MATH 372: COLLEGE ALGEBRA FOR CALCULUS (2E)



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Math 372: College Algebra for Calculus (2e)

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Templates

The following content is reused throughout the course and is held here for simplicity.

Not in Curriculum (Section)

Note for the Instructor (at Cosumnes River College)

The material in this section is considered a prerequisite for this course. As such, it is not in the curriculum and is only provided here for the interested student to read independently.

Not in Curriculum (Subsection)

Note for the Instructor (at Cosumnes River College)

The material in this subsection is considered a prerequisite for this course. As such, it is not in the curriculum and is only provided here for the interested student to read independently.

Refresher Topics

This section requires the following prerequisite knowledge. If you're enrolled in a class with a corequisite support course, these topics will be reviewed there. If you're in a class without a support course, you're expected to have already mastered these topics.

Corequisite Warning

If you are consistently struggling with these skills, or if you would feel more comfortable reviewing these skills in a structured environment, you might be better served in a course with support (if you are not already enrolled in such a course).

Foundational Topics

The following prerequisite topics are critical building blocks for this section and others in the course. They are usually too basic to be covered in a support course, though your instructor may offer quick remediation if time allows. Struggling with one or two of these skills is normal, but consistent difficulty suggests you may be in the wrong course. If so, please discuss alternative options with your instructor.

Prerequisite Topics

We're bringing back some skills from earlier courses that are essential for the rest of your math journey. If you're rusty, see **CRC's Corequisite Codex** for a quick review. (Don't rely solely on support classes, as they may not cover everything!)

Prerequisite Warning

If you are consistently struggling with these skills, you would be better served enrolling in a Math Boot Camp at Cosumnes River College. Please speak with your instructor about when the next Math Boot Camp starts.

Core Prerequisites

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

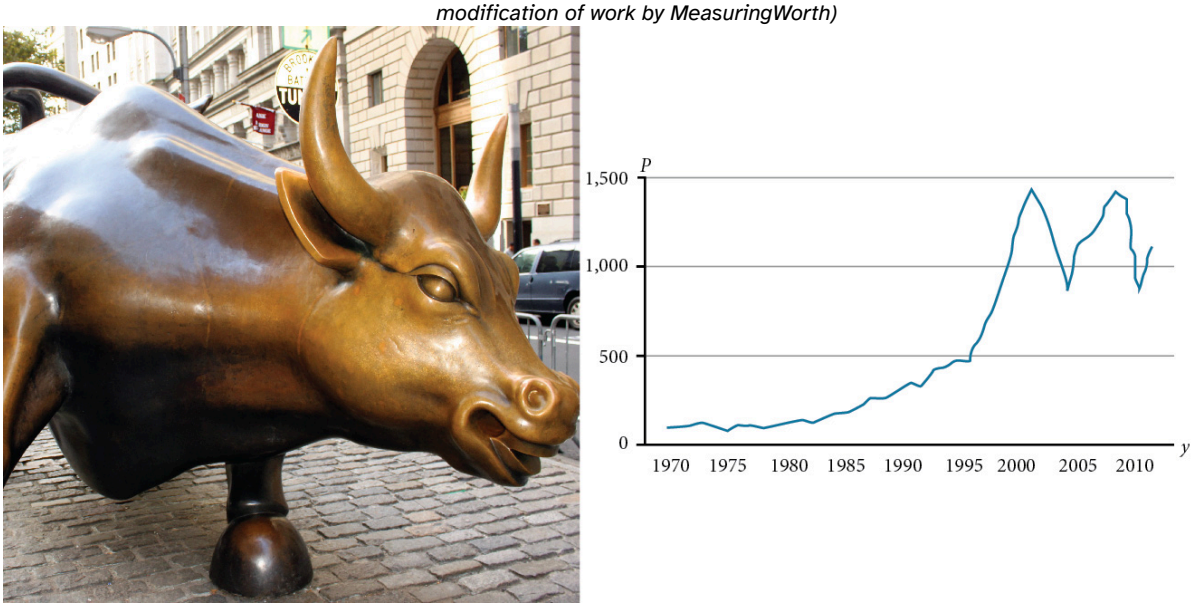
Operational Prerequisites

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

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1: A REINTRODUCTION TO FUNCTIONS

Figure 1: *Standard and Poor's Index with dividends reinvested (credit "bull": modification of work by Prayitno Hadinata; credit "graph": modification of work by MeasuringWorth)*



Toward the end of the twentieth century, the values of stocks of internet and technology companies rose dramatically. As a result, the Standard and Poor's stock market average rose as well. The chart on the right tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1,100 by the beginning of 2000. That five-year period became known as the "dot-com bubble" because so many internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000.

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

1.1: Functions and Function Notation

1.1.1: Resources and Key Concepts

1.1.2: Homework

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1.1: Functions and Function Notation

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

We begin our journey with a definition (and from this, progress to more definitions, then theorems, then more theorems, and so on until we will have derived all of College Algebra) and an exploration.

Definition: Ordered Pair

An **ordered pair** (a, b) is a listing of two objects in which the order is significant; a is called the **first coordinate** and b the **second coordinate**. Two ordered pairs, (a, b) and (c, d), are equal (written as $(a, b) = (c, d)$) if and only if $a = c$ and $b = d$.

Consider the following set of ordered pairs:

$$D = \{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)\}. \tag{1.1.1}$$

The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first. It is logical to want to describe this link between the first and second values of these ordered pairs. We do so by defining this link as a **relation**.

Definition: Relation

A **relation** from a set A to a set B is a subset of all possible ordered pairs (a, b) with $a \in A$ and $b \in B$. When $A = B$, we call it a relation **on** A.

We should spend a moment unpacking the language in that last definition.

- "A relation *from* a set A *to* a set B," implies there is a direction—we are going *from* somewhere *to* somewhere else.
- "...is a *subset* of *all possible* ordered pairs (a, b)," means that the relation might not contain every possible combination of elements from A and B—it only needs to contain a *subset* of these combinations. It is possible that it contains all

combinations of ordered pairs between these two sets, but it is not necessary.

For example, let A be the set of names of states in the United States, and let B be the set of flavors of Ben & Jerry's ice cream. We could create a (non-sensical) relation *from* A *to* B containing only the following ordered pairs: {(California, Chunky Monkey), (New York, Americone Dream), (Texas, Cherry Garcia)}. Notice that this relation definitely didn't use *all* of the states, nor did it use *all* of the Ben & Jerry's ice cream flavors—that's fine! By definition, it is still a relation between A and B.

Relations *might* be able to be described in terms of formulas (like, $y = 2x$ in the set 1.1.1), but this property need not exist. That is, we might be given a set of ordered pairs where there is no formula to the relationship between x and y.

Note

Before we go too far, it is important to mention that relations can involve more than two variables, but until you reach Calculus III, we will try our best to restrict our exploration to relations between two unknowns.

Now that we have a name for the "machine" linking the first and second values of these ordered pairs, let's introduce more language so that we can stop saying, "the first and second values of the ordered pairs."

Definition: Domain and Range (of a relation)

Let R be a relation. The **domain** of R is the set of all first coordinates that appear in R, and the **range** of R is the set of all second coordinates that appear in R:

$$\text{domain} = \{\, a \mid (a,b) \in R \text{ for some } b \,\}, \qquad \text{range} = \{\, b \mid (a,b) \in R \text{ for some } a \,\}.$$

Therefore, the domain of the relation 1.1.1 is $\{1, 2, 3, 4, 5\}$ and the range is $\{2, 4, 6, 8, 10\}$.

Definition: Input and Output Values

Each value in the domain is also known as an **input value**, and each value in the range is also known as an **output value**.

Input and output values are traditionally considered constants. That is, in 1.1.1 the input value of 3 mapped to an output value of 6. However, when referring to an *arbitrary* element from the set of input or output values, we use a slightly modified language (because, in Mathematics, the word "arbitrary" means "variable"—or non-constant).

Definition: Independent and Dependent Variables

If we are treating the input value as a variable, x, then we call x the **independent variable**. Likewise, the variable y is often used when referencing an arbitrary element from the set of output values, and y is called the **dependent variable**.

A Variable By Any Other Name...

While x and y are traditionally used in lower-level math courses as the independent and dependent variables, respectively, this need not always be the case. We can use any letter (or picture, for that matter) to represent these variables. The choice is literally limitless.

Relations are incredibly flexible—as long as there is some link between two variables, we can form a relation between them. For example, there is a relation between social security number and name. That is, the social security number 123-45-6789 (which, in truth, is not a real social security number) is related to a person's name. A natural question that arises when we do some thinking is whether the relation is unique. That is, if we grab a random social security number, will it be related to one name? This type of special relation is called a **function**.

Definition: Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say the output is a function of the input. The set of input values makes up the **domain** of the function, and the corresponding set of output values makes up the **range** of the function.

In a very real sense, functions are a special subclass of relations. While a relation can relate the input of 5 to the output 7 and then the input of 5 to the output of 10, functions are not allowed such silliness.

The relation from 1.1.1 is a function because each element in the domain, $\{1, 2, 3, 4, 5\}$, is paired with exactly one element in the range, $\{2, 4, 6, 8, 10\}$. For the social security numbers, the relation in one direction is a function—each social security number leads to a single name. However, if we reversed the direction of the relation so that the domain is the set of all names and the range is the set of all social security numbers, the relation would *not* be a function. This is because the name, Roy Simpson, leads to more than one social security number (there are more Roy Simpsons in the United States than just me!).

Let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd}, 1), (\text{even}, 2), (\text{odd}, 3), (\text{even}, 4), (\text{odd}, 5)\}.$$

Notice that each element in the domain, $\{\text{even}, \text{odd}\}$, is not paired with exactly one element in the range, $\{1, 2, 3, 4, 5\}$. For example, the term "odd" corresponds to three values from the range, $\{1, 3, 5\}$ and the term "even" corresponds to two values from the range, $\{2, 4\}$. This violates the definition of a function, so this relation is not a function. Figure $1.1.1$ compares relations that are functions and not functions.

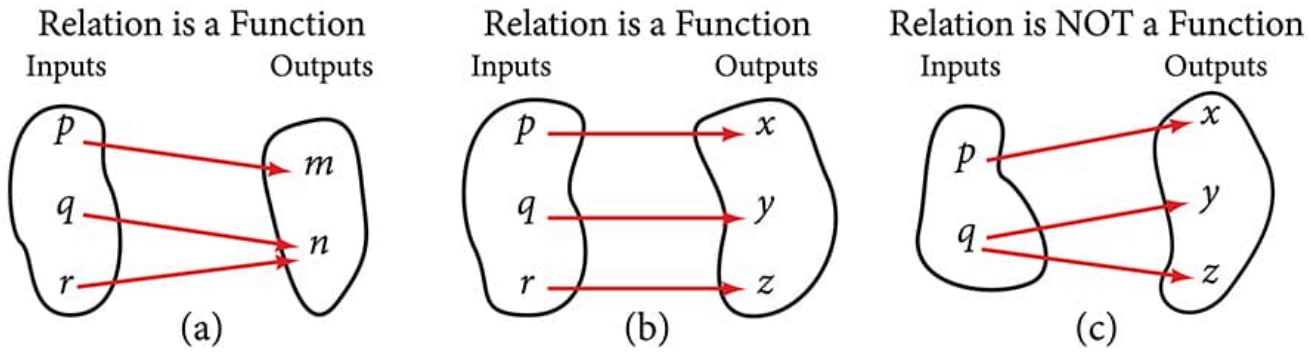


Figure $1.1.1$

Figure $1.1.1(a)$ is a function because each input is associated with a single output. Note that inputs q and r both give output n. This is perfectly fine. The definition of a function only restricts a single input from leading to two distinctly different outputs. Figure $1.1.1(b)$ is also a function. In this case, each input is associated with a single output. Finally, Figure $1.1.1(c)$ is *not* a function because input q is associated with two different outputs (y and z).

Example 1.1.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in Figure $1.1.2$, consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

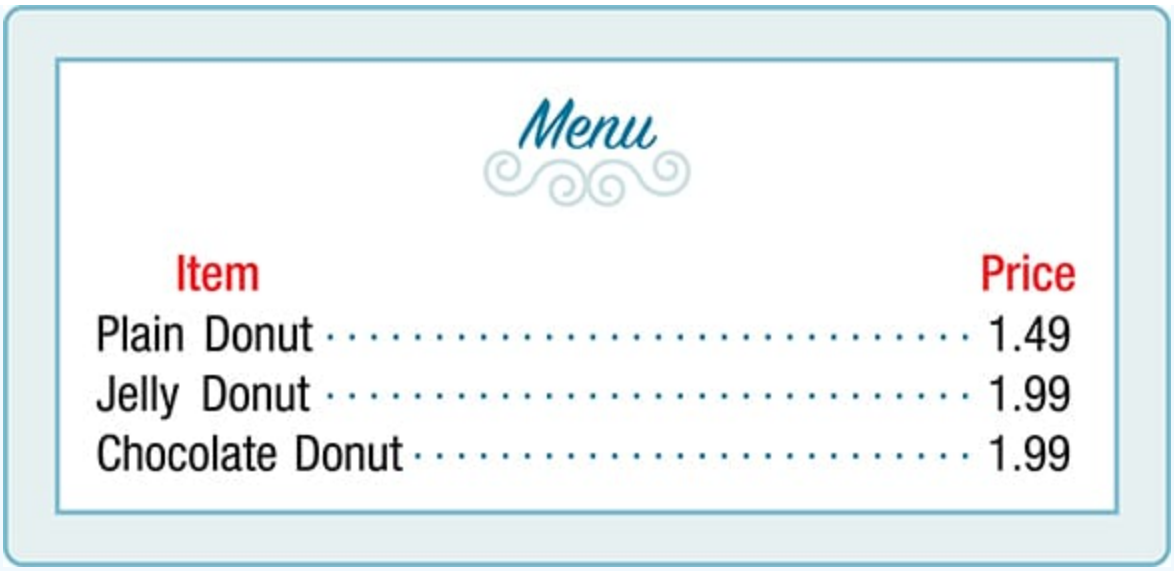


Figure 1.1.2

Solutions available at the end of this section.

Example 1.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 1.1.1 shows a possible rule for assigning grade points.

Table 1.1.1

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution available at the end of this section.

Checkpoint 1.1.2

Table 1.1.2 lists the five greatest baseball players of all time in order of rank.

Table 1.1.2

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables: h for height and a for age. The letters f, g, and h are often used to represent functions just as we use x, y, and z to represent numbers and A, B, and C to represent sets.

We then build a structure for the function notation. Since height is a function of age, we would like our notation to match.

$$\begin{array}{rccc} & \text{height} & \text{is} & \text{a function of age} \\ \Longrightarrow & h & = & f(a) \end{array}$$

Remember, we can use any letter to name the function. The notation $f(a)$ shows us that f depends on a. The value a must be put into the function f to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example, $f(a+b)$ means, "first add a and b, and then input the result into the function f." The operations must be performed in this order to obtain the correct result.

Definition: Function Notation

For the notation $y = f(x)$, f is the **name** of the function, x is the **domain value**, also known as the **argument** or the **independent variable**, and $f(x)$ is the **range value** of the function f at x. Since $y = f(x)$, we can also say that the range value of the function f at x is y. In this case, we say that y is the **dependent variable** (as its value *depends* on x). We read $f(x)$ as "f of x" or the value of f at x.

Example 1.1.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month. Assume that the domain does not include leap years.

Solution available at the end of this section.

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 1.1.4: Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Solution available at the end of this section.

Checkpoint 1.1.4

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done, especially in applied subjects that use higher math, such as Physics and Engineering. However, in exploring Mathematics itself, we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table $1.1.3$ lists the input number of each month (January = 1, February = 2, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Table $1.1.3$

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table $1.1.4$ defines a function $Q = g(n)$. Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

Table $1.1.4$

n	1	2	3	4	5
Q	8	6	7	6	8

Table $1.1.5$ displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Table $1.1.5$

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

Example $1.1.5$: Identifying Tables that Represent Functions

Which table, Table $1.1.6A$, Table $1.1.6B$, or Table $1.1.6C$, represents a function (if any)?

Table $1.1.6A$

Input	Output
2	1
5	3
8	6

Table $1.1.6B$

Input	Output
-3	5
0	1
4	5

Table $1.1.6C$

Input	Output
1	0
5	2
5	4

Solution available at the end of this section.

Checkpoint $1.1.5$

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Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we *evaluate* the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and *solve* for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

Preview of Future Material

Each of the *types* of functions in the following examples will be studied in detail as we move through the textbook. The functions in Examples $1.1.6$, $1.1.7$, and $1.1.8$ are called **quadratic functions**, and are the focus of an entire section later on. The functions in Checkpoints $1.1.7$ and $1.1.8$ are called **square root functions** (a subset of the **radical functions**). These also get their own sections later in the text.

Example $1.1.6$: Evaluating Functions at Specific Values

Evaluate $f(x) = x^2 + 3x - 4$ at:

a. 2
b. a
c. $a+h$
d. Now evaluate $\frac{f(a+h)-f(a)}{h}$

Solutions available at the end of this section.

Example 1.1.7: Evaluating Functions

Given the function $h(p)=p^2+2p$, evaluate $h(4)$.

Solution available at the end of this section.

Checkpoint 1.1.7

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Example 1.1.8: Solving Functions

Given the function $h(p)=p^2+2p$, solve $h(p)=3$.

Solution available at the end of this section.

Checkpoint 1.1.8

Given the function $g(m)=\sqrt{m-4}$, solve $g(m)=2$.

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n+6p=12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

Example 1.1.9: Finding the Algebraic Form of a Function

Express the relationship $2n+6p=12$ as a function $p=f(n)$, if possible.

Solution available at the end of this section.

Caution: Not all relations are functions

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Example 1.1.10: Expressing the Equation of a Circle as a Function

Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution available at the end of this section.

Checkpoint 1.1.10

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Q&A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table.

Pet	Memory span in hours
Puppy	0.008
Adult dog	0.083
Cat	16
Goldfish	2160
Beta fish	3600

At times, evaluating a function in table form may be more useful than using equations. For the pet-memory span information, let's call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(\text{goldfish}) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

Example 1.1.11: Evaluating and Solving a Tabular Function

Use the provided table to answer the following questions.

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

n	1	2	3	4	5
g	8	6	7	6	8

Solutions available at the end of this section.

Checkpoint 1.1.11

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Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 1.1.12: Reading Function Values from a Graph

Use the graph below to answer the following questions.

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

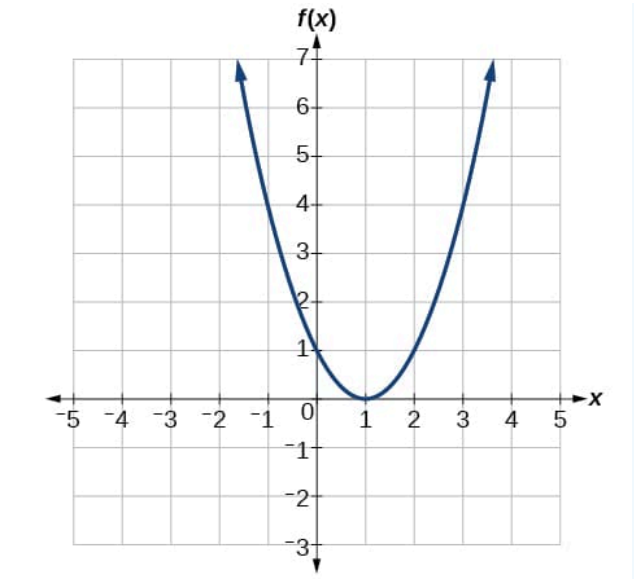


Figure 1.1.6

Solutions available at the end of this section.

Checkpoint 1.1.12

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input.

Definition: One-to-One Function

A function f is **one-to-one** (or **injective**) if distinct inputs always produce distinct outputs; that is, whenever $a \neq b$ in the domain of f, then $f(a) \neq f(b)$. Equivalently, $f(a) = f(b)$ forces $a = b$.

As an example, consider a school that uses only letter grades and decimal equivalents, as listed in the following table.

Letter grade	Grade point average
A	4.0
B	3.0
C	2.0

Letter grade	Grade point average
D	1.0

This grading system represents a one-to-one function because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figure $1.1.1(a)$ and Figure $1.1.1(b)$. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

Example $1.1.13$: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution available at the end of this section.

Checkpoint $1.1.13A$

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Checkpoint $1.1.13B$

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output value y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in the following figure tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points.

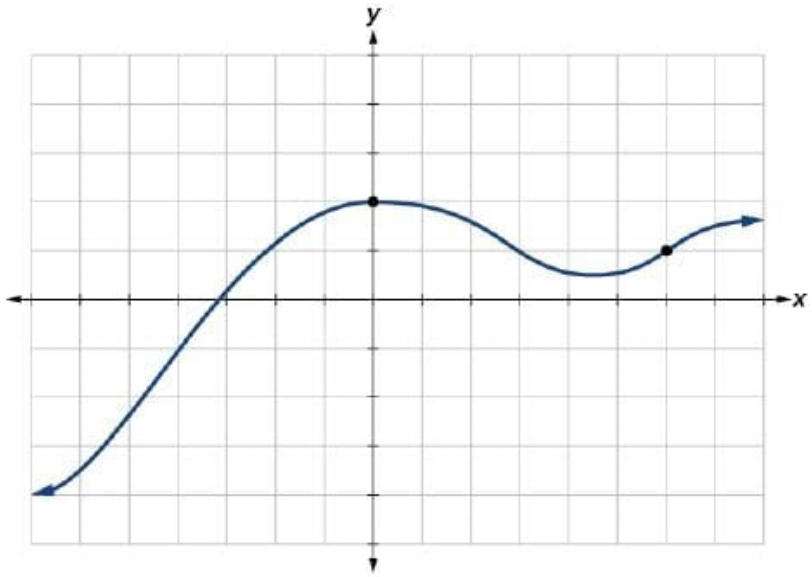


Figure 1.1.9

If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value (see the following figure).

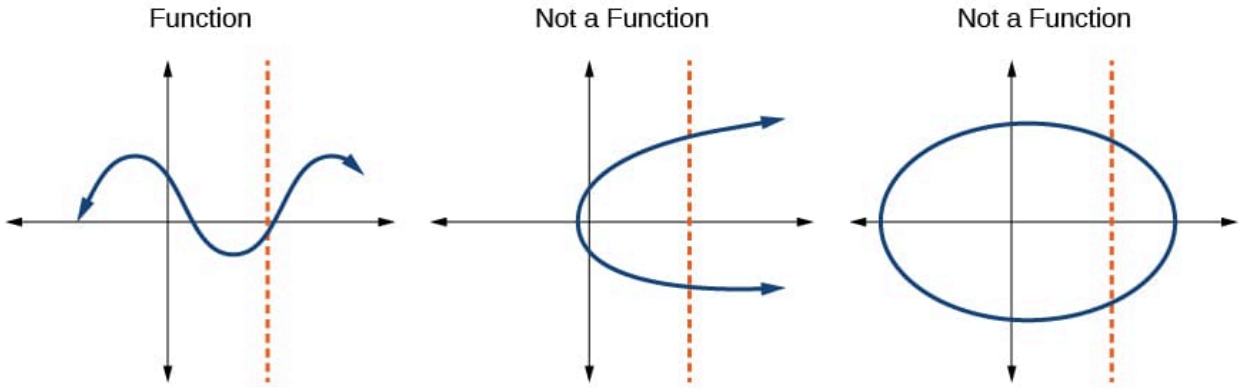


Figure 1.1.10

This leads us to the **vertical line test**.

Theorem: Vertical Line Test

A set of points in a rectangular coordinate system is the graph of a function if and only if every vertical line intersects the graph in at most one point.

Interactive Element: The Vertical Line Test

The following Interactive Element might help you visualize why a circle is not a function.

Interact: Drag the red point along the horizontal axis to sweep the vertical line across the circle.

Observation: Whenever the line passes through the interior of the circle it meets the circle at two points — one input x with two outputs — so the circle fails the Vertical Line Test and is not a function $y = f(x)$. A single vertical line with two crossings is enough to disqualify the graph.

Example 1.1.14: Applying the Vertical Line Test

Which of the graphs in the following figure represent(s) a function $y = f(x)$?

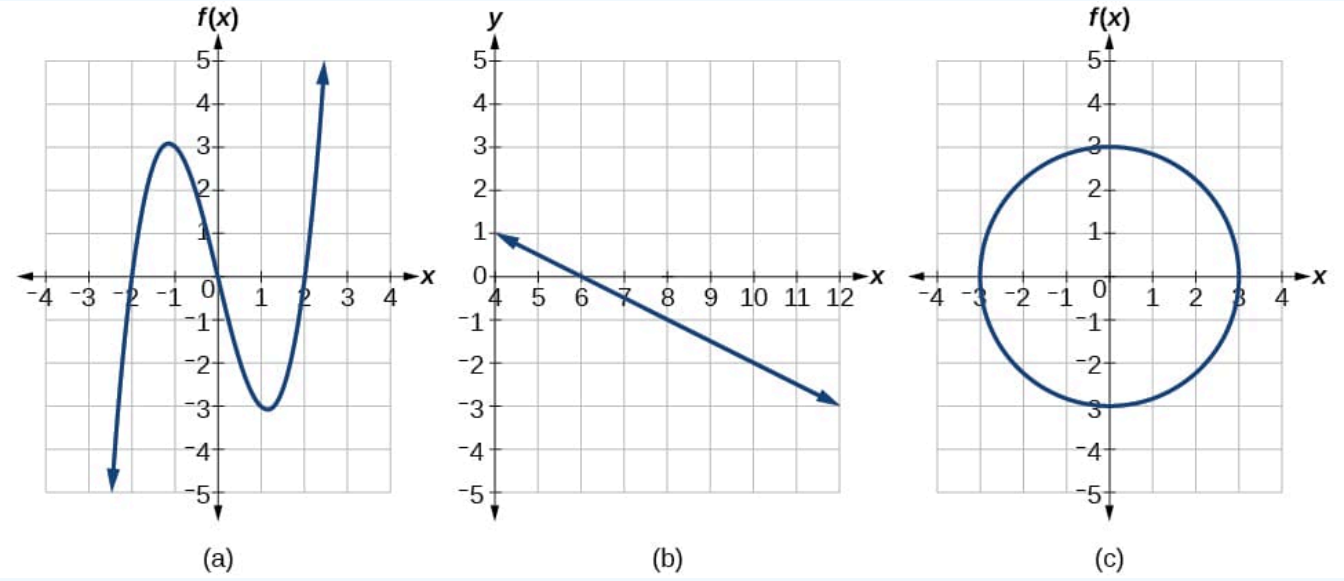


Figure 1.1.11

Solution available at the end of this section.

Checkpoint 1.1.14

Does the graph in the following figure represent a function?

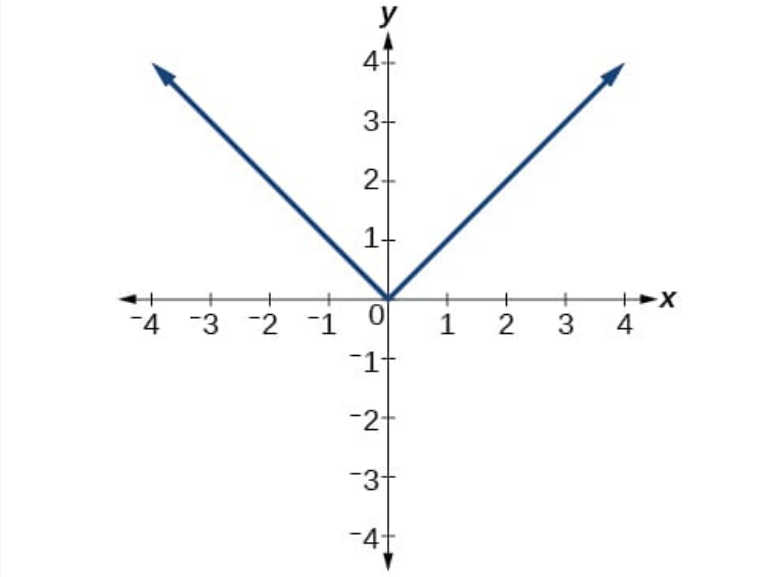


Figure 1.1.13

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the

Horizontal Line Test.

Theorem: Horizontal Line Test

A set of points representing a function in a rectangular coordinate system is the graph of a one-to-one function if and only if every horizontal line intersects the graph in at most one point.

To use the Horizontal Line Test, draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Example 1.1.15: Applying the Horizontal Line Test

Consider the functions shown in parts (a) and (b) of Example 1.1.14. Are either of the functions one-to-one?

Solution available at the end of this section.

Checkpoint 1.1.15

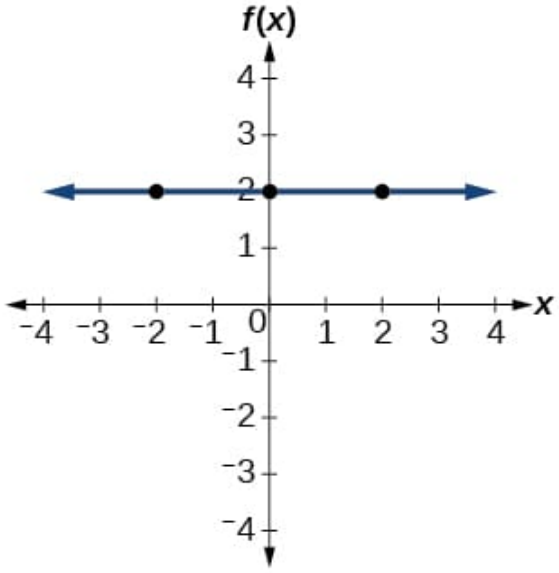
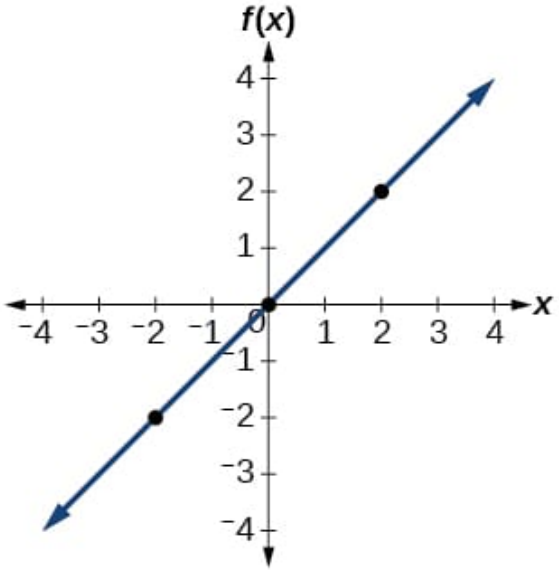
Is the graph shown in part (c) of Example 1.1.14 one-to-one?

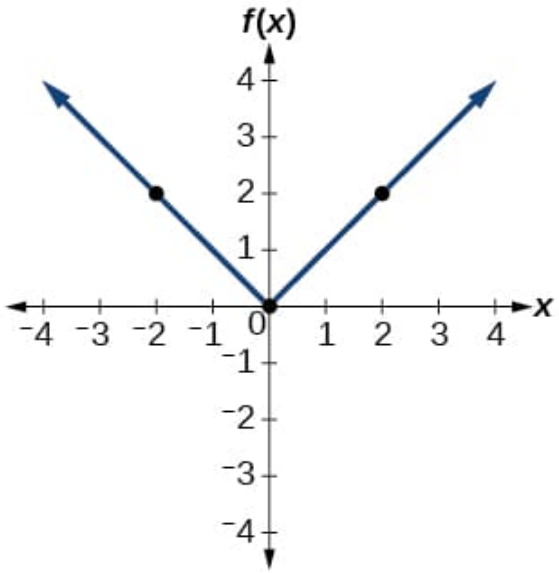
Identifying Basic Toolkit Functions

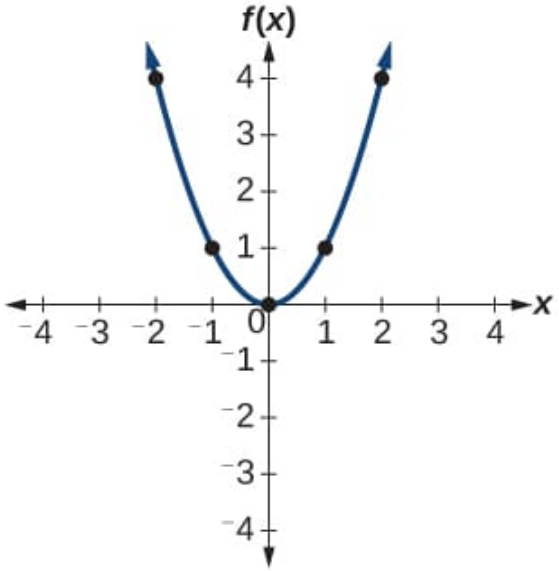
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do Arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "**toolkit functions**," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in the following table.

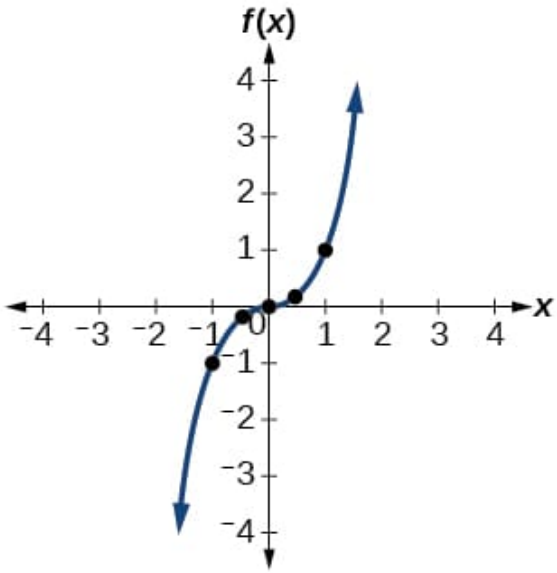
Toolkit Functions

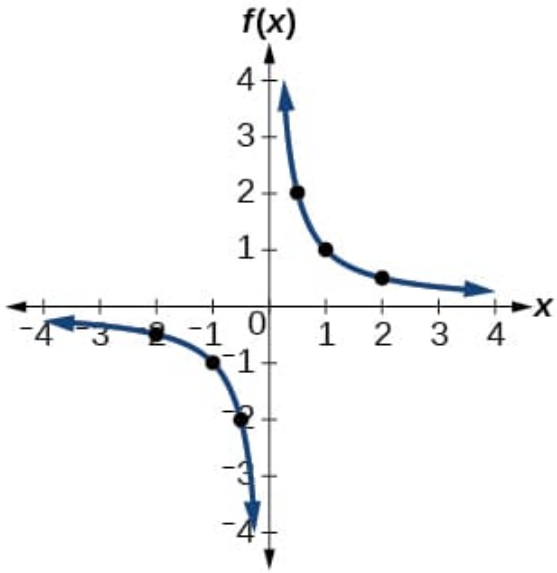
Name	Function	Graph
Constant	$f(x) = c$, where c is a constant	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>2</td></tr></table>
Identity	$f(x) = x$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table>

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Absolute value</td><td>$f(x) = |x|$</td><td> 

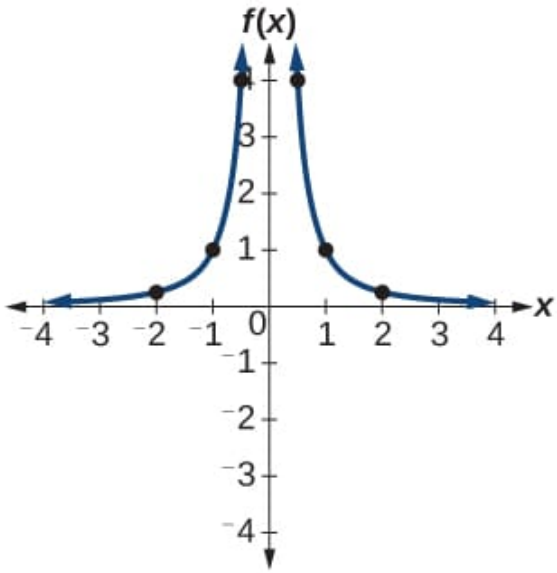
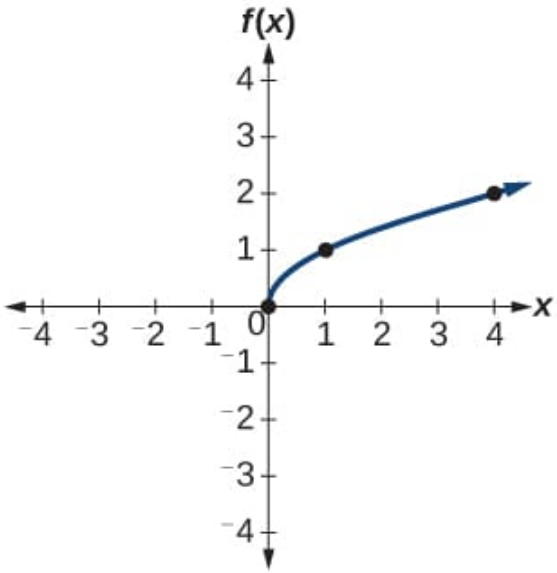
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>2</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
<tr><td>Quadratic</td><td>$f(x) = x^2$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>4</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>4</td></tr>
</table></td></tr>
</table>

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Cubic</td><td>$f(x) = x^3$</td><td> 

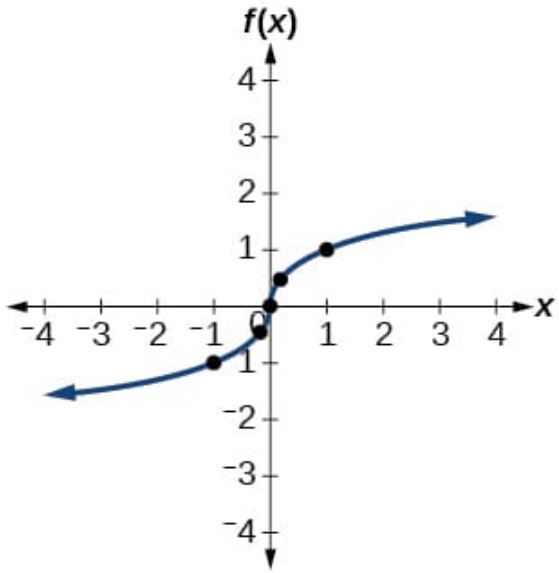
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−0.125</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.5</td><td>0.125</td></tr>
<tr><td>1</td><td>1</td></tr>
</table></td></tr>
<tr><td>Reciprocal</td><td>$f(x) = \frac{1}{x}$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−0.5</td></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−2</td></tr>
<tr><td>0.5</td><td>2</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.5</td></tr>
</table></td></tr>
</table>

Name	Function	Graph
Reciprocal squared	$f(x) = \frac{1}{x^2}$	
Square root	$f(x) = \sqrt{x}$	

x	f(x)
−2	0.25
−1	1
−0.5	4
0.5	4
1	1
2	0.25

x	f(x)
0	0
1	1
4	2

Name	Function	Graph
Cube root	$f(x) = \sqrt[3]{x}$	

x	f(x)
−1	−1
−0.125	−0.5
0	0
0.125	0.5
1	1

Hidden Resources: Borrowing Technology

Having a computer is a requirement for any college class these days; however, what do you do if you don't have a computer? Cosumnes River College (and the Los Rios Community College District) has pulled together a great list of resources for students needing a computer. Whether you want to borrow a Chromebook from the CRC library for the semester or you prefer to purchase a discounted computer, the CRC Library has all the information you need!

MyOpenMath Advice: "Message Instructor" Within Questions

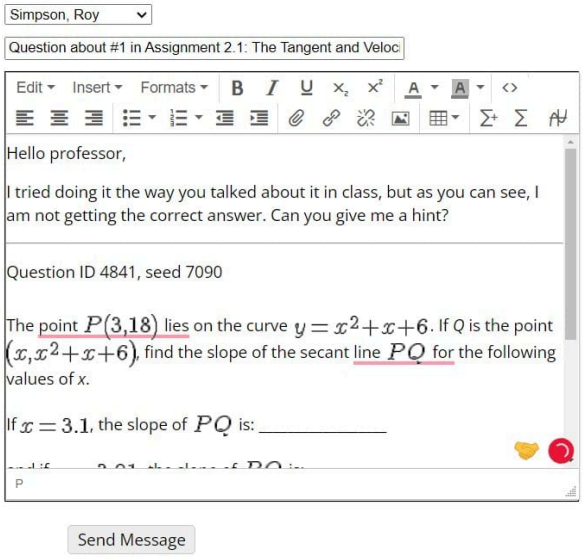
What is it?

If you are stuck on a homework question, and if your professor has set up a discussion area on Canvas, post your question to that area first; however, not all professors have enabled a discussions area. In this case, you can ask your professor for help directly by clicking on the "Message instructor" link at the bottom of the question.



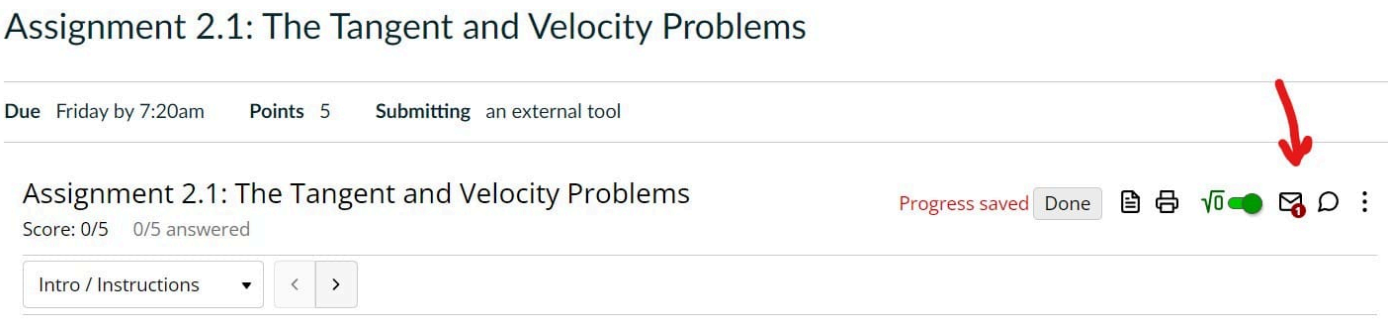
Why use it?

The nice thing about using the "Message instructor" link is that it sends *your version* of the question you are asking about directly to your professor. This helps them better understand your question. Be sure to include some information as to what might be causing your confusion.



How do you know when your professor has responded?

This is the only tricky part. When your professor responds, you will *not* get an email notification. This is a weakness of the MyOpenMath embed into Canvas (the messaging systems do not work well together); however, when your professor responds, you will see a message flag within *any* homework assignment you open.



Just click on the envelope and you will see your professor's response.

Answers

Section Preview

Treat this section as a review of functions (represented as tables, algebraic functions, word problems, and graphs). As far as computation is concerned, the algebraic functions used within this section are restricted to linear, quadratic, radical, and very simple rational functions. Finally, the nine Toolkit functions are introduced, but do not cover transformations (as this is held off for later).

Section Preview

- **Sets and Numbers**
  - Sets of Numbers: The opening relation $D=\{(1,2),(2,4),...\}$ is built from the natural numbers, and domains/ranges throughout the section are described as sets of numbers; students must be able to recognize and classify these number types to read the examples.
- **Graphs**
  - The Cartesian Coordinate System and the Quadrants: All graphical representations of functions, the Vertical Line Test, and the Horizontal Line Test presuppose a working understanding of the coordinate plane and how points live on it.
  - Ordered Pairs and Graphing Equations by Point-Plotting: The section introduces relations and functions as sets of ordered pairs (x,y); a student must already understand what an ordered pair represents and how it corresponds to a plotted point to follow the development.

Section Preview

- **Arithmetic**
  - Mathematical Operations — Powers and Square Roots: Evaluating the quadratic $f(x)=x^2$ and square-root $f(x)=\sqrt{x}$ toolkit functions, and computing values such as $h(4)=4^2+2(4)$ in Example 7, requires fluency with powers and roots.
  - The Order of Operations: Correctly evaluating expressions like $f(2)=2^2+3(2)-4$ in Example 6a depends on applying PEMDAS in the right sequence.
- **Geometry**

- Circles — Properties, Area, and Circumference: Example 13 relies on the area formula $A = \pi r^2$ to argue that area is a one-to-one function of radius; students must already know and apply this formula.
- **Factoring Techniques**
  - Greatest Common Factor: Example 6d factors the common h out of $2ah + h^2 + 3h$ in order to cancel and simplify the difference-quotient result.
- **Simplifying Expressions**
  - Algebraic Vocabulary and Evaluating Expressions: Nearly every example (6, 7, 8) hinges on substituting a value or expression for the input variable and simplifying the result, the central operational skill of the section.
  - Multiplying Polynomials and the Distributive Property: Evaluating $f(a + h)$ in Example 6c requires expanding $(a + h)^2$ and distributing $3(a + h)$.
- **Solving Equations**
  - Solving Literal Equations: Example 9 rewrites $2n + 6p = 12$ as $p = f(n)$, which requires isolating one variable while treating the other as a constant.
  - Solving Quadratic Equations by Factoring: Example 8 solves $h(p) = 3$ by rewriting it as $p^2 + 2p - 3 = 0$, factoring, and applying the Zero Product Property.
  - Solving Quadratic Equations Using Extraction of Roots: Example 10 isolates y^2 in $x^2 + y^2 = 1$ and extracts roots to obtain $y = \pm\sqrt{1 - x^2}$.
- **Graphs**
  - Reading Values from Graphs: Example 12 asks students to evaluate $f(2)$ and solve $f(x) = 4$ by reading input and output coordinates directly off a plotted curve.

Section Preview

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the Vertical Line Test to identify functions.
- Graph the functions listed in the library of functions.

Example 1.1.1: Determining If Menu Price Lists Are Functions

a. Let's begin by considering the input as the items on the menu. The output values are then the prices (see Figure 1.1.2). Each item on the menu has only one price, so the price is a function of the item.

b. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it (see the figure below).

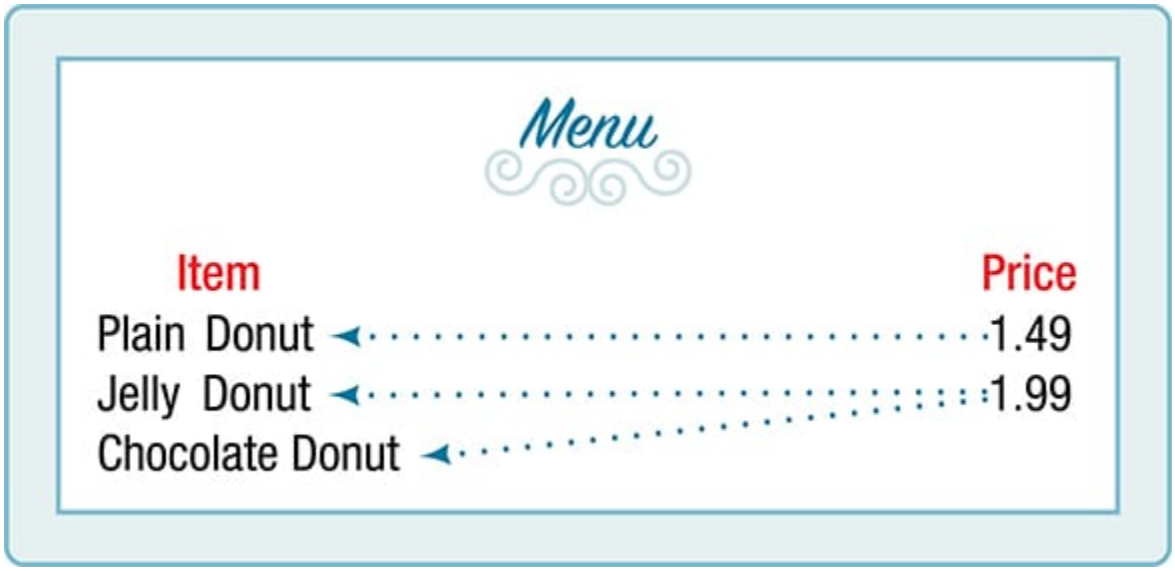


Figure 1.1.3

Example 1.1.2: Determining If Class Grade Rules Are Functions

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Example 1.1.3: Using Function Notation for Days in a Month

The number of days in a month is a function of the name of the month, so if we name the function f, we write

$$\text{days} = f(\text{month})$$

or

$$d = f(m).$$

The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

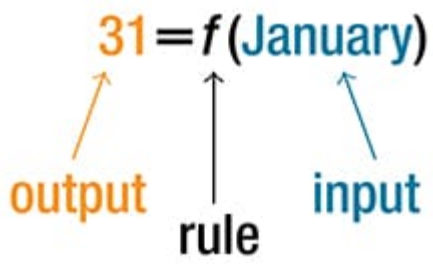


Figure 1.1.4

For example, $f(\text{March}) = 31$ because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Example 1.1.4**: Interpreting Function Notation**

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Example 1.1.5**: Identifying Tables that Represent Functions**

Table $1.1.6A$ and Table $1.1.6B$ define functions. In both, each input value corresponds to exactly one output value. Table $1.1.6C$ does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table $1.1.6A$ can be represented by writing

$$f(2) = 1, f(5) = 3, \text{ and } f(8) = 6.$$

Similarly, the statements

$$g(-3) = 5, g(0) = 1, \text{ and } g(4) = 5$$

represent the function in Table $1.1.6B$.

Table $1.1.6C$ cannot be expressed in a similar way because it does not represent a function.

Example 1.1.6**: Evaluating Functions at Specific Values**

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple Algebra to simplify.

$$\begin{aligned} f(2) &= (2)^2 + 3(2) - 4 && \text{(substituting)} \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 && \text{(substituting)} \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 && \text{(distributing)} \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4.$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} && \text{(substituting)} \\ &= \frac{a^2 + 2ah + h^2 + 3a + 3h - 4 - a^2 - 3a + 4}{h} && \text{(distributing)} \\ &= \frac{2ah + h^2 + 3h}{h} && \text{(combining like terms)} \\ &= \frac{h\,(2a + h + 3)}{h} && \text{(factoring out the GCF)} \\ &= \frac{\cancel{h}^{1}\,(2a + h + 3)}{\cancel{h}^{1}} && \text{(canceling like factors)} \\ &= 2a + h + 3 \end{aligned}$$

Example 1.1.7**: Evaluating Functions**

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(4) &= (4)^2 + 2(4) && \text{(substituting)} \\ &= 16 + 8 && \text{(simplifying)} \\ &= 24 && \text{(simplifying)} \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

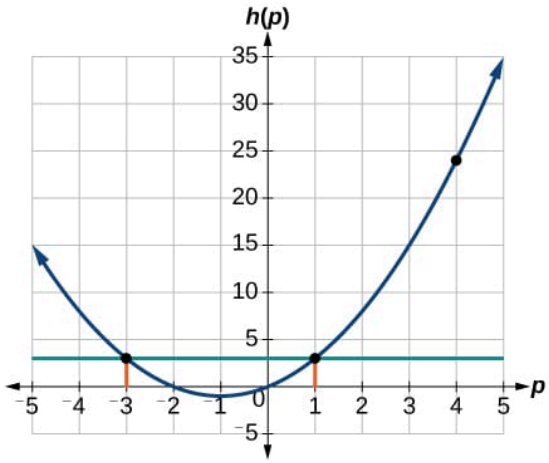
Example $1.1.8$**: Solving Functions**

$$\begin{aligned} & h(p) = 3 \\ \Longrightarrow \quad & p^2 + 2p = 3 && \text{(substituting)} \\ \Longrightarrow \quad & p^2 + 2p - 3 = 0 && \text{(subtracting 3 from both sides)} \\ \Longrightarrow \quad & (p+3)(p-1) = 0 && \text{(factoring)} \end{aligned}$$

If $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case (recall from Elementary Algebra, this is called the **Zero Factor Property**).

$$\begin{aligned} (p+3) = 0 \quad &\Longrightarrow \quad p = -3 \\ (p-1) = 0 \quad &\Longrightarrow \quad p = 1 \end{aligned}$$

This gives us two solutions, so we know that $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing (see the figure below). The graph verifies that $h(1) = h(-3) = 3$ and (for Example $1.1.7$) $h(4) = 24$.



p	-3	-2	0	1	4
$h(p)$	3	0	0	3	24

Figure $1.1.5$

Example $1.1.9$**: Finding the Algebraic Form of a Function**

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as

$$p = [\text{expression involving } n].$$

$$\begin{aligned} 2n + 6p &= 12 \\ \Longrightarrow \quad 6p &= 12 - 2n && \text{(subtracting } 2n \text{ from both sides)} \\ \Longrightarrow \quad p &= \frac{12 - 2n}{6} && \text{(dividing both sides by 6)} \\ \Longrightarrow \quad p &= \frac{12}{6} - \frac{2n}{6} \\ \Longrightarrow \quad p &= 2 - \frac{n}{3} && \text{(canceling like factors)} \end{aligned}$$

Therefore, p as a function of n is written as

$$p = f(n) = 2 - \frac{1}{3}n.$$

Example $1.1.10$**: Expressing the Equation of a Circle as a Function**

First we subtract x^2 from both sides, to get

$$y^2 = 1 - x^2.$$

We now try to solve for y in this equation.

$$\begin{aligned} y^2 &= 1 - x^2 \\ \Longrightarrow \quad y &= \pm\sqrt{1 - x^2} && \text{(Extraction of Roots)} \end{aligned}$$

Thus, either $y = \sqrt{1 - x^2}$ or $y = -\sqrt{1 - x^2}$. That is, we get two outputs corresponding to the same input. Therefore, this relationship cannot be represented as a

single function $y = f(x)$.

Example 1.1.11**: Evaluating and Solving a Tabular Function**

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.

b. Solving $g(n) = 6$ means identifying the input values, n, that produce an output value of 6. The table shows two solutions: 2 and 4. That is, when we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

Example 1.1.12**: Reading Function Values from a Graph**

a. To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$ (see the following figure).

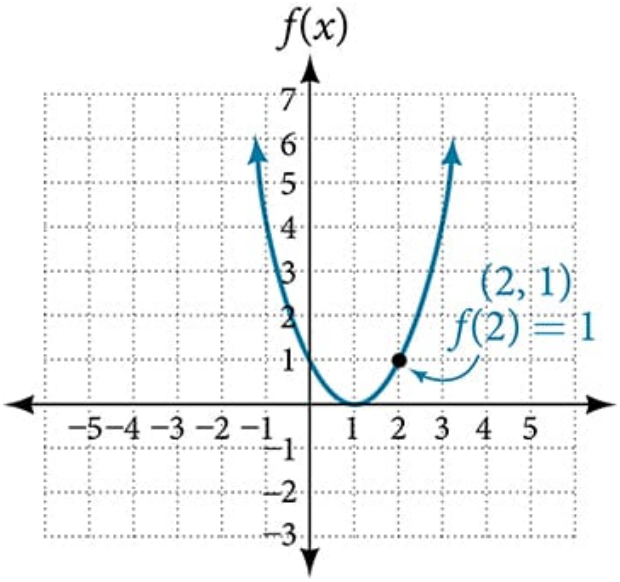


Figure 1.1.7

b. To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: -1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4 (see the figure below).

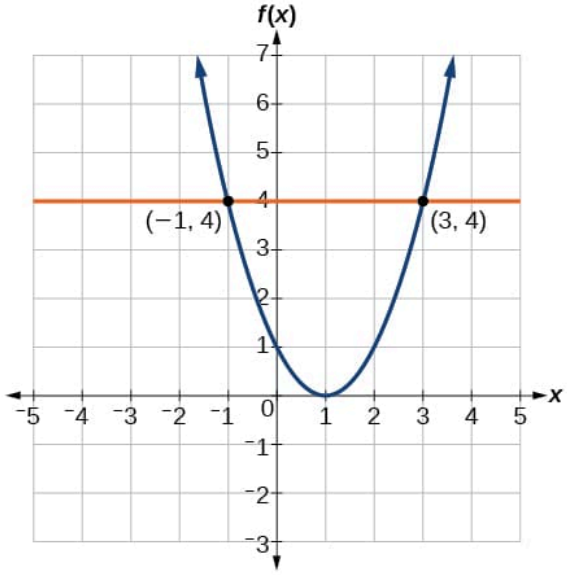


Figure 1.1.8

Example 1.1.13**: Determining Whether a Relationship Is a One-to-One Function**

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius r.

If the function is one-to-one, the output value (the area) must correspond to a unique input value (the radius). Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Example 1.1.14**: Applying the Vertical Line Test**

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of the figure. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in the figure below.

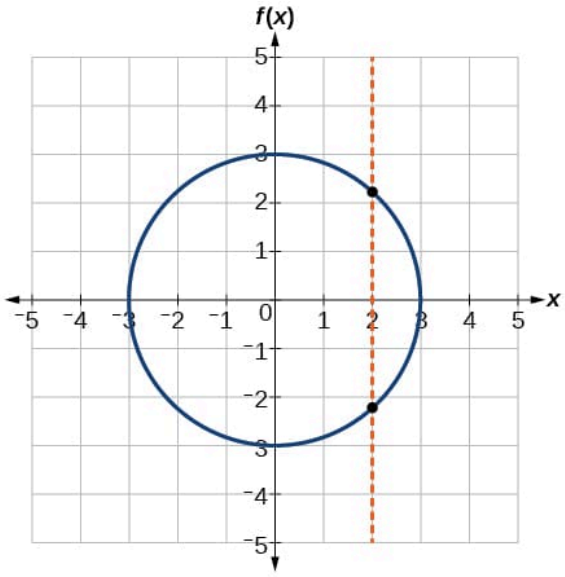


Figure 1.1.12

Example 1.1.15**: Applying the Horizontal Line Test**

The function in part (a) is not one-to-one. The horizontal line shown in the following figure intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

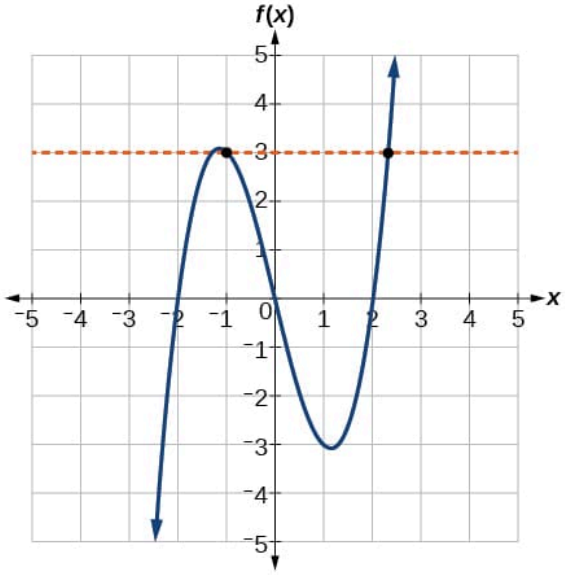


Figure 1.1.14

The function in part (b) is one-to-one. Any horizontal line will intersect a diagonal line at most once.

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1.1.1: Resources and Key Concepts

Resources

Videos

- Relations
  - Functions - Overview - A Review for College Algebra: Relations and Functions
- Functions
  - Algebra in Review: Functions - Overview
  - Intermediate Algebra - Functions: An Introduction
- Function notation
  - Algebra in Review: Functions - Notation
- Evaluating functions
  - Functions - Overview - A Review for College Algebra: Evaluating Functions
  - Functions - Overview - A Review for Precalculus: Evaluating Functions
  - Algebra in Review: Functions - Evaluation
- Representation of functions
  - Intermediate Algebra - Functions: The Four Representations
- One-to-one function
  - Functions - Overview - A Review for College Algebra: One-to-One Functions
- Vertical Line Test
  - Functions - Overview - A Review for College Algebra: The Vertical Line Test
- Horizontal Line Test
  - Functions - Overview - A Review for College Algebra: The Horizontal Line Test
- Toolkit functions
  - Intermediate Algebra - Functions: Base Graphs

Key Concepts

Definitions

- **Relation:** Any set of ordered pairs, denoted as (x, y).
- **Domain (of a relation/function):** The set of all first components (x-values) from the ordered pairs in a relation; it represents all possible input values.
- **Range (of a relation/function):** The set of all second components (y-values) from the ordered pairs in a relation; it represents all possible output values.
- **Input Value:** A value from the domain of a relation or function.
- **Output Value:** A value from the range of a relation or function, resulting from a given input value.
- **Independent Variable:** The variable (commonly x) that represents the input values of a function.
- **Dependent Variable:** The variable (commonly y) that represents the output values of a function, as its value depends on the input.
- **Function:** A specific type of relation where each possible input value is associated with exactly one output value.
- **Function Notation:** A convention, typically written as $y = f(x)$, used to denote a function, where f is the name of the function, x is the input (independent variable), and $f(x)$ is the output (dependent variable) corresponding to the input x.
- **One-to-One Function:** A function for which every distinct input value produces a distinct output value, meaning each output value corresponds to exactly one input value.

Theorems

- **Vertical Line Test:** A graph in a rectangular coordinate system represents a function if and only if no vertical line intersects the graph at more than one point. If any vertical line intersects the graph at more than one point, the graph does not represent

a function.
- **Horizontal Line Test:** The graph of a function represents a one-to-one function if and only if no horizontal line intersects the graph at more than one point. If any horizontal line intersects the graph at more than one point, the function is not one-to-one.

