{x : x \leq -1 \text{ or } x \geq 3\}$$

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2.2P: Practice

Practice Makes Progress

Solve the following equations algebraically. Then, show your process for checking the solution(s) graphically or algebraically. Clearly state the verified solution.

Exercise 1

$|x+3|=0$

Exercise 2

$2|x+7|=14$

Exercise 3

$\frac{1}{3}|x-5|-2=-4$

Exercise 4

$|x+2|-3=4$

Exercise 5

$|\frac{x}{3}-\frac{1}{4}|=\frac{1}{12}$

Solve the following absolute value equations graphically. Clearly label key points on the graph and state the solutions.

Exercise 6

$-2|x|+5=3-x$

Exercise 7

$|x-2|=\frac{1}{3}x+2$

Perform each of the following tasks.

1. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
2. Sketch the graph of each side of the inequality without the aid of a calculator. Label each graph with its equation.
3. Shade and label the solution of the inequality on the coordinate plane. Clearly state the solution.

Exercise 8

|x| > 1

Exercise 9

|x| ≤ 2

Solve the following absolute value inequalities algebraically. Then, show your process for checking the solution(s) graphically or algebraically. Clearly state the verified solution.

Exercise 10

$|4x+5| < 7$

Exercise 11

$|13 - 2x| \leq 7$

Exercise 12

$4|x + 3| < 8$

Exercise 13

$\frac{1}{2}|x - 3| - 1 \geq 1$

Exercise 14

$-2|x + 1| \leq -6$

Exercise 15

$\left|\frac{x}{4} - \frac{1}{2}\right| < \frac{2}{3}$

Self-Check

In this section, you practiced:

a. Understanding absolute value expressions as distance
b. Solving absolute value linear equations algebraically
c. Solving absolute value linear equations graphically
d. Solving absolute value linear inequalities algebraically

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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2.3: Piecewise-Defined Functions

Learning Objectives

- Write a piecewise function to model a real-world scenario
- Evaluate piecewise functions at particular points
- Sketch a graph of a piecewise-defined function

Piecewise functions are a favorite of engineers. Let's start with an example.

Example 1

Suppose that a battery provides no voltage to a circuit when a switch is open. Then, starting at time $t = 0$, the switch is closed and the battery provides a constant 5 volts from that time forward. Create a piecewise function modeling the problem constraints and sketch its graph.

Solution

This is a fairly simple exercise, but we will have to introduce some new notation. First of all, if the time t is less than zero ($t < 0$), then the voltage is 0 volts. If the time t is greater than or equal to zero ($t \geq 0$), then the voltage is a constant 5 volts. Here is the notation we will use to summarize this description of the voltage.

$$V(t) = \begin{cases} 0, & \text{if } t < 0 \\ 5, & \text{if } t \geq 0 \end{cases}$$

Some comments are in order:

- The voltage difference provided by the battery in the circuit is a function of time. Thus, $V(t)$ represents the voltage at time t.
- The notation used is universally adopted by mathematicians in situations where the function changes definition depending on the value of the independent variable. This definition of the function V is called a "piecewise definition." Because each of the pieces in this definition is constant, the function V is called a piecewise constant function.
- This particular function has two pieces. The function is the constant function $V(t) = 0$, when $t < 0$, but a different constant function, $V(t) = 5$, when $t \geq 0$.

If $t < 0, V(t) = 0$. For example, for $t = -1, t = -10$, and $t = -100$

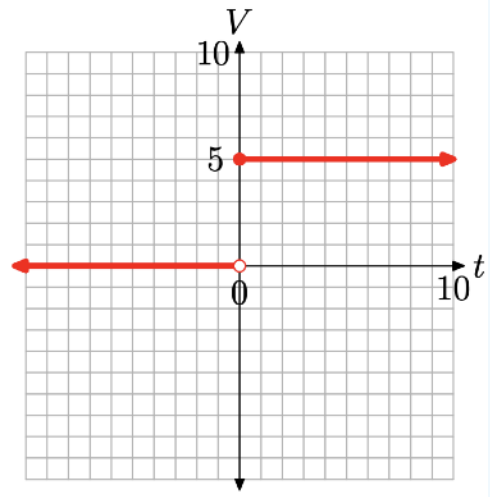
$$V(-1) = 0, \quad V(-10) = 0, \quad \text{and} \quad V(-100) = 0$$

On the other hand, if $t \geq 0$, then $V(t) = 5$. For example, for $t = 0, t = 10$, and $t = 100$

$$V(0) = 5, \quad V(10) = 5, \quad \text{and} \quad V(100) = 5$$

Let's turn our attention to the graph of the voltage defined by the equation defined earlier. When $t < 0$, then $V(t) = 0$. Normally, the graph of $V(t) = 0$ would be a horizontal line where each point on the line has V-value equal to zero. However, $V(t) = 0$ only if $t < 0$, so the graph is the horizontal ray that starts at the origin, then moves indefinitely to the left. That is, the horizontal line $V = 0$ has been restricted to the domain $\{t : t < 0\}$ and exists only to the left of the origin.

Similarly, when $t \geq 0$, then $V(t) = 5$ is the horizontal ray. Each point on the ray has a V-value equal to 5.



Two comments are in order:

- Because $V(t) = 0$ only when $t < 0$, the point $(0, 0)$ is unfilled (it is an open circle).
- Because $V(t) = 5$ when $t \geq 0$, the point $(0, 5)$ is filled (it is a filled circle).

Piecewise-Defined Functions

Now, let's look at a more generic situation involving piecewise-defined functions—one where the pieces are not necessarily constant. The best way to learn is by doing, so let's start with an example.

Example 2

Consider the piecewise-defined function

$$f(x) = \begin{cases} -x + 2, & \text{if } x < 2 \\ x - 2, & \text{if } x \geq 2 \end{cases}$$

Evaluate $f(x)$ for $x = 0, 1, 2, 3$ and 4, then sketch the graph of the piecewise-defined function.

Solution

The function changes definition at $x = 2$. If $x < 2$, then $f(x) = -x + 2$. Because both 0 and 1 are strictly less than 2, we evaluate the function with this first piece of the definition.

$$\begin{array}{lll} f(x) = -x + 2 & \text{and} & f(x) = -x + 2 \\ f(0) = -0 + 2 & & f(1) = -1 + 2 \\ f(0) = 2 & & f(1) = 1 \end{array}$$

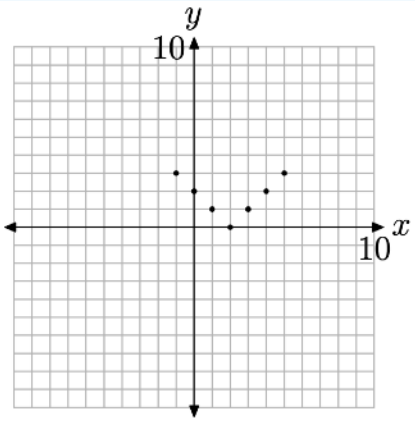
On the other hand, if $x \geq 2$, then $f(x) = x - 2$. Because 2, 3, and 4 are all greater than or equal to 2, we evaluate the function with this second piece of the definition.

$$\begin{array}{lllllll} f(x) = x - 2 & \text{and} & f(x) = x - 2 & \text{and} & f(x) = x - 2 \\ f(2) = 2 - 2 & \text{and} & f(3) = 3 - 2 & \text{and} & f(4) = 4 - 2 \\ f(2) = 0 & \text{and} & f(3) = 1 & \text{and} & f(4) = 2 \end{array}$$

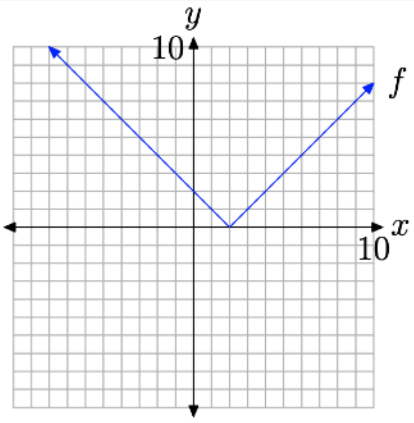
One possible approach to graphing f is to place the points we've already calculated, along with a couple of extra ones, in a table, plot them, and then infer the shape of the graph from the evidence provided by the plotted points.

x	$f(x)$
-1	3
0	2
1	1
2	0
3	1
4	2
5	3

(a)



(b)



(c)

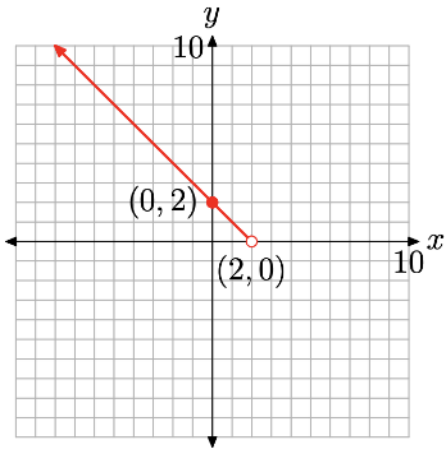
Let's use our knowledge of the equation of a line (i.e. $y = mx + b$) to help sketch the graph of the piecewise function.

Let's sketch the first piece of the function f defined. We have $f(x) = -x + 2$, provided $x < 2$. Normally, this would be a line (with slope -1 and intercept 2), but we are only to sketch a part of that line, specifically the part where $x < 2$ (i.e., x is to the left of 2). Thus, this piece of the graph will be a ray, starting at the point where $x = 2$, then moving indefinitely to the left.

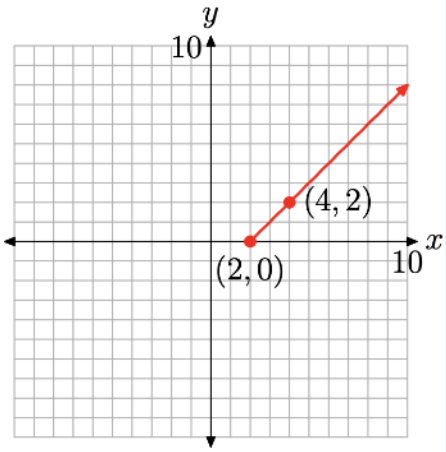
The easiest way to sketch a ray is to first calculate and plot its fixed endpoint (in this case at $x = 2$), then plot a second point on the ray having an x-value less than 2, then use a ruler to draw the ray.

With this thought in mind, to find the coordinates of the endpoint of the ray, evaluate $-x + 2$ for $x = 2$. Now, technically, we're not supposed to use this part of the function unless x is strictly less than 2, but we could use it with $x = 1.9$, or $x = 1.99$, or$x = 1.999$, and so on. So let's go ahead and use $x = 2$ in this piece of the function, but indicate that we're not actually supposed to use this point by drawing an "empty circle" at $(2, 0)$.

To complete the plot of the ray, we need a second point that lies to the left of its endpoint at $(2, 0)$. Note that $x = 0$ is to the left of $x = 2$. $f(0) = -0 + 2 = 2$ gives us the second point $(0, 2)$. Finally, draw the ray with endpoint at $(2, 0)$ and second point at $(0, 2)$.



(a)



(b)

We now repeat this process for the second piece of the function. The equation of the second piece is $f(x) = x - 2$, provided $x \geq 2$. Normally, $f(x) = x - 2$ would be a line (with slope 1 and intercept -2), but we're only supposed to sketch that part of the line that lies to the right of or at $x = 2$. Thus, the graph of this second piece is a ray, starting at the point with $x = 2$ and continuing to the right. $f(2) = 2 - 2 = 0$, so the fixed endpoint of the ray is at the point $(2, 0)$. Since we're actually supposed to use this piece with $x = 2$, we indicate this fact with a filled circle at $(2, 0)$.

We need a second point to the right of this fixed endpoint, so we evaluate $f(4) = 4 - 2 = 2$. Thus, a second point on the ray is the point $(4, 2)$. Finally, we simply draw the ray, starting at the endpoint $(2, 0)$ and passing through the second point at $(4, 2)$.

To complete the graph of the piecewise function f, simply combine the two pieces to get the finished graph found earlier from our other method.

Let's look at a very practical application of piecewise functions.

Example 3

The federal income tax rates for a single filer in 2005 are listed in the table below.

Income	Tax Rate
Up to $7,150	10%
$7,151-$29,050	15%
$29,051-$70,350	25%
$70,351-$146,750	28%
$146,751-$319,100	33%
$319,101 or more	35%

Create a piecewise definition that provides the tax rate as a function of personal income.

Solution

In reporting taxable income, amounts are rounded to the nearest dollar on the federal income tax form. Technically, the domain is discrete. You can report a taxable income of $35,000 or $35,001, but numbers between these two incomes are not used on the federal income tax form. However, we will consider the income as a continuum, allowing it to be any real number greater than or equal to zero. If we did not do this, our graph would be a series of dots — one for each dollar amount. We would have to plot lots of dots!

We will let R represent the tax rate and I represent the income. The goal is to define R as a function of I.

- If income I is any amount greater than or equal to zero, and less than or equal to $7,150, the tax rate R is 10% (i.e., $R = 0.10$). Thus, if $\$0 \le I \le \$7,150$, $R(I) = 0.10$.
- If income I is any amount that is strictly greater than $7,150 but less than or equal to $29,050, then the tax rate R is 15% (i.e., $R = 0.15$). Thus, if $\$7,150 < I \le \$29,050$, then $R(I) = 0.15$.

Continuing in this manner, we can construct a piecewise definition of the rate R as a function of taxable income I.

$$R(I) = \begin{cases} 0.10, & \text{if } \$0 \le I \le \$7,150 \\ 0.15, & \text{if } \$7,150 < I \le \$29,050 \\ 0.25, & \text{if } \$29,050 < I \le \$70,350 \\ 0.28, & \text{if } \$70,350 < I \le \$146,750 \\ 0.33, & \text{if } \$146,750 < I \le \$319,100 \\ 0.35, & \text{if } I > \$319,100 \end{cases}$$

Let's turn our attention to the graph of this piecewise-defined function. All of the pieces are constant functions, so each piece will be a horizontal line at a level indicating the tax rate. However, each of the first five pieces of the function is a segment, because the rate is defined on an interval with a starting and ending income. The sixth and last piece is a ray, as it has a starting endpoint, but the rate remains constant for all incomes above $319,100. We use this knowledge to construct the graph shown.

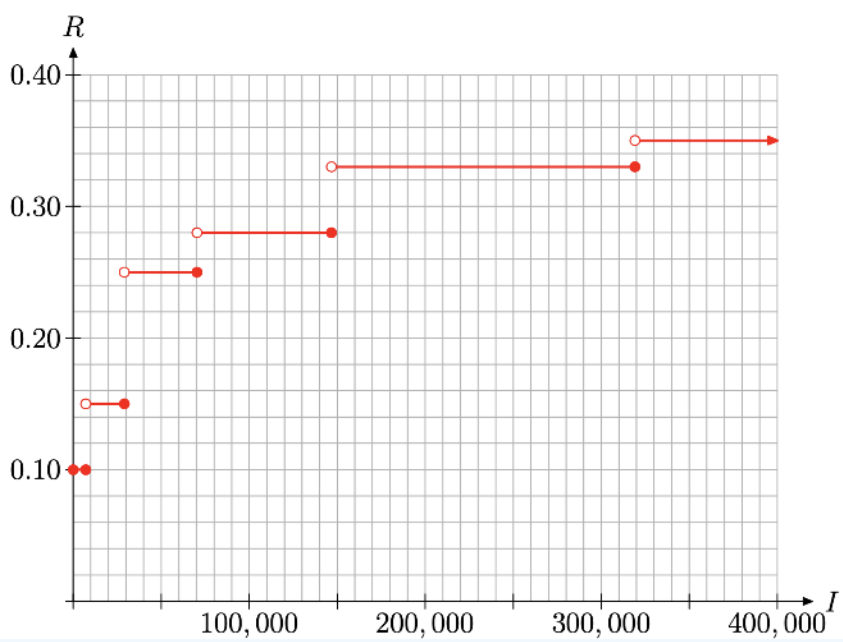
The first rate is 10%, which applies to taxable income ranging from $0 to $7,150, inclusive. Thus, note the first horizontal line segment that runs from $0 to $7,150 at a height of $R = 0.10$. Note that a filled circle represents each endpoint.

The second rate is 15% and this is assigned to taxable incomes greater than \$7,150, but less than or equal to \$29,050. The second horizontal line segment runs from \$7,150 to \$29,050 at a height of $R = 0.15$. Note that the endpoint at the left end of this horizontal segment is an open circle while the endpoint on the right end is a filled circle because the taxable incomes range on $\$7,150 < I \leq \$29,050$. Thus, we exclude the left endpoint and include the right endpoint.

The remaining segments are drawn in a similar manner.

The last piece assigns a rate of $R = 0.35$ to all taxable incomes strictly above \$319,100. Hence, the last piece is a horizontal ray, starting at $\$319100, 0.35)$ and extending indefinitely to the right. Note that the left endpoint of this ray is an open circle because the rate $R = 0.35$ applies to taxable incomes $I > \$319,100$.

Let's talk a moment about the domain and range of the function R. If we project all points on the graph onto the horizontal axis, the entire axis will "lie in shadow." Thus, at first glance, one would state that the domain of R is the set of all real numbers that are greater than or equal to zero.



However, remember that we chose to model a discrete situation with a continuum. Taxable income is always rounded to the nearest dollar on federal income tax forms. Therefore, the domain is actually all whole numbers greater than or equal to zero. In symbols,

$$\text{Domain } = \{I \in \mathbb{W} : I \geq 0\}$$

To find the range of R, we would project all points on the graph of R onto the vertical axis. The result would be that six points would be shaded on the vertical axis, one each at 0.10, 0.15, 0.25, 0.28, 0.33, and 0.35. Thus, the range is a finite discrete set, so it's best described by simply listing its members.

$$\text{Range } = \{0.10, 0.15, 0.25, 0.28, 0.33, 0.35\}$$

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2.3P: Practice

Practice Makes Progress

Exercise 1

Given the function defined by the rule

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 2, & \text{if } x \geq 0 \end{cases}$$

a. Draw the graph of f on a sheet of graph paper.
b. Evaluate $f(-2), f(0)$, and $f(3)$ and plot these points on your graph.
c. State the domain and range of f.

Exercise 2

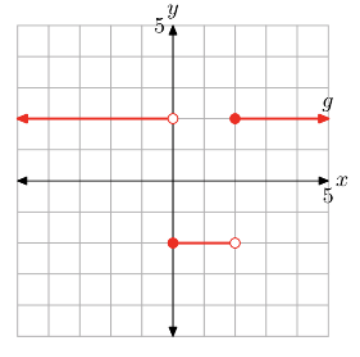
Given the function defined by the rule

$$f(x) = \begin{cases} -3, & \text{if } x < -2 \\ 1, & \text{if } -2 \leq x < 2 \\ 3, & \text{if } x \geq 2 \end{cases}$$

a. Draw the graph of g on a sheet of graph paper.
b. Evaluate $g(-3), g(-2)$, and $g(5)$ and plot these points on your graph.
c. State the domain and range of g.

Exercise 3

Determine a piecewise definition of the function described by the graph, then state the domain and range of the function.



Exercise 4

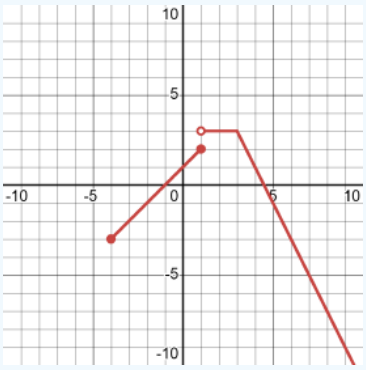
Given the piecewise function

$$g(x) = \begin{cases} 2x + 3, & \text{if } -2 \leq x < 0 \\ -2x + 1, & \text{if } x \geq 0 \end{cases}$$

a. Draw the graph of g on a sheet of graph paper.
b. Evaluate $g(0)$ and $g(3)$ and plot these points on your graph.
c. State the domain and range of the function.

Exercise 5

Write a formula for the piecewise function graphed below:



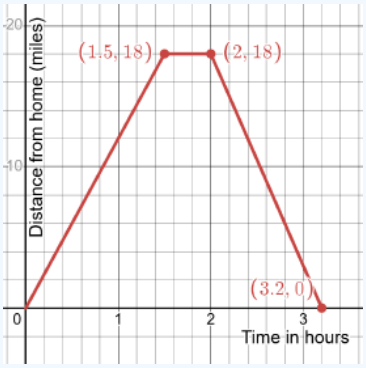
Exercise 6

A bakery has special deals on bulk orders of cookies. The first 100 cookies cost $2.00 each, the next 150 cookies cost $1.50 each, and any additional cookies beyond 250 cost $1.25 each.

a. Sketch a graph of the cost of an order of cookies as a function of the number of cookies purchased. Clearly label the axes.
b. Define variables and write a piecewise function to model this scenario.
c. You may do parts a. and b. in either order - which do you find easier, graphing first or writing the function first?

Exercise 7

Jules left his house for a bike ride and rode straight to a coffee shop. He stayed at the shop for a while, then rode home. A graph of his ride is below.



a. Define variables and write a piecewise function to model this scenario.
b. State the domain and range of the function.

Self-Check

In this section, you practiced:

a. Graphing piecewise functions given their formulas
b. Writing a piecewise function given its graph
c. Evaluating piecewise expressions

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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2.4: The Parabola

Learning Objectives

- Sketch a graph of the basic quadratic function $f(x) = x^2$
- Use scaling, shifting, and reflecting to sketch the graph of the quadratic function $f(x) = a(x - h)^2 + k$
- Use the technique of completing the square to change a quadratic function from standard form to vertex form

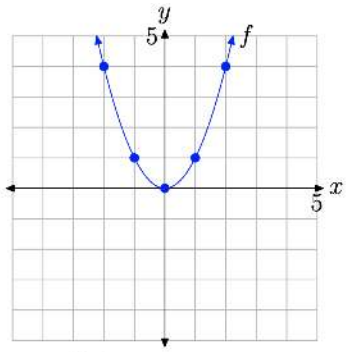
In this section, you will learn how to draw the graph of the quadratic function defined by the equation

$$f(x) = a(x - h)^2 + k$$

You will quickly learn that the graph of the quadratic function is shaped like a "U" and is called a **parabola**. This form of the quadratic function is called **vertex form**, so named because the form easily reveals the vertex or “turning point” of the parabola. Each of the constants in the vertex form of the quadratic function plays a role. As you will soon see, the constant a controls the scaling (stretching or compressing of the parabola), the constant h controls a horizontal shift and placement of the axis of symmetry, and the constant k controls the vertical shift. Let’s begin by looking at the scaling of the quadratic.

Scaling the Quadratic

The graph of the basic quadratic function $f(x) = x^2$ shown below is a parabola. We say that the parabola in the figure below “opens upward.” The point at (0, 0), the “turning point” of the parabola, is called the vertex of the parabola. We’ve tabulated a few points for reference in the table and then superimposed these points on the graph of $f(x) = x^2$.

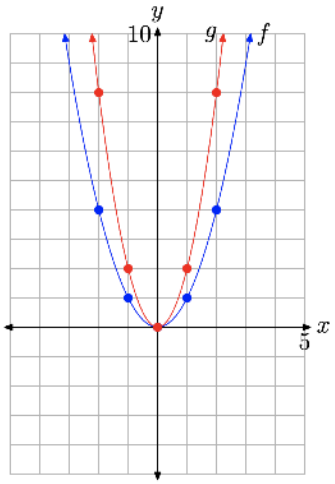


(a) A basic parabola.

x	$f(x) = x^2$
−2	4
−1	1
0	0
1	1
2	4

(b) Table of x-values and function values satisfying $f(x) = x^2$.

Now that we know the basic shape of the parabola determined by $f(x) = x^2$, let’s see what happens when we scale the graph of $f(x) = x^2$ in the vertical direction. For example, let’s investigate the graph of $g(x) = 2x^2$. The factor of 2 has a doubling effect. Note that each of the function values of g is twice the corresponding function value of f in the table.



(a) The graphs of f and g.

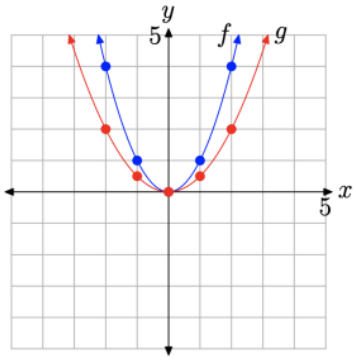
x	$f(x) = x^2$	$g(x) = 2x^2$
-2	4	8
-1	1	2
0	0	0
1	1	2
2	4	8

(b) Table of x-values and function values satisfying $f(x) = x^2$ and $g(x) = 2x^2$.

Stretching Factor

If a is a constant larger than 1, that is, if $a > 1$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is **stretched by a factor of a.**

Next, let's consider what happens when we scale by a number that is smaller than 1 (but greater than zero — we'll deal with the negative in a moment). For example, let's investigate the graph of $g(x) = (1/2)x^2$. The factor 1/2 has a halving effect. Note that each of the function values of g is half the corresponding function value of f in the table below.



(a) The graphs of f and g.

x	$f(x) = x^2$	$g(x) = (1/2)x^2$
-2	4	2
-1	1	1/2
0	0	0
1	1	1/2
2	4	2

(a) Table of x-values and function values satisfying $f(x) = x^2$ and $g(x) = (1/2)x^2$.

Compression Factor

If a is a constant smaller than 1 (but larger than zero), that is, if $0 < a < 1$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is **compressed by a factor of $\frac{1}{a}$.**

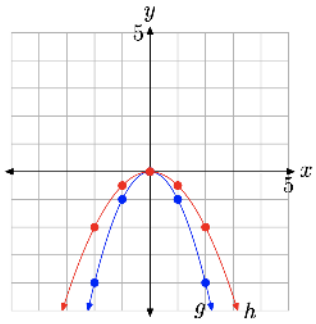
Some find this somewhat counterintuitive. However, if you compare the function $g(x) = (1/2)x^2$ with the general form $g(x) = ax^2$, you see that $a = \frac{1}{2}$. This property states that the graph will be compressed by a factor of $\frac{1}{a}$. In this case, $a = 1/2$ and

$$\frac{1}{a} = \frac{1}{1/2} = 2$$

Thus, the graph of $g(x) = (1/2)x^2$ should be compressed by a factor of 1/(1/2) or 2.

Vertical Reflections

Let's consider the graph of $g(x) = ax^2$, when a < 0. For example, consider the graphs of $g(x) = -x^2$ and $h(x) = (-1/2)x^2$ shown here.



(a) The graphs of g and h.

x	$g(x) = -x^2$	$h(x) = (-1/2)x^2$
−2	−4	−2
−1	−1	−1/2
0	0	0
1	−1	−1/2
2	−4	−2

(b) Table of x-values and function values satisfying $g(x) = -x^2$ and $h(x) = (-1/2)x^2$.

When the two tables are compared, it is easy to see that the numbers in the last two columns are the same, but they've been negated. The graphs have been reflected across the x-axis. Each of the parabolas now "opens downward."

However, it is encouraging to see that the scaling role of the constant a in $g(x) = ax^2$ remains unchanged. In the case of $h(x) = (-1/2)x^2$, the y-values are still "compressed" by a factor of two, but the minus sign negates these values, causing the graph to reflect across the x-axis. Thus, for example, one would think that the graph of $y = -2x^2$ would be stretched by a factor of two, then reflected across the x-axis. Indeed, this is correct, and this discussion leads to the following property.

Reflection Across x-axis

If $-1 < a < 0$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is compressed by a factor of 1/|a|, then reflected across the x-axis. Secondly, if a < −1, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is stretched by a factor of |a|, then reflected across the x-axis.

Again, some find this Property confusing. However, if you compare $g(x) = (-1/2)x^2$ with the general form $g(x) = ax^2$, you see that a = −1/2. Note that in this case, −1 < a < 0. The property states that the graph will be compressed by a factor of 1/|a|. In this case, a = −1/2 and

$$\frac{1}{|a|} = \frac{1}{|-1/2|} = 2$$

That is, the property states that the graph of $g(x) = (-1/2)x^2$ is compressed by a factor of 1/(| − 1/2|), or 2, then reflected across the x-axis. Note again that the vertex at the origin is unaffected by this scaling and reflection.

Horizontal Translations

The graph of $g(x) = (x + 1)^2$ below shows a basic parabola that is shifted one unit to the left. Examine the table and note that the equation $g(x) = (x + 1)^2$ produces the same y-values as does the equation $f(x) = x^2$, the only difference being that these y-values are calculated at x-values that are one unit less than those used for $f(x) = x^2$. Consequently, the graph of $g(x) = (x + 1)^2$ must shift one unit to the left of the graph of $f(x) = x^2$.

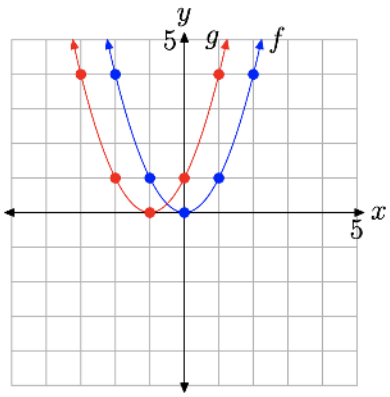
Note that this result is counterintuitive. One would think that replacing x with x + 1 would shift the graph one unit to the right, but the shift actually occurs in the opposite direction.

Finally, note that this time the vertex of the parabola has shifted 1 unit to the left and is now located at the point (−1, 0).

We are led to the following conclusion.

Shift to Left or Right

If c > 0, then the graph of $g(x) = (x + c)^2$ is shifted c units to the left of the graph of $f(x) = x^2$. However, if c > 0, then the graph of $g(x) = (x - c)^2$ is shifted c units to the right of the graph of $f(x) = x^2$.



(a) The graphs of f and g.

x	$f(x) = x^2$	x	$g(x) = (x + 1)^2$
-2	4	-3	4
-1	1	-2	1
0	0	-1	0
1	1	0	1
2	4	1	4

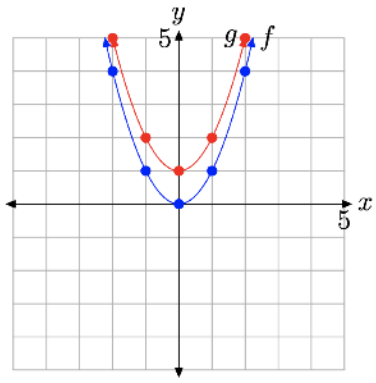
(a) Table of x-values and function values satisfying $f(x) = x^2$ and $g(x) = (x + 1)^2$.

A similar thing happens when you replace x with x − 1, only this time the graph is shifted one unit to the right.

Vertical Translations

The graph of $g(x) = x^2 + 1$ is shifted one unit upward from the graph of $f(x) = x^2$. This is easy to see as both equations use the same x-values in the table, but the function values of $g(x) = x^2 + 1$ are one unit larger than the corresponding function values of $f(x) = x^2$.

Note that the vertex of the graph of $g(x) = x^2 + 1$ has also shifted upward 1 unit and is now located at the point (0, 1).



x	$f(x) = x^2$	$g(x) = x^2 + 1$
-2	4	5
-1	1	2
0	0	1
1	1	2
2	4	5

The above discussion leads to the following property.

Shift Up or Down

If c > 0, the graph of $g(x) = x^2 + c$ is shifted c units upward from the graph of $f(x) = x^2$. However, if c > 0, the graph of $g(x) = x^2 - c$ is shifted c units downward from the graph of $f(x) = x^2$.

In a similar vein, the graph of $y = x^2 - 1$ is shifted downward one unit from the graph of $y = x^2$.

The Axis of Symmetry

In the first figure we encountered, the graph of $y = x^2$ was symmetric with respect to the y-axis. One half of the parabola is a mirror image of the other with respect to the y-axis. We say the y-axis is acting as the axis of symmetry.

If the parabola is reflected across the x-axis, as in Figure 6, the axis of symmetry doesn't change. The graph is still symmetric with respect to the y-axis. Similar comments apply to scalings and vertical translations. However, if the graph of $y = x^2$ is shifted right or

left, then the axis of symmetry will change.

Example 1

Sketch the graph of $y = -(x+2)^2 + 3$.

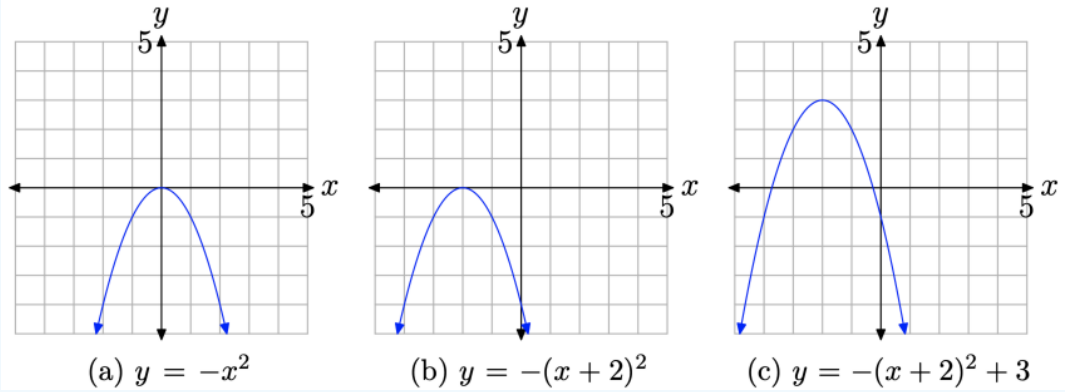
Solution

Although not required, this example is much simpler if you perform reflections before translations.

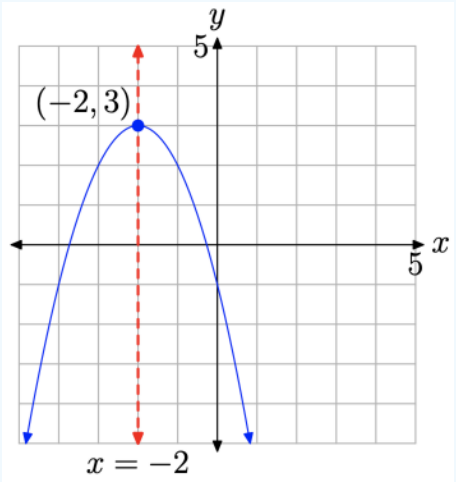
If at all possible, perform scalings and reflections before translations.

