MATH 3AC: ANALYTIC GEOMETRY AND CALCULUS I



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Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: Basic Algebra Review

”Failure is the foundation of success, and the means by which it is achieved.” - Lao Tzu

Because this book is targeted at a wide range of different student audiences, it is natural to ask where to begin. Many students attempting calculus have a strong foundation in algebra and trigonometry (the elements of precalculus), while others may be lacking in one or the other. The first chapter of this book is dedicated to students who may not have the strongest background in algebra, but may be a worthy brush up for students of all skill levels. Chapter one begins with a review of solving basic linear equations $Ax + By = C$, which we then use to help us draw the graphs of lines on the Cartesian plane. We will also discuss the three different equations of a line which we will use throughout the rest of the book. We also review solving linear inequalities and putting our answer into interval notation.

This chapter continues to discuss the topic of relationships vs. functions and function notation. We cover the family of power functions $f(x) = x^p$, and how to solve basic linear equations involving them. Alongside this exploration of the power functions are radical functions and quadratic functions. We talk about how to sketch the parabola represented by a given quadratic function as well as methods for solving quadratic equations. A student who has mastered the topics in this chapter should feel confident solving an assortment of equations from a first algebra course, linear inequalities, using function notation, and graphing a line or parabola given in a variety of different possible forms. All of these skills will be expanded as we progress through the book, but they are ultimately the building blocks of mathematics.

1.1: Solving Linear Equations
1.2: Plotting in the Cartesian Plane
1.3: Slopes and 3 Forms for Lines
1.4: Solving Linear Inequalities
1.5: Functions
1.6: Solving Non-Linear Equations
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1.9: Quadratic Equations

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1.1: Solving Linear Equations

Most linear equations are of the form Ax+B=C, where A, B and C are fixed constants.
When solving an equation, we looking for the value(s) of x which make the equation true.

Example 1.1.1

What value(s) of x satisfy the equation $2x+1=9$?

Solution

$2\cdot 0+1=1$

$2\cdot 1+1=3$

$2\cdot 2+1=5$

$2\cdot 3+1=7$

$2\cdot 4+1=9$

Therefore $x=4$

Instead of guessing numbers, we rely on two basic skills to do this:

1. Adding or subtracting a number from both sides.

Example 1.1.2

What value(s) of x satisfy the equation $x+5=11$?

Solution

Subtract 5 from both sides to get $x=6$

Example 1.1.3

What value(s) of y satisfy the equation $y-4=2$?

Solution

Add 4 to both sides to get $y=6$

Example 1.1.4

What value(s) of z satisfy the equation $z+\frac{1}{2}=\frac{3}{4}$?

Solution

Minus $\frac{1}{2}$ from both sides to get $z=\frac{1}{4}$

2. Multiplying or dividing a number from both sides.

Example 1.1.5

What value(s) of x satisfy the equation $3x=12$?

Solution

Divide both sides by 3 to get $x=4$

Example 1.1.6

What value(s) of y satisfy the equation $-2y = 7$?

Solution

Divide both sides by -2 to get $y = -\frac{7}{2}$

Example 1.1.7

What value(s) of z satisfy the equation $\frac{2}{5}z = \frac{4}{3}$?

Solution

Divide both sides by $\frac{5}{2}$ to get $z = \frac{10}{3}$

These are the only two skills you need in order to solve any linear equation. Individually they are very simple and straightforward rules, but typically you need to use more than one move to solve the more complicated equations.

Keep in mind that there is more than one way to get to the correct answer, as long as you only use those two rules: multiply both sides by the same number, or add the same number to both sides.

Example 1.1.8

What value(s) of a satisfy the equation $2a - 3 = 7$?

Solution

Add 3 to both sides to get $2a = 10$

Then divide both sides by 2 to get $a = 5$

Example 1.1.9

What value(s) of b satisfy the equation $4(b - 1) = 8$?

Solution

Divide both sides by 4 to get $b - 1 = 2$

Then add 1 to both sides to get $b = 3$

When fractions are involved, sometimes it makes things more pleasant to multiply by the denominator of any or all fractions in the equation early on:

Example 1.1.10

What value(s) of c satisfy the equation $\frac{1}{2}c - 1 = \frac{2}{3}$?

Solution

Multiply both sides by 6 to get $3c - 6 = 4$

Then add 6 to both sides to get $3c = 10$

Finally divide both sides by 3 to get $c = \frac{10}{3}$

Example 1.1.11

What value(s) of d satisfy the equation $\frac{3}{4}(d-2)=\frac{1}{3}$?

Solution

Multiply both sides by 12 to get $9d-18=4$

Then add 18 to both sides to get $9d=22$

Finally divide both sides by 9 to get $d=\frac{22}{9}$

When the variable is on both sides of the equation, you can add or subtract it as well:

Example 1.1.12

What value(s) of x satisfy the equation $2x=x+5$?

Solution

Subtract x from both sides to get $x=5$

Example 1.1.13

What value(s) of y satisfy the equation $4y-1=y+5$?

Solution

Subtract y from both sides to get $3y-1=5$

Add 1 to both sides to get $3y=6$

Finally divide both sides by 3 to get $y=2$

Example 1.1.14

What value(s) of z satisfy the equation $-3z+1=z-4$?

Solution

Add 3z to both sides to get $1=4z-4$

Add 4 to both sides to get $5=4z$

Finally divide both sides by 4 to get $z=\frac{5}{4}$

Example 1.1.15

What value(s) of t satisfy the equation $\frac{1}{2}t-2=3t-\frac{1}{3}$?

Solution

Multiply both sides by 6 to get $3t-12=18t-2$

Add 2 and subtract 3t on both sides to get $-10=15t$

Finally divide both sides by 15 to get $t=-\frac{2}{3}$

Just remember to fully distribute both sides before trying to solve.

Example 1.1.16

What value(s) of x satisfy the equation $2(x-1)+3x=x+8$?

Solution

Distribute and collect to get $5x-2=x+8$

Add 2 and subtract x on both sides to get $4x=10$

Finally divide both sides by 4 to get $x=\frac{5}{2}$

Example 1.1.17

What value(s) of t satisfy the equation $5t-3(t-4)=7t+2$?

Solution

Distribute and collect to get $2t+12=7t+2$

Subtract 2t and subtract 2 from both sides to get $10=5t$

Finally divide both sides by 5 to get $t=2$

Example 1.1.18

What value(s) of v satisfy the equation $v=3(2v-1)+6(v+1)$?

Solution

Distribute and collect to get $v=12v+3$

Subtract 12v from both sides to get $-11v=3$

Finally divide both sides by -11 to get $v=-\frac{3}{11}$

Example 1.1.19

What value(s) of q satisfy the equation $q+5=\frac{1}{2}(q-1)-\frac{1}{3}(2q+1)$?

Solution

Multiply both sides by 6 to get $6q+30=3(q-1)-2(2q+1)$

Distribute and collect to get $6q+30=-q-5$

Add q and subtract 30 from both sides to get $7q=-35$

Finally divide both sides by 7 to get $q=-5$

Some special cases can arise when all the variables cancel out.

Example 1.1.20

What value(s) of x satisfy the equation $3x+4=x+2(x+2)$?

Solution

Distribute and collect to get $3x+4=3x+4$

Subtract 3x from both sides to get $4=4$

This equation is always true, regardless of what number we choose for x.

We say **all real numbers** to represent this solution set.

Example 1.1.21

What value(s) of y satisfy the equation $6y-2(2y-3)=5(y+2)-3y$?

Solution

Distribute and collect to get $2y+6=2y+10$

Subtract 2y from both sides to get $6=10$

This equation is never true, regardless of what number we choose for y.

We say **no solution** exists for this equation, or write $\emptyset$

Just remember never to divide by a variable!

Example 1.1.22

What value(s) of x satisfy the equation $4x=3x$?

Solution

BAD IDEA: Divide both sides by x to get $4=3$

This would lead us to believe that there is no solution to this equation.

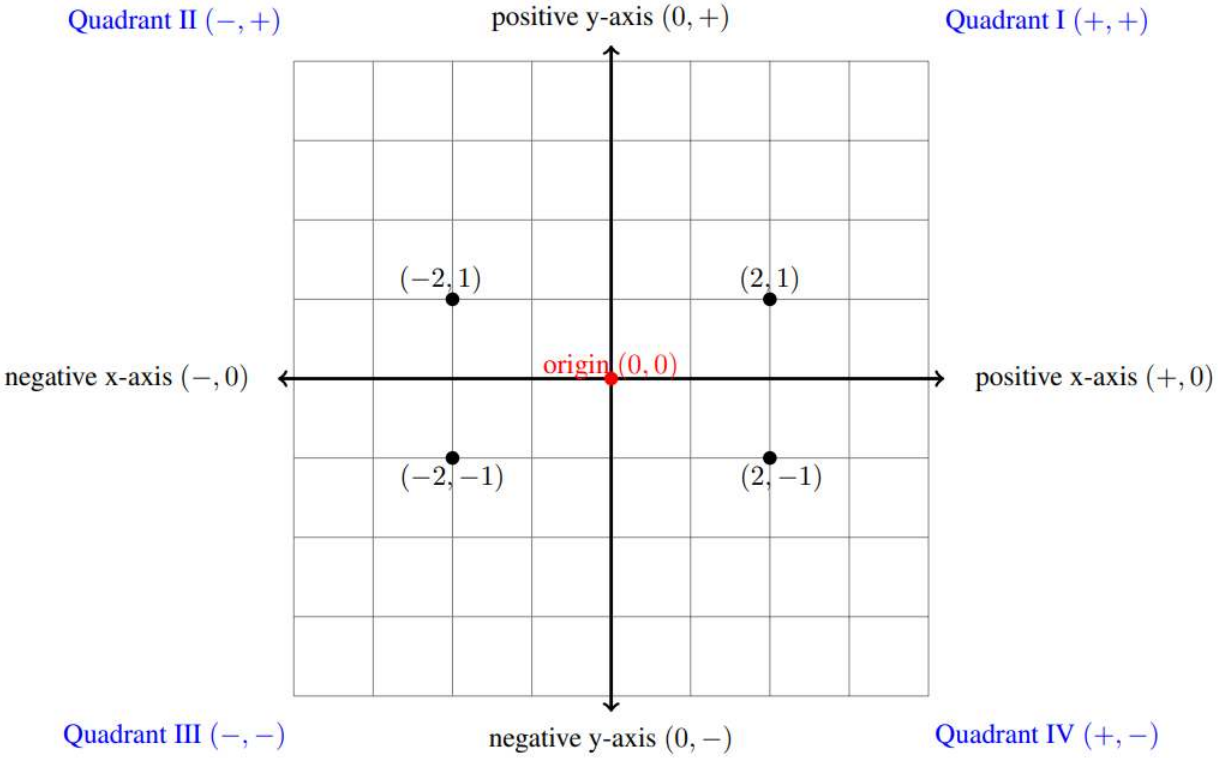
CORRECT IDEA: Subtract 3x from both sides to get $x=0$

This gives the correct solution to the equation, which you would have missed it if you divided by x

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1.2: Plotting in the Cartesian Plane

The Cartesian Plane (commonly called the xy-plane) has the following components:



While plotting points in quadrants may grow dull quickly, the Cartesian plane is a powerful tool. We can use this plane to help visualize relationships between the variables x and y by plotting a dot on every point which satisfies it. The resulting shape is a graph unique to the given relationship.

When the equation looks like $Ax + By = C$, the graph looks like a straight line.
Any straight line can be drawn by finding any two points on that line.
To find these points, you can pick any value of x or y to plug in, and then solve for the other variable in the equation. If plugging in $x = a$ gives an equation that has solution $y = b$ (or if plugging in $y = b$ gives an equation with solution $x = a$) then plot the point (a, b).

Example 1.2.1

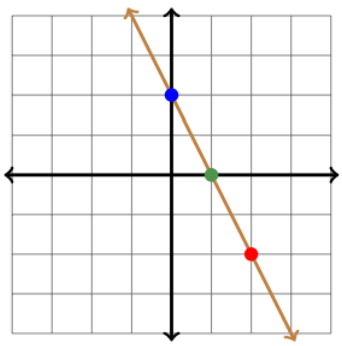
Sketch the graph of $2x + y = 2$

Solution

We plug in a few different x values, and solve the equation they form for y:

Plug in	Equation Becomes	Solution
$x = 0$	$2 \cdot 0 + y = 2$	$y = 2$
$x = 1$	$2 \cdot 1 + y = 2$	$y = 0$
$x = 2$	$2 \cdot 2 + y = 2$	$y = -2$

Now we plot these points and connect any two of them to draw the graph of our line.



Example 1.2.2

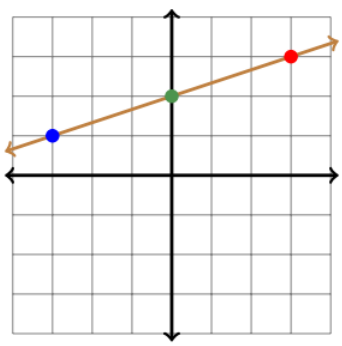
Sketch the graph of $-x+3y=6$

Solution

This time we plug in a few different y values, and solve the equation they form for x:

Plug in	Equation Becomes	Solution
$y=1$	$-x+3\cdot 1=6$	$x=-3$
$y=2$	$-x+3\cdot 2=6$	$x=0$
$y=3$	$-x+3\cdot 3=6$	$x=3$

Again we plot these points and connect any two of them to draw the graph of our line.



Example 1.2.3

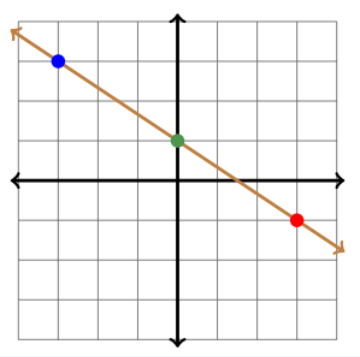
Sketch the graph of $y=-\frac{2}{3}x+1$

Solution

We plug in a few friendly x values, and solve the equation they form for y:

Plug in	Equation Becomes	Solution
$x=-3$	$y=-\frac{2}{3}\cdot(-3)+1$	$y=3$
$x=0$	$y=-\frac{2}{3}\cdot 0+1$	$y=1$
$x=3$	$y=-\frac{2}{3}\cdot 3+1$	$y=-1$

Again we plot these points and connect any two of them to draw the graph of our line.



Example 1.2.4

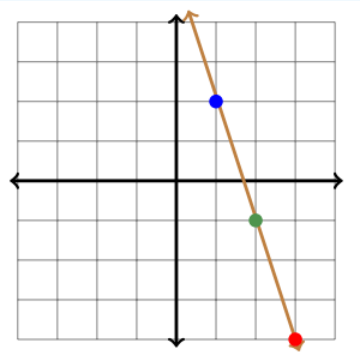
Sketch the graph of $y+1=-3(x-2)$

Solution

We plug in a few friendly x values, and solve the equation they form for y:

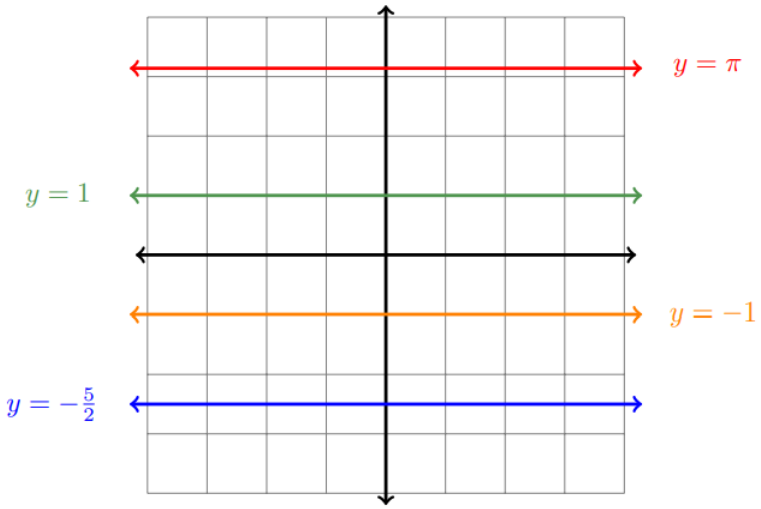
Plug in	Equation Becomes	Solution
$x=1$	$y+1=-3(1-2)$	$y=2$
$x=2$	$y+1=-3(2-2)$	$y=-1$
$x=3$	$y+1=-3(2-3)$	$y=-4$

Again we plot these points and connect any two of them to draw the graph of our line.

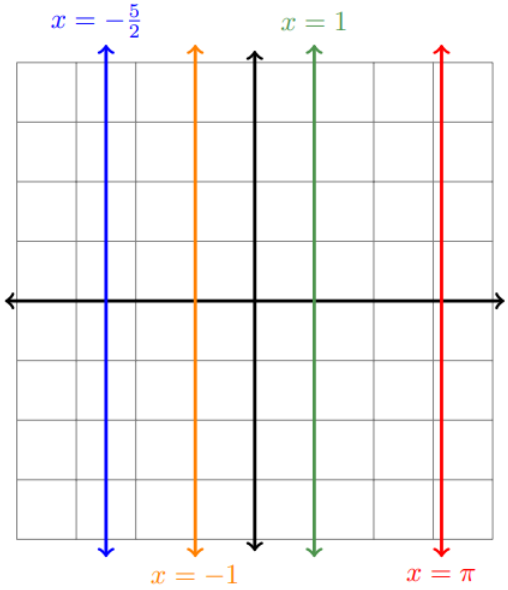


Of course in $Ax+By=C$, if A or B are zero, then one of the variables disappears, giving either the line $x=\alpha$ or $y=\beta$.

For lines of the form $y=\beta$, note that for any real number a, if you plug in $x=a$ the result is always $y=\beta$. This means the point (a,β) must be on this line for all values of a. The resulting picture will contain all points which have the fixed y-value of β, which gives a *horizontal line*.



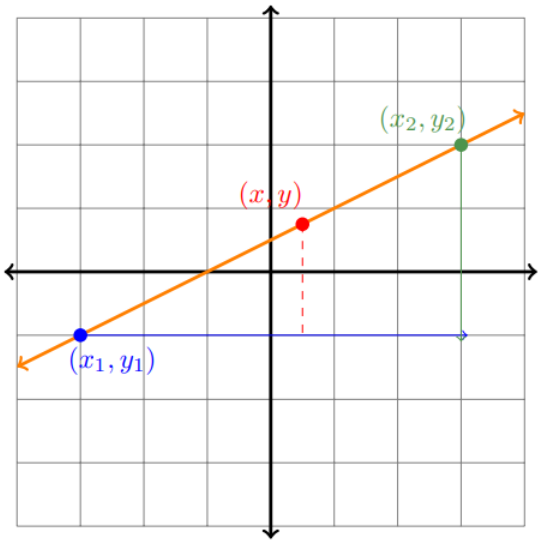
For lines of the form $x = \alpha$, note that for any real number b, if you plug in $y = b$ the result is always $x = \alpha$. This means the point (α, b) must be on this line for all values of b. The resulting picture will contain all points which have the fixed x-value of α, which gives a *vertical line*.



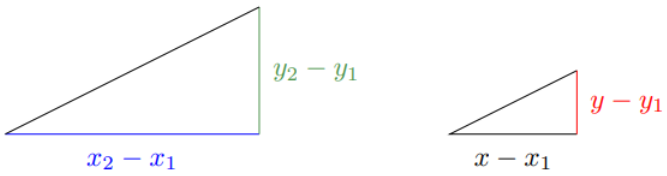
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1.3: Slopes and 3 Forms for Lines

The slope of a line between two points (x_1, y_1) and (x_2, y_2) is a number representing the ratio of the change in y direction and the change in x direction.



Notice that for any point (x, y) on this line, we form two similar triangles:



Thus, the ratio of the height over the base is always constant. We call this constant the *slope* of the line, given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ (where (x_1, y_1) and (x_2, y_2) are any two points on the line) Since (x, y) is arbitrary, we know that $m = \dfrac{y - y_1}{x - x_1}$ must hold for all points on this line.

This means the equation of the line can be given in *point-slope form:* $y - y_1 = m(x - x_1)$

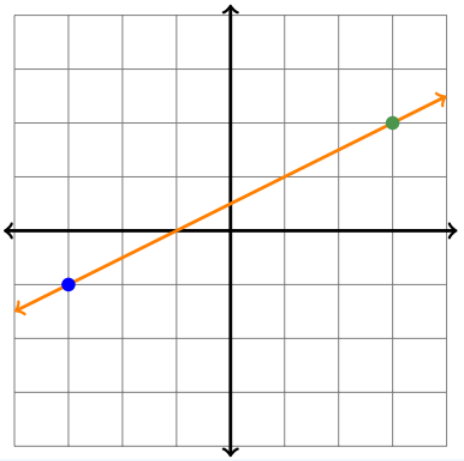
If the point (x_1, y_1) happens to be of the form $(0, b)$, then this equation simplifies to *slope-intercept form:* $y = mx + b$

These two, along with the original $Ax + By = C$ (known as *general form*) give us the three different ways to describe a line. Each of them has their own advantages and disadvantages, so it is worth being familiar with them all.

Example 1.3.1

Graph the line containing the points $(-3, -1)$ and $(3, 2)$ and find its equation in slope intercept form if possible.

Solution



We can find the slope to be $m = \frac{2-(-1)}{3-(-3)} = \frac{3}{6} = \frac{1}{2}$

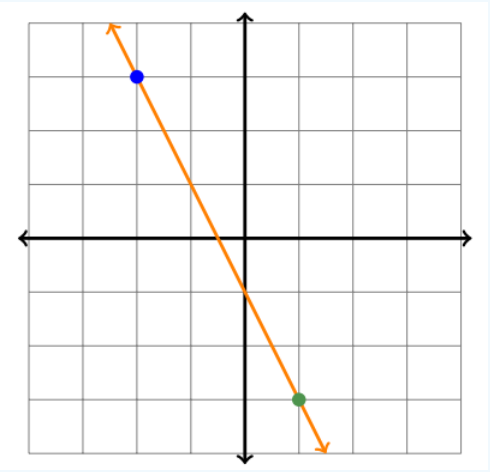
We can use either point and put this into point-slope form: $y-(-1) = \frac{1}{2}(x-(-3))$

Then distribute and solve for y to put it into slope-intercept form: $y = \frac{1}{2}x + \frac{1}{2}$

Example 1.3.2

Graph the line containing the points $(-2, -3)$ and $(1, -3)$ and find its equation in slope intercept form if possible.

Solution



We can find the slope to be $m = \frac{-3-3}{1-(-2)} = \frac{-6}{3} = -2$

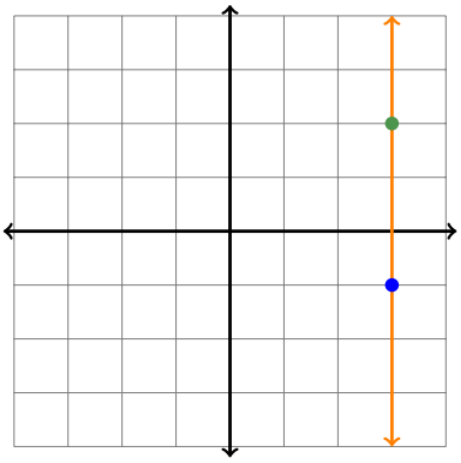
We can use either point and put this into point-slope form: $y-(-3) = -2(x-1)$

Then distribute and solve for y to put it into slope-intercept form: $y = -2x - 1$

Example 1.3.3

Graph the line containing the points $(3, -1)$ and $(3, 2)$ and find its equation in slope intercept form if possible.

Solution



We can find the slope to be $m = \frac{2-(-1)}{3-(3)} = \frac{3}{0} = undefined$

Slope doesn't exist for vertical lines! Thankfully we know that a vertical line is always of the form $x = \alpha$ for some real number α

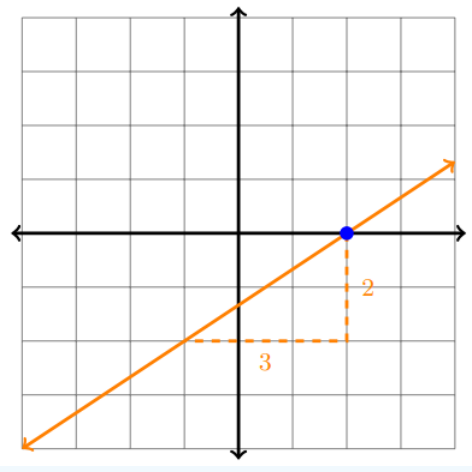
Thus our answer is $x = 3$ which cannot be put into slope-intercept form.

Instead of being given two points, all that you need to solve for the equation of a line is any one point and another piece of information that tells you the slope. This can come in various ways:

Example 1.3.4

Graph the line with slope $m = \frac{2}{3}$ and x-intercept $(2, 0)$ and find its equation in slope intercept form if possible.

Solution



We can use point-slope form immediately:

$y - 0 = \frac{2}{3}(x - 2)$

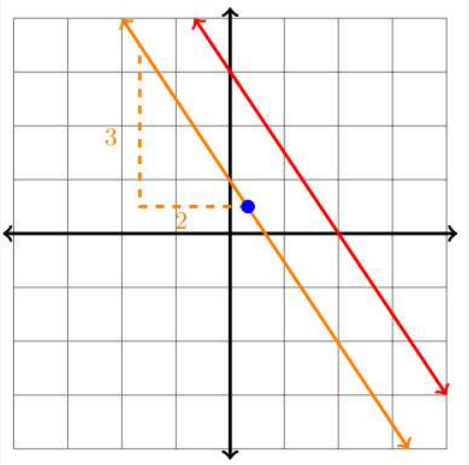
Which you can re-write as $y = \frac{2}{3}x - \frac{4}{3}$

We say two lines are *parallel* if they have the same slope. Since slope is very apparent when in slope-intercept form $y = mx + b$, solving for y is usually an easy way to find the slope of a line.

Example 1.3.5

Graph the line containing the point $\left(\frac{1}{3}, \frac{1}{2}\right)$ which is *parallel* to the line $3x + 2y = 6$ and find its equation in slope intercept form if possible.

Solution



First we find the slope of the given line by solving for y:

$y = -\frac{3}{2}x + 3$

This means it has a slope of $m = -\frac{3}{2}$

To be parallel, our line will have the same slope! Now we put it into point-slope form:

$y - \frac{1}{2} = -\frac{3}{2}\left(x - \frac{1}{3}\right)$

Which you can re-write as $y = -\frac{3}{2}x + 1$

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1.4: Solving Linear Inequalities

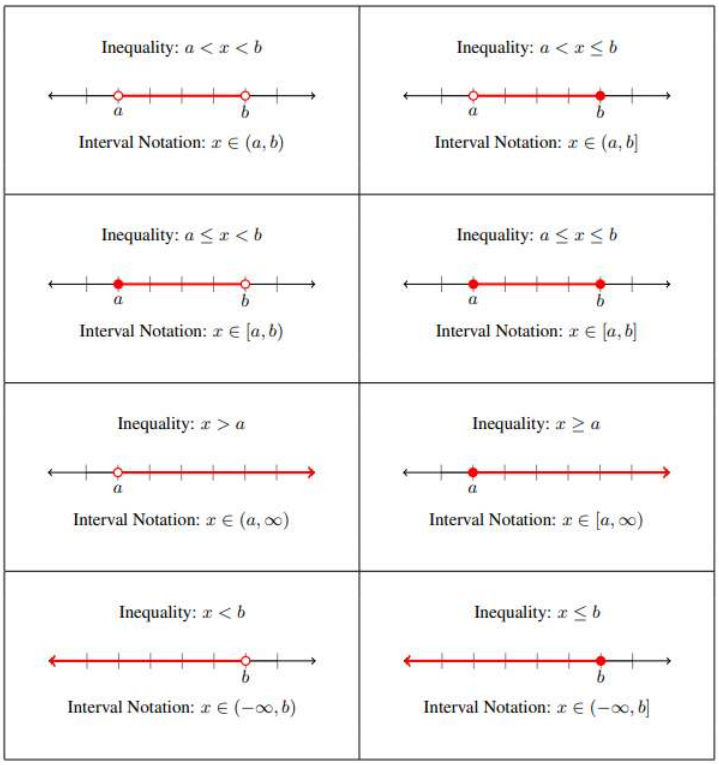
Besides the equal sign, we can compare expressions using a few other symbols.

$a < b$	a is strictly less than b
$a \leq b$	a is less than or equal to b
$a > b$	a is strictly greater than b
$a \geq b$	a is greater than or equal to b
$a \neq b$	a is not equal to b

Because an answer can sometimes look a little different despite meaning the same thing

(such as $x > 3$ and $3 < x$), we need a uniform way to describe our answers to inequalities.

This brings us to *interval notation*. Interval notation describes an set of numbers that x can equal, typically written (a, b), $(a, b]$, $[a, b)$, or $[a, b]$



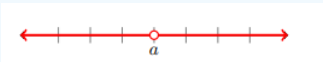
Of course, sometimes your answer is determined by more than one interval.
The *union* symbol: $\cup$ is used when x can be from either one interval **OR** another

Example 1.4.1

Write $x \neq a$ in interval notation

Solution

Sketch the interval(s) on the number line, shading every number but a



Left Interval: $(-\infty, a) \leftrightarrow x < a$ Right Interval: $(a, \infty) \leftrightarrow x > a$
(so $x \neq a$ is the same as " $x < a$ OR $x > a$ ")

So in interval notation this is: $x \in (-\infty, a) \cup (a, \infty)$

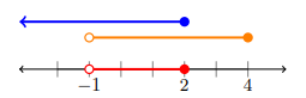
The *intersection* symbol: $\cap$ is used when x must be in both one interval **AND** another

Example 1.4.2

Evaluate $(-\infty, 2] \cap (-1, 4]$

Solution

Sketch the two intervals separately on the number line



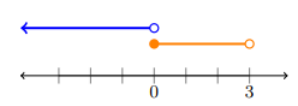
The overlap of these two intervals (shown in red) is your answer: $(-1, 2]$

Example 1.4.3

Evaluate $(-\infty, 0) \cap [0, 3)$

Solution

Sketch the two intervals separately on the number line



These two intervals do not overlap, so their intersection does not contain anything.

Thus our answer, just like when solving an impossible to satisfy equation, is $\emptyset$

Not all inequalities start off in an easy form to translate into interval notation.
Sometimes the problem initially looks like $Ax + B \neq C$ or $C \leq Ax + B \leq D$, etc.
To solve linear inequalities, you again can use two skills:

1. Add or subtract the same thing to all sides.

2. Multiply or divide all sides by the same **positive number**.
Notice we do not allow for multiplication by zero or negative numbers. This is because either of the above multiplication types can sometimes change whether an inequality is true or false.

Original Inequality	$3 < 5$	$3 \leq 5$	$3 \neq 5$
	(true)	(true)	(true)
Times Zero	$0 \cdot 3 < 0 \cdot 5$ $0 < 0$	$0 \cdot 3 \leq 0 \cdot 5$ $0 \leq 0$	$0 \cdot 3 \neq 0 \cdot 5$ $0 \neq 0$
	(false)	(true)	(false)
Times Negative	$-1 \cdot 3 < -1 \cdot 5$ $-3 < -5$	$-1 \cdot 3 \leq -1 \cdot 5$ $-3 \leq -5$	$-1 \cdot 3 \neq -1 \cdot 5$ $-3 \neq -5$

	(false)	(false)	(true)
Times Negative (and switch sign)	$-1 \cdot 3 > -1 \cdot 5$ $-3 > -5$	$-1 \cdot 3 \geq -1 \cdot 5$ $-3 \geq -5$	$-1 \cdot 3 \neq -1 \cdot 5$ $-3 \neq -5$
	(true)	(true)	(true)

Luckily, multiplying both sides of an inequality by zero is not something you will often do.

Unfortunately, there are certain instances where multiplication or division by a negative number is necessary. In these cases, please notice the bottom row above would be true in every column after we switched the direction of the inequality.

This means you are allowed to multiply both sides of an inequality by a negative as long as you switch the direction of the inequality symbol when you do so.

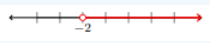
Example 1.4.4

Find the solutions to the inequality $2a > -4$ and put your answer in interval notation

Solution

We divide both sides by 2 to get $a > -2$

Sketch this interval on the number line:



Now we put it into interval notation: $a \in (-2, \infty)$

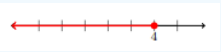
Example 1.4.5

Find the solutions to the inequality $b + 3 \leq 7$ and put your answer in interval notation

Solution

We subtract 3 from both sides to get $b \leq 4$

Sketch this interval on the number line:



Now we put it into interval notation: $b \in (-\infty, 4]$

Example 1.4.6

Find the solutions to the inequality $-4c + 1 \leq 17$ and put your answer in interval notation

Solution

We subtract 1 from both sides to get $-4c \leq 16$

We divide by -4 on both sides and switch the inequality sign to get $c \geq 4$

Sketch this interval on the number line:



Now we put it into interval notation: $c \in [4, \infty)$

Example 1.4.7

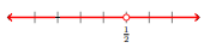
Find the solutions to the inequality $d+1 \neq 7d-2$ and put your answer in interval notation

Solution

We add 2 and subtract d from both sides to get $3 \neq 6d$

We divide by 6 on both sides to get $d \neq \frac{1}{2}$

Sketch this interval on the number line:



Now we put it into interval notation: $d \in \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

You can solve two inequalities at once as long as they are joined with an OR/AND.

Example 1.4.8

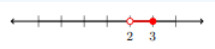
Find the solutions to the inequalities $3x > 6$ AND $x+4 \leq 7$ and put your answer in interval notation

Solution

We first solve each inequality separately:

$x > 2$ AND $x \leq 3$

Sketch these intervals on the number line:



For AND, we want the overlap and we put it into interval notation: $x \in (2, 3]$

Example 1.4.9

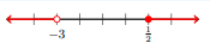
Find the solutions to the inequalities $-4t \leq -2$ OR $t+2 < -1$ and put your answer in interval notation

Solution

We first solve each inequality separately:

$t \geq \frac{1}{2}$ OR $t < -3$

Sketch these intervals on the number line:



For OR, we will take either piece and join them with the union symbol.

Thus our interval notation is: $t \in (-\infty, -3) \cup \left[\frac{1}{2}, \infty\right)$

Compound inequalities (like $a < b < c$) are a specific situation that occurs with AND.
Most of the time we can solve these by multiplying/adding to all three sides at once.

Example 1.4.10

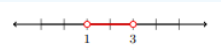
Find the solution to the inequalities $2 < 3x - 1 < 8$ and put your answer in interval notation

Solution

Add 1 to all three sides to get $3 < 3x < 9$

Divide all sides by 3 to get $1 < x < 3$

Sketch this interval on the number line:



We put our answer in interval notation: $x \in (1, 3)$

Example 1.4.11

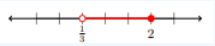
Find the solution to the inequalities $-4 \leq 2 - 3y < 1$ and put your answer in interval notation

Solution

Subtract 2 from all three sides to get $-6 \leq -3y < -1$

Divide all sides by -3 and flip the inequalities to get $2 \geq y > \frac{1}{3}$

Sketch this interval on the number line:

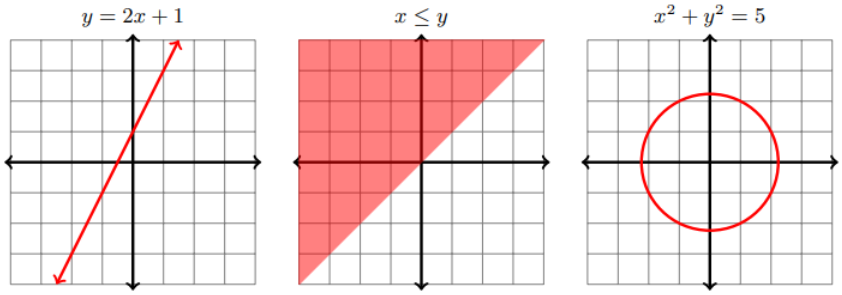


We put our answer in interval notation: $y \in \left(\frac{1}{3}, 2\right]$

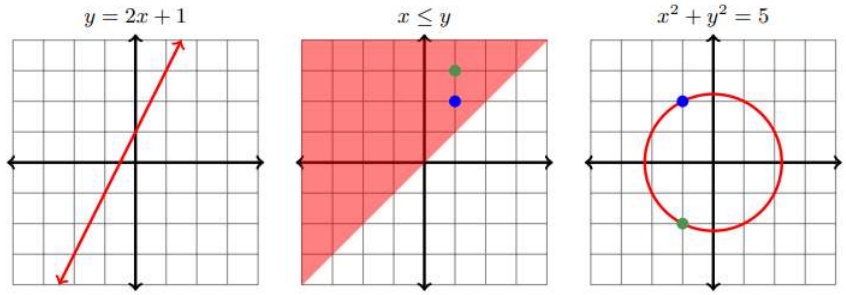
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1.5: Functions

Contrary to popular belief, calculus is not the study of numbers. It is actually the study of relationships. We say a *relation* is any equation or inequality involving variables.
We can draw the collection of all points (x, y) that satisfy the relationship to help visualize them.



A **function** is a special type of relation that is **well-defined**:
For any choice of input x, there is at most one corresponding output y value.



(each x value corresponds to only one y value)

FUNCTION

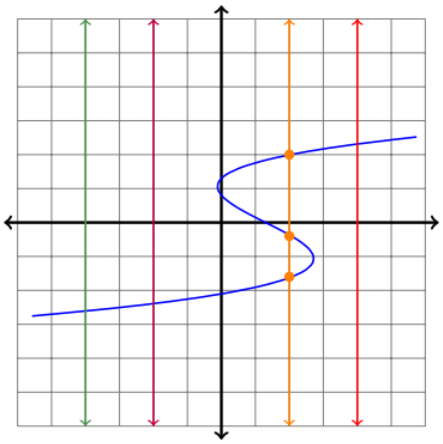
contains $(1, 2)$:
$1 \leq 2$
contains $(-1, -2)$:
$1 \leq 3$

NOT A FUNCTION

contains $(-1, 2)$
$(-1)^2 + 2^2 = 5$
contains $(1, 3)$:
$(-1)^2 + (-2)^2 = 5$

NOT A FUNCTION

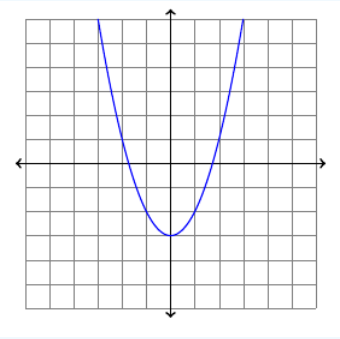
The most common way relations are defined outside of equations are graphically.
A relation is a function if its graph passes the **vertical line test**:
(any line $x = a$ intercepts the graph at most once)



All it takes is **ONE** vertical line to prevent a relation from being a function.
In the graph above, the orange line intersects the curve at three (more than one) points.
That means that for that particular value of x, there are three different y values that satisfy the relation, which is how we know it is not a function.

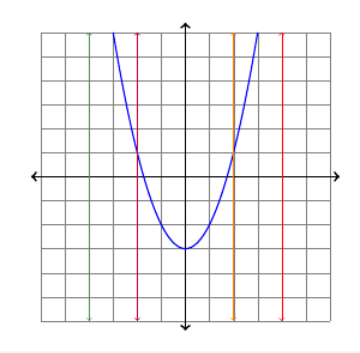
Given the graph of a relation in blue, determine if the relation defines a function or not:

Example 1.5.1



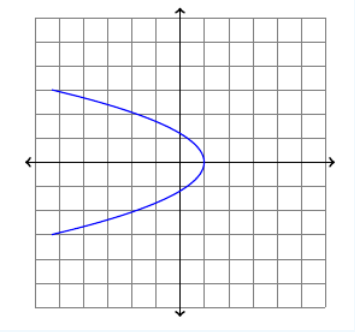
Solution

Passes Vertical Line Test



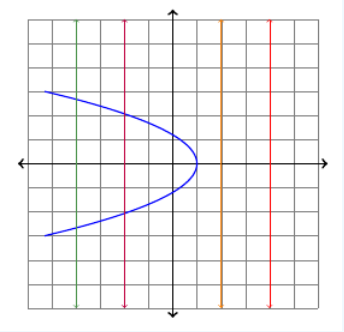
FUNCTION

Example 1.5.2



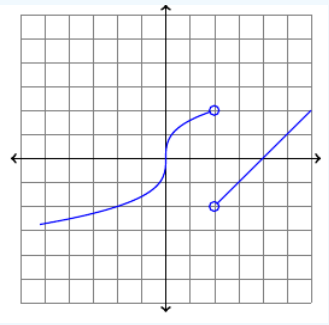
Solution

Passes Vertical Line Test



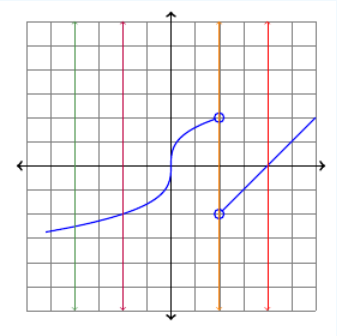
FUNCTION

Example 1.5.3



Solution

Passes Vertical Line Test



FUNCTION

Once we recognize if a relation is a function, there are a few different terms we can use.

If the function is solved for y (as in $y = f(x)$), we say it is defined **explicitly**
(If the function is not solved for y, we say it is defined *implicitly*)
This explicit function notation is used when you want to plug in a particular value of x.

We write $f(a)$ to represent $f(x)$ with $x = a$ plugged into it.

Example 1.5.4

If $f(x) = 5x + 2$ evaluate the following expressions:
(a) $f(2)$ (b) $f(4)$ (c) $f(-1)$

Solution

(a) $f(2) = 5 \cdot 2 + 2 = 12$

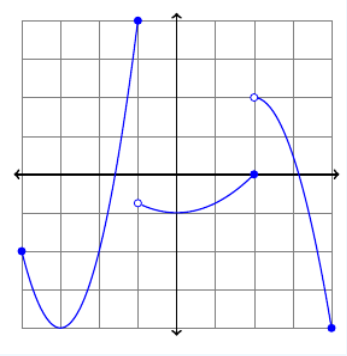
(b) $f(4) = 5 \cdot 4 + 2 = 22$

(c) $f(-1) = 5 \cdot (-1) + 2 = -3$

The same notation can be used even if you aren't given the explicit form of your function.
If the point (a, b) is on the graph of the function, then $f(a) = b$

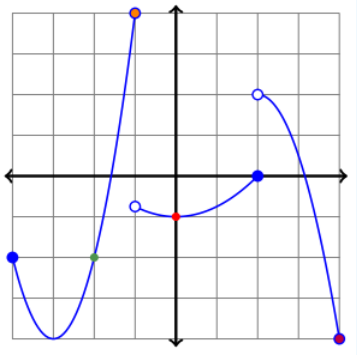
Example 1.5.5

Given the graph of $y = f(x)$ below, evaluate the following expressions



(a) $f(-2)$ (b) $f(-1)$ (c) $f(0)$ (d) $f(-4)$

Solution



(a) $f(-2) = -2$

(b) $f(-1) = 4$

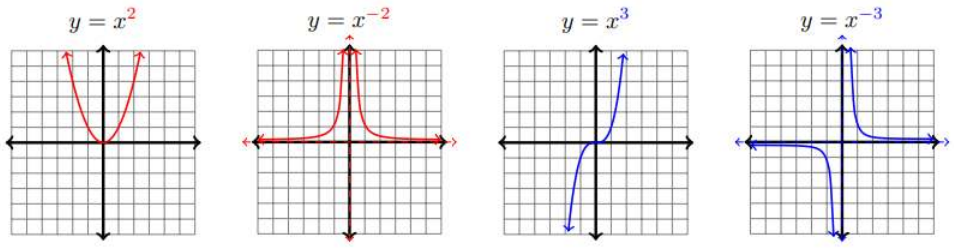
(c) $f(0) = -1$

(d) $f(4) = -4$

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1.6: Solving Non-Linear Equations

Besides linear functions $(x) = mx + b$, there are a wide variety of other functions we can study. One such family of functions is called the *power functions*: $f(x) = x^p$ for some real number p.
In the coming chapters we will discuss many specific power functions and how to sketch their graphs. Here are a few examples of different integer powers of p and the graph of $y = x^p$



Notice that when solving the equation $x^p = A$ for some constant A, we are really solving for where one of the above graphs intersects with the horizontal line $y = A$.

When p is an odd power, we will get exactly one solution: $x = A^{\frac{1}{p}}$

When p is an even power, we will get:
exactly zero solutions when $A < 0$,
one solution when $A = 0$,
or two solutions when $A > 0$ (which are $x = \pm A^{\frac{1}{p}}$)

This fractional power notation translates directly into radicals. We write $a^{\frac{1}{p}} = \sqrt[p]{a}$ to represent the pth root of a, or the number b such that $b^p = a$

(When p is even, and $a^{\frac{1}{p}}$ exists, we define it to be non-negative)

Example 1.6.1

Evaluate $25^{\frac{1}{2}}$

Solution

Even though both $5^2 = 25$ and $(-5)^2 = 25$, the even root is always positive.

Therefore $25^{\frac{1}{2}} = 5$

Example 1.6.2

Evaluate $(-8)^{\frac{1}{3}}$

Solution

We know $(-2)^3 = -8$ is our only choice.

Therefore $(-8)^{\frac{1}{3}} = -2$

Example 1.6.3

Evaluate $81^{-\frac{1}{4}}$

Solution

We can see that $3^4 = 81$, so we use reciprocals to write this as $\left(\frac{1}{3}\right)^{-4} = 81$

Therefore $81^{-\frac{1}{4}} = \frac{1}{3}$

Unfortunately, these radical expressions do not always simplify to perfect numbers.
Because of this, we need to be comfortable doing algebra with them.
Remember the following rules which apply to any power function $p \neq 1$:

$a^p \cdot b^p = (a \cdot b)^p \quad a^p \div b^p = (a \div b)^p$

This means we can still simplify many radical expressions by factoring out perfect p^{th} powers:

Example 1.6.4

Evaluate each expression below:

(a) $\sqrt{108}$ (b) $\sqrt[3]{4000}$ (c) $\sqrt{\frac{49}{8}}$

Solution

(a) $\sqrt{108} = \sqrt{6^2 \cdot 3} = 6\sqrt{3}$

(b) $\sqrt[3]{4000} = \sqrt[3]{10^3 \cdot 4} = 10 \cdot \sqrt[3]{4}$

(c) $\sqrt{\frac{49}{8}} = \sqrt{\frac{7^2}{2^2} \cdot \frac{1}{2}} = \frac{7}{2\sqrt{2}}$

Note that answers such as $\frac{7}{2\sqrt{2}}$ can be rationalized if it makes the reader more comfortable!
(Some authors require students to rationalize the denominator, which means to never have radicals in the denominator, but this author has no such requirements.)

Remember the following rules which fail for any power function $p \neq 1$:

$a^p + b^p \neq (a+b)^p \quad a^p - b^p \neq (a-b)^p$

An important example of this is $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$.

Instead, we add and subtract radicals by simplifying and collecting like terms.

Example 1.6.5

Simplify each expression below into a single term if possible:

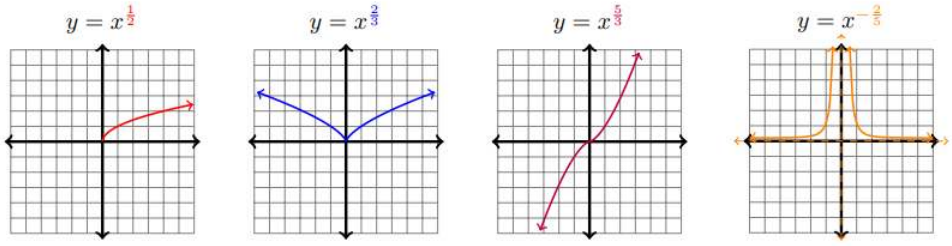
(a) $\sqrt{2} + \sqrt{32}$ (b) $\sqrt{27} - \sqrt{75}$

Solution

(a) $\sqrt{2} + \sqrt{32} = \sqrt{2} + \sqrt{4^2 \cdot 2} = \sqrt{2} + 4\sqrt{2} = 5\sqrt{2}$

(b) $\sqrt{27} - \sqrt{75} = \sqrt{3^2 \cdot 3} - \sqrt{5^2 \cdot 3} = 3\sqrt{3} - 5\sqrt{3} = -2\sqrt{3}$

Some power functions have fractional powers $f(x) = x^{\frac{p}{q}}$.



Based on the graphs above, we can see that solving equations of the form $x^{\frac{p}{q}} = A$ will yield different results depending on if p and q are even or odd.

If p is even, (look at the graphs of $y = x^{\frac{2}{3}}$ and $y = x^{-\frac{2}{5}}$) then our graph is considered *even*.
his means that it looks mirrored on the left and right side of the y-axis.
Furthermore, notice that the graph only intersects horizontal lines $x = A$ when $A > 0$,
(and when it does so you get two equal but opposite solutions.)

So if p is even and $A > 0$ we have, $x^{\frac{p}{q}} = A \rightarrow x = \pm A^{\frac{q}{p}}$
and if p is even and $A < 0$ we have, $x^{\frac{p}{q}} = A \rightarrow \emptyset$

Example 1.6.6

Solve for x in the equation $x^{\frac{2}{3}} = 9$

Solution

The exponent $\frac{p}{q} = \frac{2}{3}$ has an even $p = 2$ so we use $x = \pm 9^{\frac{3}{2}}$

Notice $9^{\frac{3}{2}} = \left(\sqrt{9}\right)^3 = 3^3 = 27$

Therefore our solutions are $x = \pm 27$

Example 1.6.7

Solve for t in the equation $t^{-\frac{4}{5}} = \frac{1}{16}$

Solution

The exponent $\frac{p}{q} = \frac{-4}{5}$ has an even $p = -4$ so we use $t = \pm\left(\frac{1}{16}\right)^{-\frac{5}{4}}$

Notice $\left(\frac{1}{16}\right)^{-\frac{5}{4}} = 16^{\frac{5}{4}} = \left(\sqrt[4]{16}\right)^5 = 2^5 = 32$

Therefore our solutions are $t = \pm 32$

Example 1.6.8

Solve for y in the equation $y^{\frac{2}{7}} = -1$

Solution

The exponent $\frac{p}{q} = \frac{2}{7}$ has an even $p = 2$, but $A = -1 < 0$

Therefore we know there is no solution, or $\emptyset$

If q is even, then when we go to solve $x^{\frac{p}{q}} = A$, we will be ultimately squaring both sides of this equation in the process. This is a situation in which we want to check our answers, as the algebra can sometimes suggest *extraneous solutions* (solutions that do not

satisfy the original equation).

Example 1.6.9

Solve for u in the equation $u^{\frac{5}{6}} = 1$

Solution

The exponent $\frac{p}{q} = \frac{5}{6}$ has an odd $p = 5$ so we know that $u = 1^{\frac{6}{5}}$

Notice that $1^{\frac{6}{5}} = \left(\sqrt[5]{1}\right)^6 = 1^6 = 1$

This means our equation can only be satisfied by $u = 1$, but since $q = 6$ was even, we need to check our answer:

$1^{\frac{5}{6}} = 1$ is true, so we know that our solution is $u = 1$

Example 1.6.10

Solve for v in the equation $v^{-\frac{3}{4}} = 8$

Solution

The exponent $\frac{p}{q} = \frac{-3}{4}$ has an odd $p = -3$ so we know that $v = 8^{-\frac{4}{3}}$

Notice that $8^{-\frac{4}{3}} = \left(\sqrt[3]{8}\right)^{-4} = 2^{-4} = \frac{1}{16}$

This means our equation can only be satisfied by $v = \frac{1}{16}$, but since $q = 4$ was even, we need to check our answer:

$\left(\frac{1}{16}\right)^{-\frac{3}{4}} = 8$ is true, so we know that our solution is $v = \frac{1}{16}$

Example 1.6.11

Solve for w in the equation $w^{\frac{3}{2}} = -8$

Solution

The exponent $\frac{p}{q} = \frac{3}{2}$ has an odd $p = 3$ so we know that $w = (-8)^{\frac{2}{3}}$

Notice that $(-8)^{\frac{2}{3}} = \left(\sqrt[3]{-8}\right)^2 = (-2)^2 = 4$

This means our equation can only be satisfied by $w = 4$, but since $q = 2$ was even, we need to check our answer:

$4^{\frac{3}{2}} = -8$ is not true, so we know that $w = 4$ is not a real solution to this equation.

There are no other possible candidates, so there is no solution to this equation, or $\emptyset$

We can combine these skills with our previous ones for solving linear equations too!

Example 1.6.12

Solve for x in the equation $(5x - 3)^{\frac{2}{3}} = 16$

Solution

The exponent $\frac{p}{q} = \frac{2}{3}$ has an even $p = 2$ so we know that $5x - 3 = \pm 16^{\frac{3}{2}}$

Notice that $16^{\frac{3}{2}} = \left(\sqrt{16}\right)^3 = 4^3 = 64$

This means our equation becomes $5x-3=64$ and $5x-3=-64$

We solve each equation separately to get $x=\frac{67}{5},-\frac{61}{5}$

The denominator $q=3$ was odd, so we do not need to check the answers

Example 1.6.13

Solve for t in the equation $1+3t^{-\frac{3}{5}}=25$

Solution

We first isolate $t^{-\frac{3}{5}}$ by subtracting 1 and then dividing both sides by 3:

$t^{-\frac{3}{5}}=8$

Now the exponent $\frac{p}{q}=\frac{-3}{5}$ has an odd $p=-3$ so we know that $t=8^{-\frac{5}{3}}$

Notice that $8^{-\frac{5}{3}}=\left(\sqrt[3]{8}\right)^{-5}=2^{-5}=\frac{1}{32}$

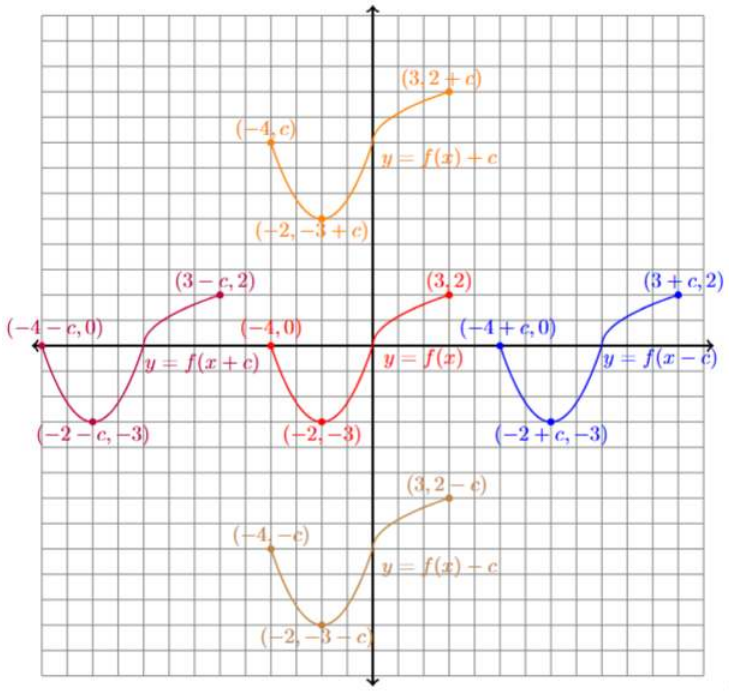
This means our solution to the equation must be $t=\frac{1}{32}$

Neither of the two examples above required us to check our answers (as q was odd in both), but it is never a terrible idea to check your solutions to any equation if you have the time.

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1.7: Graph Transformations

We now study how slight changes in a function effect the given graph of a function *[Math Processing Error]*.

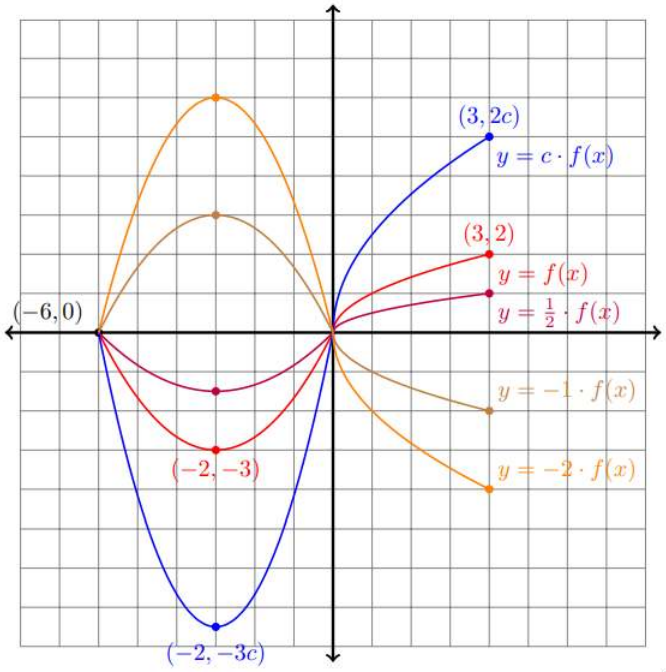


Notice that for the graphs of *[Math Processing Error]* and *[Math Processing Error]*, the values of *[Math Processing Error]* on each point remain the same, while the *[Math Processing Error]*-values at each point are increased or decreased by *[Math Processing Error]*. You can think of this as the function is doing the same work it normally does for each *[Math Processing Error]* value, but the *[Math Processing Error]* value is being shifted up or shifted down afterwards.

Instead, for the graphs of *[Math Processing Error]* and *[Math Processing Error]*, the values of *[Math Processing Error]* on each point remain the same, while the *[Math Processing Error]*-values at each point are decreased or increased by *[Math Processing Error]*. You can think of this as the function needs *[Math Processing Error]* to compensate for the *[Math Processing Error]* by being *[Math Processing Error]* smaller and compensate for the *[Math Processing Error]* by being *[Math Processing Error]* bigger than it normally would be.
This compensating is the reason that the end result feels backwards: the *[Math Processing Error]* value is being shifted left or shifted right afterwards.

The inital results here seem to suggest that adding or subtracting a constant outside of the function will result in a *vertical shift*, while doing so inside of the function will result in a *horizontal shift*.

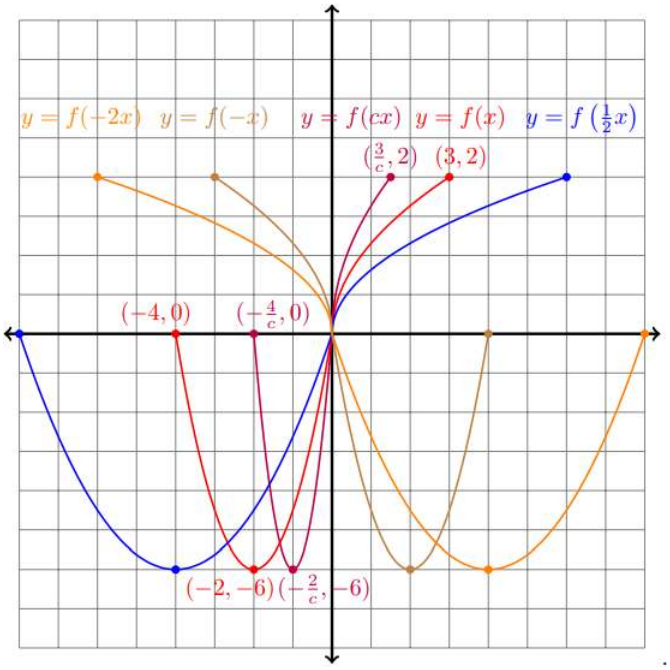
This time, we will see how multiplication by a constant can change our graph *[Math Processing Error]*.



We see from the graphs above that multiplication by a constant *[Math Processing Error]* will again have the values of *[Math Processing Error]* at each point remain the same, while the *[Math Processing Error]*-values are multiplied by *[Math Processing Error]*.

When *[Math Processing Error]*, we get a graph which is stretched *[Math Processing Error]* times as tall.
When *[Math Processing Error]*, we get a graph which is squeezed *[Math Processing Error]* times as tall.
When *[Math Processing Error]*, we reflect the graph vertically across the *[Math Processing Error]*-axis.
When *[Math Processing Error]*, we reflect vertically and stretch/squeeze by *[Math Processing Error]*.
All of these transformations of the graph are considered *vertical stretches* or *reflections*.

Now, we see how multiplication by a constant inside the parenthesis changes our graph *[Math Processing Error]*.



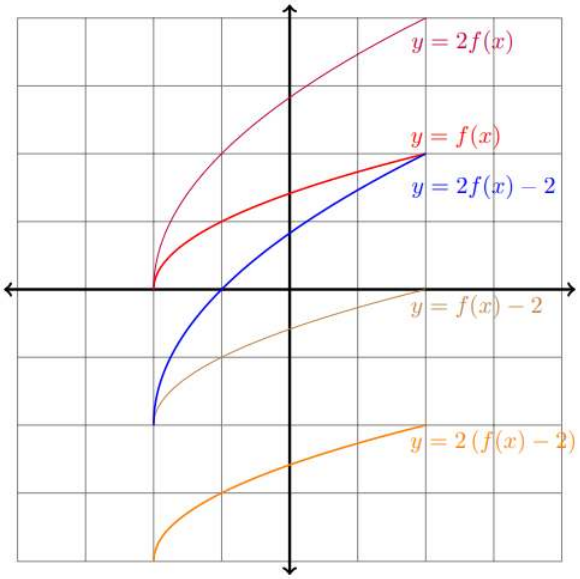
We see from the graphs above that multiplication by a constant *[Math Processing Error]* will again have the values of *[Math*

Processing Error] at each point remain the same, while the *[Math Processing Error]*-values are divided by *[Math Processing Error]*.
This is again because *[Math Processing Error]* needs to compensate for the new multiplication by *[Math Processing Error]*, again making it feel 'backwards' for big and small *[Math Processing Error]* when compared to the previous page.

When *[Math Processing Error]*, we get a graph which is stretched *[Math Processing Error]* times as wide.
When *[Math Processing Error]*, we get a graph which is squeezed *[Math Processing Error]* times as wide.
When *[Math Processing Error]*, we reflect the graph horizontally across the *[Math Processing Error]*-axis.
When *[Math Processing Error]*, we reflect horizontally and stretch/squeeze by *[Math Processing Error]*.
All of these transformations of the graph are considered *horizontal stretches* or *reflections*.

Individually, any of these operations (shift, stretch, or reflect) are straightforward to deal with. Sadly, in combination things can become quite complicated.

First, notice that a shift and then a stretch will give a different graph than a stretch then a shift:



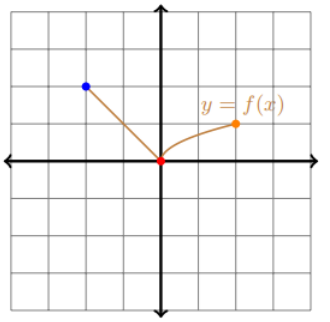
Clearly, order matters when combining these transformations of graphs.
Instead of trying to keep track of many moves in order, consider this method:

If *[Math Processing Error]* are on the graph of *[Math Processing Error]*, then *[Math Processing Error]* are on the graph of *[Math Processing Error]* as long as *[Math Processing Error]* and *[Math Processing Error]*

This means we can focus on the horizontal and vertical transformations separately. If we plug in the list of known points *[Math Processing Error]*, we can solve these equations for the new points *[Math Processing Error]*. Then we can plot these points and connect the dots with the same order and shape.

✔ Example *[Math Processing Error]*

Given the graph of *[Math Processing Error]* below, sketch *[Math Processing Error]*



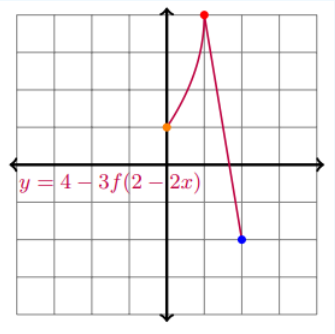
Solution

Our two transformation equations are *[Math Processing Error]* and *[Math Processing Error]*

We plug in the *[Math Processing Error]* points we know on the original graph: (-2,2), (0,0), and (2,1)

[Math Processing Error]	*[Math Processing Error]*	*[Math Processing Error]*	*[Math Processing Error]*
-2	2	2	-2
0	0	1	4
2	1	0	1

Now we plot these three new points *[Math Processing Error]* and connect the dots in the same way to get our final graph:

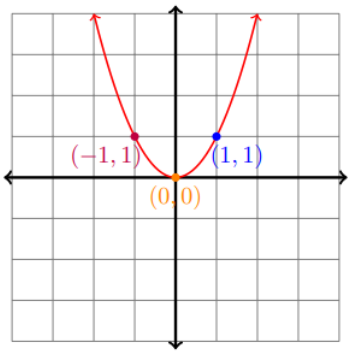


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1.8: Parabolas

Quadratic functions are those of the form $f(x) = ax^2 + bx + c$ for some real numbers a, b, c

When $a = 0$, this simplifies down to a line (which we have already studied fully), so assume $a \neq 0$
We start with the graph of the standard parabola $y = x^2$

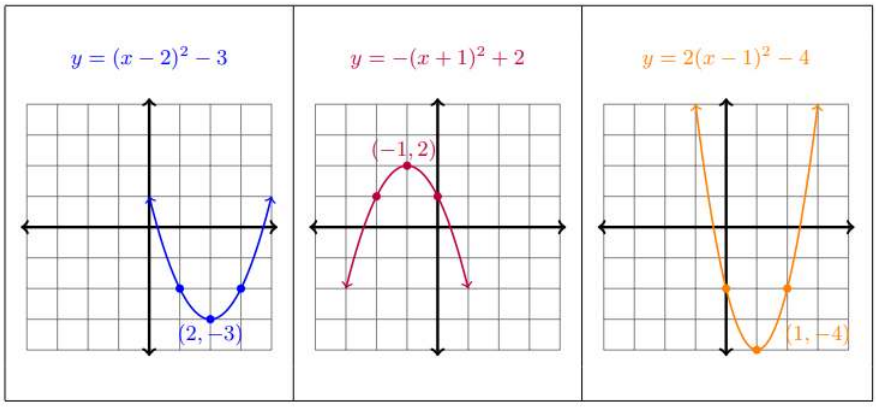


(We say the point $(0, 0)$ is the *vertex* of the parabola.)

A parabola is in **vertex form** when written $y = a(x - h)^2 + k$
For $g(x) = x^2$, this graph is just a transformation of $g(x)$: $y = ag(x - h) + k$
$y_{new} = a \cdot y_{old} + k$ and $x_{old} = x_{new} - h$

x_{old}	y_{old}	x_{new}	y_{new}
-1	1	h-1	k+a
0	0	h	k
1	1	h+1	k+a

So when graph $y = a(x - h)^2 + k$, we can always plot three simple points:
The vertex: (h, k), and the two points $(h \pm 1, k + a)$



If you are given a quadratic, you can put it into vertex form by completing the square.
(Remember that $(x + h)^2 = x^2 + 2hx + h^2$ and $(x - h)^2 = x^2 - 2hx + h^2$)

So if we start off with $x^2 + bx + c$, then b should be doing the job of $2h$:

$$b = 2h \longleftrightarrow h = \frac{b}{2}$$

Therefore we should add and subtract $\left(\frac{b}{2}\right)^2$ (which is the same as h^2) to our expression giving

$x^2+bx+c=\left(x+\frac{b}{2}\right)^2+k$ for some k

Example 1.8.1

Write the following quadratics in vertex form and find the vertex (h,k):

(a) $y=x^2+6x$ (b) $y=x^2-2x+4$

Solution

(a) Since $b=6\to h=3$ we know $h^2=9$

So we add and subtract it: $y=(x^2+6x+9)-9$

And group up the first three as a perfect square to get $y=(x+3)^2-9$

This is vertex form, and we can clearly see our vertex is at the point $(-3,-9)$

(b) Since $b=-2\to h=-1$ we know $h^2=1$

So we add and subtract it: $y=(x^2-2x+1)-1+4$

And group up the first three as a perfect square to get $y=(x-1)^2+3$

This is vertex form, and we see our vertex is the point $(1,3)$

If the leading term is ax^2 instead of x^2, your first step should be to factor out a from both x terms, then repeat the process above. (Just dont forget to distribute a when adding and subtracting h^2)

Example 1.8.2

Write the following quadratics in vertex form and find the vertex (h,k):

(a) $y=2x^2-8x+5$ (b) $y=-\frac{1}{2}x^2-6x-4$

Solution

(a) First we factor out the 2 from the x terms: $y=2(x^2-4x)+5$

Since $b=-4\to h=-2$ we know $h^2=4$

So we add and subtract it: $y=2(x^2-4x+4)-4\cdot 2+5$

Inside the parenthesis is a perfect square, so we write $y=2(x-2)^2-3$

This is vertex form, and we can clearly see our vertex is at the point $(2,-3)$

(b) First we factor out the $-\frac{1}{2}$ from the x terms: $y=-\frac{1}{2}(x^2+12x)-4$

Since $b=12\to h=6$ we know $h^2=36$

So we add and subtract it: $y=-\frac{1}{2}(x^2+12x+36)-36\cdot\left(-\frac{1}{2}\right)-4$

Inside the parenthesis is a perfect square, so we write $y=-\frac{1}{2}(x+6)^2+14$

This is vertex form, and we see our vertex is the point $(-6,14)$

As we can see, completing the square is a great tool we can use to help turn quadratic functions from their general form $f(x)=ax^2+bx+c$ into vertex form $f(x)=a(x-h)^2+k$
This is definitely useful if you ever want to draw a parabola!

Example 1.8.3

Sketch the graph of the function $y = -2x^2 + 12x - 14$

Solution

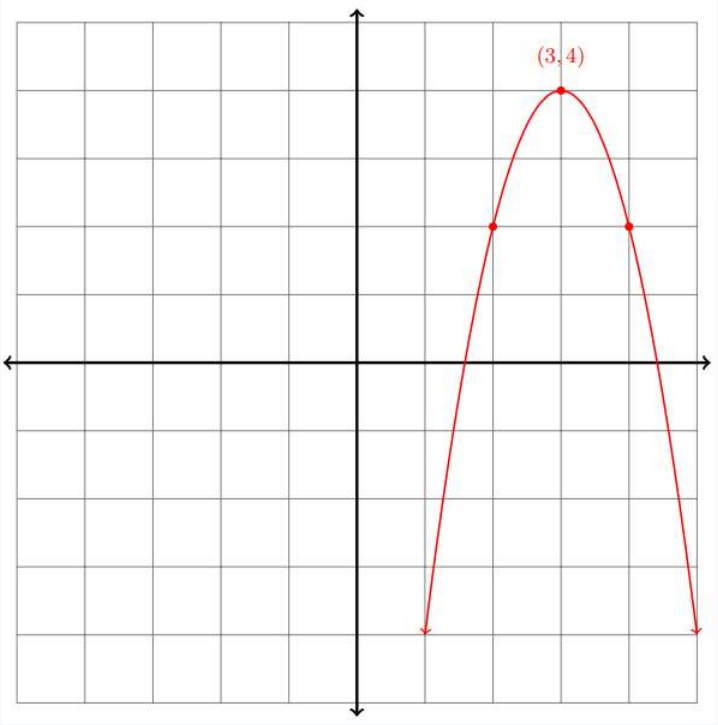
First we put it into vertex form, so we factor out the -2 from the x terms to get $y = -2(x^2 - 6x) - 14$

Then since $b = -6$, we get $h = -3$ so $h^2 = 9$

We add and subtract it: $y = -2(x^2 - 6x + 9) - 9 \cdot (-2) - 14$

Which simplifies into vertex form: $y = -2(x - 3)^2 + 4$

We have a vertex at $(3, 4)$ and can sketch using the nearby points $(2, 2)$ and $(4, 2)$

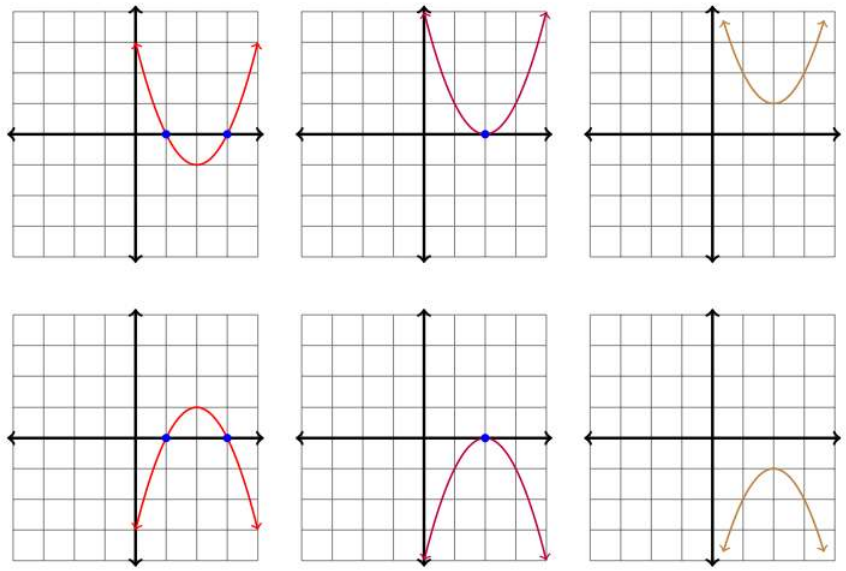


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1.9: Quadratic Equations

Now that we can visualize the graphs of quadratic equations, we can investigate how to solve *quadratic equations*, which generally look like $ax^2+bx+c=0$
(if the equation doesn't equal zero, add or subtract everything to one side)

We know that there are only six different ways for a parabola to intersect $y=0$:



There will either be two distinct real solutions, or exactly one solution, or no real solutions.

In situations where $b=0$, solving $ax^2+c=0$ can be done in the same method we learned for power functions in general. Solving for x^2 then using the $\pm$ and reciprocal power $\frac{1}{2}$)

Example 1.9.1

Find all real solutions to $x^2-16=0$

Solution

Add 16 to both sides to get $x^2=16$

Use the reciprocal power to get $x=\pm 16^{\frac{1}{2}}$

Notice $16^{\frac{1}{2}}=\sqrt{16}=4$

So our final answers are $x=\pm 4$

Example 1.9.2

Find all real solutions to $x^2=0$

Solution

Use the reciprocal power to get $x=\pm 0^{\frac{1}{2}}$

Notice $0^{\frac{1}{2}}=\sqrt{0}=0$

So our final answer is just $x=0$

Example 1.9.3

Find all real solutions to $x^2+9=0$

Solution

Subtract 9 from both sides to get $x^2=-9$

The reciprocal power does not work on negative values

There are no real numbers that are solutions to this equation $\emptyset$

But for nonzero values of b, we can still solve them by completing the square:

Example 1.9.4

Find all real solutions to $x^2+6x+8=0$

Solution

Subtract 8 from both sides to get $x^2+6x=-8$

We see $b=6\rightarrow h=3$ so $h^2=9$

Adding this gives $x^2+6x+9=-8+9$

Which simplifies down to $(x+3)^2=1$

Use the reciprocal power to get $x+3=\pm 1^{\frac{1}{2}}$

Therefore we have $x=-3\pm 1$, so the solutions to this equation are$x=-4,-2$

Example 1.9.5

Find all real solutions to $3x^2-6x-9=0$

Solution

Add 9 to both sides to get $3(x^2-2x)=9$

We see $b=-2\rightarrow h=-1$ so $h^2=1$

Adding this gives $3(x^2-2x+1)=9+1\cdot 3$

Which simplifies down to $3(x-1)^2=12$

Divide both sides by 3 to make $(x-1)^2=4$

Use the reciprocal power to get $x-1=\pm 4^{\frac{1}{2}}$

Therefore we have $x=1\pm 2$, so the solutions to this equation are$x=-1,3$

In general, we can solve $ax^2+bx+c=0$ the same way:

Subtract c to right: $ax^2+bx=-c$

Divide both sides by a: $x^2+\frac{b}{a}x=-\frac{c}{a}$

$h=\frac{b}{2a}\rightarrow h^2=\frac{b^2}{4a^2}$, add h^2 to both sides: $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$

Re-write the left side as a perfect square, and common denominator the right side: $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

Reciprocal power and $\pm$: $x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

Subtract $\frac{b}{2a}$ to right side, simplify root denominator: $x = \frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$

This is known as the *quadratic formula:* $ax^2+bx+c=0 \longleftrightarrow x = \frac{b \pm \sqrt{b^2-4ac}}{2a}$

In this formula we can see that the number of solutions we have depends entirely on b^2-4ac:

If $b^2-4ac<0$ the radical does not make sense, and no solutions for x exist.

Example 1.9.6

Find all real solutions to $x^2+4x+6=0$

Solution

$a=1, b=4, c=6$

$$x = \frac{-4 \pm \sqrt{4^2-4\cdot 1\cdot 6}}{2\cdot 1} = \frac{-4 \pm \sqrt{-8}}{2}$$

Since there is no real number $\sqrt{-8}$, then no such solutions x can exist for this equation, so $\emptyset$

If $b^2-4ac=0$ the radical becomes 0, and only one solution $x=-\frac{b}{2a}$ exists.

Example 1.9.7

Find all real solutions to $-3x^2+3x-\frac{3}{4}=0$

Solution

$a=-3, b=3, c=-\frac{3}{4}$

$$x = \frac{-3 \pm \sqrt{3^2-4\cdot(-3)\cdot\left(-\frac{3}{4}\right)}}{2\cdot(-3)} = \frac{-3 \pm \sqrt{0}}{-6}$$

Since $\sqrt{0}=0$ both the plus and minus give the same answer: $x=\frac{1}{2}$

If $b^2-4ac>0$ then radical is a positive number and two distinct solutions exist.

Example 1.9.8

Find all real solutions to $2x^2-5x=0$

Solution

$a=2, b=-5, c=0$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2-4\cdot 2\cdot 0}}{2\cdot 2} = \frac{5 \pm \sqrt{25}}{4} = \frac{5 \pm 5}{4}$$

This time both the plus and minus give different real answers: $x=\frac{5}{2}, 0$

The quadratic formula is a tool that works on **any quadratic equation** $ax^2+bx+c=0$

Although you can always use the quadratic formula on a quadratic equation, sometimes the numbers are large or unmanagable in practice. In situations like this, it is always nice to have an alternative method for solving.

Sometimes a quadratic function $f(x)=x^2+bx+c$ can be re-written as $f(x)=(x+r)(x+s)$

First we distribute the r and s version of $f(x)$:
$(x+r)(x+s)=x\cdot(x+s)+r\cdot(x+s)=x^2+sx+rx+rs=x^2+(s+r)x+rs$

For these two version of $f(x)$ to agree, we need $x^2+bx+c=x^2+(s+r)x+rs$
That means we want $b=s+r$ (so numbers next to x match) and $c=rs$

This tells us the relationship between factored form and general form for a quadratic function.
In order to factor a quadratic function $f(x)=x^2+bx+c$, into $f(x)=(x+r)(x+s)$ you need to find two numbers r and s that have a **sum of b and a product of c**.

Example 1.9.9

Factor each of the following quadratic expressions.

(a) x^2+4x+3

(b) $x^2-9x+20$

(c) x^2+11x

(d) x^2-25

Solution

(a) We want to find r and s that satisfy: $r+s=4$ and $rs=3$

We see that $r=3, s=1$ satisfies both equations, so our factored form is $(x+3)(x+1)$

(b) We want to find r and s that satisfy: $r+s=-9$ and $rs=20$

We see that $r=-4, s=-5$ satisfies both equations, so our factored form is $(x-4)(x-5)$

(c) We want to find r and s that satisfy: $r+s=11$ and $rs=0$

We see that $r=0, s=11$ satisfies both equations, so our factored form is $x(x+11)$

(d) We want to find r and s that satisfy: $r+s=0$ and $rs=-25$

We see that $r=5, s=-5$ satisfies both equations, so our factored form is $(x+5)(x-5)$

Unfortunately, not every quadratic can be factored. Consider x^2+2x+2 :
There are no two real numbers with a sum of 2 and product of 2.
(This means we cannot find the r and s needed to factor the expression.)
A quadratic expression that cannot be factored is called *prime*.

How about factoring $f(x)=ax^2+bx+c$?
In this case, the factored form looks like $f(x)=(px+r)(qx+s)$
$(px+r)(qx+s)=(pq)x^2+(ps+qr)x+(rs)$

For these two version of $f(x)$ to agree, we need $ax^2+bx+c=(pq)x^2+(ps+qr)x+(rs)$
That means we want $a=pq, b=ps+qr$, and $c=rs$

Since this is a little harder to solve, we can do so in smaller steps.
First find some v, w such that $v+w=b$ and $vw=ac$
(Notice that means $v+w=ps+qr$ and $vw=pqrs$, giving us $v=ps$ and $w=qr$)
Then, we can use this v and w to split up the term \$bx\$ into two pieces, and then factor the resulting four term expression by grouping the first two together and the last two together.
$ax^2+vx+wx+c=(pq)x^2+(ps)x+(qr)x+(rs)=px(qx+s)+r(qx+s)=(px+r)(qx+s)$

Example 1.9.10

Factor each of the following quadratic expressions.

(a) $2x^2+7x+3$

(b) $3x^2-16x+5$

(c) $2x^2+7x-4$

(d) $4x^2-8x-5$

Solution

(a) We want to find v and w that satisfy: $v+w=7$ and $vw=2\cdot 3=6$

We see that $v=1, w=6$ satisfies both equations, so we write:

$2x^2+x+6x+3=x(2x+1)+3(2x+1)=(x+3)(2x+1)$

(b) We want to find v and w that satisfy: $v+w=-16$ and $vw=3\cdot 5=15$

We see that $v=-1, w=-15$ satisfies both equations, so we write:

$3x^2-x-15x+5=x(3x-1)-5(3x-1)=(x-5)(3x-1)$

(c) We want to find v and w that satisfy: $v+w=7$ and $vw=2\cdot(-4)=-8$

We see that $v=-1, w=8$ satisfies both equations, so we write:

$2x^2-x+8x-4=x(2x-1)+4(2x-1)=(x+4)(2x-1)$

(d) We want to find v and w that satisfy: $v+w=-8$ and $vw=4\cdot(-5)=-20$

We see that $v=-10, w=2$ satisfies both equations, so we write:

$4x^2-10x+2x-5=2x(2x-5)+1(2x-5)=(2x+1)(2x-5)$

The reason we factor quadratics is to make equations easier to solve.
Consider two real numbers α and β with a product of 0.
We know that at least one of $\alpha=0$ or $\beta=0$ (or both) must be true.

This means a factored quadratic function will only equal zero if one of its factors equals zero.

How do we solve $(px+r)(qx+s)=0$?

We set $px+r=0$ or $qx+s=0$ separately and solve for x.
Just remember to move everything to one side and set equal to zero before factoring!

Example 1.9.11

Factor each of the following quadratic expressions.

(a) $x^2+x-20=0$

(b) $2x^2-5x+3=0$

(c) $x^2+49=14x$

(d) $3x(x+3)=7x+1$

Solution

(a) Factor the left side into $(x-4)(x+5)=0$

Set each factor equal to zero separately to get $x=4, x=-5$

(b) Factor the left side into $(2x-3)(x-1)=0$

Set each factor equal to zero separately to get $x=\frac{3}{2}, x=1$

(c) Move everything to the left side to get: $x^2-14x+49=0$

Factor the left side into $(x-7)(x-7)=0$

Set each factor equal to zero separately to get the same answer $x=7$

(d) Move everything to the left side to get: $3x^2+2x-1=0$

Factor the left side into $(3x-1)(x+1)=0$

Set each factor equal to zero separately to get $x=\frac{1}{3}, x=-1$

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CHAPTER OVERVIEW

2: Limits

> *"From a small seed a mighty trunk may grow."*
>
> - Aeschylus

Chapter two invites us to begin our journey into calculus. We will discuss the most powerful tool we will use in this book: limits. Limits give a lot of information about how a function acts not just at one single point, but in the neighborhood around that point. We will see examples of many different kinds of limits, including right-handed and left-handed limits and double sided limits, and lay the foundation for our future work with limits as well. At first we will find the business of calculating limits to be a slow and painful process, but soon afterwards we will find some patterns to help us avoid this calculator intensive chore. These properties of limits will naturally lead us to define functions that are well behaved inside of the limits (polynomials). These polynomials are studied in depth, including factoring them using synthetic division, solving polynomial equations, and solving polynomial inequalities.

2.1: Definition of Limit
2.2: Limit Laws
2.3: Polynomial Functions
2.4: Polynomial Inequalities

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2.1: Definition of Limit

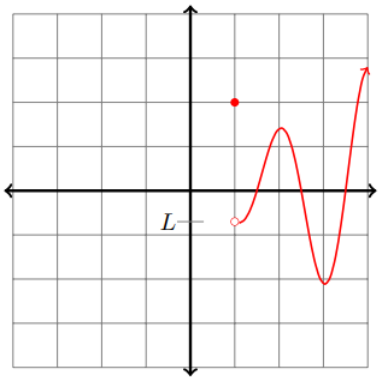
Definition of Limit

For an explicit function $y = f(x)$, when you plug in $x = a$, you get one specific output $f(a)$.

Sometimes, we are curious how a function acts nearby to a instead of directly at a.

This brings us to a very powerful tool of calculus: *limits.*

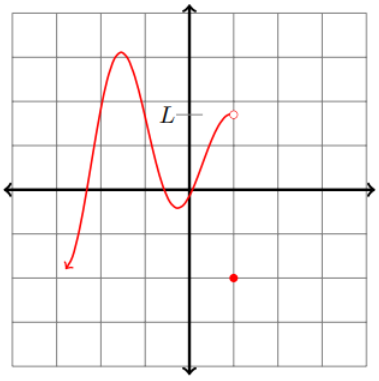
We write $\lim\limits_{x \to a^+} f(x) = L$ *if: when* $x > a$ *is close to* a*, we must have* $f(x)$ *close to* L.



Notice that in the picture above, we have $f(1) = 2$, but nothing nearby to $x = 1$ (like $x = 1.1, 1.02, 1.0000783$, etc) has an output that is close to 2.
Instead, when x is strictly greater than 1 but still very close to 1, most of the outputs are very close to some negative number L.

This number L is our limit, not the value of 2.

We write $\lim\limits_{x \to a^-} f(x) = L$ if when $x < a$ is close to a, we must have $f(x)$ close to L.



Again in the picture above, we have $f(1) = -2$, but nothing nearby to $x = 1$ (like $x = 0.9, 0.99, 0.9999783$, etc) has an output that is close to -2.
Instead, when x is strictly smaller than 1 but still very close to 1, most of the outputs are very close to some positive number L.

This number L is our limit, not the value of -2.

Remember: **the value of $f(a)$ does not influence the limit at $x = a$ from either side**

Since each of $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to a^-} f(x)$ require $x > a$ and $x < a$ respectively:

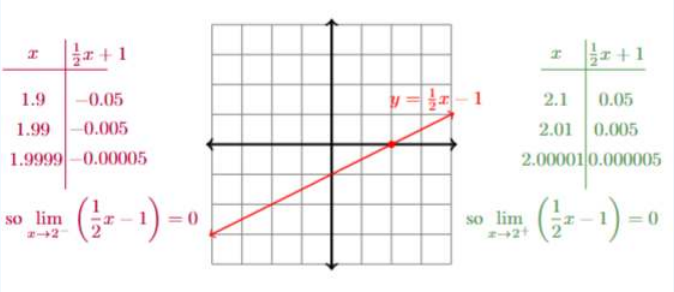
$\lim\limits_{x \to a^-} f(x)$ is the *left-handed limit*, since we are approaching x from numbers on its left side.
$\lim\limits_{x \to a^+} f(x)$ is the *right-handed limit*, since we are approaching x from numbers on the right side.

Example 2.1.1

Evaluate $\lim_{x\to 2^+}\left(\frac{1}{2}x-1\right)$ and $\lim_{x\to 2^-}\left(\frac{1}{2}x-1\right)$

Solution

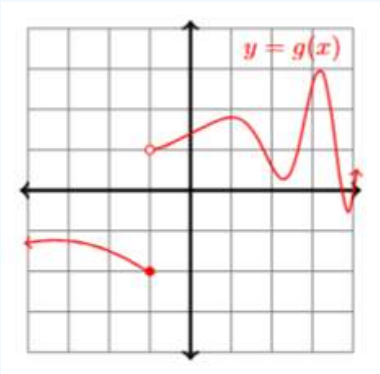
We can sketch the graph and plug in some neighboring values to 2:



In situations like the one above, where the right and left sided limit of a function agree at a given point, we just say $\lim_{x\to 2}\left(\frac{1}{2}x-1\right)=0$

Example 2.1.2

Evaluate $\lim_{x\to -1^+} g(x)$ and $\lim_{x\to -1^-} g(x)$ for the graph below:



Solution

We can see that for numbers to the left of $x=-1$, we can see that we approach the point $(-1,-2)$

Therefore we have $\lim_{x\to -1^-} g(x)=-2$

And for numbers to the right of $x=-1$, the function approaches the point $(-1,1)$

Therefore we have $\lim_{x\to -1^+} g(x)=1$

When the right and left sided limit of a function do NOT agree at a given point as they just did above, we just say $\lim_{x\to -1} g(x)=$DNE (does not exist).

So we can officially say that $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=L$ is the only way for $\lim_{x\to a} f(x)=L$ (and when they don't match, the double-sided limit does not exist.)