Common Mistakes

- **Misinterpreting Parentheses in Function Notation:** The parentheses in function notation, such as in $f(a)$, signify that a is the input to the function f. They do not indicate multiplication of f by a.
- **Assuming All Equations Define Functions:** Not every relationship expressed by an equation can be written as a single function where one variable is a function of the other. For example, the equation of a circle $x^2 + y^2 = 1$ does not define y as a single function of x because for a given x-value (other than 1 or -1), there are two corresponding y-values.

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1.1.2: Homework

Reading Questions

1. What is a relation?
2. What are the domain and range of a relation?
3. How does a function differ from a general relation?
4. In the function notation $y = f(x)$, what do x, f, and $f(x)$ represent?
5. What does it mean to "evaluate" a function at a specific value, for example, $f(2)$?
6. Describe the Vertical Line Test. What is it used for?
7. What is a one-to-one function?
8. How can the Horizontal Line Test be used?
9. Does the equation $x^2 + y^2 = 1$ represent y as a function of x? Why or why not?
10. Name three of the "toolkit functions" mentioned in the section.

Homework

1. What is the difference between a relation and a function?
2. What is the difference between the input and the output of a function?
3. Why does the Vertical Line Test tell us whether the graph of a relation represents a function?
4. How can you determine if a relation is a one-to-one function?
5. Why does the Horizontal Line Test tell us whether the graph of a function is one-to-one?

For the following exercises, determine whether the relation represents a function.

6. $\{(a, b), (c, d), (a, c)\}$
7. $\{(a, b), (b, c), (c, c)\}$

For the following exercises, determine whether the relation represents y as a function of x.

8. $5x + 2y = 10$
9. $y = x^2$
10. $x = y^2$
11. $3x^2 + y = 14$
12. $2x + y^2 = 6$
13. $y = -2x^2 + 40x$
14. $y = \frac{1}{x}$
15. $x = \frac{3y+5}{7y-1}$
16. $x = \sqrt{1 - y^2}$
17. $y = \frac{3x+5}{7x-1}$
18. $x^2 + y^2 = 9$
19. $2xy = 1$
20. $x = y^3$
21. $y = x^3$
22. $y = \sqrt{1 - x^2}$
23. $x = \pm\sqrt{1 - y}$
24. $y = \pm\sqrt{1 - x}$
25. $y^2 = x^2$
26. $y^3 = x^2$

For the following exercises, evaluate $f(-3)$, $f(2)$, $f(-a)$, $-f(a)$, and $f(a + h)$.

27. $f(x) = 2x - 5$
28. $f(x) = -5x^2 + 2x - 1$

29. $f(x) = \sqrt{2-x} + 5$
30. $f(x) = \frac{6x-1}{5x+2}$
31. $f(x) = |x-1| - |x+1|$
32. Given the function $g(x) = 5 - x^2$, evaluate

$$\frac{g(x+h) - g(x)}{h}, \text{ where } h \neq 0.$$

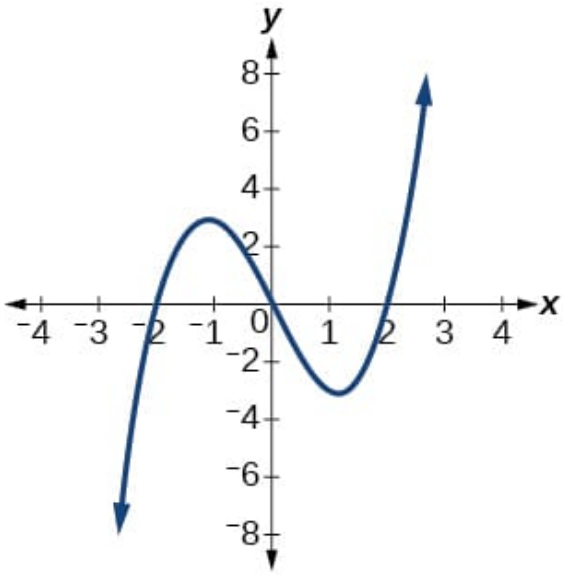
33. Given the function $g(x) = x^2 + 2x$, evaluate

$$\frac{g(x) - g(a)}{x - a}, \text{ where } x \neq a.$$

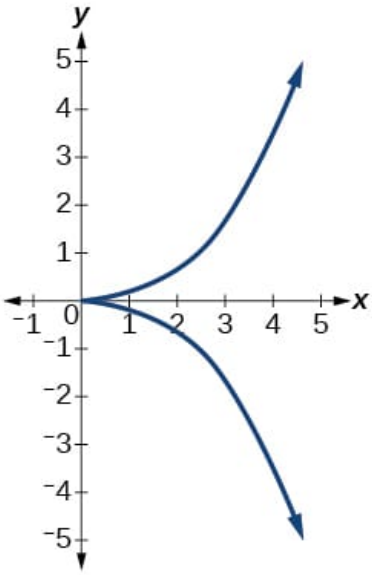
34. Given the function $k(t) = 2t - 1$:
    a. Evaluate $k(2)$.
    b. Solve $k(t) = 7$.
35. Given the function $f(x) = 8 - 3x$:
    a. Evaluate $f(-2)$.
    b. Solve $f(x) = -1$.
36. Given the function $p(c) = c^2 + c$:
    a. Evaluate $p(-3)$.
    b. Solve $p(c) = 2$.
37. Given the function $f(x) = x^2 - 3x$:
    a. Evaluate $f(5)$.
    b. Solve $f(x) = 4$.
38. Given the function $f(x) = \sqrt{x+2}$:
    a. Evaluate $f(7)$.
    b. Solve $f(x) = 4$.
39. Consider the relationship $3r + 2t = 18$.
    a. Write the relationship as a function $r = f(t)$.
    b. Evaluate $f(-3)$.
    c. Solve $f(t) = 2$.

For the following exercises, use the Vertical Line Test to determine which graphs show relations that are functions.

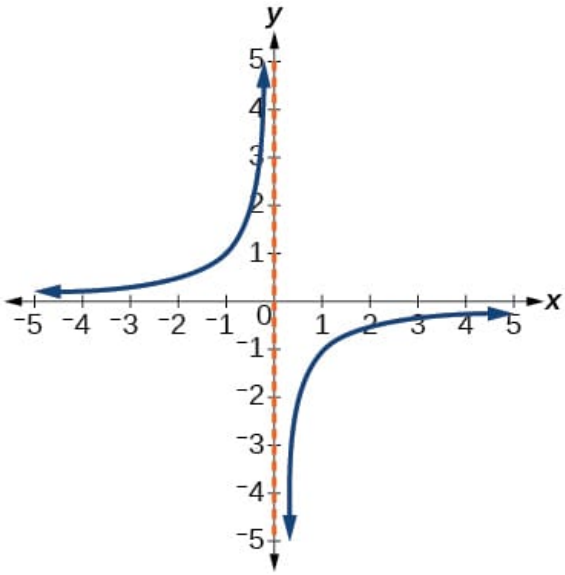
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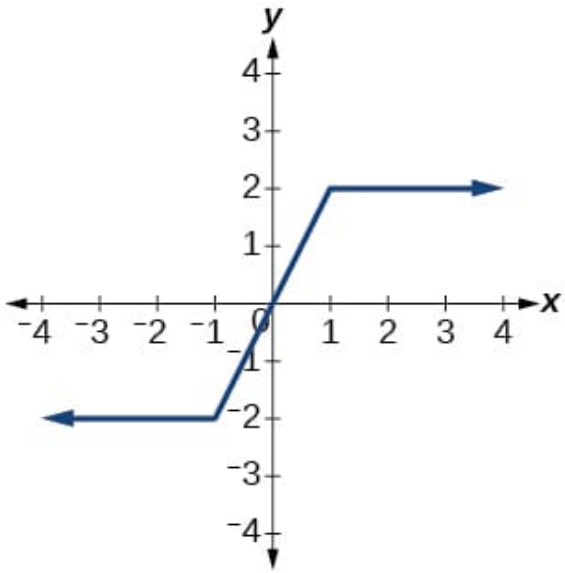


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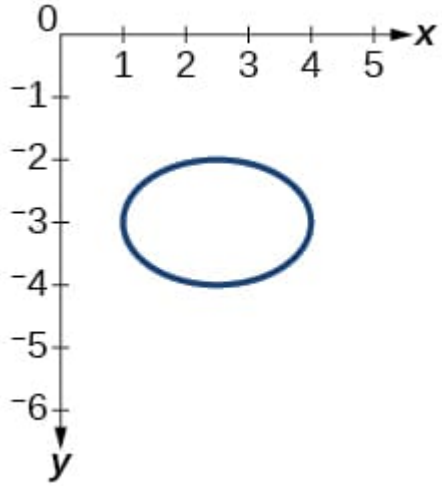


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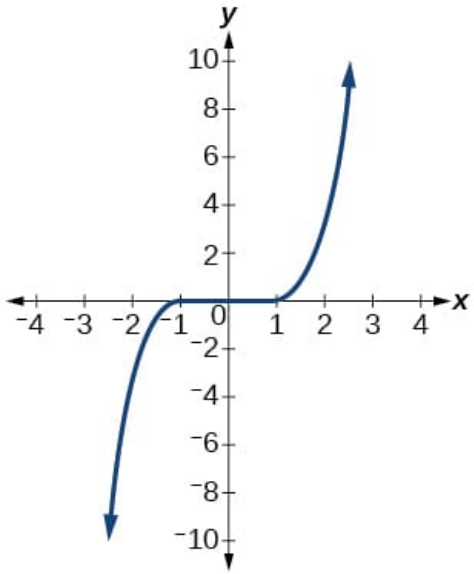
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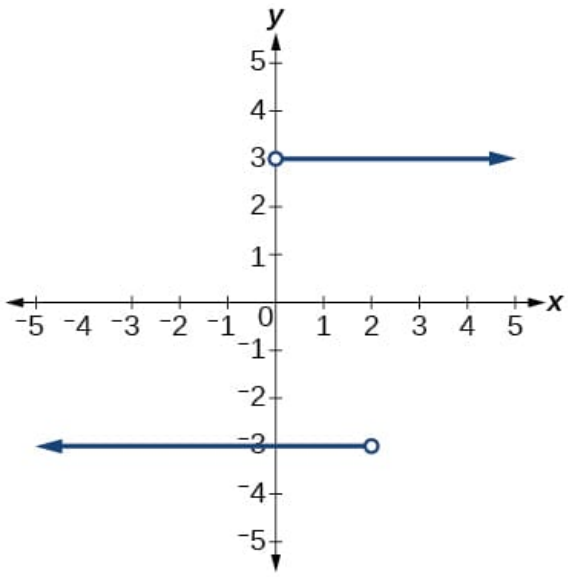

y
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15.

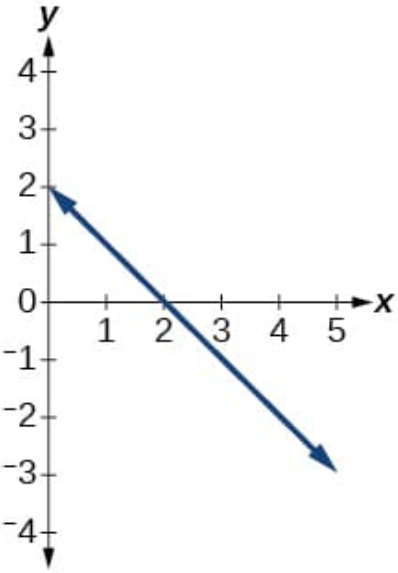

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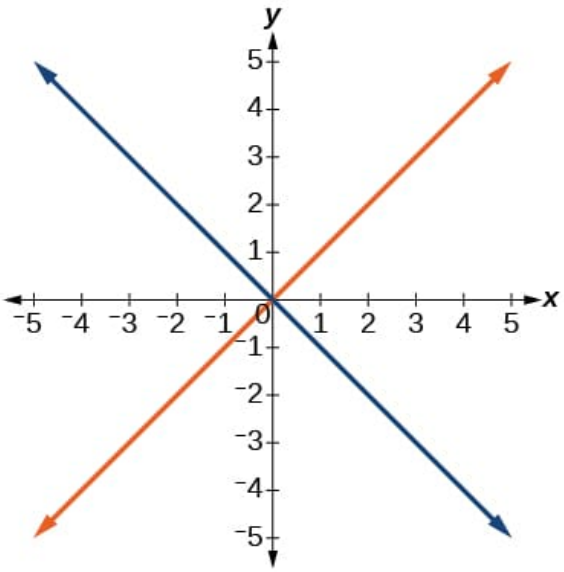

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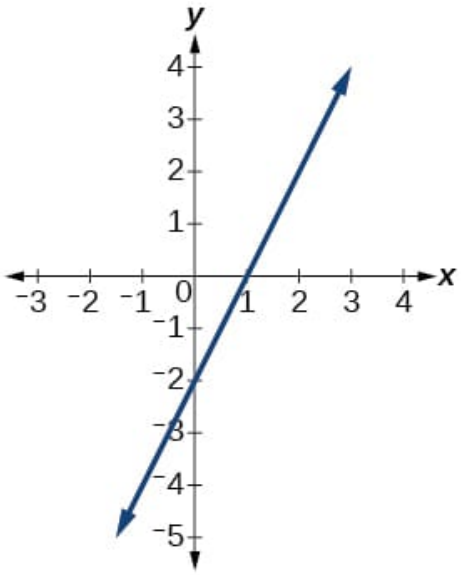
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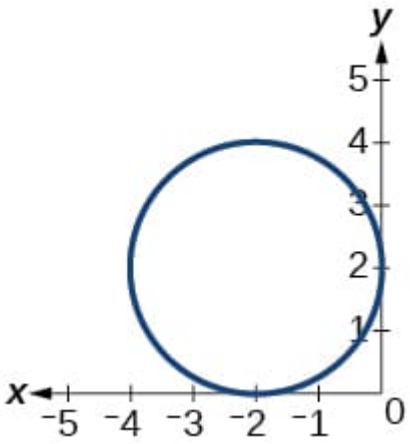
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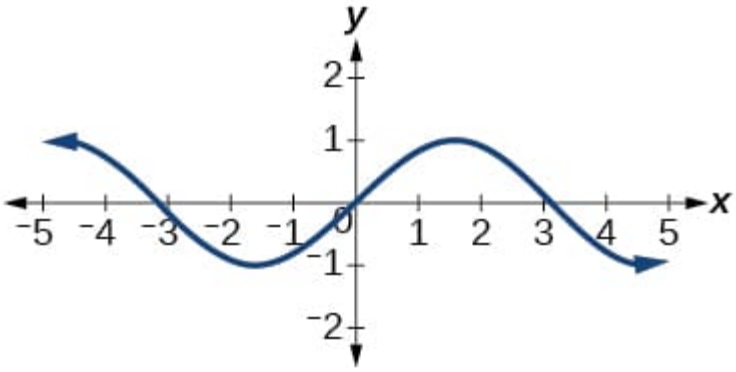
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-3 -2 -1 1 2 3 4

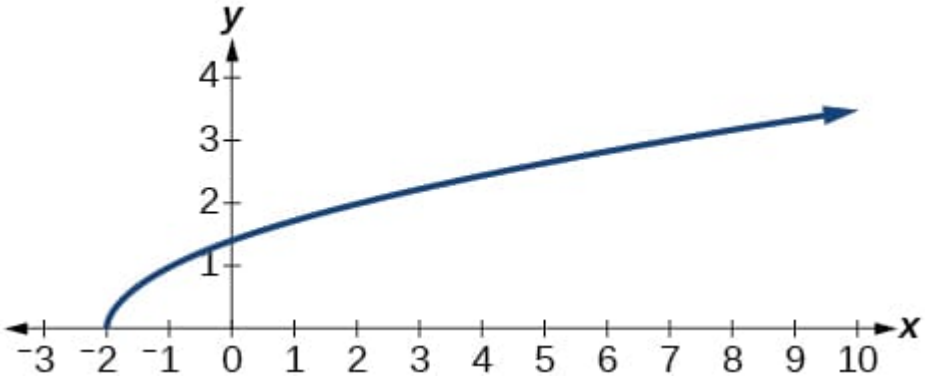
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1
0
-5 -4 -3 -2 -1

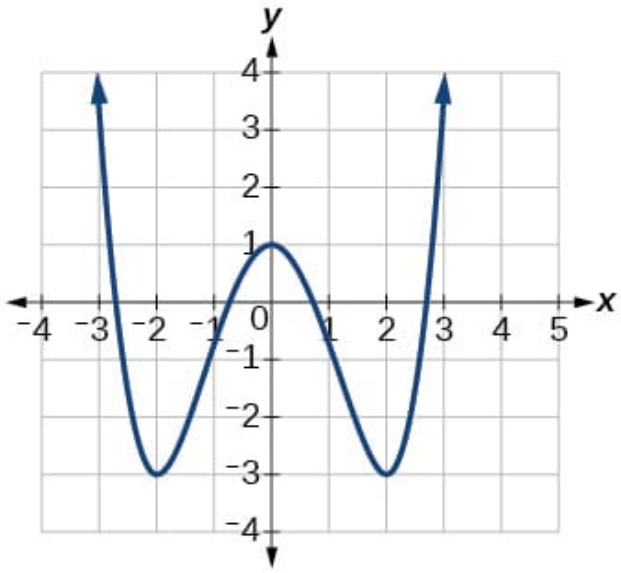
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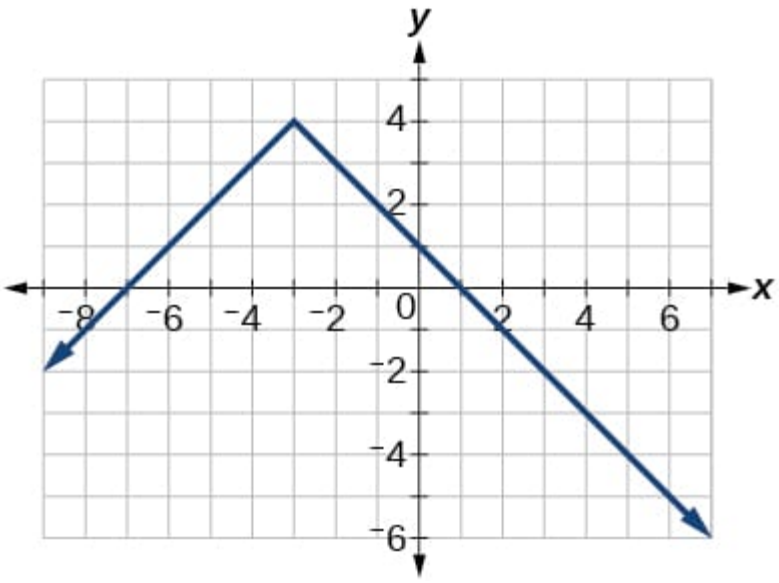
52. Given the following graph,
    a. Evaluate $f(-1)$.
    b. Solve for $f(x) = 3$.



53. Given the following graph,
    a. Evaluate $f(0)$.
    b. Solve for $f(x) = -3$.

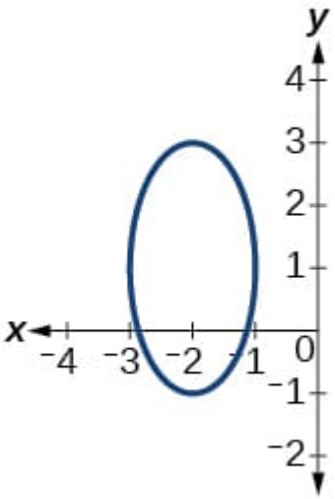


54. Given the following graph,
    a. Evaluate $f(4)$.
    b. Solve for $f(x) = 1$.

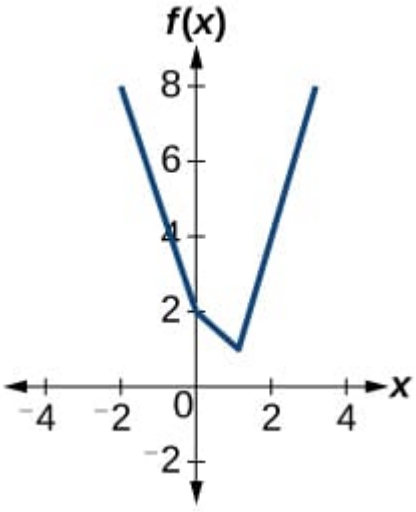


For the following exercises, determine if the given graph is a one-to-one function.

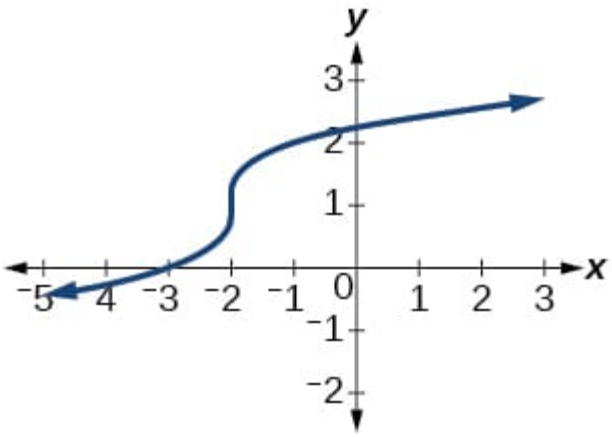
55.



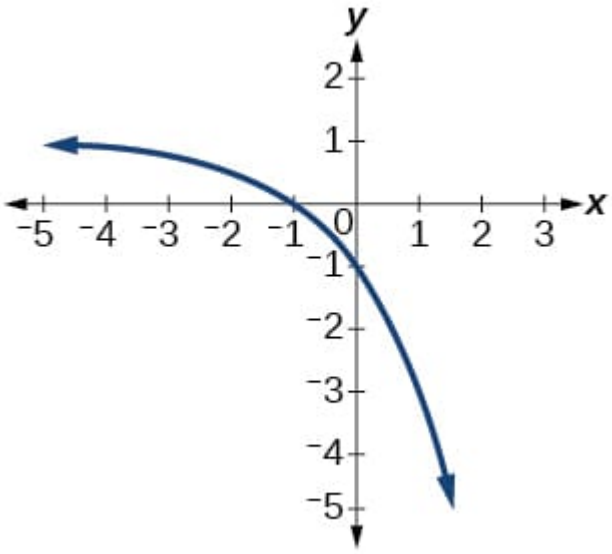
56.



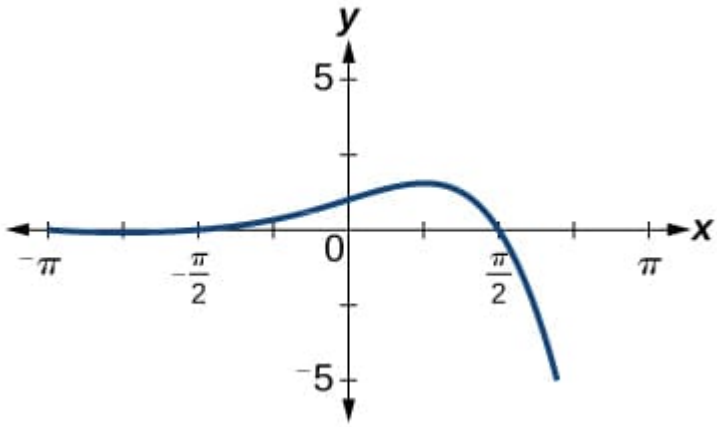
57.



58.



59.



For the following exercises, determine whether the relation represents a function.

60. $\{(-1,-1),(-2,-2),(-3,-3)\}$
61. $\{(3,4),(4,5),(5,6)\}$
62. $\{(2,5),(7,11),(15,8),(7,9)\}$

For the following exercises, determine if the relation represented in table form represents y as a function of x.

63.

x	5	10	15

y	3	8	14

54.

x	5	10	15
y	3	8	8

55.

x	5	10	10
y	3	8	14

For the following exercises, use the function f represented in the following table.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

56. Evaluate $f(3)$.
57. Solve $f(x) = 1$.

For the following exercises, evaluate the function f at the values -2, -1, 0, 1, and 2.

58. $f(x) = 4 - 2x$
59. $f(x) = 8 - 3x$
60. $f(x) = 8x^2 - 7x + 3$
61. $f(x) = 3 + \sqrt{x+3}$
62. $f(x) = \frac{x-2}{x+3}$
63. $f(x) = 3^x$

For the following exercises, evaluate the expressions, given functions f, g, and h:

- $f(x) = 3x - 2$
- $g(x) = 5 - x^2$
- $h(x) = -2x^2 + 3x - 1$

64. $3f(1) - 4g(-2)$
65. $f\left(\frac{7}{3}\right) - h(-2)$

For the following exercises, graph $y = x^2$ on the given domain. Determine the corresponding range. Show each graph.

66. $[-0.1, 0.1]$
67. $[-10, 10]$
68. $[-100, 100]$

For the following exercises, graph $y = x^3$ on the given domain. Determine the corresponding range. Show each graph.

69. $[-0.1, 0.1]$
70. $[-10, 10]$
71. $[-100, 100]$

For the following exercises, graph $y = \sqrt{x}$ on the given domain. Determine the corresponding range. Show each graph.

72. $[0, 0.01]$
73. $[0, 100]$
74. $[0, 10000]$

For the following exercises, graph $y = \sqrt[3]{x}$ on the given domain. Determine the corresponding range. Show each graph.

75. $[-0.001, 0.001]$
76. $[-1000, 1000]$

37. $[-1000000, 1000000]$
38. The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.
    a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
    b. Explain the meaning of the statement $f(5) = 2$.
39. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.
    a. A garden with area 5000 ft^2 requires 50 yd^3 of dirt. Express this information in terms of the function g.
    b. Explain the meaning of the statement $g(100) = 1$.
40. Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:
    a. $f(5) = 30$
    b. $f(10) = 40$
41. Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:
    a. $h(1) = 200$
    b. $h(2) = 350$
42. Show that the function $f(x) = 3(x - 5)^2 + 7$ is not one-to-one.

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1.2: Domain and Range

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—*I am Legend*, *Hannibal*, *The Ring*, *The Grudge*, and *The Conjuring*. Figure 1.2.1 shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

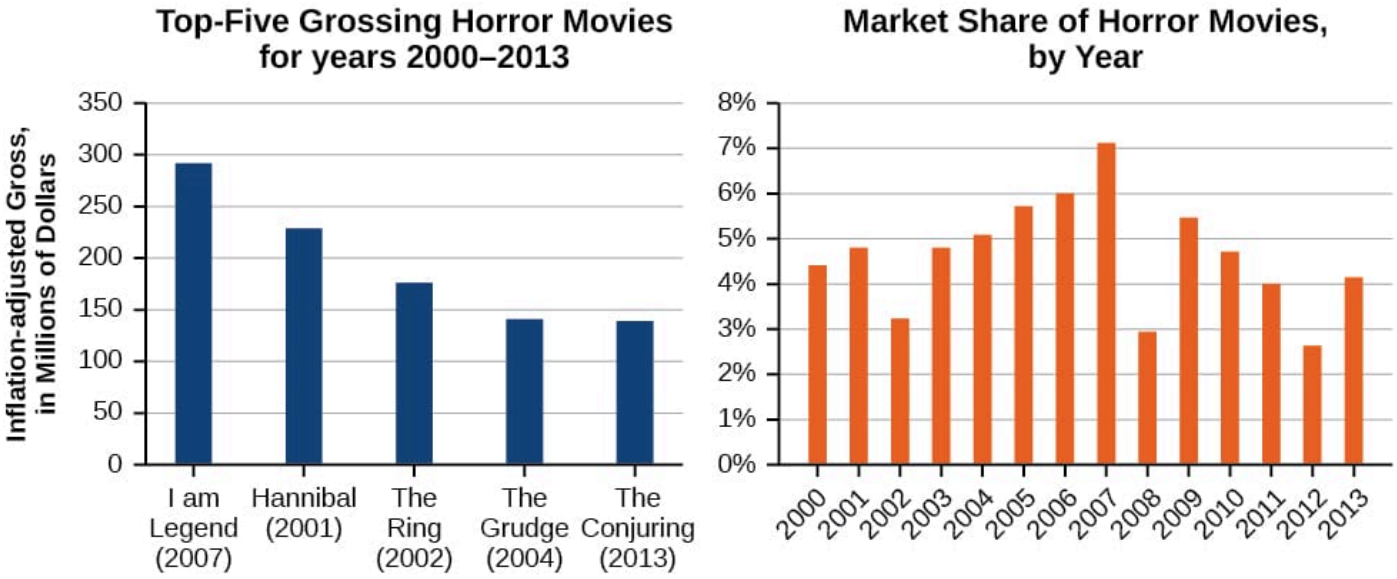


Figure 1.2.1

Finding the Domain of a Function Defined by an Equation

In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as ticket sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include

any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products (see Figure 1.2.2).

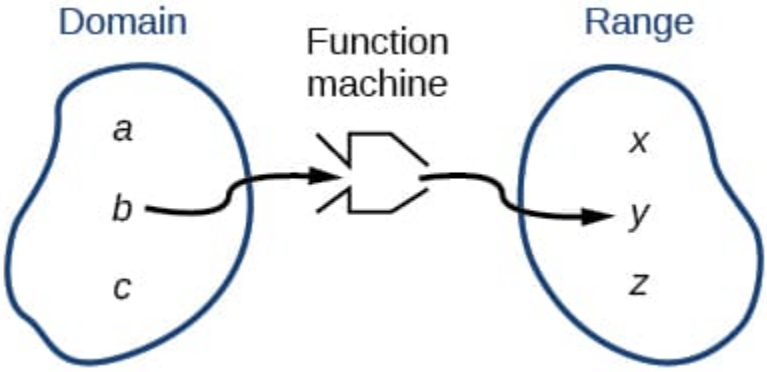


Figure 1.2.2

We can write the domain and range in interval notation, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket, [, when the set includes the endpoint and a parenthesis, (, to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will review interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an odd root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called **exclusive**.
- Brackets, [or], are used to indicate that an endpoint is included, called **inclusive**.

Figure 1.2.3 provides a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Figure 1.2.3

Example 1.2.1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2, 10), (3, 10), (4, 20), (5, 30), (6, 40)\}$.

Solution available at the end of this section.

Checkpoint 1.2.1

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View this Interactive Content online

Example 1.2.2: Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$.

Solution available at the end of this section.

Checkpoint 1.2.2

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View this Interactive Content online

Example 1.2.3: Finding the Domain of a Function Involving a Denominator

Find the domain of the function

$$f(x) = \frac{x+1}{2-x}.$$

Solution available at the end of this section.

Checkpoint 1.2.3

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View this Interactive Content online

Example 1.2.4: Finding the Domain of a Function with an Even Root

Find the domain of the function

$$f(x) = \sqrt{7-x}.$$

Solution available at the end of this section.

Checkpoint 1.2.4

Find the domain of the function

$$f(x) = \sqrt{5+2x}.$$

Preview of Future Material

Functions of the types found in Examples 1.2.2 through 1.2.4 will be explored in more detail later in the textbook. For now, any domain questions involving these functions will be fairly straightforward; however, later we will see them "composed" with other functions.

Q&A

Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function

$$f(x) = -\frac{1}{\sqrt{x}}$$

has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in **set-builder notation**.

Definition: Set-Builder Notation

Set-builder notation is a notation for specifying a set by a property that characterizes its members.

Notation: Set-Builder Notation

Set-builder notation commonly uses the following notation:

$$\{x \mid x \text{ satisfies some condition}\}.$$

The braces, $\{\ \}$, are read literally as "the set of." The vertical bar is read as, "such that." Hence, set-builder notation is read as "the set of x such that x satisfies some given condition."

For example, $\{x \mid 10 \leq x < 30\}$ describes the behavior of x in set-builder notation. So we would read $\{x \mid 10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

The following figure compares inequality notation, set-builder notation, and interval notation.

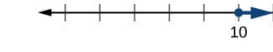
	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure 1.2.5

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$, to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one *or* the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x \mid |x| \geq 3\} = (-\infty, -3] \cup [3, \infty).$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x \mid \text{statement about } x\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x \mid 4 < x \leq 12\}.$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12].$$

Example 1.2.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in the following figure using inequality notation, set-builder notation, and interval notation.

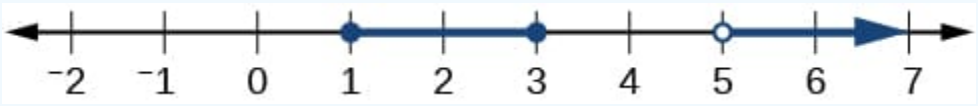


Figure 1.2.6

Solution available at the end of this section.

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Checkpoint 1.2.5

Use the following figure to specify the graphed set in

a. words
b. set-builder notation
c. interval notation



Figure 1.2.7

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values.

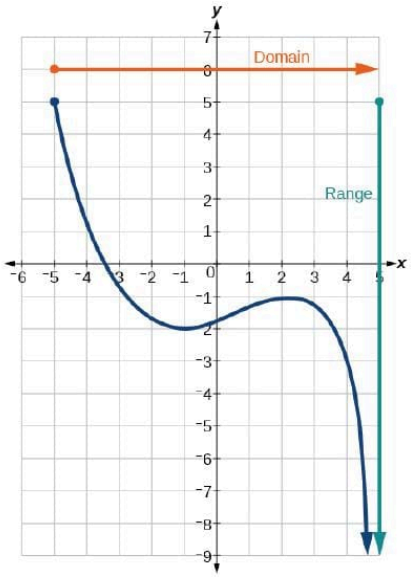


Figure 1.2.8

We can observe that the graph in the figure above extends horizontally from -5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example 1.2.6: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown below.

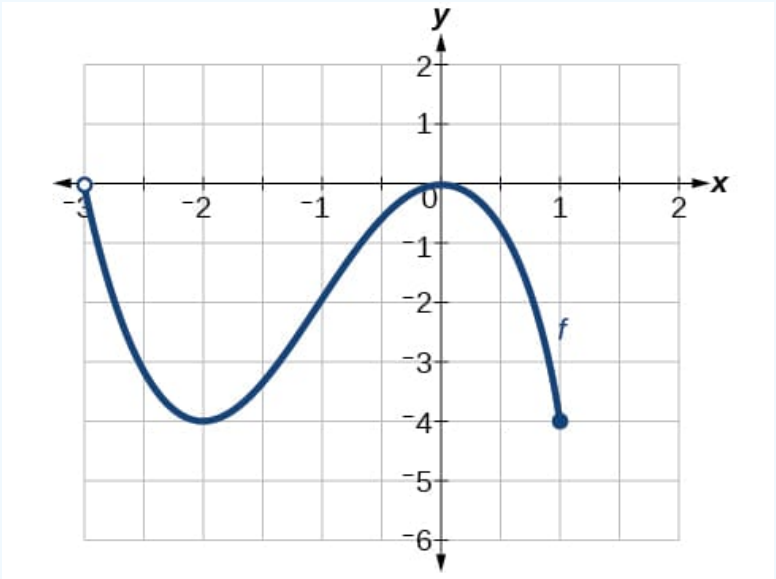


Figure 1.2.9

Solution available at the end of this section.

Example 1.2.7: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown below.

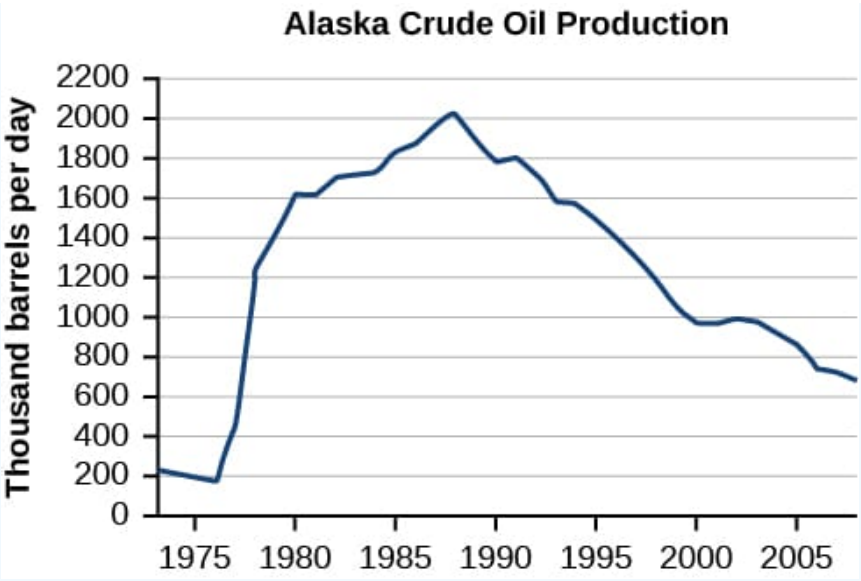


Figure 1.2.11

Solution available at the end of this section.

Checkpoint 1.2.7

Identify the domain and range of the graph in the figure below using interval notation.

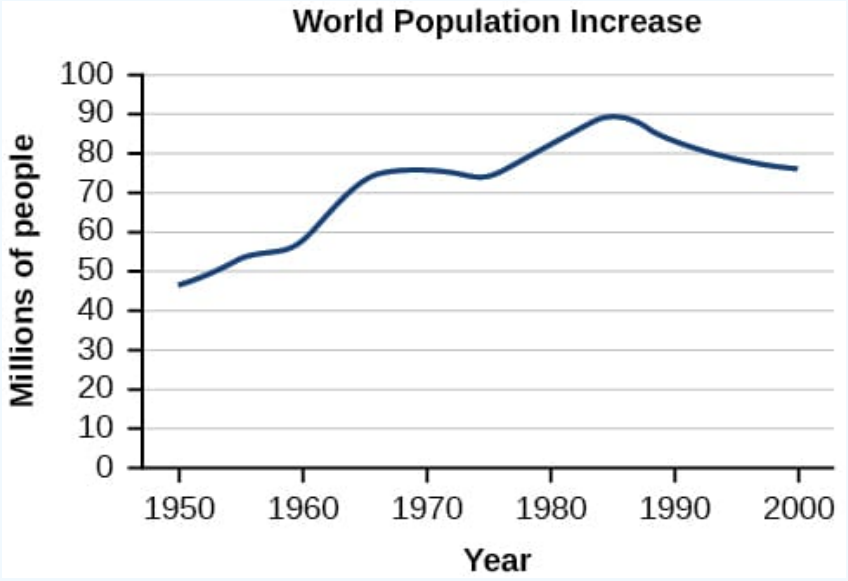


Figure 1.2.12

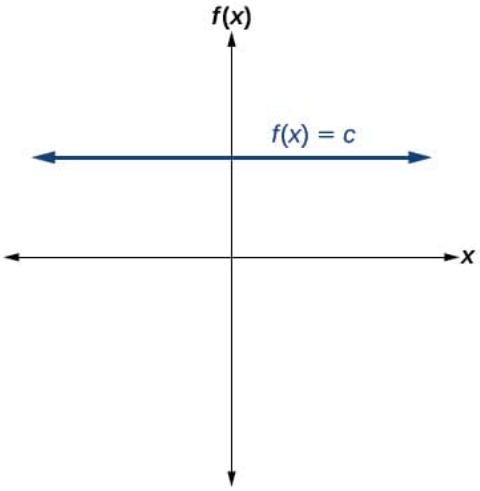
Q&A

Can a function's domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

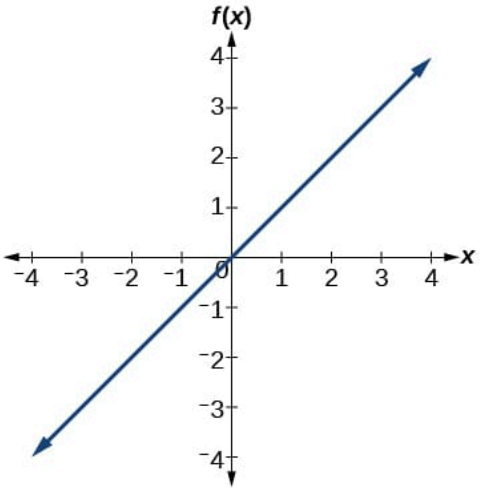
Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.



Domain: $(-\infty, \infty)$
Range: [c,c]

Figure 1.2.13**:** *For the constant function $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.*



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 1.2.14**:** *For the identity function $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.*

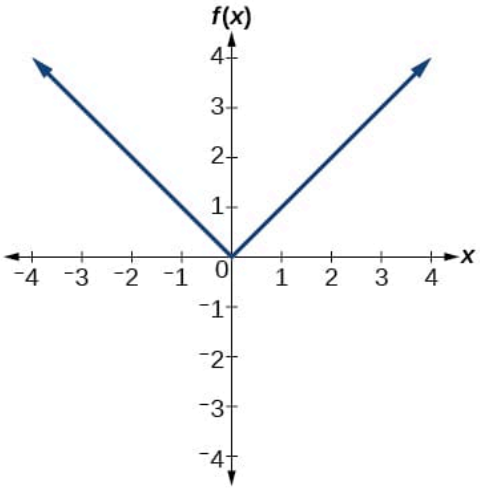


Figure 1.2.15**:** *For the absolute value function $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.*

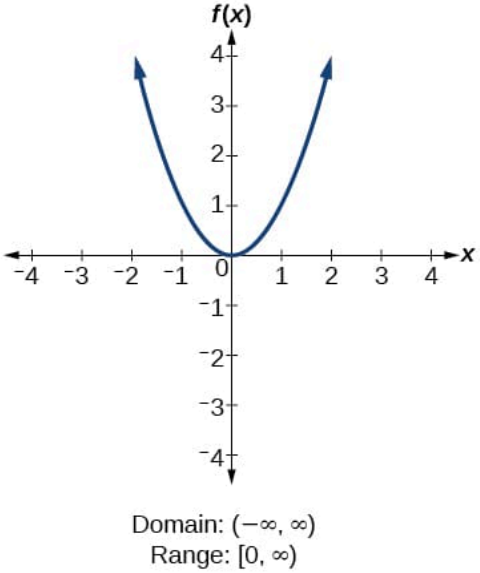


Figure 1.2.16**:** *For the quadratic function $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.*

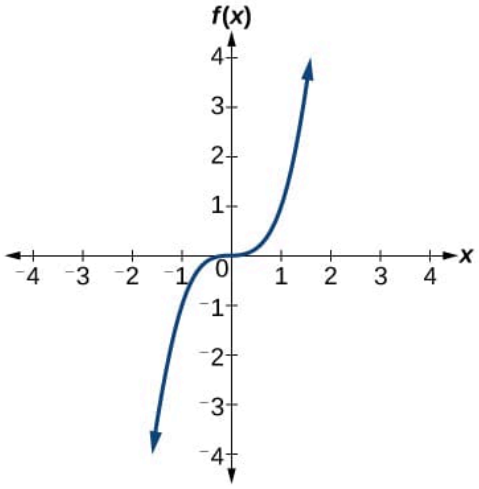


Figure 1.2.17**:** *For the cubic function $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.*

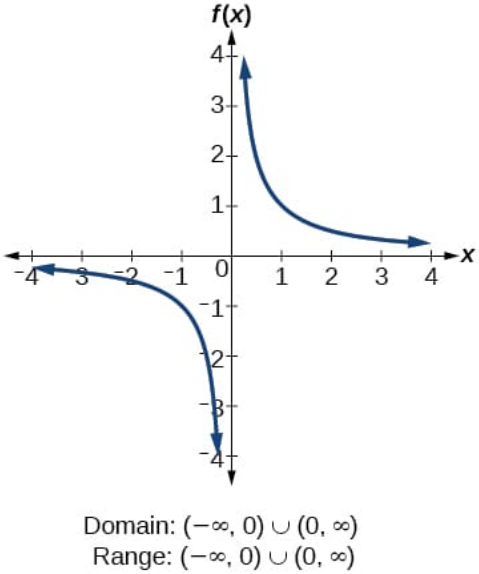
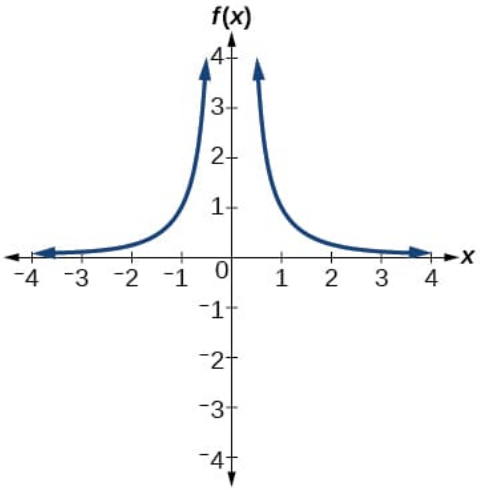
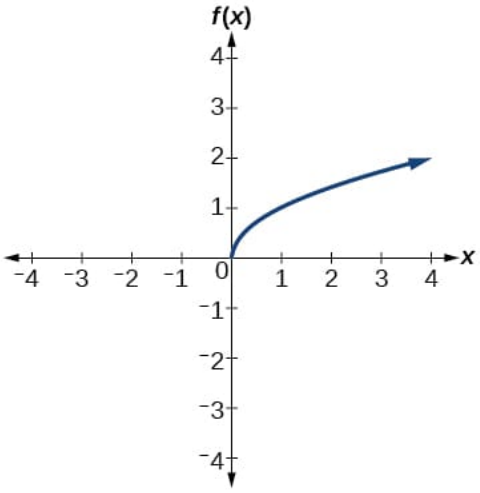


Figure 1.2.18**:** *For the reciprocal function $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x \mid x \neq 0\}$, the set of all real numbers that are not zero.*



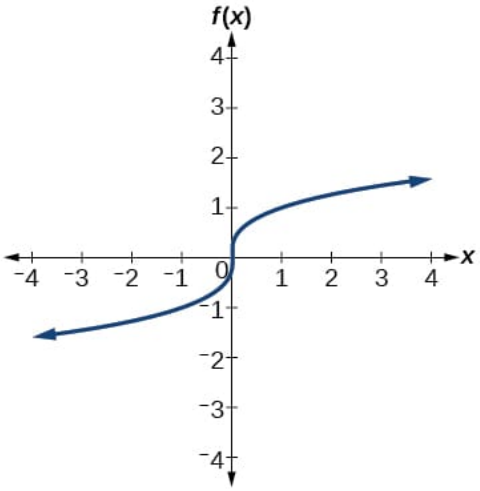
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

Figure 1.2.19**:** *For the reciprocal squared function $f(x) = \frac{1}{x^2}$, we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.*



Domain: $[0, \infty)$
Range: $[0, \infty)$

Figure 1.2.20**:** *For the square root function $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.*



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 1.2.21**:** *For the cube root function $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).*

Example 1.2.8: Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

Solution available at the end of this section.

Example 1.2.9: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution available at the end of this section.

Example 1.2.10: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution available at the end of this section.

Interactive Element: Domain and Range as Shadows of a Graph

The following Interactive Element might help you visualize how the domain and range of $f(x) = 2\sqrt{x+4}$ are just the horizontal and vertical extents of its graph.

Interact: Drag the large orange point along the curve and watch the green bar on the x-axis and the purple bar on the y -axis grow.

Observation: The green bar (the domain) always starts at the closed endpoint $x = -4$ and the purple bar (the range) always starts at $y = 0$; dragging far right shows both go on without bound, so the domain is $[-4, \infty)$ and the range is $[0, \infty)$.

Checkpoint 1.2.10

Find the domain and range of $f(x) = -\sqrt{2-x}$.

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, absolute value can be defined as the **magnitude**, or **modulus**, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0.$$

If we input a negative value, the output is the opposite of the input.

$$f(x) = -x \text{ if } x < 0.$$

Because this requires two different processes, or pieces, the absolute value function is an example of a **piecewise function**. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

Definition: Piecewise Function

A **piecewise-defined function** (also called a **piecewise function** or a **function defined by cases**) is a function specified by two or more sub-functions, where each sub-function applies on its own portion of the domain. The portion of the domain on which a given sub-function applies is called its **subdomain**.

Piecewise functions can be defined using the common functional notation, where the body of the function consists of a list of "subfunctions" and associated subdomains.

Notation: Piecewsie Function

Piecewise functions are almost always written using the following notation:

$$f(x) = \begin{cases} \text{subfunction 1}, & \text{if } x \text{ is in subdomain 1} \\ \text{subfunction 2}, & \text{if } x \text{ is in subdomain 2} \\ \text{subfunction 3}, & \text{if } x \text{ is in subdomain 3} \\ \vdots & \vdots \end{cases}$$

Some authors prefer to use a semicolon instead of a comma following the subfunctions. Moreover, the use of "if" following the comma is a stylistic choice, with some authors preferring to use the word "for" instead (e.g., "for $x < 2$). A piecewise function can also be written more succinctly by leaving out both "if" and "for," as follows:

$$f(x) = \begin{cases} x^2, & x < 2 \\ e^x, & 2 \leq x < 5 \\ \sin(x), & x \geq 5 \end{cases}$$

A value of x at which one subdomain ends and the next begins is called a **breakpoint**.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to \$10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10,000$ and $\$1000 + 0.2(S - \$10,000)$ if $S > \$10,000$.

Example 1.2.11: Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a function relating the number of people, n, to the cost, C.

Solution available at the end of this section.

Example 1.2.12: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25, & \text{if } 0 < g < 2 \\ 25 + 10(g-2), & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Solution available at the end of this section.

While I am not a huge fan of "algorithmic" instructions on how to do each problem (such habits lead to an overwhelming amount of memorization and a subpar level of understanding), such instruction is incredibly helpful when it comes to graphing piecewise functions.

How To

Given a piecewise function, sketch a graph.

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example 1.2.13: Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ 3, & \text{if } 1 < x \leq 2 \\ x, & \text{if } x > 2 \end{cases}$$

Solution available at the end of this section.

Checkpoint 1.2.13

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3, & \text{if } x \leq -1 \\ -2, & \text{if } -1 < x \leq 4 \\ \sqrt{x}, & \text{if } x > 4 \end{cases}$$

Before we leave this section, it's best to preview the *real* definition of the absolute value as a piecewise function.

Definition: Absolute Value Function

The absolute value function is defined to be $f(x) = |x|$, where

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

Q&A

Can more than one formula from a piecewise function be applied to a value in the domain?

No. Each value corresponds to one equation in a piecewise formula.

Time-Sensitive Advice: HawkConnect

What is HawkConnect?

In the past, navigating through all the student resources (EOPS, TRIO, MI CASA, etc.) required you to go to individual sites and apply individually for each program. HawkConnect now unifies the entire process into a single, guided application process.

To learn more about HawkConnect and to apply, go to the CRC Support Services website.

Success in Mathematics: Show Your Work

Many mathematics exams include the statement, "Show Your Work" (if it's not explicitly written, then it is implied). The exact meaning of these words can change from one professor to another; however, they usually mean that you need to show more than just the answer to a problem. You need to explain or show how you found the answer. Your responsibility is to learn what "Show Your Work" means to your instructor.

Why do you need to show your work?

Your *authentic* work (e.g., without use of A.I., tutors, or any other external help) demonstrates what you know. It shows what you understand and what you do not understand. By looking at your work and the work of others, an instructor can help you improve your understanding.

Everyone makes mistakes. To learn from these mistakes, you need to be able to find them. If you show your work, you can go back and find mistakes. This helps you identify misunderstandings and bad habits. Without your work, all you can do is start over.

Finally, although you may be able to do a problem "in your head," this will not always be possible. The problems get more challenging and involve more detail as we move forward. Showing your work now develops the habits and procedures you will need to do math in the future.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. A student did a homework exercise. She checked her answer with the back of the book. It was wrong. So, she erased her work and started over. Why should she not erase her work?
2. A student did a homework exercise. He checked his answer with the back of the book. It was wrong. He marked the problem on his homework with an asterisk. He went on with the assignment. He never went back to the problem. Why is it essential for this student to finish the problem?

If you are unsure about the answers to either of these questions, feel free to ask your instructor - that's what they are there for!

Answers

Section Preview

The algebraic functions encountered in this section should *not* involve compositions of functions (e.g., the inverse square function composed with a square root composed with a quadratic). Focus on the domains of basic functions (involving shifts). Piecewise functions and the absolute value represented as a piecewise function are critical topics.

Section Preview

- **Sets and Numbers**
  - Inequalities and Inequality Notation: Domain conditions (radicand ≥ 0, denominator $\neq 0$) and the horizontal/vertical extents of graphs are stated with inequalities, so students must already read order relations fluently.
  - Intervals and Interval Notation: Every domain and range answer is expressed in interval notation; understanding brackets vs. parentheses (inclusion vs. exclusion) is essential to the entire section.
  - Set-Builder Notation: The section specifies domains and ranges in set-builder form and relies on students reading $\{x \mid P(x)\}$.
- **Arithmetic**
  - Mathematical Operations—Higher-Index Roots: The domain rule explicitly distinguishes odd roots (domain all reals) from even roots (restricted), as illustrated by the cube root vs. square root toolkit functions.
  - The Absolute Value of a Number: Absolute value as distance from zero underlies the $\{x \mid |x| \geq 3\}$ set and motivates the piecewise definition of $|x|$ previewed at the end of the section.
- **Graphing Functions**
  - Intercepts and Base Graphs: Finding domains/ranges of the toolkit functions and sketching the pieces in Example 13 relies on already recognizing parent-function shapes; the section states that "each of the component functions is from our library of toolkit functions, so we know their shapes."

Section Preview

- **Sets and Numbers**
  - Sets and Set Notation: Example 1 lists a domain inside braces, and Examples 3 and 5 use the union symbol $\cup$ to combine sets/intervals.
- **Solving Equations**
  - Solving Linear Equations—One or Two-Steps: Example 3 finds the excluded input by setting the denominator $2 - x = 0$ and solving.
- **Solving Inequalities**
  - Solving Linear Inequalities: Examples 4 and 10 obtain the domain by setting the radicand ≥ 0 and solving the resulting linear inequality.
- **Graphs**
  - Closed- and Open-Circle Notations: Graphing the piecewise function in Example 13 requires open circles for excluded endpoints and closed circles for included ones.

Section Preview

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

Example $1.2.1$**: Finding the Domain of a Function as a Set of Ordered Pairs**

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs: $\{2, 3, 4, 5, 6\}$.

Example $1.2.2$**: Finding the Domain of a Function**

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers. In interval form, the domain of f is $(-\infty, \infty)$.

Example $1.2.3$**: Finding the Domain of a Function Involving a Denominator**

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2 - x &= 0 \\ \Longrightarrow \quad 2 &= x \qquad (\text{adding } x \text{ to both sides}) \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$, to combine the two sets. In interval notation, we write the solution: $(-\infty, 2) \cup (2, \infty)$.

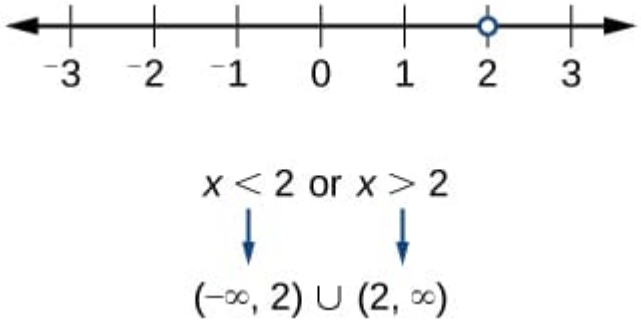


Figure $1.2.4$

Thus, in interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

Example $1.2.4$**: Finding the Domain of a Function with an Even Root**

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand. To do this, we set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7-x &\geq 0 && \text{(even-indexed radicals must have nonnegative arguments)} \\ \Rightarrow \quad 7 &\geq x && \text{(adding } x \text{ to both sides)} \end{aligned}$$

Therefore, x must be less than or equal to 7. Hence, we need to exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

Example $1.2.5$**: Describing Sets on the Real-Number Line**

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality	$1 \leq x \leq 3 \text{ or } x > 5$
Set-builder notation	$\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$
Interval notation	$[1, 3] \cup (5, \infty)$

Example $1.2.6$**: Finding Domain and Range from a Graph**

We can observe that the horizontal extent of the graph is -3 (exclusive) to 1 (inclusive), so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 (inclusive) to -4 (inclusive), so the range is $[-4, 0]$. See the figure below.

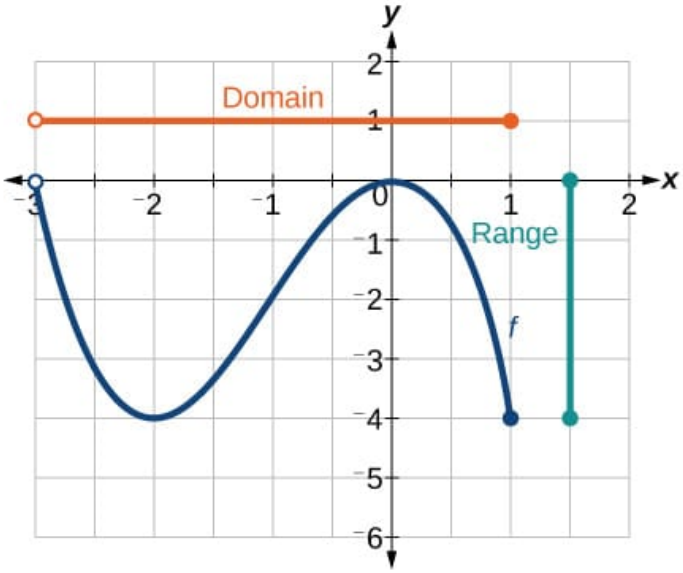


Figure $1.2.10$

Example $1.2.7$**: Finding Domain and Range from a Graph of Oil Production**

The input quantity along the horizontal axis is "years," which we represent with the variable t for time. The output quantity is "thousands of barrels of oil per day," which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is $[1973, 2008]$, and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Example $1.2.8$**: Finding the Domain and Range Using Toolkit Functions**

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result. Therefore, the domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example $1.2.9$**: Finding the Domain and Range**

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example $1.2.10$**: Finding the Domain and Range**

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$$x + 4 \geq 0 \text{ when } x \geq -4.$$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

The following figure represents the function f.

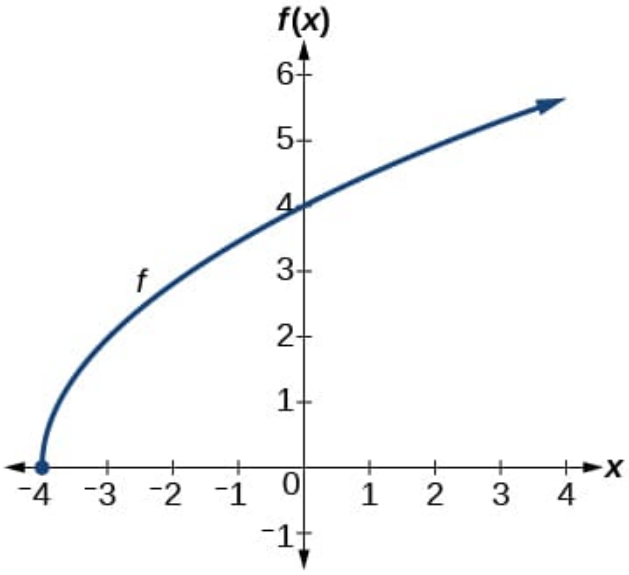


Figure 1.2.22

Example 1.2.11**: Writing a Piecewise Function**

Two different formulas will be needed. For n-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n, & \text{if } 0 < n < 10 \\ 50, & \text{if } n \geq 10 \end{cases}$$

The function is represented in the figure below. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.

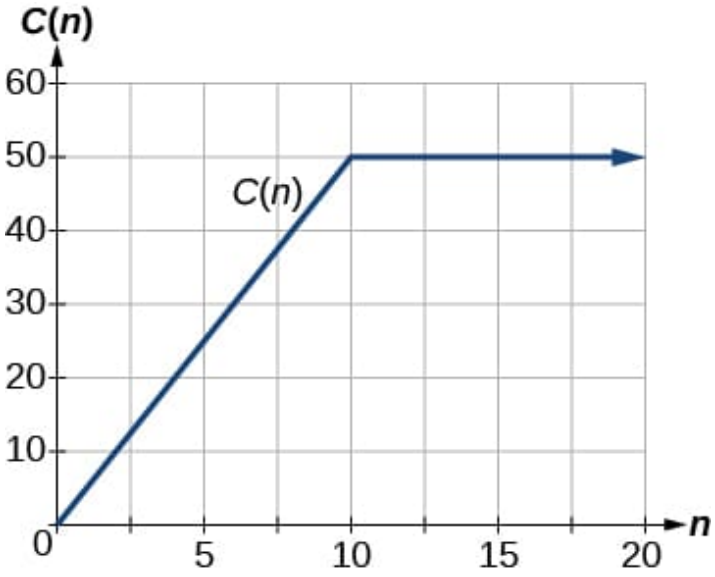


Figure 1.2.23

Example 1.2.12**: Working with a Piecewise Function**

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25.$$

To find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4 - 2) = \$45.$$

The function is represented in the figure below. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

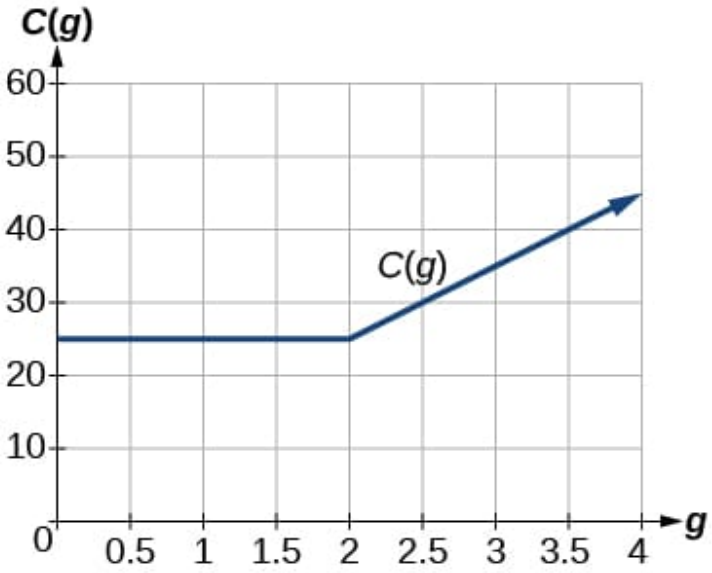


Figure 1.2.24

Example 1.2.13**: Graphing a Piecewise Function**

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

The figure below shows the three components of the piecewise function graphed on separate coordinate systems.