In the series shown in the next figure, we first perform a reflection, then a horizontal translation, followed by a vertical translation.

- The graph of $y = -x^2$ is a reflection of the graph of $y = x^2$ across the x-axis and opens downward. Note that the vertex is still at the origin.
- We've replaced x with x+ 2 in the equation $y = -x^2$ to obtain the equation $y = -(x+2)^2$. The effect is to shift the graph of $y = -x^2$ two units to the left to obtain the graph of $y = -(x+2)^2$. Note that the vertex is now at the point (−2, 0).
- We've added 3 to the equation $y = -(x+2)^2$ to obtain the equation $y = -(x+2)^2 + 3$. The effect is to shift the graph of $y = -(x+2)2$ upward 3 units to obtain the graph of $y = -(x+2)^2 + 3$. Note that the vertex is now at the point (−2, 3).



In practice, we can proceed more quickly. Analyze the equation $y = -(x+2)^2 + 3$. The minus sign indicates that the parabola opens downward. The presence of x + 2 indicates a shift of 2 units to the left. Finally, adding the 3 will shift the graph upward 3 units. Thus, we have a parabola that "opens downward" with a vertex at (−2, 3).



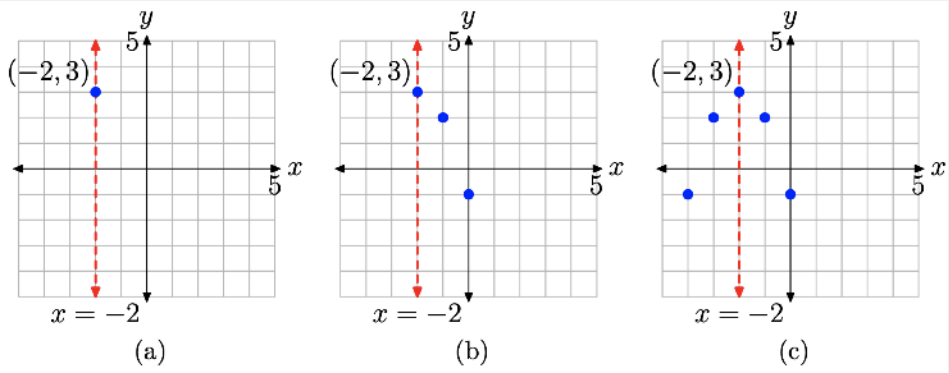
The axis of symmetry passes through the vertex (−2, 3) and has equation $x = -2$. Note that the right half of the parabola is a mirror image of its left half across this axis of symmetry. We can use the axis of symmetry to gain an accurate plot of the parabola with minimal plotting of points.

Guidelines for Using the Axis of Symmetry

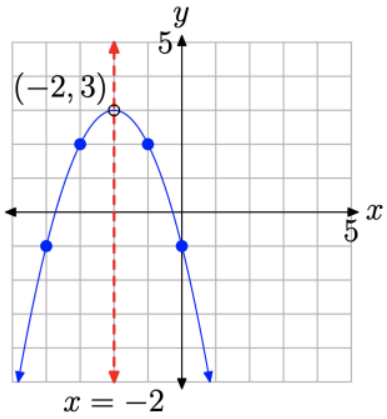
- Start by plotting the vertex and axis of symmetry.
- Next, compute two points on either side of the axis of symmetry. We choose x = −1 and x = 0 and compute the corresponding y-values using the equation $y = -(x+2)^2 + 3$

x	$y = -(x+2)^2 + 3$
-1	2
0	-1

- Plot the points from the table.
- Finally, plot the mirror images of these points across the axis of symmetry, as shown in the figure.



The image in the far right graph clearly contains enough information to complete the graph of the parabola having equation $y = -(x+2)^2 + 3$.



Let's summarize what we've seen thus far.

Summary

The form of the quadratic function

$$f(x) = a(x-h)^2 + k$$

is called vertex form. The graph of this quadratic function is a parabola.

1. The graph of the parabola opens upward if a > 0, downward if a < 0.
2. If the magnitude of a is larger than 1, then the graph of the parabola is stretched by a factor of a. If the magnitude of a is smaller than 1, then the graph of the parabola is compressed by a factor of 1/a.
3. The parabola is translated h units to the right if h > 0, and h units to the left if h < 0.

4. The parabola is translated k units upward if k > 0, and k units downward if k < 0.
5. The coordinates of the vertex are (h, k).
6. The axis of symmetry is a vertical line through the vertex whose equation is x = h.

Complete The Square

So far, you learned that it is a simple task to sketch the graph of a quadratic function if it is presented in **vertex form**

$$f(x) = a(x-h)^2 + k$$

The goal of the current section is to start with the most general form of the quadratic function, namely

$$f(x) = ax^2 + bx + c$$

and manipulate the equation into vertex form. Once you have your quadratic function in vertex form, the technique of the previous section should allow you to construct the graph of the quadratic function.

We will use special product formulas for perfect square trinomials:

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \text{or} \quad (a-b)^2 = a^2 - 2ab + b^2$$

We use these formulas to help us solve by completing the square.

We first begin by completing the square and rewriting the trinomial in factored form using the perfect square trinomial formulas.

Example 2

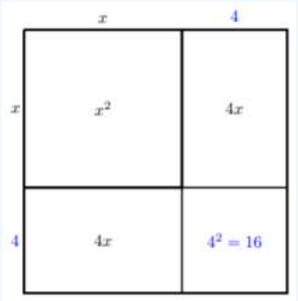
Complete the square by finding c: $x^2 + 8x + c$

Solution

There are several ways to complete the square. The first way is to mentally think about a number for c such that we can factor the trinomial as a perfect square trinomial, i.e.,

$$x^2 + 8x + c = (x + \_\_)^2$$

Some might see that this number $c = 16$ if they are keen at factoring. Notice, $(x+4)^2 = x^2 + 8x + 16$. Another way is to, literally, complete the square:



Notice the square has all components of the perfect square trinomial. Hence, we can see the dimensions of this square to be

$$(x+4) \times (x+4)$$

which is

$$(x+4)^2$$

and the missing constant coefficient is 16, the square of 4. Using one of these methods will suffice, depending on the learner. Some students enjoy the geometric relationship between the quadratic equation and the square, and some enjoy the algebraic method. It is up to the discretion of the student.

Thus, we see $c = 16$ and the perfect square trinomial $x^2 + 8x + 16$ is factored into $(x+4)^2$.

Note

To complete the square of any trinomial, we always square half of the linear term's coefficient, i.e.,

$$\left(\frac{b}{2}\right)^2 \quad \text{or} \quad \left(\frac{1}{2}b\right)^2$$

We usually use the second expression when the middle term's coefficient is a fraction.

Example 3

Complete the square by finding c: $x^2 - 7x + c$

Solution

To obtain c, we use the formula above $\left(\frac{b}{2}\right)^2$.

$$x^2 - 7x + c \quad b = -7;\ \text{apply formula } \left(\frac{b}{2}\right)^2$$

$$x^2 - 7x + \left(\frac{-7}{2}\right)^2 \quad \text{Simplify } c$$

$$x^2 - 7x + \frac{49}{4} \quad \text{Perfect square trinomial}$$

Thus, $c = \frac{49}{4}$. Rewriting this perfect square trinomial in factored form, we get

$$x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$$

Working with $f(x) = x^2 + bx + c$

The examples in this section will have the form $f(x) = x^2 + bx + c$. Note that the coefficient of x^2 is 1. In the next section, we will work with a harder form, $f(x) = ax^2 + bx + c$, where $a \neq 1$.

Algorithm for Completing the Square

The procedure for completing the square involves three key steps.

1. Take half of the coefficient of x and square the result.
2. Add and subtract the quantity from step one so that the right-hand side of the equation does not change.
3. Factor the resulting perfect square trinomial and combine constant terms.

Let's follow this procedure and place $f(x) = x^2 + 4x + 7$ in vertex form.

1. Take half of the coefficient of x. Thus, (1/2)(4) = 2. Square this result. Thus, $2^2 = 4$.
2. Add and subtract 4 on the right side of the equation $f(x) = x^2 + 4x + 7$

$$f(x) = x^2 + 4x + 4 - 4 + 7$$

3. Group the first three terms on the right-hand side. These form a perfect square trinomial.

$$f(x) = (x^2 + 4x + 4) - 4 + 7$$

Now factor the perfect square trinomial and combine the constants at the end to get

$$f(x) = (x + 2)^2 + 3$$

That's it, we're done! We've returned the general quadratic $f(x) = x^2 + 4x + 7$ back to vertex form $f(x) = (x + 2)^2 + 3$.

Example 4

Complete the square to place $f(x) = x^2 - 8x + 21$ in vertex form and sketch its graph.

Solution

First, take half of the coefficient of x and square; i.e., $[(1/2)(-8)]^2 = 16$. On the right side of the equation, add and subtract this amount so as to not change the equation.

$$f(x) = x^2 - 8x + 16 - 16 + 21$$

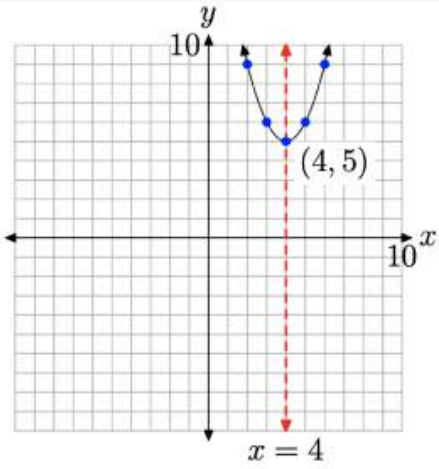
Group the first three terms on the right-hand side of the equation.

$$f(x) = (x^2 - 8x + 16) - 16 + 21$$

The first three terms form a perfect square trinomial that is easily factored. Also, combine constants at the end.

$$f(x) = (x - 4)^2 + 5$$

This is a parabola that opens upward. We need to shift the parabola 4 units to the right and then 5 units upward, placing the vertex at (4, 5). The table calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.



(a)

x	$y = (x - 4)^2 + 5$
5	6
6	9

(b)

Working with $f(x) = ax^2 + bx + c$

In the last two examples, the coefficient of x^2 was 1. In this section, we will learn how to complete the square when the coefficient of x^2 is some number other than 1.

Example 5

Complete the square to place $f(x) = -x^2 + 6x - 2$ in vertex form and sketch its graph.

Solution

If we factor out the coefficient of x^2, this will leave us with a trinomial having a leading coefficient of 1, which enables us to proceed much as we did before: halve the middle coefficient and square, add and subtract this amount, and factor the resulting perfect square trinomial. Since we were successful with this technique in the last example, let's begin by factoring out the leading coefficient, in this case −1.

$$f(x) = -1\left[x^2 - 6x + 2\right]$$

If we ignore the factor of −1 out front, the coefficient of x^2 in the trinomial expression inside the parentheses is a 1. Again, familiar ground! We will proceed as we did before, but we will carry the factor of −1 outside the parentheses in each step. Start by taking half of the coefficient of x and squaring the result; i.e., $[(1/2)(-6)]^2 = 9$.

Add and subtract this amount inside the parentheses so as to not change the equation.

$$f(x) = -1\left[x^2 - 6x + 9 - 9 + 2\right]$$

Group the first three terms inside the parentheses and combine constants.

$$f(x) = -1\left[(x^2 - 6x + 9) - 7\right]$$

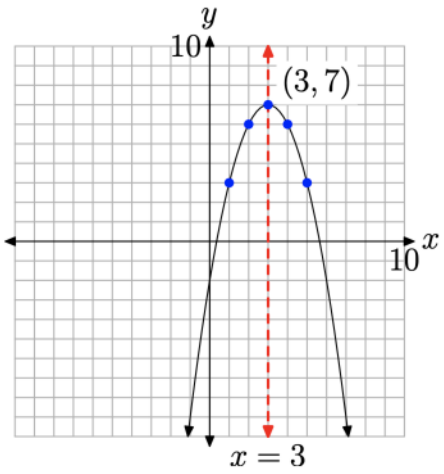
The grouped terms inside the parentheses form a perfect square trinomial that is easily factored.

$$f(x) = -1\left[(x-3)^2 - 7\right]$$

Finally, redistribute the −1.

$$f(x) = -(x-3)^2 + 7$$

This is a parabola that opens downward. The parabola is also shifted 3 units to the right, then 7 units upward, placing the vertex at (3, 7). The table calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.



(a)

x	$y = -(x-3)^2 + 7$
4	6
5	3

(b)

Let's try one more example.

Example 6

Complete the square to place $f(x) = 3x^2 + 4x - 8$ in vertex form and sketch its graph.

Solution

Let's begin again by factoring out the leading coefficient, in this case a 3.

$$f(x) = 3\left[x^2 + \frac{4}{3}x - \frac{8}{3}\right]$$

Fractions add a degree of difficulty, but, if you follow the same routine as in the previous examples, you should be able to get the needed result. Take half of the coefficient of x and square the result; i.e., $[(1/2)(4/3)]^2 = [2/3]^2 = 4/9$.

Add and subtract this amount inside the parentheses so as to not change the equation.

$$f(x) = 3\left[x^2 + \frac{4}{3}x + \frac{4}{9} - \frac{4}{9} - \frac{8}{3}\right]$$

Group the first three terms inside the parentheses. You'll need a common denominator to combine constants.

$$f(x) = 3\left[\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) - \frac{4}{9} - \frac{24}{9}\right]$$

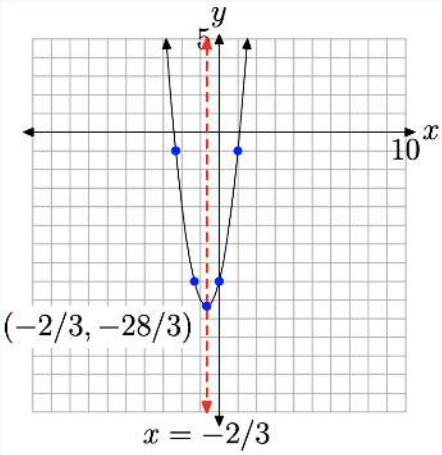
The grouped terms inside the parentheses form a perfect square trinomial that is easily factored.

$$f(x) = 3\left[\left(x + \frac{2}{3}\right)^2 - \frac{28}{9}\right]$$

Finally, redistribute the 3.

$$f(x) = 3\left(x + \frac{2}{3}\right)^2 - \frac{28}{3}$$

This is a parabola that opens upward. It is also stretched by a factor of 3, so it will be narrower than all of our previous examples. The parabola is also shifted 2/3 units to the left, then 28/3 units downward, placing the vertex at (−2/3, −28/3). The table calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.



(a)

x	$y = 3(x+2/3)^2 - 28/3$
0	−8
1	−1

(b)

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2.4P: Practice

Practice Makes Progress

Perform each of the following tasks for the given quadratic function.

1. Set up a coordinate system on graph paper. Label and scale each axis.
2. Plot the vertex of the parabola and label it with its coordinates.
3. Draw the axis of symmetry and label it with its equation.
4. Set up a table near your coordinate system that contains the exact coordinates of two points on either side of the axis of symmetry. Plot them on your coordinate system and their "mirror images" across the axis of symmetry.
5. Sketch the parabola and label it with its equation.
6. Use interval notation to describe both the domain and range of the quadratic function.

Exercise 1

$f(x) = (x+2)^2 - 3$

Exercise 2

$f(x) = -(x-2)^2 + 5$

Exercise 3

$f(x) = -\frac{1}{2}(x+1)^2 + 5$

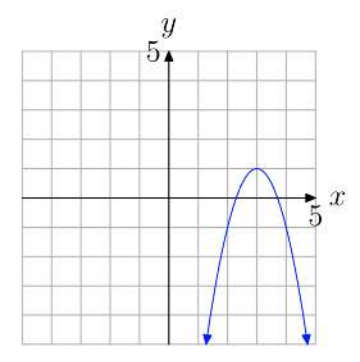
Exercise 4

$f(x) = 2(x - \frac{5}{2})^2 - \frac{15}{2}$

Using the given value of a, find the specific quadratic function of the form $f(x) = a(x-h)^2 + k$ that has the graph shown. Note: h and k are integers. Check your solution with a graphing utility such as Desmos.

Exercise 5

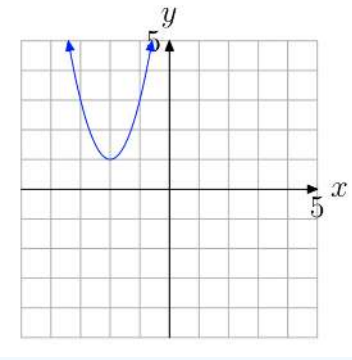
a = −2



Exercise 6

a = 2

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For each of the quadratic functions, perform each of the following tasks.

1. Use the technique of completing the square to place the given quadratic function in vertex form.
2. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
3. Draw the axis of symmetry and label it with its equation. Plot the vertex and label it with its coordinates.
4. Set up a table near your coordinate system that calculates the coordinates of two points on either side of the axis of symmetry. Plot these points and their mirror images across the axis of symmetry. Draw the parabola and label it with its equation
5. Use the graph of the parabola to determine the domain and range of the quadratic function. Describe the domain and range using interval notation.

? Exercise 7

$f(x) = x^2 - 8x + 12$

? Exercise 8

$f(x) = x^2 - 2x - 6$

? Exercise 9

$f(x) = -2x^2 - 12x - 13$

? Exercise 10

$f(x) = -\frac{1}{2}x^2 - 3x + \frac{1}{2}$

? Exercise 11

$f(x) = x^2 - x + 8$

? Exercise 12

$f(x) = 5x^2 + 5x + 5$

Self-Check

In this section, you practiced:

a. Identifying the vertex of a quadratic function written in vertex form
b. Identifying the axis of symmetry of a quadratic function written in vertex form
c. Sketching parabolas by using the vertex, axis of symmetry, and symmetric points
d. Describing the domain and range of quadratic functions
e. Completing the square to write a quadratic function in vertex form

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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2.5: Optimization

Learning Objectives

- Use the vertex to find the absolute maximum or minimum of a quadratic function
- Explain how the extreme value formula $f\left(-\frac{b}{2a}\right)$ gives the absolute maximum or minimum of a quadratic function
- Model real world optimization problems with a quadratic function

In this section, we will explore the science of optimization. Suppose you are trying to find a pair of numbers with a fixed sum so that the product of the two numbers is maximized. This is an example of an optimization problem. However, optimization is not limited to finding a maximum. For example, consider a manufacturer that wants to minimize its costs based on certain criteria. This is another example of an optimization problem. As you can see, optimization can encompass finding either a maximum or a minimum.

Optimization can be applied to a broad family of different functions. However, in this section, we will concentrate on finding the maximums and minimums of quadratic functions. There is a large body of real-life applications that quadratic functions can model, so we will find that this is an excellent entry point into the study of optimization.

Finding the Maximum or Minimum of a Quadratic Function

Consider the quadratic function

$$f(x) = -x^2 + 4x + 2$$

Let's complete the square to place this quadratic function in vertex form. First, factor out a minus sign.

$$f(x) = -\left[x^2 - 4x - 2\right]$$

Take half of the coefficient of x and square, as in $[(1/2)(-4)]^2 = 4$. Add and subtract this amount to keep the equation balanced.

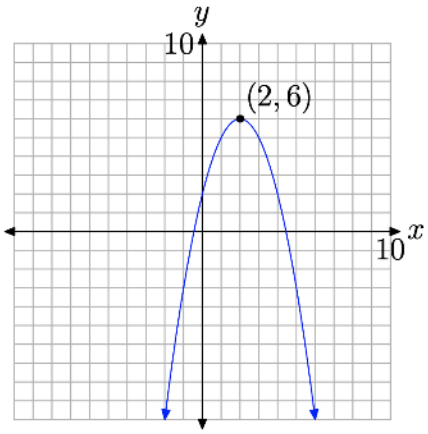
$$f(x) = -\left[x^2 - 4x + 4 - 4 - 2\right]$$

Factor the perfect square trinomial, combine the constants at the end, and then redistribute the minus sign to place the quadratic function in vertex form.

$$f(x) = -\left[(x-2)^2 - 6\right]$$
$$f(x) = -(x-2)^2 + 6$$

This is a parabola that opens downward, has been shifted 2 units to the right and 6 units upward. This places the vertex of the parabola at (2, 6), as shown below. Note that the maximum function value (y-value) occurs at the vertex of the parabola. A mathematician would say that the function "attains a maximum value of 6 at x equals 2."

Note that 6 is greater than or equal to any other y-value (function value) that occurs on the parabola. This gives rise to the following definition.



Maximum

The function f is said to achieve a **maximum** at $x = c$ if $f(c) \geq f(x)$ for all x.

Next, let's look at a quadratic function that attains a minimum on its domain.

Example 1

Find the minimum of $f(x) = 2x^2 + 12x + 12$.

Solution

Factor out a 2.

$$f(x) = 2\left[x^2 + 6x + 6\right]$$

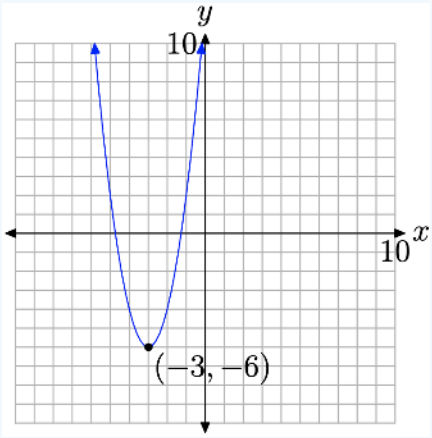
Take half of the coefficient of x and square, as in $[(1/2)(6)]^2 = 9$. Add and subtract this amount to keep the equation balanced.

$$f(x) = 2\left[x^2 + 6x + 9 - 9 + 6\right]$$

Factor the trinomial and combine the constants, and then redistribute the 2 in the next step.

$$f(x) = 2\left[(x+3)^2 - 3\right]$$
$$f(x) = 2(x+3)^2 - 6$$

The graph is a parabola that opens upward, shifted 3 units to the left and 6 units downward. This places the vertex at (−3, −6). Note that the minimum function value (y-value) occurs at the vertex of the parabola. A mathematician would say that the function "attains a minimum value of −6 at x equals −3."



Note that −6 is less than or equal to any other y-value (function value) that occurs on the parabola.

This last example gives rise to the following definition.

Minimum

The function f is said to achieve a **minimum** at $x = c$ if $f(c) \leq f(x)$ for all x.

A Shortcut for the Vertex

It should now be clear that the vertex of the parabola plays a crucial role when optimizing a quadratic function. We also know that we can complete the square to find the coordinates of the vertex. However, it would be nice if we had a quicker way of finding the coordinates of the vertex. Let's look at the general quadratic function

$$y = ax^2 + bx + c$$

and complete the square to find the coordinates of the vertex. First, factor out the a.

$$y = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

Take half of the coefficient of x and square, as in $[(1/2)(b/a)]^2 = [b/(2a)]^2 = b^2/(4a^2)$. Add and subtract this amount to keep the equation balanced.

$$y = a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

Factor the perfect square trinomial and make equivalent fractions for the constant terms with a common denominator.

$$y = a\left[\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{b^2}{4a^2} + \tfrac{4ac}{4a^2}\right]$$
$$y = a\left[\left(x + \tfrac{b}{2a}\right)^2 + \tfrac{4ac-b^2}{4a^2}\right]$$

Finally, redistribute that a. Note how multiplying by a cancels one a in the denominator of the constant term.

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Now, here's the key idea. The results depend upon the values of a, b, and c, but it should be clear that the coordinates of the vertex are

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$

The y-value of the vertex is a bit hard to memorize, but the x-value of the vertex is easy to memorize.

Find the Extreme Value

Extreme Value

The ***extreme value*** of a quadratic function $f(x)$ is either the *minimum* or *maximum* value of the quadratic function $f(x)$. The extreme value is given as

$$f\left(-\frac{b}{2a}\right)$$

The *minimum* value is located at the lowest point of an upwards parabola, and the *maximum* value is located at the highest point of a downwards parabola.

Note

Notice in the definition that the extreme value is the y-coordinate of the vertex. Recall, the vertex is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

and we can see the y-coordinate is just the extreme value. Hence, when we use the words vertex, minimum or maximum value, or extreme value, they are all associated with the vertex of a parabola.

Example 2

Find the vertex and the extreme value of the function $f(x) = x^2 - 4x + 5$.

Solution

Notice $a = 1$, which means $a > 0$. Hence, $f(x)$ is an upwards parabola and, from the definition, we expect $f(x)$ to have a minimum value. Let's use the formula to find the vertex, where $a = 1$ and $b = -4$.

$$\begin{aligned} x &= -\frac{b}{2a} \\ x &= -\frac{-4}{2(1)} && \text{Simplify} \\ x &= 2 && \text{The } x\text{-coordinate of the vertex} \end{aligned}$$

Next, we find the y-coordinate of the vertex by obtaining $f(2)$.

$$\begin{aligned} f(x) &= x^2 - 4x + 5 \\ f(2) &= (2)^2 - 4(2) + 5 && \text{Evaluate} \\ f(2) &= 1 && \text{The } y\text{-coordinate of the vertex} \end{aligned}$$

Hence, the vertex is $(2, 1)$. Next, we find the extreme value of $f(x)$. From the vertex calculation, we see $f\left(-\frac{b}{2a}\right) = f(2) = 1$. Thus, the extreme value is 1, the y-coordinate of the vertex.

Example 3

Find the vertex and the extreme value of the function $q(n) = -3n^2 - 5n + 3$.

Solution

Notice $a = -3$, which means $a < 0$. Hence, $q(n)$ is a downward parabola and, from the definition, we expect $q(n)$ to have a maximum value. Let's use the formula to find the vertex, where $a = -3$ and $b = -5$.

$$\begin{aligned} n &= -\frac{b}{2a} \\ n &= -\frac{-5}{2(-3)} && \text{Simplify} \\ n &= -\frac{5}{6} && \text{The } x\text{-coordinate of the vertex} \end{aligned}$$

Next, we find the y-coordinate of the vertex by obtaining $q\left(-\frac{5}{6}\right)$.

$$\begin{aligned} q(n) &= -3n^2 - 5n + 3 \\ q\left(-\frac{5}{6}\right) &= -3\left(-\frac{5}{6}\right)^2 - 5\left(-\frac{5}{6}\right) + 3 && \text{Evaluate} \\ q\left(-\frac{5}{6}\right) &= \frac{61}{12} && \text{The } y\text{-coordinate of the vertex} \end{aligned}$$

Hence, the vertex is $\left(-\frac{5}{6}, \frac{61}{12}\right)$. Next, we find the extreme value of $q(n)$. From the vertex calculation, we see $q\left(-\frac{b}{2a}\right) = q\left(-\frac{5}{6}\right) = \frac{61}{12}$. Thus, the extreme value is $\frac{61}{12}$, the y-coordinate of the vertex.

It may seem redundant to find the vertex and the extreme value, but remember that the goal is to apply this concept to real-world applications involving quadratic functions. Let's examine a few applications where we encounter the extreme value in the context of a real-world model.

Projectile Motion

Example 4

A rocket is launched at $t = 0$ seconds. Its height, in meters above sea-level, as a function of time is given by $h(t) = -4.9t^2 + 46t + 157$. At what time does the rocket reach its maximum height? At what height does the rocket reach its maximum height above the water? *Round the answers to 2 decimal places.*

Solution

When we read the word *maximum*, we should think about the vertex of $h(t)$. Since we need to find the time in which the maximum height occurs, then we can find the x-coordinate of the vertex.

$$\begin{aligned} t &= -\frac{b}{2a} \\ t &= -\frac{46}{2(-4.9)} \\ t &\approx 4.69 \end{aligned} \tag{2.5.1}$$

Thus, the maximum height occurs after 4.69 seconds. Next, we find the height of the rocket when it reaches its maximum height above the water. Since we need to find the maximum height, then we need to find the y-coordinate of the vertex, or $h(4.69)$.

$$h(4.69) = -4.9(4.69)^2 + 46(4.69) + 157 \approx 264.96$$

Thus, the maximum height of the rocket is 264.96 meters after 4.69 seconds the rocket is launched.

Revenue and Cost Functions

Example 5

The revenue, $R(x)$, of producing and selling x *Awesome Hearing Aids* is modeled by the function $R(x) = -6x^2 + 108x$. What is the maximum revenue?

Solution

To find the maximum revenue, we need to find $R\left(-\frac{b}{2a}\right)$, where $a = -6$ and $b = 108$. Let's plug-n-chug this into $R(x)$ to find the maximum revenue.

$$\begin{aligned} R(x) &= -6x^2 + 108x \\ R\left(-\frac{b}{2a}\right) &= -6\left(-\frac{b}{2a}\right)^2 + 108\left(-\frac{b}{2a}\right) \\ R\left(-\frac{108}{2(-6)}\right) &= -6\left(-\frac{108}{2(-6)}\right)^2 + 108\left(-\frac{108}{2(-6)}\right) \\ R\left(-\frac{108}{2(-6)}\right) &= 486 \end{aligned} \tag{2.5.2}$$

Thus, the maximum revenue is $486.

Example 6

The cost, $C(x)$, of producing x *Totally Cool Coolers* is modeled by the function $C(x) = 0.005x^2 - 0.3x + 17$. What is the minimum cost?

Solution

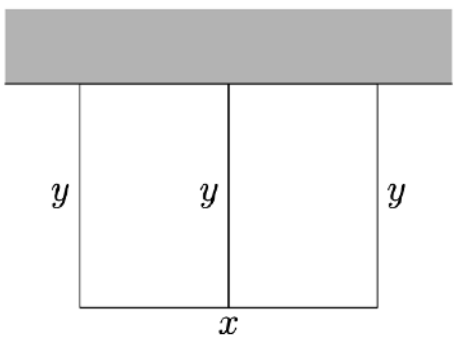
To find the minimum cost, we need to find $C\left(-\frac{b}{2a}\right)$, where $a = 0.005$ and $b = -0.3$. Let's plug-n-chug this into $C(x)$ to find the minimum cost.

$$\begin{aligned} C(x) &= 0.005x^2 - 0.3x + 17 \\ C\left(-\frac{b}{2a}\right) &= 0.005\left(-\frac{b}{2a}\right)^2 - 0.3\left(-\frac{b}{2a}\right) + 17 \\ C\left(-\frac{-0.3}{2(0.005)}\right) &= 0.005\left(-\frac{-0.3}{2(0.005)}\right)^2 - 0.3\left(-\frac{-0.3}{2(0.005)}\right) + 17 \\ C\left(-\frac{-0.3}{2(0.005)}\right) &= 12.5 \end{aligned} \tag{2.5.3}$$

Area and Perimeter Functions

Example 7

Mary wants to fence a rectangular garden to keep the deer from eating her fruit and vegetables. One side of her garden abuts her shed wall, so she will not need to fence that side. However, she also wants to use material to separate the rectangular garden into two sections. She can afford to buy 80 total feet of fencing to use for the perimeter and the section dividing the rectangular garden. What dimensions will maximize the total area of the rectangular garden?



Solution

Again, before we take an algebraic approach, let's just experiment. Note that we've labeled the width with the letter x and the height with the letter y in our sketch of the garden.

There is a total of 80 feet of fence material. Suppose that we let $y = 5$ ft. Because there are three sides of length $y = 5$ ft, we've used 15 feet of material. That leaves 65 feet of material, which will be used to fence the width of the garden. That is, the width is $x = 65$ ft. Thus, the dimensions of the garden are $x = 65$ ft by $y = 5$ ft. The area equals the product of these two measures, so $A = 325\text{ft}^2$. Let's put this result into a table.

x	y	$A = xy$
65 ft	5 ft	325 ft^2

Suppose instead that we let the height be $y = 10$ ft. Again, there are three sections of this length, so this will take 30 ft of material. That leaves 50 ft of material, so the width $x = 50$ ft. The area is the product of these two measures, so $A = 500\text{ft}^2$. As a third experiment, let the height $y = 15$ ft. Subtracting three of these lengths from 80 ft, we see that the width $x = 35$ ft. The area is the product of these measures, so $A = 525\text{ft}^2$. Let's add these last two number experiments to our table.

x	y	$A = xy$
65 ft	5 ft	325 ft^2
50 ft	10 ft	500 ft^2
35 ft	15 ft	525 ft^2

At this point, the last set of dimensions yields the maximum area, but is it possible that another choice of x and y will yield a larger area? Experiment further with numbers of your choice to see if you can find dimensions that will yield an area larger than the current maximum in the table, namely 525 ft^2.

Let's now call on what we've learned in this section to attack this model. First, we're constrained by the amount of material we have for the job, a total of 80 ft of fencing. This constraint requires that 3 times the height of the garden, plus the width of the garden, should equal the available amount of fencing material. In symbols,

$$x + 3y = 80$$

We're asked to maximize the area, so we focus our efforts on finding a formula for the area of the rectangular garden. Because the area A of the rectangular garden is the product of the width and the height,

$$A = xy$$

We now have a formula for the area of the rectangular garden, but unfortunately, we have the area A as a function of two variables. We need to eliminate one or the other of these variables. This is easily done by solving the fence equation for x.

$$x = 80 - 3y$$

Next, substitute this result in the area equation to get

$$A = (80 - 3y)y$$

or, equivalently,

$$A = -3y^2 + 80y$$

Note that we have expressed the area A as a function of a single variable y. Also, the area function is quadratic. Compare $A = -3y^2 + 80y$ with the general form $A = ay^2 + by + c$ and note that $a = -3$ and $b = 80$ (we have no need of the fact that $c = 0$). Therefore, if we plot A versus y, the graph is a parabola that opens downward, so the maximum value of A will occur at the vertex. The y-coordinate of the vertex is found with

$$y_{\text{vertex}} = -\frac{b}{2a} = -\frac{80}{2(-3)} = \frac{80}{6} = \frac{40}{3}$$

To find the width of the rectangular garden, substitute $y = 40/3$ into equation (20) and solve for x.