Lets try a few quick examples before determining some shortcuts in the next section:

Example 2.1.3

Evaluate $\lim_{x\to 4}(7-3x)$

Solution

$x<4$	$f(x)=7-3x$		$x>4$	$f(x)=7-3x$
3.9	-4.7		4.1	-5.3
3.99	-4.97		4.01	-5.03
3.99999	-4.99997		4.00001	-5.00003

The left columns show us $\lim_{x\to 4^-}(7-3x)=-5$ and the right columns give $\lim_{x\to 4^+}(7-3x)=-5$

Since these two match, we know that $\lim_{x\to 4}(7-3x)=-5$

Example 2.1.4

Evaluate $\lim_{x\to 0}(x^2+2x)$

Solution

$x<0$	$f(x)=x^2+2x$		$x>0$	$f(x)=x^2+2x$
-0.1	-0.19		0.1	0.21
-0.01	-0.0199		0.01	0.0201
-0.0001	-0.000199..		0.0001	0.00020001

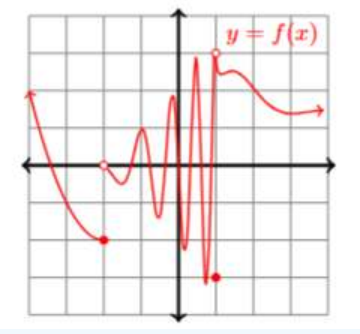
The left columns show us $\lim_{x\to 0^-}(x^2+2x)=0$ and the right columns give $\lim_{x\to 0^+}(x^2+2x)=0$

Since these two match, we know that $\lim_{x\to 0}(x^2+2x)=0$

Sadly for more complicated functions and numbers, this strategy relies a calculator to be effective.

Example 2.1.5

For the function graphed below, evaluate (a) $\lim_{x\to -2} f(x)$ and (b) $\lim_{x\to 1} f(x)$



Solution

(a) Near $x=-2$ we see the graph approaches the point $(-2,-2)$ from the left so $\lim_{x\to -2^-} f(x)=-2$

and the graph approaches the point $(-2,0)$ from the right side, so $\lim_{x\to -2^+} f(x)=0$

Since these two do not match, we know $\lim_{x \to -2} f(x) =$DNE

(b) Near $x = 1$ we see the graph approaches the point $(1, 3)$ from the left so $\lim_{x \to 1^-} f(x) = 3$

and the graph approaches the point $(1, 3)$ from the right side, so $\lim_{x \to 1^+} f(x) = 3$

Since these two match, we can say that $\lim_{x \to 1} f(x) = 3$

When given a graph, calculators can be avoided, but it is difficult to determine your answer if it isn't an integer or if the graph is not perfectly drawn.

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2.2: Limit Laws

Taking limits is a bit cumbersome, and taking something like $\lim_{x\to 4} x^5$ can be daunting without a calculator. Lets examine the properties of limits to see if we can save ourselves some of this work.

Assume that we have two functions $f(x)$ and $g(x)$ and we know $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$

Then we can evaluate a few related limits:

$\lim_{x\to a}[f(x)+g(x)] = F+G$
when x is close to a: the sum of outputs close to F and close to G should be close to $F+G$

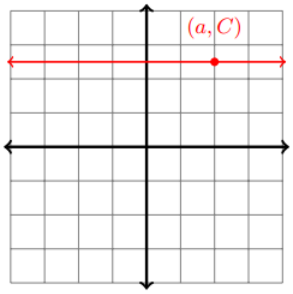
$\lim_{x\to a}[f(x)-g(x)] = F-G$
when x is close to a: the difference of outputs close to F and close to G should be close to $F-G$

$\lim_{x\to a}[f(x)\cdot g(x)] = F\cdot G$
when x is close to a: the product of outputs close to F and close to G should be close to $F\cdot G$

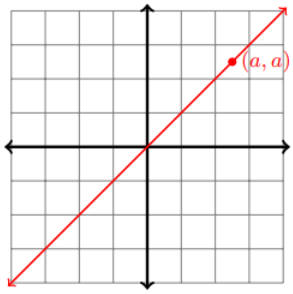
These properties should feel intuitive, and confirm great news: you can distribute a limit across a sum, difference, or product of any number of functions as long as each individual limit exists.
(Note: The statements above hold for any limit: right-handed, left-handed, or two-sided.)

To jump start these limit laws, lets calculate a couple of very useful building blocks for limits by looking at the graphs of $y = C$ (where C is a constant) and $y = x$

$\lim_{x\to a} C = C$



$\lim_{x\to a} x = a$



For these two functions, we can see that taking $\lim_{x\to a} f(x)$ is the same as $f(a)$.

This clearly does not always occur for every function, but it has happened before.

It turns out there are a few more functions we can find that also share this property.

Consider the power function $f(x) = x^n$ for any positive integer n:

$$\lim_{x\to a} x^n = \lim_{x\to a}(x\cdot x\cdots x) = \left(\lim_{x\to a} x\right)\cdot\left(\lim_{x\to a} x\right)\cdots\left(\lim_{x\to a} x\right) = \left(\lim_{x\to a} x\right)^n = a^n$$

So for any a, $\lim_{x\to a} f(x) = f(a)$ for these functions (for both one-sided limits as well)

Example 2.2.1

Evaluate the following limits by plugging in a:

(a) $\lim_{x\to 3} x^4$

(b) $\lim_{x\to -5^+} x^3$

(c) $\lim_{x\to -\sqrt{2}^-} x^{20}$

Solution

These all become very easy:

(a) $\lim_{x\to 3} x^4 = 3^4 = 81$

(b) $\lim_{x\to -5^+} x^3 = (-5)^3 = -125$

(c) $\lim_{x\to -\sqrt{2}^-} x^{20} = \left(-\sqrt{2}\right)^{20} = 1024$

If the function is $f(x) = x^{\frac{m}{n}}$ instead, we can use a similar argument to solve its limit:

Assume $\lim_{x\to a} x^{\frac{m}{n}}$ exists, then consider that $\left(x^{\frac{m}{n}}\right)^n = x^m$

$\left(\lim_{x\to a} x^{\frac{m}{n}}\right)^n = \lim_{x\to a}\left(x^{\frac{m}{n}}\right)^n = \lim_{x\to a} x^m = a^m$

So we n^{th} root both sides to get $\lim_{x\to a} x^{\frac{m}{n}} = a^{\frac{m}{n}}$ (as long as it exists).

Example 2.2.2

Evaluate the following limits by plugging in a and seeing if it exists:

(a) $\lim_{x\to 9} x^{\frac{3}{2}}$

(b) $\lim_{x\to -\frac{1}{8}^+} \sqrt[3]{x}$

(c) $\lim_{x\to -1} x^{\frac{9}{4}}$

Solution

(a) $\lim_{x\to 9} x^{\frac{3}{2}} = 9^{\frac{3}{2}} = \left(\sqrt{9}\right)^3 = (3)^3 = 27$

(b) $\lim_{x\to -\frac{1}{8}^+} \sqrt[3]{x} = \left(-\frac{1}{8}\right)^{\frac{1}{3}} = \sqrt[3]{-\frac{1}{8}} = -\frac{1}{2}$

(c) $\lim_{x\to -1} x^{\frac{9}{4}} = (-1)^{\frac{9}{4}} = \left(\sqrt[4]{-1}\right)^9$ and since $\sqrt[4]{-1}$ does not exist, the limit DNE

Now we will take these power function results and apply the distributive properties of the limit.

As long as $\lim_{x\to a} x^p$ and $\lim_{x\to a} x^q$ both exist (and therefore equal a^p and a^q respectively) we will have

$$\lim_{x\to a} Cx^p = \left(\lim_{x\to a} C\right)\cdot\left(\lim_{x\to a} x^p\right) = C\cdot a^p$$

$$\lim_{x\to a}(x^p + x^q) = \left(\lim_{x\to a} x^p\right) + \left(\lim_{x\to a} x^q\right) = a^p + a^q$$

$$\lim_{x\to a}(x^p - x^q) = \left(\lim_{x\to a} x^p\right) - \left(\lim_{x\to a} x^q\right) = a^p - a^q$$

Example 2.2.3

Evaluate the following limits by plugging in a and seeing if it exists:

(a) $\lim_{x\to 2}\left(x^5 - x^4\right)$

(b) $\lim_{x\to 64^-}\left(\frac{3}{8}x^{\frac{1}{3}}\right)$

(c) $\lim_{x\to -1}\left(x^{\frac{2}{3}}+x^{\frac{3}{4}}\right)$

Solution

(a) $\lim_{x\to 2}\left(x^5-x^4\right)=2^5-2^4=16$

(b) $\lim_{x\to 64^-}\left(\frac{3}{8}x^{\frac{1}{3}}\right)=\frac{3}{8}\cdot 64^{\frac{1}{3}}=\frac{3}{2}$

(c) $\lim_{x\to -1}\left(x^{\frac{2}{3}}+x^{\frac{3}{4}}\right)=(-1)^{\frac{2}{3}}+(-1)^{\frac{3}{4}}$ and since $(-1)^{\frac{3}{4}}$ does not exist, the limit DNE

As a more general rule, we can actually say that:

$\lim_{x\to a} C_1x^{p_1}+C_2x^{p_2}+\cdots+C_nx^{p_n}=C_1a^{p_1}+C_2a^{p_2}+\ldots C_na^{p_n}$
(as long as all parts of the answer exist)

One good way to ensure the limit does exist is to ensure all powers p_i are non-negative integers. In other words, if we have any function $f(x)=c_0+c_1x^1+c_2x^2+\cdots+c_{n-1}x^{n-1}+c_nx^n$ where $c_0,c_1,\ldots,c_n$ are all constants, then we know $\lim_{x\to a} f(x)=f(a)$

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2.3: Polynomial Functions

A *polynomial* is a function $f(x) = c_0 + c_1x^1 + c_2x^2 + \cdots + c_{n-1}x^{n-1} + c_nx^n$
(where $c_0, c_1, \ldots, c_n$ are all constants.)

Notice that a polynomial can be described as any sum or difference of finitely many constant multiples of power functions (with non-negative integer powers).

We say the largest of these powers is called the *degree* of the polynomial.

Example 2.3.1

Determine if the functions below are polynomials, and if so, find their degree.

(a) $a(x) = 10x^6 + 3x^7 - \sqrt[3]{2}$

(b) $b(x) = 2x^2 - x^{-1}$

(c) $c(x) = 9\sqrt{x} - x$

Solution

(a) The powers of x that show up in $a(x)$ are $6, 7, 0$, all non-negative integers.

Therefore, $a(x)$ is a polynomial of degree 7.

(b) The powers of x that show up in $b(x)$ are $2, -1$, all integers but not all non-negative!

Therefore, $b(x)$ is NOT a polynomial.

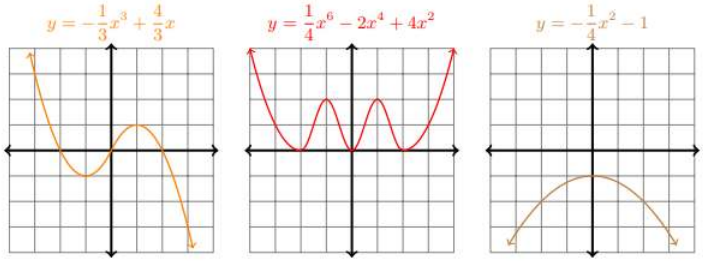
(c) The powers of x that show up in $c(x)$ are $\frac{1}{2}, 1$, not all integers!

Therefore, $c(x)$ is NOT a polynomial.

Notice that all linear functions (which look like $f(x) = mx + b$) are polynomials of degree 1,
and all quadratic functions ($f(x) = ax^2 + bx + c$) are polynomials of degree 2.

The reason we care about any of these functions, is because as we discovered in the previous section, we can ensure that for any real number a we have $\lim_{x \to a} f(x) = f(a)$.

There are many different shapes of graphs for polynomials:



The larger the degree of a polynomial, the more chances its graph has to change direction.
we will get into the specifics of how in the future).

For a given polynomial if plugging in $x = r$ makes $f(r) = 0$, we say that r is a *root* of $f(x)$.
In this case we know that $(x - r)$ is a factor of $f(x)$, just like when we solved quadratic equations.

Solving polynomial equations is almost always done via factoring.
This can be more challenging when the degree is large, but the strategy is always the same.

We use polynomial long division to check if $(x - r)$ divides $f(x)$ perfectly (with no remainder).

Is $x = 3$ a root of $f(x) = x^2 + 2x - 15$? Lets check a few different ways:

Method 1: Plug it in

Plugging in $x = 3$ gives $f(3) = 3^2 + 2 \cdot 3 - 15 = 0$
That means that yes, 3 is a root of $f(x)$, so $(x-3)$ is a factor of $f(x)$

Method 2: Polynomial Long Division

We attempt $f(x) \div (x-3)$ with long division:

$$
\begin{array}{rl}
 & 1x+5 \\
x-3 \;\big| & 1x^2+2x-15 \\
 & -(1x^2-3x) \\
 & \quad 5x-15 \\
 & \quad -(5x-15) \\
 & \quad\quad 0
\end{array}
$$

(what multiple of $x-3$ will cancel out $1x^2$?)
(it was $1x$, so subtract that multiple)
(what multiple of $x-3$ will cancel $5x$?)
(it was 5, so subtract that multiple)
(remainder zero means it divided perfectly!)

Since the remainder was zero, we know that $f(x) \div (x-3) = x+5$ exactly,
r equivalently we can say that $f(x) = (x+5)(x-3)$

Method 3: Synthetic division

Synthetic division is a shortcut for polynomial long division when you are dividing by $x - r$.
The idea is you put your 'guess' for the root r in the box on the top left, and then the coefficients of the polynomial you are dividing in decreasing power order along the top row.

After it is set up, you bring down the first coefficient, multiply it by the number in the box, and add that to the next coefficient in line. Then you repeat the process until you get to the far right side, and the final number should be your remainder.

So to synthetically divide $(1x^2+2x-15) \div (x-3)$ this looks like:

3	1	2	−15
		$3 \cdot 1 = 3$	$3 \cdot 5 = 15$
	1	$2+3=5$	$-15+15=0$

Please take some time to compare synthetic division and polynomial long division above.
The numbers have been color coded to help show you how the two are related)

We can see that the numbers that end up on the very bottom row correspond to the coefficients of the answer when we did long division, and the bottom right number 0 is the remainder.
So our answer is once again $f(x) \div (x-3) = 1x+5$

Lets try a practice example of solving polynomial equations by factoring:

Example 2.3.2

Solve $x^3 - 3x^2 - 10x + 24 = 0$

Solution

We set up synthetic division, and guess a root until we succeed.

Starting with $x = 1$:

1	1	−3	−10	24
		1	−2	−12
	1	−2	−12	12

Looks like $x = 1$ is not a root, so $(x-1)$ is not a factor of this polynomial.

Try $x = -1$ next:

−1	1	−3	−10	24
		−1	4	6
	1	−4	−6	30

Looks like $x = -1$ is not a root, so $(x+1)$ is not a factor of this polynomial.

Try $x = 2$ next:

2	1	−3	−10	24
		2	−2	24
	1	−1	−12	0

Since the remainder is 0, we know $x = 2$ is a root, and $(x - 2)$ is a factor:

$x^3 - 3x^2 - 10x + 24 = (x^2 - x - 12)(x - 2)$

We can continue using synthetic division if we like, but once we reach a quadratic polynomial it is often easier to factor those with our other methods.

$x^2 - x - 12 = (x - 4)(x + 3)$

This means our original equation can be re-written as $(x - 4)(x + 3)(x - 2) = 0$

Solving each factored equation separately gives us solutions $x = 4, -3, -2$

Example 2.3.3

Given the function $f(x) = x^4 - 2x^3 - 6x^2 + 18x$ for which value(s) of a do we know $\lim_{x \to a} f(x) = 27$?

Solution

Since $f(x)$ is a polynomial, we know $\lim_{x \to a} f(x) = f(a)$

So really we are trying to solve $a^4 - 2a^3 - 6a^2 + 18a = 27$

Subtract the 27 to the other side to make it all equal to zero

$a^4 - 2a^3 - 6a^2 + 18a - 27 = 0$

Again we guess a root until we succeed. $a = 1, -1, 2, -2$ all fail (try them yourself), but for $a = 3$:

3	1	−2	−6	18	−27
		3	3	−9	27
	1	1	−3	9	0

Now we have $(a^3 + a^2 - 3a + 9)(a - 3) = 0$

We keep factoring the $a^3 + a^2 - 3a + 9$ polynomial with synthetic division. Since $a = \pm 1, \pm 2$ failed earlier, we know they wont work here. You can guess $a = 3$ again, but unfortunately in this instance it does not work a second time. Next we try $a = -3$:

−3	1	1	−3	9
		−3	6	−9
	1	−2	3	0

Now we have $(a^2 - 2a + 3)(a + 3)(a - 3) = 0$

The remaining quadratic $(a^2 - 2a + 3)$ is prime (does not factor) but we can still solve for when it equals zero using the quadratic formula.

$$a^2 - 2a + 3 = 0 \longrightarrow a = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 3}}{2}$$

Inside the radical is negative, so no real solutions exist for this factor of our polynomial. The other two factors can be set equal to zero and solved to give our final answer, which is $a = \pm 3$

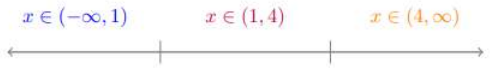
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2.4: Polynomial Inequalities

Factoring polynomials is an important skill, not just for solving equations, but also when solving inequalities involving polynomials of degree 2 or larger.

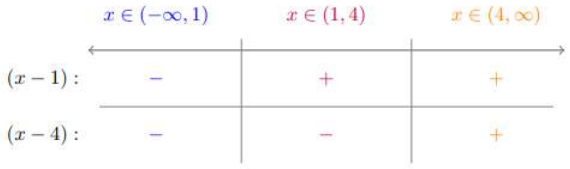
If we wanted to solve an inequality like $(x-1)(x-4)>0$,

The number line shows that for $x \neq 1, 4$ there are only 3 different things that can happen:



The factors $(x-1)$ and $(x-4)$ will each individually remain positive or negative for any value of x in a given interval.

So we turn our number line above into a *sign-chart*, by denoting if each factor is positive or negative on each interval like so:



Since we can see sign for each individual factor, that tells us the sign of their product.

In $x \in (-\infty, 1)$, both factors are negative so the product $(x-1)(x-4)$ must be positive.
In $x \in (1, 4)$, one factor is +, the other -, so the product $(x-1)(x-4)$ must be negative.
In $x \in (4, \infty)$, both factors are positive so the product $(x-1)(x-4)$ must be positive.

Lastly, we can tell for the actual roots $x=1$ and $x=4$ whether the inequality is true or not very easily (it is not, since plugging in either would give $0>0$ in this example)
So we can finally answer the question of when $(x-1)(x-4)>0$, and that is: $x \in (-\infty, 1) \cup (4, \infty)$

This strategy works for any degree polynomial. As long as one side of your inequality is zero, you can factor the other side and make a sign chart to determine where the function is positive or negative. Once you have the sign chart, you can manually check any roots in your inequality and collect all of the successful values at the end to put your answer into interval notation.

As with everything in life, the more you practice this the easier it becomes.

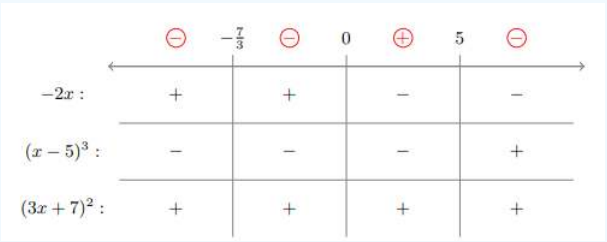
Example 2.4.1

Solve $-2x(x-5)^3(3x+7)^2<0$

Solution

Again we will make a sign chart, with lines at each root of the polynomial

We can easily see that $x=0, 5, -\frac{7}{3}$ are the roots for each factor.



Since the inequality is strict, none of the roots will satisfy it

That means our solution should be any interval which is overall − in the top row.

$x \in \left(-\infty, -\frac{7}{3}\right) \cup \left(-\frac{7}{3}, 0\right) \cup (5, \infty)$

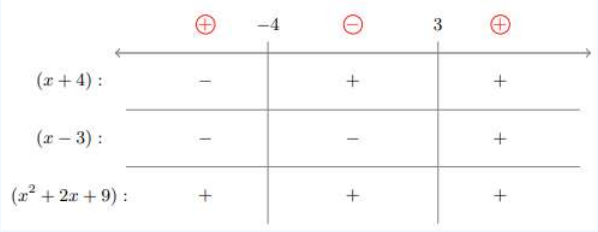
Example 2.4.2

Solve $(x+4)(x-3)(x^2+2x+9) \leq 0$

Solution

Again we will make a sign chart, with lines at each root of the polynomial

We can easily see the roots $x = -4, 3$ but for $x^2+2x+9=0$, we must the quadratic formula. Spoiler: there are no other real roots



Since the inequality is not strict, all of the roots will satisfy it $(0 \leq 0)$

That means our solution should include any root and any interval which is overall $-$ in the top row.

$x \in [-4, 3]$

If you are unlucky, you have to do the factoring yourself first:

Example 2.4.3

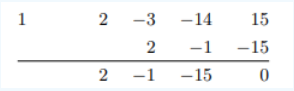
Solve $2x^3+15 \geq 3x^2+14x$

Solution

Move everything to one side to compare to zero

$2x^3-3x^2-14x+15 \geq 0$

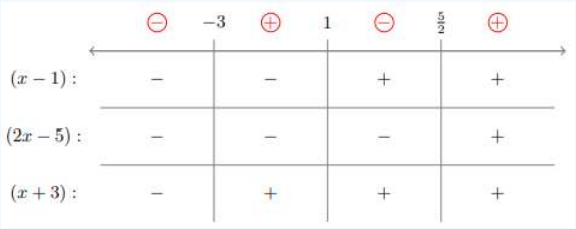
We factor by using synthetic division. We get lucky with $x=1$:



So we factor to $(x-1)(2x^2-x-15) \geq 0$

The quadratic term factors nicely to give $(x-1)(2x-5)(x+3) \geq 0$

We can easily see that $x = 1, \frac{5}{2}, -3$ are the roots



Since the inequality is not strict, each of the roots will satisfy it.
Our solution should be any interval which is overall $+$ in the top row.

$x \in [-3, 1] \cup \left[\frac{5}{2}, \infty\right)$

Any polynomial inequality can be solved as long as you are able to factor it, solve for the roots, and draw a sign chart. Just remember to check if each endpoint of your interval satisfies the given inequality to determine if you should use "[" and "]" or "(" and ")"}

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CHAPTER OVERVIEW

3: Rational Functions

> *"In everything one thing is impossible: rationality." - Friedrich Nietzsche*

Chapter three attempts to tackle the slightly sticky interaction between limits and division. Since division by zero is a generally frowned upon mathematical operation, we run into a few new wrinkles that limits can help us to understand better.

We begin with the limit law for division of two functions, which immediately inspires the fraction version of polynomials: rational functions. We explore how to add, subtract, multiply, divide, and find domains of these functions. This allows us to also solve equations and inequalities involving rational functions

Limits of rational functions can be quite easy as long as we avoid division by zero. In the situations where the limit brings a denominator to zero, we discuss different methods to decipher the limit value.

These limits tell us about the shape of the graph for a given rational function (which will be useful later), but also help us better define when a given function has vertical or horizontal asymptote(s).

Click through the following links to read this chapter.

3.1: Domains of Functions
3.2: Rational Functions
3.3: Using Rational Functions
3.4: Limits and Rational Functions
3.5: Asymptotes

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3.1: Domains of Functions

For any function $f(x)$, we use $f(a)$ to denote plugging in the specific value of $x = a$. Depending on the function (and the value of a), this may not make sense!

Consider the power function $f(x) = x^{\frac{1}{2}}$

$f(4) = 4^{\frac{1}{2}} = 2$ makes sense, $f(0) = 0^{\frac{1}{2}} = 0$ makes sense, but $f(-1) = (-1)^{\frac{1}{2}}$ does not!

We say the collection of all x values that make sense to plug into $f(x)$ is called the *domain* of $f(x)$

Example 3.1.1

Determine the domain of the following functions in interval notation:

(a) $f(x) = 2x - 1$

(b) $g(x) = x^{\frac{2}{3}}$

(c) $h(x) = x^{\frac{3}{4}}$

Solution

(a) We know that $f(a) = 2a - 1$ exists for all real numbers a, so the domain is all real numbers: $D_f = (-\infty, \infty)$

(b) We know that $g(a) = a^{\frac{2}{3}}$ exists for all real numbers a, so the domain is all real numbers: $D_g = (-\infty, \infty)$

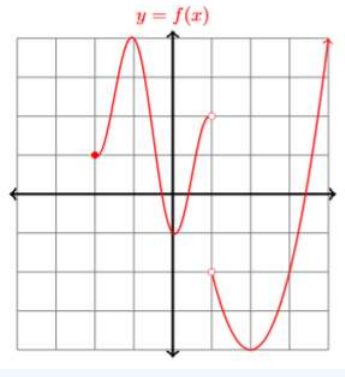
(c) We know that $h(a) = a^{\frac{3}{4}}$ exists only for real numbers $a \geq 0$, so the domain is the interval: $D_h = [0, \infty)$

Similarly, if you are given a graph of $y = f(x)$, you can deduce the domain by realizing that every closed dot on your graph is a point where $(a, f(a))$ exists.

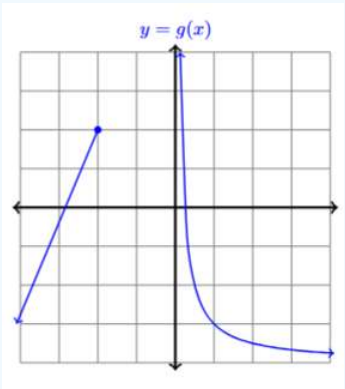
Example 3.1.2

For the graphs below, find the domain of each function below in interval notation.

(a)



(b)



Solution

(a) We see the from its graph that $f(a)$ exists for all $a \geq -2$ except for $a = 1$

Therefore its domain is: $D_f = [-2, 1) \cup (1, \infty)$

(b) We see the from its graph that $g(a)$ exists for all $a \leq -2$ or $a > 0$

Therefore its domain is: $D_g = (-\infty, -2] \cup (0, \infty)$

Remember that polynomials were functions $f(x)$ that satisfied $\lim_{x \to a} f(x) = f(a)$ for any real number a.

That means that the domain of any polynomial is always $(-\infty, \infty)$

Sometimes a function can be forced to be restricted onto a set $x \in D$.
(we call this set D the *restricted domain*)

This just means that even if a number outside of D makes sense to plug into $f(x)$, we wont do so.

Example 3.1.3

Sketch the graphs of the following functions with restricted domains:

(a) $f(x) = x^2 - 1$ for $x \in [-2, 1)$

(b) $g(x) = -\frac{1}{3}x + 2$ for $x \in (-3, \infty)$

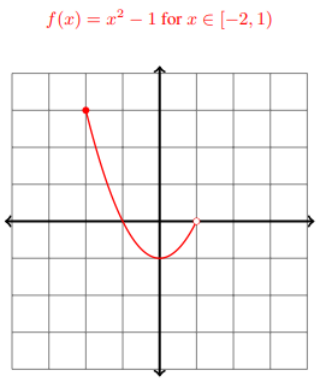
(c) $h(x) = 2x^2 + 4x$ for $x \leq -1$

(d) $j(x) = 2x$ for $x \neq \frac{1}{2}$

Solution

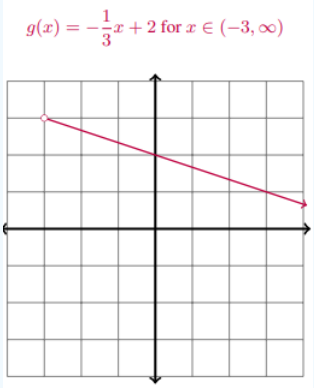
(a) We first draw the parabola with vertex at $(0, -1)$ and then erase all points left of $x = -2$ and all points right of $x = 1$

Since $x = 1$ is not in the domain, we replace the point $(1, 0)$ with an empty dot.



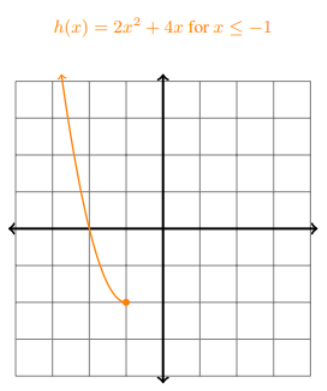
(b) We first draw the line $y = -\frac{1}{3}x + 2$ and then erase all points left of $x = -3$

Since $x = -3$ is not in the domain, we replace the point $(-3, 3)$ with an empty dot.

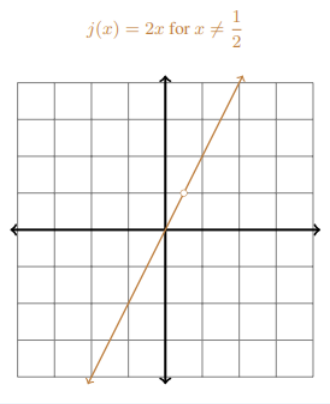


(c) We first put $y = 2x^2 + 4x$ into vertex form $y = 2(x + 1)^2 - 2$

Now we graph it with vertex at $(-1, -2)$ and then erase all points right of $x = -1$



(d) We first draw the line $y = 2x$ and then erase only the point $\left(\frac{1}{2}, 1\right)$ and replace it with an open dot



Every function has a (natural or restricted) domain, and finding it is typically a great starting point when dealing with any new type of function we encounter.

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3.2: Rational Functions

Remember, the limit laws for addition, subtraction, and multiplication led us to the definition of polynomials (and allowed us to say $\lim_{x\to a} p(x) = p(a)$ holds for all polynomials $p(x)$)

We can add another limit law, this time concerning division:

If $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G \neq 0$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{F}{G}$

(don't worry, we will cover what happens when $G = 0$ later in this chapter)

Notice that if both $f(x)$ and $g(x)$ are polynomials, then we know both limits will exist for any choice of a, so we should consider the functions of the form $R(x) = \dfrac{f(x)}{g(x)}$. Functions of this type (a polynomial divided by a polynomial) are called *rational functions*.

Remember the domain of any polynomial is always $(-\infty, \infty)$, so the only way for $R(x)$ not to exist is for the denominator $g(x)$ to equal zero.

The domain of any rational function $R(x) = \dfrac{f(x)}{g(x)}$ is always $D = \{g(x) \neq 0\}$

Example 3.2.1

Find the domains of the rational functions below:

(a) $a(x) = \dfrac{x^2+3x+2}{x^2-4}$

(b) $b(x) = \dfrac{x}{2x^3+3x^2-8x+3}$

Solution

(a) We solve for when the denominator is not equal to zero: $x^2 - 4 \neq 0$

Just like other inequalities, we do this by factoring the left side

$(x-2)(x+2) \neq 0$

Clearly then, we need $x \neq 2, x \neq -2$ as our domain.

In interval notation, this looks like $D_a = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

(b) We solve for when the denominator is not equal to zero: $2x^3 + 3x^2 - 8x + 3 \neq 0$

After some synthetic division this factors into $(x-1)(x+3)(2x-1) \neq 0$

This means our domain requires $x \neq 1, -3, \dfrac{1}{2}$

In interval notation, this looks like $D_b = (-\infty, -3) \cup \left(-3, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \cup (1, \infty)$

For $a(x)$ above, notice that the numerator can also be factored:

$a(x) = \dfrac{(x+2)(x+1)}{(x-2)(x+2)}$ has domain $D_a = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$,

but reducing it to $A(x) = \dfrac{x+1}{x-2}$ would change its domain to $D_A = (-\infty, 2) \cup (2, \infty)$.
This is a different function.

This is worth remembering: **reducing a rational function changes its domain**

Since rational functions are just like fractions, we can multiply/divide them using the same rules:

$$\frac{A}{B}\cdot\frac{C}{D}=\frac{A\cdot C}{B\cdot D} \text{ and } \frac{A}{B}\div\frac{C}{D}=\frac{A\cdot D}{B\cdot C}$$

Just remember that whenever you reduce $(x-r)$ from the top and bottom, you may change your domain at $x=r$. Because of this, it is good practice to write down $x\neq r$ after reducing.

Example 3.2.2

Define $f(x)=\dfrac{x^2+5x+6}{x^2+5x+4}$ and $g(x)=\dfrac{x^2+4x+3}{x^2+6x+8}$. Evaluate

(a) $f(x)\cdot g(x)$

(b) $f(x)\div g(x)$

Solution

(a) First we factor all terms in $f(x)\cdot g(x)$:

$$f(x)\cdot g(x)=\frac{(x+2)(x+3)}{(x+4)(x+1)}\cdot\frac{(x+1)(x+3)}{(x+2)(x+4)}=\frac{(x+1)(x+2)(x+3)^2}{(x+1)(x+2)(x+4)^2}$$

Reduce away $(x+1)(x+2)$

$$f(x)\cdot g(x)=\frac{(x+3)^2}{(x+4)^2} \text{ for } x\neq -1,-2$$

(b) First we factor all terms in $f(x)\div g(x)$:

$$f(x)\div g(x)=\frac{(x+2)(x+3)}{(x+4)(x+1)}\div\frac{(x+1)(x+3)}{(x+2)(x+4)}=\frac{(x+2)^2(x+3)(x+4)}{(x+1)^2(x+3)(x+4)}$$

Reduce away $(x+3)(x+4)$

$$f(x)\div g(x)=\frac{(x+2)^2}{(x+1)^2} \text{ for } x\neq -3,-4$$

Similarly, we add or subtract rational functions by using common denominators:

Example 3.2.3

Simplify each expression into a single fraction:

(a) $\dfrac{2}{x+5}+\dfrac{x}{x-3}$

(b) $\dfrac{3}{x^2+x-2}-\dfrac{2}{x^2+2x}$

Solution

(a) The common denominator is $(x+5)(x-3)$, so multiply each fraction by the term that reaches this denominator:

$$\frac{2}{x+5}\cdot\frac{(x-3)}{(x-3)}+\frac{x}{x-3}\cdot\frac{(x+5)}{(x+5)}$$

$$=\frac{2(x-3)+x(x+5)}{(x+5)(x-3)}=\frac{x^2+7x-6}{(x+5)(x-3)}$$

(b) First we factor the denominators of the original expression

We see the common denominator is $x(x+2)(x-1)$, so multiply each fraction by the term that reaches this denominator:

$$\frac{3}{(x+2)(x-1)}-\frac{2}{x(x+2)}=\frac{3}{(x+2)(x-1)}\cdot\frac{x}{x}-\frac{2}{x(x+2)}\cdot\frac{(x-1)}{(x-1)}$$

$$= \frac{3x-2(x-1)}{x(x+2)(x-1)} = \frac{x+2}{x(x+2)(x-1)}$$

We can reduce away the $(x+2)$ as long as we update our domain

So ultimately $\frac{3}{x^2+x-2} - \frac{2}{x^2+2x} = \frac{1}{x(x-1)}$ for $x \neq -2$

Generally the sum, difference, product, or quotient of two rational functions will equal a rational function, but they can look a bit messy when written as fractions within fractions.

We call these types of expressions *complex fractions*.

For example, $\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{2}{5}}$ is a complex fraction, and you can separately simplify the numerator $\frac{1}{3}+\frac{1}{2} = \frac{5}{6}$ and the denominator $1-\frac{2}{5} = \frac{3}{5}$

And then replace them in the original expression, leading to fraction division:

$$\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{2}{5}} = \frac{\frac{5}{6}}{\frac{3}{5}} = \frac{5}{6} \div \frac{3}{5} = \frac{25}{18}$$

Alternatively, you can multiply the original complex fraction by the common denominator of all of the interior fractions to eliminate them (like we do for linear equations with fractions)

$$\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{2}{5}} \cdot \frac{30}{30} = \frac{\frac{1}{3}\cdot 30 + \frac{1}{2}\cdot 30}{1\cdot 30 - \frac{2}{5}\cdot 30} = \frac{10+15}{30-12} = \frac{25}{18}$$

Example 3.2.4

Simplify the following complex fractions into rational functions and state their domain.

(a) $f(x) = \dfrac{\frac{6}{x+1}-2}{x-2}$

(b) $g(x) = \dfrac{x+\frac{3}{x+4}}{\frac{1}{x}+1}$

Solution

(a) Our first step is to multiply the top and bottom of the big fraction by $(x+1)$:

$$\frac{\frac{6}{x+1}-2}{x-2} \cdot \frac{(x+1)}{(x+1)}$$

$$= \frac{\frac{6}{x+1}\cdot(x+1) - 2\cdot(x+1)}{(x-2)\cdot(x+1)} \text{ for } x \neq -1$$

$$= \frac{6-2(x+1)}{(x-2)(x+1)} = \frac{-2x+4}{(x-2)(x+1)} = \frac{-2(x-2)}{(x-2)(x+1)}$$

Lastly, we reduce away the $(x-2)$ term and remind ourself of our domain:

$= \frac{-2}{x+1}$ for $x \neq -1, 2$, or in interval notation: $D_f = (-\infty,-1)\cup(-1,2)\cup(2,\infty)$

(b) Our first step is to multiply the top and bottom of the big fraction by $x(x+4)$:

$$= \frac{x+\frac{3}{x+4}}{\frac{1}{x}+1} \cdot \frac{x(x+4)}{x(x+4)}$$

$$= \frac{x \cdot x(x+4) + \frac{3}{x+4} \cdot x(x+4)}{\frac{1}{x} \cdot x(x+4) + 1 \cdot x(x+4)}$$

$$= \frac{x^2(x+4)+3x}{(x+4)+x(x+4)} = \frac{x^3+4x^2+3x}{x^2+5x+4} = \frac{x(x+3)(x+1)}{(x+1)(x+4)}$$

Lastly, we reduce away the $(x+1)$ term and remind ourself of our domain:

$= \frac{x(x+3)}{x+4}$ for $x \neq -4, -1, 0$, or in interval notation: $D_g = (-\infty, -4) \cup (-4, -1) \cup (-1, 0) \cup (0, \infty)$

So when taking the limit of the rational function $R(x) = \frac{f(x)}{g(x)}$ we have:

$$\lim_{x \to a} R(x) = \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)} = R(a) \quad \text{as long as } g(a) \neq 0$$

This means that for any a in its domain, we know $\lim_{x \to a} R(x) = R(a)$. The same can be said about complex fractions:

Example 3.2.5

Evaluate each limit below exactly.

(a) $\lim_{x \to 3} \frac{x^2+1}{x-2}$

(b) $\lim_{x \to -1^+} \frac{3-x^3}{5x^7+3x^2}$

(c) $\lim_{x \to 0^-} \frac{\frac{9}{x+3} - \frac{2}{x-1}}{x^2 + \frac{1}{x^2+1}}$

(d) $\lim_{t \to \frac{1}{2}} \frac{\frac{1}{2t+1} - 1}{\frac{1}{t} + 6t}$

Solution

(a) Note $\alpha(x) = \frac{x^2+1}{x-2}$ is a rational function and its domain is $D_\alpha = (-\infty, 2) \cup (2, \infty)$

Since 3 is in the domain of $\alpha(x)$ we know that $\lim_{x \to 3} \alpha(x) = \alpha(3) = \frac{3^2+1}{3-2} = 10$

(b) Note $\beta(x) = \frac{3-x^3}{5x^7+3x^2}$ is a rational function and its domain is $D_\beta = \{5x^6+3x^2 \neq 0\}$

Since $\beta(-1) = \frac{3-(-1)^3}{5(-1)^7+3(-1)^2} = -2$ is in the domain of $\beta(x)$ we know that $\lim_{x \to -1^+} \beta(x) = \beta(-1) = -2$

(c) Since $\gamma(x) = \frac{\frac{9}{x+3} - \frac{2}{x-1}}{x^2 + \frac{1}{x^2+1}}$ is a complex fraction, and

$\gamma(0) = \frac{\frac{9}{0+3} - \frac{2}{0-1}}{0^2 + \frac{1}{0^2+1}} = \frac{3-(-2)}{0+1} = 5$ exists, we know that

$\lim_{x \to 0^-} \gamma(x) = \gamma(0) = 5$

(d) Finally $\delta(t) = \dfrac{\frac{1}{2t+1} - 1}{\frac{1}{t} + 6t}$ is a complex fraction, and

$\delta\left(\dfrac{1}{2}\right) = \dfrac{\frac{1}{2\cdot\frac{1}{2}+1} - 1}{\frac{1}{\frac{1}{2}} + 6\cdot\frac{1}{2}} = \dfrac{\frac{1}{2} - 1}{2+3} = -\dfrac{1}{5}$ exists, we know that

$\lim\limits_{t\to\frac{1}{2}} \delta(t) = \delta\left(\dfrac{1}{2}\right) = -\dfrac{1}{5}$

Limits of rational and complex functions are very simple as long as you don't end up dividing by zero!

For any function we can construct using the sum, difference, product, or quotient of integer powers of x, we have $\lim\limits_{x\to a} f(x) = f(a)$ whenever a is in the domain of $f(x)$.

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3.3: Using Rational Functions

While there is still more to explore when taking the limits of rational functions, we must first become familiar with how we can use them in equations and inequalities.

Remember earlier when solving a linear equations containing any fractions, we multiplied both sides by the same number (the common denominator) to eliminate all of the fractions:

$$\frac{1}{2}x+1=\frac{1}{3}x-2$$

$$\frac{1}{2}x\cdot 6+1\cdot 6=\frac{1}{3}x\cdot 6-2\cdot 6$$

$$3x+6=2x-12$$

The end result is a new (nicer) equation with the same solutions as the original (which has solution $x=-18$)

A *rational equation* is any equation involving rational functions.

We solve these just like we do above, by multiplying by the common denominator to eliminate all fraction terms. The only thing we must be careful of, is to ensure that our solutions are in the domain of any function involved in the original equation.

Example 3.3.1

Find all solutions to $\frac{2}{x+3}-\frac{3}{x-2}=2$

Solution

We start by multiplying all terms by the common denominator: $(x+3)(x-2)$

$$\frac{2}{x+3}\cdot(x+3)(x-2)-\frac{3}{x-2}\cdot(x+3)(x-2)=2\cdot(x+3)(x-2)$$

This simplifies mightily into:

$$2(x-2)-3(x+3)=2(x+3)(x-2)$$

Distribute and collect like terms to get

$$-x-13=2x^2+2x-12$$

Since this is quadratic, we move everything to one side and then factor

$$0=2x^2+3x+1$$

$$0=(2x+1)(x+1)$$

Therefore our final solutions are $x=-\frac{1}{2},-1$

Since both of these values make sense in the original rational equation, they are both solutions.

In this example, the domain of the functions on the left side of the original equation did not effect our final answer, but it is always important to check!

Here are some examples where the domain is important:

Example 3.3.2

Find all solutions to $\frac{3x}{x+4}+\frac{4x}{x-3}=\frac{84}{x^2+x-12}$

Solution

We start by multiplying all terms by the common denominator: $(x-3)(x+4)$

$$\frac{3x}{x+4}(x-3)(x+4)+\frac{4x}{x-3}(x-3)(x+4)=\frac{84}{(x+4)(x-3)}(x-3)(x+4)$$

This simplifies into:

$3x(x-3)+4x(x+4)=84$

Distribute and collect like terms to one side to get

$7x^2+7x-84=0$

Finally we factor

$7(x-3)(x+4)=0$

Therefore our possible solutions are $x=-4,3$

Since neither of these values make sense in the original rational equation, there are no solutions: $\emptyset$

Example 3.3.3

Find all solutions to $\frac{x}{x+4}+\frac{4}{x-4}=\frac{x^2+16}{x^2-16}$

Solution

We start by multiplying all terms by the common denominator: $(x-2)(x+2)$

$$\frac{x}{x+4}(x-4)(x+4)+\frac{4}{x-4}(x-4)(x+4)=\frac{x^2+16}{(x-4)(x+4)}(x-4)(x+4)$$

This simplifies into:

$x(x-4)+4(x+4)=x^2+16$

Distribute and collect like terms to get

$x^2+16=x^2+16$

This equation is always true regardless of the value of x

Since $x=4,-4$ are not allowed in the original equation, these are the only numbers not in our solution set:

$x\in(-\infty,-4)\cup(-4,4)\cup(4,\infty)$

Besides equations, we will also sometimes need to solve inequalities involving rational functions. Thankfully, these are done in a very similar fashion to quadratic inequalities: sign charts!

Example 3.3.4

Find the solutions for the following inequalities

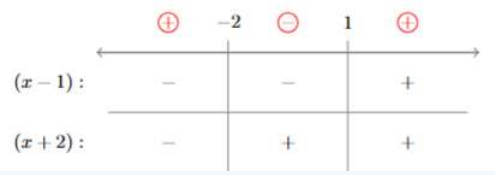
(a) $\frac{x-1}{x+2}>0$

(b) $\frac{x-1}{x+2}\leq 0$

Solution

(a) First we can see that the roots of the numerator are $x=1$ and the roots of the denominator are $x=-2$

Now we set up a sign chart separated by these roots

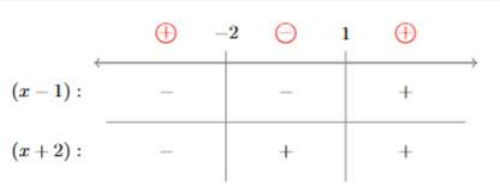


Since the inequality is strict, the roots of the top do not satisfy it.
The roots of the bottom are not in the domain, so they will not satisfy it either.

Our solution should be any interval which is overall $+$ in the top row.
$x \in (-\infty, -2) \cup (1, \infty)$

(b) Again we can see that the roots of the numerator are $x = 1$ and the roots of the denominator are $x = -2$

Now we set up a sign chart separated by these roots



Since the inequality is not strict, the roots of the top do satisfy it.
The roots of the bottom are not in the domain, so they will not satisfy the inequality.

Our solution should be any interval which is overall $-$ in the top row and any roots of the top.

$x \in (-2, 1]$

Just make sure you are always comparing to zero on one side, and that you can completely factor the top and bottom of the other side. Once you have them factored, you can make a sign chart separated by the roots of the top and bottom polynomials like we did in the last examples.

Example 3.3.5

Find all solutions to $\dfrac{20x - 8x^2}{x^2 - 4} \geq -x$

Solution

First we subtract everything to the left side and find a common denominator:

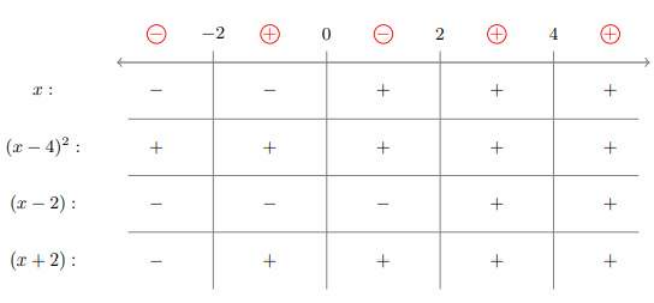
$$\frac{(20x - 8x^2) + x(x^2 - 4)}{x^2 - 4} \geq 0$$

We can simplify and factor the top and bottom of this fraction

$$\frac{x(x-4)^2}{(x-2)(x+2)} \geq 0$$

The roots of the numerator are $x = 0, 4$ the roots of the denominator are $x = 2, -2$

We make a sign chart separated by these roots:



Since the inequality is not strict, the roots of the top do satisfy it.
The roots of the bottom are not in the domain, so they will not satisfy it either.

Our solution should be any interval which is overall $+$ in the top row and the roots of our numerator.

$x \in (-2, 0] \cup (2, \infty)$

Solving equations and inequalities of rational functions is very similar to our methods of solving them for polynomials. The only thing we must remember, is to make sure that we never use any answers that cause us to divide by zero in any expression along the way.

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3.4: Limits and Rational Functions

So we have seen that for any rational function $R(x)$ with domain D, if $a \in D$ we must have $\lim_{x \to a} R(x) = R(a)$

This still leaves the question of what happens to this limit when $a \notin D$?

When $R(x) = \dfrac{f(x)}{g(x)}$, we know $a \notin D$ means that $g(a) = 0$.

That means $R(a)$ does not make sense, so $\lim_{x \to a} R(x) \neq R(a)$, but it may still exist!

We will consider two different cases for limits of these functions: $f(a) = 0$ and $f(a) \neq 0$

Case 1: $f(a) \neq 0$

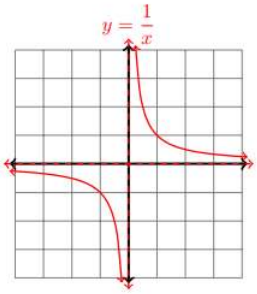
Lets consider a simple rational function $R(x) = \dfrac{1}{x}$.

Clearly its domain is $D = (-\infty, 0) \cup (0, \infty)$, so lets explore $\lim_{x \to 0} \dfrac{1}{x}$

We start the same way we originally learned to solve our limits, by plugging in x values:

x	$\frac{1}{x}$
-0.1	-10
-0.01	-100
-0.00001	-100000

so $\lim_{x \to 0^-} \dfrac{1}{x} = -\infty$



x	$\frac{1}{x}$
0.1	10
0.01	100
0.00001	100000

so $\lim_{x \to 0^+} \dfrac{1}{x} = +\infty$

Notice that for this function we have $\lim_{x \to 0^-} \dfrac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \dfrac{1}{x} = +\infty$, which means that $\lim_{x \to 0} \dfrac{1}{x}$ does not exist.

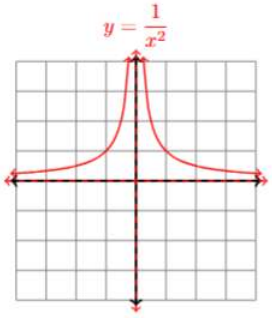
(This leads us to a short-hand notation: $\dfrac{1}{0^+} = +\infty$ and $\dfrac{1}{0^-} = -\infty$)

Any one-sided limit you take of a rational function which sends the denominator to zero but the numerator to a non-zero number will always yield positive or negative infinity. You just have to figure out whether the denominator is approaching zero as a positive or negative number!

What about $\lim_{x \to 0} \dfrac{1}{x^2}$

x	$\frac{1}{x}$
-0.1	100
-0.01	10000
-0.00001	10000000000

so $\lim_{x \to 0^-} \dfrac{1}{x^2} = +\infty$



x	$\frac{1}{x}$
0.1	100
0.01	10000
0.00001	10000000000

so $\lim_{x \to 0^+} \dfrac{1}{x^2} = +\infty$

For this function we have $\lim_{x \to 0^-} \dfrac{1}{x^2} = \lim_{x \to 0^+} \dfrac{1}{x^2} = +\infty$, so $\lim_{x \to 0} \dfrac{1}{x^2} = +\infty$.

We could use our short-hand notation from earlier to reach the same results:

$$\lim_{x\to 0^-}\frac{1}{x^2}=\frac{1}{(0^-)^2}=\frac{1}{0^+}=+\infty$$

Note: $(0^-)^2$ represents the square of a tiny negative number, which is a tiny positive number 0^+

$$\lim_{x\to 0^+}\frac{1}{x^2}=\frac{1}{(0^+)^2}=\frac{1}{0^+}=+\infty$$

Also, $(0^+)^2$ represents the square of a tiny positive number, which is a tiny positive number 0^+

Factoring always makes it easier to determine if you have 0^+ or 0^-:

Example 3.4.1

Evaluate each limit below.

(a) $\lim\limits_{x\to 3^+}\dfrac{x}{9-x^2}$

(b) $\lim\limits_{x\to -2^-}\dfrac{x+1}{x^3+6x^2+8x}$

Solution

(a) We begin by factoring the denominator completely into $\lim\limits_{x\to 3^+}\dfrac{x}{(3-x)(3+x)}$

We know x becomes a 3, and $(3+x)$ becomes a 6, and $(3-x)$ becomes 0
We care about what "sign" 0 has when it shows up, and since $x\to 3^+$ we know that $(3-x)$ will be a negative number, so its 0^-

Therefore $\lim\limits_{x\to 3^+}\dfrac{x}{(3-x)(3+x)}=\dfrac{3}{0^-\cdot 6}=-\infty$

(b) We begin by factoring the denominator completely into $\lim\limits_{x\to -2^-}\dfrac{x+1}{x(x+4)(x+2)}$

We know $(x+1)$ becomes a -1, and x becomes -2, and $(x+4)$ becomes a 2, and $(x+2)$ becomes 0
We care about what "sign" 0 has when it shows up, and since $x\to -2^-$ we know that $(x+2)$ will be a negative number, so its 0^-

Therefore $\lim\limits_{x\to -2^-}\dfrac{x+1}{x(x+4)(x+2)}=\dfrac{-1}{(-2)\cdot 2\cdot 0^-}=-\infty$

Case 2: $f(a)=0$

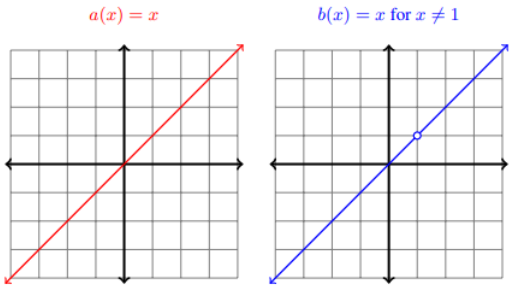
Remember $f(a)=0$ for the polynomial $f(x)$ only holds when $(x-a)$ is a factor.

So if $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{0}{0}$, that means $(x-a)$ is a factor of both the top and bottom polynomials.

Even though $(x-a)$ is a factor of the top and bottom, we must be careful about reducing it away!

Remember: reducing a rational function changes its domain.

Consider the graphs of the two functions below:



We can clearly see that $\lim_{x\to 1} a(x) = a(1) = 1$, but $\lim_{x\to 1} b(x) = 1$ also holds, even though $b(1)$ does not exist.

These two functions are exactly the same for every value of x except when $x = 1$.
Luckily, when taking the limit as x approaches 1, that is the only value of x that does not effect our answer!

This generalizes to the following statement:

If $f(x)$ and $g(x)$ are functions such that $f(x) = g(x)$ for all $x \neq a$, then \(\displaystyle{\lim_{x\to a} f(x)= \lim_{x\to a} g(x)}\)

In the context of rational functions, this is great news: Since reducing away $(x-a)$ on top and bottom will only change our function's value at $x = a$, it will not effect the outcome of our limit. **We are allowed to reduce rational functions inside of a limit**.

And that is exactly the plan of attack whenever we run into a $\frac{0}{0}$ limit: Factor, cancel, and try the limit again.

Example 3.4.2

Evaluate the following limits

(a) $\lim_{x\to 5} \dfrac{x^2-25}{x^2-5x}$

(b) $\lim_{x\to -1^+} \dfrac{x^3+3x^2-2}{x^2+3x+2}$

Solution

(a) We start by factoring both the numerator and denominator $\lim_{x\to 5} \dfrac{(x-5)(x+5)}{x(x-5)}$

Next we reduce inside of the limit to get $\lim_{x\to 5} \dfrac{x+5}{x}$

Since $x = 5$ is in the domain of this new rational function, we know the limit will match plugging it in!

$$\lim_{x\to 5} \frac{x^2-25}{x^2-5x} = \frac{5+5}{5} = 2$$

(b) We start by factoring both the numerator and denominator $\lim_{x\to -1^+} \dfrac{(x+1)(x^2+2x-2)}{(x+1)(x+2)}$

Next we reduce inside of the limit to get $\lim_{x\to -1^+} \dfrac{x^2+2x-2}{x+2}$

Since $x = -1$ is in the domain of this new rational function, we know the limit will match plugging it in!

$$\lim_{x\to -1^+} \frac{x^3+3x^2-2}{x^2+3x+2} = \frac{(-1)^2+2(-1)-2}{(-1)+2} = -3$$

Even after reducing, you may end up with 0 in the denominator.
(Good thing we already learned how to deal with that!)

Example 3.4.3

Evaluate the following limits

(a) $\lim_{x\to \frac{1}{2}^-} \dfrac{2x^2-3x+1}{4x^2-4x+1}$

(b) $\lim_{x\to 1} \dfrac{x^2-x}{x^3-3x^2+3x-1}$

Solution

(a) We start by factoring both the numerator and denominator $\lim_{x\to\frac{1}{2}^-} \dfrac{(2x-1)(x-1)}{(2x-1)^2}$

Next we reduce inside of the limit to get $\lim_{x\to\frac{1}{2}^-} \dfrac{x-1}{2x-1}$

Since $x=5$ is in the domain of this new rational function, we know the limit will match plugging it in!

$$\lim_{x\to\frac{1}{2}^-} \frac{2x^2-3x+1}{4x^2-4x+1} = \frac{-\frac{1}{2}}{0^-} = +\infty$$

(b) We start by factoring both the numerator and denominator $\lim_{x\to -1} \dfrac{x(x-1)}{(x-1)^3}$

Next we reduce inside of the limit to get $\lim_{x\to -1^+} \dfrac{x}{(x-1)^2}$

Since $x=-1$ is in the domain of this new rational function, we know the limit will match plugging it in!

$$\lim_{x\to 1} \frac{x^2-x}{x^3-3x^2+3x-1} = \frac{-1}{(0^+)^2} = \frac{-1}{0^+} = -\infty$$

The same rules apply when dealing with limits of complex fractions.

You are allowed to rewrite it as a rational function and reduce whenever necessary:

Example 3.4.4

Evaluate the limit $\lim_{x\to 4} \dfrac{\frac{8}{x}-2}{\frac{1}{x-5}+1}$

Solution

We start by multiplying the numerator and denominator of the large fraction by the common denominator $x(x-5)$

$$= \lim_{x\to 4} \frac{\frac{8}{x}-2}{\frac{1}{x-5}+1} \cdot \frac{x(x-5)}{x(x-5)}$$

$$= \lim_{x\to 4} \frac{8(x-5)-2x(x-5)}{x+x(x-5)}$$

$$= \lim_{x\to 4} \frac{-2x^2+18x-40}{x^2-4x}$$

$$= \lim_{x\to 4} \frac{-2(x-4)(x-5)}{x(x-4)}$$

Now we can reduce since we are inside of the limit and this becomes

$$= \lim_{x\to 4} \frac{-2(x-5)}{x}$$

This new function has $x=4$ in its domain, so we just plug it in to get our final answer $\dfrac{-2(4-5)}{4} = \dfrac{1}{2}$

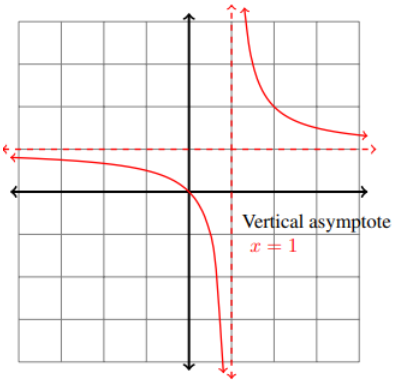
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3.5: Asymptotes

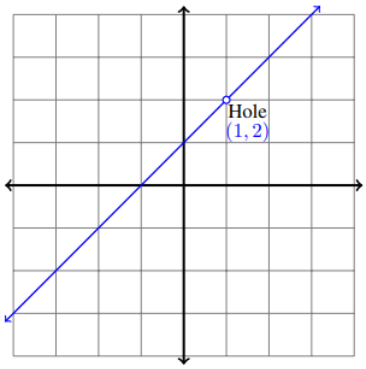
Now that we know how to handle limits at values of x outside of the domain of a given rational function, we can describe how they effect its graph.

Let $R(x)$ be a rational function with domain D, and consider some real number $a \notin D$

Case 1: $R(x)$ will have a vertical asymptote at $x = a$ when:
$\lim_{x \to a^+} R(x) = \pm\infty$ or $\lim_{x \to a^-} R(x) = \pm\infty$



Case 2: $R(x)$ will have a hole at (a, L) when:
$\lim_{x \to a} R(x) = L$



These are the only two possible outcomes for limits outside of the domain of a rational function.

Example 3.5.1

Find all vertical asymptotes and holes for the function $p(x) = \dfrac{2x^2 - 10x + 8}{x^3 - 16x}$

Solution

We start by factoring the numerator and denominator completely to get $p(x) = \dfrac{2(x-1)(x-4)}{x(x-4)(x+4)}$

We see our domain contains all x values except for $x = 0, 4, -4$, so we take limits at these values.

$$\lim_{x \to 0^+} p(x) = \lim_{x \to 0^+} \frac{2(x-1)}{x(x+4)} = \frac{2 \cdot (-1)}{0^+ \cdot 4} = -\infty$$

This means we have a **vertical asymptote** at $x = 0$

(the limit for $x \to 0^-$ is $+\infty$, if you try it)

$$\lim_{x \to 4^+} p(x) = \lim_{x \to 4^+} \frac{2(x-1)}{x(x+4)} = \frac{2 \cdot 3}{4 \cdot 8} = \frac{3}{16}$$

$\lim_{x\to 4^-} p(x) = \lim_{x\to 4^-} \dfrac{2(x-1)}{x(x+4)} = \dfrac{2\cdot 3}{4\cdot 8} = \dfrac{3}{16}$

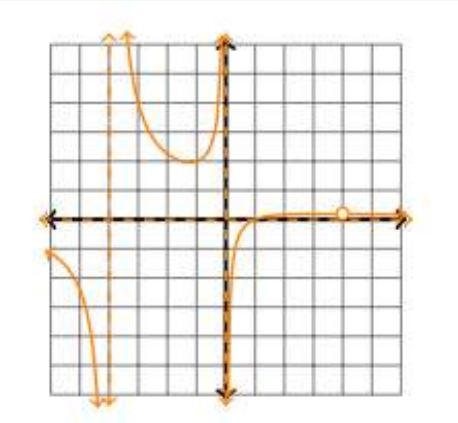
Since these match and are finite, we have a **hole** at $\left(4, \dfrac{3}{16}\right)$

$\lim_{x\to -4^+} p(x) = \lim_{x\to -4^+} \dfrac{2(x-1)}{x(x+4)} = \dfrac{2\cdot(-5)}{-4\cdot 0^+} = +\infty$

This means we have a **vertical asymptote** at $x = -4$

(the limit for $x \to -4^-$ is $-\infty$, if you try it)

In case you are curious, below is the graph of $y = p(x)$
Notice how it agrees with the results of our limits above.



You may notice in the graphs we make for rational functions, that there are sometimes another type of asymptote that runs horizontally instead of vertically. These are called *horizontal asymptotes*, and they are quite different than their vertical counterparts.

A horizontal asymptote is a line $y = a$ that the graph of your function $y = f(x)$ gets close to when x is very large (far to the right) or very large and negative (far to the left).

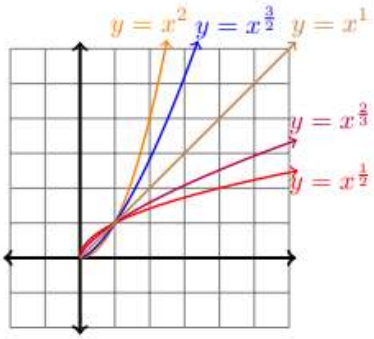
Another way of saying this is we have a horizontal asymptote at $y = a$ when:
$\lim_{x\to\infty} f(x) = a$ or $\lim_{x\to -\infty} f(x) = a$

It is important to realize that the symbols ∞ and $-\infty$ are not real numbers.

(That means you cannot say things like $0\cdot\infty = 0$ or $\infty - \infty = 0$ or $\dfrac{\infty}{-\infty} = -1$)

When taking limits to positive or negative infinity, we cannot just plug them into $f(x)$ (like we would have done for limits at real numbers)

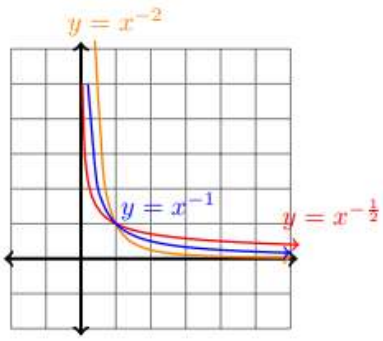
Obviously then, before we can find horizontal asymptotes of functions reliably, we should practice taking these infinite limits.



If $p > 0$ is a fixed exponent for x, then $\lim_{x\to\infty} x^p = +\infty$

(since when x is very large and positive, x^p will be very large and positive.)

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If $p < 0$ is a fixed exponent for x, then $\lim_{x\to\infty} x^p = 0$

(since when x is very large and positive, x^p will be the reciprocal of a very large number.)

Furthermore, as long as large negative numbers are allowed in the domain of x^p:
$\lim_{x\to-\infty} x^p$ is 0 when $p < 0$ and either ∞ or $-\infty$ when $p > 0$

This makes a few limits pretty straightforward:

$\lim_{x\to\infty} x^2 = +\infty$ and $\lim_{x\to\infty} x^3 = +\infty$

It is fairly obvious then that $\lim_{x\to\infty} x^3 + x^2 = +\infty$

(since a large positive number plus a large positive number gives a large positive number)

But what about $\lim_{x\to\infty} x^3 - x^2$?

Remember we cannot say this limit is just $\infty - \infty$, since this does not represent a number.
(a large positive number minus a large positive number could be positive, negative, or zero)

Instead, we re-write this as $\lim_{x\to\infty} x^3\left(1 - \frac{1}{x}\right) = \infty \cdot 1 = \infty$

(a large positive number times a number close to 1 is a large positive number)

We know since $\lim_{x\to\infty} \frac{1}{x} = 0$, that the x^2 term is actually infinitely small compared to the x^3 term.

In fact, for any polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ we can see that

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n \left(1 + \frac{\frac{a_{n-1}}{a_n}}{x} + \ldots \frac{\frac{a_1}{a_n}}{x^{n-1}} + \frac{\frac{a_0}{a_n}}{x^n}\right) = \lim_{x\to\infty} a_n x^n \cdot 1$$

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n \left(1 + \frac{\frac{a_{n-1}}{a_n}}{x} + \ldots \frac{\frac{a_1}{a_n}}{x^{n-1}} + \frac{\frac{a_0}{a_n}}{x^n}\right) = \lim_{x\to-\infty} a_n x^n \cdot 1$$

Again we see that the value of this limit is completely dependent on the largest power term: $a_n x^n$

This translates to the fact that when taking a limit to positive or negative infinity, only the fastest growing/largest power term is important. The author denotes this property as the **"Lebron James Rule"**: when taking infinite limits we only pay attention to the biggest strongest term.

This makes infinite limits of polynomials very easy.

Example 3.5.2

Evaluate each infinite limit below.

(a) $\lim_{x\to\infty} 12x^2 + 4x - 59$

(b) $\lim_{x\to\infty} -\frac{1}{3}x^3 + 2x + 1$

(c) $\lim_{x\to-\infty} 8x^9 + 6000x^7 - 3x^2$

Solution

Remember in polynomials, the largest power does all the work:

(a) $\lim_{x\to\infty} 12x^2 + 4x - 59 = \lim_{x\to\infty} 12x^2 = 12(+\infty)^2 = +\infty$

(b) $\lim_{x\to\infty} -\frac{1}{3}x^3 + 2x + 1 = \lim_{x\to\infty} -\frac{1}{3}x^3 = -\frac{1}{3}\cdot(+\infty)^3 = -\infty$

(c) $\lim_{x\to-\infty} 8x^9 + 6000x^7 - 3x^2 = \lim_{x\to-\infty} 8x^9 = 8(-\infty)^9 = -\infty$

We can see that $\lim_{x\to\infty} p(x) = \pm\infty$ and $\lim_{x\to-\infty} p(x) = \pm\infty$ will always occur for any polynomial.

Since these limits are never a real number, a polynomial will never have a horizontal asmyptote.

Rational functions, on the other hand, can definitely end up with horizontal asymptotes.

We can use the Lebron James rule on them as well!

Example 3.5.3

Evaluate each infinite limit below.

(a) $\lim_{x\to\infty} \frac{6x^2 + 2x + 1}{3x^2 + 5}$

(b) $\lim_{x\to-\infty} \frac{4x^6 + 5x - 1}{8x^6 + 15}$

(c) $\lim_{x\to\infty} \frac{x^3 + 2x^2}{3x^5 + 5x - 1}$

(d) $\lim_{x\to-\infty} \frac{-2x^4 + 5x - 1}{x^3 + 7x + 3}$

Solution

(a) $\lim_{x\to\infty} \frac{6x^2 + 2x + 1}{3x^2 + 5} = \lim_{x\to\infty} \frac{6x^2}{3x^2} = \lim_{x\to\infty} 2 = 2$

(b) $\lim_{x\to-\infty} \frac{4x^6 + 5x - 1}{8x^6 + 15} = \lim_{x\to-\infty} \frac{4x^6}{8x^6} = \lim_{x\to-\infty} \frac{1}{2} = \frac{1}{2}$

(c) $\lim_{x\to\infty} \dfrac{x^3+2x^2}{3x^5+5x-1} = \lim_{x\to\infty} \dfrac{x^3}{3x^5} = \lim_{x\to\infty} \dfrac{1}{3x^2} = 0$

(d) $\lim_{x\to-\infty} \dfrac{-2x^4+5x-1}{x^3+7x+3} = \lim_{x\to-\infty} \dfrac{-2x^4}{x^3} = \lim_{x\to-\infty} -2x = +\infty$

We just have to be careful that we correctly choose which term is largest:

$$\lim_{x\to\infty} (2x+1)^2 - (2x-1)^2$$

Wrong Way	Right Way
$\lim_{x\to\infty} (2x)^2 - (2x)^2$	$\lim_{x\to\infty} (4x^2+4x+1) - (4x^2-4x+1)$
$= \lim_{x\to\infty} 0$	$= \lim_{x\to\infty} 8x$
$= 0$	$= +\infty$

As a general rule, if it is not obvious which term is biggest, simplify first!

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CHAPTER OVERVIEW

4: Basic Trigonometry

> *"The whole universe is based on rhythms. Everything happens in circles. . . " - John Hartford*

The functions we have worked with so far have all been created by adding, subtracting, multiplying, and dividing power functions. While these cover a wide array of different curves, there are other more specific tools that we can use when dealing with circles and triangles. Chapter four is dedicated to two circular functions with big implications.

We begin with the Pythagorean Theorem for right triangles and use it to define the distance formula in the Cartesian plane. The distance formula immediately leads us to the equation for the graph of a circle in addition to its circumference and area formulas.

We define the sine and cosine functions in terms of the unit circle, and discuss their domains and graphs, as well as some easy to evaluate points for each, and how they interact with limits.

Trigonometry is the art of using these two functions to solve problems inside of triangles. We can define four additional trigonometric functions in terms of sine and cosine, (named secant, cosecant, tangent and cotangent) and their domains and graphs as well. Lastly we look into some basic identities to relate these trigonometric functions to one another in different ways.

Click through the following links to read this chapter.

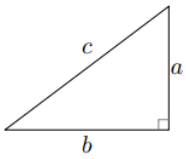
- 4.1: Circles and Triangles
- 4.2: Functions of the Circle
- 4.3: Ratios in Triangles
- 4.4: Graphing Other Trigonometric Functions
- 4.5: Trigonometric Identities

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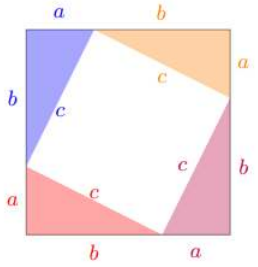
4.1: Circles and Triangles

A triangle is a shape made of three non-linear points connected by three straight lines.

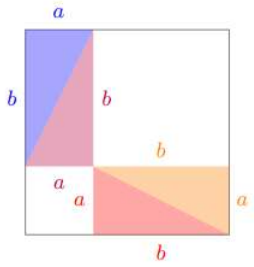
If any two of these lines are perpendicular to one another, we say it is a *right triangle*, (and we call the longest of the three sides the *hypotenuse* of the triangle.)



If we draw four copies of this triangle in two different arrangements we can find a clever relationship between the values of a, b, and c:



Empty (white) area is c^2



Empty (white) area is $a^2 + b^2$

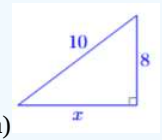
Since these triangles were only moved around, the total area in white should not be changed.

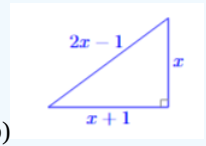
That means we know that $a^2 + b^2 = c^2$
(this is known as the *Pythagorean Theorem*})

As long as you know two sides of a right triangle, you can find the third:

Example 4.1.1

Solve for x in each triangle pictured below.

(a) 

(b) 

Solution

(a) We set up the Pythagorean identity as an equation for x and then solve it:

$x^2 + 8^2 = 10^2$

$x^2 + 64 = 100$

$x^2 - 36 = 0$

$(x - 6)(x + 6) = 0$

Note that $x = -6$ does not make sense since it makes a length negative, so our solution must be only $x = 6$

(b) We set up the Pythagorean identity as an equation for x and then solve it:

$x^2 + (x+1)^2 = (2x-1)^2$

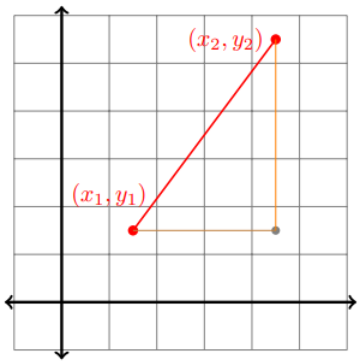
$2x^2 + 2x + 1 = 4x^2 - 4x + 1$

$0 = 2x^2 - 6x$

$0 = 2x(x-3)$

Note that $x = 0$ does not make sense since it makes a length negative, so our solution must be only $x = 3$

An immediate consequence of the Pythagorean Theorem is the distance formula on the xy-plane:



Here we can see that the distance between the two points (x_1, y_1) and (x_2, y_2) is the length of the hypotenuse of a right triangle. The bottom has length $x_2 - x_1$ and the right has length $y_2 - y_1$

So we have $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, and if we solve for d:

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example 4.1.2

What is the distance between the points $(-4, 5)$ and $(2, -3)$?

Solution

$d = \sqrt{(2-(-4))^2 + (-3-5)^2} = \sqrt{6^2 + (-8)^2} = \sqrt{100} = 10$ units apart

Example 4.1.3

What value(s) of t make $(1, 2)$ and (t, t) exactly 5 units apart?

Solution

$5^2 = (t-1)^2 + (t-2)^2$

$25 = 2t^2 - 6t + 5$

$0 = 2t^2 - 6t - 20$

$0 = 2(t-5)(t+2)$

This gives us two answers $t = 5$ and $t = -2$ both of which are valid

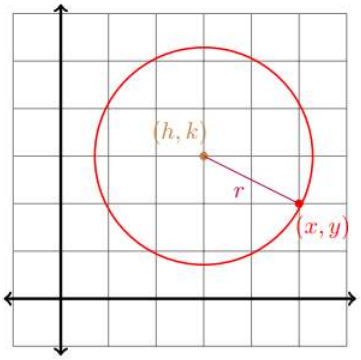
Consider a point on the xy-plane (h, k).

If we wanted to look at a collection of points (x, y) that are all exactly r units away from (h, k):

$r = \sqrt{(x-h)^2 + (y-k)^2}$

Squaring both sides gives us the equation of a circle with radius r and center (h, k):

$(x-h)^2 + (y-k)^2 = r^2$



If you ever want to graph a circle, it is best to write it in the above form first.

Example 4.1.4

Sketch the graph of $x^2 + y^2 = 2y - 6x - 6$

Solution

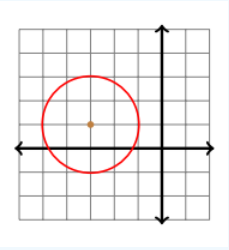
We start by moving all of the variables onto one side to get

$x^2 + 6x + y^2 - 2y = -6$

Next we complete the square for both x and y by adding the missing terms to both sides

$(x^2 + 6x + 9) + (y^2 - 2y + 1) = -6 + 9 + 1$

This gives us $(x+3)^2 + (y-1)^2 = 4$, so our center is at $(-3, 1)$ and the radius is $r = 2$



Just like any other shape, circles have both an area and perimeter.
The perimeter of a circle (called the *circumference*) is always directly proportional to its radius.

Unfortunately that proportion is a very unique number that cannot be expressed as a fraction: $C = 2\pi \cdot r$

Similarly, the area of a circle is proportional to the square of the radius: $A = \pi r^2$

In both of these equations, we use π to represent the number below:

3.14159265358979323846264338327950288419716939937510$
$58209749445923078164062862089986280348253421170679...
(Since this decimal does not repeat or end, it is easiest just to use the symbol π)

It is important to notice that circles are **not** functions.
(They fail the vertical line test at $x = h$ every time.)

Consider the circle of radius 1 centered at the origin (we call this the unit circle)

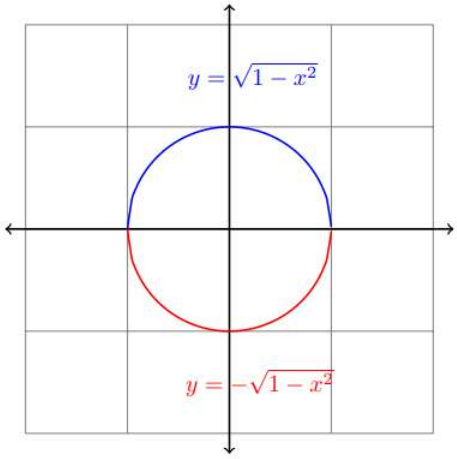
$x^2 + y^2 = 1$

In order to use our function notation, we can solve for y:

$y^2 = 1 - x^2$

$y = \pm\sqrt{1 - x^2}$

These two functions $f(x) = \sqrt{1 - x^2}$ and $g(x) = -\sqrt{1 - x^2}$ combine to give us the unit circle.



We can see that the domain for each function $f(x)$ and $g(x)$ is the interval $[-1, 1]$

Within this domain there are a few values of x that we like to keep track of:

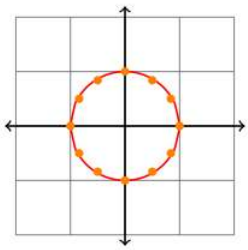
$f(0) = \sqrt{1 - 0^2} = 1$ and $g(0) = -1$

$f\left(\pm\frac{1}{2}\right) = \sqrt{1 - \left(\pm\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ and $g\left(\pm\frac{1}{2}\right) = -\frac{\sqrt{3}}{2}$

$f\left(\pm\frac{\sqrt{3}}{2}\right) = \sqrt{1 - \left(\pm\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}$ and $g\left(\pm\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}$

$f(\pm 1) = \sqrt{1 - (\pm 1)^2} = 0$ and $g(\pm 1) = 0$

While these x-values may not seem special, if we plot them on the unit circle we can see that they actually split up the circumference into twelve pieces of equal length:



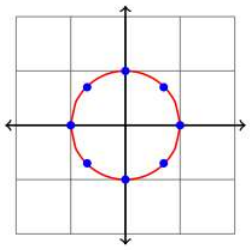
Since the radius $r = 1$, the total circumference is $C = 2\pi$,
So dividing it into 12 pieces (called *arcs*) means they are each $\frac{\pi}{6}$ units long.

Alternatively, we could focus on a few different x values. (These split the circumference into eight arcs of equal length)

$f(0) = \sqrt{1 - 0^2} = 1$ and $g(0) = -1$

$f\left(\pm\frac{\sqrt{2}}{2}\right) = \sqrt{1 - \left(\pm\frac{\sqrt{2}}{2}\right)^2} = \frac{\sqrt{2}}{2}$ and $g\left(\pm\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}$

$f(\pm 1) = \sqrt{1-(\pm 1)^2} = 0$ and $g(\pm 1) = 0$



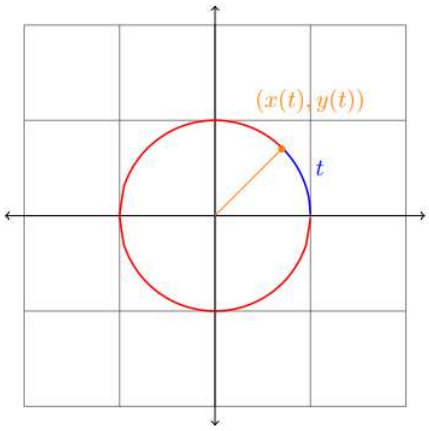
This time dividing it into 8 arcs means they are each $\frac{\pi}{4}$ units long.

These values will show up again in the near future!

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4.2: Functions of the Circle

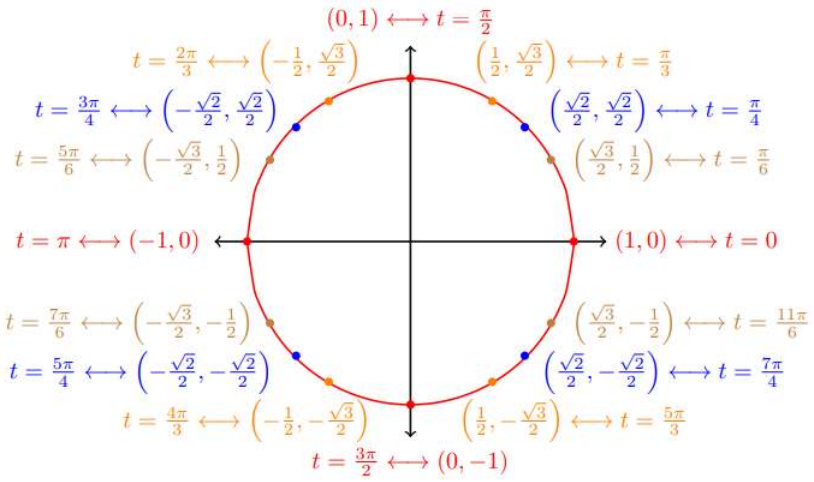
Consider the unit circle $x^2+y^2=1$ which has a circumference of 2π



Then for each value of $0 \leq t \leq 2\pi$, if a person walked around the edge of the circle for exactly t units, they would end up at a specific location (x, y)

We use this picture to define the two functions $x(t)$ and $y(t)$.

Since we start at $(1, 0)$ when $t = 0$, we can use the evenly divided arcs we found earlier to evaluate $x(t)$ and $y(t)$ for some special values of t:



There is a lot of information shown here, but it definitely is worth memorizing it.

Realize that there is a pattern: If you can remember the first quadrant, then symmetry should let you find the points in the other three quadrants relatively easily.

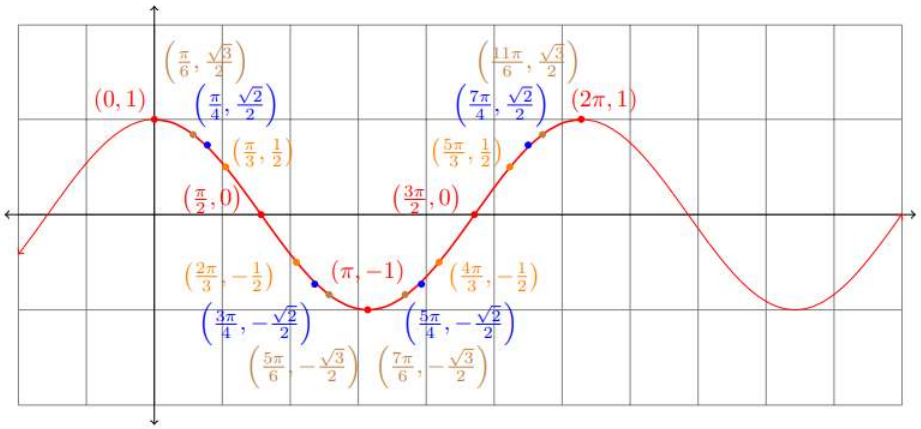
In order to avoid confusion with the variables x and y, we will give these functions new names:

$x(t)$ will be replaced by $\cos(t)$, referred to as the *cosine function*.

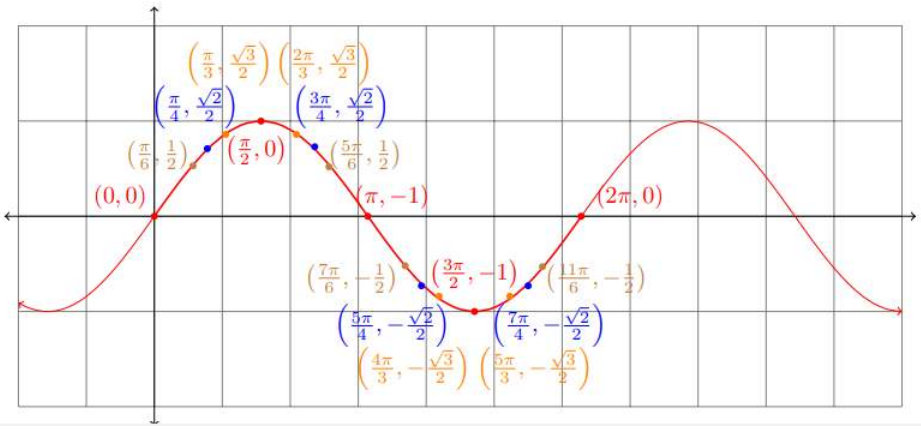
$y(t)$ will be replaced by $\sin(t)$, referred to as the *sine function*.

Lets see what their graphs look like based on what we know about these two functions so far:

$y = \cos(x)$



$y = \sin(x)$



Notice that if you walk for a distance longer than 2π, you end up repeating all of the same values of x and y again. This means the domain for these two functions can include any positive number. Similarly, you could "walk backward" and consider that a negative value of t, and the values of $\cos(t)$ and $\sin(t)$ are still defined. Thus the domain of each of these functions will be $(-\infty, \infty)$.

While that may sound intimidating, remember that the values of $\cos(t)$ and $\sin(t)$ will repeat themselves every full rotation, which means every 2π. Functions like this are called *periodic*. Once you find how they act on one period, you can simply repeat it forever in both directions.

Even though we only know about 16 exact points on each of the graphs for the sine and cosine functions, we can use the fact that they are periodic to evaluate a wide variety of inputs:

Example 4.2.1

Evaluate each sine or cosine expression below exactly.

(a) $\sin\left(\frac{21\pi}{4}\right)$

(b) $\cos\left(\frac{7\pi}{2}\right)$

(c) $\sin\left(-\frac{11\pi}{3}\right)$

Solution

(a) $\sin\left(\frac{21\pi}{4}\right) = \sin\left(\frac{7\pi}{2} - 2\pi\right) = \sin\left(\frac{3\pi}{2}\right) = -1$

(b) $\cos\left(\frac{7\pi}{2}\right) = \cos\left(\frac{21\pi}{4} - 2 \cdot 2\pi\right) = \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

(c) $\sin\left(-\frac{11\pi}{3}\right) = \sin\left(-\frac{100\pi}{3} + 17 \cdot 2\pi\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

A quick glance at the graphs we drew earlier should convince you that

$\lim_{x\to a} \cos(x) = \cos(a)$ and $\lim_{x\to a} \sin(x) = \sin(a)$ for any real number a.

Example 4.2.2

Evaluate each limit below exactly.

(a) $\lim_{x\to\pi} \sin\left(-\frac{97x}{4}\right)$

(b) $\lim_{t\to 6^-} \cos\left(\frac{3\pi}{t}\right)$

Solution

(a) $\lim_{x\to\pi} \sin\left(-\frac{97x}{4}\right) = \sin\left(-\frac{97\pi}{4}\right) = \sin\left(-\frac{97\pi}{4} + 13 \cdot 2\pi\right) = \sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

(b) $\lim_{t\to 6^-} \cos\left(\frac{3\pi}{t}\right) = \cos\left(\frac{3\pi}{6}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

It is also worth noting that there are no horizontal asymptotes for any non-constant periodic graph.

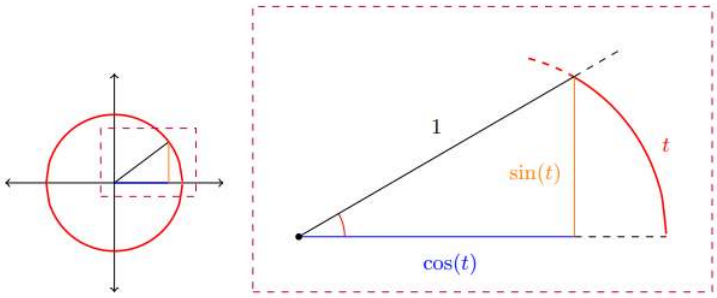
That means that none of the following limits exist:

$\lim_{x\to\infty} \cos(x)$ $\lim_{x\to-\infty} \cos(x)$ $\lim_{x\to\infty} \sin(x)$ $\lim_{x\to-\infty} \sin(x)$

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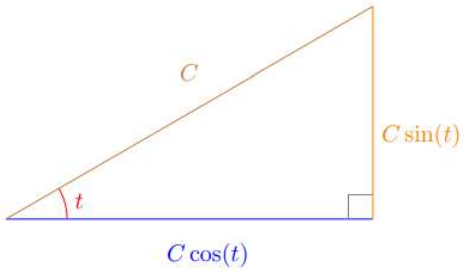
4.3: Ratios in Triangles

Consider the unit circle $x^2 + y^2 = 1$, lets explore the properties of a triangle inside of quadrant 1:



This triangle will always have a hypotenuse of 1, but for different values of t (which is just the length of the arc on the unit circle) the values of $\cos(t)$ and $\sin(t)$ will obviously change.

We can create a new triangle by taking that same one above and multiplying all three sides by the same constant C.
his makes all three sides maintain the same ratio:



For any right triangle of the form above, we say that the angle in red is t *radians*.

(Radians are a way to measure angles in terms of the arc they trace out on the unit circle.)

We also call the blue leg the *adjacent side* to the angle t, and the orange leg is called the *opposite side* to the angle t.