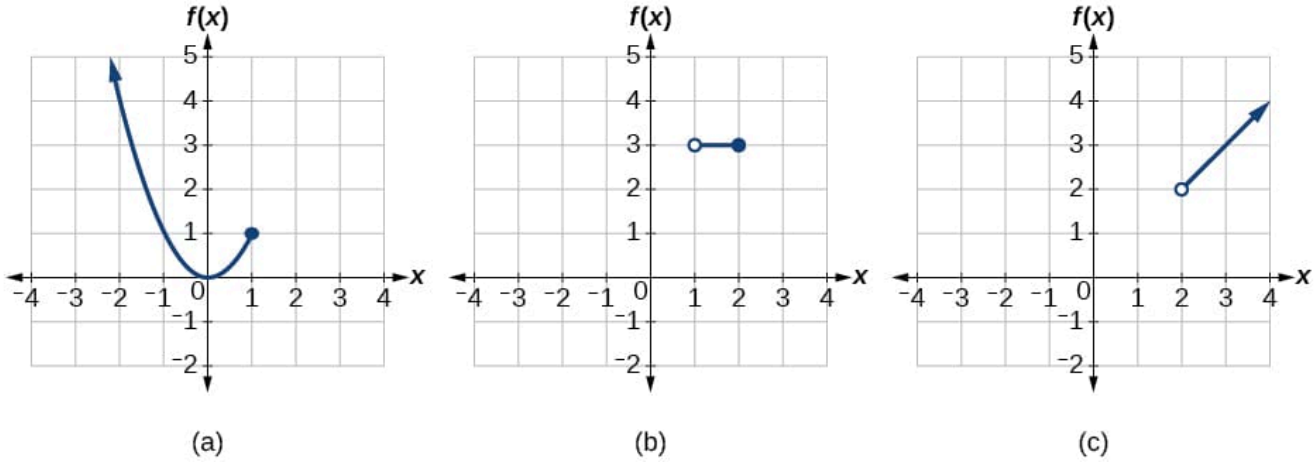


Figure 1.2.25

Now that we have sketched each piece individually, we combine them in the same coordinate plane.

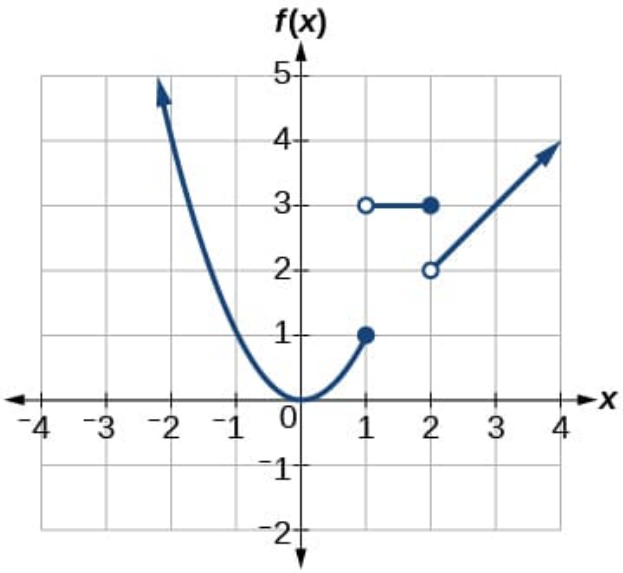


Figure 1.2.26

Note that the graph does pass the Vertical Line Test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

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1.2.1: Resources and Key Concepts

Resources

Videos

- Domain and Range of a function
  - Algebra in Review: Functions - Determining Domain
  - Domain and Range - Finding Domain and Range from Graphs of Functions
  - Domain and Range - Finding the Domain of a Function Defined by an Equation
- Interval notation
  - Logic & Set Theory in Review: Sets - Notation (Interval Notation)
- Graphing piecewise functions
  - Domain and Range - Graphing Piecewise-Defined Functions
  - Functions - Overview - A Review for Precalculus: Piecewise-Defined Functions - Graphing

Key Concepts

Definitions

- **Interval Notation:** A method of writing sets of numbers using parentheses $(\ )$ to indicate endpoints that are not included (exclusive) and square brackets $[\ ]$ to indicate endpoints that are included (inclusive). Unbounded intervals use ∞ or $-\infty$.
- **Exclusive Endpoint:** An endpoint of an interval that is not included in the set, denoted by a parenthesis in interval notation.
- **Inclusive Endpoint:** An endpoint of an interval that is included in the set, denoted by a square bracket in interval notation.
- **Union ($\cup$):** A symbol used to combine two or more sets, representing all elements that belong to any of the original sets. For intervals, it connects two or more disjoint segments of the number line.
- **Set-Builder Notation:** A notation used to describe a set by stating the properties that its members must satisfy, typically in the form $\{x \mid \text{condition(s) involving } x\}$, read as "the set of all x such that x satisfies some given condition."
- **Piecewise Function:** A function defined by multiple sub-functions, where each sub-function applies to a different interval in the domain.
- **Magnitude (or Modulus):** For a real number, its absolute value, representing its distance from 0 on the number line, regardless of sign.
- **Absolute Value Function (as a piecewise function):** The function $f(x) = |x|$ defined as $f(x) = x$ if $x \geq 0$ and $f(x) = -x$ if $x < 0$.

Common Mistakes

- **Graphing Multiple Function Pieces Over the Same Interval:** When sketching a piecewise function, graphing more than one of its defining equations over the same segment of the domain violates the definition of a function (as each input would have multiple outputs).
- **Misinterpreting Open and Closed Circles in Piecewise Graphs:** Incorrectly determining whether boundary points are included in a piece of a piecewise function can lead to errors. For example, in Example 13, points like $(1, 3)$ and $(2, 2)$ are excluded (open circles) while $(1, 1)$ and $(2, 3)$ are included (closed circle or part of a continuous segment).
- **Applying Multiple Formulas to a Single Input in a Piecewise Function:** For any given input value in the domain of a piecewise function, only one of its defining formulas/equations applies.

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1.2.2: Homework

Reading Questions

1. What are the two main algebraic restrictions you must consider when determining the domain of a function consisting of real numbers given by an equation?
2. Explain the difference between a parenthesis and a square bracket in interval notation.
3. If a graph of a function has an open circle at $x = 3$, is $x = 3$ included in the domain? What if it has a closed circle?
4. What is a piecewise function?
5. How do you find the domain of a function like $f(x) = \frac{5}{x-7}$?
6. How do you find the domain of a function like $g(x) = \sqrt{x+4}$?
7. Can a single input value in the domain of a piecewise function correspond to more than one output value or be defined by more than one of its formulas?
8. What is the domain and range of the identity function, $f(x) = x$?
9. What is the domain and range of the absolute value function, $f(x) = |x|$?
10. When graphing a piecewise function, what do you draw at an endpoint of an interval if the inequality is strict (e.g., $<$ or $>$)? What if it is inclusive (e.g., $\leq$ or $\geq$)?

Homework

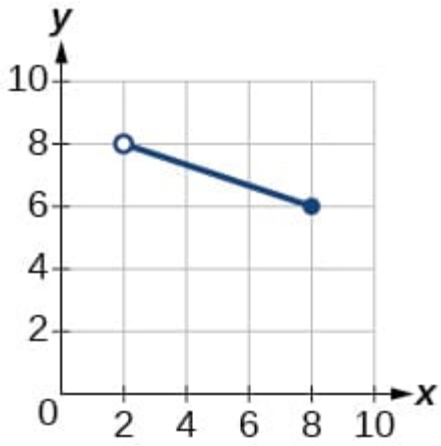
1. Why does the domain differ for different functions?
2. How do we determine the domain of a function defined by an equation?
3. Explain why the domain of $f(x) = \sqrt[3]{x}$ is different from the domain of $f(x) = \sqrt{x}$.
4. When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?
5. How do you graph a piecewise function?

For the following exercises, find the domain of each function using interval notation.

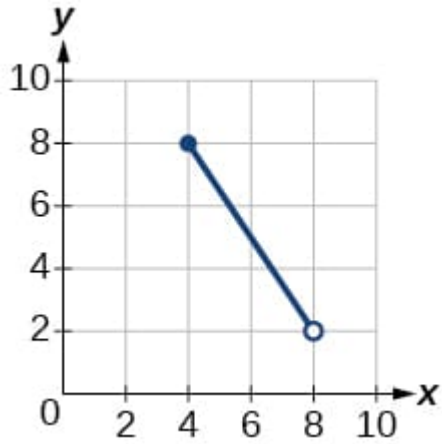
6. $f(x) = -2x(x-1)(x-2)$
7. $f(x) = 5 - 2x^2$
8. $f(x) = 3\sqrt{x-2}$
9. $f(x) = 3 - \sqrt{6-2x}$
10. $f(x) = \sqrt{4-3x}$
11. $f(x) = \sqrt{x^2+4}$
12. $f(x) = \sqrt[3]{1-2x}$
13. $f(x) = \sqrt[3]{x-1}$
14. $f(x) = \frac{9}{x-6}$
15. $f(x) = \frac{3x+1}{4x+2}$
16. $f(x) = \frac{\sqrt{x+4}}{x-4}$
17. $f(x) = \frac{x-3}{x^2+9x-22}$
18. $f(x) = \frac{1}{x^2-x-6}$
19. $f(x) = \frac{2x^3-250}{x^2-2x-15}$
20. $f(x) = \frac{5}{\sqrt{x-3}}$
21. $f(x) = \frac{2x+1}{\sqrt{5-x}}$
22. $f(x) = \frac{\sqrt{x-4}}{\sqrt{x-6}}$
23. $f(x) = \frac{\sqrt{x-6}}{\sqrt{x-4}}$
24. $f(x) = \frac{x}{x}$
25. $f(x) = \frac{x^2-9x}{x^2-81}$

For the following exercises, write the domain and range of each function using interval notation.

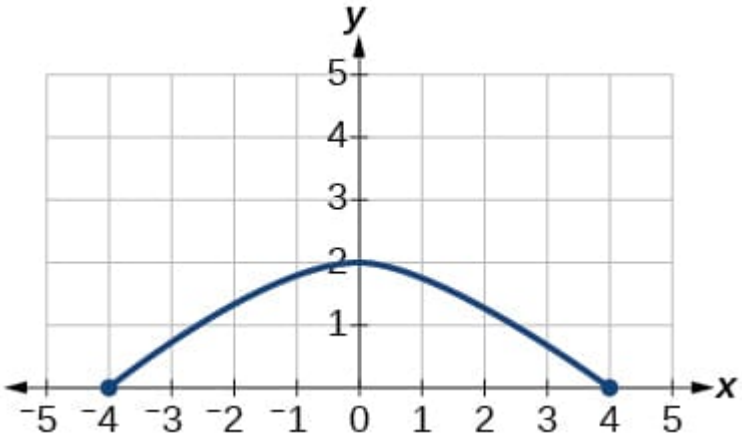
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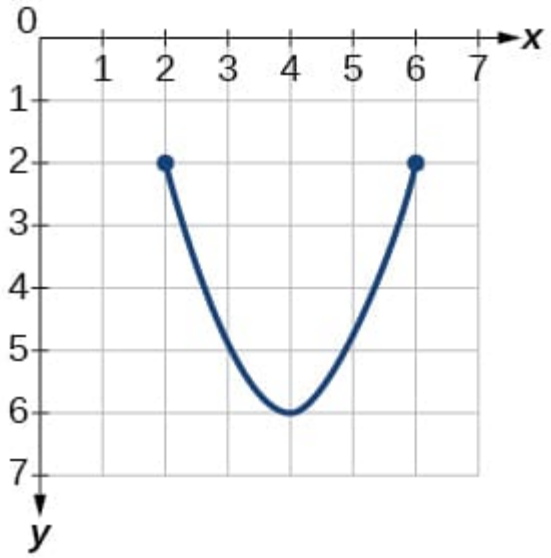
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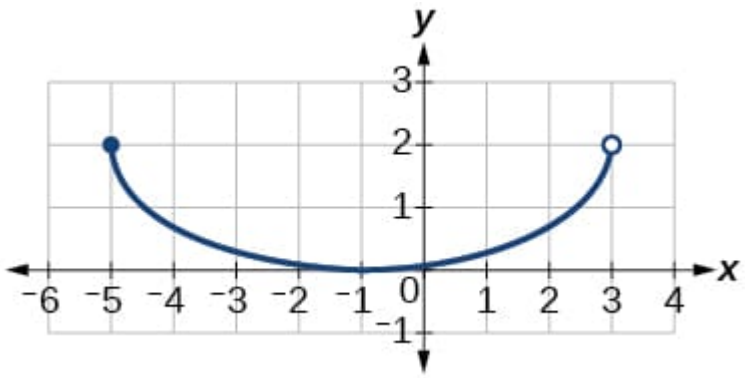
28.



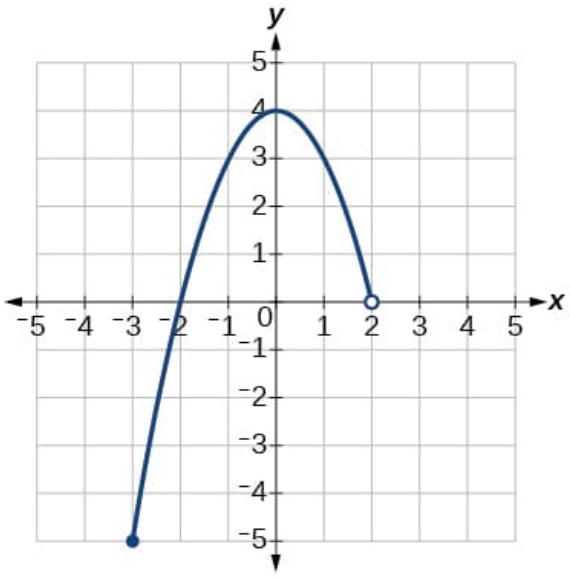
29.


0
1 2 3 4 5 6 7
x
1
2
3
4
5
6
7
y

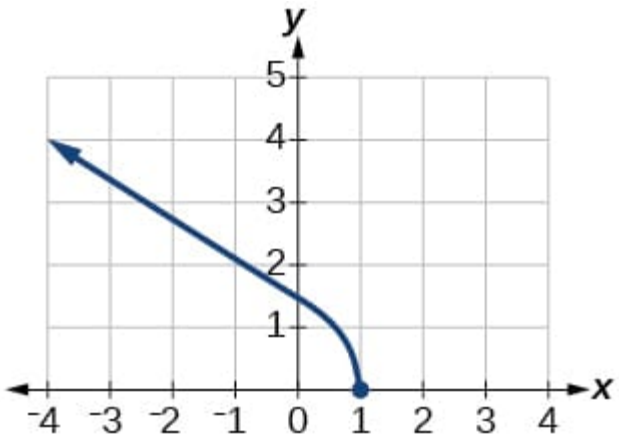
30.


y
3
2
1
-6 -5 -4 -3 -2 -1 0 1 2 3 4
x
-1

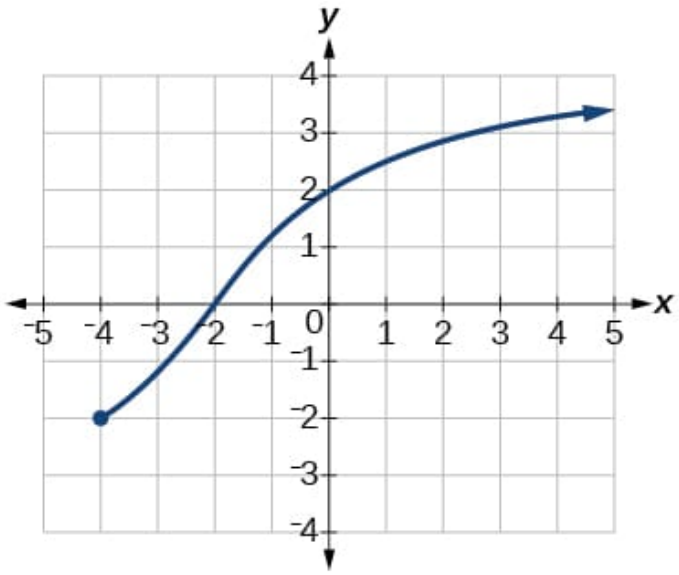
31.


y
5
4
3
2
1
-5 -4 -3 -2 -1 0 1 2 3 4 5
x
-1
-2
-3
-4
-5

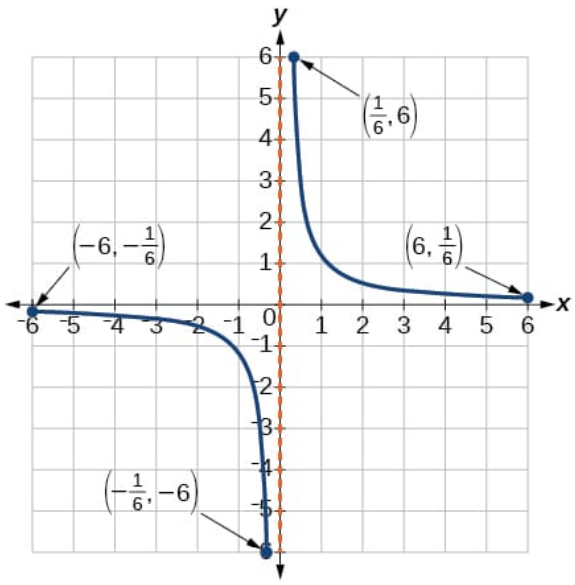
32.


y
x
5
4
3
2
1
-4 -3 -2 -1 0 1 2 3 4

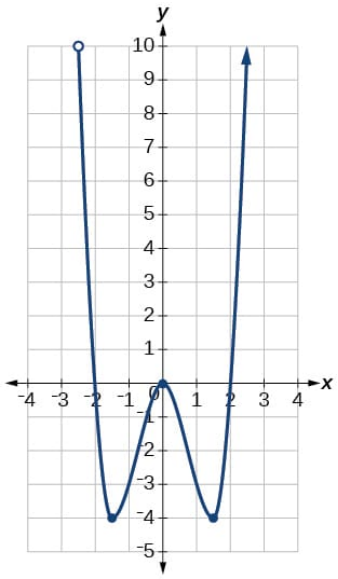
33.


y
x
4
3
2
1
-5 -4 -3 -2 -1 0 1 2 3 4 5
-1
-2
-3
-4

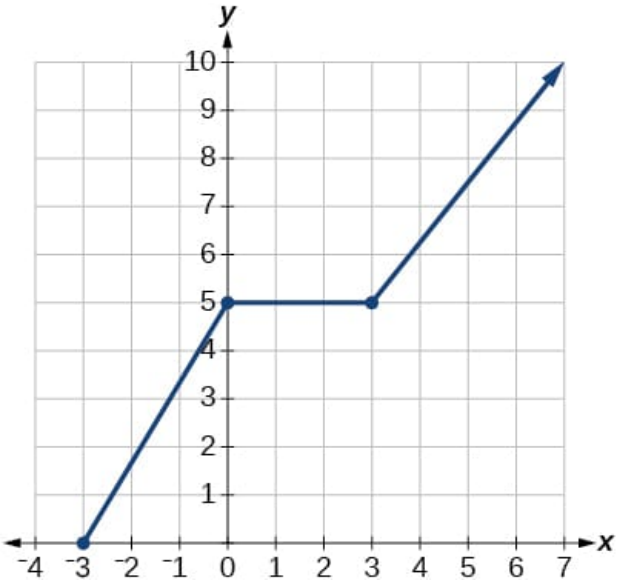
34.


y
x
$\left(\frac{1}{6}, 6\right)$
$\left(-6, -\frac{1}{6}\right)$
$\left(6, \frac{1}{6}\right)$
$\left(-\frac{1}{6}, -6\right)$

35.



36.



For the following exercises, sketch a graph of the piecewise function. Write the domain in interval notation.

37. $f(x) = \begin{cases} x+1, & \text{if } x < -2 \\ -2x-3, & \text{if } x \geq -2 \end{cases}$

38. $f(x) = \begin{cases} 2x-1, & \text{if } x < 1 \\ 1+x, & \text{if } x \geq 1 \end{cases}$

39. $f(x) = \begin{cases} x+1, & \text{if } x < 0 \\ x-1, & \text{if } x > 0 \end{cases}$

40. $f(x) = \begin{cases} 3, & \text{if } x < 0 \\ \sqrt{x}, & \text{if } x \geq 0 \end{cases}$

41. $f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ 1-x, & \text{if } x > 0 \end{cases}$

42. $f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ x+2, & \text{if } x \geq 0 \end{cases}$

43. $f(x) = \begin{cases} x+1, & \text{if } x < 1 \\ x^3, & \text{if } x \geq 1 \end{cases}$

44. $f(x) = \begin{cases} |x|, & \text{if } x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}$

For the following exercises, given each function f, evaluate $f(-3), f(-2), f(-1)$, and $f(0)$.

45. $f(x) = \begin{cases} x+1, & \text{if } x < -2 \\ -2x-3, & \text{if } x \geq -2 \end{cases}$

46. $f(x) = \begin{cases} 1, & \text{if } x \leq -3 \\ 0, & \text{if } x > -3 \end{cases}$

47. $f(x) = \begin{cases} -2x^2+3, & \text{if } x \leq -1 \\ 5x-7, & \text{if } x > -1 \end{cases}$

For the following exercises, given each function f, evaluate $f(-1), f(0), f(2)$, and $f(4)$.

48. $f(x) = \begin{cases} 7x+3, & \text{if } x < 0 \\ 7x+6, & \text{if } x \geq 0 \end{cases}$

49. $f(x) = \begin{cases} x^2-2, & \text{if } x < 2 \\ 4+|x-5|, & \text{if } x \geq 2 \end{cases}$

50. $f(x) = \begin{cases} 5x, & \text{if } x < 0 \\ 3, & \text{if } 0 \leq x \leq 3 \\ x^2, & \text{if } x > 3 \end{cases}$

For the following exercises, write the domain for the piecewise function in interval notation.

51. $f(x) = \begin{cases} x+1, & \text{if } x < -2 \\ -2x-3, & \text{if } x \geq -2 \end{cases}$

52. $f(x) = \begin{cases} x^2-2, & \text{if } x < 1 \\ -x^2+2, & \text{if } x > 1 \end{cases}$

53. $f(x) = \begin{cases} 2x-3, & \text{if } x < 0 \\ -3x^2, & \text{if } x \geq 2 \end{cases}$

54. Graph $y = \frac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

55. Graph $y = \frac{1}{x}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

56. Suppose the range of a function f is $[-5, 8]$. What is the range of $|f(x)|$?

57. Create a function in which the range is all nonnegative real numbers.

58. Create a function in which the domain is $x > 2$.

59. The height h of a projectile is a function of the time t it is in the air. The height in feet for t seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?

60. The cost in dollars of making x items is given by the function $C(x) = 10x + 500$.
    a. The fixed cost is determined when zero items are produced. Find the fixed cost for this item.
    b. What is the cost of making 25 items?
    c. Suppose the maximum cost allowed is $1500. What are the domain and range of the cost function, $C(x)$?

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1.3: The Arithmetic and Composition of Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. This cost will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, $\text{Cost} = C(T(d))$ means that the cost depends on the temperature, which in turn depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the cost function at that temperature. We would write $C(T(5))$.

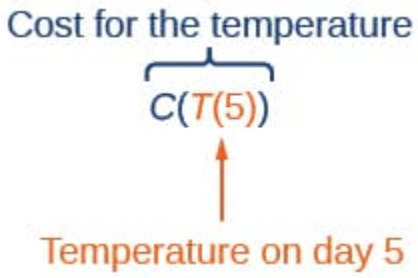


Figure 1.3.1

By combining these two relationships into one function, we have performed **function composition**, which is the focus of this section.

Combining Functions Using Algebraic Operations

Before we get to function composition, we need to backtrack a little to learn how to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y).$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w.$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of Algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

Definition: Function Arithmetic

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ by the relations

$$\begin{aligned} (f+g)(x) &= f(x)+g(x) \\ (f-g)(x) &= f(x)-g(x) \\ (fg)(x) &= f(x)g(x) \\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}, \text{ if } g(x) \neq 0 \end{aligned}$$

Example 1.3.1: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x) = x-1$ and $g(x) = x^2-1$. Are they the same function?

Solution available at the end of this section.

In Example 1.3.1, for $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ is necessary because when $x = 1$, the denominator is equal to 0, which makes the function undefined.

Checkpoint 1.3.1

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$ given

$$f(x) = x-1 \text{ and } g(x) = x^2-1.$$

Are they the same function?

Focus on Calculus - Difference Quotients

Now that we know how to perform arithmetic with functions, it's time to turn our attention to the primary building block of a huge concept in Calculus. This primary building block is called the **difference quotient**.

Definition: Difference Quotient

For a function f, the **difference quotient** of f at x with increment h, where $h \neq 0$, is

$$\frac{f(x+h)-f(x)}{h}.$$

It is the average rate of change of f over the interval $[x, x+h]$, whose length is h.

You might see this written online and in other texts as

$$\frac{f(a+h)-f(a)}{h},$$

but fear not—this is the same expression (just with a instead of x).

We will revisit this concept in more detail in a future section, but for now we use it as a way to practice function notation and function arithmetic. For reasons that will become clear in Calculus, "simplifying" a difference quotient means rewriting it in a form where the h cancels from the denominator. Once that happens, we consider our work to be done.

Example 1.3.2

Find and simplify the difference quotient for the following function.

$$f(x) = x^2 - x - 2$$

Solution available at the end of this section.

Interactive Element: The Difference Quotient as a Slope

The following Interactive Element might help you visualize what the difference quotient $\frac{f(x+h)-f(x)}{h}$ measures for the function $f(x) = x^2 - x - 2$ from Example 1.3.2: the slope of the line through two points on the graph.

Interact: Drag the orange point B along the curve toward the gray point A to shrink the run h; drag A to change the starting point.

Observation: The difference quotient is exactly the slope of the red line through A and B. As you slide B toward A, that slope settles toward a single value—a preview of an idea you will meet in Calculus.

A word of caution before we continue: watch your negatives! Notice that at some point in the solution of Example 1.3.2 we had the expression $(x^2 + 2xh + h^2 - x - h - 2) - (x^2 - x - 2)$ in the numerator. Those parentheses on the second group are critically important and necessary. We are subtracting *the entire function* $f(x)$. As such, we needed $f(x + h) - f(x) = f(x + h) - (x^2 - x - 2)$. If we didn't have those parentheses, we would (incorrectly) have written $f(x + h) - f(x) = f(x + h) - x^2 - x - 2$, which I hope you would agree is not the same thing.

Let's state that one more time for clarity.

Caution

The most common mistake made on a difference quotient problem is not using grouping symbols properly!

Checkpoint 1.3.2

Find and simplify the difference quotient for $g(x) = 1 - 3x$.

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**.

Definition: Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a **composition of functions**. Given functions f and g, the **composite function**, written as $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x)).$$

The function g is the **inner function** (applied first), and f is the **outer function** (applied second).

We read the left-hand side as "f composed with g at x," and the right-hand side as "f of g of x." The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the **composition operator**. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value.

Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication take two numbers and give a new number. However, it is important not to confuse function composition with multiplication. It is also important to understand the Order of Operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

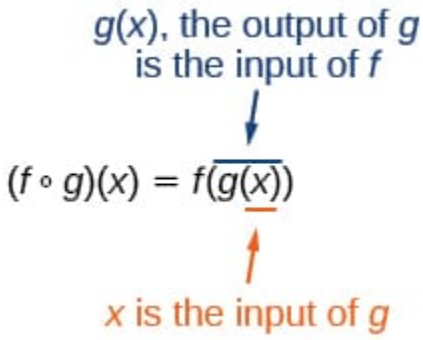


Figure 1.3.2

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) && \text{(substituting)} \\ &= (x+2)^2 && \text{(substituting)} \\ &= x^2 + 4x + 4 && \text{(distributing)} \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) && \text{(substituting)} \\ &= x^2 + 2 && \text{(substituting)} \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Example 1.3.3: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is commutative.

$$f(x) = 2x + 1, \quad g(x) = 3 - x.$$

Solution available at the end of this section.

Caution: Composition Is NOT Commutative

In general, $f(g(x)) \neq g(f(x))$; reversing the order of composition usually produces a different function. Before composing, identify which function is the inner function (applied first) and which is the outer function (applied second).

Example 1.3.4: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution available at the end of this section.

Example 1.3.5: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution available at the end of this section.

Q&A

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order.

Checkpoint 1.3.5

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 1.3.6: Using a Table to Evaluate a Composite Function

Using the following table, evaluate $f(g(3))$ and $g(f(3))$.

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

Solution available at the end of this section.

Checkpoint 1.3.6

Use the table from Example 1.3.6 to evaluate $f(g(1))$ and $g(f(4))$.

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

Example 1.3.7: Using a Graph to Evaluate a Composite Function

Use the following figure to evaluate $f(g(1))$.

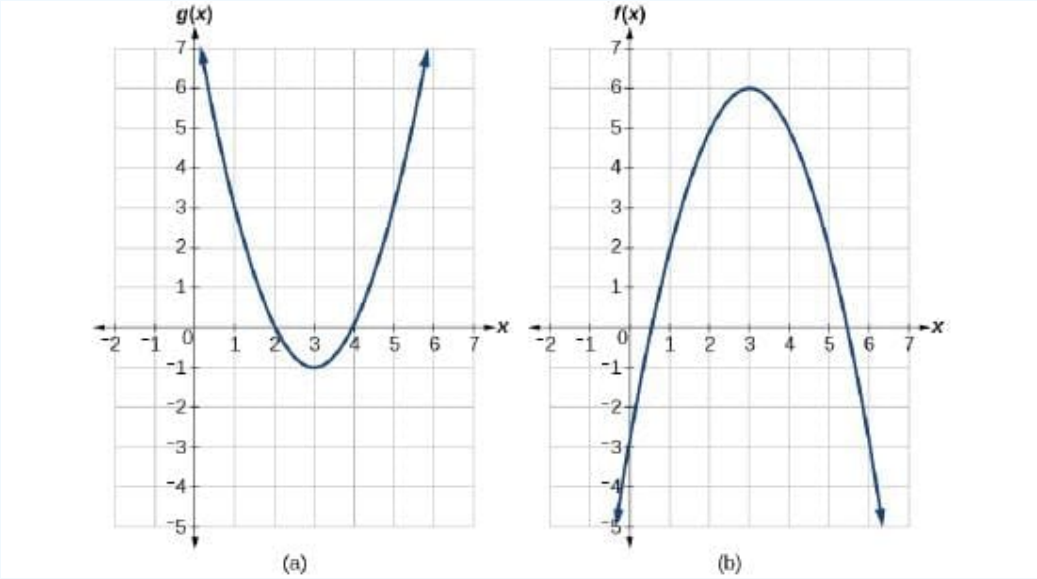


Figure 1.3.3

Solution available at the end of this section.

Checkpoint 1.3.7

Use the graph from Example 1.3.7 to evaluate $g(f(2))$.

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

Example 1.3.8: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution available at the end of this section.

Checkpoint 1.3.8

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Finding the Domain of a Composite Function

As we discussed previously, the domain of a composite function such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot. That is, it is important to know the domain of a function such as $f \circ g$.

Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful. Otherwise, we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed and the expression is still undefined. Thus, the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

Theorem: Domain of a Composition of Functions

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f:

$$\{\, x \mid x \text{ is in the domain of } g \text{ and } g(x) \text{ is in the domain of } f \,\}.$$

To find the domain of the composite function $f(g(x))$, we start by finding the domain of g. Those values of x for which g is defined will be the largest *possible* domain of the composition; however, this set might get restricted once we take the function f into consideration. That said, note the domain of f. Our goal is to find the elements, x, in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$. The following "Tip for Success" summarizes this process.

Tip for Success

To find the domain of $f \circ g$, keep only the values of x that satisfy both conditions: first, x must lie in the domain of the inner function g; second, the output $g(x)$ must lie in the domain of the outer function f. Determine the domain from these conditions *before* simplifying the composed expression, because algebraic simplification can erase a restriction that the original composition requires.

Example 1.3.9: Finding the Domain of a Composite Function

Find the domain of $(f \circ g)(x)$ where $f(x) = \frac{5}{x-1}$ and $g(x) = \frac{4}{3x-2}$.

Solution available at the end of this section.

Example 1.3.10: Finding the Domain of a Composite Function Involving Radicals

Find the domain of $(f \circ g)(x)$ where $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{3-x}$.

Solution available at the end of this section.

If we change the function f very slightly in Example 1.3.10 to $f(x) = \sqrt{x-2}$, then the composition, $(f \circ g)(x)$ becomes

$$(f \circ g)(x) = \sqrt{\sqrt{3-x} - 2}.$$

Again, since the radicand is never negative, we know

$$\begin{aligned} \sqrt{3-x} - 2 &\geq 0 \\ \implies \sqrt{3-x} &\geq 2 \qquad \text{(adding 2 to both sides)} \end{aligned}$$

This makes the problem of finding the domain more complex. If $\sqrt{3-x} \geq 2$, then

$$\left(\sqrt{3-x}\right)^2 \geq 2^2 \implies 3 - x \geq 4 \implies 3 \geq 4 + x \implies -1 \geq x.$$

Hence, we would require $x \leq 3$ (from the domain of g) and, simultaneously, $-1 \geq x$. The common intersection of these two domain restrictions is $x \leq -1$. Thus, the domain (in this adjusted case) would be $(-\infty, -1]$.

Checkpoint 1.3.10

Find the domain of $(f \circ g)(x)$ where $f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x+4}$.

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to decompose a composite function, so we may choose the decomposition that appears to be most expedient.

Example 1.3.11: Decomposing a Function

Write $f(x) = \sqrt{5-x^2}$ as the composition of two functions.

Solution available at the end of this section.

Checkpoint 1.3.11

Write $f(x) = \frac{4}{3-\sqrt{4+x^2}}$ as the composition of two functions.

Time-Sensitive Advice: "Full Refund" and "No Notation" Drop Deadlines

"Full Refund" Drop Deadline

Within the Los Rios Community College District, the last day you can drop a class for a full refund is the Friday of the second week of class. Any drops processed after that point will only receive a *partial* refund.

"No Notation" Drop Deadline

The Sunday following the "full refund" drop deadline is the last day to drop without a "W" (withdrawal) notation on your transcript. You can only take a course, and receive a notation, a maximum of three times within a given community college district in California - this includes notations of D, F, *and* W.

Once you withdraw (W) or fail (D or F) a course three times within the Los Rios Community College District (ARC, CRC, FLC, or SCC), you are *never* allowed to take that course again at any of the colleges within the District. If you find yourself in that unfortunate position, you would need to take this course outside the Los Rios Community College District.

Take a moment of self-reflection to determine if you feel ready to continue this journey for the rest of this long semester. If not, consider dropping by this course by the "full refund" deadline or the "no notation" deadline, at the latest.

Success in College: What is Tutoring?

Tutoring is an age-old practice. The dictionary definition describes a tutor as a person who gives individual or, in some cases, small group instruction. The purpose of tutoring is to help students help themselves or assist or guide them to the point at which they become independent learners and thus no longer need a tutor.

Content knowledge is an essential ingredient for a tutor; however, to be truly effective, a tutor must combine content knowledge with empathy, honesty, and humor. Empathy requires tutors to "read" their students' emotional states, attitudes, and perceptions. Empathy is the ability to see others from their frame of reference and communicate this understanding to the person involved. To establish a supportive relationship with their students, tutors must be open and honest. Students are often reluctant to talk with a stranger about their academic problems. If a tutor is perceived as genuine and strongly desires to listen, students will be more willing to open up and discuss their problems. Humor can also play an essential part in a tutoring session. Humor can reduce tension. Shared laughter is a powerful way to reinforce learning, and humor can set students at ease and increase rapport. Tutors can also use humor to compliment, guide, or positively provide feedback.

In addition, a successful tutor demonstrates a caring attitude. Caring consists of being punctual and organized for the tutoring session, establishing a learning relationship with the student, developing unique teaching strategies, and becoming familiar with the learning process. Ultimately, tutoring is sharing yourself with another student in a way that makes a difference in both your lives.

There are many benefits to tutoring.

- Heightens a sense of competency/adequacy in conforming to a new role.
- Encourages higher levels of thinking.
- Permits more advanced students to study below-level material without embarrassment.
- Increases motivation to learn to maintain the new role.
- Increases ability to manage own learning and study strategies.
- Increases subject-specific knowledge.

- Increases related general knowledge.
- Increases understanding of the subject area.
- Improves attitude toward the subject area.
- Provides more empathy with students.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Does your college have a place such as a tutoring center where you can get help?
2. If so, what are the hours that this tutoring center is open?
3. If you are unsure about the answers to either of these questions, feel free to ask your professor - that's what they are there for!

Answers

Section Preview

In addition to introducing compositions (and arithmetic combinations) of functions, you should focus on finding the domains of these new functions. Furthermore, the difference quotient is introduced (and will be revisited throughout the remainder of the text whenever a new function is introduced).

Section Preview

- **Factoring Techniques**
  - Factoring Difference of Squares: Example 1 simplifies $\left(\frac{g}{f}\right)(x) = \frac{x^2-1}{x-1}$ by recognizing $x^2 - 1 = (x-1)(x+1)$.
- **Simplifying Expressions**
  - Adding and Subtracting Polynomials: Subtracting an entire function and combining like terms drives Example 1 $(g-f)(x)$, the numerator of the difference quotient in Example 2, and the linear compositions in Example 3.
  - Simplifying Ratios of Polynomials: Example 1 cancels the common $(x-1)$ factor to reduce $\frac{x^2-1}{x-1}$ to $x+1$.
  - Multiplying and Dividing Rational Expressions: When the combined functions are themselves rational (as in the exercise set), forming $\left(\frac{f}{g}\right)(x)$ or $(fg)(x)$ requires multiplying and dividing rational expressions; this surfaces in the homework rather than a worked example.
  - Adding and Subtracting Rational Expressions: Forming $(f \pm g)(x)$ for rational functions in the exercises requires a common denominator; supports the homework, not a numbered example.
  - Simplifying Compound Rational Expressions: Composing two rational functions (e.g., $f(g(x))$ with f, g rational, as in the exercises) produces a fraction-within-a-fraction that must be cleared and simplified.
- **Solving Equations**
  - Solving Rational Equations: Example 9 finds the excluded input of $f \circ g$ by solving $\frac{4}{3x-2} = 1$.
  - Solving Radical Equations: The radical-domain discussion following Example 10 squares both sides of $\sqrt{3-x} \geq 2$ to solve it, the standard technique for undoing a root.

Section Preview

- Perform arithmetic on functions
- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.
- Compute the Difference Quotient for a given function.

Example 1.3.1: Performing Algebraic Operations on Functions

Begin by writing the general form, and then substitute the given functions. For $(g-f)(x)$, we have

$$\begin{aligned}(g-f)(x) &= g(x) - f(x) \\ &= x^2 - 1 - (x-1) && \text{(substituting)} \\ &= x^2 - 1 - x + 1 && \text{(distributing)} \\ &= x^2 - x && \text{(combining like terms)} \\ &= x(x-1) && \text{(factoring out the GCF)}\end{aligned}$$

For $\left(\frac{g}{f}\right)(x)$, we have

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} \\ &= \frac{x^2-1}{x-1} && \text{(substituting)} \\ &= \frac{(x-1)(x+1)}{x-1} && \text{(factoring)} \\ &= \frac{\cancel{(x-1)}^{1}(x+1)}{\cancel{(x-1)}^{1}} && \text{(canceling like terms and } \textbf{assuming } x \neq 1\text{)} \\ &= x+1 && \text{(where } x \neq 1\text{)}\end{aligned}$$

Therefore, the functions are not the same.

Example 1.3.2

To find $f(x+h)$, we replace every occurrence of x in the formula $f(x)=x^2-x-2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2-(x+h)-2 && \text{(substituting)} \\ &= x^2+2xh+h^2-x-h-2 && \text{(distributing)} \end{aligned}$$

So the difference quotient is

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{(x^2+2xh+h^2-x-h-2)-(x^2-x-2)}{h} \\ &= \frac{x^2+2xh+h^2-x-h-2-x^2+x+2}{h} && \text{(distributing)} \\ &= \frac{2xh+h^2-h}{h} && \text{(combining like terms)} \\ &= \frac{h(2x+h-1)}{h} && \text{(factoring out the GCF)} \\ &= \frac{\cancel{h}(2x+h-1)}{\cancel{h}} && \text{(canceling like factors)} \\ &= 2x+h-1. \end{aligned}$$

We know we are done once the original h in the denominator is canceled out.

Example 1.3.3**: Determining whether Composition of Functions is Commutative**

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= f(3-x) && \text{(substituting)} \\ &= 2(3-x)+1 && \text{(substituting)} \\ &= 6-2x+1 && \text{(distributing)} \\ &= 7-2x && \text{(combining like terms)} \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= g(2x+1) && \text{(substituting)} \\ &= 3-(2x+1) && \text{(substituting)} \\ &= 3-2x-1 && \text{(distributing)} \\ &= 2-2x && \text{(combining like terms)} \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

Example 1.3.4**: Interpreting Composite Functions**

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t=3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example 1.3.5**: Investigating the Order of Function Composition**

The function $y=f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles} = f(\text{number of hours}).$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons} = g(\text{number of miles}).$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Example 1.3.6**: Using a Table to Evaluate a Composite Function**

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g: $g(3)=2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2)=8$.

$$\begin{aligned} g(3) &= 2 \\ \implies f(g(3)) &= f(2)=8 \end{aligned}$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3)=3$. Then, using the table for g, we can evaluate

$$g(f(3))=g(3)=2.$$

The following table shows the composite functions $f\circ g$ and $g\circ f$ as tables.

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$

3	2	8	3	2

Example 1.3.7**: Using a Graph to Evaluate a Composite Function**

To evaluate $f(g(1))$, we start with the inside evaluation.

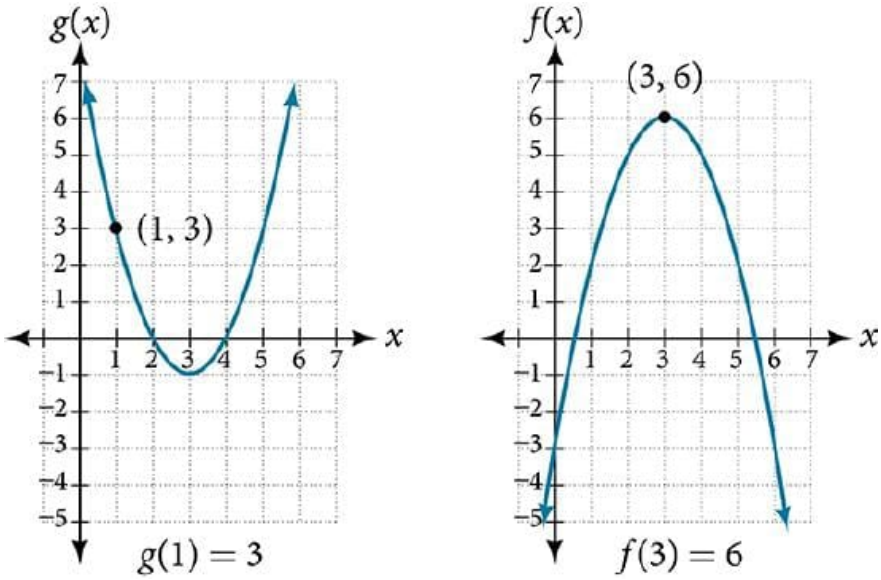


Figure 1.3.4

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

The following figure shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

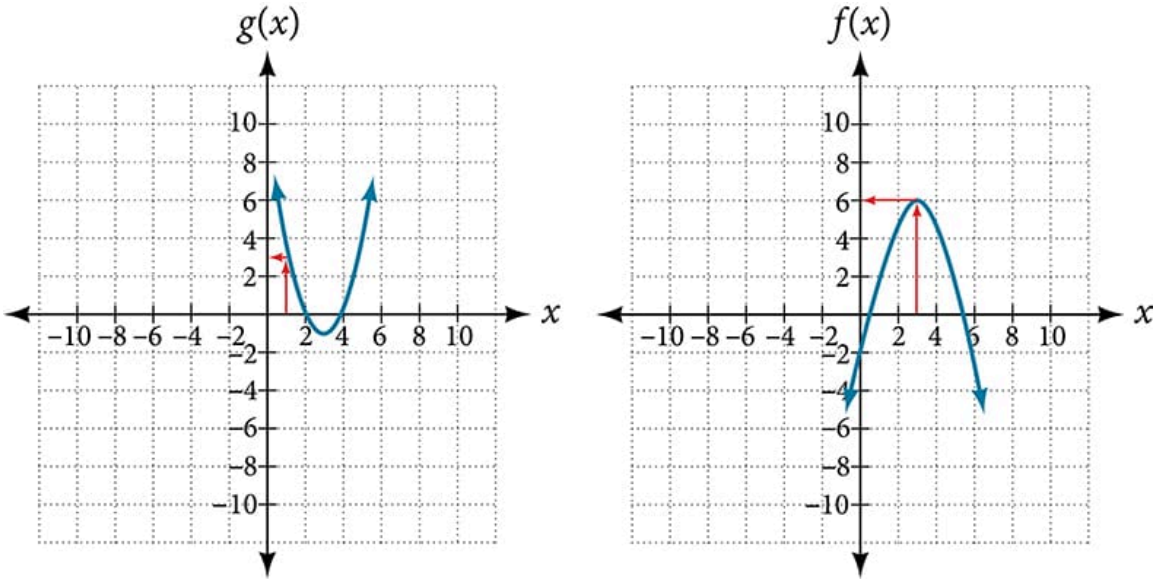


Figure 1.3.5

Example 1.3.8**: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input**

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$\begin{aligned} h(1) &= 3(1) + 2 && \text{(substituting)} \\ &= 5 && \text{(simplifying)} \end{aligned}$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$$\begin{aligned} f(h(1)) &= f(5) && \text{(substituting)} \\ &= 5^2 - 5 && \text{(substituting)} \\ &= 25 - 5 && \text{(squaring)} \\ &= 20 && \text{(simplifying)} \end{aligned}$$

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Example 1.3.9**: Finding the Domain of a Composite Function**

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x) = 1$.

$$\begin{aligned} & g(x) = 1 \\ \implies & \frac{4}{3x-2} = 1 && \text{(substituting)} \\ \implies & 4 = 1(3x-2) && \text{(multiplying both sides by the LCD)} \\ \implies & 4 = 3x-2 && \text{(distributing)} \\ \implies & 6 = 3x && \text{(adding 2 to both sides)} \\ \implies & 2 = x && \text{(dividing both sides by 3)} \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that $x \neq \frac{2}{3}$ and $x \neq 2$. We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty).$$

Example 1.3.10**: Finding the Domain of a Composite Function Involving Radicals**

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f \circ g)(x) = \sqrt{\sqrt{3-x} + 2}.$$

Since the radicand of a square root must be nonnegative, $\sqrt{3-x} + 2 \geq 0$. Hence,

$$\begin{aligned} \sqrt{3-x} + 2 &\geq 0 \\ \implies \sqrt{3-x} &\geq -2 && \text{(subtracting 2 from both sides)} \end{aligned}$$

Since square roots never return negative numbers, the inequality $\sqrt{3-x} \geq -2$ is always true. Thus, our only restriction on this composition is the original restriction from the domain of g—namely that the values of x must be in the interval $(-\infty, 3]$. This gives a domain of $(-\infty, 3]$.

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

Example 1.3.11**: Decomposing a Function**

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5 - x^2$ is the inside of the square root. We could then decompose the function as

$$h(x) = 5 - x^2 \text{ and } g(x) = \sqrt{x}.$$

We can check our answer by recomposing the functions.

$$g(h(x)) = g(5 - x^2) = \sqrt{5 - x^2}.$$

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1.3.1: Resources and Key Concepts

Resources

Videos

- Difference quotient
  - Algebra Mini-Lecture: The Difference Quotient
  - Functions - Overview - A Review for Precalculus: The Difference Quotient
- Composition of functions
  - Combining Functions - Domains of Compositions of Functions
  - Combining Functions - Applications of Compositions

Key Concepts

Definitions

- **Function Arithmetic (Sum, Difference, Product, Ratio):**
  - $(f+g)(x) = f(x) + g(x)$ (Sum of functions)
  - $(f-g)(x) = f(x) - g(x)$ (Difference of functions)
  - $(fg)(x) = f(x)g(x)$ (Product of functions)
  - $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$ (Quotient or Ratio of functions
- **Difference Quotient:** For a given function f, the expression $\frac{f(x+h)-f(x)}{h}$ (or $\frac{f(a+h)-f(a)}{h}$).
- **Composition of Functions:** The process of combining functions such that the output of one function becomes the input of another. The resulting function is known as a composite function.
- **Composite Function (Notation):** For functions f and g, the composite function, denoted $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$.
- **Composition Operator ($\circ$):** The open circle symbol used to denote function composition, as in $f \circ g$.

Theorems

- **Domain of a Composite Function:** The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

Common Mistakes

- **Incorrectly Using Grouping Symbols in Difference Quotients:** When calculating $f(x+h) - f(x)$, failing to use parentheses around $f(x)$ if $f(x)$ consists of multiple terms will lead to errors in subtraction. For example, for $f(x) = x^2 - x - 2$, writing $f(x+h) - x^2 - x - 2$ instead of $f(x+h) - (x^2 - x - 2)$.
- **Confusing Function Composition with Function Multiplication:** The product of functions $(fg)(x) = f(x)g(x)$ is not the same as the function composition $(f \circ g)(x) = f(g(x))$. In general, $f(x)g(x) \neq f(g(x))$.

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1.3.2: Homework

Reading Questions

1. If $f(x) = 2x + 3$ and $g(x) = x - 5$, what is $(f + g)(x)$?
2. If $f(x) = x^2$ and $g(x) = x + 1$, what is $(fg)(x)$?
3. What is the formula for the difference quotient of a function $f(x)$?
4. When simplifying a difference quotient, what is the main goal regarding the variable h in the denominator?
5. What does the notation $(f \circ g)(x)$ mean, and how is it evaluated?
6. Is function composition generally commutative? Explain with an example if $f(x) = x^2$ and $g(x) = x + 2$.
7. If $C(T)$ is the cost of heating a house at temperature T, and $T(d)$ is the temperature on day d, what does $C(T(d))$ represent?
8. What two conditions must be met for an input x to be in the domain of the composite function $(f \circ g)(x)$?
9. How would you decompose the function $H(x) = \sqrt{x^2 + 5}$ into two simpler functions $f(x)$ and $g(x)$ such that $H(x) = f(g(x))$?
10. If you are given functions as tables, how do you evaluate $(f \circ g)(3)$?
11. How does one find the domain of the quotient of two functions, $\frac{f}{g}$?
12. What is the composition of two functions, $f \circ g$?
13. If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.
14. How do you find the domain for the composition of two functions, $f \circ g$?

Homework

Algebraic

For the following exercises, use the pair of functions f and g to find the following values if they exist. Simplify your answers.

- $(f + g)(2)$
- $(f - g)(-1)$
- $(g - f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

15. $f(x) = 3x + 1$ and $g(x) = 4 - x$
16. $f(x) = x^2$ and $g(x) = -2x + 1$
17. $f(x) = x^2 - x$ and $g(x) = 12 - x^2$
18. $f(x) = 2x^3$ and $g(x) = -x^2 - 2x - 3$
19. $f(x) = \sqrt{x + 3}$ and $g(x) = 2x - 1$
20. $f(x) = \sqrt{4 - x}$ and $g(x) = \sqrt{x + 2}$
21. $f(x) = 2x$ and $g(x) = \frac{1}{2x+1}$
22. $f(x) = x^2$ and $g(x) = \frac{3}{2x-3}$
23. $f(x) = x^2$ and $g(x) = \frac{1}{x^2}$
24. $f(x) = x^2 + 1$ and $g(x) = \frac{1}{x^2+1}$

For the following exercises, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f + g)(x)$
- $(f - g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

25. $f(x) = 2x + 1$ and $g(x) = x - 2$
26. $f(x) = 1 - 4x$ and $g(x) = 2x - 1$
27. $f(x) = x^2$ and $g(x) = 3x - 1$
28. $f(x) = x^2 - x$ and $g(x) = 7x$
29. $f(x) = x^2 - 4$ and $g(x) = 3x + 6$
30. $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$
31. $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$
32. $f(x) = x - 1$ and $g(x) = \frac{1}{x-1}$
33. $f(x) = x$ and $g(x) = \sqrt{x+1}$
34. $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$
35. Given $f(x) = x^2 + 2x$ and $g(x) = 6 - x^2$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
36. Given $f(x) = -3x^2 + x$ and $g(x) = 5$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
37. Given $f(x) = 2x^2 + 4x$ and $g(x) = \frac{1}{2x}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
38. Given $f(x) = \frac{1}{x-4}$ and $g(x) = \frac{1}{6-x}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
39. Given $f(x) = 3x^2$ and $g(x) = \sqrt{x-5}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
40. Given $f(x) = \sqrt{x}$ and $g(x) = |x - 3|$, find $\frac{g}{f}$. Determine the domain of the function in interval notation.
41. Given $f(x) = 2x^2 + 1$ and $g(x) = 3x - 5$, find the following:
    a. $f(g(2))$
    b. $f(g(x))$
    c. $g(f(x))$
    d. $(g \circ g)(x)$
    e. $(f \circ f)(-2)$

For the following exercises, find and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for the given function.

42. $f(x) = 2x - 5$
43. $f(x) = -3x + 5$
44. $f(x) = 6$
45. $f(x) = 3x^2 - x$
46. $f(x) = -x^2 + 2x - 1$
47. $f(x) = 4x^2$
48. $f(x) = x - x^2$
49. $f(x) = x^3 + 1$
50. $f(x) = mx + b$ where $m \neq 0$
51. $f(x) = ax^2 + bx + c$ where $a \neq 0$
52. $f(x) = \frac{2}{x}$
53. $f(x) = \frac{3}{1-x}$
54. $f(x) = \frac{1}{x^2}$
55. $f(x) = \frac{2}{x+5}$
56. $f(x) = \frac{1}{4x-3}$
57. $f(x) = \frac{3x}{x+1}$
58. $f(x) = \frac{x}{x-9}$
59. $f(x) = \frac{x^2}{2x+1}$
60. $f(x) = \sqrt{x-9}$
61. $f(x) = \sqrt{2x+1}$
62. $f(x) = \sqrt{-4x+5}$
63. $f(x) = \sqrt{4-x}$
64. $f(x) = \sqrt{ax+b}$, where $a \neq 0$.
65. $f(x) = x\sqrt{x}$
66. $f(x) = \sqrt[3]{x}$.
    HINT: $(a-b)\left(a^2+ab+b^2\right) = a^3 - b^3$

For the following exercises, use each pair of functions to find $f(g(x))$ and $g(f(x))$. Simplify your answers.

67. $f(x) = x^2 + 1$, $g(x) = \sqrt{x+2}$
68. $f(x) = \sqrt{x} + 2$, $g(x) = x^2 + 3$

69. $f(x) = |x|$, $g(x) = 5x + 1$
70. $f(x) = \sqrt[3]{x}$, $g(x) = \frac{x+1}{x^3}$
71. $f(x) = \frac{1}{x-6}$, $g(x) = \frac{7}{x} + 6$
72. $f(x) = \frac{1}{x-4}$, $g(x) = \frac{2}{x} + 4$

For the following exercises, use each set of functions to find $f(g(h(x)))$. Simplify your answers.

73. $f(x) = x^4 + 6$, $g(x) = x - 6$, and $h(x) = \sqrt{x}$
74. $f(x) = x^2 + 1$, $g(x) = \frac{1}{x}$, and $h(x) = x + 3$
75. Given $f(x) = \frac{1}{x}$ and $g(x) = x - 3$, find the following:
    a. $(f \circ g)(x)$
    b. the domain of $(f \circ g)(x)$ in interval notation
    c. $(g \circ f)(x)$
    d. the domain of $(g \circ f)(x)$
    e. $\left(\frac{f}{g}\right) x$
76. Given $f(x) = \sqrt{2 - 4x}$ and $g(x) = -\frac{3}{x}$, find the following:
    a. $(g \circ f)(x)$
    b. the domain of $(g \circ f)(x)$ in interval notation
77. Given the functions $f(x) = \frac{1-x}{x}$ and $g(x) = \frac{1}{1+x^2}$, find the following:
    a. $(g \circ f)(x)$
    b. $(g \circ f)(2)$
78. Given functions $p(x) = \frac{1}{\sqrt{x}}$ and $m(x) = x^2 - 4$, state the domain of each of the following functions using interval notation:
    a. $p(x)m(x)$
    b. $p(m(x))$
    c. $m(p(x))$
79. Given functions $q(x) = \frac{1}{\sqrt{x}}$ and $h(x) = x^2 - 9$, state the domain of each of the following functions using interval notation.
    a. $\frac{q(x)}{h(x)}$
    b. $q(h(x))$
    c. $h(q(x))$
80. For $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x - 1}$, write the domain of $(f \circ g)(x)$ in interval notation.

For the following exercises, find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x) = f(g(x))$.

81. $h(x) = (x + 2)^2$
82. $h(x) = (x - 5)^3$
83. $h(x) = \frac{3}{x-5}$
84. $h(x) = \frac{4}{(x+2)^2}$
85. $h(x) = 4 + \sqrt[3]{x}$
86. $h(x) = \sqrt[3]{\frac{1}{2x-3}}$
87. $h(x) = \frac{1}{(3x^2-4)^{-3}}$
88. $h(x) = \sqrt[4]{\frac{3x-2}{x+5}}$
89. $h(x) = \left(\frac{8+x^3}{8-x^3}\right)^4$
90. $h(x) = \sqrt{2x + 6}$
91. $h(x) = (5x - 1)^3$
92. $h(x) = \sqrt[3]{x - 1}$
93. $h(x) = |x^2 + 7|$
94. $h(x) = \frac{1}{(x-2)^3}$
95. $h(x) = \left(\frac{1}{2x-3}\right)^2$
96. $h(x) = \sqrt{\frac{2x-1}{3x+4}}$

Graphical

For the following exercises, use the graphs of f, shown in Figure 1.3.2.1, and g, shown in Figure 1.3.2.2, to evaluate the expressions.

Figure 1.3.2.1

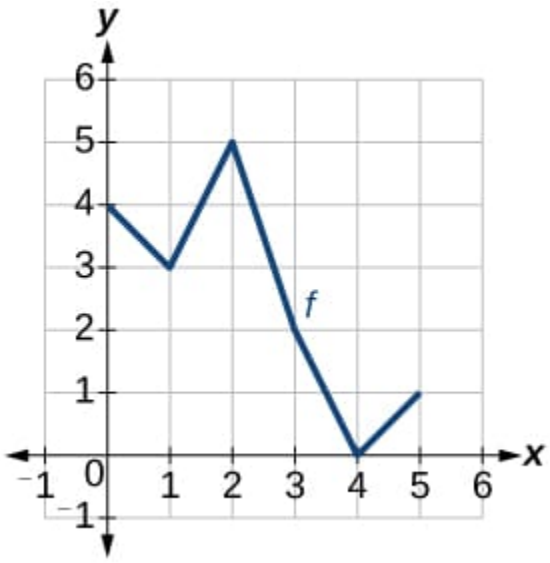
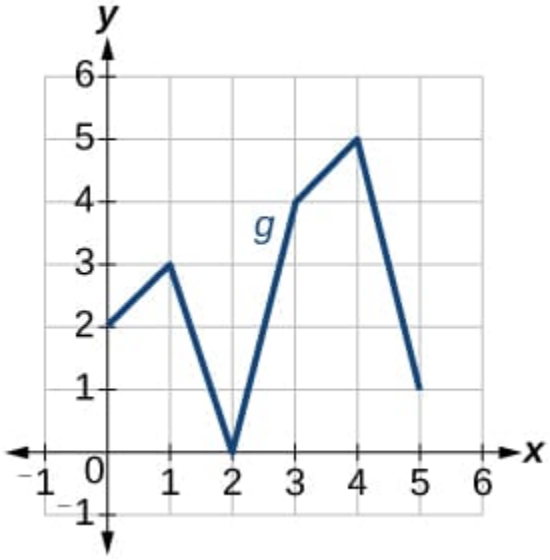


Figure 1.3.2.2



97. $f(g(3))$
98. $f(g(1))$
99. $g(f(1))$
100. $g(f(0))$
101. $f(f(5))$
102. $f(f(4))$
103. $g(g(2))$
104. $g(g(0))$

For the following exercises, use graphs of $f(x)$, shown in Figure 1.3.2.3, $g(x)$, shown in Figure 1.3.2.4, and $h(x)$, shown in Figure 1.3.2.5, to evaluate the expressions.