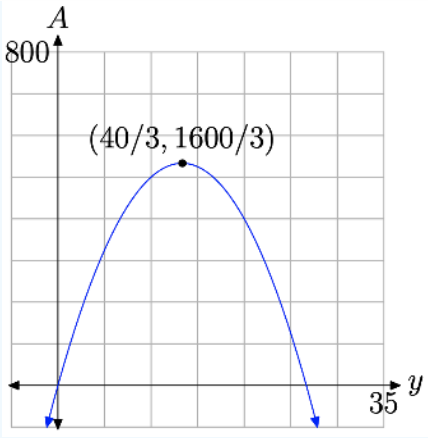
$$x = 80 - 3y = 80 - 3\left(\frac{40}{3}\right) = 80 - 40 = 40$$

Thus, the width of the rectangular garden is 40 ft. We can find the area of the garden by multiplying its width by its height.

$$A = xy = (40)\left(\frac{40}{3}\right) = \frac{1600}{3} = 533\frac{1}{3}$$

Note that the resulting area, $A = 533\frac{1}{3}\text{ft}^2$, is only slightly bigger than the last tabular entry found with our numerical experiments.

You can also find the area of the rectangular region by substituting $y = \dfrac{40}{3}$ into the quadratic equation. We'll leave it to our readers to check that this provides the same measure for the area. You will also notice that the second coordinate of the vertex is the maximum area $A = 1600/3\text{ft}^2$.



Note

From all the examples, we see the variety of methods in obtaining the extreme value of a quadratic function. We can either graph the function, find each coordinate of the vertex, or calculate the extreme value directly. It is at the student's discretion to use any method. However, directly calculating the extreme value is recommended when only the extreme value is needed.

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2.5P: Practice

Practice Makes Progress

Exercise 1

Find the exact maximum value of the function $f(x) = -x^2 - 3x$.

Exercise 2

Find the vertex of the graph of the function $f(x) = -3x^2 - x - 6$.

Exercise 3

Find the exact maximum value of the function $f(x) = -3x^2 - 9x - 4$.

Exercise 4

Find the vertex of the graph of the function $f(x) = 3x^2 + 3x + 9$.

The height after t seconds of an object launched with velocity v_0 from a height of h_0 is given by $h(t) = -16t^2 + v_0 t + h_0$. Use a function in this form to answer questions 5 and 6.

Exercise 5

A ball is thrown upward at a speed of 8 ft/s from the top of a 182-foot-high building. How many seconds does it take for the ball to reach its maximum height? Round your answer to the nearest hundredth of a second.

Exercise 6

A ball is thrown upward at a speed of 52 ft/s from the top of a 293-foot-high building. What is the maximum height of the ball? Round your answer to the nearest hundredth of a foot.

Exercise 7

Among all rectangles having a perimeter of 40 feet, find the dimensions (length and width) of the one with the greatest area.

Exercise 8

A farmer with 1700 meters of fencing wants to enclose a rectangular plot that borders on a river. If no fence is required along the river, what is the largest area that can be enclosed?

Exercise 9

Let x represent the demand (the number the public will buy) for an object and let p represent the object's unit price (in dollars). Suppose that the unit price and the demand are linearly related by the equation $p = -\frac{1}{3}x + 40$.

a. Express the revenue R (the amount earned by selling the objects) as a function of the demand x.
b. Find the demand that maximizes revenue.
c. Find the unit price that maximizes revenue.
d. What is the maximum revenue?

Self-Check

In this section, you practiced:

a. Using the vertex formula to find the vertex of a quadratic function without completing the square
b. Finding the exact maximum or minimum value of a quadratic function
c. Modeling projectile motion problems using a quadratic function
d. Modeling perimeter and area problems using a quadratic function
e. Modeling revenue problems using a quadratic function

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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2.6: Zeros of the Quadratic

Learning Objectives

- Explain how the factors of a quadratic expression relate to the zeros of the quadratic function
- Use various factoring techniques to change a quadratic expression from standard form to factored form
- Use factoring techniques to solve quadratic equations

We've seen how vertex form and intelligent use of the axis of symmetry can help draw an accurate graph of the quadratic function defined by the equation $f(x) = ax^2 + bx + c$. When drawing the graph of the parabola, it is helpful to know where the graph crosses the x-axis. That is the primary goal of this section: to find the zero crossings or x-intercepts of the parabola.

Before we begin, you'll need to review the techniques that will enable you to factor the quadratic expression $ax^2 + bx + c$.

Factoring $ax^2 + bx + c$ when $a = 1$

Our intent in this section is to provide a quick review of techniques used to factor quadratic trinomials. We begin by showing how to factor trinomials having the form $ax^2 + bx + c$, where the leading coefficient is $a = 1$; that is, trinomials having the form $x^2 + bx + c$. In the next section, we will address the technique used to factor $ax^2 + bx + c$ when $a \neq 1$.

Let's begin with an example.

Example 1

Factor $x^2 + 16x - 36$

Solution

Note that the leading coefficient, the coefficient of x^2, is $a = 1$. This is an important observation because the technique presented here will not work when the leading coefficient does not equal 1.

Note the constant term of the trinomial $x^2 + 16x - 36$ is -36. List all integer pairs whose product equals -36.

$1, -36$	$-1, 36$
$2, -18$	$\boxed{-2, 18}$
$3, -12$	$-3, 12$
$4, -9$	$-4, 9$
$6, -6$	$-6, 6$

Note that we've framed the pair -2, 18. We've done this because the sum of this pair of integers equals the coefficient of x in the trinomial expression $x^2 + 16x - 36$. Use this framed pair to factor the trinomial.

$$x^2 + 16x - 36 = (x - 2)(x + 18)$$

It is important that you check your result. Use the distributive property to multiply.

$$\begin{aligned}(x - 2)(x + 18) &= x(x + 18) - 2(x + 18)\\ &= x^2 + 18x - 2x - 36\\ &= x^2 + 16x - 36\end{aligned}$$

Thus, our factorization is correct.

Let's summarize the technique.

Algorithm

To factor the quadratic $x^2 + bx + c$, proceed as follows:

1. List all the integer pairs whose product equals c.
2. Circle or frame the pair whose sum equals the coefficient of x, namely b. Use this pair to factor the trinomial.

Let's look at another example.

Example 2

Factor the trinomial $x^2 - 25x - 84$

Solution

List all the integer pairs whose product is -84.

$1, -84$	$-1, 84$
$2, -42$	$-2, 42$
$\boxed{3, -28}$	$-3, 28$
$4, -21$	$-4, 21$
$6, -14$	$-6, 14$
$7, -12$	$-7, 12$

We've framed the pair whose sum equals the coefficient of x, namely -25. Use this pair to factor the trinomial.

$$x^2 - 25x - 84 = (x + 3)(x - 28)$$

Check

$$\begin{aligned}(x + 3)(x - 28) &= x(x - 28) + 3(x - 28)\\ &= x^2 - 28x + 3x - 84\\ &= x^2 - 25x - 84\end{aligned}$$

Thus, our factorization is correct.

With experience, several ideas can accelerate the process.

- As you list the integer pairs, if you happen to note that the current pair has the appropriate sum, there is no need to list the remaining integer pairs. Simply halt the process of listing the integer pairs and use the current pair to factor the trinomial.
- Some students are perfectly happy being asked, "Can you think of an integer pair whose product is c and whose sum is b (where b and c refer to the coefficients of $x^2 + bx + c$)?" If you can pick the pair "out of the air" like this, all is well and good. Use the integer pair to factor the trinomial, and don't bother listing any integer pairs.

Now, let's investigate how to proceed when the leading coefficient is not 1.

Factoring $ax^2 + bx + c$ when $a \neq 1$

When $a \neq 1$, we use a technique called the $a \cdot c$-test to factor the trinomial $ax^2 + bx + c$. The process is best explained with an example.

Example 3

Factor $2x^2 + 13x - 24$.

Solution

Note that the leading coefficient does not equal 1. Indeed, the coefficient of x^2 in this example is $a = 2$. Therefore, the technique of the previous examples will not work. Thus, we turn to a similar technique called the ac-test.

First, compare

$$2x^2 + 13x - 24 \qquad \text{and} \qquad ax^2 + bx + c$$

and note that $a = 2$, $b = 13$, and $c = -24$. Compute the product of a and c. This is how the technique earns its name, "ac-test."

$$ac = (2)(-24) = -48$$

List all integer pairs whose product is ac = -48.

1, −48	−1, 48
2, −24	−2, 24
3, −16	−3, 16
4, −12	−4, 12
6, −8	−6, 8

We've framed the pair whose sum is $b = 13$. The next step is to rewrite the trinomial $2x^2 + 13x - 24$, splitting the middle term into a sum, using our framed integer pair.

$$2x^2 + 13x - 24 = 2x^2 - 3x + 16x - 24$$

We factor an x out of the first two terms, then an 8 out of the last two terms. This process is called factoring by grouping.

$$2x^2 - 3x + 16x - 24 = x(2x - 3) + 8(2x - 3)$$

We now factor out a common factor of $2x - 3$.

$$x(2x - 3) + 8(2x - 3) = (x + 8)(2x - 3)$$

It's helpful to see the complete process as a coherent unit.

$$\begin{aligned} 2x^2 + 13x - 24 &= 2x^2 - 3x + 16x - 24 \\ &= x(2x - 3) + 8(2x - 3) \\ &= (x + 8)(2x - 3) \end{aligned}$$

Check

Again, it is essential to verify the answer through multiplication.

$$\begin{aligned} (x + 8)(2x - 3) &= x(2x - 3) + 8(2x - 3) \\ &= 2x^2 - 3x + 16x - 24 \\ &= 2x^2 + 13x - 24 \end{aligned}$$

Because this is the original trinomial, our solution checks.

Let's summarize this process.

Algorithm: ac-Test

To factor the quadratic $ax^2 + bx + c$, proceed as follows:

1. List all integer pairs whose product equals ac.
2. Circle or frame the pair whose sum equals the coefficient of x, namely b.
3. Use the circled pair to express the middle term bx as a sum.
4. Factor by "grouping."

Let's look at another example.

Example 4

Factor $3x^2 + 34x - 24$

Solution

Compare

$$3x^2 + 34x - 24 \quad \text{and} \quad ax^2 + bx + c$$

and note that $a = 3$, $b = 34$ and $c = -24$. List all integer pairs whose product equals ac = (3)(-24) = -72.

1, −72	−1, 72
2, −36	−2, 36
3, −24	−3, 24
4, −18	−4, 18
6, −12	−6, 12
8, −9	−8, 9

We've framed the pair whose sum is the same as $b = 34$, the coefficient of x in $3x^2 + 34x - 24$. Again, shortcuts are possible. If you can "think" of a pair whose product is $a \cdot c$ = -72 and whose sum is $b = 34$, then it is not necessary to list any integer pairs. Alternatively, if you come across the needed pair as you are listing them, then you can halt the process. There is no need to list the remaining pairs if you have the one you need.

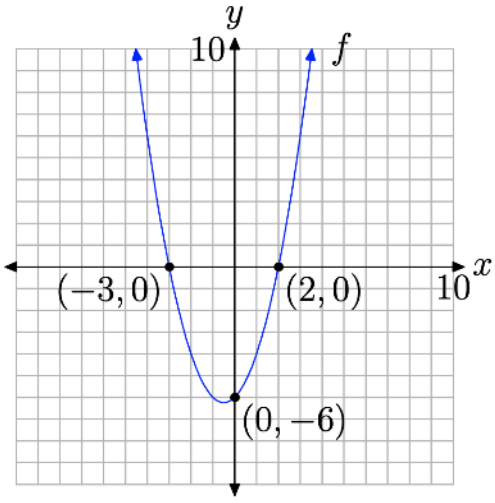
Use the framed pair to express the middle term as a sum, then factor by grouping.

$$\begin{aligned} 3x^2 + 34x - 24 &= 3x^2 - 2x + 36x - 24 \\ &= x(3x - 2) + 12(3x - 2) \\ &= (x + 12)(3x - 2) \end{aligned}$$

We leave it to the reader to check this result.

Intercepts

The points where the graph of a function crosses the x-axis are called the x-intercepts of the graph of the function. Consider the graph of the quadratic function f in the figure below.



Note that the graph of f crosses the x-axis at (-3, 0) and (2, 0). These are the x-intercepts of the parabola. Note that the y-coordinate of each x-intercept is zero.

In function notation, the solutions of $f(x) = 0$ (note the similarity to y = 0) are the x-coordinates of the points where the graph of f crosses the x-axis. Analyzing the graph of f, we see that both -3 and 2 are solutions of $f(x) = 0$.

Thus, the process for finding the x-intercepts is straightforward.

Finding x-intercepts

To find the x-intercepts of the graph of any function, set $y = 0$ and solve for x. Alternatively, if function notation is used, set $f(x) = 0$ and solve for x.

Let's look at an example.

Example 5

Find the x-intercepts of the graph of the quadratic function defined by $y = x^2 + 2x - 48$.

Solution

To find the x-intercepts, first set y = 0.

$$0 = x^2 + 2x - 48$$

Next, factor the trinomial on the right. Note that the coefficient of x^2 is 1. We need only think of two integers whose product equals the constant term -48 and whose sum equals the coefficient of x, namely 2. The numbers 8 and -6 come to mind, so the trinomial factors as follows (readers should check this result).

$$0 = (x + 8)(x - 6)$$

To complete the solution, we need to utilize a crucial property of the real numbers known as the zero product property.

In our case, we have $0 = (x + 8)(x - 6)$. Therefore, it must be the case that either

$$x + 8 = 0 \quad \text{or} \quad x - 6 = 0$$

These equations can be solved independently to produce

$$x = -8 \quad \text{or} \quad x = 6$$

Thus, the x-intercepts of the graph of $y = x^2 + 2x - 48$ are located at (-8, 0) and (6, 0).

Zero Product Property

If a and b are any real numbers such that $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Let's look at another example.

Example 6

Find the x-intercepts of the graph of the quadratic function $f(x) = 2x^2 - 7x - 15$.

Solution

To find the x-intercepts of the graph of the quadratic function f, we begin by setting

$$f(x) = 0$$

Of course, $f(x) = 2x^2 - 7x - 15$, so we can substitute to obtain

$$2x^2 - 7x - 15 = 0$$

We will now use the $a \cdot c$-test to factor the trinomial on the left. Note that $a \cdot c$ = (2)(-15) = -30. List the integer pairs whose products equal -30.

$1, -30$		$-1, 20$
$2, -15$		$-2, 15$
$\boxed{3, -10}$		$-3, 10$
$5, -6$		$-5, 6$

Note that the framed pair sums to the coefficient of x in $2x^2 - 7x - 15$. Use the framed pair to express the middle term as a sum, then factor by grouping

$$\begin{aligned} 2x^2 - 7x - 15 &= 0 \\ 2x^2 + 3x - 10x - 15 &= 0 \\ x(2x + 3) - 5(2x + 3) &= 0 \\ (x - 5)(2x + 3) &= 0 \end{aligned}$$

Now we can use the zero product property. Either

$$x - 5 = 0 \quad \text{or} \quad 2x + 3 = 0$$

Each of these can be solved independently to obtain

$$x = 5 \quad \text{or} \quad x = -3/2$$

Thus, the x-intercepts of the graph of the quadratic function $f(x) = 2x^2 - 7x - 15$ are located at (-3/2, 0) and (5, 0).

One more definition is in order.

Zeros of a Function

The solutions of $f(x) = 0$ are called the **zeros** of the function f.

Thus, in the last example, both -3/2 and 5 are zeros of the quadratic function $f(x) = 2x^2 - 7x - 15$. Note the intimate relationship between the zeros of the quadratic function and the x-intercepts of the graph. Note that -3/2 is a zero and (-3/2, 0) is an x-intercept. Similarly, 5 is a zero and (5, 0) is an x-intercept.

Finding y-Intercepts

To find the y-intercepts of the graph of any function, set $x = 0$ and solve for y. Alternatively, if function notation is used, evaluate $f(0)$.

Example 7

Find the y-intercept of the quadratic function defined by $f(x) = x^2 - 3x - 11$.

Solution

Evaluate the function at $x = 0$.

$$f(0) = (0)^2 - 3(0) - 11 = -11$$

The coordinates of the y-intercept are (0, -11).

Putting it All Together

We will find both x- and y-intercepts extremely useful when graphing a quadratic function.

Example 8

Place the quadratic function $y = x^2 + 2x - 24$ in vertex form. Plot the vertex and axis of symmetry and label them with their coordinates and equation, respectively. Find and plot the x- and y-intercepts of the parabola and label them with their coordinates.

Solution

Take half of the coefficient of x, square it, then add and subtract this amount to balance the equation. Factor and combine coefficients.

$$y = x^2 + 2x + 1 - 1 - 24$$
$$y = (x + 1)^2 - 25$$

The graph is a parabola that opens upward; it is shifted 1 unit to the left and 25 units downward. This information is enough to plot and label the vertex, then plot and label the axis of symmetry.

To find the x-intercepts, let y = 0 in $y = x^2 + 2x - 24$.

$$0 = x^2 + 2x - 24$$

The leading coefficient is $a = 1$. The integer pair -4 and 6 has a product of -24 and a sum of 2. Thus, the right-hand side factors as follows.

$$0 = (x + 6)(x - 4)$$

In order that this product equals zero, either

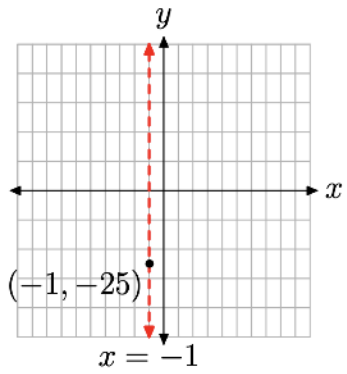
$$x + 6 = 0 \quad \text{or} \quad x - 4 = 0$$

Solve each of these linear equations independently.

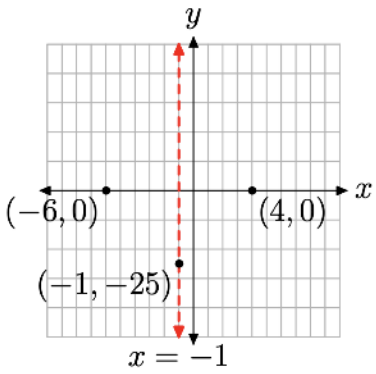
$$x = -6 \quad \text{or} \quad x = 4$$

Recall that we let $y = 0$. We've found two solutions, $x = -6$ and $x = 4$. Thus, we have x-intercepts at (-6, 0) and (4, 0).

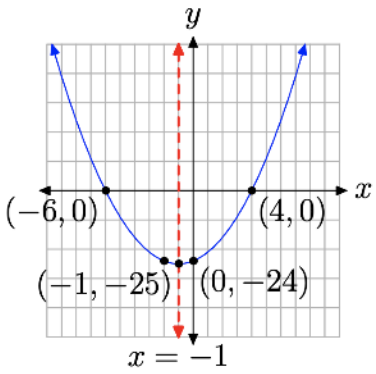
Finally, to find the y-intercept, let $x = 0$ in $y = x^2 + 2x - 24$. With this substitution, $y = -24$. Thus, the y-intercept is (0, -24). Note that we've also included the mirror image of the y-intercept across the axis of symmetry.



(a) Plotting the vertex and axis of symmetry.



(b) Adding the x-intercepts provides added accuracy.



(c) Adding the y-intercept and its mirror image provides an excellent final graph.

Let's look at one more example.

Example 9

Plot the parabola represented by the equation $f(x) = -2x^2 - 7x + 15$. Plot and label the vertex, axis of symmetry, and the x- and y-intercepts.

Solution

First, factor out a -2.

$$f(x) = -2\left[x^2 + \frac{7}{2}x - \frac{15}{2}\right]$$

Half of 7/2 is 7/4. Squared, this amounts to 49/16. Add and subtract this last amount to keep the equation balanced.

$$f(x) = -2\left[x^2 + \frac{7}{2}x + \frac{49}{16} - \frac{49}{16} - \frac{15}{2}\right]$$

The first three terms inside the parentheses form a perfect square trinomial. The last two constants are combined with a common denominator.

$$f(x) = -2\left[\left(x^2 + \tfrac{7}{2}x + \tfrac{49}{16}\right) - \tfrac{49}{16} - \tfrac{120}{16}\right]$$
$$f(x) = -2\left[\left(x + \tfrac{7}{4}\right)^2 - \tfrac{169}{16}\right]$$

Finally, redistribute the -2.

$$f(x) = -2\left(x + \frac{7}{4}\right)^2 + \frac{169}{8}$$

The graph of this last equation is a parabola that opens downward, translated 7/4 units to the left and 169/8 units upward. This is enough information to plot and label the vertex and axis of symmetry.

To find the y-intercepts, set $f(x) = 0$ in $f(x) = -2x^2 - 7x + 15$. We will also multiply both sides of the resulting equation by -1.

$$0 = -2x^2 - 7x + 15$$
$$0 = 2x^2 + 7x - 15$$

After comparing $2x^2 + 7x - 15$ with $ax^2 + bx + c$, we note that the integer pair -3 and 10 have product equal to $a \cdot c$ = -30 and sum equal to $b = 7$. Use this pair to express the middle term of $2x^2 + 7x - 15$ as a sum and then factor by grouping.

$$0 = 2x^2 - 3x + 10x - 15$$
$$0 = x(2x - 3) + 5(2x - 3)$$
$$0 = (x + 5)(2x - 3)$$

By the zero product property, either

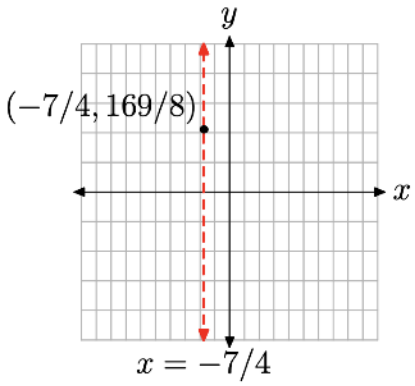
$$x + 5 = 0 \qquad \text{or} \qquad 2x - 3 = 0$$

Solve these linear equations independently.

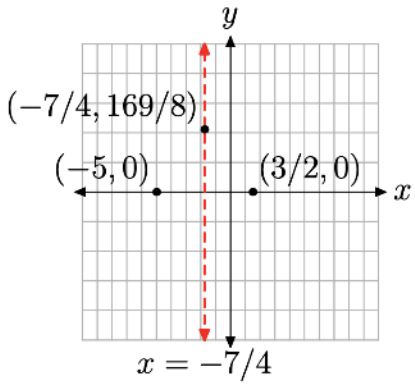
$$x = -5 \qquad \text{or} \qquad x = \frac{3}{2}$$

These x-values are the zeros of f (they make $f(x) = 0$), so we have x-intercepts at (-5, 0) and (3/2, 0).

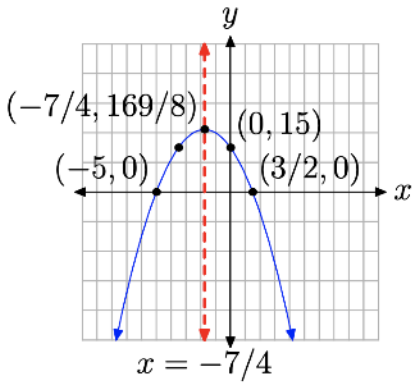
Finally, to find the y-intercept, set $x = 0$ in $f(x) = -2x^2 - 7x + 15$ to get f(0) = 15. Note the positioning of the y-intercept (0, 15) and its mirror image across the axis of symmetry.



(a) Plotting the vertex and axis of symmetry.



(b) Adding the x-intercepts provides added accuracy.



(c) Adding the y-intercept and its mirror image provide an excellent final graph.

Solving Quadratic Inequalities by Graphing

We can recall an earlier section where we obtained solutions for linear inequalities, which are intervals of numbers. With quadratic inequalities, the process is similar to that of linear inequalities, but now, instead of lines, we have parabolas.

Note

We can use the zeros (or roots) of a graph of a quadratic equation to solve for quadratic inequalities. Recall, the zeros (or roots) of a graph are the x-intercepts, i.e., $y = 0$.

Example 10

Solve $x^2 - 3x - 4 \geq 0$ by using the graphing method.

Solution

We first begin by graphing $x^2 - 3x - 4$ as we would in the previous section. Recall, the x-coordinate of the vertex is $-\frac{b}{2a}$. Hence,

$$-\frac{b}{2a} = -\frac{-3}{2(1)} = \frac{3}{2} = 1.5$$

The y-coordinate of the vertex is $x^2 - 3x - 4$ evaluated for $x = 1.5$:

$$(1.5)^2 - 3(1.5) - 4 = -6.25$$

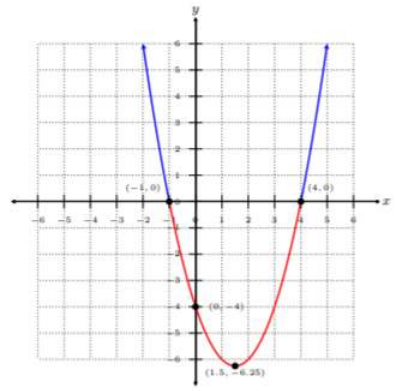
The vertex is at $(1.5, -6.25)$. The y-intercept is when $x = 0$:

$$0^2 - 3(0) - 4 = -4$$

and the x-intercepts are when $x^2 - 3x - 4 = 0$. Solving for the roots we get

$$\begin{aligned} x^2 - 3x - 4 &= 0 \\ (x-4)(x+1) &= 0 \\ x - 4 = 0 \quad &\text{or} \quad x + 1 = 0 \\ x = 4 \quad &\text{or} \quad x = -1 \end{aligned} \tag{2.6.1}$$

Thus, the intercepts are $(0, -4)$, $(4, 0)$, and $(-1, 0)$ with a vertex at $(1.5, -6.25)$. Now we draw the graph:



Given $x^2 - 3x - 4 \geq 0$ says we are specifically looking for the values of $x^2 - 3x - 4$ that are greater than or equal to zero. In other words, we are looking for all y-values that are above the x-axis because that is where all the y-values are positive. Looking at the graph above, these are all the values on the blue parts of the graph. Thus, using the graphing method, the solution is $(-\infty, -1] \cup [4, \infty)$. Note, we use brackets since it is $\geq$.

Notice, we ignore the red part of the parabola since this is where $x^2 - 3x - 4$ is negative, i.e., all the y-values are negative, because we were only looking for when $x^2 - 3x - 4 \geq 0$, the parts of the graph above the x-axis.

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2.6P: Practice

Practice Makes Progress

Factor the given quadratic polynomial.

Exercise 1

$x^2 + 9x + 14$

Exercise 2

$x^2 - 4x - 5$

Exercise 3

$x^2 - 7x + 12$

Exercise 4

$5x^2 - 19x + 12$

Exercise 5

$-4x^2 + 9x - 5$

Exercise 6

$2x^2 - 3x - 35$

Find the zeros of the given quadratic function.

Exercise 7

$f(x) = x^2 - 2x - 15$

Exercise 8

$f(x) = x^2 + 10x - 39$

Exercise 9

$f(x) = 3x^2 - 20x + 12$

Exercise 10

$f(x) = -4x^2 + 4x + 15$

Perform each of the following tasks for the given quadratic function.

1. Set up a coordinate system on graph paper. Label and scale each axis. Remember to draw all lines with a ruler.
2. Use the technique of completing the square to place the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.

3. Use a strictly algebraic technique (no calculators) to find the x-intercepts of the graph of the given quadratic function. Plot them on your coordinate system and label them with their coordinates.
4. Find the y-intercept of the graph of the quadratic function. Plot the y-intercept on your coordinate system and its mirror image across the axis of symmetry, then label these points with their coordinates.
5. Using all the information plotted, draw the graph of the quadratic function and label it with the vertex form of its equation. Use interval notation to describe the domain and range of the quadratic function.

? Exercise 11

$f(x) = x^2 + 2x - 8$

? Exercise 12

$f(x) = -2x^2 - 4x + 16$

? Exercise 13

$f(x) = 3x^2 + 18x - 48$

Solve the inequality. Write the solution in interval notation.

? Exercise 14

$x^2 - 9x + 18 > 0$

? Exercise 15

$4x^2 + 11x - 20 \geq 0$

? Exercise 16

$-4x^2 + 7x \geq 0$

? Exercise 17

$x^2 - 12x + 36 < 0$

Self-Check

In this section, you practiced:

a. Factoring quadratic expressions with $a = 1$
b. Factoring quadratic expressions with $a \neq 1$
c. Finding x-intercepts of quadratic functions by factoring
d. Graphing quadratic functions by finding the vertex, x-intercepts, and y-intercept

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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2.7: The Quadratic Formula

Learning Objectives

- Explain how you can solve a quadratic equation in vertex form without having to factor
- Use the quadratic formula to find the zeros of a quadratic function
- Use the quadratic formula to find the solutions to a quadratic equation

Consider the general quadratic function $f(x) = ax^2 + bx + c$. In the previous section, we learned that we can find the zeros of this function by solving the equation $f(x) = 0$. If we substitute $f(x) = ax^2 + bx + c$, then the resulting equation

$$ax^2 + bx + c = 0$$

is called a quadratic equation. In the previous section, we solved equations of this type by factoring and using the zero product property. However, it is not always possible to factor the trinomial on the left-hand side of the quadratic equation as a product of factors with integer coefficients. For example, consider the quadratic equation

$$2x^2 + 7x - 3 = 0$$

Comparing $2x^2 + 7x - 3$ with $ax^2 + bx + c$, let's list all integer pairs whose product is $a \cdot c = (2)(-3) = -6$.

$1, -6$	$-1, 6$
$2, -3$	$-2, 3$

Not a single one of these integer pairs adds to b = 7. Therefore, the quadratic trinomial $2x^2 + 7x - 3$ does not factor over the integers. Consequently, we'll need another method to solve this quadratic equation.

The purpose of this section is to develop a formula that will consistently provide solutions to the general quadratic equation. However, before we can develop the "Quadratic Formula," we need to lay some groundwork involving the square roots of numbers.

Square Roots

We begin our discussion of square roots by investigating the solutions of the equation $x^2 = a$. Consider the rather simple equation

$$x^2 = 25$$

Because $(-5)^2 = 25$ and $(5)^2 = 25$, $x^2 = 25$ has two solutions, $x = -5$ or $x = 5$. We usually denote these solutions simultaneously, using a "plus or minus" sign:

$$x = \pm 5$$

These solutions are called the square roots of 25. Since there are two solutions, we require distinct notation for each. We will denote the positive square root of 25 with the notation $\sqrt{25}$ and the negative square root of 25 with the notation $-\sqrt{25}$. Thus,

$$\sqrt{25} = 5 \qquad \text{and} \qquad -\sqrt{25} = -5$$

In a similar vein, the equation $x^2 = 36$ has two solutions, $x = \pm\sqrt{36}$, or alternatively, $x = \pm 6$. The notation $\sqrt{36}$ calls for the positive square root, while the notation $-\sqrt{36}$ calls for the negative square root. That is,

$$\sqrt{36} = 6 \qquad \text{and} \qquad -\sqrt{36} = -6$$

It is not necessary that the right-hand side of the equation $x^2 = a$ be a "perfect square." For example, the equation

$$x^2 = 7 \quad \text{has solutions} \quad x = \pm\sqrt{7}$$

There is no rational square root of 7. That is, there is no way to express the square root of 7 in the form p/q, where p and q are integers. Therefore, $\sqrt{7}$ is an example of an irrational number. However, $\sqrt{7}$ is a perfectly valid real number, and we're perfectly comfortable leaving our answer in the form shown in equation (4).

However, if an approximation is needed for the square root of 7, we can reason that because 7 lies between 4 and 9, the square root of 7 will lie between 2 and 3. Because 7 is closer to 9 than 4, a reasonable approximation might be

$$\sqrt{7} \approx 2.6$$

A calculator can provide a more accurate approximation.

$$\sqrt{7} \approx 2.645751311$$

There are two degenerate cases involving the equation $x^2 = a$ that demand our attention.

1. The equation $x^2 = 0$ has only one solution, namely $x = 0$. Thus, $\sqrt{0} = 0$.
2. The equation $x^2 = -4$ has no real solutions. It is not possible to square a real number and get −4. In this situation, we will simply state that "the equation $x^2 = -4$ has no real solutions (no solutions that are real numbers)."

Example 1

Find all real solutions of the equations $x^2 = 30$, $x^2 = 0$, and $x^2 = -14$.

Solution

The solutions follow.

- The equation $x^2 = 30$ has two real solutions, namely $x = \pm\sqrt{30}$.
- The equation $x^2 = 0$ has one real solution, namely $x = 0$.
- The equation $x^2 = -14$ has no real solutions.

Development of the Quadratic Formula

We now have all the groundwork in place to pursue a solution to the general quadratic equation

$$ax^2 + bx + c = 0$$

We're going to use a form of "completing the square" to solve this equation for x. Let's begin by subtracting c from both sides of the equation.

$$ax^2 + bx = -c$$

Next, divide both sides of the equation by a.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Take half of the coefficient of x, as in $\frac{1}{2} \cdot \frac{b}{a} = \frac{b}{2a}$. Square this result to get $\frac{b^2}{4a^2}$. Add this amount to both sides of the equation.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

On the left, we factor the perfect square trinomial. On the right, we get a common denominator and add the resulting equivalent fractions.

$$\left(x + \tfrac{b}{2a}\right)^2 = -\tfrac{4ac}{4a^2} + \tfrac{b^2}{4a^2}$$

$$\left(x + \tfrac{b}{2a}\right)^2 = \tfrac{b^2-4ac}{4a^2}$$

Provided the right-hand side of this last equation is positive, we have two real solutions.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

On the right, we take the square root of the top and the bottom of the fraction.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

To complete the solution, we need only subtract $\frac{b}{2a}$ from both sides of the equation.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Although this last answer is a perfectly good solution, we customarily rewrite the solution with a single common denominator.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This last result gives the solution to the general quadratic equation.

The Quadratic Formula

The solutions to the quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Although the development of the quadratic formula can be intimidating, in practice, its application is quite simple. Let's look at some examples.