This gives us a new way to find $\cos(t)$ and $\sin(t)$ in a right triangle with angle measuring t radians:

The ratio of the adjacent side over the hypotenuse will always be $\cos(t)$

The ratio of the opposite side over the hypotenuse will always be $\sin(t)$

Since there are three legs, there are actually six ratios that can be found:

The ratio of the opposite side over the adjacent side is $\tan(t)$ (known as the *tangent* function)

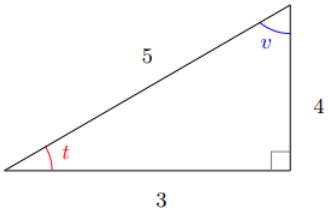
The ratio of the adjacent side over the opposite side is $\cot(t)$ (known as the *cotangent* function)

The ratio of the hypotenuse over the adjacent side is $\sec(t)$ (known as the *secant* function)

The ratio of the hypotenuse over the opposite side is $\csc(t)$ (known as the *cosecant* function)

Example 4.3.1

Find all six trigonometric functions for the angles t and v pictured



Solution

For angle t we see the adjacent side is 3, opposite side is 4, and hypotenuse is 5 so:

$$\cos(t) = \frac{3}{5}$$
$$\sin(t) = \frac{4}{5}$$
$$\tan(t) = \frac{4}{3}$$
$$\cot(t) = \frac{3}{4}$$
$$\sec(t) = \frac{5}{3}$$
$$\csc(t) = \frac{5}{4}$$

And for angle v we see the adjacent side is 4, opposite side is 3, and hypotenuse is 5 so:

$$\cos(v) = \frac{4}{5}$$
$$\sin(v) = \frac{3}{5}$$
$$\tan(v) = \frac{3}{4}$$
$$\cot(v) = \frac{4}{3}$$
$$\sec(v) = \frac{5}{4}$$
$$\csc(v) = \frac{5}{3}$$

It may seem a bit overzealous to name the same ratio and its reciprocal differently (like tangent and cotangent), but this is because we will later use the symbol $\tan^{-1}(x)$ to mean something else.

It would be incredibly confusing to try to tell the difference between a reciprocal and this symbol.
Even though there are a total of six ratios here, the good news is they are all related to one another.

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\sin(t)}{\cos(t)}$$

$$\cot(t) = \frac{\text{adjacent}}{\text{opposite}} = \frac{\frac{\text{adjacent}}{\text{hypotenuse}}}{\frac{\text{opposite}}{\text{hypotenuse}}} = \frac{\cos(t)}{\sin(t)}$$

$$\sec(t) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\frac{\text{adjacent}}{\text{hypotenuse}}} = \frac{1}{\cos(t)}$$

$$\csc(t) = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\frac{\text{opposite}}{\text{hypotenuse}}} = \frac{1}{\sin(t)}$$

This means that as long as you know the values of $\sin(t)$ and $\cos(t)$, you can calculate any of the other four trigonometric functions in a right triangle!

We take these relationships and extend them to define tangent, cotangent, secant, and cosecant for angles outside of quadrant one. Once again, as long as we can evaluate sine and cosine at a given angle, we can solve for any of the other four functions.

Example 4.3.2

Evaluate the following expressions by using the unit circle:

(a) $\tan\left(\frac{\pi}{3}\right)$

(b) $\cot\left(\frac{7\pi}{6}\right)$

(c) $\sec\left(\frac{5\pi}{4}\right)$

Solution

(a) $\tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

(b) $\cot\left(\frac{7\pi}{6}\right) = \frac{\cos\left(\frac{7\pi}{6}\right)}{\sin\left(\frac{7\pi}{6}\right)} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$

(c) $\sec\left(\frac{5\pi}{4}\right) = \frac{1}{\cos\left(\frac{5\pi}{4}\right)} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}$

Example 4.3.3

Calculate the limits below.

(a) $\lim\limits_{z\to\frac{\pi}{2}} \cot(z)$

(b) $\lim\limits_{x\to 2^-} \csc\left(\frac{\pi}{4+x}\right)$

Solution

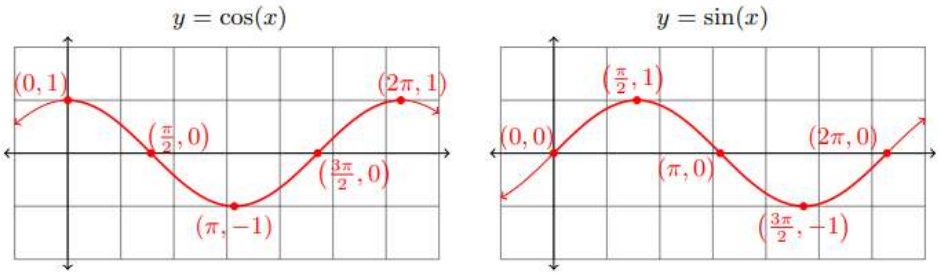
(a) $\lim\limits_{z\to\frac{\pi}{2}} \cot(z) = \lim\limits_{z\to\frac{\pi}{2}} \frac{\cos(z)}{\sin(z)} = \frac{\lim\limits_{z\to\frac{\pi}{2}} \cos(z)}{\lim\limits_{z\to\frac{\pi}{2}} \sin(z)} = \frac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)} = \frac{0}{1} = 0$

(b) $\lim\limits_{x\to 2^-} \csc\left(\frac{\pi}{4+x}\right) = \lim\limits_{x\to 2^-} \frac{1}{\sin\left(\frac{\pi}{4+x}\right)} = \frac{1}{\lim\limits_{x\to 2^-} \sin\left(\frac{\pi}{4+x}\right)} = \frac{1}{\sin\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

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4.4: Graphing Other Trigonometric Functions

Remember the graphs of the sine and cosine functions below:



When graphing transformations of either of these functions, we can just keep track of where these important points are moved to and connect the dots with the wave shape above.

Example 4.4.1

Sketch $y = 2\cos\left(\frac{1}{2}x + \frac{\pi}{3}\right) + 1$

Solution

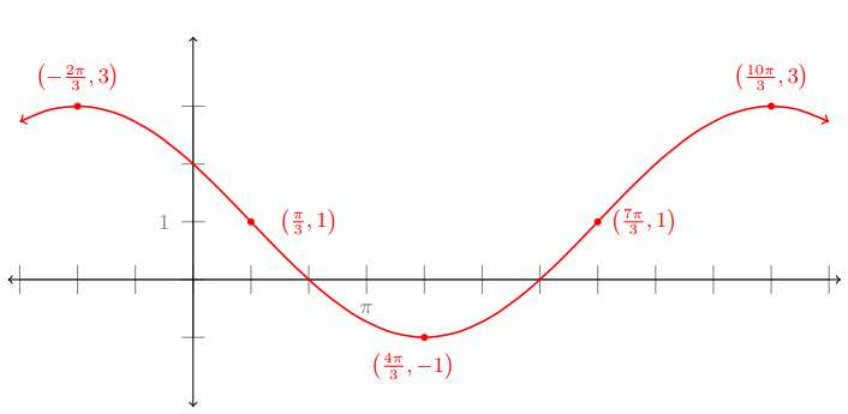
We find the relationships between (x_{old}, y_{old}) and (x_{new}, y_{new}):

$$y_{new} = 2y_{old} + 1 \text{ and } x_{old} = \frac{1}{2}x_{new} + \frac{\pi}{3} \longleftrightarrow x_{new} = 2x_{old} - \frac{2\pi}{3}$$

We plug in our known old points and solve for their new locations:

x_{old}	y_{old}	x_{new}	y_{new}
0	1	$-\frac{2\pi}{3}$	3
$\frac{\pi}{2}$	0	$\frac{\pi}{3}$	1
π	-1	$\frac{4\pi}{3}$	-1
$\frac{3\pi}{2}$	0	$\frac{7\pi}{3}$	1
2π	1	$\frac{10\pi}{3}$	3

Finally we plot each point and connect the dots with the shape of a cosine graph.



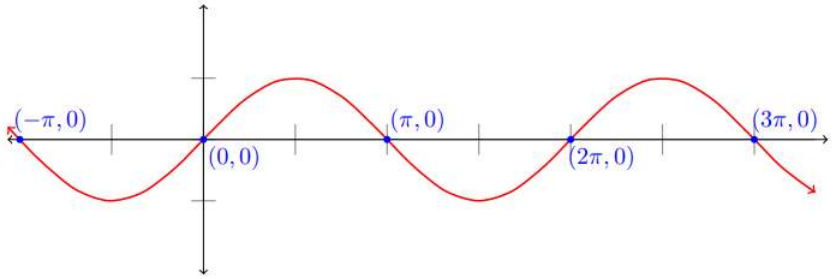
Remember again that each of the sine and cosine functions are periodic with domain $(-\infty, \infty)$, so they will repeat themselves forever in both directions.

Since the other four functions can be expressed as fractions of $\sin(x)$ and $\cos(x)$, we can determine their graphs by using our limit skills.

We know that for any real number a, we have $\lim_{x\to a}\cos(x)=\cos(a)$ and $\lim_{x\to a}\sin(x)=\sin(a)$

That means any of the remaining four functions involving fractions of these will have $\lim_{x\to a}f(x)=f(a)$ as long as it does not cause us to divide by zero!

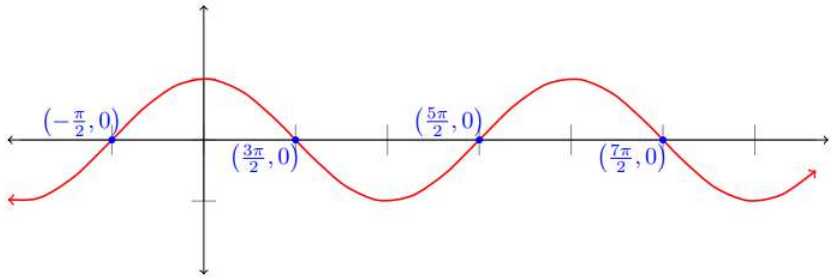
When does $\sin(x)=0$?



We can see that there are actually infinitely many x-intercepts for the graph of the sine function.
They are all easy to describe though: the integer multiples of π.

We say the roots of $\sin(x)$ are $x=k\pi$ for any integer k

When does $\cos(x)=0$?



Again we can see that there are infinitely many x-intercepts for the graph of the cosine function.
We can describe these a little tougher: the odd integer multiples of $\frac{\pi}{2}$.

We say the roots of $\cos(x)$ are $x=\dfrac{(2k+1)\pi}{2}$ for any integer k

This means we can say $\lim_{x\to a}\cot(x)=\cot(a)$ and $\lim_{x\to a}\csc=\csc(a)$ for any $a\neq k\pi$ and furthermore $\lim_{x\to a}\tan(x)=\tan(a)$ and $\lim_{x\to a}\sec(x)=\sec(a)$ for any $a\neq\dfrac{(2k+1)\pi}{2}$

Lets start with the graph of the cosecant function $f(x)=\csc(x)$.

We have already shown that this can be re-written as $f(x)=\dfrac{1}{\sin(x)}$, which allows us to find the exact height of our function at many of the nice angles we know on the unit circle. However, we cannot plug in a couple of them: $x=0$ and $x=\pi$ (and $x=2\pi$, but we know it will just repeat whatever occurs at $x=0$)

Consider the graph of $y=\sin(x)$ near $x=0$. We know on the right side of $x=0$ the value of $\sin(x)$ is positive, and on the left side of $x=0$ the value of $\sin(x)$ is negative. Therefore:

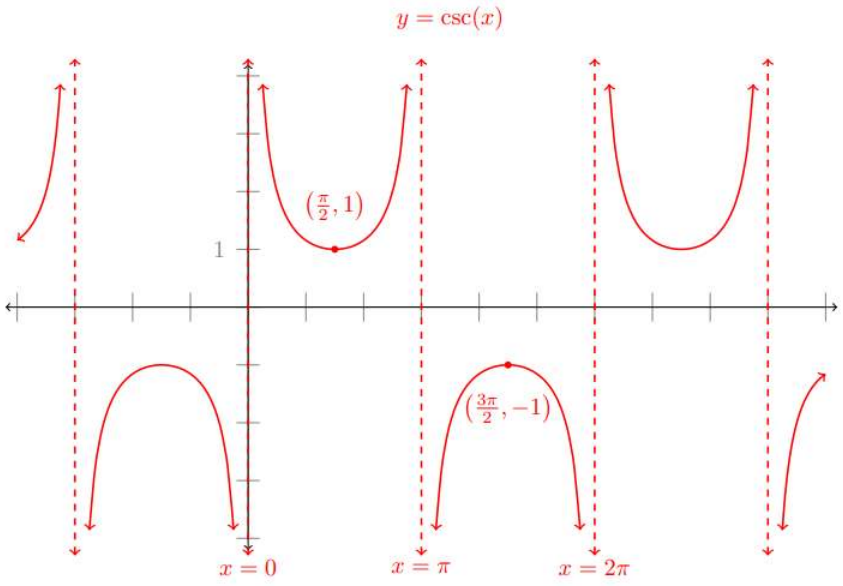
$$\lim_{x\to 0^+}\csc(x)=\lim_{x\to 0^+}\frac{1}{\sin(x)}=\frac{1}{0^+}=+\infty \quad\text{and}\quad \lim_{x\to 0^-}\csc(x)=\lim_{x\to 0^-}\frac{1}{\sin(x)}=\frac{1}{0^-}=-\infty$$

Similarly, for $y=\sin(x)$ near $x=\pi$: the value of $\sin(x)$ is negative on the right side of $x=\pi$ and the value of $\sin(x)$ is positive on the left side of $x=\pi$. So once again our limits are:

$$\lim_{x\to\pi^+}\csc(x)=\lim_{x\to\pi^+}\frac{1}{\sin(x)}=\frac{1}{0^-}=-\infty \text{ and } \lim_{x\to\pi^-}\csc(x)=\lim_{x\to\pi^-}\frac{1}{\sin(x)}=\frac{1}{0^+}=+\infty$$

Together these limits say that this graph will have vertical asymptotes at $x=0$ and $x=\pi$ (and therefore at every $x=k\pi$ since it is periodic)

So we plot these asymptotes as well as all of the nice angles from the unit circle to get:



The domain of this function is hard to write in interval notation, so we just say $D=\{x\neq k\pi\}$

For transformations of cosecant, we keep track of these two points and three asymptotes.

Next will be the graph of the secant function $f(x)=\sec(x)$.

This can be re-written as $f(x)=\dfrac{1}{\cos(x)}$, which again allows us to find the exact height of our function at many of the nice angles we know on the unit circle. Once again, we cannot plug in a couple of them: $x=\dfrac{\pi}{2}$ and $x=\dfrac{3\pi}{2}$

Consider the graph of $y=\cos(x)$ near $x=\dfrac{\pi}{2}$. We know on the right side of $x=\dfrac{\pi}{2}$ the value of $\cos(x)$ is negative, and on the left side of $x=\dfrac{\pi}{2}$ the value of $\cos(x)$ is positive. Therefore:

$$\lim_{x\to\frac{\pi}{2}^+}\sec(x)=\lim_{x\to\frac{\pi}{2}^+}\frac{1}{\cos(x)}=\frac{1}{0^-}=-\infty \text{ and } \lim_{x\to\frac{\pi}{2}^-}\sec(x)=\lim_{x\to\frac{\pi}{2}^-}\frac{1}{\cos(x)}=\frac{1}{0^+}=+\infty$$

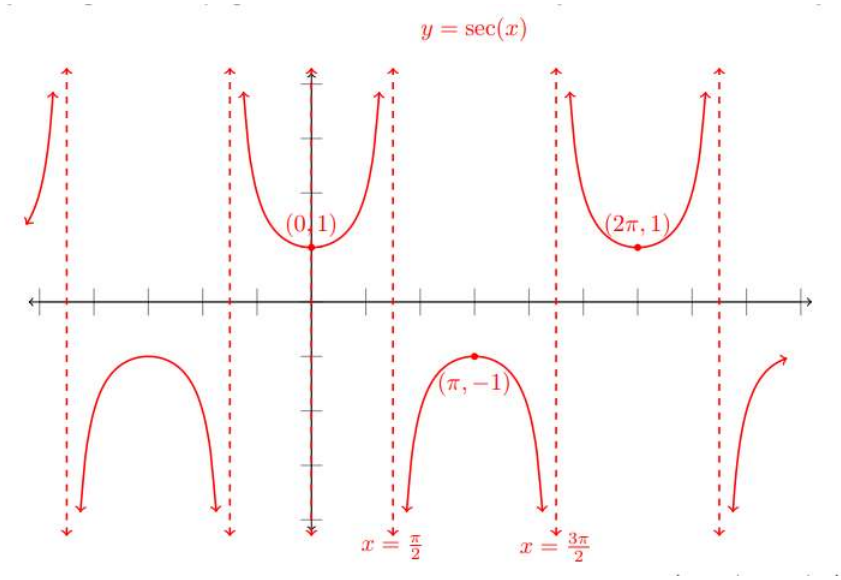
Similarly, for $y=\cos(x)$ near $x=\dfrac{3\pi}{2}$: the value of $\cos(x)$ is positive on the right side of $x=\dfrac{3\pi}{2}$ and the value of $\cos(x)$ is negative on the left side of $x=\dfrac{3\pi}{2}$. So once again our limits are:

$$\lim_{x\to\frac{3\pi}{2}^+}\sec(x)=\lim_{x\to\frac{3\pi}{2}^+}\frac{1}{\cos(x)}=\frac{1}{0^+}=+\infty \text{ and } \lim_{x\to\frac{3\pi}{2}^-}\sec(x)=\lim_{x\to\frac{3\pi}{2}^-}\frac{1}{\cos(x)}=\frac{1}{0^-}=-\infty$$

Together these limits say that this graph will have vertical asymptotes at $x=\dfrac{\pi}{2}$ and $x=\dfrac{3\pi}{2}$

(and therefore at every $x=\dfrac{(2k+1)\pi}{2}$ since it is periodic)

Again we plot these asymptotes as well as all of the nice angles from the unit circle to get:



The domain of this function is hard to write, so again we just say $D = \left\{ x \neq \dfrac{(2k+1)\pi}{2} \right\}$

For transformations of secant, we keep track of these three points and two asymptotes.

Example 4.4.2

Sketch $y = 2 - 3\csc(\pi - 2x)$

Solution

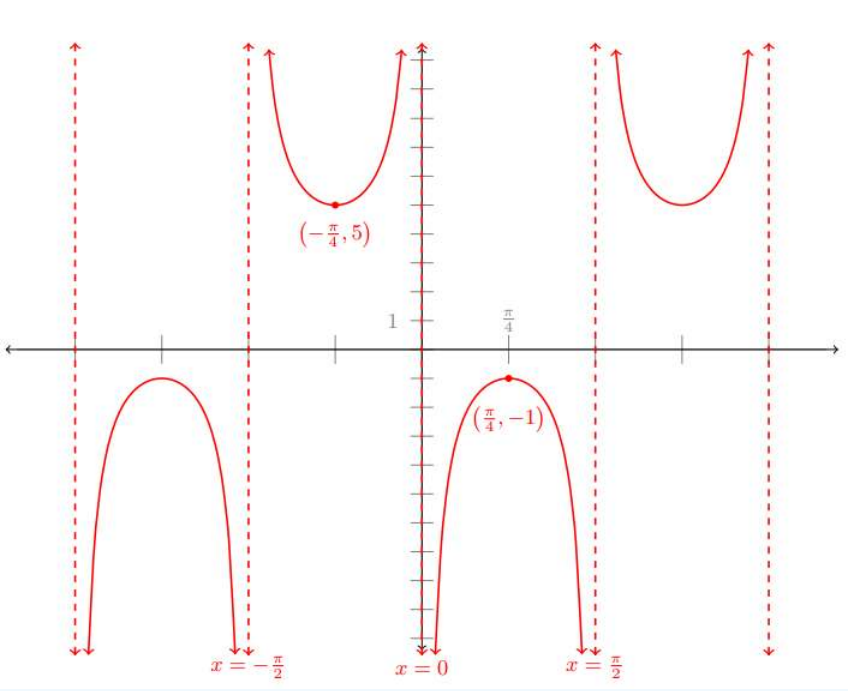
We find the relationships between (x_{old}, y_{old}) and (x_{new}, y_{new}):

$y_{new} = 2 - 3y_{old}$ and $x_{old} = \pi - 2x_{new} \longleftrightarrow x_{new} = -\dfrac{1}{2}x_{old} + \dfrac{\pi}{2}$

We plug in our known old points and solve for their new locations:

x_{old}	y_{old}	x_{new}	y_{new}
0	VA	$\frac{\pi}{2}$	VA
$\frac{\pi}{2}$	1	$\frac{\pi}{4}$	-1
π	VA	0	VA
$\frac{3\pi}{2}$	-1	$-\frac{\pi}{4}$	5
2π	VA	$-\frac{\pi}{2}$	VA

Finally we plot each point and connect the dots with the shape of a secant graph.



As long as you can track these five points and/or vertical asymptotes, you should be able to create the graph of a transformation of either of these two functions.

Now we look to the graph of the cotangent function $f(x) = \cot(x)$.

This can be re-written as $f(x) = \dfrac{\cos(x)}{\sin(x)}$, which again allows us to find the exact height of our function at many of the nice angles we know on the unit circle. Just like for cosecant, we cannot plug in $x = 0$ and $x = \pi$

Remember again the graph of $y = \sin(x)$ near $x = 0$. On the right side of $x = 0$, $\sin(x)$ is positive, and on the left side of $x = 0$, $\sin(x)$ is negative.

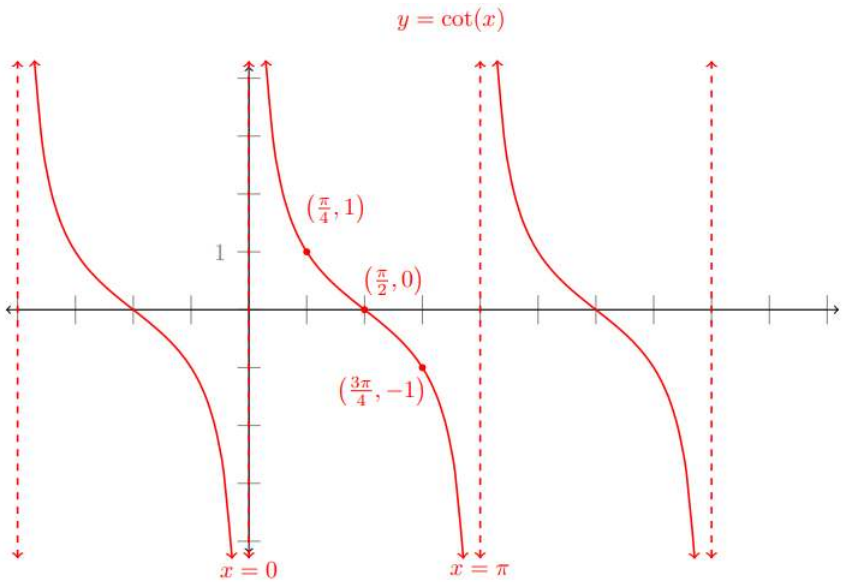
$$\lim_{x\to 0^+} \cot(x) = \lim_{x\to 0^+} \frac{\cos(x)}{\sin(x)} = \frac{1}{0^+} = +\infty \text{ and } \lim_{x\to 0^-} \cot(x) = \lim_{x\to 0^-} \frac{\cos(x)}{\sin(x)} = \frac{1}{0^-} = -\infty$$

On the right side of $x = \pi$, $\sin(x)$ is negative, and on the left side of $x = \pi$, $\sin(x)$ is positive.

$$\lim_{x\to \pi^+} \cot(x) = \lim_{x\to \pi^+} \frac{\cos(x)}{\sin(x)} = \frac{-1}{0^-} = +\infty \text{ and } \lim_{x\to \pi^-} \cot(x) = \lim_{x\to \pi^-} \frac{\cos(x)}{\sin(x)} = \frac{-1}{0^+} = -\infty$$

Once again these limits say that this graph will have vertical asymptotes at $x = 0$ and $x = \pi$
(and therefore at every $x = k\pi$ since it is periodic)

So we plot these asymptotes as well as all of the nice angles from the unit circle to get:



The domain of this function is the same as cosecant's: $D = \{x \neq k\pi\}$

For transformations of cotangent, we keep track of these three points and two asymptotes.

Lastly, we look to the graph of the tangent function $f(x) = \tan(x)$.

This can be re-written as $f(x) = \dfrac{\sin(x)}{\cos(x)}$, which again allows us to find the exact height of our function at many of the nice angles we know on the unit circle. Just like for secant, we cannot plug in $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$

Remember again the graph of $y = \cos(x)$ near $x = \dfrac{\pi}{2}$. On the right side of $x = \dfrac{\pi}{2}$, $\cos(x)$ is negative, and on the left side of $x = \dfrac{\pi}{2}$, $\cos(x)$ is positive.

$$\lim_{x \to \frac{\pi}{2}^+} \tan(x) = \lim_{x \to \frac{\pi}{2}^+} \frac{\sin(x)}{\cos(x)} = \frac{1}{0^-} = -\infty$$

$$\lim_{x \to \frac{\pi}{2}^-} \tan(x) = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin(x)}{\cos(x)} = \frac{1}{0^+} = +\infty$$

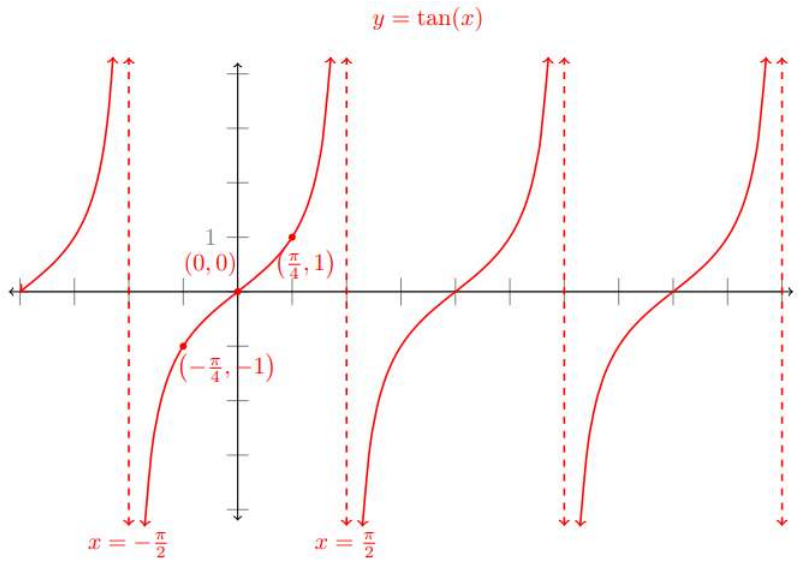
On the right side of $x = \dfrac{3\pi}{2}$, $\cos(x)$ is positive, and on the left side of $x = \dfrac{3\pi}{2}$, $\cos(x)$ is negative.

$$\lim_{x \to \frac{3\pi}{2}^+} \tan(x) = \lim_{x \to \frac{3\pi}{2}^+} \frac{\sin(x)}{\cos(x)} = \frac{-1}{0^+} = -\infty$$

$$\lim_{x \to \frac{3\pi}{2}^-} \tan(x) = \lim_{x \to \frac{3\pi}{2}^-} \frac{\sin(x)}{\cos(x)} = \frac{-1}{0^-} = +\infty$$

Once again these limits say that this graph will have vertical asymptotes at $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$ and therefore at every $x = \dfrac{(2k+1)\pi}{2}$ since it is periodic)

So we plot these asymptotes as well as all of the nice angles from the unit circle to get:



The domain of this function is the same as secant's: $D = \left\{ x \neq \dfrac{(2k+1)\pi}{2} \right\}$

For transformations of tangent, we keep track of these three points and two asymptotes.

Example 4.4.3

Sketch $y = \dfrac{1}{2}\tan\left(\dfrac{\pi(x-3)}{4}\right) + \dfrac{1}{2}$

Solution

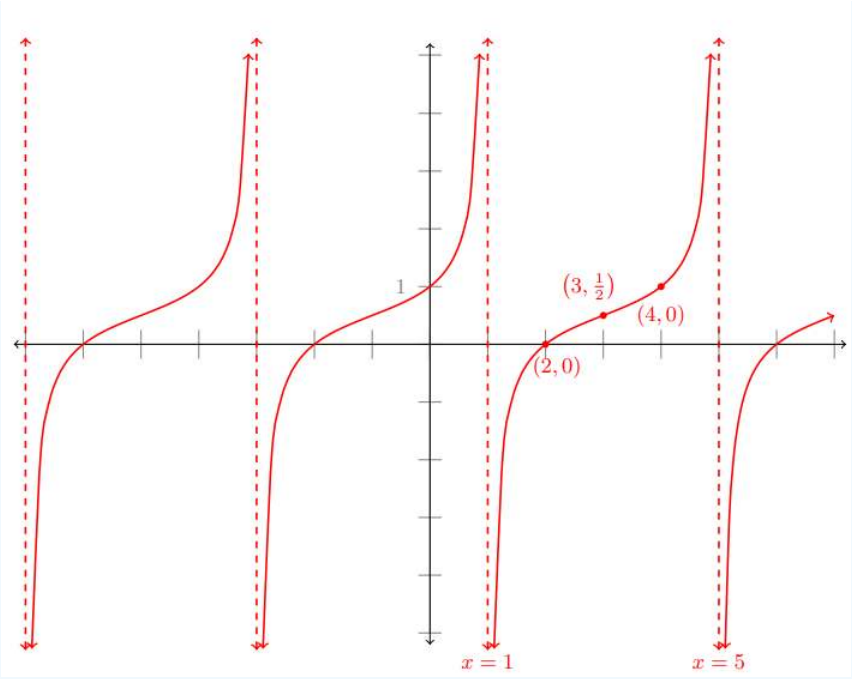
We find the relationships between (x_{old}, y_{old}) and (x_{new}, y_{new}):

$$y_{new} = \frac{1}{2}y_{old} + \frac{1}{2} \text{ and } x_{old} = \frac{\pi(x_{new}-3)}{4} \longleftrightarrow x_{new} = \frac{4}{\pi}x_{old} + 3$$

We plug in our known old points and solve for their new locations:

x_{old}	y_{old}	x_{new}	y_{new}
$-\frac{\pi}{2}$	VA	1	VA
$-\frac{\pi}{4}$	1	2	0
0	0	3	$\frac{1}{2}$
$\frac{\pi}{4}$	-1	4	1
$\frac{\pi}{2}$	VA	5	VA

Finally we plot each point and connect the dots with the shape of a secant graph.



Again, as long as you can track these five points and/or vertical asymptotes, you should be able to create the graph of a transformation of either of these trigonometric functions.

Notice that none of these graphs have any horizontal asmptotes, so none of these limits exist:

$\lim_{x\to\infty} \sec(x)$ $\lim_{x\to-\infty} \sec(x)$

$\lim_{x\to\infty} \csc(x)$ $\lim_{x\to-\infty} \csc(x)$

$\lim_{x\to\infty} \tan(x)$ $\lim_{x\to-\infty} \tan(x)$

$\lim_{x\to\infty} \cot(x)$ $\lim_{x\to-\infty} \cot(x)$

Of course knowing the graphs of these functions means that we can solve even more limits!

Example 4.4.4

Evaluate the following limits by considering the graphs of the trigonometric functions.

(a) $\lim_{x\to\pi} \sin\left(\frac{x}{6}\right) \tan\left(\frac{x}{4}\right)$

(b) $\lim_{t\to\frac{1}{4}^-} \frac{\cot(3\pi t)}{\tan(2\pi t)+1}$

(c) $\lim_{y\to\pi^+} \sec(y) \tan\left(\frac{y}{2}\right)$

(d) $\lim_{w\to\frac{4\pi}{3}^-} \frac{3w}{2\sin(w)+1}$

Solution

(a) $\lim_{x\to\pi} \sin\left(\frac{x}{6}\right) \tan\left(\frac{x}{4}\right) = \sin\left(\frac{\pi}{6}\right) \tan\left(\frac{\pi}{4}\right) = \frac{1}{2} \cdot 1 = \frac{1}{2}$

(b) $\lim_{t\to\frac{1}{4}^-} \frac{\cot(3\pi t)}{\tan(2\pi t)+1} = \frac{\cot(\frac{3\pi}{4})}{\lim_{z\to\frac{\pi}{2}^-} \tan(z)+1}$

$\tan\left(\frac{\pi}{2}\right)$ does not exist, but left of the vertical asymptote at $x = \frac{\pi}{2}$, the graph approaches $+\infty$

So our limit becomes $\frac{-1}{+\infty} = 0$

(c) $\lim\limits_{y \to \pi^+} \sec(y)\tan\left(\frac{y}{2}\right) = \sec(\pi) \cdot \lim\limits_{z \to \frac{\pi}{2}^+} \tan(z)$

$\tan\left(\frac{\pi}{2}\right)$ does not exist, but right of the vertical asymptote at $x = \frac{\pi}{2}$, the graph approaches $-\infty$

So our limit becomes $(-1) \cdot (-\infty) = +\infty$

(d) $\lim\limits_{w \to \frac{4\pi}{3}^-} \dfrac{3w}{2\sin(w)+1} = \dfrac{3 \cdot \frac{4\pi}{3}}{2 \lim\limits_{z \to \frac{4\pi}{3}^-} \left(\sin(z) + \frac{1}{2}\right)}$

Left of $\frac{4\pi}{3}$ sine goes to $-\frac{1}{2}$ from above, so $\sin(x) + \frac{1}{2} = 0^+$

This makes our overall limit $= \frac{4\pi}{2 \cdot 0^+} = +\infty$

Just like for rational functions, if the limit makes the bottom zero, just determine if it is a 0^+ or 0^-

Similarly if the limit is infinite, just make sure to pay attention to the $+$ and $-$ signs!

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4.5: Trigonometric Identities

Since these six trigonometric functions are all related to one another, there are often times we can describe the same function in multiple different ways. When limits with fractions of trigonometric functions, we may also want to simplify or reduce expressions. These are just a couple of reasons why it is important to investigate all of the different relationships between our six new functions.

We have already shown that each trig function can be written in terms of sine and cosine:

$$\tan(t)=\frac{\sin(t)}{\cos(t)}\quad \cot(t)=\frac{\cos(t)}{\sin(t)}\quad \sec(t)=\frac{1}{\cos(t)}\quad \csc(t)=\frac{1}{\sin(t)}$$

Also, remember that every point on the circle $x^2+y^2=1$ is of the form $(\cos(t),\sin(t))$ for some angle t. Plugging in $x=\cos(t)$ and $y=\sin(t)$ this gives us:

$\cos^2(t)+\sin^2(t)=1$ for all angles t

This equation can be divided by $\cos^2(t)$ or $\sin^2(t)$ on both sides:

$$\frac{\cos^2(t)}{\cos^2(t)}+\frac{\sin^2(t)}{\cos^2(t)}=\frac{1}{\cos^2(t)}$$

$$1+\left(\frac{\sin(t)}{\cos(t)}\right)^2=\left(\frac{1}{\cos(t)}\right)^2$$

$$1+\tan^2(t)=\sec^2(t)$$

$$\frac{\cos^2(t)}{\sin^2(t)}+\frac{\sin^2(t)}{\sin^2(t)}=\frac{1}{\sin^2(t)}$$

$$\left(\frac{\cos(t)}{\sin(t)}\right)^2+1=\left(\frac{1}{\sin(t)}\right)^2$$

$$\cot^2(t)+1=\csc^2(t)$$

All three of these equations are referred to as the *Pythagorean Identities*.

Using these identities, we can simplify functions to help with solving limits.

Example 4.5.1

Use the Pythagorean Identities to evaluate the limits below

(a) $\lim\limits_{x\to\pi}\dfrac{1+\cos(x)}{\sin(x)}$

(b) $\lim\limits_{w\to0^-}\dfrac{\cot(w)}{\csc(w)\left(\sec^2(w)-\tan^2(w)\right)}$

Solution

(a) Plugging in π gives $\frac{0}{0}$, we must reduce.

We do this by multiplying top and bottom by $1-\cos(x)$ to create a Pythagorean identity

$$\lim_{x\to\pi}\frac{1+\cos(x)}{\sin(x)}=\lim_{x\to\pi}\frac{1+\cos(x)}{\sin(x)}\cdot\frac{1-\cos(x)}{1-\cos(x)}=\lim_{x\to\pi}\frac{1-\cos^2(x)}{\sin(x)(1-\cos(x))}$$

$$=\lim_{x\to\pi}\frac{\sin^2(x)}{\sin(x)(1-\cos(x))}=\lim_{x\to\pi}\frac{\sin(x)}{1-\cos(x)}=\frac{\sin(\pi)}{1-\cos(\pi)}=\frac{0}{1-(-1)}=0$$

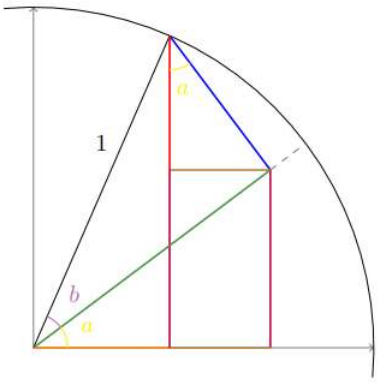
(b) Use the identity $\sec^2(w)=\tan^2(w)+1$ to see that $\sec^2(w)-\tan^2(w)=1$ and rewrite everything as sines and cosines:

$$\lim_{w\to0^-}\frac{\cot(w)}{\csc(w)\left(\sec^2(w)-\tan^2(w)\right)}=\lim_{w\to0^-}\frac{\frac{\cos(w)}{\sin(w)}}{\frac{1}{\sin(w)}}$$

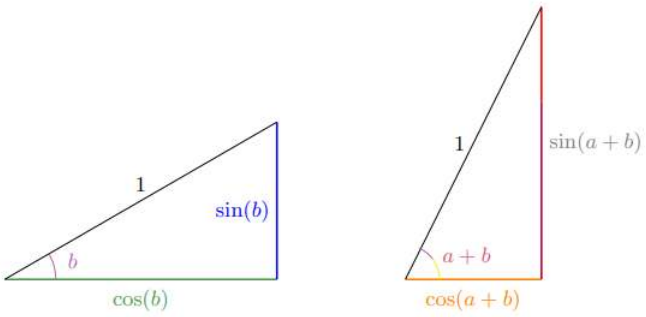
We now can multiply the top and bottom by $\sin(w)$ inside of the limit to get

$$= \lim_{w \to 0^-} \frac{\cos(w)}{1} = \cos(0) = 1$$

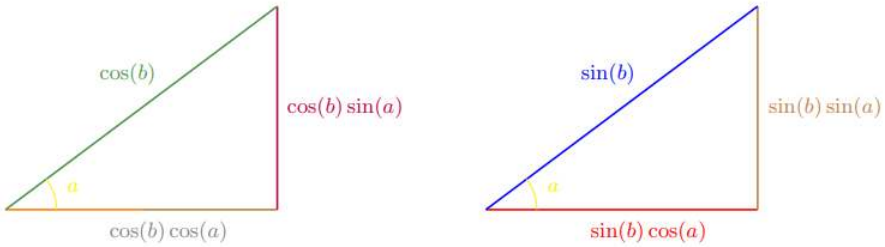
Beyond the standard and Pythagorean identities, there are a few others that can be useful in very specific situations. In order to derive them, consider the diagram in the unit circle below:



There is a lot going in in here, so lets draw the four right triangles separately. We will start with the two of them that have hypotenuse \$1\$ and solve the missing sides:



This gives us the hypotenuse of the two triangles with angle a, and again we find the rest:



Look back at the original diagram to see that the red and purple sides should add up to $\sin(a+b)$, so:
$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$

And we also see that the brown and orange sides must add up to $\cos(b)\cos(a)$, which gives us:
$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

These are known as the additive angle identities for sine and cosine.

These formulas can be used to evaluate trig functions for any multiple of $\frac{\pi}{12}$ radians.

The key is to write it as a sum or difference of angles that reduce to be on the unit circle.

Example 4.5.1

Evaluate each of the following trigonometric expressions by using the sum and difference identities for sine and cosine.

(a) $\sin\left(\frac{5\pi}{12}\right)$

(b) $\cos\left(\frac{11\pi}{12}\right)$

(c) $\tan\left(\frac{7\pi}{12}\right)$

(d) $\csc\left(\frac{17\pi}{12}\right)$

Solution

(a) First we note that $\frac{5\pi}{12} = \frac{3\pi}{12} + \frac{2\pi}{12}$

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(b) First we note that $\frac{11\pi}{12} = \frac{15\pi}{12} + \frac{-4\pi}{12}$

$$\cos\left(\frac{11\pi}{12}\right) = \cos\left(\frac{5\pi}{4} + \frac{-\pi}{3}\right) = \cos\left(\frac{5\pi}{4}\right)\cos\left(-\frac{\pi}{3}\right) - \sin\left(\frac{5\pi}{4}\right)\sin\left(-\frac{\pi}{3}\right)$$

$$= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = \frac{-\sqrt{2}-\sqrt{6}}{4}$$

(c) First we note that $\frac{7\pi}{12} = \frac{3\pi}{12} + \frac{4\pi}{12}$

$$\tan\left(\frac{7\pi}{12}\right) = \frac{\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)} = \frac{\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)-\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)}$$

$$= \frac{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)} = \frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}}$$

(d) First we note that $\frac{17\pi}{12} = \frac{8\pi}{12} + \frac{9\pi}{12}$

$$\csc\left(\frac{17\pi}{12}\right) = \frac{1}{\sin\left(\frac{2\pi}{3}+\frac{3\pi}{4}\right)} = \frac{1}{\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{3\pi}{4}\right)+\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{2\pi}{3}\right)}$$

$$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)} = \frac{4}{-\sqrt{6}-\sqrt{2}}$$

In the case where $a = b = x$, we can use the additive identities to get a double-angle formula:

$$\sin(2x) = \sin(x+x) = \sin(x)\cos(x) + \sin(x)\cos(x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos(x+x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x)$$

While we could technically repeat this process to find the triple or quadruple angle formula for either sine or cosine, they are not nearly as useful as this double angle formula. Specifically for cosine, since it can be re-written two alternative ways:

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2(x) - (1 - \cos^2(x)) = 2\cos^2(x) - 1$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = (1 - \sin^2(x)) - \sin^2(x) = 1 - 2\sin^2(x)$$

For each of these two identities, we can solve for sine or cosine to get:

$$\cos(2x) = 2\cos^2(x) - 1 \implies \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\cos(2x) = 1 - 2\sin^2(x) \implies \sin^2(x) = \frac{1 - \cos(2x)}{2}$$

Now we square root both sides and plug in $x = \dfrac{t}{2}$ to get the *half-angle identities*:

$$\cos\left(\frac{t}{2}\right) = \pm\sqrt{\frac{1 + \cos(t)}{2}}$$

$$\sin\left(\frac{t}{2}\right) = \pm\sqrt{\frac{1 - \cos(t)}{2}}$$

Please note that the $\pm$ is determined by which quadrant angle $\dfrac{t}{2}$ is in!

Just like for the additive identities, there is no need to find these formulas for any of the other four trigonometric functions. Everything can be reduced down in terms of sines and cosines!

Example 4.5.1

Use identities to evaluate the following limits

(a) $\lim\limits_{t\to\pi} \dfrac{\sin(2t)}{\sin(t)}$

(b) $\lim\limits_{v\to 0^-} \dfrac{\sin(\pi + v)}{1 - \cos^2(v)}$

Solution

(a) We start off using the double angle identity for sine, and the rest flows easily:

$$\lim_{t\to\pi} \frac{\sin(2t)}{\sin(t)} = \lim_{t\to\pi} \frac{2\sin(t)\cos(t)}{\sin(t)} = \lim_{t\to\pi} 2\cos(t) = 2\cos(\pi) = -2$$

(b) Once we use the additive angle identity for sine, again this simplifies easily:

$$\lim_{v\to 0^-} \frac{\sin(\pi + v)}{1 - \cos^2(v)} = \lim_{v\to 0^-} \frac{\sin(\pi)\cos(v) + \sin(v)\cos(\pi)}{\sin^2(v)} = \lim_{v\to 0^-} \frac{-\sin(v)}{\sin^2(v)} = \lim_{v\to 0^-} \frac{-1}{\sin(v)}$$

Since $\sin(0) = 0$ we look at at the graph of sine left of $x = 0$ and see it is negative, thus:

$$= \frac{-1}{0^-} = +\infty$$

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CHAPTER OVERVIEW

5: Continuity

> ”Continuous learning is the minimum requirement for success in any field.” - Brian Tracy

We now have a wide variety of functions at our disposal, and the ability to take limits of them. Chapter five will introduce us to the definition of continuity as well as discuss what being a continuous or discontinuous function will look like when graphed.

We say that functions that agree with their limit at a given point are continuous. We start off by defining the four different types of discontinuities: removable, infinite, jump, and oscillating. This allows us to appreciate fully what it means for a function to be continuous.

This naturally leads us to discuss the only exciting continuity examples: piecewise defined functions. These Frankenstein functions are made by putting together different parts of different functions. These will allow us to blend together some of the previous functions we have discovered, and still use limits to understand them.

We have a brief review for solving systems of equations, and are introduced to the most famous piecewise function: the absolute value function. We ensure we can solve equations and inequalities using the absolute value, as well as take limits of functions which use them as a component.

We finish out with our first theorem in the book, the Intermediate Value Theorem. This helps us understand why being a continuous function is important, and lets us know if a solution exists to an equation in certain situations.

5.1: Continuous Functions
5.2: Piecewise Functions
5.3: Absolute Value Functions
5.4: The Intermediate Value Theorem

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5.1: Continuous Functions

We have defined many functions so far: polynomials, rational functions, and the trigonometric functions. For these functions we can find domain D, and take limits for all values of $a \in D$.

When we do (for these specific functions mentioned) we have $\lim_{x \to a} f(x) = f(a)$

This is a property called *continuity*.

We say a function $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$

We also say that $f(x)$ is continuous on an interval I if $\lim_{x \to a} f(x) = f(a)$ for every $a \in I$

As long as you can take a limit of $f(x)$ at a, then you can determine if it is continuous at $x = a$:

Example 5.1.1

Where is $f(x) = \dfrac{x-1}{x^2-1}$ continuous?

Solution

First find its domain by factoring denominator and setting it equal to zero

$(x-1)(x+1) = 0 \to x = 1, -1$ so domain $D = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

For any $a \in D$ we know that $\lim_{x \to a} f(x) = \lim_{x \to a} \dfrac{x-1}{x^2-1} = \dfrac{a-1}{a^2-1} = f(a)$
so the limit exists and matches $f(a)$, we are continuous at $x = a$

For $a = 1$: $\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{x-1}{(x-1)(x+1)} = \lim_{x \to 1} \dfrac{1}{x+1} = \dfrac{1}{2}$
so the limit exists but $f(1)$ does not, so we are not continuous at $x = 1$

For $a = -1$: $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \dfrac{1}{x+1} = \dfrac{1}{0^+} = +\infty$
and also: $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \dfrac{1}{x+1} = \dfrac{1}{0^-} = -\infty$
so $\lim_{x \to -1} f(x)$ does not exist, and also $f(-1)$ does not exist

DNE=DNE does not count, so we are not continuous at $x = -1$

Therefore our function $f(x)$ is continuous on its domain D, and not continuous for any $a \notin D$

This is not unique to the example function above. Every function we have encountered thus far is continuous on its entire domain, and is not continuous for any numbers outside of that domain.

Even better than that, if we have two functions $f(x)$ and $g(x)$ that are continuous at $x = a$, then their sum, difference, or product will be continuous at $x = a$ as well:

$$\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a)$$

$$\lim_{x \to a} f(x) - g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = f(a) - g(a)$$

$$\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = f(a) \cdot g(a)$$

The same cannot be said about division, but its close:

As long as $f(x)$ and $g(x)$ are continuous at $x = a$ **and** $g(a) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)}$

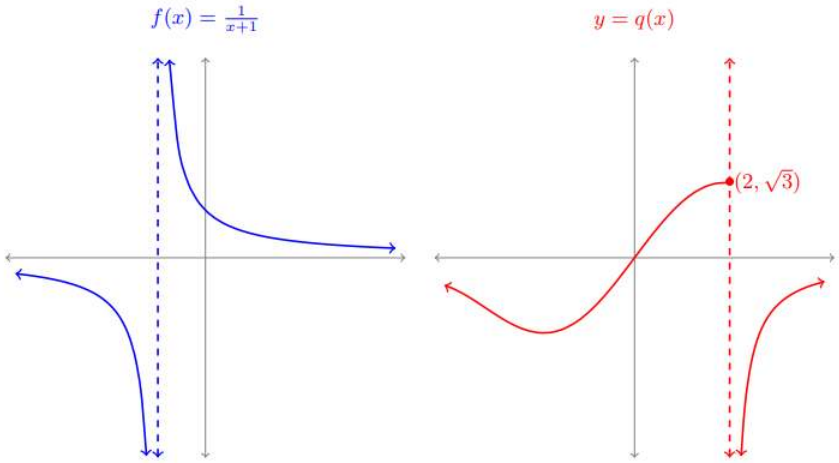
So we can say the quotient of two continuous functions is continuous wherever it exists.
(and if it does not exist, it cannot be continuous)

To better understand what it means to be a continuous function, let us explore how to fail at being continuous instead. If $\lim_{x\to a} f(x) \neq f(a)$, we say there is a *discontinuity* at $x = a$.

These come in a few different flavors, depending on how they fail the definition of continuity.

If $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$, then it is impossible for $f(a)$ to match it (since $f(a)$ must always be a number). This type of discontinuity is called an *infinite discontinuity*.

(As we have seen previously, these are exactly what give our graph a vertical asymptote at $x = a$)



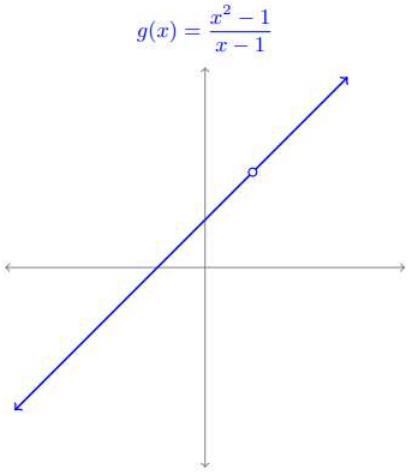
$\lim_{x\to -1^+} f(x) = +\infty$
$\lim_{x\to -1^-} f(x) = -\infty$
$f(-1) = DNE$
Infinite Discontinuity at $x = -1$

$\lim_{x\to 2^+} q(x) = -\infty$
$\lim_{x\to 2^-} q(x) = \sqrt{3}$
$q(2) = \sqrt{3}$
Infinite Discontinuity at $x = 2$

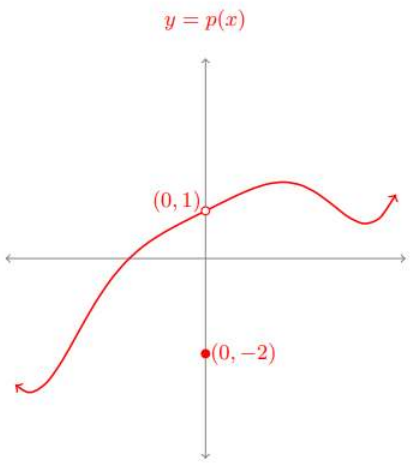
Another possibility that we have seen already is when $\lim_{x\to a} f(x)$ exists, but does not match $f(a)$. (This can be because $f(a)$ is the wrong value or does not exist)

In this situation we say that $f(x)$ has a *removable discontinuity* at $x = a$.

(Previously this is exactly what caused a rational function to have a hole in its graph)



$\lim_{x\to 1^+} g(x) = 2$
$\lim_{x\to 1^-} g(x) = 2$
$g(1) = DNE$
Removable Discontinuity at $x = 1$



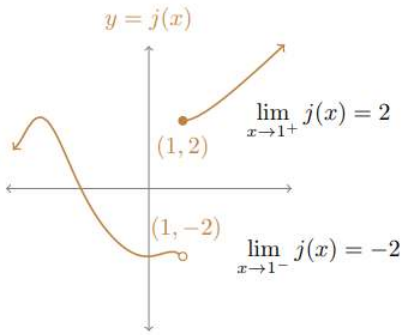
$\lim_{x\to 0^+} p(x) = 1$
$\lim_{x\to 0^-} p(x) = 1$
$q(0) = -2$
Removable Discontinuity at $x = 0$

There are a couple of other types of discontinuities, but for functions we haven't encountered yet.

It is possible that $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist (and are finite) but do not match.

This would cause $\lim_{x\to a} f(x) = DNE$ which can never equal $f(a)$

We call this a *jump discontinuity*. We can visualize an example below:



The last type of discontinuity we will discuss is a very rare one.
Although we haven't seen it occur yet, it is technically possible for a one-sided limit to not exist.

Clearly if this is the case, then the double sided limit cannot exist, and we will once again fail to have $\lim_{x\to a} f(x) = f(a)$ no matter what $f(a)$ is (or even if it does not exist). These are called *oscillating discontinuities* for reasons that will become apparent shortly.

Consider the function $f(x) = \sin\left(\frac{\pi}{2x}\right)$

There is only one number that cannot be plugged into this function, and that is $x = 0$.

So to investigate the discontinuity at $x = 0$, we start by taking a one-sided limit: $\lim_{x\to 0^+} \sin\left(\frac{\pi}{2x}\right)$

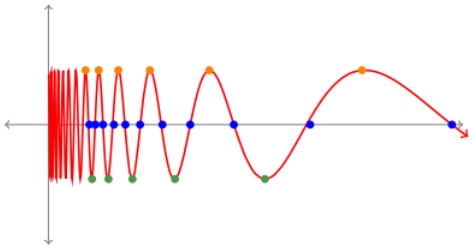
x	$f(x)$
$\frac{1}{5}$	$\sin\left(\frac{5\pi}{2}\right)=1$
$\frac{1}{9}$	$\sin\left(\frac{9\pi}{2}\right)=1$
$\frac{1}{13}$	$\sin\left(\frac{13\pi}{2}\right)=1$
$\vdots$	$\vdots$
$\frac{1}{4n+1}$	$\sin\left(\frac{(4n+1)\pi}{2}\right)=1$

x	$f(x)$
$\frac{1}{2}$	$\sin(\pi)=0$
$\frac{1}{4}$	$\sin(2\pi)=0$
$\frac{1}{6}$	$\sin(3\pi)=0$
$\vdots$	$\vdots$
$\frac{1}{2n}$	$\sin(n\pi)=0$

x	$f(x)$
$\frac{1}{3}$	$\sin\left(\frac{3\pi}{2}\right)=-1$
$\frac{1}{7}$	$\sin\left(\frac{7\pi}{2}\right)=-1$
$\frac{1}{11}$	$\sin\left(\frac{11\pi}{2}\right)=-1$
$\vdots$	$\vdots$
$\frac{1}{4n-1}$	$\sin\left(\frac{(4n-1)\pi}{2}\right)=-1$

Notice that each of these three columns of numbers approach $x=0$ from the right, but they cause the function to approach different answers!

Below is a graph of $f(x)$ so you can see how this is possible:



As $x \to 0^+$, this function does not get close to any one specific number.
Instead it oscillates faster and faster within the interval $[-1,1]$.

This is why we say $\lim\limits_{x\to 0^+} f(x) = DNE$

Therefore the function $f(x)=\sin\left(\dfrac{\pi}{2x}\right)$ has an oscillating discontinuity at $x=0$

When checking a given function for discontinuities, remember to look at: $\lim\limits_{x\to a^+} f(x)$, $\lim\limits_{x\to a^-} f(x)$, and $f(a)$

If they all match, $f(x)$ is continuous at $x=a$.
If they do not, it will be exactly one of these four types of discontinuities.

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5.2: Piecewise Functions

While continuity is pretty boring for most of the functions we have discussed thus far, there is one family of functions which can lead to any of the four different types of discontinuities.

Piecewise functions are created by taking different pieces of functions on different parts of the domain. They are typically written like something below:

$$f(x)=\begin{cases} A(x) & x>c \\ B(x) & x\le c \end{cases} \quad g(x)=\begin{cases} z(x) & x\neq d \\ 0 & x=d \end{cases} \quad h(x)=\begin{cases} p(x) & x\le \alpha \\ q(x) & \alpha<x\le\beta \\ r(x) & x>\beta \end{cases}$$

(Each line inside tells the rule to follow first, and the values of x that follow that rule second.)

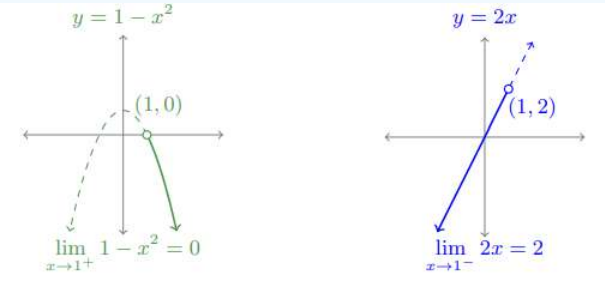
We can use our limit taking skills for the individual pieces in order to graph a piecewise function.

Example 5.2.1

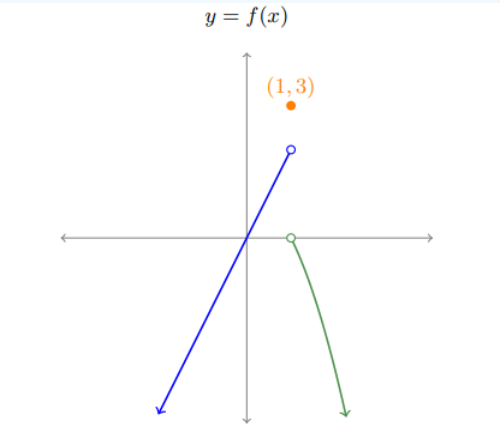
Sketch $f(x)=\begin{cases} 1-x^2 & x>1 \\ 3 & x=1 \\ 2x & x<1 \end{cases}$

Solution

We start by graphing the individual functions inside of our piecewise



Then plot the parts of them that are used together on on graph:



Notice the graph above has a jump discontinuity at $x=1$ since:
$\lim_{x\to1^-} f(x)=\lim_{x\to1^-} 2x=2$ and $\lim_{x\to1^+} f(x)=\lim_{x\to1^+} 1-x^2=0$

To determine where a piecewise function is continuous, we check each of the pieces separately on their given intervals, and then we check the points where the functions (should) meet:

Example 5.2.2

Find all discontinuities for $f(x) = \begin{cases} \sin(x) & x < 0 \\ 3x & 0 < x \leq 2 \\ \dfrac{x-2}{x^2-4} & x > 2 \end{cases}$

Solution

For $a > 2$: $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} \dfrac{x-2}{x^2-4} = \dfrac{a-2}{a^2-4} = f(a)$
So $f(x)$ is continuous for $x > 2$

For $0 < a < 2$: $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} 3x = 3a = f(a)$
So $f(x)$ is continuous for $0 < x < 2$

For $a < 0$: $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} \sin(x) = \sin(a) = f(a)$
So $f(x)$ is continuous for $x < 0$

Now we check the points where these pieces meet:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x-2}{x^2-4} = \lim_{x \to 2^+} \frac{x-2}{(x-2)(x+2)} = \lim_{x \to 2^+} \frac{1}{x+2} = \frac{1}{4}$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 3x = 6$$

Therefore $f(x)$ has a jump discontinuity at $x = 2$

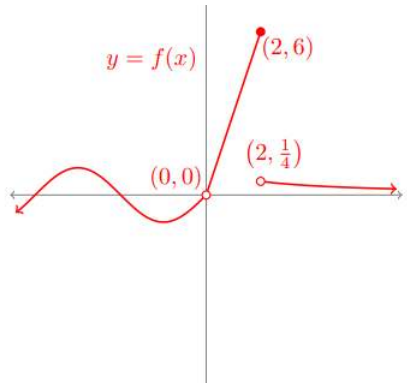
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 3x = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sin(x) = 0$$

But $f(0)$ is not defined, so it does not match this limit.

Therefore $f(x)$ has a removable discontinuity at $x = 0$

Here is the graph of the function above, which confirms our answers.



We can also solve for constants in piecewise functions to ensure continuity:

Example 5.2.3

For what α is $f(x)=\begin{cases}\dfrac{x^2+4x-5}{x^2-1} & x\neq 1\\ \alpha & x=1\end{cases}$ continuous at $x=1$?

Solution

To be continuous we want $\lim\limits_{x\to 1} f(x)=f(1)$

We can see that $f(1)=\alpha$, so we find the limit:

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x^2+4x-5}{x^2-1}=\lim_{x\to 1}\frac{(x+5)(x-1)}{(x+1)(x-1)}=\lim_{x\to 1}\frac{x+5}{x+1}=3$$

Thus, we need $\alpha=3$

This can even be done with more unknowns. (You'll just need more equations)

Example 5.2.4

For what c and d is $g(x)=\begin{cases}x^2+cx-2 & x>3\\ 5 & x=3\\ x+d\cos(\pi x) & x<3\end{cases}$ continuous at $x=3$

Solution

To be continuous we want $\lim\limits_{x\to 3} g(x)=g(3)$

We can see that $g(3)=5$, so we find both sides of the limit:

$$\lim_{x\to 3^+} g(x)=\lim_{x\to 3^+} x^2+cx-2=3^2+c\cdot 3-2=3c+7$$

Thus, we need $3c+7=5$, which means $c=-\dfrac{2}{3}$

$$\lim_{x\to 3^-} g(x)=\lim_{x\to 3^-} x+d\cos(\pi x)=3+d\cos(3\pi)=3-d$$

Thus, we need $3-d=5$, which means $d=-2$

So $g(x)$ is continuous at $x=3$ when $c=-\dfrac{2}{3}$ and $d=-2$

As we get more unknowns to solve for, we may end up with equations that involve more than one variable at a time. These are called *systems of equations*.

There are a few different methods for solving a system of equations, but we will focus on just two: **substitution** and **elimination**

Example 5.2.5

Find all solutions to the system of equations $\begin{cases}a+b=5\\ 2a-b=7\end{cases}$

Substitution Solution

First, we solve for one letter in terms of the other in any equation:

$a+b=5\to b=5-a$

Then we substitute it into the other equation:

$2a-(5-a)=7$

$3a-5=7$

$3a=12$

$a=4$

Now we use $b=5-a$ to get $b=1$, and our solution must be $a=4, b=1$

Elimination Solution

Add both equations together:

$(a+b=5)+(2a-b=7)=(3a=12)$

$3a=12 \to a=4$

Then plug in $a=4$ into *either equation* to solve for $b=1$, so our solution is again $a=4, b=1$

Example 5.2.6

Find all solutions to the system of equations $\begin{cases} 2u+3v=1 \\ u=v+3 \end{cases}$

Substitution Solution

This time we start off with one letter in terms of the other: $u=v+3$

Then we substitute it into the other equation:

$2(v+3)+3v=1$

$5v+6=1$

$5v=-5$

$v=-1$

Now we use $u=v+3$ to get $u=2$, and our solution must be $u=2, v=-1$

Elimination Solution

Subtract twice the bot equation from the top:

$(2u+3v=1)-2(u=v+3)=(3v=-2v-5)$

$3v=-2v-5 \to v=-1$

Then plug in $v=-1$ into *either equation* to solve for $u=2$, so our solution is again $u=2, v=-1$

Lets try to use this in another continuity example:

Example 5.2.7

For what m and b is $f(x)=\begin{cases} x^2 & x \geq 2 \\ mx+b & 1 \leq x < 2 \\ \dfrac{x-1}{x^2-3x+2} & x<1 \end{cases}$ continuous?

Solution

To be continuous we want $\lim\limits_{x\to 1} f(x)=f(1)$ and $\lim\limits_{x\to 2} f(x)=f(2)$

First we check both sides at $x=1$:

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+} mx+b=m\cdot 1+b=m+b$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x-1}{x^2-3x+2} = \lim_{x\to 1^-} \frac{x-1}{(x-2)(x-1)} = \lim_{x\to 1^-} \frac{1}{x-2} = -1$

$f(1) = m\cdot 1 + b = m + b$

We need all three of these to match, so $m + b = -1$

Now we check both sides at $x = 2$:

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 = 2^2 = 4$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} mx + b = m\cdot 2 + b = 2m + b$

$f(2) = 2^2 = 4$

We need all three of these to match, so $2m + b = 4$

Now we solve the system of equations $\begin{cases} m + b = 1 \\ 2m + b = 4 \end{cases}$

$(2m + b = 4) - (m + b = 1) = (m = 3)$

Plugging this into either equation gives $b = -2$

Therefore we need $m = 3$ and $b = -2$ to make $f(x)$ continuous everywhere

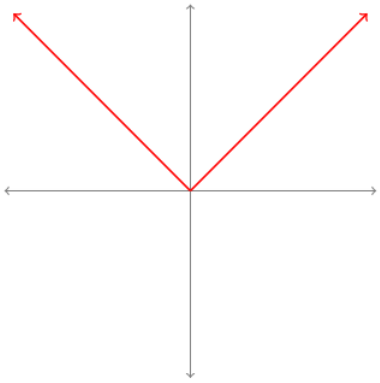
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5.3: Absolute Value Functions

The most popular piecewise defined function is the *absolute value* function.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Clearly the domain of this function is $(-\infty, \infty)$, and we can see its graph below:



Solving equations with absolute values require us to solve two separate equations (replacing the absolute value term with $+$ or $-$ itself) and then check our answers.

Example 5.3.1

Solve the following equations involving absolute values.

(a) $|x+2| = 3$

(b) $|t+1| = 2t-3$

Solution

(a) Since the inside could be positive or negative, we have two equations to solve:

$x+2=3$ and $-(x+2)=3$

This gives us two possible solutions:

$x=1$ and $x=-5$

We must check our answers to see if they indeed satisfy the original equation:

$|1+2| \stackrel{?}{=} 3 \implies 3 \stackrel{?}{=} 3$ True

$|-5+2| \stackrel{?}{=} 3 \implies 3 \stackrel{?}{=} 3$ True

Since they are both true, we have two solutions to the equation: $x = 1, -5$

(b) Since the inside could be positive or negative, we have two equations to solve:

$t+1=2t-3$ and $-(t+1)=2t-3$

This gives us two possible solutions:

$t=4$ and $t=\frac{2}{3}$

We must check our answers to see if they indeed satisfy the original equation:

$|4+1| \stackrel{?}{=} 2\cdot 4-3 \implies 5 \stackrel{?}{=} 5$ True

$\left|\frac{2}{3}+1\right| \stackrel{?}{=} 2\cdot\frac{2}{3}-3 \implies \frac{5}{3} \stackrel{?}{=} -\frac{5}{3}$ False

Therefore we have only one solution to the equation: $t=4$

Solving inequalities can be a little more work, but we can create something similar to a sign chart to get our answers. First, solve the inequality as if it were an equation, then plot the solutions to the equation on a number line, and finally check each interval to see if the original inequality holds.

Example 5.3.2

Solve each inequality involving absolute values below and put your answer into interval notation.

(a) $|1-\frac{1}{2}u| \geq 2$

(b) $|y| > 2y-3$

Solution

(a) First, we pretend the $\geq$ symbol is an $=$ instead and solve for u:

$|1-\frac{1}{2}u| = 2$

This turns into two equations $1-\frac{1}{2}u = 2$ and $-(1-\frac{1}{2}u) = 2$

We solve them each separately to get $u=-2$ and $u=6$

Then we check that our answers satisfy the absolute value equation (which they do in this instance)

We will make a number line for a sign chart, and split it up at these two solutions:



In each interval between these roots, we pick any number and see if it satisfies the original inequality

In the left most interval $(-\infty,-2)$, we will pick $u=-4$

$|1-\frac{1}{2}(-4)| \stackrel{?}{\geq} 2 \implies 3 \stackrel{?}{\geq} 2$ which is True

This means everything in $(-\infty,-2)$ will make the inequality true.

In the middle interval $(-2,6)$, we will pick $u=0$

$|1-\frac{1}{2}(0)| \stackrel{?}{\geq} 2 \implies 1 \stackrel{?}{\geq} 2$ which is False

This means nothing in $(-2,6)$ will make the inequality true.

In the right most interval $(6,\infty)$, we will pick $u=8$

$|1-\frac{1}{2}(8)| \stackrel{?}{\geq} 2 \implies 3 \stackrel{?}{\geq} 2$ which is True

This means everything in $(-2,6)$ will make the inequality true.

Lastly, since our inequality was not strict, we know our solutions to the equation are also included in our final answer:

$(-\infty,-2]\cup[6,\infty)$

(b) Again we start by pretending the $<$ symbol is an $=$ and solving:

$|y| = 2y-3$

This turns into two equations $y = 2y-3$ and $-y = 2y-3$

We solve them each separately to get $y=3$ and $y=1$

Then we check that our answers satisfy the absolute value equation (which $y=1$ does not)

This time when we make a number line for a sign chart, we split it at only the legitimate solution $y=3$:



In each interval between these roots, we pick any number and see if it satisfies the original inequality

In the left interval $(-\infty, 3)$, we will pick $y=0$

$|0| \overset{?}{>} 2\cdot 0-3 \implies 0 \overset{?}{>} -3$ which is True

This means everything in $(-\infty, 3)$ will make the inequality true.

In the right most interval $(3, \infty)$, we will pick $y=10$

$|10| \overset{?}{>} 2\cdot 10-3 \implies 10 \overset{?}{>} 17$ which is False

This means nothing in $(3, \infty)$ will make the inequality true.

Lastly, since our inequality was strict, we know our solutions to the equation are not included in our final answer:

$(-\infty, 3)$

This is similar to the sign charts we used for polynomials and rational functions. Those sign charts worked because the only way for the functions to switch from positive to negative was by having it equal to zero. These truth charts work because the only way the inequality can change between being true and being false is at a point where the equation (with the $=$ in it) would hold.

The good news is that the absolute value function is continuous everywhere. Even better, we can say that $\lim\limits_{x\to a}|f(x)| = \left|\lim\limits_{x\to a} f(x)\right|$ whenever the latter exists.

When in doubt, just write it as a piecewise function and you'll be fine!

Example 5.3.3

Evaluate each limit below involving absolute values

(a) $\lim\limits_{t\to -3} \dfrac{t^2+1}{1+|t+5|}$

(b) $\lim\limits_{w\to 2^-} \dfrac{2-|2w-6|}{w-2}$

Solution

(a) We start by writing the absolute value as a piecewise function $|t+5| = \begin{cases} t+5 & t\geq -5 \\ -t-5 & t<-5 \end{cases}$

Then use it to write the entire function as piecewise: $\dfrac{t^2+1}{1+|t+5|} = \begin{cases} \dfrac{t^2+1}{1+(t+5)} & t>-5 \\ \dfrac{t^2+1}{1+(-t-5)} & t<-5 \end{cases}$

For all values of t near -3, we will be using the top piece:

$$\lim_{t\to -3} \frac{t^2+1}{1+|t+5|} = \lim_{t\to -3} \frac{t^2+1}{1+(t+5)} = \lim_{t\to -3} \frac{t^2+1}{t+6} = \frac{(-3)^2+1}{(-3)+6} = \frac{10}{3}$$

(b) We start by writing the absolute value as a piecewise function $|2w-6| = \begin{cases} 2w-6 & w \geq 3 \\ 6-2w & w < 3 \end{cases}$

Then use it to write the entire function as piecewise: $\dfrac{2-|2w-6|}{w-2} = \begin{cases} \dfrac{2-(2w-6)}{w-2} & w \geq 3 \\ \dfrac{2-(6-2w)}{w-2} & w < 3 \end{cases}$

For all values of w left of 2, we will be using the bottom piece:

$$\lim_{w\to 2^-} \frac{2-|2w-6|}{w-2} = \lim_{w\to 2^-} \frac{2-(6-2w)}{w-2} = \lim_{w\to 2^-} \frac{2w-4}{w-2} = \lim_{w\to 2^-} \frac{2(w-2)}{w-2} = \lim_{w\to 2^-} 2 = 2$$

Limits of absolute values can always be done in this way, and with practice you won't need to write out the piecewise definition every time. (But it never hurts to do so!)

We can use also limits to find any discontinuities for functions involving the absolute value.

Example 5.3.4

Find all discontinuities for each function below.

(a) $f(x) = \dfrac{|x|}{x}$

(b) $g(x) = \dfrac{x^2-1}{|x|-1}$

Solution

(a) Since $\lim_{x\to a} |x| = |a|$ and $\lim_{x\to a} x = a$ for all a

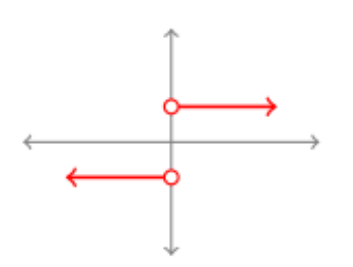
then $\lim_{x\to a} f(x) = \lim_{x\to a} \dfrac{|x|}{x} = \dfrac{|a|}{a} = f(a)$ for $a \neq 0$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{(-x)}{x} = \lim_{x\to 0^+} -1 = -1$$

Lastly, $f(0) = DNE$

Thus we have a jump discontinuity at $x = 0$ which we can see in the graph of $y = f(x)$ below



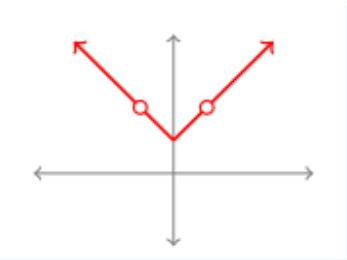
(b) Again $\lim_{x\to a} g(x) = \lim_{x\to a} \dfrac{x^2-1}{|x|-1} = \dfrac{a^2-1}{|a|-1} = g(a)$ for $|a| \neq 1$

$$\lim_{x\to 1} \frac{x^2-1}{|x|-1} = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} x+1 = 2$$

But $g(1) = DNE$ so we have a removable discontinuity at $x = 1$

$$\lim_{x\to -1} \frac{x^2-1}{|x|-1} = \lim_{x\to -1} \frac{x^2-1}{-x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{-(x+1)} = \lim_{x\to -1} -(x-1) = 2$$

But $g(-1) = DNE$ so we have a removable discontinuity at $x = -1$ which we can see in the graph of $y = g(x)$ below



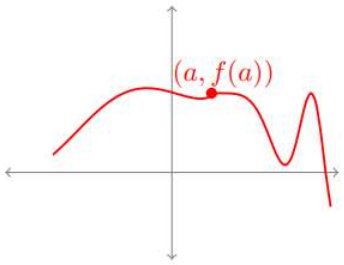
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5.4: The Intermediate Value Theorem

We are now well practiced at the art of finding out if a function is continuous or not.

We have seen the four different flavors of discontinuity, but what does it look like to be continuous?

If $\lim_{x \to a} f(x)$ exists and matches $f(a)$, then we must have a filled in hole at $(a, f(a))$

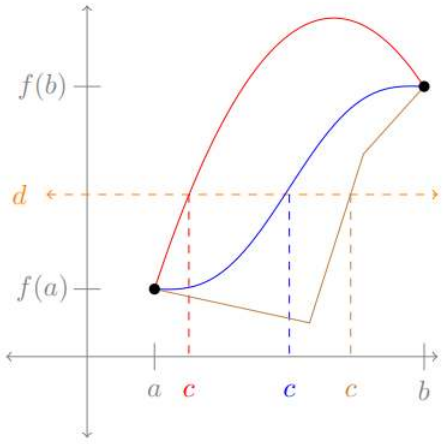


A common way of describing a function that is continuous at $x = a$ is that you can draw the curve through the point $(a, f(a))$ without lifting your pencil.

Remember a function is continuous on the interval $[a, b]$ if it continuous at every point in the interval, meaning you can connect $(a, f(a))$ to $(b, f(b))$ without lifting your pencil.

There is something very nice about a function being continuous on a closed interval, and that is that it must achieve all outputs between $f(a)$ and $f(b)$.
This result is known as the *Intermediate Value Theorem*:

If $f(x)$ is continuous on $[a, b]$ then for any value d between $f(a)$ and $f(b)$, there exists some $c \in (a, b)$ such that $f(c) = d$



This theorem says that for any orange line at height d, the graph must cross that line to connect the two dots $(a, f(a))$ to $(b, f(b))$ at least once in our interval, regardless of what $f(x)$ looks like.

This theorem does not apply when d is bigger or smaller than both of $f(a)$ and $f(b)$
(Or if not continuous, we can cross the line using a the hole, a jump, or an infinite discontinuity.)

Lets try a few examples:

Example 5.4.1

Prove there is at least one solution to the equation $x^4 - 2x^3 - 4x^2 - 4x = -4$

Solution

Notice first that $f(x) = x^4 - 2x^3 - 4x^2 - 4x$ is a polynomial which means that it is continuous everywhere

We can plug in any input we want at first (so pick an easy one):

$f(0) = 0^4 - 2 \cdot 0^3 - 4 \cdot 0^2 - 4 \cdot 0 = 0$
This is larger than the output we want $f(0) > -4$

Now we want to plug in a number that makes $f(x) < -4$ (keep guessing until you find one that works!)

$f(1) = 1^4 - 2 \cdot 1^3 - 4 \cdot 1^2 - 4 \cdot 1 = -9$
This is smaller than the output we want $f(1) < -4$

Since $f(1) < -4 < f(0)$ and $f(x)$ is continuous on $[0, 1]$, there must be some $c \in (0, 1)$ such that $f(c) = -4$ by IVT.

Remember IVT can only tell you a function equals a number, so you need all your variables on one side of an equation to use it.

Example 5.4.2

Prove there is at least one solution to the equation $t = \cos(t)$

Solution

This is equivalent to showing that $t - \cos(t) = 0$ has a solution

The function $g(t)=t-\cos(t)$ is a continuous everywhere

$g(0) = 0 - \cos(0) = -1 < 0$

Now we want to plug in a number that makes $g(t) > 0$

$g(\pi) = \pi - \cos(\pi) = \pi + 1 > 0$

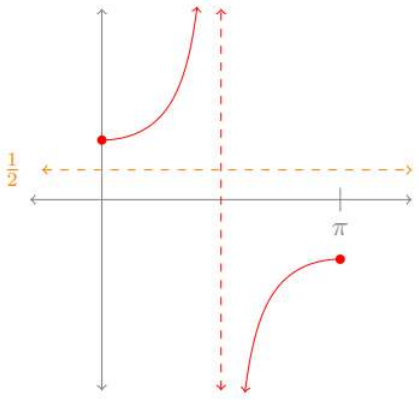
Since $g(0) < 0 < g(\pi)$ and $g(x)$ is continuous on $[0, \pi]$, there must be some $c \in (0, \pi)$ such that $g(c) = 0$ by IVT.

There are many different choices of inputs you could make that would still give sufficient proof in the questions above. Just make sure the outputs have the value of d you want between them!

Most importantly, you need to ensure your function is continuous or else IVT does not apply:

$f(x) = \sec(x)$ has $f(0) = \sec(0) = 1$ and $f(\pi) = \sec(\pi) = -1$
Since $f(\pi) < \frac{1}{2} < f(0)$, you might believe that there is some $c \in (0, \pi)$ such that $f(c) = \frac{1}{2}$ by IVT.

But look at the graph of secant on $[0, \pi]$ below:



Since $f(x)$ is not continuous at $x = \dfrac{\pi}{2}$ (infinite discontinuity), it is not continuous on the interval $[0, \pi]$ and thus IVT cannot tell us that $f(x) = \dfrac{1}{2}$ must have a solution (in fact we see that it doesnt)

IVT can even be (semi) useful in real life scenarios:

Example 5.4.3

A man goes on a two day hiking trip up a mountain. He starts at 9 AM on Saturday and reaches the summit by 5 PM. He spends the night at the top of the hill, and then begins hiking down at 9 AM and reaches the bottom by 5 PM. Prove that there is at least one time of day that he is at the exact same elevation on both Saturday and Sunday.

Solution

This seems far away from functions and the IVT, but it still applies:

Define $f(t)$ to be the elevation at time t on Saturday minus the elevation at time t on Sunday
(we will let $t = 0$ be 9 AM and $t = 8$ represent 5 PM)

If the top of the mountain has elevation $H > 0$, then we have:
$f(0) = 0 - H = -H < 0$ and $f(8) = H - 0 = H > 0$

Most importantly, while hiking your elevation should be a continuous function of time. (unless you can teleport up the mountain somehow)

Therefore $f(t)$ is continuous on $[0, 8]$

Since $f(0) < 0 < f(8)$, there exists some $c \in (0, 8)$ such that $f(c) = 0$ by IVT

Therefore c hours after 9 AM we know he is at the exact same elevation on Saturday and Sunday.

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CHAPTER OVERVIEW

6: Function Algebra

> *"As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection." - Joseph-Louis Lagrange*

Chapter six takes a step back to look at how new functions are formed out of old ones. We look at how addition, subtraction, multiplication, and division have inspired almost all of the functions we have seen thus far, and how they impact their domains as well.

We turn our attention to a fifth operation between two functions: composition. This naturally leads to a discussion about one-to-one and inverse functions. The reader is given the tools to confirm two functions are inverses, and discover the inverse of a given function.

The easiest application of the inverse is the radical function, which is already in our vocabulary. We discuss some basic properties of this family of inverse functions and demonstrate taking limits involving radicals, and how to use the difference of power formula to help factor and reduce expressions.

6.1: Basic Function Algebra

6.2: Inverses

6.3: Radical Functions

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6.1: Basic Function Algebra

We have learned about many different types of functions including:

linear, quadratic, polynomial, rational, trigonometric, and piecewise (like the absolute value).
We should be comfortable finding their domains, and taking limits involving any of these functions.

Each of these functions is just machine that turns a real number x from its domain into a real number y in its range. Since their outputs are numbers, we can define any operation on functions that we can do to numbers.

Let $f(x)$ have domain D_f and $g(x)$ have domain D_g.

We say the function $(f+g)(x)$ takes any $x \in D_f \cap D_g$ and outputs $f(x)+g(x)$
If f and g are continuous at $x=a$, then $\lim\limits_{x\to a}(f+g)(x) = \lim\limits_{x\to a} f(x)+g(x) = f(a)+g(a) = (f+g)(a)$
(therefore, the sum of two continuous functions is continuous)

Similarly, the function $(f-g)(x)$ takes any $x \in D_f \cap D_g$ and outputs $f(x)-g(x)$
If f and g are continuous at $x=a$, then $\lim\limits_{x\to a}(f-g)(x) = \lim\limits_{x\to a} f(x)-g(x) = f(a)-g(a) = (f-g)(a)$
(therefore, the difference of two continuous functions is continuous)

Again, the function $(f\cdot g)(x)$ takes any $x \in D_f \cap D_g$ and outputs $f(x)\cdot g(x)$
If f and g are continuous at $x=a$, then $\lim\limits_{x\to a}(f\cdot g)(x) = \lim\limits_{x\to a} f(x)\cdot g(x) = f(a)\cdot g(a) = (f\cdot g)(a)$
(therefore, the product of two continuous functions is continuous)

Lastly, the function $(f \div g)(x)$ takes any $x \in D_f \cap D_g \cap \{g(x) \neq 0\}$ and outputs $\dfrac{f(x)}{g(x)}$
If f and g are continuous at $x=a$, (and $g(a) \neq 0$) then $\lim\limits_{x\to a}(f \div g)(x) = \lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f(a)}{g(a)} = (f \div g)(a)$
(therefore, the quotient of two continuous functions is continuous *when it exists*)

In short, we can say that given any two continuous functions, their sum, difference, product and quotient are all continuous functions whenever they exist.

There is a fifth way to create a new function out of f and g called *composition*.

We say $(f\circ g)(x)$ is the function that takes any $x \in D_g$ such that $g(x) \in D_f$ and outputs $f(g(x))$

Example 6.1.1

For $f(x) = x^2+1$ and $g(x) = 2x-1$

(a) Find $(f\circ g)(x)$

(b) Find $(g\circ f)(x)$

Solution

(a) $(f\circ g)(x) = f(g(x)) = f(2x-1) = (2x-1)^2+1 = 4x^2-4x+2$

(b) $(g\circ f)(x) = g(f(x)) = g(x^2+1) = 2(x^2+1)-1 = 2x^2+1$

As we can see, in general $(f\circ g)(x) \neq (g\circ f)(x)$, so **order matters when using composition.**

Sometimes we can do this in reverse, and find two functions that compose to be a given function.
Notice that this can usually be done multiple different ways:

For instance, the function $h(x) = \sqrt{2x+4}$ can be written as $f\circ g$ for each of the following pairs (and more):

$f(x) = \sqrt{x}$ and $g(x) = 2x+4$

$f(x) = \sqrt{x+4}$ and $g(x) = 2x$

$f(x) = \sqrt{2x}$ and $g(x) = x+2$

$f(x) = \sqrt{2x+4}$ and $g(x) = x$

A nice thing about composition is that if $\lim_{x\to a} g(x) = b$ and $f(x)$ is continuous at $x = b$, then $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$

We say this means that you can *pass a limit through a continuous function.*

(so if $g(x)$ is continuous at a and $f(x)$ is continuous at $g(a)$ then $f \circ g$ is continuous at a)

Example 6.1.2

Evaluate the following limits involving composition.

(a) $\lim_{t\to 0} \sin(\pi \cos(t))$

(b) $\lim_{x\to \frac{\pi}{3}} \sqrt{5 + 2\sec(x)}$

Solution

(a) The inside function has limit $\lim_{t\to 0} \pi \cos(t) = \pi \cos(0) = \pi$

Since the outside function (sine) is continuous at π, we have:

$$\lim_{t\to 0} \sin(\pi \cos(t)) = \sin\left(\lim_{t\to 0} \pi \cos(t)\right) = \sin(\pi) = 0$$

(b) The inside function has limit $\lim_{x\to \frac{\pi}{3}} 5 + 2\sec(x) = 5 + 2\sec\left(\frac{\pi}{3}\right) = 9$

Since the outside function (the square root) is continuous at 9, we have:

$$\lim_{x\to \frac{\pi}{3}} \sqrt{5 + 2\sec(x)} = \sqrt{\lim_{x\to \frac{\pi}{3}} 5 + 2\sec(x)} = \sqrt{9} = 3$$

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6.2: Inverses

Sometimes two functions can be perfectly matched in a way such that their compositions cancel each other out completely. We say the functions $f(x)$ and $g(x)$ are *inverses* if:
$(f \circ g)(x) = x$ and $(g \circ f)(x) = x$

If this is the case, say that $g(x) = f^{-1}(x)$ (called "f inverse").

(Keep in mind this is not $[f(x)]^{-1}$ which would be $\dfrac{1}{f(x)}$, the reciprocal of f)

It is very easy to check if two functions are inverses: just compose them!

Example 6.2.1

For each pair of functions $f(x)$ and $g(x)$, determine if they are inverses or not.

(a) $f(x) = 3x + 1$ and $g(x) = \dfrac{1}{3}x - \dfrac{1}{3}$

(b) $f(x) = \dfrac{x+5}{x}$ and $g(x) = \dfrac{x}{x+5}$

(c) $f(x) = x^2$ and $g(x) = \sqrt{x}$

Solution

(a) $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{1}{3}x - \dfrac{1}{3}\right) = 3\left(\dfrac{1}{3}x - \dfrac{1}{3}\right) + 1 = x - 1 + 1 = x$

$(g \circ f)(x) = g(f(x)) = g(3x+1) = \dfrac{1}{3}(3x+1) - \dfrac{1}{3} = x + \dfrac{1}{3} - \dfrac{1}{3} = x$

So $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ both hold, meaning these are inverse functions

(b) $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{x}{x+5}\right) = \dfrac{\left(\frac{x}{x+5}\right)+5}{\left(\frac{x}{x+5}\right)} = \dfrac{6x+25}{x} \neq x$

$(g \circ f)(x) = g(f(x)) = g\left(\dfrac{x+5}{x}\right) = \dfrac{\left(\frac{x+5}{x}\right)}{\left(\frac{x+5}{x}\right)+5} = \dfrac{x+5}{6x+5} \neq x$

These are clearly not inverses since neither of $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ hold.

(c) $(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = x$

$(g \circ f)(x) = g(f(x)) = g\left(x^2\right) = \sqrt{x^2} = |x| \neq x$

Even though $(f \circ g)(x) = x$, since $(g \circ f)(x) \neq x$ these are not inverse functions.

Remember $f \circ g \neq g \circ f$ usually, so you always have to check both compositions!

While checking if two functions are inverses is straightforward, sometimes you need to find the inverse of a given function (and guessing isn't a strong strategy).

To do this, we start with $y = f(x)$, then apply $f^{-1}(x)$ to both sides giving:

$f^{-1}(y) = f^{-1}(f(x)) \longrightarrow f^{-1}(y) = x$

That means that the inverse function f^{-1} switches the jobs of x and y.

Given a function $y = f(x)$, to find $f^{-1}(x)$ we just switch x and y (and then solve for y)

Example 6.2.2

Find the inverse of each function below.