Figure 1.3.2.3

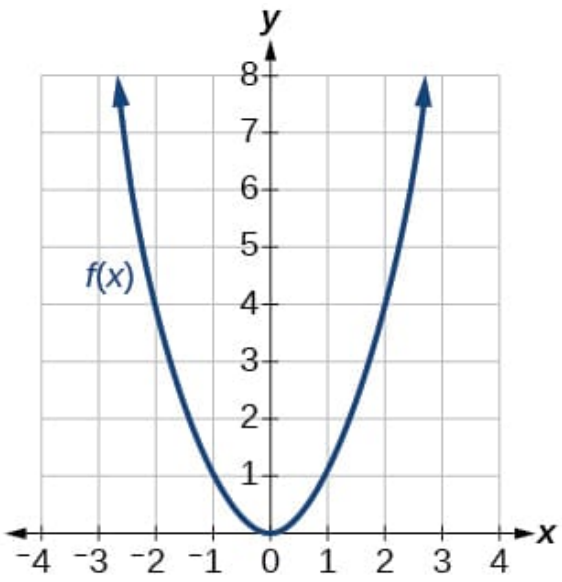


Figure 1.3.2.4

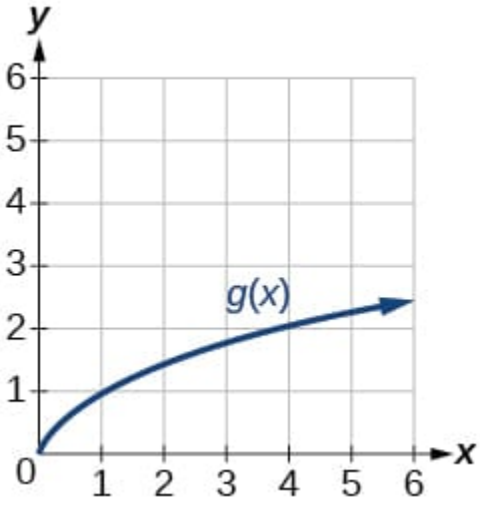
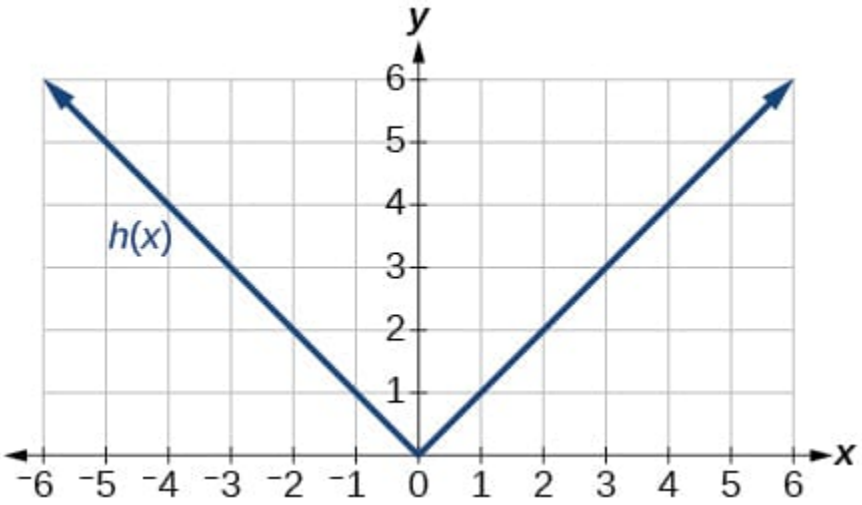


Figure 1.3.2.5



95. $g(f(1))$
96. $g(f(2))$
97. $f(g(4))$
98. $f(g(1))$
99. $f(h(2))$
100. $h(f(2))$
101. $f(g(h(4)))$
102. $f(g(f(-2)))$

Numeric

For the following exercises, use the function values for f and g shown in Table 1.3.2.1 to evaluate each expression.

Table 1.3.2.1

x	$f(x)$	$g(x)$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

103. $f(g(8))$
104. $f(g(5))$
105. $g(f(5))$
106. $g(f(3))$
107. $f(f(4))$
108. $f(f(1))$
109. $g(g(2))$
110. $g(g(6))$

For the following exercises, use the function values for f and g shown in Table 1.3.2.2 to evaluate the expressions.

Table 1.3.2.2

x	$f(x)$	$g(x)$
-3	11	-8
-2	9	-3
-1	7	0
0	5	1
1	3	0
2	1	-3

x	$f(x)$	$g(x)$
3	-1	-8

21. $(f \circ g)(1)$
22. $(f \circ g)(2)$
23. $(g \circ f)(2)$
24. $(g \circ f)(3)$
25. $(g \circ g)(1)$
26. $(f \circ f)(3)$

For the following exercises, use each pair of functions to find $f(g(0))$ and $g(f(0))$.

27. $f(x) = 4x + 8$, $g(x) = 7 - x^2$
28. $f(x) = 5x + 7$, $g(x) = 4 - 2x^2$
29. $f(x) = \sqrt{x+4}$, $g(x) = 12 - x^3$
30. $f(x) = \frac{1}{x+2}$, $g(x) = 4x + 3$

For the following exercises, use the functions $f(x) = 2x^2 + 1$ and $g(x) = 3x + 5$ to evaluate or find the composite function as indicated.

31. $f(g(2))$
32. $f(g(x))$
33. $g(f(-3))$
34. $(g \circ g)(x)$

Extensions

For the following exercises, let f be the function defined by

$$f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$$

and let g be the function defined

$$g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$$

Compute the indicated value if it exists.

35. $(f+g)(-3)$
36. $(f-g)(2)$
37. $(fg)(-1)$
38. $(g+f)(1)$
39. $(g-f)(3)$
40. $(gf)(-3)$
41. $\left(\frac{f}{g}\right)(-2)$
42. $\left(\frac{f}{g}\right)(-1)$
43. $\left(\frac{f}{g}\right)(2)$
44. $\left(\frac{g}{f}\right)(-1)$
45. $\left(\frac{g}{f}\right)(3)$
46. $\left(\frac{g}{f}\right)(-3)$

For the following exercises, use $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$.

47. Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Compare the two answers.
48. Find $(f \circ g)(2)$ and $(g \circ f)(2)$.
49. What is the domain of $(g \circ f)(x)$?
50. What is the domain of $(f \circ g)(x)$?
51. Let $f(x) = \frac{1}{x}$.
    a. Find $(f \circ f)(x)$.
    b. Is $(f \circ f)(x)$ for any function f the same result as the answer to part (a) for any function? Explain.

For the following exercises, let $F(x) = (x+1)^5$, $f(x) = x^5$, and $g(x) = x+1$.

52. True or False: $(g \circ f)(x) = F(x)$.
53. True or False: $(f \circ g)(x) = F(x)$.

For the following exercises, find the composition when $f(x) = x^2 + 2$ for all $x \geq 0$ and $g(x) = \sqrt{x-2}$.

54. $(f \circ g)(6)$; $(g \circ f)(6)$
55. $(g \circ f)(a)$; $(f \circ g)(a)$
56. $(f \circ g)(11)$; $(g \circ f)(11)$

Real-World Applications

57. The function $D(p)$ gives the number of items that will be demanded when the price is p. The production cost $C(x)$ is the cost of producing x items. To determine the cost of production when the price is $6, you would do which of the following?
    a. Evaluate $D(C(6))$.
    b. Evaluate $C(D(6))$.
    c. Solve $D(C(x)) = 6$.
    d. Solve $C(D(p)) = 6$.
58. The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient's system after t minutes is modeled by $m(t)$. Which of the following would you do in order to determine when the patient will be at a pain level of 4?
    a. Evaluate $A(m(4))$.
    b. Evaluate $m(A(4))$.
    c. Solve $A(m(t)) = 4$.
    d. Solve $m(A(d)) = 4$.
59. A store offers customers a 30% discount on the price x of selected items. Then, the store takes off an additional 15% at the cash register. Write a price function $P(x)$ that computes the final price of the item in terms of the original price x. (Hint: Use function composition to find your answer.)
60. A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25\sqrt{t+2}$, find the area of the ripple as a function of time. Find the area of the ripple at $t = 2$.
61. A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula $r(t) = 2t + 1$, express the area burned as a function of time, t (minutes).
62. Use the function you found in the previous exercise to find the total area burned after 5 minutes.
63. The radius r, in inches, of a spherical balloon is related to the volume, V, by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V(t) = 10 + 20t$.
    a. Find the composite function $r(V(t))$.
    b. Find the *exact* time when the radius reaches 10 inches.
64. The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, \(3
65. Find the composite function $N(T(t))$.
66. Find the time (round to two decimal places) when the bacteria count reaches 6752.

For the following exercises, $C(x)$ denotes the cost to produce x items and $p(x)$ denotes the price-demand function in the given economic scenario. In each Exercise, do the following:

- Find and interpret $C(0)$.
- Find and interpret $\overline{C}(10)$. ($\overline{C}$ is called the **average cost**)
- Find and interpret $p(5)$.
- Find and simplify $R(x)$. ($R(x)$ is the **revenue function**)
- Find and simplify $P(x)$. ($P(x)$ is the **profit function**)
- Solve $P(x) = 0$ and interpret.

65. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, $0 \leq x \leq 15$.
66. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x, 0 \leq x \leq 35$.
67. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x, 0 \leq x \leq 30$.

58. The daily cost, in dollars, to produce x Sasquatch Berry Pies $C(x) = 3x + 36,\ x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x,\ 0 \leq x \leq 24$.
59. The monthly cost, in hundreds of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000,\ x \geq 0$ and the price-demand function, in hundreds of dollars per scooter, is $p(x) = 140 - 2x,\ 0 \leq x \leq 70$.

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1.4: Properties of Function Graphs

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Gasoline costs have experienced some wild fluctuations over the last several decades. Table 1.4.1 lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

Table 1.4.1

y	2005	2006	2007	2008	2009	2010	2011	2012
$C(y)$	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from \$2.31 to \$3.68, an increase of \$1.37. While this is interesting, it might be more useful to look at how much the price changed *per year*. In this section, we will investigate changes such as these.

Average Rate of Change

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity.

Definition: Rate of Change

Given points (x_1, y_1) and (x_2, y_2), and defining $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$, the ratio

$$m = \frac{\Delta y}{\Delta x}$$

is defined to be the **rate of change** of y with respect to x.

The Greek letter Δ (Delta) signifies the change in a quantity; we read the ratio as "delta-y over delta-x" or "the change in y divided by the change in x." Occasionally we write Δf instead of Δy, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

We can see that the price of gasoline in Table 1.4.1 did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\text{Average rate of change} = \frac{\text{Change in output}}{\text{Change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Definition: Average Rate of Change

The **average rate of change** of a function f over an interval $[a, b]$, where $a \neq b$, is the quotient

$$\frac{f(b) - f(a)}{b - a}.$$

In our example, the gasoline price increased by $1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7\,\text{years}} \approx 0.196 \text{ dollars per year.}$$

On average, the price of gas increased by about 19.6 cents each year.

Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by $4,000 per quarter

Example 1.4.1: Computing an Average Rate of Change

Using the data in Table 1.4.1, find the average rate of change of the price of gasoline between 2007 and 2009.

Solution available at the end of this section.

Note that a decrease is expressed by a negative change or "negative increase." A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

Checkpoint 1.4.1

Using the data in Table 1.4.1, find the average rate of change between 2005 and 2010.

Example 1.4.2: Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in the following figure, find the average rate of change on the interval $[-1, 2]$.

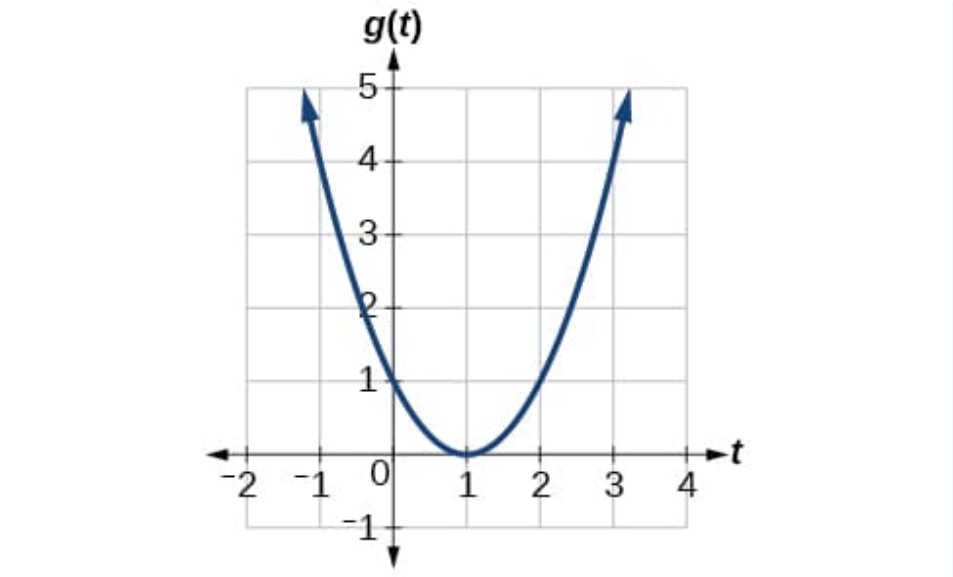


Figure 1.4.1

Solution available at the end of this section.

Example 1.4.2 illustrates the fact that the order we choose is very important. If, for example, we used $\frac{y_2-y_1}{x_1-x_2}$, we would not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as (x_1, y_1) and (x_2, y_2).

Interactive Element: Average Rate of Change as the Slope of a Secant Line

The following Interactive Element might help you visualize how the average rate of change (ARoC) of a function is the slope of the line joining two points on its graph, and how that value depends on the interval you choose.

Interact: Drag the two red points along the curve to set the endpoints of the interval $[x_1, x_2]$.

Observation: The green line through the points is the secant line, and its slope equals the average rate of change $\frac{f(x_2)-f(x_1)}{x_2-x_1}$ over the interval. Moving the points to different parts of the curve changes the value—it can be positive, negative, or zero—so the average rate of change depends on the interval you pick.

Example 1.4.3: Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in the following table. Find her average speed over the first 6 hours.

t **(hours)**	0	1	2	3	4	5	6	7
$D(t)$ **(miles)**	10	55	90	153	214	240	292	300

Solution available at the end of this section.

In Example 1.4.3, because the speed is not constant, the average speed depends on the interval chosen. For the interval $[2, 3]$, the average speed is 63 miles per hour.

Example 1.4.4: Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval $[2, 4]$.

Solution available at the end of this section.

Checkpoint 1.4.4

Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval $[1, 9]$.

Example 1.4.5: Finding the Average Rate of Change of a Force

The electrostatic force F, measured in newtons, between two charged particles can be related to the distance between the particles d, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

Solution available at the end of this section.

Example 1.4.6: Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a.

Solution available at the end of this section.

The answer in Example 1.4.6 is a repeated concept from the previous section—it is an alternative form of the difference quotient!

Checkpoint 1.4.6

Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$.

Focus on Calculus - Local Extrema and Intervals of Increase and Decrease

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is **increasing** on an interval if the function values increase as the input values increase within that interval. Similarly, a function is **decreasing** on an interval if the function values decrease as the input values increase over that interval. The following definitions codify these statements.

Definition: Increasing

Let f be a function defined on an interval. The function f is **increasing** on that interval if, for every pair of numbers a and b in the interval with $a < b$, it follows that $f(a) < f(b)$.

Definition: Decreasing

Let f be a function defined on an interval. The function f is **decreasing** on that interval if, for every pair of numbers a and b in the interval with $a < b$, it follows that $f(a) > f(b)$.

The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. Figure $1.4.3$ shows examples of increasing and decreasing intervals on a function.

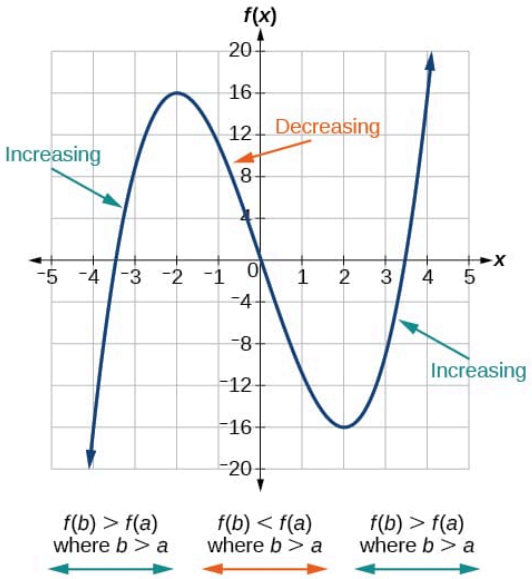


Figure $1.4.3$**:** *The function* $f(x) = x^3 - 12x$ *is increasing on* $(-\infty, -2) \cup (2, \infty)$ *and is decreasing on* $(-2, 2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is the location of a **local maximum**. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is the location of a **local minimum**.

Definition: Local Extrema

A function f has a **local maximum** at $x = b$ in an open interval (a, c) if $f(b) \geq f(x)$ for all $x \neq b$ in the interval. Likewise, f has a **local minimum** at $x = b$ in (a, c) if $f(b) \leq f(x)$ for all $x \neq b$ in the interval.

If f has more than one local maximum on a given interval, we call the collection of these maximums the **local maxima**. Likewise, if f has more than one local minimum on a given interval, we call the collection of these minimums the **local minima**.

Together, local maxima and minima are called **local extrema** (or **local extreme values**).

Some texts use the word "relative" instead of "local." We will use the latter throughout this text.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of *local* extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown below, the local maximum is 16, and it occurs at $x = -2$. The local minimum is -16 and it occurs at $x = 2$.

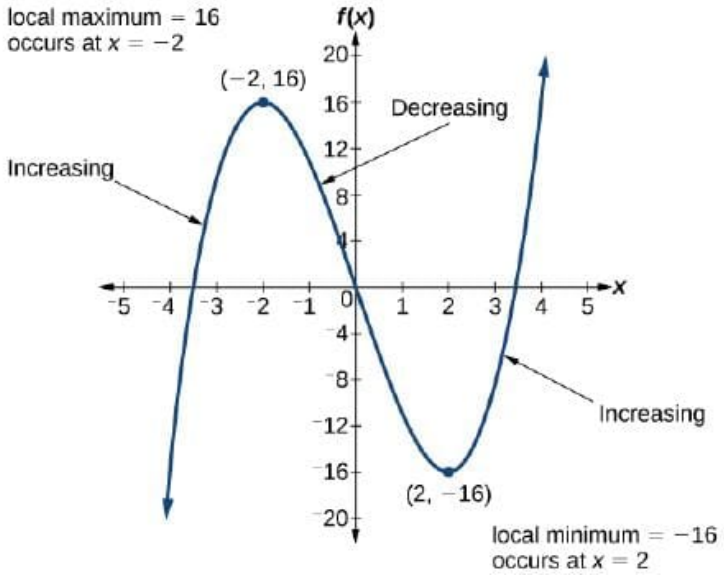


Figure 1.4.4

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an *open interval*. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. The following figure illustrates these ideas for a local maximum.

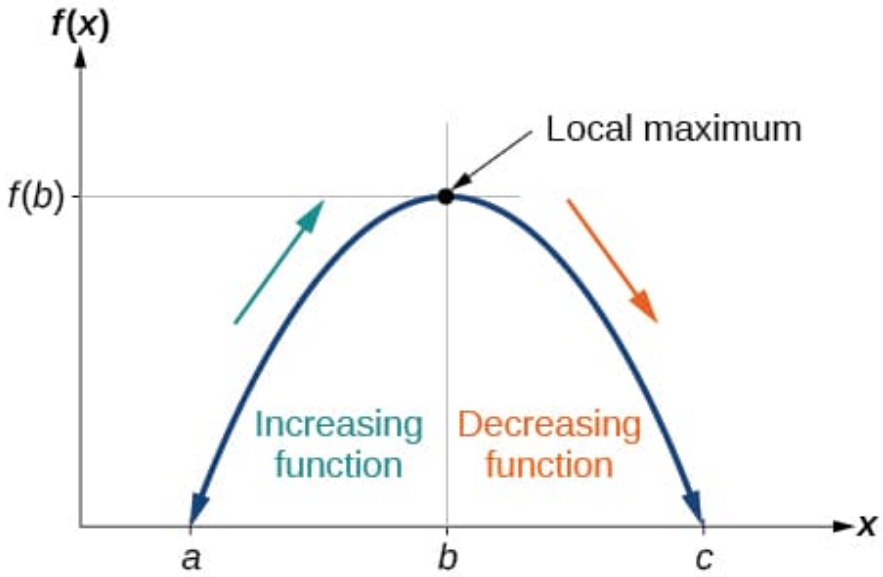


Figure 1.4.5

Example 1.4.7: Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in the following figure, identify the intervals on which the function appears to be increasing.

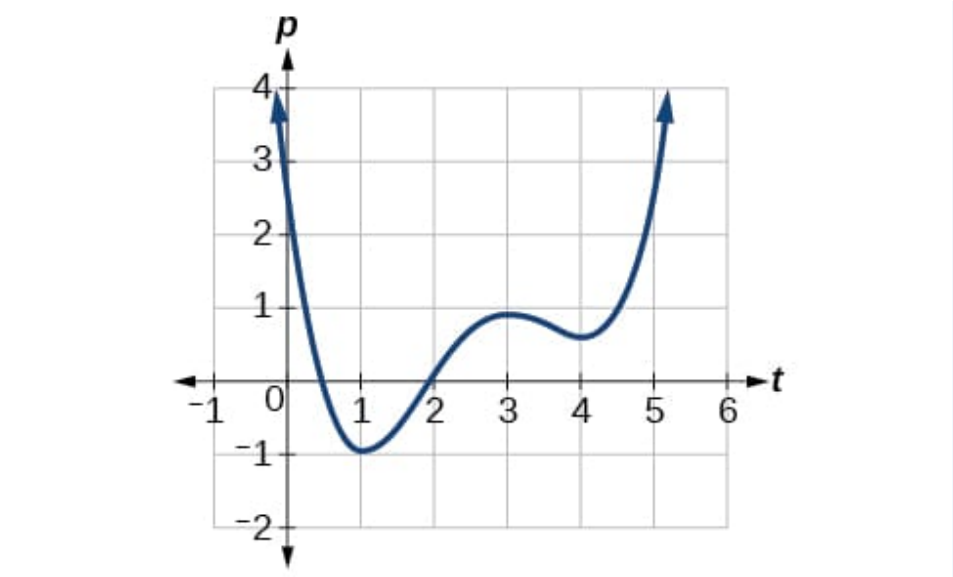


Figure 1.4.6

Solution available at the end of this section.

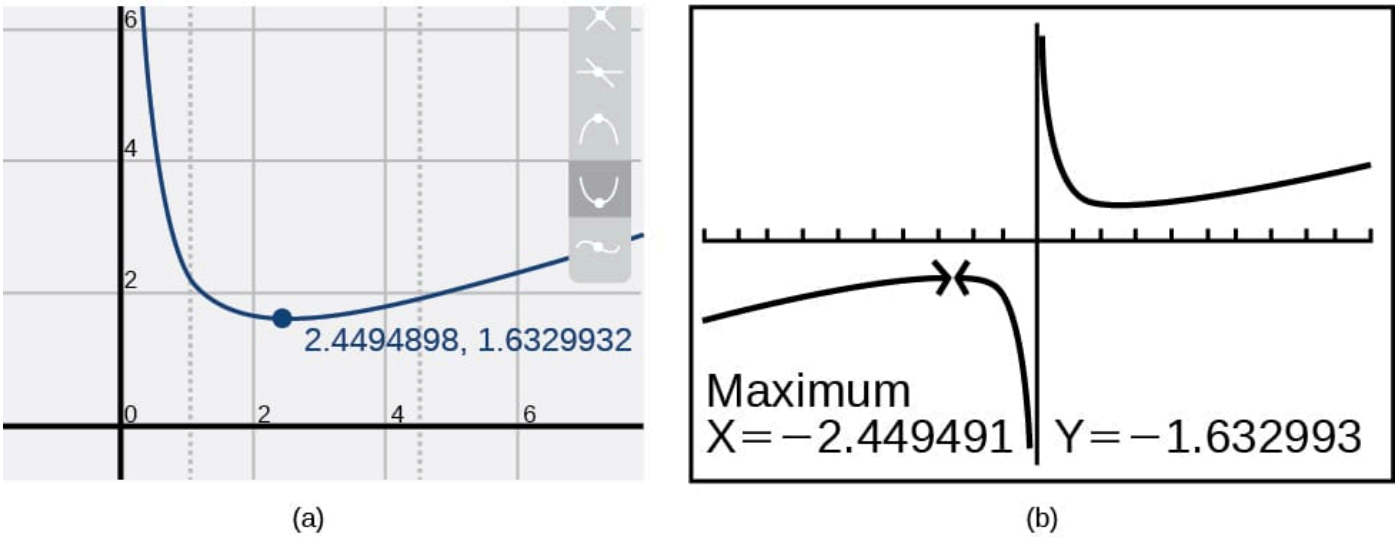
Notice in Example 1.4.7 that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

Example 1.4.8: Finding Local Extrema from a Graph

Use graphing technology to graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

Solution available at the end of this section.

Most graphing calculators and graphing utilities can *estimate* the location of maxima and minima. The following figure provides screen images from two different technologies, showing the estimate for the local maximum and minimum.



(a) (b)

Figure 1.4.8

Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that, while we expect

the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires Calculus.)

Checkpoint 1.4.8

Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Example 1.4.9: Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown below, find all local maxima and minima.

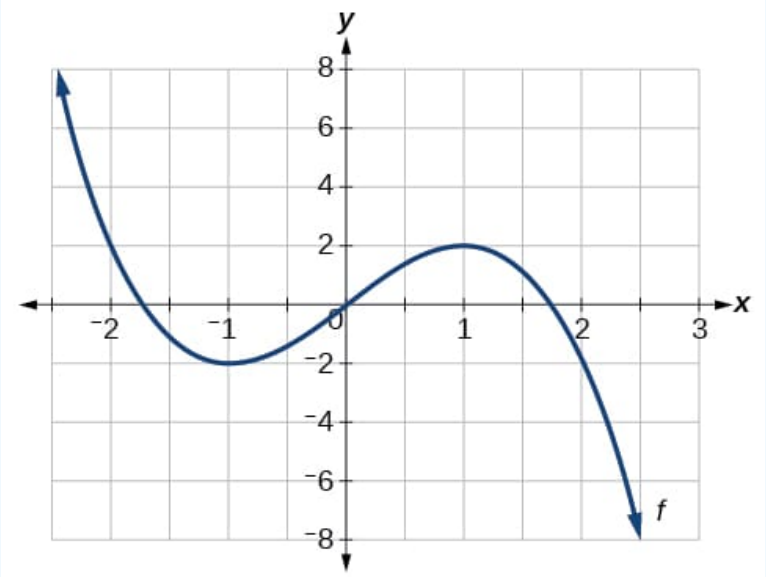


Figure 1.4.9

Solution available at the end of this section.

Revisiting the Toolkit Functions

We will now return to our toolkit functions and discuss their graphical behavior.

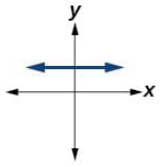
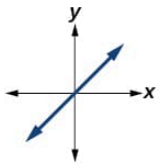
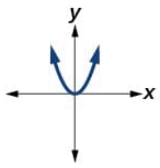
Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

Figure 1.4.10

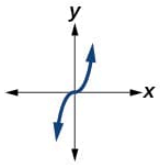
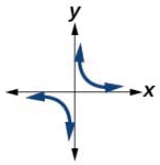
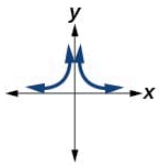
Function	Increasing/Decreasing	Example
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	

Figure 1.4.11

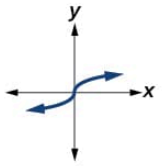
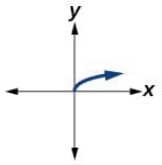
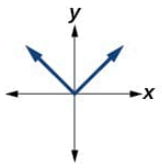
Function	Increasing/Decreasing	Example
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \|x\|$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$	

Figure 1.4.12

Focus on Calculus - Absolute Extrema

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively.

Definition: Absolute Extrema

A function f has an **absolute maximum** at $x = b$ if $f(b) \geq f(x)$ for all x in the domain of f. Likewise, a function f has an **absolute minimum** at $x = b$ if $f(b) \leq f(x)$ for all x in the domain of f.

If f has more than one local absolute maximum, we call the collection of these maximums the **absolute maxima**. Likewise, if f has more than one absolute minimum, we call the collection of these minimums the **absolute minima**.

Together, absolute maxima and minima are called **absolute extrema**.

Some texts use the word "global" instead of "absolute." We will use the latter throughout this text.

To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points on the domain of the function.

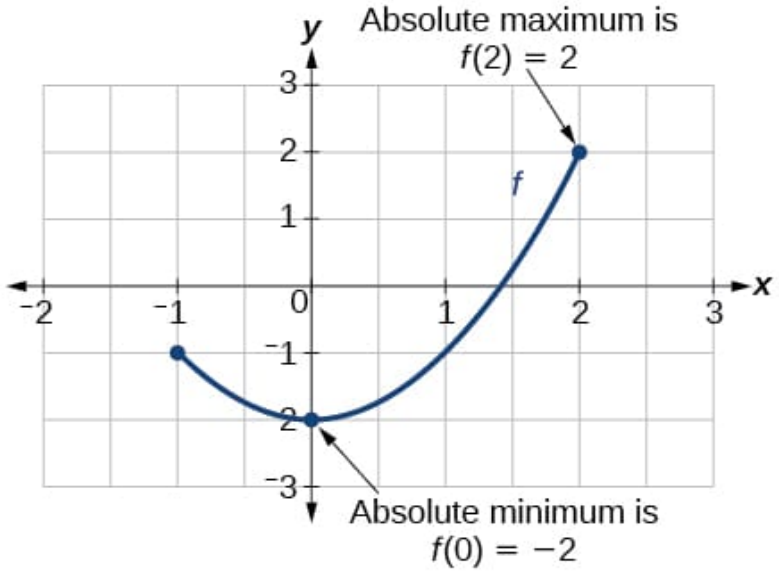


Figure 1.4.13

Not every function has an absolute maximum or minimum value. The toolkit function $f(x) = x^3$ is one such function.

Example 1.4.10: Finding Absolute Maxima and Minima from a Graph

For the function f shown below, find all absolute maxima and minima.

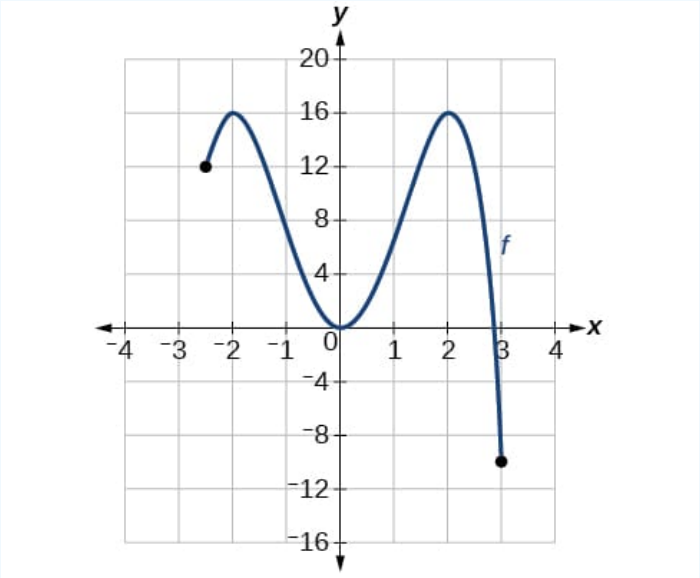


Figure 1.4.14

Solution available at the end of this section.

Symmetry - Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are **symmetric about the y -axis**. Functions whose graphs are symmetric about the y-axis are called **even functions**.

Definition: Even and Odd Functions

The function f is called an **even function** if for every input x in the domain of f

$$f(-x) = f(x).$$

The function f is called an **odd function** if for every input x in the domain of f

$$f(-x) = -f(x).$$

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over *both* axes, the result would be the original graph, as shown in Figure 1.4.15.

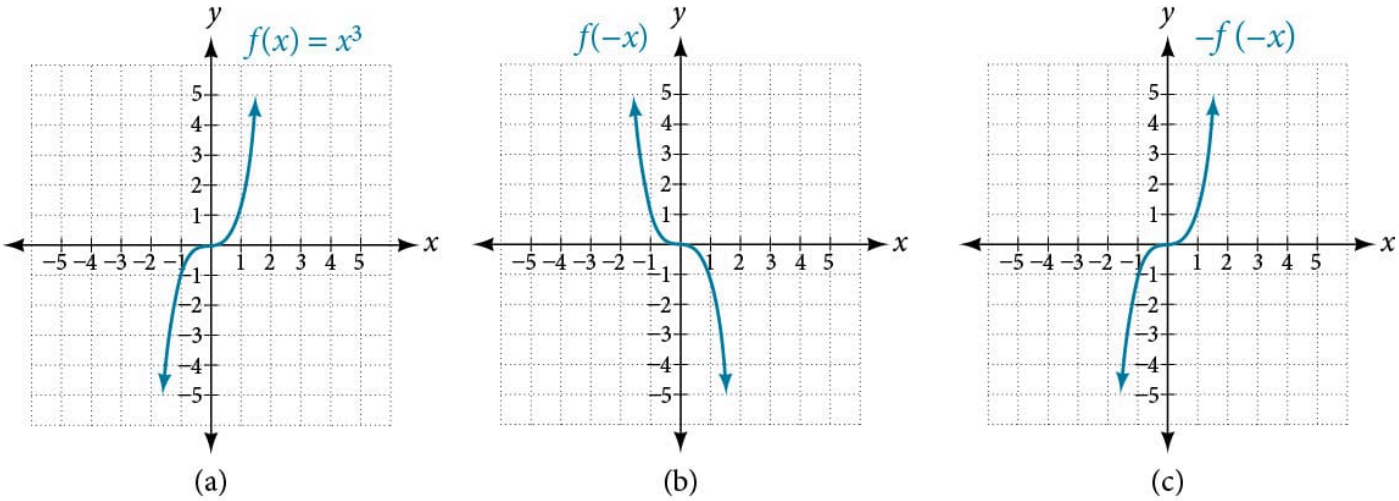


Figure 1.4.15**:** *(a) The cubic toolkit function. (b) Horizontal reflection of the cubic toolkit function. (c) Horizontal and vertical reflections reproduce the original cubic function.*

We say that these graphs are **symmetric about the origin**. A function with a graph that is symmetric about the origin is called an **odd function**.

Theorem: Symmetry of Even and Odd Functions

Let f be a real function.

- f is even if and only if its graph is symmetric with respect to the y-axis; that is, the graph is unchanged under reflection across the y-axis.
- f is odd if and only if its graph is symmetric with respect to the origin; that is, the graph is unchanged under a $180°$ rotation about the origin.

Equivalently, f is even when the point $(-x, y)$ lies on the graph whenever (x, y) does, and f is odd when the point $(-x, -y)$ lies on the graph whenever (x, y) does.

A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Example 1.4.11: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution available at the end of this section.

The graph of the function from Example 1.4.11 is shown below. Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.

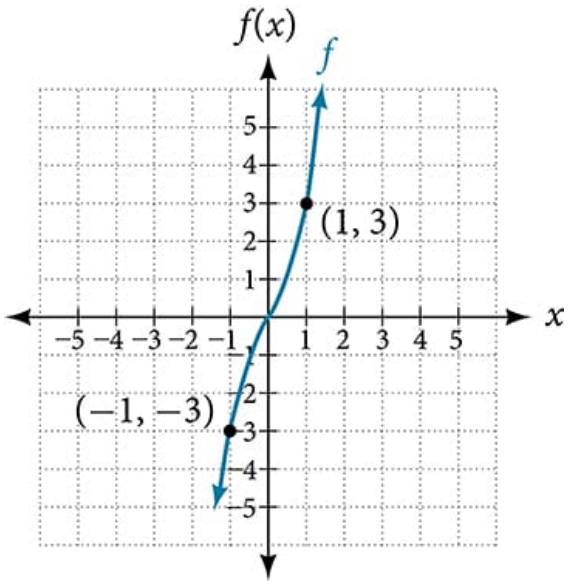


Figure 1.4.16

Checkpoint 1.4.11

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

MyOpenMath Advice: Check For Feedback

When grading your uploaded (handwritten) work, your professor might have left feedback. Sometimes this feedback can be as simple as, "Please read the written solution," but it can be as robust as a breakdown of your mistake and how you can avoid such a mistake in the future. To check a past due assignment for any feedback, go to the Grades area in Canvas, click on the assignment in question, and you should be able to see any feedback there. If you cannot see feedback, please let your professor know.

Success in College: Meeting With Your Professor

Meeting with a faculty member might be intimidating. Students are encouraged to use the following guidelines to make the faculty meeting productive. These strategies were developed by the Retention Office at U. of Toledo (OH) and first appeared in National On-Campus Report's sister publication, Recruitment and Retention in Higher Education.

Step 1: Make an appointment

- Review the syllabus for the professor's office hours and make an appointment.
- If you're uncertain of your professor's office hours, ask.
- If your schedule conflicts with the office hours, tell your professor why you can't make the scheduled office hours. Then tell them about your specific concern.
- Set up a specific time to meet with your professor. Be sure to indicate the amount of time you will need.

Step 2: Establish rapport

- Be on time for your appointment.
- When you arrive, be pleasant, smile, introduce yourself again (include your name and class), and shake hands if appropriate.

Step 3: Present your concern

- Focus on the specific questions you've identified as problematic.
- Have your problem(s) written out so that the professor can see where your difficulties are.

Step 4: Provide background information

- Briefly tell your professor about your high school background and preparation for the course – if relevant to solving the problem.
- Explain the study strategies you've used to understand the material.

Step 5: Redirect for clarification

- If the professor's explanation isn't clear, redirect their attention to the specific point where you became confused.
- "Talk through" your problem.

Step 6: Summarize Resolution of the Problem

- "I was missing this step in... I need to apply this formula..."

Step 7: Thank your Professor

- Ask to come back if necessary.

Armed with this information, think about the following questions:

1. How would you make an appointment to talk to your professor if your schedule prevents you from visiting during office hours? If it is not possible to make such an appointment or your professor does not have office hours, think about another way you can get extra help.
2. Some professors prefer you talk to them using their title and last name. For example, you might say Dr. Rodriguez or Ms. Smith. Others prefer that you use just their first name. When asking your professor a question, what name or title do they prefer that you use?

If you are unsure about the answers to either of these questions, feel free to ask your professor - that's what they are there for!

Answers

Section Preview

The first half of this section is meant to get the student used to (and to compute) the average rate of change of a function. Use this as an opportunity to impress upon the student the fact that this concept is a fundamental building block for Calculus. The remainder of the section is dedicated to interpreting the behavior of the graph of a function—specifically, increasing, decreasing, and local/global extrema. It is important to note that we adopt the convention that intervals of increase and decrease are always open. Finally, we finish the section with the important conversation about symmetry.

Section Preview

- **Linear Functions**
  - The Slope of a Line: Average rate of change is presented as a generalization of slope, so the student must already understand slope as a constant rate of change ("rise over run") to grasp why $\frac{y_2-y_1}{x_2-x_1}$ measures change.

Section Preview

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.
- Determine whether a function is even, odd, or neither from its graph.

Example 1.4.1**: Computing an Average Rate of Change**

In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$2.41 - \$2.84}{2009 - 2007} && \text{(substituting)} \\ &= \frac{-\$0.43}{2\,\text{years}} && \text{(simplifying)} \\ &\approx -\$0.22 \text{ per year}\end{aligned}$$

Example 1.4.2**: Computing Average Rate of Change from a Graph**

The following figure shows $g(-1) = 4$. At $t = 2$, the graph shows $g(2) = 1$.

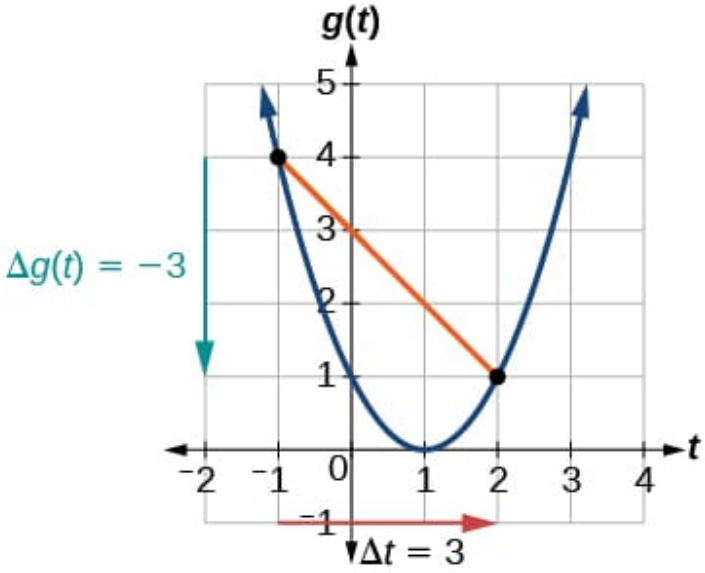


Figure 1.4.2

The horizontal change $\Delta t = 3$ is shown by the red arrow, and the vertical change $\Delta g(t) = -3$ is shown by the turquoise arrow. The output changes by -3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)} = \frac{-3}{3} = -1.$$

Example 1.4.3**: Computing Average Rate of Change from a Table**

Here, the average speed is the average rate of change. She traveled 282 miles in 6 hours, for an average speed of

$$\frac{292 - 10}{6 - 0} = \frac{282}{6} = 47.$$

The average speed is 47 miles per hour.

Example 1.4.4**: Computing Average Rate of Change for a Function Expressed as a Formula**

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2 - \frac{1}{2} & \qquad f(4) &= 4^2 - \frac{1}{4} \\ &= 4 - \frac{1}{2} & &= 16 - \frac{1}{4} \\ &= \frac{7}{2} & &= \frac{63}{4}\end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned}\text{Average rate of change} &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{\frac{63}{4} - \frac{7}{2}}{4 - 2} && \text{(substituting)} \\ &= \frac{4}{4} \cdot \left(\frac{\frac{63}{4} - \frac{7}{2}}{2}\right) && \text{(simplifying the compound fraction)} \\ &= \frac{63 - 14}{8} && \text{(distributing)} \\ &= \frac{49}{8} && \text{(simplifying)}\end{aligned}$$

Example 1.4.5**: Finding the Average Rate of Change of a Force**

We are computing the average rate of change of $\frac{2}{d^2}$ on the interval $[2, 6]$.

$$\begin{aligned}
\text{Average rate of change} &= \frac{F(6) - F(2)}{6 - 2} \\
&= \frac{\frac{2}{6^2} - \frac{2}{2^2}}{4} && \text{(substituting and simplifying)} \\
&= \frac{\frac{2}{36} - \frac{2}{4}}{4} && \text{(simplifying)} \\
&= \frac{\frac{1}{18} - \frac{1}{2}}{4} && \text{(simplifying)} \\
&= \frac{18}{18} \cdot \left(\frac{\frac{1}{18} - \frac{1}{2}}{4} \right) && \text{(simplifying the compound fraction)} \\
&= \frac{1 - 9}{72} && \text{(distributing)} \\
&= -\frac{8}{72} && \text{(simplifying)} \\
&= -\frac{1}{9} && \text{(simplifying)}
\end{aligned}$$

The average rate of change is $-\frac{1}{9}$ newton per centimeter.

Example 1.4.6**: Finding an Average Rate of Change as an Expression**

We use the average rate of change formula.

$$\begin{aligned}
\text{Average rate of change} &= \frac{g(a) - g(0)}{a - 0} \\
&= \frac{(a^2 + 3a + 1) - (0^2 + 3(0) + 1)}{a} && \text{(substituting)} \\
&= \frac{a^2 + 3a + 1 - 1}{a} && \text{(distributing)} \\
&= \frac{a^2 + 3a}{a} && \text{(combining like terms)} \\
&= \frac{a(a + 3)}{a} && \text{(factoring } a \text{ out of the numerator)} \\
&= \frac{\cancel{a}^{1}(a + 3)}{\cancel{a}^{1}} && \text{(canceling like factors)} \\
&= a + 3 && \text{(simplifying)}
\end{aligned}$$

This result tells us the average rate of change in terms of a between $t = 0$ and any other point $t = a$. For example, on the interval $[0, 5]$, the average rate of change would be $5 + 3 = 8$.

Example 1.4.7**: Finding Increasing and Decreasing Intervals on a Graph**

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on. In interval notation, we would say the function appears to be increasing on the interval $(1, 3) \cup (4, \infty)$.

Example 1.4.8**: Finding Local Extrema from a Graph**

Using technology, we find that the graph of the function looks like that in the figure below. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$.

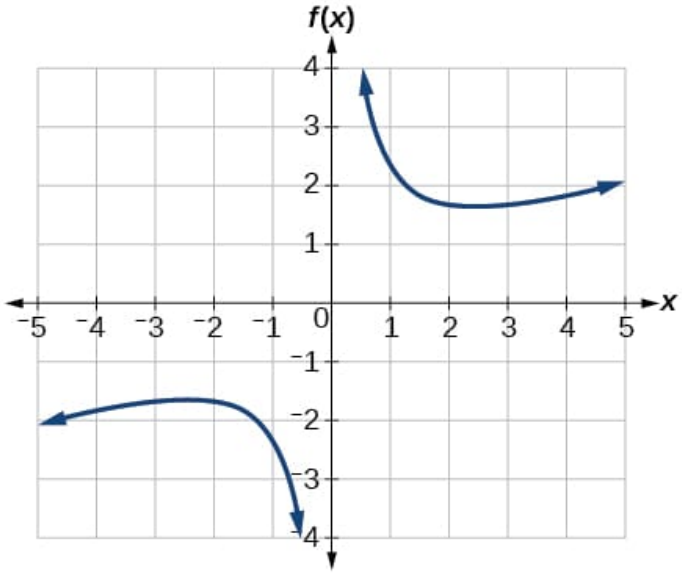


Figure 1.4.7

Example 1.4.9**: Finding Local Maxima and Minima from a Graph**

The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$. The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is -2.

Example 1.4.10**: Finding Absolute Maxima and Minima from a Graph**

The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the y-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at $x = 3$, because it is the lowest point on the domain of the function's graph. The absolute minimum is the y-coordinate at $x = 3$, which is -10.

Example 1.4.11**: Determining whether a Function Is Even, Odd, or Neither**

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin by substituting $-x$ into the argument of the function.

$$\begin{aligned} f(-x) &= (-x)^3 + 2(-x) && \text{(substituting)} \\ &= -x^3 - 2x && \text{(simplifying)} \\ &= -(x^3 + 2x) && \text{(factoring out } -1) \\ &= -f(x) \end{aligned}$$

Because $f(-x) = -f(x)$, this is an odd function.

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1.4.1: Resources and Key Concepts

Resources

Videos

- Average rate of change
  - CRC Mathematics in Review: Linears - Slope and Rate of Change
- Intervals of Increase and Decrease
  - Rates of Change and Behavior of Graphs - Intervals of Increase/Decrease and Local Extrema
  - Intermediate Algebra - Functions: Intro to Graphs
- Even and Odd functions
  - Transformations of Functions - Determining Even and Odd Functions
- Using Technology to Graph
  - Functions - Overview - A Review for Precalculus: Graphs - Using Desmos to Graph

Key Concepts

Definitions

- **Rate of Change:** A description of how an output quantity changes relative to a change in the input quantity. The units are "output units per input units."
- **Average Rate of Change:** The total change in function values (output values) divided by the change in the input values over a specified interval. Formula: $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.
- **Increasing Function (on an interval):** A function f is increasing on an open interval I if $f(b) > f(a)$ for every a and b in I where $b > a$.
- **Decreasing Function (on an interval):** A function f is decreasing on an open interval I if $f(b) < f(a)$ for every a and b in I where $b > a$.
- **Local Maximum (Relative Maximum):** A function f has a local maximum at $x = b$ in an open interval (a, c) if $f(b) \geq f(x)$ for all $x \neq b$ in the interval. It is a point where the function changes from increasing to decreasing.
- **Local Minimum (Relative Minimum):** A function f has a local minimum at $x = b$ in an open interval (a, c) if $f(b) \leq f(x)$ for all $x \neq b$ in the interval. It is a point where the function changes from decreasing to increasing.
- **Local Maxima:** The collection of all local maximum values on a given interval.
- **Local Minima:** The collection of all local minimum values on a given interval.
- **Local Extrema (Local Extreme Values):** The collective term for all local maxima and local minima.
- **Absolute Maximum (Global Maximum):** A function f has an absolute maximum at $x = b$ if $f(b) \geq f(x)$ for all x in the entire domain of f.
- **Absolute Minimum (Global Minimum):** A function f has an absolute minimum at $x = b$ if $f(b) \leq f(x)$ for all x in the entire domain of f.
- **Absolute Maxima:** The collection of all absolute maximum values of a function.
- **Absolute Minima:** The collection of all absolute minimum values of a function.
- **Absolute Extrema (Global Extrema):** The collective term for all absolute maxima and absolute minima.
- **Even Function:** A function f for which $f(-x) = f(x)$ for every input x in its domain. Its graph is symmetric about the y-axis.
- **Odd Function:** A function f for which $f(-x) = -f(x)$ for every input x in its domain. Its graph is symmetric about the origin.
- **Symmetric about the y-axis:** A characteristic of the graph of an even function, where reflecting the graph horizontally across the y-axis results in the original graph.
- **Symmetric about the origin:** A characteristic of the graph of an odd function, where reflecting the graph over both the x-axis and y-axis (or rotating 180 degrees about the origin) results in the original graph.

Theorems

- **Symmetry of Even and Odd Functions:**

- The graph of an even function is symmetric about the y-axis.
- The graph of an odd function is symmetric about the origin.

Common Mistakes

- **Incorrect Order of Subtraction in Average Rate of Change:** Using an inconsistent order for coordinates when calculating average rate of change, for example, $\frac{y_2-y_1}{x_1-x_2}$ instead of $\frac{y_2-y_1}{x_2-x_1}$.
- **Confusing Local Extrema with Absolute Extrema:** A local maximum/minimum is the highest/lowest point in an open interval around it, but not necessarily the highest/lowest point over the entire domain of the function.
- **Assuming Endpoints of Intervals are Extrema for Increasing/Decreasing Intervals:** When defining intervals of increase or decrease, open intervals are used because the function is neither increasing nor decreasing *at* the local extrema points themselves.

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1.4.2: Homework

Reading Questions

1. What does the average rate of change of a function represent, and how is it calculated?
2. If a function is increasing on an interval, what can you say about its average rate of change on that interval?
3. Describe the visual characteristic of a local maximum on a graph.
4. What is the difference between a local minimum and an absolute minimum?
5. How can you determine algebraically if a function $f(x)$ is an even function?
6. What type of symmetry does the graph of an odd function exhibit?
7. Can a function be both even and odd? If so, give an example.
8. Give an example of a toolkit function that is decreasing on the interval $(-\infty, 0)$ and increasing on $(0, \infty)$.
9. What does it mean if the average rate of change of a function over an interval is zero?
10. If a graph has a local maximum at $x = a$, what is the value of the local maximum?
11. Can the average rate of change of a function be constant?
12. If a function f is increasing on (a, b) and decreasing on (b, c), then what can be said about the local extremum of f on (a, c)?
13. How are the absolute maximum and minimum similar to and different from the local extrema?
14. How does the graph of the absolute value function compare to the graph of the quadratic function, $y = x^2$, in terms of increasing and decreasing intervals?
15. How can you determine whether a function is odd or even from the formula of the function?

Homework

Algebraic

For the following exercises, find the average rate of change of each function on the interval specified for real numbers b or h.

16. $f(x) = 4x^2 - 7$ on $[1, b]$
17. $g(x) = 2x^2 - 9$ on $[4, b]$
18. $p(x) = 3x + 4$ on $[2, 2 + h]$
19. $k(x) = 4x - 2$ on $[3, 3 + h]$
20. $f(x) = 2x^2 + 1$ on $[x, x + h]$
21. $g(x) = 3x^2 - 2$ on $[x, x + h]$
22. $a(t) = \frac{1}{t+4}$ on $[9, 9 + h]$
23. $b(x) = \frac{1}{x+3}$ on $[1, 1 + h]$
24. $j(x) = 3x^3$ on $[1, 1 + h]$
25. $r(t) = 4t^3$ on $[2, 2 + h]$
26. $f(x) = 2x^2 - 3x$ on $[x, x + h]$

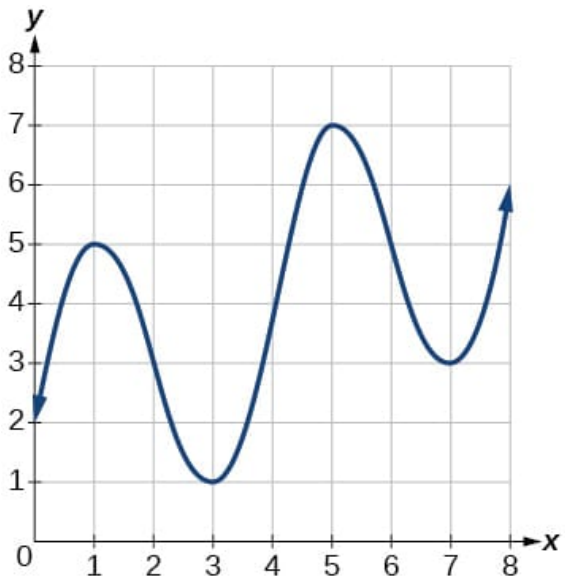
For the following exercises, determine whether the function is odd, even, or neither.

27. $f(x) = 3x^4$
28. $g(x) = \sqrt{x}$
29. $h(x) = \frac{1}{x} + 3x$
30. $f(x) = (x - 2)^2$
31. $g(x) = 2x^4$
32. $h(x) = 2x - x^3$

Graphical

For the following exercises, consider the graph of f shown in Figure $1.4.2.1$.

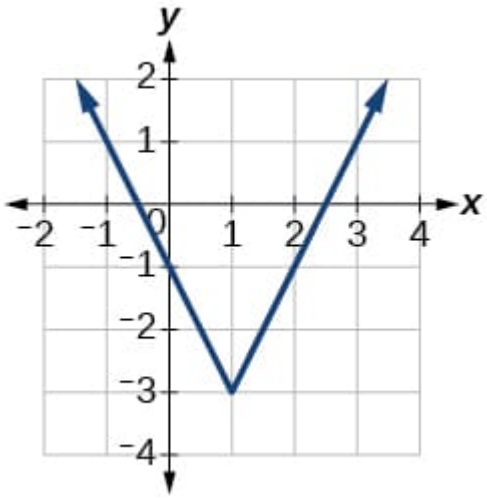
Figure 1.4.2.1



33. Estimate the average rate of change from $x = 1$ to $x = 4$.
34. Estimate the average rate of change from $x = 2$ to $x = 5$.

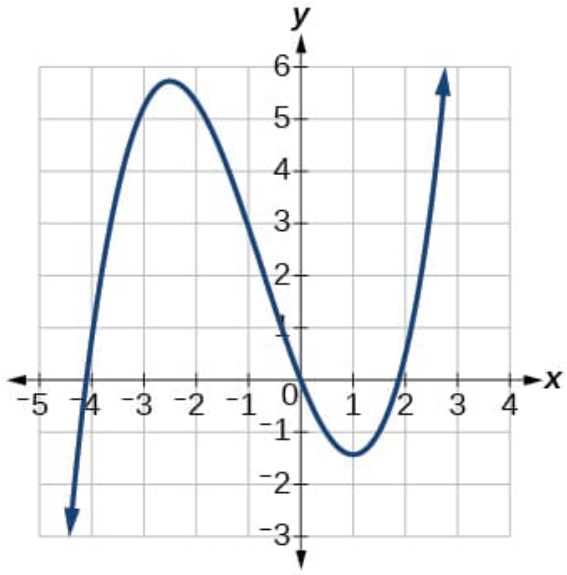
For the following exercises, use the graph of each function to estimate the intervals on which the function is increasing or decreasing.

35.

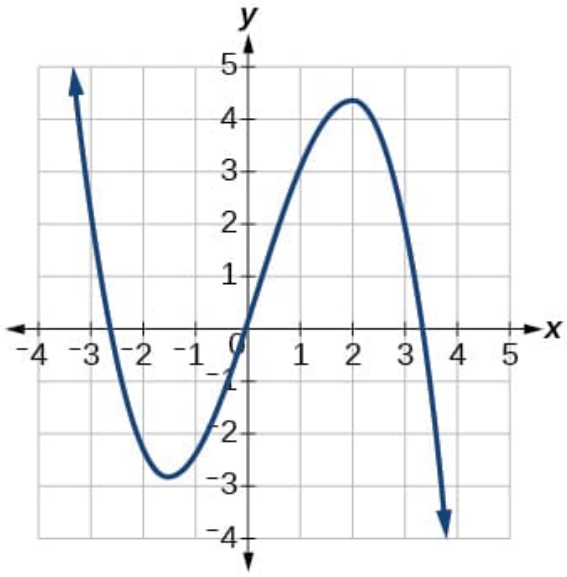


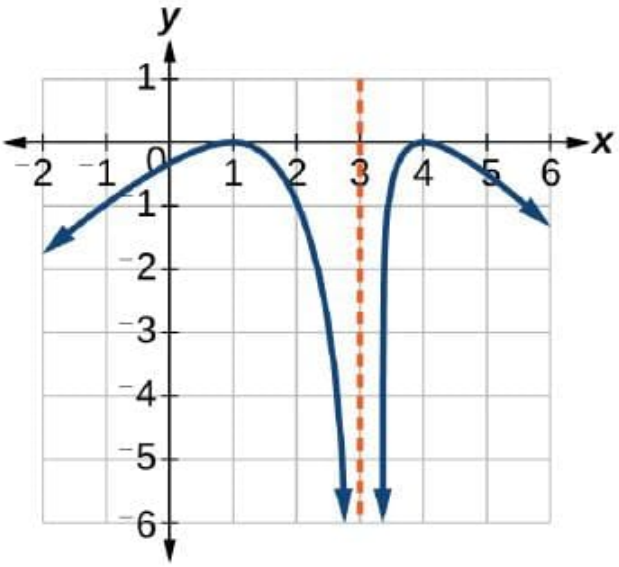
36.

37.



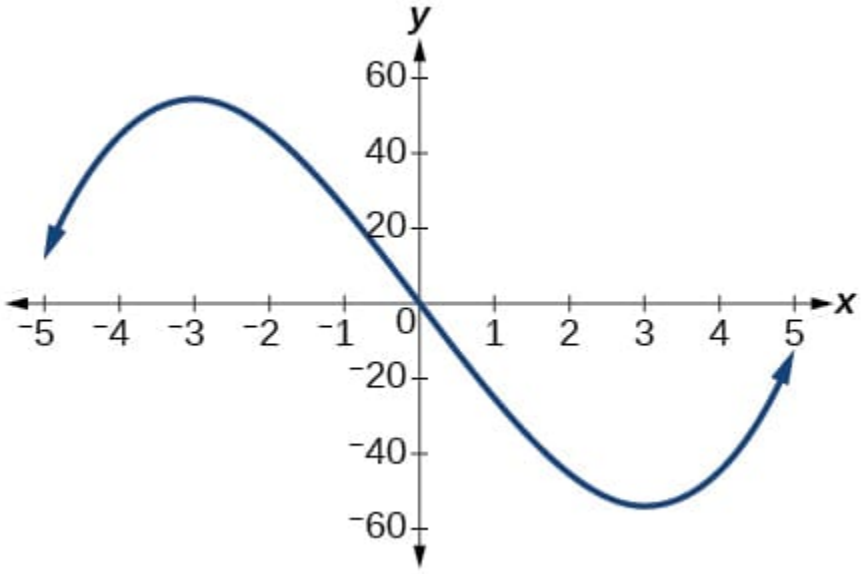
38.





For the following exercises, consider the graph shown in Figure 1.4.2.2**.**

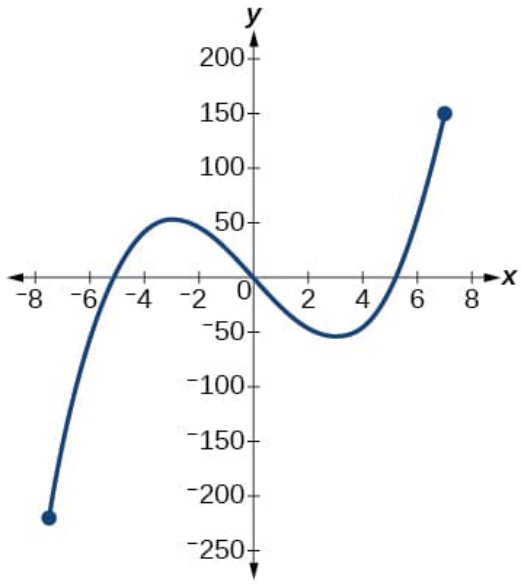
Figure 1.4.2.2



10. Estimate the intervals where the function is increasing or decreasing.
11. Estimate the point(s) at which the graph of f has a local maximum or a local minimum.

For the following exercises, consider the graph in Figure 1.4.2.3**.**

Figure 1.4.2.3



12. If the complete graph of the function is shown, estimate the intervals where the function is increasing or decreasing.
13. If the complete graph of the function is shown, estimate the absolute maximum and absolute minimum.

Numeric

14. Table 1.4.2.1 gives the annual sales (in millions of dollars) of a product from 1998 to 2006. What was the average rate of change of annual sales (a) between 2001 and 2002, and (b) between 2001 and 2004?

Table 1.4.2.1

Year	Sales (millions of dollars)
1998	201
1999	219
2000	233
2001	243
2002	249
2003	251
2004	249
2005	243
2006	233

15. Table 1.4.2.2 gives the population of a town (in thousands) from 2000 to 2008. What was the average rate of change of population (a) between 2002 and 2004, and (b) between 2002 and 2006?