Example 2

Use the quadratic formula to solve the equation

$$x^2 = 27 - 6x$$

Solution

The first step is to place the equation in the form $ax^2 + bx + c = 0$ by moving every term to one side of the equation, arranging the terms in descending powers of x.

$$x^2 + 6x - 27 = 0$$

Next, compare $x^2 + 6x - 27 = 0$ with the general form of the quadratic equation $ax^2 + bx + c = 0$ and note that $a = 1$, $b = 6$, and $c = -27$. Copy down the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute $a = 1$, $b = 6$, and $c = -27$ and simplify.

$$x = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(-27)}}{2(1)}$$
$$x = \frac{-6 \pm \sqrt{36 + 108}}{2}$$
$$x = \frac{-6 \pm \sqrt{144}}{2}$$

In this case, 144 is a perfect square. That is, $\sqrt{144} = 12$, so we can continue to simplify

$$x = \frac{-6 \pm 12}{2}$$

It's important to note that there are two real answers, namely

$$x = \frac{-6 - 12}{2} \quad \text{or} \quad x = \frac{-6 + 12}{2}$$

Simplifying,

$$x = -9 \qquad \text{or} \qquad x = 3$$

It's interesting to note that this problem could have been solved by factoring. Indeed,

$$x^2 + 6x - 27 = 0$$
$$(x - 3)(x + 9) = 0$$

So the zero product property requires that either $x - 3 = 0$ or $x + 9 = 0$, which leads to $x = 3$ or $x = -9$, answers identical to those found by the quadratic formula.

We'll have more to say about the "discriminant" soon, but it's no coincidence that the quadratic $x^2 + 6x - 27$ factored. Here is the relevant fact.

When the Discriminant is a Perfect Square

In the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

the number under the radical, $b^2 - 4ac$, is called the discriminant. When the discriminant is a perfect square, the quadratic function will always factor.

However, it is not always the case that we can factor the given quadratic. Let's look at another example.

Example 3

Given the quadratic function $f(x) = x^2 - 2x$, find all real solutions of $f(x) = 2$.

Solution

Because $f(x) = x^2 - 2x$, the equation f(x) = 2 becomes

$$x^2 - 2x = 2$$

Set one side of the equation equal to zero by subtracting 2 from both sides of the equation.

$$x^2 - 2x - 2 = 0$$

Compare $x^2 - 2x - 2 = 0$ with the general quadratic equation $ax^2 + bx + c = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Next, substitute $a = 1$, $b = -2$, and $c = -2$. Note the careful use of parentheses.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$

Simplify.

$$x = \frac{2 \pm \sqrt{4+8}}{2}$$

$$x = \frac{2 \pm \sqrt{12}}{2}$$

We can approximate these solutions with the aid of a calculator.

$$x \approx -0.7320508076 \qquad x \approx 2.732050808$$

We will find these approximations useful in what follows.

These examples represent a fundamental shift in our traditional approach to solving equations. In the past, we've tried to "isolate" the terms containing x (or whatever unknown we are solving for) on one side of the equation, and all other terms on the other side of the equation. Now, we find ourselves moving everything to one side of the equation, making one side equal to zero. This bears some explanation.

Linear or Nonlinear

Let's assume that the unknown we are solving for is x.

- If the highest power of x present in the equation is x to the first power, then the equation is linear. Thus, for example, each of the equations

$$2x+3=7, \quad 3-4x=5x+9, \quad \text{and} \quad ax+b=cx+d$$

  is linear.
- If there are powers of x higher than x to the first power in the equation, then the equation is nonlinear. Thus, for example, each of the equations

$$x^2-4x=9, \quad x^3=2x+3, \quad \text{and} \quad ax^2+bx=cx+d$$

  is nonlinear.

The strategy for solving an equation will shift, depending on whether the equation is linear or nonlinear.

Solution Strategy—Linear Versus Nonlinear

When solving equations, you must first ask if the equation is linear or nonlinear. Again, let's assume the unknown we wish to solve for is x.

- If the equation is linear, move all terms containing x to one side of the equation, and all the remaining terms to the other side of the equation.
- If the equation is nonlinear, move all terms to one side of the equation, making the other side of the equation zero.

Intercepts

In a previous example, we used the quadratic formula to find the solutions of $x^2-2x-2=0$. These solutions, along with their approximations, were presented as the final step. It is important to make the connection that the solutions are the zeros of the quadratic function $g(x)=x^2-2x-2$. The zeros also provide the x-coordinates of the x-intercepts of the graph of g (a parabola). To emphasize this point, let's draw the graph of the parabola having the equation $g(x)=x^2-2x-2$.

First, complete the square to place the quadratic function in vertex form. Take half the middle coefficient and square it, as in $\left[\frac{1}{2}\cdot(-2)\right]^2=1$; then add and subtract this term so that the equation remains balanced.

$$g(x)=x^2-2x-2$$
$$g(x)=x^2-2x+1-1-2$$

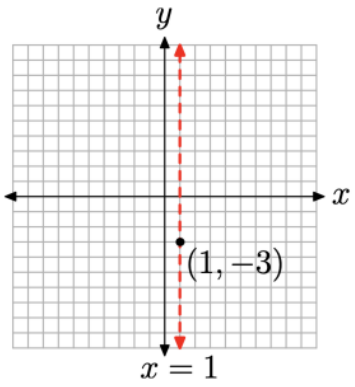
Factor the perfect square trinomial, then combine the constants at the end.

$$g(x)=(x-1)^2-3$$

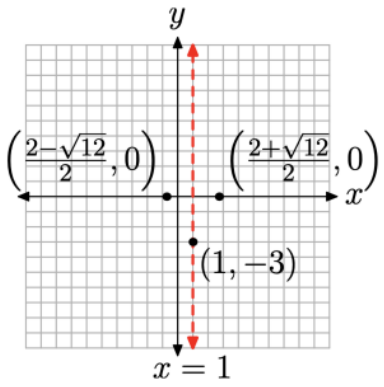
This is a parabola that opens upward. It is shifted to the right 1 unit and down 3 units. This makes it easy to identify the vertex and draw the axis of symmetry.

It will now be apparent why we used our calculator to approximate the solutions. These are the x-coordinates of the x-intercepts. One x-intercept is located at approximately (−0.73, 0), the other at approximately (2.73, 0). These approximations are used to plot the location of the intercepts. However, the actual values of the intercepts are $((2-\sqrt{12})/2,0)$ and $((2+\sqrt{12})/2,0)$, and these exact values should be used to annotate the intercepts.

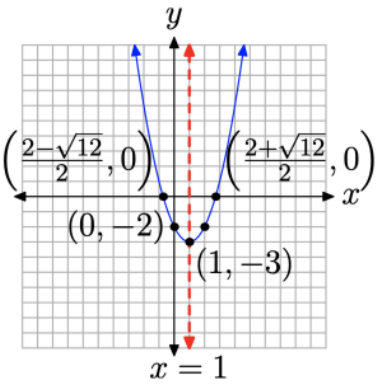
Finally, to find the y-intercept, let $x = 0$ in $g(x) = x^2 - 2x - 2$. Thus, g(0) = −2 and the y-intercept is (0, −2). The y-intercept and its mirror image across the axis of symmetry are both plotted, where the final graph of the parabola is also shown.



(a) Plotting the vertex and axis of symmetry.



(b) Adding the x-intercepts provides added accuracy.



(c) Adding the y-intercept and its mirror image provides an excellent final graph.

We've made an important point, and we pause to provide emphasis.

Zeros and Intercepts

Whenever you use the quadratic formula to solve the quadratic equation

$$ax^2 + bx + c = 0$$

the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are the zeros of the quadratic function

$$f(x) = ax^2 + bx + c$$

The solutions also provide the x-coordinates of the x-intercepts of the graph of f.

We need to discuss one final concept.

The Discriminant

Consider again the quadratic equation $ax^2 + bx + c = 0$ and the solutions (zeros) provided by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression under the radical, $b^2 - 4ac$, is called the discriminant, which we denote by the letter D. That is, the formula for the discriminant is given by

$$D = b^2 - 4ac$$

The discriminant is used to determine the nature and number of solutions to the quadratic equation $ax^2 + bx + c = 0$. This is done without actually calculating the solutions.

Let's look at three key examples.

Example 4

Consider the quadratic equation $x^2 - 4x - 4 = 0$. Calculate the discriminant and use it to determine the nature and number of the solutions.

Solution

Compare $x^2 - 4x - 4 = 0$ with $ax^2 + bx + c = 0$ and note that a = 1, b = −4, and c = −4. The discriminant is given by the calculation

$$D = b^2 - 4ac = (-4)^2 - 4(1)(-4) = 32$$

Note that the discriminant D is positive; i.e., $D > 0$.

Consider the quadratic function $f(x) = x^2 - 4x - 4$, which can be written in vertex form

$$f(x) = (x - 2)^2 - 8$$

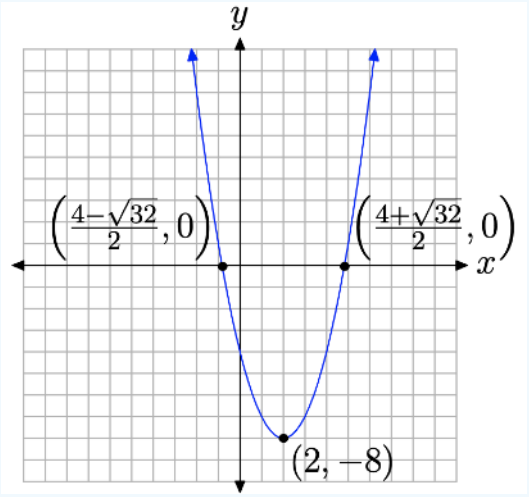
This is a parabola that opens upward. It is shifted to the right 2 units, then downward 8 units. Therefore, it will intersect the x-axis at two points. Hence, one would expect that the quadratic formula would provide two real solutions (x-intercepts). Indeed,

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-4)}}{2(1)} = \frac{4 \pm \sqrt{32}}{2}$$

Note that the discriminant, $D = 32$ as calculated above, is the number under the square root. These solutions have approximations

$$x = \frac{4 - \sqrt{32}}{2} \approx -0.8284271247 \qquad \text{and} \qquad x = \frac{4 + \sqrt{32}}{2} \approx 4.828427125$$

which aid in plotting an accurate graph of $f(x) = (x - 2)^2 - 8$.



Thus, if the discriminant is positive, the parabola will have two real x-intercepts.

Next, let's look at an example where the discriminant equals zero.

Example 5

Consider again the quadratic equation $ax^2 + bx + c = 0$ and the solutions (zeros) provided by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression under the radical, $b^2 - 4ac$, is called the discriminant, which we denote by the letter D. That is, the formula for the discriminant is given by

$$D = b^2 - 4ac$$

The discriminant is used to determine the nature and number of solutions to the quadratic equation $ax^2 + bx + c = 0$. This is done without actually calculating the solutions. Consider the quadratic equation

$$x^2 - 4x + 4 = 0$$

Calculate the discriminant and use it to determine the nature and number of the solutions.

Solution

Compare $x^2 - 4x + 4 = 0$ with $ax^2 + bx + c = 0$ and note that $a = 1$, $b = -4$, and $c = 4$. The discriminant is given by the calculation

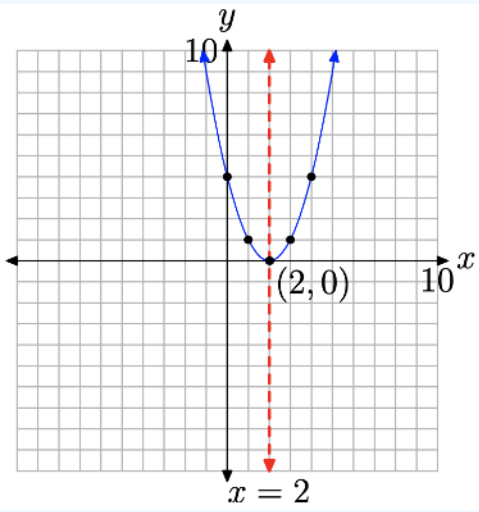
$$D = b^2 - 4ac = (-4)^2 - 4(1)(4) = 0$$

Note that the discriminant equals zero.

Consider the quadratic function $f(x) = x^2 - 4x + 4$, which can be written in vertex form

$$f(x) = (x - 2)^2$$

This is a parabola that opens upward and is shifted 2 units to the right. Note that there is no vertical shift, so the vertex of the parabola will rest on the x-axis, as shown in the figure. In this case, we found it necessary to plot two points to the right of the axis of symmetry and then mirror them across the axis of symmetry to obtain an accurate plot of the parabola.



x	$f(x) = (x - 2)^2$
3	1
4	4

If we set $f(x) = 0$ in this equation, then we get $0 = (x - 2)^2$. This could be written $0 = (x - 2)(x - 2)$ and we could say that the solutions are 2 and 2 again. However, mathematicians prefer to say that "2 is a solution of multiplicity 2" or "2 is a double solution." Note how the parabola is tangent to the x-axis at the location of the "double solution." That is, the parabola comes down from positive infinity, touches (but does not cross) the x-axis at $x = 2$, then rises again to positive infinity. Of course, the situation would be reversed in the parabola opened downward, as in $g(x) = -(x - 2)^2$, but the graph would still "kiss" the x-axis at the location of the "double solution."

Still, the key point to note here is that the discriminant $D = 0$, and the parabola has only one x-intercept. That is, the equation $x^2 - 4x + 4 = 0$ has a single real solution.

Next, let's look at what happens when the discriminant is negative.

Example 6

Consider the quadratic equation

$$x^2 - 4x + 8 = 0$$

Calculate the discriminant and use it to determine the nature and number of the solutions.

Solution

Compare $x^2 - 4x + 8 = 0$ with $ax^2 + bx + c = 0$ and note that $a = 1$, $b = -4$, and $c = 8$. The discriminant is given by the calculation

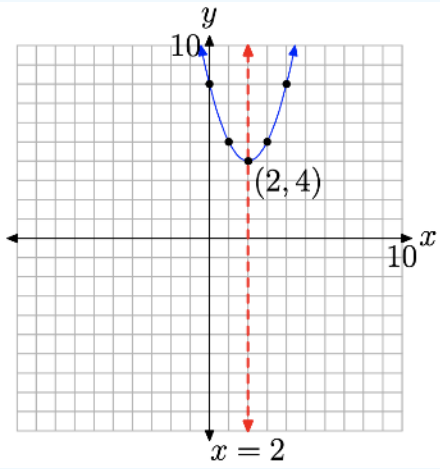
$$D = b^2 - 4ac = (-4)^2 - 4(1)(8) = -16$$

Note that the discriminant is negative.

Consider the quadratic function $f(x) = x^2 - 4x + 8$, which can be written in vertex form

$$f(x) = (x - 2)^2 + 4$$

This is a parabola that opens upward. Moreover, it has to be shifted 2 units to the right and 4 units upward, so there can be no x-intercepts. Again, we found it necessary in this example to plot two points to the right of the axis of symmetry, then mirror them, in order to get an accurate plot of the parabola.



x	$f(x) = (x-2)^2+4$
3	5
4	8

Once again, the key point in this example is the fact that the discriminant is negative and there are no real solutions of the quadratic equation (equivalently, there are no x-intercepts). Let's see what happens if we actually try to find the solutions of $x^2 - 4x + 8 = 0$ using the quadratic formula. Again, $a = 1$, $b = -4$, and $c = 8$, so

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(8)}}{2(1)}$$

Simplifying,

$$x = \frac{4 \pm \sqrt{-16}}{2}$$

Again, remember that the number under the square root is the discriminant. In this case, the discriminant is −16. It is not possible to square a real number and get −16. Thus, the quadratic equation $x^2 - 4x + 8 = 0$ has no real solutions, as predicted.

Let's summarize the findings in our last three examples.

Summary

Consider the quadratic equation $ax^2 + bx + c = 0$. The discriminant is defined as $D = b^2 - 4ac$. There are three possibilities:

1. If D > 0, then the quadratic equation has two real solutions.
2. If D = 0, then the quadratic equation has one real solution.
3. If D < 0, then the quadratic equation has no real solutions.

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2.7P: Practice

Practice Makes Progress

Find all real solutions of the given equation. Use a calculator to approximate the answers, correct to the nearest hundredth.

Exercise 1

$x^2 = 36$

Exercise 2

$x^2 = 17$

Exercise 3

$x^2 = 0$

Exercise 4

$x^2 = -12$

Exercise 5

$(x-1)^2 = 25$

Exercise 6

$(x+2)^2 = 0$

Exercise 7

$(x-8)^2 = 15$

Perform each of the following tasks for the given quadratic function.

1. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Remember to draw all lines with a ruler.
2. Place the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.
3. Use the quadratic formula to find the x-intercepts of the parabola. Use a calculator to approximate each intercept, correct to the nearest tenth, and use these approximations to plot the x-intercepts on your coordinate system. However, label each x-intercept with its **exact** coordinates.
4. Plot the y-intercept on your coordinate system and its mirror image across the axis of symmetry and label each with their coordinates.
5. Using all of the information on your coordinate system, draw the graph of the parabola, then label it with the vertex form of the function. Use interval notation to state the domain and range of the quadratic function.

Exercise 8

$f(x) = x^2 - 4x - 8$

? Exercise 9

$f(x) = -x^2 + 2x + 10$

Perform each of the following tasks for the given quadratic equation.

1. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Remember to draw all lines with a ruler.
2. Show that the discriminant is negative.
3. Use the technique of completing the square to put the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.
4. Plot the y-intercept and its mirror image across the axis of symmetry on your coordinate system and label each with their coordinates.
5. Because the discriminant is negative (did you remember to show that?), there are no x-intercepts. Use the given equation to calculate one additional point, then plot the point and its mirror image across the axis of symmetry and label each with their coordinates.
6. Using all of the information on your coordinate system, draw the graph of the parabola, then label it with the vertex form of function. Use interval notation to describe the domain and range of the quadratic function.

? Exercise 10

$f(x) = x^2 + 4x + 8$

? Exercise 11

$f(x) = -x^2 + 6x - 11$

Find all values of k so that the graph of the quadratic function given has exactly two x-intercepts (hint: use the discriminant).

? Exercise 12

$f(x) = kx^2 - 3x + 5$

? Exercise 13

$f(x) = 2x^2 - x + 5k$

Find all real solutions, if any, of the quadratic equation.

? Exercise 14

$x^2 - 2 = -3x$

? Exercise 15

$9x^2 + 81 = -54x$

Find all of the x-intercepts, if any, of the given function.

? Exercise 16

$f(x) = -4x^2 - 4x - 5$

Exercise 17

$f(x) = -56x^2 + 47x + 18$

Self-Check

In this section, you practiced:

a. Solving quadratic equations using the square root method
b. Solving quadratic equations using the quadratic formula
c. Graphing quadratic equations by finding the vertex, x-intercepts, and y-intercept
d. Using the discriminant to determine the number of x-intercepts of a quadratic function

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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CHAPTER OVERVIEW

Unit 3: Ratios, Rates, and Rational Functions

This unit is about making sense of relationships built from ratios. You'll turn statements like "y is proportional to x" or "inversely proportional" into simple equations, find the constant that fits the data, and use the model to make predictions. We'll treat rational functions as "fraction-of-polynomials" graphs with sensible inputs and built-in limits—vertical and horizontal asymptotes act like guardrails that show what can't happen and what the long-term trend looks like. You'll learn to shift and stretch these graphs, simplify and combine expressions just like numeric fractions, and solve equations and inequalities while watching for extraneous answers. By the end, you'll model everyday scenarios—shared work, motion, revenue—explain what your results mean, and clearly state when the model applies and when it doesn't.

3.1: Direct and Inverse Variation

3.1P: Practice

3.2: Graphs of Rational Functions

3.2P: Practice

3.3: Reducing Rational Expressions

3.3P: Practice

3.4: Sums and Differences of Rational Functions

3.4P: Practice

3.5: Solving Rational Equations

3.5P: Practice

3.6: Rational Applications

3.6P: Practice

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3.1: Direct and Inverse Variation

Learning Objectives

- Model direct and inverse variation with appropriate equations
- Analyze a real-world scenario using direct or inverse variation

We start with a definition of the phrase “is proportional to.”

Proportional

We say that y is proportional to x if and only if

$$y = kx$$

where k is a constant called the constant of proportionality. The phrase “y varies directly as x” is an equivalent way of saying “y is proportional to x.”

Here are a few examples that translate the phrase “is proportional to.”

- Given that d is proportional to t, we write $d = kt$, where k is a constant.
- Given that y is proportional to the cube of x, we write $y = kx^3$, where k is a constant.
- Given that s is proportional to the square of t, we write $s = kt^2$, where k is a constant.

We are not restricted to always using the letter k for our constant of proportionality.

Example 1

Given that y is proportional to x and the fact that $y = 12$ when $x = 5$, determine the constant of proportionality, then determine the value of y when $x = 10$.

Solution

Given the fact that y is proportional to x, we know immediately that

$$y = kx$$

where k is the proportionality constant. Because we are given that $y = 12$ when $x = 5$, we can substitute 12 for y and 5 for x to determine k.

$$\begin{aligned} y &= kx && y \text{ is proportional to } x \\ 12 &= k(5) && \text{Substitute 12 for } y, 5 \text{ for } x \\ \frac{12}{5} &= k && \text{Divide both sides by 5} \end{aligned}$$

Next, substitute the constant of proportionality $12/5$ for k in $y = kx$, then substitute 10 for x to determine y when $x = 10$.

$$\begin{aligned} y &= \frac{12}{5}x && \text{Substitute } 12/5 \text{ for } k \\ y &= \frac{12}{5}(10) && \text{Substitute 10 for } x \\ y &= 24 && \text{Cancel and simplify.} \end{aligned}$$

Example 2

A ball is dropped from a balloon floating above the Earth's surface. The distance s the ball falls is proportional to the square of the time t that has passed since the ball’s release. If the ball falls 144 feet during the first 3 seconds, how far does the ball fall in 9 seconds?

Solution

Given the fact that s is proportional to the square of t, we know immediately that

$$s = kt^2$$

where k is the proportionality constant. Because we are given that the ball falls 144 feet during the first 3 seconds, we can substitute 144 for s and 3 for t to determine the constant of proportionality.

$$\begin{aligned} s &= kt^2 && s \text{ is proportional to the square of } t \\ 144 &= k(3)^2 && \text{Substitute 144 for } s, 3 \text{ for } t \\ 144 &= 9k && \text{Simplify: } 3^2 = 9 \\ 16 &= k && \text{Divide both sides by 9} \end{aligned}$$

Next, substitute the constant of proportionality 16 for k in $s = kt^2$, and then substitute 9 for t to determine the distance fallen when $t = 9$ seconds.

$$\begin{aligned} s &= 16t^2 && \text{Substitute 16 for } k \\ s &= 16(9)^2 && \text{Substitute 9 for } t \\ s &= 1296 && \text{Simplify} \end{aligned}$$

Thus, the ball falls $1,296$ feet during the first 9 seconds.

Example 3

Tony and Paul are hanging weights on a spring in the physics lab. Each time a weight is hung, they measure the distance the spring stretches. They discover that the distance y that the spring stretches is proportional to the weight hung on the spring (Hooke's Law). If a 0.5 pound weight stretches the spring 3 inches, how far will a 0.75 pound weight stretch the spring?

Solution

Let W represent the weight hung on the spring. Let y represent the distance the spring stretches. We're told that the distance y the spring stretches is proportional to the amount of weight W hung on the spring. Hence, we can write:

$$y = kW \quad y \text{ is proportional to } W$$

Substitute 3 for y, 0.5 for W, then solve for k.

$$\begin{aligned} 3 &= k(0.5) && \text{Substitute 3 for } y, 0.5 \text{ for } W \\ \frac{3}{0.5} &= k && \text{Divide both sides by 0.5} \\ k &= 6 && \text{Simplify.} \end{aligned}$$

Substitute 6 for k in $y = kW$ to produce:

$$y = 6W \quad \text{Substitute 6 for } k \text{ in } y = kW$$

To determine the distance the spring will stretch when 0.75 pounds are hung on the spring, substitute 0.75 for W.

$$\begin{aligned} y &= 6(0.75) && \text{Substitute 0.75 for } W \\ y &= 4.5 && \text{Simplify.} \end{aligned}$$

Thus, the spring will stretch 4.5 inches.

Inversely Proportional

In the previous examples, where one quantity was proportional to a second quantity, you may have noticed that when one quantity increased, the second quantity also increased. Vice versa, when one quantity decreased, the second quantity also decreased.

However, not all real-world situations follow this pattern. There are times when, as one quantity increases, the related quantity decreases. For example, consider the situation where you increase the number of workers on a job and note that the time to finish the job decreases. This is an example of a quantity being inversely proportional to a second quantity.

Inversely proportional

We say the y is inversely proportional to x if and only if

$$y = \frac{k}{x}$$

where k is a constant called the constant of proportionality. The phrase "y varies inversely as x" is an equivalent way of saying "y in inversely proportional to x."

Here are a few examples that translate the phrase "is inversely proportional to."

- Given that d is inversely proportional to t, we write $d = \frac{k}{t}$, where k is a constant.
- Given that y is inversely proportional to the cube of x, we write $y = \frac{k}{x^3}$, where k is a constant.
- Given that s is inversely proportional to the square of t, we write $s = \frac{k}{t^2}$, where k is a constant.

We are not restricted to always using the letter k for our constant of proportionality.

Example 4

Given that y is inversely proportional to x and the fact that $y = 4$ when $x = 2$, determine the constant of proportionality, then determine the value of y when $x = 4$.

Solution

Given the fact the y is inversely proportional to x, we know immediately that

$$y = \frac{k}{x}$$

where k is the proportionality constant. Because we are given that $y = 4$ when $x = 2$, we can substitute 4 for y and 2 for x to determine k.

$$\begin{aligned} y &= \frac{k}{x} && y \text{ is inversely proportional to } x. \\ 4 &= \frac{k}{2} && \text{Substitute 4 for } y, 2 \text{ for } x. \\ 8 &= k && \text{Multiply both sides by 2.} \end{aligned}$$

Substitute 8 for k in $y = k/x$, then substitute 4 for x to determine y when $x = 4$.

$$\begin{aligned} y &= \frac{8}{x} && \text{Substitute 8 for } k. \\ y &= \frac{8}{4} && \text{Substitute 4 for } x. \\ y &= 2 && \text{Reduce.} \end{aligned}$$

Note that as x increased from 2 to 4, y decreased from 4 to 2.

Example 5

The intensity I of light is inversely proportional to the square of the distance d from the light source. If the light intensity 5 feet from the light source is 3 foot-candles, what is the intensity of the light 15 feet from the light source?

Solution

Given the fact that the intensity I of the light is inversely proportional to the square of the distance d from the light source, we know immediately that

$$I = \frac{k}{d^2}$$

where k is the proportionality constant. Because we are given that the intensity is $I = 3$ foot-candles at $d = 5$ feet from the light source, we can substitute 3 for I and 5 for d to determine k.

$$\begin{aligned} I &= \frac{k}{d^2} && I \text{ is inversely proportional to } d^2. \\ 3 &= \frac{k}{5^2} && \text{Substitute 3 for } I, 5 \text{ for } d. \\ 3 &= \frac{k}{25} && \text{Simplify.} \\ 75 &= k && \text{Multiply both sides by 25.} \end{aligned}$$

Substitute 75 for k in $I = k/d^2$, then substitute 15 for d to determine I when $d = 15$.

$$\begin{aligned} I &= \frac{75}{d^2} && \text{Substitute 75 for } k. \\ I &= \frac{75}{15^2} && \text{Substitute 15 for } d. \\ I &= \frac{75}{225} && \text{Simplify.} \\ I &= \frac{1}{3} && \text{Reduce.} \end{aligned}$$

Thus, the intensity of the light 15 feet from the light source is $1/3$ foot-candle.

Example 6

Suppose the price per person for a camping experience is inversely proportional to the number of people who sign up for it. If 10 people sign up, the price per person is $\$350$. What will be the price per person if 50 people sign up?

Solution

Let p represent the price per person and let N be the number of people who sign up for the camping experience. Because we are told that the price per person is inversely proportional to the number of people who sign up for the camping experience, we can write:

$$p = \frac{k}{N}$$

where k is the proportionality constant. Because we are given that the price per person is $\$350$ when 10 people sign up, we can substitute 350 for p and 10 for N to determine k.

$$\begin{aligned} p &= \frac{k}{N} && p \text{ is inversely proportional to } N. \\ 350 &= \frac{k}{10} && \text{Substitute 350 for } p, 10 \text{ for } N. \\ 3500 &= k && \text{Multiply both sides by 10.} \end{aligned}$$

Substitute 3500 for k in $p = k/N$, then substitute 50 for N to determine p when $N = 50$.

$$\begin{aligned} p &= \frac{3500}{N} && \text{Substitute 3500 for } k. \\ p &= \frac{3500}{50} && \text{Substitute 50 for } N. \\ p &= 70 && \text{Simplify.} \end{aligned}$$

Thus, the price per person is $\$70$ if 50 people sign up for the camping experience.

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3.1P: Practice

Practice Makes Progress

Exercise 1

Given that s is proportional to t and the fact that $s = 632$ when $t = 79$, determine the value of s when $t = 50$.

Exercise 2

Given that q is proportional to the square of c and the fact that $q = 13448$ when $c = 82$, determine the value of q when $c = 29$.

Exercise 3

Given that F is inversely proportional to x and the fact that $F = 19$ when $x = 22$, determine the value of F when $x = 16$.

Exercise 4

Given that y is inversely proportional to the square of x and the fact that $y = 14$ when $x = 4$, determine the value of y when $x = 10$.

Exercise 5

Given that d is inversely proportional to the cube of t and the fact that $d = 18$ when $t = 2$, determine the value of d when $t = 3$.

Exercise 6

Given that q is inversely proportional to the cube of c and the fact that $q = 16$ when $c = 5$, determine the value of q when $c = 6$.

For the application problems below,

a. Define variables and find a formula that relates the two quantities
b. Answer the question in the problem, rounding your solution to 2 decimal places.

Exercise 7

Joe and Mary are hanging weights on a spring in the physics lab. Each time a weight is hung, they measure the distance the spring stretches. They discover that the distance that the spring stretches is proportional to the weight hung on the spring. If a 2 pound weight stretches the spring 16 inches, how far will a 5 pound weight stretch the spring?

Exercise 8

The intensity I of light is inversely proportional to the square of the distance d from the light source. If the light intensity 4 feet from the light source is 20 foot-candles, what is the intensity of the light 18 feet from the light source?

Exercise 9

Suppose that the price per person for a camping experience is inversely proportional to the number of people who sign up for the experience. If 18 people sign up, the price per person is $\$204$. What will be the price per person if 35 people sign up? Round your answer to the nearest dollar.

Self-Check

In this section, you practiced:

a. Interpreting the phrases “proportional” and “inversely proportional” into algebraic equations
b. Solving for a constant of proportionality
c. Modeling a real world scenario involving proportional or inversely proportional relationships

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don’t understand these yet:

It’s okay to not be fully confident with every skill yet! But it’s essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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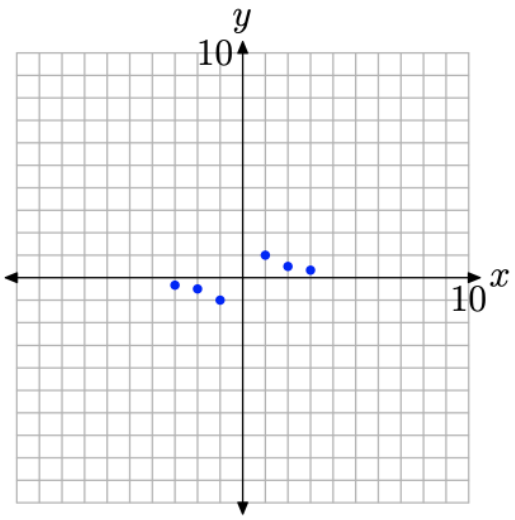
3.2: Graphs of Rational Functions

Learning Objectives

- Sketch a graph of the basic rational functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$
- Create a graphical representation of a rational function through transformations
- Identify horizontal and vertical asymptotes of a rational function

The Graph of $y = \frac{1}{x}$

In all new situations, when we are presented with an equation whose graph we've not considered or do not recognize, we begin the process of drawing the graph by creating a table of points that satisfy the equation. It's important to remember that the graph of an equation is the set of all points that satisfy the equation. We note that zero is not in the domain of $y = \frac{1}{x}$ (division by zero is undefined and makes no sense), and create a table of points satisfying the equation.



x	$y = 1/x$
-3	$-1/3$
-2	$-1/2$
-1	-1
1	1
2	$1/2$
3	$1/3$

At this point, it's pretty clear what the graph is doing between $x = -3$ and $x = -1$. Likewise, it's clear what is happening between x = 1 and x = 3. However, there are some areas of concern that remain open.

1. What happens to the graph as x increases without bound? That is, what happens to the graph as x moves toward ∞?
2. What happens to the graph as x decreases without bound? That is, what happens to the graph as x moves toward $-\infty$?
3. What happens to the graph as x approaches zero from the right?
4. What happens to the graph as x approaches zero from the left?

Let's answer each of these questions in turn. We'll begin by discussing the "end behavior" of the rational function defined by $y = \frac{1}{x}$. First, the right end. What happens as x increases without bound? That is, what happens as x increases toward ∞? In the table below, we computed $y = \frac{1}{x}$ for x equaling 100, 1000, and 10000. Note how the y-values are all positive and approach zero.

Students in calculus use the following notation to represent this concept.