(a) $f(x) = 5x^3 + 2$

(b) $g(x) = \dfrac{x+3}{2x-7}$

Solution

(a) First we switch all the x and y terms: $y = 5x^3 + 2 \longrightarrow x = 5y^3 + 2$

Then solve for y:

$x - 2 = 5y^3$

$\dfrac{x-2}{5} = y^3$

$y = \sqrt[3]{\dfrac{x-2}{5}}$

Therefore we have $f^{-1}(x) = \sqrt[3]{\dfrac{x-2}{5}}$

(b) We switch all the x and y terms: $y = \dfrac{x+3}{2x-7} \longrightarrow x = \dfrac{y+3}{2y-7}$

Then solve for y:

$x(2y-7) = y+3$

$2xy - 7x = y + 3$

$2xy - y = 7x + 3$

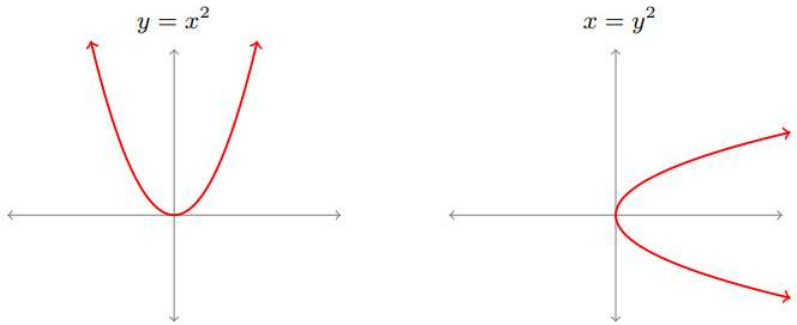
$(2x-1)y = 7x + 3$

$y = \dfrac{7x+3}{2x-1}$

Therefore we have $f^{-1}(x) = g^{-1}(x) = \dfrac{7x+3}{2x-1}$

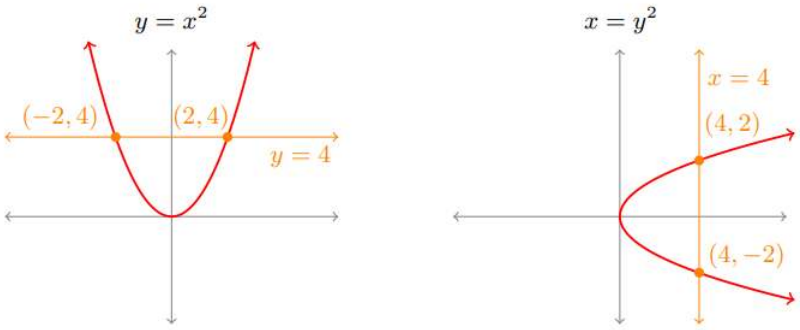
For $f(x) = x^2$, we know $f^{-1}(x) \neq \sqrt{x}$ so what is it's inverse?

Look at the graph for $y = f(x)$ and then $f(y) = x$:



The graph of $x = y^2$ is not a function! That means that there is no $f^{-1}(x)$ for this particular choice of $f(x) = x^2$. Not all functions will have an inverse, because just like we see above, it is possible that switching the jobs of x and y will lead to a relationship that is not a well-defined function.

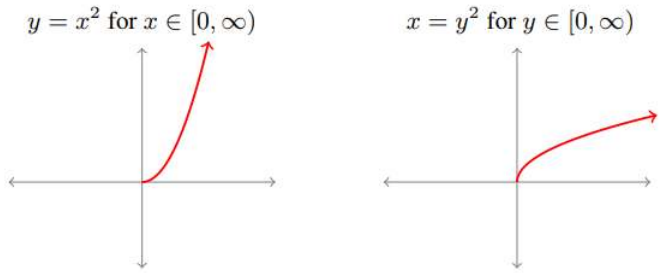
In order for the inverse of $f(x)$ to exist, it must pass the vertical line test after you switch x and y. As you can see though, each vertical line on $x = y^2$ corresponds to a horizontal line on $y = x^2$:



On the left we have two different x values that give the same output, which means the graph on the right will have two different y values for the same x. Instead of forcing us to draw the second picture and use the vertical line test, we can instead just check to see if any horizontal lines intersect the original graph $y = f(x)$ more than once. We call this the *horizontal line test*.

Functions with graphs that pass the horizontal line test are called *one-to-one* functions.
(These are the functions that can have an inverse function.)

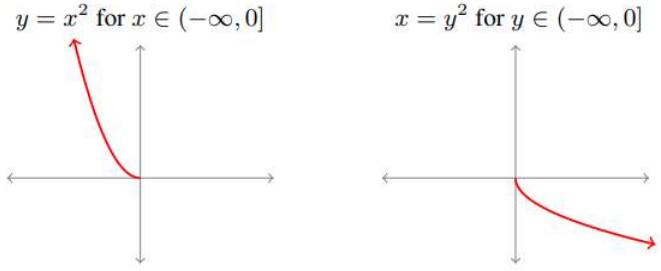
Since $f(x) = x^2$ does not pass the horizontal line test, it does not have an inverse. However, we can get around that by simply restricting the domain of $f(x)$ until it becomes one-to-one:



It is worth noting that for $x = y^2$ and $y \geq 0$, we can square root both sides and get $y = \sqrt{x}$

Remember we saw when $f(x) = x^2$ and $g(x) = \sqrt{x}$, we had $(f \circ g)(x) = x$ and $(g \circ f)(x) = |x|$, but when $x \geq 0$ the absolute value of x is the same as just x itself!

This is not the only way to restrict the domain of $f(x) = x^2$ to make a one-to-one function, we could also have used domain $D = (-\infty, 0]$:



Here we see that the inverse function simplifies to $f^{-1}(x) = -\sqrt{x}$ instead.

This further drives home the point that a function's domain is an important part of its identity! Changing it may change what the inverse of f looks like, or if it even exists at all.

Example 6.2.3

Find the inverse of $h(x) = x^2 - 2x$ for $x \in (-\infty, 1]$

Solution

Switching x and y: $y = x^2 - 2x$ with $x \leq 1$ becomes $x = y^2 - 2y$ with $y \leq 1$

Now we move everything to one side and use the quadratic formula for y

$$y^2 - y - x = 0 \longrightarrow y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-x)}}{2 \cdot 1}$$

$$= \frac{2 \pm \sqrt{4 + 4x}}{2} = \frac{2}{2} \pm \frac{2\sqrt{1+x}}{2} = 1 \pm \sqrt{1+x}$$

since $y \leq 1$ we should pick the $-$ from the $\pm$

So $h^{-1}(x) = 1 - \sqrt{1+x}$

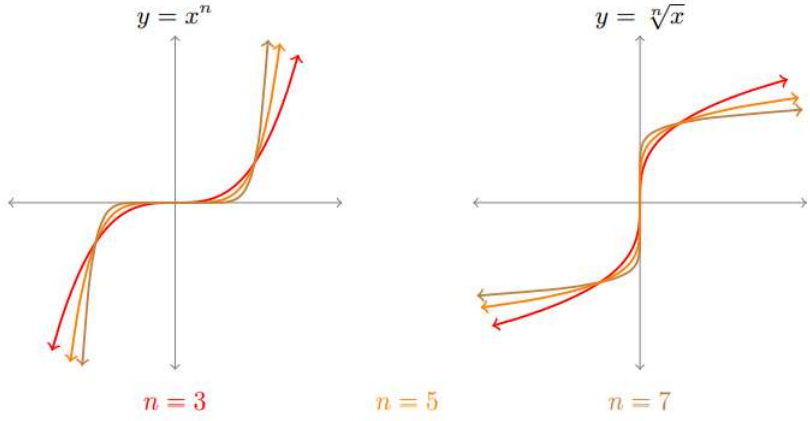
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6.3: Radical Functions

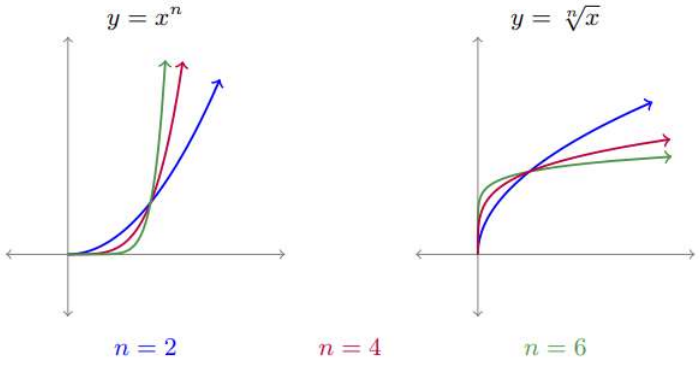
Radical functions are the most famous example of an inverse function.

For $f(x) = x^n$, we say that $f^{-1}(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$

When n is an odd positive integer, $f(x)$ is 1-to-1 and the domain of f^{-1} is $(-\infty, \infty)$:



But when n is an even positive integer, $f(x)$ fails the horizontal line test. To create an inverse for these functions, we restrict the domain of $f(x)$ to $[0, \infty)$, this results in $f^{-1}(x)$ existing and having domain $[0, \infty)$ as well:



While we have already seen and used radicals previously (such as in the quadratic formula), we have not yet investigated them as functions.

From the graphs we can see that they are clearly continuous on their domain.

This generalizes to $\lim_{x \to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \to a} g(x)}$ (whenever the radical makes sense)

Remember: radicals are relatively well behaved when it comes to multiplication and division:

$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ and $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ whenever all terms exist.

But they almost never distribute across a plus or minus sign:

$\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$ and $\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$

Sometimes radicals end up in places that make us uncomfortable (like as part of a fraction) and we want to re-write them in a different way. This can be done by using a difference of nth's formula:

$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^1 b^{n-1} + b^{n-1}\right)$

when $n = 2$, this is called the difference of squares $a^2 - b^2 = (a-b)(a+b)$

These can be used to *rationalize* (get rid of radicals in) a denominator or numerator if desired:

Example 6.3.1

(a) Rationalize the denominator for $f(x)=\dfrac{x+1}{\sqrt{x+2}-3}$

(b) Rationalize the numerator for $g(x)=\dfrac{\sqrt[3]{x}-2}{x-8}$

Solution

(a) To eliminate $\sqrt{x+2}$, we multiply top and bottom by $\sqrt{x+2}+3$

$$f(x)=\frac{x+1}{\sqrt{x+2}-3}\cdot\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}=\frac{(x+1)(\sqrt{x+2}+3)}{(\sqrt{x+2})^2-3^2}=\frac{(x+1)(\sqrt{x+2}+3)}{(x+2)-9}=\frac{(x+1)(\sqrt{x+2}+3)}{x-7}$$

(b) To eliminate $\sqrt[3]{x}$, we multiply top and bottom by $(\sqrt[3]{x})^2+(\sqrt[3]{x})\cdot 2+2^2$

$$\frac{(\sqrt[3]{x})^3-2^3}{(x-8)\left(\sqrt[3]{x^2}+2\sqrt[3]{x}+4\right)}=\frac{x-8}{(x-8)\left(\sqrt[3]{x^2}+2\sqrt[3]{x}+4\right)}=\frac{1}{\sqrt[3]{x^2}+2\sqrt[3]{x}+4}\quad\text{(for } x\neq 8\text{)}$$

As we can see above, sometimes rationalizing a numerator or denominator can help with reducing, which means it is definitely a great tool for taking limits.

Evaluate the following limits involving radical functions:

Example 6.3.2

Evaluate each of the limits below by rationalizing the numerator or denominator.

(a) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{x+1}-1}$

(b) $\lim\limits_{t\to 4}\dfrac{\sqrt{t^2+9}-5}{t-4}$

(c) $\lim\limits_{y\to -2}\dfrac{y^2+9y+8}{\sqrt[3]{y}+2}$

Solution

(a) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{x+1}-1}=\lim\limits_{x\to 0}\dfrac{x}{\sqrt{x+1}-1}\cdot\dfrac{\sqrt{x+1}+1}{\sqrt{x+1}+1}=\lim\limits_{x\to 0}\dfrac{x(\sqrt{x+1}+1)}{(\sqrt{x+1})^2-1^2}$

$=\lim\limits_{x\to 0}\dfrac{x(\sqrt{x+1}+1)}{x}=\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}+1}{1}=\dfrac{\sqrt{0+1}+1}{1}=2$

(b) $\lim\limits_{t\to 4}\dfrac{\sqrt{t^2+9}-5}{t-4}=\lim\limits_{t\to 4}\dfrac{\sqrt{t^2+9}-5}{t-4}\cdot\dfrac{\sqrt{t^2+9}+5}{\sqrt{t^2+9}+5}=\lim\limits_{t\to 4}\dfrac{(\sqrt{t^2+9})^2-5^2}{(t-4)(\sqrt{t^2+9}+5)}$

$=\lim\limits_{t\to 4}\dfrac{t^2-16}{(t-4)(\sqrt{t^2+9}+5)}=\lim\limits_{t\to 4}\dfrac{(t-4)(t+4)}{(t-4)(\sqrt{t^2+9}+5)}=\lim\limits_{t\to 4}\dfrac{t+4}{\sqrt{t^2+9}+5}=\dfrac{4+4}{\sqrt{4^2+9}+5}=\dfrac{4}{5}$

(c) $\lim\limits_{y\to -2}\dfrac{y^2+9y+8}{\sqrt[3]{y}+2}=\lim\limits_{y\to -8}\dfrac{(y+8)(y+1)}{\sqrt[3]{y}-(-2)}\cdot\dfrac{\sqrt[3]{y^2}+(-2)\sqrt[3]{y}+(-2)^2}{\sqrt[3]{y^2}+(-2)\sqrt[3]{y}+(-2)^2}$

$=\lim\limits_{y\to -8}\dfrac{(y+8)(y+1)\left(\sqrt[3]{y^2}-2\sqrt[3]{y}+4\right)}{\left(\sqrt[3]{y}\right)^3-(-2)^3}=\lim\limits_{y\to -8}\dfrac{(y+8)(y+1)\left(\sqrt[3]{y^2}-2\sqrt[3]{y}+4\right)}{y+8}$

$$= \lim_{y\to -8}(y+1)\left(\sqrt[3]{y^2}-2\sqrt[3]{y}+4\right) = (-8+1)\left(\sqrt[3]{(-8)^2}-2\sqrt[3]{-8}+4\right) = -7\,(4+4+4) = -84$$

Remember that $\sqrt[n]{x^n} = x$ when n is odd, but $\sqrt[n]{x^n} = |x|$ when n is even!

This can make a big difference when using the Lebron James Rule:

Example 6.3.3

(a) $\lim_{x\to\infty} \dfrac{\sqrt{9x^2+4}+5x}{2x+7}$

(b) $\lim_{x\to-\infty} \dfrac{\sqrt{9x^2+4}+5x}{2x+7}$

Solution

(a) Get rid of terms that are clearly smaller

$$\lim_{x\to\infty} \frac{\sqrt{9x^2+4}+5x}{2x+7} = \lim_{x\to\infty} \frac{\sqrt{9x^2}+5x}{2x} = \lim_{x\to\infty} \frac{|3x|+5x}{2x}$$

Since $x \to +\infty$ we know $x > 0$ so $|x| = x$, turning our limit into:

$$= \lim_{x\to\infty} \frac{3x+5x}{2x} = \lim_{x\to\infty} \frac{8x}{2x} = \lim_{x\to\infty} 4 = 4$$

(b) $\lim_{x\to-\infty} \dfrac{\sqrt{9x^2+4}+5x}{2x+7} = \lim_{x\to-\infty} \dfrac{\sqrt{9x^2}+5x}{2x} = \lim_{x\to-\infty} \dfrac{|3x|+5x}{2x}$

Since $x \to -\infty$ we know $x < 0$ so $|x| = -x$, turning our limit into:

$$= \lim_{x\to-\infty} \frac{-3x+5x}{2x} = \lim_{x\to-\infty} \frac{2x}{2x} = \lim_{x\to-\infty} 1 = 1$$

Just remember, whenever the limit is $\infty - \infty$ and youre not sure who is biggest, you should rewrite it first:

Example 6.3.4

$\lim_{w\to\infty} \sqrt{4w^2+2w-1} - 2w$

Solution

Tempting to write as $\sqrt{4w^2} - 2w$, but this would be wrong!

$$\lim_{w\to\infty} \sqrt{4w^2+2w-1} - 2w = \lim_{w\to\infty} \left(\sqrt{4w^2+2w-1} - 2w\right) \cdot \frac{\sqrt{4w^2+2w-1}+2w}{\sqrt{4w^2+2w-1}+2w}$$

$$= \lim_{w\to\infty} \frac{\left(\sqrt{4w^2+2w-1}\right)^2 - (2w)^2}{\sqrt{4w^2+2w-1}+2w} = \lim_{w\to\infty} \frac{2w-1}{\sqrt{4w^2+2w-1}+2w}$$

Now use the Lebron James rule

$$= \lim_{w\to\infty} \frac{2w}{\sqrt{4w^2}+2w} = \lim_{w\to\infty} \frac{2w}{|2w|+2w} = \lim_{w\to\infty} \frac{2w}{2w+2w} = \lim_{w\to\infty} \frac{1}{2} = \frac{1}{2}$$

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CHAPTER OVERVIEW

7: Derivatives

> *"There is nothing permanent except change." - Heraclitus*

Chapter seven is where limits deliver their biggest performance yet: the derivative. We will see how a specific limit can tell us the slope of the line tangent to a curve at a given point. In loose terms, it is the slope of the curve at one point. While this does not sound overly impressive on its own, consider that we will see the word derivative in every future chapter of this book!

We start by learning the formula for finding the derivative of a function at a point, and get plenty of practice taking a special limit. When this limit gets too hard, we offer a different (but equivalent) limit option. While these limits are sometimes of the more difficult variety, we explore a few different challenges and how to overcome them by using either of the two different aforementioned limit methods.

We finish the chapter with a more formal discussion on what a derivative is, and when it should or should not exist for a given function. We also discuss how a derivative will effect/be effected by the graph of a function.

This is the chapter where a student should be able to put their algebra skills learned thus far to the test to help solve a problem by using limits. Being able to evaluate a derivative or find a tangent line in this way is critical to understanding why we are allowed the shortcuts and patterns we will use in the future to take derivatives.

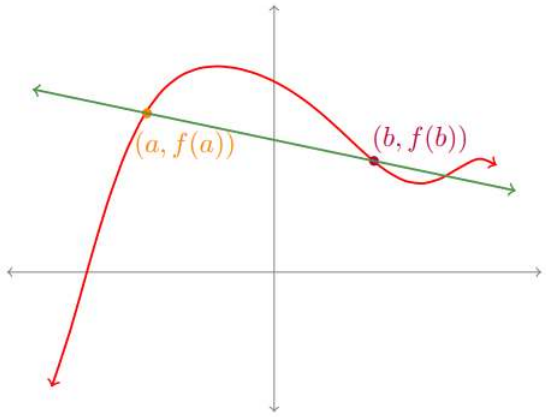
7.1: Tangent Lines and the Derivative
7.2: An Alternate Definition of Derivative
7.3: Derivative as a Function
7.4: Differentiability
7.5: Derivatives and Graphs

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7.1: Tangent Lines and the Derivative

For a given graph $y = f(x)$, we used both vertical and horizontal lines to determine properties of the function $f(x)$. There are other types of lines that we can study as well.

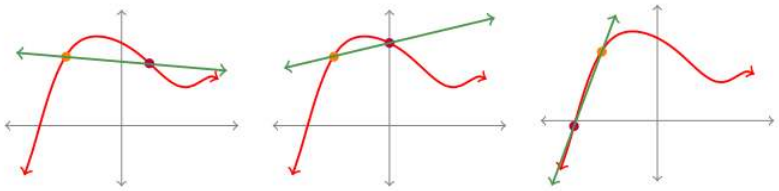
For any two points $(a, f(a))$ and $(b, f(b))$, the line connecting them is called a *secant line*.



Finding the slope of secant line is straightforward: $m = \dfrac{f(b) - f(a)}{b - a}$

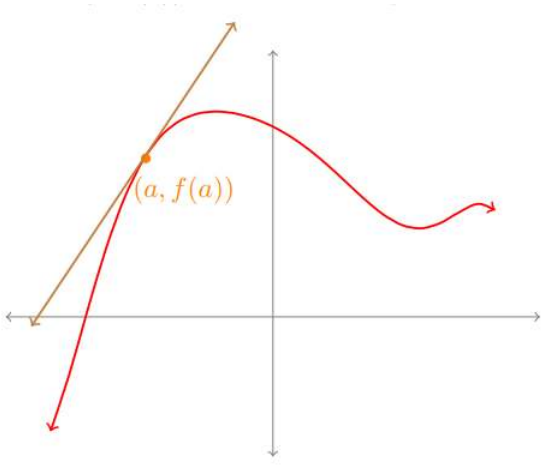
(and with the slope it is easy to find the equation of any such line)

In fact, if we force one of these points $(a, f(a))$ to remain fixed, we can see that the slope of the secant line forms a function depending on b.



This function is defined for all values of b in the domain of f except for one: $b = a$

If you were to visualize the line that would occur as you drag the point $(b, f(b))$ closer and closer to $(a, f(a))$, you should end up with something like this:



This is called the *tangent line* to $y = f(x)$ at $(a, f(a))$.

Since the slope formula $m = \dfrac{f(b) - f(a)}{b - a}$ only works when $a \neq b$, we need a different formula to find the slope of a tangent line.

The slope of a tangent line is the limit of the slope of the secant lines formed as $b \to a$

$$m_{tan} = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

It is clear that this slope is dependent on the location of a, so we give this function a name: the *derivative* of $f(x)$ at $x = a$, which we write as $f'(a)$.

Once you know the slope, the equation of the tangent line can be found using point slope form: $y - f(a) = f'(a)(x - a)$

Example 7.1.1

Find the equation of the line tangent to $y = 3x + 1$ at $(2, 7)$

Solution

Our function is $f(x) = 3x + 1$ and we are given $f(2) = 7$ so we just need to find the slope of this line: $f'(2)$

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{(3x + 1) - 7}{x - 2}$$

$$= \lim_{x \to 2} \frac{3x - 6}{x - 2} = \lim_{x \to 2} \frac{3(x - 2)}{x - 2} = \lim_{x \to 2} 3 = 3$$

Now we plug it into the tangent line formula $y - f(2) = f'(2)(x - 2)$

Giving our final answer: $y - 7 = 3(x - 2)$

Notice that the line $y - 7 = 3(x - 2)$ simplifies down to $y = 3x + 1$.

This is not a coincidence, the secant line on any linear function is always itself. Therefore the tangent line to a given line at any point will always match the original equation of the line.

That just means that linear functions make for boring tangent line questions! Lets try one that's a little more exciting.

Example 7.1.2

Find the equation of the line tangent to $y = x^2 - 4x + 3$ at $(1, 0)$

Solution

Our function is $f(x) = x^2 - 4x + 3$ and we are given $f(1) = 0$, so we need find the slope of this line: $f'(1)$

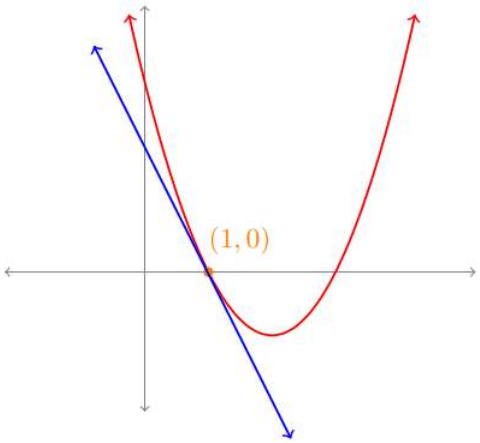
$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{(x^2 - 4x + 3) - 0}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 3)(x - 1)}{x - 1} = \lim_{x \to 1} (x - 3) = (1 - 3) = -2$$

Now we plug it into the tangent line formula $y - f(1) = f'(1)(x - 1)$

Giving our final answer: $y - 0 = -2(x - 1)$

We can graph both $y = x^2 - 4x + 3$ and $y = -2(x - 1)$, to see that this looks correct:



Whenever the function is a bigger polynomial, remember your synthetic division skills!

Example 7.1.3

Find the equation of the line tangent to $y = x^3 + 3x^2$ when $x = -2$

Solution

Our function is $f(x) = x^3 + 3x^2$ and we can manually solve for $f(-2) = (-2)^3 + 3(-2)^2 = 4$,

Again we need find the slope of this line: $f'(-2)$

$$f'(-2) = \lim_{x \to -2} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \to -2} \frac{x^3 + 3x^2 - 4}{x + 2}$$

We do synthetic division to reduce this fraction:

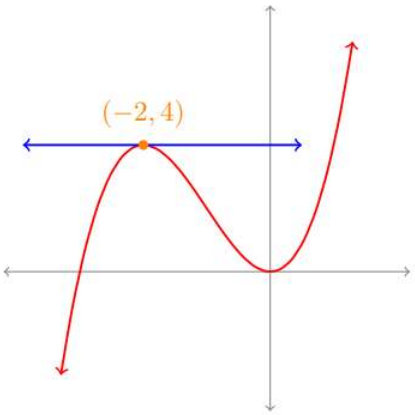
-2	1	3	0	-4
		-2	-2	4
	1	1	-2	0

$$= \lim_{x \to -2} \frac{(x^2 + x - 2)(x + 2)}{x + 2} = \lim_{x \to -2} (x^2 + x - 2) = (-2)^2 + (-2) - 2) = 0$$

Now we plug it into the tangent line formula $y - f(-2) = f'(-2)(x - (-2))$

Giving our final answer: $y - 4 = 0(x + 2)$

Again we can graph both $y = x^3 + 3x^2$ and $y = 4$, to see that this looks correct:



For rational functions, $f'(a)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ involves a complex fraction:

Example 7.1.4

Find (a) $f'(3)$ for $f(x)=\dfrac{2x+5}{x-2}$ and (b) $g'(-1)$ for $g(x)=\dfrac{x}{x^2+1}$

Solution

(a) $f'(3)=\lim\limits_{x\to3}\dfrac{f(x)-f(3)}{x-3}=\lim\limits_{x\to3}\dfrac{\frac{2x+5}{x-2}-11}{x-3}\cdot\dfrac{x-2}{x-2}=\lim\limits_{x\to3}\dfrac{(2x+5)-11(x-2)}{(x-3)(x-2)}$

$=\lim\limits_{x\to3}\dfrac{-9x+27}{(x-3)(x-2)}=\lim\limits_{x\to3}\dfrac{-9(x-3)}{(x-3)(x-2)}=\lim\limits_{x\to3}\dfrac{-9}{x-2}=\dfrac{-9}{3-2}=-9$

(b) $g'(-1)=\lim\limits_{x\to-1}\dfrac{g(x)-g(-1)}{x-(-1)}=\lim\limits_{x\to-1}\dfrac{\frac{x}{x^2+1}-\left(-\frac{1}{2}\right)}{x+1}\cdot\dfrac{2(x^2+1)}{2(x^2+1)}=\lim\limits_{x\to-1}\dfrac{2x+(x^2+1)}{2(x+1)(x^2+1)}$

$=\lim\limits_{x\to-1}\dfrac{x^2+2x+1}{2(x+1)(x^2+1)}=\lim\limits_{x\to-1}\dfrac{(x+1)^2}{2(x+1)(x^2+1)}=\lim\limits_{x\to-1}\dfrac{x+1}{2(x^2+1)}=\dfrac{-1+1}{2((-1)^2+1)}=0$

For n^{th} radical functions, just remember to use difference of nth powers:

Example 7.1.5

Find (a) $s'(2)$ for $s(t)=\sqrt{2t^2+1}$ and (b) $w'(-8)$ for $w(x)=\sqrt[3]{x}$

Solution

(a) $s'(2)=\lim\limits_{t\to2}\dfrac{s(t)-s(2)}{t-2}=\lim\limits_{t\to2}\dfrac{\sqrt{2t^2+1}-3}{t-2}\cdot\dfrac{\sqrt{2t^2+1}+3}{\sqrt{2t^2+1}+3}=\lim\limits_{t\to2}\dfrac{\left(\sqrt{2t^2+1}\right)^2-3^2}{(t-2)\left(\sqrt{2t^2+1}+3\right)}$

$=\lim\limits_{t\to2}\dfrac{2t^2-8}{(t-2)\left(\sqrt{2t^2+1}+3\right)}=\lim\limits_{t\to2}\dfrac{2(t-2)(t+2)}{(t-2)\left(\sqrt{2t^2+1}+3\right)}=\lim\limits_{t\to2}\dfrac{2(t+2)}{\sqrt{2t^2+1}+3}=\dfrac{2(2+2)}{\sqrt{2\cdot2^2+1}+3}=\dfrac{4}{3}$

(b) $w'(-8)=\lim\limits_{x\to-8}\dfrac{w(x)-w(-8)}{x-(-8)}=\lim\limits_{x\to-8}\dfrac{\sqrt[3]{x}-(-2)}{x+8}\cdot\dfrac{\left(\sqrt[3]{x}\right)^2+(-2)\sqrt[3]{x}+(-2)^2}{\left(\sqrt[3]{x}\right)^2+(-2)\sqrt[3]{x}+(-2)^2}$

$=\lim\limits_{x\to-8}\dfrac{\left(\sqrt[3]{x}\right)^3-(-2)^3}{(x+8)\left(\sqrt[3]{x^2}-2\sqrt[3]{x}+4\right)}=\lim\limits_{x\to-8}\dfrac{x+8}{(x+8)\left(\sqrt[3]{x^2}-2\sqrt[3]{x}+4\right)}$

$=\lim\limits_{x\to-8}\dfrac{1}{\sqrt[3]{x^2}-2\sqrt[3]{x}+4}=\dfrac{1}{\sqrt[3]{(-8)^2}-2\sqrt[3]{-8}+4}=\dfrac{1}{4+4+4}=\dfrac{1}{12}$

If you're really (un)lucky, you can combine these tricks in the same problem:

Example 7.1.6

Find the equation of the line tangent to $y=\dfrac{x}{\sqrt{x-1}}$ at $a=10$

Solution

$y'(10)=\lim\limits_{x\to10}\dfrac{y(x)-y(10)}{x-10}=\lim\limits_{x\to10}\dfrac{\frac{x}{\sqrt{x-1}}-\frac{10}{3}}{x-10}\cdot\dfrac{3\sqrt{x-1}}{3\sqrt{x-1}}=\lim\limits_{x\to10}\dfrac{3x-10\sqrt{x-1}}{3(x-10)\sqrt{x-1}}\cdot\dfrac{3x+10\sqrt{x-1}}{3x+10\sqrt{x-1}}$

$$= \lim_{x\to 10} \frac{(3x)^2 - \left(10\sqrt{x-1}\right)^2}{3(x-10)\sqrt{x-1}\,(3x+10\sqrt{x-1})} = \lim_{x\to 10} \frac{9x^2 - 100x + 100}{3(x-10)\sqrt{x-1}\,(3x+10\sqrt{x-1})} =$$

$$\lim_{x\to 10} \frac{(9x-10)(x-10)}{3(x-10)\sqrt{x-1}\,(3x+10\sqrt{x-1})}$$

$$= \lim_{x\to 10} \frac{9x-10}{3\sqrt{x-1}\,(3x+10\sqrt{x-1})} = \frac{9\cdot 10 - 10}{3\sqrt{10-1}\,(3\cdot 10 + 10\sqrt{10-1})} = \frac{80}{9\,(30+30)} = \frac{4}{27}$$

Plugging this into $y - y(10) = y'(10)(x-10)$ we get our answer: $y - \frac{10}{3} = \frac{4}{27}(x-10)$

All of our limit practice has been setting us up for taking these limits! You can take the derivative of (and even find the line tangent to) any function for which you can evaluate the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ (as long as it exists)

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7.2: An Alternate Definition of Derivative

To find the slope of a tangent line to the curve $y = f(x)$ at $(a, f(a))$ we use the derivative limit $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

No matter what type of function $f(x)$ seems to be, this limit always seems to end up with factoring the top and reducing away $(x - a)$.

Unfortunately, if your factoring skills aren't great (or a isn't very nice), this may be quite challenging for certain functions or values of a.

Example 7.2.1

Consider $f(x) = x^2$ and find $f'(\sqrt{3})$

Solution

$$f'(\sqrt{3}) = \lim_{x \to \sqrt{3}} \frac{f(x) - f(\sqrt{3})}{x - \sqrt{3}} = \lim_{x \to \sqrt{3}} \frac{x^2 - 3}{x - \sqrt{3}}$$

(factoring the top isn't obvious, but its possible!)

$$= \lim_{x \to \sqrt{3}} \frac{(x - \sqrt{3})(x + \sqrt{3})}{x - \sqrt{3}} = \lim_{x \to \sqrt{3}} (x + \sqrt{3}) = 2\sqrt{3}$$

Clearly if you cannot factor the numerator, you won't be able to reach this answer.

Instead, we can do a little substitution inside of the derivative formula to avoid this issue:

Define $h = x - a$, which can be re-written as $x = a + h$.

So $f(x)$ becomes $f(a + h)$ and $\lim_{x \to a} h = 0$ so $x \to a$ can be replaced by $h \to 0$

Putting this all together we get $f'(a) = \lim_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$

If we try to find $f'(\sqrt{3})$ as we did above with this new definition:

Example 7.2.1

Reconsider $f(x) = x^2$ and find $f'(\sqrt{3})$

Solution

$$f'(\sqrt{3}) = \lim_{h \to 0} \frac{f(\sqrt{3} + h) - f(\sqrt{3})}{h} = \lim_{h \to 0} \frac{(\sqrt{3} + h)^2 - 3}{h} = \lim_{h \to 0} \frac{(3 + 2h\sqrt{3} + h^2) - 3}{h}$$

$$= \lim_{h \to 0} \frac{2h\sqrt{3} + h^2}{h} = \lim_{h \to 0} (2\sqrt{3} + h) = 2\sqrt{3}$$

This alternative limit definition of the derivative will always yield the same answer as the original definition.
Pick whichever one is easiest for you to use on a given question.

Lets just try a few example problems using this new definition (the old one works fine if you prefer it, but we will get some practice using this new one)

Example 7.2.2

Find the line tangent to $g(x) = \dfrac{x + 1}{x - 1}$ at the point $\left(\dfrac{1}{2}, -3\right)$

Solution

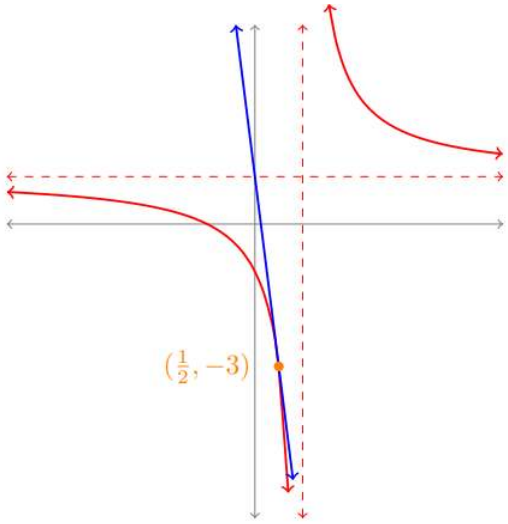
$$g'\left(\frac{1}{2}\right)=\lim_{h\to 0}\frac{g\left(\frac{1}{2}+h\right)-g\left(\frac{1}{2}\right)}{h}=\lim_{h\to 0}\frac{\frac{\frac{1}{2}+h+1}{\frac{1}{2}+h-1}-(-3)}{h}=\lim_{h\to 0}\frac{\frac{h+\frac{3}{2}}{h-\frac{1}{2}}+3}{h}\cdot\frac{h-\frac{1}{2}}{h-\frac{1}{2}}$$

$$=\lim_{h\to 0}\frac{\left(h+\frac{3}{2}\right)+3\left(h-\frac{1}{2}\right)}{h\left(h-\frac{1}{2}\right)}=\lim_{h\to 0}\frac{4h}{h\left(h-\frac{1}{2}\right)}=\lim_{h\to 0}\frac{4}{h-\frac{1}{2}}=\frac{4}{0-\frac{1}{2}}=-8$$

We plug this into the tangent line formula $y-g\left(\frac{1}{2}\right)=g'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$

Which gives our final answer, $y+3=-8\left(x-\frac{1}{2}\right)$

A quick glance at the graph of makes this look believable.



The same tricks works when dealing with radicals for both versions of the derivative limit:

Example 7.2.3

Find the line tangent to $f(x)=\sqrt{x^2-x-2}$ at the point where $x=3$

Solution

First we see that $f(3)=\sqrt{3^2-3-2}=\sqrt{4}=2$, now we find $f'(3)$:

$$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{\sqrt{(3+h)^2-(3+h)-2}-2}{h}$$

We rationalize our numerator to help us reduce this:

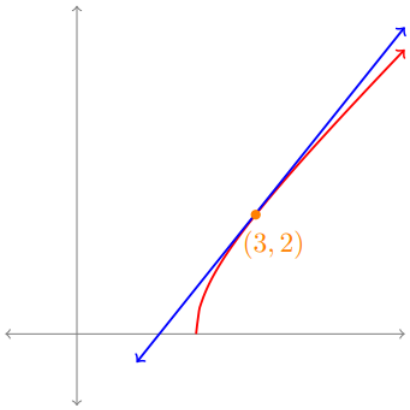
$$=\lim_{h\to 0}\frac{\sqrt{h^2+5h+4}-2}{h}\cdot\frac{\sqrt{h^2+5h+4}+2}{\sqrt{h^2+5h+4}+2}=\lim_{h\to 0}\frac{\left(\sqrt{h^2+5h+4}\right)^2-2^2}{h\left(\sqrt{h^2+5h+4}+2\right)}=\lim_{h\to 0}\frac{h^2+5h+4-4}{h\left(\sqrt{h^2+5h+4}+2\right)}$$

$$=\lim_{h\to 0}\frac{h^2+5h}{h\left(\sqrt{h^2+5h+4}+2\right)}=\lim_{h\to 0}\frac{h+5}{\sqrt{h^2+5h+4}+2}=\frac{0+5}{\sqrt{0^2+5\cdot 0+4}+2}=\frac{5}{4}$$

We plug this into the tangent line formula $y-f(3)=f'(3)(x-3)$

Which gives our final answer, $y-2=\frac{5}{4}(x-3)$

Again here is some graphical evidence that we are correct:



This definition helps to avoid factoring, but be warned that evaluating $f(a+h)$ usually takes an equivalent amount of work (but many students think its easier work):

Example 7.2.4

Find $s'(1-\sqrt{2})$ for $s(t)=\dfrac{1}{(1-t)^3}$

Solution

First we find $s(1-\sqrt{2})=\dfrac{1}{(1-(1-\sqrt{2}))^3}=\dfrac{1}{(\sqrt{2})^3}=\dfrac{1}{2\sqrt{2}}$, now we find $s'(1-\sqrt{2})$:

$$s'\left(1-\sqrt{2}\right)=\lim_{h\to 0}\frac{s\left(1-\sqrt{2}+h\right)-s\left(1-\sqrt{2}\right)}{h}=\lim_{h\to 0}\frac{\frac{1}{(1-(1-\sqrt{2}+h))^3}-\frac{1}{2\sqrt{2}}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{(\sqrt{2}-h)^3}-\frac{1}{2\sqrt{2}}}{h}=\lim_{h\to 0}\frac{\frac{1}{(\sqrt{2})^3-3(\sqrt{2})^2\cdot h+3\sqrt{2}\cdot h^2-h^3}-\frac{1}{2\sqrt{2}}}{h}$$

We multiply the top and bottom by the common denominator of the smaller fractions:

$$=\lim_{h\to 0}\frac{\frac{1}{2\sqrt{2}-6h+3h^2\sqrt{2}-h^3}-\frac{1}{2\sqrt{2}}}{h}\cdot\frac{2\sqrt{2}(2\sqrt{2}-6h+3h^2\sqrt{2}-h^3)}{2\sqrt{2}(2\sqrt{2}-6h+3h^2\sqrt{2}-h^3)}$$

$$=\lim_{h\to 0}\frac{2\sqrt{2}-(2\sqrt{2}-6h+3h^2\sqrt{2}-h^3)}{2h\sqrt{2}(2\sqrt{2}-6h+3h^2\sqrt{2}-h^3)}=\lim_{h\to 0}\frac{6h-3h^2\sqrt{2}+h^3}{2h\sqrt{2}(2\sqrt{2}-6h+3h^2\sqrt{2}-h^3)}$$

$$=\lim_{h\to 0}\frac{6-3h\sqrt{2}+h^2}{2\sqrt{2}(2\sqrt{2}-6h+3h^2\sqrt{2}-h^3)}=\frac{6-3\cdot 0\cdot\sqrt{2}+0^2}{2\sqrt{2}(2\sqrt{2}-6\cdot 0+3\cdot 0^2\cdot\sqrt{2}-0^3)}=\frac{3}{4}$$

This looks ugly, but consider that if you try the $\lim_{t\to 1-\sqrt{2}}$ version of derivative for this question, at some point you would have had to factor the term $(t-(1-\sqrt{2}))$ out of the cubic polynomial $-t^3+3t^2-3t+1-2\sqrt{2}$

It is always good to have an alternative approach when factoring is hard!

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7.3: Derivative as a Function

Finding a tangent line to a function at a given point requires us to evaluate the limit:

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

But what if we wanted to find a few different tangent lines to the same function?

If we wanted the tangent line at $(a, f(a))$ and $(b, f(b))$ and $(c, f(c)), \ldots$ we would need to solve for $f'(a), f'(b), f'(c), \ldots$
Each of these would be a different limit, so a lot of similar work would need to be repeated.

We can avoid this by just finding the explicit form of the *function* $f'(x)$. In other words, we avoid plugging in a specific value of a into the derivative limit until after all of the limit work is done!

Example 7.3.1

Find $f'(a)$ for $f(x)=2x^2+5x+1$

Solution

We will take this as an opportunity to show the calculation for both limit definitions of derivative.

First we show it done with the $x\to a$ limit:

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{(2x^2+5x+1)-(2a^2+5a+1)}{x-a}=\lim_{x\to a}\frac{2x^2-2a^2+5x-5a}{x-a}$$

$$=\lim_{x\to a}\frac{2(x+a)(x-a)+5(x-a)}{x-a}=\lim_{x\to a}\frac{(2(x+a)+5)(x-a)}{x-a}=\lim_{x\to a}2(x+a)+5=2(a+a)+5=2a+5$$

Alternatively we show how it would look using the $h\to 0$ limit:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\left((a+h)^2+5(a+h)+1\right)-(a^2+5a+1)}{h}$$

$$=\lim_{h\to 0}\frac{\left(a^2+2ah+h^2+5a+5h+1\right)-(a^2+5a+1)}{h}=\lim_{h\to 0}\frac{2ah+h^2+5h}{h}=\lim_{h\to 0}(2a+h+5)-2a+0+5$$
$$=2a+5$$

These will always agree, so you only needed to do one of them to get the answer!

Once you have the function $f'(a)$, you can plug in any number a in its domain to find the slope of the tangent line at $(a, f(a))$

Example 7.3.2

For $f(x)=3x^2-x$, at what point does the tangent line to $y=f(x)$ at} $(1,2)$ intersect
with the tangent line to $y=f(x)$ at $(-1,4)$?

Solution

Clearly we should start by finding these tangent lines:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\left(3(a+h)^2-(a+h)\right)-(3a^2-a)}{h}$$

$$=\lim_{h\to 0}\frac{\left(3a^2+6ah+3h^2-a-h\right)-(3a^2-a)}{h}=\lim_{h\to 0}\frac{6ah+3h^2-h}{h}=\lim_{h\to 0}(6a+3h-1)=6a-1$$

Plugging in $a=1$ and $a=-1$ gives: $f'(1)=6\cdot 1-1=5$ and $f'(-1)=6\cdot(-1)-1=-7$

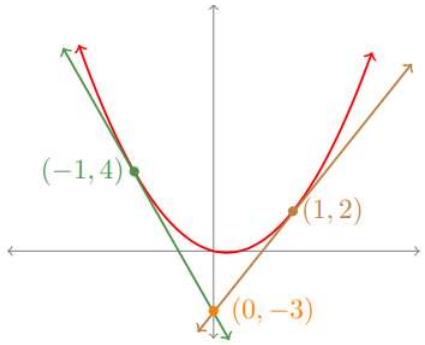
Using $y-f(1)=f'(1)(x-1)$ gives tangent line at $a=1$: $y-2=5(x-1)$

Using $y-f(-1)=f'(-1)(x-(-1))$ gives tangent line at $a=-1$: $y-4=-7(x+1)$

Now we find their intersection by solving $\begin{cases} y-2=5(x-1) \\ y-4=-7(x+1) \end{cases}$

The solution to this system of equations is our answer: $(0,-3)$

This looks accurate when you draw them on the graph:



We can also look for points on a curve that yield a tangent line with a specific slope:

Example 7.3.3

What point(s) on the graph of $f(x)=\dfrac{2x-1}{x-3}$ have a tangent line that is parallel to the line $y=-5x+1$?

Solution

The slope of $y=-5x+1$ is $m=-5$ and to be parallel, we need $f'(a)=-5$

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\frac{2x-1}{x-3}-\frac{2a-1}{a-3}}{x-a}\cdot\frac{(x-3)(a-3)}{(x-3)(a-3)}$$

$$=\lim_{x\to a}\frac{(2x-1)(a-3)-(2a-1)(x-3)}{(x-a)(x-3)(a-3)}=\lim_{x\to a}\frac{(2ax-a-6x+3)-(2ax-6a-x+3)}{(x-a)(x-3)(a-3)}$$

$$=\lim_{x\to a}\frac{-5x+5a}{(x-a)(x-3)(a-3)}=\lim_{x\to a}\frac{-5(x-a)}{(x-a)(x-3)(a-3)}=\lim_{x\to a}\frac{-5}{(x-3)(a-3)}=\frac{-5}{(a-3)^2}$$

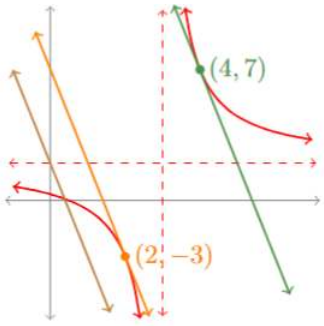
Setting $f'(a)=-5$ we get a rational equation: $\dfrac{-5}{(a-3)^2}=-5$

Which we can solve to get $a=2$ and $a=4$

$f(2)=\dfrac{2\cdot 2-1}{2-3}=-3$ and $f(4)=\dfrac{2\cdot 4-1}{4-3}=7$

Giving our final answers: $(2,-3)$ and $(4,7)$

Once again we can visualize our results by graphing the curve and lines: \\



Remember a horizontal line is one with slope equal to zero.

Example 7.3.4

What point(s) on the graph of $f(x)=\sqrt{3x-x^3}$ have a horizontal tangent line?

Solution

We need to find when $f'(a)=0$, so find $f'(a)$ first:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\sqrt{3(a+h)-(a+h)^3}-\sqrt{3a-a^3}}{h}\cdot\frac{\sqrt{3(a+h)-(a+h)^3}+\sqrt{3a-a^3}}{\sqrt{3(a+h)-(a+h)^3}+\sqrt{3a-a^3}}$$

$$=\lim_{h\to 0}\frac{\left(\sqrt{3(a+h)-(a+h)^3}\right)^2-\left(\sqrt{3a-a^3}\right)^2}{h\left(\sqrt{3(a+h)-(a+h)^3}+\sqrt{3a-a^3}\right)}=\lim_{h\to 0}\frac{(3(a+h)-(a+h)^3)-(3a-a^3)}{h\left(\sqrt{3(a+h)-(a+h)^3}+\sqrt{3a-a^3}\right)}$$

$$=\lim_{h\to 0}\frac{(3a+3h-a^3-3a^2h-3ah^2-h^3)-(3a-a^3)}{h\left(\sqrt{3(a+H)-(a+h)^3}+\sqrt{3a-a^3}\right)}=\lim_{h\to 0}\frac{3h-a^2h-3ah^2-h^3}{h\left(\sqrt{3(a+h)-(a+h)^3}+\sqrt{3a-a^3}\right)}$$

$$=\lim_{h\to 0}\frac{3-3a^2-3ah-h^2}{\sqrt{3(a+h)-(a+h)^3}+\sqrt{3a-a^3}}=\frac{3-3a^2-3a\cdot 0-0^2}{\sqrt{3(a+0)-(a+0)^3}+\sqrt{3a-a^3}}=\frac{3-3a^2}{2\sqrt{3a-a^3}}$$

Setting $f'(a)=0$ we get the equation: $\dfrac{3-3a^2}{2\sqrt{3a-a^3}}=0$

This can only be satisfied if $3-3a^2=0$, which yields $a=1$ and $a=-1$

$f(1)=\sqrt{3\cdot 1-1^2}=\sqrt{2}$ but $f(-1)=\sqrt{3\cdot(-1)-(-1)^2}=\sqrt{-4}$ does not exist

and since the domain of f does not include -1, it cannot be an answer!

So our final answer is only the first point: $(1,\sqrt{2})$

We can even solve for tangent lines in trickier settings this way:

Example 7.3.1

What value(s) of c make the line $3x+y=c$ tangent to $y=x^3+3x^2+5$?

Solution

First lets find the general equation of the tangent line at $(a,f(a))$:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\left((a+h)^3+3(a+h)^2+5\right)-\left(a^3+3a^2+5\right)}{h}$$

$$=\lim_{h\to 0}\frac{3a^2h+3ah^2+h^3+6ah+3h^2}{h}=\lim_{h\to 0}\left(3a^2+3ah+h^2+6a+3h\right)=3a^2+6a$$

Plugging this into the formula $y-f(a)=f'(a)(x-a)$ gives: $y-\left(a^3+3a^2+5\right)=(3a^2+6a)(x-a)$

This looks nicer rewritten as $y=(3a^2+6a)x+(5-2a^3-3a^2)$

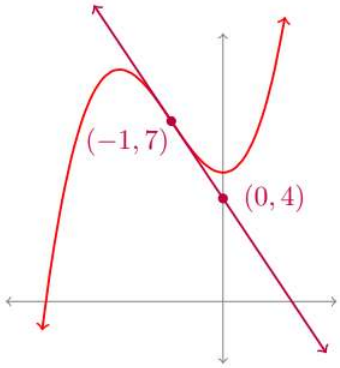
We want this to be the same as $3x+y=c$, which we can write as $y=-3x+c$

This means the lines must have the same slope and same y-intercept, which gives us a system of equations:

$$\begin{cases}-3=3a^2+6a\\ c=5-2a^3-3a^2\end{cases}$$

$-3=3a^2+6a$ has only one solution: $a=-1$, so that means $c=5-2(-1)^3-3(-1)^2=4$ is our answer

Once again, a graph confirms this solution:



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7.4: Differentiability

As we gain more confidence in our ability to take the derivative limit, a natural question arises: will this limit always exist?

In $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, the denominator always goes to zero: $\lim\limits_{x \to a}(x - a) = (a - a) = 0$

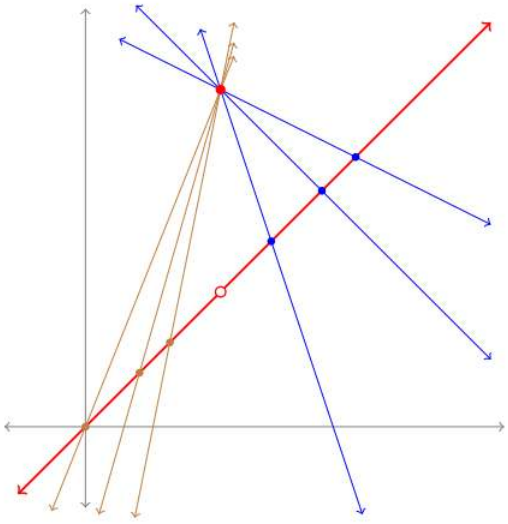
That means if the numerator goes to anything besides zero, then $f'(a)$ cannot be finite.

In order for $f(x)$ to be *differentiable at* a, (that just means $f'(a)$ exists and is finite), we must have
$\lim\limits_{x \to a} f(x) - f(a) = 0$

Adding $f(a)$ to both sides turns this into a familiar sight: $\lim\limits_{x \to a} f(x) = f(a)$

A function must be continuous at a to be differentiable at a

For example if $f(x) = \begin{cases} x & x \neq 2 \\ 5 & x = 2 \end{cases}$



As you drag a blue dot toward $x = 2$, the slope becomes very large and negative

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{x - 5}{x - 2} = \frac{-3}{0^+} = -\infty$$

As you drag a brown dot toward $x = 2$, the slope becomes very large and positive

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{x - 5}{x - 2} = \frac{-3}{0^-} = +\infty$$

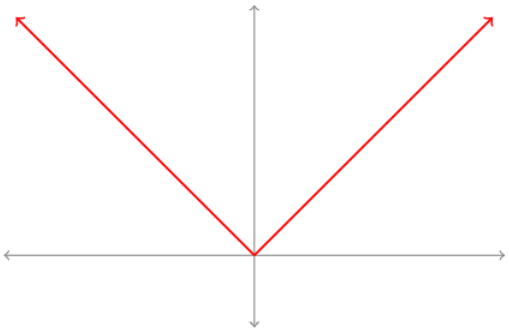
Since $\lim\limits_{x \to 2^+} \dfrac{f(x) - f(2)}{x - 2} \neq \lim\limits_{x \to 2^-} \dfrac{f(x) - f(2)}{x - 2}$ we know $f'(2) = \lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} = DNE$

Any type of discontinuity (removable, jump, infinite, or oscillating) at $x = a$ will ensure that $f'(a)$ does not exist, so $f(x)$ is not differentiable at that point.

But does this mean that all continuous functions are differentiable?

Consider the absolute value function $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

We can see from the graph that this function is continuous everywhere (including $x = 0$)



But is it differentiable? We can check by taking some limits:

For $a > 0$, we get $f'(a) = \lim_{x \to a} \frac{|x| - |a|}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = \lim_{x \to a} 1 = 1$
This is because for positive a, $f(x)$ acts like $y = x$ which has slope 1

For $a < 0$, $f'(a) = \lim_{x \to a} \frac{|x| - |a|}{x - a} = \lim_{x \to a} \frac{(-x) - (-a)}{x - a} = \lim_{x \to a} \frac{-(x - a)}{x - a} = \lim_{x \to a} -1 = -1$
This is because for negative a, $f(x)$ acts like $y = -x$ which has slope -1

That only leaves $a = 0$: $f'(0) = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$

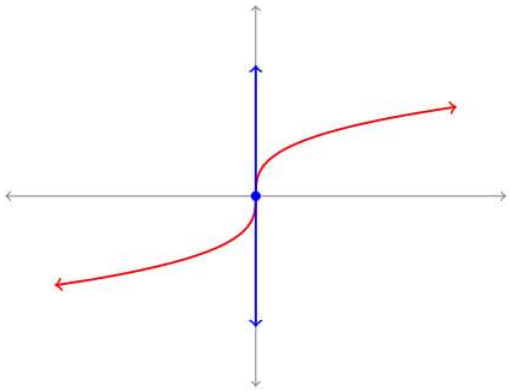
But $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$ and $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1$
Therefore $f'(0) = DNE$ and $f(x)$ is not differentiable at $x = 0$.

So **not all continuous functions are differentiable** (but all differentiable functions must be continuous!)
Anytime your function $f(x)$ has a sharp turn where the slope on the left and right side of $(a, f(a))$ changes instantly, the limit for $f'(a)$ will not exist.
Thus, any function with a sharp turn will not be differentiable at that point.

Another example of a continuous but not differentiable function is $f(x) = \sqrt[3]{x}$:

For $a \neq 0$ we can calculate:
$$f'(a) = \lim_{x \to a} \frac{\sqrt[3]{x} - \sqrt[3]{a}}{x - a} \cdot \frac{\sqrt[3]{x^2} + \sqrt[3]{ax} + \sqrt[3]{a^2}}{\sqrt[3]{x^2} + \sqrt[3]{ax} + \sqrt[3]{a^2}} = \lim_{x \to a} \frac{(\sqrt[3]{x})^3 - (\sqrt[3]{a})^3}{(x - a)\left(\sqrt[3]{x^2} + \sqrt[3]{ax} + \sqrt[3]{a^2}\right)}$$
$$= \lim_{x \to a} \frac{x - a}{(x - a)\left(\sqrt[3]{x^2} + \sqrt[3]{ax} + \sqrt[3]{a^2}\right)} = \lim_{x \to a} \frac{1}{\sqrt[3]{x^2} + \sqrt[3]{ax} + \sqrt[3]{a^2}} = \frac{1}{3\sqrt[3]{a^2}}$$

But when $a = 0$, this looks like:
$$f'(0) = \lim_{x \to a} \frac{\sqrt[3]{x} - \sqrt[3]{0}}{x - 0} = \lim_{x \to 0} \frac{\sqrt[3]{x}}{x} = \lim_{x \to 0} \frac{1}{\sqrt[3]{x^2}} = \frac{1}{0^+} = +\infty$$



While there is still a tangent line to this function at $(0,0)$, it is a vertical line (which has no slope).
In this scenario we still say that $f'(0)$ is undefined, since functions must output finite values.

A function is not differentiable at $(a, f(a))$ if it has an infinite slope.

Therefore a function is differentiable at a point as long as both:

(i) it is continuous,
(ii) it does not have a sharp turn or infinite slope.

Keep those rules in mind when taking the derivative of a piecewise function:

Example 7.4.1

Find the derivative of $f(x)=\begin{cases} x^2+x & x\geq 1 \\ 3x-1 & x<1 \end{cases}$

Solution

First consider $a>1$:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\left((a+h)^2+(a+h)\right)-(a^2+a)}{h}$$

$$=\lim_{h\to 0}\frac{\left(a^2+2ah+h^2+a+h\right)-(a^2+a)}{h}=\lim_{h\to 0}\frac{2ah+h^2+h}{h}=\lim_{h\to 0}(2a+h+1)=2a+1$$

Then consider $a<1$:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{(3(a+h)-1)-(3a-1)}{h}=\lim_{h\to 0}\frac{3h}{h}=\lim_{h\to 0}3=3$$

Lastly when $a=1$:

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{(x^2+x)-2}{x-1}=\lim_{x\to 1^+}\frac{(x+2)(x-1)}{x-1}=\lim_{x\to 1^+}(x+2)=3$$

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{(3x-1)-2}{x-1}=\lim_{x\to 1^+}\frac{3(x-1)}{x-1}=\lim_{x\to 1^+}3=3$$

Since these limits are finite and match, we know there is no infinite slope or sharp turn and $f'(1)=3$

So our final answer is $f'(a)=\begin{cases} 2a+1 & a>1 \\ 3 & a\leq 1 \end{cases}$

Similar to continuity, we can try to make a piecewise function differentiable:

Example 7.4.1

For what α, β does $f'(3)$ exist in $f(x) = \begin{cases} \sqrt{x+\alpha} & x < 3 \\ -\frac{3}{2x} + \beta & x \geq 3 \end{cases}$

Solution

We need to make sure that two pairs of limits match. First we need continuity at $x = 3$:

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} -\frac{3}{2x} + \beta = -\frac{1}{2} + \beta = f(3)$$
$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{x+\alpha} = \sqrt{3+\alpha}$$

So to be continuous, we must have $-\frac{1}{2} + \beta = \sqrt{3+\alpha}$

To be differentiable, we also need the slope to match on both sides of $x = 3$: $\lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3}$

$$\lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3^+} \frac{\left(-\frac{3}{2x}+\beta\right)-\left(-\frac{1}{2}+\beta\right)}{x-3} = \lim_{x\to 3^+} \frac{-\frac{3}{2x}+\frac{1}{2}}{x-3} \cdot \frac{2x}{2x}$$
$$= \lim_{x\to 3^+} \frac{-3+x}{2x(x-3)} = \lim_{x\to 3^+} \frac{x-3}{2x(x-3)} = \lim_{x\to 3^+} \frac{1}{2x} = \frac{1}{6}$$

$$\lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3^-} \frac{\sqrt{x+\alpha}-\left(-\frac{1}{2}+\beta\right)}{x-3} = \lim_{x\to 3^-} \frac{\sqrt{x+\alpha}-\sqrt{3+\alpha}}{x-3} \cdot \frac{\sqrt{x+\alpha}+\sqrt{3+\alpha}}{\sqrt{x+\alpha}+\sqrt{3+\alpha}}$$

$$= \lim_{x\to 3^-} \frac{(x+\alpha)-(3+\alpha)}{(x-3)(\sqrt{x+\alpha}+\sqrt{3+\alpha})} = \lim_{x\to 3^-} \frac{x-3}{(x-3)(\sqrt{x+\alpha}+\sqrt{3+\alpha})} = \lim_{x\to 3^-} \frac{1}{\sqrt{x+\alpha}+\sqrt{3+\alpha}} = \frac{1}{2\sqrt{3+\alpha}}$$

Setting these slopes equal to each other, we need $\frac{1}{2\sqrt{3+\alpha}} = \frac{1}{6}$, which gives $\alpha = 6$

This makes the first equation we found turn into $-\frac{1}{2} + \beta = \sqrt{3+6}$ giving $\beta = \frac{7}{2}$

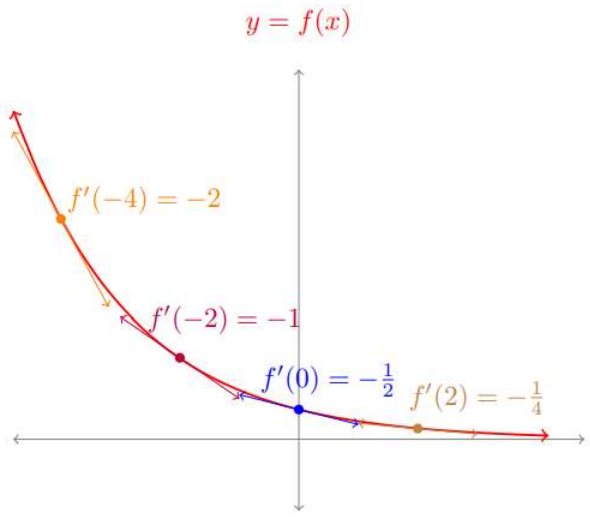
So if $\alpha = 6$ and $\beta = \frac{7}{2}$ then $f'(3) = \frac{1}{6}$

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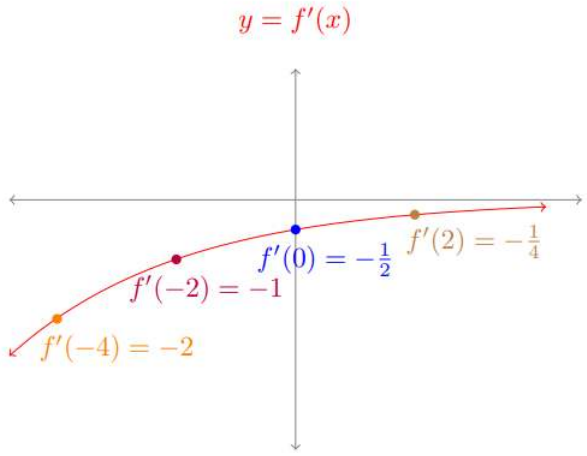
7.5: Derivatives and Graphs

The derivative of a function is just the slope of the tangent line at a given point, so if we have the graph of $y = f(x)$, we can use it to estimate the graph of $y = f'(x)$ (even without knowing the explicit definition of f)

Simply draw the tangent line at a few points on $y = f(x)$ and estimate the slope.

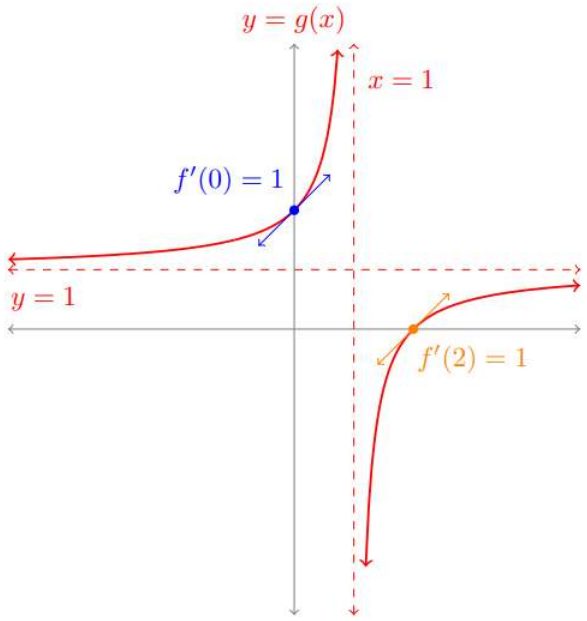


Then plot those values (the slope of the tangent line) as points on the graph of $y = f'(x)$:



The height of the graph of $f'(x)$ should match the slope on the graph of $y = f(x)$
You can see that $f(x)$ has a negative slope that becomes less negative as the value of x gets bigger.
That is why we connect these points so that the values of $f'(x)$ on the graph are negative but becoming less negative and approaching zero as x moves toward the right.

Lets try another, this time with some asymptotes involved:

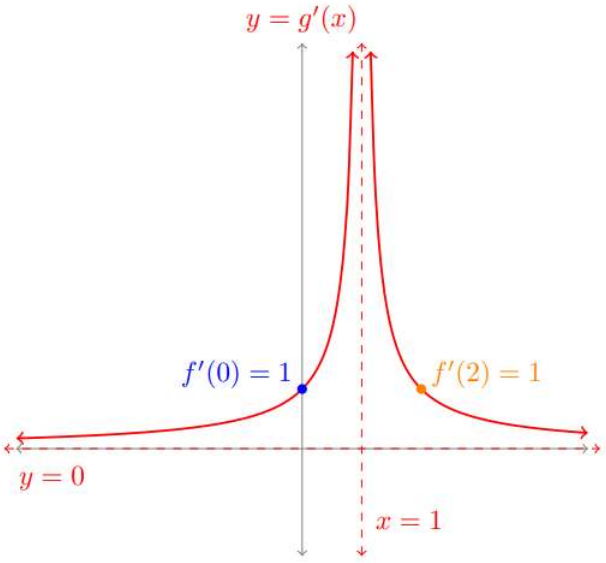


We don't need as many points, just a enough to have some points to plot, and the rest can be done by determining if the slope is getting bigger or smaller as we move around.

In $(-\infty, 1)$ the slope starts off as a tiny positive, eventually reaches a slope of 1 at $x = 0$, and then becomes extremely large and positive as we approach the vertical asymptote at $x = 1$.

In $(1, \infty)$, the slope starts off extremely positive near the vertical asymptote $x = 1$ and then becomes less positive as we move to the right, with our slope approaching 0 as $x \to \infty$

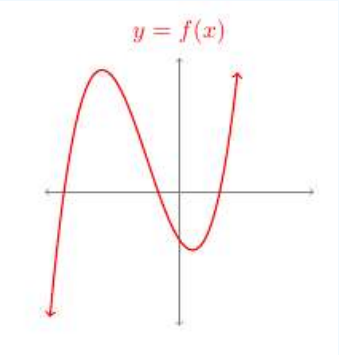
Lastly $g'(1)$ does not exist, because $g(x)$ was not continuous at $x = 1$



In general, the easiest places to evaluate or estimate f' are:
(i) where the tangent line is horizontal, so you know the derivative must equal zero
(ii) where the tangent lines are steepest (negative or positive) so you know the max and min values of f'

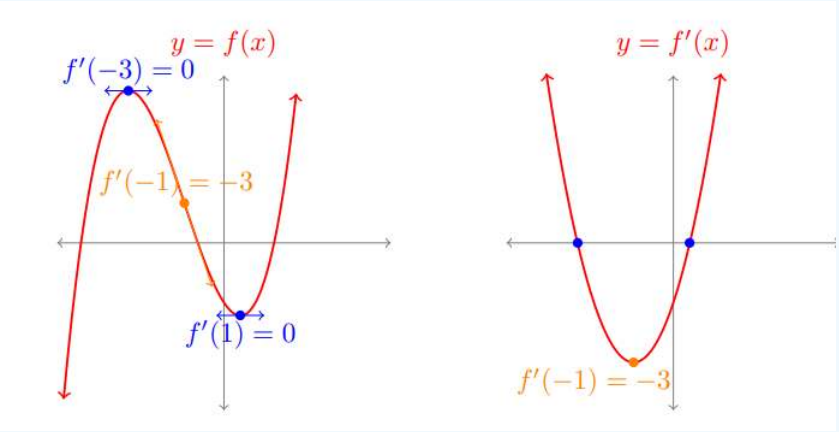
Example 7.5.1

Sketch the graph of the $y = f'(x)$ for the function $y = f(x)$ provided below.



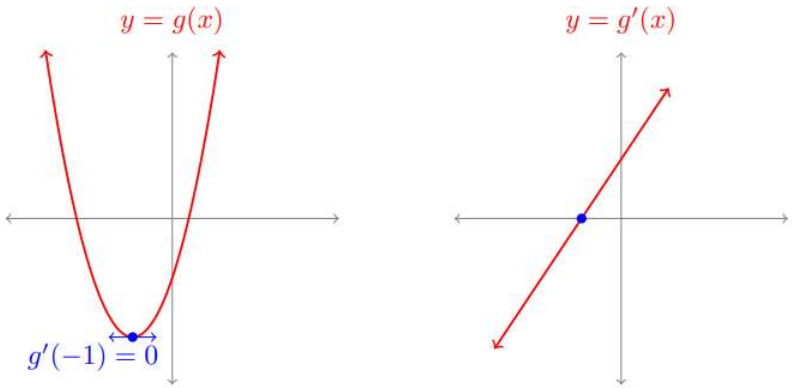
Solution

In $(-\infty, -3)$ the slope begins very large and positive, but approaches 0 as $x \to -3^-$
In $(-3, 1)$ the slope starts at zero, becomes negative, and then returns back to zero.
In $(1, \infty)$ the slope begins at zero and becomes very large and positive as $x \to +\infty$



Let $g(x) = f'(x)$ from the previous example. Since we have its graph now, we are be able to sketch the graph of $y = g'(x)$:

In $(-\infty, -1)$ the slope begins very large and negative, but approaches 0 as $x \to -1^-$
In $(-1, \infty)$ the slope begins at zero and becomes very large and positive as $x \to +\infty$



In case you missed it, we just sketched the graph of the derivative of $f'(x)$. This is called the *second derivative* of $f(x)$. We typically write this as $f''(x)$
(If you took a derivative again, it would be the third derivative: $f'''(x)$)

Generally we write $f^{(n)}(x)$ to represent the nth derivative of a function, especially when n is large (so we don't have to count the number of ' we see next to f)

If you want to find $f''(x)$ explicitly, you can use the same formula: $f''(a) = \lim_{x \to a} \dfrac{f'(x) - f'(a)}{x - a}$

Example 7.5.2

Find the second derivative of $f(x) = \dfrac{x^2+1}{x}$

Solution

We start by finding $f'(a)$:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\frac{x^2+1}{x} - \frac{a^2+1}{a}}{x - a} \cdot \frac{ax}{ax} = \lim_{x \to a} \frac{(x^2+1)a - (a^2+1)x}{(x-a)ax}$$

$$= \lim_{x \to a} \frac{a - x}{(x-a)ax} = \lim_{x \to a} \frac{-(x-a)}{(x-a)ax} = \lim_{x \to a} \frac{-1}{ax} = -\frac{1}{a^2}$$

Now we repeat to find $f''(a)$:

$$f''(a) = \lim_{x \to a} \frac{f'(x) - f'(a)}{x - a} = \lim_{x \to a} \frac{\left(-\frac{1}{x^2}\right) - \left(-\frac{1}{a^2}\right)}{x - a} \cdot \frac{a^2x^2}{a^2x^2} = \lim_{x \to a} \frac{-a^2 + x^2}{(x-a)a^2x^2}$$

$$= \lim_{x \to a} \frac{(x-a)(x+a)}{(x-a)a^2x^2} = \lim_{x \to a} \frac{x+a}{a^2x^2} = \frac{2a)}{a^4} = \frac{2}{a^3}$$

So our second derivative is the function $f''(x) = \dfrac{2}{x^3}$

For a function $f(x)$, we know that $f'(a)$ represents the slope of the tangent line at the point $(a, f(a))$. This is often referred to as the instantaneous rate of change of $f(x)$ at $x = a$.

The second derivative is therefore the rate of change of the slope of the tangent line, so it determines if the slope is increasing or decreasing at $(a, f(a))$.

Later we will see how the values of $f''(x)$ effect the shape of the graph of $f(x)$, but for now we will settle for just being able to find it or sketch it!

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CHAPTER OVERVIEW

8: More Derivatives

> *"No one is original. Everyone is derivative." - Sonny Rollins*

Chapter eight will cut down the laborious task of taking a derivative into an efficient pattern recognition activity.

First we find the derivatives of constant and linear functions, then stumble into the pattern for the derivative of a power function. We also see how the derivative limit interacts with function addition, subtraction, multiplication, and division. The calculations for each shortcut are justified between statements, but the lazy reader only needs to memorize the bolded results.

We also find the derivatives of each of the trigonometric functions (after a little help from a new theorem called the Squeeze Theorem). These combined with the power functions give many different ways to practice the differentiation rules mentioned above.

We define the exponential function, including its domain and range and how to sketch it depending on the base of the function. The properties of exponential functions and taking their limits are also discussed.

We look into how our new exponential functions interact with the differentiation process to discover that one base in particular causes it to be its own derivative. This number is referred to as e, Euler's number. We can once again get practice using our new exponential derivative shortcuts in the middle of more and more elaborate examples.

8.1: Derivative Shortcuts
8.2: Trig Derivatives
8.3: Exponential Functions
8.4: Derivatives of Exponential Functions
8.5: Derivative Examples

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8.1: Derivative Shortcuts

We have been introduced to the derivative of a function and 2 different forms of the limit to find it:

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

As we will be using derivatives for many future topics, it is in our best interest to find some shortcuts for evaluating these limits. This section will focus on finding the relationship between the functions $f(x)$ and $f'(x)$ so that we can find a pattern and avoid having to these limits repeatedly.

The easiest of these patterns is taking the derivative of a constant function:

If $f(x)=C$ for all x, then $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{C-C}{x-a}=\lim_{x\to a}0=0$

The derivative of any constant function is always zero.

Similarly, we can find a shortcut for the derivative of a linear function:

If $f(x)=mx+b$, then $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{(mx+b)-(ma+b)}{x-a}$

$$=\lim_{x\to a}\frac{mx-ma}{x-a}=\lim_{x\to a}\frac{m(x-a)}{x-a}=\lim_{x\to a}m=m$$

The derivative of a linear function is constant and equal to the slope m

We can also find the derivative of a power function with positive rational power: $f(x)=x^{\frac{m}{n}}$

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{x^{\frac{m}{n}}-a^{\frac{m}{n}}}{x-a}$$

$$=\lim_{x\to a}\frac{\sqrt[n]{x^m}-\sqrt[n]{a^m}}{x-a}\cdot\frac{\left[(\sqrt[n]{x^m})^{n-1}+(\sqrt[n]{x^m})^{n-2}\cdot(\sqrt[n]{a^m})+\cdots+(\sqrt[n]{a^m})^{n-1}\right]}{\left[(\sqrt[n]{x^m})^{n-1}+(\sqrt[n]{x^m})^{n-2}\cdot(\sqrt[n]{a^m})+\cdots+(\sqrt[n]{a^m})^{n-1}\right]}$$

$$=\lim_{x\to a}\frac{x^m-a^m}{(x-a)\left[(\sqrt[n]{x^m})^{n-1}+(\sqrt[n]{x^m})^{n-2}\cdot(\sqrt[n]{a^m})+\cdots+(\sqrt[n]{a^m})^{n-1}\right]}$$

$$=\lim_{x\to a}\frac{(x-a)\left(x^{m-1}+x^{m-2}\cdot a+\cdots+a^{m-1}\right)}{(x-a)\left[(\sqrt[n]{x^m})^{n-1}+(\sqrt[n]{x^m})^{n-2}\cdot(\sqrt[n]{a^m})+\cdots+(\sqrt[n]{a^m})^{n-1}\right]}$$

$$=\lim_{x\to a}\frac{x^{m-1}+x^{m-2}\cdot a+\cdots+a^{m-1}}{(\sqrt[n]{x^m})^{n-1}+(\sqrt[n]{x^m})^{n-2}\cdot(\sqrt[n]{a^m})+\cdots+(\sqrt[n]{a^m})^{n-1}}$$

$$=\frac{a^{m-1}+a^{m-2}\cdot a+\cdots+a^{m-1}}{(\sqrt[n]{a^m})^{n-1}+(\sqrt[n]{a^m})^{n-2}\cdot(\sqrt[n]{a^m})+\cdots+(\sqrt[n]{a^m})^{n-1}}$$

$$=\frac{ma^{m-1}}{n(\sqrt[n]{a^m})^{n-1}}=\frac{m}{n}a^{(m-1)-\frac{(n-1)m}{n}}=\frac{m}{n}a^{\frac{m}{n}-1}$$

This result is called the *power rule*: $f(x)=x^p$ has $f'(x)=px^{p-1}$
(**This formula works for all powers, not just positive or rational ones!**)

These shortcuts are nice, but they become much more powerful if we can determine how the sum, difference, product, and quotient of two functions are related to differentiation.

Assume that $f(x)$ and $g(x)$ are differentiable functions:

(so $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ and $g'(a)=\lim_{x\to a}\frac{g(x)-g(a)}{x-a}$ are both finite and exist.)

If $h(x) = f(x) + g(x)$, then

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{(f(x) + g(x)) - (f(a) + g(a))}{x - a}$$
$$= \lim_{x \to a} \frac{(f(x) - f(a)) + (g(x) - g(a))}{x - a}$$
$$= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) + \left(\frac{g(x) - g(a)}{x - a} \right)$$
$$= f'(a) + g'(a)$$

The derivative of the sum of functions is the sum of their derivatives.

If $h(x) = f(x) - g(x)$, then

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{(f(x) - g(x)) - (f(a) - g(a))}{x - a}$$
$$= \lim_{x \to a} \frac{(f(x) - f(a)) - (g(x) - g(a))}{x - a}$$
$$= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) - \left(\frac{g(x) - g(a)}{x - a} \right)$$
$$= f'(a) - g'(a)$$

The derivative of the difference of functions is the difference of their derivatives.

If $h(x) = C \cdot f(x)$ for some constant C, then

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{C \cdot f(x) - C \cdot f(a)}{x - a}$$
$$= \lim_{x \to a} C \cdot \frac{f(x) - f(a)}{x - a}$$
$$= C \cdot f'(a)$$

The derivative of a constant multiple of a function is that multiple of its derivative.

This is enough for us to find the derivative of any polynomial function (and even a few others):

Example 8.1.1

Find the derivative of each polynomial below

$p(x) = 2x^3 + 5x^2 - 9$

$s(t) = t^9 - 6\sqrt{t} + 11t$

Solution

(a) $p'(x) = 2 \cdot 3x^2 + 5 \cdot 2x^1 - 9 \cdot 0 = 6x^2 + 10x$

(b) $s'(t) = 9t^8 - 6 \cdot \frac{1}{2} t^{-\frac{1}{2}} + 11 = 9t^8 - 3t^{-\frac{1}{2}} + 11$

Sadly, the rules for multiplication and division are not as simple:

If $h(x) = f(x) \cdot g(x)$, then

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{(f(x) \cdot g(x)) - (f(a) \cdot g(a))}{x - a}$$

Here we add and subtract $f(a)g(x)$ to the numerator

$$= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x - a}$$

$$= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) g(x) - f(a) \left(\frac{g(x) - g(a)}{x - a} \right)$$

$$= f'(a)g(a) + f(a)g'(a)$$

This is called the *product rule* for derivatives

If $h(x) = \dfrac{f(x)}{g(x)}$, then

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x - a} \cdot \frac{g(x)g(a)}{g(x)g(a)}$$

$$= \lim_{x \to a} \frac{f(x) \cdot g(a) - f(a) \cdot g(x)}{(x - a)g(x)g(a)}$$

This time we add and subtract $f(a)g(a)$ to the numerator

$$= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a) + f(a)g(a) - f(a)g(a)}{(x - a)g(x)g(a)}$$

$$= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) \frac{1}{g(a)} - \frac{f(a)}{g(x)g(a)} \left(\frac{g(x) - g(a)}{x - a} \right)$$

$$= \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$$

This is called the *quotient rule* for derivatives

These rules combine to give us shortcuts to take the derivative of any sum, difference, product, or quotient of power functions:

Example 8.1.2

Find the derivative of each function below

(a) $h(x) = (3x^2 + 5)(5x - \sqrt[3]{x})$

(b) $v(y) = \dfrac{6y^3 - y + 1}{y^5 + 7}$

Solution

(a) $h(x) = f(x) \cdot g(x)$

$f(x) = 3x^2 + 5$ yields $f'(x) = 3(2x^1) + 0 = 6x$

$g(x) = 5x - x^{\frac{1}{3}}$ yields $g'(x) = 5 + \frac{1}{3}x^{-\frac{2}{3}}$

$$h'(x) = f'(x)g(x) + f(x)g'(x) = (6x)\left(5x - x^{\frac{1}{3}}\right) + \left(3x^2 + 5\right)\left(5 + \tfrac{1}{3}x^{-\frac{2}{3}}\right)$$

(b) $v(y) = \dfrac{f(y)}{g(y)}$

$f(y) = 6y^3 - y + 1$ yields $f'(y) = 6(3y^2) - 1 + 0 = 18y^2 - 1$

$g(y) = y^5 + 7$ yields $g'(y) = 5y^4 + 0 = 5y^4$

$$v'(y) = \frac{f'(y)g(y) - f(y)g'(y)}{[g(y)]^2} = \frac{(18y^2 - 1)(y^5 + 7) - (6y^3 - y + 1)(5y^4)}{(y^5 + 7)^2}$$

While using this formula works, sometimes the letter f or g is already being used. It is best to simply copy the pattern instead of naming functions:

Example 8.1.2

Find the derivative of each function below

(a) $h(x) = (3x^2+5)(5x-\sqrt[3]{x})$

(b) $v(y) = \dfrac{6y^3-y+1}{y^5+7}$

Solution

(a) We see that $h(x) = (3x^2+5)(5x-\sqrt[3]{x})$ is a product, so we start with the product rule:

$$h'(x) = (3x^2+5)'(5x-\sqrt[3]{x}) + (3x^2+5)(5x-\sqrt[3]{x})' = (6x)\left(5x-x^{\frac{1}{3}}\right) + \left(3x^2+5\right)\left(5+\tfrac{1}{3}x^{-\frac{2}{3}}\right)$$

(b) We see that $v(y) = \dfrac{6y^3-y+1}{y^5+7}$ is a fraction, so we start with the quotient rule:

$$v'(y) = \frac{(6y^3-y+1)'(y^5+7) - (6y^3-y+1)(y^5+7)'}{(y^5+7)^2} = \frac{(18y^2-1)(y^5+7) - (6y^3-y+1)(5y^4)}{(y^5+7)^2}$$

The more you practice this, the easier it will be to take derivatives.

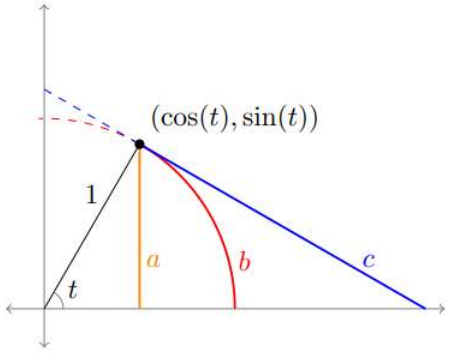
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8.2: Trig Derivatives

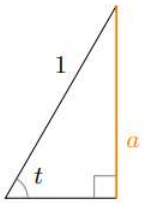
The goal of this section is to find the derivatives of the six trigonometric functions. In order to do this, we will need to be able to evaluate $\frac{0}{0}$ limits involving these trig functions.

Thankfully most of these can be calculated as long as we can determine $\lim_{t\to 0}\frac{\sin(t)}{t}$

To find this limit, consider the diagram in the unit circle below for $t \in \left(0, \frac{\pi}{2}\right)$:

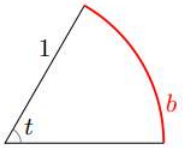


Its pretty clear that in terms of length: $0 \le a \le b \le c$
Lets focus on the three shapes individually to solve for each:



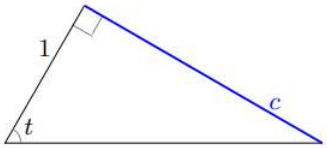
$$\sin(t) = \frac{a}{1}$$

$$a = \sin(t)$$



$$b = \frac{t}{2\pi} \cdot (2\pi \cdot 1)$$

$$b = t$$



$$\tan(t) = \frac{c}{1}$$

$$c = \tan(t)$$

This gives us $\sin(t) \le t \le \tan(t)$ for all $t \in \left(0, \frac{\pi}{2}\right)$

If we divide all three sides by $\sin(t)$ (which is positive), this becomes:

$$\frac{\sin(t)}{\sin(t)} \le \frac{t}{\sin(t)} \le \frac{\tan(t)}{\sin(t)}$$

Then we take the reciprocal of all terms (which switches the direction of the inequality):

$$\frac{\sin(t)}{\sin(t)} \ge \frac{\sin(t)}{t} \ge \frac{\sin(t)}{\tan(t)}$$

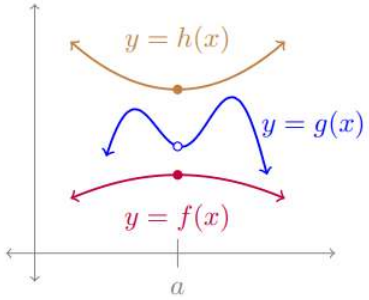
Lastly we simplify the outside two functions:

$$1 \ge \frac{\sin(t)}{t} \ge \cos(t)$$

This inequality is going to be very useful when trying to take $\lim_{t\to 0}\frac{\sin(t)}{t}$, but we still need one more limit property to evaluate it!

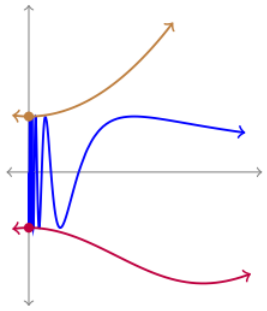
First, if we know that $f(x), g(x)$ and $h(x)$ are functions that satisfy $f(x) \le g(x) \le h(x)$, **and** we know that $\lim\limits_{x \to a} f(x)$, $\lim\limits_{x \to a} g(x)$, and $\lim\limits_{x \to a} h(x)$ all exist, then we must have $\lim\limits_{x \to a} f(x) \le \lim\limits_{x \to a} g(x) \le \lim\limits_{x \to a} h(x)$

$$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x) \le \lim_{x \to a} h(x)$$



Unfortunately, if we don't know the middle limit exists, this becomes useless:

Consider $f(x) = \cos(x) - 2$, $g(x) = \sin\left(\frac{\pi}{x}\right)$ and $h(x) = x^2 + 1$ near $a = 0$:

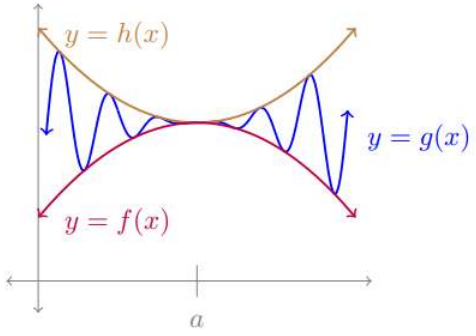


Clearly $f(x) \le g(x) \le h(x)$, and $\lim\limits_{x \to 0^+} f(x) = -1$, $\lim\limits_{x \to 0^+} h(x) = 1$, but $\lim\limits_{x \to 0^+} g(x) = DNE$ and we cannot say $-1 \le DNE \le 1$, so the inequality fails!

Luckily if $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x)$, there is no space for $\lim\limits_{x \to a} g(x) = DNE$ to occur.

This result is called the *Squeeze Theorem*:

When $f(x), g(x)$ and $h(x)$ are functions that satisfy $f(x) \le g(x) \le h(x)$, **and** we know that $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x)$, then we must have $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} h(x)$



The Squeeze Theorem is useful in situations when the limit of some component in a function does not exist, but is multiplied by a component that has a limit of zero:

Example 8.2.1

Evaluate the following limits by using the Squeeze Theorem:

(a) $\lim\limits_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$

(b) $\lim\limits_{x\to \infty} \frac{5x+\cos(x^3)}{\sqrt{x^2+4}}$

Solution

(a) This cannot turn into $\left(\lim\limits_{x\to 0} x^2\right)\left(\lim\limits_{x\to 0} \sin\left(\frac{\pi}{x}\right)\right) = 0 \cdot DNE$

(besides what even is $0 \cdot DNE$ anyway?)

Instead we see that $-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$ for all $x \neq 0$

If we multiply all three sides by x^2, this becomes:

$-x^2 \le x^2 \sin\left(\frac{\pi}{x}\right) \le x^2$

Since $\lim\limits_{x\to 0} -x^2 = -0^2 = 0$ and $\lim\limits_{x\to 0} x^2 = 0^2 = 0$

by the Squeeze Theorem, we must have $\lim\limits_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0$

(b) Since $\lim\limits_{x\to \infty} \cos(x^3) = DNE$ we cannot use the division limit law

Instead we again use the fact that $-1 \le \cos(x^3) \le 1$ for all x

If we add $5x$ to all three sides, then divide by $\sqrt{x^2+4}$ we get:

$$\frac{5x-1}{\sqrt{x^2+4}} \le \frac{5x+\cos(x^3)}{\sqrt{x^2+4}} \le \frac{5x+1}{\sqrt{x^2+4}}$$

We have $\lim\limits_{x\to \infty} \frac{5x-1}{\sqrt{x^2+4}} = \lim\limits_{x\to \infty} \frac{5x}{\sqrt{x^2}} = \lim\limits_{x\to \infty} \frac{5x}{x} = \lim\limits_{x\to \infty} 5 = 5$

and have $\lim\limits_{x\to \infty} \frac{5x+1}{\sqrt{x^2+4}} = \lim\limits_{x\to \infty} \frac{5x}{\sqrt{x^2}} = \lim\limits_{x\to \infty} \frac{5x}{x} = \lim\limits_{x\to \infty} 5 = 5$

Therefore by the Squeeze Theorem, $\lim\limits_{x\to \infty} \frac{5x+\cos(x^3)}{\sqrt{x^2+4}} = 5$

Note that the Squeeze Theorem works for any type of limit (one-sided, double sided, or infinite) as long as the resulting limits on the two outside functions match.

Since we know $1 \ge \frac{\sin(t)}{t} \ge \cos(t)$, and see $\lim\limits_{t\to 0^+} 1 = 1$ and $\lim\limits_{t\to 0^+} \cos(t) = \cos(0) = 1$

We know that $\lim\limits_{t\to 0^+} \frac{\sin(t)}{t} = 1$ must hold by the Squeeze Theorem!