Table 1.4.2.2

Year	Population (thousands)
2000	87
2001	84

Year	Population (thousands)
2002	83
2003	80
2004	77
2005	76
2006	78
2007	81
2008	85

For the following exercises, find the average rate of change of each function on the interval specified.

46. $f(x) = x^2$ on $[1, 5]$
47. $h(x) = 5 - 2x^2$ on $[-2, 4]$
48. $q(x) = x^3$ on $[-4, 2]$
49. $g(x) = 3x^3 - 1$ on $[-3, 3]$
50. $y = \frac{1}{x}$ on $[1, 3]$
51. $p(t) = \frac{(t^2-4)(t+1)}{t^2+3}$ on $[-3, 1]$
52. $k(t) = 6t^2 + \frac{4}{t^3}$ on $[-1, 3]$

Technology

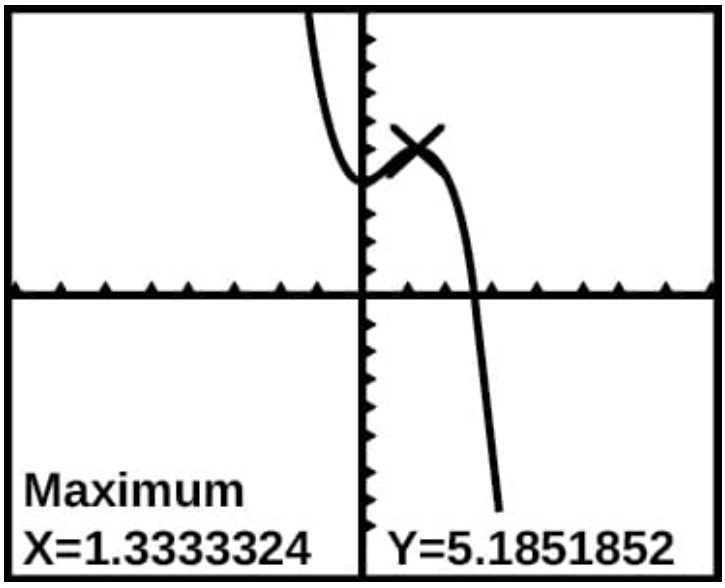
For the following exercises, use graphing technology to estimate the local extrema of each function and to estimate the intervals on which the function is increasing and decreasing.

53. $f(x) = x^4 - 4x^3 + 5$
54. $h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$
55. $g(t) = t\sqrt{t+3}$
56. $k(t) = 3t^{2/3} - t$
57. $m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$
58. $n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

Extension

59. The graph of the function f is shown in Figure $1.4.2.4$. Based on the calculator screen shot, the point $(1.333, 5.185)$ is which of the following?
    a. a relative (local) maximum of the function
    b. the vertex of the function
    c. the absolute maximum of the function
    d. a zero of the function

Figure 1.4.2.4

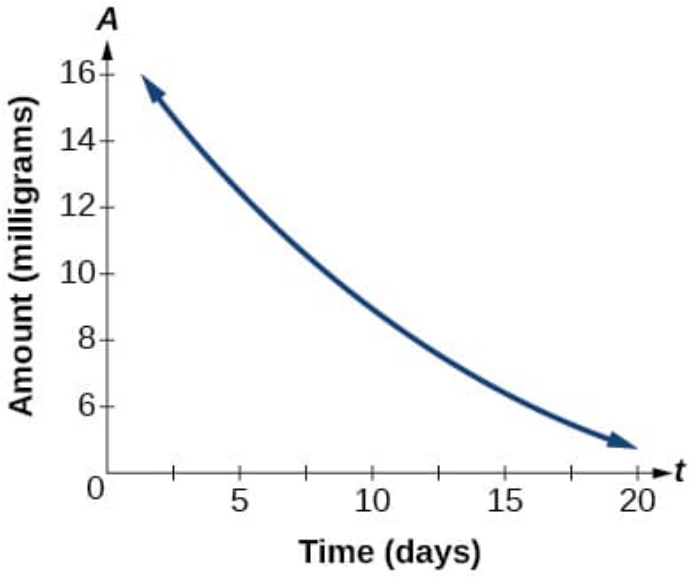


60. Let $f(x) = \frac{1}{x}$. Find a number c such that the average rate of change of the function f on the interval $(1, c)$ is $-\frac{1}{4}$.
61. Let $f(x) = \frac{1}{x}$. Find the number b such that the average rate of change of f on the interval $(2, b)$ is $-\frac{1}{10}$.

Real-World Applications

62. At the start of a trip, the odometer on a car read 21,395. At the end of the trip, 13.5 hours later, the odometer read 22,125. Assume the scale on the odometer is in miles. What is the average speed the car traveled during this trip?
63. A driver of a car stopped at a gas station to fill up his gas tank. He looked at his watch, and the time read exactly 3:40 p.m. At this time, he started pumping gas into the tank. At exactly 3:44, the tank was full and he noticed that he had pumped 10.7 gallons. What is the average rate of flow of the gasoline into the gas tank?
64. Near the surface of the moon, the distance that an object falls is a function of time. It is given by $d(t) = 2.6667t^2$, where t is in seconds and $d(t)$ is in feet. If an object is dropped from a certain height, find the average velocity of the object from $t = 1$ to $t = 2$.
65. The graph in Figure 1.4.2.5 illustrates the decay of a radioactive substance over t days.

Figure 1.4.2.5



Use the graph to estimate the average decay rate from $t = 5$ to $t = 15$.

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1.5: Transformations of Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure 1.5.1**:** *(credit: "Misko"/Flickr)*

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

Graphing Functions Using Vertical and Horizontal Shifts

Often when given a problem, we try to model the scenario using Mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest of these is a **vertical shift**, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See Figure 1.5.2 for an example.

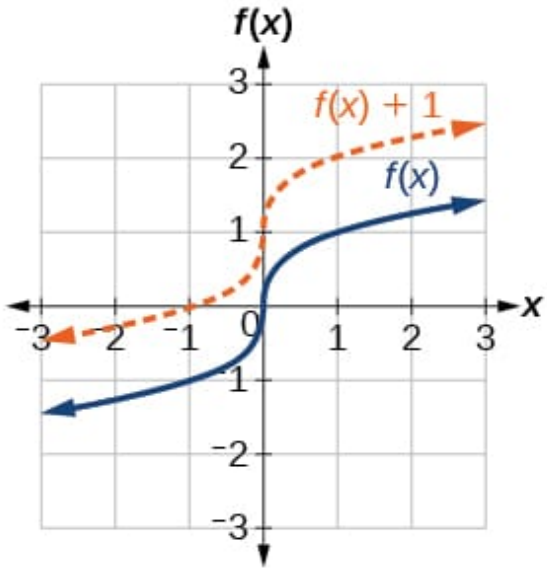


Figure 1.5.2**:** *Vertical shift by* $k = 1$ *of the cube root function* $f(x) = \sqrt[3]{x}$

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Theorem: Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a vertical shift of $f(x)$. All the output values change by k units. If k is positive, the graph will shift up by k units. If k is negative, the graph will shift down by $|k|$ units.

Example 1.5.1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 1.5.3 shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

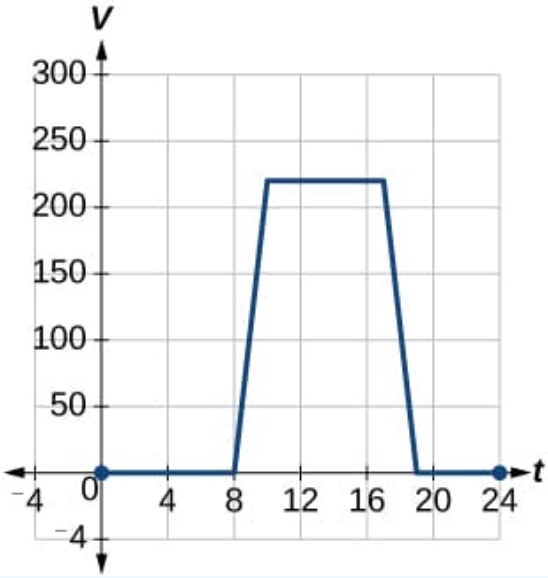


Figure 1.5.3

Solution available at the end of this section.

Example 1.5.2: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 1.5.2. Create a table for the function $g(x) = f(x) - 3$.

Table 1.5.2

x	2	4	6	8
$f(x)$	1	3	7	11

Solution available at the end of this section.

As with Example 1.5.1, the input values in Example 1.5.2 stay the same and only the output values change.

Checkpoint 1.5.2

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 1.5.5.

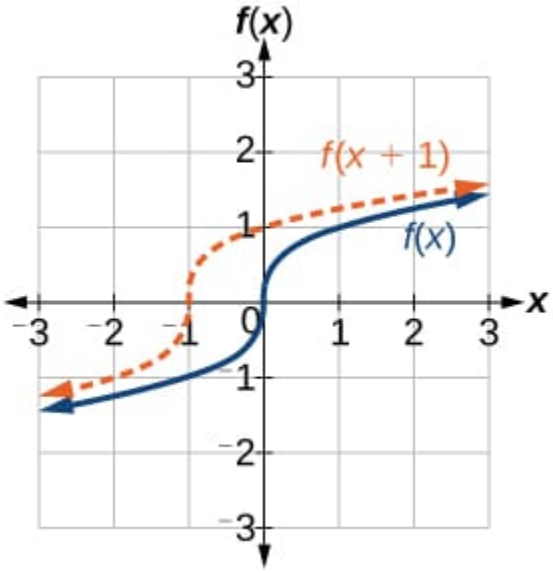


Figure 1.5.5**:** *Horizontal shift of the function $f(x) = x^3$. Note that $(x + 1)$ means $h = -1$ which shifts the graph to the left, that is, towards negative values of x.*

For example, if $f(x) = x^2$, then $g(x) = (x - 2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Theorem: Horizontal Shift

Given a function $f(x)$, a new function $g(x) = f(x - h)$, where h is a constant, is a horizontal shift of $f(x)$. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example 1.5.3: Adding a Constant to an Input

Returning to our building airflow example from Figure 1.5.3, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution available at the end of this section.

It is important to note that, in Example 1.5.3, $V(t + 2)$ has the effect of shifting the graph to the *left*.

Horizontal changes, or "inside changes," affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for $F(t) = V(t + 2)$.

Example 1.5.4: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 1.5.4. Create a table for the function $g(x) = f(x - 3)$.

Table 1.5.4

x	2	4	6	8
$f(x)$	1	3	7	11

Solution available at the end of this section.

Figure 1.5.7 represents both of the functions. We can see the horizontal shift in each point.

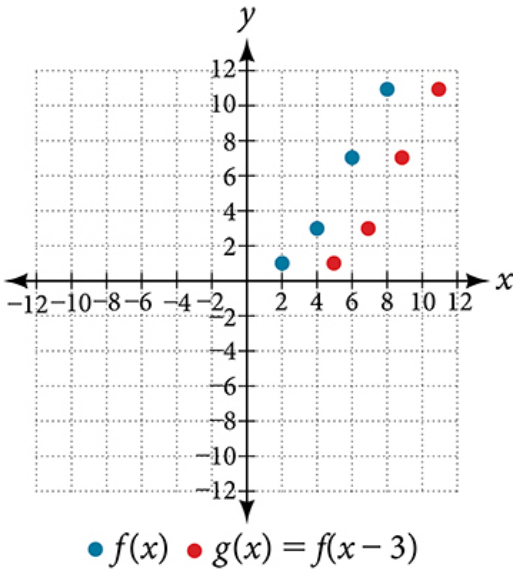


Figure 1.5.7

Example 1.5.5: Identifying a Horizontal Shift of a Toolkit Function

Figure 1.5.8 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

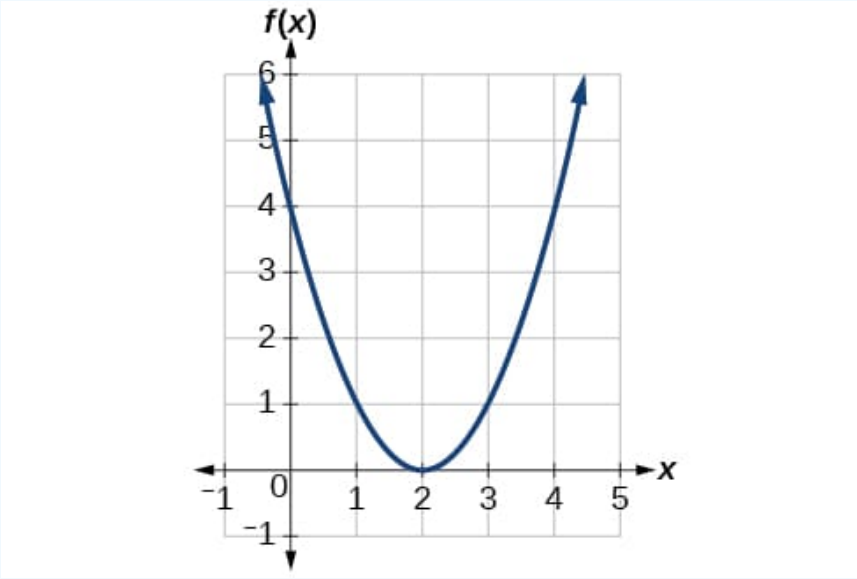


Figure 1.5.8

Solution available at the end of this section.

In Example 1.5.5, to determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the **vertex** point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x - 2) = f(0) = 0$. For this to work, we will need to *subtract* 2 units from our input values.

Example 1.5.6: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m) + 10$ and $G(m + 10)$.

Solution available at the end of this section.

Checkpoint 1.5.6

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output y-axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input x-axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down *and* right or left. Note that these shifts can be performed in either order.

Example 1.5.7: Graphing Combined Vertical and Horizontal Shifts

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x + 1) - 3$.

Solution available at the end of this section.

Checkpoint 1.5.7

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x - 2) + 4$.

Example 1.5.8: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure 1.5.11, which is a transformation of the toolkit square root function.

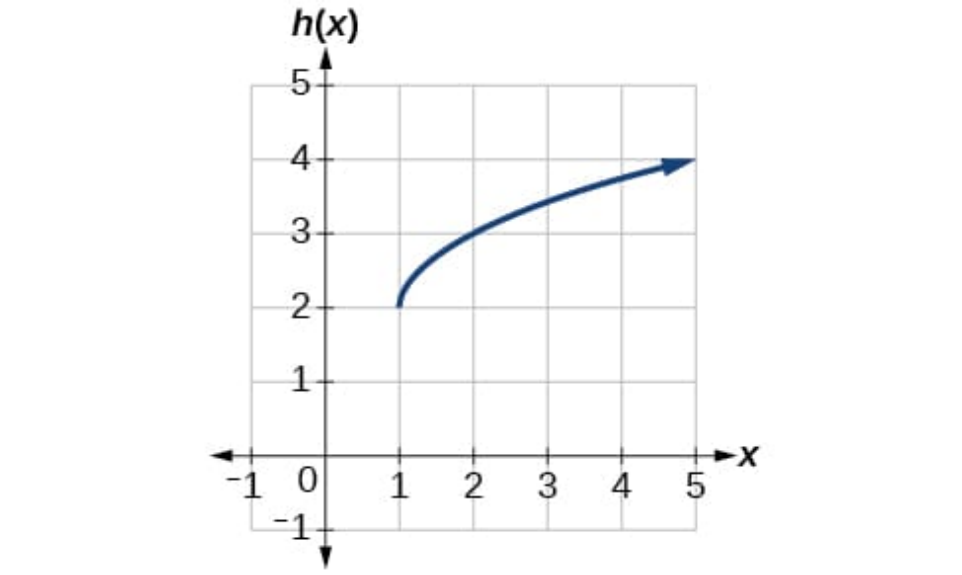


Figure 1.5.11

Solution available at the end of this section.

Note that the transformation in Example 1.5.8 has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

Checkpoint 1.5.8

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 1.5.12.

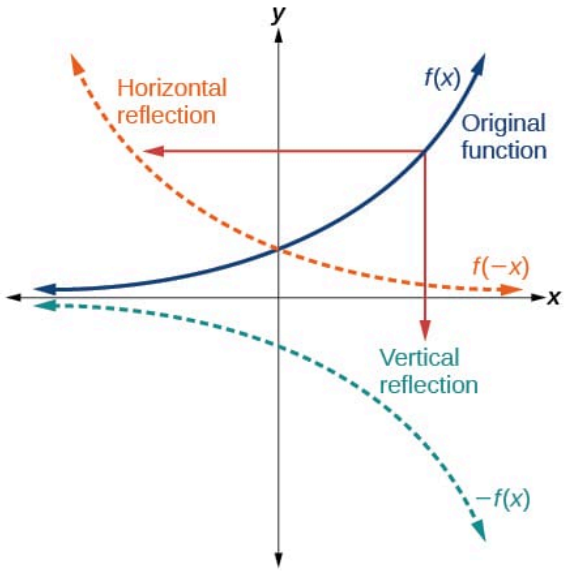


Figure 1.5.12**:** *Vertical and horizontal reflections of a function*

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

Theorem: Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a vertical reflection of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a horizontal reflection of the function $f(x)$, sometimes called a reflection about the y-axis.

Thus, we multiply all outputs by -1 for a vertical reflection, and we multiply all inputs by -1 for a horizontal reflection.

Example 1.5.9: Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution available at the end of this section.

Note that transformations like those in Example 1.5.9 can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

Checkpoint 1.5.9

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

Example 1.5.10: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table 1.5.6. Create a table for the functions below.

a. $g(x) = -f(x)$

b. $h(x) = f(-x)$

Table $1.5.6$

x	2	4	6	8
$f(x)$	1	3	7	11

Solution available at the end of this section.

Checkpoint $1.5.10$

A function $f(x)$ is given as Table $1.5.9$. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table $1.5.9$

x	-2	0	2	4
$f(x)$	5	10	15	20

Example $1.5.11$: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure $1.5.15$. Sketch a graph of $k(t)$.

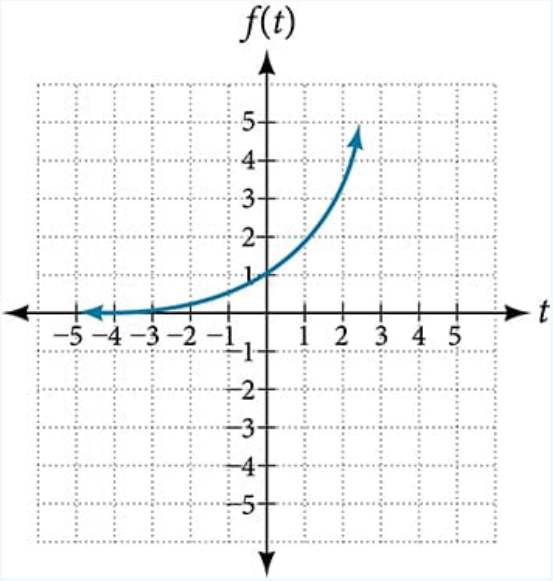


Figure $1.5.15$

Solution available at the end of this section.

As a model for learning, the function in Example $1.5.11$ would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

Checkpoint 1.5.11

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 1.5.17 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

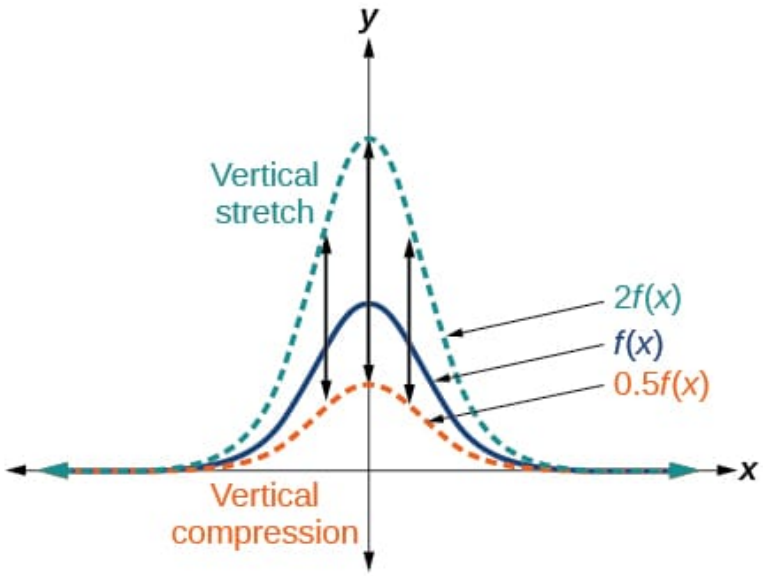


Figure 1.5.17: *Vertical stretch and compression*

Theorem: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = a\ f(x)$, where a is a constant, is a vertical stretch or vertical compression of the function $f(x)$.

- If $a > 1$, then the graph will be stretched vertically by a factor of a.
- If \(0
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

A good summary is that, if you have a function multiplied by a, you multiply all range values by a.

Example 1.5.12: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure 1.5.18.

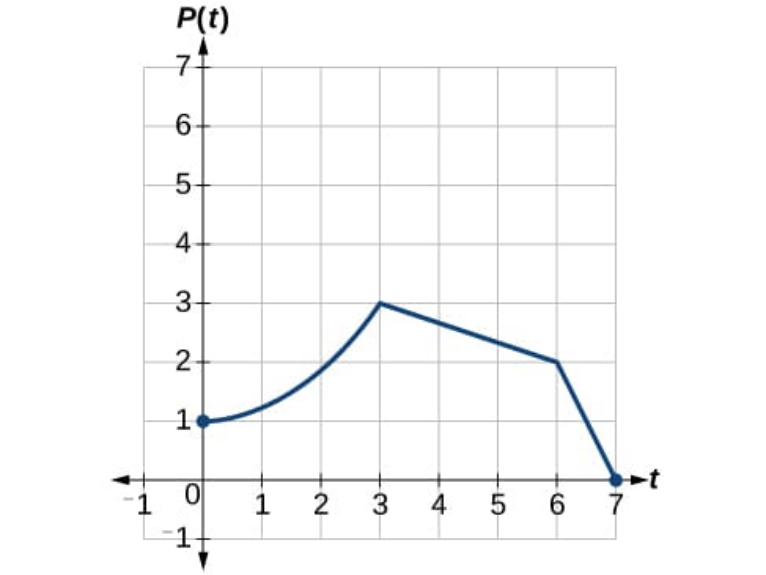


Figure 1.5.18

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution available at the end of this section.

Example 1.5.13: Finding a Vertical Compression of a Tabular Function

A function f is given as Table 1.5.10. Create a table for the function $g(x) = \frac{1}{2}f(x)$.

Table 1.5.10

x	2	4	6	8
$f(x)$	1	3	7	11

Solution available at the end of this section.

The result in Example 1.5.13 is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Checkpoint 1.5.13

A function f is given as Table 1.5.12. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

Table 1.5.12

x	2	4	6	8
$f(x)$	12	16	20	0

Example 1.5.14: Recognizing a Vertical Stretch

The graph in Figure 1.5.20 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

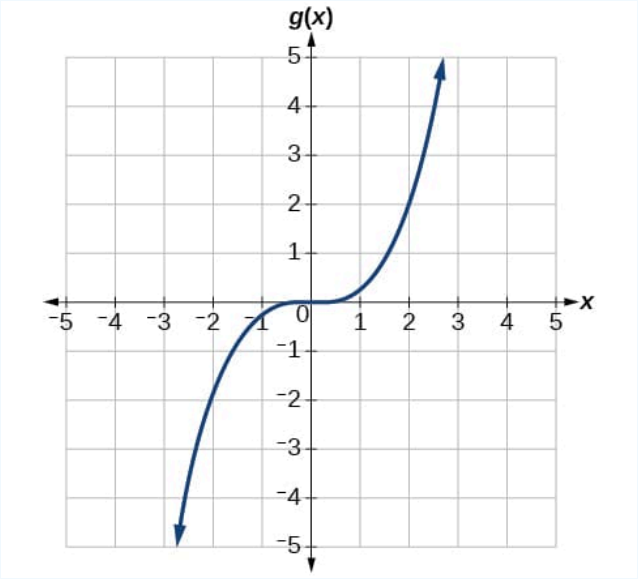


Figure 1.5.20

Solution available at the end of this section.

Checkpoint 1.5.14

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

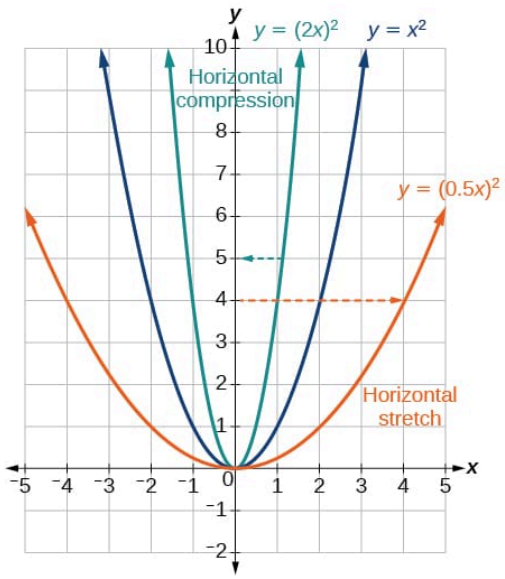


Figure 1.5.21

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 1.5.21. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of $\frac{1}{2}$.

Theorem: Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a horizontal stretch or horizontal compression of the function $f(x)$.

- If $b > 1$, then the graph will be compressed horizontally by a factor of $\frac{1}{b}$.
- If \(0
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

In effect, we multiply all x-values by $\frac{1}{b}$.

Example 1.5.15: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution available at the end of this section.

Note that, in Example 1.5.15, the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 1.5.16: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 1.5.13. Create a table for the function $g(x) = f\left(\frac{1}{2}x\right)$.

Table 1.5.13

x	2	4	6	8
$f(x)$	1	3	7	11

Solution available at the end of this section.

Because each input value has been doubled, the result in Example 1.5.16 is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Example 1.5.17: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 1.5.24.

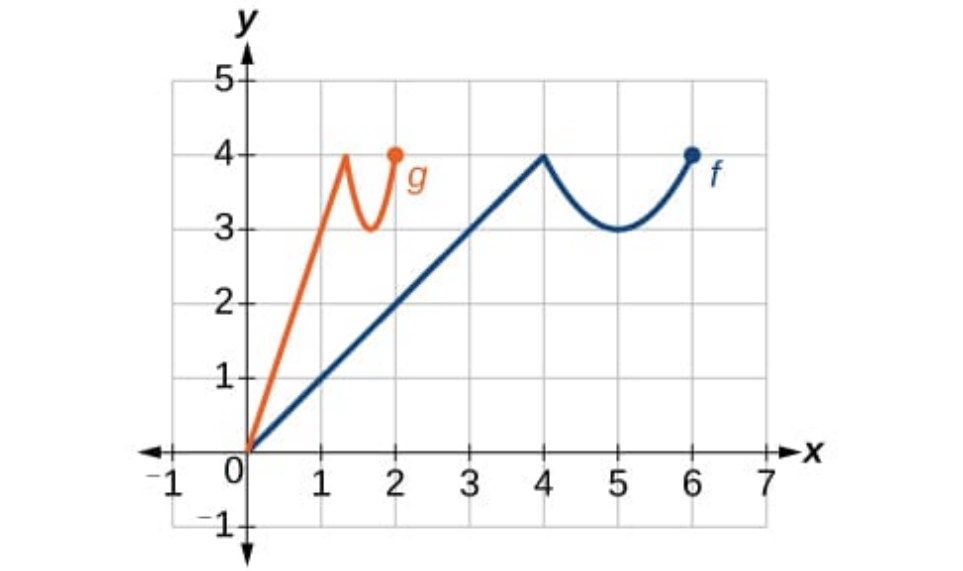


Figure 1.5.24

Solution available at the end of this section.

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f\left(\frac{1}{4}\right)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Checkpoint 1.5.17

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Performing a Sequence of Transformations

When combining transformations, it is *very* important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x) + 3$, which transformation should we start with? The answer here follows nicely from the

Order of Operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x + 3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7) = 12$. What input to g would produce that output? In other words, what value of x will allow $g(x) = f(2x + 3) = 12$? We would need $2x + 3 = 7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f\left(bx + p\right) = f\left(b\left(x + \frac{p}{b}\right)\right).$$

Let's work through an example. If

$$f(x) = (2x + 4)^2,$$

then we can factor out a 2 to get

$$f(x) = (2(x + 2))^2.$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Interactive Element: Horizontal Stretch, Compression, and Shift

The following Interactive Element might help you visualize how the coefficient b and the shift h act together in $g(x) = f\left(b(x - h)\right)$.

Interact: Drag the red point along the x-axis to change the shift h; drag the green handle to change the horizontal factor b.

Observation: The curve's start always sits at $x = h$ no matter what b does, and $b > 1$ squeezes the graph *toward* the line $x = h$ while $0 < b < 1$ spreads it *away*—the opposite of what the number b might suggest.

Combining Transformations

Given a function $g(x)$ of the form

$$g(x) = af(bx + p) + k,$$

where the base graph of $y = f(x)$ is known, we factor b from the argument $bx + p$ to get $b\left(x + \frac{p}{b}\right)$. This gives

$$g(x) = af(bx + p) + k = af\left(b\left(x + \frac{p}{b}\right)\right) + k.$$

With the function rewritten in this specific order, we perform transformations on the graph of f in the order written - from left to right - to arrive at the graph of $g(x)$.

If your original function is not in the form $af(bx + p) + k$, then you will have to "precondition" the function using arithmetic. For example,

$$g(x) = -2 - \frac{1}{3}f\left(7x - 8\right) = -\frac{1}{3}f\left(7x - 8\right) - 2 = -\frac{1}{3}f\left(7\left(x - \frac{8}{7}\right)\right) - 2.$$

Example 1.5.18: Finding a Triple Transformation of a Tabular Function

Given Table 1.5.15 for the function $f(x)$, create a table of values for the function $g(x) = 2f(3x) + 1$.

Table 1.5.15

x	6	12	18	24

$f(x)$	10	14	15	17

Solution available at the end of this section.

Example 1.5.19: Graphing a Function with Many Transformations

Sketch a graph of the function

$$f(x) = -4 - \frac{1}{2}\sqrt{12 - 3x}.$$

Solution available at the end of this section.

Example 1.5.20: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 1.5.30 to sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3$.

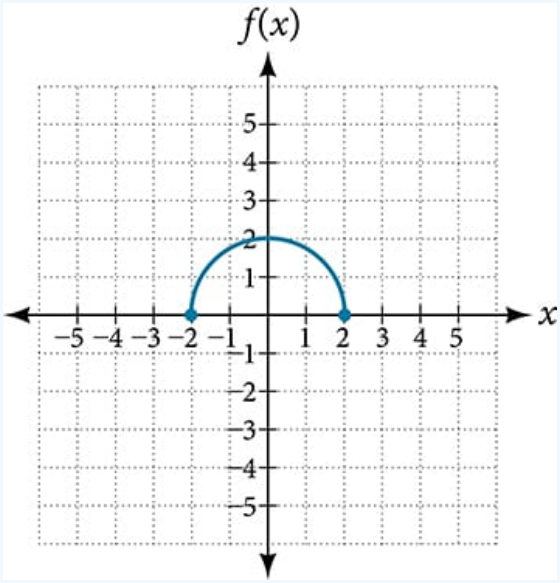


Figure 1.5.30

Solution available at the end of this section.

Hidden Resources: Tutoring Services

Technically, this is not a "hidden resource," per se, but the tutoring services at CRC are a bit spread out between programs. We have the Tutoring Center, Math Center at CRC, Math Center at Elk Grove, Reading and Writing Center, Accounting Lab, Embedded Tutoring (Supplemental Instruction), and NetTutor. If you're an athlete, you might have additional access to tutors through the athletics program. Needless to say, being overwhelmed by the number of overlapping tutoring services is completely understandable. Luckily, CRC has a single webpage with all of the links to these services. Check out the details on this CRC webpage.

MyOpenMath Advice: Latepasses and File Uploads

Sometimes, life can get overwhelmingly busy, and missing a due date happens. In those instances, you can use a Latepass to

extend the original due date of the homework (see the syllabus for how long a single Latepass extends the original due date by).

Warning: Hand-Written Solutions

Read the syllabus to find out what happens to the questions that must be graded by hand (file uploads) - most professors are not likely to grade the late submitted hand-written homework solutions due to time constraints and fairness to other students.

Answers

Section Preview

This is the typical section on transformations of functions; however, it is important that students get a strong exposure to understanding horizontal stretches and compressions that are combined with horizontal shifts (everything is assumed to have been seen before).

Section Preview

- **Arithmetic**
  - Multiplying and Dividing Fractions: Computing scaled outputs requires fraction arithmetic: $g(x) = \frac{1}{2}f(x)$ producing $\frac{3}{2}, \frac{7}{2}$ (Example 13), the $\frac{1}{4}$ stretch factor (Example 14), and the reciprocal coefficients used for horizontal stretches/compressions (Examples 16–17). (Used in a handful of specific examples rather than throughout.)

Section Preview

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Graph functions using compressions and stretches.
- Combine transformations.

Example 1.5.1: Adding a Constant to a Function

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 1.5.4.

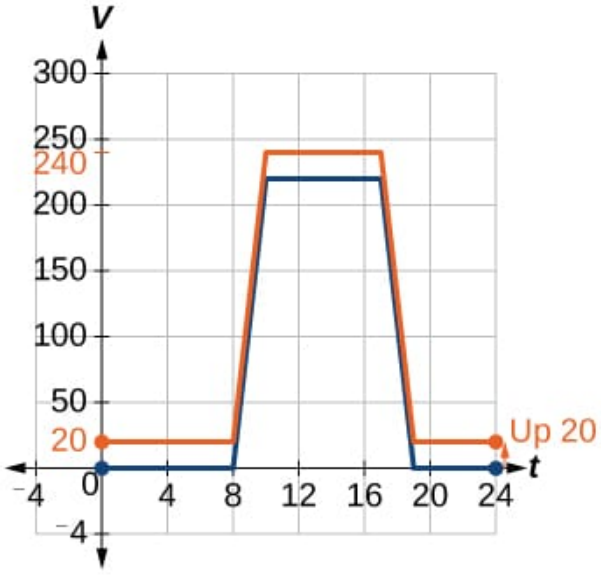


Figure 1.5.4

Notice that in Figure 1.5.4, for each input value, the output value has increased by 20, so if we call the new function $S(t)$, we could write

$$S(t) = V(t) + 20.$$

This notation tells us that, for any value of t, $S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 1.5.1.

***Table* 1.5.1**

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

Example 1.5.2: Shifting a Tabular Function Vertically

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example, we know (from Table 1.5.2)

that $f(2) = 1$, and we were given $g(x) = f(x) - 3$. Therefore,

$$\begin{aligned} g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 1.5.3.

Table 1.5.3

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	-2	0	4	8

Example 1.5.3: Adding a Constant to an Input

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$\begin{aligned} V(t) &= \text{the original venting plan} \\ F(t) &= \text{starting 2 hours sooner} \end{aligned}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$ (see Figure 1.5.6). Notice also that the vents first opened to 220 ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220 ft^2 at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t - (-2)) = V(t + 2)$.

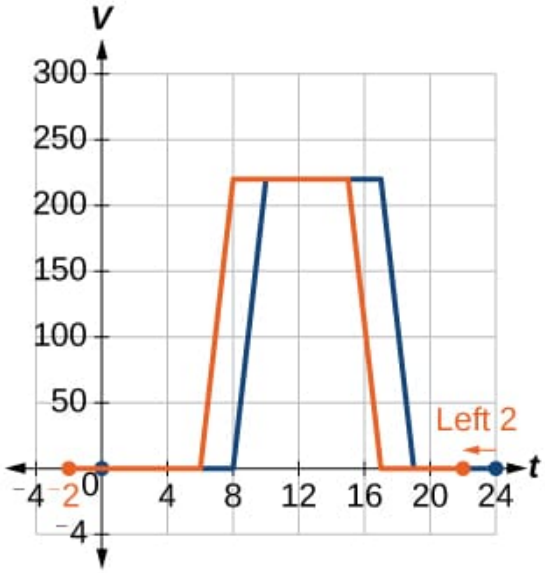


Figure 1.5.6

Example 1.5.4: Shifting a Tabular Function Horizontally

The formula $g(x) = f(x - 3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 *larger*. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5 - 3) \\ &= f(2) \end{aligned}$$

We continue with the other values to create Table 1.5.5.

Table 1.5.5

x	5	7	9	11
$x - 3$	2	4	6	8
$f(x - 3)$	1	3	7	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Example $1.5.5$**: Identifying a Horizontal Shift of a Toolkit Function**

Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0, 0)$, but now the vertex is at $(2, 0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x - 2).$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x - 2)$. That is, since $f(x) = x^2$, $g(x) = f(x - 2)$. Therefore,

$$g(x) = f(x - 2) = (x - 2)^2.$$

Example $1.5.6$**: Interpreting Horizontal versus Vertical Shifts**

$G(m) + 10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m + 10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Example $1.5.7$**: Graphing Combined Vertical and Horizontal Shifts**

The function f is our toolkit absolute value function. We know that this graph has a "V" shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x + 1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x + 1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure $1.5.9$.

Let us follow one point of the graph of $f(x) = |x|$.

- The point $(0, 0)$ is transformed first by shifting left 1 unit: $(0, 0) \to (-1, 0)$
- The point $(-1, 0)$ is transformed next by shifting down 3 units: $(-1, 0) \to (-1, -3)$

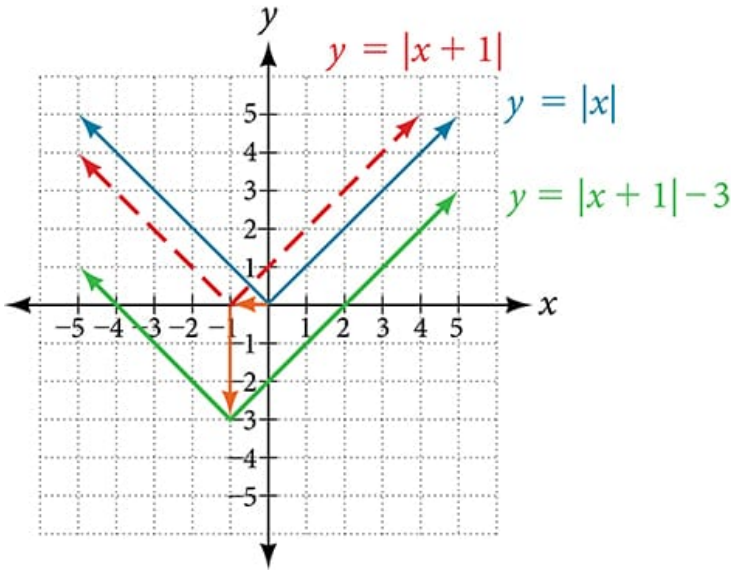


Figure $1.5.9$

Figure $1.5.10$ shows the graph of h.

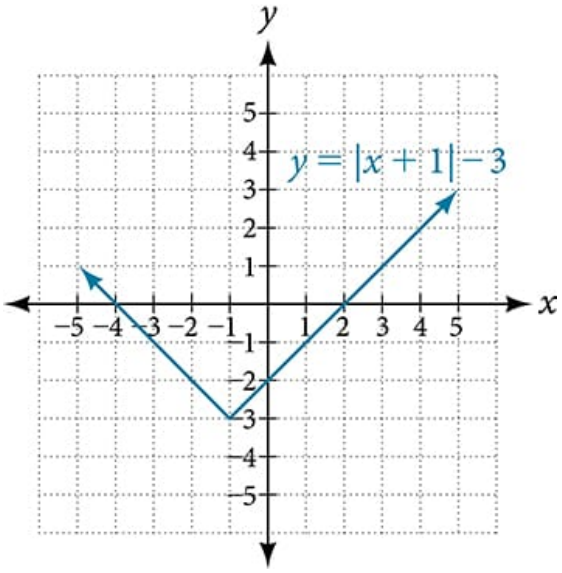


Figure $1.5.10$

Example 1.5.8**: Identifying Combined Vertical and Horizontal Shifts**

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1) + 2.$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1} + 2.$$

Example 1.5.9**: Reflecting a Graph Horizontally and Vertically**

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure 1.5.13.

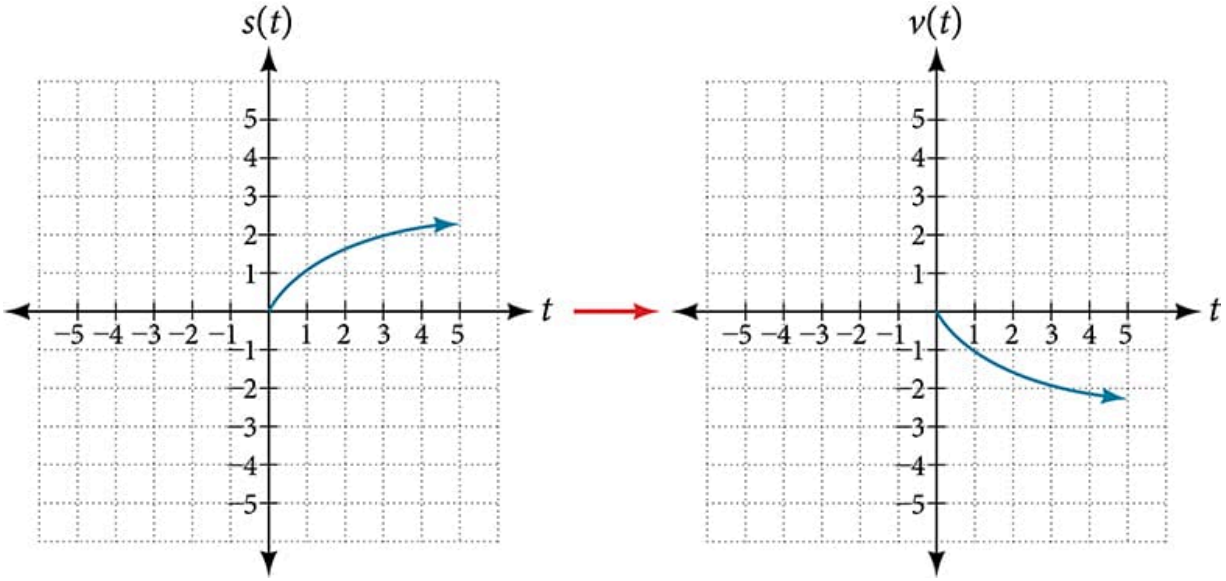


Figure 1.5.13**:** *Vertical reflection of the square root function*

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}.$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 1.5.14.

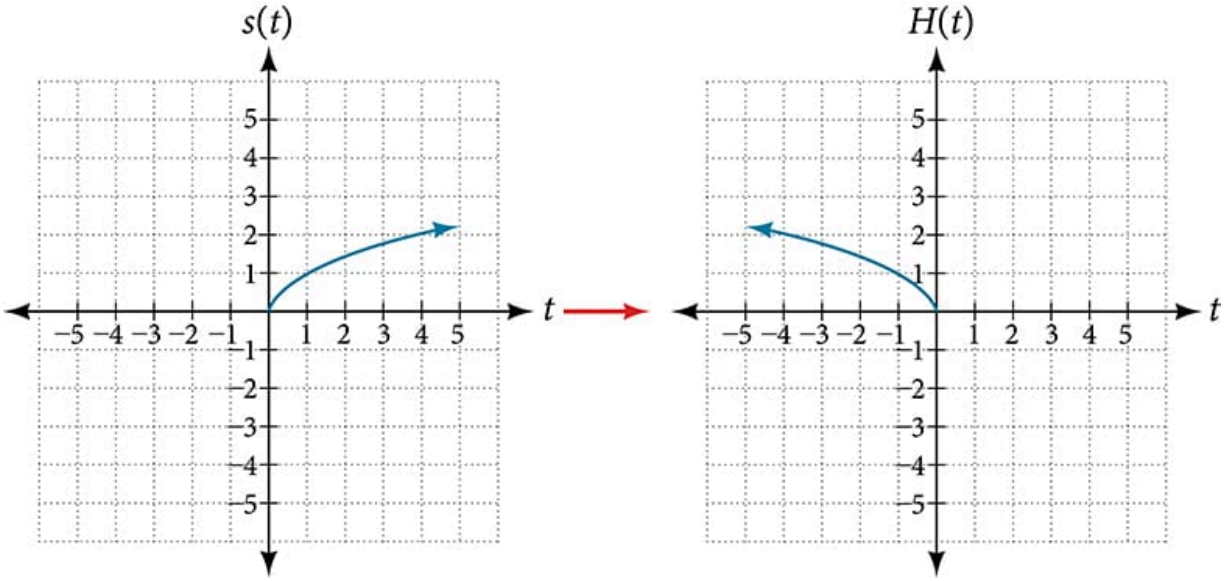


Figure 1.5.14**:** *Horizontal reflection of the square root function*

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}.$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Example 1.5.10**: Reflecting a Tabular Function Horizontally and Vertically**

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 1.5.7.

Table 1.5.7

x	2	4	6	8
$g(x)$	-1	-3	-7	-11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 1.5.8.

Table 1.5.8

x	-2	-4	-6	-8
$h(x)$	1	3	7	11

Example 1.5.11**: Applying a Learning Model Equation**

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$
- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t) + 1 = -2^{-t} + 1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points $(0, 1)$ and $(1, 2)$.

1. First, we apply a horizontal reflection: $(0, 1) \to (0, 1)$ and $(1, 2) \to (-1, 2)$.
2. Then, we apply a vertical reflection: $(0, 1) \to (0, -1)$ and $(-1, 2) \to (-1, -2)$.
3. Finally, we apply a vertical shift: $(0, -1) \to (0, 0)$ and $(-1, -2) \to (-1, -1)$.

This means that the original points, $(0, 1)$ and $(1, 2)$ become $(0, 0)$ and $(-1, -1)$ after we apply the transformations.

In Figure 1.5.16, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

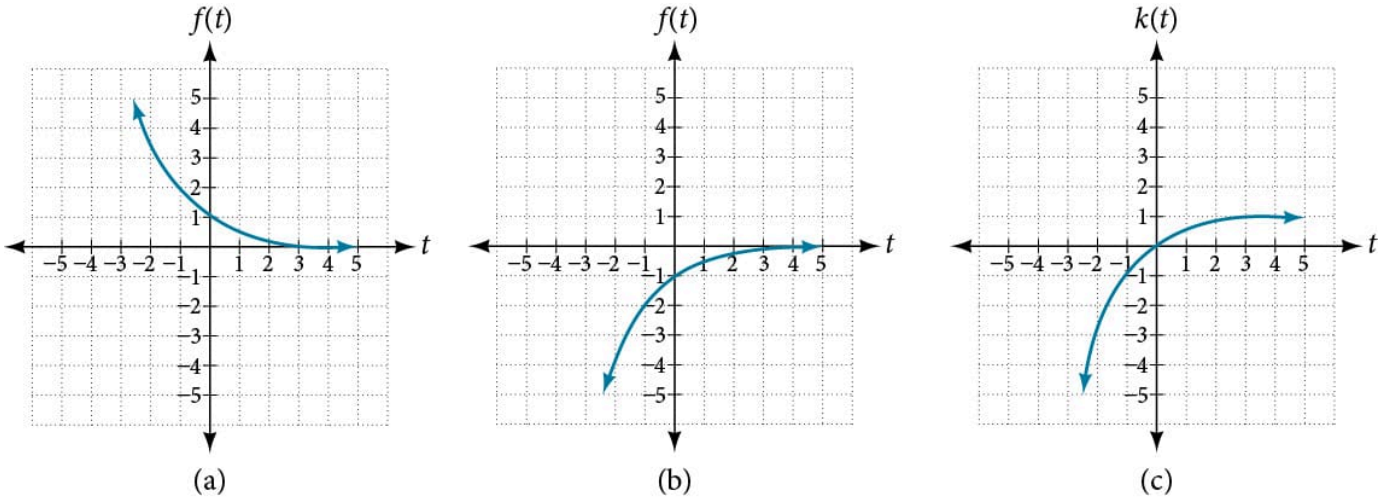


Figure 1.5.16

Example 1.5.12**: Graphing a Vertical Stretch**

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure 1.5.19.

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$, and $(7, 0)$, we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \to (0, 2)$$
$$(3, 3) \to (3, 6)$$
$$(6, 2) \to (6, 4)$$
$$(7, 0) \to (7, 0)$$

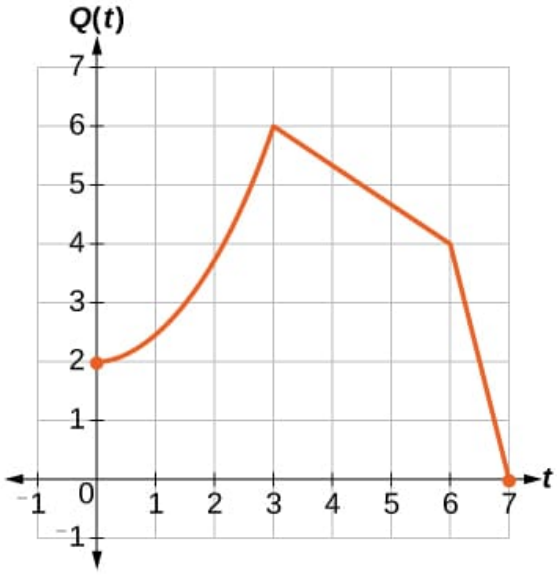


Figure 1.5.19

Symbolically, the relationship is written as

$$Q(t) = 2P(t).$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

Example 1.5.13**: Finding a Vertical Compression of a Tabular Function**

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}.$$

We do the same for the other values to produce Table 1.5.11.

Table 1.5.11

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Example 1.5.14**: Recognizing a Vertical Stretch**

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can (somewhat) safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3.$$

Example 1.5.15**: Graphing a Horizontal Compression**

Symbolically, we could write

$$R(1) = P(2),$$
$$R(2) = P(4),$$
$$\text{and in general,}$$
$$R(t) = P(2t).$$

See Figure 1.5.22 for a graphical comparison of the original population and the compressed population.

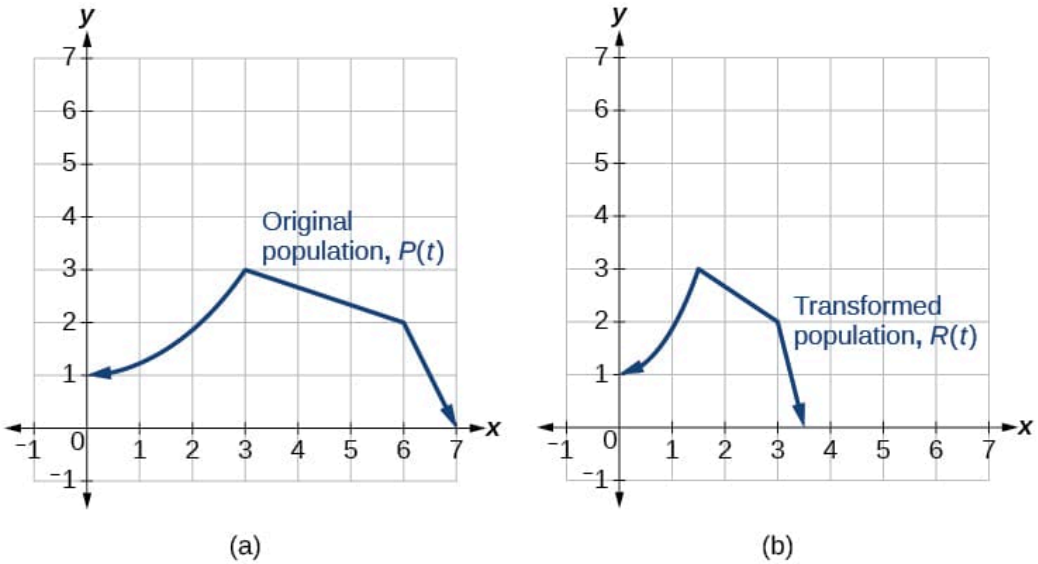


Figure 1.5.22**:** *(a) Original population graph (b) Compressed population graph*

Example 1.5.16**: Finding a Horizontal Stretch for a Tabular Function**

The formula $g(x) = f\left(\frac{1}{2}x\right)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f\left(\frac{1}{2} \cdot 2\right) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f\left(\frac{1}{2} \cdot 4\right) = f(2) = 1.$$

We do the same for the other values to produce Table 1.5.14.

Table 1.5.14

x	4	8	12	16
$g(x)$	1	3	7	11

Figure 1.5.23 shows the graphs of both of these sets of points.

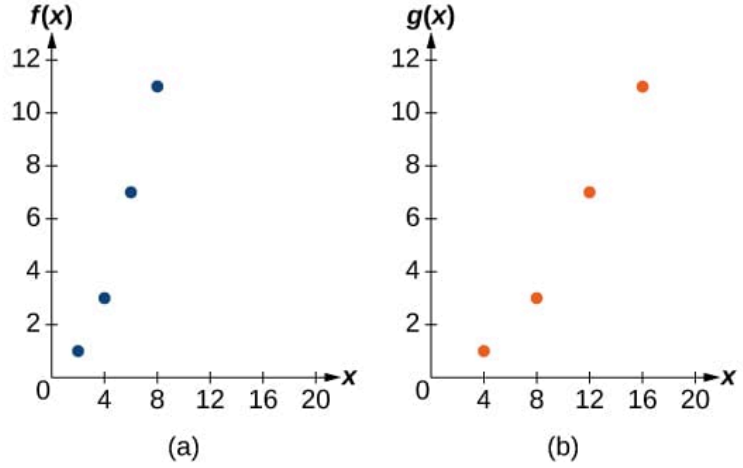


Figure 1.5.23

Example 1.5.17**: Recognizing a Horizontal Compression on a Graph**

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at $(6, 4)$ and $g(x)$ ends at $(2, 4)$, we can see that the x-values have been compressed by $\frac{1}{3}$, because $6\left(\frac{1}{3}\right) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Example 1.5.18**: Finding a Triple Transformation of a Tabular Function**

Using the advice given previously, we make sure the function is written in the form:

$$af(bx + p) + k$$

In this case, we were given $g(x) = 2f(3x) + 1$, which is already in this form. We then start applying the transformations by reading from left to right.

We start with the vertical stretch, which will multiply the output values by 2. Table 1.5.16.

Table 1.5.16

x	6	12	18	24
$2f(x)$	20	28	30	34

We then move to the right in the function $g(x) = 2f(3x) + 1$ and deal with the 3. This causes a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$. We apply these transformations to the x-values from Table 1.5.16 to arrive at Table 1.5.17.

Table 1.5.17

x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 1.5.18.

Table 1.5.18

x	2	4	6	8
$g(x) = 2f(3x) + 1$	21	29	31	35

Example 1.5.19**: Graphing a Function with Many Transformations**

The function given to us needs to be rewritten in the form $af(bx + p) + k = af\left(b\left(x + \frac{p}{b}\right)\right) + k$. Let's start by simply rearranging terms.

$$f(x) = -4 - \frac{1}{2}\sqrt{12 - 3x} = -\frac{1}{2}\sqrt{-3x + 12} - 4.$$

We now take the time to factor the lead coefficient from both terms inside the function.

$$f(x) = -\frac{1}{2}\sqrt{-3\,(x - 4)} - 4.$$

The parent function is $y = \sqrt{x}$. It's best to label two "anchor" points (which will almost always be $(0, 0)$ and $(1, 1)$ for the Toolkit functions).

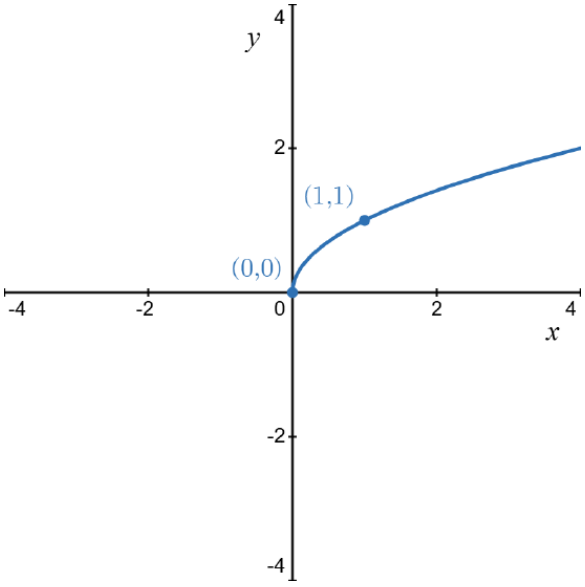


Figure 1.5.25

Reading our "cleaned" function from left to right, the coefficient $-\frac{1}{2}$ reflects the graph of $y = \sqrt{x}$ vertically about the x-axis and compresses it vertically (squishing towards the x-axis) by a factor of $\frac{1}{2}$. Another way to think of this is that all of the y-values on the graph of $y = \sqrt{x}$ are multiplied by $-\frac{1}{2}$.

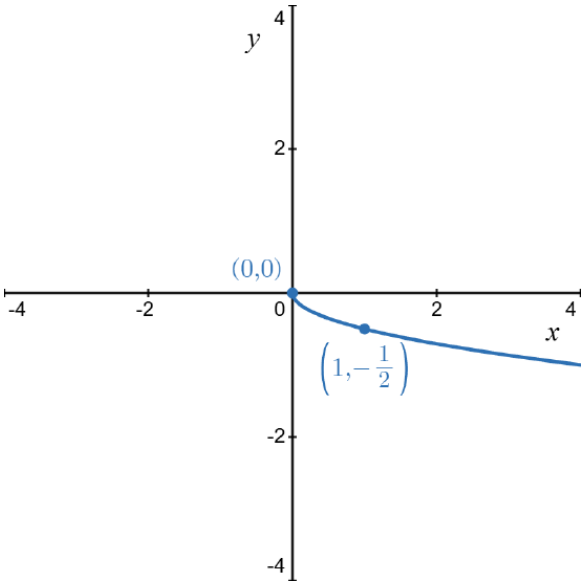


Figure 1.5.26

Continuing reading from left to right, we have the factor of -3 on the *inside* of the function. Remember, anything on the inside of the function affects the x-values (and in an opposite way from how they affect the outside of the function). Specifically, the factor of -3 reflects the graph of $y=-\frac{1}{2}\sqrt{x}$ about the y-axis and *compresses* the graph horizontally (squishing it) towards the y-axis by a factor of $\frac{1}{3}$. Another way to think of this is that the factor -3 *divides* all of the x-values on the graph of $y=-\frac{1}{2}\sqrt{x}$ by -3.

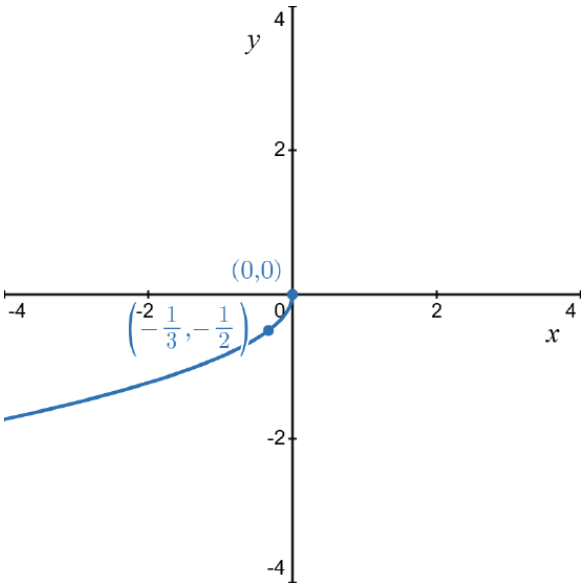


Figure 1.5.27

Continuing reading our original function from left to right, we have $(x-4)$ on the *inside* of the function. This causes a shift of 4 units horizontally to the right for the graph of $y=-\frac{1}{2}\sqrt{3x}$. Another way to think of this is that we *add* 4 units to each of the x-values.

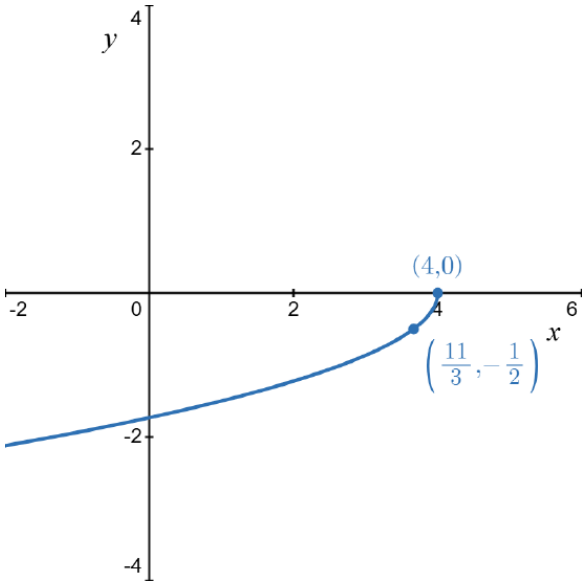


Figure 1.5.28

Finally, we have the subtraction of 4 with which to contend. This subtract occurs *outside* the function (the square root), so it shifts the graph of $y=-\frac{1}{2}\sqrt{3(x-4)}$ vertically *down* 4 units. Our final graph is given below.