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{1}{x} = 0$$

They say "the limit of y as x approaches infinity is zero." That is, as x approaches infinity, y approaches zero.

x	$y = \frac{1}{x}$
100	0.01
1000	0.001
10000	0.0001

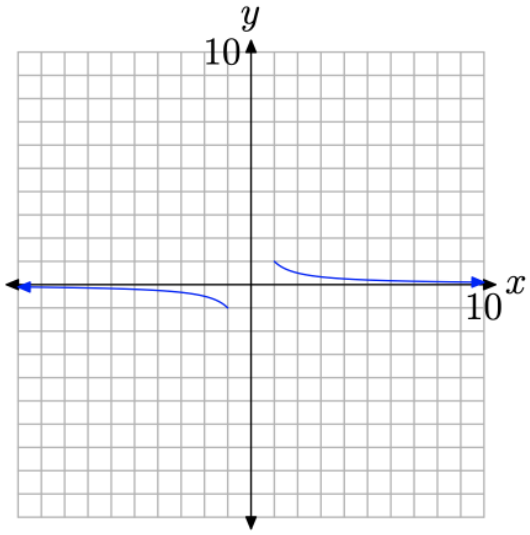
A completely similar event happens at the left end. As x decreases without bound, that is, as x decreases toward $-\infty$, note that the y -values in the table are all negative and approach zero. Calculus students have a similar notation for this idea.

$$\lim_{x \to -\infty} y = \lim_{x \to -\infty} \frac{1}{x} = 0$$

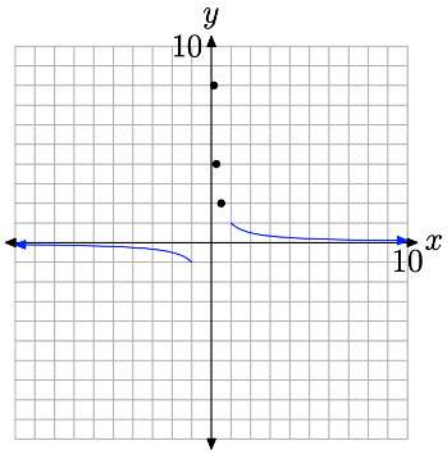
They say "the limit of y as x approaches negative infinity is zero." That is, as x approaches negative infinity, y approaches zero.

x	$y = \frac{1}{x}$
-100	-0.01
-1000	-0.001
-10000	-0.0001

These numbers in the tables and the ideas described above accurately predict the correct end behavior of the graph of $y = \frac{1}{x}$. At each end of the x-axis, the y-values must approach zero. This means that the graph of $y = \frac{1}{x}$ must approach the x-axis for x-values at the far right and left ends of the graph. In this case, we say that the x-axis acts as a horizontal asymptote for the graph of $y = \frac{1}{x}$. As x approaches either positive or negative infinity, the graph of $y = \frac{1}{x}$ approaches the x-axis. This behavior is shown in the figure.



Our next investigation will cover the interval from $x = -1$ to $x = 1$. Readers are again reminded that the function $y = \frac{1}{x}$ is undefined at $x = 0$. Consequently, we will break this region in half, first investigating what happens in the region between $x = 0$ and $x = 1$. We evaluate $y = \frac{1}{x}$ at $x = 1/2$, $x = 1/4$, and $x = 1/8$, as shown in the table below, then plot the resulting points.

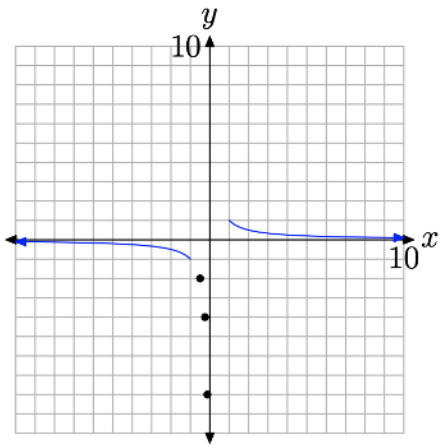


x	$y = 1/x$
$1/2$	2
$1/4$	4
$1/8$	8

Note that the x-values in the table approach zero from the right; then, note that the corresponding y-values increase in size. We could continue in this vein, adding points. For example, if $x = 1/16$, then $y = 16$. If $x = 1/32$, then $y = 32$. If $x = 1/64$, then $y = 64$. Each time we halve our value of x, the resulting value of x is closer to zero, and the corresponding y-value doubles in size. Calculus students describe this behavior with the notation

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} \frac{1}{x} = \infty$$

That is, as "x approaches zero from the right, the value of y grows to infinity." This is evident in the graph, where the plotted points move closer to the vertical axis while simultaneously moving upward without bound. A similar thing happens on the other side of the vertical axis.



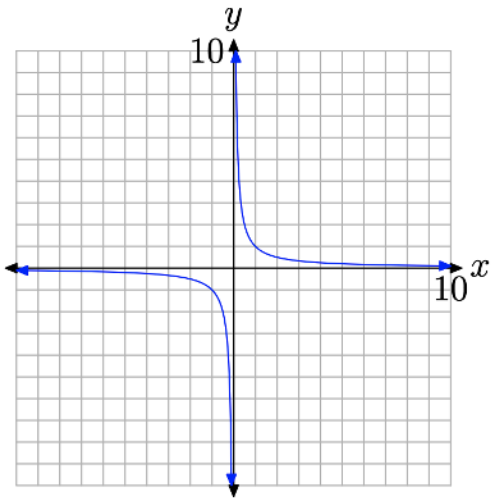
x	$y = 1/x$
$-1/2$	-2
$-1/4$	-4
$-1/8$	-8

Again, calculus students would write

$$\lim_{x \to 0^-} y = \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

That is, "as x approaches zero from the left, the values of y decrease to negative infinity." In the figure, it is clear that as points move closer to the vertical axis (as x approaches zero) from the left, the graph decreases without bound.

The evidence gathered to this point indicates that the vertical axis is acting as a vertical asymptote. As x approaches zero from either side, the graph approaches the vertical axis, either rising to infinity or falling to negative infinity. The graph cannot cross the vertical axis because the function is undefined at that point. The completed graph is shown in the figure.



The complete graph of $y = \frac{1}{x}$ is called a hyperbola and serves as a fundamental starting point for all subsequent discussion in this section.

We noted earlier that the domain of the function defined by the equation $y = \frac{1}{x}$ is the set $D = \{x : x \neq 0\}$. Zero is excluded from the domain because division by zero is undefined. It's no coincidence that the graph has a vertical asymptote at $x = 0$. This relationship will be further reinforced in subsequent examples.

Let's investigate these functions a little further. We know the domain is all real numbers except for $x = 0$, but let's look at the graph more closely. Notice that the graph doesn't intersect the y axis. Why? Well, let's set $y = 0$ and solve:

$$0 = \frac{1}{x}$$

When is this fraction zero? We know from previous sections that a fraction is zero when the numerator is zero. Will the numerator ever be zero, i.e.,

$$1 \stackrel{?}{=} 0$$

No, never! This means there are no values of x such that $y = 0$, and that $y = 0$ is not in the range of the function.

This is not a coincidence. The fact that $x \neq 0$ and $y \neq 0$ for the function $f(x) = \frac{1}{x}$ means that $f(x)$ has ***vertical*** and ***horizontal*** asymptotes at $x = 0$ and $y = 0$, respectively.

Definition: Horizontal Asymptote & Vertical Asymptote

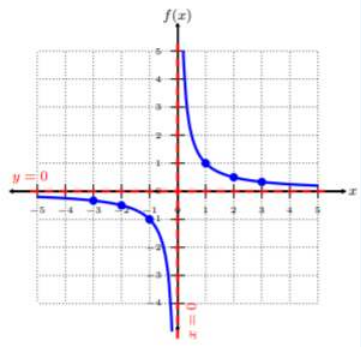
A function, $R(x)$, has a ***horizontal asymptote*** at $y = 0$ and a ***vertical asymptote*** at $x = 0$ when $R(x)$ is of the form

$$\frac{1}{x^n}$$

We denote these asymptotes by drawing *dashed* lines for lines $x = 0$ and $y = 0$.

Example 1

Let's regraph $f(x) = \frac{1}{x}$ showing the horizontal and vertical asymptotes at $y = 0$ and $x = 0$, respectively.



So, we see the asymptotes, in red, are dashed lines on the x and y-axis and are the lines $y = 0$ and $x = 0$. The only case in which the horizontal and vertical asymptotes move left or right, and up or down, respectively, is if there are shifts to the parent function $f(x)$.

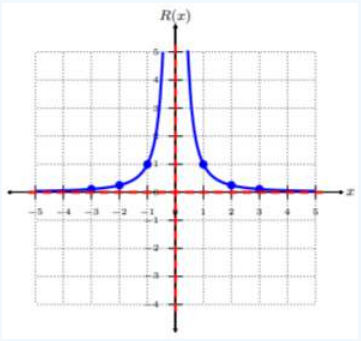
Example 2

Graph $R(x) = \frac{1}{x^2}$.

Solution

Let's pick x-coordinates, and find corresponding y-values.

x	$R(x) = \frac{1}{x^2}$	$(R, g(x))$
-3	$f(-3) = \frac{1}{(-3)^2}$	$(-3, \frac{1}{9})$
-2	$f(-2) = \frac{1}{(-2)^2}$	$(-2, \frac{1}{4})$
-1	$f(-1) = 1$	$(-1, 1)$
0	$f(0) =$ undefined	no point
1	$f(1) = 1$	$(1, 1)$
2	$f(2) = \frac{1}{2^2}$	$(2, \frac{1}{4})$
3	$f(3) = \frac{1}{3^2}$	$(3, \frac{1}{9})$



Plot the ordered pairs from the table. To connect the points, ensure that you connect them from the smallest x-value to the largest x-value, i.e., from left to right. The domain of $R(x)$ is all real numbers except for $x = 0 : \{x | x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$. Since $R(x)$ has ***horizontal*** and ***vertical asymptotes*** at $y = 0$ and $x = 0$, respectively, let's draw dashed lines that represent these asymptotes.

Graphing Rational Functions Using Shifts

Let's take a look at when rational functions' graphs contain horizontal and vertical shifts. It will be interesting to see the horizontal and vertical asymptotes change due to these shifts.

Horizontal and Verticals Shifts of Rational Functions

Given $f(x)$ is a rational function, a horizontal shift and vertical shift of $f(x)$ are described below:

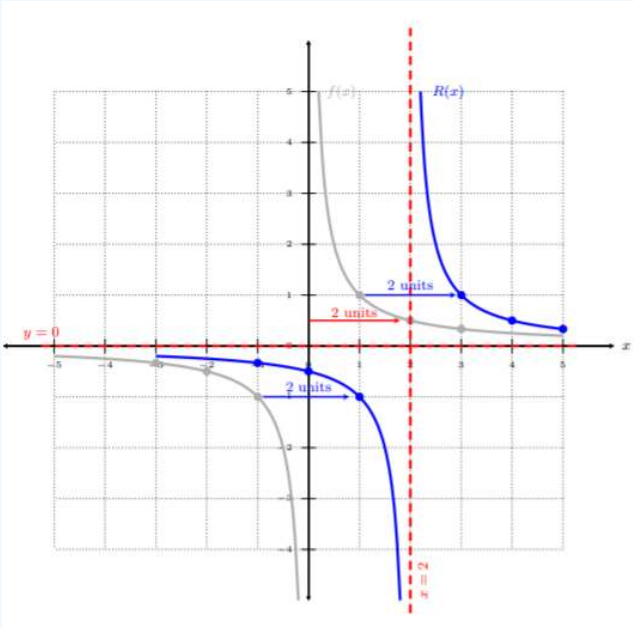
	$f(x-h)$	$f(x+h)$	$f(x)-k$	$f(x)+k$
Shift	Horizontal shift	Horizontal shift	Vertical shift	Vertical shift
Units	Shift h units to the right	Shift h units to the left	Shift k units downward	Shift k units upward
Asymptotes	Vertical asymptote is $x = h$ Horizontal asymptote is $y = 0$	Vertical asymptote is $x = h$ Horizontal asymptote is $y = 0$	Vertical asymptote is $x = 0$ Horizontal asymptote is $y = k$	Vertical asymptote is $x = 0$ Horizontal asymptote is $y = k$

Example 3

Graph $R(x) = \dfrac{1}{x-2}$.

Solution

Let's start by taking the *parent function* $f(x) = \dfrac{1}{x}$. We see that $R(x) = f(x-2)$ because we replaced x with the factor $(x-2)$. Looking at the table above, we see this is a horizontal shift with $h = 2$, moving 2 units to the right, and the vertical asymptote changes to $x = 2$.



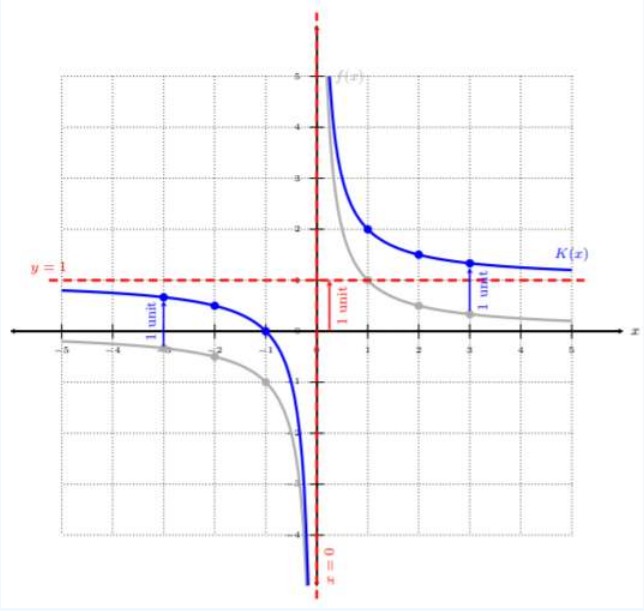
We can see the gray graph, $f(x)$, moved two units to the right, in addition to the vertical asymptote. Recall from the table above that the horizontal asymptote remains $y = 0$. Hence, the blue graph, $R(x)$, is the final graph after applying the shifts.

Example 4

Graph $K(x) = \frac{1}{x} + 1$.

Solution

Let's start by taking the *parent function* $f(x) = \frac{1}{x}$. We see that $K(x) = f(x) + 1$ because we added 1 to $f(x)$. Looking at the table above, we see this is a vertical shift with $k = 1$, moving 1 unit upward, and the horizontal asymptote changes to $y = 1$.



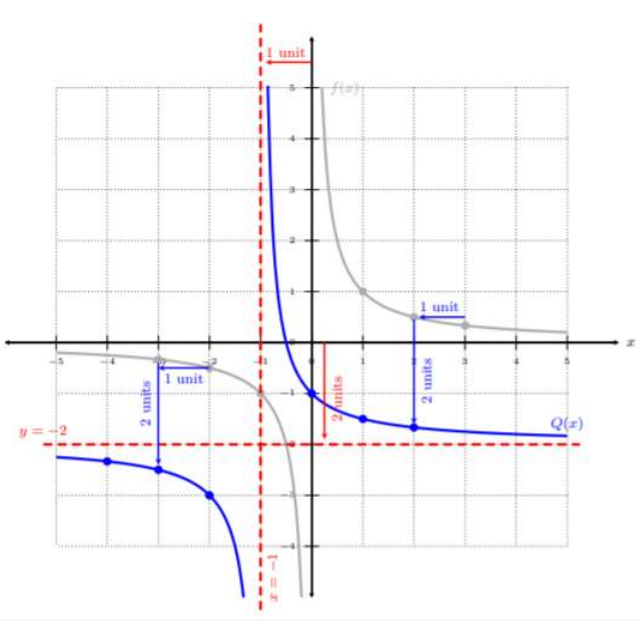
We can see the gray graph, $f(x)$, moved one unit upward, in addition to the horizontal asymptote. Recall from the table above that the vertical asymptote remains at $x = 0$. Hence, the blue graph, $K(x)$, is the final graph after applying the shifts.

Example 5

Graph $Q(x) = \frac{1}{x+1} - 2$.

Solution

Let's start by taking the *parent function* $f(x) = \frac{1}{x}$. We see that $Q(x) = f(x+1) - 2$ because we replaced x with the factor $(x + 1)$ and we subtracted 2 from $f(x)$. Looking at the table above, we see $Q(x)$ has a few shifts: a horizontal shift with $h = -1$, moving 1 unit to the left, a vertical shift with $k = -2$, moving 2 units downward, and the vertical and horizontal asymptotes change to $x = -1$ and $y = -2$, respectively.



We can see the gray graph, $f(x)$, moved one unit to the left and 2 units downward in addition to the horizontal and vertical asymptotes. Notice that we had a vertical and horizontal shift. We moved $f(x)$ one unit left, then 2 units down for all points. These shifts cause the asymptotes to move, too. In fact, the vertical asymptote moved one unit to the left, and the horizontal asymptote moved 2 units downward. Hence, the blue graph, $Q(x)$, is the final graph after applying the shifts.

Scaling and Reflection

In this section, we will both scale and reflect the graph of $y=\frac{1}{x}$. For extra measure, we also include translations in both horizontal and vertical directions.

Example 6

Sketch the graph of

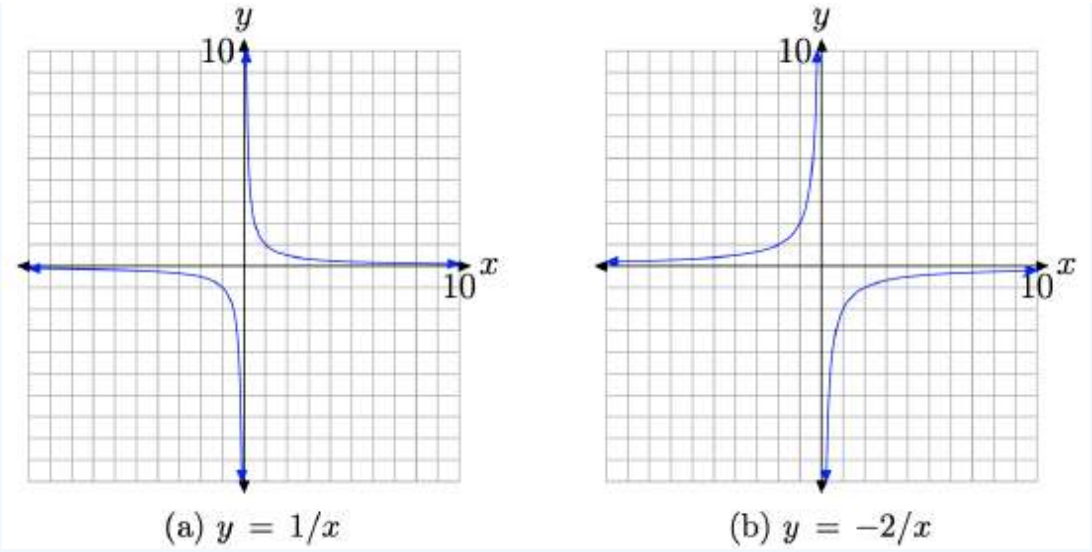
$$y=-\frac{2}{x-4}+3$$

Solution

Let's start by taking the *parent function* $f(x)=\frac{1}{x}$. First, we multiply the equation by -2 to get

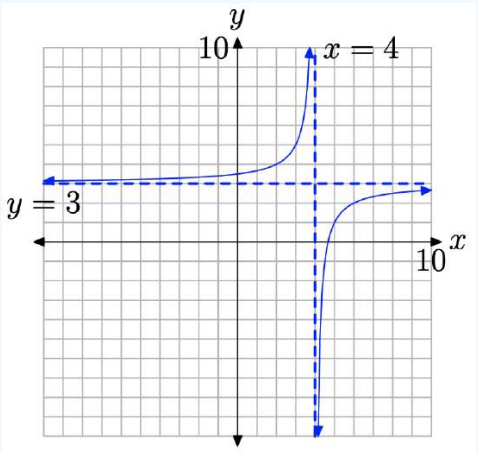
$$-2f(x)=-\frac{2}{x}$$

Multiplying by 2 should stretch the graph in the vertical direction (both positive and negative) by a factor of 2. Note that points that are very near the x-axis, when doubled, are not going to stray too far from the x-axis, so the horizontal asymptote will remain the same. Finally, multiplying by -2 will not only stretch the graph, it will also reflect the graph across the x-axis, as shown here.



Replacing x with $x-4$ will shift the graph 4 units to the right. Then, adding 3 will shift the graph 3 units up, as shown below. Note again that $x=4$ makes the denominator of $y=-2\frac{1}{(x-4)}+3$ equal to zero and there is a vertical asymptote at $x=4$. The domain of this function is $D=\{x:x\neq 4\}$.

As x approaches positive or negative infinity, points on the graph of $y=-2\frac{1}{(x-4)}+3$ get arbitrarily close to the horizontal asymptote $y=3$ but never touch it. Therefore, there is no point on the graph that has a y-value of 3. Thus, the range of the function is the set $R=\{y:y\neq 3\}$.



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3.2P: Practice

Practice Makes Progress

Find all vertical asymptotes, if any, of the graph of the given function.

Exercise 1

$f(x) = -\frac{5}{x+1} - 3$

Exercise 2

$f(x) = -\frac{9}{x+2} - 6$

Exercise 3

$f(x) = -\frac{8}{x-4} - 5$

Exercise 4

$f(x) = \frac{3}{x} + 2$

Find all horizontal asymptotes, if any, of the graph of the given function.

Exercise 5

$f(x) = \frac{5}{x+7} + 9$

Exercise 6

$f(x) = \frac{8}{x+5} - 1$

Exercise 7

$f(x) = \frac{7}{x-1}$

Perform each of the following tasks for the given rational function.

a. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
b. Use transformations to draw the graphs of each of the following rational functions. Draw the vertical and horizontal asymptotes as dashed lines and label each with its equation.
c. Use set-builder notation to describe the domain and range of the given rational function.

Exercise 8

$f(x) = -\frac{2}{x}$

Exercise 9

$f(x) = \frac{1}{x-4}$

Exercise 10

$f(x) = \frac{1}{x} - 2$

Exercise 11

$f(x) = \frac{1}{x-2} - 3$

Exercise 12

$f(x) = -\frac{2}{x-3} - 4$

Self-Check

In this section, you practiced:

a. Finding the vertical asymptote of a rational function
b. Finding the horizontal asymptote of a rational function
c. Graphing the basic rational functions $\frac{1}{x}$ and $\frac{1}{x^2}$
d. Graphing rational functions using transformations
    i. Horizontal shifts (left and right)
    ii. Vertical shifts (up and down)
    iii. Vertical stretching
    iv. Vertical reflections
e. Identifying the domain and range of a rational function

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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3.3: Reducing Rational Expressions

Learning Objectives

- Explain the difference between a *term* and a *factor*, and how these relate to simplifying fractions
- Reduce rational expressions by identifying common factors in the numerator and denominator
- Clearly state any restrictions on a rational expression that result from reducing common factors

The goal of this section is to learn how to reduce a rational expression to "lowest terms." Of course, that means that we will have to understand what is meant by the phrase "lowest terms." With that thought in mind, we begin with a discussion of the greatest common divisor of a pair of integers.

First, we define what we mean by "divisibility."

Definition

Suppose that we have a pair of integers a and b. We say that "a is a divisor of b," or "a divides b" if and only if there is another integer k so that $b = ak$. Another way of saying the same thing is to say that a divides b if, upon dividing b by a, the remainder is zero.

Let's look at an example.

Example 1

What are the factors of 12?

Solution

Because $12 = 1 \times 12$, both 1 and 12 are factors of 12. Because $12 = 2 \times 6$, both 2 and 6 are factors of 12. Finally, because $12 = 3 \times 4$, both 3 and 4 are factors of 12. If we list them in ascending order, the factors of 12 are

$$1, 2, 3, 4, 6, \text{ and } 12$$

Let's look at another example.

Example 2

What are the factors of 18?

Solution

Because $18 = 1 \times 18$, both 1 and 18 are factors of 18. Similarly, $18 = 2 \times 9$ and $18 = 3 \times 6$, so in ascending order, the factors of 18 are

$$1, 2, 3, 6, 9, \text{ and } 18$$

The greatest common divisor of two or more integers is the largest divisor that the integers share in common. An example should make this clear.

Example 3

What is the greatest common divisor of 12 and 18?

Solution

In Examples 2 and 3, we observed the following.

Divisors of 12: *1*, *2*, *3*, 4, *6*, 12

Divisors of 18: *1*, *2*, *3*, *6*, 9, 18

We've italicized the factors that 12 and 18 have in common. They are 1, 2, 3, and 6. The "greatest" of these "common" factors is 6. Hence, we say that "the greatest common divisor of 12 and 18 is 6."

Definition

The greatest common divisor of two integers a and b is the largest divisor they have in common. We will use the notation GCF (a, b) to represent the greatest common divisor of a and b.

Thus, as we saw in Example 3, GCF$(12, 18)$ = 6.

When the greatest common divisor of a pair of integers is one, we give that pair a special name.

Definition

Let a and b be integers. If the greatest common divisor of a and b is one, that is, if GCF(a, b) = 1, then we say that a and b are relatively prime.

For example:

- 9 and 12 are not relatively prime because GCF$(9, 12) = 3$.
- 10 and 15 are not relatively prime because GCF$(10, 15) = 5$.
- 8 and 21 are relatively prime because GCF$(8, 21) = 1$.

We can now define what is meant when we say that a rational number is reduced to its lowest terms.

Definition

A rational number in the form $\frac{p}{q}$, where p and q are integers, is said to be reduced to lowest terms if and only if GCF$(p, q) = 1$. That is, $\frac{p}{q}$ is reduced to lowest terms if the greatest common divisor of both numerator and denominator is 1.

As we saw in the third example, the greatest common divisor of 12 and 18 is 6. Therefore, the fraction 12/18 is not reduced to its lowest terms. However, we can reduce 12/18 to its lowest terms by dividing both the numerator and the denominator by their greatest common divisor. That is,

$$\frac{12}{18} = \frac{12 \div 6}{18 \div 6} = \frac{2}{3}$$

Note that GCF$(2, 3) = 1$, so $2/3$ is reduced to lowest terms.

When it is difficult to ascertain the greatest common divisor, we'll find it more efficient to proceed as follows:

• Prime factor both the numerator and the denominator.

• Cancel common factors.

Thus, to reduce $12/18$ to its lowest terms, first express both the numerator and the denominator as a product of prime numbers, then cancel common primes.

$$\frac{12}{18} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 3} = \frac{\cancel{2} \cdot 2 \cdot \cancel{3}}{\cancel{2} \cdot 3 \cdot \cancel{3}} = \frac{2}{3}$$

When you cancel a 2, you're actually dividing both the numerator and the denominator by 2. When you cancel a 3, you're actually dividing both the numerator and the denominator by 3. Note that doing both (dividing by 2 and then dividing by 3) is equivalent to dividing both the numerator and the denominator by 6.

We will favor this latter technique, precisely because it is identical to the technique we will use to reduce rational functions to lowest terms. However, this "cancellation" technique has some pitfalls, so let's take a moment to discuss some common mistakes

associated with cancellation.

Cancellation

You can spark some pretty heated debate amongst mathematics educators by innocently mentioning the word “cancellation.” There seem to be two diametrically opposed camps: those who don’t mind when their students use the technique of cancellation, and on the other side, those who refuse to even use the term “cancellation” in their classes.

Both sides of the argument have merit. As we showed, we can reduce 12/18 quite efficiently by simply canceling common factors. On the other hand, instructors from the second camp prefer to use the phrase “factor out a 1” instead of the phrase “cancel,” encouraging their students to reduce 12/18 as follows.

$$\frac{12}{18}=\frac{2\cdot 2\cdot 3}{2\cdot 3\cdot 3}=\frac{2}{3}\cdot\frac{2\cdot 3}{2\cdot 3}=\frac{2}{3}\cdot 1=\frac{2}{3}$$

This is a perfectly valid technique and one that, quite honestly, avoids the quicksand of “cancellation mistakes.” Instructors who grow weary of watching their students “cancel” when they shouldn’t are quite likely to promote this latter technique.

However, if we can help our students avoid “cancellation mistakes,” we prefer to allow them to cancel common factors when reducing fractions, such as 12/18, to their lowest terms. So, with these thoughts in mind, let’s discuss some of the most common cancellation mistakes.

Warning

Many of the ensuing calculations are incorrect. They are examples of common mistakes that are made when performing cancellations. Make sure that you read carefully and avoid just “scanning” these calculations.

As a first example, consider the rational expression

$$\frac{2+6}{2}$$

which clearly equals 8/2, or 4. However, if you cancel in this situation, as in

$$\frac{2+6}{2}=\frac{\cancel{2}+6}{\cancel{2}}$$

you certainly do not get the same result. So, what happened?

Note that in the numerator, the 2 and the 6 are separated by a plus sign. Thus, they are not factors; they are terms! You are not allowed to cancel terms, only factors.

The correct answer should have been 8/2, or 4. We mistakenly found the answer to be 6 because we canceled the terms. A workaround would be to factor the numerator first, then cancel, as follows.

$$\frac{2+6}{2}=\frac{2(1+3)}{2}=\frac{\cancel{2}(1+3)}{\cancel{2}}=1+3=4$$

Note that we cancelled factors in this approach, which is permissible, and got the correct answer 4.

Suppose, for comparison, that the rational expression had been

$$\frac{2\cdot 6}{2}$$

which clearly equals 12/2, or 6. In this case, the 2 and the 6 in the numerator are separated by a multiplication symbol, so they are factors and cancellation is allowed, as in

$$\frac{2\cdot 6}{2}=\frac{\cancel{2}\cdot 6}{\cancel{2}}=6$$

Now, before you dismiss these examples as trivial, consider the following examples, which are identical in structure. First, consider

$$\frac{x+(x+2)}{x} = \frac{\cancel{x}+(x+2)}{\cancel{x}} = x+2$$

This cancellation is not permitted. In the numerator, note that x and $(x+2)$ are separated by an addition symbol, so they are terms. You are not allowed to cancel terms!

Conversely, consider the following example.

$$\frac{x(x+2)}{x} = \frac{\cancel{x}(x+2)}{\cancel{x}} = x+2$$

In the numerator of this example, x and $(x+2)$ are separated by implied multiplication. Hence, they are factors, and cancellation is permissible.

Warning

We have now finished discussing common cancellation mistakes, and you may continue reading with confidence that all mathematics is presented correctly.

Reducing Rational Expressions in x

Now that we've discussed some fundamental ideas and techniques, let's apply what we've learned to rational expressions that are functions of an independent variable (usually x). Let's start with a simple example.

How to Avoid Cancellation Mistakes

You may only cancel factors, not terms. To avoid cancellation mistakes, factor completely before you begin to cancel.

Example 4

Reduce the rational expression $\dfrac{2x-6}{x^2-7x+12}$ to lowest terms. For what values of x is your result valid?

Solution

With both the numerator and the denominator factored, we can cancel common factors.

$$\begin{aligned}\frac{2x-6}{x^2-7x+12} &= \frac{2(x-3)}{(x-3)(x-4)} \\ &= \frac{2\,\cancel{(x-3)}}{\cancel{(x-3)}\,(x-4)} \\ &= \frac{2}{x-4}\end{aligned}$$

Thus, we have shown that

$$\frac{2x-6}{x^2-7x+12} = \frac{2}{x-4}$$

However, there is one notable exception, $x=3$. If we substitute $x=3$ into the original expression, we get

$$\frac{2x-6}{x^2-7x+12} = \frac{2(3)-6}{(3)^2-7(3)+12} = \frac{0}{0}$$

But our simplified expression $\dfrac{2}{x-4}$ evaluated at $x=3$ is equal to -2!

This contradiction suggests that our simplified expression is not valid for $x=3$.

We are led to the following key result.

Factor Cancelation Restrictions

In general, when you reduce a rational expression to its lowest terms, the expression obtained should be **identical** to the original expression for all values of the variables in each expression, excluding those values of the variables that make the canceled factors equal to zero. We will refer to these excluded values of the variable as **restrictions**.

Let's look at another example.

Example 5

Reduce the expression $\dfrac{2x^2+5x-12}{4x^3+16x^2-9x-36}$ to lowest terms and state any restrictions.

Solution

With both the numerator and the denominator factored, we cancel common factors.

$$\begin{aligned}\frac{2x^2+5x-12}{4x^3+16x^2-9x-36} &= \frac{(x+4)(2x-3)}{(2x+3)(2x-3)(x+4)} \\ &= \frac{\cancel{(x+4)}\ \cancel{(2x-3)}}{(2x+3)\ \cancel{(2x-3)}\ \cancel{(x+4)}} \\ &= \frac{1}{2x+3}\end{aligned}$$

We must now determine the restrictions. This means that we must find those values of x that make the canceled terms equal to zero.

$$2x-3\neq 0 \quad \text{or} \quad x+4\neq 0$$

Each of these linear factors can be solved independently, generating two restrictions

$$x\neq 3/2 \quad \text{or} \quad x\neq -4$$

Thus, for all values of x except $3/2$, or -4:

$$\frac{2x^2+5x-12}{4x^3+16x^2-9x-36}=\frac{1}{2x+3}$$

Sign Changes

It is not uncommon that you will have to manipulate the signs in a fraction in order to obtain common factors that can be then cancelled. Consider, for example, the rational expression

$$\frac{3-x}{x-3}$$

One possible approach is to factor -1 out of the numerator to obtain

$$\frac{3-x}{x-3}=\frac{-(x-3)}{x-3}$$

You can now cancel common factors.

$$\frac{3-x}{x-3}=\frac{-(x-3)}{x-3}=\frac{-\cancel{(x-3)}}{\cancel{(x-3)}}=-1$$

This result is valid for all values of x except $x=3$.

Let's look at another example.

Example 6

Reduce the rational expression $\dfrac{2x-2x^3}{3x^3+4x^2-3x-4}$ to lowest terms and state any restrictions.

Solution

Factoring the numerator and denominator,

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4}=\frac{2x(1+x)(1-x)}{(x+1)(x-1)(3x+4)}$$

Because we have $1-x$ in the numerator and $x-1$ in the denominator, we will factor out a -1 from $1-x$, and because the order of factors does not affect their product, we will move the -1 out to the front of the numerator. We can now cancel common factors.

$$\begin{aligned}\frac{2x-2x^3}{3x^3+4x^2-3x-4}&=\frac{-2x(1+x)(x-1)}{(x+1)(x-1)(3x+4)}\\&=\frac{-2x\,\cancel{(1+x)}\,\cancel{(x-1)}}{\cancel{(x+1)}\,\cancel{(x-1)}\,(3x+4)}\\&=\frac{-2x}{3x+4}\quad\text{for } x\neq -1,1\end{aligned}$$

Note that the variable restrictions generated by the cancellation are included in the last line of the simplification process.