(A similar picture and inequalities in quadrant 4 will give us $\lim\limits_{t\to 0^-} \frac{\sin(t)}{t} = 1$ as well.)

Therefore we can officially say that $\lim\limits_{t\to 0} \frac{\sin(t)}{t} = 1$

This can be generalized to say that $\sin(t) \approx t$ when t is close to zero.

This can be useful when taking other $\frac{0}{0}$ limits involving trigonometric functions:

Example 8.2.2

Evaluate each of the following limits.

(a) $\lim_{x\to 0}\dfrac{\sin(2x)}{3x}$

(b) $\lim_{z\to 0}\dfrac{\tan(8z)}{\sin(4z)}$

(c) $\lim_{w\to \pi}\dfrac{\sin(w)}{5w}$

(d) $\lim_{h\to 0}\dfrac{\cos(h)-1}{h}$

Solution

(a) $\lim_{x\to 0}\dfrac{\sin(2x)}{3x}=\lim_{x\to 0}\dfrac{\sin(2x)}{3x}\cdot\dfrac{2x}{2x}=\lim_{x\to 0}\dfrac{\sin(2x)}{2x}\cdot\dfrac{2x}{3x}$

$=\left(\lim_{x\to 0}\dfrac{\sin(2x)}{2x}\right)\cdot\left(\lim_{x\to 0}\dfrac{2}{3}\right)=1\cdot\dfrac{2}{3}=\dfrac{2}{3}$

(b) $\lim_{z\to 0}\dfrac{\tan(8z)}{\sin(4z)}=\lim_{z\to 0}\dfrac{\sin(8z)}{\cos(8z)\sin(4z)}\cdot\dfrac{8z}{8z}\cdot\dfrac{4z}{4z}=\lim_{z\to 0}\dfrac{\sin(8z)}{8z}\cdot\dfrac{4z}{\sin(4z)}\cdot\dfrac{8z}{4z\cos(8z)}$

$=\left(\lim_{z\to 0}\dfrac{\sin(8z)}{8z}\right)\cdot\left(\lim_{z\to 0}\dfrac{4z}{\sin(4z)}\right)\cdot\left(\lim_{z\to 0}\dfrac{2}{\cos(z)}\right)=1\cdot 1\cdot\dfrac{2}{1}=2$

(c) $\lim_{w\to \pi}\dfrac{\sin(w)}{5w}$ this limit is not $\frac{0}{0}$, we can legally plug in $w=\pi$:

$=\dfrac{\lim_{w\to\pi}\sin(w)}{\lim_{w\to\pi}5w}=\dfrac{\sin(\pi)}{5\pi}=0$

(d) $\lim_{h\to 0}\dfrac{\cos(h)-1}{h}=\lim_{h\to 0}\dfrac{\cos(h)-1}{h}\cdot\dfrac{\cos(h)+1}{\cos(h)+1}=\lim_{h\to 0}\dfrac{\cos^2(h)-1}{h(\cos(h)+1)}$

$=\lim_{h\to 0}\dfrac{-\sin^2(h)}{h(\cos(h)+1)}\cdot\dfrac{h}{h}=\lim_{h\to 0}\dfrac{\sin(h)}{h}\cdot\dfrac{\sin(h)}{h}\cdot\dfrac{-h}{\cos(h)+1}$

$=\left(\lim_{h\to 0}\dfrac{\sin(h)}{h}\right)\cdot\left(\lim_{h\to 0}\dfrac{\sin(h)}{h}\right)\cdot\left(\lim_{h\to 0}\dfrac{-h}{\cos(h)+1}\right)=1\cdot 1\cdot\dfrac{-0}{1+1}=0$

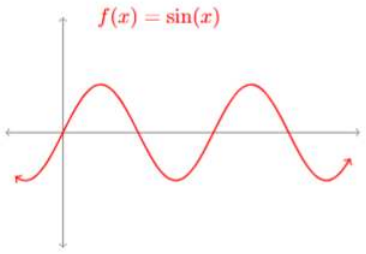
Finally we have all the tools we need to find the derivatives of the six trigonometric functions!

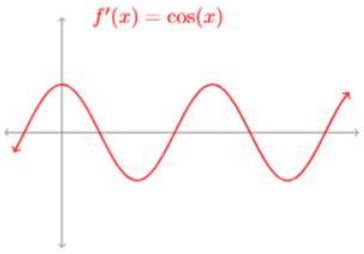
We start with $f(x)=\sin(x)$:

$f'(a)=\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\dfrac{\sin(a+h)-\sin(a)}{h}=\lim_{h\to 0}\dfrac{(\sin(a)\cos(h)-\sin(h)\cos(a))-\sin(a)}{h}$

$=\lim_{h\to 0}\sin(a)\left(\dfrac{\cos(h)-1}{h}\right)-\left(\dfrac{\sin(h)}{h}\right)\cos(a)=\sin(a)\cdot 0-1\cdot\cos(a)=\cos(a)$

This makes sense if you look at the graphs!



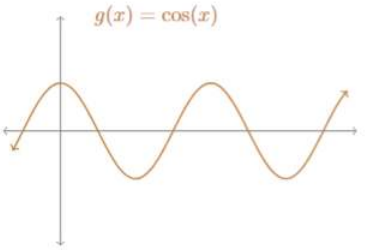


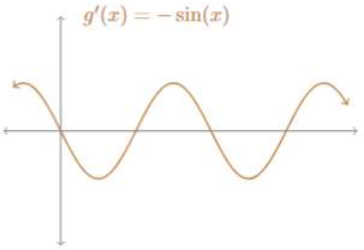
Next is $g(x) = \cos(x)$:

$$g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h\to 0} \frac{\cos(a+h) - \cos(a)}{h} = \lim_{h\to 0} \frac{(\cos(a)\cos(h) + \sin(h)\sin(a)) - \cos(a)}{h}$$

$$= \lim_{h\to 0} \cos(a)\left(\frac{\cos(h)-1}{h}\right) - \left(\frac{\sin(h)}{h}\right)\sin(a) = \cos(a)\cdot 0 - 1\cdot\sin(a) = -\sin(a)$$

And once again this makes sense if you look at the graphs!





Since the other four trigonometric functions can be written as fractions with sine and cosine, we can use the quotient rule to evaluate their derivatives:

If $f(x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$, then

$$f'(x) = \frac{(\sin(x))'(\cos(x)) - (\sin(x))(\cos(x))'}{[\cos(x)]^2}$$

$$= \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\sin^2(x)+\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

If $f(x) = \cot(x) = \dfrac{\cos(x)}{\sin(x)}$

$$f'(x) = \frac{(\cos(x))'(\sin(x)) - (\cos(x))(\sin(x))'}{[\sin(x)]^2}$$

$$= \frac{(-\sin(x))\cdot\sin(x) - \cos(x)\cdot\cos(x)}{\cos^2(x)}$$

$$= \frac{-(\sin^2(x)+\cos^2(x))}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)$$

If $f(x) = \sec(x) = \dfrac{1}{\cos(x)}$, then

$$f'(x) = \frac{(1)'(\cos(x)) - (1)(\cos(x))'}{[\cos(x)]^2}$$

$$= \frac{0 \cdot \cos(x) - 1 \cdot (-\sin(x))}{\cos^2(x)}$$

$$= \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)$$

If $f(x) = \csc(x) = \dfrac{1}{\sin(x)}$, then

$$f'(x) = \frac{(1)'(\sin(x)) - (1)(\sin(x))'}{[\sin(x)]^2}$$

$$= \frac{0 \cdot \sin(x) - 1 \cdot (\cos(x))}{\sin^2(x)}$$

$$= \frac{-\cos(x)}{\sin^2(x)} = \frac{-1}{\sin(x)} \cdot \frac{\cos(x)}{\sin(x)} = -\csc(x)\cot(x)$$

These six derivatives can be used in combination with our other differentiation formulas!

Example 8.2.3

Find the derivative of each function below

(a) $f(x) = x^3 \sec(x)$

(b) $p(t) = \dfrac{\tan(t)}{t^4 + 1}$

Solution

(a) $f'(x) = (x^3)'(\sec(x)) + (x^3)(\sec(x))' = (3x^2)(\sec(x)) + (x^3)(\sec(x)\tan(x)) = 3x^2\sec(x) + x^3\sec(x)\tan(x)$

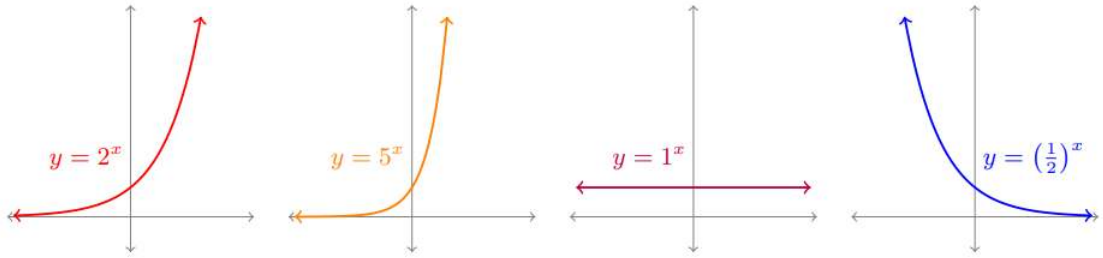
(b)

$$p'(t) = \frac{(\tan(t))'(t^4+1) - (\tan(t))(t^4+1)'}{(t^4+1)^2} = \frac{(\sec^2(t))(t^4+1) - (\tan(t))(4t^3)}{(t^4+1)^2} = \frac{(t^4+1)\sec^2(t) - (4t^3)\tan(t)}{(t^4+1)^2}$$

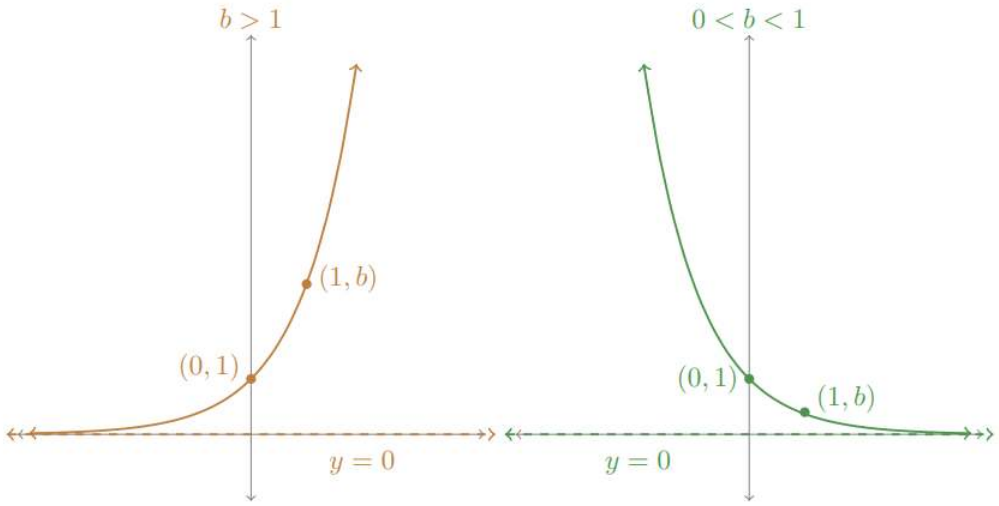
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8.3: Exponential Functions

Since derivatives are so fun, and we are running out of functions, we will introduce a new type.
For any constant $b > 0$ we say $f(x) = b^x$ is an *exponential function*.
Depending on the value of b, the graph can look a bit different:



In general, the graph of $y = b^x$ has two possible shapes for $b \neq 1$:



The good news is that for any choice of base $b > 0$, the domain of $f(x) = b^x$ is $(-\infty, \infty)$

And besides when $b = 1$, we can see that this function is one-to-one with a range of $(0, \infty)$

Notice that they both use the horizontal asymptote $y = 0$, but at different sides!

When $b > 1$ this means: $\lim_{x \to \infty} b^x = +\infty$ and $\lim_{x \to -\infty} b^x = 0$

When $0 < b < 1$ this means: $\lim_{x \to \infty} b^x = 0$ and $\lim_{x \to -\infty} b^x = +\infty$

We can easily sketch transformations of exponential functions by keeping track of the two points $(0, 1), (1, b)$ and the horizontal asymptote $y = 0$:

Example 8.3.1

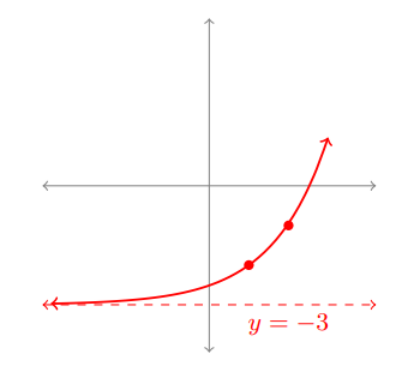
Sketch the graphs of each exponential function below.

(a) $y = 2^{x-1} - 3$

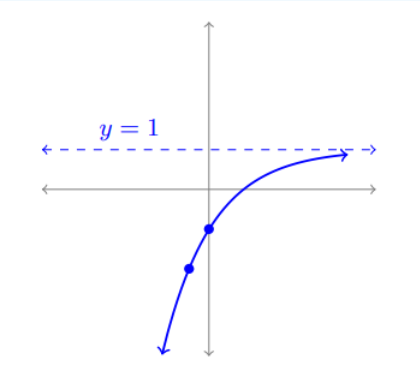
(b) $y = 1 - 3\left(\frac{2}{3}\right)^{2x+1}$

Solution

(a) Our transformation equations are $y_{new} = y_{old} - 3$ and $x_{new} = x_{old} + 1$



(b) Our transformation equations are $y_{new} = 1 - 3y_{old}$ and $x_{new} = \dfrac{x_{old} - 1}{2}$



We can see that any such transformation of an exponential function will yield a continuous graph.

To help us during limits, lets quickly review the laws of exponents:

$b^0 = 1$ and $b^1 = b$ for any b

$b^x \cdot b^y = b^{x+y}$ $b^x \div b^y = b^{x-y}$

$(b^x)^y = b^{x \cdot y}$ $a^x \cdot b^x = (a \cdot b)^x$

There are some things that you wish exponents did, but they dont for most a and b:

$(a+b)^x \neq a^x + b^x$ $(a-b)^x \neq a^x - b^x$ $a^x \cdot b^y \neq (a \cdot b)^{x+y}$

Since $f(x) = b^x$ is continuous everywhere, taking limits is usually straightforward plugging in.

Example 8.3.2

Evaluate the following limits involving exponential functions.

(a) $\lim\limits_{x \to 4} \dfrac{2^x - 8}{x^2 - 12}$

(b) $\lim\limits_{x \to \pi} 8^{\cos(x)}$

Solution

(a) $\lim_{x\to 4}\dfrac{2^x-8}{x^2-12}=\dfrac{2^4-8}{4^2-12}=\dfrac{8}{4}=2$

(b) $\lim_{x\to \pi}8^{\cos(x)}=8^{\cos(\pi)}=8^{-1}=\dfrac{1}{8}$

When taking limits to positive or negative infinity, just remember that the Lebron James Rule only applies when things are growing very large:

Example 8.3.3

Find the horizontal asymptotes of $y=\dfrac{3^x+8}{3^x+2}$

Solution

We start by taking $\lim_{x\to\infty}\dfrac{3^x+8}{3^x+2}$

In the top: $\lim_{x\to\infty}(3^x+8)=+\infty$ and bottom: $\lim_{x\to\infty}(3^x+2)=+\infty$

So we use the Lebron Rule to simplify the limit:

$y=\dfrac{3^x+8}{3^x+2}=\lim_{x\to\infty}\dfrac{3^x}{3^x}=\lim_{x\to\infty}1=1$

Therefore $y=1$ is a horizontal asymptote for this graph on the right side.

Next we check $\lim_{x\to-\infty}\dfrac{3^x+8}{3^x+2}$

In the top: $\lim_{x\to-\infty}(3^x+8)=8$ and bottom: $\lim_{x\to-\infty}(3^x+2)=2$

We cant use the Lebron Rule here, so use limit laws instead:

$\lim_{x\to-\infty}\dfrac{3^x+8}{3^x+2}=\dfrac{\lim_{x\to-\infty}3^x+8}{\lim_{x\to-\infty}3^x+2}=\dfrac{0+8}{0+2}=4$

Therefore $y=4$ is a horizontal asymptote for this graph on the left side.

This function has horizontal asymptotes at $y=1, y=4$

You can even include exponentials in piecewise function problems now:

Example 8.3.4

Find all discontinuities for the function $f(x)=\begin{cases}2^{x+1}-x & x\geq 2\\ \dfrac{3^x}{3^x-3} & x<2\end{cases}$

Solution

For $x>2$, we know $f(x)=2^{x+1}-x$ is continuous everywhere

For $x<2$, we know $f(x)=\dfrac{3^x}{3^x-3}$ is continuous whenever $3^x-3\neq 0$

(note that $3^1=3$, so $f(x)$ is continuous for all $x<2$ except $x=1$)

$\lim_{x\to 1^+}f(x)=\lim_{x\to 1+}\dfrac{3^x}{3^x-3}=\dfrac{3}{0^+}=+\infty$

This means there is an infinite discontinuity at $x=1$

And at the point $x = 2$ we check both sides:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2+} 2^{x+1} - x = 2^3 - 2 = 6$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2-} \frac{3^x}{3^x - 3} = \frac{3^2}{3^2 - 3} = \frac{3}{2}$$

Therefore there is a jump discontinuity at $x = 2$

Example 8.3.5

For what values of α, β is $g(x) = \begin{cases} \alpha^{2\cos(x)} & x > \pi \\ \beta & x = \pi \\ \cos^4\left(\frac{x}{3}\right) & x < \pi \end{cases}$ continuous at $x = \pi$?

Solution

$$\lim_{x \to \pi^+} g(x) = \lim_{x \to \pi^+} \alpha^{2\cos(x)} = \alpha^{2\cos(\pi)} = \alpha^{-2}$$

$$\lim_{x \to \pi^-} g(x) = \lim_{x \to \pi^-} \cos^4\left(\frac{x}{3}\right) = \cos^4\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

$g(\pi) = \beta$

To be continuous all three must match: $\alpha^{-2} = \frac{1}{16} = \beta$

Since $\alpha > 0$ must hold to be a exponential base, we get:

$\alpha = 4$ and $\beta = \frac{1}{16}$ are what make $g(x)$ continuous at $x = \pi$

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8.4: Derivatives of Exponential Functions

Now that we can take limits with exponential functions, the next step is to find its derivative:

If $f(x) = b^x$, and $b > 0$ then:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{b^{a+h}-b^a}{h} = \lim_{h\to 0} \frac{b^a \cdot b^h - b^a}{h} = b^a \cdot \lim_{h\to 0} \frac{b^h-1}{h}$$

The remaining limit is not easy to solve, but lets name it something: $L(b) = \lim_{h\to 0} \frac{b^h-1}{h}$

There is at least one value of b that we can easily plug into L:

$$L(1) = \lim_{h\to 0} \frac{1^h-1}{h} = \lim_{h\to 0} \frac{1-1}{h} = \lim_{h\to 0} 0 = 0$$

(Thats not too surprising, considering $f(x) = 1^x = 1$ is a constant function, so $f'(x) = 0$)

Sadly, there are no other *easy* values of b to plug in, but $L(b)$ is still a continuous function.

Lets plug in a few nice numbers and see what comes out:

$L(2)$

h	$\frac{2^h-1}{h}$
0.1	0.7177
0.01	0.6955
0.001	0.6934
0.0001	0.6931

$L(3)$

h	$\frac{3^h-1}{h}$
0.1	1.1612
0.01	1.1047
0.001	1.0992
0.0001	1.0986

Clearly $L(2) < 1$ and $L(3) > 1$, therefore by the Intermediate Value Theorem, we know there is some value $c \in (2,3)$ such that $L(c) = 1$

Since $\lim_{h\to 0} \frac{c^h-1}{h} = 1$, we can estimate it by picking $h = \frac{1}{n}$ to get:

$$\frac{c^{\frac{1}{n}}-1}{\frac{1}{n}} \approx 1 \longleftrightarrow c \approx \left(1+\frac{1}{n}\right)^n$$

This value of c is referred to as Euler's Number, written as e:

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \approx 2.7182$$

Another property of $L(b)$ can help us when $b = e^r$ for some constant $r \neq 0$:

$$L(e^r) = \lim_{h\to 0} \frac{(e^r)^h-1}{h} = \lim_{h\to 0} \frac{e^{rh}-1}{h} \cdot \frac{r}{r} = r \cdot \lim_{rh\to 0} \frac{e^{rh}-1}{rh} = r \cdot L(e) = r$$

The end result then is for $f(x) = (e^r)^x = e^{rx}$ we have $f'(a) = (e^r)^a L(e^r) = re^{ar}$

We will determine how to deal with bases that aren't of the form e^r in a later chapter.

For now we can combine these functions with others for more derivative practice:

Example 8.4.1

Find the derivative of each function below involving exponentials.

(a) $q(t) = e^{2t}\tan(t)$

(b) $u(s) = \frac{e^s-1}{e^{-s}+s}$

Solution

(a) $q'(t) = \left(e^{2t}\right)'(\tan(t)) + \left(e^{2t}\right)(\tan(t))' = \left(2e^{2t}\right)(\tan(t)) + \left(e^{2t}\right)(\sec^2(t))e^{2t}\left(2\tan(t) + \sec^2(t)\right)$

(b) $u'(s) = \dfrac{(e^s-1)'(e^{-s}+s) - (e^s-1)(e^{-s}+s)'}{(e^{-s}+s)^2} = \dfrac{(e^s)(e^{-s}+s) - (e^s-1)(-e^{-s}+1)}{(e^{-s}+s)^2} = \dfrac{3+(s-1)e^s - e^{-s}}{(e^{-s}+s)^2}$

These exponential functions can sometimes end up making easily recognizable patterns when taking multiple derivatives (which is something that will be useful in the distant future):

Example 8.4.2

Find the formula for $f^{(n)}(x)$ when $f(x) = xe^x$

Solution

We start by taking the first few derivatives:

$f'(x) = (x)'(e^x) + (x)(e^x)' = 1 \cdot e^x + x \cdot e^x = (x+1)e^x$
$f^{(2)}(x) = (x+1)'(e^x) + (x+1)(e^x)' = 1 \cdot e^x + (x+1) \cdot e^x = (x+2)e^x$
$f^{(3)}(x) = (x+2)'(e^x) + (x+2)(e^x)' = 1 \cdot e^x + (x+2) \cdot e^x = (x+3)e^x$
$f^{(4)}(x) = (x+3)'(e^x) + (x+3)(e^x)' = 1 \cdot e^x + (x+3) \cdot e^x = (x+4)e^x$

It is clear to see the pattern is $f^{(n)}(x) = (x+n)e^x$

And we can even use these new exponential derivatives to make piecewise functions differentiable:

Example 8.4.3

For what α, β is $g(x) = \begin{cases} e^{\alpha x} + \beta e^x & x > 0 \\ 5x - 6 & x \le 0 \end{cases}$ differentiable at $x = 0$?

Solution

First, we need $g(x)$ to be continuous:

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} e^{\alpha x} + \beta e^x = e^{\alpha \cdot 0} + \beta e^0 = 1 + \beta$$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} 5x - 6 = 5 \cdot 0 - 6 = -6$$

and since $g(0) = -6$, we just need $1 + \beta = -6$
Therefore if $\beta = -7$, $g(x)$ will be continuous at $x = 0$

We can find $g'(x)$ everywhere except at $x = 0$ by differentiating each piece:

$$g'(x) = \begin{cases} \alpha e^{\alpha x} + \beta e^x & x > 0 \\ 5 & x < 0 \\ ? & x = 0 \end{cases}$$

To be differentiable at $x = 0$, the slope needs to match on both sides of 0:

$$\lim_{x\to 0^+} g'(x) = \lim_{x\to 0^+} \alpha e^{\alpha x} + \beta e^x = \alpha \cdot e^{\alpha \cdot 0} + \beta \cdot e^0 = \alpha + \beta$$

$$\lim_{x\to 0^-} g'(x) = \lim_{x\to 0^-} 5 = 5$$

So to be differentiable we need $\alpha + \beta = 5$

Since we already solved $\beta = -7$, this means we need $\alpha = 12$

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8.5: Derivative Examples

Lets quickly recap all of the shortcuts we have for derivatives:

$(f(x)+g(x))' = f'(x)+g'(x)$
$(f(x)-g(x))' = f'(x)-g'(x)$
$(C\cdot f(x))' = C\cdot f'(x)$
$(f(x)\cdot g(x))' = f'(x)g(x)+f(x)g'(x)$
$\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$
$(C)' = 0$
$(mx+b)' = m$
$(x^p)' = px^{p-1}$
$(e^{rx})' = re^{rx}$
$(\sin(x))' = \cos(x)$
$(\cos(x))' = -\sin(x)$
$(\tan(x))' = \sec^2(x)$
$(\cot(x))' = -\csc^2(x)$
$(\sec(x))' = \sec(x)\tan(x)$
$(\csc(x))' = -\csc(x)\cot(x)$

If all you do is memorize the information above, you will still be a successful calculus student. There is no shame in doing practice problems while looking at a cheat sheet at first, but *hopefully* the more you practice the less you will be looking back at the formulas.

Example 8.5.1

Find the line tangent to $f(x) = e^{8-2x}\cdot\sqrt{x}$ at $x = 4$

Solution

We want to find $y - f(4) = f'(4)(x-4)$

Start with $f(4) = e^{8-2\cdot4}\cdot\sqrt{4} = e^0\cdot 2 = 2$

Next we use the product rule to differentiate (and note $e^{8-2x} = e^8\cdot e^{-2x}$):

$$f'(x) = \left(e^8\cdot e^{-2x}\right)'\left(x^{\frac{1}{2}}\right) + \left(e^{8-2x}\right)\left(x^{\frac{1}{2}}\right)'$$

$$= \left(e^8\cdot\left(-2e^{-2x}\right)\right)\left(x^{\frac{1}{2}}\right) + \left(e^{8-2x}\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= -2e^{8-2x}\cdot\sqrt{x} + e^{8-2x}\cdot\frac{1}{2\sqrt{x}}$$

$$f'(4) = -2e^{8-2\cdot4}\cdot\sqrt{4} + e^{8-2\cdot4}\cdot\frac{1}{2\sqrt{4}} = -4 + \frac{1}{4} = -\frac{15}{4}$$

So the equation of the line is: $y - 2 = -\dfrac{15}{4}(x-4)$

Example 8.5.2

Find the line tangent to $g(x)=\dfrac{e^x\cos(x)}{x^3-3x+1}$ at $(0,1)$

Solution

We want to find $y-g(0)=g'(0)(x-0)$

Since $g(0)=1$ is already given, we find $g'(x)$ by using the quotient rule:

$$g'(x)=\frac{(e^x\cos(x))'\left(x^3-3x+1\right)-(e^x\cos(x))\left(x^3-3x+1\right)'}{(x^3-3x+1)^2}$$

When taking $(e^x\cos(x))'$ we use the product rule.

$$=\frac{[(e^x)'(\cos(x))+(e^x)(\cos(x))']\left(x^3-3x+1\right)-(e^x\cos(x))\left(3x^2-3\right)}{(x^3-3x+1)^2}$$

$$=\frac{[(e^x)(\cos(x))+(e^x)(-\sin(x))]\left(x^3-3x+1\right)-e^x\cos(x)\left(3x^2-3\right)}{(x^3-3x+1)^2}$$

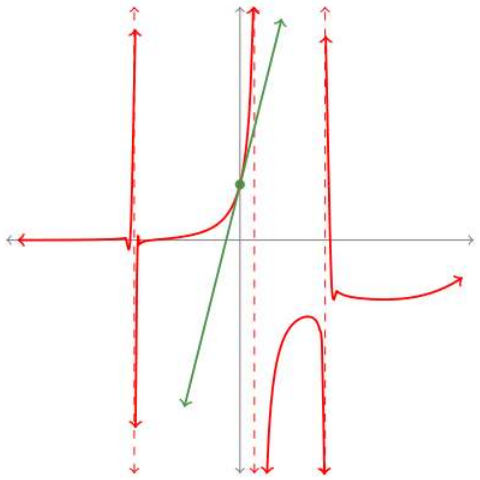
Now we plug in $x=0$:

$$g'(0)=\frac{[(e^0)(\cos(0))+(e^0)(-\sin(0))]\left(0^3-3\cdot0+1\right)-e^0\cos(0)\left(3\cdot0^2-3\right)}{(0^3-3\cdot0+1)^2}$$

$$=\frac{1\cdot1-1\cdot(-3)}{1^2}=4$$

Therefore the tangent line is: $y-1=4(x-0)$

Its a hard graph to draw, but the answer looks believable!



You can be given information about a function and or its derivative as part of a bigger function:

Example 8.5.3

Let $f(t)$ be a function with $f(\pi)=5$ and $f'(\pi)=3$. If $g(t)=\dfrac{f(t)}{\sec(t)+4}$, what is $g'(\pi)$?

Solution

We use the quotient rule formula to find g':

$$g'(t)=\frac{f'(t)(\sec(t)+4)-f(t)(\sec(t)+4)'}{(\sec(t)+4)^2}=\frac{f'(t)(\sec(t)+4)-f(t)(\sec(t)\tan(t))}{(\sec(t)+4)^2}$$

$$\text{So } g'(\pi)=\frac{f'(\pi)(\sec(\pi)+4)-f(\pi)\cdot\sec(\pi)\tan(\pi)}{(\sec(\pi)+4)^2}=\frac{3\cdot 3-5\cdot 0}{3^2}=1$$

Sometimes the information is given in a less straight-forward manner:

Example 8.5.4

Let $h(x)$ be a function tangent to $y=-3x+10$ at $(2,4)$. If $r(x)=e^{\frac{1}{2}x}h(x)$, what is $r'(2)$?

Solution

We are given $h(2)=4$ since that is the tangent point, and from the tangent line slope we can see that $h'(2)=-3$

We use the product rule formula to find r':

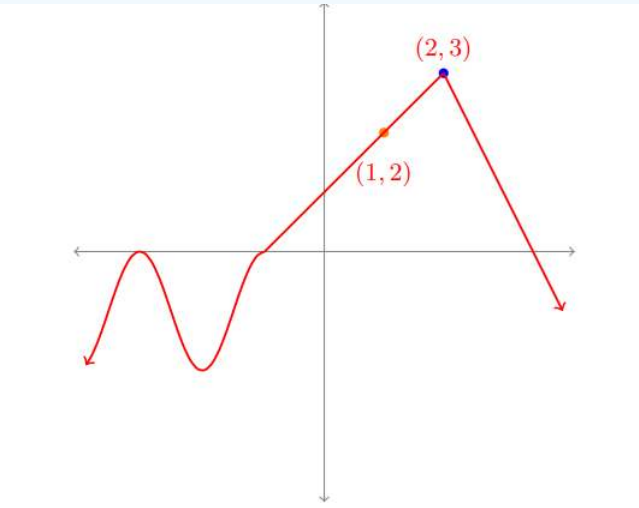
$$r'(x)=\left(e^{\frac{1}{2}x}\right)'\cdot h(x)+\left(e^{\frac{1}{2}x}\right)\cdot h'(x)=\left(\frac{1}{2}e^{\frac{1}{2}x}\right)\cdot h(x)+\left(e^{\frac{1}{2}x}\right)\cdot h'(x)$$

$$\text{So } r'(2)=\frac{1}{2}e^{\frac{1}{2}\cdot 2}\cdot h(2)+e^{\frac{1}{2}\cdot 2}\cdot h'(2)=\frac{1}{2}e\cdot 4+e\cdot(-3)=-e$$

You can also use the graph of a function to determine derivatives:

Example 8.5.5

Consider the graph of $y=q(x)$ below:



If $\ell(x) = \dfrac{x^2}{q(x)}$, find (a) $\ell'(1)$ and (b) $\ell'(2)$:

Solution

For both parts, we use the quotient rule formula to find ℓ':

$$\ell'(x) = \frac{(x^2)' \cdot q(x) - (x^2) \cdot q'(x)}{[q(x)]^2} = \frac{2xq(x) - x^2 q'(x)}{[q(x)]^2}$$

(a) We can see on the graph at $(1, 2)$ that $q(1) = 2$ and the slope is $q'(1) = 1$

So $\ell'(1) = \dfrac{2 \cdot 1 \cdot q(1) - 1^2 \cdot q'(1)}{[q(1)]^2} = \dfrac{2 \cdot 2 - 1 \cdot 1}{2^2} = \dfrac{3}{4}$

(b) We can see on the graph at $(2, 3)$ that $q(2) = 3$ but there is a sharp turn so the slope is $q'(2) = DNE$

So $\ell'(2) = \dfrac{2 \cdot 2 \cdot q(2) - 2^2 \cdot q'(2)}{[q(2)]^2} = \dfrac{4 \cdot 3 - 4 \cdot DNE}{3^2} = DNE$

Remember the derivative shortcut rules only apply when all of the derivatives and function values of the smaller components exist! If any part of the function is not continuous or differentiable, it can cause the whole expression is to be not differentiable!

You can also use the general formula for tangent lines to answer trickier problems:

Example 8.5.6

Find a line that is tangent to both $y = x^2$ and $y = x^2 - 2x + 3$

Solution

First we find the tangent line at $(a, f(a))$ for $f(x) = x^2$:

$f(a) = a^2$ and $f'(x) = 2x$ so $f'(a) = 2a$, which turns $y - f(a) = f'(a)(x - a)$ into $y - a^2 = 2a(x - a)$

We can re-write this as $y = 2ax + (-a^2)$

Next we find the tangent line at $(b, g(b))$ for $g(x) = x^2 - 2x + 3$:

$g(b) = b^2 - 2b + 3$ and $g'(x) = 2x - 2$ so $g'(b) = 2b - 2$, which turns $y - g(b) = g'(b)(x - b)$ into $y - (b^2 - 2b + 3) = (2b - 2)(x - b)$

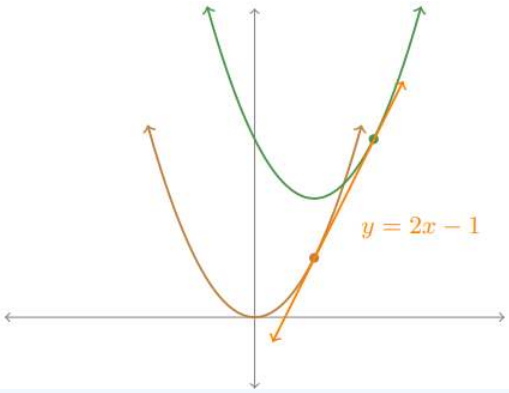
We can re-write this as $y = (2b - 2)x + (3 - b^2)$

For these two to match, we need the slopes and intercepts to be the same: $\begin{cases} 2a = 2b - 2 \\ -a^2 = 3 - b^2 \end{cases}$

The top equation becomes $a = b - 1$ and we can substitute it into the bottom to get: $-(b - 1)^2 = 3 - b^2$

This has solution $b = 2$, so we have $a = 2 - 1 = 1$

Plugging these into either tangent line equation gives the same result: $y = 2x - 1$



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CHAPTER OVERVIEW

9: Chain Rule and Inverse Trig Functions

> *"I am in the world only for the purpose of composing." - Franz Schubert*

Just as before, we have fully explored the relationship between derivatives and the four major operations, and forgotten about the fifth: composition. Chapter nine is an exploration on how differentiation interacts with composition.

First we see the nuts and bolts of the operation by explicitly stating the formula for the chain rule. After a few examples, the reader should feel competent enough to take the derivative of most combinations of functions they could imagine.

Of course, one cant mention composition without thinking about inverse functions again. We discover a formula for taking the derivative of an inverse function as a direct result of the chain rule.

Since inverse functions are so popular, we introduce the six new inverse trigonometric functions. These functions have specific restricted domains/ranges, but all ultimately are used to reverse one of the six standard trigonometric functions. We find all of their graphs, and learn how to evaluate them at points or in composition with other trigonometric functions.

Since these six functions are all continuous and smooth, they also have derivatives. We use our new inverse derivative formula to find out exactly what the derivative of each of our six new inverse trigonometric friends will be. As always, we get a little practice in using these new derivatives in combination with our old derivative shortcuts that we have collected along the way.

9.1: Chain Rule
9.2: Inverse Functions and Derivatives
9.3: The Inverse Trigonometric Functions
9.4: Applications of Inverse Trig Functions
9.5: Derivatives of Inverse Trig Functions

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9.1: Chain Rule

There is another rule that we must use when taking the derivative of the composition of functions.

If $h(x) = f(g(x))$, then $h'(a) = \lim_{x \to a} \dfrac{h(x) - h(a)}{x - a}$

$= \lim_{x \to a} \dfrac{f(g(x)) - f(g(a))}{x - a} \cdot \dfrac{g(x) - g(a)}{g(x) - g(a)} = \left(\lim_{x \to a} \dfrac{f(g(x)) - f(g(a))}{g(x) - g(a)} \right) \cdot \left(\lim_{x \to a} \dfrac{g(x) - g(a)}{x - a} \right)$

Consider $f'(b) = \lim_{y \to b} \dfrac{f(y) - f(b)}{y - b}$ matches the left limit when $y = g(x)$ and $b = g(a)$, so $h'(a) = f'(g(a)) \cdot g'(a)$

This is known as the *chain rule*: $(f(g(x)))' = f'(g(x))g'(x)$

Example 9.1.1

Differentiate the following functions:

(a) $h(x) = \sqrt{\cos(x) + 1}$

(b) $q(t) = \sec(5t^2)$

Solution

(a) $h(x) = f(g(x))$ for $f(x) = \sqrt{x}$ and $g(x) = \cos(x) + 1$

Individually their derivatives are $f'(x) = \dfrac{1}{2} x^{-\frac{1}{2}}$ and $g'(x) = -\sin(x)$

so $h'(x) = f'(g(x))g'(x) = f'(\cos(x) + 1) \cdot (-\sin(x)) = \dfrac{1}{2}(\cos(x) + 1)^{-\frac{1}{2}} \cdot (-\sin(x))$

(b) $q(t) = f(g(t))$ for $f(t) = \sec(t)$ and $g(t) = 5t^2$

Individually their derivatives are $f'(t) = \sec(t)\tan(t)$ and $g'(t) = 10t$

so $q'(t) = f'(g(t)g'(t) = f'\left(5t^2\right) \cdot (10t) = \sec\left(5t^2\right)\tan\left(5t^2\right) \cdot (10t)$

Just like with the product and quotient rules, instead of naming f and g every time, we should focus on the pattern: take the derivative of the outside function (and leave the inside function the same inside of it) then multiply by the derivative of the inside function.

Example 9.1.1 again

Differentiate the following functions:

(a) $h(x) = \sqrt{\cos(x) + 1}$

(b) $q(t) = \sec(5t^2)$

Solution

(a) For $h(x) = \sqrt{\cos(x) + 1}$, we see the outside function is $(\star)^{\frac{1}{2}}$, so

$h'(x) = \dfrac{1}{2}(\cos(x) + 1)^{-\frac{1}{2}} \cdot (\cos(x) + 1)' = \dfrac{1}{2}(\cos(x) + 1)^{-\frac{1}{2}} \cdot (-\sin(x))$

(b) For $q(t) = \sec(5t^2)$, we see the outside function is $\sec(\star)$, so

$q'(t) = \sec\left(5t^2\right)\tan(5t^2) \cdot \left(5t^2\right)' = \sec\left(5t^2\right)\tan\left(5t^2\right) \cdot (10t)$

Its worth noting that our formula for $(e^{rx})' = re^{rx}$ is actually the result of the chain rule: $h(x) = f(g(x))$ for $f(x) = e^x$ and $g(x) = rx$

$f'(x) = e^x$ and $g'(x) = r$ so $h'(x) = f'(g(x))g'(x) = f'(rx) \cdot r = e^{rx} \cdot r$

The chain rule can be combined with any of the previous functions that we know how to differentiate, or combined with previous derivative rules like the product/quotient rule:

Example 9.1.2

Find the derivative of each function below.

(a) $f(x) = e^{x\tan(x)}$

(b) $g(x) = \sqrt[3]{6x+1} \cdot \csc(x^2)$

(c) $h(x) = \sin\left(\pi e^x (5x^3-4)^7\right)$

(d) $\Omega(z) = \dfrac{\cos(5z)}{\sec(e^{3z})}$

(e) $Q(t) = \sqrt[4]{\dfrac{t^4+1}{e^t-1}}$

(f) $m(u) = \cot\left(\dfrac{u}{(u^3+7)^5}\right)$

Solution

(a) This is a product inside of another function so chain rule first, then product rule:

$f'(x) = e^{x\tan(x)} \cdot (x\tan(x))' = e^{x\tan(x)}\left[(x)' \cdot \tan(x) + x \cdot (\tan(x))'\right]$

$= e^{x\tan(x)}\left[1 \cdot \tan(x) + x \cdot \sec^2(x)\right] = e^{x\tan(x)}\left(\tan(x) + x\sec^2(x)\right)$

(b) This is a product rule first, with chain rule inside of the smaller pieces:

$g'(x) = (\sqrt[3]{6x+1})' \cdot \left(\csc(x^2)\right) + (\sqrt[3]{6x+1}) \cdot \left(\csc(x^2)\right)'$

$= \left[\frac{1}{3}(6x+1)^{-\frac{2}{3}} \cdot (6x+1)'\right] \cdot \left(\csc(x^2)\right) + (\sqrt[3]{6x+1}) \cdot \left[-\csc(x^2)\cot(x^2) \cdot (x^2)'\right]$

$= \frac{1}{3}(6x+1)^{-\frac{2}{3}} \cdot 6 \cdot \csc(x^2) - (6x+1)^{\frac{1}{3}} \cdot \csc(x^2)\cot(x^2) \cdot 2x$

(c) This is a chain rule first, then a product rule with a small chain rule inside:

$h'(x) = \cos\left(\pi e^x (5x^3-4)^7\right) \cdot \left(\pi e^x (5x^3-4)^7\right)'$

$= \cos\left(\pi e^x (5x^3-4)^7\right) \cdot \left[(\pi e^x)'(5x^3-4)^7 + (\pi e^x)\left((5x^3-4)^7\right)'\right]$

$= \cos\left(\pi e^x (5x^3-4)^7\right) \cdot \left[\pi e^x \cdot (5x^3-4)^7 + \pi e^x \cdot \left(7(5x^3-4)^6 \cdot (5x^3-4)'\right)\right]$

$= \cos\left(\pi e^x (5x^3-4)^7\right) \cdot \left[\pi e^x \cdot (5x^3-4)^7 + \pi e^x \cdot 7(5x^3-4)^6 \cdot 15x^2\right]$

(d) This is a quotient rule first, with small chain rules inside of it:

$\Omega'(z) = \dfrac{(\cos(5z))'(\sec(e^{3z})) - (\cos(5z))(\sec(e^{3z}))'}{(\sec(e^{3z}))^2}$

$$= \frac{(-\sin(5z)\cdot(5z)')\sec(e^{3z}) - \cos(5z)(\sec(e^{3z})\tan(e^{3z})(e^{3z})')}{\sec^2(e^{3z})}$$

$$= \frac{-\sin(5z)\cdot 5\cdot\sec(e^{3z}) - \cos(5z)\cdot\sec(e^{3z})\tan(e^{3z})\cdot 3e^{3z}}{\sec^2(e^{3z})}$$

(e) This is a chain rule first with a quotient rule for the inside derivative second:

$$Q'(t) = \frac{1}{4}\left(\frac{t^4+1}{e^t-1}\right)^{-\frac{3}{4}}\cdot\left(\frac{t^4+1}{e^t-1}\right)'$$

$$= \frac{1}{4}\left(\frac{t^4+1}{e^t-1}\right)^{-\frac{3}{4}}\cdot\left[\frac{(t^4+1)'(e^t-1)-(t^4+1)(e^t-1)'}{(e^t-1)^2}\right]$$

$$= \frac{1}{4}\left(\frac{t^4+1}{e^t-1}\right)^{-\frac{3}{4}}\cdot\left[\frac{4t^3\cdot(e^t-1)-(t^4+1)\cdot e^t}{(e^t-1)^2}\right]$$

(f) This is chain rule first, with a quotient rule inside that has a small chain rule in it:

$$m'(u) = -\csc^2\left(\frac{u}{(u^3+7)^5}\right)\cdot\left(\frac{u}{(u^3+7)^5}\right)'$$

$$= -\csc^2\left(\frac{u}{(u^3+7)^5}\right)\cdot\left(\frac{(u)'\left((u^3+7)^5\right)-(u)\left((u^3+7)^5\right)'}{\left((u^3+7)^5\right)^2}\right)$$

$$= -\csc^2\left(\frac{u}{(u^3+7)^5}\right)\cdot\left(\frac{(1\cdot(u^3+7)^5 - u\cdot\left(5(u^3+7)^4\cdot(u^3+7)'\right)}{(u^3+7)^{10}}\right)$$

$$= -\csc^2\left(\frac{u}{(u^3+7)^5}\right)\cdot\left(\frac{(1\cdot(u^3+7)^5 - u\cdot\cdot 5(u^3+7)^4\cdot 3u^2}{(u^3+7)^{10}}\right)$$

Alternatively, you may come across a function that is made by composing more than two functions together. This just means multiple layers of chain rule:

Example 9.1.3

Find the derivative of each function below.

(a) $\zeta(x) = (\csc(e^x+1)-1)^{\,8}$

(b) $h(\theta) = \sin(\cos(\tan(\theta^2)))$

Solution

(a) $\zeta'(x) = 8(\csc(e^x+1)-1)^{\,7}\cdot(\csc(e^x+1)-1)'$

$$= 8(\cot(e^x+1)-1)^{\,7}\cdot[(-\csc(e^x+1)\cot(e^x+1))\cdot(e^x+1)']$$

$$= -8(\cot(e^x+1)-1)^{\,7}\cdot\csc(e^x+1)\cot(e^x+1)\cdot e^x$$

(b) $h'(\theta) = \cos(\cos(\tan(\theta^2)))\cdot\left[\cos(\tan(\theta^2))\right]'$

$$= \cos(\cos(\tan(\theta^2)))\cdot\left[-\sin(\tan(\theta^2))\cdot\left(\tan(\theta^2)\right)'\right]$$

$$= -\cos(\cos(\tan(\theta^2))) \cdot \sin(\tan(\theta^2)) \cdot \left[\sec^2(\theta^2) \cdot (\theta^2)'\right]$$

$$= -\cos(\cos(\tan(\theta^2))) \cdot \sin(\tan(\theta^2)) \cdot \sec^2(\theta^2) \cdot 2\theta$$

The chain rule is not a difficult rule, but it is the one that students forget to use most often!

Whenever taking a derivative, remember that its like peeling an onion:
Start working from the outside toward the inside, and try to finish without crying too much.

Just like the previous differentiation rules, sometimes you can be given information about a component of your function instead of being directly given its explicit form:

Example 9.1.4

Let $f(x)$ be a function with $f(1) = 5, f(2) = 1, f'(1) = -3, f'(2) = 9$

If $h(x) = f(f(x))$, find the line tangent to $y = h(x)$ when $x = 2$

Solution

First we plug in $x = 2$ directly: $h(2) = f(f(2)) = f(1) = 5$

And for our slope, $h'(x) = f'(f(x)) \cdot f'(x)$ so plugging in $x = 2$ gives:

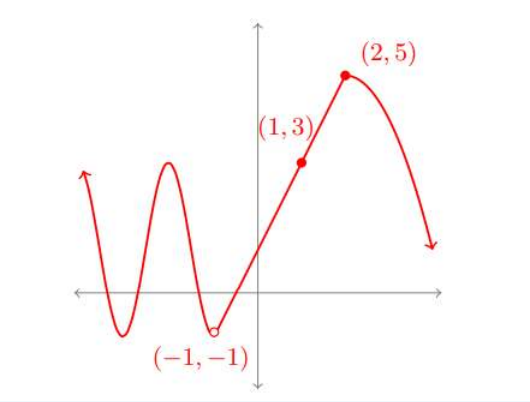
$h'(2) = f'(f(2))f'(2) = f'(1) \cdot 9 = -3 \cdot 9 = -27$

So the tangent line $y - h(2) = h'(2)(x - 2)$ becomes $y - 5 = -27(x - 2)$

The component function(s) could also be given to you graphically:

Example 9.1.5

Let $g(x)$ be the function with graph below.



If $k(x) = x^3 g(x^2)$, find the line tangent to $y = k(x)$ at $x = -1$

Solution

First we want $k(-1) = (-1)^3 g((-1)^2) = -g(1)$

Looking at the graph above, we see that $g(1) = 3$, so $k(-1) = -3$

Next we differentiate $k(x)$ to get: $k'(x) = (x^3)'(g(x^2)) + (x^3)(g(x^2))'$

$= 3x^2 \cdot g(x^2) + x^3 \cdot \left(g'(x^2) \cdot (x^2)'\right) = 3x^2 \cdot g(x^2) + x^3 \cdot g'(x^2) \cdot 2x$

Plugging in $x=-1$ this becomes:

$k'(-1)=3(-1)^2\cdot g((-1)^2)+(-1)^3\cdot g'((-1)^2)\cdot 2(-1)=3g(1)+2g'(1)$

On the graph of g provided, the slope at $x=1$ is clearly $g'(1)=2$

so $k'(-1)=3\cdot 3+2\cdot 2=13$

Thus our tangent line $y-k(-1)=k'(-1)(x-(-1))$ becomes $y+3=13(x+1)$

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9.2: Inverse Functions and Derivatives

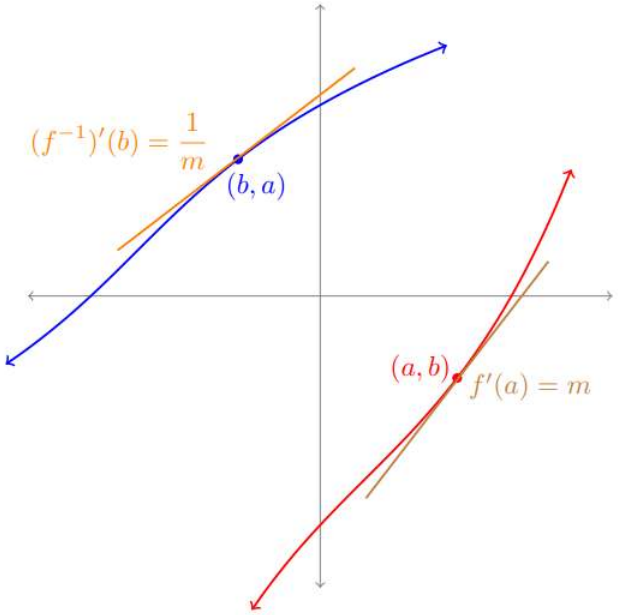
If $f(x)$ and $g(x)$ are inverses (remember that means $f(g(x)) = g(f(x)) = x$), then we can find out how their derivatives are related by using the chain rule!

$(f(g(x)))' = (x)' \implies f'(g(x))g'(x) = 1$

Solving for $g'(x)$ gives us $g'(x) = \dfrac{1}{f'(g(x))}$

Rewriting $g(x) = f^{-1}(x)$, this tells us that: $\left(f^{-1}\right)'(x) = \dfrac{1}{f'(f^{-1}(x))}$

If we look at a graph, this should make some sense:



Since the roles of x and y are switched for f and f^{-1}, the slope of the tangent line to f^{-1} (which is the change in y over the change in x) at (b, a) is the reciprocal of the slope of the tangent line to f at (a, b).

Special note: if $f'(a) = 0$, then $(f^{-1})'(b)$ would be undefined.

We can verify this formula for an example function. Consider $f(x) = 2x^3 + 1$ with $f'(x) = 6x^2$

We can find f^{-1}:

$x = 2y^3 + 1$
$x - 1 = 2y^3$
$\dfrac{x-1}{2} = y^3$
$y = \sqrt[3]{\dfrac{x-1}{2}}$

Thus $f^{-1}(x) = \sqrt[3]{\dfrac{x-1}{2}}$

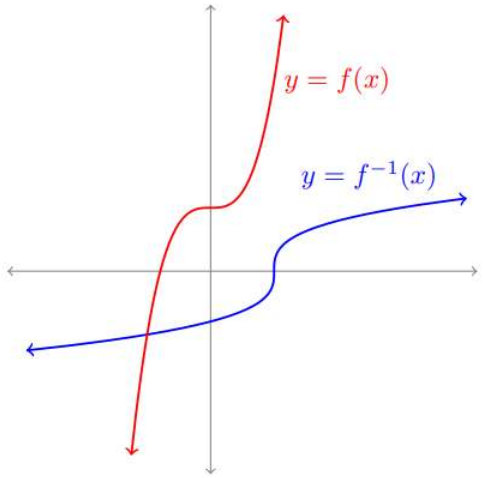
According to the derivative of inverse formula:

$$(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{f'\left(\sqrt[3]{\frac{x-1}{2}}\right)} = \frac{1}{6\left(\sqrt[3]{\frac{x-1}{2}}\right)^2} = \frac{1}{6}\left(\frac{x-1}{2}\right)^{-\frac{2}{3}}$$

If we use the chain rule directly on $f^{-1}(x)$ instead we get:

$$\left(\sqrt[3]{\frac{x-1}{2}}\right)' = \frac{1}{3}\left(\frac{x-1}{2}\right)^{-\frac{2}{3}} \cdot \left(\frac{x-1}{2}\right)' = \frac{1}{3}\left(\frac{x-1}{2}\right)^{-\frac{2}{3}} \cdot \frac{1}{2} = \frac{1}{6}\left(\frac{x-1}{2}\right)^{-\frac{2}{3}}$$

These definitely match, as promised. We can see the graphs of these curves below:



Example 9.2.1

Find the line tangent to $y = g^{-1}(x)$ at $x = 3$ when $g(x) = x^3 + x + 1$

Solution

$g'(x) = 3x^2 + 1$ which is always positive, so $g(x)$ is increasing everywhere.

That means the graph of $y = g(x)$ clearly passes the horizontal line test, and g^{-1} should exist.

(But if we try to solve for it we get stuck immediately at: $x = y^3 + y$)

To find the tangent line we need $g^{-1}(3)$ and $(g^{-1})'(3)$.

$g^{-1}(3) = b$ for whichever b makes $g(b) = 3$

$b^3 + b + 1 = 3 \quad \longleftrightarrow \quad b^3 + b - 2 = 0 \quad \longleftrightarrow \quad (b-1)(b^2 + b + 2) = 0$

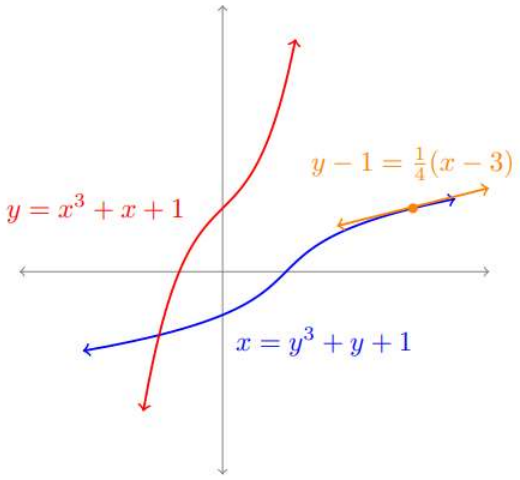
Therefore we know $g^{-1}(3) = 1$

Now we can use the derivative of inverse formula:

$$(g^{-1})'(3) = \frac{1}{g'\left(g^{-1}(3)\right)} = \frac{1}{g'(1)} = \frac{1}{3 \cdot 1^2 + 1} = \frac{1}{4}$$

So our tangent line $y - g^{-1}(3) = \left(g^{-1}\right)'(3)(x-3)$ becomes $y - 1 = \frac{1}{4}(x-3)$

Once again everything looks right when we sketch the graphs:



Just like in previous sections, you can do this without the explicit form of a component function:

Example 9.2.2

Let $f(x)$ be a function with $f(1)=2, f(2)=4, f(4)=1, f'(1)=3, f'(2)=5, f'(4)=-2$
If $g(x)=f(x^2)\cdot f^{-1}(x)$, find the line tangent to $y=g(x)$ at $x=2$.

Solution

We are looking for $y-g(2)=g'(2)(x-2)$

We see that since $f(1)=2$, then $f^{-1}(2)=1$ so

$g(2)=f(2^2)\cdot f^{-1}(2)=f(4)\cdot f^{-1}(2)=1\cdot 1=1$

Then we differentiate to find $g'(x)$:

$$g'(x)=\left(f(x^2)\right)'\left(f^{-1}(x)\right)+\left(f(x^2)\right)\left(f^{-1}(x)\right)'$$

$$=\left(f'(x^2)\cdot(x^2)'\right)\cdot f^{-1}(x)+f(x^2)\cdot\left(f^{-1}\right)'(x)$$

$$=f'(x^2)\cdot 2x\cdot f^{-1}(x)+f(x^2)\cdot\frac{1}{f'\left(f^{-1}(x)\right)}$$

Plugging in $x=2$ we get:

$$g'(2)=f'(4)\cdot 4\cdot f^{-1}(2)+f(4)\cdot\frac{1}{f'\left(f^{-1}(2)\right)}$$

$$=-2\cdot 4\cdot 1+1\cdot\frac{1}{f'(1)}=-8+\frac{1}{3}=-\frac{23}{3}$$

Therefore the tangent line is: $y-1=-\frac{23}{3}(x-2)$

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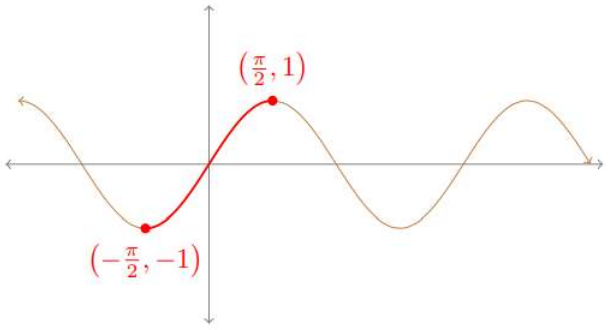
9.3: The Inverse Trigonometric Functions

Remember that not all functions have an inverse. $y = f(x)$ needs to pass the horizontal line test on its domain in order for the function $f^{-1}(x)$ to exist. Sadly, every trigonometric function we have seen fails the horizontal line test spectacularly. Since these functions repeat every 2π, we will need to restrict their domain appropriately if we want to make them invertible.

There are many different ways we could do this, but since we were born a few centuries too late to be the first people trying this, we will just go with what is considered the standard domain for each:

SINE

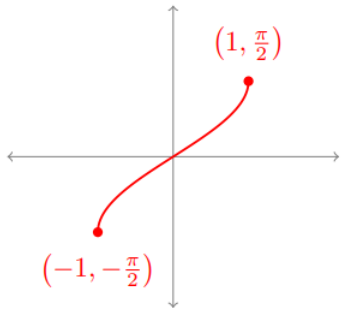
$f(x) = \sin(x)$ with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



The red portion of this graph clearly passes the horizontal line test, so f^{-1} should exist:

$f^{-1}(x) = \sin^{-1}(x)$

$$f^{-1}(x) = \sin^{-1}(x)$$



We can easily see the domain of $\sin^{-1}(x)$ is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Remember inverses are supposed to compose with each other and "cancel out".
In other words, that means $\sin^{-1}(x)$ is the value of $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that makes $\sin(t) = x$

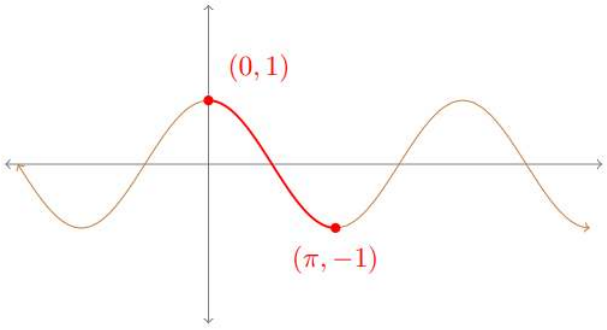
For example: $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

Even though $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$, since $\frac{5\pi}{6}$ is not in the range of $\sin^{-1}(x)$, it is not the right answer! We need to be careful and pick the answer from the correct interval in order to be well defined.

COSINE

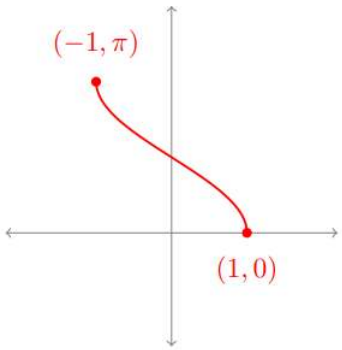
Since using $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ would not make cosine one-to-one, we use a different restricted domain:

$f(x) = \cos(x)$ with domain $[0, \pi]$



Again the red portion of this graph passes the horizontal line test, so f^{-1} should exist:

$f^{-1}(x) = \cos^{-1}(x)$



We can easily see the domain of $\cos^{-1}(x)$ is $[-1, 1]$ and the range is $[0, \pi]$

Therefore this means that $\cos^{-1}(x)$ is the angle $t \in [0, \pi]$ that satisfies $\cos(t) = x$

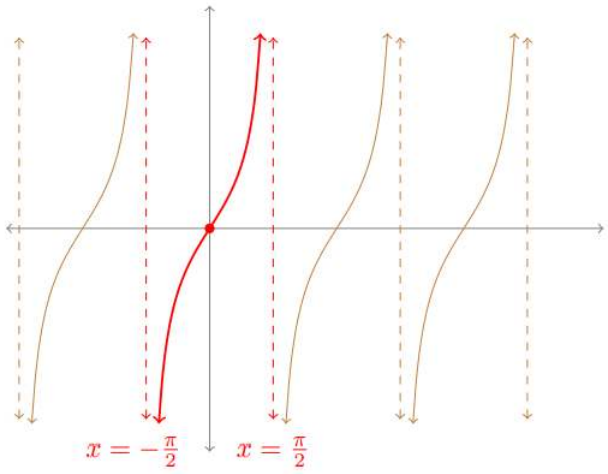
For example: $\cos^{-1}(0) = \frac{\pi}{2}$ since $\cos\left(\frac{\pi}{2}\right) = 0$

It is worth noting that for each of the last two graphs, since $\sin(x)$ and $\cos(x)$ were continuous on the restricted domains, their inverses were also continuous functions. This will always be the case: **the inverse of a continuous function is continuous.**

TANGENT

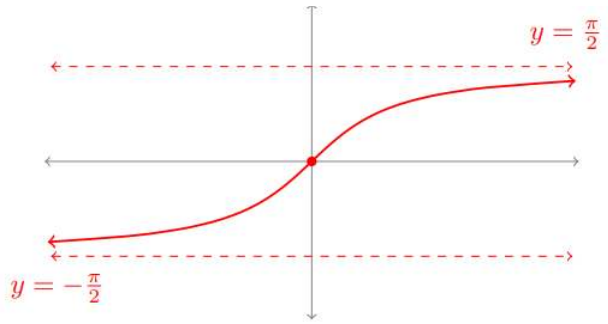
This time using $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ again *does* make tangent one-to-one, but $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ are not technically in the domain of tangent at all, so we use $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$f(x) = \tan(x)$ with domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Once again the red portion of this graph passes the horizontal line test, so f^{-1} should exist:

$f^{-1}(x) = \tan^{-1}(x)$



We can clearly see the domain of $\tan^{-1}(x)$ is $(-\infty, \infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Again this means that $\tan^{-1}(x)$ is the angle $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that satisfies $\tan(t) = x$

For example: $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$ since $\tan\left(-\frac{\pi}{3}\right) = \frac{\sin\left(-\frac{\pi}{3}\right)}{\cos\left(-\frac{\pi}{3}\right)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$

It is a good idea to become familiar with the values of $\tan(t)$ for t in this range:

$\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$ $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\tan\left(-\frac{\pi}{4}\right) = -1$ $\tan\left(\frac{\pi}{4}\right) = 1$

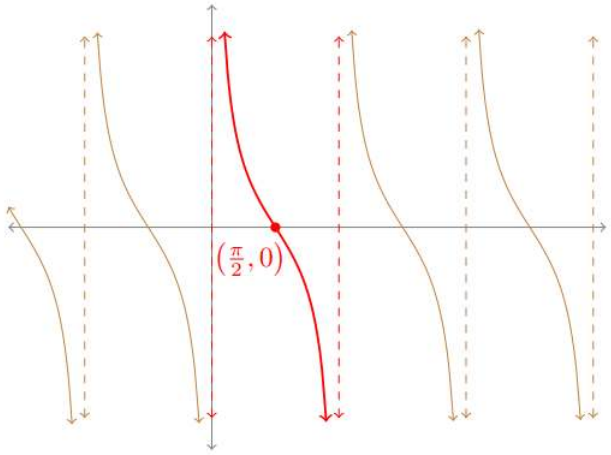
$\tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$ $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

and of course $\tan(0) = 0$

COTANGENT

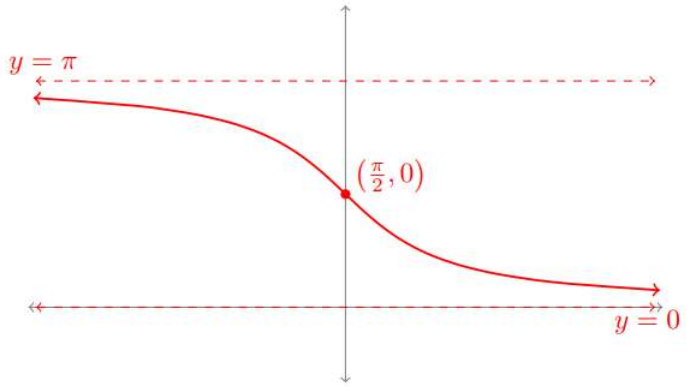
For cotangent to be one-to-one, we use $(0, \pi)$ as the restricted domain:

$f(x) = \cot(x)$ with domain $(0, \pi)$



Once more the red portion of this graph passes the horizontal line test, so f^{-1} should exist:

$f^{-1}(x) = \cot^{-1}(x)$



We can see the domain of $\cot^{-1}(x)$ is $(-\infty, \infty)$ and the range is $(0, \pi)$

Therefore this means that $\cot^{-1}(x)$ is the angle $t \in (0, \pi)$ that satisfies $\cot(t) = x$

For example: $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \frac{2\pi}{3}$ since $\cot\left(\frac{2\pi}{3}\right) = \frac{\cos\left(\frac{2\pi}{3}\right)}{\sin\left(\frac{2\pi}{3}\right)} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$

It is also worth it to become familiar with the values of $\cot(t)$ for t in this range:

$\cot\left(\frac{\pi}{3}\right) = \sqrt{3}$ $\cot\left(\frac{2\pi}{3}\right) = -\sqrt{3}$

$\cot\left(\frac{\pi}{4}\right) = 1$ $\cot\left(\frac{3\pi}{4}\right) = -1$

$\cot\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$ $\cot\left(\frac{5\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

and of course $\cot\left(\frac{\pi}{2}\right) = 0$

SECANT

The last two trigonometric functions (secant and cosecant) are the most interesting.

It would be reasonable to pick $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ for secant, since these intervals make $\sec(x)$ one-to-one and closely align with the restrictions for cosine we used earlier

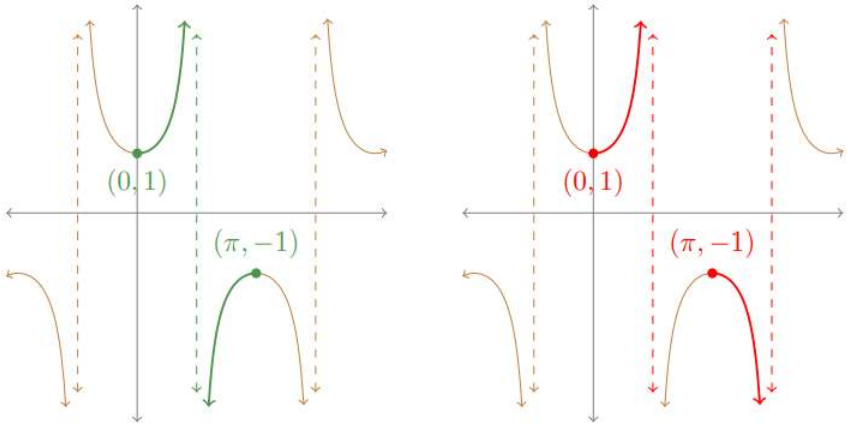
(with the vertical asymptote $x = \frac{\pi}{2}$ removed)

In fact, many books and online resources actually use this restriction.

But some books (like this one) choose a different domain: $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

This makes a difference because these domains effect what the derivative of these graphs look like:

$y = \sec(x)$

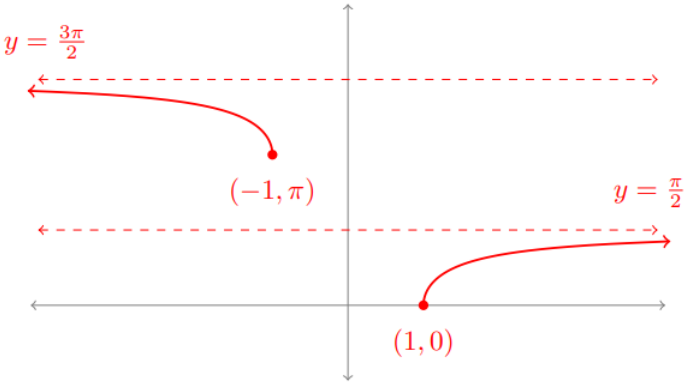


In the green regions, $\sec(x)$ is always increasing, so its derivative will always be positive.
(This means its inverse will always have a positive derivative too since they are reciprocals!)

But for the red regions, $\sec(x)$ can be increasing or decreasing, so its derivative (and thus the derivative of its inverse) will have both positive and negative values.

When we solve for the derivatives of these inverses soon, it will be easier to deal with the red version. That is why this author chooses the restricted domain $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ for $\sec(x)$.

This leads us to the graph of $y = \sec^{-1}(x)$:



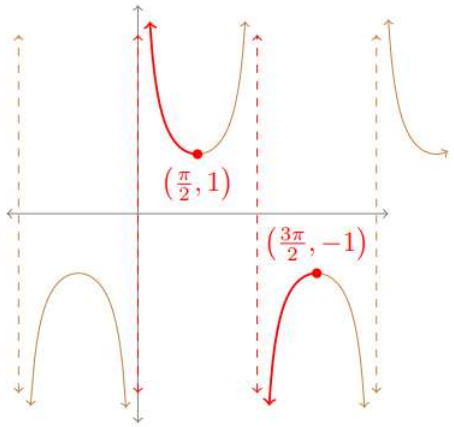
We can see the domain of $\sec^{-1}(x)$ is $(-\infty, -1] \cup [1, \infty)$ and the range is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

Therefore this means that $\sec^{-1}(x)$ is the angle $t \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ that satisfies $\sec(t) = x$

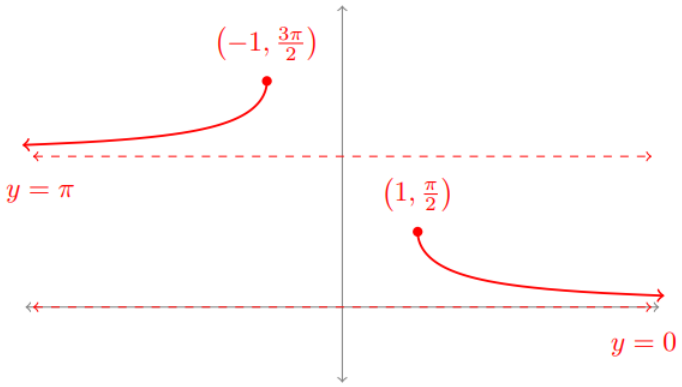
COSECANT

For the same reasons as we saw in secant, we will choose to restrict cosecant's domain to $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ instead of $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ (which some books choose to use)

$f(x) = \csc(x)$ with domain $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$



This is clearly invertible and leads us to the graph of $y = \csc^{-1}(x)$:



We can see the domain of $\csc^{-1}(x)$ is $(-\infty, -1] \cup [1, \infty)$ and the range is $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$

Therefore this means that $\csc^{-1}(x)$ is the angle $t \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ that satisfies $\csc(t) = x$

Many students feel overwhelmed by what seems like a large amount of information for these six new inverse functions. The good news is that besides the specific choice of domain/range, everything here is exactly mirrored by information about the original six trigonometric functions (just with outputs and inputs reversed!)
Focus on remembering the ranges of the inverse trig functions and you should be fine!

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9.4: Applications of Inverse Trig Functions

We have now officially defined the six inverse trig functions and their domains/ranges:

Function	Domain	Range
$\sin^{-1}(x)$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}(x)$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}(x)$	$(-\infty,\infty)$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$\cot^{-1}(x)$	$(-\infty,\infty)$	$(0,\pi)$
$\sec^{-1}(x)$	$(-\infty,-1]\cup[1,\infty)$	$\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$
$\csc^{-1}(x)$	$(-\infty,-1]\cup[1,\infty)$	$\left(0,\frac{\pi}{2}\right]\cup\left(\pi,\frac{3\pi}{2}\right]$

We use the unit circle to evaluate many expressions involving these functions:

Example 9.4.1

Evaluate each inverse trigonometric expression.

(a) $\csc^{-1}(2)$

(b) $\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

(c) $\cos^{-1}\left(\frac{5}{4}\right)$

Solution

(a) Which t makes $\csc(t)=2$?

Same as $\sin(t)=\frac{1}{2}$

$\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$ and $\sin\left(\frac{5\pi}{6}\right)=\frac{1}{2}$

Pick the $t\in\left(0,\frac{\pi}{2}\right]\cup\left(\pi,\frac{3\pi}{2}\right]$

So $\csc^{-1}(2)=\frac{\pi}{6}$

(b) Which t makes $\sec(t)=-\frac{2}{\sqrt{3}}$?

Same as $\cos(t)=-\frac{\sqrt{3}}{2}$

$\cos\left(\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}$ and $\cos\left(\frac{7\pi}{6}\right)=-\frac{\sqrt{3}}{2}$

Pick the $t\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$

So $\sec^{-1}(2) = \dfrac{7\pi}{6}$

(c) Which t makes $\cos(t) = \dfrac{5}{4}$?

This is impossible, there is no t that does this. Because $\dfrac{5}{4} \notin [-1, 1]$

$\cos^{-1}\left(\dfrac{5}{4}\right) = DNE$

We know that inverses are supposed to cancel each other, but remember that the original trigonometric functions domains had to be restricted, which makes this a delicate situation.

For example, remember $f(x) = x^2$ with domain $[0, \infty)$ had inverse $f^{-1}(x) = \sqrt{x}$

So $f^{-1}\left(f(x)\right) = \sqrt{x^2} = |x| = x$ for all $x \in [0, \infty)$

But $f^{-1}\left(f(-3)\right) = \sqrt{(-3)^2} = \sqrt{9} = 3 \neq -3$ because $-3 \notin [0, \infty)$

$f^{-1} \circ f$ only cancels when the input is in the restricted domain of f.

$\left(f^{-1} \circ f\right)(x)$ can be evaluated (sometimes) for x values outside of that domain, but wont output x.

Example 9.4.2

Evaluate each composition of inverses below.

(a) $\sin^{-1}\left(\sin\left(\dfrac{\pi}{6}\right)\right)$

(b) $\sin^{-1}\left(\sin\left(\dfrac{7\pi}{6}\right)\right)$

Solution

(a) We know $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$, so this becomes $\sin^{-1}\left(\dfrac{1}{2}\right)$

We want to know what t makes $\sin(t) = \dfrac{1}{2}$?

$\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$ and $\sin\left(\dfrac{5\pi}{6}\right) = \dfrac{1}{2}$

We need $t \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ so pick $t = \dfrac{\pi}{6}$, so $\sin^{-1}\left(\sin\left(\dfrac{\pi}{6}\right)\right) = \dfrac{\pi}{6}$

Things cancelled here because $\dfrac{\pi}{6} \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

(b) We know $\sin\left(\dfrac{7\pi}{6}\right) = -\dfrac{1}{2}$, so this becomes $\sin^{-1}\left(-\dfrac{1}{2}\right)$

We want to know what t makes $\sin(t) = -\dfrac{1}{2}$?

$\sin\left(\dfrac{7\pi}{6}\right) = -\dfrac{1}{2}$ and $\sin\left(\dfrac{11\pi}{6}\right) = -\dfrac{1}{2}$$

We need $t \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ so neither works immediately, but subtract a full rotation from each:

$\sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}$ and $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

Now we see to pick $t = -\frac{\pi}{6}$, so $\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right) = -\frac{\pi}{6}$

Things did not cancel here because $\frac{7\pi}{6} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Sometimes we can compose the inverses of one trig function with a different trig function:

Example 9.4.3

Evaluate each expression below

(a) $\tan^{-1}(\cos(\pi))$

(b) $\sec^{-1}\left(\tan\left(\frac{5\pi}{6}\right)\right)$

(c) $\cos^{-1}\left(\cot\left(-\frac{\pi}{2}\right)\right)$

Solution

(a) We know $\cos(\pi) = -1$ so this becomes $\tan^{-1}(-1)$

We want to know what t makes $\tan(t) = -1$?

$\tan\left(\frac{3\pi}{4}\right) = -1$ and $\tan\left(\frac{7\pi}{4}\right) = -1$

We need $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so neither works immediately, so subtract a full rotation from each:

$\tan\left(-\frac{5\pi}{4}\right) = -1$ and $\tan\left(-\frac{\pi}{4}\right) = -1$

Now we see to pick $t = -\frac{\pi}{4}$, so $\tan^{-1}(\cos(\pi)) = -\frac{\pi}{4}$

(b) We know $\tan\left(\frac{5\pi}{6}\right) = -\frac{1}{\sqrt{3}}$ so this becomes $\sec^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

We want to know what t makes $\sec(t) = -\frac{1}{\sqrt{3}}$?

The answer is that no such t can exist, since the domain of $\sec^{-1}(x)$ is $(-\infty, -1] \cup [1, \infty)$

We know $-\frac{1}{\sqrt{3}} \notin (-\infty, -1] \cup [1, \infty)$, so $\sec^{-1}\left(\tan\left(\frac{5\pi}{6}\right)\right) = DNE$

We know $\cot\left(-\frac{\pi}{2}\right) = 0$ so this becomes $\cos^{-1}(0)$

We want to know what t makes $\cos(t) = 0$?

$\cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{3\pi}{2}\right) = 0$

We need $t \in [0, \pi]$, so $\cos^{-1}\left(\cot\left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{2}$

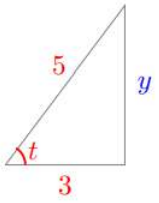
Composition can also go in the opposite order, which is easy for values on the unit circle, but still possible for other values in the domain. The trick is to draw a *helper triangle*:

For example, lets evaluate $\tan\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

We know $\cos^{-1}\left(\frac{3}{5}\right) = t$ for some angle $t \in [0, \pi]$ that makes $\cos(t) = \frac{3}{5}$

$\cos(t)$ is only positive in $\left[0, \frac{\pi}{2}\right)$ so we know angle t would be in quadrant 1.

Remember $\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}}$ so make these two legs have that ratio:



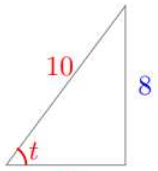
We can find the missing side y by using the Pythagorean Theorem:

$3^2 + y^2 = 5^2$

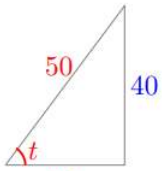
This gives us $y = \pm 4$, and since t is in quadrant 1, $y = 4$

Now we can use this new triangle to get our answer: $\tan(t) = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$

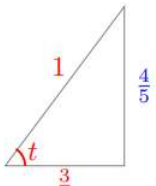
This helper triangle would have worked with any choice of adjacent and hypotenuse that have the same ratio, since the missing side will scale at the same rate, as you can see below



$\tan(t) = \frac{8}{6} = \frac{4}{3}$



$\tan(t) = \frac{40}{30} = \frac{4}{3}$



$\tan(t) = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$

(so just pick the most obvious one)

Lets try an example that isn't in quadrant 1:

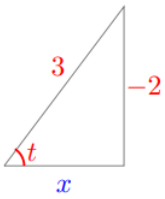
Evaluate $\cot\left(\csc^{-1}\left(-\frac{3}{2}\right)\right)$

We know $\csc^{-1}\left(-\frac{3}{2}\right) = t$ for some $t \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ that makes $\csc(t) = -\frac{3}{2}$

$\csc(t)$ is only negative in the $\left[\pi, \frac{3\pi}{2}\right)$ part of this range, so we know angle t will be in quadrant 3.

Remember $\csc(t) = \dfrac{\text{hypotenuse}}{\text{opposite}}$ so make these two legs in our helper triangle have that ratio.

Notice that for t in quadrant 3 both sine and cosine are negative, so both legs should have negative values (**but the hypotenuse will always be positive in any triangle no matter what t is**)



We can find the missing side x by using the Pythagorean Theorem:

$(-2)^2 + x^2 = 3^2$

This gives us $x = \pm\sqrt{5}$, and we must pick x to be negative so, $x = -\sqrt{5}$

Finally we can use this helper triangle to get our answer:

$$\cot(t) = \frac{\text{adjacent}}{\text{opposite}} = \frac{-\sqrt{5}}{-2} = \frac{\sqrt{5}}{2}$$

Any composition of an inverse trigonometric function inside of a standard trigonometric function can be evaluated by using a helper triangle. Just make sure you are careful as to which quadrant the inverse trig function puts you into so you don't make any $\pm$ errors.

These inverse trig functions are continuous on their domains, so we can use them in limits:

Example 9.4.4

Evaluate each limit below

(a) $\lim\limits_{x\to\frac{1}{2}} \dfrac{\sin^{-1}(x)}{\cos^{-1}(x)}$

(b) $\lim\limits_{u\to\pi} \cos^{-1}(\sin(u))$

Solution

(a) These are both continuous at $x = \frac{1}{2}$ so plug it in:

$$\lim_{x\to\frac{1}{2}} \frac{\sin^{-1}(x)}{\cos^{-1}(x)} = \frac{\sin^{-1}\left(\frac{1}{2}\right)}{\cos^{-1}\left(\frac{1}{2}\right)} = \frac{\frac{\pi}{6}}{\frac{\pi}{3}} = 2$$

(b) Both pieces are continuous so plug it in:

$$\lim_{u\to\pi}\cos^{-1}(\sin(u)) = \sin^{-1}(\sin(\pi)) = \cos^{-1}(0) = \frac{\pi}{2}$$

Knowing the graphs of these inverse trig functions can be helpful too:

Example 9.4.5

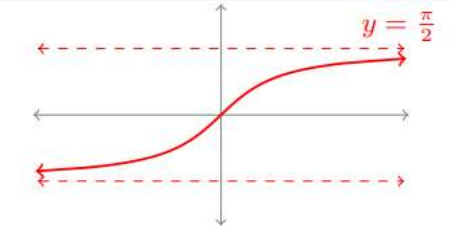
Evaluate each of the following limits

(a) $\lim_{x\to\infty} \dfrac{\tan^{-1}(x)}{2x}$

(b) $\lim_{x\to 1^-} \dfrac{x}{2\sin^{-1}(x)-\pi}$

Solution

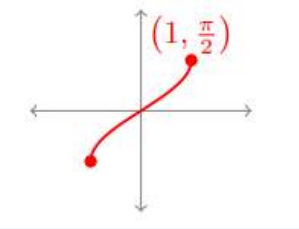
(a) Remember the graph of $y = \tan^{-1}(x)$



Clearly $\lim_{x\to\infty} \tan^{-1}(x) = \dfrac{\pi}{2}$

Therefore our limit $\lim_{x\to\infty} \dfrac{\tan^{-1}(x)}{2x} = \dfrac{\frac{\pi}{2}}{+\infty} = 0$

(b) Remember the graph of $y = \sin^{-1}(x)$:



We see that $2\sin^{-1}(x) - \pi$ is always negative and $\lim_{x\to 1^-} 2\sin^{-1}(x) - \pi = 0$

So $\lim_{x\to 1^-} \dfrac{x}{2\sin^{-1}(x)-\pi} = \dfrac{1}{0^-} = -\infty$

We can even try some more exciting tangent line problems:

Example 9.4.6

Let $f(x) = x + \cos(x)$ with domain $[0, \pi]$.
What point(s) on $y = f(x)$ have a tangent line parallel to $5y - 2x = 7$?

Solution

To be parallel to $y = \frac{2}{5}x + \frac{7}{5}$, we need to have same slope $m = \frac{2}{5}$

So we differentiate and get $f'(x) = 1 - \sin(x)$ and solve $f'(a) = \frac{2}{5}$

$1 - \sin(a) = \frac{2}{5}$

$\sin(a) = \frac{3}{5}$

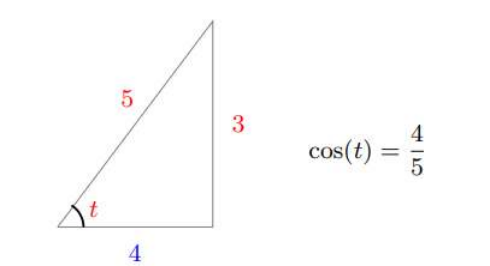
$a = \sin^{-1}\left(\frac{3}{5}\right)$

This value of a is in $\left[0, \frac{\pi}{2}\right]$ so its in our domain.

Since the answer we want is the point $(a, f(a))$ we just plug it in to get:

$$f\left(\sin^{-1}\left(\frac{3}{5}\right)\right) = \sin^{-1}\left(\frac{3}{5}\right) + \cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$$

And use the helper triangle below to simplify:



$\cos(t) = \frac{4}{5}$

So $f\left(\sin^{-1}\left(\frac{3}{5}\right)\right) = \sin^{-1}\left(\frac{3}{5}\right) + \frac{4}{5}$

And our final answer is the point: $\left(\sin^{-1}\left(\frac{3}{5}\right), \sin^{-1}\left(\frac{3}{5}\right) + \frac{4}{5}\right)$

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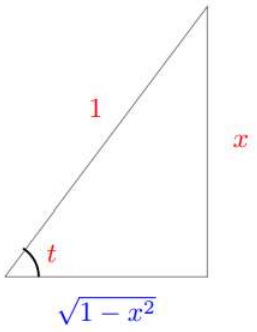
9.5: Derivatives of Inverse Trig Functions

We now have all of the tools we need to differentiate the six inverse trig functions:

For $f(x) = \sin(x)$ and $f^{-1}(x) = \sin^{-1}(x)$ we know:

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\cos(\sin^{-1}(x))}$$

We can use a helper triangle for $t = \sin^{-1}(x)$:



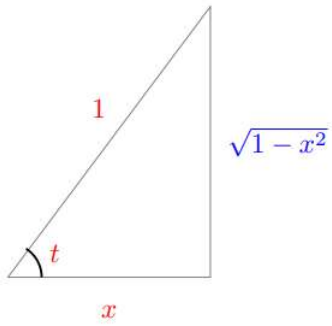
$$\cos(t) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

Therefore we get $\left(\sin^{-1}(x)\right)' = \frac{1}{\sqrt{1-x^2}}$

For $f(x) = \cos(x)$ and $f^{-1}(x) = \cos^{-1}(x)$ we know:

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{-\sin(\cos^{-1}(x))}$$

We can use a helper triangle for $t = \cos^{-1}(x)$:



$$\sin(t) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

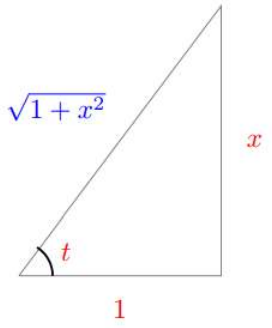
Therefore we get $\left(\cos^{-1}(x)\right)' = -\frac{1}{\sqrt{1-x^2}}$

The same works for tangent and cotangent:

For $f(x) = \tan(x)$ and $f^{-1}(x) = \tan^{-1}(x)$ we know:

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\sec^2(\tan^{-1}(x))}$$

We can use a helper triangle for $t = \tan^{-1}(x)$:



$$\sec(t) = \frac{\sqrt{1+x^2}}{1} = \sqrt{1-x^2}$$

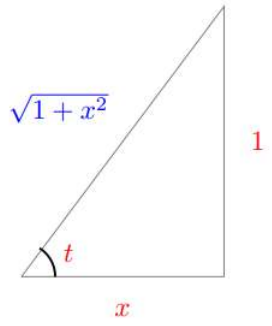
$$\text{so } \sec^2(t) = (\sqrt{1+x^2})^2 = 1+x^2$$

Therefore we get $\left(\tan^{-1}(x)\right)' = \frac{1}{1+x^2}$

For $f(x) = \cot(x)$ and $f^{-1}(x) = \cot^{-1}(x)$ we know:

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{-\csc^2(\cot^{-1}(x))}$$

We can use a helper triangle for $t = \cot^{-1}(x)$:



$$\csc(t) = \frac{\sqrt{1+x^2}}{1} = \sqrt{1-x^2}$$

$$\text{so } \csc^2(t) = (\sqrt{1+x^2})^2 = 1+x^2$$

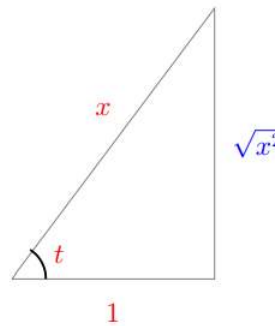
Therefore we get $\left(\cot^{-1}(x)\right)' = -\frac{1}{1+x^2}$

And finally, it works for secant and cosecant:

For $f(x) = \sec(x)$ and $f^{-1}(x) = \sec^{-1}(x)$ we know:

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\sec(\sec^{-1}(x))\tan(\sec^{-1}(x))}$$

We can use a helper triangle for $t = \sec^{-1}(x)$:



$$\tan(t) = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$$

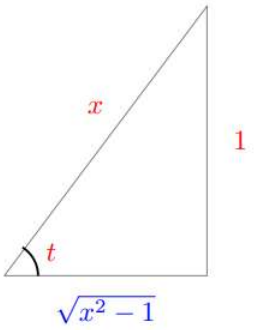
$$\text{and } \sec(t) = \frac{x}{1} = x$$

Therefore we get $\left(\sec^{-1}(x)\right)' = \frac{1}{x\sqrt{x^2-1}}$

For $f(x)=\csc(x)$ and $f^{-1}(x)=\csc^{-1}(x)$ we know:

$$\left(f^{-1}\right)'(x)=\frac{1}{f'\left(f^{-1}(x)\right)}=\frac{1}{-\csc(\csc^{-1}(x))\cot(\csc^{-1}(x))}$$

We can use a helper triangle for $t=\csc^{-1}(x)$:



$$\cot(t)=\frac{\sqrt{x^2-1}}{1}=\sqrt{x^2-1}$$

$$\text{and } \csc(t)=\frac{x}{1}=x$$

Therefore we get $\left(\csc^{-1}(x)\right)'=-\dfrac{1}{x\sqrt{x^2-1}}$

Had we used the different restricted domain of $\sec(x)$ and $\csc(x)$ mentioned earlier in this chapter, these derivatives would have had to be defined piecewise (where x is replaced by $|x|$). This is why we chose the somewhat unobvious range for these functions!