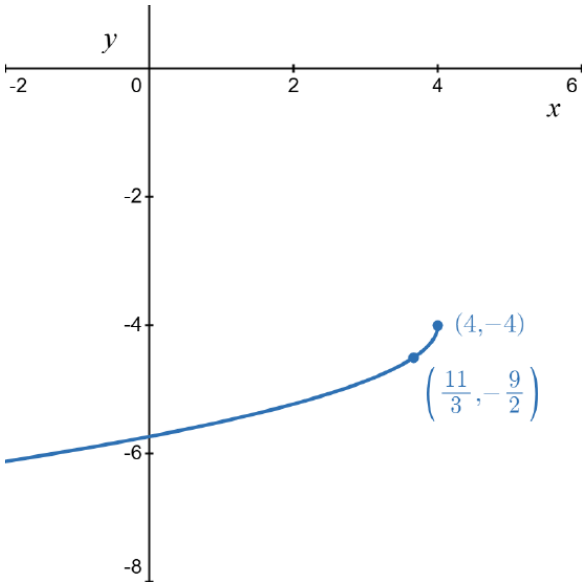


Figure 1.5.29

Example 1.5.20**: Finding a Triple Transformation of a Graph**

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3.$$

Reading from left to right, we first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0,2)$ remains at $(0,2)$ while the point $(2,0)$ will stretch to $(4,0)$. See Figure 1.5.31.

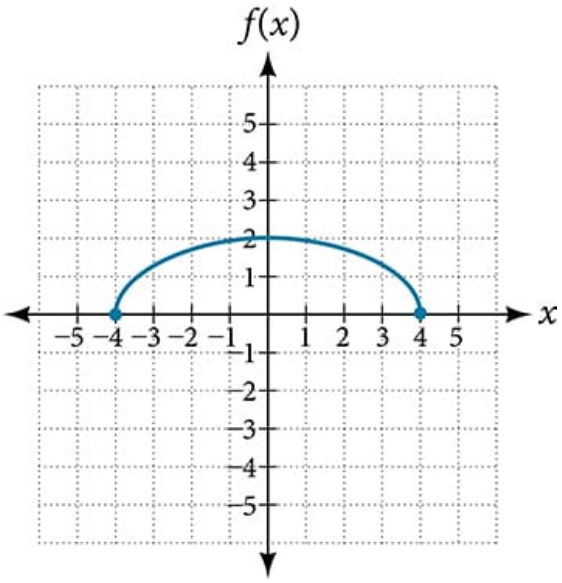


Figure 1.5.31

Next, we horizontally shift left by 2 units, as indicated by $x + 2$. See Figure 1.5.32.

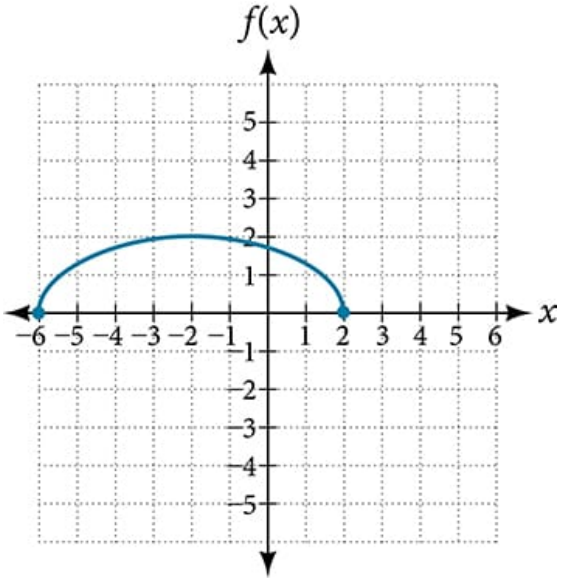


Figure 1.5.32

Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See Figure 1.5.33.

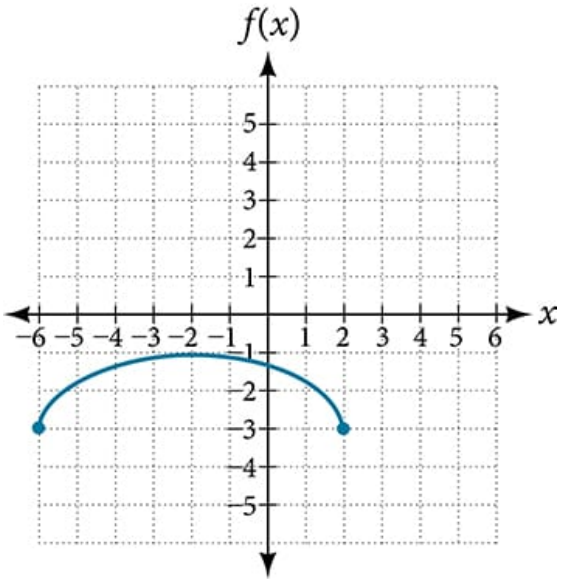


Figure 1.5.33

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1.5.1: Resources and Key Concepts

Resources

Videos

- Transformations of Functions
  - Functions - Graphs - Rigid Transformations - Horizontal and Vertical Shifts
  - Functions - Graphs - Rigid Transformations - Horizontal and Vertical Reflections
  - Functions - Graphs - Non-Rigid Transformations - Stretches and Compressions
  - Functions - Graphs - Performing a Sequence of Rigid and Non-Rigid Transformations (Standard)
  - Functions - Graphs - Sequences of Rigid and Non-Rigid Transformations on Non-Standard Functions
  - Intermediate Algebra - Functions: Transformations (older video)

Key Concepts

Definitions

- **Vertical Shift:** A transformation that moves the graph of a function up or down by adding a constant k to the output: $g(x) = f(x) + k$. If $k > 0$, the shift is upward; if $k < 0$, the shift is downward.
- **Horizontal Shift:** A transformation that moves the graph of a function left or right by adding or subtracting a constant h from the input: $g(x) = f(x - h)$. If $h > 0$, the shift is to the right; if $h < 0$ (e.g., $f(x + c)$ where $c > 0$), the shift is to the left.
- **Vertical Reflection:** A transformation that reflects the graph of a function vertically across the x-axis, given by $g(x) = -f(x)$.
- **Horizontal Reflection:** A transformation that reflects the graph of a function horizontally across the y-axis, given by $g(x) = f(-x)$.
- **Vertical Stretch:** A transformation where the graph of a function is stretched away from the x-axis, caused by multiplying the output by a constant a where $|a| > 1$. If $a > 1$, it's a pure stretch.
- **Vertical Compression:** A transformation where the graph of a function is compressed toward the x-axis, caused by multiplying the output by a constant a where $0 < |a| < 1$. If $0 < a < 1$, it's a pure compression.
- **Horizontal Stretch:** A transformation where the graph of a function is stretched away from the y-axis, caused by multiplying the input by a constant b where $0 < |b| < 1$. If $0 < b < 1$, it's a pure stretch (e.g., $f(\frac{1}{2}x)$).
- **Horizontal Compression:** A transformation where the graph of a function is compressed toward the y-axis, caused by multiplying the input by a constant b where $|b| > 1$. If $b > 1$, it's a pure compression (e.g., $f(2x)$).

Theorems

- **Vertical Shift:** Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a vertical shift of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.
- **Horizontal Shift:** Given a function $f(x)$, a new function $g(x) = f(x - h)$, where h is a constant, is a horizontal shift of $f(x)$. If h is positive, the graph will shift right. If h is negative, the graph will shift left.
- **Reflections:**
  - Given a function $f(x)$, a new function $g(x) = -f(x)$ is a vertical reflection of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.
  - Given a function $f(x)$, a new function $g(x) = f(-x)$ is a horizontal reflection of the function $f(x)$, sometimes called a reflection about the y-axis
- **Vertical Stretches and Compressions:** Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a vertical stretch or vertical compression of the function $f(x)$.
  - If $a > 1$, then the graph will be stretched vertically by a factor of a.
  - If $0 < a < 1$, then the graph will be compressed vertically by a factor of a.
  - If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection
- **Horizontal Stretches and Compressions:** Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a horizontal stretch or horizontal compression of the function $f(x)$.

- If $b > 1$, then the graph will be compressed horizontally by a factor of $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched horizontally by a factor of $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

Common Mistakes

- **Counterintuitive Nature of Horizontal Shifts:** Horizontal changes often seem counterintuitive. For example, $f(x - h)$ shifts right if $h > 0$, and $f(x + h)$ (which is $f(x - (-h))$) shifts left if $h > 0$.
- **Order of Transformations:** When combining multiple transformations, the order in which they are applied is very important, especially when mixing shifts with stretches/compressions. For horizontal transformations, factoring inside the function (e.g., writing $f(b(x + p))$) can make the order easier to manage.
- **Misinterpreting Coefficients for Horizontal Stretch/Compression:** The coefficient b in $f(bx)$ leads to a stretch/compression by a factor of $\frac{1}{b}$, not b. For example, $f(2x)$ is a horizontal compression by a factor of $\frac{1}{2}$, and $f(\frac{1}{2}x)$ is a horizontal stretch by a factor of 2.

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1.5.2: Homework

Reading Questions

1. How does the transformation $g(x) = f(x) + 5$ affect the graph of $f(x)$?
2. If the graph of $f(x)$ is shifted 3 units to the left, what is the equation of the new function in terms of $f(x)$?
3. What is the difference in the graphical effect of $g(x) = -f(x)$ versus $h(x) = f(-x)$?
4. If $f(x) = x^2$, describe the transformation that results in $g(x) = 3x^2$. Is this a stretch or a compression? In which direction?
5. If $f(x) = \sqrt{x}$, describe the transformation that results in $g(x) = \sqrt{2x}$. Is this a stretch or a compression? In which direction? By what factor?
6. When transforming a function using the form $y = af(bx + p) + k$, why is it recommended to first factor out b from the term $(bx + p)$?
7. Explain why $f(x - 2)$ shifts the graph of $f(x)$ to the right, even though a subtraction is involved.
8. If you are given the graph of $y = f(x)$ and want to graph $y = 2f(x) - 3$, describe the sequence of transformations you would apply.
9. What is the domain and range of $g(t) = \sqrt{-t}$ if the original function is $s(t) = \sqrt{t}$ with domain $[0, \infty)$ and range $[0, \infty)$?
10. Consider the function $k(t) = -2^{-t} + 1$. Identify the base function and list the transformations applied to it in the correct order to obtain $k(t)$.
11. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?
12. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?
13. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?
14. When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the x-axis from a reflection with respect to the y-axis?

Homework

Algebraic

15. Write a formula for the function obtained when the graph of $f(x) = \sqrt{x}$ is shifted up 1 unit and to the left 2 units.
16. Write a formula for the function obtained when the graph of $f(x) = |x|$ is shifted down 3 units and to the right 1 unit.
17. Write a formula for the function obtained when the graph of $f(x) = \frac{1}{x}$ is shifted down 4 units and to the right 3 units.
18. Write a formula for the function obtained when the graph of $f(x) = \frac{1}{x^2}$ is shifted up 2 units and to the left 4 units.

For the following exercises, describe how the graph of the function is a transformation of the graph of the original function f.

19. $y = f(x - 49)$
20. $y = f(x + 43)$
21. $y = f(x + 3)$
22. $y = f(x - 4)$
23. $y = f(x) + 5$
24. $y = f(x) + 8$
25. $y = f(x) - 2$
26. $y = f(x) - 7$
27. $y = f(x - 2) + 3$
28. $y = f(x + 4) - 1$

For the following exercises, determine the interval(s) on which the function is increasing and decreasing.

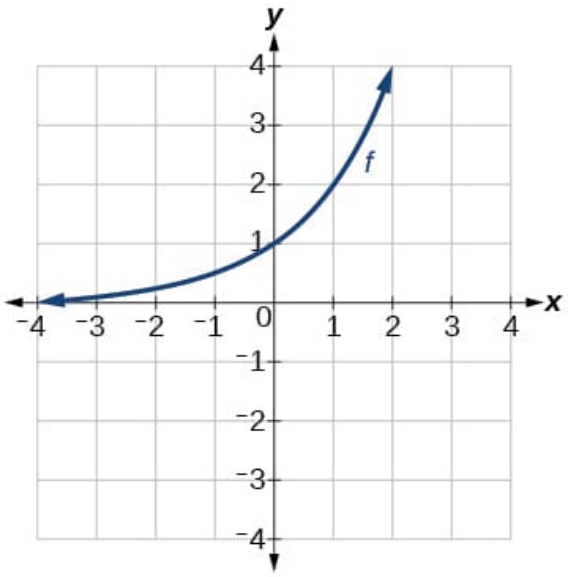
29. $f(x) = 4(x + 1)^2 - 5$

30. $g(x) = 5(x+3)^2 - 2$
31. $a(x) = \sqrt{-x+4}$
32. $k(x) = -3\sqrt{x} - 1$

Graphical

For the following exercises, use the graph of $f(x) = 2^x$ shown in Figure 1.5.2.1 to sketch a graph of each transformation of $f(x)$.

Figure 1.5.2.1



33. $g(x) = 2^x + 1$
34. $h(x) = 2^x - 3$
35. $w(x) = 2^{x-1}$

For the following exercises, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

36. $f(t) = (t+1)^2 - 3$
37. $h(x) = |x-1| + 4$
38. $k(x) = (x-2)^3 - 1$
39. $m(t) = 3 + \sqrt{t+2}$

Numeric

40. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	–2	–1	0	1	2
$f(x)$	–2	–1	–3	1	2

x	–1	0	1	2	3
$g(x)$	–2	–1	–3	1	2

x	–2	–1	0	1	2
$h(x)$	–1	0	–2	2	3

41. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

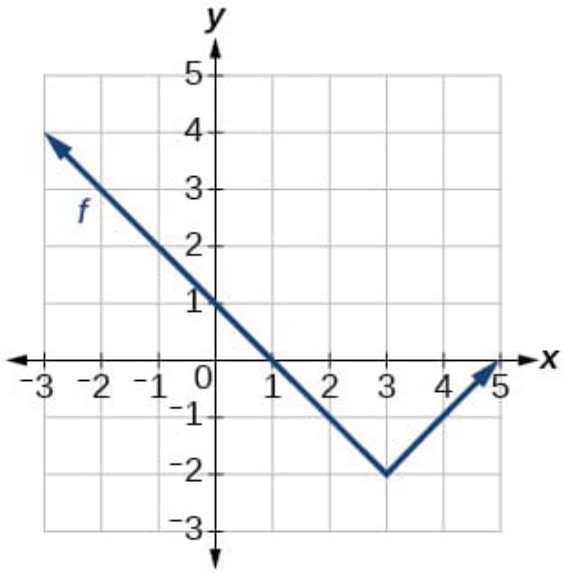
x	–2	–1	0	1	2
$f(x)$	–1	–3	4	2	1

x	–3	–2	–1	0	1
$g(x)$	–1	–3	4	2	1

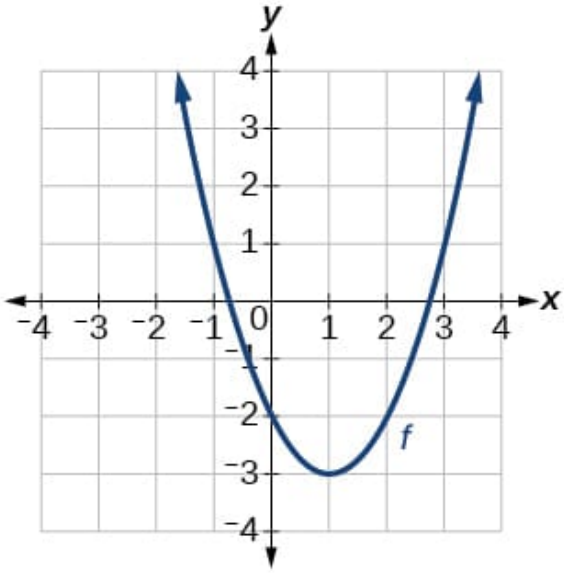
x	–2	–1	0	1	2
$h(x)$	–2	–4	3	1	0

For the following exercises, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.

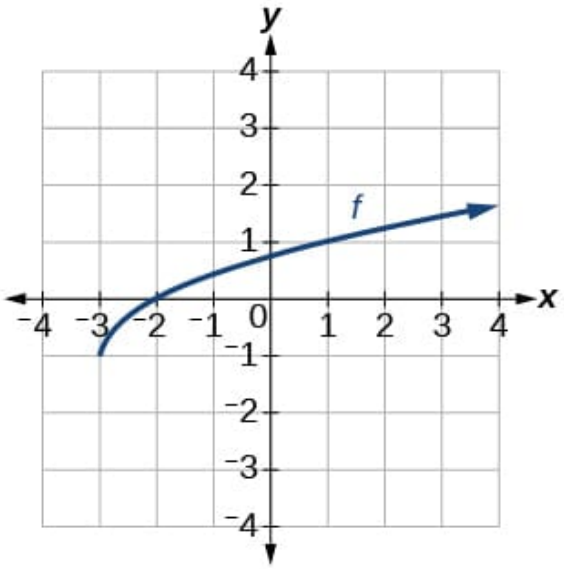
42.



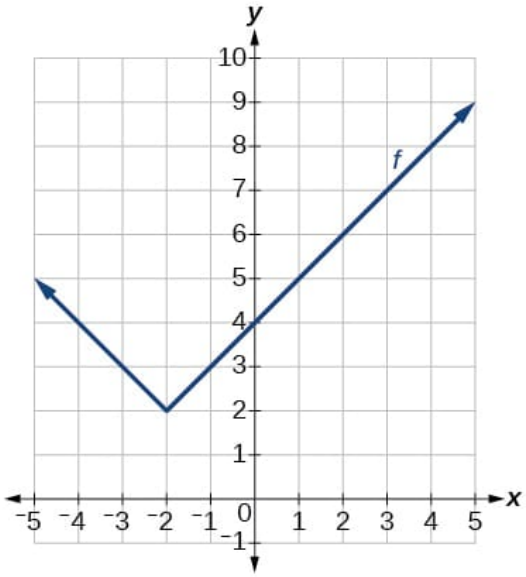
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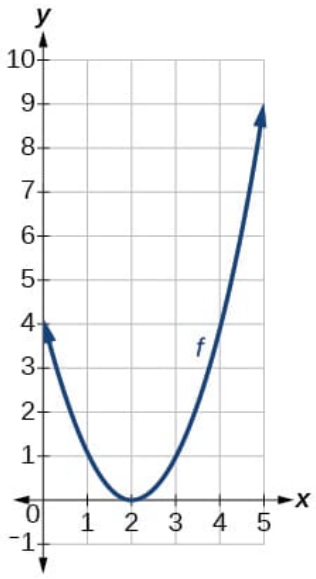
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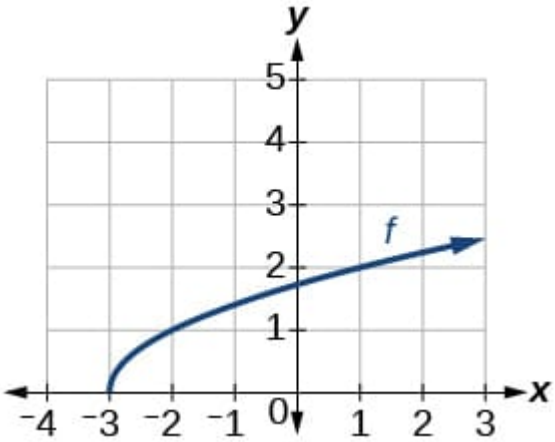
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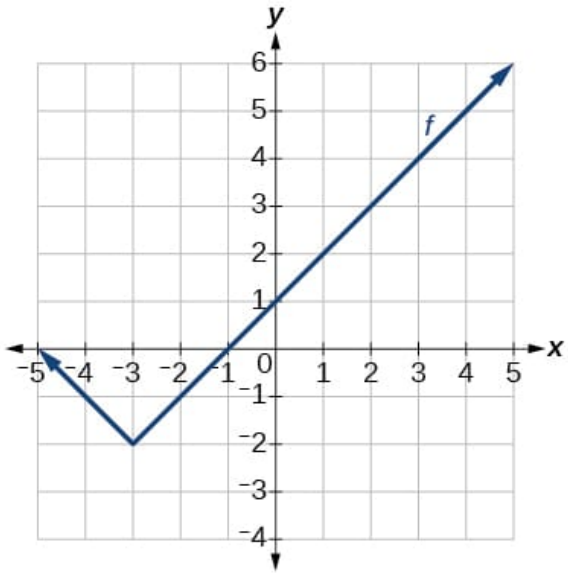
16.


y
10
9
8
7
6
5
4
3
2
1
0
-1
1
2
3
4
5
x
f

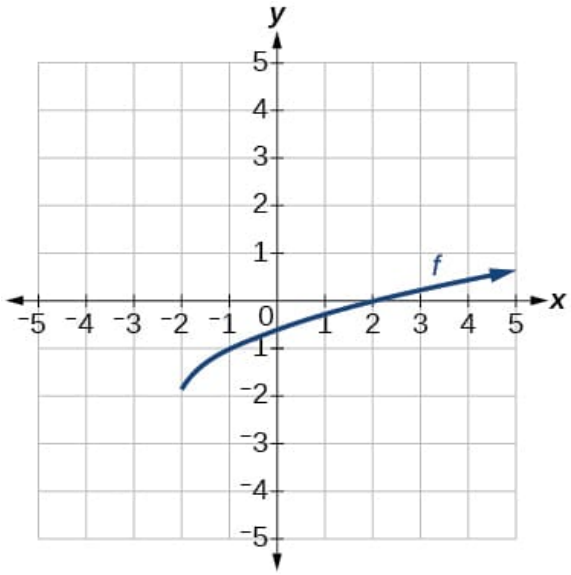
47.


y
5
4
3
2
1
0
-4
-3
-2
-1
1
2
3
x
f

48.

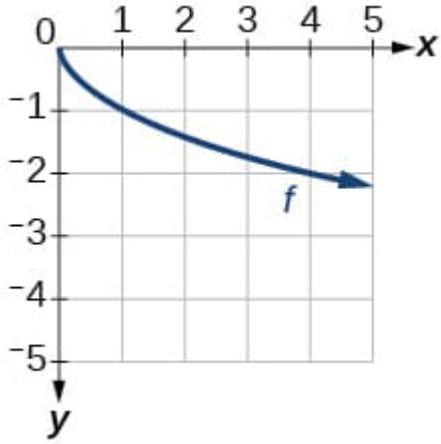

y
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-4
-3
-2
-1
1
2
3
4
5
x
f

49.

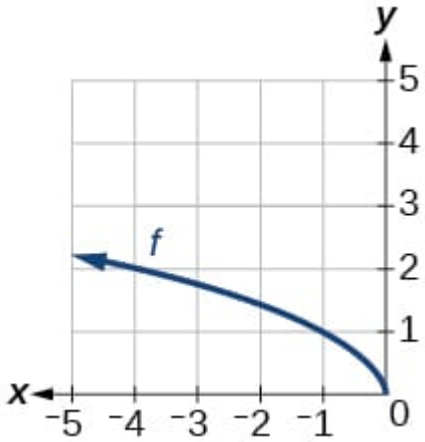


For the following exercises, use the graphs of transformations of the square root function to find a formula for each of the functions.

50.

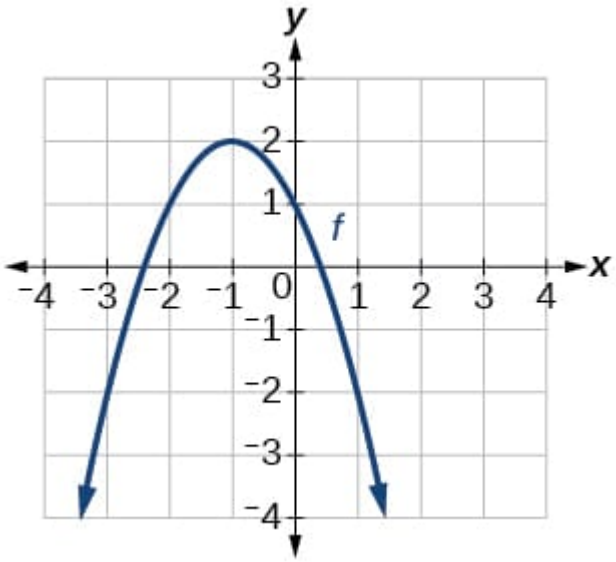


51.

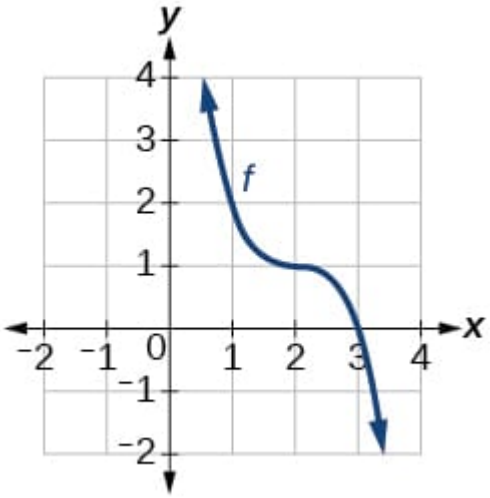


For the following exercises, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

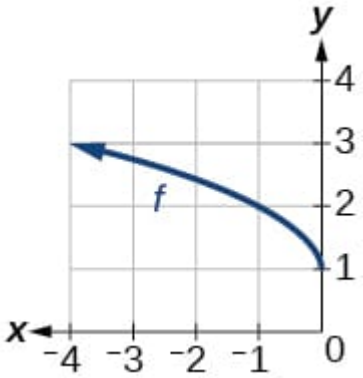
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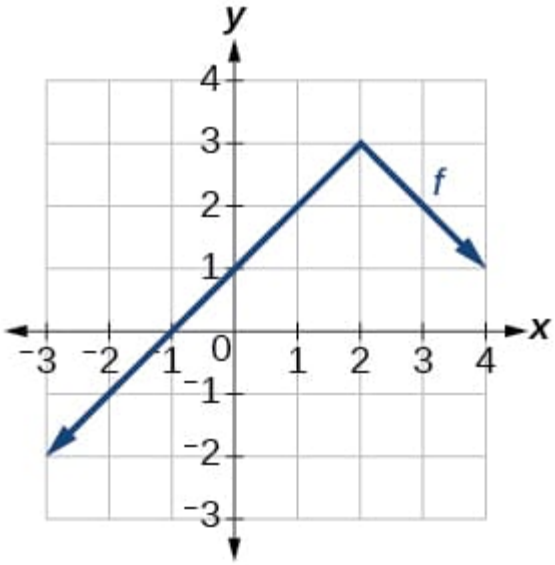
53.



54.



55.



For the following exercises, describe how the graph of each function is a transformation of the graph of the original function f.

56. $g(x) = -f(x)$
57. $g(x) = f(-x)$
58. $g(x) = 4f(x)$
59. $g(x) = 6f(x)$
60. $g(x) = f(5x)$
61. $g(x) = f(2x)$
62. $g(x) = f\left(\frac{1}{3}x\right)$
63. $g(x) = f\left(\frac{1}{5}x\right)$
64. $g(x) = 3f(-x)$
65. $g(x) = -f(3x)$

For the following exercises, write a formula for the function g that results when the graph of a given toolkit function is transformed as described.

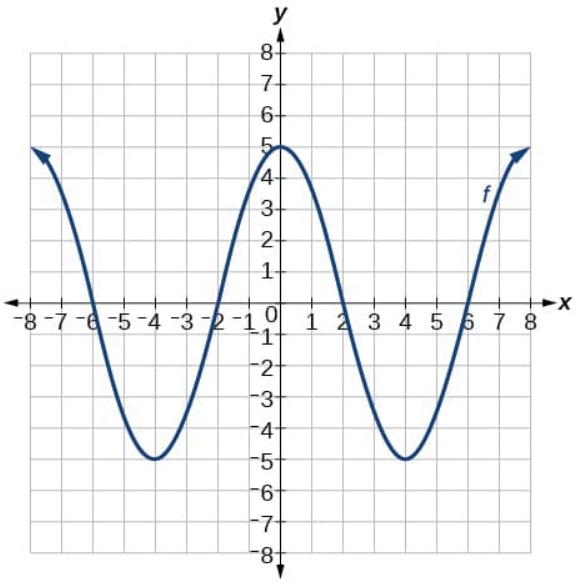
66. The graph of $f(x) = |x|$ is reflected over the y-axis and horizontally compressed by a factor of $\frac{1}{4}$.
67. The graph of $f(x) = \sqrt{x}$ is reflected over the x-axis and horizontally stretched by a factor of 2.
68. The graph of $f(x) = \frac{1}{x^2}$ is vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.
69. The graph of $f(x) = \frac{1}{x}$ is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.
70. The graph of $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$, then shifted to the right 5 units and up 1 unit.
71. The graph of $f(x) = x^2$ is horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

For the following exercises, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

72. $g(x) = 4(x+1)^2 - 5$
73. $g(x) = 5(x+3)^2 - 2$
74. $h(x) = -2|x-4| + 3$
75. $k(x) = -3\sqrt{x} - 1$
76. $m(x) = \frac{1}{2}x^3$
77. $n(x) = \frac{1}{3}|x-2|$
78. $p(x) = \left(\frac{1}{3}x\right)^3 - 3$
79. $q(x) = \left(\frac{1}{4}x\right)^3 + 1$
80. $a(x) = \sqrt{-x+4}$

For the following exercises, use the graph in Figure $1.5.2.2$ to sketch the given transformations.

Figure 1.5.2.2



81. $g(x) = f(x) - 2$
32. $g(x) = -f(x)$
33. $g(x) = f(x + 1)$
34. $g(x) = f(x - 2)$

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1.6: Absolute Value Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

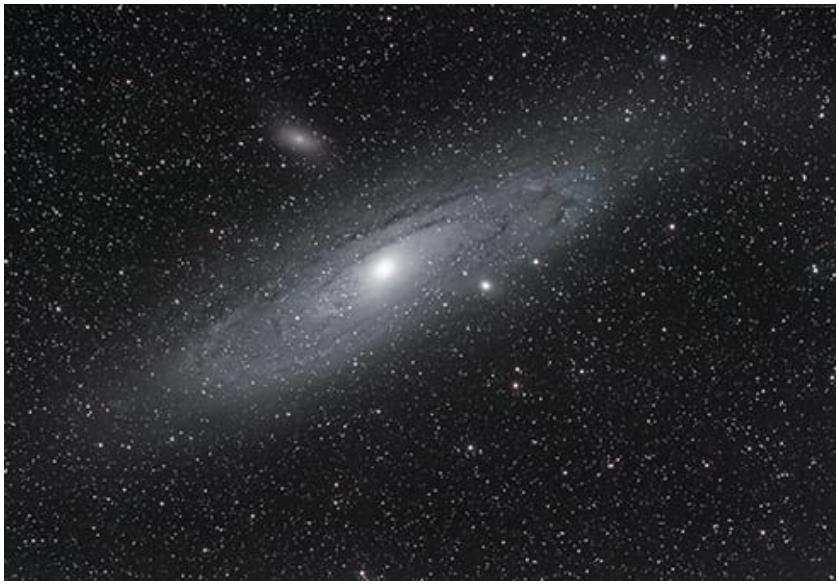


Figure 1.6.1**:** *Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)*

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate absolute value functions.

Focus on Calculus - Piecewise Definition of the Absolute Value

Recall that in its basic form $f(x) = |x|$, the absolute value function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

Definition: Absolute Value Function

The absolute value function is defined to be $f(x) = |x|$, where

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

Example 1.6.1: Determine a Number within a Prescribed Distance

Describe all values x within or including a distance of 4 from the number 5.

Solution available at the end of this section.

Note that

$$\begin{aligned} -4 &\leq x-5 & \qquad x-5 &\leq 4 \\ 1 &\leq x & \qquad x &\leq 9. \end{aligned}$$

So $|x-5| \leq 4$ is equivalent to $1 \leq x \leq 9$.

However, mathematicians generally prefer absolute value notation.

Checkpoint 1.6.1

Describe all values x within a distance of 3 from the number 2.

Example 1.6.2: Resistance of a Resistor

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often $\pm 1\%$, $\pm 5\%$, or $\pm 10\%$.

Suppose we have a resistor rated at 680 ohms, $\pm 5\%$. Use the absolute value function to express the range of possible values of the actual resistance.

Solution available at the end of this section.

Checkpoint 1.6.2

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

A theme of this textbook is to loop back to previous concepts that constantly reappear in Calculus. The difference quotient is one such concept.

Example 1.6.3: Revisiting the Difference Quotient

Find, and simplify, the difference quotient for $f(x) = -\frac{1}{2}|4x-5|+3$ assuming that $x < \frac{5}{4}$ and $h < 0$.

Solution available at the end of this section.

Checkpoint 1.6.3

Find, and simplify, the difference quotient for $f(x) = -\frac{1}{2}|4x-5|+3$ assuming that $x > \frac{5}{4}$ and $h > 0$.

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in Figure 1.6.3.

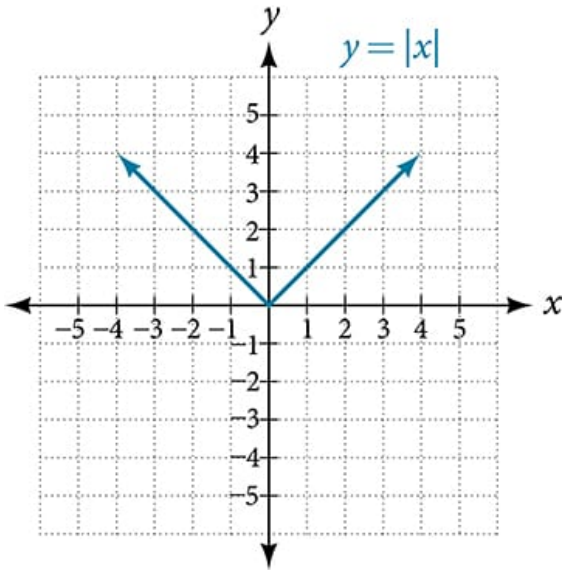


Figure 1.6.3

Figure 1.6.4 shows the graph of $y = 2|x - 3| + 4$. The graph of $y = |x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3, 4)$ for this transformed function.

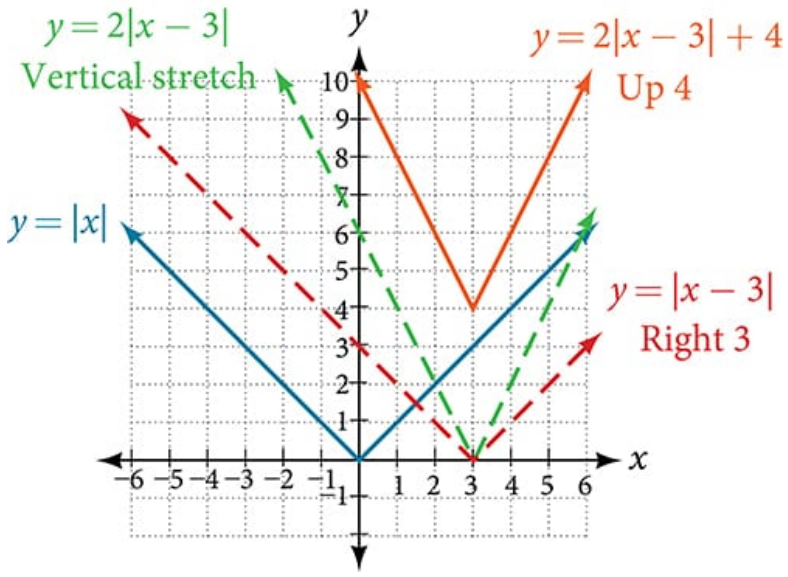


Figure 1.6.4

Interactive Element: Transformations of the Absolute Value Function

The following Interactive Element might help you visualize how the numbers a, h, and k in $y = a|x - h| + k$ shift, stretch, and reflect the basic graph of $|x|$.

Interact: Drag the red corner point to reposition the vertex (h, k); drag the green handle on the left up or down to change the stretch a (moving it below the axis flips the graph).

Observation: The vertex of $y = a|x - h| + k$ always lands exactly at (h, k); larger $|a|$ makes the V steeper, and a negative a reflects it downward.

Example 1.6.4: Writing an Equation for an Absolute Value Function

Write an equation for the function graphed in Figure 1.6.5.

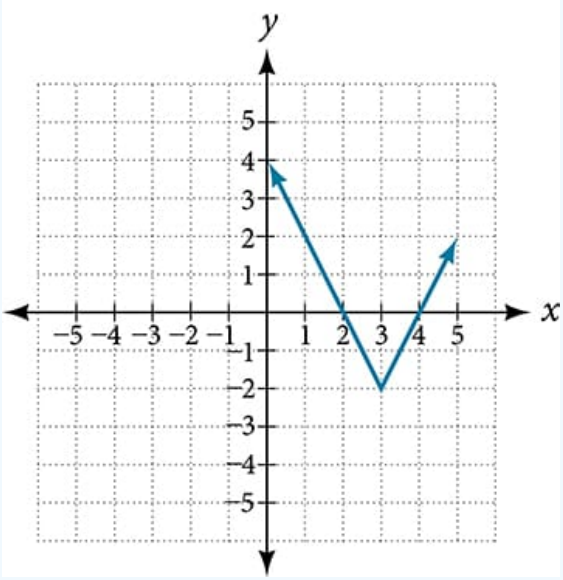


Figure 1.6.5

Solution available at the end of this section.

In Example 1.6.4, we could have opted to treat the stretch as a horizontal compression. In that case, the function would have been $f(x) = |2(x - 3)| - 2$. Note that this equation is algebraically equivalent to what we arrived at in our solution. This result, that the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression, is not a rule for all functions—it just happens to work for absolute value (and linear) functions. Note also that if the vertical stretch factor is negative, there is also a reflection about the x-axis.

Q&A

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for x and $f(x)$.

$$f(x) = a|x - 3| - 2.$$

Now substituting in the point $(1, 2)$, we get

$$\begin{aligned} 2 &= a|1 - 3| - 2 && \text{(substituting)} \\ \implies 4 &= 2a && \text{(simplifying and adding 2 to both sides)} \\ \implies 2 &= a && \text{(dividing both sides by 2)} \end{aligned}$$

Checkpoint 1.6.4A

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

Q&A

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (see Figure 1.6.8).

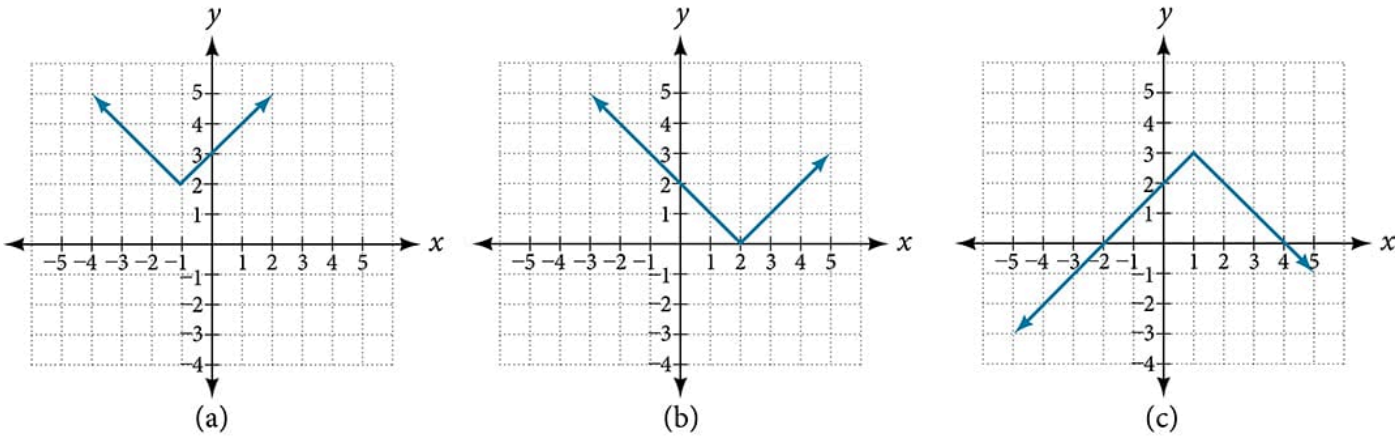


Figure 1.6.8: *(a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.*

Checkpoint 1.6.4B

Is the absolute value function one-to-one?

Solving an Absolute Value Equation

Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8 = |2x - 6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$\begin{aligned} 2x - 6 &= 8 \quad \text{or} \quad 2x - 6 = -8 \\ \implies 2x &= 14 \qquad\quad 2x = -2 \\ \implies x &= 7 \qquad\quad x = -1 \end{aligned}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

Definition: Absolute Value Equation

An **absolute value equation** is an equation in which the unknown variable appears within the absolute value bars.

For example,

$$\begin{aligned} |x| &= 4, \\ |2x-1| &= 3, \text{ and} \\ |5x+2|-4 &= 9. \end{aligned}$$

How To: Solve Absolute Value Equations

To solve an absolute value equation, $|\blacksquare| = k$, where $\blacksquare$ is an algebraic expression involving a variable, we rely on the piecewise function definition of the absolute value. Namely,

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

This means that to solve $|\blacksquare| = k$, we have two cases to consider:

If $\blacksquare \geq 0$,	If $\blacksquare < 0$,
then we solve	then we solve
$\blacksquare = k$	$-\blacksquare = k$

However, if $k < 0$, we can safely say (without any effort) that the original absolute value equation has no solution. Remember, an absolute value cannot be a negative number.

I specifically chose to use $\blacksquare$ in this version of the definition instead of x because most students new to the concept of the piecewise definition of the absolute value tend to think that the conditions must always be $x \geq 0$ and $x < 0$—**this is not true**.

Let's illustrate the process through an example.

Example 1.6.5: Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x+1| - 7$, find the values of x such that $f(x) = 0$.

Solution available at the end of this section.

Checkpoint 1.6.5

For the function $f(x) = |2x-1| - 3$, find the values of x such that $f(x) = 0$.

Q&A

Should we always expect two answers when solving $|A| = B$?

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x-5| = 1$.

Example 1.6.6: Solving an Absolute Value Equation

Solve $1 = 4|x - 2| + 2$.

Solution available at the end of this section.

Q&A

In Example 1.6.6, if $f(x) = 1$ and $g(x) = 4|x - 2| + 2$ were graphed on the same set of axes, would the graphs intersect?

No. The graphs of f and g would not intersect, as shown in Figure 1.6.10. This confirms, graphically, that the equation $1 = 4|x - 2| + 2$ has no solution.

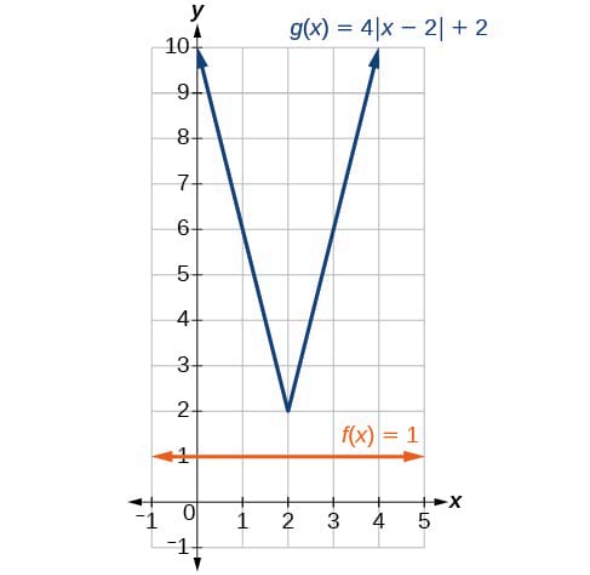


Figure 1.6.10

Checkpoint 1.6.6

Find where the graph of the function $f(x) = -|x + 2| + 3$ intersects the horizontal and vertical axes.

Focus on Calculus - Solving Absolute Value Inequalities

Absolute value equations may not always involve equalities. Instead, we may need to solve an equation within a range of values. We would use an absolute value inequality to solve such an equation. An **absolute value inequality** is an equation of the form\[|A|B,\quad \text{or }|A| \geq B,\nonumber\]where an expression A (and possibly, but not usually, B) depends on a variable x.

Definition: Absolute Value Inequality

An **absolute value inequality** is an inequality in which the unknown variable appears within the absolute value bars.

Solving the inequality means finding the set of all x that satisfy the inequality. Usually this set will be an interval or the union of two intervals.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical

approach is we can read the solution by interpreting the graphs of two functions. The disadvantage of this approach is that, when graphing by hand, the solutions are likely not going to be accurate. The advantage of the algebraic approach is it yields solutions that may be difficult to read from the graph.

For example, we know that all numbers within 200 units of 0 may be expressed as\[| x |<200 \text{ or } -200 & -200 & < & x -600 & < & 200 & & \[6pt]
\implies & 400 & < & x & < & 800 & \quad & \left(\text{adding }600 \text{ to all three "sides"} \right) \[6pt]
\end{array} \nonumber\]This means our returns would be between $400 and $800.

Sometimes an absolute value inequality problem will be presented to us in terms of a shifted and/or stretched or compressed absolute value function, where we must determine for which values of the input the function's output will be negative or positive.

Example 1.6.7: Solving an Absolute Value Inequality

Solve $|x-5|<4$.

Solution available at the end of this section.

The approach we used in our solution to Example 1.6.7 is what I consider to be a "preferred method." This is because it reinforces the piecewise definition of the absolute value (which is something you must know to be successful in Calculus); however, it is best to show you another approach used by many other instructors and students.

First, we will need to know where the corresponding equality is true. That is, we need to solve $|x-5|=4$. The process is almost identical to what we did in Example 1.6.7.

$$|x-5| = \begin{cases} x-5, & \text{if } x-5 \geq 0 \implies x \geq 5 \\ -(x-5), & \text{if } x-5 < 0 \implies x < 5 \end{cases}$$

This tells us that our absolute value equation splits into two possibilities:

$$\begin{array}{rrcl} & \text{For } x \geq 5 & & \\ & |x-5| & = & 4 \\ \implies & x-5 & = & 4 \\ \implies & x & = & 9 \end{array} \qquad \begin{array}{rrcl} & \text{For } x < 5 & & \\ & |x-5| & = & 4 \\ \implies & -(x-5) & = & 4 \\ \implies & x-5 & = & -4 \\ \implies & x & = & 1 \end{array}$$

After determining that the absolute value is equal to 4 at $x=1$ and $x=9$, we know the graph can change only from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals:\[x<1,\quad 19. \nonumber\]To determine when the function is less than 4, we could choose a value in each interval and see if the output is less than or greater than 4, as shown in Table 1.6.1.

Table 1.6.1

Interval test x	Test value for x	<4 or >4?	
$x<1$	0	$\|0-5\|=5$	Greater than
\(1	6	$\|6-5\|=1$	Less than
$x>9$	11	$\|11-5\|=6$	Greater than

Because \(1

To use a graph, we can sketch the function $f(x)=|x-5|$. To help us see where the outputs are 4, the line $g(x)=4$ could also be sketched as in Figure 1.6.11.

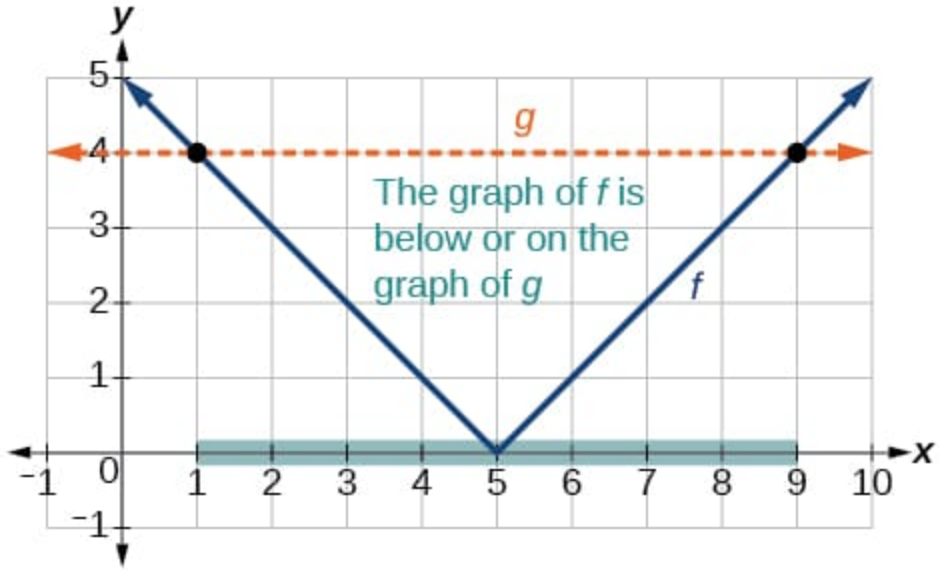


Figure 1.6.11**:** *Graph to find the points satisfying an absolute value inequality.*

We can see the following:

- The output values of the absolute value are equal to 4 at $x = 1$ and $x = 9$.
- The graph of f is below the graph of g on \(1
- The absolute value is less than or equal to 4 between these two points, when \(1

For absolute value inequalities,

$$\begin{array}{rrrrrrrrrrr} & |x-A| & < & C & & & |x-A| & > & C & \\ \Longrightarrow & -C & < & x-A & < & C & & x-A & < & -C & \text{or} \quad x-A > C \end{array}$$

The $<$ or $>$ symbol may be replaced by $\leq$ or $\geq$.

So, for Example 1.6.7, we could use this alternative approach.

$$\begin{array}{rrrrrrl} & & |x-5| & < & 4 & & \\ \Longrightarrow & -4 & < & x-5 & < & 4 & \text{(rewriting by removing the absolute value bars)} \\ \Longrightarrow & -4+5 & < & x-5+5 & < & 4+5 & \text{(adding 5 to all three "sides")} \\ \Longrightarrow & 1 & < & x & < & 9 & \end{array}$$

Checkpoint 1.6.7

Solve $|x + 2| \leq 6$.

Example 1.6.8: Using a Graphical Approach to Solve Absolute Value Inequalities

Given the function $f(x) = -\frac{1}{2}|4x - 5| + 3$, determine the x-values for which the function values are negative.

Solution available at the end of this section.

Checkpoint 1.6.8

Solve $-2|k - 4| \leq -6$.

Hidden Resources: The Hawk Spot Food Pantry

The Hawk Spot Food Pantry is a program at CRC that gives food once per week to CRC students.

How Does CalFresh Work?

Details may change, so use the link below to verify what is stated here; however, you show your Student ID, fill out the intake form (on first visit), and pick-up your premade bag. Each student is allowed to be served once per week.

For more information, please see this page on the CRC website.

Hidden Resources: Student Health and Wellness Services

Cosumnes River College has amazing support services for students in need of emotional and/or mental health services. Our Student Health and Wellness Services include the following:

Physical Health Services (no cost)

- Health and illness assessments
- First aid for minor injuries & illness
- Over-the-counter medications
- Party Safe Kits - Narcan, fentanyl & drink spike test strips
- Sexual health education
- Condoms (internal & external) & lubricant
- Opill (birth control)
- Emergency contraception (Plan B)
- STI testing
- Pregnancy testing
- Flu A/B & Covid testing
- Flu vaccine
- Strep throat testing
- TB risk assessments
- Health requirements for academic programs

Physical Health Services (low cost)

- Immunizations/titers
- TB skin test

Mental Health Services (no cost)

- **Togetherall:** This is a safe online space to connect anonymously with other students to get and give support.
- **Therapy:** Meet with a Los Rios Student Health and Wellness Center (SHWC) therapist or advocate. Support is available in person and virtually.
- **BetterMynd:** Online Therapy: Talk to an online licensed therapist during the day, evenings or weekends. Or call 24/7 support line: (844) BTR-MYND.

For more information, please see this page on the CRC website.

Answers

Section Preview

Although it was introduced when we introduced piecewise functions, the definition of the absolute value function as a piecewise function is reintroduced (it is that important). The middle part of this section focuses on graphing transformations of the absolute value function, which is a nice bridge from the previous section. The remainder of the section is focused on solving absolute value equations and inequalities (needed for Calculus); however, they only should involve linear expressions (other types are treated in a future section).

Section Preview

- **Solving Inequalities**
  - Solving Compound Inequalities: The AND/OR (intersection vs. union) logic explains why $|A| > B$ produces a union of two intervals; students need it to interpret and assemble the final solution sets.

Section Preview

- Graph an absolute value function.
- Solve an absolute value equation.
- Solve an absolute value inequality.

Example 1.6.1: Determine a Number within a Prescribed Distance

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as the one in Figure 1.6.2, to represent the condition to be satisfied.

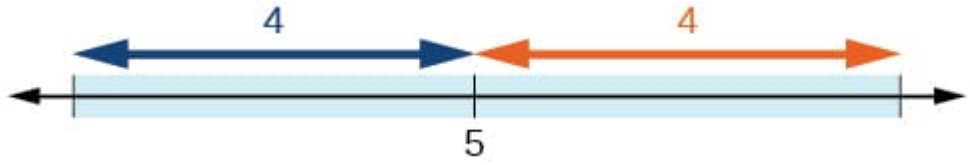


Figure 1.6.2

The distance from x to 5 can be represented using the absolute value as $|x-5|$. We want the values of x that satisfy the condition $|x-5| \le 4$.

Example 1.6.2: Resistance of a Resistor

5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,

$$|R-680| \le 34.$$

Example 1.6.3: Revisiting the Difference Quotient

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\left(-\frac{1}{2}|4(x+h)-5|+3\right)-\left(-\frac{1}{2}|4x-5|+3\right)}{h} && \text{(substituting)}\\
&= \frac{-\frac{1}{2}|4(x+h)-5|+3+\frac{1}{2}|4x-5|-3}{h} && \text{(distributing)}\\
&= \frac{-\frac{1}{2}|4(x+h)-5|+\frac{1}{2}|4x-5|}{h} && \text{(simplifying)}\\
&= \frac{2}{2}\cdot\frac{-\frac{1}{2}|4(x+h)-5|+\frac{1}{2}|4x-5|}{h} && \text{(simplifying the compound fraction)}\\
&= \frac{-|4(x+h)-5|+|4x-5|}{2h} && \text{(distributing)}\\
&= \frac{-|4x+4h-5|+|4x-5|}{2h} && \text{(distributing)}
\end{aligned}$$

At this point, we are stuck and need to refer to the piecewise definition of the absolute value function.

$$|4x-5| = \begin{cases} 4x-5, & \text{if } 4x-5 \ge 0 \implies x \ge \frac{5}{4} \\ -(4x-5), & \text{if } 4x-5 < 0 \implies x < \frac{5}{4} \end{cases}$$

Since we were told $x < \frac{5}{4}$, $|4x-5| = -(4x-5) = -4x+5$. Turning our attention to the other absolute value, we get the following:

$$|4x+4h-5| = \begin{cases} 4x+4h-5, & \text{if } 4x+4h-5 \ge 0 \implies 4x \ge 5-4h \implies x \ge \frac{5}{4}-h \\ -(4x+4h-5), & \text{if } 4x+4h-5 < 0 \implies 4x < 5-4h \implies x < \frac{5}{4}-h \end{cases}$$

Since we are told that $x < \frac{5}{4}$ and $h < 0$, it must be true that $x+h < \frac{5}{4}$. Hence, $x < \frac{5}{4}-h$ and so $|4x+4h-5| = -(4x+4h-5) = -4x-4h+5$. Combining these two results, we get the following:

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{-|4x+4h-5|+|4x-5|}{2h}\\
&= \frac{-(-4x-4h+5)+(-4x+5)}{2h} && \text{(substituting)}\\
&= \frac{4x+4h-5-4x+5}{2h} && \text{(distributing)}\\
&= \frac{4h}{2h} && \text{(combining like terms)}\\
&= \frac{4\cancel{h}^{1}}{2\cancel{h}^{1}} && \text{(canceling like factors)}\\
&= 2
\end{aligned}$$

Example 1.6.4**: Writing an Equation for an Absolute Value Function**

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See Figure 1.6.6.

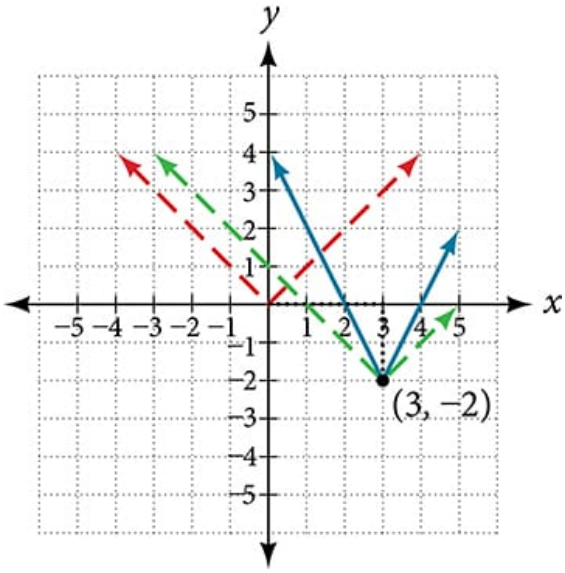


Figure 1.6.6

We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in Figure 1.6.7.

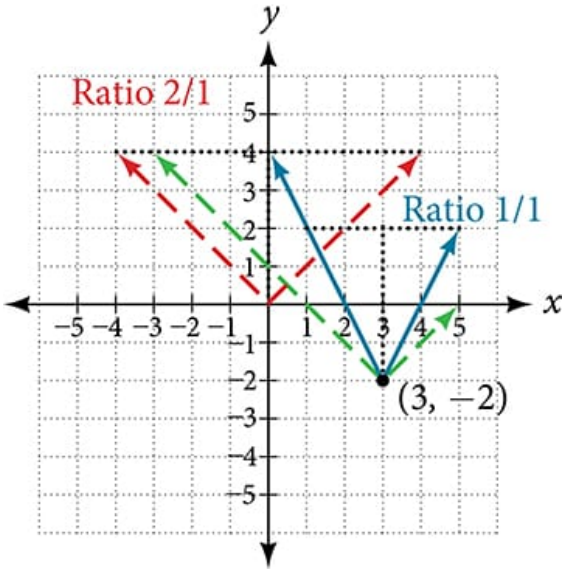


Figure 1.6.7

From this information we can write the equation as $f(x) = 2|x - 3| - 2$, treating the stretch as a vertical stretch.

Example 1.6.5**: Finding the Zeros of an Absolute Value Function**

We set $f(x) = 0$ and solve.

$$\begin{aligned} 0 &= |4x + 1| - 7 && \text{(setting } f(x) = 0\text{)} \\ \implies 7 &= |4x + 1| && \text{(adding 7 to both sides)} \end{aligned}$$

Now that we have the absolute value isolated, let's recall the piecewise function definition of the absolute value.

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

In our case, $\blacksquare = 4x + 1$. Hence,

$$|4x + 1| = \begin{cases} 4x + 1, & \text{if } 4x + 1 \geq 0 \\ -(4x + 1), & \text{if } 4x + 1 < 0 \end{cases}$$

That is,

$$|4x+1| = \begin{cases} 4x+1, & \text{if } x \geq -\frac{1}{4} \\ -(4x+1), & \text{if } x < -\frac{1}{4} \end{cases}$$

This tells us that our absolute value equation splits into two possibilities:

For $x \geq -\frac{1}{4}$			For $x < -\frac{1}{4}$		
	7	$= \quad \lvert 4x+1 \rvert$		7	$= \quad \lvert 4x+1 \rvert$
$\implies$	7	$= \quad 4x+1$	$\implies$	7	$= \quad -(4x+1)$
$\implies$	6	$= \quad 4x$	$\implies$	-7	$= \quad 4x+1$
$\implies$	$\frac{3}{2}$	$= \quad x$	$\implies$	-8	$= \quad 4x$
			$\implies$	-2	$= \quad x$

The function outputs 0 when $x = 1.5$ or $x = -2$. See Figure 1.6.9.

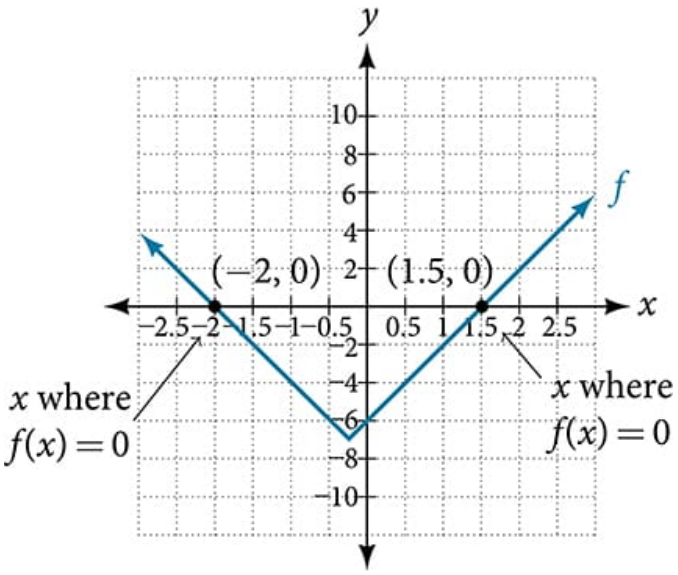


Figure 1.6.9

Example 1.6.6**: Solving an Absolute Value Equation**

Isolating the absolute value on one side of the equation gives the following.

$$\begin{aligned} 1 &= 4|x-2|+2 \\ \implies -1 &= 4|x-2| \\ \implies -\frac{1}{4} &= |x-2| \end{aligned}$$

The absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value. At this point, we notice that this equation has no solutions.

Example 1.6.7**: Solving an Absolute Value Inequality**

Our approach will duplicate our work in Example 1.6.6. Since the inequality is already isolated, we immediately dive into the piecewise definition of the absolute value:

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

In our case, $\blacksquare = x - 5$. Hence,

$$|x-5| = \begin{cases} x-5, & \text{if } x-5 \geq 0 \implies x \geq 5 \\ -(x-5), & \text{if } x-5 < 0 \implies x < 5 \end{cases}$$

This tells us that our absolute value inequality splits into two possibilities:

For $x \geq 5$			For $x < 5$		
	$\lvert x-5 \rvert$	$< \quad 4$		$\lvert x-5 \rvert$	$< \quad 4$
$\implies$	$x-5$	$< \quad 4$	$\implies$	$-(x-5)$	$< \quad 4$
$\implies$	x	$< \quad 9$	$\implies$	$x-5$	$> \quad -4$
			$\implies$	x	$> \quad 1$

To summarize, we are okay with $x \geq 5$ as long as $x < 9$. Additionally, we are okay with $x < 5$ as long as $x > 1$. Combining this information, our solution interval is $(1, 9)$.