The Sign Change Rule for Fractions

Let's examine an alternative approach to the previous example. First, let's share the precept that every fraction has three signs, one on the numerator, one on the denominator, and a third on the fraction bar. Thus,

$$\frac{-2}{3}\quad\text{has understood signs}\quad+\frac{-2}{+3}$$

Let's state the sign change rule for fractions.

The Sign Change Rule for Fractions

Every fraction has three signs, one on the numerator, one on the denominator, and one on the fraction bar. If you don't see an explicit sign, then a plus sign is understood. If you negate any two of these parts,

- numerator and denominator, or
- numerator and fraction bar, or
- fraction bar and denominator,

then the fraction remains unchanged.

For example, let's start with -2/3, then do two negations: the numerator and the fraction bar. Then,

$$+\frac{-2}{+3}=-\frac{+2}{+3},\quad\text{or with understood plus signs,}\quad\frac{-2}{3}=-\frac{2}{3}$$

This is a familiar result, as negative two divided by a positive three equals a negative two-thirds.

On another note, we might decide to negate the numerator and denominator. Then $-2/3$ becomes

$$+\frac{-2}{+3}=\frac{+2}{-3},\quad\text{or with understood plus signs,}\quad\frac{-2}{3}=\frac{2}{-3}$$

Again, a familiar result. Certainly, negative two divided by positive three is the same as positive two divided by negative three. They both equal minus two-thirds.

Products and Quotients of Rational Expressions

Now we deal with products and quotients of rational expressions. Before we begin, we'll need to establish some fundamental definitions and techniques. We begin with the definition of the product of two rational numbers.

Definition

Let $\frac{a}{b}$ and $\frac{c}{d}$ be rational numbers. The product of these rational numbers is defined by

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}, \quad \text{or more compactly,} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

The definition simply states that you should multiply the numerators of each rational number to obtain the numerator of the product, and you also multiply the denominators of each rational number to obtain the denominator of the product. For example,

$$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$$

Of course, you should also verify that your final answer is reduced to its lowest terms.

Let's look at an example.

Example 7

Simplify the product of rational numbers $\frac{6}{231} \cdot \frac{35}{10}$

Solution

We could first multiply the numerators and denominators together as follows.

$$\frac{6}{231} \cdot \frac{35}{10} = \frac{6 \cdot 35}{231 \cdot 10} = \frac{210}{2310}$$

However, the answer is not reduced to its lowest terms.

A better approach is to factor the smaller numerators and denominators immediately, as follows.

$$\frac{6}{231} \cdot \frac{35}{10} = \frac{2 \cdot 3}{3 \cdot 7 \cdot 11} \cdot \frac{5 \cdot 7}{2 \cdot 5}$$

Now, we can complete our computation by dividing common factors.

$$\frac{6}{231} \cdot \frac{35}{10} = \frac{2 \cdot 3}{3 \cdot 7 \cdot 11} \cdot \frac{5 \cdot 7}{2 \cdot 5} = \frac{\cancel{2} \cdot \cancel{3}}{\cancel{3} \cdot \cancel{7} \cdot 11} \cdot \frac{\cancel{5} \cdot \cancel{7}}{\cancel{2} \cdot \cancel{5}} = \frac{1}{11}$$

Thus, we have two choices when multiplying rational expressions. Let's look at another example.

Example 8

Simplify the expression $\frac{x^2 - x - 6}{x^2 + 2x - 15} \cdot \frac{x^2 - x - 30}{x^2 - 2x - 8}$ and state any restrictions.

Solution

Factor the numerator and denominator and then cancel as shown.

$$\frac{x^2-x-6}{x^2+2x-15}\cdot\frac{x^2-x-30}{x^2-2x-8}=\frac{(x+2)(x-3)}{(x-3)(x+5)}\cdot\frac{(x+5)(x-6)}{(x+2)(x-4)}$$
$$=\frac{(x+2)(x-3)(x+5)(x-6)}{(x-3)(x+5)(x+2)(x-4)}$$
$$=\frac{\cancel{(x+2)}\ \cancel{(x-3)}\ \cancel{(x+5)}\ (x-6)}{\cancel{(x-3)}\ \cancel{(x+5)}\ \cancel{(x+2)}\ (x-4)}$$
$$=\frac{x-6}{x-4}\quad \text{for } x\neq -5,-2,3$$

Let's look at another example.

Example 9

Simplify $\frac{9-x^2}{x^2+3x}\cdot\frac{6x-2x^2}{x^2-6x+9}$ and state any restrictions.

Solution

Factoring the numerator and denominator we obtain

$$\frac{9-x^2}{x^2+3x}\cdot\frac{6x-2x^2}{x^2-6x+9}\cdot=\frac{(3+x)(3-x)}{x(x+3)}\cdot\frac{2x(3-x)}{(x-3)^2}$$

We'll need to execute a sign change or two to create common factors in the numerators and denominators. So, in both the first and second numerator, factor a -1 from the factor $3-x$ to obtain $3-x=-1(x-3)$. Because the order of factors in a product doesn't matter, we'll slide the -1 to the front in each case. We can now cancel common factors.

$$\frac{9-x^2}{x^2+3x}\cdot\frac{6x-2x^2}{x^2-6x+9}=\frac{-(3+x)(x-3)}{x(x+3)}\cdot\frac{-2x(x-3)}{(x-3)^2}$$
$$=\frac{-(3+x)(x-3)(-2x)(x-3)}{x(x+3)(x-3)^2}$$
$$=\frac{-\cancel{(3+x)}\ \cancel{(x-3)}\ (-2\cancel{x})\ \cancel{(x-3)}}{\cancel{x}\ \cancel{(x+3)}\ \cancel{(x-3)^2}}$$
$$=2\quad \text{for } x\neq -3,0,3$$

A few things to notice:

- The factors $3+x$ and $x+3$ are identical, so they may be cancelled, one on the top for one on the bottom.
- Two factors of $x-3$ on the top are cancelled for $(x-3)^2$ (which is equivalent to $(x-3)(x-3)$) on the bottom.
- We're left with two minus signs (two -1's) and a 2. Therefore, the solution is positive 2.

Division of Rational Expressions

A simple definition will change a problem involving the division of two rational expressions into one involving the multiplication of two rational expressions. Then there's nothing left to explain, for we already know how to multiply two rational expressions.

So, let's motivate our definition of division. Suppose we ask the question: How many halves are in a whole? The answer is easy, as two halves make a whole. Thus, when we divide 1 by 1/2, we should get 2. There are two halves in one whole.

Let's raise the stakes a bit and ask how many halves are in six? To make the problem more precise, imagine you've ordered 6 pizzas and you cut each in half. How many halves do you have? Again, this is easy when you think about the problem in this manner, the answer is 12. Thus,

$$6\div\frac{1}{2}$$

(how many halves are in six?) is identical to

$$6\cdot 2$$

which, of course, is 12. Hopefully, thanks to this opening motivation, the following definition will not seem too strange.

Definition

To perform the division

$$\frac{a}{b} \div \frac{c}{d}$$

invert the second fraction and multiply, as in

$$\frac{a}{b} \cdot \frac{d}{c}$$

Thus, if we want to know how many halves are in 12, we change the division into multiplication ("invert and multiply").

$$12 \div \frac{1}{2} = 12 \cdot 2 = 24$$

This makes sense, as there are 24 "halves" in 12. Let's look at a harder example.

Example 10

Simplify $\frac{33}{15} \div \frac{14}{10}$

Solution

Invert the second fraction and multiply. After that, all we need to do is factor numerators and denominators, then cancel common factors.

$$\frac{33}{15} \div \frac{14}{10} = \frac{33}{15} \cdot \frac{10}{14} = \frac{3 \cdot 11}{3 \cdot 5} \cdot \frac{2 \cdot 5}{2 \cdot 7} = \frac{\cancel{3} \cdot 11}{\cancel{3} \cdot \cancel{5}} \cdot \frac{\cancel{2} \cdot \cancel{5}}{\cancel{2} \cdot 7} = \frac{11}{7}$$

Alternative Notation

Note that the fractional expression $\frac{m}{n}$ means "m divided by n," so we can use this equivalent notation for $m \div n$. For example, if both m and n are fractions, we can write

$$\frac{a}{b} \div \frac{c}{d} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)}$$

The latter format is helpful when dividing rational expressions involving variables.

Example 11

Simplify $\frac{x^2-1}{x+4} \div \frac{x+1}{x-1}$ and state any restrictions.

Solution

$$\begin{aligned}\frac{x^2-1}{x+4} \div \frac{x-1}{x+1} &= \frac{\left(\frac{x^2-1}{x+4}\right)}{\left(\frac{x-1}{x+1}\right)} \\ &= \frac{x^2-1}{x+4} \cdot \frac{x+1}{x-1} \\ &= \frac{(x-1)(x+1)^2}{(x+4)(x-1)} \\ &= \frac{\cancel{(x-1)}\,(x+1)^2}{(x+4)\,\cancel{(x-1)}} \\ &= \frac{(x+1)^2}{(x+4)} \quad \text{for } x \neq 1\end{aligned}$$

However, there is one notable exception, $x = -1$. If we substitute $x = -1$ into the original expression, we get

$$\frac{\left(\frac{(-1)^2-1}{(-1)+4}\right)}{\left(\frac{(-1)-1}{(-1)+1}\right)} = \frac{\left(\frac{0}{3}\right)}{\left(\frac{-2}{0}\right)}$$

which is invalid as the rational expression in the denominator has division by zero.

But our simplified expression $\frac{(x+1)^2}{x+4}$ evaluated at $x = -1$ is equal to 0!

This contradiction suggests that our simplified expression is not valid for $x = -1$ in addition to the restriction of $x = 1$ generated by factor cancellation.

Rational Division Restrictions

In addition to any restrictions generated by the cancellation of factors during simplification, values of the variables that cause division by zero in the denominator expression must also be restricted. When dividing rational expressions, including factor cancellation, the set of excluded values will be those created by both factor cancellation *and* potential division by zero.

Let's look at another example.

Example 12

Simplify $\frac{3x+6}{-2x^2+6x} \div \frac{x^2-4}{x^2-3x+2}$ and state any restrictions.

Solution

We can switch division formats and factor to identify additional restrictions before proceeding with simplifying by inverting to multiply and then canceling factors.

$$\begin{aligned}\frac{3x+6}{-2x^2+6x} \div \frac{x^2-4}{x^2-3x+2} &= \frac{\left(\frac{3(x+2)}{-2x(x-3)}\right)}{\left(\frac{(x-2)(x+2)}{(x-1)(x-2)}\right)} \\ &= \frac{3(x+2)}{-2x(x-3)} \cdot \frac{(x-1)(x-2)}{(x-2)(x+2)} \\ &= \frac{3\,\cancel{(x+2)}\,(x-1)\,\cancel{(x-2)}}{-2x(x-3)\,\cancel{(x-2)}\,\cancel{(x+2)}} \\ &= \frac{3(x-1)}{-2x(x-3)} \quad \text{for } x \neq -2, 1, 2\end{aligned}$$

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3.3P: Practice

Practice Makes Progress

Reduce the given expression to lowest terms. State all restrictions.

Exercise 1

$$\frac{x^2-10x+9}{5x-5}$$

Exercise 2

$$\frac{x^2-2x-35}{x^2-7x}$$

Exercise 3

$$\frac{x+2}{-x-2}$$

Exercise 4

$$\frac{2x-6}{3-x}$$

Exercise 5

$$\frac{3x^2+6x}{-x-2}$$

Exercise 6

$$\frac{x^2-x-20}{25-x^2}$$

Exercise 7

$$\frac{x^2-3x-10}{-9x-18}$$

Multiply and simplify. State all restrictions.

Exercise 8

$$\frac{x+6}{x^2+16x+63}\cdot\frac{x^2+7x}{x+4}$$

Exercise 9

$$\frac{x^2+7x+10}{x^2-1}\cdot\frac{-9+10x-x^2}{x^2+9x+20}$$

Exercise 10

$$\frac{x^2+2x-15}{x^2-10x+16}\cdot\frac{x^2-7x+10}{3x^2+13x-10}$$

Exercise 11

$$\frac{5-x}{x+3} \cdot \frac{x^2+3x-18}{2x^2-7x-15}$$

Divide and simplify. State all restrictions.

Exercise 12

$$\frac{x^2-1}{x^2-7x+12} \div \frac{x^2+6x+5}{-24+10x-x^2}$$

Exercise 13

$$\frac{x^2-25}{x+1} \div \frac{5x^2+23x-10}{x-3}$$

Exercise 14

$$\frac{x^2+x-30}{x^2+5x-36} \div \frac{-6-x}{x+8}$$

Exercise 15

$$\frac{x^2-11x+28}{x^2+5x+6} \div \frac{7x^2-30x+8}{x^2-x-6}$$

Self-Check

In this section, you practiced:

a. Factoring quadratic expressions
b. Simplifying rational expressions using factoring
c. Stating restrictions on simplified rational expressions to preserve equality
d. Multiplying and simplifying rational expressions
e. Dividing and simplifying rational expressions
f. Stating restrictions on a rational expression that is the result of division to preserve equality

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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3.4: Sums and Differences of Rational Functions

Learning Objectives

- Explain why fractions can only be added or subtracted if they have the same denominator
- Use the concept of a least common multiple to find a least common denominator of two fractions
- And and subtract rational expressions by finding a least common denominator, and reduce the result if possible

In this section we concentrate on finding sums and differences of rational expressions. However, before we begin, we need to review some fundamental ideas and technique.

Definition

Let a and b be integers. The **least common multiple** of a and b, denoted LCM(a, b), is the smallest positive multiple that a and b have in common.

Lets find the LCM$(8, 6)$.

Multiples of 6: 6, 12, 18, 24, 30, 36, ...

Multiples of 8: 8, 16, 24, 32, 40, ...

The smallest common multiple of 8 and 6 is 24.

For larger numbers, listing multiples until you find one in common can be impractical and time-consuming. Let's find the least common multiple of 8 and 6 a second time, only this time let's use a different technique.

First, write each number as a product of primes in exponential form.

$$8 = 2^3$$
$$6 = 2 \cdot 3$$

In our example, the primes that occur are 2 and 3. The highest power of 2 that occurs is 2^3. The highest power of 3 that occurs is 3^1. Thus, the LCM$(8, 6)$ is

$$\text{LCM}(8, 6) = 2^3 \cdot 3^1 = 24$$

Note that this result is identical to the result found above by listing all common multiples and choosing the smallest.

Here's the rule.

A Procedure to Find the LCM

To find the LCM of two integers, proceed as follows.

1. Express the prime factorization of each integer in exponential format.
2. To find the least common multiple, write down every prime number that appears, then affix the largest exponent of that prime that appears.

Let's try a harder example.

Example 1

Find the least common multiple of 24 and 36.

Solution

Using the first technique, we list the multiples of each number, framing the multiples in common.

Multiples of 24: 24, 48, 72, 96, 120, 144, 168, ...

Multiples of 36: 36, 72, 108, 144, 180, ...

The multiples in common are 72, 144, etc., and the **least** common multiple is $\text{LCM}(24, 36) = 72$.

Now, let's use our second technique to find the least common multiple. First, express each number as a product of primes in exponential format.

$$24 = 2^3 \cdot 3$$
$$36 = 2^2 \cdot 3^2$$

To find the least common multiple, write down every prime that occurs and affix the highest power of that prime that occurs. Thus, the highest power of 2 that occurs is 2^3, and the highest power of 3 that occurs is 3^2. Thus, the least common multiple is

$$\text{LCM}(24, 36) = 2^3 \cdot 3^2 = 8 \cdot 9 = 72$$

Addition and Subtraction Defined

Definition

To add or subtract two fractions with a **common denominator**, such as $\frac{a}{c}$ and $\frac{b}{c}$, add the numerators and divide by the common denominator. In symbols,

$$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$$

For example:

$$\begin{aligned} \frac{5}{21} + \frac{3}{21} &= \frac{5+3}{21} \\ &= \frac{8}{21} \end{aligned}$$

Another example:

$$\begin{aligned} \frac{2}{x+2} - \frac{x-3}{x+2} &= \frac{2-(x-3)}{x+2} \\ &= \frac{2-x+3}{x+2} \\ &= \frac{5-x}{x+2} \end{aligned}$$

In the second example, note that it is extremely important to use grouping symbols when subtracting numerators. Note that the minus sign in front of the parenthetical expression changes the sign of each term inside the parentheses.

There are times when a sign change will provide a common denominator.

Example 2

Simplify $\frac{x}{x-3} - \frac{2}{3-x}$ and state all restrictions.

Solution

At first glance, it appears that we do not have a common denominator. On second glance, we notice that $3 - x = -(x - 3)$ so if we make a sign change on the second fraction, it might help. So, on the second fraction, let's negate the denominator and fraction bar to obtain

$$\begin{aligned}\frac{x}{x-3}-\frac{2}{3-x}&=\frac{x}{x-3}-\frac{2}{-(x-3)}\\&=\frac{x}{x-3}+\frac{2}{x-3}\\&=\frac{x+2}{x-3}\end{aligned}$$

Note that because we did not cancel any terms nor are we dividing a rational expression by a rational expression, there are no restrictions on the "all Real Numbers" domain of x.

Adding and Subtracting Fractions with Different Denominators

In this section, we show our readers how to add and subtract fractions having *different* denominators. For example, suppose we are asked to add the following fractions.

$$\frac{5}{12}+\frac{5}{18}$$

First, we must find a “common denominator.” Fortunately, the machinery to find the “common denominator” is already in place. It turns out that the least common denominator for 12 and 18 is the least common multiple of 12 and 18.

$$\begin{aligned}18&=2\cdot 3^2\\12&=2^2\cdot 3\end{aligned}\tag{3.4.1}$$

Therefore, LCD$(12,18)=2^2\cdot 3^2=36$ and we can add the fractions by rewriting each as equivalent fractions with a denominator of 36.

$$\begin{aligned}\frac{5}{12}+\frac{5}{18}&=\frac{5}{12}\cdot\frac{3}{3}+\frac{5}{18}\cdot\frac{2}{2}\\&=\frac{15}{36}+\frac{10}{36}\\&=\frac{25}{36}\end{aligned}$$

Let’s look at a more complicated example.

Example 3

Simplify $\frac{x+3}{x+2}-\frac{x+2}{x+3}$ and state all restrictions.

Solution

The denominators are already factored. If we take each factor that appears to the highest exponential power that appears, our least common denominator is (x+2)(x+3). Our first task is to make equivalent fractions having this common denominator.

$$\begin{aligned}\frac{x+3}{x+2}-\frac{x+2}{x+3}&=\frac{x+3}{x+2}\cdot\frac{x+3}{x+3}-\frac{x+2}{x+3}\cdot\frac{x+2}{x+2}\\&=\frac{x^2+6x+9}{(x+2)(x+3)}-\frac{x^2+4x+4}{(x+2)(x+3)}\\&=\frac{(x^2+6x+9)-(x^2+4x+4)}{(x+2)(x+3)}\\&=\frac{x^2+6x+9-x^2-4x-4}{(x+2)(x+3)}\\&=\frac{2x+5}{(x+2)(x+3)}\end{aligned}$$

Note the use of parentheses when we subtracted the numerators. Note further how the minus sign negates each term in the parenthetical expression that follows the minus sign.

Example 4

Simplify $\frac{6}{8a+4}+\frac{3a}{8}$ and state all restrictions.

Solution

$$\begin{aligned}\frac{6}{8a+4}+\frac{3a}{8} &= \frac{6}{4(2a+1)}+\frac{3a}{8}\\ &= \frac{2}{2}\cdot\frac{6}{4(2a+1)}+\frac{3a}{8}\cdot\frac{(2a+1)}{(2a+1)}\\ &= \frac{12}{8(2a+1)}+\frac{3a(2a+1)}{8(2a+1)}\\ &= \frac{12+6a^2+3a}{8(2a+1)}\\ &= \frac{3(2a^2+a+4)}{8(2a+1)}\\ &= \frac{3(2a^2+a+4)}{8(2a+1)}\end{aligned}$$

Notice there is a GCF of 3 in the numerator, but 3 isn't a common factor. However, we still need to factor the GCF where possible from the numerator to check whether the expression can be reduced.

Let's look at another example.

Example 5

Simplify $\frac{4}{x^2+6x+5}-\frac{2}{x^2+8x+15}$ and state all restrictions.

Solution

To find the least common denominator, or least common multiple (LCM), requires that we factor each denominator completely and write down each factor that occurs, then affix the highest power of that factor that occurs. Then, we make equivalent fractions having this common denominator and simplify our expression.

$$\begin{aligned}\frac{4}{x^2+6x+5}-\frac{2}{x^2+8x+15} &= \frac{4}{(x+1)(x+5)}-\frac{2}{(x+3)(x+5)}\\ &= \frac{4}{(x+1)(x+5)}\cdot\frac{x+3}{x+3}-\frac{2}{(x+3)(x+5)}\cdot\frac{x+1}{x+1}\\ &= \frac{4x+12}{(x+3)(x+5)(x+1)}-\frac{2x+2}{(x+3)(x+5)(x+1)}\\ &= \frac{(4x+12)-(2x+2)}{(x+3)(x+5)(x+1)}\\ &= \frac{4x+12-2x-2}{(x+3)(x+5)(x+1)} \qquad (3.4.2)\\ &= \frac{2x+10}{(x+3)(x+5)(x+1)}\\ &= \frac{2(x+5)}{(x+3)(x+5)(x+1)}\\ &= \frac{2\,\cancel{(x+5)}}{(x+3)\,\cancel{(x+5)}\,(x+1)}\\ &= \frac{2}{(x+3)(x+1)} \quad \text{if } x\neq -5\end{aligned}$$

Example 6

Simplify $\frac{x+1}{x-4} - \frac{x+1}{x^2-7x+12}$ and state all restrictions.

Solution

$$\begin{aligned}
\frac{x+1}{x-4} - \frac{x+1}{x^2-7x+12} &= \frac{x+1}{(x-4)} - \frac{x+1}{(x-3)(x-4)} \\
&= \frac{(x-3)}{(x-3)} \cdot \frac{(x+1)}{(x-4)} - \frac{(x+1)}{(x-3)(x-4)} \\
&= \frac{(x-3)(x+1)}{(x-3)(x-4)} - \frac{(x+1)}{(x-3)(x-4)} \\
&= \frac{x^2-2x-3-x-1}{(x-3)(x-4)} \\
&= \frac{x^2-3x-4}{(x-3)(x-4)} \\
&= \frac{(x-4)(x+1)}{(x-3)(x-4)} \\
&= \frac{\cancel{(x-4)}\,(x+1)}{(x-3)\,\cancel{(x-4)}} \\
&= \frac{(x+1)}{(x-3)} \quad \text{if } x \neq 4
\end{aligned}$$

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3.4P: Practice

Practice Makes Progress

Add or subtract the rational expressions, as indicated, and simplify your answer. State all restrictions.

Exercise 1

$$\frac{7x^2-49x}{x-6}+\frac{42}{x-6}$$

Exercise 2

$$\frac{9x^2}{x-1}+\frac{72x-63}{1-x}$$

Exercise 3

$$\frac{4x^2-60x}{x-7}+\frac{224}{x-7}$$

Exercise 4

$$\frac{3x^2}{x-2}-\frac{48-30x}{2-x}$$

Exercise 5

$$\frac{3x}{x^2-6x+5}+\frac{15}{x^2-14x+45}$$

Exercise 6

$$\frac{5x}{x^2-21x+98}-\frac{35}{7x-x^2}$$

Exercise 7

$$\frac{-7x}{x^2-8x+15}-\frac{35}{x^2-12x+35}$$

Exercise 8

$$\frac{-9x}{x^2-6x+8}-\frac{18}{x^2-2x}$$

Exercise 9

$$\frac{8x}{5x-x^2}+\frac{120}{x^2+5x-50}$$

Self-Check

In this section, you practiced:

a. Identifying least common denominators (LCD)
b. Completing addition or subtraction of two rational expressions with the same denominator
c. Completing addition or subtraction of two rational expressions with different denominators using the LCD
d. Identifying excluded values of a rational expression

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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3.5: Solving Rational Equations

Learning Objectives

- Simplify a rational equation by clearing fractions
- Determine excluded values of a rational equation
- Solve a rational equation algebraically and verify if the solution is valid
- Solve a rational inequality by using a graph or a number line

In this section, we'll look at some strategies for solving equations involving fractions or rational terms.

Clear the Fractions from a Rational Equation

If your equation has rational expressions, multiply both sides of the equation by the least common denominator to clear the equation of rational expressions.

Let's look at an example.

Example 1

Solve $\frac{x}{2}-\frac{2}{3}=\frac{3}{4}$ for x.

Solution

To clear this equation of fractions, we will multiply both sides by LCM$(2,3,4)$, which is 12, distribute 12, then cancel and solve.

$$\begin{aligned}\frac{x}{2}-\frac{2}{3}&=\frac{3}{4}\\ 12\left(\frac{x}{2}-\frac{2}{3}\right)&=\left(\frac{3}{4}\right)12\\ 12\left(\frac{x}{2}\right)-12\left(\frac{2}{3}\right)&=\left(\frac{3}{4}\right)12\\ 6x-8&=9\\ 6x&=17\\ x&=\frac{17}{6}\end{aligned}$$

We'll leave it to our readers to check this solution.

Let's try another example.

Example 2

Solve $6=\frac{5}{x}+\frac{6}{x^2}$ for x.

Solution

In this equation, the denominators are 1, x, and x^2, and the common denominator for both sides of the equation is x^2. Consequently, we begin the solution by first multiplying both sides of the equation by x^2 and note the restriction $x\neq 0$ to check our later solutions against.

$$
\begin{aligned}
6 &= \frac{5}{x} + \frac{6}{x^2} \\
x^2(6) &= \left(\frac{5}{x} + \frac{6}{x^2}\right) x^2 \\
x^2(6) &= \left(\frac{5}{x}\right) x^2 + \left(\frac{6}{x^2}\right) x^2 \\
6x^2 &= 5x + 6 \\
6x^2 - 5x - 6 &= 0 \\
6x^2 + 4x - 9x - 6 &= 0 \\
2x(3x+2) - 3(3x+2) &= 0 \\
(2x-3)(3x+2) &= 0
\end{aligned}
$$

The zero product property forces either

$$2x - 3 = 0 \quad \text{or} \quad 3x + 2 = 0$$

Each of these linear equations is easily solved.

$$x = \frac{3}{2} \quad \text{or} \quad x = -\frac{2}{3}$$

Let's look at another example.

Example 3

Solve $\frac{5x+5}{x+2} + 3x = \frac{x^2}{x+2}$ for x.

Solution

The denominators in this equation are $x+2$ and 1, so the common denominator is $x+2$. Consequently, we begin the solution by multiplying both sides of the equation by $x+2$ and note the restriction $x \neq -2$ to check our later solutions against.

$$
\begin{aligned}
\frac{5x+5}{x+2} + 3x &= \frac{x^2}{x+2} \\
(x+2)\left(\frac{5x+5}{x+2} + 3x\right) &= (x+2)\left(\frac{x^2}{x+2}\right) \\
\frac{(x+2)(5x+5)}{x+2} + 3x(x+2) &= \frac{(x+2)x^2}{x+2} \\
5x+5 + 3x(x+2) &= x^2 \\
5x+5 + 3x^2 + 6x &= x^2 \\
3x^2 + 11x + 5 &= x^2 \\
3x^2 + 11x + 5 - x^2 &= 0 \\
2x^2 + 11x + 5 &= 0
\end{aligned}
\tag{3.5.1}
$$

Now we factor the quadratic:

$$
\begin{aligned}
2x^2 + 11x + 5 &= 0 \\
2x^2 + x + 10x + 5 &= 0 \\
(2x^2 + x) + (10x + 5) &= 0 \\
x(2x+1) + 5(2x+1) &= 0 \\
(2x+1)(x+5) &= 0
\end{aligned}
$$

The zero product property forces either

$$2x + 1 = 0 \quad \text{or} \quad x + 5 = 0$$

Each of these linear equations is easily solved.

$$x = -\tfrac{1}{2} \quad \text{or} \quad x = -5$$

Since neither solution violates the restriction $x \neq -2$, both solutions are valid.

Final Answer: $x = -\frac{1}{2},\ -5$

Let's look at another example.

Example 4

Solve $\frac{2}{x^2} = 1 - \frac{2}{x}$ for x.

Solution

The denominators in this equation are x^2, x, and 1, so the common denominator is x^2. Consequently, we begin the solution by multiplying both sides of the equation by x^2 and note the restriction $x \neq 0$ to check our later solutions against.

$$\begin{aligned} \frac{2}{x^2} &= 1 - \frac{2}{x} \\ x^2\left(\frac{2}{x^2}\right) &= x^2(1) \ - \ x^2\left(\frac{2}{x}\right) \\ \frac{2x^2}{x^2} &= x^2 - \frac{2x^2}{x} \\ 2 &= x^2 - 2x \\ 0 &= x^2 - 2x - 2 \end{aligned} \tag{3.5.2}$$

We now solve the quadratic equation:

$$x^2 - 2x - 2 = 0$$

Using the quadratic formula:

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{2 \pm \sqrt{4+8}}{2} \\ &= \frac{2 \pm \sqrt{12}}{2} \\ &= \frac{2 \pm 2\sqrt{3}}{2} \\ &= 1 \pm \sqrt{3} \end{aligned}$$

Since neither solution violates the restriction $x \neq 0$, both are valid.

Final Answer: $x = 1 + \sqrt{3},\ 1 - \sqrt{3}$

Let's look at another example.

Example 5

Solve $\frac{1}{x+2} - \frac{x}{2-x} = \frac{x+6}{x^2-4}$ for x

Solution

First note the restrictions from the denominators: $x \neq -2, 2$.
Factor the denominators: $x^2 - 4 = (x-2)(x+2)$ and $2 - x = -(x-2)$.
Now multiply the entire equation by the common denominator $(x-2)(x+2)$:

$$\left(\frac{1}{x+2}-\frac{x}{2-x}\right)=\left(\frac{x+6}{(x-2)(x+2)}\right)$$
$$(x-2)(x+2)\left(\frac{1}{x+2}+\frac{x}{x-2}\right)=(x-2)(x+2)\left(\frac{x+6}{(x-2)(x+2)}\right)$$
$$\left(\frac{(x-2)(x+2)}{x+2}+\frac{x(x-2)(x+2)}{x-2}\right)=\left(\frac{(x-2)(x+2)(x+6)}{(x-2)(x+2)}\right)$$
$$(x-2)+x(x+2)=x+6 \tag{3.5.3}$$
$$x-2+x(x+2)=x+6$$
$$x^2+3x-2=x+6$$
$$x^2+2x-8=0$$
$$(x+4)(x-2)=0$$

Thus $x+4=0$ or $x-2=0$, so $x=-4$ or $x=2$.
Checking the restrictions $x\neq -2, 2$, we must discard $x=2$.

Final Answer: $x=-4$

Rational Inequalities

In earlier chapters, we discussed many types of inequalities, and a key aspect of inequalities is that we know the solutions are always intervals of numbers. Hence, we now discuss rational inequalities where the solutions are intervals, but we take one critical detail into account. Since rational inequalities and equations contain excluded values, we must consider this when obtaining the solution. In fact, we use the excluded values and x-intercepts, if any, to determine the intervals that make the inequality true. Again, with inequalities, we are trying to find intervals of numbers in which the statement is true.

Steps for solving rational inequalities

Step 1. Rewrite the inequality so that only zero is on the right side.

Step 2. Determine where the rational expression is *undefined* or *equals zero*.

Step 3. Graph the values found in **Step 2.** on a number line split into intervals.

Step 4. Take a test point within each interval and determine the sign of the result.

Step 5. Determine the solution, where the solution is the interval in which makes the inequality true.

Note

Be careful when determining the intervals. Recall, where the expression is undefined is not included in the interval, i.e., *always* use parenthesis for the values in which the expression is undefined.

Example 6

Solve $\frac{x-3}{x+1}>0$.

Solution

Step 1. Rewrite the inequality so that only zero is on the right side.
Since $\frac{x-3}{x+1}>0$ already has zero on the right side, this step is done.

Step 2. Determine where the rational expression is *undefined* or *equals zero*.
To obtain where the rational expression **equals zero**, we set the numerator equal to zero:

$$x-3=0 \tag{3.5.4}$$
$$x=3$$

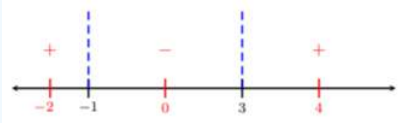
To obtain where the expression is **undefined**, we find its excluded value(s) by setting the denominator equal to zero:

$$\begin{aligned} x+1 &= 0 \\ x &= -1 \end{aligned} \tag{3.5.5}$$

Step 3. Graph the values found in Step 2 on a number line, and split into intervals.
Label -1 and 3 on a blank real number line:



Step 4. Take a test point within each interval and determine the sign of the result.
We take test values on each side of -1 and 3. Let's choose relatively easy numbers such as -2, 0, and 4. We evaluate the right-hand side at each of these numbers:



x	$\frac{x-3}{x+1}$	Sign
-2	5	$+$
0	-3	$-$
4	$\frac{1}{5}$	$+$

Step 5. Determine the solution, where the solution is the interval that makes the inequality true.
Since $\frac{x-3}{x+1} > 0$ (from **Step 1.**), then we are looking for where the test values are positive. Looking at the number line above, we see these are the values to the left of -1 and to the right of 3. Thus, the solution is $(-\infty, -1) \cup (3, \infty)$.

Note

We always use a **parenthesis** for the value that is **excluded** since this value makes the rational expression undefined. Using a bracket or parenthesis for where the expression equals zero is determined by the original inequality sign, e.g., $<, >$ use parenthesis, and $\geq, \leq$ use brackets.

Example 7

Solve $\frac{2x+3}{x-2} \leq 1$.