We can now add these to our list of known derivatives, and use them in combination with our differentiation rules.

Example 9.5.1

Find the derivative of each function below.

(a) $f(x)=x^5\tan^{-1}(x)$

(b) $g(x)=\dfrac{\sin^{-1}(x)}{e^x+2}$

(c) $h(t)=\sec^{-1}(2t^3)$

(d) $F(q)=e^{\cot^{-1}(4q)}$

Solution

(a) $f'(x)=(x^5)'(\tan^{-1}(x))+(x^5)(\tan^{-1}(x))'=5x^4\cdot\tan^{-1}(x)+x^5\cdot\dfrac{1}{1+x^2}$

(b) $g'(x)=\dfrac{(\sin^{-1}(x))'(e^x+2)-(\sin^{-1}(x))(e^x+2)'}{(e^x+2)^2}=\dfrac{\frac{1}{\sqrt{1-x^2}}\cdot(e^x+2)-\sin^{-1}(x)\cdot e^x}{(e^x+2)^2}$

(c) $h'(t)=\dfrac{1}{(2t^3)\sqrt{(2t^3)^2-1}}\cdot\left(2t^3\right)'=\dfrac{1}{2t^3\sqrt{4t^6-1}}\cdot 6t^2=\dfrac{3}{\sqrt{4t^6-1}}$

(d) $F'(q)=e^{\cot^{-1}(4q)}\cdot\left(\cot^{-1}(4q)\right)'=e^{\cot^{-1}(4q)}\cdot\left(-\dfrac{1}{1+(4q)^2}\right)\cdot(4q)'=-e^{\cot^{-1}(4q)}\cdot\dfrac{1}{1+16q^2}\cdot 4$

Example 9.5.2

Find the line tangent to $y=6x^2\cos^{-1}\left(\frac{x}{2}\right)$ at $x=\sqrt{3}$

Solution

The tangent line will be $y-y(\sqrt{3})=y'(\sqrt{3})(x-\sqrt{3})$

We see $y\left(\sqrt{3}\right)=6(\sqrt{3})^2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=6\cdot 3\cdot\frac{\pi}{6}=3\pi$

And $y'(x)=(6x^2)'\left(\cos^{-1}\left(\frac{x}{2}\right)\right)+(6x^2)\left(\cos^{-1}\left(\frac{x}{2}\right)\right)'$

$=12x\cdot\cos^{-1}\left(\frac{x}{2}\right)+6x^2\cdot\left[-\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{1}{2}\right]$

Thus $y'\left(\sqrt{3}\right)=12\sqrt{3}\cdot\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)-6(\sqrt{3})^2\cdot\frac{1}{\sqrt{1-\frac{(\sqrt{3})^2}{4}}}\cdot\frac{1}{2}=2\pi\sqrt{3}-18$

So our tangent line is: $y-3\pi=(2\pi\sqrt{3}-18)\,(x-\sqrt{3})$

Any type of problem we have seen before can be made a little spicier by using inverse trigonometric functions. Don't be scared, the only difference is the algebra can be a little uglier (and you should memorize your unit circle if you haven't done so already!)

Example 9.5.3

For what value(s) of c does $y+2x=c$ lie tangent to $f(x)=4\tan^{-1}\left(\frac{1}{x}\right)$?

Solution

The line $y=-2x+c$ has slope $m=2$, so we want to know when $f'(a)=2$

$f'(x)=\frac{4}{1+\left(\frac{1}{x}\right)^2}\cdot\left(\frac{1}{x}\right)'=\frac{4}{1+x^{-2}}\cdot(-x^{-2})=\frac{-4x^{-2}}{1+x^{-2}}$

Setting $f'(a)=-2$ we get the equation: $\frac{-4a^{-2}}{1+a^{-2}}=-2$

$-4a^{-2}=-2-2a^{-2}$

Multiply everything by a^2 to get:

$-4=-2a^2-2$

$a^2=1$

$a=\pm 1$

So there are two tangent lines that work:

$f(1)=4\tan^{-1}\left(\frac{1}{1}\right)=4\cdot\frac{\pi}{4}=\pi$

Gives $y-\pi=-2(x-1)$ which we can write as $y+2x=2+\pi$

$f(-1) = 4\tan^{-1}\left(\frac{1}{-1}\right) = 4 \cdot \left(-\frac{\pi}{4}\right) = -\pi$

Gives $y + \pi = -2(x+1)$ which we can write as $y + 2x = -2 - \pi$

Therefore our final answers are $c = 2 + \pi$ and $c = -2 - \pi$

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CHAPTER OVERVIEW

10: Logarithms

> *"You have no idea, how much poetry there is in the calculation of a table of logarithms!" - Carl Friedrich Gauss*

Chapter ten is dedicated to a fan favorite: the logarithm. This feisty function is defined entirely as an inverse, and yet it is incredibly powerful in calculus.

We give the official definition of a logarithmic function as the inverse of an exponential function. This immediately tells us the graph of any member of the log family, as well as its domain and range. We also inherit properties of logarithms based on properties we know of exponential functions. Since the exponential functions are continuous, so are the logarithmic functions on their entire domain. Taking limits involving logs is usually pretty painless.

One powerful application of the logarithm is that we can use it to solve exponential equations (which are equations with variables in the exponent). Mastering the properties of the logarithm often makes solving these equations easier. There are even real life examples of exponential growth and decay, or temperature change models where using logarithms is essential to find a solution.

Once again our inverse derivative formula comes in handy to help us learn the derivative of the logarithmic functions, as well as how to use them in combination with our previously memorized derivative shortcuts. In fact the properties of the logarithm can be used in a special process called logarithmic differentiation to help us take derivatives of some even more difficult and extreme functions.

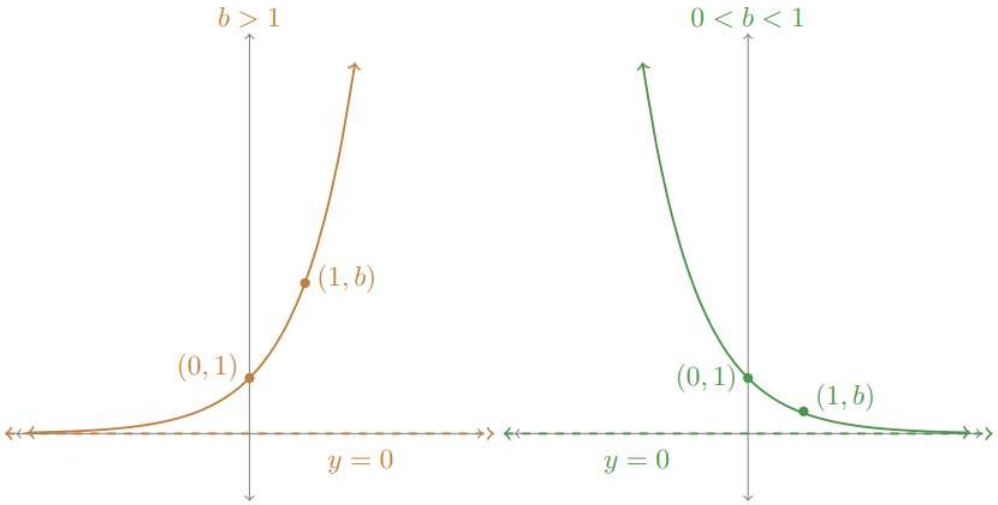
10.1: Logarithms
10.2: Exponential Growth and Decay
10.3: Derivatives of Logarithms
10.4: Logarithmic Differentiation

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10.1: Logarithms

There is one last family of functions we will study, and they are the inverses of $f(x) = b^x$

As we recall, the graph of exponential functions come in two main varieties:



Clearly each of these graphs passes the horizontal line test, so $f^{-1}(x)$ exists.

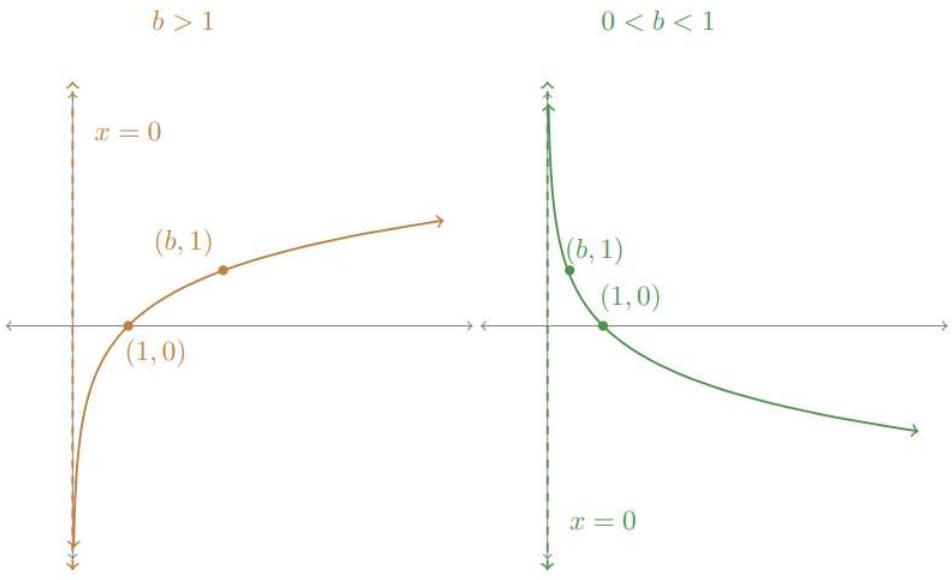
If we try to solve for it: $y = b^x \longrightarrow x = b^y$

But we cannot seem to solve for y in this equation!

Since we lack the algebraic skills to solve it, we will just invent a new function that does its job: $x = b^y$ is equivalent to $y = \log_b(x)$

In other words, we say $f^{-1}(x) = \log_b(x)$ the *logarithm function*.

Switch x and y in the graphs above to see what the graph of $y = \log_b(x)$ looks like:



The domain for either version is $(0, \infty)$ and the range is $(-\infty, \infty)$

If you want to graph a translation of a logarithmic function, just pay attention to the two points $(1, 0)$, $(b, 1)$ and the vertical asymptote $x = 0$:

Example 10.1.1

Sketch the graph of each logarithmic function below.

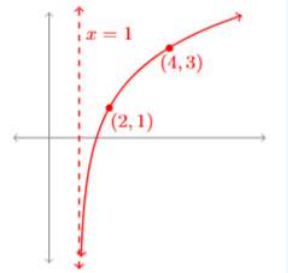
(a) $y = 2\log_3(x-1)+1$

(b) $y = 3 - \log_{\frac{1}{2}}\left(2 - \frac{1}{2}x\right)$

Solution

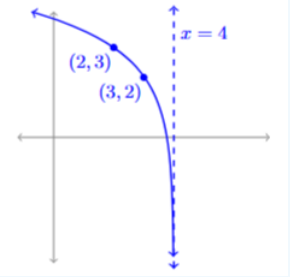
(a) We use the equations $y_{new} = 2y_{old}+1$ and $x_{new} = x_{old}+1$

x_{old}	y_{old}	x_{new}	y_{new}
1	0	2	1
3	1	4	3
0	VA	1	VA



(b) We use the equations $y_{new} = 3 - y_{old}$ and $x_{new} = 4 - 2x_{old}$

x_{old}	y_{old}	x_{new}	y_{new}
1	0	2	3
$\frac{1}{2}$	1	3	2
0	VA	4	VA



While it is nice to be able to "see" the graph of a logarithm, it does not help us understand what the function "does". All we know about $g(x) = \log_b(x)$ is that it is the inverse of $f(x) = b^x$:

$(f \circ g)(x) = x \longrightarrow b^{\log_b(x)} = x$

$(g \circ f)(x) = x \longrightarrow \log_b(b^x) = x$

This means that $\log_b(x)$ is the **exponent of b required to reach x**.

If you want to evaluate $\log_b(x)$, just ask yourself "b to what power equals x?"

Example 10.1.2

Simplify each logarithm below

(a) $\log_2(32)$

(b) $\log_7\left(\frac{1}{49}\right)$

(c) $\log_{\frac{3}{8}}(1)$

Solution

(a) $32 = 2^5$, so $\log_2(32) = 5$

(b) $49 = 7^2 \implies \frac{1}{49} = \frac{1}{7^2} = 7^{-2}$, so $\log_7\left(\frac{1}{49}\right) = -2$

(c) $y^0 = 1$ for any number y, so $\log_{\frac{3}{8}}(1) = 0$

Just remember that b and x must be positive, or the entire expression does not make sense!

For example: $\log_5(-1) = DNE$ and $\log_{-3}(2) = DNE$

Since the graph of $y = \log_b(x)$ is continuous on its domain $(0, \infty)$, we can say for any $a > 0$ that $\lim_{x \to a} \log_b(x) = \log_b(a)$

Near $x = 0$ we use the vertical asymptote so $\lim_{x \to 0^+} \log_b(x) = \begin{cases} -\infty & \text{when } b > 1 \\ +\infty & \text{when } 0 < b < 1 \end{cases}$

And there is no horizontal asymptote, so we see $\lim_{x \to \infty} \log_b(x) = \begin{cases} \infty & \text{when } b > 1 \\ -\infty & \text{when } 0 < b < 1 \end{cases}$

Just keep these in mind, and you should be able to tackle most limits involving logs:

Example 10.1.3

Evaluate each limit involving logarithms below.

(a) $\lim_{x \to 3} \frac{\log_2\left(x^2 - 1\right)}{x}$

(b) $\lim_{z \to \infty} z \log_{\frac{2}{3}}(z)$

Solution

(a) $\lim_{x \to 3} \frac{\log_2\left(x^2 - 1\right)}{x} = \frac{\log_2\left(3^2 - 1\right)}{3} = \frac{\log_2(8)}{3} = \frac{3}{3} = 1$

(b) $\lim_{z \to \infty} z = +\infty$ and $\lim_{z \to \infty} \log_{\frac{2}{3}}(z) = -\infty$

So, $\lim_{z \to \infty} z \log_{\frac{2}{3}}(z) = (+\infty) \cdot (-\infty) = -\infty$

Since logarithms are defined as inverses of exponential functions, the properties of exponents translate directly into properties of logarithms:

Exponent Laws	Logarithm Laws
$b^x \cdot b^y = b^{x+y}$	$\log_b(XY) = \log_b(X) + \log_b(Y)$ (log of a product is a sum of logs)
$\frac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\frac{X}{Y}\right) = \log_b(X) - \log_b(Y)$ (log of a fraction is a difference of logs)
$(b^x)^p = b^{p \cdot x}$	$\log_b(X^p) = p \cdot \log_b(X)$ (powers inside a log can be pulled to the front)
$b^0 = 1$ for any $b > 0$	$\log_b(1) = 0$ for any $b > 0$
$b^1 = b$ for any $b > 0$	$\log_b(b) = 1$ for any $b > 0$
$d = c^{\log_c(d)}$ and $d = b^{\log_b(d)} = \left(c^{\log_c(b)}\right)^{\log_b(d)}$	$\log_b(d) = \frac{\log_c(d)}{\log_c(b)}$ for any $c \in (0,1) \cup (1,\infty)$ (change of base formula)

We can use these rules to expand one big log into a bunch of smaller ones:

Example 10.1.4

Expand each logarithmic expression below completely.

(a) $\log_4\left(\frac{x^2\sqrt{y}}{16z^3}\right)$

(b) $\log_{\sqrt{3}}\left(\frac{27(x^2+4)^3}{\sqrt[5]{2+\cos(x)}}\right)$

Solution

(a) We can use the division rule to turn this into the difference of two logs

$= \log_4\left(x^2\sqrt{y}\right) - \log_4\left(16z^3\right)$

And then turn each product into a sum

$= \left[\log_4\left(x^2\right) + \log_4\left(y^{\frac{1}{2}}\right)\right] - \left[\log_4(16) + \log_4\left(z^3\right)\right]$

Then we bring down the powers on each term (and notice $16 = 4^2$)

$= \left[2\log_4(x) + \frac{1}{2}\log_4(y)\right] - [2 + 3\log_4(z)]$

Then distribute the minus sign to get your final answer

$$= 2\log_4(x) + \frac{1}{2}\log_4(y) - 2 - 3\log_4(z)$$

(b) We can use the division rule to turn this into the difference of two logs

$$= \log_{\sqrt{3}}\left(27(x^2+4)^3\right) - \log_{\sqrt{3}}\left(\sqrt[5]{2+\cos(x)}\right)$$

And then turn the first product into a sum

$$= \left[\log_{\sqrt{3}}(9) + \log_{\sqrt{3}}\left((x^2+4)^3\right)\right] - \log_{\sqrt{3}}\left((2+\cos(x))^{\frac{1}{5}}\right)$$

Then we use change of base on the first term and bring down the other powers

$$= \frac{\log_3(27)}{\log_3(\sqrt{3})} + 3\log_{\sqrt{3}}\left(x^2+4\right) - \frac{1}{5}\log_{\sqrt{3}}(2+\cos(x))$$

Then notice $27 = 3^3$ and $\sqrt{3} = 3^{\frac{1}{2}}$ so we can write the first term as:

$$= \frac{3}{\frac{1}{2}} + 3\log_{\sqrt{3}}\left(x^2+4\right) - \frac{1}{5}\log_{\sqrt{3}}(2+\cos(x))$$

Then we simplify to get our final answer

$$= 6 + 3\log_{\sqrt{3}}\left(x^2+4\right) - \frac{1}{5}\log_{\sqrt{3}}(2+\cos(x))$$

This is called *expanding a logarithm*, which will be a useful skill to have in the near future.

Going the opposite way (turning many logs into one big log) is called *condensing the logarithm*.

You can use these properties of logarithms when calculating limits as well:

Example 10.1.5

Evaluate each limit below.

(a) $\lim\limits_{x\to\infty} \dfrac{\log_5(3x)}{\log_5(x^2)}$

(b) $\lim\limits_{t\to-\infty} \log_7(7t^4+3t^2+5) - 2\log_7(t^2+1)$

Solution

(a) Expand the top logarithm and pull down the exponent for the bottom log to get:

$$= \lim_{x\to\infty} \frac{\log_5(3)+\log_5(x)}{2\log_5(x)}$$

Then since $\lim\limits_{x\to\infty} \log_5(x) = +\infty$ we can use the Lebron Rule to get rid of $\log_5(3)$:

$$= \lim_{x\to\infty} \frac{\log_5(x))}{2\log_5(x)} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$$

(b) Each of these two logs approach $+\infty$ as $t \to -\infty$, giving us $\infty - \infty$

This unfortunately doesn't mean our answer is zero, (since ∞ isn't a number $\infty - \infty$ doesn't necessarily cancel)

We can condense these two logs into one by bringing the 2 inside of the log second log:

$$= \lim_{t \to -\infty} \log_7(7t^4 + 3t^2 + 5) - \log_7\left((t^2+1)^2\right)$$

And then turn this difference of logs into a fraction inside of one log

$$= \lim_{t \to -\infty} \log_7\left(\frac{7t^4 + 3t^2 + 5}{(t^2+1)^2}\right)$$

$$= \lim_{t \to -\infty} \log_7\left(\frac{7t^4 + 3t^2 + 5}{t^4 + 2t^2 + 1}\right)$$

Since the logarithm function is continuous, we can pass the limit through.

$$= \log_7\left(\lim_{t \to -\infty} \frac{7t^4 + 3t^2 + 5}{t^4 + 2t^2 + 1}\right)$$

Now we have an $\frac{\infty}{\infty}$ limit, so we can use the Lebron Rule:

$$= \log_7\left(\lim_{t \to -\infty} \frac{7t^4}{t^4}\right) = \log_7\left(\lim_{t \to -\infty} 7\right) = \log_7(7) = 1$$

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10.2: Exponential Growth and Decay

Now that we have some practice using logarithms and their properties, we can use them for their most important application: solving exponential equations.

For example, if you wanted to solve an equation like $7^{3x+1} = \frac{1}{7}$

We can apply $\log_7$ to both sides of the equation to get:

$$\log_7\left(7^{3x+1}\right) = \log_7\left(\frac{1}{7}\right)$$

Then we bring down the power to the outside of the log on the left and get

$$(3x+1)\log_7(7) = \log_7\left(7^{-1}\right)$$

Since the remaining things inside of the log simplify nicely this becomes:

$$(3x+1)\cdot 1 = -1$$

Which easily solves to $x = -\frac{2}{3}$

You can pick any base you want for the log, but its often a good idea to pick a base that simplifies with the base of one or both exponents!

Example 10.2.1

Solve $4^{5x-2} = 8^{2x+1}$

Solution

Apply $\log_2$ to both sides:

$\log_2\left(4^{5x-2}\right) = \log_2\left(8^{2x+1}\right)$

Then bring down the exponents from both sides

$(5x-2)\log_2(4) = (2x+1)\log_2(8)$

And since $4 = 2^2$ and $8 = 2^3$ these simplify nicely to:

$$(5x-2)\cdot 2 = (2x+1)\cdot 3$$

$$10x - 4 = 6x + 3$$

$$4x = 7$$

$$x = \frac{7}{4}$$

One such situation where we need to solve exponential equations is when dealing with the function: $A(t) = Pe^{rt}$ (where P and r are constants).

This function is useful for modeling two very common real-life scenarios: *exponential growth* and *exponential decay*.

This is because $A'(t) = P \cdot re^{rt} = r \cdot A(t)$

In other words, the rate of change of $A(t)$ is directly proportional to the value of $A(t)$.

For example, the number of babies that a population of 500 rabbits would make in a year should be 5 times as many as the number of babies that a population of 100 rabbits would make. So the rate of increase of the rabbit population should be some multiple of the current number of rabbits!

All exponential growth/decay problems will have two pieces of information to help you solve for P and r, and ask a question for you to answer once you've found $A(t)$.

Example 10.2.2

The population of a town in 1950 was 2500 people. By the year 1980, there were 3000 people living there. At this rate of exponential growth, What will the population be in 2040?

Solution

The two pieces of given information are $A(1950) = 2500$ and $A(1980) = 3000$

Writing these out with the formula $A(t) = Pe^{rt}$ we get:

$Pe^{r\cdot 1950} = 2500$ and $Pe^{r\cdot 1980} = 3000$

We can divide these two equations on each side to get:

$$\frac{Pe^{1980r}}{Pe^{1950r}} = \frac{3000}{2500}$$

Which reduces (and cancels the P) to give:

$$e^{30r} = \frac{6}{5}$$

Use $\log_e$ on both sides to get:

$$30r = \log_e\left(\frac{6}{5}\right) \rightarrow r = \frac{1}{30}\log_e\left(\frac{6}{5}\right)$$

And plugging this into either of our previous equations lets us solve for P:

$$3000 = Pe^{\frac{1}{30}\log_e\left(\frac{6}{5}\right)\cdot 1980} \rightarrow P = 3000e^{-66\log_e\left(\frac{6}{5}\right)}$$

Therefore the equation that models the population is:

$$A(t) = 3000e^{-66\log_e\left(\frac{6}{5}\right)} \cdot e^{\frac{1}{30}\log_e\left(\frac{6}{5}\right)t} = 3000e^{\left(\frac{t}{30}-66\right)\log_e\left(\frac{6}{5}\right)}$$

Finally we can answer the question asked, what is the population in 2040?

$$A(2040) = 3000e^{\left(\frac{2040}{30}-66\right)\log_e\left(\frac{6}{5}\right)} = 3000e^{2\log_e\left(\frac{6}{5}\right)}$$

$$= 3000e^{\log_e\left(\left[\frac{6}{5}\right]^2\right)} = 3000\left(\frac{6}{5}\right)^2 = 3000\cdot\frac{36}{25} = 120\cdot 36 = 4320 \text{ people}$$

Since e is the most popular base for differentiating exponential functions, $\log_e(x)$ is used very often as well.
We call this the *natural logarithm* and it is shortened to $\ln(x)$.

This next one is an example of exponential decay, but there is no difference in approach:

Example 10.2.3

A 256 gram sample of Pizzaranium is discovered, but six hours later it has decayed to 64 grams. How long will it take for the original sample to decay down to 4 grams?

Solution

The given information is the starting amount: $A(0) = 256$ and the amount six hours later: $A(6) = 64$

We can translate these into equations with $A(t) = Pe^{rt}$

$256 = Pe^{r\cdot 0}$ and $64 = Pe^{r\cdot 6}$

The first equation simplifies immediately to $P = 256$

Substituting this into the other equation gives $64 = 256e^{6r}$, so we divide both sides by 256:

$$\frac{1}{4} = e^{6r}$$

Now we can ln both sides (remember, thats the inverse of e^x) to get:

$$\ln\left(\frac{1}{4}\right) = 6r$$

So $r = \frac{1}{6}\ln\left(\frac{1}{4}\right)$ and $A(t) = 256e^{\frac{1}{6}\ln\left(\frac{1}{4}\right)t}$

Now we want to solve for which t makes $A(t) = 4$:

$$256e^{\frac{1}{6}\ln\left(\frac{1}{4}\right)t} = 4$$
$$e^{\frac{1}{6}\ln\left(\frac{1}{4}\right)t} = \frac{1}{64}$$
$$\frac{1}{6}\ln\left(\frac{1}{4}\right)t = \ln\left(\frac{1}{64}\right)$$
$$t = \frac{\ln\left(\frac{1}{64}\right)}{\frac{1}{6}\ln\left(\frac{1}{4}\right)} = \frac{\ln\left(2^{-6}\right)}{\frac{1}{6}\ln\left(2^{-2}\right)} = \frac{(-6)\ln(2)}{\frac{1}{6}\cdot(-2)\ln(2)} = 18 \text{ hours}$$

Another real world example that yields exponential equations is *Newton's Law of Cooling*.

The idea is that the difference between an object and its surroundings will decay exponentially as it warms or cools over time. So if we let T_o be the **initial temperature** of the object and T_e be the **temperature of the environment**, we know the function of the temperature is:

$T(t) = T_e + (T_o - T_e)\, e^{rt}$

Example 10.2.4

A turkey is set out to defrost overnight. At 8 PM it is taken out of the freezer with a temperature of -18° and placed onto the kitchen counter in a room that is 22°. By 6 AM the meat is 12°. At what time will it reach 17°?

Solution

We have the initial temperature $T_o = -18$ and the room temperature $T_e = 22$

As for the time, if we count 8 PM as $t = 0$, then 6 AM is $t = 10$ hours later.

This means we know that $T(10) = 12$ and we want to know when $T(t) = 17$

We start by plugging in T_o and T_e to get $T(t) = 22 + (-18 - 22)e^{rt} = 22 - 40e^{rt}$

So $T(10) = 12$ becomes $22 - 40e^{r \cdot 10} = 12$

$$-40e^{10r} = -10$$

$$e^{10r} = \frac{1}{4}$$

We ln both sides again to get:

$$10r = \ln\left(\frac{1}{4}\right)$$

Thus we have $r = \frac{1}{10}\ln\left(\frac{1}{4}\right)$ and so $T(t) = 22 - 40e^{\frac{1}{10}\ln\left(\frac{1}{4}\right)t}$

Now we try to solve $T(t) = 17$:

$$17 = 22 - 40e^{\frac{1}{10}\ln\left(\frac{1}{4}\right)t}$$

$$-5 = -40e^{\frac{1}{10}\ln\left(\frac{1}{4}\right)t} \longrightarrow \frac{1}{8} = e^{\frac{1}{10}\ln\left(\frac{1}{4}\right)t}$$

$$\ln\left(\frac{1}{8}\right) = \frac{1}{10}\ln\left(\frac{1}{4}\right)t$$

$$t = \frac{\ln\left(\frac{1}{8}\right)}{\frac{1}{10}\ln\left(\frac{1}{4}\right)} = \frac{\ln\left(2^{-3}\right)}{\frac{1}{10}\ln\left(2^{-2}\right)} = 15$$

So 15 hours after 8 PM gives us 11 AM as our final answer.

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10.3: Derivatives of Logarithms

Remember we have already shown the derivative of certain exponential functions:

$(e^{rx})' = re^{rx}$

Now the properties of logarithms tell us that $b^x = \left(e^{\ln(b)}\right)^x = e^{x\ln(b)}$, which means $(b^x)' = b^x \ln(b)$

Consider the function $f(x) = b^x$, who has inverse $f^{-1}(x) = \log_b(x)$.

We can once again use the inverse derivative formula to find $\left(f^{-1}\right)'(x)$:

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{b^{\log_b(x)}\ln(b)} = \frac{1}{x\ln(b)}$$

Therefore $(\log_b(x))' = \frac{1}{x\ln(b)}$ and if $b = e$, this simplifies to $(\ln(x))' = \frac{1}{x}$

These are the last functions we will add to our list of derivatives to memorize for a while.

With these we can officially differentiate pretty much any explicitly defined function.

Example 10.3.1

Find the derivative of each function below.

(a) $h(x) = \log_2(x^2+4)$

(b) $g(t) = \ln(2\ln(t))$

(c) $\ell(y) = \dfrac{3^y}{4^y+1}$

(d) $w(z) = 5^z \log_5(z)$

Solution

(a) $h'(x) = \dfrac{1}{(x^2+4)\ln(2)} \cdot (x^2+4)' = \dfrac{1}{(x^2+4)\ln(2)} \cdot 2x$

(b) $g'(t) = \dfrac{1}{(2\ln(t))} \cdot (2\ln(t))' = \dfrac{1}{(2\ln(t))} \cdot \dfrac{2}{t}$

(c) $\ell'(y) = \dfrac{(3^y)'(4^y+1) - (3^y)(4^y+1)'}{(4^y+1)^2} = \dfrac{3^y\ln(3)\cdot(4^y+1) - 3^y\cdot 4^y\ln(4)}{(4^y+1)^2}$

(d) $w'(z) = (5^z)'(\log_5(z)) + (5^z)(\log_5(z))' = 5^z\ln(5)\cdot\log_5(z) + 5^z\cdot\dfrac{1}{z\ln(5)}$

And we can also use them to do tangent line problems:

Example 10.3.2

Find the equation of the tangent line to $f(x) = \log_2(2^x+8)$ at $x = 3$.

Solution

The tangent line will be $y - f(3) = f'(3)(x-3)$

$f(3) = f(x) = \log_2(2^3+8) = \log_2(16) = \log_2(2^4) = 4$

And we can differentiate $f(x)$ to get:

$$f'(x) = \frac{1}{(2^x+8)\ln(2)} \cdot (2^x+8)' = \frac{1}{(2^x+8)\ln(2)} \cdot 2^x\ln(2) = \frac{2^x}{2^x+8}$$

So $f'(3) = \dfrac{2^3}{2^3+8} = \dfrac{8}{16} = \dfrac{1}{2}$

Thus the tangent line is $y - 4 = \dfrac{1}{2}(x-3)$

Example 10.3.3

If the function $g(x) = b^x$ is tangent to $y = 2x$, what is b?

Solution

The tangent line at $x = a$ would be $y - g(a) = g'(a)(x-a)$

Clearly $g(a) = b^a$ and $g'(x) = b^x \ln(b)$ so $g'(a) = b^a \ln(b)$

This makes the tangent line $y - b^a = (b^a \ln(b)) \cdot (x-a)$

Which can be written $y = (b^a \ln(b))x + (b^a - ab^a \ln(b)))$

For this to be the line $y = 2x$ we need the slope and intercept to match: $\begin{cases} b^a \ln(b) = 2 \\ b^a - ab^a \ln(b) = 0 \end{cases}$

$b^a - ab^a \ln(b) = 0$

$b^a(1 - a\ln(b)) = 0$

$b^a(1 - \ln(b^a)) = 0$

Since exponentials never equal zero, $b^a = 0$ is impossible. This leaves us with $\ln(b^a) = 1$, and thus $b^a = e$

Substituting this into the top equation gives: $e\ln(b) = 2$

So $\ln(b) = \dfrac{2}{e} \longrightarrow b = e^{\frac{2}{e}}$

Sometimes the properties of log can help you to simplify a function before differentiating.

Example 10.3.4

Find the tangent line at $x = 2$ for $f(x) = \log_3\left(\dfrac{(2x-1)^2 \cdot \sqrt{2x+5}}{9 \cdot 27^{x+1}}\right)$

Solution

To find this line we need $y - f(2) = f'(2)(x-2)$

$f(2) = \log_3\left(\dfrac{(2\cdot 2-1)^2 \cdot \sqrt{2\cdot 2+5}}{9 \cdot 27^{2+1}}\right)$

$= \log_3\left(\dfrac{3^2 \cdot \sqrt{9}}{9 \cdot (3^3)^3}\right) = \log_3\left(\dfrac{3^3}{3^2 \cdot 3^9}\right) = \log_3\left(3^{-8}\right) = -8$

If you thought that was ugly, think about $f'(x)$: Its a chain rule with a quotient rule and product rule with smaller chain rules inside of it.

Thankfully, instead you can re-write $f(x)$ by using the log rules first:

$f(x) = \log_3\left[(2x-1)^2\right] + \log_3\left[(2x+5)^{\frac{1}{2}}\right] - \log_3[9] - \log_3\left[27^{x+1}\right]$

$= 2\log_3(2x-1) + \frac{1}{2}\log_3(2x+5) - 2 - (x+1)\cdot 3$

This would be a bit easier to find $f(2)$ and a lot easier for $f'(2)$:

$f(2) = 2\log_3(2\cdot 2-1) + \frac{1}{2}\log_3(2\cdot 2+5) - 2 - (2+1)\cdot 3$

$= 2\log_3(3) + \frac{1}{2}\log_3(9) - 2 - 3\cdot 3 = 2\cdot 1 + \frac{1}{2}\cdot 2 - 11 = -8$

$f'(x) = 2\cdot\frac{1}{(2x-1)\ln(3)}\cdot(2x-1)' + \frac{1}{2}\cdot\frac{1}{(2x+5)\ln(3)}\cdot(2x+5)' - 0 - 1\cdot 3$

$= \frac{2}{(2x-1)\ln(3)}\cdot 2 + \frac{1}{2\,(2x+5)\ln(3)}\cdot 2 - 3$

so $f'(2) = \frac{4}{(2\cdot 2-1)\ln(3)} + \frac{1}{(2\cdot 2+5)\ln(3)} - 3$

$= \frac{4}{3\ln(3)} + \frac{1}{9\ln(3)} - 3 = \frac{13}{9\ln(3)} - 3$

And we get our final answer: $y+8 = \left(\frac{13}{9\ln(3)} - 3\right)(x-2)$

Despite defined entirely as an inverse, **the logarithm is a powerful tool.**

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10.4: Logarithmic Differentiation

This section focuses on one specific trick that allows us to use the properties of logarithms on a function we wish to differentiate. Let $f(x)$ be a positive function we want the derivative of, but instead of attacking it head on, we consider two other functions $h(x) = e^x$ and $g(x) = \ln(f(x))$

Since $e^{\ln(a)} = a$ for any $a > 0$, we clearly have $f(x) = h\left(g(x)\right)$

So $f'(x) = (h \circ g)'(x) = h'(g(x)) \cdot g'(x) = e^{\ln(f(x))} \cdot [\ln(f(x))]' = f(x) \cdot [\ln(f(x))]'$

This formula, $f'(x) = f(x) \cdot [\ln(f(x))]'$, is called the *logarithmic differentiation formula*.

It works on the previous functions we already learned to differentiate:

If $f(x) = x^p$ then $f'(x) = x^p \cdot [\ln(x^p)]'$

And we can use our log properties to simplify the second term before differentiating:

$$= x^p \cdot [p \ln(x)]' = x^p \cdot \left[\frac{p}{x}\right] = px^{p-1}$$

Similarly, if $f(x) = x^4 \cdot e^x$ then $f'(x) = x^4 \cdot e^x \cdot \left[\ln\left(x^4 \cdot e^x\right)\right]'$

$$= x^4 \cdot e^x \cdot \left[\ln(x^4) + \ln(e^x)\right]' = x^4 \cdot e^x \cdot [4\ln(x) + x]'$$

$$= x^4 \cdot e^x \cdot \left[\frac{4}{x} + 1\right] = x^4 \cdot e^x \cdot \frac{4}{x} + x^4 \cdot e^x = 4x^3 \cdot e^x + x^4 \cdot e^x$$

This is the same result you would have gotten using the product rule.

While this strategy works for many functions $f(x)$, it is only useful when $\ln(f(x))$ simplifies nicely using log properties. Without that, it makes taking the derivative harder instead of easier!

If $f(x) = x^2 + x^3$ then $f'(x) = \left(x^2 + x^3\right) \cdot \left[\ln(x^2 + x^3)\right]'$

$$= \left(x^2 + x^3\right) \cdot \left[\frac{1}{x^2 + x^3} \cdot (x^2 + x^3)'\right] = (x^2 + x^3)' = 2x + 3x^2$$

Clearly logarithmic differentiation is a useful tool, but not for every function!

Example 10.4.1

Find the derivatives of the functions below using logarithmic differentiation.

(a) $f(x) = \dfrac{x^2(5x+1)^3}{e^{7x}\sqrt[3]{x-1}}$

(b) $g(t) = \sqrt{t^3(t-3)^6(t-7)^5(t+8)^4}$

Solution

(a) $f'(x) = \dfrac{x^2(5x+1)^3}{e^{7x} \cdot \sqrt[3]{x-1}} \cdot \left[\ln\left(\dfrac{x^2(5x+1)^3}{e^{7x}\sqrt[3]{x-1}}\right)\right]'$

$$= \frac{x^2(5x+1)^3}{e^{7x}\sqrt[3]{x-1}} \cdot \left[\ln[x^2] + \ln\left[(5x+1)^3\right] - \ln\left[e^{7x}\right] - \ln\left[(x-1)^{\frac{1}{3}}\right]\right]'$$

$$= \frac{x^2(5x+1)^3}{e^{7x}\sqrt[3]{x-1}} \cdot \left[2\ln(x) + 3\ln(5x+1) - 7x - \frac{1}{3}\ln(x-1)\right]'$$

$$= \frac{x^2(5x+1)^3}{e^{7x}\sqrt[3]{x-1}} \cdot \left[2 \cdot \frac{1}{x} + 3 \cdot \frac{1}{5x+1} \cdot 5 - 7 - \frac{1}{3} \cdot \frac{1}{x-1} \cdot 1\right]$$

$$= \frac{x^2(5x+1)^3}{e^{7x}\sqrt[3]{x-1}} \cdot \left[\frac{2}{x} + \frac{15}{5x+1} - 7 - \frac{1}{3(x-1)}\right]$$

(b) First, lets simplify the log term, $\ln(g(t))$:

$$\ln\left(\sqrt{t^3(t-3)^6(t-7)^5(t+8)^4}\right) = \frac{1}{2}\ln\left(t^3(t-3)^6(t-7)^5(t+8)^4\right)$$

$$= \frac{1}{2}\left(\ln[t^3] + \ln[(t-3)^6] + \ln[(t-7)^5] + \ln[(t+8)^4]\right)$$

$$= \frac{1}{2}(3\ln(t) + 6\ln(t-3) + 5\ln(t-7) + 4\ln(t+8))$$

$$= \frac{3}{2}\ln(t) + 3\ln(t-3) + \frac{5}{2}\ln(t-7) + 2\ln(t+8)$$

Now the logarithmic differentiation formula gives us:

$$g'(t) = g(t)\left[\frac{3}{2}\ln(t) + 3\ln(t-3) + \frac{5}{2}\ln(t-7) + 2\ln(t+8)\right]'$$

$$= g(t)\left[\frac{3}{2} \cdot \frac{1}{t} + 3 \cdot \frac{1}{t-3} + \frac{5}{2} \cdot \frac{1}{t-7} + 2 \cdot \frac{1}{t+8}\right]$$

$$= \sqrt{t^3(t-3)^6(t-7)^5(t+8)^4} \cdot \left[\frac{3}{2t} + \frac{3}{t-3} + \frac{5}{2(t-7)} + \frac{2}{t+8}\right]$$

Both of the previous examples *could have* been done with straightforward derivative formulas, but the log helped to avoid a lot of messy work!

There are some functions which can **only** be differentiated with logarithmic differentiation:

Functions with variables in the base and exponent: $f(x) = [g(x)]^{h(x)}$

Example 10.4.2

Find the derivative of each function below by using logarithmic differentiation.

(a) $f(x) = x^x$

(b) $g(t) = (t^2+1)^{\sin(t)}$

(c) $A(v) = (v+1)^{\frac{1}{\ln(v+1)}}$

Solution

(a) $f'(x) = x^x \cdot [\ln(x^x)]'$

$$= x^x \cdot [x\ln(x)]'$$

$$= x^x \cdot \left[1 \cdot \ln(x) + x \cdot \frac{1}{x}\right]$$

$= x^x \cdot [\ln(x)+1]$

(b) $g'(t) = (t^2+1)^{\sin(t)} \cdot \left[\ln\left((t^2+1)^{\sin(t)}\right)\right]'$

$= (t^2+1)^{\sin(t)} \cdot \left[\sin(t)\ln(t^2+1)\right]'$

$= (t^2+1)^{\sin(t)} \cdot \left[\cos(t)\cdot\ln(t^2+1) + \sin(t)\cdot\frac{1}{t^2+1}\cdot 2t\right]$

$= (t^2+1)^{\sin(t)} \cdot \left[\cos(t)\ln(t^2+1) + \frac{2t\sin(t)}{t^2+1}\right]$

(c) $A'(v) = (v+1)^{\frac{1}{\ln(v+1)}} \cdot \left[\ln\left((v+1)^{\frac{1}{\ln(v+1)}}\right)\right]'$

$= (v+1)^{\frac{1}{\ln(v+1)}} \cdot \left[\frac{1}{\ln(v+1)}\ln(v+1)\right]'$

$= (v+1)^{\frac{1}{\ln(v+1)}} \cdot [1]'$

$= (v+1)^{\frac{1}{\ln(v+1)}} \cdot 0 = 0$

(Thats because $A(v) = \left(e^{\ln(v+1)}\right)^{\frac{1}{\ln(v+1)}} = e^1 = e$ is a constant)

You don't have to use logarithmic differentiation on every piece of a function. Instead, you can also pick and choose which parts to use it on:

Example 10.4.3

Find the equation of the tangent line to $f(x) = x^{2x} - 4x^2$ at $x = 2$

Solution

The tangent line will be $y - f(2) = f'(2)(x-2)$

$f(2) = 2^{2\cdot 2} - 4\cdot 2^2 = 2^4 - 4\cdot 4 = 0$

The derivative of $4x^2$ is easy, but x^{2x} requires log differentiation:

$f'(x) = x^{2x}\cdot\left[\ln(x^{2x})\right]' - (4x^2)'$

$= x^{2x}\cdot[2x\ln(x)]' - 8x$

$= x^{2x}\cdot\left[2\cdot\ln(x) + 2x\cdot\frac{1}{x}\right] - 8x$

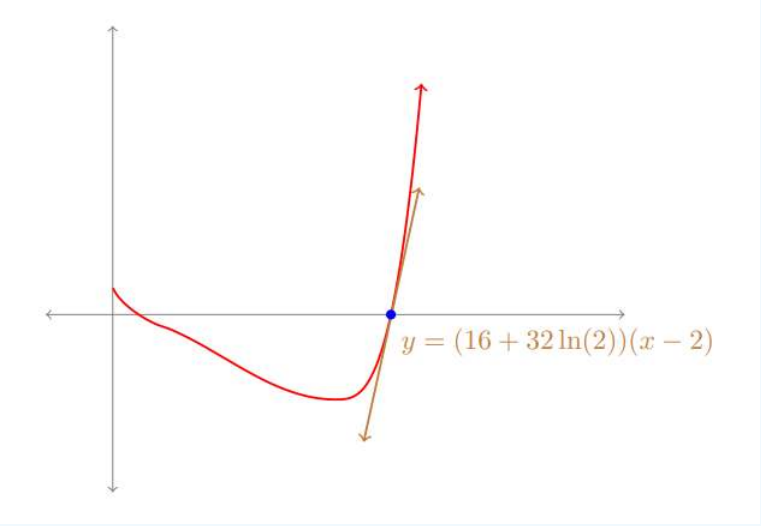
$= x^{2x}\cdot[2\ln(x)+2] - 8x$

This means $f'(2) = 2^{2\cdot 2}\cdot[2\ln(2)+2] - 8\cdot 2$

$= 16\,[2\ln(2)+2] - 16 = 16 + 32\ln(2)$

So our tangent line must be: $y - 0 = (16 + 32\ln(2))(x - 2)$

This graph of the function is not an easy one to draw by hand, but we can use a computer to see that this is indeed the correct answer:



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CHAPTER OVERVIEW

11: Applications of Derivatives

> ”In differentiation, not in uniformity, lies the path of progress” - Louis Brandeis

Chapter eleven brings us our first batch of applications of derivatives. Since we can now differentiate almost any function we are given, we should be able to answer questions that can be answered using tangent lines. Calculus theorems are propositions which can be shown to be true based on some logical chain of reasoning called a proof. Each theorem covered in this chapter has direct applications to a specific type of problem or helps us to further describe the relationship between a function and its derivative.

Linearization is the process of using a tangent line to a curve to approximate its value. This relatively simple observation can help to give surprisingly accurate estimates if you don’t have a calculator handy. One of the theorems is the Extreme Value Theorem. This promises us that a continuous function on a closed interval must achieve a maximum and minimum value (and also tells us where to look for them). We use it to help us discover the extrema for any given function on an interval.

We arm ourselves with one last differentiation skill: implicit differentiation. This allows us to take derivatives and find tangent lines to curves which are not functions or cannot be written explicitly. While not technically a theorem, L’Hopital’s rule is a very useful application of derivatives when taking limits of indeterminate forms. There is something satisfying about using limits to define derivatives in order to further solve limits.

Our final subchapter here is curve sketching. This is an important life skill to have, as it will let you draw out very accurate graphs of functions (as long as the derivatives and algebra aren’t too overwhelming).

11.1: Theorems of Calculus
11.2: Linearization
11.3: Extreme Values
11.4: Implicit differentiation
11.5: L’Hopital’s Rule
11.6: Curve Sketching

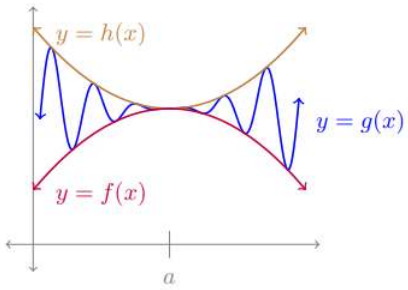
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11.1: Theorems of Calculus

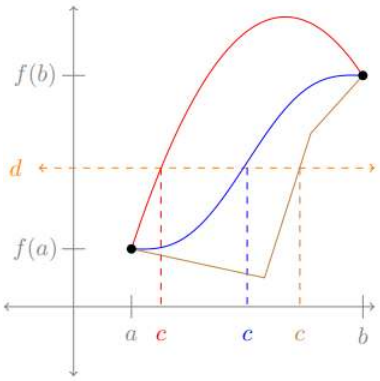
So far, we have learned two theorems to help us on our calculus journey.

We learned the **Squeeze Theorem** as a tool to help us to take certain limits whenever a component limit does not exist. In case you don't remember, it said:

When $f(x), g(x)$ and $h(x)$ are functions that satisfy $f(x) \leq g(x) \leq h(x)$, **and** we know that $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$, then we must have $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x)$



We also learned the **Intermediate Value Theorem**, which tells us that a continuous function much achieve all possible y-values between any two points it connects. As a reminder, it said:
If $f(x)$ is continuous on $[a, b]$ then for any value d between $f(a)$ and $f(b)$, there exists some $c \in (a, b)$ such that $f(c) = d$

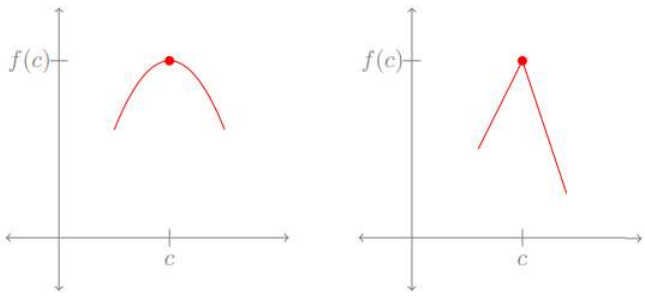


These two have been great, but there are a few more theorems of calculus that can help us better understand the relationship between functions and their derivatives.

First, we have a theorem about maximums and minimums.

Before we discuss it, consider what it means for a function $f(x)$ to have a maximum at $(c, f(c))$:

This means that $f(c) \geq f(x)$ for all x values, especially the ones that are "very close" to c



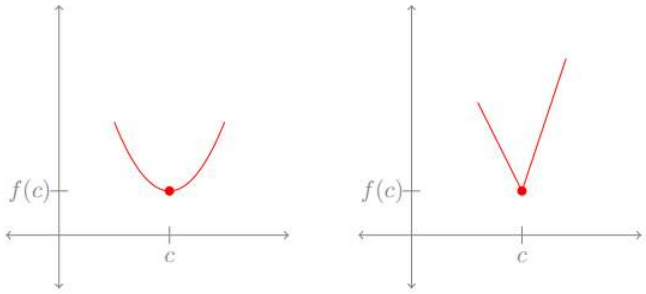
That means the slope for any point to the immediate left of c should be positive, and the slope for a point to the immediate right of c should be negative.

For $f'(c)$ to exist, we need $\lim_{x\to c^+} f'(x) = \lim_{x\to c^-} f'(x)$, (and the only number that can be approached by both positive and negative slopes is 0)

So for $f(x)$ to have a maximum at $x = c$, either $f'(c) = 0$ (like the left picture) or $f'(c) = DNE$ (like the right picture)

Now consider what it means for a function $f(x)$ to have a minimum at $(c, f(c))$:

This means that $f(c) \leq f(x)$ for all x values, especially the ones that are "very close" to c



That means the slope for any point to the immediate left of c should be negative, and the slope for a point to the immediate right of c should be positive.

Again $f'(c)$ only exists when $\lim_{x\to c^+} f'(x) = \lim_{x\to c^-} f'(x)$,

So for $f(x)$ to have a minimum at $x = c$, either $f'(c) = 0$ (like the left picture) or $f'(c) = DNE$ (like the right picture)

We define a *critical point* of $f(x)$ to be any point such that $f'(x) = 0$ or $f'(x) = DNE$.

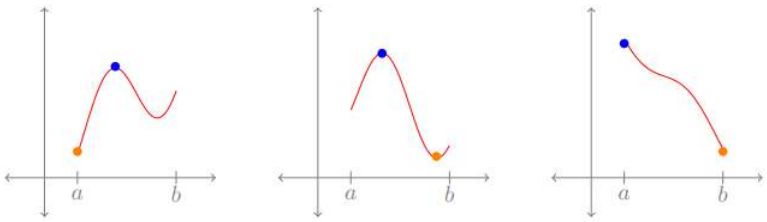
Therefore **all maximums and minimums must occur at a critical point**.

This leads us to the **Extreme Value Theorem**:

If $f(x)$ is continuous on $[a, b]$, then it must achieve a maximum and minimum value in that interval.

Thus, there exist some $c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$

Furthermore, we know that c and d each must be critical points of $f(x)$ (or equal to one of a or b)



The graphs above should demonstrate why we need to allow for these *extreme values* to occur at either the endpoints or at a critical point inside of the interval.

This theorem is helpful as a guide for what to do when searching for the maximum or minimum values of a function, but unlike the Squeeze Theorem and Intermediate Value Theorem, it does not actively tell us something about a specific function. In other words, you won't be solving any problems and saying "by the Extreme Value Theorem.."

However, it is the backbone to proving the next theorem, **Rolle's Theorem**:

If $f(x)$ is continuous and differentiable on $[a,b]$ and $f(a)=f(b)$, then there must be some $c\in(a,b)$ such that $f'(c)=0$

This becomes apparent when you realize there are only three possible cases to consider:

Case 1: $f(x)$ achieves a minimum value at some $c\in(a,b)$

In this case, we know that $f'(c)$ must exist, so $f'(c)=0$

Case 2: $f(x)$ achieves a maximum value at some $d\in(a,b)$

In this case, we know that $f'(c)$ must exist, so $f'(d)=0$

Case 3: $f(x)$ achieves both its maximum and minimum values at a or b

In this case, since $f(a)=f(b)$ the maximum and minimum values are the same! That means $f(x)$ must be a constant function, so its derivative is zero everywhere in (a,b)

This theorem tells us that **in order for a continuous function to connect two points with the same height, there must be at least one critical point between them.**

Together with the Intermediate Value Theorem, this allows us to prove that there is exactly one solution to certain equations (even without solving them)

For example, we can prove that there is exactly one solution to: $3x+2\cos(x)=7$

Define the function $f(x)=3x+2\cos(x)$, which is continuous everywhere.

We know that $f(0)=2<7$ and $f(\pi)=3\pi-2>7$

So by IVT there must exist some $c\in(0,\pi)$ such that $f(c)=7$

That means there **at least one solution** to $f(x)=7$

But can there be more than one? According to Rolle's Theorem, we can only have two solutions to $f(x)=7$, if $f'(x)=0$ somewhere between them. So if $f'(x)=0$ has no solutions, you know your function is one-to-one.

For this example function we have $f'(x)=3-2\sin(x)$ and $3-2\sin(x)=0\longleftrightarrow\sin(x)=\dfrac{3}{2}$ has no solutions, then it is impossible for $f(x)=7$ to occur twice!

Therefore there is **exactly one solution** to $f(x)=7$.

Lets try another similar example:

Example 11.1.1

Prove there is exactly one solution to $e^{2x}=1-e^{x}$

Solution

Define $g(x)=e^{2x}+e^{x}$, we want to prove $g(x)=1$ has exactly one solution

Since $g(x)$ is continuous everywhere, and we have $g(0)=e^{0}+e^{0}=1+1=2>1$

and $g(-\ln(2))=e^{-2\ln(2)}+e^{-\ln(2)}=e^{\ln(\frac{1}{4})}+e^{\ln(\frac{1}{2})}=\dfrac{1}{4}+\dfrac{1}{2}=\dfrac{3}{4}<1$

So by IVT there must be at least one solution to $g(x)=1$

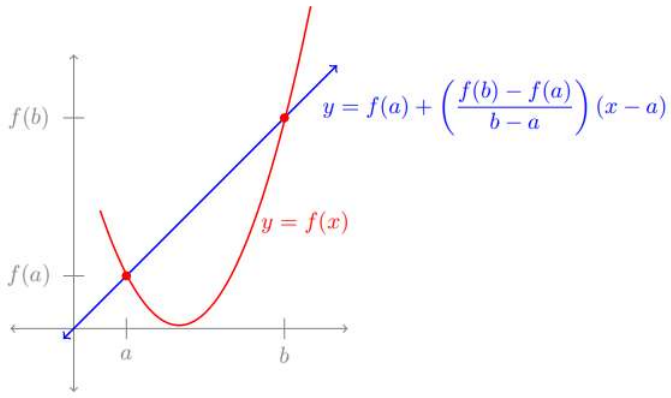
And $g'(x)=2e^{2x}+e^{x}$, which can never equal zero since $e^{\star}$ is always positive

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So by Rolle's Theorem there cannot be two solutions to $g(x) = 1$

That means there is exactly one solution to $g(x) = 1$ as desired.

Rolle's Theorem only applies to functions with $f(a) = f(b)$, but if we remove that restriction we can still say something about the derivative of f:



To compare the function $f(x)$ to the secant line connecting $(a, f(a))$ and $(b, f(b))$, we can define the new function $g(x) = f(x) - \left[f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)\right]$

This function is continuous and differentiable, and we can see that:

$g(a) = f(a) - \left[f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(a-a)\right] = 0$

$g(b) = f(b) - \left[f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(b-a)\right] = 0$

Since $g(a) = g(b)$, by Rolle's Theorem we know $g'(c) = 0$ must hold for some $c \in (a, b)$

And since $g'(x) = f'(x) - \left(\frac{f(b)-f(a)}{b-a}\right)$

Therefore $f'(c) = \frac{f(b)-f(a)}{b-a}$

This result is called the **Mean Value Theorem**:

If f is differentiable on $[a, b]$ then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

In other words, as long as a function is differentiable between two points $(a, f(a))$ and $(b, f(b))$, then there must be a point between them were the slope of the tangent line is to that point equal to the slope of the secant line connecting those two endpoints.

Alternatively if $f(t)$ is considered to be a position function (so its derivative $f'(t)$ is velocity), the MVT states that an object must achieve its average velocity at at least one specific time.

Sometimes we can solve for the c value in the MVT directly:

For example, verify $f(x) = x^2$ satisfies the MVT on $[1, 5]$

The secant line slope will be $\frac{f(5)-f(1)}{5-1} = \frac{24}{4} = 6$

So the derivative $f'(x) = 2x$ must equal 6 at some $c \in (1, 5)$

$2c = 6 \longrightarrow c = 3 \in (1,5)$

Therefore $f'(3) = \dfrac{f(5)-f(1)}{5-1}$ and MVT holds.

Example 11.1.2

Determine the value of c in each interval which satisfies the Mean Value Theorem on the given interval.

(a) $g(x) = \dfrac{2}{x^2}$ on $[1,2]$

(b) $h(t) = \sin^{-1}(\sqrt{t})$ on $\left[\dfrac{1}{2}, \dfrac{3}{4}\right]$

Solution

(a) The secant line slope will be $\dfrac{g(2)-g(1)}{2-1} = \dfrac{\frac{1}{2}-2}{1} = -\dfrac{3}{2}$

So the derivative $g'(x) = -4x^{-3} = -\dfrac{4}{x^3}$ must equal $-\dfrac{3}{2}$ at some $c \in (1,2)$

$-\dfrac{4}{c^3} = -\dfrac{3}{2} \longrightarrow 3c^3 = 8 \longrightarrow c = \sqrt[3]{\dfrac{8}{3}} = \dfrac{2}{\sqrt[3]{3}} \in (1,2)$

Therefore $g'\left(\dfrac{2}{\sqrt[3]{3}}\right) = \dfrac{g(2)-g(1)}{2-1}$ and MVT holds.

(b) The secant line slope will be $\dfrac{h(\frac{3}{4})-h(\frac{1}{2})}{\frac{3}{4}-\frac{1}{2}} = \dfrac{\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{4}} = \dfrac{\frac{\pi}{3}-\frac{\pi}{4}}{\frac{1}{4}} = \dfrac{\pi}{3}$

So $h'(t) = \dfrac{1}{\sqrt{1-(\sqrt{t})^2}} \cdot \dfrac{1}{2}t^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{t-t^2}}$ must equal $\dfrac{\pi}{3}) at some \backslash (c \in \left(\dfrac{1}{2}, \dfrac{3}{4}\right)$

$\dfrac{1}{2\sqrt{c-c^2}} = \dfrac{\pi}{3} \longrightarrow 0 = c^2 - c + \dfrac{9}{4\pi^2}$

We can use the quadratic formula to solve this:

$c = \dfrac{1 \pm \sqrt{1-\frac{9}{\pi^2}}}{2}$

We pick the solution in our desired interval $\left(\dfrac{1}{2}, \dfrac{3}{4}\right)$

Therefore $h'\left(\dfrac{1}{2} + \dfrac{\sqrt{1-\frac{9}{\pi^2}}}{2}\right) = \dfrac{h(\frac{3}{4})-h(\frac{1}{2})}{\frac{3}{4}-\frac{1}{2}}$ and MVT holds.

The Mean Value Theorem can also be used in reverse, so if you know something about $f'(x)$ and one point on $f(x)$, you can determine something about another point on the curve.

Example 11.1.3

If $f(x)$ is a function such that $f(3) = 7$ and $-1 \leq f'(x) \leq 2$, then what are the possible values of $f(8)$?

Solution

Consider $\dfrac{f(8) - f(3)}{8 - 3} = \dfrac{f(8) - 7}{5} = f'(c)$ for some c by MVT

And since $-1 \leq f'(c) \leq 2$

$$-1 \leq \frac{f(8) - 7}{5} \leq 2$$

$$-5 \leq f(8) - 7 \leq 10$$

$$2 \leq f(8) \leq 17$$

Therefore we see that the only possible values are $f(8) \in [2, 17]$

MVT is also used in real life, like when you see the signs that say *speed checked by aircraft*:

This is not done with a radar gun like make other speeding tickets, but instead by taking photos. A plane or drone will fly over a long road and take photos and note the location and time of day.

For example, assume a photo taken of your car at 2 pm next to mile marker number 7 and then another photo taken at 5 pm at mile marker number 283.

Then the aircraft sends the data to an analyst who states that the car's position at time t is a function $f(t)$ that must be continuous (unless you have a teleporting car, which you don't) and differentiable (unless your car can instantaneously change its speed, which it doesn't).

Therefore MVT applies, and there is some c with $f'(c) = \dfrac{f(5) - f(2)}{5 - 2} = \dfrac{283 - 7}{3} = 92$

Since $f'(t)$ represents the velocity of the car, that means there was at least one moment on the journey that the car was going exactly 92 miles/hour.

That is how fast the ticket you get in the mail will say! You may have been going much faster at times, but MVT only promises that you much achieve your average speed at some moment.

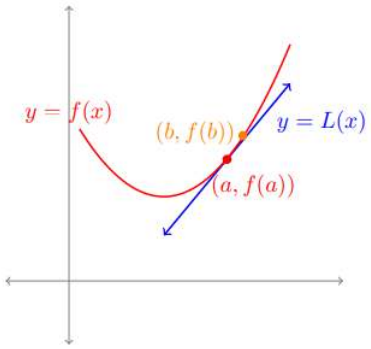
Side note: the ticket will not say "by MVT" so this analyst would lose points for not showing work on a math test. (Unfortunately, you will still need to pay the ticket)

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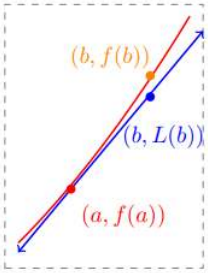
11.2: Linearization

We have been many functions and each has points that are "easy" to plug in other other points that are "nasty" to plug in.
For example, if $f(x) = \sin(x)$ we would be much happier plugging in $x = \pi$ than $x = 3$, while for $g(x) = x^5$, your preferences would probably be switched.

In this section, we will assume that we want to evaluate $f(b)$ where b is an ugly number for f, but b is nearby to a (where a is some nice number for f). Consider the sketch below of $y = f(x)$ and the line tangent to f at a, called $y = L(x)$



Lets zoom in a bit and focus only on the region between $(a, f(a))$ and $(b, f(b))$:



As we can see, when a is close to b, then $f(b)$ is close to $L(b)$.

Since the tangent line at a is given by: $y - f(a) = f'(a)(x - a)$

We can say that $L(x) = f(a) + f'(a)(x - a)$

To summarize our finding, when $a \approx b$ (that means they are close), then $f(b) \approx f(a) + f'(a)(b - a)$

This is *linearization*: approximating a function by using a nearby tangent line.

Almost all linearization questions say something along the lines of *estimate* or *approximate* some value.

Example 11.2.1

Approximate each value below without a calculator by using linearization

(a) $e^{.04}$

(b) $\sqrt{25.1}$

Solution

(a) First we need to write $e^{.04}$ as $f(b)$ for some b

The most obvious way to do so is to set $f(x) = e^x$ and $b = .04$

Next we need to choose a nice value of a that is nearby to $b = .04$

Clearly $a = 0$ works here, and is close to b, so we need to find the linearization function:

$L(x) = f(0) + f'(0)(x - 0)$

$f(x) = e^x \longrightarrow f(0) = e^0 = 1$

$f'(x) = e^x \longrightarrow f'(0) = e^0 = 1$

So $L(x) = 1 + 1(x - 0)$, and thus $e^{.04} = f(.04) \approx L(.04) = 1 + 1(.04 - 0) = 1.04$

(b) Set $f(x) = \sqrt{x}$ and $b = 25.1$

$a = 25$ works here, and is close to b, so we find the linearization function:

$L(x) = f(25) + f'(25)(x - 25)$

$f(x) = \sqrt{x} \longrightarrow f(25) = \sqrt{25} = 5$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \longrightarrow f'(25) = \frac{1}{2} \cdot 25^{-\frac{1}{2}} = \frac{1}{10}$

So $L(x) = 5 + \frac{1}{10}(x - 25)$, and thus $\sqrt{25.1} = f(25.1) \approx L(25.1) = 5 + \frac{1}{10}(25.1 - 25) = 5.01$

Just remember your differentiation rules when the function gets complicated.

Example 11.2.2

Estimate each value below without a calculator by using linearization.

(a) $\sqrt[3]{(2.96)^2 - 1}$

(b) $\sin^2\left(\frac{\pi}{4.01}\right)$

Solution

(a) Set $f(x) = \sqrt[3]{x^2 - 1}$ and $b = 2.96$

$a = 3$ works here, and is close to b, so we find $L(x) = f(3) + f'(3)(x - 3)$

$f(x) = \sqrt[3]{x^2 - 1} \longrightarrow f(3) = \sqrt[3]{3^2 - 1} = 2$

$f'(x) = \frac{1}{3}(x^2 - 1)^{-\frac{2}{3}} \cdot 2x \longrightarrow f'(3) = \frac{1}{3}(3^2 - 1)^{-\frac{2}{3}} \cdot 2 \cdot 3 = \frac{1}{2}$

So $L(x) = 2 + \frac{1}{2}(x - 3)$, and we get $\sqrt[3]{(2.96)^2 - 1} = f(2.96) \approx L(2.96) = 2 + \frac{1}{2}(2.96 - 3) = 1.98$

(b) Set $f(x) = \sin^2\left(\frac{\pi}{x}\right)$ and $b = 4.01$

$a = 4$ works here, and is close to b, we find $L(x) = f(4) + f'(4)(x - 4)$

$f(x) = \sin^2\left(\frac{\pi}{x}\right) \longrightarrow f(4) = \sin^2\left(\frac{\pi}{4}\right) = \frac{1}{2}$

$$f'(x) = 2\sin\left(\frac{\pi}{x}\right) \cdot \cos\left(\frac{\pi}{x}\right) \cdot \left(-\pi x^{-2}\right) = -2\sin\left(\frac{\pi}{x}\right)\cos\left(\frac{\pi}{x}\right) \cdot \frac{\pi}{x^2}$$

$$\longrightarrow f'(4) = -2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) \cdot \frac{\pi}{4^2} = -\frac{\pi}{16}$$

So $L(x) = \frac{1}{2} - \frac{\pi}{16}(x-4)$, and we get $\sin^2\left(\frac{\pi}{4.01}\right) = f(4.01) \approx L(4.01) = \frac{1}{2} - \frac{\pi}{16}(4.01-4) = \frac{800-\pi}{1600}$

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11.3: Extreme Values

Remember the Extreme Value Theorem: a continuous function on a closed interval must achieve its maximum and minimum values on that interval (either at its endpoints or at a critical point).

Lets find these extrema the function $f(x) = x^3 - 3x$ on the interval $[-3, 3]$:

We start by finding our critical points, so we differentiate: $f'(x) = 3x^2 - 3$

Remember that critical points are where the derivative equals zero or doesnt exist. Since this f' is a polynomial, we know it always exists and therefore only need to look when $f'(x) = 0$:

$3x^2 - 3 = 0 \longrightarrow x = \pm 1$

Now to help us visualize the graph, we will make a sign chart for f' on $[-3, 3]$:

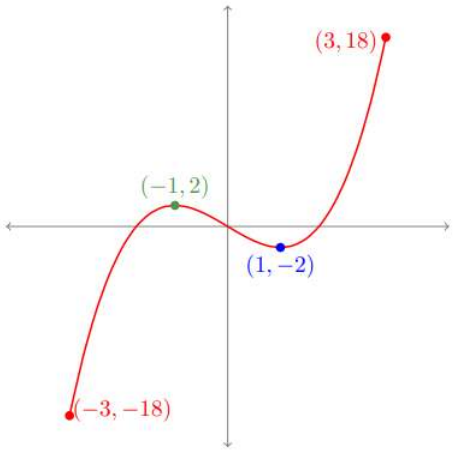


We can clearly see that the only possible extreme values can occur at the following points:

$f(-3) = (-3)^3 - 3 \cdot (-3) = -18 \longrightarrow (-3, -18)$
$f(-1) = (-1)^3 - 3 \cdot (-1) = 2 \longrightarrow (-1, 2)$
$f(1) = 1^3 - 3 \cdot 1 = -2 \longrightarrow (1, -2)$
$f(3) = 3^3 - 3 \cdot 3 = 18 \longrightarrow (3, 18)$

Here is where we find our answers: **maximum** at $(3, 18)$, **minimum** at $(-3, -18)$

But at $x = 1$ and $x = -1$ this graph still *acts like* it has a minimum and maximum respectively:



While not the largest or smallest overall value of f, these points are the largest and smallest when compared to only nearby points. We call these *local maximums* and *local minimums*.

When on a closed interval, the sign chart is not necessary to solve for extrema, (but it can be helpful if it is hard to tell which is bigger between two points, and will be a skill we rely on in the near future so its good practice!)

Example 11.3.1

Find the extrema for $g(x) = (x-8)\sqrt[3]{x}$ on $[-1, 8]$

Solution

First we rewrite $g(x) = x^{\frac{4}{3}} - 8x^{\frac{1}{3}}$

So $g'(x) = \frac{4}{3}x^{\frac{1}{3}} - \frac{8}{3}x^{-\frac{2}{3}} = \frac{4x-8}{3x^{\frac{2}{3}}}$

$g'(x) = 0$ occurs when $4x - 8 = 0 \longrightarrow x = 2$

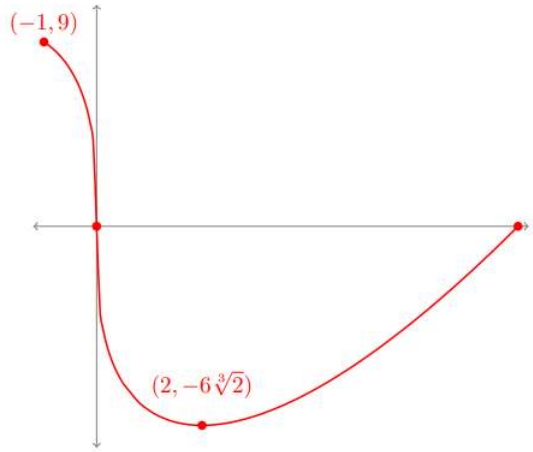
$g'(x) = DNE$ occurs when $3x^{\frac{2}{3}} = 0 \longrightarrow x = 0$

These critical points $x = 0, 2$ and our endpoints $x = -1, 8$ are the only candidates for an extreme value of $g(x)$ on this interval

$g(-1) = (-1-8)\cdot\sqrt[3]{-1} = 9$
$g(0) = (0-8)\cdot\sqrt[3]{0} = 0$
$g(2) = (2-8)\cdot\sqrt[3]{2} = -6\sqrt[3]{2}$
$g(8) = (8-8)\cdot\sqrt[3]{8} = 0$

We clearly see our **maximum** at $(-1, 9)$ and **minimum** at $(2, -6\sqrt[3]{2})$

Here is the graph (which we will learn to sketch soon) of the function in the above example:



In some situations, you may be given a function without an interval.

This may be because there is a natural domain to consider, like for $f(x) = x\sqrt{4-x^2}$

Clearly this function is only defined when $4 - x^2 \geq 0$, which is $x \in [-2, 2]$

Also $f'(x) = 1\cdot(4-x^2)^{\frac{1}{2}} + x\cdot\frac{1}{2}(4-x^2)^{-\frac{1}{2}}\cdot(-2x) = \frac{4-2x^2}{\sqrt{4-x^2}}$

So $f'(x) = 0$ when $4 - x^2 = 0 \longrightarrow x = \pm\sqrt{2}$

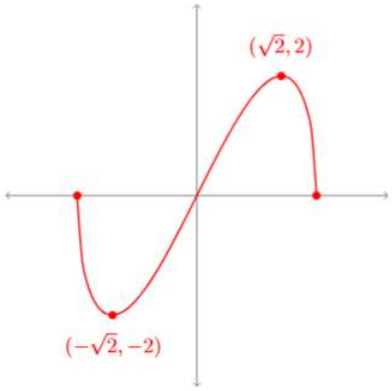
and $f'(x) = DNE$ when $\sqrt{4-x^2} = 0 \longrightarrow x = \pm 2$

Since our interval endpoints ($x = \pm 2$) are also critical points of f, we see these four are our only possible locations for extrema.

$f(-2) = -2 \cdot \sqrt{4-(-2)^2} = 0$
$f(-\sqrt{2}) = -\sqrt{2} \cdot \sqrt{4-(-\sqrt{2})^2} = -2$
$f(\sqrt{2}) = \sqrt{2} \cdot \sqrt{4-(\sqrt{2})^2} = 2$
$f(2) = 2 \cdot \sqrt{4-2^2} = 0$

So our **maximum** is at $(\sqrt{2}, 2)$ and **minimum** is at $(-\sqrt{2}, -2)$

Again here is the graph of $f(x)$ that confirms our findings:



As we can see, in these cases, there is no practical difference between being given a closed interval as a domain and discovering it yourself. The maximums and minimums of the function will occur at critical points or endpoints of the interval either way!

Of course not all functions have a closed interval as a domain, so this won't work for everything. If we are unlucky like this, then a sign chart will help immensely.

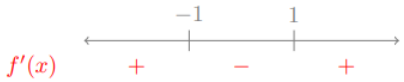
For example, lets find all local extrema for $f(x) = 4\tan^{-1}(x) - 2x$

This function has domain $(-\infty, \infty)$ which is not a closed interval.

Again we start by differentiating $f'(x) = \dfrac{4}{1+x^2} - 2 = \dfrac{2-2x^2}{1+x^2}$

We have $f'(x) = 0$ when $2 - 2x^2 = 0 \longrightarrow x = \pm 1$

And $f'(x) = DNE$ when $1 + x^2 = 0$ (which has no real solutions, so never)



Since we have no endpoints to check, we make a sign chart:
$f(-1) = 4\tan^{-1}(-1) - 2 \cdot (-1) = 2 - \pi$
Since $f'(x)$ changes from $+$ to $-$ at $x = -1$, we have a **local maximum** at $(-1, 2-\pi)$.

$f(1) = 4\tan^{-1}(1) - 2 \cdot 1 = \pi - 2$
Since $f'(x)$ changes from $-$ to $+$ at $x = 1$, we have a **local minimum** at $(1, \pi - 2)$.

The natural question is "are these two points the absolute maximum and minimum for $f(x)$ on $(-\infty, \infty)$?

The sign chart says we are increasing when $(1, \infty)$, so as we travel right, the values of $f(x)$ get larger, and the largest value in this interval would be reached as $x \to \infty$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 4\tan^{-1}(x) - 2x = 4\cdot\frac{\pi}{2} - 2\cdot\infty = -\infty$$

Similarly, the sign chart says we are increasing on $(-\infty, -1)$ so as we travel left, the values of $f(x)$ get smaller, and our smallest value in the interval would occur as $x \to -\infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 4\tan^{-1}(x) - 2x = 4\cdot\left(-\frac{\pi}{2}\right) - 2\cdot(-\infty) = +\infty$$

Clearly there is no biggest or smallest value for $f(x)$ on its domain, so we say it **does not** have an absolute maximum or minimum.

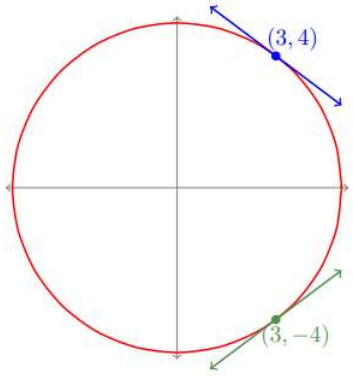
Remember the Extreme Value Theorem only promises an absolute maximum and minimum when the domain is a closed interval! If this is not the case, we can only look for local maximums and minimums (but they might be lucky enough to still be absolute extrema)

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11.4: Implicit differentiation

Although we know how to take derivatives of all types of *explicitly defined* functions, there are still other curves out there that have tangent lines!

For example, lets find the slopes of lines that are tangent to the circle $x^2+y^2=25$ when $x=3$:



Since the circle is not a function, there is more than one possible point to consider. Lets solve each one separately.

First, the point $(3,4)$ is on the top half of the circle, which we can consider as being on the function $y_{top}(x)=\sqrt{25-x^2}$

This gives $y'_{top}(x)=\frac{1}{2}(25-x^2)^{-\frac{1}{2}}\cdot(-2x)=\frac{-x}{\sqrt{25-x^2}}$

So the slope of the blue line is $y'_{top}(3)=\frac{-3}{\sqrt{25-3^2}}=-\frac{3}{4}$

Next, the point $(3,-4)$ is on the bottom half of the circle, which we can consider as being on the function $y_{bot}(x)=-\sqrt{25-x^2}$

This gives $y'_{bot}(x)=-\frac{1}{2}(25-x^2)^{-\frac{1}{2}}\cdot(-2x)=\frac{x}{\sqrt{25-x^2}}$

So the slope of the green line is $y'_{bot}(3)=\frac{3}{\sqrt{25-3^2}}=\frac{3}{4}$

These results seem to match with our picture, but notice that for each of the explicit functions y_{top} and y_{bot} there is a nice pattern in their derivatives:

$$y'_{top}(x)=-\frac{x}{\sqrt{25-x^2}}=-\frac{x}{y_{top}(x)} \text{ and } y'_{bot}(x)=\frac{x}{\sqrt{25-x^2}}=-\frac{x}{y_{bot}(x)}$$

This is not a coincidence!

Instead of solving for the top and bottom functions that defined the circle, we could have taken the derivative of the implicit equation. The way that it works, is you assume that y is a secret function of x, and take the derivative of the entire equation at once using our differentiation laws:

$$\left(x^2+[y(x)]^2=25\right)'$$

$$2x+2y(x)\cdot y'(x)=0$$

Then we can use algebra to solve for $y'(x)$ in terms of x and $y(x)$:

$$y'(x) = -\frac{x}{y(x)}$$

Notice that is the same as our two derivatives we solved previously! The best part about this, is that you never have to actually solve for the secret function $y(x)$. (Which is impossible to do anyway for some other curves you'll see).

This is *implicit differentiation*: taking the derivative of an implicitly defined curve. Of course, writing $y(x)$ everywhere can be tedious, so the short-hand way to do this looks like:

$$\left(x^2 + y^2 = 25\right)'$$

$$2x + 2y \cdot y' = 0$$

$$y' = -\frac{x}{y}$$

The important thing to remember, is that for implicit curves you need to have both the x and y values of the point you want to find the tangent line at (since we have seen on our circle that just the same x value may have multiple y values that satisfy the implicit equation)

Special note: When taking a derivative, you are finding the ratio of how much one variable changes when the other is increased by an infinitely small amount. Because of this, it is very important to keep track of which variable you are differentiating with respect to. The example above used the derivative with respect to variable x. When it is obvious which letter is the changing variable and which is the function, writing y' is clear. In the future, you may have more than two letters, variables, or functions in a problem, so feel free to write $y'(x)$ instead of y' if you want to stress the derivative of y is with respect to x!

Lets try a few quick examples.

Example 11.4.1

Find y' for each implicitly defined curve below.

(a) $x^2y^3 = \sin(y)$

(b) $\dfrac{e^{xy}}{x+2y} = x - 2y$

Solution

(a) Taking the implicit derivative, we see the left side starts with a product rule.

$$\left(x^2y^3 = \cos(y)\right)'$$

$$(x^2)' \cdot y^3 + x^2 \cdot (y^3)' = (\sin(y))'$$

$$2x \cdot y^3 + x^2 \cdot 3y^2 \cdot y' = \cos(y) \cdot y'$$

Now we do some algebra to solve for y':

$$2xy^3 = \cos(y) \cdot y' - 3x^2y^2 \cdot y'$$

$$2xy^3 = \left(\cos(y) - 3x^2y^2\right) y'$$

$$y' = \frac{2xy^3}{\cos(y) - 3x^2y^2}$$

(b) Before we differentiate, lets do some algebra to make this equation nicer:

$e^{xy} = (x-2y)\cdot(x+2y) \longrightarrow e^{xy} = x^2 - 4y^2$

Now we can take the implicit derivative more easily:

$\left(e^{xy} = x^2 - 4y^2\right)'$

$(e^{xy})' = (x^2)' - (4y^2)'$

$e^{xy}\cdot[1\cdot y + x\cdot y'] = 2x - 8y\cdot y'$

Then a bit more algebra to solve for y':

$ye^{xy} + xe^{xy}\cdot y' = 2x - 8y\cdot y'$

$xe^{xy}\cdot y' + 8y\cdot y' = 2x + ye^{xy}$

$(xe^{xy} + 8y)y' = 2x + ye^{xy}$

$y' = \dfrac{2x + ye^{xy}}{xe^{xy} + 8y}$

We can of course use these derivatives to find tangent lines.

Example 11.4.2

Find the line tangent to $\cos\left(\dfrac{\pi x}{y}\right) = y - 2x$ at $(3, 6)$

Solution

We start by taking the implicit derivative:

$\left(\cos\left(\dfrac{\pi x}{y}\right) = y - 2x\right)'$

$-\sin\left(\dfrac{\pi x}{y}\right)\cdot\left(\dfrac{\pi\cdot y - \pi x\cdot y'}{y^2}\right) = y' - 2$

Instead of solving for y' and then plugging in $x = 3, y = 6$

We can plug in those values and then solve for y' (which is easier)

$-\sin\left(\dfrac{\pi\cdot 3}{6}\right)\cdot\left(\dfrac{\pi\cdot 6 - \pi\cdot 3\cdot y'}{6^2}\right) = y' - 2$

$-\sin\left(\dfrac{\pi}{2}\right)\cdot\left(\dfrac{6\pi - 3\pi y'}{36}\right) = y' - 2$

$-1\cdot\left(\dfrac{6\pi - 3\pi y'}{36}\right) = y' - 2$

$6\pi - 3\pi y' = -36(y' - 2)$

$6\pi - 3\pi y' = -36y' + 72$

$6\pi - 72 = (3\pi - 36)y'$

$y' = \dfrac{6\pi - 72}{3\pi - 36} = 2$

Now we use the standard tangent line formula to get our answer: $y - 6 = 2\,(x - 3)$

While sketching this curve is not something we will learn how to do, we still have a way to find the equation of tangent lines to any given point (as long as the derivative exists at that point).

If the equation you get for y' after differentiating and plugging in the given x and y values ends up with no solution, that just means the curve is not differentiable at that point.

Notice that when given a specific point as we were above, it is often easier to solve for a tangent line than to find the formula for the implicit derivative in terms of the variables x and y.

Similarly, you can sometimes find a point on an implicit curve with a certain slope:

Example 11.4.3

What points(s) on each curve below have a horizontal tangent line?

(a) $x^2 + xy + y^2 = 12$

(b) $x = y^3 + 5y$

Solution

(a) $\left(x^2 + xy + y^2 = 12\right)'$

$2x + 1 \cdot y + x \cdot y' + 2y \cdot y' = 0$

We want horizontal tangent lines, so plug in $y' = 0$:

$2x + y + x \cdot 0 + 2y \cdot 0 = 0$

$y = -2x$

This tells us any point on this curve with $y' = 0$ must have $y = -2x$. So substitute this into the original implicit equation:

$x^2 + x(-2x) + (-2x)^2 = 12$

$3x^2 = 12 \longrightarrow x = \pm 2$

When $x = 2$: $y = -2 \cdot 2 = -4$, and $x = -2$: $y = -2 \cdot (-2) = 4$

So our final answers are $(2, -4)$ and $(-2, 4)$

(b) $(x = y^3 + 5y)'$

$1 = 3y^2 \cdot y' + 5y'$

Again we want a horizontal tangent line, so $y' = 0$

Plugging in $y' = 0$ makes this equation into: $1 = 0$

There are no solutions to this equation, which means **there are no points on this curve with $y' = 0$**

Example 11.4.4

Which points on $x^2 + 2x + y^2 - 4y = 20$ have a tangent line which is parallel to the line $3x + 4y = 0$?

Solution

We start once again by taking the implicit derivative:

$$\left(x^2 + 2x + y^2 - 4y = 20\right)'$$

$$2x + 2 + 2y \cdot y' - 4y' = 0$$

The given line $3x + 4y = 0$ has slope $m = -\frac{3}{4}$, so we want $y' = -\frac{3}{4}$

$$2x + 2 + 2y \cdot \left(-\frac{3}{4}\right) - 4 \cdot \left(-\frac{3}{4}\right) = 0$$

$$2x + 5 - \frac{3}{2}y = 0$$

$$y = \frac{4}{3}x + \frac{10}{3}$$

Once more we substitute this relationship into the original curve:

$$x^2 + 2x + \left(\frac{4}{3}x + \frac{10}{3}\right)^2 - 4\left(\frac{4}{3}x + \frac{10}{3}\right) = 20$$

$$x^2 + 2x + \frac{16}{9}x^2 + \frac{80}{9}x + \frac{100}{9} - \frac{16}{3}x - \frac{40}{3} = 20$$

$$\frac{25}{9}x^2 + \frac{50}{9}x - \frac{200}{9} = 0$$

$$\frac{25}{9}(x^2 + 2x - 8) = 0$$

$$\frac{25}{9}(x + 4)(x - 2) = 0$$

When $x = -4$ we have $y = \frac{4}{3} \cdot (-4) + \frac{10}{3} = \frac{-6}{3} = -2$

And $x = 2$ gives $y = \frac{4}{3} \cdot 2 + \frac{10}{3} = \frac{18}{3} = 6$

So our **final answers are the points $(-4, -2)$ and $(2, 6)$**

Implicit differentiation allows us to answer derivative and tangent line questions for pretty much any curve we are given. Just

remember your explicit differentiation rules and dont forget the chain rule that yields y' any time you take the derivative of a y term.

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11.5: L'Hopital's Rule

With all this talk of derivatives, it is easy to lose sight of the reason they exist: **limits**.

When we are finding the derivative of $f(x)$ at $x=a$, we are taking: $\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

This limit is one example of the more general situation: $\lim_{x\to a}\dfrac{f(x)}{g(x)}$

The results of this type of limit (when we know $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist) clearly depend on these individual limits of the top and bottom functions.

We break things down into a few cases, starting with the easiest:

Case 1: Assume $\lim_{x\to a} g(x) = G$ equals some real number that isnt zero.

Case 1a: $\lim_{x\to a} f(x) = 0$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{0}{G} = 0$$

Case 1b: $\lim_{x\to a} f(x) = F$ for some real number F that isnt zero.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{F}{G} = \text{ some real number (that isnt zero)}$$

Case 1c: $\lim_{x\to a} f(x) = \infty$ or $-\infty$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{\pm\infty}{G} = \pm\infty$$

Here we write $\pm\infty$ to denote the limit could be either positive or negative infinity (notice this can depend on if G is positive or negative as well)

A crude summary of what we discovered here:

$$\frac{0}{\#} = 0 \quad \frac{\#}{\#} = 0 \quad \frac{\pm\infty}{\#} = \pm\infty$$

Next up, **Case 2:** Assume $\lim_{x\to a} g(x) = \pm\infty$

Case 2a: $\lim_{x\to a} f(x) = 0$

$\lim_{x\to a}\dfrac{f(x)}{g(x)}$ is numbers getting closer to zero being divided by bigger and bigger values.
*This clearly means the overall fraction is going to **zero** even faster.*

Case 2b: $\lim_{x\to a} f(x) = F$ for some real number F that isnt zero.

$\lim_{x\to a}\dfrac{f(x)}{g(x)}$ is numbers getting closer to F being divided by bigger and bigger values.
This also means *the overall fraction is going to **zero***.

Case 2c: $\lim_{x\to a} f(x) = \pm\infty$

This case of $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ is very interesting.

We have seen examples before like $\lim_{x\to-\infty}\frac{2x^2+1}{x^2-5}$ and $\lim_{x\to\infty}\frac{e^x}{1+e^x}$ or even $\$\lim_{t\to0^+}\frac{\ln(t)+2}{(\ln(t))^2}$

In case you forgot, for each of these, we use the **Lebron James Rule**:

$\lim_{x\to-\infty}\frac{2x^3+1}{x^2-5}=\lim_{x\to-\infty}\frac{2x^3}{x^2}=\lim_{x\to-\infty}2x=-\infty$
The biggest powers of a polynomial are the terms that grow the fastest to infinity.