Example 1.6.8**: Using a Graphical Approach to Solve Absolute Value Inequalities**

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x-5|+3<0$. We begin by isolating the absolute value.

$$\begin{aligned} -\frac{1}{2}|4x-5| &< -3 && \text{(subtracting 3 from both sides)} \\ \implies |4x-5| &> 6 && \text{(multiplying both sides by } -2 \text{ and switching the inequality sign)} \end{aligned}$$

Using the definition of the absolute value, we get

$$|4x-5| = \begin{cases} 4x-5, & \text{if } 4x-5 \geq 0 \implies x \geq \frac{5}{4} \\ -(4x-5), & \text{if } 4x-5 < 0 \implies x < \frac{5}{4} \end{cases}$$

Thus,

For $x \geq \frac{5}{4}$			For $x < \frac{5}{4}$		
	$\lvert 4x-5\rvert$	> 6		$\lvert 4x-5\rvert$	> 6
$\implies$	$4x-5$	> 6	$\implies$	$-(4x-5)$	> 6
$\implies$	$4x$	> 11	$\implies$	$4x-5$	< -6
$\implies$	x	$> \frac{11}{4}$	$\implies$	$4x$	< -1
			$\implies$	x	$< -\frac{1}{4}$

The leftmost solution tells us that as long as $x \geq \frac{5}{4}$, the original inequality is satisfied for $x > \frac{11}{4}$. The rightmost solution tells us that as long as $x < \frac{5}{4}$, the original inequality is satisfied for $x < -\frac{1}{4}$. Thus, our solution is $\left(-\infty, -\frac{1}{4}\right) \cup \left(\frac{11}{4}, \infty\right)$.

If using the graphing method, we can examine the graph of f to observe where the output is negative. We will observe where the branches are below the x-axis. Notice that it is not even important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$ and that the graph has been reflected vertically. See Figure 1.6.12**.**

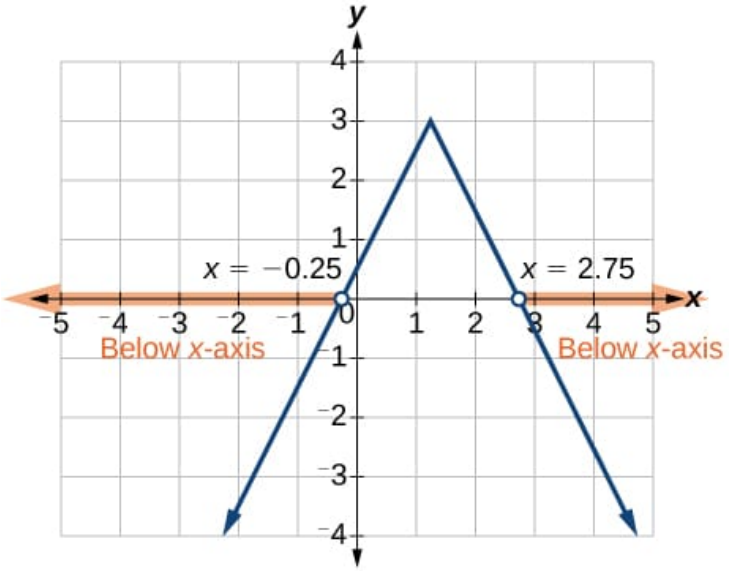


Figure 1.6.12

We observe that the graph of the function is below the x-axis left of $x = -\frac{1}{4}$ and right of $x = \frac{11}{4}$. This means the function values are negative to the left of the first horizontal intercept at $x = -\frac{1}{4}$, and negative to the right of the second intercept at $x = \frac{11}{4}$. This gives us the solution to the inequality.

$$x < -\frac{1}{4} \text{ or } x > \frac{11}{4}.$$

In interval notation, this would be $\left(-\infty, -\frac{1}{4}\right) \cup \left(\frac{11}{4}, \infty\right)$.

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1.6.1: Resources and Key Concepts

Resources

Videos

- Absolute value as a concept
  - Intermediate Algebra - Absolute Value: The Concepts
- Creating absolute value equations
  - Intermediate Algebra - Absolute Value: Forming Absolute Value Equations
- Solving absolute value equations
  - Equations - Overview - A Review for College Algebra - Solving Absolute Value Equations
- Solving absolute value inequalities
  - Inequalities - Nonlinear - A Review for College Algebra: Solving Absolute Value Inequalities
- Interval notation for solutions
  - Logic & Set Theory in Review: Sets - Notation (Interval Notation)

Key Concepts

Definitions

- **Absolute Value Function:** The function $f(x) = |x|$, which gives the distance of x from zero on a number line. Algebraically, $f(x) = x$ if $x \geq 0$ and $f(x) = -x$ if $x < 0$.
- **Absolute Value Equation:** An equation in which the unknown variable appears within absolute value bars.
- **Absolute Value Inequality:** An inequality in which the unknown variable appears within the absolute value bars, typically of the form $|A| < B$, $|A| \leq B$, $|A| > B$, or $|A| \geq B$.

Common Mistakes

- **Assuming Absolute Value Equations Always Have Two Solutions:** While many absolute value equations yield two solutions (e.g., $|2x - 6| = 8$ in the introduction to solving equations), they can also have one solution (e.g., $|x - 3| = 0$) or no solutions (e.g., $|x - 2| = -\frac{1}{4}$ in Example 5).
- **Incorrectly Handling Negative Numbers in Absolute Value Equations/Inequalities:**
  - An equation of the form $|A| = k$ where $k < 0$ has no solution, because an absolute value cannot be negative.
  - An inequality of the form $|A| < k$ where $k < 0$ has no solution.
  - An inequality of the form $|A| > k$ where $k < 0$ is true for all values of A for which A is defined, because an absolute value is always non-negative and thus always greater than a negative number.
- **Forgetting to Switch Inequality Sign When Multiplying/Dividing by a Negative Number:** When solving absolute value inequalities algebraically, if a step involves multiplying or dividing both sides by a negative number, the direction of the inequality sign must be reversed.
- **Misinterpreting Conditions for Piecewise Definition:** When using the piecewise definition of absolute value, $|■| = ■$ if $■ \geq 0$ and $|■| = -■$ if $■ < 0$, students might incorrectly assume the condition always refers to $x \geq 0$ or $x < 0$ instead of the expression inside the absolute value bars (■).

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1.6.2: Homework

Reading Questions

1. How is the absolute value function $f(x) = |x|$ defined as a piecewise function? What does each piece represent?
2. What does the expression $|x - a|$ represent geometrically on a number line?
3. Explain how the parameters a, h, and k in the function $g(x) = a|x - h| + k$ transform the graph of the basic absolute value function $f(x) = |x|$.
4. What is the "corner point" of an absolute value function's graph, and how can its coordinates be determined from the equation $g(x) = a|x - h| + k$?
5. When solving an absolute value equation such as $|ax + b| = c$, where c is a positive number, what are the two distinct linear equations that arise?
6. Under what condition does an absolute value equation of the form $|E| = c$ have no solution? Provide an example.
7. Explain the algebraic steps to solve an absolute value inequality of the form $|E| < c$ where $c > 0$. How does this differ from solving $|E| > c$?
8. What does it mean if the graph of an absolute value function $f(x)$ is below the x-axis for a certain interval of x-values? How does this relate to solving $f(x) < 0$?
9. If you are solving an absolute value inequality and you multiply or divide both sides by -1, what critical adjustment must be made to the inequality symbol?
10. In the piecewise definition of $|E|$, where E is an algebraic expression, when is $|E|$ equal to E, and when is it equal to $-E$?
11. How do you solve an absolute value equation?
12. How can you tell whether an absolute value function has two x-intercepts without graphing the function?
13. When solving an absolute value function, the isolated absolute value term is equal to a negative number. What does that tell you about the graph of the absolute value function?
14. How can you use the graph of an absolute value function to determine the x-values for which the function values are negative?
15. How do you solve an absolute value inequality algebraically?

Homework

Algebraic

16. Describe all numbers x that are at a distance of 4 from the number 8. Express this using absolute value notation.
17. Describe all numbers x that are at a distance of $\frac{1}{2}$ from the number -4. Express this using absolute value notation.
18. Describe the situation in which the distance that point x is from 10 is at least 15 units. Express this using absolute value notation.
19. Find all function values $f(x)$ such that the distance from $f(x)$ to the value 8 is less than 0.03 units. Express this using absolute value notation.

For the following exercises, solve the equations below and express the answer using set notation.

20. $|x + 3| = 9$
21. $|6 - x| = 5$
22. $|5x - 2| = 11$
23. $|4x - 2| = 11$
24. $2|4 - x| = 7$
25. $3|5 - x| = 5$
26. $3|x + 1| - 4 = 5$
27. $5|x - 4| - 7 = 2$
28. $0 = -|x - 3| + 2$
29. $2|x - 3| + 1 = 2$
30. $|3x - 2| = 7$
31. $|3x - 2| = -7$
32. $\left|\frac{1}{2}x - 5\right| = 11$

33. $\left|\frac{1}{3}x+5\right|=14$
34. $-\left|\frac{1}{3}x+5\right|+14=0$

For the following exercises, find the x- and y-intercepts of the graphs of each function.

35. $f(x)=2|x+1|-10$
36. $f(x)=4|x-3|+4$
37. $f(x)=-3|x-2|-1$
38. $f(x)=-2|x+1|+6$

For the following exercises, solve each inequality and write the solution in interval notation.

39. $|x-2|>10$
40. $2|v-7|-4\geq 42$
41. $|3x-4|\leq 8$
42. $|x-4|\geq 8$
43. $|3x-5|\geq 13$
44. $|3x-5|\geq -13$
45. $\left|\frac{3}{4}x-5\right|\geq 7$
46. $\left|\frac{3}{4}x-5\right|+1\leq 16$

Graphical

For the following exercises, graph the absolute value function. Plot at least five points by hand for each graph.

47. $y=|x-1|$
48. $y=|x+1|$
49. $y=|x|+1$

For the following exercises, graph the given functions by hand.

50. $y=|x|-2$
51. $y=-|x|$
52. $y=-|x|-2$
53. $y=-|x-3|-2$
54. $f(x)=-|x-1|-2$
55. $f(x)=-|x+3|+4$
56. $f(x)=2|x+3|+1$
57. $f(x)=3|x-2|+3$
58. $f(x)=|2x-4|-3$
59. $f(x)=|3x+9|+2$
60. $f(x)=-|x-1|-3$
61. $f(x)=-|x+4|-3$
62. $f(x)=\frac{1}{2}|x+4|-3$

Technology

63. Use graphing technology to graph $f(x)=10|x-2|$ on the viewing window $[0,4]$. Identify the corresponding range. Show the graph.
64. Use graphing technology to graph $f(x)=-100|x|+100$ on the viewing window $[-5,5]$. Identify the corresponding range. Show the graph.

For the following exercises, graph each function using graphing technology. Specify the viewing window.

65. $f(x)=-0.1|0.1(0.2-x)|+0.3$
66. $f(x)=4\times 10^9|x-(5\times 10^9)|+2\times 10^9$

Extensions

For the following exercises, solve the inequality.

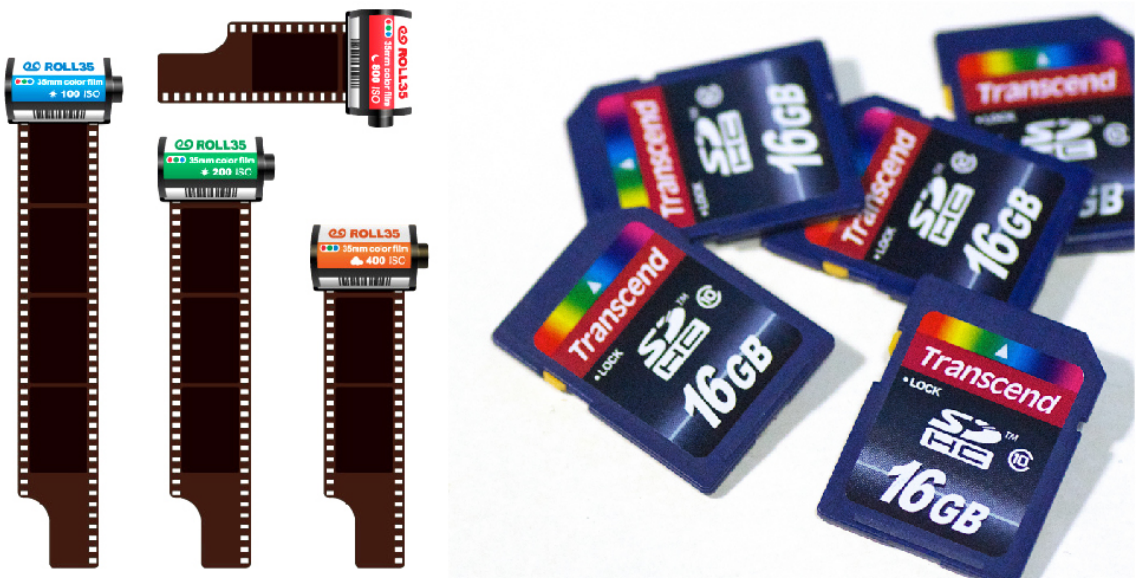
67. $\left|-2x-\frac{2}{3}(x+1)\right|+3>-1$
68. If possible, find all values of a such that there are no x-intercepts for $f(x)=2|x+1|+a$.
69. If possible, find all values of a such that there are no y-intercepts for $f(x)=2|x+1|+a$.

Real-World Applications

70. Cities A and B are on the same east-west line. Assume that city A is located at the origin. If the distance from city A to city B is at least 100 miles and x represents the distance from city B to city A, express this using absolute value notation.
71. The true proportion p of people who give a favorable rating to Congress is 8% with a margin of error of 1.5%. Describe this statement using an absolute value equation.
72. Students who score within 18 points of the number 82 will pass a particular test. Write this statement using absolute value notation and use the variable x for the score.
73. A machinist must produce a bearing that is within 0.01 inches of the correct diameter of 5.0 inches. Using x as the diameter of the bearing, write this statement using absolute value notation.
74. The tolerance for a ball bearing is 0.01. If the true diameter of the bearing is to be 2.0 inches and the measured value of the diameter is x inches, express the tolerance using absolute value notation.

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2: POLYNOMIAL AND RATIONAL FUNCTIONS



35-mm film, once the standard for capturing photographic images, has been made largely obsolete by digital photography. (credit "film": modification of work by Horia Varlan; credit "memory cards": modification of work by Paul Hudson)

Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by Mathematics. An image becomes a series of numbers, representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects. Inverse functions make it possible to convert from one file format to another. In this chapter, we will learn about these concepts and discover how Mathematics can be used in such applications.

2.1: Complex Numbers

2.1.1: Resources and Key Concepts

2.1.2: Homework

2.2: Quadratic Functions

2.2.1: Resources and Key Concepts

2.2.2: Homework

2.3: Power Functions and Polynomial Functions

2.3.1: Resources and Key Concepts

2.3.2: Homework

2.4: Graphs of Polynomial Functions

2.4.1: Resources and Key Concepts

2.4.2: Homework

2.5: Dividing Polynomials

2.5.1: Resources and Key Concepts

2.5.2: Homework

2.6: Zeros of Polynomial Functions

2.6.1: Resources and Key Concepts

2.6.2: Homework

2.7: Rational Functions

2.7.1: Resources and Key Concepts

2.7.2: Homework

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2.1: Complex Numbers

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The study of Mathematics continuously builds upon itself. Negative integers, for example, fill a void left by the set of positive integers. The set of rational numbers, in turn, fills a void left by the set of integers. The set of real numbers fills a void left by the set of rational numbers. Not surprisingly, the set of real numbers has voids as well. For example, we still have no solution to equations such as $x^2 + 4 = 0$. Our best guesses might be 2 or -2. But if we test 2 in this equation, it does not work. If we test -2 , it does not work. If we want to have a solution for this equation, we will have to go farther than we have so far. After all, to this point we have described the square root of a negative number as undefined. Fortunately, there is another system of numbers that provides solutions to problems such as these. In this section, we will explore this number system and how to work within it.

Expressing Square Roots of Negative Numbers as Multiples of i

We know how to find the square root of any positive real number. In a similar way, we can find the square root of a negative number. The difference is that the root is not real. We begin our exploration of this new beast by defining the **unit imaginary number**.

Definition: Unit Imaginary Number

The **unit imaginary number**, denoted as i, is defined as the number whose square root is -1. That is,

$$\sqrt{-1} = i.$$

The unit imaginary number is often referred to as the **imaginary unit**. This definition leads directly to the following theorem.

Theorem: The Square of the Imaginary Unit

$$i^2 = -1$$

Proof available at the end of this section.

We can write the square root of any negative number as a multiple of i. Consider the square root of -25.

$$\sqrt{-25} = \sqrt{25 \cdot (-1)} = \sqrt{25}\sqrt{-1} = 5i.$$

We use $5i$ and not $-5i$ because the principal root of 25 is the positive root.

By defining the imaginary unit, we have access to an entirely new type of number—the **complex number**.

Definition: Complex Number

A **complex number** is a number that can be written in the form $a+bi$ where a is called the **real part** and b is called the **imaginary part** of the complex number. If $b=0$, then $a+bi$ is simply a real number. If $a=0$ and $b\neq 0$, the complex number is called an **imaginary number**. The form, $a+bi$ is called the **standard form** of the complex number.

Complex numbers are a combination of real and imaginary numbers. For example, $5+2i$ is a complex number. The real part is 5, and it happens that 5 is a real number. The imaginary part is 2, and $2i$ is called an imaginary number.

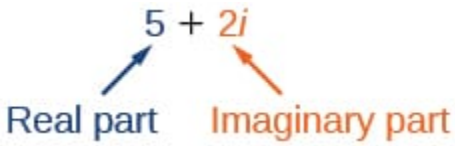


Figure $2.1.1$

Recall, when a positive real number is squared, the result is a positive real number and when a negative real number is squared, again, the result is a positive real number. Imaginary numbers are distinguished from real numbers because a squared imaginary number produces a negative real number!

If we were to gather all of the complex numbers in existence, we would have a completely new number system.

Definition: Complex Number System

The set of all numbers of the form $a+bi$ is called the **complex number system** (symbolically denoted as $\mathbb{C}$).

Example $2.1.1$: Expressing an Imaginary Number in Standard Form

Express $\sqrt{-9}$ in standard form.

Solution available at the end of this section.

Checkpoint $2.1.1$

Express $\sqrt{-24}$ in standard form.

Plotting a Complex Number on the Complex Plane

We cannot plot complex numbers on a number line as we might real numbers. However, we can still represent them graphically. To represent a complex number we need to address the two components of the number. Two components means we need two number lines—one for the real part and one for the imaginary part. Thus is borne the **complex plane**.

Definition: Complex Plane

The **complex plane** is a coordinate system in which the horizontal axis represents the real part of a complex number and the vertical axis represents the imaginary part. In this sense, we define the horizontal axis to be the **real axis** and the vertical axis to be the **complex axis**. Complex numbers are the points on the plane, expressed as ordered pairs (a, b), where a represents the coordinate for the horizontal axis and b represents the coordinate for the vertical axis.

Let's consider the number $-2 + 3i$. The real part of the complex number is -2 and the imaginary part is 3. We plot the ordered pair $(-2, 3)$ to represent the complex number $-2 + 3i$ as shown in Figure $2.1.2$.

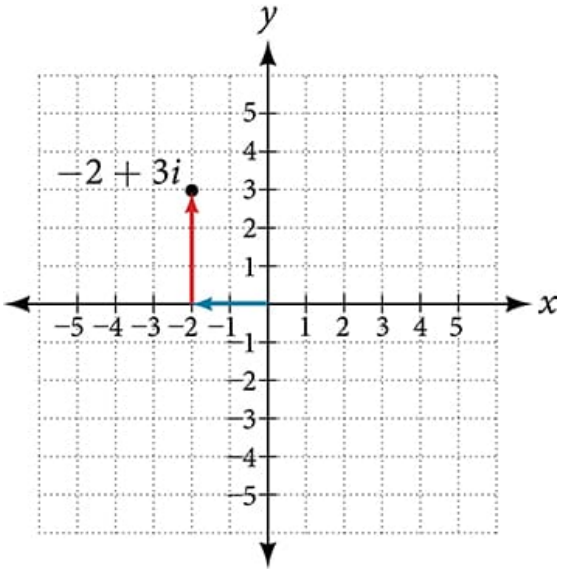


Figure $2.1.2$

Example $2.1.2$: Plotting a Complex Number on the Complex Plane

Plot the complex number $3 - 4i$ on the complex plane.

Solution available at the end of this section.

Interactive Element: A Complex Number as a Point

The following Interactive Element might help you visualize how a complex number $a + bi$ corresponds to a single point in the complex plane.

Interact: Drag the red point anywhere in the plane.

Observation: The horizontal position gives the real part and the vertical position gives the imaginary part, so $a + bi$ is exactly the point (a, b). Land on the horizontal axis and the number is purely real; land on the vertical axis and it is purely imaginary.

Checkpoint $2.1.2$

Plot the complex number $-4 - i$ on the complex plane.

Adding and Subtracting Complex Numbers

Just as with real numbers, we can perform arithmetic operations on complex numbers; however, we need to define what it means to add and subtract complex numbers.

Definition: Addition and Subtraction of Complex Numbers

Given two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$, we define addition and subtraction as follows:

Addition:

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i.$$

Subtraction:

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i.$$

That is, to add or subtract complex numbers, we combine the real parts and combine the imaginary parts.

Example 2.1.3: Adding Complex Numbers

Add $3 - 4i$ and $2 + 5i$.

Solution available at the end of this section.

Checkpoint 2.1.3

Subtract $2 + 5i$ from $3 - 4i$.

Multiplying Complex Numbers

When defining multiplication of complex numbers, we borrow heavily from binomial products.

Definition: Multiplication of Complex Numbers

Given two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$, we define multiplication as follows:

$$\begin{aligned} z_1 \cdot z_2 &= (z_1)(z_2) \\ &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + (ad + bc)i + bc(-1) \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

While it might be tempting, you should *never* memorize the result of this definition. Instead, treat multiplying complex numbers like multiplying binomials.

Let's begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. So, for example,

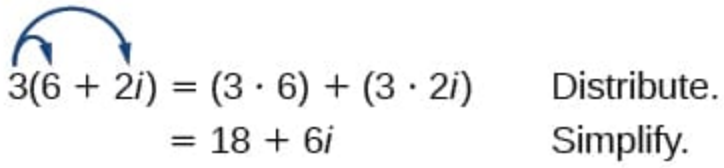


Figure 2.1.4

Example 2.1.4: Multiplying a Complex Number by a Real Number

Find the product $4(2+5i)$.

Solution available at the end of this section.

Checkpoint 2.1.4

Find the product $-4(2+6i)$.

Example 2.1.5: Multiplying a Complex Number by a Complex Number

Multiply $(4+3i)(2-5i)$.

Solution available at the end of this section.

Checkpoint 2.1.5

Multiply $(3-4i)(2+3i)$.

Dividing Complex Numbers

When building a definition for division of complex numbers, we must come to some agreement concerning allowable forms of complex numbers. At this time, a complex number in standard form must look like $a+bi$, where both a and b are real numbers. Thus, it is safe to claim that a and b, if fractions, cannot have complex numbers in their denominators. Said differently, we require that any fraction must have a real-number denominator. This is so important, it bears repeating.

> Any fraction must have a real-valued denominator.

Therefore, when dividing a complex number by another complex number, we need to find a number by which we can multiply the numerator and the denominator that will force the denominator to be a pure real number. That is, given $z_1 = a+bi$ and $z_2 = c+di$, we need to find a number, let's call this k for now, such that

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{(a+bi)}{(c+di)} \cdot \frac{k}{k} = \frac{k(a+bi)}{k(c+di)},$$

where $k(c+di)$ is a pure real number.

A great choice for k is $c-di$. This is because

$$k(c+di) = (c-di)(c+di) = c^2 + cdi - cdi - d^2i^2 = c^2 + \cancel{cdi} - \cancel{cdi} - d^2(-1) = c^2 + d^2.$$

Since both c and d are real numbers, this result is a real number. Our choice for k has a special name (and is *incredibly* important as we move through the rest of Mathematics).

Definition: Complex Conjugate

The **complex conjugate** of $z = a + bi$ is defined to be $\overline{z} = a - bi$. Likewise, the complex conjugate of $z = a - bi$ is defined to be $\overline{z} = a + bi$.

Prior to this definition, we proved the following theorem.

Theorem: Products of Complex Conjugates

Let $z = a + bi$, so that its complex conjugate is $\overline{z} = a - bi$. Then

$$z \cdot \overline{z} = a^2 + b^2.$$

Proof available at the end of this section.

We now have everything we need to create a definition for division of complex numbers that will result in denominators being real numbers.

Definition: Division of Complex Numbers

Given two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$, we define division as follows:

$$\begin{aligned} z_1 \div z_2 &= \frac{z_1}{z_2} \\ &= \frac{a+bi}{c+di} \\ &= \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} \\ &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\ &= \frac{ac-adi+bci-bdi^2}{c^2+d^2} \\ &= \frac{ac+(-ad+bc)i-bd(-1)}{c^2+d^2} \\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i \\ &= r + si \end{aligned}$$

where r and s are real numbers.

As was mentioned with multiplication of complex numbers, it is best not to memorize this resulting formula. You should be able to just "do the math."

Example 2.1.6: Finding Complex Conjugates

Find the complex conjugate of each number.

a. $2 + i\sqrt{5}$

b. $-\frac{1}{2}i$

Solution available at the end of this section.

Although we have seen that we can find the complex conjugate of an imaginary number, in practice we generally find the complex conjugates of only complex numbers with both a real and an imaginary component. To obtain a real number from an imaginary number, we can simply multiply by i.

Example 2.1.7: Dividing Complex Numbers

Divide $(2+5i)$ by $(4-i)$.

Solution available at the end of this section.

Example 2.1.8: Substituting a Complex Number into a Polynomial Function

Let $f(x)=x^2-5x+2$. Evaluate $f(3+i)$.

Solution available at the end of this section.

Checkpoint 2.1.8

Let $f(x)=2x^2-3x$. Evaluate $f(8-i)$.

Example 2.1.9: Substituting an Imaginary Number in a Rational Function

Let $f(x)=\frac{2+x}{x+3}$. Evaluate $f(10i)$.

Solution available at the end of this section.

Checkpoint 2.1.9

Let $f(x)=\frac{x+1}{x-4}$. Evaluate $f(-i)$.

Simplifying Powers of i

The powers of i are cyclic. Let's look at what happens when we raise i to increasing powers.

$$\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= i^2 \cdot i = -1 \cdot i = -i \\ i^4 &= i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1 \\ i^5 &= i^4 \cdot i = 1 \cdot i = i \end{aligned}$$

We can see that when we get to the fifth power of i, it is equal to the first power. As we continue to multiply i by itself for increasing powers, we will see a cycle of 4. Let's examine the next 4 powers of i.

$$\begin{aligned} i^6 &= i^5 \cdot i = i \cdot i = i^2 = -1 \\ i^7 &= i^6 \cdot i = i^2 \cdot i = i^3 = -i \\ i^8 &= i^7 \cdot i = i^3 \cdot i = i^4 = 1 \\ i^9 &= i^8 \cdot i = i^4 \cdot i = i^5 = i \end{aligned}$$

Theorem: Powers of the Imaginary Unit

Let $p \in \mathbb{W}$ (the Whole Number system) and r be the remainder when p is divided by 4. Then

$$i^p = i^r.$$

Proof available at the end of this section.

Example 2.1.10: Simplifying Powers of i

Evaluate i^{35}.

Solution available at the end of this section.

Q&A

Can we write i^{35} in other helpful ways?

As we saw in Example 2.1.10, we reduced i^{35} to i^3 by dividing the exponent by 4 and using the remainder to find the simplified form. But perhaps another factorization of i^{35} may be more useful. Table 2.1.1 shows some other possible factorizations.

Table 2.1.1

Factorization of i^{35}	$i^{34} \cdot i$	$i^{33} \cdot i^2$	$i^{31} \cdot i^4$	$i^{19} \cdot i^{16}$
Reduced form	$(i^2)^{17} \cdot i$	$i^{33} \cdot (-1)$	$i^{31} \cdot 1$	$i^{19} \cdot (i^4)^4$
Simplified form	$(-1)^{17} \cdot i$	$-i^{33}$	i^{31}	i^{19}

Each of these will eventually result in the answer we obtained above but may require several more steps than our earlier method.

Success in College: Study Time

Believe it or not, there is a direct link between the number of units a class is worth and how many hours you should be studying (outside of class). For each hour of class time, the California state education code suggests at least two hours of studying outside of class. If your math background is weak, you may need to spend more time studying.

At Cosumnes River College, each unit of a math course corresponds to one hour of in-class time per week. Thus, a five-unit math course requires five hours of in-class interaction per week and *ten* hours of study time outside of class per week.

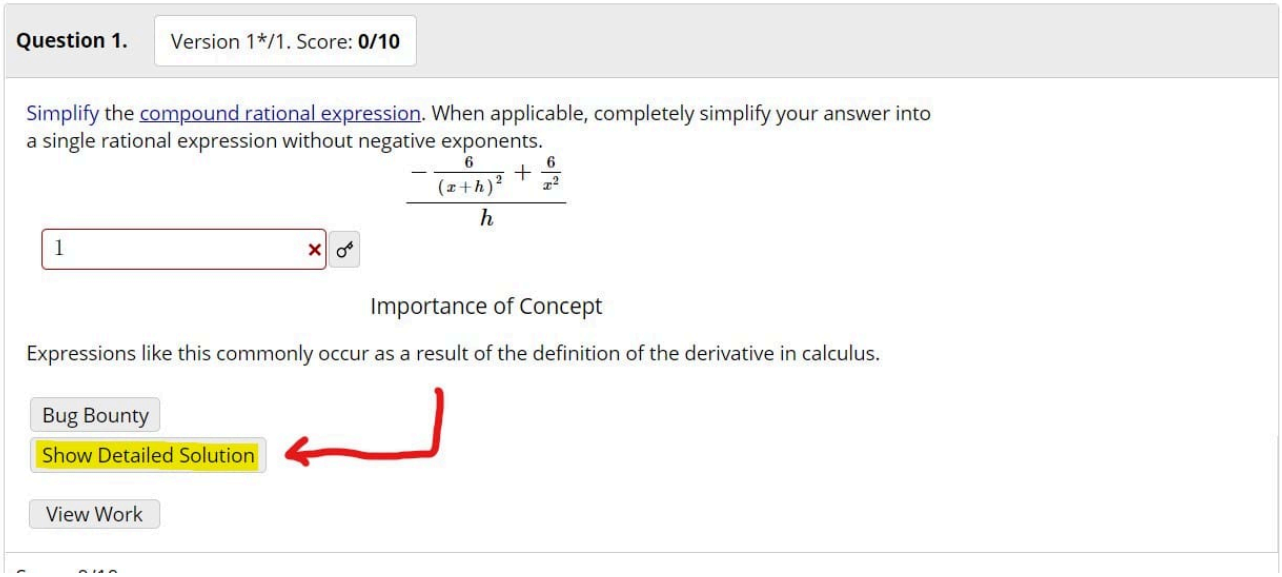
One way to study is to do what you're doing now - read! Class time is an excellent opportunity to introduce new concepts and see how they fit with those previously learned; however, there is usually not enough time to address details with the depth that a textbook can. In this way, your professor and the textbook should be considered as a combined primary source of information in your course.

Here are some questions to consider regarding how you study.

1. Do you study at least two hours for every one hour of class time? If not, why?
2. How many hours do you dedicate to TV, YouTube, and social media every week (on average)?
3. How many hours do you dedicate to leisure activities per week (e.g., hanging out with friends, going to the movies, gaming)?
4. Would it be too difficult to trade a few hours for study time?

MyOpenMath Advice: Reading Solutions to Exercises

Some problems have complete worked-out solutions. If your feedback has a comment like, "Please see the solutions," then you should probably look through these solutions. You can see solutions by going into the Grades area, finding the assignment, and clicking "Show Detailed Solution" on an exercise with a solution. Solutions are generally not shown until after the due date has passed.



I usually leave a comment of the form, "Please see the written solutions," if there were too many mistakes for me to cover by typing or if I could not follow your logic.

- Professor Simpson, Cosumnes River College

Answers

Section Preview

This required topic is needed for students to work with imaginary solutions of quadratic equations (coming up in the next section).

Section Preview

- **Sets and Numbers**
  - Properties of Real Numbers: Addition, subtraction, and multiplication of complex numbers are defined by extending the commutative, associative, and distributive properties of the reals (e.g., combining real parts and imaginary parts separately relies on these laws).

Section Preview

- **Simplifying Expressions**
  - Laws of Exponents — Product and Quotient Rules: Needed to establish $i^2 = -1$ and to simplify higher powers of i. In Example 10, $i^{35} = i^{4 \cdot 8+3} = (i^4)^8 \cdot i^3$ requires splitting exponents across a product; the same skill drives the homework on powers of i.
  - Simplifying Products of Radical Expressions: Expressing a square root of a negative number uses the product rule and perfect-square extraction, e.g., $\sqrt{-25} = \sqrt{25}\sqrt{-1} = 5i$ (Example 1). Homework such as $\sqrt{-2} \cdot \sqrt{-8}$ requires rewriting each term with i and then multiplying radicals.
  - Rationalizing Denominators and Numerators; Simplifying Quotients of Radical Expressions: Multiplying numerator and denominator by a conjugate to clear radicals is directly analogous to multiplying by the complex conjugate to force a real-valued denominator when dividing complex numbers, as in Examples 7 and 9.

Section Preview

- Express square roots of negative numbers as multiples of i.
- Plot complex numbers on the complex plane.
- Add and subtract complex numbers.
- Multiply and divide complex numbers.

Theorem: The Square of the Imaginary Unit

$$i^2 = (i)^2 = \left(\sqrt{-1}\right)^2 = -1$$

Example 2.1.1**: Expressing an Imaginary Number in Standard Form**

$$\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$$

In standard form, this is $0 + 3i$.

Example 2.1.2**: Plotting a Complex Number on the Complex Plane**

The real part of the complex number is 3, and the imaginary part is -4. We plot the ordered pair $(3, -4)$ as shown in Figure 2.1.3.

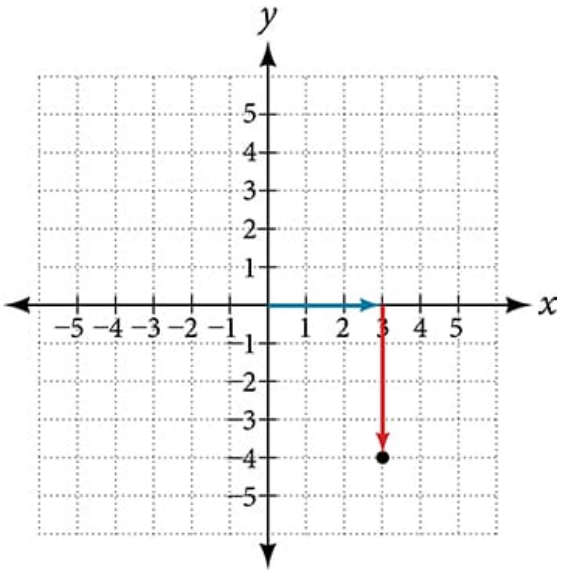


Figure 2.1.3

Example 2.1.3**: Adding Complex Numbers**

We add the real parts and add the imaginary parts.

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i && \text{(regrouping)}\\ &= (3+2)+(-4+5)i && \text{(substituting)}\\ &= 5+i && \text{(simplifying)}\end{aligned}$$

Example 2.1.4**: Multiplying a Complex Number by a Real Number**

Distribute the 4.

$$\begin{aligned}4(2+5i) &= (4\cdot 2)+(4\cdot 5i) && \text{(distributing)}\\ &= 8+20i && \text{(simplifying)}\end{aligned}$$

Example 2.1.5**: Multiplying a Complex Number by a Complex Number**

$$\begin{aligned}(4+3i)(2-5i) &= (4\cdot 2)-(4\cdot 5i)+(3i\cdot 2)-(3i\cdot 5i) && \text{(distributing)}\\ &= 8-20i+6i-15i^2 && \text{(simplifying)}\\ &= 8-14i-15(-1) && \text{(combining like terms, and } i^2=-1)\\ &= 8-14i+15 && \text{(simplifying)}\\ &= 23-14i && \text{(simplifying)}\end{aligned}$$

Theorem: Products of Complex Conjugates

Let $z = a + bi$. Then

$$\begin{aligned}z\cdot\overline{z} &= (a+bi)(a-bi)\\ &= a^2-abi+abi-b^2i^2\\ &= a^2-\cancel{abi}+\cancel{abi}-b^2(-1)\\ &= a^2+b^2\end{aligned}$$

Example 2.1.6**: Finding Complex Conjugates**

a. The number is already in the form $a + bi$. The complex conjugate is $a - bi$, or $2 - i\sqrt{5}$.
b. We can rewrite this number in the form $a + bi$ as $0 - \frac{1}{2}i$. The complex conjugate is $a - bi$, or $0 + \frac{1}{2}i$. This can be written simply as $\frac{1}{2}i$.

Example 2.1.7**: Dividing Complex Numbers**

We begin by writing the problem as a fraction:

$$\frac{2+5i}{4-i}.$$

Then we multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)}.$$

To multiply two complex numbers, we expand the product as we would with polynomials.

$$\begin{aligned}
\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)} &= \frac{8+2i+20i+5i^2}{(4)^2+(-1)^2} && \text{(distributing and using the fact that } (a+bi)(a-bi)=a^2+b^2) \\
&= \frac{8+22i-5}{16+1} && \text{(simplifying)} \\
&= \frac{3+22i}{17} && \text{(simplifying)} \\
&= \frac{3}{17}+\frac{22}{17}i && \text{(separating the real and imaginary parts)}
\end{aligned}$$

Note that this expresses the quotient in standard form.

Example 2.1.8**: Substituting a Complex Number into a Polynomial Function**

$$\begin{aligned}
f(3+i) &= (3+i)^2-5(3+i)+2 && \text{(substituting)} \\
&= (9+6i+i^2)-15-5i+2 && \text{(distributing)} \\
&= 9+6i+(-1)-15-5i+2 && (i^2=-1) \\
&= -5+i && \text{(combining like terms)}
\end{aligned}$$

Example 2.1.9**: Substituting an Imaginary Number in a Rational Function**

$$\begin{aligned}
f(10i) &= \frac{2+10i}{10i+3} && \text{(substituting)} \\
&= \frac{2+10i}{3+10i} && \text{(rewriting the denominator in standard form)} \\
&= \frac{(2+10i)}{(3+10i)} \cdot \frac{(3-10i)}{(3-10i)} && \text{(multiplying by the conjugate of the denominator)} \\
&= \frac{6-20i+30i-100i^2}{(3)^2+(10)^2} && \text{(distributing)} \\
&= \frac{6-20i+30i-100(-1)}{(3)^2+(10)^2} && (i^2=-1) \\
&= \frac{106+10i}{109} && \text{(simplifying)} \\
&= \frac{106}{109}+\frac{10}{109}i && \text{(separating the real and imaginary parts)}
\end{aligned}$$

Theorem: Powers of the Imaginary Unit

Let $p \in \mathbb{W}$ and r be the remainder when p is divided by 4. Then r is either 0, 1, 2, or 3. The Divison Algorithm tells us that

$$\frac{p}{4} = q + \frac{r}{4},$$

where q is the quotient and r is the remainder. Multiplying both sides by 4, we get

$$p = 4q + r.$$

Hence,

$$\begin{aligned}
i^p &= i^{4q+r} && \text{(substituting)} \\
&= i^{4q} \cdot i^r && \text{(Laws of Exponents)} \\
&= (i^4)^q \cdot i^r && \text{(Laws of Exponents)} \\
&= (1)^q \cdot i^r && (i^4=1) \\
&= 1 \cdot i^r \\
&= i^r
\end{aligned}$$

Therefore, all whole-numbered powers of i can be simplified to

$$\begin{aligned}
i^0 &= 1 \\
i^1 &= i \\
i^2 &= -1 \\
i^3 &= i^2 \cdot i = (-1) \cdot i = -i
\end{aligned}$$

Example 2.1.10**: Simplifying Powers of** i

Since $i^4 = 1$, we can simplify the problem by factoring out as many factors of i^4 as possible. To do so, first determine how many times 4 goes into 35: $35 = 4 \cdot 8 + 3$. Therefore,

$$i^{35} = i^{4 \cdot 8+3} = i^{4 \cdot 8} \cdot i^3 = (i^4)^8 \cdot i^3 = 1^8 \cdot i^3 = i^3 = -i.$$

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2.1.1: Resources and Key Concepts

Resources

Videos

- Complex numbers
  - Arithmetic - Sets - Complex Numbers - Definitions, Theorem, and Example
- Complex plane
  - Arithmetic - Sets - Complex Numbers - The Complex Plane and Plotting Complex Numbers
- Adding and Subtracting complex numbers
  - Arithmetic - Sets - Complex Numbers - Adding and Subtracting Complex Numbers
- Multiplying complex numbers
  - Arithmetic - Sets - Complex Numbers - Multiplying Complex Numbers
- Dividing complex numbers
  - Arithmetic - Sets - Complex Numbers - Dividing Complex Numbers
- Powers of i
  - Arithmetic - Sets - Complex Numbers - Simplifying Powers of i

Key Concepts

Definitions

- **Unit Imaginary Number**: The number, denoted as i, which is defined as the square root of negative one ($\sqrt{-1} = i$).
- **Complex Number**: A number that can be expressed in the form $a + bi$, where a and b are real numbers.
  - **Real Part**: For a complex number $a + bi$, a is the real part.
  - **Imaginary Part**: For a complex number $a + bi$, b is the imaginary part.
  - **Imaginary Number**: A complex number $a + bi$ where $a = 0$ and $b \neq 0$.
  - **Standard Form**: The representation of a complex number as $a + bi$.
- **Complex Number System**: The entire set of numbers of the form $a + bi$, symbolically denoted as $\mathbb{C}$.
- **Complex Plane**: A coordinate system used for graphing complex numbers, where the horizontal axis (real axis) represents the real part and the vertical axis (complex axis) represents the imaginary part.
  - **Real Axis**: The horizontal axis in the complex plane.
  - **Complex Axis**: The vertical axis in the complex plane.
- **Addition of Complex Numbers**: If $z_1 = a + bi$ and $z_2 = c + di$, their sum is $z_1 + z_2 = (a + c) + (b + d)i$.
- **Subtraction of Complex Numbers**: If $z_1 = a + bi$ and $z_2 = c + di$, their difference is $z_1 - z_2 = (a - c) + (b - d)i$.
- **Multiplication of Complex Numbers**: If $z_1 = a + bi$ and $z_2 = c + di$, their product is $z_1 \cdot z_2 = (ac - bd) + (ad + bc)i$. (Treat this like binomial multiplication rather than memorizing the formula.)
- **Complex Conjugate**: For a complex number $z = a + bi$, its complex conjugate is $\overline{z} = a - bi$. Similarly, the complex conjugate of $z = a - bi$ is $\overline{z} = a + bi$.
- **Division of Complex Numbers**: For two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ (where $z_2 \neq 0$), their quotient $\frac{z_1}{z_2}$ is found by multiplying the numerator and denominator by the conjugate of the denominator, resulting in $\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$. (Perform the multiplication by the conjugate rather than memorizing this formula.)

Theorems

- **The Square of the Imaginary Unit**: $i^2 = -1$.
- **Products of Complex Conjugates**: Let $z = a + bi$, so that its complex conjugate is $\overline{z} = a - bi$. Then $z \cdot \overline{z} = a^2 + b^2$.
- **Powers of the Imaginary Unit**: Let $p \in \mathbb{W}$ (the Whole Number system) and r be the remainder when p is divided by 4. Then $i^p = i^r$. (This implies $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i$).

Common Mistakes

- Memorizing the formula for the product of two complex numbers instead of treating the multiplication like that of two binomials. The text advises: "you should never memorize the result of this definition. Instead, treat multiplying complex numbers like multiplying binomials."
- Leaving a complex number in the denominator of a fraction when expressing a complex number. The text states: "Any fraction must have a real-valued denominator."
- Memorizing the formula for the division of two complex numbers instead of understanding and applying the process of multiplying the numerator and denominator by the complex conjugate of the denominator. The text advises: "it is best not to memorize this resulting formula. You should be able to just 'do the math.'"

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2.1.2: Homework

Reading Questions

1. What is the definition of the imaginary unit i? What is the value of i^2?
2. How do you express the square root of a negative number, like $\sqrt{-49}$, in terms of i?
3. What are the two main components of a complex number when written in standard form $a + bi$?
4. Describe the complex plane. What does the horizontal axis represent, and what does the vertical axis represent?
5. How do you add two complex numbers, for example, $(3 - 2i) + (1 + 5i)$?
6. How do you subtract two complex numbers, for example, $(5 + i) - (2 - 3i)$?
7. Explain the process of multiplying two complex numbers, such as $(2 + 3i)(1 - i)$. What crucial substitution is made involving i?
8. What is the complex conjugate of a complex number $a + bi$? How is it found?
9. What is the result when a complex number is multiplied by its complex conjugate? (e.g., $(a + bi)(a - bi)$)
10. Why is the complex conjugate used when dividing complex numbers? What is the goal?
11. Outline the steps to divide $\frac{2+5i}{4-i}$, as shown in Example $2.1.2.7$.
12. What are the first four powers of i (i.e., i^1, i^2, i^3, i^4)?
13. How can you simplify a higher power of i, such as i^{35}, using the cyclic nature of its powers? (Refer to Example $2.1.2.10$)
14. If a complex number is $0 + 3i$, what is its real part and what is its imaginary part? Is it considered a pure imaginary number?
15. Explain how to add complex numbers.
16. What is the basic principle in multiplication of complex numbers?
17. Give an example to show the product of two imaginary numbers is not always imaginary.
18. What is a characteristic of the plot of a real number in the complex plane?

Homework

Algebraic

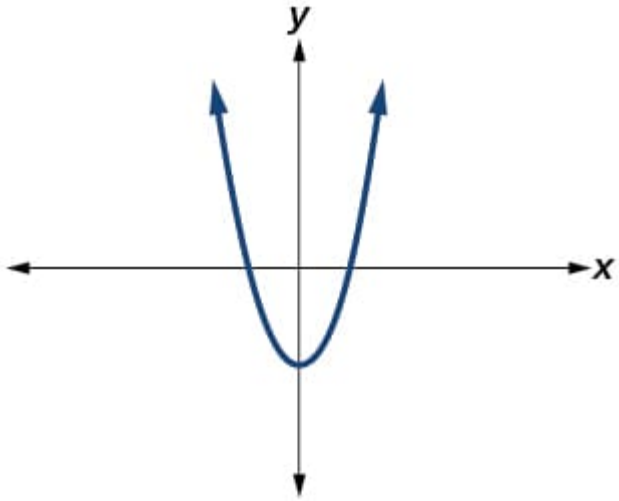
For the following exercises, evaluate the algebraic expressions.

19. If $f(x) = x^2 + x - 4$, evaluate $f(2i)$.
20. If $f(x) = x^3 - 2$, evaluate $f(i)$.
21. If $f(x) = x^2 + 3x + 5$, evaluate $f(2 + i)$.
22. If $f(x) = 2x^2 + x - 3$, evaluate $f(2 - 3i)$.
23. If $f(x) = \frac{x+1}{2-x}$, evaluate $f(5i)$.
24. If $f(x) = \frac{1+2x}{x+3}$, evaluate $f(4i)$.

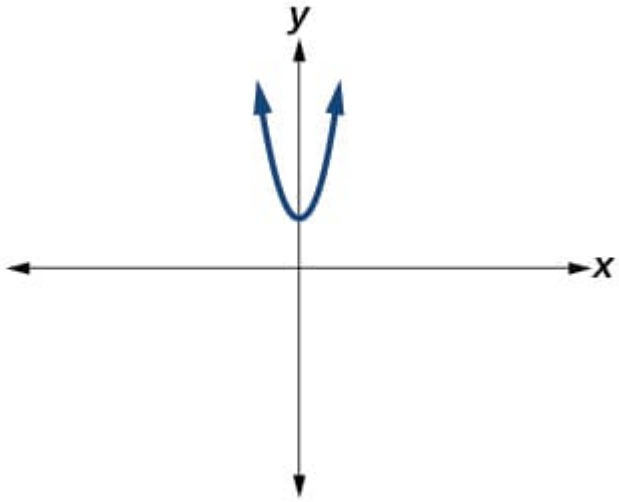
Graphical

For the following exercises, determine the number of real and nonreal solutions for each quadratic function shown.

25.



26.



For the following exercises, plot the complex numbers on the complex plane.

27. $1-2i$
28. $-2+3i$
29. i
30. $-3-4i$

Numeric

For the following exercises, perform the indicated operation and express the result as a simplified complex number.

31. $(3+2i)+(5-3i)$
32. $(-2-4i)+(1+6i)$
33. $(-5+3i)-(6-i)$
34. $(2-3i)-(3+2i)$
35. $(-4+4i)-(-6+9i)$
36. $(2+3i)(4i)$
37. $(5-2i)(3i)$
38. $(6-2i)(5)$
39. $(-2+4i)(8)$
40. $(2+3i)(4-i)$
41. $(-1+2i)(-2+3i)$

42. $(4-2i)(4+2i)$
43. $(3+4i)(3-4i)$
44. $\frac{3+4i}{2}$
45. $\frac{6-2i}{3}$
46. $\frac{-5+3i}{2i}$
47. $\frac{6+4i}{i}$
48. $\frac{2-3i}{4+3i}$
49. $\frac{3+4i}{2-i}$
50. $\frac{2+3i}{2-3i}$
51. $\sqrt{-9}+3\sqrt{-16}$
52. $-\sqrt{-4}-4\sqrt{-25}$
53. $\frac{2+\sqrt{-12}}{2}$
54. $\frac{4+\sqrt{-20}}{2}$
55. i^8
56. i^{15}
57. i^{22}

Technology

For the following exercises, use a calculator to help answer the questions.

58. Evaluate $(1+i)^k$ for $k=4$, 8, and 12. Predict the value if $k=16$.
59. Evaluate $(1-i)^k$ for $k=2$, 6, and 10. Predict the value if $k=14$.
60. Evaluate $(1+i)^k-(1-i)^k$ for $k=4$, 8, and 12. Predict the value if $k=16$.
61. Show that a solution of $x^6+1=0$ is $\frac{\sqrt{3}}{2}+\frac{1}{2}i$.
62. Show that a solution of $x^8-1=0$ is $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$.

Extensions

For the following exercises, evaluate the expressions, writing the result as a simplified complex number.

63. $\frac{1}{i}+\frac{4}{i^3}$
64. $\frac{1}{i^{11}}-\frac{1}{i^{21}}$
65. $i^7(1+i^2)$
66. $i^{-3}+5i^7$
67. $\frac{(2+i)(4-2i)}{(1+i)}$
68. $\frac{(1+3i)(2-4i)}{(1+2i)}$
69. $\frac{(3+i)^2}{(1+2i)^2}$
70. $\frac{3+2i}{2+i}+(4+3i)$
71. $\frac{4+i}{i}+\frac{3-4i}{1-i}$
72. $\frac{3+2i}{1+2i}-\frac{2-3i}{3+i}$

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2.2: Quadratic Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

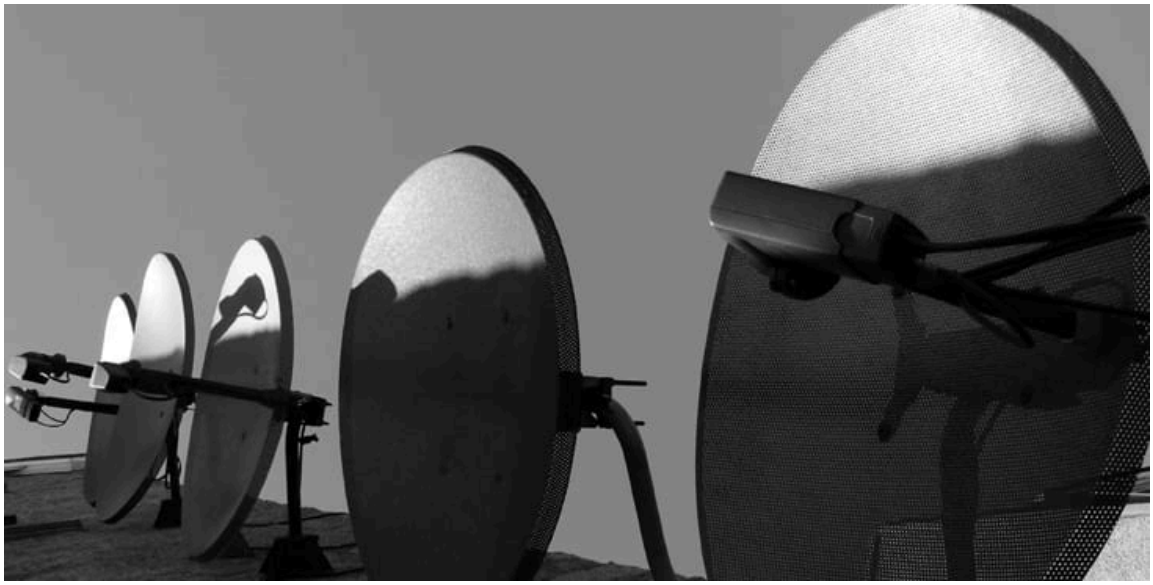


Figure 2.2.1**:** *An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)*

Curved antennas, such as the ones shown in Figure 2.2.1, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

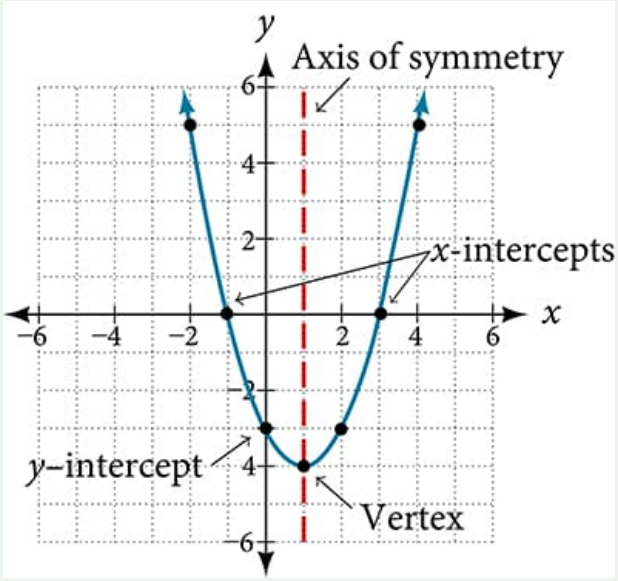
It is assumed that you have already worked with quadratic functions in your prerequisite Algebra course. Therefore, a lot of the theory is mentioned in brief.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a **parabola**.

Definition: Parabola

The shape in the following figure is called a **parabola**. Every parabola has one extreme point called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the **minimum value** of the parabola. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in following figure.



The y-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the **zeros** of the quadratic function, the values of x at which $y = 0$.

Definition: Zeros of Functions

The value c in the domain of the function f is called a **zero** of f if $f(c) = 0$.

While we try our best in Mathematics to avoid multiple definitions for the same thing, you will sometimes hear someone say "**roots**" instead of zeros.

Definition: Roots of Equations

If c solves a given equation, we say that c is a **root** of the equation.

The reason for the ambiguity is that, if c is a zero of the function f, then $f(c) = 0$. That is, given the *equation* $f(x) = 0$, c would be a solution because $f(c) = 0$. Hence, zeros of a function *are* roots of the associated equation (and vice versa).

In any case, I will try my best to use the word "zeros" when speaking of functions and "roots" when speaking of solutions to equations; however, to most professors, either is acceptable.

Example 2.2.1: Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure 2.2.2.

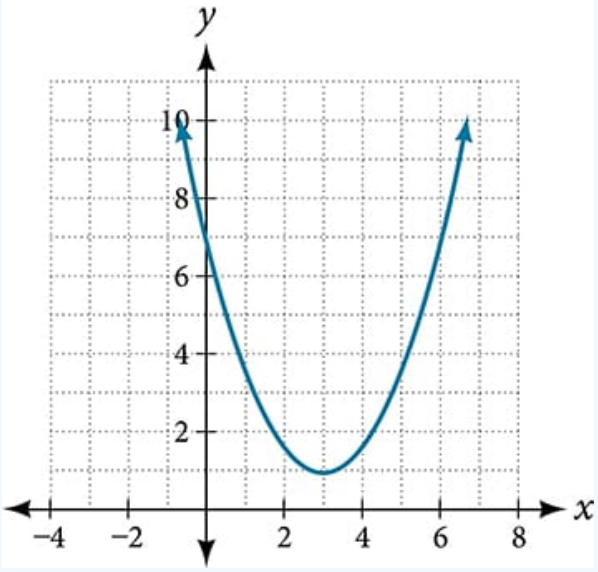


Figure 2.2.2

Solution available at the end of this section.

Quadratic Functions in General Form

Recall from Algebra that the parabola is the graph of a quadratic function. There are two forms for quadratic functions (and polynomials, in general) — **general form** and **standard form**. We begin our review with the general form of a quadratic function.

Definition: General Form of a Quadratic Function

The **general form** of a quadratic function is

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are real numbers and $a \neq 0$.

It was shown in your prerequisite Algebra course that the parabola opens upward if $a > 0$. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry. To do so, we first reintroduce a core theorem in all Algebra courses.

Theorem: Quadratic Formula

The roots of the equation

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers and $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Proof available at the end of this section.

With the Quadratic Formula in hand, we now can find the equation for the **axis of symmetry**.

Theorem: Axis of Symmetry for a Quadratic Function

The vertical line, $x = -\frac{b}{2a}$, is the axis of symmetry for the graph of the quadratic function $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$.

Proof available at the end of this section.

Figure $2.2.3$ represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense, because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

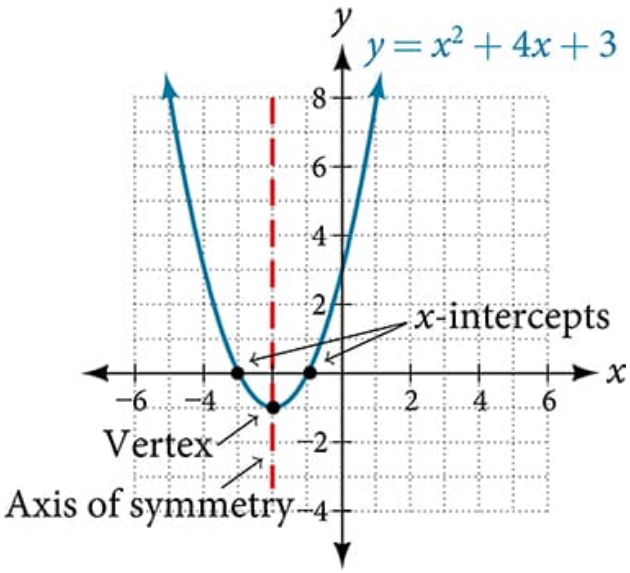


Figure $2.2.3$

Since the vertex always occurs where the axis of symmetry intersects the parabola, we immediately get the following theorem.

Theorem: Coordinates of the Vertex

The vertex of the quadratic function $f(x) = ax^2 + bx + c$ is located at the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Proof available at the end of this section.

The general form of a quadratic function is commonly encountered in applications and Physics. While most Algebra textbooks focus on the second form (standard form), you should strive to understand both forms.

Quadratic Functions in Standard Form

The general form of a quadratic can be incredibly useful; however, we cannot escape the need to talk about a quadratic function's alternate identity — the **standard form**.

Definition: Standard Form of a Quadratic Function

The **standard form** (also called **vertex form**) of a quadratic function is

$$f(x) = a\,(x-h)^2 + k,$$

where a, h, and k are real numbers and $a \neq 0$.

The standard form of a quadratic function has a very appealing feature.

Theorem: Vertex of a Quadratic Function in Standard Form

The vertex of the quadratic function given in the standard form

$$f(x) = a\,(x-h)^2 + k$$

is (h, k).

That is, if given a quadratic function in standard form, you have little-to-no work involved in finding its vertex.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure $2.2.4$ represents the graph of the quadratic function written in standard form as $y = -3(x+2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

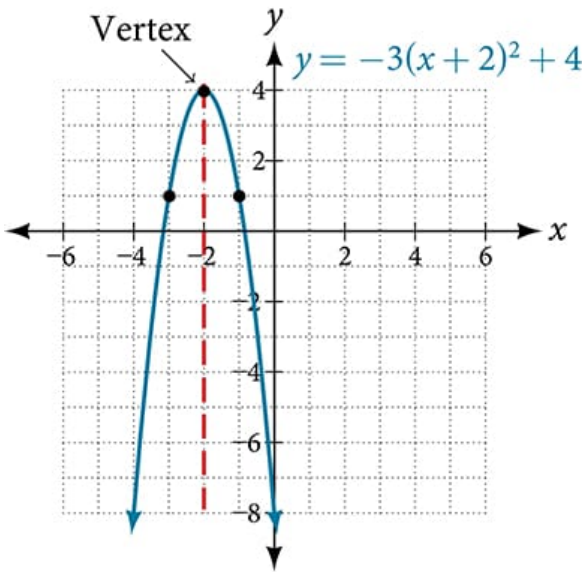


Figure $2.2.4$

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. Figure $2.2.5$ is the graph of this basic function.