Solution

Step 1. Rewrite the inequality so that only zero is on the right side.
We rewrite $\frac{2x+3}{x-2} \leq 1$ so that there is zero on the right side, and as one fraction:

$$\begin{aligned} \frac{2x+3}{x-2} &\leq 1 \\ \frac{2x+3}{x-2} - 1 &\leq 0 \\ \frac{2x+3}{x-2} - \frac{x-2}{x-2} &\leq 0 \\ \frac{2x+3-x+2}{x-2} &\leq 0 \\ \frac{x+5}{x-2} &\leq 0 \end{aligned}$$

Step 2. Determine where the rational expression is *undefined* or *equals zero*.
To obtain where the rational expression **equals zero**, we set the numerator equal to zero:

$$x+5=0$$
$$x=-5 \tag{3.5.6}$$

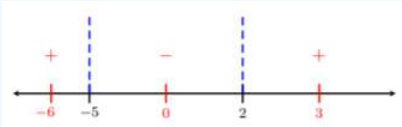
To obtain the **excluded** values, we set the denominator equal to zero:

$$x-2=0$$
$$x=2 \tag{3.5.7}$$

Step 3. Graph the values found in Step 2 on a number line, and split into intervals.
Label -5 and 2 on a blank number line:



Step 4. Take a test point within each interval and determine the sign of the result.
We take test values on each side of -5 and 2. Let's choose fairly easy numbers such as -6, 0, and 3. We plug these numbers into $\frac{x+5}{x-2}$ and determine whether the value is positive or negative:



x	$\frac{x+5}{x-2}$	Sign
-6	$\frac{1}{8}$	$+$
0	$-\frac{5}{2}$	$-$
3	8	$+$

Step 5. Determine the solution, where the solution is the interval that makes the inequality true.
Since $\frac{x+5}{x-2} \leq 0$ (from **Step 1.**), then we are looking for where the test values are negative or equal to zero. Looking at the number line above, we see these are the values between -5 and 2. Thus, the solution is $[-5, 2)$.

We used a bracket on -5 since the original inequality was $\leq$ and a parenthesis on 2 since 2 was an excluded value.

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3.5P: Practice

Practice Makes Progress

For each of the rational functions given,

a. Solve the equation $f(x) = k$

b. Check your solution by graphing $y = f(x)$ and $y = k$ using a graphing utility such as Desmos.

Exercise 1

$f(x) = \dfrac{x-1}{x+2}$; $k = 3$

Exercise 2

$f(x) = \dfrac{2x+3}{x-1}$; $k = -3$

Use a strictly algebraic technique to solve the equation given. Check your answers with a graphing utility such as Desmos.

Exercise 3

$\dfrac{16x-9}{2x-1} = 8$

Exercise 4

$\dfrac{8x+2}{x-11} = 11$

Use a strictly algebraic technique to solve the given equation.

Exercise 5

$\dfrac{x}{7} + \dfrac{8}{9} = -\dfrac{8}{7}$

Exercise 6

$\dfrac{7}{x} = 4 - \dfrac{3}{x^2}$

Exercise 7

$\dfrac{2}{x+9} + \dfrac{2}{x+6} = -1$

Exercise 8

$\dfrac{3}{x+3} + \dfrac{6}{x+2} = -2$

Exercise 9

$\dfrac{x}{x+1} + \dfrac{8}{x^2-2x-3} = \dfrac{2}{x-3}$

Exercise 10

$$\frac{x}{x-2}+\frac{3}{x+2}=\frac{8}{4-x^2}$$

Exercise 11

$$\frac{3x}{x+2}-\frac{7}{x}=-\frac{1}{2x+4}$$

Exercise 12

$$\frac{2x}{x-4}+\frac{5}{2-x}=\frac{x+8}{x^2-6x+8}$$

Exercise 13

$$\frac{2x}{x+5}-\frac{2}{6-x}=\frac{x-2}{x^2-x-30}$$

Exercise 14

$$\frac{4x}{x+6}-\frac{5}{7-x}=\frac{x-5}{x^2-x-42}$$

Use the techniques shown in the textbook to solve the inequalities.

Exercise 15

$$\frac{x-3}{x+1}<0$$

Exercise 16

$$\frac{x-3}{x+1}<1$$

Exercise 17

$$\frac{x+6}{x-4}\geq 0$$

Self-Check

In this section, you practiced:

a. Solving equations involving rational terms with the same denominator
b. Solving equations involving rational terms with different denominators using the LCD
c. Solving rational equations graphically
d. Checking solutions to rational equations and identifying extraneous solutions
e. Solving rational inequalities with linear numerator and denominator

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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3.6: Rational Applications

Learning Objectives

- Explain basic relations such as Distance = Rate x Time or Revenue = Quantity x Price
- Use these basic relations to model and solve real-world scenarios using rational equations

In this section, we will investigate the use of rational functions in several applications.

Work Problems

A nice application of rational functions involves the amount of work a person (or team of persons) can do in a certain amount of time. We can handle these applications involving work in a manner similar to the method we used to solve distance, speed, and time problems. Here is the guiding principle.

Note

The amount of work done is equal to the product of the rate at which work is being done and the amount of time required to do the work. That is,

$$\text{Work} = \text{Rate} \times \text{Time}.$$

For example, suppose that Emilia can mow lawns at a rate of 3 lawns per hour. After 6 hours,

$$\text{Work} = \frac{3 \text{ lawns}}{1 \text{ hr}} \times 6 \text{ hr} = 18 \text{ lawns}.$$

A second important concept is the fact that rates add. For example, if Emilia can mow lawns at a rate of 3 lawns per hour and Michele can mow the same lawns at a rate of 2 lawns per hour, then together they can mow the lawns at a combined rate of 5 lawns per hour.

Let's look at an example.

Example 1

Bill can finish a report in 2 hours. Maria can finish the same report in 4 hours. How long will it take them to finish the report if they work together?

Solution

A common misconception is that the times add in this case. That is, it takes Bill 2 hours to complete the report and it takes Maria 4 hours to complete the same report, so if Bill and Maria work together it will take 6 hours to complete the report. A little thought reveals that this result is nonsense. Clearly, if they work together, it will take them less time than it takes Bill to complete the report alone; that is, the combined time will surely be less than 2 hours.

However, as we saw above, the rates at which they are working will add. To take advantage of this fact, we set up what we know in a Work, Rate, and Time table.

• It takes Bill 2 hours to complete 1 report. This is reflected in the entries in the first row of the table.

• It takes Maria 4 hours to complete 1 report. This is reflected in the entries in the second row of the table.

• Let t represent the time it takes them to complete 1 report if they work together. This is reflected in the entries in the last row of the table.

	w (reports)	r (reports/h)	t (h)
Bill	1	?	2
Maria	1	?	4
Together	1	?	t

We can calculate the rate at which Bill is working by solving the equation Work = Rate × Time for the Rate, then substitute Bill's data from row one of the table.

$$\text{Rate} = \frac{\text{Work}}{\text{Time}} = \frac{1 \text{ report}}{2 \text{ hr}}$$

Thus, Bill is working at a rate of 1/2 report per hour. Note how we've entered this result in the first row of the table. Similarly, Maria is working at a rate of 1/4 report per hour, which we've also entered in the table.

We've let t represent the time it takes them to write 1 report if they are working together, so the following calculation gives us the combined rate.

$$\text{Rate} = \frac{\text{Work}}{\text{Time}} = \frac{1 \text{ report}}{t \text{ hr}}$$

That is, together they work at a rate of $\frac{1}{t}$ reports per hour. This result is also recorded in the table.

	w (reports)	r (reports/h)	t (h)
Bill	1	1/2	2
Maria	1	1/4	4
Together	1	1/t	t

In our discussion above, we pointed out the fact that rates add. Thus, the equation we seek lies in the Rate column of the table. Bill is working at a rate of 1/2 report per hour and Maria is working at a rate of 1/4 report per hour. Therefore, their combined rate is 1/2 + 1/4 reports per hour. However, the last row of the table indicates that the combined rate is also 1/t reports per hour. Thus,

$$\begin{aligned} \frac{1}{2} + \frac{1}{4} &= \frac{1}{t} \\ (4t)\left[\frac{1}{2} + \frac{1}{4}\right] &= \left[\frac{1}{t}\right](4t) \\ 2t + t &= 4 \\ 3t &= 4 \\ t &= 4/3 \end{aligned}$$

Thus, it will take 4/3 of an hour to complete 1 report if Bill and Maria work together.

It is very important that we check this result.

• We know that Bill does 1/2 reports per hour. In 4/3 of an hour, Bill will complete

$$\text{Work} = \frac{1}{2}\frac{\text{reports}}{\text{h}} \times \frac{4}{3}\text{h} = \frac{2}{3} \text{ reports.}$$

That is, Bill will complete 2/3 of a report.

• We know that Maria does 1/4 reports per hour. In 4/3 of an hour, Maria will complete

$$\text{Work} = \frac{1}{4}\frac{\text{reports}}{\text{h}} \times \frac{4}{3}\text{h} = \frac{1}{3}\text{reports}$$

That is, Maria will complete 1/3 of a report.

Clearly, working together, Bill and Maria will complete 2/3 + 1/3 reports, that is, one full report.

A bit of advice about work, rate, and time tables.

Work, Rate, and Time Tables

Because work, rate, and time are related by the equation

$$\text{Work} = \text{Rate} \times \text{Time}$$

whenever you have two boxes in a row completed, the third box in that row can be calculated by means of the relation Work = Rate × Time.

Let's look at another example.

Example 2

It takes Liya 7 more hours to paint a kitchen than it takes Hank to complete the same job. Together, they can complete the same job in 12 hours. How long does it take Hank to complete the job if he works alone?

Solution

Let H represent the time it takes Hank to complete the job of painting the kitchen when he works alone. Because it takes Liya 7 more hours than it takes Hank, let $H + 7$ represent the time it takes Liya to paint the kitchen when she works alone. This leads to the entries in the following table.

	w (kitchens)	r (kitchens/h)	t (h)
Hank	1	?	H
Liya	1	?	H+7
Together	1	?	12

We can calculate the rate at which Hank is working alone by solving the equation Work = Rate × Time for the Rate, then substituting Hank's data from row one of the table.

$$\text{Rate} = \frac{\text{Work}}{\text{Time}} = \frac{1 \text{ kitchen}}{H \text{ hour}}$$

Thus, Hank is working at a rate of $\frac{1}{H}$ kitchens per hour. Similarly, Liya is working at a rate of $\frac{1}{H+7}$ kitchens per hour. Because it takes them 12 hours to complete the task when working together, their combined rate is 1/12 kitchens per hour. Each of these rates is entered in the table.

	w (kitchens)	r (kitchens/h)	t (h)
Hank	1	$\frac{1}{H}$	H
Liya	1	$\frac{1}{H+7}$	H+7
Together	1	1/12	12

Because the rates add, we can write

$$\begin{aligned}
\frac{1}{H}+\frac{1}{H+7}&=\frac{1}{12}\\
12H(H+7)\left(\frac{1}{H}+\frac{1}{H+7}\right)&=\left(\frac{1}{12}\right)12H(H+7)\\
12(H+7)+12H&=H(H+7)\\
12H+84+12H&=H^2+7H\\
24H+84&=H^2+7H\\
0&=H^2+7H-24H-84\\
0&=H^2-17H-84
\end{aligned}\tag{3.6.1}$$

Note that ac = (1)(−84) = −84. The integer pair {4, −21} has product −84 and sums to −17. Hence,

$$0=(H+4)(H-21)$$

Using the zero product property, either

$$H+4=0 \quad \text{or} \quad H-21=0$$

leading to the solutions

$$H=-4 \quad \text{or} \quad H=21$$

We eliminate the solution H = −4 from consideration (it doesn't take Hank negative time to paint the kitchen), so we conclude that it takes Hank 21 hours to paint the kitchen.

Does our solution make sense?

- It takes Hank 21 hours to complete the kitchen, so he is finishing 1/21 of the kitchen per hour.
- It takes Liya 7 hours longer than Hank to complete the kitchen, namely 28 hours, so she is finishing 1/28 of the kitchen per hour.

Together, they are working at a combined rate of

$$\frac{1}{21}+\frac{1}{28}=\frac{4}{84}+\frac{3}{84}=\frac{7}{84}=\frac{1}{12}$$

or 1/12 of a kitchen per hour. This agrees with the combined rate in the table.

Distance, Speed, and Time Problems

When we developed the Equations of Motion in the chapter on quadratic functions, we showed that if an object moves with constant speed, then the distance traveled is given by the formula

$$d=rt$$

where d represents the distance traveled, r represents the speed, and t represents the time of travel.

For example, if a car travels down a highway at a constant speed of 50 miles per hour (50 mi/h) for 4 hours (4 h), then it will travel

$$\begin{aligned}
d&=rt\\
d&=50\frac{\text{mi}}{\text{h}}\times 4\text{h}\\
d&=200\text{mi}
\end{aligned}$$

Let's put this relation to use in some applications.

Example 3

A boat travels at a constant speed of 3 miles per hour in still water. In a river with an unknown current, it takes the boat twice as long to travel 60 miles upstream (against the current) as it does for the 60-mile return trip (with the current). What is the speed of the current in the river?

Solution

The speed of the boat in still water is 3 miles per hour. When the boat travels upstream, the current is against the direction the boat is traveling and works to reduce the actual speed of the boat. When the boat travels downstream, then the actual speed of the boat is its speed in still water increased by the speed of the current. If we let c represent the speed of the current in the river, then the boat's speed upstream (against the current) is $3-c$, while the boat's speed downstream (with the current) is $3+c$. Let's summarize what we know in a distance-speed-time table.

	d (mi)	r (mi/h)	t (h)
Upstream	60	$3-c$	?
Downstream	60	$3+c$	?

Note that each row of the table has two entries. The third entry in each row is time. Solve the equation $d=rt$ for t to obtain

$$t=\frac{d}{r}$$

The relation $t=d/r$ can be used to compute the time entry in each row of the table.

For example, in the first row, $d=60$ miles and $r=3-c$ miles per hour. Therefore, the time of travel is

$$t=\frac{d}{r}=\frac{60}{3-c}$$

Note how we've filled in this entry in the table. In a similar fashion, the time to travel downstream is calculated with

$$t=\frac{d}{r}=\frac{60}{3+c}$$

We've also added this entry to the time column in the table.

	d (mi)	r (mi/h)	t (h)
Upstream	60	$3-c$	$\frac{60}{3-c}$
Downstream	60	$3+c$	$\frac{60}{3+c}$

To set up an equation, we need to use the fact that the time to travel upstream is twice the time to travel downstream. This leads to the result

$$\begin{aligned}\frac{60}{3-c}&=2\left(\frac{60}{3+c}\right)\\ \frac{60}{3-c}&=\frac{120}{3+c}\\ (3-c)(3+c)\left[\frac{60}{3-c}\right]&=\left[\frac{120}{3+c}\right](3-c)(3+c)\\ 60(3+c)&=120(3-c)\\ 180+60c&=360-120c\\ 60c+120c&=360-180\\ 180c&=180\\ c&=1\end{aligned}$$

Hence, the speed of the current is 1 mile per hour.

It is important to check that the solution satisfies the constraints of the problem statement.

- If the speed of the boat in still water is 3 miles per hour and the speed of the current is 1 mile per hour, then the speed of the boat upstream (against the current) will be 2 miles per hour. It will take 30 hours to travel 60 miles at this rate.
- The speed of the boat as it goes downstream (with the current) will be 4 miles per hour. It will take 15 hours to travel 60 miles at this rate.

Note that the time to travel upstream (30 hours) is twice the time to travel downstream (15 hours), so our solution is correct.

Here is a helpful piece of advice regarding distance, speed, and time tables.

Distance, Speed, and Time Tables

Because the equation relates distance, speed, and time $d = rt$, whenever you have two boxes in a row of the table completed, the third box in that row can be calculated using the formula $d = rt$.

Let's look at another example.

Example 4

Greg went to a conference in a city 120 miles away. On the way back, due to road construction, he had to drive 10 mph slower, which resulted in the return trip taking 2 hours longer. How fast did he drive on the way to the conference?

Solution

First, we can make a table to organize the given information and then create an equation. Let r represent the rate in which he drove to the conference.

	rate	time	distance
To the conference	r	t	120
From the conference	$r-10$	$t+2$	120

Now we can set up each equation.

$$t_{to} = \frac{120}{r} \quad t_{from} = \frac{120}{r-10} - 2$$

Since we solved for t in each equation, we can set the t's equal to each other and solve for r:

$$\begin{gathered} t_{to} = t_{from} \\ \frac{120}{r} = \frac{120}{r-10} - 2 \\ r(r-10)\left[\frac{120}{r}\right] = \left[\frac{120}{r-10} - 2\right] r(r-10) \\ r(r-10)\cdot\frac{120}{r} = r(r-10)\cdot\frac{120}{r-10} - r(r-10)\cdot 2 \\ 120(r-10) = 120r - 2r(r-10) \\ 120r - 1200 = 120r - 2r^2 + 20r \\ 120r - 1200 = 140r - 2r^2 \\ 2r^2 - 20r - 1200 = 0 \\ r^2 - 10r - 600 = 0 \\ (r+20)(r-30) = 0 \\ r+20 = 0 \quad \text{or} \quad r-30 = 0 \\ r = -20 \quad \text{or} \quad r = 30 \end{gathered}$$

Since Greg could not drive at a negative speed, he must have driven at 30 miles per hour on the way to the conference.

Let's look at another class of problems.

Revenue Applications

Revenue problems are problems where a person buys a certain number of items for a certain price per item. If we multiply the number of items by the price per item, we will get the total value.

$$\textbf{Amount} \cdot \textbf{Value} = \textbf{Total}$$

We can rewrite it to model with rational equations as

$$\text{Value} = \frac{\text{Total}}{\text{Amount}}$$

Example 6

A man buys several fish for $56. He keeps three fish and decides to sell the rest for $5 more than he bought them. His total profit was $4. How many fish did he buy to begin with?

Solution

First, we can make a table to organize the given information and then create an equation. Let n represent the number of fish and p be the price of each fish.

	Amount	Price	Total Value
Buy	n	p	$56
Sell	$n-3$	$p+5$	$56+$4

Let's discuss the table for a moment. When the man purchased the fish, the total value of the fish purchased was $56. Since the price of each fish and the quantity purchased are unknown, we leave them as p and n, respectively. The man wants to sell the fish, but keeps three; hence, the amount left to sell is $n-3$. Since he wants to make $5 per fish, then we take the price the man bought the fish for and add $5, $p+5$. It is given that his total profit was $4, so his total value from selling the fish was the original value, $56, plus the $4 profit; hence, a total of $60.

Finally, let's set up the equations and solve:

$$p_{Buy} = \frac{56}{n} \quad p_{Sell} = \frac{60}{n-3} - 5$$

Since we solved for p in each equation, we can set the p's equal to each other:

$$\begin{gathered} p_{Buy} = p_{Sell} \\ \frac{56}{n} = \frac{60}{n-3} - 5 \\ n(n-3)\cdot\frac{56}{n} = n(n-3)\cdot\frac{60}{n-3} - n(n-3)\cdot 5 \\ 56(n-3) = 60n - 5n(n-3) \\ 56n - 168 = 60n - 5n^2 + 15n \\ 56n - 168 = 75n - 5n^2 \\ 5n^2 - 19n - 168 = 0 \\ (5n+21)(n-8) = 0 \\ 5n+21 = 0 \quad \text{or} \quad n-8 = 0 \\ n = -\frac{21}{5} \quad \text{or} \quad n = 8 \end{gathered}$$

Since the quantity of fish is always positive, we omit the solution $n = -\frac{21}{5}$. Thus, the man purchased 8 fish.

Example 7

A group of students bought a couch for their dorm that cost $96. However, 2 students failed to pay their share, so each student had to pay $4 more. How many students were in the original group?

Solution

First, we can make a table to organize the given information and then create an equation. Let n represent the number of students and p be the price of each share.

	Amount	Price	Total Value
Original deal	n	p	$96
Actual deal	$n-2$	$p+4$	$96

Let's discuss the table for a moment. The original deal was every student in the original group with n number of students were going to split the total value of the couch valued at $96. Since the price of each share and the number of students is unknown, we leave it as p and n, respectively. When it came to actually paying for the couch, 2 students didn't pay their share; hence, the number of students left to pay is $n-2$. Since this increases each share from the rest of the group, then we take the original share and add $4, $p+4$.

Finally, let's set up the equations and solve:

$$p_o = \frac{96}{n} \quad p_A + 4 = \frac{96}{(n-2)}$$
$$p_o = \frac{96}{n} \quad p_A = \frac{96}{(n-2)} - 4$$

Since we solved for p in each equation, we can set the p's equal to each other:

$$\frac{96}{n} = \frac{96}{n-2} - 4$$
$$n(n-2)\cdot\frac{96}{n} = n(n-2)\cdot\frac{96}{n-2} - n(n-2)\cdot 4$$
$$96(n-2) = 96n - 4n(n-2)$$
$$96n - 192 = 96n - 4n^2 + 8n$$
$$96n - 192 = 104n - 4n^2$$
$$4n^2 - 8n - 192 = 0$$
$$n^2 - 2n - 48 = 0$$
$$(n+6)(n-8) = 0$$
$$n+6=0 \quad \text{or} \quad n-8=0$$
$$n=-6 \quad \text{or} \quad n=8$$

Since the quantity of students is always positive, we omit the solution $n=-6$ and there were 8 students in the original group.

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3.6P: Practice

Practice Makes Progress

? Exercise 1

It takes Fatima 15 hours longer to complete an inventory report than it takes Dae-jung. If they work together, it takes them 10 hours. How many hours would it take Dae-jung if he worked alone?

? Exercise 2

Dan can paint a room in 4 hours. It takes Lina 7 hours to paint the same room. How many hours will it take if they work together?

? Exercise 3

Amara can paddle her kayak at a speed of 6 mph in still water. She paddles 5 miles upstream against the current and then returns to the starting location. The total trip time is 5 hours. What is the speed (in mph) of the current? Round your answer to the nearest hundredth.

? Exercise 4

Hiroshi is canoeing in a river with a 5 mph current. Suppose that they can canoe 4 miles upstream in the same amount of time as it takes them to canoe 8 miles downstream. Find the speed (mph) of Hiroshi's canoe in still water.

? Exercise 5

A merchant named Luis bought several barrels of apples for $\$120$. He kept two barrels and sold the remainder at a profit of $\$2$ per barrel, making a total profit of $\$34$. How many barrels did he originally buy?

Self-Check

In this section, you practiced:

a. Setting up and solving rational equations to model Work-rate problems given $\text{Work} = \text{Rate} \times \text{Time}$
b. Setting up and solving rational equations to model Uniform Motion problems given $\text{Distance} = \text{Rate} \times \text{Time}$
c. Setting up and solving rational equations to model Revenue problems given $\text{Total} = \text{Amount} \times \text{Value}$

From the list of skills above:

1. I can complete these independently without notes:
2. I can complete these, but need to reference an example or other resource to help:
3. I don't understand these yet:

It's okay to not be fully confident with every skill yet! But it's essential to continue making progress. Identify at least one step you will take to improve your understanding of the *topics you listed in parts 2 or 3* above:

- Visit the math lab
- Contact my instructor
- Re-work practice problems
- Ask a classmate or friend for help

Which step(s) from above did you complete after last week's textbook practice?

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CHAPTER OVERVIEW

Unit 4: Practice Problem Answers

This unit provides the answers to all the practice problems from the textbook. It's here to help you confirm whether your solutions are correct and to support your learning by giving you immediate feedback. But remember: knowing the final answer is only part of doing math. You're still responsible for showing clear, step-by-step work that demonstrates your thinking and understanding.

With great power comes great responsibility—so use these answers wisely. They are meant to *check* your work, not *replace* it. If you find a mistake, go back, try again, and figure out where things went off track. That process is how real learning happens.

4.1: Unit 1 Practice Answers

4.2: Unit 2 Practice Answers

4.3: Unit 3 Practice Answers

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4.1: Unit 1 Practice Answers

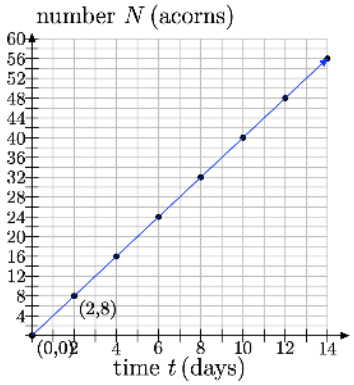
Unit 1 Answers

Section 1.1P: Solving Equations - Essential Skills

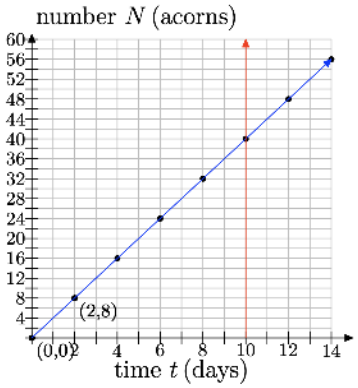
1. $x = -11$
2. $x = -5$
3. $x = -\frac{4}{21}$
4. $x = \frac{85}{6}$
5. $x = -\frac{5}{4}$
6. $x = \frac{1}{5}$
7. $x = \frac{3}{2}$
8. $x = -\frac{6}{7}$

Section 1.2P: Linear Models and Rates of Change

1. a–e)



f) After 10 days, Earl will have collected 40 acorns.



g) $\frac{8}{2} = 4$ acorns/day

h)

t	N
0	0
1	4
2	8
3	12
4	16
6	24
8	32
10	40
12	48
14	56

i) $N = 4t$

j) When $t = 10$, $N = 4(10) = 40$. Yes, this matches the graph from part (f).

2. a. The slope is negative. It is the same everywhere along the line.

   b. Slope = -1

   c. $y = -x + 2$

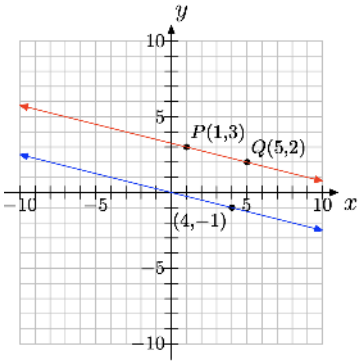
3. $y = 3x - 2$

4. a. slope = 1: (b)

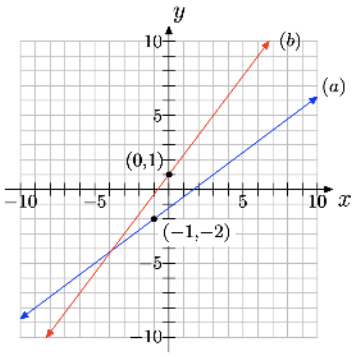
   b. slope = $\frac{2}{3}$: (c)

   c. slope = -2: (a)

5. Graphs of the two parallel lines are below. The equation of the first line is $y = -\frac{1}{4}x$ and the equation of the parallel line is $y = -\frac{1}{4}x + \frac{13}{4}$



6. a-b)

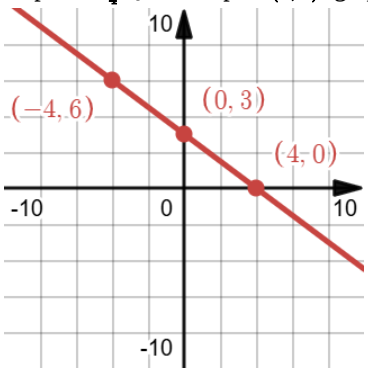


c) The lines are neither parallel nor perpendicular.

7. a. $\frac{8}{100} = \frac{2}{25}$ is the slope.
   b. $\frac{16}{180} = \frac{4}{45} \approx 8.9\%$ is the grade.
   c. A 0% grade represents a completely flat road (slope of 0).

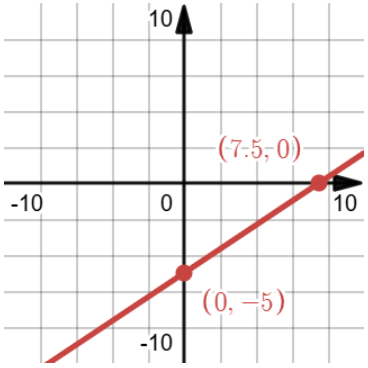
Section 1.3P: Linear Equations and Systems

1. slope = $-\frac{3}{4}$, y-intercept = $(0,3)$; graph passes through $(4,0)$



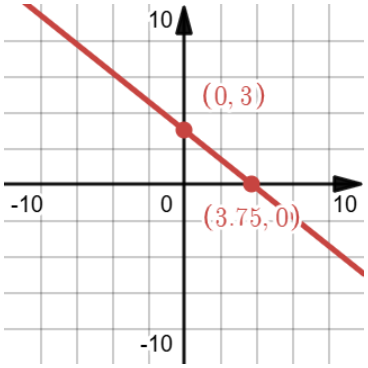
2. slope = $\frac{2}{3}$, y-intercept = $(0,-2)$; equation: $y = \frac{2}{3}x - 2$
3. a. t is the independent variable; S is the dependent variable.
   b. Intercept: $(0,1000)$; slope: 25
   c. $S = 25t + 1000$
   d. $S = 25(12) + 1000 = 1300$
   e. $2000 = 25t + 1000 \Rightarrow t = 40$
   f. Line through $(0,1000)$ with slope 25
   g. Line through $(0,1200)$ with same slope
   h. Same slope; Anne-Marie's line starts higher
   i. $S = 25t + 1200$
   j. Same slope = same rate; graphs are parallel
   k. Sooner (because of higher starting amount)
   l. $2000 = 25t + 1200 \Rightarrow t = 32$
4. a. (7.5,0) and (0,-5)
   b. Slope = $\frac{2}{3}$

c.

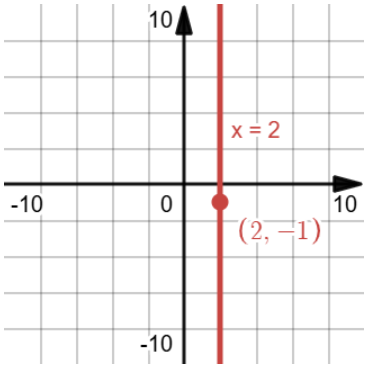


5. a. \(y=-\dfrac{4}{5}x+3

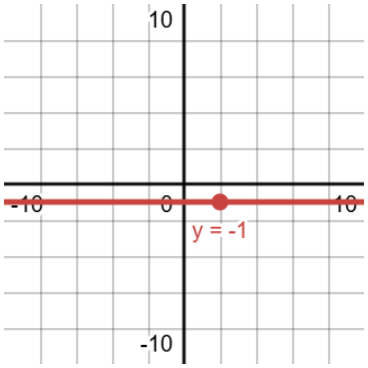
b.



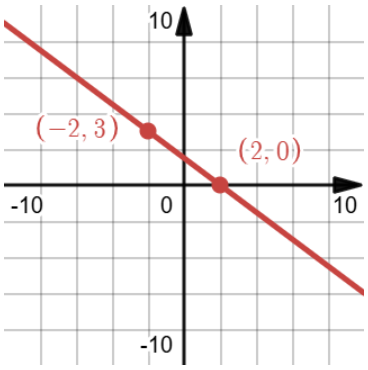
6.



7.



8.



Point-slope: $y-3=-\frac{3}{4}(x+2)$

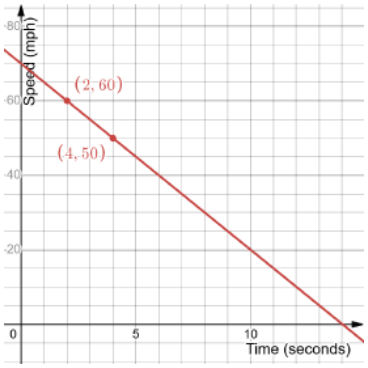
Slope-intercept: $y=-\frac{3}{4}x+\frac{3}{2}$

9. Point-slope: $y-9=-\frac{2}{3}(x-1)$

Slope-intercept: $y=-\frac{2}{3}x+\frac{29}{3}$

10. $y=-\frac{5}{3}x-\frac{2}{3}$

11. a.



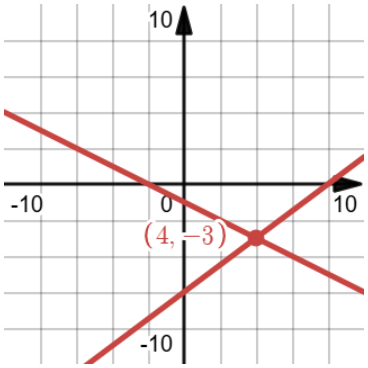
b. Slope: $-5\ \frac{\text{km/h}}{\text{s}}$, meaning the speed of the car decreases by 5 km/h per second.

c. Equation: $v(t)=-5t+70$

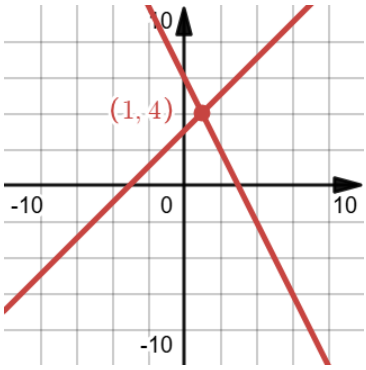
d. Assuming the constant rate of change continues, the car stops after $t=14$ seconds

e. The v-intercept of (0,70) tells us that the car was traveling at 70 km/h before applying the brakes.

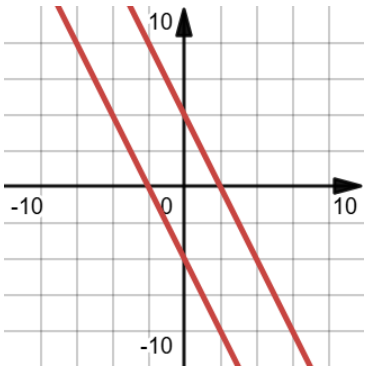
12. $(4,-3)$



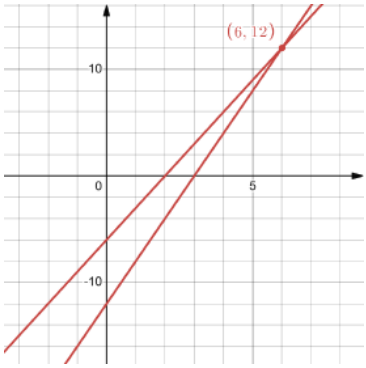
13. $(1, 4)$



14. No solution. Lines are parallel.



15. $(6, 12)$



Section 1.4P: Linear Inequalities

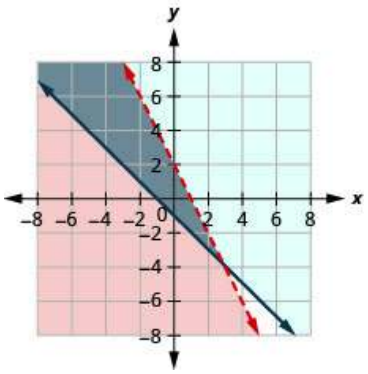
1. 