$\lim_{x\to\infty}\frac{e^x}{1+e^x}=\lim_{x\to\infty}\frac{e^x}{e^x}=\lim_{x\to\infty}1=1$
Since $e^{+\infty}=+\infty$, it grows faster than the constant 1 (which can be ignored)

$\lim_{t\to0^+}\frac{\ln(t)+2}{(\ln(t))^2}=\lim_{t\to0^+}\frac{\ln(t)}{(\ln(t))^2}\lim_{t\to0^+}\frac{1}{\ln(t)}=\frac{1}{-\infty}=0$
Since $\ln(0^+)=-\infty$, it grows faster than the constant 2 (which can be ignored)

Clearly in Case 2c, any outcome is possible. **It depends on what "size" of infinity each of the limits for $f(x)$ and $g(x)$ are, and how they compare to each other.**

Another crude summary of what we discovered here:

$\frac{0}{\pm\infty}=0$ $\frac{\#}{\pm\infty}=0$ $\frac{\pm\infty}{\pm\infty}=$ *needs further investigation*

Finally **Case 3:** $\lim_{x\to a}g(x)=0$

Case 3a: $\lim_{x\to a}f(x)=F$ for some real number F that isn't zero.

Since $g(x)$ approaches zero, the limit laws do not apply to $\lim_{x\to a}\frac{f(x)}{g(x)}$

But we know that $\frac{F}{0^+}=\pm\infty$ and $\frac{F}{0^-}=\pm\infty$, so we can say that this limit will always be infinite *if it exists*.

Case 3b: $\lim_{x\to a}f(x)=\pm\infty$

This time $\lim_{x\to a}\frac{f(x)}{g(x)}$ represents taking a larger and larger number and dividing it by a smaller and smaller positive or negative number will always make it even bigger (or bigger and negative).

Case 3c: $\lim_{x\to a}f(x)=0$

This is once again the interesting case. We have seen limits of the form $\frac{0}{0}$ throughout our studies in calculus. (Remember the derivative is typically one such example of this form).

We have seen examples before like $\lim_{x\to0}\frac{6x}{x^3}$ and $\lim_{w\to2}\frac{w^2-w-2}{w^2-4}$ or even $\lim_{h\to0}\frac{h^2}{\sqrt{1+h}-1}$

Each of these is different:

$\lim_{x\to0^+}\frac{6x}{x^3}=\lim_{x\to0^+}\frac{6}{x^2}=\frac{6}{(0^+)^2}=\frac{6}{0^+}=\infty$
$\lim_{x\to0^-}\frac{6x}{x^3}=\lim_{x\to0^-}\frac{6}{x^2}=\frac{6}{(0^-)^2}=\frac{6}{0^+}=\infty$ so $\lim_{x\to0}\frac{6x}{x^3}=\infty$
When zeros in the denominator dont reduce away, we check both sides.

$$\lim_{w\to 2}\frac{w^2-w-2}{w^2-4}=\lim_{w\to 2}\frac{(w-2)(w+1)}{(w-2)(w+2)}=\lim_{w\to 2}\frac{w+1}{w+2}=\frac{2+1}{2+2}=\frac{3}{4}$$

Factor and reduce works great when everything cancels out though!

$$\lim_{h\to 0}\frac{h^2}{\sqrt{1+h}-1}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}=\lim_{h\to 0}\frac{h^2(\sqrt{1+h}+1)}{(\sqrt{1+h})^2-1^2}=\lim_{h\to 0}h(\sqrt{1+h}+1)=0$$

Sadly, factoring and reducing sometimes requires clever algebraic tricks (like conjugates)

Again in Case 3c, any outcome is possible. **It depends on on what side of zero they are approaching from, how "quickly" each of the limits for $f(x)$ and $g(x)$ are approaching zero, and how they compare to each other.**

Our last crude summary of what we discovered here:

$$\frac{\#}{0}=\pm\infty\quad\frac{\pm\infty}{0}=\pm\infty\quad\frac{0}{0}=\textit{needs further investigation}$$

The two cases that needed further investigation were the $\frac{0}{0}$ and $\frac{\infty}{\infty}$ limits.

These ambiguous cases are commonly referred to as **indeterminate forms**. They are limits that you cannot evaluate without doing a little bit of legwork.

The good news is that they both can be solved in the same way. When the limit of the top and bottom function are both zero, we care about how "quickly" they approach zero.

When the limit of the top and bottom were both infinite, we cared about the "fastest growing" terms. These both point to the same thing: the rate of change of this function near the limit. That is the intuition behind **L'Hopital's Rule**: $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ for these two cases.

The actual proof of why L'Hopital's Rule works relies on the Mean Value Theorem, and is a bit tedious due to how many different styles of limit it applies to. You can safely believe that it works for any one-sided or double-sided limits where the top and bottom are both zero or both infinite and the resulting limit exists!

Look at how it works on those $\frac{\infty}{\infty}$ examples we saw previously:

$$\lim_{x\to-\infty}\frac{2x^3+1}{x^2-5}\overset{LH}{=}\lim_{x\to-\infty}\frac{6x^2}{2x}=\lim_{x\to-\infty}3x=-\infty$$

$$\lim_{x\to\infty}\frac{e^x}{1+e^x}\overset{LH}{=}\lim_{x\to\infty}\frac{e^x}{e^x}=\lim_{x\to\infty}1=1$$

$$\lim_{t\to 0^+}\frac{\ln(t)+2}{(\ln(t))^2}\overset{LH}{=}\lim_{t\to 0^+}\frac{\frac{1}{t}}{2\ln(t)\cdot\frac{1}{t}}=\lim_{t\to 0^+}\frac{1}{2\ln(t)}=\frac{1}{-\infty}=0$$

And see that it works on those $\frac{0}{0}$ examples we saw as well:

$$\lim_{x\to 0^+}\frac{6x}{x^3}\overset{LH}{=}\lim_{x\to 0^+}\frac{6}{3x^2}=\frac{6}{3(0^+)^2}=\frac{6}{0^+}=\infty$$
$$\lim_{x\to 0^-}\frac{6x}{x^3}\overset{LH}{=}\lim_{x\to 0^-}\frac{6}{3x^2}=\frac{6}{3(0^-)^2}=\frac{6}{0^+}=\infty$$
so $\lim_{x\to 0}\frac{6x}{x^3}=\infty$

$$\lim_{w\to 2}\frac{w^2-w-2}{w^2-4}\overset{LH}{=}\lim_{w\to 2}\frac{2w-1}{2w}=\frac{2\cdot 2-1}{2\cdot 2}=\frac{3}{4}$$

$$\lim_{h\to 0}\frac{h^2}{\sqrt{1+h}-1}\overset{LH}{=}\lim_{h\to 0}\frac{2h}{\frac{1}{2}(1+h)^{-\frac{1}{2}}}=\lim_{h\to 0}4h\sqrt{1+h}=0$$

There are some limits where L'Hopital is your only chance to evaluate:

Example 11.5.1

Use L'Hopital's Rule to evaluate each limit below.

(a) $L=\lim_{x\to 0}\dfrac{e^x-1}{\sin(x)}$

(b) $L=\lim_{t\to\infty}\dfrac{\ln(t)}{\sqrt{t}}$

(c) $L=\lim_{y\to 0}\dfrac{\cos(3y)-1}{e^y-1-y}$

Solution

(a) We see $\lim_{x\to 0}e^x-1=e^0-1=0$ and $\lim_{x\to 0}\sin(x)=\sin(0)=0$

The original limit clearly goes to $\dfrac{0}{0}$, so we can use L'Hopital:

$$L\overset{LH}{=}\lim_{x\to 0}\frac{e^x}{\cos(x)}=\frac{e^0}{\cos(0)}=\frac{1}{1}=1$$

(b) We see $\lim_{t\to\infty}\ln(t)=+\infty$ and $\lim_{t\to\infty}\sqrt{t}=+\infty$

The original limit clearly goes to $\dfrac{\infty}{\infty}$, so we can use L'Hopital:

$$L\overset{LH}{=}\lim_{t\to\infty}\frac{\frac{1}{t}}{\frac{1}{2}t^{-\frac{1}{2}}}=\lim_{t\to\infty}\frac{2}{t^{\frac{1}{2}}}=\frac{2}{\infty}=0$$

(c) We see $\lim_{y\to 0}\cos(3y)-1=\cos(0)-1=0$ and $\lim_{y\to 0}e^y-1-y=e^0-1-0=0$

The original limit clearly goes to $\dfrac{0}{0}$, so we can use L'Hopital:

$$L\overset{LH}{=}\lim_{y\to 0}\frac{-3\sin(3y)}{e^y-1}$$

Now $\lim_{y\to 0}\sin(3y)=\sin(0)=0$ and $\lim_{y\to 0}e^y-1=e^0-1=0$

This is **still zero over zero**, so we can *use L'Hopital again*:

$$L\overset{LH}{=}\lim_{y\to 0}\frac{-9\cos(3y)}{e^y}=\frac{-9\cos(0)}{e^0}=-9$$

You can use L'Hopital's Rule as many times as is necessary **as long as the limit is $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$**

There are a few other indeterminate forms that we can use L'Hopital's Rule for (as long as we do a little bit of algebra to manipulate them into $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ first).

For example, $0 \cdot c = 0$ for any real number c, but we cant say the same about $0 \cdot \infty$:

Consider $L = \lim_{x \to \infty} x^2 e^{-3x}$

We know that $\lim_{x \to \infty} x^2 = +\infty$ and $\lim_{x \to \infty} e^{-3x} = 0$, so we try to re-write the original limit as a fraction. There are two straightforward ways to do so:

$\frac{x^2}{e^{3x}}$ (which becomes $\frac{\infty}{\infty}$) and $\frac{e^{-3x}}{x^{-2}}$ (which becomes $\frac{0}{0}$).

Lets see how L'Hopital reacts to either choice:

$$L = \lim_{x \to \infty} \frac{x^2}{e^{3x}} \overset{LH}{=} \lim_{x \to \infty} \frac{2x}{3e^{3x}} \overset{LH}{=} \lim_{x \to \infty} \frac{2}{9e^{3x}} = \frac{2}{\infty} = 0$$

This is the correct answer and we reach it relatively painlessly.

$$L = \lim_{x \to \infty} \frac{e^{-3x}}{x^{-2}} \overset{LH}{=} \lim_{x \to \infty} \frac{-3e^{-3x}}{-2x^{-3}} \overset{LH}{=} \lim_{x \to \infty} \frac{9e^{-3x}}{6x^{-4}} \overset{LH}{=} \lim_{x \to \infty} \frac{-27e^{-3x}}{-24x^{-5}} = \cdots$$

This second choice is technically correct, but does not lead you to an answer. Instead the limit gets worse and worse with every iteration of L'Hopital's Rule.

Always try to pick the fractional version of your function who would **benefit** from a derivative.
(If you try one version and things are looking worse and worse, try doing the reciprocal instead.)

Example 11.5.2

Evaluate $L = \lim_{t \to 0^+} t \log_2(t)$

Solution

We know $\lim_{t \to 0^+} t = 0$ and $\lim_{t \to 0^+} \log_2(t) = -\infty$

If you try $L = \lim_{t \to 0^+} \frac{t}{(\log_2(t))^{-1}} \overset{LH}{=} \lim_{t \to 0^+} \frac{1}{-(\log_2(t))^{-2} \cdot \frac{1}{t \ln(2)}} = \lim_{t \to 0^+} -t(\log_2(t))^2 \cdot \ln(2)$ it gets worse!

Instead we go with the reciprocal and get:

$$L = \lim_{t \to 0^+} \frac{\log_2(t)}{t^{-1}} \overset{LH}{=} \lim_{t \to 0^+} \frac{\frac{1}{t \ln(2)}}{-t^{-2}} = \lim_{t \to 0^+} \frac{-t}{\ln(2)} = \frac{0}{\ln(2)} = 0$$

Another one that may look familiar is a limit of the form $\infty - \infty$. We have seen previously that the "bigger" infinity does all the work, but when it is not immediately obvious which is bigger, we can combine the expression into one fraction and use L'Hopital:

For example, $L = \lim_{x \to 0^+} \frac{1}{x} - \csc(x)$

We know $\lim_{x \to 0^+} \frac{1}{x} = \frac{1}{0^+} = +\infty$ and $\lim_{x \to 0^+} \csc(x) = +\infty$, so we write

$$L = \lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin(x)} = \lim_{x \to 0^+} \frac{\sin(x) - x}{x \sin(x)} \overset{LH}{=} \lim_{x \to 0^+} \frac{\cos(x) - 1}{1 \cdot \sin(x) + x \cdot \cos(x)} = \lim_{x \to 0^+} \frac{\cos(x) - 1}{\sin(x) + x \cos(x)}$$

Notice $\lim_{x \to 0^+} \cos(x) - 1 = 0$ and $\lim_{x \to 0^+} \sin(x) + x \cos(x) = 0$, so again:

$$\overset{LH}{=} \lim_{x\to 0^+} \frac{-\sin(x)}{\cos(x)+1\cdot\cos(x)+x\cdot(-\sin(x))} = \lim_{x\to 0^+} \frac{-\sin(x)}{2\cos(x)-x\sin(x))} = \frac{-0}{2-0} = 0$$

It turns out that both of these functions approach positive infinity on the right side of $x=0$ in almost the exact same way, so their difference goes to zero.

Example 11.5.3

Evaluate $L = \lim_{t\to\infty} t - \ln(3^t+1)$

Solution

We see $L = \lim_{t\to\infty} t = \infty$ and $\lim_{t\to\infty} \ln(3^t+1) = +\infty$

So we re-write the original limit in a way that combines these together: $L = \lim_{t\to\infty} \ln(e^t) - \ln(3^t+1) = \lim_{t\to\infty} \ln\left(\frac{e^t}{3^t+1}\right)$

Remember that $\ln(x)$ is a continuous function everywhere, so we can pass the limit through:

$$L = \ln\left(\lim_{t\to\infty} \frac{e^t}{3^t+1}\right) \overset{LH}{=} \lim_{t\to\infty} \ln\left(\frac{e^t}{3^t\ln(3)}\right)$$

This can be re-written to avoid $\frac{\infty}{\infty}$ and using more L'Hopital's Rule:

$$= \lim_{t\to\infty} \ln\left(\frac{1}{\left(\frac{3}{e}\right)^t \ln(3)}\right) = \ln\left(\frac{1}{+\infty}\right) = \ln(0^+) = -\infty$$

The most important skill for limits is *still re-writing them*, but L'Hopital's Rule gives you options when you get stuck in certain spots!

The last two indeterminate forms of limits are 1^∞ and ∞^0

To evaluate a limit of one of these two forms directly is difficult, but you can always use the log properties to first reduce them to one of our previous cases:

For example, you may recall the limit $L = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

If we start by applying ln to both sides and passing the limit through to get:

$$\ln(L) = \ln\left[\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n\right] = \lim_{n\to\infty} \ln\left[\left(1+\frac{1}{n}\right)^n\right]$$

And now the properties of the logarithm help us bring down the power to the front:

$$\ln(L) = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln(1+n^{-1})}{n^{-1}} \overset{LH}{=} \lim_{n\to\infty} \frac{\frac{1}{1+n^{-1}}\cdot(-n^{-2})}{-n^{-2}}$$

$$= \lim_{n\to\infty} \frac{1}{1+n^{-1}} = \lim_{n\to\infty} \frac{n}{n+1} \overset{LH}{=} \lim_{n\to\infty} \frac{1}{1} = 1$$

Since $\ln(L) = 1$, the value of the original limit $L = e^1 = e$

Remember this was the definition of e that we described when we first discussed its derivative!

Example 11.5.4

Evaluate $L = \lim_{x\to\infty} (x^2+1)^{\frac{1}{x}}$

Solution

Again we start by applying ln and passing through the limit:

$$\ln(L) = \lim_{x\to\infty} \ln\left[(x^2+1)^{\frac{1}{x}}\right] = \lim_{x\to\infty} \frac{1}{x}\ln(x^2+1)$$

$$= \lim_{x\to\infty} \frac{\ln(x^2+1)}{x} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{x^2+1}\cdot 2x}{1} = \lim_{x\to\infty} \frac{2x}{x^2+1} \overset{LH}{=} \lim_{x\to\infty} \frac{2}{2x} = \frac{2}{\infty} = 0$$

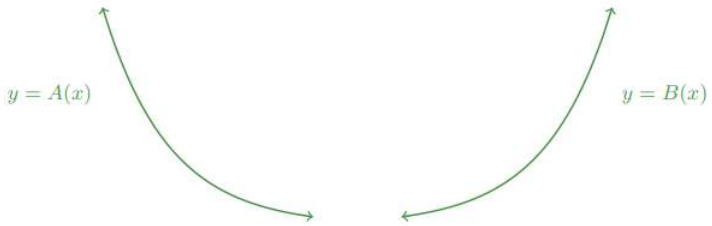
Since $\ln(L) = 0$, then $L = e^0 = 1$

L'Hopital's Rule is a powerful tool for taking limits that end up as indeterminate forms. The goal is always the same: re-write your limit so that it can be expressed as $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then differentiate the top and bottom in hopes of making the limit easier.

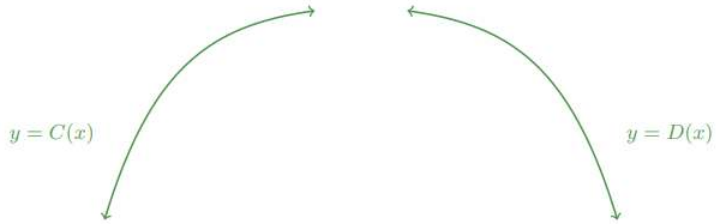
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11.6: Curve Sketching

Most differentiable functions are made out of four different pieces all connected in various locations and various orders.



We say these two curves above with positive second derivative are **concave up.**



We say these two curves above with negative second derivative are **concave down.**

As long as you know where your function has a positive or negative first derivative (where it is *increasing and decreasing*), and you know where it has positive or negative second derivative (where it is *concave up or concave down*), then you can tell what shape the curve makes there.

Remember **critical points** (where $f'(x) = 0$ or DNE) separated where a function is increasing and decreasing. Similarly, we say that where $f''(x) = 0$ or DNE we have an **inflection point**, which separates where f'' can change signs and make the concavity change.

Memorizing these four shapes will help us to draw almost any function (as long as we can differentiate it and handle any algebra that gets in our way).

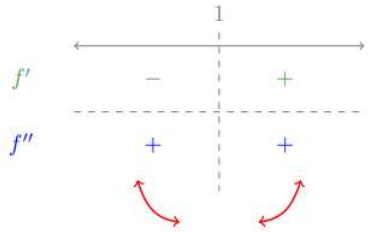
1. Find $f'(x)$ and critical points ($f'(x) = 0$ or $f'(x) =$DNE)
2. Find $f''(x)$ and inflection points ($f''(x) = 0$ or $f''(x) =$DNE)
3. Create a sign chart for both of f' and f''
4. Plot any inflection or critical points.
5. Consider any horizontal or vertical asymptotes the graph may have.
6. Connect the dots.

Lets start with an easy example, $f(x) = x^2 - 2x$

For step 1, we differentiate: $f'(x) = 2x - 2$.
This always exists and $2x - 2 = 0 \longrightarrow x = 1$
So the critical points are just $x = 1$

For step 2, we differentiate again: $f''(x) = 2$
This always exists and $2 = 0 \longrightarrow \emptyset$ is never satisfied.
So there are no inflection points.

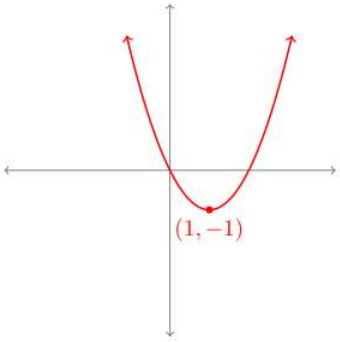
Step 3 is to make a sign chart, we make a dividing line for all critical and inflection points.
Once we fill in the $+$ and $-$, below we draw which of the four shapes go there.



Step 4, we plug in $x = 1$ to find our height: $f(1) = 1^2 - 2 \cdot 1 = -1$

There are no asymptotes for polynomials, so we can skip Step 5.

Lastly we draw our sketch by plotting the point $(1, -1)$ and using the curves from the sign chart:



Lets try another polynomial before making things more difficult:

Example 11.6.1

Sketch $f(x) = 3x^4 - 4x^3 - 12x^2 + 16$

Solution

Start with $f'(x) = 12x^3 - 12x^2 - 24x$

$12x^3 - 12x^2 - 24x = 0 \longrightarrow 12x(x^2 - x - 2) = 0 \longrightarrow 12x(x-2)(x+1) = 0 \longrightarrow x = 0, 2, -1$

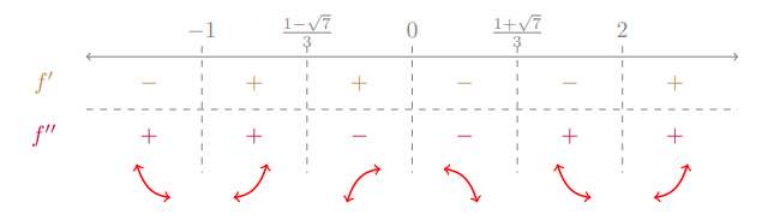
f' always exists, so these are all of the critical points.

Next we see $f''(x) = 36x^2 - 24x - 24$

$36x^2 - 24x - 24 = 0 \longrightarrow 12(3x^2 - 2x - 2) = 0 \longrightarrow x = \dfrac{1 \pm \sqrt{7}}{3}$ (quadratic formula)

f'' always exists, so these are our two inflection points.

Now we put these all *in order* on the sign chart again:



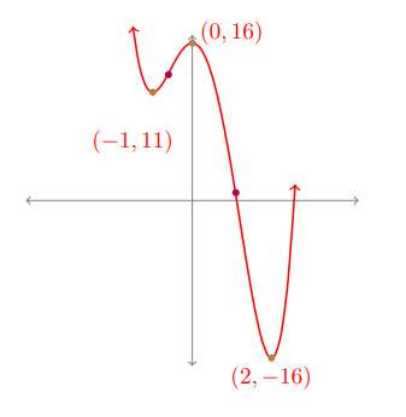
Plugging each of these critical points in give us:

$f(-1) = 3 \cdot (-1)^4 - 4 \cdot (-1)^3 - 12 \cdot (-1)^2 + 16 = 11$

$f(0) = 3 \cdot 0^4 - 4 \cdot 0^3 - 12 \cdot 0^2 + 16 = 16$

$f(2) = 3 \cdot 2^4 - 4 \cdot 2^3 - 12 \cdot 2^2 + 16 = -16$

The inflection points look a bit too ugly to find the exact height of, and there are no asymptotes since this is again a polynomial. Lets sketch:



Lets try a rational function.

Example 11.6.2

Sketch $g(x) = \dfrac{2x}{x^2 - 4}$

Solution

First derivative: $g'(x) = \dfrac{(x^2-4)\cdot 2 - 2x \cdot (2x)}{(x^2-4)^2} = \dfrac{-2x^2 - 8}{(x^2-4)^2}$

$g'(x) = 0 \longrightarrow -2x^2 - 8 = 0 \longrightarrow \emptyset$

$g'(x) = DNE \longrightarrow (x^2-4)^2 = 0 \longrightarrow x = \pm 2$

CP: $x = 2, -2$

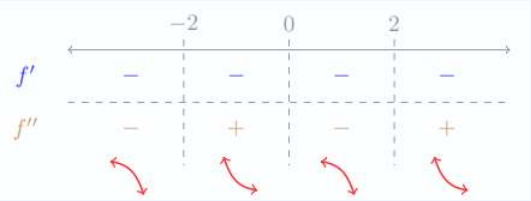
Second derivative: $g''(x) = \dfrac{(x^2-4)^2 \cdot (-4x) - (-2x^2-8) \cdot \left[2 \cdot (x^2-4) \cdot 2x\right]}{(x^2-4)^4} = \dfrac{4x^3 + 48x}{(x^2-4)^3}$

$g''(x) = 0 \longrightarrow 4x^3 + 48x = 0 \longrightarrow 4x(x^2+12) = 0 \longrightarrow x = 0$

$g''(x) = DNE \longrightarrow (x^2-4)^3 = 0 \longrightarrow x = \pm 2$

IP: $x = 0, 2, -2$

Once again we make a sign chart and determine which shape goes to each region:



Plugging in $x = 0$ is simple: $g(0) = \dfrac{2 \cdot 0}{0^2 - 4} = 0$

$g(2)$ and $g(-2)$ do not exist, but limits on each side determine how our graph acts near each.

$\lim_{x\to 2^+} g(x) = \frac{4}{0^+} = +\infty$ and $\lim_{x\to 2^-} g(x) = \frac{4}{0^-} = -\infty$
so **vertical asymptote at** $x = 2$

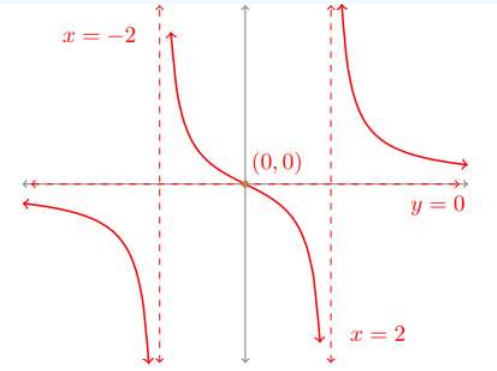
$\lim_{x\to -2^+} g(x) = \frac{-4}{0^-} = +\infty$ and $\lim_{x\to -2^-} g(x) = \frac{-4}{0^+} = -\infty$
so **vertical asymptote at** $x = -2$

While we are at it, lets check for horizontal asymptotes:

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{2x}{x^2} = \lim_{x\to\infty} \frac{2}{x} = \frac{2}{\infty} = 0$$
$$\lim_{x\to -\infty} g(x) = \lim_{x\to -\infty} \frac{2x}{x^2} = \lim_{x\to -\infty} \frac{2}{x} = \frac{2}{-\infty} = 0$$

So **horizontal asymptote at** $y = 0$

Finally we draw all this and connect the dots:



Lets try an exotic one.

Example 11.6.3

Sketch $h(x) = \frac{2e^x + 3e}{e^x - e}$

Solution

$h'(x) = \frac{-5e \cdot e^x}{(e^x - e)^2}$

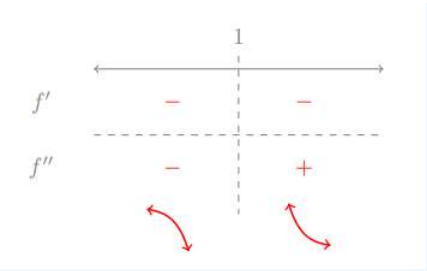
Setting top and bottom equal to zero, we get critical points: $x = 1$

$h''(x) = \frac{5e \cdot e^x (e^x + e)}{(e^x - e)^3}$

Setting top and bottom equal to zero, we get inflection points: $x = 1$

Lots of algebra but just quotient rule then set the top and bottom equal to zero

Now a sign chart:



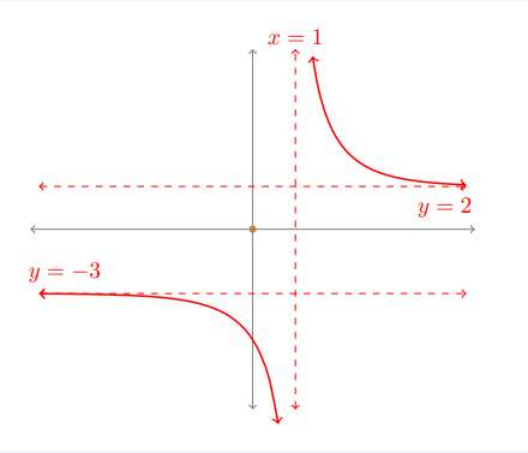
$h(1) = \dfrac{2e^1 + 3e}{e^1 - e} = \dfrac{5e}{0} = DNE$ so there are no points to plot

$\lim\limits_{x \to 1^+} h(x) = \dfrac{5e}{0^+} = +\infty$ and $\lim\limits_{x \to 1^-} h(x) = \dfrac{5e}{0^-} = -\infty$
Therefore there is a **vertical asymptote at** $x = 1$

$\lim\limits_{x \to \infty} h(x) \overset{LH}{=} \lim\limits_{x \to \infty} \dfrac{2e^x}{e^x} = \lim\limits_{x \to \infty} 2 = 2$

$\lim\limits_{x \to -\infty} h(x) = \dfrac{2 \cdot 0 + 3e}{0 - e} = \dfrac{3e}{-e} = -3$
Notice: for $x \to -\infty$, you cannot use L'Hopital's Rule since it is not an indeterminate form

This means there are **horizontal asymptotes at** $y = 2$ **and** $y = -3$



Some functions have a natural domain to consider.

Example 11.6.4

Sketch $y = x\sqrt{4 - 2x^2}$

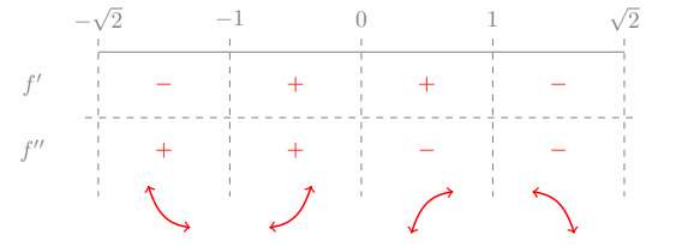
Solution

$y' = \dfrac{4 - 4x^2}{\sqrt{4 - 2x^2}}$, means we have critical points: $x = \pm 1, \pm\sqrt{2}$

$y'' = \dfrac{8x^3 - 24x}{(4 - 2x^2)^{\frac{3}{2}}}$, means we have inflection points; $x = 0, \pm\sqrt{2}, \pm\sqrt{3}$

The original function only exists when $4 - 2x^2 \geq 0$, or $x \in \left[-\sqrt{2}, \sqrt{2}\right]$

So we ignore everything outside of this domain for our sign chart:



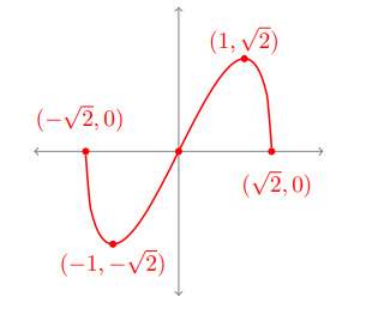
We plug in each critical and inflection point:

$$y(\sqrt{2}) = \sqrt{2} \cdot \sqrt{4 - 2 \cdot (\sqrt{2})^2} = 0$$
$$y(1) = 1 \cdot \sqrt{4 - 2 \cdot 1^2} = \sqrt{2}$$
$$y(0) = 0 \cdot \sqrt{4 - 2 \cdot 0^2} = 0$$
$$y(-1) = -1 \cdot \sqrt{4 - 2 \cdot (-1)^2} = -\sqrt{2}$$
$$y(-\sqrt{2}) = -\sqrt{2} \cdot \sqrt{4 - 2 \cdot (-\sqrt{2})^2} = 0$$

This is continuous on its domain, so no vertical asymptotes.

Since the domain is closed, there is no limit to either infinity (so no horizontal asymptotes either)

We just plot each point and connect the dots:



Just remember to use limits whenever anything gets weird!

Example 11.6.5

Sketch $f(x) = \tan^{-1}\left(\frac{1}{x^2}\right)$

Solution

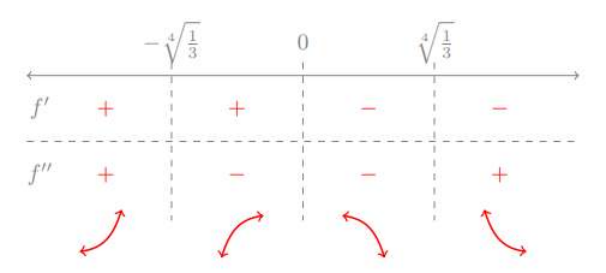
$$f'(x) = \frac{1}{1 + (x^{-2})^2} \cdot (-2x^{-3}) = -\frac{2x}{x^4 + 1}$$

So we have critical point $x = 0$

$$f''(x) = \frac{6x^4 - 2}{(x^4 + 1)^2}$$

So there are inflection points $x = \pm\sqrt[4]{3}$

Even though $x = 0$ is not in the domain of f, the derivative exists on either side of it so we will include it in the sign chart (and explore exactly what happens with limits shortly).



We have a discontinuity at $x = 0$, but what kind?

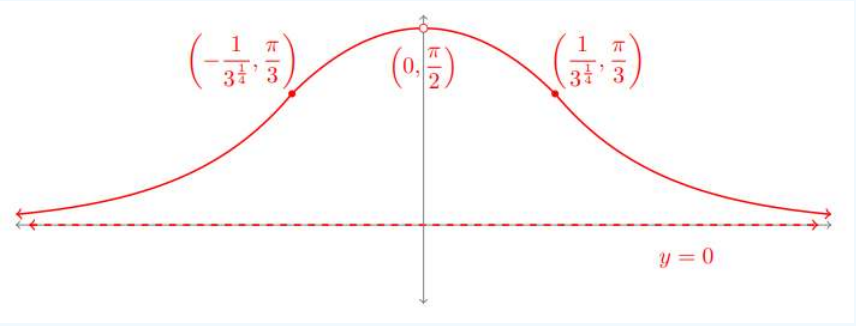
$$\lim_{x\to 0} f(x) = \tan^{-1}\left(\frac{1}{0^+}\right) = \tan^{-1}(+\infty) = \frac{\pi}{2}$$

$$f\left(\pm\sqrt[4]{\frac{1}{3}}\right) = \tan^{-1}\left(\frac{1}{\left(\pm 3^{-\frac{1}{4}}\right)^2}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

There is a removable discontinuity at $x = 0$, forming a **hole at** $\left(0, \frac{\pi}{2}\right)$

$$\lim_{x\to\infty} f(x) = \tan^{-1}\left(\frac{1}{\infty}\right) = \tan^{-1}(0) = 0$$
$$\lim_{x\to-\infty} f(x) = \tan^{-1}\left(\frac{1}{\infty}\right) = \tan^{-1}(0) = 0$$

This means we have a **horizontal asymptote at** $y = 0$



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CHAPTER OVERVIEW

12: Word Problems with Derivatives

> ”If your only tool is a hammer, every problem looks like a nail” - Abraham Maslow

Most of the mathematical applications we saw in the previous chapter have a real world application. Chapter twelve is focused primarily on these types of questions. We start with a brief demonstration on how linearization (and tangent lines) can be used to approximate values in the real world.

This leads us into related rates problems. These questions typically provide the rate of change of one quantity, and ask you to find the rate of change of a second quantity. The hardest part is generally seeing how these two quantities are related. Beyond that, organization and some implicit differentiation will always help you solve these. Optimization is another big topic, which generally asks us to find the biggest, smallest, shortest, tallest, fastest, slowest, furthest, closest, something-est that a particular value can be. If given a single variable function, this just boils down to finding extrema as we did in the previous chapter.

Finding and taking derivatives are obviously involved in these processes, but students will also have to solve equations involving these derivatives. Because of these we take a quick lesson in solving some uglier trigonometric equations, and quickly apply it to many of our recent application problem types. We can solve more questions by understanding more relationships in triangles. We discuss the law of sines and cosines and how it helps us solve for sides and angles in general triangles. We also find a nice formula for the area of a triangle in terms of two adjacent sides and the angle between them. Each of these new relationships or formulas can help us to answer a wider array of related rates and optimization questions.

The chapter concludes with two subchapters dedicated to hyperbolic trigonometric functions. This material does not appear in future chapters, but is presented as an option for interested parties.

12.1: Linearization
12.2: Related Rates
12.3: Optimization
12.4: Solving Trigonometric Equations
12.5: Trigonometric Applications
12.6: Hyperbolic Trigonometric Functions
12.7: Inverse Hyperbolic Trigonometric Functions

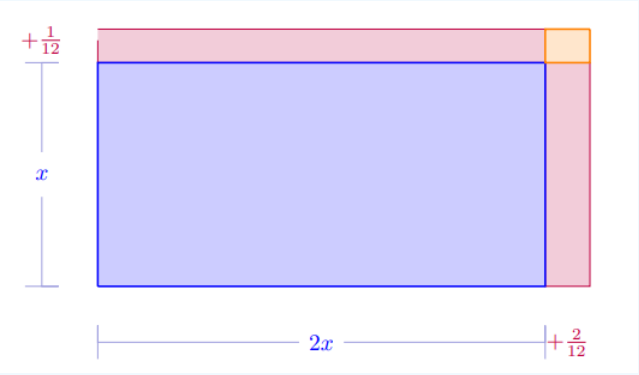
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12.1: Linearization

Remember linearization was the subtle art of using a nearby tangent line to approximate a function. This can be done to approximate real world functions as well.

Example 12.1.1

Assume you are designing a pool that needs to be twice as long as it is wide, and exactly 8 feet deep. The current volume of the planned pool is 1600 feet^3. Approximately how much more volume would you gain by extending the width by one inch (and adjusting the length accordingly)?



Solution

Clearly the volume of the pool will be length times width times depth. For the pool with top view above, we then have
$V(x) = x \cdot 2x \cdot 8 = 16x^2$

Since the pool volume was planned for 1600 feet^3, we get
$16x^2 = 1600 \longrightarrow x^2 = 100 \longrightarrow x = 10$ feet

But the question asks for *how much volume you gain* by increasing x by one inch (which is $\frac{1}{12}$ since x is measured in feet)

Instead of taking $V\left(10 + \frac{1}{12}\right)$, we use linearization: $V(x) \approx V(10) + V'(10)(x - 10)$

$V(10) = 1600$ and $V'(x) = 32x \longrightarrow V'(10) = 320$

$$V\left(10 + \frac{1}{12}\right) \approx 1600 + 320\left(\left[10 + \frac{1}{12}\right] - 10\right) = 1600 + \frac{80}{3}$$

We estimate that the volume of the pool would increase by $\frac{80}{3}$ feet^3

Checkings our work: $V\left(10 + \frac{1}{12}\right) = 16\left(10 + \frac{1}{12}\right)^2 = 1600 + \frac{80}{3} + \frac{25}{36}$

Using linearization to approximate the area of the new rectangle only focuses on the linear growth or decay of the function. This can help us to determine we have an underestimate.

Example 12.1.2

You are drag racing a car, which goes from 0 to 60 miles per hour in exactly 4 seconds. If you have moved a total of $\frac{1}{10}$ mile from the starting line during these 4 seconds, approximately how far will you have gone after 6 seconds total have elapsed?

Solution

Define $f(t) =$ distance traveled (in miles) during the first t seconds.

We are given $f(4) = \frac{1}{10}$, and $f'(4)$ should be the rate of change of f (but in miles per second to match units).

$$\frac{60\ miles}{1\ hour} \cdot \frac{1\ hour}{60\ minutes} \cdot \frac{1\ minute}{60\ seconds} = \frac{1}{60}\frac{miles}{second} \longrightarrow f'(4) = \frac{1}{60}$$

Since that's all we have to go on, we can say $f(t) \approx f(4) + f'(4)(t-4) = \frac{1}{10} + \frac{1}{60}(t-4)$

So $f(6) \approx \frac{1}{10} + \frac{1}{60}(6-4) = \frac{2}{15}$ miles

Example 12.1.3

A balloon expands in the shape of a perfect sphere as air is pumped into it. If the radius is 3 inches before an additional π inches^3 of air is pumped in, use linearization to approximate how much the radius of the balloon increases.

Hint: The volume of a sphere is $V = \frac{4\pi}{3}r^3$ and surface area $SA(r) = 4\pi r^2$

Solution

Our radius should be near its start of 3 inches, so lets find the linearization of $V(r)$ near $r = 3$:

$V(r) \approx V(3) + v'(3)(r-3)$

$V(3) = \frac{4\pi}{3} \cdot 3^3 = 36\pi$

$V'(r) = 4\pi r^2 \longrightarrow V'(3) = 4\pi \cdot 3^2 = 36\pi$

Therefore, $V(r) \approx 36\pi + 36\pi(r-3)$

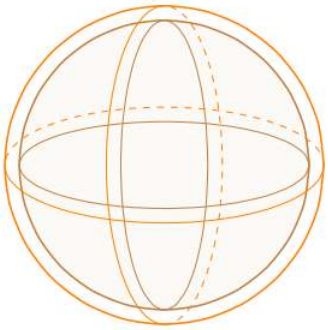
Now we can set this equal to the volume increased by π:

$36\pi + \pi = 36\pi + 36\pi(r-3)$

$r = 3 + \frac{1}{36}$

So we have the radius increase by about $\frac{1}{36}$ inches

Isn't it interesting that for $V(r) = \frac{4\pi}{3}r^3$ the volume of the sphere, we have $V'(r) = 4\pi r^2$ which is the formula for the surface area of a sphere.



One way to visualize this is that an infinitely small increase in the value of r will expand the volume of that sphere $V(r)$ by an infinitely thin shell around it's surface. This thin sheet has an infinitely thin depth, but has the same area as the surface area of the sphere $V'(r) = SA(r)$.

Many future applications in calculus will occur when we taking infinitely many things or infinitely small and/or large things. These things are all ultimately understood by taking the proper limits, which (thanks to L'Hopital's Rule) means all can relate to the derivative operation!

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12.2: Related Rates

Another type of real world application for differentiation are word problems that tell you the rate of change of one value at a given time, and ask for the rate of change of something else (related to that value) at that time. These types of problems are called *related rates* questions.

For example, if we know that a square has all four sides expanding at 2 inches per second, how quickly is the area of that square changing when the sides are exactly 5 inches?

We will go through the explicit logic of this problem first, then show the short version afterwards

We are given that the squares sides are all expanding, so there is some function $s(t)$ that tells you the length of each side of the square at time t.
Furthermore, since they are expanding at 2 inches per second, $s'(t) = 2$

Thankfully, we know the only type of function which has a constant derivative is a line. So $s(t)$ is a linear function with slope 2.
$s(t) = 2t + K$ for some constant K

The area of this square should then be $A(t) = [s(t)]^2 = (2t + K)^2 = 4t^2 + 4Kt + K^2$

We want to know the rate of change of this area, so differentiating gives: $A'(t) = 8t + 4K$

We want to know this derivative at a very specific time, and that is "when the sides are exactly 5 inches", so that means we want the value of t that makes $s(t) = 5$

$$2t + K = 5 \longrightarrow t = \frac{5 - K}{2}$$

Plugging this into $A'(t)$ gives: $A'\left(\frac{5-K}{2}\right) = 8 \cdot \left(\frac{5-K}{2}\right) + 4K = 20 - 4K + 4K = 20$

We have reached our final answer: **the area is increasing at 20 inches per second**

This was a lot of work to do explicitly, but its much easier implicitly:

Since $A(t) = [s(t)]^2$, we can write this as $A = s^2$ where A and s are secret functions of time t.
If we take the implicit derivative with respect to t (measured in *seconds*) we get:

$$\left(A = s^2\right)'$$

$$A' = 2s \cdot s'$$

Then we plug in the given info ($s' = 2$ and $s = 5$) and solve for A':

$$A' = 2 \cdot 5 \cdot 2 = 20$$

We get the same answer as we did before, with way less work!

All related rates questions will be similar to our previous example. There are three major ingredients hiding in a problem: the given rate, the wanted rate, and the "when" information.

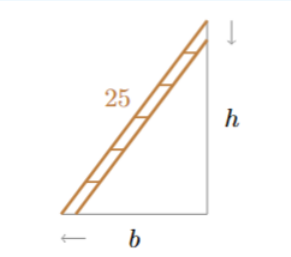
Beyond that, the only thing you need is to find the equation that relates the variables in the given and wanted rate so you can take its implicit derivative. Let's try another example.

Example 12.2.1

A 25 foot ladder leans against a wall, causing the base of the ladder slides away from the wall at 2 feet per hour. At what rate is the top of the ladder sliding down the wall when the base of the ladder is exactly 15 feet away from the wall?

Solution

In these problems, a diagram is usually helpful.
Just remember to label the values that are changing as letters and the values that are constant as numbers!



First we gather the three given ingredients:

Given Rate: $b' = 2$
Wanted Rate: h'
When: $b = 15$

Next we find the relationship between b and h. In this case, the Pythagorean theorem works:

$$b^2 + h^2 = 25^2$$

So we take the implicit derivative with respect to time (measured in *hours*):

$$\left(b^2 + h^2 = 25^2\right)'$$

$$2b \cdot b' + 2h \cdot h' = 0$$

Plug in the ingredients above ($b' = 2, b = 15$):

$$2 \cdot 15 \cdot 2 + 2 \cdot h \cdot h' = 0$$

We still need to find h to find our answer. This can be found by using the "when" information and the original equation before it was differentiated:

$$15^2 + h^2 = 25^2 \longrightarrow h = 20$$

Now we get $60 + 2 \cdot 20 \cdot h' = 0$, which when solved becomes $h' = -\frac{3}{2}$

Therefore h **is decreasing at** $\frac{3}{2}$ **feet per hour**

At the end of a related rates problem, always remember to include the units of the derivative you solved. It is also good practice to check and see if your derivative being positive or negative matches with your intuition from the diagram. In this example it makes sense that h would get smaller (decrease) as b gets bigger (increases), so h' should indeed be negative.

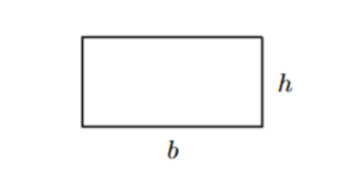
After you differentiate, anything you are missing should be in the "when" information.

Example 12.2.2

Assume a rectangle maintains a constant area of 360 meters^2. if the base is decreasing at 2 meters per day, how quickly is the height changing when the base is exactly 20 meters?

Solution

So we have a rectangle with changing base and height:



Again gather the three given ingredients:

Given Rate: $b' = -2$
Wanted Rate: h'
When: $b = 20$

Notice that the given rate is negative because b is *decreasing* by that amount!

The relationship between b and h is based on the constant area:

$$b \cdot h = 360$$

We take the implicit derivative with respect to time (measured in *days*):

$$(b \cdot h = 360)'$$

Remember to use the product rule for this one:

$$b' \cdot h + b \cdot h' = 0$$

Plug in b' and b:

$$-2 \cdot h + 20 \cdot h' = 0$$

Looks like we need to find h. We use the original equation with the *when info*:

$$20 \cdot h = 360 \longrightarrow h = 18$$

So our differentiated equation is now:

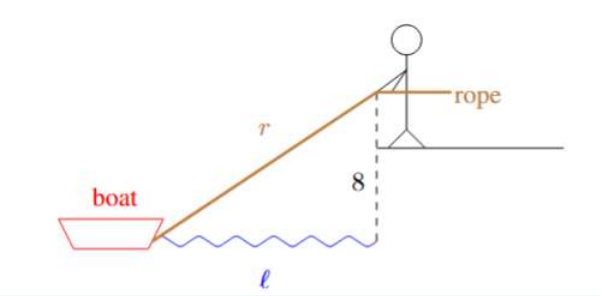
$$-2 \cdot 18 + 20h' = 0 \longrightarrow h' = \frac{9}{5}$$

We have our final answer: **the height is increasing at $\frac{9}{5}$ meters per day**.

Some questions even come with their own diagrams included.

Example 12.2.3

You are on a dock pulling a rope attached to a boat through a pulley. If the pulley is exactly 8 feet above water level and you are pulling the rope at 36 feet per minute, how quickly is the boat approaching the dock along the water when it is exactly 12 feet away?



Solution

There is clearly a right triangle in the diagram that determines everything we need to know.

Gather the three given ingredients:

Given Rate: $r' = -36$
Wanted Rate: ℓ'
When: $\ell = 12$

The relationship between r and ℓ is the Pythagorean Theorem:

$$\ell^2 + 8^2 = r^2$$

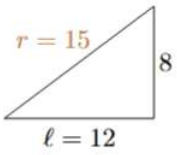
Take the implicit derivative with respect to time (measured in *minutes*):

$$\left(\ell^2 + 8^2 = r^2\right)'$$

$$2\ell \cdot \ell' + 0 = 2r \cdot r'$$

We know r' and ℓ, but will still need to find r to solve for ℓ'.
We can use the *when info* and original equation to draw a "when picture":

$$\ell^2 + 8^2 = r^2$$
$$12^2 + 8^2 = r^2$$
$$r = 15$$



Plugging all of this into our differentiated equation gives:

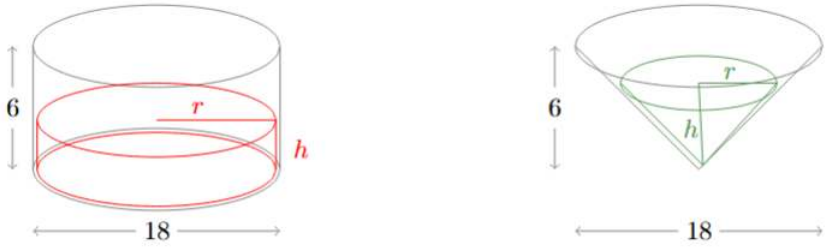
$$2 \cdot 12 \cdot \ell' = 2 \cdot 15 \cdot (-36)$$

$$24\ell' = -1080$$

$$\ell' = -45$$

The distance between the boat and the dock is decreasing at 45 feet per minute

Two neighbors are throwing a pool party on the same day. Neighbor A has a pool that is a perfect cylinder with a diameter of 18 feet and depth of 6 feet. Neighbor B has a pool that is a perfect cone, also with diameter 18 feet and a deepest depth of 6 feet. They each begin to fill up their pools at a rate of 5 cubic feet per minute. How quickly is the water level rising when it is exactly 4 feet deep? (Answer for each neighbor and compare)}



In both cases, we need to pay attention to the shape the water forms in each pool.

The main ingredients are the same for both:

Given Rate: $V' = 5$
Wanted Rate: h'
When: $h = 4$

For Neighbor A with the cylinder pool, we know the volume formula is $V = \pi r^2 h$

Clearly the value of h is changing over time, but r remains constant: $r = 9$

So we plug this constant in to get our equation: $V = \pi \cdot 9^2 \cdot h = 81\pi h$

We can differentiate implicitly with respect to time then plug in what we know to solve for h':

$$(V = 81\pi h)' \quad \longrightarrow \quad V' = 81\pi \cdot h' \quad \longrightarrow \quad 5 = 81\pi h' \quad \longrightarrow \quad h' = \frac{5}{81\pi}$$

So **the water in Pool A is rising at $\frac{5}{81\pi}$ feet per minute**.

But in Pool B the value of r is not constant, it also depends on time! Sadly, we are not given r' so if we were to differentiate the volume formula for a cone ($V = \frac{\pi}{3} r^2 h$), we still cant solve for h'

Instead, we see that because of the cone shaped container, the ratio of the height and radius of the water cone formed remains constant: $\frac{r}{h} = \frac{9}{6}$ which re-writes as $r = \frac{3}{2} h$

Plugging this into the volume of a cone formula gives our equation: $V = \frac{\pi}{3} \left(\frac{3}{2} h \right)^2 h = \frac{3\pi}{4} h^3$

We differentiate this with respect to t and then plug in what we know again:

$$\left(V = \frac{3\pi}{4} h^3 \right)' \quad \longrightarrow \quad V' = \frac{9\pi}{4} h^2 \cdot h' \quad \longrightarrow \quad 5 = \frac{9\pi}{4} \cdot 4^2 \cdot h' \quad \longrightarrow \quad h' = \frac{5}{36\pi}$$

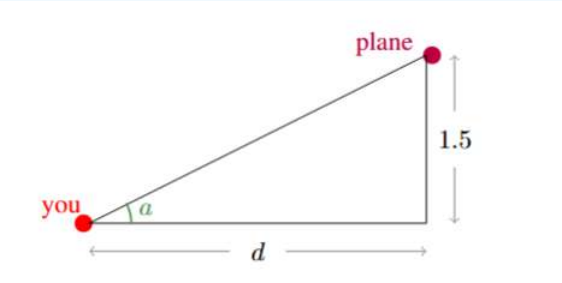
So **the water in Pool B is rising at $\frac{5}{36\pi}$ feet per minute**.

This should match your intuition, the cone holds a third as much water, so it should fill faster!

If you are lucky, you might end up with a trigonometric (or inverse trig) function involved!

Example 12.2.4

An airplane flies at 400 mph in a straight line over your head at a constant elevation of 1.5 miles. How quickly is your angle of elevation (angle upward from the ground) to see the plane changing when the plane has flown exactly 2 miles past you horizontally?



Solution

Again this story is defined by a right triangle for angle a and adjacent leg d

Given Rate: $d' = 400$
Wanted Rate: a'
When: $d = 2$

Next we want to find the equation that relates a and d. There are six trig functions, but only four of them involve the adjacent leg d. Two of those relate d to the hypotenuse of this triangle (which we don't know much about, so we should avoid!) That leaves only two trigonometric functions that will get us to our desired answer:

$$\tan(a) = \frac{1.5}{d} \text{ and } \cot(a) = \frac{d}{1.5}$$

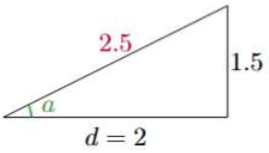
These are ultimately the same equation, so it does not matter which one you pick. In fact, if you want to you can re-write this as

$$a = \tan^{-1}\left(\frac{1.5}{d}\right) \text{ or } a = \cot^{-1}\left(\frac{d}{1.5}\right)$$

Differentiate whichever one of these four equations you like:

$$\left(\tan(a) = \frac{1.5}{d}\right)' \quad \longrightarrow \quad \sec^2(a) \cdot a' = -1.5d^{-2} \cdot d'$$

We draw our "when diagram" to gain a helper triangle:



$$\sec(a) = \frac{2.5}{2}$$

Therefore $\left(\frac{5}{4}\right)^2 \cdot a' = -1.5 \cdot 2^{-2} \cdot 400 \longrightarrow a' = -96$

So our final answer: **the angle of elevation is decreasing at 96 radians per hour**

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12.3: Optimization

Many times we want to find a maximum or minimum in a real life scenario. These mostly show up in the form of word problems, which will ask for the *biggest, smallest, shortest, tallest, etc.* value of some function. This can be very straightforward, but typically these functions depend on more than one parameter (or variable). The hard work is turning these into a function of one variable!

The most common example of an optimization problem is the fence question:

A farmer wants to build a rectangular pig pen for his pigs, but only has 200 feet of fence material available. What is the largest area he can enclose with this fence?

To solve this, we know that a rectangular pig pen's area is the product of its length and width. We write this as $A(\ell, w) = \ell \cdot w$

This function alone poses a lot of issues for us. Firstly, it has more than once variable (and this is single variable calculus), so we don't know how to find its extrema. Secondly, we can clearly make this as large as we desire by just making ℓ and w bigger and bigger.

This brings us to the second major ingredient to an optimization problem: the constraint. These types of problems always provide some relationship that must be obeyed. In this example its the perimeter of the rectangle being limited by the 200 feet of fence.

$2\ell + 2w = 200$

We can use this constraint to solve for one variable in terms of the other:

$\ell = 100 - w$

Then we can plug this into $A(\ell, w)$ to get:

$A(w) = (100 - w) \cdot w = 100w - w^2$

This is the single variable function we want to find the *maximum* of!

Notice that it has a natural domain of $w \in [0, 100]$ (since $w, \ell \geq 0$ must hold).

$A'(w) = 100 - 2w$

So $A'(w) = 0$ when $w = 50$

Finally we check our endpoints and critical points to get:

$A(0) = 100 \cdot 0 - 0^2 = 0$
$A(50) = 100 \cdot 50 - 50^2 = 2500$
$A(100) = 100 \cdot 100 - 100^2 = 0$

Thus our **maximum possible area is 2500 feet^2**

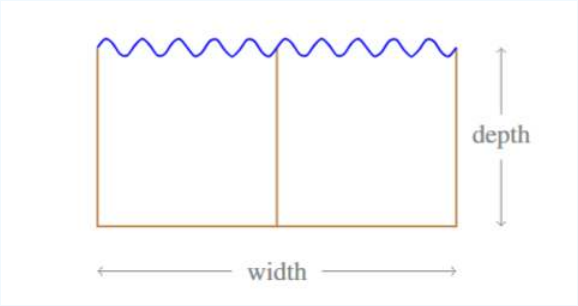
This is the blueprint for solving any optimization question:

Find the **optimizing function** and constraint, then use the constraint to solve for one variable and substitute to create a single variable function so you can differentiate and find extrema. Just dont forget to answer the question being asked and include units when necessary!

Another farming example awaits us:

Example 12.3.1

This time a farmer wants to create two identical adjacent rectangular paddocks for his cattle. To save in fence material, he puts one side of the rectangle along the edge of a river as shown in the diagram below (since cows cant swim). What width of the paddock will enclose the largest possible area using exactly 60 meters of fence material?



Solution

Here we are again trying to find the maximum possible area, so our optimizing function is $A(w,d)=w\cdot d$ where w is the width and d is the depth in the diagram above.

The constraint is that the amount of fence material must total to 60$ meters. Since we need 3 walls of length d and one of length w, we need $3d+w=60$

Notice that it is easy to solve for either variable here in terms of the other, and it does not matter which you pick as it will ultimately lead to the same final answer. To make our lives easier we will just solve for w, giving $w=60-3d$

Now substituting this into A we get $A(d)=(60-3d)\cdot d=60d-3d^2$

This function has a natural domain since $w\geq 0$ and $d\geq 0$ must hold, so we get $d\in[0,20]$

Now we can differentiate, giving $A'(d)=60-6d$

This always exists, so our critical points are when $A'(d)=0$, giving only $d=10$

We check our critical points and endpoints to find the maximum for A:

$A(0)=60\cdot 0-3\cdot 0^2=0$
$A(10)=60\cdot 10-3\cdot 10^2=200$
$A(20)=60\cdot 20-3\cdot 20^2=0$

Clearly the maximum possible area is 200 meters^2, but the question asks for the *width*.

We can plug in $d=10$ to the equation $w=60-3d$ to get $w=30$ meters.

Therefore our answer is **the maximum area occurs when the width is 30 meters**.

Always remember to answer the question being asked!

Lets try a non-farming example for some variety:

Example 12.3.2

The army wants to make a cylindrical soup can that can hold 250π yards^3 of soup to feed a battalion. Since they are making such a large can, they want to be efficient and be sure to use the smallest possible surface area (so that they can use less metal when creating the can). What is the smallest possible surface area of such a can?

Solution

Remember a cylinder with base radius r and height h has a volume of $V = \pi r^2 h$ and a surface area of $S = 2\pi r^2 + 2\pi rh$

We want to find the smallest surface area, so that should be our optimizing function: $S(r,h) = 2\pi r^2 + 2\pi rh$

Our constraint is that we want the volume to be 250π, therefore $250\pi = \pi r^2 h$

We can solve for h more easily than r here, giving $h = \dfrac{250}{r^2}$

We substitute this into our optimizing function to get:

$$S(r) = 2\pi r^2 + 2\pi r \cdot \frac{250}{r^2} = 2\pi r^2 + 500\pi r^{-1}$$

The domain here is again $r, h \geq 0$ must hold, but this only tells us $r \in [0, \infty)$.

Differentiating this gives $S'(r) = 4\pi r - 500\pi r^{-2} = \dfrac{4\pi r^3 - 500\pi}{r^2}$

Thus $S'(r) = 0$ when $4\pi r^3 - 500\pi = 0 \longrightarrow r = 5\$$

and $S'(r) =$DNE when $r^2 = 0 \longrightarrow r = 0$

Since our domain is not a closed interval, we make a sign chart:



Our local minimum must therefore occur at $r = 5$ yards. This question asked us to find the smallest possible surface area, so we plug this into $S(r)$ to get:

$$S(5) = 2\pi \cdot 5^2 + 500\pi \cdot 5^{-1} = 50\pi + 100\pi = 150\pi$$

Our answer is officially that **the minimum surface area is 150π yards^2**.

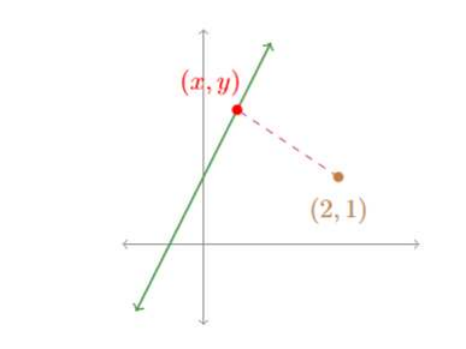
Not all optimization problems are from the real world. Many times we can use optimization to answer questions about graphs!

Example 12.3.3

Find the point on the line $y - 2x = 1$ which is closest to $(2, 1)$

Solution

Here it helps to draw a sketch of what is going on:



Our goal is to minimize the length of the purple line connecting (x, y) to $(2, 1)$.

We can find this length by using the distance formula: $d(x, y) = \sqrt{(x-2)^2 + (y-1)^2}$

The constraint is that the point (x, y) must be on the line $y - 2x = 1$, which we can easily solve for y in to get: $y = 2x + 1$

Now we substitute that into our optimizing function d to get:

$d(x) = \sqrt{(x-2)^2 + ((2x+1)-1)^2} = \sqrt{5x^2 - 4x + 4}$

The domain is clearly $(-\infty, \infty)$, but you can check by hand if needed.

Next, we find $d'(x) = \frac{1}{2}(5x^2 - 4x + 4)^{-\frac{1}{2}} \cdot (10x - 4) = \frac{5x - 2}{\sqrt{5x^2 - 4x + 4}}$

We have $d'(x) = 0$ when $5x - 2 = 0 \longrightarrow x = \frac{2}{5}$

And $d'(x) =$DNE when $\sqrt{5x^2 - 4x + 4} = 0$ (which has no real solutions, so never)

A quick check of the sign chart confirms we indeed have a minimum at $x = \frac{2}{5}$:



We want to find the *closest point*, so we need to find y next:

Since $y = 2x + 1$, we plug in $x = \frac{2}{5}$ to get $y = \frac{9}{5}$

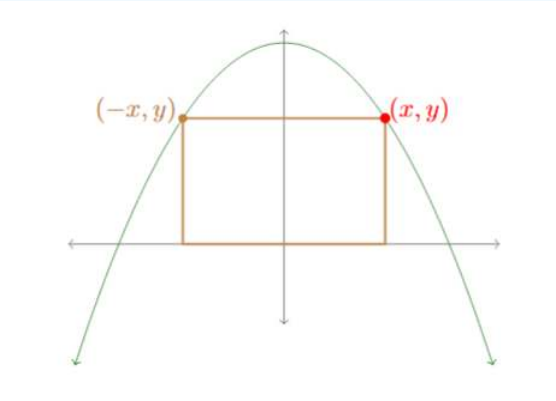
So our final answer is **the closest point is at** $\left(\frac{2}{5}, \frac{9}{5}\right)$

Example 12.3.4

What is the largest possible area of a rectangle which can be inscribed inside of the parabola $y = 12 - x^2$ with one side resting on the x-axis?

Solution

Once again we start with a sketch to help see what is going on:



We can parameterize the rectangle in terms of the corner that lands in quadrant one at (x, y). Moving this point along the parabola will create infinitely many different rectangles. We want to maximize the area of the blue rectangle, which has a base of $2x$ and a height of y. This gives our optimizing function as $A(x, y) = 2xy$

The constraint is that the point (x, y) must be on the parabola $y = 12 - x^2$, (which is already set up to be substituted into the optimizing function A)

$A(x) = 2x(12 - x^2) = 24x - 2x^3$

The domain here is that x must put us into the first quadrant, so $x \in [0, 2\sqrt{3}]$

Differentiating gives us $A'(x) = 24 - 6x^2$, which always exists.

So our only critical points can be when $A'(x) = 0 \longrightarrow x = \pm 2$

Since our domain $[0, 2\sqrt{3}]$ we can ignore $x = -2$ and just plug in the remaining critical point and domain endpoints to find our maximum for $A(x)$:

$A(0) = 24 \cdot 0 - 2 \cdot 0^3 = 0$
$A(2) = 24 \cdot 2 - 2 \cdot 2^3 = 32$
$A(2\sqrt{3}) = 24 \cdot 2\sqrt{3} - 2 \cdot (2\sqrt{3})^3 = 0$

Therefore our **maximum possible area of the rectangle is 32**
(there no no units provided, so we don't need to use them in our answer)

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12.4: Solving Trigonometric Equations

In life, we may find ourselves with an equation that needs to be solved involving trig functions.
These can vary in difficulty, so we will start by recapping the easier cases:

For $-1 \leq \alpha \leq 1$, the equation $\sin(t) = \alpha$ has solutions $t = \sin^{-1}(\alpha)$ and $\pi - \sin^{-1}(\alpha)$

For $\alpha \in (-\infty, -1] \cup [1, \infty)$, the equation $\csc(t) = \alpha \longleftrightarrow \sin(t) = \dfrac{1}{\alpha}$
so it has solutions $t = \sin^{-1}\left(\dfrac{1}{\alpha}\right)$ and $\pi - \sin^{-1}\left(\dfrac{1}{\alpha}\right)$

For $-1 \leq \alpha \leq 1$, the equation $\cos(t) = \alpha$ has solutions $t = \cos^{-1}(\alpha)$ and $-\cos^{-1}(\alpha)$

For $\alpha \in (-\infty, -1] \cup [1, \infty)$, the equation $\sec(t) = \alpha \longleftrightarrow \cos(t) = \dfrac{1}{\alpha}$
so it has solutions $t = \cos^{-1}\left(\dfrac{1}{\alpha}\right)$ and $-\cos^{-1}\left(\dfrac{1}{\alpha}\right)$

For any real α, the equation $\tan(t) = \alpha$ has solutions $t = \tan^{-1}(\alpha)$ and $\pi + \tan^{-1}(\alpha)$

For any real α, the equation $\cot(t) = \alpha \longleftrightarrow \tan(t) = \dfrac{1}{\alpha}$
so it has solutions $t = \tan^{-1}\left(\dfrac{1}{\alpha}\right)$ and $\pi + \tan^{-1}\left(\dfrac{1}{\alpha}\right)$

By the Intermediate Value Theorem, we know that a solution exists to any of the equations above.
Solving general trigonometric equations rely on your ability to solve these forms first. As long as we can factor an equation into these linear types of trig equations, we can solve it like polynomials!

Since all six trigonometric functions are periodic (they repeat every 2π), we write most solutions with $+2\pi k$ to represent "plus any full rotation from this angle". For example:

$\cos(x) = 0 \longrightarrow x = \pm\cos^{-1}(0) = \pm\dfrac{\pi}{2} + 2\pi k$
(This is the same as saying $x = \ldots, -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$)

Lets try an example with less nice angles:

Example 12.4.1

Find the local extrema of $f(x) = \ln(\sec(x) + \tan(x)) - 3x$ on $\left[0, \dfrac{\pi}{4}\right]$

Solution

$$f'(x) = \frac{1}{\sec(x) + \tan(x)} \cdot (\sec(x)\tan(x) + \sec^2(x)) - 3 = \sec(x) - 3$$

This always exists on $\left[0, \dfrac{\pi}{4}\right]$, so we solve $f'(x) = 0$

$$\sec(x) = 3 \longrightarrow \cos(x) = \frac{1}{3} \longrightarrow x = \cos^{-1}\left(\frac{1}{3}\right)$$

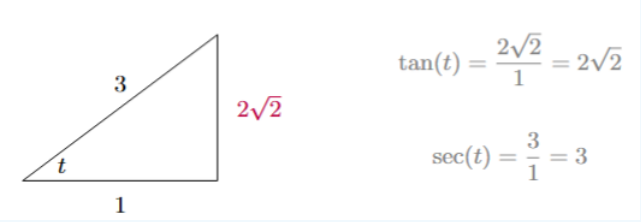
So we check this critical point and the endpoints:

$$f(0) = \ln(\sec(0) + \tan(0)) - 3 \cdot 0 = \ln(1) - 0 = 0$$

$f\left(\frac{\pi}{4}\right) = \ln\left(\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right) - 3 \cdot \frac{\pi}{4} = \ln(\sqrt{2}+1) - \frac{3\pi}{4}$

$f\left(\cos^{-1}\left(\frac{1}{3}\right)\right) = \ln\left(\sec\left(\cos^{-1}\left(\frac{1}{3}\right)\right) + \tan\left(\cos^{-1}\left(\frac{1}{3}\right)\right)\right) - 3 \cdot \cos^{-1}\left(\frac{1}{3}\right)$

Lets draw a quick helper triangle for this critical point:



So $f\left(\cos^{-1}\left(\frac{1}{3}\right)\right) = \ln(2\sqrt{2}+3) - 3\cos^{-1}\left(\frac{1}{3}\right)$

So our **maximum is at** $\left(\cos^{-1}\left(\frac{1}{3}\right), \ln(2\sqrt{2}+3) - 3\cos^{-1}\left(\frac{1}{3}\right)\right)$
and **minimum is at** $\left(\frac{\pi}{4}, \ln(\sqrt{2}+1) - \frac{3\pi}{4}\right)$

Although having an answer written in terms of $\cos^{-1}\left(\frac{1}{3}\right)$ seems tedious, it is still useful for us to recognize that there is indeed a maximum at a particular location in this interval.

Sometimes the trigonometric equation will look like a polynomial in terms of one trigonometric function. To solve such an equation, we will do substitution and factoring.

For example, if we try to solve $\sec^2(x) + 2 = 3\sec(x)$

We can define $u = \sec(x)$ so the equation becomes easier to solve:

$u^2 + 2 = 3u \longrightarrow u^2 - 3u + 2 = 0 \longrightarrow (u-1)(u-2) = 0 \longrightarrow u = 1, 2$

Now we switch back: $\sec(x) = 1$ and $\sec(x) = 2$

Reciprocal both of these to become: $\cos(x) = 1$ and $\cos(x) = \frac{1}{2}$

So we get $x = \pm\cos^{-1}(1), \pm\cos^{-1}\left(\frac{1}{2}\right) + 2\pi k$

Equivalent to $x = 0, \frac{\pi}{3}, -\frac{\pi}{3} + 2\pi k$

(Or if you prefer: $x = \cdots - 2\pi, -\frac{5\pi}{3}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi, \frac{7\pi}{3}, \ldots)$

Example 12.4.2

Find all solutions to $2\sin^2(w) + \sin(w) = 1$ in $[0, 2\pi]$

Solution

Let $u = \sin(w)$ to get $2u^2 + u - 1 = 0$

$(2u-1)(u+1)=0$

This has solutions $u=\frac{1}{2},-1$, giving

$\sin(w)=\frac{1}{2}$ gives $w=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$ and $w=\pi-\sin^{-1}\left(\frac{1}{2}\right)=\frac{5\pi}{6}$

$\sin(w)=-1$ gives $w=\sin^{-1}(-1)=-\frac{\pi}{2}$ and and $w=\pi-\sin^{-1}(-1)=\frac{3\pi}{2}$

So our solutions are $w=\frac{\pi}{6},\frac{5\pi}{6},\frac{3\pi}{2}$

If there are more than one trigonometric function involved in your equation, you can use the Pythagorean Identities to re-write it.

Example 12.4.3

Find all solutions in $[0,2\pi)$ for each equation below by doing an appropriate substitution

(a) $\sec^2(t)=\tan(t)+1$

(b) $1+3\cos(z)=\sin^2(z)$

Solution

(a) We start by using $\sec^2(t)=\tan^2(t)+1$ to turn this into:

$\tan^2(t)+1=\tan(t)+1$
$\tan^2(t)-\tan(t)=0$

Let $u=\tan(t)$ and this becomes $u^2-u=0$

$u(u-1)=0$
$u=0,1$

Therefore we solve $\tan(t)=0$ and $\tan(t)=1$

$t=\tan^{-1}(0),\pi+\tan^{-1}(0)$ and $t=\tan^{-1}(1),\pi+\tan^{-1}(1)$

Thus our final answers are: $t=0,\pi,\frac{\pi}{4},\frac{5\pi}{4}$

(b) We start by replacing $\sin^2(z)=1-\cos^2(z)$:

$1+3\cos(z)=1-\cos^2(z)\longrightarrow\cos^2(z)+3\cos(z)=0$

Let $u=\cos(z)$ to get $u^2+3u=0$

$u(u+3)=0$
Has solutions: $u=0,-3,$

Therefore we solve $\cos(z)=0$ and $\cos(z)=-3$

$z=\cos^{-1}(0)=\frac{\pi}{2}$ and $z=-\cos^{-1}(0)=-\frac{\pi}{2}$ and $\emptyset$

So our solutions are $z=\frac{\pi}{2},\frac{3\pi}{2}$

We can use this to solve equations for critical points:

Example 12.4.4

Sketch the graph of $f(x) = \dfrac{\cos(x)}{\sin(x) - 2}$

Solution

$$f'(x) = \frac{(\sin(x)-2)(-\sin(x)) - \cos(x)\cdot\cos(x)}{(\sin(x)-2)^2} = \frac{2\sin(x)-1}{(\sin(x)-2)^2}$$

$$2\sin(x) - 1 = 0 \longrightarrow \sin(x) = \frac{1}{2} \longrightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$(\sin(x)-2)^2 = 0 \longrightarrow \sin(x) = 2 \longrightarrow \emptyset$$

So our critical points are $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$$f''(x) = \frac{-2\cos(x)(\sin(x)+1)}{(\sin(x)-2)^3}$$

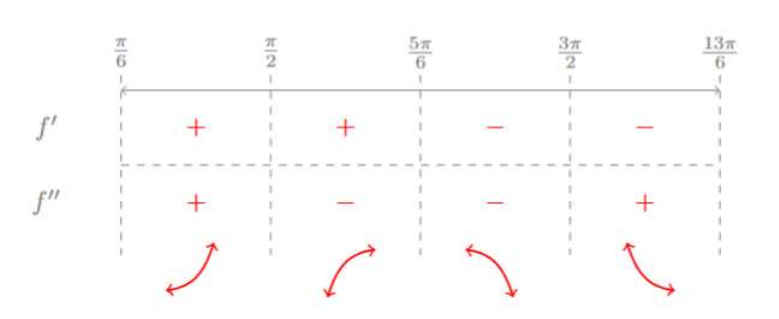
The solving and simplifying of f' and f'' both require the student to use the Pythagorean Identity here. Be sure to take your time when simplifying derivatives, or you will get wrong answers!

$$-2\cos(x)(\sin(x)+1) = 0 \longrightarrow \sin(x) = -1, \cos(x) = 0 \longrightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$(\sin(x)-2)^3 = 0 \longrightarrow \sin(x) = 2 \longrightarrow \emptyset$$

So our inflection points are $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

We can make a sign chart for any interval of length 2π, so it makes sense to start at a critical or inflection point and adjust any points by 2π when necessary:



So we plug in these two critical and inflection endpoints:

$$f\left(\frac{\pi}{6}\right) = f\left(\frac{13\pi}{6}\right) = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}-2} = -\frac{1}{\sqrt{3}}$$

$$f\left(\frac{5\pi}{6}\right) = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}-2} = \frac{1}{\sqrt{3}}$$

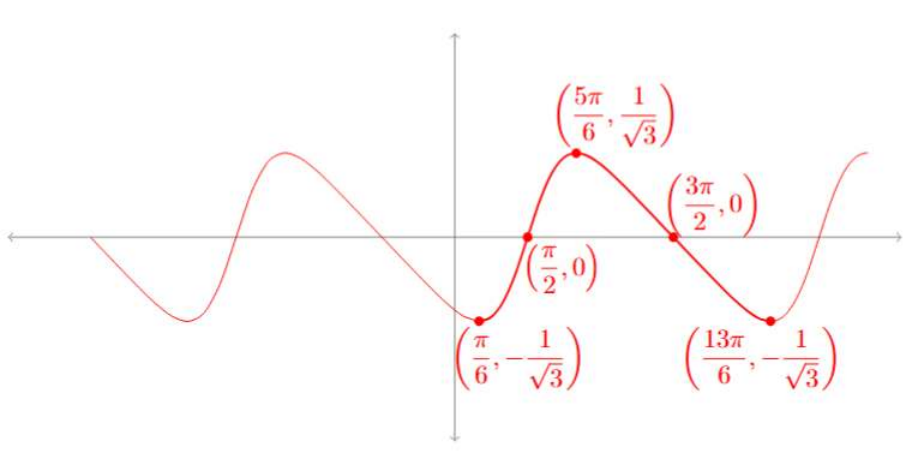
$$f\left(\frac{\pi}{2}\right) = \frac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)-2} = \frac{0}{1-2} = 0$$

$$f\left(\frac{3\pi}{2}\right) = \frac{\cos\left(\frac{3\pi}{2}\right)}{\sin\left(\frac{3\pi}{2}\right)-2} = \frac{0}{-1-2} = 0$$

This function is continuous everywhere, so there are **no vertical asymptotes**.
The entire function repeats every 2π, so it cannot get close to any one number on either side.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\cos(x)}{\sin(x)-2} = \text{DNE}$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} \frac{\cos(x)}{\sin(x)-2} = \text{DNE}$$

Therefore, we will have **no horizontal asymptotes.**
Lastly we can draw one period (between $\frac{\pi}{6} \le x \le \frac{13\pi}{6}$) and then repeat it in both directions.



Another style of trigonometric equation comes when two trig functions are added together but the Pythagorean theorem cannot help. For example:

$\sin(x)+\cos(x)=0$

We can simply multiply or divide through an equation by sine or cosine. In this case, dividing by cosine works:

$$\frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\cos(x)} = 0$$

$\tan(x)+1=0$

$\tan(x)=-1$

This gives $x = \frac{\pi}{4}, \frac{5\pi}{4} + 2\pi k$

Example 12.4.5

Solve each equation below for x in the interval $[0, 2\pi)$

(a) $\cot(x) = 3\tan(x)$

(b) $\cos^2(x) = \sec^2(x)$

(c) $8\sin^2(x) = -\csc(x)$

Solution

(a) $1 = \dfrac{3\tan(x)}{\cot(x)} = 3\tan^2(x) \longrightarrow \tan(x) = \pm\dfrac{1}{\sqrt{3}}$

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

(b) $1 = \dfrac{\sec^2(x)}{\cos^2(x)} = \sec^4(x) \longrightarrow \sec(x) = \pm 1$

$x = 0, \pi$

(c) $8\sin^3(x) = -1 \longrightarrow \sin(x) = -\dfrac{1}{2}$

$x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

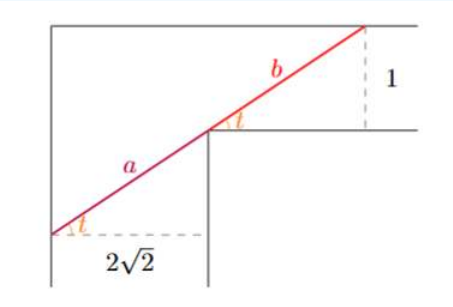
We can use this when solving optimization questions involving angles:

Example 12.4.6

A long pole must be carried horizontally level around a hallway corner. The width of the current hallway is exactly 1 meter while the width of the perpendicular hallway is $2\sqrt{2}$ meters. What is the longest possible pole that can make the turn?

Solution

We start with a diagram:



The pole length $L(a, b)$ will be the sum of a and b

But they both relate to the angle t:

$\cos(t) = \dfrac{2\sqrt{2}}{a} \longleftrightarrow a = 2\sqrt{2}\sec(t)$

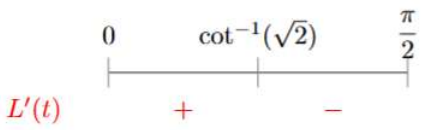
$\sin(t) = \dfrac{1}{b} \longleftrightarrow b = \csc(t)$

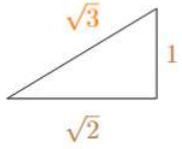
So our optimizing function $L(t) = 2\sqrt{2}\sec(t) + \csc(t)$ for $t \in \left(0, \dfrac{\pi}{2}\right)$

$$L'(t) = 2\sqrt{2}\sec(t)\tan(t) - \csc(t)\cot(t) = \frac{2\sqrt{2}\sin^3(t) - \cos^3(t)}{\sin^2(t)\cos^2(t)}$$

$L'(t) = DNE \longleftrightarrow \sin^2(t)\cos^2(t) = 0 \longleftrightarrow$ no solution in this domain.
$L'(t) = 0 \longleftrightarrow 2\sqrt{2}\sin^3(t) - \cos^3(t) = 0 \longleftrightarrow 2\sqrt{2} = \cot^3(t)$
Which means our critical points are when $\cot(t) = \sqrt{2}$, giving $t = \cot^{-1}\left(\sqrt{2}\right)$

So here is our sign chart and helper triangle:



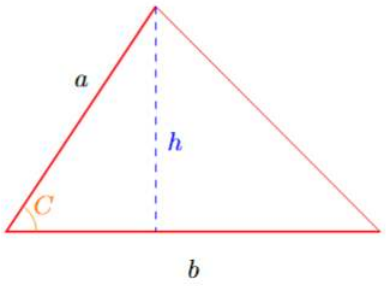


The longest pole will then be $L\left(\cot^{-1}(\sqrt{2})\right) = 2\sqrt{2} \cdot \frac{\sqrt{3}}{\sqrt{2}} + \frac{\sqrt{3}}{1} = 3\sqrt{3}$ meters long.

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12.5: Trigonometric Applications

While the area of a triangle formula $A = \frac{1}{2}bh$ is simple, it only applies when the base and height are perpendicular. There is a more general formula to find the area of a triangle as a function of two legs and the angle t between them (with $0 < t < \pi$):



We can see that $\sin(C) = \frac{h}{a}$, so we can solve for $h = a\sin(C)$ to get our new area formula: $A_{\triangle} = \frac{1}{2}ab\sin(C)$

This works with any permutation of letters in the triangle: $A_{\triangle} = \frac{1}{2}bc\sin(A) = \frac{1}{2}ac\sin(B)$

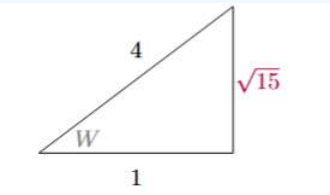
Notice that when $t = \frac{\pi}{2}$ is a right angle, since $\sin\left(\frac{\pi}{2}\right) = 1$ then this becomes the classic right triangle area formula $A = \frac{1}{2}bh$

Example 12.5.1

What is the area of triangle UVW with $u = 4, v = 3, W = \sec^{-1}(4)$

Solution

Start with a helper triangle for the angle W:

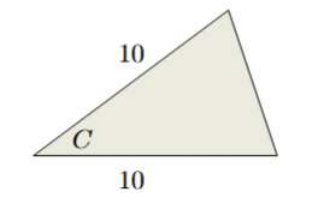


$A = \frac{1}{2} \cdot u \cdot v \cdot \sin(W) = \frac{1}{2} \cdot 4 \cdot 3 \cdot \frac{\sqrt{15}}{4} = 3\sqrt{15}$

Here is a related rates example using this area formula:

Example 12.5.2

An isosceles triangle has two legs 10 meters long, and the angle between them is expanding at $\frac{1}{3}$ radian per minute. How quickly is the area changing when the angle is $\cos^{-1}\left(\frac{3}{5}\right)$?



Solution

We start by collecting the easy things:

Given Rate: $C' = \frac{1}{3}$

Wanted Rate: A'

When: $C = \cos^{-1}\left(\frac{3}{5}\right)$

For the equation, remember we have $A = \frac{1}{2}ab\sin(C)$.

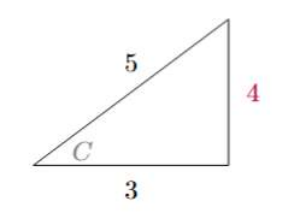
In this problem, the values of a and b are both constant and equal to 10.

$$A = \frac{1}{2} \cdot 10 \cdot 10 \cdot \sin(C)$$

$$(A = 50\sin(C))'$$

$$A' = 50\cos(C) \cdot C'$$

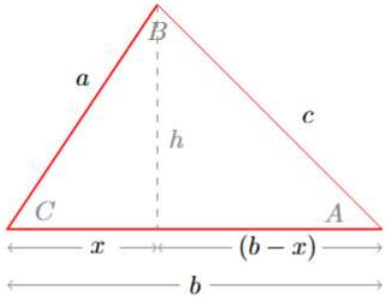
We use the *when diagram* for our helper triangle to find $\cos(C)$:



$$A' = 50 \cdot \frac{3}{5} \cdot \frac{1}{3} = 10$$

So the **area is increasing at 10 meters^2 per minute.**

You may have noticed that many diagrams boil down to a triangle, and it isn't always a right triangle. There are two powerful laws we can use to solve for missing pieces of a non-right triangle.



h here can be written using the left triangle: $\sin(C) = \frac{h}{a}$ or the right triangle: $\sin(A) = \frac{h}{a}$

Solving each for h we can set them equal to each other and get $a\sin(C) = C\sin(A)$.

Doing this for all three cases we see that:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

This is known as **The Law Of Sines**. Anytime you know one matching pair in a triangle (angle and side across from it), the law of sines can be used.

Alternatively, we could use the Pythagorean Theorem on the left and right triangles to get:

$x^2 + h^2 = a^2$ and $(b-x)^2 + h^2 = c^2$

Solving for h^2 we can set these equal to each other:

$a^2 - x^2 = c^2 - (b-x)^2$

$c^2 = a^2 + b^2 - 2bx$

And we can use the left triangle to get $\cos(C) = \frac{x}{a}$ so we plug in $x = a\cos(C)$ and get:

$c^2 = a^2 + b^2 - 2ab\cos(C)$

This is known as **The Law of Cosines**. If you know 2 or 3 sides of any triangle, you can use the law of cosines to find the third (or any angle you want to focus on by interchanging the variables)

$a^2 = b^2 + c^2 - 2bc\cos(A)$ and $b^2 = a^2 + c^2 - 2ac\cos(B)$

In any triangle, as long as you know three out of the six possible sides and angles, you should be able to find the other three. (The only exception to this is if you know three angles and no sides, since there would be infinitely many answers to this case).

Example 12.5.3

Triangle ABC has constant side $c = 6$ inches, and angle $A = \csc^{-1}(3)$. If side a is increasing by 2 inches per hour, how much is angle C changing by when $a = 12$ inches?

Solution

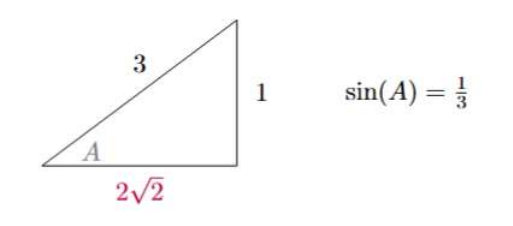
We start this related rates question by collecting:
Given Rate: $a' = 2$
Wanted Rate: C'
When: $a = 12$

Then we need to find the equation that relates a and C.

By the law of sines we know $\dfrac{\sin(A)}{a} = \dfrac{\sin(C)}{c}$

Therefore we can plug in the constant values to get: $\dfrac{\sin\left(\csc^{-1}(3)\right)}{a} = \dfrac{\sin(C)}{6}$

The helper triangle simplies this further:



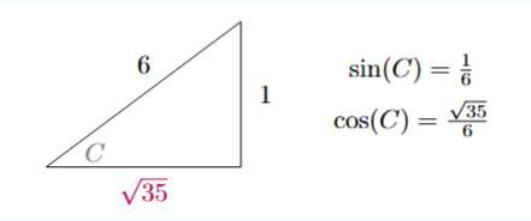
Finally we have our equation: $\dfrac{\frac{1}{3}}{a} = \dfrac{\sin(C)}{6} \longrightarrow 2 = a\sin(C)$

Now we differentiate with respect to time (measured in hours):

$(2 = a\sin(C))'$
$0 = a' \cdot \sin(C) + a \cdot \cos(C) \cdot C'$
$0 = 2 \cdot \sin(C) + 12 \cdot \cos(C) \cdot C'$

We use then when, $a = 12$ in our equation to get $2 = 12\sin(C) \longrightarrow \sin(C) = \frac{1}{6}$

This gives another helper triangle (*neither of these has been a when diagram*)



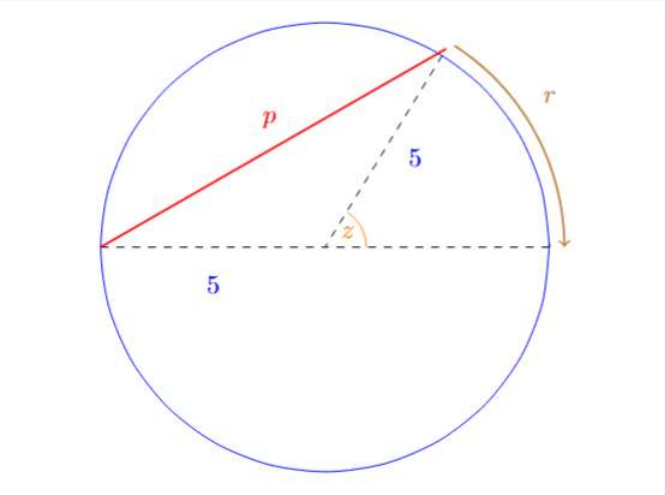
Plugging these in finally gives us $0 = 2 \cdot \dfrac{1}{6} + 12 \cdot \dfrac{\sqrt{35}}{6} \cdot C'$

Since $C' = -\dfrac{1}{6\sqrt{35}}$, we know C **is decreasing at** $\dfrac{1}{6\sqrt{35}}$ **radians per hour.**

Example 12.5.4

A man is attempting to go from one side of the circular pond to the other. If he were to paddle directly across the 10 meter diameter, it would take him 20 seconds. If he were to run around half of the circumference instead, it would take him 20

seconds. Instead of paddling directly across, if he directs his boat at an angle, he can boat a diagonal chord of the circle and then run the rest of the way. What is the shortest amount of time he can make it from one side to the other?



Solution

We want to minimize the total travel time, which would be the sum of each leg's distance over rate.

The rate directly across is $\frac{10\ meters}{20\ seconds} = \frac{1}{2}$ meters per second.

That means the time in the boat is $\frac{p}{\frac{1}{2}} = 2p$ seconds.

The rate to run around the edge is $\frac{10\pi\ meters}{20\ seconds} = \frac{\pi}{2}$ meters per second.

That means the time running is $\frac{r}{\frac{\pi}{2}} = \frac{2r}{\pi}$ seconds

Our optimizing function is thus $T(p, r) = 2p + \frac{2r}{\pi}$

We can relate both of these quantities to the value of angle z in the diagram:

$r = 5z$ is the arc length formula for a fraction of a circle and the law of cosines gives us:

$p^2 = 5^2 + 5^2 - 2 \cdot 5 \cdot 5 \cdot \cos(\pi - z)$

(Fun fact: $\cos(\pi - z) = \cos(\pi)\cos(z) + \sin(\pi)\sin(z) = -\cos(z)$)

$p^2 = 50 + 50\cos(z) \longrightarrow p = 5\sqrt{2 + 2\cos(z)}$

Plugging these in give us our single variable version of T:

$T(z) = 2 \cdot 5\sqrt{2 + 2\cos(z)} + \frac{2 \cdot 5z}{\pi} = 10\sqrt{2 + 2\cos(z)} + \frac{10}{\pi}z$

This has domain $[0, \pi]$ based on our diagram. Now we can differentiate to find our critical points:

$$T'(z) = 10 \cdot \frac{1}{2}(2+2\cos(z))^{-\frac{1}{2}} \cdot (-2\sin(z)) + \frac{10}{\pi} = \frac{10}{\pi} - \frac{10\sin(z)}{\sqrt{2+2\cos(z)}}$$

First $T'(z) = 0 \longrightarrow \frac{10}{\pi} = \frac{10\sin(z)}{\sqrt{2+2\cos(z)}}$

$\pi \sin(z) = \sqrt{2+2\cos(z)}$

$\pi^2 \sin^2(z) = 2 + 2\cos(z)$

$\pi^2 - \pi^2 \cos^2(z) = 2 + 2\cos(z)$

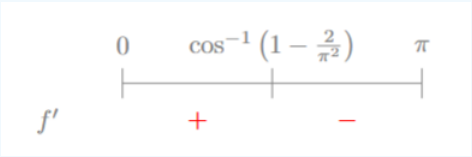
$0 = \pi^2 \cos^2(z) + 2\cos(z) + (2 - \pi^2)$

$\cos(z) = \frac{-2 \pm \sqrt{4 - 4\pi^2(2-\pi^2)}}{2\pi^2} = \frac{-1}{\pi^2} \pm \left(1 - \frac{1}{\pi^2}\right)$

$\cos(z) = 1 - \frac{2}{\pi^2} \longrightarrow z = \cos^{-1}\left(1 - \frac{2}{\pi^2}\right)$ or $\cos(z) = -1 \longrightarrow z = \pi$

Also $T'(z) = \text{DNE}$ when $2 + 2\cos(z) = 0 \longrightarrow z = \pi$

Before we plug in the critical and endpoints, lets make a quick sign chart:

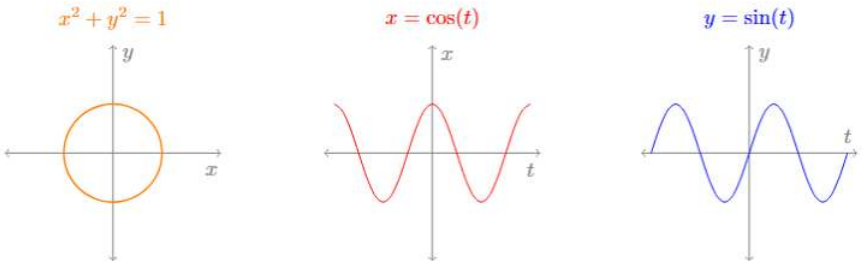


We can see that the minimum time would definitely not occur at the interior critical point, so it must be one of our endpoints. Furthermore, from the start we knew that these both take the same amount of time: $T(0) = T(\pi) =$ **20 seconds**!

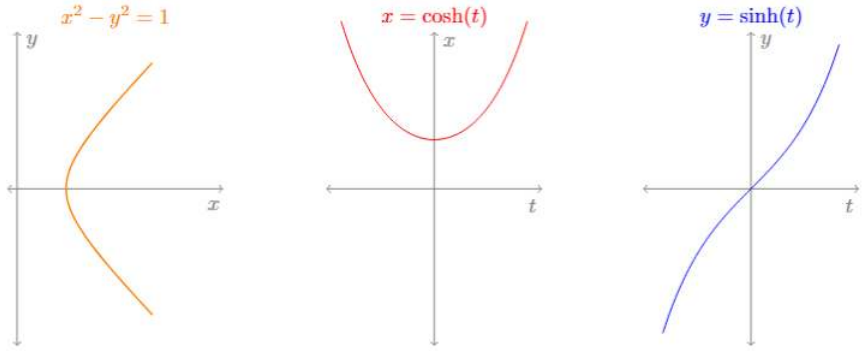
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12.6: Hyperbolic Trigonometric Functions

Its been a while, but remember that we invented the sine and cosine functions to be the x and y coordinates on the unit circle $x^2+y^2=1$, ensuring that: $\cos^2(t)+\sin^2(t)=1$



We can similarly define functions on the unit hyperbola $x^2-y^2=1$, call $x=\cosh(t)$ the function **hyperbolic cosine** and $y=\sinh(t)$ the function **hyperbolic sine**.



Thankfully, these are not incredibly foreign functions.

We see that for $\cosh(t)=\dfrac{e^t+e^{-t}}{2}$ and $\sinh(t)=\dfrac{e^t-e^{-t}}{2}$ we will have

$[\cosh(t)]^2-[\sinh(t)]^2=1$, since:

$$\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2=\frac{e^{2x}+2+e^{-2x}}{4}-\frac{e^{2x}-2+e^{-2x}}{4}=\frac{4}{4}=1$$

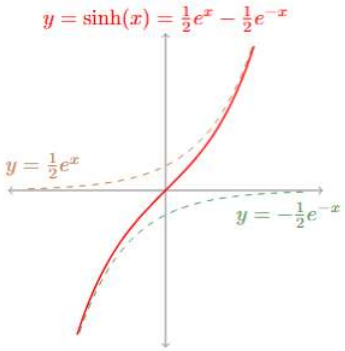
Notice that $\cosh(0)=\dfrac{e^0+e^0}{2}=1$ and $\sinh(0)=\dfrac{e^0-e^0}{2}=0$

(Just like $\cos(0)=1$ and $\sin(0)=0$, there will be more similarities to come!)

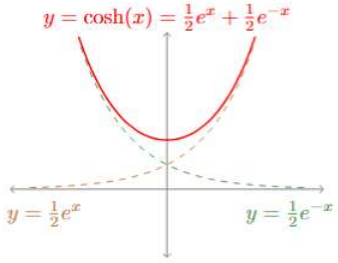
Notice that $\lim\limits_{x\to\infty}e^x=\infty$, $\lim\limits_{x\to-\infty}e^x=0$, $\lim\limits_{x\to\infty}e^{-x}=0$, and $\lim\limits_{x\to-\infty}e^{-x}=+\infty$

Because of this, we find there are no horizontal asymptotes for $\sinh(x)$ or $\cosh(x)$:

$\lim\limits_{x\to\infty}\sinh(x)=\lim\limits_{x\to\infty}\dfrac{1}{2}e^x=+\infty$ and $\lim\limits_{x\to-\infty}\sinh(x)=\lim\limits_{x\to-\infty}-\dfrac{1}{2}e^{-x}=-\infty$



$$\lim_{x\to\infty} \cosh(x) = \lim_{x\to\infty} \frac{1}{2}e^x = +\infty \text{ and } \lim_{x\to-\infty} \cosh(x) = \lim_{x\to-\infty} \frac{1}{2}e^{-x} = +\infty$$



Given these definitions, we can find the derivatives for each of these two:

If $f(t) = \sinh(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t}$, then $f'(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t)$

If $g(t) = \cosh(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$, then $g'(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh(t)$

These two functions are each other's derivative! This is one of the major differences between the *hyperbolic* sine and cosine and their *circular* counterparts. (which are off by a minus sign)

Since we have a hyperbolic sine and cosine, lets go ahead and define the other four:

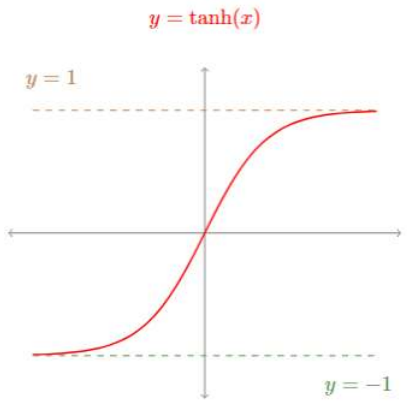
Hyperbolic Tangent: $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

This function is continuous on its entire domain $(-\infty, \infty)$ and has no vertical asymptotes. There are however two horizontal asymptotes:

$$\lim_{x\to\infty} \tanh(x) \overset{LH}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = \lim_{x\to\infty} 1 = 1$$

$$\lim_{x\to-\infty} \tanh(x) \overset{LH}{=} \lim_{x\to-\infty} \frac{e^{-x}}{-e^{-x}} = \lim_{x\to-\infty} -1 = -1$$

Therefore, we have horizontal asymptotes at: $y = 1, y = -1$



The derivative can be found using the quotient rule on $f(x) = \tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$

$$f'(x) = \frac{\cosh(x)\cdot\cosh(x) - \sinh(x)\cdot\sinh(x)}{(\cosh(x))^2} = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}$$

If we go with the intuitive definition of $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)}$ (which we will in a few pages), then we find that our hyperbolic tangent derivative matches that of our circular tangent function:

$(\tan(x))' = \sec^2(x)$ and $(\tanh(x))' = \operatorname{sech}^2(x)$

Hyperbolic Cotangent: $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

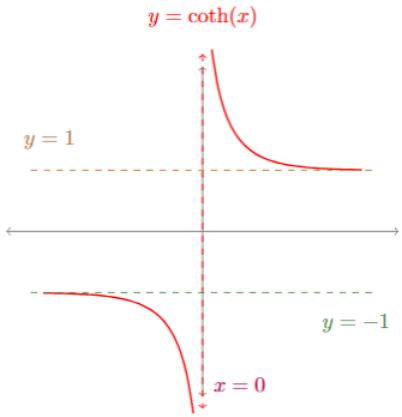
This function is continuous on its entire domain $(-\infty, 0)\cup(0, \infty)$ and has $\lim\limits_{x\to 0^+} \coth(x) = \dfrac{2}{0^+} = +\infty$ and $\lim\limits_{x\to 0^-} \coth(x) = \dfrac{2}{0^-} = -\infty$
so vertical asymptote at $x = 0$.

There are again two horizontal asymptotes:

$$\lim_{x\to\infty} \coth(x) \overset{LH}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = \lim_{x\to\infty} 1 = 1$$

$$\lim_{x\to-\infty} \coth(x) \overset{LH}{=} \lim_{x\to-\infty} \frac{-e^{-x}}{e^{-x}} = \lim_{x\to-\infty} -1 = -1$$

Therefore, we have horizontal asymptotes at: $y = 1, y = -1$



The derivative can be found using the quotient rule on $f(x) = \coth(x) = \dfrac{\cosh(x)}{\sinh(x)}$

$$f'(x) = \frac{\sinh(x)\cdot\sinh(x) - \cosh(x)\cdot\cosh(x)}{(\sinh(x))^2} = \frac{\sinh^2(x) - \cosh^2(x)}{\sinh^2(x)} = \frac{-1}{\sinh^2(x)}$$

If we go with the intuitive definition of $\operatorname{csch}(x) = \dfrac{1}{\sinh(x)}$ (which again, we will in a few pages), then we find that our hyperbolic cotangent derivative matches that of our circular tangent function:

$(\cot(x))' = -\csc^2(x)$ and $(\coth(x))' = -\operatorname{csch}^2(x)$

Hyperbolic Secant: $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)} = \dfrac{2}{e^x + e^{-x}}$

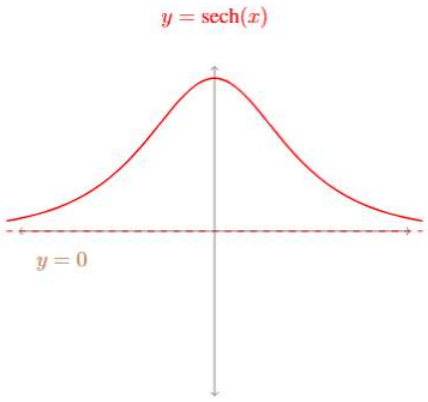
This function is continuous on its entire domain $(-\infty, \infty)$ and has no vertical asymptotes.

We can check for the horizontal asymptotes:

$$\lim_{x\to\infty} \operatorname{sech}(x) = \frac{2}{\infty + 0} = 0$$

$$\lim_{x\to-\infty} \operatorname{sech}(x) = \frac{2}{0 + \infty} = 0$$

Therefore, we have one horizontal asymptote at: $y = 0$



The derivative can be found using the quotient rule on $f(x) = \operatorname{sech}(x) = \dfrac{1}{\cosh(x)}$

$$f'(x) = \frac{\cosh(x) \cdot 0 - 1 \cdot \sinh(x)}{(\cosh(x))^2} = \frac{-\sinh(x)}{\cosh^2(x)} = -\operatorname{sech}(x)\tanh(x)$$

Notice that for secant (and cosine) the our hyperbolic derivative is off by a minus sign from the pattern for the derivative of our circular cosine or secant function:

$(\cos(x))' = -\sin(x)$ and $(\cosh(x))' = \sinh(x)$

$(\sec(x))' = \sec(x)\tan(x)$ and $(\operatorname{sech}(x))' = -\operatorname{sech}(x)\tanh(x)$

Hyperbolic Cosecant: $\operatorname{csch}(x) = \dfrac{1}{\sinh(x)} = \dfrac{2}{e^x - e^{-x}}$

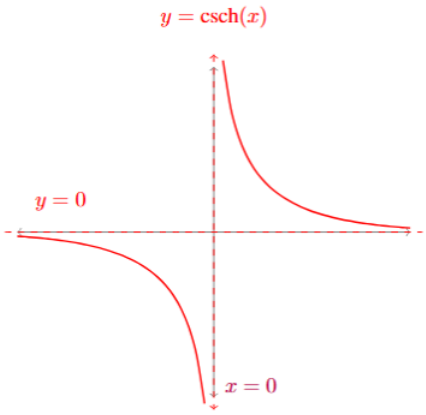
This function is continuous on its entire domain $(-\infty, 0) \cup (0, \infty)$ and has $\lim\limits_{x \to 0^+} \operatorname{csch}(x) = \dfrac{2}{0^+} = +\infty$ and $\lim\limits_{x \to 0^-} \operatorname{csch}(x) = \dfrac{2}{0^-} = -\infty$
so vertical asymptote at $x = 0$}.

There is again one horizontal asymptote

$$\lim_{x \to \infty} \operatorname{csch}(x) = \frac{2}{\infty - 0} = 0$$

$$\lim_{x \to -\infty} \operatorname{csch}(x) = \frac{2}{0 - \infty} = 0$$

Thus, the horizontal asymptote is: $y = 0$



The derivative can be found using the quotient rule on $f(x) = \operatorname{csch}(x) = \dfrac{1}{\sinh(x)}$

$$f'(x) = \frac{\sinh(x)\cdot 0 - 1\cdot\cosh(x)}{(\sinh(x))^2} = \frac{-\cosh(x)}{\sinh^2(x)} = -\operatorname{csch}(x)\coth(x)$$

Finally, our hyperbolic cosecant derivative matches that of our circular cosecant function:

$(\csc(x))' = -\csc(x)\cot(x)$ and $(\operatorname{csch}(x))' = -\operatorname{csch}(x)\coth(x)$

Each of these functions can be used in combination with the other functions we have learned as well as the differentiation rules.

Example 12.6.1

Find the line tangent to $f(x) = (\cosh(x))^3$ at $x = \ln\left(\dfrac{1}{2}\right)$

Solution

$$f\left(\ln\left(\frac{1}{2}\right)\right) = \left[\cosh\left(\ln\left(\frac{1}{2}\right)\right)\right]^3 = \left[\frac{e^{\ln(\frac{1}{2})} + e^{-\ln(\frac{1}{2})}}{2}\right]^3 = \left[\frac{5}{4}\right]^3 = \frac{125}{64}$$

$$f'(x) = 3[\cosh(x)]^2\cdot\sinh(x) \longrightarrow f'\left(\ln\left(\frac{1}{2}\right)\right) = 3\cdot\left[\frac{5}{4}\right]^2\cdot\left(-\frac{3}{4}\right) = -\frac{225}{64}$$

Thus the tangent line is $y - f\left(\ln\left(\dfrac{1}{2}\right)\right) = f'\left(\ln\left(\dfrac{1}{2}\right)\right)\left(x - \ln\left(\dfrac{1}{2}\right)\right)$

$$y - \frac{125}{64} = -\frac{225}{64}\left(x - \ln\left(\frac{1}{2}\right)\right)$$

And we can use them in any of our previous application settings as well.

Example 12.6.2

Estimate $\sinh(\ln(1.92))$

Solution

Using $f(x) = \sinh(\ln(x))$ we see this is $f(1.92)$ and close to $a = 2$

So we want to find $L(x) = f(2) + f'(2)(x-2)$

$f(2) = \sinh(\ln(2)) = \dfrac{e^{\ln(2)} - e^{-\ln(2)}}{2} = \dfrac{2 - \frac{1}{2}}{2} = \dfrac{3}{4}$

$f'(x) = \cosh(\ln(x)) \cdot \dfrac{1}{x}$ so we get:

$f'(2) = \cosh(\ln(2)) \cdot \dfrac{1}{2} = \dfrac{e^{\ln(2)} + e^{-\ln(2)}}{4} = \dfrac{2 + \frac{1}{2}}{4} = \dfrac{5}{8}$

$f(x) \approx f(2) + f'(2)(x-2) = \dfrac{3}{4} + \dfrac{5}{8}(x-2)$

$f(1.92) \approx \dfrac{3}{4} + \dfrac{5}{8}(1.92-2) = \dfrac{3}{4} - \dfrac{5}{100} = \dfrac{70}{100} = 0.7$

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12.7: Inverse Hyperbolic Trigonometric Functions

Just like the six circular trigonometric functions, we can find inverses for each of the six hyperbolic trig functions (even if we have to adjust a domain or two)

We saw in the previous section that the graph of $f(x) = \sinh(x)$ is a continuous function which passes the horizontal and vertical line tests. The domain and range of this function are both clearly $(-\infty, \infty)$. This means that $f^{-1}(x)$ exists for all real x, can we find it?

$x = \sinh(y) \longrightarrow 2x = e^y - e^{-y}$

Multiply both sides by e^y to get: $2xe^y = e^{2y} - 1$

Moving everything to one side this looks like $e^{2y} - 2xe^y - 1 = 0$

Substitute $u = e^y$ and this becomes: $u^2 - 2xu - 1 = 0$

Then we use the quadratic formula for u:

$$u = \frac{2x \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Switching back $u = e^y$ this gives: $e^y = x \pm \sqrt{x^2 + 1}$

Since $x < \sqrt{x^2 + 1}$ for all x, this only works for the $+$, giving us: $y = \ln(x + \sqrt{x^2 + 1})$

Thus we have found explicitly $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$

If you want to check this answer, we can "easily" compose the two:

$$\sinh\left(\ln(x + \sqrt{x^2 + 1})\right) = \frac{e^{\ln(x+\sqrt{x^2+1})} - e^{-\ln(x+\sqrt{x^2+1}}}{2} = \frac{x + \sqrt{x^2 + 1} - \frac{1}{x+\sqrt{x^2+1}}}{2}$$

$$= \frac{(x + \sqrt{x^2 + 1})^2 - 1}{2(x + \sqrt{x^2 + 1})} = \frac{\left[x^2 + 2x\sqrt{x^2 + 1} + (x^2 + 1)\right] - 1}{2(x + \sqrt{x^2 + 1})} = \frac{2x^2 + 2x\sqrt{x^2 + 1}}{2(x + \sqrt{x^2 + 1})} = x$$

What does $f(x) = \sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$ look like? We can start with critical points:

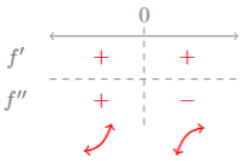
$$f'(x) = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \cdot 2x\right) = \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

Thus $f'(x)$ always exists and never equals zero. *There are no critical points.*

Taking another derivative we get: $f''(x) = -\frac{1}{2}(x^2 + 1)^{-\frac{3}{2}} \cdot 2x = -\frac{x}{(x^2 + 1)^{\frac{3}{2}}}$

We see $f''(x)$ always exists, and $f''(x) = 0$ at $x = 0$, so there is one inflection point: $x = 0$

This means the sign chart will be simple:



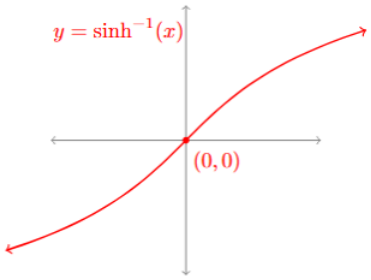
Since we are continuous and defined everywhere, there are *no vertical asymptotes*

To check for horizontal asymptotes, we take each infinite limit:

$$\lim_{x\to\infty} \ln(x+\sqrt{x^2+1}\,) = \ln(+\infty+\infty) = \ln(+\infty) = +\infty$$

$$\lim_{x\to-\infty} \ln(x+\sqrt{x^2+1}\,) = \ln\left(\lim_{x\to-\infty} \frac{x+\sqrt{x^2+1}}{1}\cdot\frac{x-\sqrt{x^2+1}}{x-\sqrt{x^2-1}}\right)$$
$$= \ln\left(\lim_{x\to-\infty} \frac{-1}{x-\sqrt{x^2+1}}\right) = \ln\left(\frac{-1}{-\infty-\infty}\right) = \ln(0^+) = -\infty$$

There are *no horizontal asymptotes*, so we plot our only point $f(0)=0$ and get:



Similarly, we can derive the inverse of $f(x)=\cosh(x)$ as long as we restrict its domain to $[0,\infty)$.

$x=\cosh(y) \longrightarrow 2x = e^y + e^{-y}$

$2xe^y = e^{2y}+1$

$0 = e^{2y} - 2xe^y + 1$

$0 = u^2 - 2xu + 1$

$$u = \frac{2x\pm\sqrt{4x^2-4}}{2} = x\pm\sqrt{x^2-1}$$

$e^y = x+\sqrt{x^2-1}$

$y = \ln\left(x+\sqrt{x^2-1}\right)$

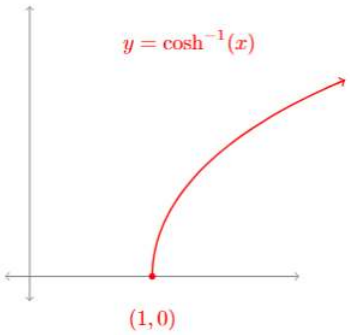
Therefore we know: $\cosh^{-1}(x) = \ln(x+\sqrt{x^2-1}\,)$ with domain $[1,\infty)$

To sketch $\cosh^{-1}(x)$ we can go through the same steps as we did for $\sinh^{-1}(x)$ to get:

$$(\cosh^{-1}(x))' = \frac{1}{\sqrt{x^2-1}} \longrightarrow \text{ critical point: } x=1$$

$(\cosh^{-1}(x))'' = -\dfrac{x}{(x^2-1)^{\frac{3}{2}}} \longrightarrow$ inflection point: $x = 1$

We don't need a sign chart, since there is only one interval $(1, \infty)$ and $f' > 0$ while $f'' < 0$



Next, we will derive the inverse of $f(x) = \tanh(x)$ on its natural domain $(-\infty, \infty)$.

$x = \tanh(y) \longrightarrow x = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$

$x(e^y + e^{-y}) = e^y - e^{-y}$

$xe^{2y} + x = e^{2y} - 1$

$(x-1)e^{2y} = -x - 1$

$e^{2y} = \dfrac{1+x}{1-x}$

$2y = \ln\left(\dfrac{1+x}{1-x}\right)$

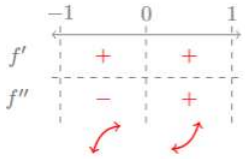
$y = \ln\sqrt{\dfrac{1+x}{1-x}}$

Therefore we know: $\tanh^{-1}(x) = \ln\sqrt{\dfrac{1+x}{1-x}}$ with domain $(-1, 1)$

To sketch $\tanh^{-1}(x)$ we can go through the same steps as we did for the others:

$(\tanh^{-1}(x))' = \dfrac{1}{1-x^2} \longrightarrow$ critical points: $x = \pm 1$

$(\tanh^{-1}(x))'' = \dfrac{2x}{(1-x^2)^2} \longrightarrow$ inflection points: $x = 0, \pm 1$

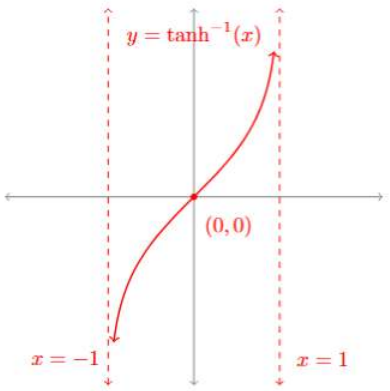


This is clearly continuous on its domain $(-1, 1)$, but we can check the behavior at each endpoint with limits:

$$\lim_{x\to 1^-}\tanh^{-1}(x)=\frac{1}{2}\ln\left(\lim_{x\to 1^-}\frac{1+x}{1-x}\right)=\frac{1}{2}\ln\left(\frac{2}{0^+}\right)=+\infty$$

$$\lim_{x\to -1^+}\tanh^{-1}(x)=\frac{1}{2}\ln\left(\lim_{x\to -1^+}\frac{1+x}{1-x}\right)=\frac{1}{2}\ln\left(\frac{0^+}{2}\right)=-\infty$$

Therefore the graph looks like



These three results actually yield a few nice identities for us. Similar to the results of a helper triangle, we want to be able to compose hyperbolic trigonometric functions and their inverses, but there is no triangle here for hyperbolic trigonometric functions. Remember the inverse function derivative formula:

First, the function $f(x)=\sinh(x)$ has derivative $f'(x)=\cosh(x)$ and inverse $f^{-1}(x)=\ln(x+\sqrt{x^2+1})$, with $\left(f^{-1}\right)'(x)=\dfrac{1}{\sqrt{x^2+1}}$

$$\left(f^{-1}\right)'(x)=\frac{1}{f'\left(f^{-1}(x)\right)}=\frac{1}{\cosh\left(\sinh^{-1}(x)\right)}=\frac{1}{\sqrt{x^2+1}}$$

So we get $\cosh\left(\sinh^{-1}(x)\right)=\sqrt{x^2+1}$

We already know $\sinh\left(\sinh^{-1}(x)\right)=x$ for all x, so we can find any of the six hyperbolic trig functions with $\sinh^{-1}(x)$ inside.

Also, the function $g(x)=\cosh(x)$ has derivative $g'(x)=\sinh(x)$ and inverse $g^{-1}(x)=\ln(x+\sqrt{x^2-1})$, with $\left(g^{-1}\right)'(x)=\dfrac{1}{\sqrt{x^2-1}}$

$$\left(g^{-1}\right)'(x)=\frac{1}{g'\left(g^{-1}(x)\right)}=\frac{1}{\sinh\left(\cosh^{-1}(x)\right)}=\frac{1}{\sqrt{x^2-1}}$$

So we get $\sinh\left(\cosh^{-1}(x)\right)=\sqrt{x^2-1}$

We already know $\cosh\left(\cosh^{-1}(x)\right)=x$ for all $x\geq 1$, so we can find any of the six hyperbolic trig functions with \cosh^{-1}(x) inside.

Last, the function $h(x)=\tanh(x)$ has derivative $h'(x)=\operatorname{sech}^2(x)$ and inverse $h^{-1}(x)=\ln\sqrt{\dfrac{1+x}{1-x}}$, with $\left(h^{-1}\right)'(x)=\dfrac{1}{1-x^2}$

$$\left(h^{-1}\right)'(x)=\frac{1}{h'\left(h^{-1}(x)\right)}=\frac{1}{\operatorname{sech}^2\left(\tanh^{-1}(x)\right)}=\frac{1}{1-x^2}$$

Square rooting here gives $\operatorname{sech}\left(\tanh^{-1}(x)\right)=\sqrt{1-x^2}$

Taking the reciprocal finally gets us to: $\cosh\left(\tanh^{-1}(x)\right)=\dfrac{1}{\sqrt{1-x^2}}$

And since $x=\tanh\left(\tanh^{-1}(x)\right)=\dfrac{\sinh\left(\tanh^{-1}(x)\right)}{\cosh\left(\tanh^{-1}(x)\right)}=\sinh\left(\tanh^{-1}(x)\right)\sqrt{1-x^2}$

This gives $\sinh\left(\tanh^{-1}(x)\right)=\dfrac{x}{\sqrt{1-x^2}}$ for all $-1<x<1$ so we can find any of the six hyperbolic trig functions with $\tanh^{-1}(x)$ inside

The remaining three can also be differentiated by using basic chain rule and identities:

$$y=\operatorname{csch}^{-1}(x)\longleftrightarrow\operatorname{csch}(y)=x\longleftrightarrow\sinh(y)=\frac{1}{x}\longleftrightarrow y=\sinh^{-1}\left(\frac{1}{x}\right)$$

Thus, we have $\operatorname{csch}^{-1}(x)=\sinh^{-1}\left(\dfrac{1}{x}\right)=\ln\left(\dfrac{1}{x}+\sqrt{\dfrac{1}{x^2}+1}\right)$ for $x\neq 0$

$$\left(\operatorname{csch}^{-1}(x)\right)'=\left(\sinh^{-1}(x^{-1})\right)'=\frac{1}{\sqrt{(x^{-1})^2+1}}\cdot(-x^{-2})=-\frac{1}{\sqrt{x^2+x^4}}$$

$$\sinh\left(\operatorname{csch}^{-1}(x)\right)=\sinh\left(\sinh^{-1}\left(\frac{1}{x}\right)\right)=\frac{1}{x}=x^{-1}$$

$$\cosh\left(\operatorname{csch}^{-1}(x)\right)=\cosh\left(\sinh^{-1}\left(\frac{1}{x}\right)\right)=\sqrt{\left(\frac{1}{x}\right)^2+1}=\sqrt{x^{-2}+1}$$

$$y=\operatorname{sech}^{-1}(x)\longleftrightarrow\operatorname{sech}(y)=x\longleftrightarrow\cosh(y)=\frac{1}{x}\longleftrightarrow y=\cosh^{-1}\left(\frac{1}{x}\right)$$

Thus, we have $\operatorname{sech}^{-1}(x)=\cosh^{-1}\left(\dfrac{1}{x}\right)=\ln\left(\dfrac{1}{x}+\sqrt{\dfrac{1}{x^2}-1}\right)$ for $0<x<1$

$$\left(\operatorname{sech}^{-1}(x)\right)'=\left(\cosh^{-1}(x^{-1})\right)'=\frac{1}{\sqrt{(x^{-1})^2-1}}\cdot(-x^{-2})=-\frac{1}{\sqrt{x^2-x^4}}$$

$$\sinh\left(\operatorname{sech}^{-1}(x)\right)=\sinh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)=\sqrt{\left(\frac{1}{x}\right)^2-1}=\sqrt{x^{-2}-1}$$

$$\cosh\left(\operatorname{sech}^{-1}(x)\right)=\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)=\frac{1}{x}=x^{-1}$$

$$y=\coth^{-1}(x)\longleftrightarrow\coth(y)=x\longleftrightarrow\tanh(y)=\frac{1}{x}\longleftrightarrow y=\tanh^{-1}\left(\frac{1}{x}\right)$$

Thus, we have $\coth^{-1}(x)=\tanh^{-1}\left(\dfrac{1}{x}\right)=\ln\sqrt{\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}}=\ln\sqrt{\dfrac{x+1}{x-1}}$ for $x>1$

$$\left(\coth^{-1}(x)\right)'=\left(\tanh^{-1}(x^{-1})\right)'=\frac{1}{1-(x^{-1})^2}\cdot(-x^{-2})=\frac{1}{1-x^2}$$

$$\sinh\left(\coth^{-1}(x)\right)=\sinh\left(\tanh^{-1}\left(\frac{1}{x}\right)\right)=\frac{\frac{1}{x}}{\sqrt{1-\left(\frac{1}{x}\right)^2}}=\frac{1}{\sqrt{x^2-1}}$$

$$\cosh\left(\coth^{-1}(x)\right) = \cosh\left(\tanh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} = \frac{x}{\sqrt{x^2-1}}$$

Now you can solve most hyperbolic trigonometric equations by using inverses, as well as take any limits or derivatives involving their inverses.

Example 12.7.1

Solve $4\tanh^2(x^3) = 1$

Solution

This easily becomes two equations by undoing the square: $\tanh(x^3) = \pm\frac{1}{2}$

Apply $\tanh^{-1}(x) = \ln\sqrt{\frac{1+x}{1-x}}$ to both sides:

$\tanh(x^3) = \frac{1}{2}$ and $\tanh(x^3) = -\frac{1}{2}$

$x^3 = \tanh^{-1}\left(\frac{1}{2}\right)$ and $x^3 = \tanh^{-1}\left(-\frac{1}{2}\right)$

$x^3 = \ln\sqrt{3}$ and $x^3 = -\ln\sqrt{3}$

$x = \sqrt[3]{\ln\sqrt{3}}$ and $x = \sqrt[3]{-\ln\sqrt{3}}$

So $x = \pm\sqrt[3]{\ln\sqrt{3}}$

Example 12.7.2

Find all horizontal asymptotes of $y = \frac{1}{\coth^{-1}(x)} \cdot \operatorname{csch}^{-1}(x)$

Solution

Notice that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both agree at every step below, so we shorten our work to $\lim_{x\to\pm\infty} f(x)$

$$\lim_{x\to\pm\infty} \operatorname{csch}^{-1}(x) = \lim_{x\to\pm\infty} \sinh^{-1}\left(\frac{1}{x}\right) = \sinh^{-1}(0) = 0$$

$$\lim_{x\to\pm\infty} \coth^{-1}(x) = \lim_{x\to\pm\infty} \tanh^{-1}\left(\frac{1}{x}\right) = \tanh^{-1}(0) = 0$$

This limit is $\frac{0}{0}$, so we can use L'Hopital:

$$\lim_{x\to\pm\infty} f(x) \overset{LH}{=} \lim_{x\to\pm\infty} \frac{-\frac{1}{\sqrt{x^4+x^2}}}{\frac{1}{1-x^2}} = \lim_{x\to\pm\infty} \frac{x^2-1}{\sqrt{x^4+x^2}}$$

This time its $\frac{\infty}{\infty}$, and we can save some time by using the Lebron James Rule:

$$\lim_{x\to\pm\infty} f(x) \overset{LJ}{=} \lim_{x\to\pm\infty} \frac{x^2}{\sqrt{x^4}} = \lim_{x\to\pm\infty} 1 = 1$$

So the **horizontal asymptote is** $y = 1$

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CHAPTER OVERVIEW

13: Integrals

> *"Success is the sum of details" - Harvey Firestone*

In a seemingly unrelated turn, we shift our focus to a new question: how to find the area under a curve. Chapter thirteen explores exactly how we can use limits to find exact areas. We start by refreshing ourselves on our area formulas from geometry, and soon run into limitations for some functions. We start by trying to estimate the area under the curve by using rectangles, and soon after learn that if we incorporate a limit we have defined a new object: the definite integral. This tool (much like the limit definition of derivative) is a cumbersome tool for finding the area under a curve.

Just like for derivatives, we quickly look for patterns to help avoid doing as much work. We discuss the properties of integrals, and how they are strangely similar to some of the properties of derivatives. This is no coincidence, as we discover the Fundamental Theorem of Calculus: derivatives and integrals are opposites. We hijack the integral sign to create the indefinite integral which stands for anti-derivatives in general. After a little bit of practice, we realize that indeed many of our differentiation skills can double as anti-derivative skills.

Lastly we turn back to the Fundamental Theorem of Calculus and how it applies to us solving definite integrals. We can see the wide variety of different functions we can integrate, and even some real world applications to suggest why we would like to do so. Just be cautious that there are many more tricks we will need before we are professional anti-derivative takers.

13.1: Areas and Distances
13.2: Definite Integrals
13.3: The Fundamental Theorem of Calculus
13.4: Anti-Derivatives and the Indefinite Integral
13.5: More Definite Integrals
13.6: Anti-derivative Applications

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13.1: Areas and Distances

Before we get too deep into finding areas, we will discuss **sigma notation**.

We write $\sum_{k=1}^{n} f(k)$ to represent the sum $f(1)+f(2)+\cdots+f(n-1)+f(n)$

The notation on the $\sum$ at the bottom tells you the index variable (k) that you are counting by and the starting number to count from (1).
The number at the top of the $\sum$ represents where to stop counting ($k=n$).
The function inside of the $\sum$ tells you what to do to each value of k before you add them up.

For example, $\sum_{k=3}^{6} k^2$ becomes $3^2+4^2+5^2+6^2=86$

While this is reasonable to handle for small finite sums, it would not be fun to find $\sum_{k=1}^{5000} k^2$ by hand.

Thankfully, there are a few formulas that we can use to help:

$$\sum_{k=1}^{n} 1 = n \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Not that you would, but you could use this for our previous example:

$$\sum_{k=3}^{6} k^2 = \left(\sum_{k=1}^{6} k^2\right) - \left(\sum_{k=1}^{2}\right) = \frac{6\cdot 7\cdot 13}{6} - \frac{2\cdot 3\cdot 5}{6} = 86$$

Example 13.1.1

Evaluate $\sum_{k=1}^{10} 4k^3-(k+1)^2$

Solution

$$\sum_{k=1}^{10} 4k^3-(k+1)^2 = \sum_{k=1}^{10} \left[4k^3-k^2-2k-1\right]$$

$$= 4\left(\sum_{k=1}^{10} k^3\right) - \left(\sum_{k=1}^{10} k^2\right) - 2\left(\sum_{k=1}^{10} k\right) - \left(\sum_{k=1}^{10} 1\right)$$

$$= 4\left(\frac{10^2\cdot 11^2}{4}\right) - \left(\frac{10\cdot 11\cdot 21}{6}\right) - 2\left(\frac{10\cdot 11}{2}\right) - (10) = 11595$$

The big topic of this section is finding the area underneath a function $y=f(x)$ for $a\le x\le b$.

For now, we can safely skip over the "why" we need to find the area under a curve, and just assure you that we will indeed want it for some reason in the future.

If you are lucky, you will be asked to find an area that you have a geometric formula for:

Example 13.1.2

Find the area under each function for the provided interval.

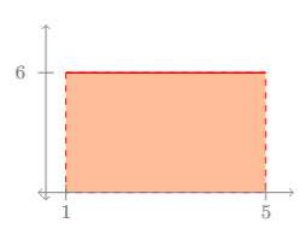
(a) $y = 6$ for $1 \le x \le 5$

(b) $y = \dfrac{2x+7}{3}$ for $-2 \le x \le 7$

(c) $y = \sqrt{9-x^2}$ for $-3 \le x \le 3$

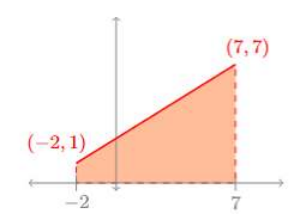
Solution

(a) We sketch the area:



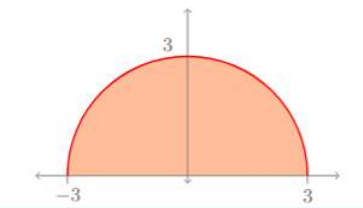
The area of this rectangle is $A = b \cdot h = 4 \cdot 6 = 24$

(b) We sketch the area:



The area of this trapezoid is $A = \dfrac{b_1+b_2}{2} \cdot h = \dfrac{1+7}{2} \cdot 9 = 36$

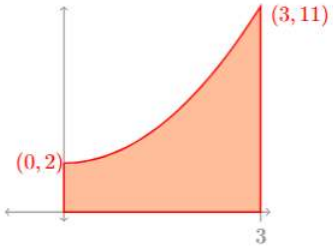
(c) We sketch the area:



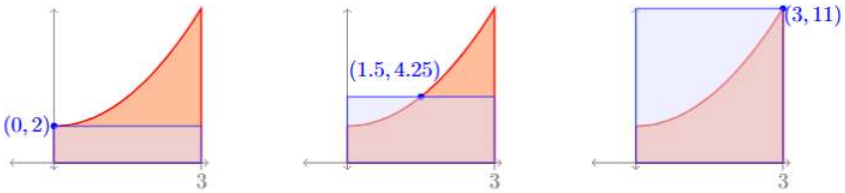
The area of this semi-circle is $A = \dfrac{1}{2}\pi r^2 = \dfrac{\pi}{2} \cdot 3^2 = \dfrac{9\pi}{2}$

Sadly, we dont have a formula for the area of many famous functions (in fact most). In these cases, we will settle for estimating the area under the curves.

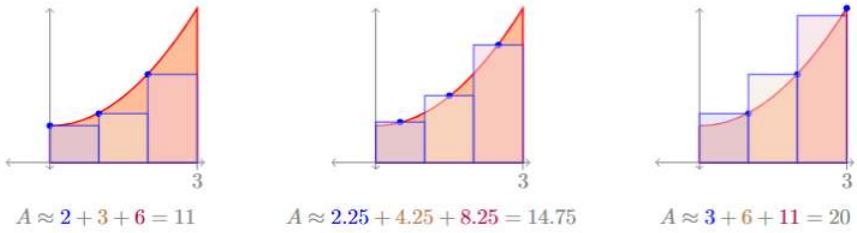
What is the area under $y = x^2 + 2$ for $0 \le x \le 3$?



We can estimate this area by using one rectangle, but obviously the estimate would depend on where we determined the height of that rectangle in the interval $[0, 3]$:

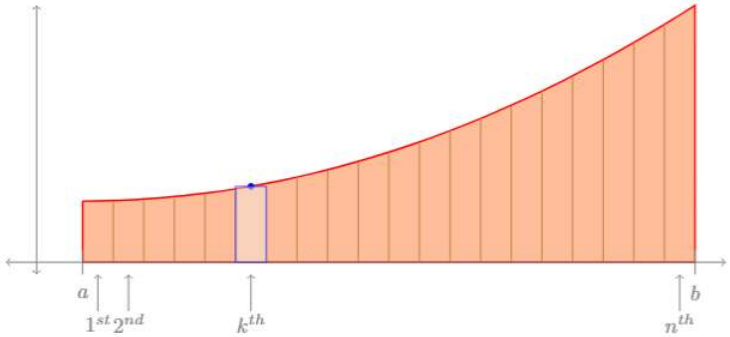


None of these seem very accurate (we can see how far off the left and right sided rectangles were visually), but we can improve this approximation by using *more rectangles*. Lets try using three:



No matter how you pick "where" the rectangle height is determined, one thing is certain:
more rectangles will almost always increase the accuracy of your approximation.

Lets derive formulas to keep things more organized when we use a large number of rectangles:



Each of these n rectangles have a width of $\Delta x = \dfrac{b-a}{n}$

The right endpoint of the kth rectangle is $x_k = a + k\Delta x$

The midpoint of the kth rectangle is $x_k = a + \left(k - \dfrac{1}{2}\right)\Delta x$

The left endpoint of the kth rectangle is $x_k = a + (k-1)\Delta x$

Depending on which x_k you choose, of the area will be approximately $\sum_{k=1}^{n} f(x_k)\Delta x$

Lets estimate the area under $y = x^2 + 2$ for $0 \le x \le 3$ by using 6 rectangles and right endpoints:

$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$ and $x_k = a + k\Delta x = 0 + k \cdot \frac{1}{2} = \frac{k}{2}$

Plugging in $k = 1, 2, \ldots 6$ gives $x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2, x_5 = \frac{5}{2}, x_6 = 3$ and so

$$A \approx \sum_{k=1}^{n} f(x_k)\Delta x = f\left(\frac{1}{2}\right) \cdot \frac{1}{2} + f(1) \cdot \frac{1}{2} + f\left(\frac{3}{2}\right) \cdot \frac{1}{2} + f(2) \cdot \frac{1}{2} + f\left(\frac{5}{2}\right) \cdot \frac{1}{2} + f(3) \cdot \frac{1}{2}$$

$$= \left[\frac{9}{4} + 3 + \frac{17}{4} + 5 + \frac{33}{4} + 10\right] \cdot \frac{1}{2} = \frac{131}{8}$$

If we use 6 rectangles and left endpoints instead this looks like:

$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$ and $x_k = a + (k-1)\Delta x = 0 + (k-1) \cdot \frac{1}{2} = \frac{k-1}{2}$

Plugging in $k = 1, 2, \ldots 6$ gives $x_1 = 0, x_2 = \frac{1}{2}, x_3 = 1, x_4 = \frac{3}{2}, x_5 = 2, x_6 = \frac{5}{2}$ and so

$$A \approx \sum_{k=1}^{n} f(x_k)\Delta x = f(0) \cdot \frac{1}{2} + f\left(\frac{1}{2}\right) \cdot \frac{1}{2} + f(1) \cdot \frac{1}{2} + f\left(\frac{3}{2}\right) \cdot \frac{1}{2} + f(2) \cdot \frac{1}{2} + f\left(\frac{5}{2}\right) \cdot \frac{1}{2}$$

$$= \left[2 + \frac{9}{4} + 3 + \frac{17}{4} + 5 + \frac{33}{4}\right] \cdot \frac{1}{2} = \frac{99}{8}$$

Lets try a few more examples:

Example 13.1.3

Estimate the area under each function on the provided interval by using the given number of rectangles.

(a) $f(x) = \frac{1}{x}$ over $1 \le x \le 5$ using 4 rectangles and midpoints.

(b) $f(x) = 4\cos^2(\pi x)$ from $\frac{1}{3} \le x \le 1$ using 4 rectangles and left endpoints.

Solution

(a) $\Delta x = \frac{5-1}{4} = 1$ and $x_k = 1 + \left(k - \frac{1}{2}\right) 1 = k + \frac{1}{2}$

Thus $x_1 = \frac{3}{2}, x_2 = \frac{5}{2}, x_3 = \frac{7}{2}, x_4 = \frac{9}{2}$

So $A \approx f\left(\frac{3}{2}\right) \cdot 1 + f\left(\frac{5}{2}\right) \cdot 1 + f\left(\frac{7}{2}\right) \cdot 1 + f\left(\frac{9}{2}\right) \cdot 1$

$$= \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} = \frac{496}{315}$$

(b) $\Delta x = \frac{1-\frac{1}{3}}{4} = \frac{1}{6}$ and $x_k = \frac{1}{3} + (k-1)\frac{1}{6} = \frac{k+1}{6}$

Thus $x_1 = \frac{1}{3}, x_2 = \frac{1}{2}, x_3 = \frac{2}{3}, x_4 = \frac{5}{6}$

So $A \approx f\left(\frac{1}{3}\right)\cdot\frac{1}{6} + f\left(\frac{1}{2}\right)\cdot\frac{1}{6} + f\left(\frac{2}{3}\right)\cdot\frac{1}{6} + f\left(\frac{5}{6}\right)\cdot\frac{1}{6}$

$= (1+0+1+3)\cdot\frac{1}{6} = \frac{5}{6}$

If n gets large, we can use the shortcuts we saw earlier:

Example 13.1.4

Estimate the area under $f(x) = 3x^2$ over $0 \le x \le 1$ using 100 rectangles and right endpoints.

Solution

$\Delta x = \frac{1-0}{100} = \frac{1}{100}$ and $x_k = 0 + k\cdot\frac{1}{100} = \frac{k}{100}$

Listing all the x_k is tedious, but we can find $f(x_k) = f\left(\frac{k}{100}\right) = 3\left(\frac{k}{100}\right)^2 = \frac{3k^2}{10000}$

So $A \approx \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{100} \frac{3k^2}{10000}\cdot\frac{1}{100} = \frac{3}{1000000}\sum_{k=1}^{100} k^2$

$= \frac{3}{1000000}\cdot\frac{100\cdot 101\cdot 201}{6} = \frac{20301}{20000}$

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13.2: Definite Integrals

To denote the exact area under a curve, we introduce *integral notation*. The **definite integral** $\int_a^b f(x)dx$ represents the area between $y = f(x)$, the x-axis, and the two lines $x = a$ and $x = b$.

This means many of the same calculations we used before can be asked again in different clothing:

Example 13.2.1

Evaluate the following definite integrals

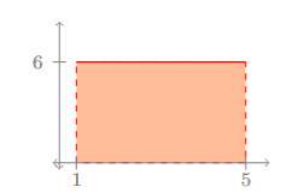
(a) $\int_1^5 6dx$

(b) $\int_{-2}^7 \frac{2x+7}{3}dx$

(c) $\int_{-3}^3 \sqrt{9-x^2}\,dx$

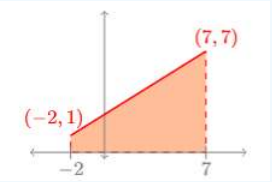
Solution

(a) We sketch the graph of $y = 6$ over $[1, 5]$:



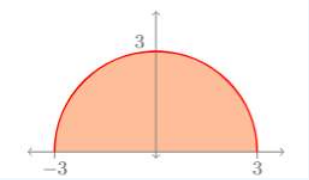
The area of this rectangle is $A = b \cdot h = 4 \cdot 6 = 24$

(b) We sketch the graph of $y = \frac{2x+7}{3}$ over $[-2, 7]$:



The area of this trapezoid is $A = \frac{b_1+b_2}{2} \cdot h = \frac{1+7}{2} \cdot 9 = 36$

(c) We sketch the graph of $y = \sqrt{9-x^2}$ over $[-3, 3]$



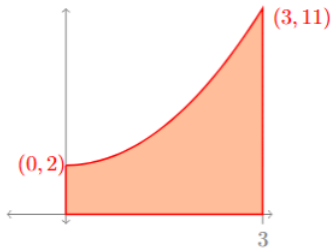
The area of this semi-circle is $A = \frac{1}{2}\pi r^2 = \frac{\pi}{2} \cdot 3^2 = \frac{9\pi}{2}$

Since more rectangles make for better approximations, it makes sense that the exact area is the limit as the number of rectangles goes to infinity.

We call this a **Riemann Sum**: $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$

So for the area we were working on in the previous section:

$\int_0^3 (x^2+2)dx$



We leave n as a variable, and find $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$

We use the right endpoint formula to get $x_k = a + k\cdot\Delta x = 0 + k\cdot\frac{3}{n} = \frac{3k}{n}$

Since we cannot possibly list out all of these values, we again find $f(x_k)$ explicitly:

$$f(x_k) = f\left(\frac{3k}{n}\right) = \left(\frac{3k}{n}\right)^2 + 2 = \frac{9k^2}{n^2} + 2$$

Now we just follow the limit definition above:

$$\int_0^3 (x^2+2)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{9k^2}{n}+2\right)\cdot\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{27k^2}{n^3}+\frac{6}{n}\right)$$

$$= \lim_{n\to\infty}\frac{27}{n^3}\sum_{k=1}^{n}k^2 + \frac{6}{n}\sum_{k=1}^{n}1$$

$$= \lim_{n\to\infty}\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{6}{n}\cdot n$$

$$= \lim_{n\to\infty}\frac{9(n+1)(2n+1)}{2n^2} + 6$$

$$= \lim_{n\to\infty}\frac{18n^2}{2n^2} + 6 = 9 + 6 = 15$$

This is the exact area under this curve, and a template for evaluating future definite integrals.

Remembering the formulas below is often helpful for such calculations.

$$\sum_{k=1}^{n} 1 = n,\ \sum_{k=1}^{n} k = \frac{n(n+1)}{2},\ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},\ \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Example 13.2.2

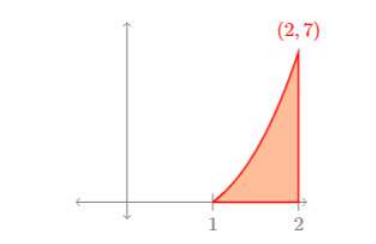
Evaluate $\int_1^2 (x^3 - 1)dx$

Solution

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$x_k = a + k\Delta x = 1 + k \cdot \frac{1}{n} = 1 + \frac{k}{n}$$

$$f(x_k) = f\left(1 + \frac{k}{n}\right) = \left(1 + \frac{k}{n}\right)^3 - 1 = \frac{3k}{n} + \frac{3k^2}{n^2} + \frac{k^3}{n^3}$$



$$\int_1^2 (x^3 - 1)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{3k}{n} + \frac{3k^2}{n^2} + \frac{k^3}{n^3}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{3k}{n^2} + \frac{3k^2}{n^3} + \frac{k^3}{n^4}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n^2}\sum_{k=1}^{n} k + \frac{3}{n^3}\sum_{k=1}^{n} k^2 + \frac{1}{n^4}\sum_{k=1}^{n} k^3$$

$$= \lim_{n\to\infty} \frac{3}{n^2} \cdot \frac{n(n+1)}{2} + \frac{3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{1}{n^4} \cdot \frac{n^2(n+1)^2}{4}$$

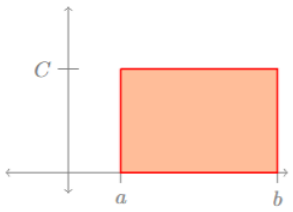
$$= \lim_{n\to\infty} \frac{3(n+1)}{2n} + \frac{(n+1)(2n+1)}{2n^2} + \frac{(n+1)^2}{4n^2}$$

$$= \frac{3}{2} + 1 + \frac{1}{4} = \frac{11}{4}$$

Evaluating Riemann sums in this manner works for any function that we know the necessary sum of powers of k.
(For us this means any polynomial of degree 3 or less)

But even if we don't know how to immediately evaluate a specific integral, we can use this limit definition to inspire the properties of a definite integral:

As a shortcut, we can integrate any constant function by using geometry:



$\int_a^b C\,dx = C \cdot (b-a)$ is the area of a rectangle.

We also can show $\int_a^a f(x)dx = 0$ for any function $f(x)$.

Any integral that starts and ends at the same number must have $\Delta x = \frac{0}{n} = 0$, which makes the entire sum and limit zero at every step along the way

Similarly, if we compare $\int_a^b f(x)dx$ to $\int_b^a f(x)dx$

The values of Δx will be $\frac{b-a}{n}$ and $\frac{a-b}{n}$ respectively (equal but opposite).

Meanwhile the values of x_k will all be the same (just in a different order), which makes the rest of the Riemann Sum identical. Therefore we can conclude that $\int_a^b f(x)dx = -\int_b^a f(x)dx$

The values of a and b are called the *limits of integration* for this definite integral. It is **not necessary** for the limits of integration to have the bigger number on top and the smaller number on the bottom, but this rule allows us to swap them (in exchange for a minus sign) whenever we want to.

Properties of sums and limits can directly translate into integral properties:

$$\int_a^b Cf(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} Cf(x_k)\Delta x$$

$$= C \cdot \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$$

$$= C \cdot \int_a^b f(x)dx$$

This means you can factor out any constants to the outside of an integral.

Intuitively this makes sense as well, if you double the height of a function you will double the area under it.

$$\int_a^b [f(x)+g(x)]dx = \lim_{n\to\infty}\sum_{k=1}^{n}[f(x_k)+g(x_k)]\Delta x$$

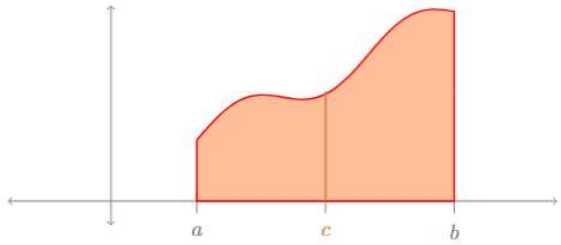
$$= \left(\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \right) + \left(\lim_{n\to\infty} \sum_{k=1}^{n} g(x_k)\Delta x \right)$$

$$= \int_a^b f(x)dx + \int_a^b g(x)dx$$

This means you can distribute the integral across a sum (or difference).

Lastly, we have $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

This is easier to see with a picture:



Clearly for any $c \in (a, b)$ this just says the area we want can be split up into two smaller pieces added together. The interesting thing is that this formula *also works* for c outside of this interval!

These properties are useful when you can use geometry on certain pieces of the function inside of an integral, or even in situations where you are given the value of definite integrals but not told the explicit function inside.

Example 13.2.3

Let $g(x)$ be a function satisfying: $\int_0^3 g(x)dx = 5$, $\int_3^5 g(x)dx = 4$, and $\int_0^7 g(x)dx = 11$

(a) $\int_7^7 g(x)dx = ?$

(b) $\int_0^5 g(x)dx = ?$

(c) $\int_5^7 6g(x)dx = ?$

(d) $\int_3^5 [10 - g(x)]dx = ?$

(e) $\int_0^3 [x^2 + \sqrt{144 - 16x^2}]dx = ?$

Solution

(a) Since we start and stop at the same number: $\int_7^7 g(x)dx = 0$

(b) $\int_0^5 g(x)dx = \int_0^3 g(x)dx + \int_3^5 g(x)dx = 5 + 4 = 9$

(c) $\int_5^7 6g(x)dx = 6\int_5^7 g(x)dx = 6\left[\int_0^7 g(x)dx - \int_0^3 g(x)dx - \int_3^5 g(x)dx\right] = 6[11 - 5 - 4] = 12$

(d) $\int_3^5 [10-g(x)]dx = \int_3^5 10dx - \int_3^5 g(x)dx = 10\cdot(5-3) - 4 = 16$

(e) $\int_0^3 [x^2+\sqrt{144-16x^2}]dx = \int_0^3 x^2dx + \int_0^3 4\sqrt{9-x^2}\,dx$

We can use a Riemann sum for the first integral:

$$\int_0^3 x^2dx = \lim_{n\to\infty}\sum_{k=1}^n \left(\frac{3k}{n}\right)^2\cdot\frac{3}{n} = \lim_{n\to\infty}\frac{27}{n^3}\sum_{k=1}^n k^2 = \lim_{n\to\infty}\frac{9(n+1)(2n+1)}{2n^2} = 9$$

And geometry to find the second integral (which represents the area of a quarter circle)

$$\int_0^3 \sqrt{36-4x^2}\,dx = \int_0^3 4\sqrt{9-x^2}\,dx = 4\int_0^3 \sqrt{9-x^2}\,dx = 4\cdot\frac{1}{4}\pi\cdot 3^2 = 9\pi$$

So our final answer is $\int_0^3 [x^2+\sqrt{36-4x^2}]dx = 9+9\pi$

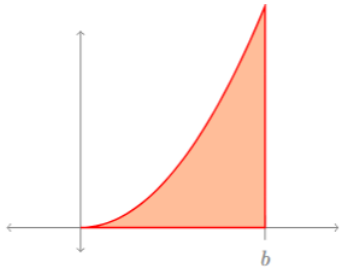
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13.3: The Fundamental Theorem of Calculus

If we have a function $f(x)$ and we wanted to know what $\int_a^b f(x)dx$ was for a wide variety of different a and b values (without using geometry), it would take a lot of different Riemann Sums.

Instead, we could try to derive a formula for $F(b)=\int_0^b f(x)dx$, and use it to find $\int_a^b f(x)dx=\int_0^b f(x)dx-\int_0^a f(x)dx=F(b)-F(a)$

For example, define $F(b)=\int_0^b x^2dx$



This function changes as b increases (moves to the right) or decreases (moves to the left).

We can find its exact value by setting up just one Riemann Sum:

$$F(b)=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{bk}{n}\right)^2\cdot\frac{b}{n}=\lim_{n\to\infty}\frac{b^3}{n^3}\sum_{k=1}^{n}k^2=\lim_{n\to\infty}\frac{b^3(n+1)(2n+1)}{6n^2}=\frac{b^3}{3}$$

Now we can use that to find things like $\int_{\sqrt[3]{2}}^{\sqrt[3]{3}}x^2dx=F\left(\sqrt[3]{3}\right)-F\left(\sqrt[3]{2}\right)=\frac{3}{3}-\frac{2}{3}=\frac{1}{3}$
Note that this would have been incredibly ugly to do directly with Riemann sums!

$F(b)=\frac{b^3}{3}$ is obviously directly dependent on the function $f(x)=x^2$ inside of the integral.
We quickly realize that $F'(x)=f(x)$, which is not a coincidence.
This result is known as the **Fundamental Theorem of Calculus**:

If $F(x)=\int_a^x f(t)dt$ for any choice of a, then $F'(x)=f(x)$

This relationship between the integral function and the function inside (called the *integrand*) is useful in two directions.
The first direction is for directly taking derivatives of functions defined as integrals:

Example 13.3.1

Find the derivative of each function below

(a) $F(x)=\int_2^x\sqrt{t^2+1}\,dt$

(b) $G(x)=\int_{3x}^{x^3}\sqrt{t^2+1}\,dt$

Solution

(a) By an immediate use of the Fundamental Theorem of Calculus (FTC) we get $F'(x) = \sqrt{x^2+1}$

(b) We cant use FTC directly, but $G(x)$ is closely related to $F(x)$ above:

$$G(x) = \int_2^{x^3} \sqrt{t^2+1}\,dt - \int_2^{3x} \sqrt{t^2+1}\,dt = F(x^3) - F(3x)$$

So $G'(x) = \big(F(x^3) - F(3x)\big)' = F'(x^3)\cdot 3x^2 + F'(3x)\cdot 3$
$= \sqrt{(x^3)^2+1}\cdot 3x^2 - \sqrt{(3x)^2+1}\cdot 3 = 3x^2\sqrt{x^6+1} - 3\sqrt{9x^2+1}$

The previous example shows that we can generalize the Fundamental Theorem of Calculus to say:

If $F(x) = \int_{a(x)}^{b(x)} g(t)dt$, then $F'(x) = g(b(x))b'(x) - g(a(x))a'(x)$

Example 13.3.2

Find line tangent to $H(x) = \int_{4x}^{\tan(\pi x)} \frac{1}{t}dt$ at $x = \frac{1}{4}$

Solution

First $H\left(\frac{1}{4}\right) = \int_1^1 \frac{1}{t}dt = 0$

Also $H'(x) = \frac{1}{\tan(\pi x)}\cdot \pi\sec^2(\pi x) - \frac{1}{4x}\cdot 4 = \frac{\pi\sec^2(\pi x)}{\tan(\pi x)} - \frac{1}{x}$

So $H'\left(\frac{1}{4}\right) = \frac{\pi\cdot\frac{1}{2}}{1} - 4 = \frac{\pi-8}{2}$

So $y - H\left(\frac{1}{4}\right) = H'\left(\frac{1}{4}\right)\left(x - \frac{1}{4}\right)$ becomes $y - 0 = \frac{\pi-8}{2}\left(x - \frac{1}{4}\right)$

While this is very useful for taking the derivatives of functions defined as integrals, that is a very specific situation. The FTC can also be used to evaluate integrals!

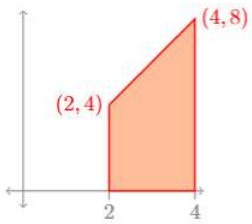
Remember if we wanted to evaluate $\int_a^b g(t)dt$, we can use the function $G(x) = \int_0^x g(t)dt$

And since $G'(x) = g(x)$ must hold, we can hope to guess the function $G(x)$ that has derivative $g(x)$!

This is sometimes easier said than done, but lets focus on an example where it is easier done than said: $\int_4^8 2x\,dx$

The integrand is $g(x) = 2x$, and thankfully its easy to see that $G(x) = x^2$ is one possible function which satisfies $G'(x) = 2x$

So $\int_4^8 2x\,dx = G(8) - G(4) = 64 - 16 = 48$



This is the same answer you would get by using geometry or Riemann sums

You may notice that we said $G(x) = x^2$ was *one possible function* with this derivative.
There are actually infinitely many different functions with this property.

For example $H(x) = x^2 - 4$ also has $H'(x) = 2x$

But we still have $\int_4^8 2x\,dx = H(8) - H(4) = 60 - 12 = 48$

In fact for any constant C we know $F(x) = x^2 + C$ has $F'(x) = 2x$

And still $\int_4^8 2x\,dx = F(8) - F(4) = (64 + C) - (16 + C) = 48$

We got pretty lucky trying to guess functions with derivative $2x$, but it may be more challenging with uglier functions as integrands.
We will work on refining this skill in the next section.

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13.4: Anti-Derivatives and the Indefinite Integral

The Fundamental Theorem of Calculus told us that $F(x) = \int_a^x g(t)dt$ has derivative $g(x)$.

So if we are given some definite integral of some $f(x)$, it would be very useful to be able to reliably find out the function who's derivative is that integrand. This is called taking an **anti-derivative**.

(The anti-derivative of $f(x)$ is the function $F(x)$ with $F'(x) = f(x)$)

Since these anti-derivatives are mostly used for solving definite integrals, we say that the **indefinite integral** $F(x) = \int f(x)dx$ denotes the general anti-derivative of $f(x)$

For example, since $f(x) = 2x$ is the derivative of the family of functions $F(x) = x^2 + C$, (where C is any constant)
we say $\int 2xdx = x^2 + C$

By MVT we know that any two functions with the same derivative must be off by a constant, so all indefinite integrals should have a " $+C$ " at the end!

While our goal is to attempt to take the anti-derivatives of much more complicated functions, lets start with power functions $f(x) = x^p$.
These can be calculated by using *guess and check*:

Example 13.4.1

Find $\int 20x^3dx$

Solution

By the power rule for derivatives, we know that if $f'(x) = 20x^3$, then $f(x) = Kx^4$ for some coefficient K

Taking the derivative of this guess we get: $f'(x) = 4Kx^3$

So for these to match we need $4K = 20$

Solving this yields $K = 5$ and we can check to see that

$f(x) = 5x^4$ has derivative $f'(x) = 20x^3$ like we wanted.

Therefore $\int 20x^3dx = 5x^4 + C$

That was pretty painless, but things will quickly get messier.

This time lets throw in some fractions for extra flavor and excitement.

Example 13.4.2

Find $\int \frac{10}{21}\sqrt[3]{x}dx$

Solution

Since $\sqrt[3]{x} = x^{\frac{1}{3}}$, the original functions power must be 1 greater,

So this time we guess $f(x) = Kx^{\frac{4}{3}}$ for some coefficient K

Taking the derivative of this guess we get: $f'(x) = \frac{4}{3}Kx^{\frac{1}{3}}$

So for these to match we need $\frac{4K}{3} = \frac{10}{21}$

Solving this yields $K = \frac{5}{14}$ and we can check to see that

$f(x) = \frac{5}{14}x^{\frac{4}{3}}$ has derivative $f'(x) = \frac{10}{21}x^{\frac{1}{3}}$ like we wanted.

Therefore $\int \frac{10}{21}\sqrt[3]{x}dx = \frac{5}{14}x^{\frac{4}{3}} + C$

If we expand our anti-derivative search to include polynomials, the work quickly gets out of hand:

Example 13.4.3

Find $\int (30x^9 - 25x^4 + 4x)dx$

Solution

Each of these terms are the result of a power rule from a power that was 1 greater

So this time we guess $f(x) = \alpha x^{10} + \beta x^5 + \gamma x^2$ for some coefficients α, β, γ

Taking the derivative of this guess we get: $f'(x) = 10\alpha x^9 + 5\beta x^4 + 2\gamma x$

So for these to match we need all three of the following to hold:
$10\alpha = 30$ and $5\beta = -25$ and $2\gamma = 4$

Solving these each gives $\alpha = 3, \beta = -5, \gamma = 2$

We can check to see that $f(x) = 3x^{10} - 5x^5 + 2x^2$ has derivative $f'(x) = 30x^9 - 25x^4 + 4x$ like we wanted.

Therefore $\int (30x^9 - 25x^4 + 4x)dx = 3x^{10} - 5x^5 + 2x^2 + C$

Clearly this method works, but will get messier as the number of terms (and therefore unknown coefficients) increases. Instead, lets see if we can find the pattern.

In general if $f'(x) = x^p$ as the result of a power rule, then $f(x) = Kx^{p+1}$ would be our guess.

The value of K that works would then be $\frac{1}{p+1}$ (as long as $p \neq -1$),

So $\int x^p dx = \frac{1}{p+1}x^{p+1} + C$ for all $p \neq -1$

What about when $p = -1$? Notice that this is the one power of x that cannot be reached by taking the derivative of another power function. But it *is* the derivative of another function we know: $f(x) = \ln(x)$

Interestingly enough though, $g(x) = \ln(-x)$ also has $g'(x) = \frac{1}{x}$ thanks to the chain rule.

Since these two functions have non overlapping domains, ($D_f = (0, \infty)$ and $D_g = (-\infty, 0)$)it should be clear based on if x is negative or positive which of these two functions you should use.

To avoid dealing with this (especially if you dont know if x is positive or negative in a given problem), we can combine them to say that $\int \frac{1}{x} dx = \ln|x| + C$

In fact, every derivative that you know is an anti-derivative formula waiting to be discovered!

$f(x)$	$\int f(x)dx$
x^p for $(p \neq -1)$	$\frac{1}{p+1}x^{p+1} + C$
x^{-1}	$\ln\|x\| + C$
e^x	$e^x + C$
b^x	$\frac{1}{\ln(b)}b^x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x) + C$
$\frac{1}{1+x^2}$	$\tan^{-1}(x) + C$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1}(x) + C$

$f(x)$	$\int f(x)dx$
1	$x + C$
$\cos(x)$	$\sin(x) + C$
$\sin(x)$	$-\cos(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\csc^2(x)$	$-\cot(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$
$\csc(x)\cot(x)$	$-\csc(x) + C$

Not only these, but any function that you learned the derivative of can now be read backwards as an anti-derivative (including hyperbolic trigonometric functions, etc).

That is nice, but it still doesn't help us find the anti-derivative of anything that isn't exactly listed. But remember, we learned how derivatives are effected by adding, subtracting, multiplying, and dividing functions. Those laws can be translated into similar properties for taking anti-derivatives!

We know that $[f(x) + g(x)]' = f'(x) + g'(x)$, if we integrate both sides this becomes:

$$\int [f'(x) + g'(x)]dx = \int f'(x)dx + \int g'(x)dx$$

We know that $[f(x) - g(x)]' = f'(x) - g'(x)$, integrating both sides this becomes:

$$\int [f'(x) - g'(x)]dx = \int f'(x)dx - \int g'(x)dx$$

For any constant K we know that $[K \cdot f(x)]' = K \cdot f'(x)$, which turns into:

$$\int K \cdot f'(x)dx = K \cdot \int f'(x)dx$$

Look familiar? We discovered these properties hold for definite integrals earlier this chapter as the result of the Riemann Sum definition of an integral! Its also a result of this anti-derivative property

The more complicated derivative rules also have equivalent anti-derivative rules, but they are significantly more difficult. These will be explored in the future, but for this chapter we will only focus on using the three properties above!

Lets revisit a previous example:

Example 13.4.3

Find $\int (30x^9 - 25x^4 + 4x)dx$

Solution

First we distribute the integral across each sum and difference:

$$= \int 30x^9 dx - \int 25x^4 dx + \int 4x dx$$

Then we pull each constant to the front of the integral:

$$= 30\int x^9 dx - 25\int x^4 dx + 4\int x dx$$

Lastly we use the power rule for anti-derivatives on the previous page:

$$= 30 \cdot \frac{x^{10}}{10} - 25 \cdot \frac{x^5}{5} + 4 \cdot \frac{x^2}{2} + C$$

$$= 3x^{10} - 5x^5 + 2x^2 + C$$

Note that you only need **one** $+C$ for the anti-derivative, as they can all combine into one constant!

Any constant multiple, sum, or difference of things in our list are easy to anti-differentiate. If something is not immediately on our list, we just have to re-write it first!

Example 13.4.4

Evaluate each integral below

(a) $\int 2\cos(x) - \sec^2(x) + \frac{1}{2x} dx$

(b) $\int \frac{7t^3 + 4t - 3}{\sqrt{t}} dt$

(c) $\int e^y \sin^2(y) + e^y \cos^2(y) dy$

Solution

(a) $\int 2\cos(x) - \sec^2(x) + \frac{1}{2x} dx$

$$= 2\int \cos(x)dx - \int \sec^2(x)dx + \frac{1}{2}\int \frac{1}{x} dx$$

$$= 2\sin(x) - \tan(x) + \frac{1}{2}\ln|x| + C$$

(b) $\int \frac{7t^3 + 4t - 3}{\sqrt{t}} dt$

$$= \int \left[\frac{7t^3}{\sqrt{t}} + \frac{4t}{\sqrt{t}} - \frac{3}{\sqrt{t}}\right] dt = \int \left[7t^{\frac{5}{2}} + 4t^{\frac{1}{2}} - 3t^{-\frac{1}{2}}\right] dt$$

$$= \int 7t^{\frac{5}{2}} dt + \int 4t^{\frac{1}{2}} dt - \int 3t^{-\frac{1}{2}} dt$$

$$= 7\int t^{\frac{5}{2}}dt + 4\int t^{\frac{1}{2}}dt - 3\int t^{-\frac{1}{2}}dt$$

$$= 7\cdot\frac{t^{\frac{7}{2}}}{\frac{7}{2}} + 4\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 3\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2t^{\frac{7}{2}} + \frac{8}{3}t^{\frac{3}{2}} - 6t^{\frac{1}{2}} + C$$

(c) $\int e^y \sin^2(y) + e^y \cos^2(y)dy$

$$= \int e^y(\sin^2(y) + \cos^2(y))dy$$

$$= \int e^y \cdot 1 dy = \int e^y dy = e^y + C$$

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13.5: More Definite Integrals

Now that we have mastered taking anti-derivatives, we can use the Fundamental Theorem of Calculus (FTC) to solve definite integrals.

The way that we write down our anti-derivatives when solving a definite integral looks like:

$$\int_a^b f'(x)dx = f(x)\Big|_a^b = f(b) - f(a)$$

That middle step is where you find the anti-derivative of the integrand (the bar with the numbers on it means "plug in the top number into this function minus plug in the bottom number").

For instance, remember that we solved $\int_4^8 2x\,dx$ in a previous section.

$$\int_4^8 2x\,dx = x^2\Big|_4^8 = 8^2 - 4^2 = 48$$

Example 13.5.1

Evaluate each definite integral below

(a) $\int_0^2 (8x^3 + 3e^x)dx$

(b) $\int_0^{\frac{\pi}{4}} \dfrac{\sec(t) - \cos(t)}{\tan(t)}dt$

Solution

(a) We take the anti-derivative of the function inside

$$\int_0^2 (8x^3 + 3e^x)dx = 2x^4 + 3e^x\Big|_0^2$$

Then plug in the top and bottom numbers

$$= \left(2 \cdot 2^4 + 3e^2\right) - \left(2 \cdot 0^4 + 3e^0\right) = 29 + 3e^2$$

(b) This one we have to re-write first using trig identities, then take the anti-derivative:

$$= \int_0^{\frac{\pi}{4}} \frac{\sec(t) - \cos(t)}{\tan(t)} \cdot \frac{\cos(t)}{\cos(t)}dt = \int_0^{\frac{\pi}{4}} \frac{1 - \cos^2(t)}{\sin(t)}dt$$

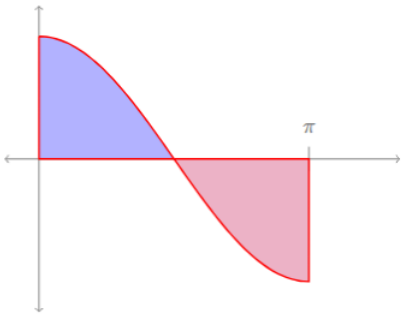
$$= \int_0^{\frac{\pi}{4}} \frac{\sin^2(t)}{\sin(t)}dt = \int_0^{\frac{\pi}{4}} \sin(t)dt = -\cos(t)\Big|_0^{\frac{\pi}{4}} = -\cos\left(\frac{\pi}{4}\right) + \cos(0) = 1 - \frac{\sqrt{2}}{2}$$

Next we will consider a fairly simple example:

$$\int_0^{\pi} \cos(x)dx = \sin(x)\Big|_0^{\pi} = \sin(\pi) - \sin(0) = 0$$

It may seem strange to say that *the area under* $y = \cos(x)$ *in this interval* is zero.

If we take a look at the graph of $y = \cos(x)$, we can shed some light on this:



If we split the above integral at $x = \frac{\pi}{2}$ we can see that:

$$\int_0^{\pi} \cos(x)dx = \int_0^{\frac{\pi}{2}} \cos(x)dx + \int_{\frac{\pi}{2}}^{\pi} \cos(x)dx$$

$$= \sin(x)\Big|_0^{\frac{\pi}{2}} + \sin(x)\Big|_{\frac{\pi}{2}}^{\pi} = \left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right) + \left(\sin(\pi) - \sin\left(\frac{\pi}{2}\right)\right)$$

$$= 1 + (-1) = 0$$

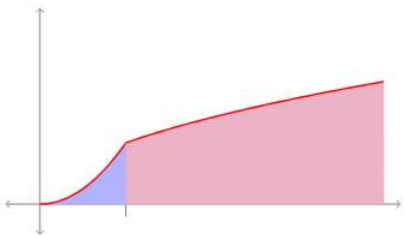
The blue area on the left is positive 1 because $\cos(x)$ is positive and above the x-axis in this region. The purple area on the right is considered negative 1 since $\cos(x)$ is negative and below the x-axis.

The lesson to learn from this example is that a definite integral is an area, but it does not always have to be positive. Instead it represents the net area between the curve and the x-axis, with areas above the axis considered positive and areas beneath the axis considered negative.

To make things a little more interesting, lets try taking a definite integral of a piecewise function.

Let $f(x) = \begin{cases} x^2 & x \leq 1 \\ \sqrt{x} & x > 1 \end{cases}$ and find $\int_0^4 f(x)dx$

We do not know how to anti-derivative a piecewise function, but consider the graph below:



Once again, we can see that this definite integral can be split into two areas: the left region under $y = x^2$ for $x \in [0, 1]$ and the right region under $y = \sqrt{x}$ for $x \in (1, 4]$

Technically we could split the definite integral up *at any number*, but it makes the most sense to split at $x = 1$, since that is where

the piecewise function changes rules.

$$\int_0^4 f(x)dx = \int_0^1 f(x)dx + \int_1^4 f(x)dx = \int_0^1 x^2 dx + \int_1^4 x^{\frac{1}{2}} dx$$

$$= \frac{1}{3}x^3\Big|_0^1 + \frac{2}{3}x^{\frac{3}{2}}\Big|_1^4 = \left(\frac{1}{3} - 0\right) + \left(\frac{16}{3} - \frac{2}{3}\right) = 5$$

In general, as long as you can take the anti-derivative of each piece of the piecewise function, this strategy will work to evaluate definite integrals. (Don't worry about what happens at the endpoints)

Example 13.5.2

For $g(x) = \begin{cases} \sin(x) & x < \pi \\ -2 & x \geq \pi \end{cases}$ and find $\int_0^{2\pi} g(x)dx$

Solution

$$\int_0^{2\pi} g(x)dx = \int_0^{\pi} \sin(x)dx + \int_{\pi}^{2\pi} -2dx = \sin(x)\Big|_0^{\pi} + -2x\Big|_{\pi}^{2\pi}$$

$$= (1+1) + (-4\pi + 2\pi) = 2 - 2\pi$$

Remember, the most popular piecewise function is still the absolute value function.

Example 13.5.3

Evaluate each integral below

(a) $\int_1^4 |2t-6|dt$

(b) $\int_0^{2\pi} \big|\sin(x) - \cos(x)\big|dx$

Solution

(a) We know $|2t-6| = \begin{cases} 2t-6 & t \geq 3 \\ -2t+6 & t < 3 \end{cases}$, so we should split the integral at $t = 3$:

$$\int_1^4 |2t-6|dt = \int_1^3 (-2t+6)dt + \int_3^4 (2t-6)dt$$

$$= (-t^2+6t)\Big|_1^3 + (t^2-6t)\Big|_3^4 = [9-5] + [(-8)-(-9)] = 5$$

(b) Writing this one out as a piecewise is a bit hairy, but we know that the rules will only change when $\sin(x) = \cos(x)$

In $[0, 2\pi]$ this occurs at $x = \frac{\pi}{4}, \frac{5\pi}{4}$

$$\int_0^{\frac{\pi}{4}} \big|\sin(x) - \cos(x)\big|dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \big|\sin(x) - \cos(x)\big|dx + \int_{\frac{5\pi}{4}}^{2\pi} \big|\sin(x) - \cos(x)\big|dx$$

In each interval, we can determine if $\sin(x) - \cos(x)$ is positive or negative. If its positive, ignore the absolute values. If its negative, flip the sign.

$$= \int_0^{\frac{\pi}{4}} -\sin(x) + \cos(x)dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin(x) - \cos(x)dx + \int_{\frac{5\pi}{4}}^{2\pi} -\sin(x) + \cos(x)dx$$

$$= (\cos(x)+\sin(x))\Big|_0^{\frac{\pi}{4}} + (-\cos(x)-\sin(x))\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + (\cos(x)+\sin(x))\Big|_{\frac{5\pi}{4}}^{2\pi}$$

$$= \left[\sqrt{2}-1\right] + \left[\sqrt{2}+\sqrt{2}\right] + \left[1+\sqrt{2}\right] = 4\sqrt{2}$$

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13.6: Anti-derivative Applications

Now that we have practiced taking integrals and anti-derivatives, the natural question is *why?*

While finding the areas of mathematically defined shapes is cool, its not that practical. The actual applications of anti-derivatives are much more prevalent in the real world.

Anytime you are given the rate of change of a function, you are being told its derivative. Taking the anti-derivative will find the infinitely many different possible functions it could be (and hopefully you have more information to help you determine which value of C specifically is involved.)

For example: A painter begins a 60 square foot mural at some time in the morning, takes a lunch break and resumes painting at a rate of $r(t) = 4\sqrt[3]{t}$ square feet per hour starting at noon. If at 1 PM they have completed exactly 15 square feet, at what time will they finish?

If $f(t)$ represents the total amount of square feet painted after t hours, then we have $f'(t) = r(t)$

Thus $f(t) = \int r(t)dt = \int 2t^{\frac{1}{3}}dt = 3t^{\frac{4}{3}} + C$

But without C we cannot solve for when $f(t) = 60$.

Thankfully, we are given that $f(1) = 15$, so:

$f(1) = 3 \cdot 1^{\frac{4}{3}} + C \longrightarrow 15 = 3 + C$

So $C = 12$, and thus $f(t) = 3t^{\frac{4}{3}} + 12$

Now we can solve for when $f(t) = 60$ to get our final answer:

$3t^{\frac{4}{3}} + 12 = 60 \longrightarrow t^{\frac{4}{3}} = 16 \longrightarrow t = \pm 8$

Since $t = -8$ does not make sense, we know our answer is at $t = 8$. **The mural will be completed at 8 PM.**

Many real world examples are similar to this, where you are given the derivative of a function and some information that will help you solve for the $+C$ when you take the anti-derivative. Before diving into more such examples, lets get some practice solving for these constants.

Example 13.6.1

Find $f(x)$ satisfying $f'(x) = 4e^x - 3\sin(x)$ and $f(0) = 2$

Solution

$f(x) = \int 4e^x - 3\sin(x)dx = 4e^x + 3\cos(x) + C$

$f(0) = 4e^0 + 3\cos(0) + C \longrightarrow 2 = 7 + C \longrightarrow C = -5$

Therefore $f(x) = 4e^x + 3\cos(x) - 5$

In extreme situations, we may be given the second derivative of the function we want. This just means we go through the same process twice:

Example 13.6.2

Find $w(t)$ satisfying $w''(t) = 12t + 2$ and $w'(2) = 0, w(1) = 5$

Solution

$$w'(t) = \int w''(t)dt = \int (12t+2)dt = 24t^2 + 2t + C$$

$$w'(2) = 24 \cdot 2^2 + 2 \cdot 2 + C = 100 + C \longrightarrow 100 + C = 0 \longrightarrow C = -100$$

So $w'(t) = 24t^2 + 2t - 100$

$$w(t) = \int w'(t)dt = \int (24t^2 + 2t - 100)dt = 8t^3 + t^2 - 100t + D$$

$$w(1) = 8 \cdot 1^3 + 1^2 - 100 \cdot 1 + D = D - 91 \longrightarrow D - 91 = 5 \longrightarrow D = 96$$

So our answer is $w(t) = 8t^3 + t^2 - 100t + 96$

Sometimes you are given the information to solve for C and D differently.

Example 13.6.3

Find $g(x)$ satisfying $g''(x) = \cos(x) - 2\sin(x)$ and $g(0) = 1, g(\pi) = \pi + 3$

Solution

$$g'(x) = \int g''(x)dx = \int [\cos(x) - 2\sin(x)]dx = \sin(x) + 2\cos(x) + C$$

We dont have any information about g' at a number, so just keep going

$$g(x) = \int g'(x)dx = \int [\sin(x) + 2\cos(x) + C]dx = -\cos(x) + 2\sin(x) + Cx + D$$

$$g(0) = -\cos(0) + 2\sin(0) + C \cdot 0 + D = D - 1$$

$$D - 1 = 1 \longrightarrow D = 2$$

$$g(\pi) = -\cos(\pi) + 2\sin(\pi) + C \cdot \pi + D = C\pi + D + 1$$

Plugging in $D = 2$ this becomes $C\pi + 3 = \pi + 3 \longrightarrow C = 1$

Thus our function is $g(x) = -\cos(x) + 2\sin(x) + x + 2$

The most common application of these anti-derivatives is for motion problems in physics. If we are given the velocity $v(t)$ or acceleration $a(t)$ of an object, we can use anti-derivatives to find its position function $s(t)$, since $s'(t) = v(t)$ and $s''(t) = v'(t) = a(t)$ must hold.

For example: The gravity on Earth is 32 feet/sec^2 in the downward direction. If a ball is thrown upward into the air from an initial height of 80 feet above the ground and with an upward velocity of 64 feet/sec, how long will it take to hit the ground?

Since the only acceleration on this ball is due to gravity, we know the function $a(t)$ must be constant. The natural orientation for most situations is to consider *up* to be positive and *down* to be negative. Since gravity pulls the ball downward, we have $a(t) = -32$

Thus $v(t) = \int a(t)dt = \int -32dt = -32t + C$

We also are given the initial velocity is 64 feet/sec upward, so $v(0) = 64$.

$v(0) = -32 \cdot 0 + C = C \longrightarrow C = 64$

Now we have the velocity function $v(t) = -32t + 64$

Next we have $s(t) = \int v(t)dt = \int (-32t + 64)dt = -16t^2 + 64t + D$

The initial height of the ball is 80 feet upward, so $s(0) = 80$

$s(0) = -16 \cdot 0^2 + 64 \cdot 0 + D = D \longrightarrow D = 80$

Thus our position function is $s(t) = -16t^2 + 64t + 80$

We are looking for *when the ball hits the ground* which is when $s(t) = 0$:
$-16t^2 + 64t + 80 = 0$
$-16(t-5)(t+1) = 0$
There are two solutions: $t = -1$ (which doesn't make sense) and $t = 5$

Therefore we know **the ball will hit the ground after 5 seconds.**

The question above could have instead asked *what is the maximum height of the ball?* and we would still have had to do all the same work to find $v(t)$ and $s(t)$.
The only difference of course is that instead of solving when $s(t) = 0$, we are looking for the maximum of $s(t)$.

We would do this by setting $s'(t) = 0$ to find critical points: $-32t + 64 = 0 \longrightarrow t = 2$
(feel free to make a sign chart to confirm this is a maximum)

Thus the **maximum height will be** $s(2) = -16 \cdot 2^2 + 64 \cdot 2 + 80 = 144$ **feet.**

One small wrinkle when dealing with these questions is how we define *distance*.

This can mean two different things, so to avoid ambiguity we say that the difference between an objects starting and stopping position for $t \in [a, b]$ is its **displacement**: $s(b) - s(a)$

Displacement can be either positive or negative, and can be found in terms of velocity as $\int_a^b v(t)dt$

Alternatively, the **distance traveled** by the object must always be positive. This is because it is increases whether the velocity is positive or negative. We can find it using $\int_a^b |v(t)|dt$

If the velocity is always positive, these things will match (since $|v(t)| = v(t)$). But in any other case, be certain to calculate the correct one!

Example 13.6.4

A particle travels along a number line with velocity function $v(t) = 12 - 3t^2$. Find the displacement as well as distance traveled for $0 \leq t \leq 3$

Solution

The displacement is just $\int_0^3 v(t)dt = \int_0^3 (12-3t^2)dt = (12t-t^3)\Big|_0^3 = 9$

The total distance travelled would be $\int_0^3 |v(t)|dt = \int_0^3 |12-3t^2|dt$

This integral must be split whenever $12-3t^2 = 0$ in $[0,3]$, so $t=2$

$= \int_0^2 |12-3t^2|dt + \int_2^3 |12-3t^2|dt$

$= \int_0^2 (12-3t^2)dt + \int_2^3 (-12+3t^2)dt$

$= (12t-t^3)\Big|_0^2 + (-12t+t^3)\Big|_2^3$

$= [16-0]+[-9+16] = 23$

So our displacement is 9 but the total distance travelled is 23

This is because the particle moves 16 in the positive direction during $0 \le t \le 2$, and then it moves backward (in the negative direction) 7 during $2 \le t \le 3$

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