2. Interval: $(-\infty, 9)$
   Set-Builder: $\{x | x < 9\}$
3. Interval: $(-\infty, 2)$
   Set-Builder: $\{x | x < 2\}$
4. $[1, \infty)$
   $\{y | y \geq 1\}$
5. $\left[\frac{5}{2}, \infty\right)$
   $\left\{n | n \geq \frac{5}{2}\right\}$
6. $\left(-\infty, -\frac{1}{3}\right)$
   $\left\{x | x < -\frac{1}{3}\right\}$

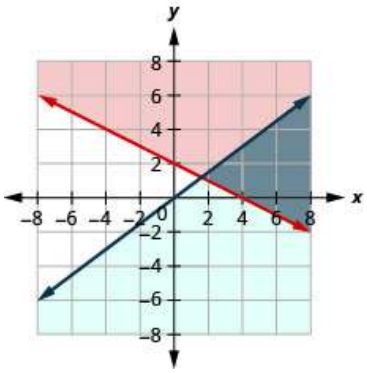
7. $\left(\frac{3}{4}, \infty\right)$
   $\left\{x | x > \frac{3}{4}\right\}$
8. $\left(-\infty, \frac{3}{10}\right)$
   $\left\{x | x < \frac{3}{10}\right\}$
9. $\left[\frac{5}{7}, \infty\right)$
   $\left\{x | x \geq \frac{5}{7}\right\}$
10. $\left(-\frac{7}{8}, \infty\right)$
    $\left\{y | y > -\frac{7}{8}\right\}$
11. $(-\infty, \infty)$
    $\{n | n \in \mathbb{R}\}$
12. No Solution
    $\{\}$
13. Let T be the dollar amount of Kimuyen's total sales. Then $3475 + 0.04T \geq 4150$
    Kimuyen must sell at least $\$16,875$
14. If Nataly makes S total sales per year, then $66500 + 0.15S > 83000$
    Nataly's total sales would need to be greater than $\$110,000$
15. Let n be the number of people at the shower, then $350 + 32.5n \leq 1500$
    Kellen can invite at most 35 people

Section 1.5P: Two Variable Linear Inequalities and Systems

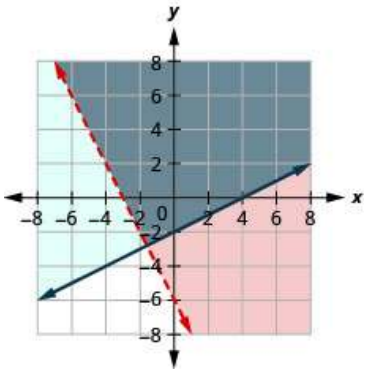
1. a. false
   b. true
2. a. true
   b. false
3. See graph here: https://www.desmos.com/calculator/4gxnu8lwva
4. See graph here: https://www.desmos.com/calculator/4gxnu8lwva
5. See graph here: https://www.desmos.com/calculator/4gxnu8lwva
6. a. false
   b. true
7. a. false
   b. true
8. The solution is the grey region:



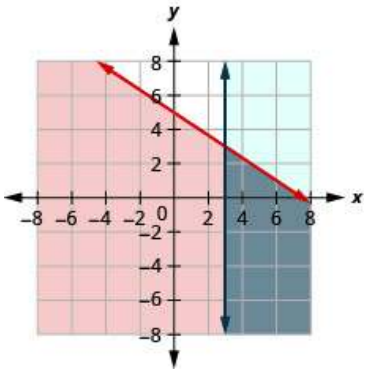
9. The solution is the grey region:



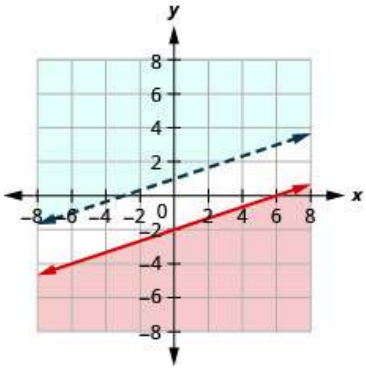
10. The solution is the grey region:



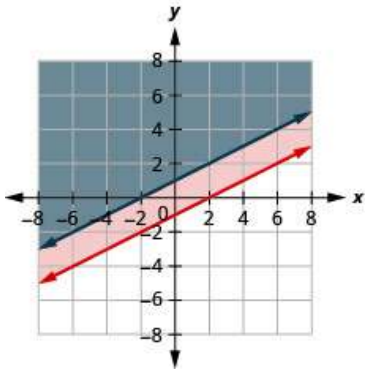
11. The solution is the grey region:



12. No solution:



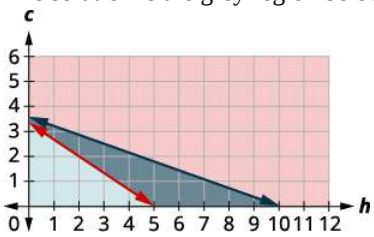
13. The solution is the grey region:



14. a. Letting h be the number of hamburgers and c be the number of cookies, the system of inequalities is

$$\begin{cases} 240h + 160c \geq 800 \\ 1.40h + 0.50c \leq 5 \end{cases}$$

b. The solution is the grey region below:

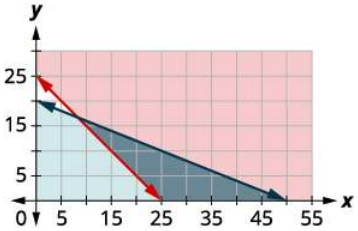


c. and d. Test whether each point is in the solution (the grey region).

15. a. Letting x equal the number of small photos and y equal the number of large photos, the system of inequalities is

$$\begin{cases} x + y \geq 25 \\ 4x + 10y \leq 200 \end{cases}$$

b. The solution to the system is the grey region below:



c. No, that point is not in the grey region.

d. Yes, that point is in the grey region.

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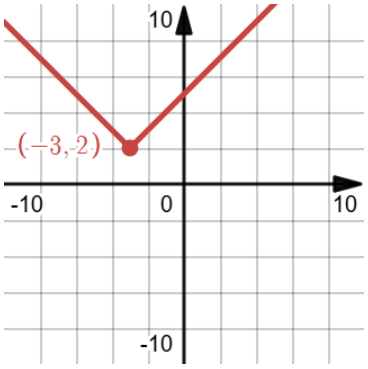
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4.2: Unit 2 Practice Answers

Unit 2 Answers

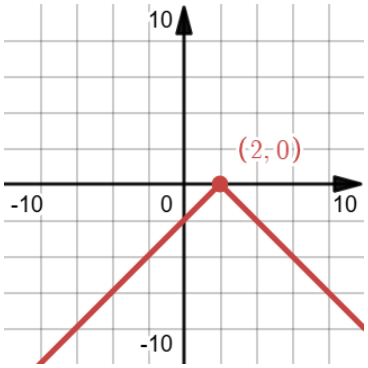
Section 2.1P: Absolute Value

1. a. The function is shifted 3 units to the left, and 2 units up.



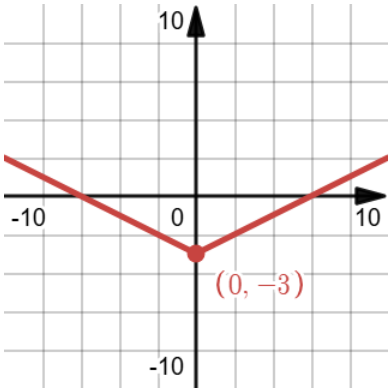
b. The domain is $(-\infty, \infty)$ and the range is $[2, \infty)$.

2. a. The function is shifted 2 units to the right, and reflected over the x-axis.



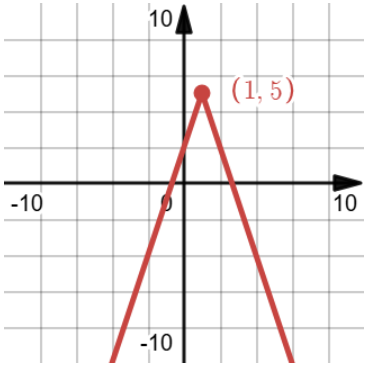
b. The domain is $(-\infty, \infty)$ and the range is $(-\infty, 0]$.

3. a. The function is compressed vertically by $\frac{1}{2}$, then shifted 3 units down.



b. The domain is $(-\infty, \infty)$ and the range is $[-3, \infty)$.

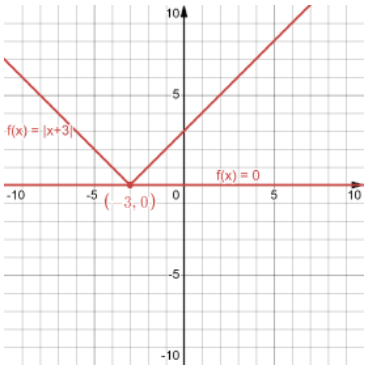
4. a. The function is reflected over the x-axis, stretched vertically by 3, shifted 1 unit to the right, and 5 units up.



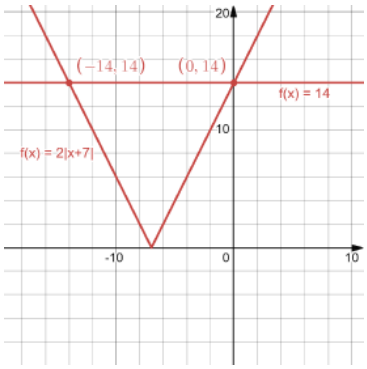
b. The domain is $(-\infty, \infty)$ and the range is $(-\infty, 5]$.

Section 2.2P: Absolute Value Equations

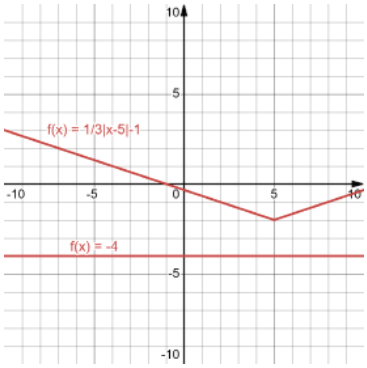
1. $x = -3$



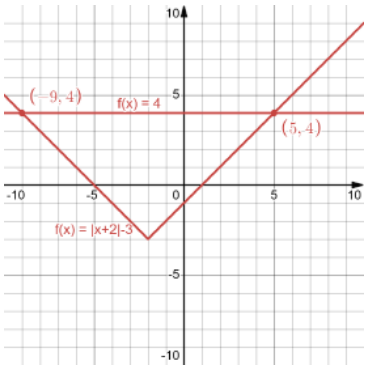
2. $x = -14$ or $x = 0$



3. No solution

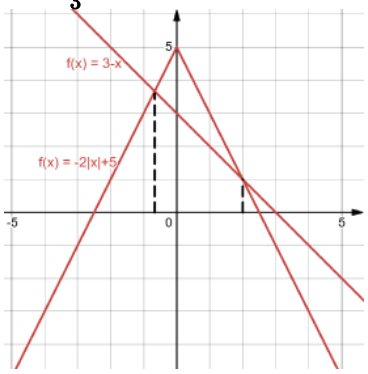


4. $x = -9$ or $x = 5$

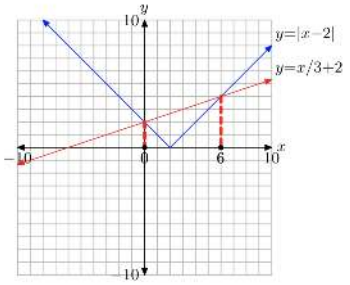


5. $x = \frac{1}{2}$ or $x = 1$

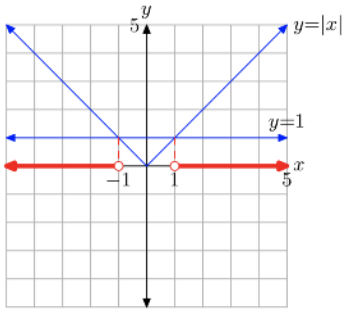
6. $x = -\frac{2}{3}$ or $x = 2$



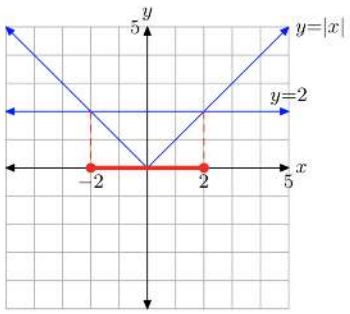
7. $x = 0$ or $x = 6$



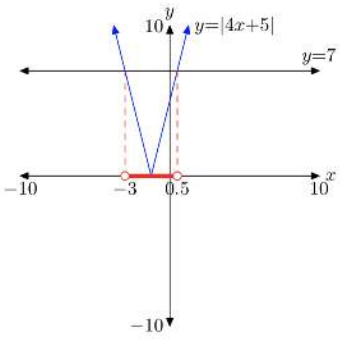
8. Solution: $(-\infty, -1) \cup (1, \infty)$



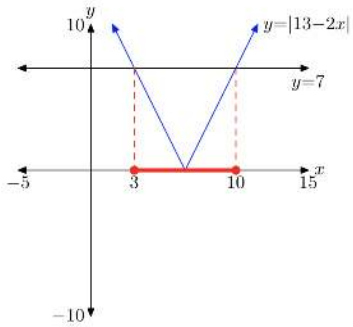
9. Solution: $[-2, 2]$



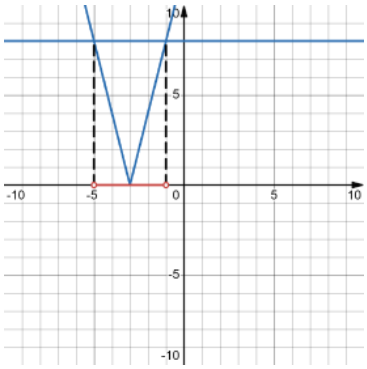
10. Solution: $(-3, 0.5)$



11. Solution: $[3, 10]$

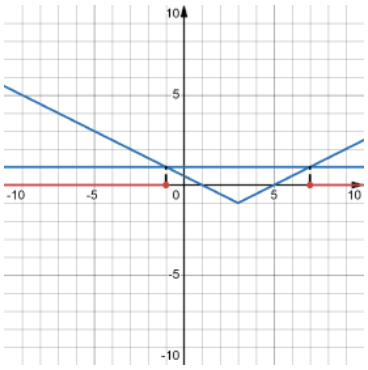


12. $(-5, -1)$

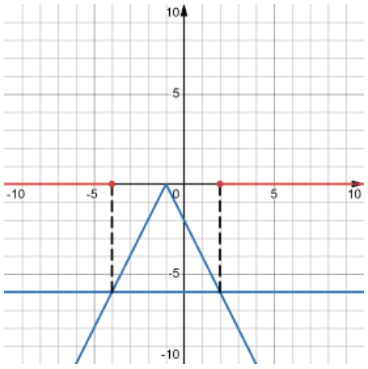


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13. $(-\infty, -1] \cup [7, \infty)$



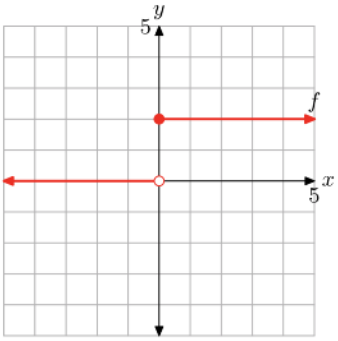
14. $(-\infty, -4] \cup [2, \infty)$



15. $\left(-\frac{2}{3}, \frac{14}{3}\right)$

Section 2.3P: Piecewise-Defined Functions

1. a.

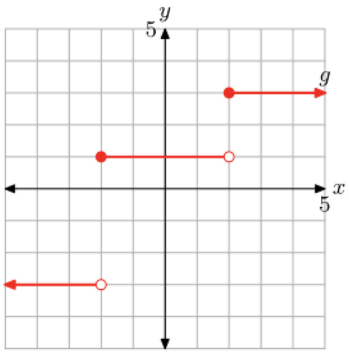


b. $f(-2) = 0$, $f(0) = 2$, $f(3) = 2$

c. Domain: $(-\infty, \infty)$

Range: $0, 2$

2. a.



b. $g(-3) = -3$, $g(-2) = 1$, $g(5) = 3$

c. Domain: ((-\infty,\infty)\)

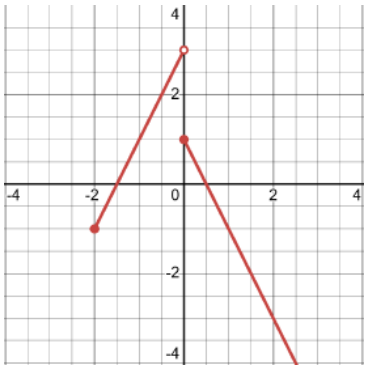
Range: {−3, 1, 3}

3. $g(x) = \begin{cases} 2, & \text{if } x < 0 \\ -2, & \text{if } 0 \leq x < 2 \\ 2, & \text{if } x > 2 \end{cases}$

Domain: all real numbers

Range: {−2, 2}

4. a.



b. $g(0) = 1$ and $g(3) = -2$

Domain: $\{x | x \geq -2\}$ or $[-2, \infty)$

Range: $\{y | y < 3\}$ or $(-\infty, 3)$

5. $f(x) = \begin{cases} x + 1, & \text{if } -4 \leq x \leq 1 \\ 3, & \text{if } 1 < x \leq 3 \\ -2x + 9, & \text{if } x > 3 \end{cases}$

6. a.



b. Let c be the number of cookies purchased and T be the total cost of the order, in dollars.

$$T(c) = \begin{cases} 2c, & \text{if } 0 \le c \le 100 \\ 1.5c + 50, & \text{if } 100 < c \le 250 \\ 1.25c + 112.5, & \text{if } c > 250 \end{cases}$$

c. Answers will vary. Personally, this author prefers finding specific points first, then sketching the graph, and finally writing the function by using the graph as a reference.

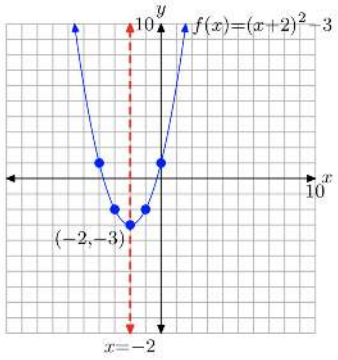
7. Let t be the time since Jules left on his ride, and D be his distance from home in miles.

$$D(t) = \begin{cases} 12t, & \text{if } 0 \le t < 1.5 \\ 18, & \text{if } 1.5 \le t \le 2 \\ -15t + 48, & \text{if } 2 < t \le 3.2 \end{cases}$$

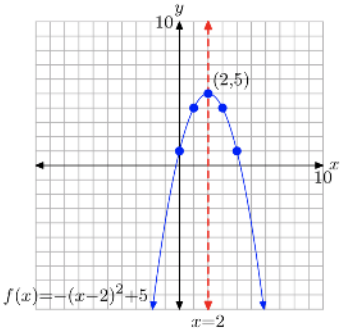
b. The domain of $D(t)$ is $[0, 3.2]$ hours, and the range of $D(t)$ is $[0, 18]$ miles.
These could also be written as $\{t | 0 \le t \le 3.2\}$ and $\{D | 0 \le D \le 18\}$

Section 2.4P: The Parabola

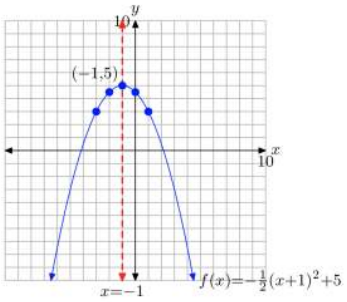
1. Domain = $(-\infty, \infty)$; Range = $[-3, \infty)$



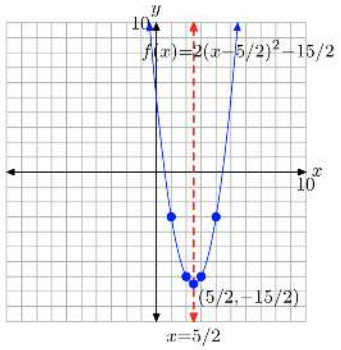
2. Domain = $(-\infty, \infty)$; Range = $(-\infty, 5]$



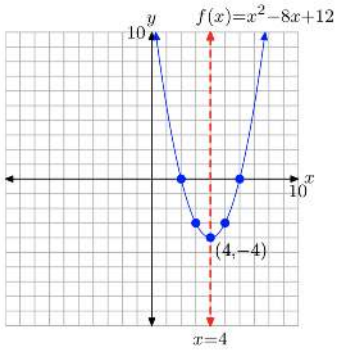
3. Domain = $(-\infty, \infty)$; Range = $(-\infty, 5]$



4. Domain = $(-\infty, \infty)$; Range = $[-\frac{15}{2}, \infty)$

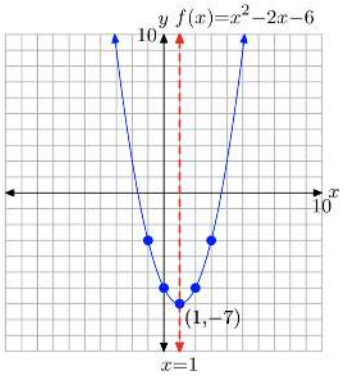


5. $f(x) = -2(x-3)^2 + 1$
6. $f(x) = 2(x+2)^2 + 1$
7. $f(x) = (x-4)^2 - 4$



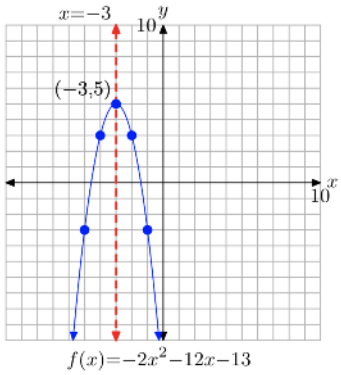
Domain = $\mathbb{R}$, Range = $[-4, \infty)$

8. $f(x) = (x-1)^2 - 7$



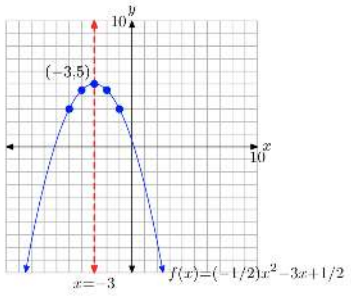
Domain = $\mathbb{R}$, Range = $[-7, \infty)$

9. $f(x) = -2(x+3)^2 + 5$



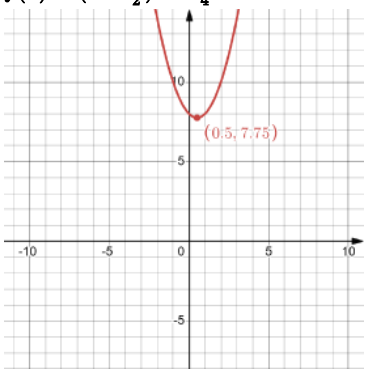
Domain = $\mathbb{R}$, Range = $(-\infty, 5]$

10. $f(x) = -\frac{1}{2}(x+3)^2 + 5$

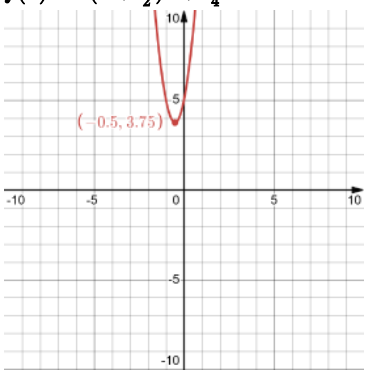


Domain = $\mathbb{R}$, Range = $(-\infty, 5]$

11. $f(x) = (x - \frac{1}{2})^2 + \frac{31}{4}$



12. $f(x) = 5(x + \frac{1}{2})^2 + \frac{15}{4}$

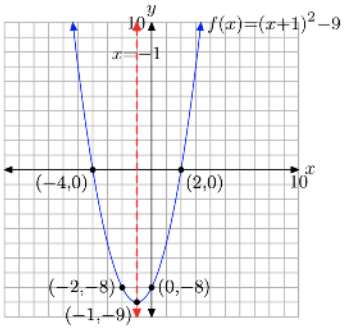


Section 2.5P: Optimization

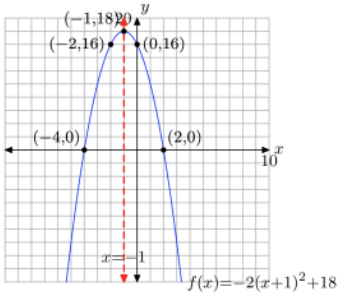
1. $\frac{9}{4}$
2. $(-\frac{1}{6}, -\frac{71}{12})$
3. $\frac{11}{4}$
4. $(-\frac{1}{2}, \frac{33}{4})$
5. The ball reaches its maximum height after a quarter of a second (t = 0.25 seconds).
6. The maximum height of the ball is 335.25 feet.
7. 10 feet by 10 feet
8. 361250 square meters
9. a. Use the fact that Revenue = (Unit Price) * (Demand). Then $R = -\frac{1}{3}x^2 + 40x$
   b. x = 60 objects
   c. p = 20 dollars
   d. R = $1200

Section 2.6P: Zeros of the Quadratic

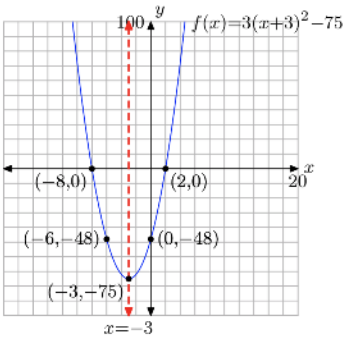
1. $(x+2)(x+7)$
2. $(x-5)(x+1)$
3. $(x-4)(x-3)$
4. $(x-3)(5x-4)$
5. $-(4x-5)(x-1)$
6. $(2x+7)(x-5)$
7. Zeros: $x = -3$, $x = 5$
8. Zeros: $x = -13$, $x = 3$
9. Zeros: $x = \frac{2}{3}$, $x = 6$
10. Zeros: $x = -\frac{3}{2}$, $x = \frac{5}{2}$
11. Domain = $(-\infty, \infty)$, Range = $[-9, \infty)$



12. Domain = $(-\infty, \infty)$, Range = $(-\infty, 18]$



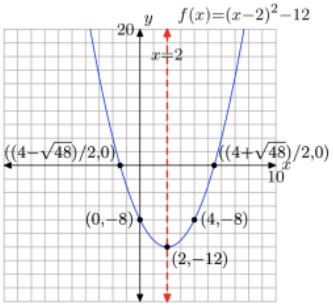
13. Domain = $(-\infty, \infty)$, Range = $[-75, \infty)$



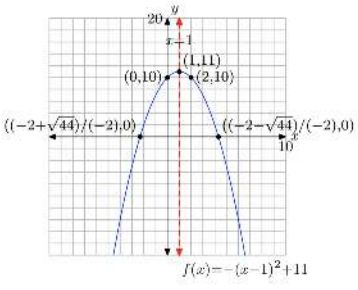
14. $(-\infty, 3) \cup (6, \infty)$
15. $(-\infty, -4] \cup \left[\frac{5}{4}, \infty\right)$
16. $\left[0, \frac{7}{4}\right]$
17. No solution

Section 2.7P: The Quadratic Formula

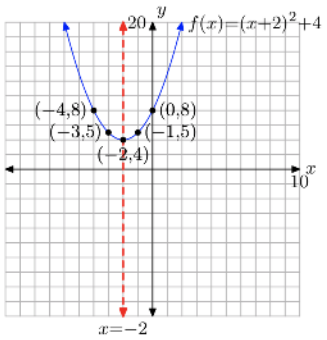
1. $x = \pm 6$
2. $x = \pm\sqrt{17} \approx \pm 4.12$
3. $x = 0$
4. No real solution
5. $x = -4$ or $x = 6$
6. $x = -2$
7. $x = 8 \pm \sqrt{15} \approx 4.13, 11.87$
8. Domain = $(-\infty, \infty)$, Range = $[-12, \infty)$



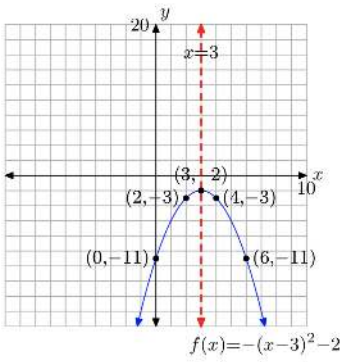
9. Domain = $(-\infty, \infty)$, Range = $(-\infty, 11]$



10. Domain = $(-\infty, \infty)$, Range = $[4, \infty)$



11. Domain = $(-\infty, \infty)$, Range = $(-\infty, -2]$



12. $\{k \mid k < \frac{9}{20}\}$
13. $\{k \mid k > \frac{1}{40}\}$
14. $x = -\frac{3-\sqrt{17}}{2}, -\frac{3+\sqrt{17}}{2}$
15. $x = -3$
16. No x-intercepts
17. $(\frac{9}{8}, 0), (-\frac{2}{7}, 0)$

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4.3: Unit 3 Practice Answers

Unit 3 Answers

Section 3.1P: Direct and Inverse Variation

1. 400
2. 1682
3. $\frac{209}{8}$ or 26.125
4. $\frac{56}{25}$ or 2.24
5. $\frac{16}{3}$
6. $\frac{250}{27}$
7. a. Let d be the distance the spring stretches, and w be the weight on the spring. Then $d = 8w$
   b. 40 inches
8. a. Let I be the intensity of light and d the distance from the light source. Then $I = \frac{320}{d^2}$
   b. 0.99 foot-candles
9. a. Let n be the number of people who sign up and P be the price per person. Then $P = \frac{3672}{n}$
   b. $\$104.91$

Section 3.2P: Graphs of Rational Functions

1. Vertical asymptote: $x = -1$
2. Vertical asymptote: $x = -2$
3. Vertical asymptote: $x = 4$
4. Vertical asymptote: $x = 0$
5. Horizontal asymptote: $y = 9$
6. Horizontal asymptote: $y = -1$
7. Horizontal asymptote: $y = 0$
8. D = {x: $x \neq 0$}, R = {y: $y \neq 0$}
9. D = {x: $x \neq 4$}, R = {y: $y \neq 0$}
10. D = {x: $x \neq 0$}, R = {y: $y \neq -2$}
11. D = {x: $x \neq 2$}, R = {y: $y \neq -3$}
12. D = {x: $x \neq 3$}, R = {y: $y \neq -4$}

Section 3.3P: Reducing Rational Expressions

1. $\frac{x-9}{5}; x \neq 1$
2. $\frac{x+5}{x}; x \neq 7$
3. $-1; x \neq -2$
4. $-2; x \neq 3$
5. $-3x; x \neq -2$
6. $-\frac{x+4}{x+5}; x \neq 5$
7. $-\frac{x-5}{9}; x \neq -2$
8. $\frac{x(x+6)}{(x+9)(x+4)}; x \neq -7$
9. $-\frac{(x+2)(x-9)}{(x+1)(x+4)}; x \neq 1, -5$
10. $\frac{(x-3)(x-5)}{(3x-2)(x-8)}; x \neq 2, -5$
11. $-\frac{(x+6)(x-3)}{(2x+3)(x+3)}; x \neq 5$
12. $-\frac{(x-1)(x-6)}{(x-3)(x+5)}; x \neq -1, 4, 6$

13. $\frac{(x-5)(x-3)}{(5x-2)(x+1)}$; $x \neq -5, 3$
14. $-\frac{(x-5)(x+8)}{(x-4)(x+9)}$; $x \neq -8, -6$
15. $\frac{(x-7)(x-3)}{(7x-2)(x+3)}$; $x \neq -2, 3, 4$

Section 3.4P: Sums and Differences of Rational Functions

1. 7(x−1); $x \neq 6$
2. 9(x−7); $x \neq 1$
3. 4(x−8); $x \neq 7$
4. 3(x−8); $x \neq 2$
5. $\frac{3(x+1)}{(x-1)(x-9)}$; $x \neq 5$
6. $\frac{5(x+14)}{x(x-14)}$; $x \neq 7$
7. $\frac{-7(x+3)}{(x-3)(x-7)}$; $x \neq 5$
8. $\frac{-9(x+4)}{x(x-4)}$; $x \neq 2$
9. $\frac{-8}{x+10}$; $x \neq 5, 0, -10$

Section 3.5P: Solving Rational Equations

1. $x = -\frac{7}{2}$
   See graph here for checking your solution: https://www.desmos.com/calculator/p2zs8clqmf
2. $x = 0$
3. No solution
4. $x = 41$
5. $x = -\frac{128}{9}$
6. $x = \frac{7+\sqrt{97}}{8}, \frac{7-\sqrt{97}}{8}$
7. $x = -7, -12$
8. $\frac{-19+\sqrt{73}}{4}, \frac{-19-\sqrt{73}}{4}$
9. $x = 2$ (note that $x = 3$ is an extraneous solution)
10. $x = \frac{5-\sqrt{17}}{2}, \frac{-5-\sqrt{17}}{2}$
11. $x = \frac{7}{2}, -\frac{4}{3}$
12. $x = 3$ (note that $x = 2$ is an extraneous solution)
13. $x = 4, \frac{3}{2}$
14. $x = \frac{7}{2}, \frac{5}{2}$
15. $(-1, 3)$
16. $(-1, \infty)$
17. $(-\infty, -6] \cup (4, \infty)$

Section 3.6P: Rational Applications

1. 15 hours
2. $\frac{28}{11}$ hours
3. 4.90 mph
4. 15 mph
5. 19 barrels

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