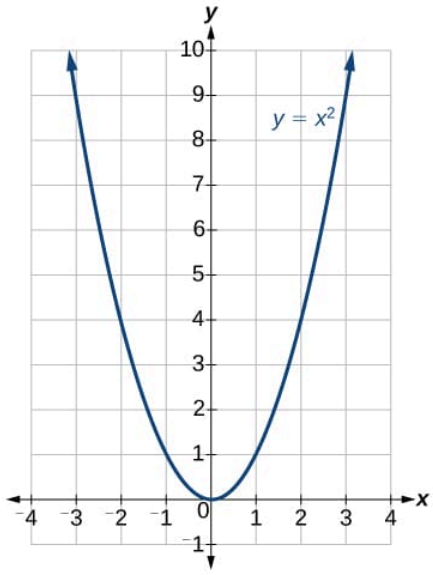


Figure 2.2.5

From our previous work with transformations of functions, if $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In Figure 2.2.4, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In Figure 2.2.4, $h < 0$, so the graph is shifted 2 units to the left. Again, from transformations, we know that the magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph *appears* to become narrower (this is an illusion), and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph *appears* to become wider (another illusion), but in fact there is a vertical compression. In Figure 2.2.4, $|a| > 1$, so the graph *appears* narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$\begin{aligned} a(x-h)^2 + k &= ax^2 + bx + c \\ \implies \quad ax^2 - 2ahx + (ah^2 + k) &= ax^2 + bx + c \end{aligned}$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah = b \implies h = -\frac{b}{2a}.$$

This is the axis of symmetry we defined earlier. Setting the constant terms equal, we get

$$\begin{aligned} ah^2 + k &= c \\ \implies \quad k &= c - ah^2 \\ &= c - a\left(-\frac{b}{2a}\right)^2 \\ &= c - \frac{b^2}{4a}. \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h) = k$.

Interactive Element: Vertex Form and Translation

The following Interactive Element might help you visualize how h and k in $y = (x-h)^2 + k$ position the parabola.

Interact: Drag the orange vertex point anywhere on the plane.

Observation: The vertex is always (h, k); changing h slides the parabola horizontally and changing k slides it vertically, with the shape unchanged from $y = x^2$.

Example 2.2.2: Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in Figure 2.2.6 as a transformation of $f(x) = x^2$, and then expand the formula and simplify terms to write the equation in general form.

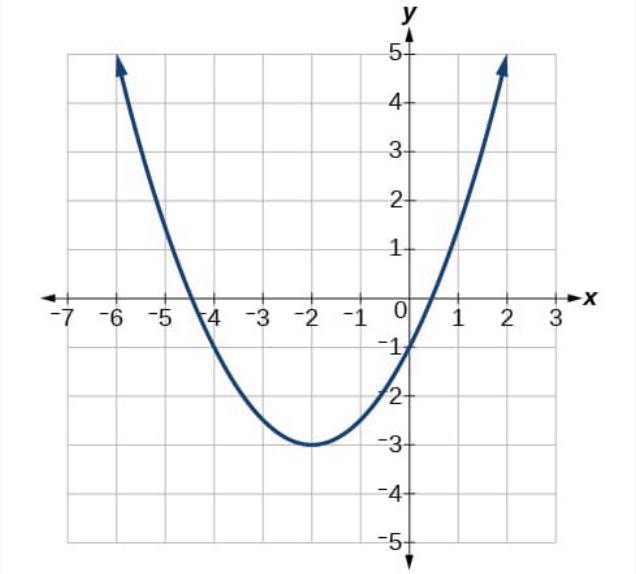


Figure 2.2.6

Solution available at the end of this section.

Checkpoint 2.2.2

A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 2.2.7. Find an equation for the path of the ball. Does the shooter make the basket?

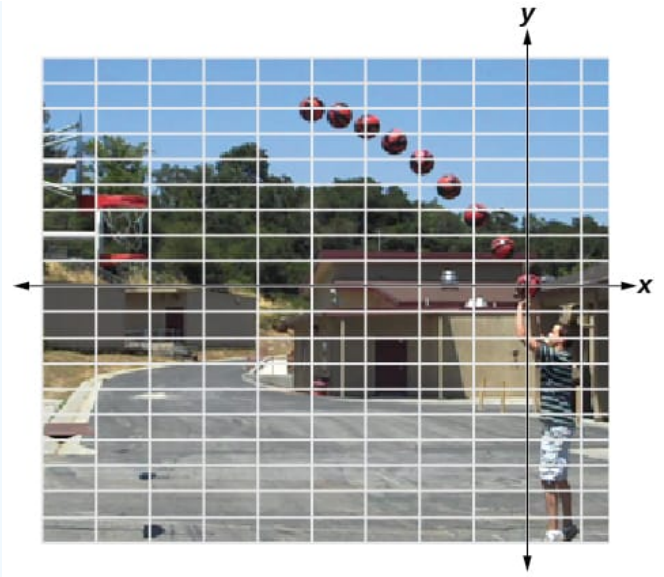


Figure 2.2.7**:** *(credit: modification of work by Dan Meyer)*

Example 2.2.3: Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution available at the end of this section.

If given a quadratic function in general form, most students are going to use the formula $-\frac{b}{2a}$ to find the x-value of the vertex (and substitute this into the original quadratic function to find the corresponding y-value for the vertex). This is perfectly fine, and it will always lead you to the vertex of a given quadratic function; however, another approach, which is inevitably going to be necessary in this course, is to **complete the square** like you did in your Elementary Algebra course (and as we did in the proof of the Quadratic Formula).

Example 2.2.4: Rewriting in Standard Form Using Completing the Square

Rewrite the quadratic function $f(x) = 2x^2 - 6x + 7$ in standard (vertex) form by Completing the Square.

Solution available at the end of this section.

Important Skill: Completing the Square

The purpose of being introduced to the process of Completing the Square in your Algebra courses was to prove the Quadratic Formula. We now see that this same process can be used to rewrite a quadratic function in vertex form; however, these are not the only uses of Completing the Square. We will use this skill again in this course when working with conic sections. In Calculus II, you will find many opportunities to use this very same skill when manipulating a seemingly impossible-to-work-with expression into something more simple.

Checkpoint 2.2.4

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

At the beginning of Calculus, the difference quotient is used on several different types of functions; however, one of the most common encounters is with quadratic functions.

Example 2.2.5: Revisiting the Difference Quotient

Find the difference quotient for the function.

$$f(x) = -3x^2 + 2x + 1$$

Solution available at the end of this section.

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers; however, because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

Theorem: Domain and Range of a Quadratic Function

The domain of any quadratic function is all real numbers.

The range of a quadratic function written in general form, $f(x) = ax^2 + bx + c$, with a positive a value is $f(x) \geq f\left(-\frac{b}{2a}\right)$, or $\left[f\left(-\frac{b}{2a}\right), \infty\right)$.

The range of a quadratic function written in general form with a negative a value is $f(x) \leq f\left(-\frac{b}{2a}\right)$, or $\left(-\infty, f\left(-\frac{b}{2a}\right)\right]$.

The range of a quadratic function written in standard form, $f(x) = a(x-h)^2 + k$, with a positive a value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \leq k$.

The theorem above is only provided for completeness — it should not be memorized.

Example 2.2.6: Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution available at the end of this section.

Checkpoint 2.2.6

Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the parabola. We can see the maximum and minimum values in Figure 2.2.8.

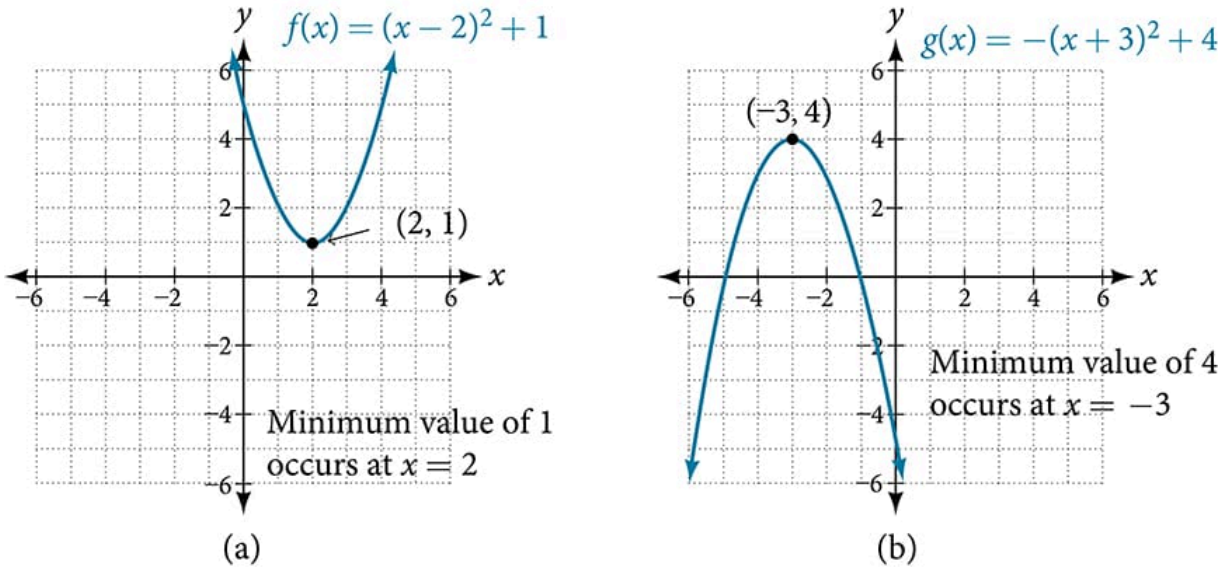


Figure 2.2.8

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Example 2.2.7: Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
b. What dimensions should she make her garden to maximize the enclosed area?

Solutions available at the end of this section.

Example 2.2.7 also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 2.2.10.

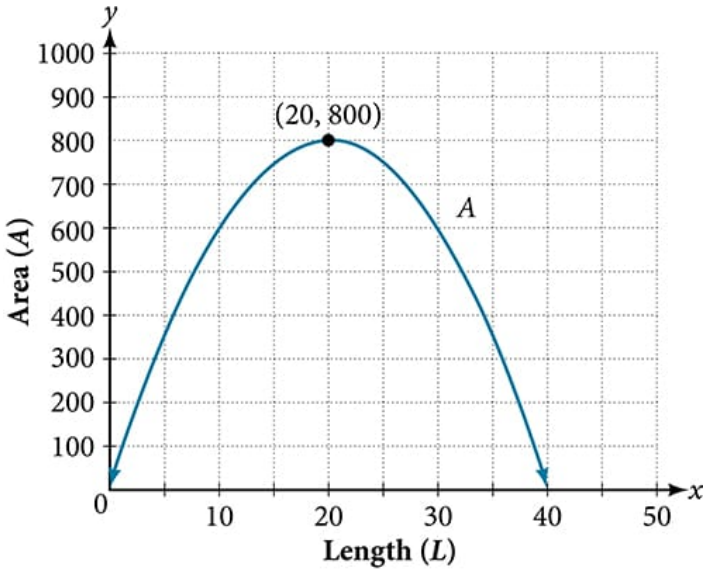


Figure 2.2.10

Example 2.2.8: Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution available at the end of this section.

As before, Example 2.2.8 could also be solved by graphing the quadratic as in Figure 2.2.11. We can see the maximum revenue on a graph of the quadratic function.

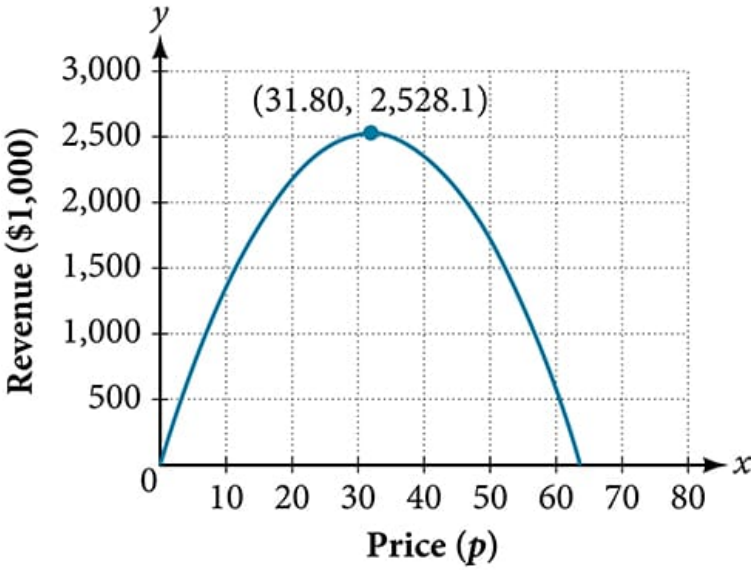


Figure 2.2.11

Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure $2.2.12$ that the number of x-intercepts can vary depending upon the location of the graph.

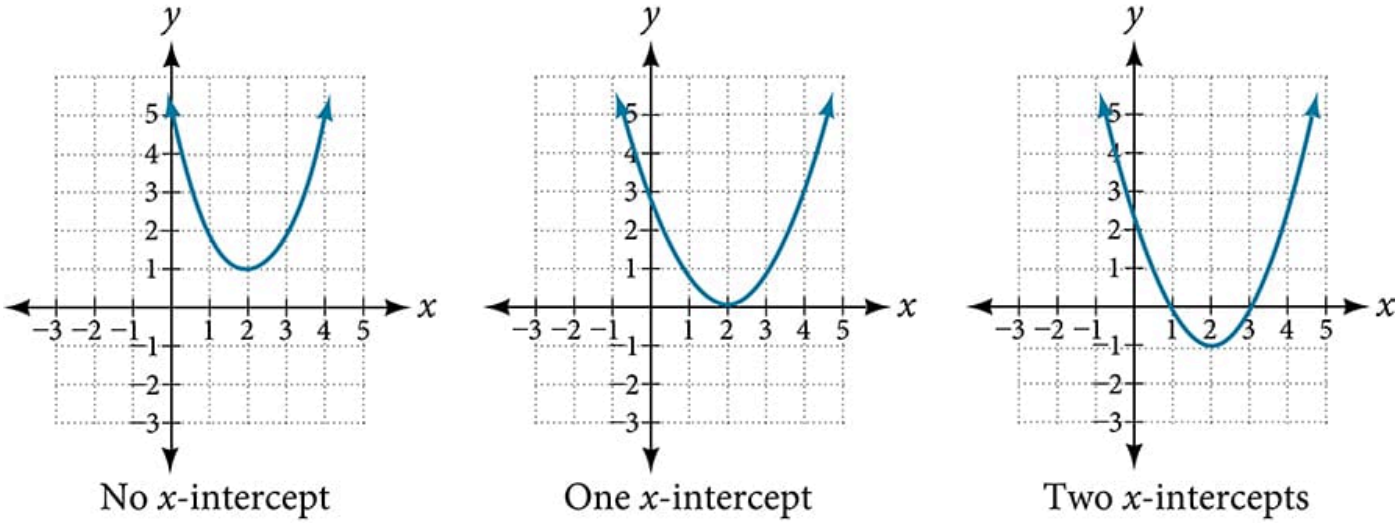


Figure $2.2.12$**:** *Number of x-intercepts of a parabola*

Example $2.2.9$: Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution available at the end of this section.

By graphing the function, we can confirm that the graph crosses the y-axis at $(0, -2)$. We can also confirm that the graph crosses the x-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$. See Figure $2.2.13$.

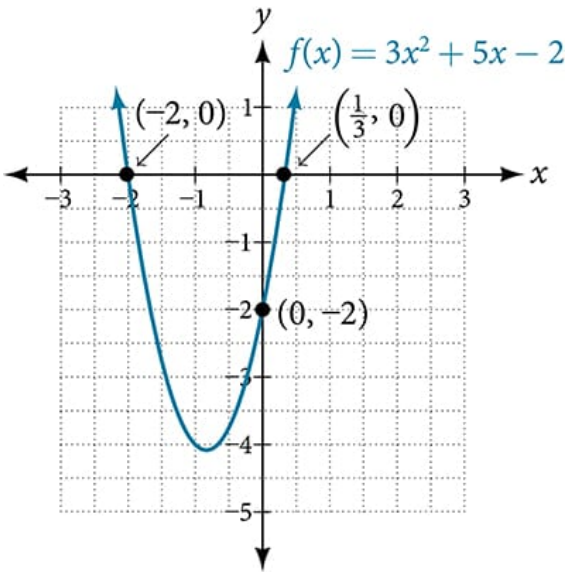


Figure $2.2.13$

Example 2.2.9 illustrates a strength of having the general form of a quadratic function — the y-intercept is the constant term. That is, given $f(x) = ax^2 + bx + c$, the y-intercept is $f(0) = c$. This is only true, however, when the quadratic is given in general form.

Another lesson to be learned from Example 2.2.9 is that to solve $ax^2 + bx + c = 0$, we don't have to always resort to the Quadratic Formula. In fact, whenever you need to solve a quadratic equation, you should spend about 30 seconds or so trying to factor the given quadratic before plunging into the algorithm called the Quadratic Formula.

As Figure 2.2.12 illustrates, there are some quadratic functions that do not have x-intercepts. Algebraically, this occurs when the given quadratic function is **prime**. Just like a prime number, a prime quadratic is not factorable (across the real number system). That is, no matter how ingenious we are, we will never be able to write a prime quadratic in the form $(c_1x + d_1)(c_2x + d_2)$ (where c_1, c_2, d_1, and d_2 are real numbers).

In Mathematics, we always strive to develop methods to reduce our workload. It would be nice to know ahead of time that the quadratic function $f(x) = ax^2 + bx + c$ is prime. Knowing this would allow us to skip trying to find any x-intercepts. Luckily, we have such a method.

The x-intercepts of a quadratic function occur when $ax^2 + bx + c = 0$. We know from the Quadratic Formula that the solutions to this equation are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. However, the x-axis is a *real number* axis. That is, only real numbers exist along the x-axis. Hence, our x-intercepts *cannot* be imaginary numbers.

From our previous work, we know we have an imaginary number if our expression contains the square root of a negative number. Therefore, we don't want the radicand in the numerator of the Quadratic Formula to be negative. That is, we require $b^2 - 4ac \geq 0$. This expression is so important in Mathematics, it deserves its own name.

Definition: Discriminant

The **discriminant** of the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, is the quantity

$$D = b^2 - 4ac.$$

From our discussion above, if the discriminant is negative, the quadratic function will not have any x-intercepts. Likewise, if the discriminant is negative, the equation $ax^2 + bx + c = 0$ will not have real solutions.

If the discriminant is 0, then (from the Quadratic Formula) the x-intercepts for the quadratic function would occur at

$$x = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}.$$

Since this is a single value, a discriminant of 0 implies that the graph of the quadratic function has only one x-intercept (as seen in the middle frame of Figure 2.2.12).

When the discriminant is positive, the quadratic function has two real roots, which means its graph crosses the x-axis twice.

Theorem: Zeros of a Quadratic Function and the Discriminant

Let

$$f(x) = ax^2 + bx + c.$$

The graph of f has

- two x-intercepts if the discriminant is positive,
- one x-intercept if the discriminant is 0, and
- no x-intercepts if the discriminant is negative.

Another way to say this is the equation, $ax^2 + bx + c = 0$, has

- two real roots if the discriminant is positive,
- one real root if the discriminant is 0, and

- two non-real roots if the discriminant is negative.

Interactive Element: The Discriminant and x-Intercepts

The following Interactive Element might help you visualize how the number of x-intercepts depends on where the vertex sits relative to the x-axis.

Interact: Drag the red vertex point up and down along the y-axis.

Observation: Two x-intercepts appear when the vertex is below the axis, exactly one when it touches, and none when it rises above — the geometric meaning of a positive, zero, or negative discriminant.

Example 2.2.10: Using the Discriminant to Save Time

Find the y- and x-intercepts of the quadratic $f(x) = -3x^2 + 4x - 2$.

Solution available at the end of this section.

It is important to clarify that finding the x-intercepts (or roots) of a quadratic function and *solving* a quadratic equation are often, but not always, the same thing. How are these processes different? Since the horizontal intercepts of a graph can only be real numbers, we stop our search for such intercepts if the discriminant of the quadratic function is negative. *In contrast*, the solutions to a random quadratic equation *can* be imaginary, and therefore the discriminant can be negative.

Example 2.2.11: Solving a Quadratic Equation

Find all the solutions of the quadratic equation.

$$0 = -3x^2 + 4x - 2$$

Solution available at the end of this section.

Example 2.2.12: Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution available at the end of this section.

Again, we can check our work for Example 2.2.12 by graphing the given function on a graphing utility and observing the x-intercepts. See Figure 2.2.14.

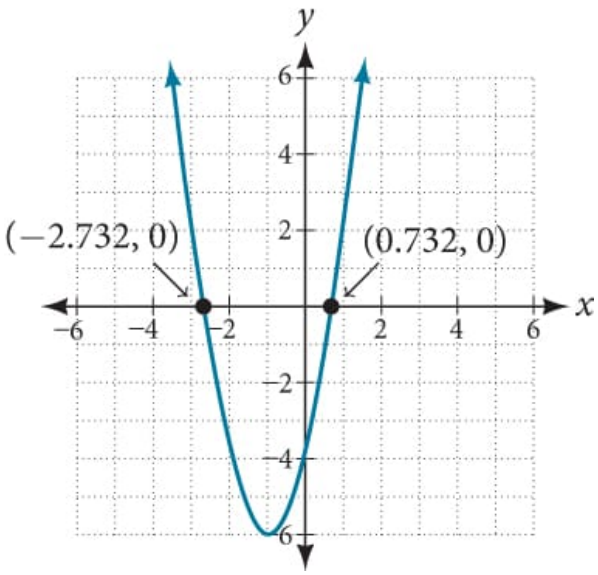


Figure 2.2.14

Checkpoint 2.2.12

In a separate Checkpoint, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the y- and x -intercepts (if any).

Solving Quadratic Inequalities

We now turn our attention to **quadratic inequalities**.

Definition: Quadratic Inequality

A **quadratic inequality** in x is an inequality that can be written in one of the forms

$$ax^2 + bx + c < 0, \quad ax^2 + bx + c \leq 0, \quad ax^2 + bx + c > 0, \quad \text{or} \quad ax^2 + bx + c \geq 0,$$

where a, b, and c are real numbers with $a \neq 0$.

If we consider $f(x) = x^2 - x - 6$, then solving $x^2 - x - 6 < 0$ corresponds graphically to finding the values of x for which the graph of $y = f(x) = x^2 - x - 6$ (the parabola) is below the graph of $y = 0$ (the x-axis). See Figure 2.2.15.

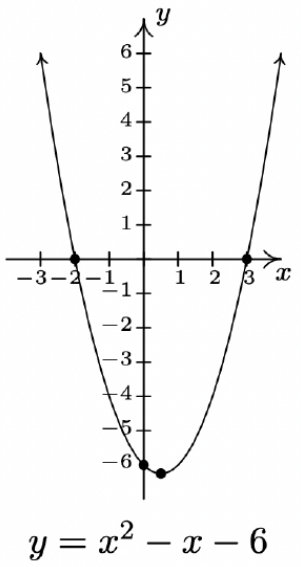


Figure 2.2.15

We can see that the graph of f dips below the x-axis between its two x-intercepts. The zeros of f are $x = -2$ and $x = 3$ in this case and they divide the domain (the x-axis) into three intervals: $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$. For every number in $(-\infty, -2)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty, -2)$. Similarly, $f(x) < 0$ for all x in $(-2, 3)$, and $f(x) > 0$ for all x in $(3, \infty)$. Thus, the solution to $x^2 - x - 6 < 0$ is all x in the interval $(-2, 3)$.

The method introduced above (graphing the quadratic function and determining when the graph satisfies the inequality) is called the **graphical method** of solving a quadratic inequality. This is the method I prefer to teach in my courses because it simultaneously reinforces graphing and it is *much* faster (in the long run) than the alternative method. Before discussing the alternative method, let's look at a couple examples.

Example 2.2.13: Solving Quadratic Inequalities Using the Graphing Method

Solve the following inequalities using the graphing method.

a. $2x^2 \leq 3 - x$
b. $x^2 - 2x > 1$

Solutions available at the end of this section.

The alternative method is called the **sign chart method** of solving a quadratic inequality. Like the graphical method, we start by finding the x-intercepts. We then mark those intercepts on a number line. For example, considering our original inequality $x^2 - x - 6 < 0$, we would create the sign chart in Figure 2.2.18.

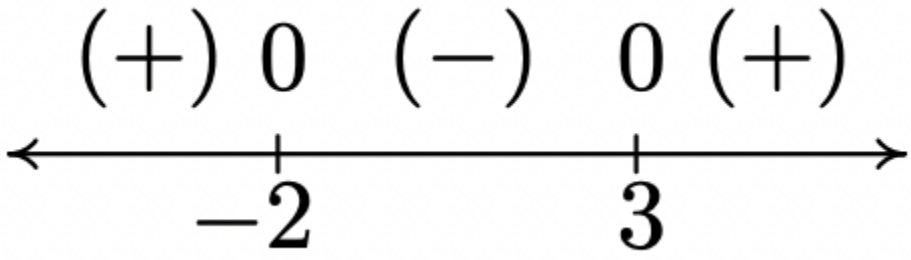


Figure 2.2.18

Here, the $(+)$ above a portion of the number line indicates $f(x) > 0$ for those values of x; the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the number line are the zeros of f, so we place 0 above them. We see at once that the solution to $f(x) < 0$ is $(-2, 3)$.

An important property of quadratic functions is that if the function is positive at one point and negative at another, the function must have at least one zero in between. Graphically, this means that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis. This allows us to determine the sign of *all* of the function values on a given interval by testing the function at just *one* value in the interval. We use this idea in the following example.

Example 2.2.14: Solving Quadratic Inequalities Using the Sign Chart Method

Solve the following inequalities analytically using sign diagrams. Verify your answer graphically.

a. $2x^2 \leq 3 - x$
b. $x^2 - 2x > 1$

Solutions available at the end of this section.

Applications Involving Quadratic Functions

It should come as no surprise that quadratic functions appear in a lot of applications — from business to physics. You will continue to encounter them in such contexts when you get to Calculus. Therefore, let's take a look at a couple common applications.

Example 2.2.15: Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

a. When does the ball reach the maximum height?
b. What is the maximum height of the ball?
c. When does the ball hit the ground?
d. When will the ball be above 104 feet?

Solutions available at the end of this section.

It's important to note that Figure 2.2.22 in Example 2.2.15 doesn't represent the path of the ball. We arrived at the graph in Figure 2.2.22 by manipulating an inequality based on the function for the path of the ball's travel.

Checkpoint 2.2.15

A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Solving Absolute Value Equations Involving Quadratics

We finish this section with two synthesis topics:

1. Solving equations involving the composition of the absolute value function and a quadratic function, and
2. Graphing the absolute value of a quadratic function.

Both of these topics will seem more complex than our previous work because you are combining previously-learned skills (called synthesizing skills). The crucial thing to remember is the piecewise definition of the absolute value (repeated below because, yes, it's that important).

Definition: Absolute Value Function

The absolute value function is defined to be $f(x) = |x|$, where

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

We will be combining this knowledge with our graphing skills for quadratic functions.

Example 2.2.16: Solve Absolute Value Equations Involving Quadratic Expressions

Solve each absolute value equation.

a. $\left|2x^2 + x - 3\right| = 0$
b. $\left|2x^2 + x - 3\right| = -6$
c. $\left|x^2 - 5x\right| = 6$

Solutions available at the end of this section.

It is very important to check that the solutions from Example 2.2.16 occur in the proper intervals. That is, we arrive at the solutions $x = -1$ and $x = 6$ whenever $x \in (-\infty, 0] \cup [5, \infty)$. Both of these solutions exist in these intervals, so we are fine. I leave it to you to check that the second solution set exists within the interval $(0, 5)$.

We conclude this section with a graph of a more complicated absolute value function.

Example 2.2.17: Graph the Absolute Value of a Quadratic Function

Graph $f(x) = |x^2 - x - 6|$.

Solution available at the end of this section.

If we take a step back and look at the graphs of g and f in the Example 2.2.17, we notice that to obtain the graph of f from the graph of g, we reflect a *portion* of the graph of g about the x-axis. We can see this analytically by substituting $g(x) = x^2 - x - 6$ into the formula for $f(x)$ and calling to mind the concept of reflections from transformations of functions.

$$f(x) = \begin{cases} g(x), & \text{if} \quad g(x) \geq 0 \\ -g(x), & \text{if} \quad g(x) < 0 \end{cases}$$

The function f is defined so that when $g(x)$ is negative (i.e., when its graph is below the x-axis), the graph of f is its reflection across the x-axis. This is a general template to graph functions of the form $f(x) = |g(x)|$. From this perspective, the graph of $f(x) = |x|$ can be obtained by reflecting the portion of the line $g(x) = x$ which is below the x-axis back above the x-axis creating the characteristic "vee" shape.

Interactive Element: The Absolute Value of a Quadratic

The following Interactive Element might help you visualize how the graph of $f(x) = |g(x)|$ is obtained from a parabola g.

Interact: Drag the red vertex to reposition the dashed parabola g.

Observation: The solid graph of $|g(x)|$ reflects every below-axis piece of g up across the x-axis, while leaving the pieces on or above the axis alone.

Focus on Calculus — Solving Equations that are Quadratic-in-Form

You have already solved quite a few quadratic equations throughout your academic journey in Mathematics; however, as you approach Calculus, you will need to focus your attention on pattern recognition instead of expecting equations identical in form to $ax^2 + bx + c = 0$. We must begin to recognize that

$$a\blacksquare^2 + b\blacksquare + c = 0$$

is nearly this same equation.

Definition: Quadratic-in-Form

An equation in the variable x is **quadratic-in-form** if it can be rewritten as

$$au^2 + bu + c = 0,$$

where a, b, and c are constants with $a \neq 0$, and u represents an algebraic expression in x.

It turns out that any equation that is quadratic-in-form can be solved using methods similar to those from our Elementary and Intermediate Algebra courses. Let's illustrate the process with a few examples.

Example 2.2.18: Solving an Equation that is Quadratic-in-Form

Solve $3(3x-1)^2 + 13(3x-1) - 10 = 0$.

Solution available at the end of this section.

You will be using the u-substitution process illustrated in Example 2.2.18 often throughout Calculus. As such, it's best to start looking for patterns like this from this point forward.

Example 2.2.19: Solving an Equation that is Quadratic-in-Form

Solve $x^4 - 3x^2 = 45 + x^2$.

Solution available at the end of this section.

Success in College: I Understood It... Until I Left Class

You understand a concept when the professor does it in class, but you're lost when trying it on your own - this is very common. There are two general remedies to this problem.

The first revolves around how the information is delivered and how you engage with it. We tend to retain the least when only *listening* (maybe this is why those who passively sit in the back of a classroom and zone out end up having the highest failure rate). We gain a little more retention when we *see* someone do something (e.g., watching your professor, but not taking notes). A substantial leap in retention and understanding occurs when we *write* concepts down (e.g., taking notes during lectures). Finally, the biggest gains in understanding and retention occur when we *do* work on our own (e.g., practicing, doing homework, independently investigating a concept).

This kind of makes sense, right? This is why you see the most successful students in class attempting the examples before the professor starts them. It forces the information into their memory.

It gets even better if we involve "real-world" examples with manipulatives. Imagine learning about mathematical models that deal with finance (exponential and simple interest formulas) by *actually* investing. How cool would it be to build a fence that borders a river by using minimization techniques learned in Algebra? The problem with math and how it is taught is that we hear about the applications but nobody wants to go out and do them. If you had more physical or engaging examples, your chance of success would skyrocket. Unfortunately, there is only a limited amount of time in class, and this kind of teaching would require students to allot several more hours a day to their math class (however, the benefit would be enormous).

The second remedy is squarely on your shoulders. You will never master a skill by putting only half of your effort into it. To excel and enjoy mathematics, you must immerse yourself in it. Imagine yourself as a mathematician and look at the world differently. As you drive your car to work, make a mental graph of the distance you have traveled versus the time you have been driving (if you're in Calculus, then interpret the slope of the tangent line to this graph at any given point as the derivative). When you purchase items, do the math in your head... yes, even the tax! When you buy a car, make a function that describes the cost of owning the car! You'd be surprised at how helpful mathematics is.

In college, class time is limited. Often, class is over before students feel confident about new concepts. They leave class worried they won't be able to do the homework. These are just some suggestions, but others in your class might have some wonderful additions to this list. Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. How often does this happen to you (understanding in class but completely lost out of class)? When it does, what do you do about it?
2. How often do you think this happens to the other class members?

Success in Mathematics: Holding Onto Old, Broken Techniques

It's common for students to learn a problem-solving technique years ago and not want to learn a different method because they are comfortable with their old habits. Unfortunately, we often fail to recognize that if our old, comfortable technique worked well, we should be much further along in our mathematical journey.

This adherence to comfort is widespread for all human beings in various situations. Whether it's a dead-end job, an abusive relationship, or a bad habit, we tend to ignore signals—remaining in a known situation rather than taking the chance to improve and evolve.

In general, when your professor gives you advice or a different approach to doing something, they are doing so from the lens of years of experience seeing students repeat old, broken techniques or habits that tend to lead students to make mistakes. Paying attention to these words of wisdom is critical.

EXAMPLE

Simplify the compound rational expression

$$\frac{-\frac{2}{(x+h)^2}+\frac{2}{x^2}}{h}$$

Students stuck in old habits tend to find the common denominator of the fractions in the numerator, make equivalent fractions, add fractions, divide the result by h, then simplify. This entire process looks like the following:

$$\begin{aligned}
\frac{-\frac{2}{(x+h)^2}+\frac{2}{x^2}}{h} &= \frac{-\frac{2x^2}{x^2(x+h)^2}+\frac{2(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \frac{\frac{-2x^2+2(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \frac{-2x^2+2(x+h)^2}{x^2(x+h)^2} \div h \\
&= \frac{-2x^2+2(x+h)^2}{x^2(x+h)^2} \cdot \frac{1}{h} \\
&= \frac{-2x^2+2(x+h)^2}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2(x^2+2hx+h^2)}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2x^2+4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{h(4x+2h)}{hx^2(x+h)^2} \\
&= \frac{\cancel{h}(4x+2h)}{\cancel{h}x^2(x+h)^2} \\
&= \frac{(4x+2h)}{x^2(x+h)^2}
\end{aligned}$$

Compare this to the advice given by your math professor to begin by multiplying both numerator and denominator of the original expression by the LCD of all minor fractions within the compound rational expression.

$$\begin{aligned}
\frac{-\frac{2}{(x+h)^2}+\frac{2}{x^2}}{h} &= \frac{-2x^2+2(x+h)^2}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2(x^2+2hx+h^2)}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2x^2+4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{h(4x+2h)}{hx^2(x+h)^2} \\
&= \frac{\cancel{h}(4x+2h)}{\cancel{h}x^2(x+h)^2} \\
&= \frac{(4x+2h)}{x^2(x+h)^2}
\end{aligned}$$

As you can see, the second method is cleaner and less complex.

The moral of the story is, when learning, try to be open to... learn.

Answers

Section Preview

This *very dense* section covers everything the student will need to know about quadratic functions. Plan to spend a bit of time here as the topics go through graphing quadratics, optimization problems, the discriminant, solving equations that are quadratic-in-form, solving absolute value equations involving quadratics and quadratic inequalities, and graphing the absolute value of a quadratic function. Again, this is a *very* dense section.

Section Preview

- **Arithmetic**
  - Adding and Subtracting Fractions: Combining the constant terms over a common denominator while completing the square in Example 4 ($\frac{14}{2} - \frac{9}{2} = \frac{5}{2}$).
- **Solving Equations**
  - Solving Absolute Value Equations: The case-splitting used to solve $|x^2 - 5x| = 6$ in Example 16 rests on already knowing how a basic absolute value equation is broken into separate equations.
  - Solving Quadratic Equations by Completing the Square: Rewriting $f(x) = 2x^2 - 6x + 7$ in vertex form in Example 4, which the section explicitly invokes as an "older skill" rather than re-teaching.
  - Solving Quadratic Equations Using the Quadratic Formula: Producing irrational and complex roots in Examples 11, 12, and 15; the section reintroduces the formula but assumes the student can already execute the procedure.

Section Preview

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.
- Use the discriminant to determine the number of solutions to a quadratic equation.
- Solve equations that are quadratic-in-form.
- Solve a quadratic inequality using the graphical and sign chart methods.
- Solve an absolute value equation involving a quadratic.
- Graph the absolute value of a quadratic function.

Example 2.2.1: Identifying the Characteristics of a Parabola

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at $(0, 7)$, so this is the y-intercept.

Theorem: Quadratic Formula

Suppose a, b, and c are real numbers, where $a \neq 0$. Then

$$
\begin{aligned}
& & ax^2 + bx + c &= 0 & & \\
\implies & & ax^2 + bx &= -c & & \text{(subtracting } c \text{ from both sides)} \\
\implies & & x^2 + \frac{b}{a}x &= -\frac{c}{a} & & \text{(dividing both sides by } a, \text{ where } a \neq 0) \\
\implies & x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 & & \left(\text{completing the square on the left side by adding } \left(\frac{b}{2a}\right)^2 \text{ to both sides}\right) \\
\implies & & \left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 & & \text{(factoring the left side)} \\
\implies & & \left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} + \frac{b^2}{4a^2} & & \text{(Laws of Exponents)} \\
\implies & & \left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} & & \text{(getting common denominators by multiplying numerator and denominator of the first term by } 4a) \\
\implies & & \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} & & \text{(combining fractions with like denominators)} \\
\implies & & x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} & & \text{(Extraction of Roots)} \\
\implies & & x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2\sqrt{a^2}} & & \text{(Properties of Radicals)} \\
\implies & & x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} & & \left(\sqrt{a^2} = \pm a, \text{ but this gets "absorbed" into the } \pm \text{ out front}\right) \\
\implies & & x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} & & \left(\text{subtracting } \frac{b}{2a} \text{ from both sides}\right) \\
\implies & & x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} & & \text{(combining fractions with like denominators)}
\end{aligned}
$$

Theorem: Axis of Symmetry for a Quadratic Function

To avoid complications, we only submit the proof for two cases: the quadratic function has one zero or it has two real zeros (the case for no real zeros can be derived geometrically from the latter case).

Case 1 (1 Zero): Suppose the quadratic function has only one zero. Then the Quadratic Formula must return $x = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}$ (if the radicand in the numerator was not 0, the Quadratic Formula would return *two* zeros, $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$). Hence, the vertex of the parabola (where the axis of symmetry crosses)

occurs when $x = -\frac{b}{2a}$. That is, the axis of symmetry is $x = -\frac{b}{2a}$.

Case 2 (2 Real Zeros): Suppose the quadratic function has two real zeros. Then the Quadratic Formula shows us that these occur at $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$. Since a parabola is symmetric about its axis of symmetry, the axis of symmetry must be halfway between these two x-values. Averaging the zeros for our function, we get

$$\begin{aligned}
\frac{\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}}{2} &= \frac{\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}}{2} \cdot \frac{2a}{2a} && \text{(simplifying the compound rational expression)} \\
&= \frac{-b+\sqrt{b^2-4ac} - b - \sqrt{b^2-4ac}}{4a} && \text{(distributing)} \\
&= \frac{-2b}{4a} && \text{(simplifying)} \\
&= -\frac{b}{2a} && \text{(simplifying)}
\end{aligned}$$

Hence, the axis of symmetry is $x = -\frac{b}{2a}$.

Theorem: Coordinates of the Vertex

Since the vertex occurs where the axis of symmetry intersects the graph of the quadratic function, the x-coordinate of this intersection is $x = -\frac{b}{2a}$. The corresponding y-coordinate is $f\left(-\frac{b}{2a}\right)$.

Example 2.2.2: Writing the Equation of a Quadratic Function from the Graph

We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$. Substituting the coordinates of any point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$\begin{aligned}
-1 &= a(0+2)^2 - 3 \\
\implies 2 &= 4a \\
\implies \frac{1}{2} &= a
\end{aligned}$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x+2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned}
g(x) &= \frac{1}{2}(x+2)^2 - 3 \\
&= \frac{1}{2}(x+2)(x+2) - 3 \\
&= \frac{1}{2}(x^2 + 4x + 4) - 3 \\
&= \frac{1}{2}x^2 + 2x + 2 - 3 \\
&= \frac{1}{2}x^2 + 2x - 1
\end{aligned}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Example 2.2.3: Finding the Vertex of a Quadratic Function

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} = -\frac{-6}{2(2)} = \frac{6}{4} = \frac{3}{2}.$$

The vertical coordinate of the vertex will be at

$$k = f(h) = f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 = \frac{5}{2}.$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic. Therefore, using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}.$$

One reason we may want to identify the vertex of the parabola is that this point will inform us where the maximum or minimum value of the output occurs (k) and where it occurs (x).

Example 2.2.4: Rewriting in Standard Form Using Completing the Square

Since Completing the Square is an older skill, we will just use it here rather than explaining the entire process.

$$
\begin{aligned}
f(x) &= 2x^2 - 6x + 7 \\
&= 2(x^2 - 3x) + 7 && \text{(factoring the lead coefficient from both variable terms)} \\
&= 2\left(x^2 - 3x + \left(\frac{-3}{2}\right)^2\right) + 7 - 2\left(\frac{-3}{2}\right)^2 && \left(\text{Completing the Square and offsetting the } 2\left(\frac{-3}{2}\right)^2 \text{ term with } -2\left(\frac{-3}{2}\right)^2\right) \\
&= 2\left(x - \frac{3}{2}\right)^2 + 7 - 2\left(\frac{9}{4}\right) && \text{(factoring and simplifying)} \\
&= 2\left(x - \frac{3}{2}\right)^2 + \frac{14}{2} - \frac{9}{2} && \text{(common denominator)} \\
&= 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2} && \text{(simplifying)}
\end{aligned}
$$

Example 2.2.5: Revisiting the Difference Quotient

$$
\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(-3(x+h)^2 + 2(x+h) + 1) - (-3x^2 + 2x + 1)}{h} && \text{(substituting)} \\
&= \frac{-3(x^2 + 2hx + h^2) + 2x + 2h + 1 + 3x^2 - 2x - 1}{h} && \text{(distributing)} \\
&= \frac{-3x^2 - 6hx - 3h^2 + 2x + 2h + 1 + 3x^2 - 2x - 1}{h} && \text{(distributing)} \\
&= \frac{-6hx - 3h^2 + 2h}{h} && \text{(simplifying)} \\
&= \frac{h(-6x - 3h + 2)}{h} && \text{(factoring)} \\
&= -6x - 3h + 2 && \text{(cancelling like factors)}
\end{aligned}
$$

Example 2.2.6: Finding the Domain and Range of a Quadratic Function

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$h = -\frac{b}{2a} = -\frac{9}{2(-5)} = \frac{9}{10}.$$

The maximum value is given by $f(h)$.

$$f\left(\frac{9}{10}\right) = -5\left(\frac{9}{10}\right)^2 + 9\left(\frac{9}{10}\right) - 1 = \frac{61}{20}.$$

The range is $\left(-\infty, \frac{61}{20}\right]$.

Example 2.2.7: Finding the Maximum Value of a Quadratic Function

Let's use a diagram such as Figure 2.2.9 to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

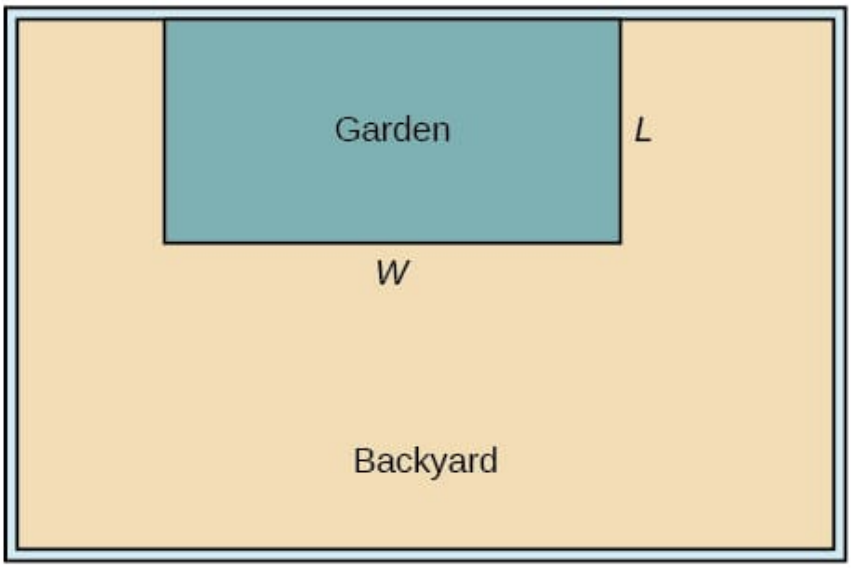


Figure 2.2.9

a. We know we have only 80 feet of fence available, and $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W, in terms of L:

$$W = 80 - 2L.$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80 - 2L) = 80L - 2L^2.$$

This formula represents the area of the fence in terms of the variable length L. The function, written in general form, is

$$A(L) = -2L^2 + 80L.$$

b. The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a = -2$, $b = 80$, and $c = 0$.

To find the vertex:

$$\begin{aligned} h &= -\frac{80}{2(-2)} & k &= A(20) \\ &= 20 \quad \text{and} & &= 80(20) - 2(20)^2 \\ & & &= 800 \end{aligned}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Example 2.2.8: Finding Maximum Revenue

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation

$$\text{Revenue} = pQ.$$

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to \$32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30} = \frac{-5,000}{2} = -2,500.$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$\begin{aligned} Q &= -2500p + b \\ \implies 84,000 &= -2500(30) + b \qquad \text{(substituting)} \\ \implies b &= 159,000 \qquad \text{(isolating the variable)} \end{aligned}$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\begin{aligned} \text{Revenue} &= pQ \\ \implies \text{Revenue} &= p(-2,500p + 159,000) \\ \implies \text{Revenue} &= -2,500p^2 + 159,000p \end{aligned}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$h = -\frac{159,000}{2(-2,500)} = 31.8.$$

The model tells us that the maximum revenue will occur if the newspaper charges \$31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\text{maximum revenue} = -2,500(31.8)^2 + 159,000(31.8) = 2,528,100.$$

Example 2.2.9: Finding the y- and x-Intercepts of a Parabola

We find the y-intercept by evaluating $f(0)$.

$$f(0) = 3(0)^2 + 5(0) - 2 = -2.$$

So the y-intercept is at $(0, -2)$.

For the x-intercepts, we find all solutions of $f(x) = 0$. That is, we want to solve

$$0 = 3x^2 + 5x - 2.$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = (3x - 1)(x + 2).$$

Thus, $x = \frac{1}{3}$ or $x = -2$. So the x-intercepts are at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.

Example 2.2.10: Using the Discriminant to Save Time

Since the quadratic function is already in general form, finding the y-intercept is as simple as looking for the constant term. In this case, $(0, -2)$ is the y-intercept.

As for any x-intercepts, per my advice earlier, we spend roughly 30 seconds trying to factor the function, but have no success. Before running straight to the full Quadratic Formula, let's only compute one part — the discriminant (heck, we need to anyway if we are eventually going to use the Quadratic Formula).

$$b^2 - 4ac = 4^2 - 4(-3)(-2) = 16 - 24 = -8 < 0.$$

Since the discriminant is negative, the function doesn't have any x-intercepts.

Example 2.2.11: Solving a Quadratic Equation

From Example 2.2.10, we know the discriminant is -8. Thus, the solutions of this equation are going to be imaginary. An *incredibly* important piece of advice is to start our solution process by multiplying both sides of the given equation by -1. We do this because the original lead coefficient is negative, which only serves to

complicate using the Quadratic Formula. Therefore, let's focus on solving

$$0 = 3x^2 - 4x + 2.$$

Note that the discriminant,

$$b^2 - 4ac = (-4)^2 - 4(3)(2) = 16 - 24 = -8$$

doesn't change when multiplying both sides of the quadratic equation by -1. Now we can solve.

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{(Quadratic Formula)} \\
&= \frac{-(-4) \pm \sqrt{\text{discriminant}}}{2(3)} && \text{(substituting)} \\
&= \frac{4 \pm \sqrt{-8}}{6} && \text{(substituting and simplifying)} \\
&= \frac{4 \pm \sqrt{4 \cdot -2}}{6} \\
&= \frac{4 \pm 2\sqrt{-2}}{6} && \text{(simplifying the radical)} \\
&= \frac{4 \pm 2\sqrt{-1}\sqrt{2}}{6} \\
&= \frac{4 \pm 2i\sqrt{2}}{6} && \left(\sqrt{-1} = i\right) \\
&= \frac{4}{6} \pm \frac{2i\sqrt{2}}{6} && \text{(splitting the fraction)} \\
&= \frac{2}{3} \pm \frac{i\sqrt{2}}{3} && \text{(simplifying fractions)} \\
&= \frac{2}{3} \pm \frac{\sqrt{2}}{3}i && \text{(rewriting in the standard form of a complex number)}
\end{aligned}$$

Example 2.2.12**: Finding the x-Intercepts of a Parabola**

We begin by solving for when the output will be zero; however, because this quadratic is not easily factorable, we start by checking the discriminant to see if there are any x-intercepts at all.

$$b^2 - 4ac = 4^2 - 4(2)(-4) = 16 + 32 = 48 > 0.$$

Thus, there will be two x-intercepts. Our time computing the discriminant is not wasted since we will need its value when solving the equation $2x^2 + 4x - 4 = 0$. Using the Quadratic Formula, we get the following:

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{(Quadratic Formula)} \\
&= \frac{-4 \pm \sqrt{\text{discriminant}}}{2(2)} && \text{(substituting)} \\
&= \frac{-4 \pm \sqrt{48}}{4} && \text{(substituting)} \\
&= \frac{-4 \pm \sqrt{16 \cdot 3}}{4} \\
&= \frac{-4 \pm 4\sqrt{3}}{4} && \text{(simplifying the radical expression)} \\
&= \frac{4(-1 \pm \sqrt{3})}{4} && \text{(factoring out the GCF from the numerator)} \\
&= \frac{\cancel{4}^{1}(-1 \pm \sqrt{3})}{\cancel{4}^{1}} && \text{(canceling like factors)} \\
&= -1 \pm \sqrt{3}
\end{aligned}$$

The graph has x-intercepts at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$.

Example 2.2.13**: Solving Quadratic Inequalities Using the Graphing Method**

a. To solve $2x^2 \leq 3 - x$, we first get 0 on one side of the inequality which yields $2x^2 + x - 3 \leq 0$. Therefore, we want to know when the function $f(x) = 2x^2 + x - 3$ is at or below the x-axis. Since the lead coefficient is positive, we know the parabola representing f opens upward. We just need to find the zeros of f by solving $2x^2 + x - 3 = 0$ for x. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$ or $x = 1$. A simple and quick sketch yields the following:

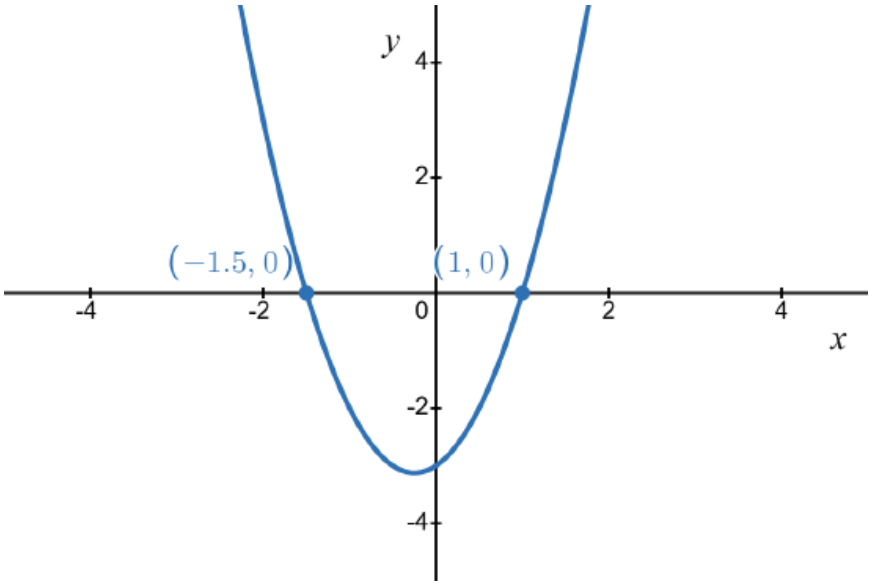


Figure 2.2.16

By our sketch, we can see $f(x)$ is at or below the x-axis on the interval $\left[-\frac{3}{2}, 1\right]$. This is the solution to our inequality.

b. Once again, we rewrite $x^2 - 2x > 1$ as $x^2 - 2x - 1 > 0$ and we identify $f(x) = x^2 - 2x - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $x^2 - 2x - 1$ doesn't factor nicely. Hence, we resort to the Quadratic Formula to solve $x^2 - 2x - 1 = 0$, and arrive at $x = 1 \pm \sqrt{2}$. As before, our parabola is opening upward. We spend a moment sketching the graph of f and asking ourselves when the parabola is strictly above the x-axis (because we are solving $f(x) > 0$).

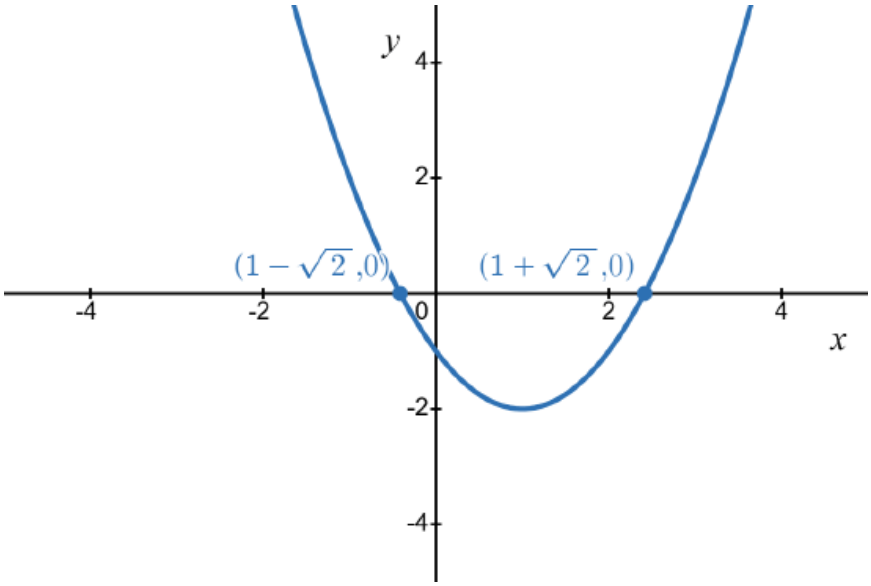


Figure 2.2.17

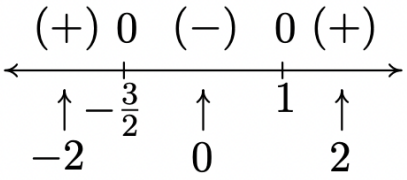
So, we see that the interval notation $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$ solves our inequality.

Example 2.2.14**: Solving Quadratic Inequalities Using the Sign Chart Method**

a. To solve $2x^2 \leq 3 - x$, we first get 0 on one side of the inequality which yields $2x^2 + x - 3 \leq 0$. We find the zeros of $f(x) = 2x^2 + x - 3$ by solving $2x^2 + x - 3 = 0$ for x. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$ or $x = 1$. We place these values on the number line with 0 above them and choose test values in the intervals $(-\infty, -\frac{3}{2})$, $(-\frac{3}{2}, 1)$ and $(1, \infty)$. For the interval $(-\infty, -\frac{3}{2})$, we choose $x = -2$; for $(-\frac{3}{2}, 1)$, we pick $x = 0$; and for $(1, \infty)$, $x = 2$. Evaluating the function at the three test values gives us $f(-2) = 3 > 0$, so we place $(+)$ above $(-\infty, -\frac{3}{2})$; $f(0) = -3 < 0$, so $(-)$ goes above the interval $(-\frac{3}{2}, 1)$; and, $f(2) = 7$, which means $(+)$ is placed above $(1, \infty)$.

Since we are solving $2x^2 + x - 3 \leq 0$, we look for solutions to $2x^2 + x - 3 < 0$ as well as solutions for $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which we have a $(-)$. Checking the sign diagram, we see this is $(-\frac{3}{2}, 1)$. We know $2x^2 + x - 3 = 0$ when $x = -\frac{3}{2}$ and $x = 1$, so our final answer is $\left[-\frac{3}{2}, 1\right]$.

To verify our solution graphically, we refer to the original inequality, $2x^2 \leq 3 - x$. We let $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the x values where the graph of g is below that of h (the solution to $g(x) < h(x)$) as well as the points of intersection (the solutions to $g(x) = h(x)$).



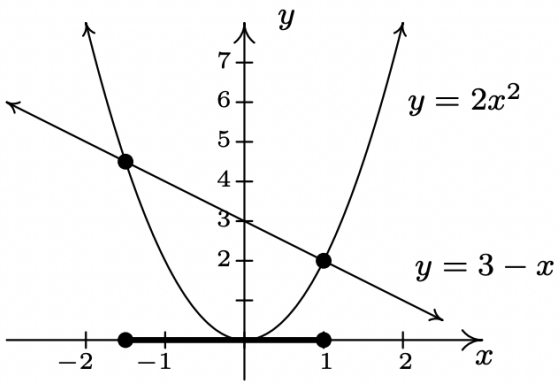
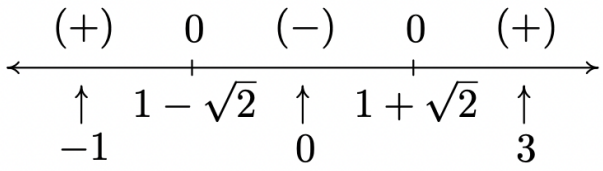


Figure 2.2.19

b. Once again, we re-write $x^2 - 2x > 1$ as $x^2 - 2x - 1 > 0$ and we identify $f(x) = x^2 - 2x - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $x^2 - 2x - 1$ doesn't factor nicely. Hence, we resort to the Quadratic Formula to solve $x^2 - 2x - 1 = 0$, and arrive at $x = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $x = -1$, $x = 0$ and $x = 3$ as our test values and find $f(-1) = 2$, which is $(+)$; $f(0) = -1$ which is $(-)$; and $f(3) = 2$ which is $(+)$ again. Our solution to $x^2 - 2x - 1 > 0$ is where we have $(+)$, so, in interval notation $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$.

To check the inequality $x^2 - 2x > 1$ graphically, we set $g(x) = x^2 - 2x$ and $h(x) = 1$. We are looking for the x values where the graph of g is above the graph of h.



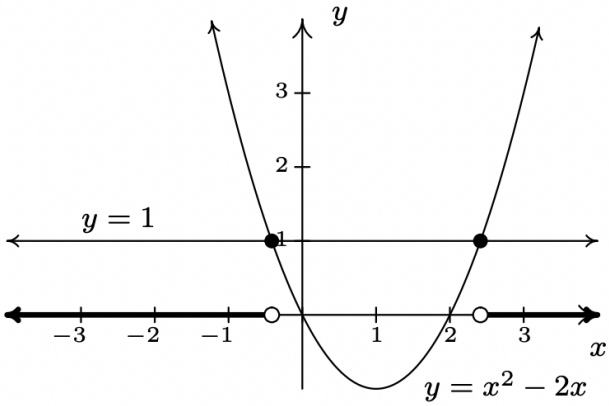


Figure 2.2.20

Example 2.2.15**: Applying the Vertex and x-Intercepts of a Parabola**

a. The ball reaches the maximum height at the vertex of the parabola.

$$h = -\frac{80}{2(-16)} = \frac{80}{32} = \frac{5}{2} = 2.5.$$

The ball reaches a maximum height after 2.5 seconds.

b. To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$k = H\left(-\frac{b}{2a}\right) = H(2.5) = -16(2.5)^2 + 80(2.5) + 40 = 140.$$

The ball reaches a maximum height of 140 feet.

c. We need to solve

$$0 = -16t^2 + 80t + 40$$

which has the same solution as

$$0 = 16t^2 - 80t - 40.$$

Since the coefficients are all divisible by 8, we divide both sides by 8 to get

$$0 = 2t^2 - 10t - 5.$$

Try as you might, the quadratic on the right side doesn't factor. Therefore, we use the Quadratic Formula. Before going any further, let's compute the discriminant (if it's negative, then we can stop because this would mean the ball would never touch the ground).

$$\text{discriminant} = b^2 - 4ac = (-10)^2 - 4(2)(-5) = 100 + 40 = 140 > 0.$$

Thus, we continue with the Quadratic Formula.

$$\begin{aligned} t &= \frac{-b \pm \sqrt{\text{discriminant}}}{2a} && \text{(Quadratic Formula)} \\ &= \frac{10 \pm \sqrt{140}}{2(2)} && \text{(substituting)} \\ &= \frac{10 \pm \sqrt{140}}{4} && \text{(simplifying)} \end{aligned}$$

Because this is an application and the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t \approx 5.458 \quad \text{or} \quad t \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 2.2.21.

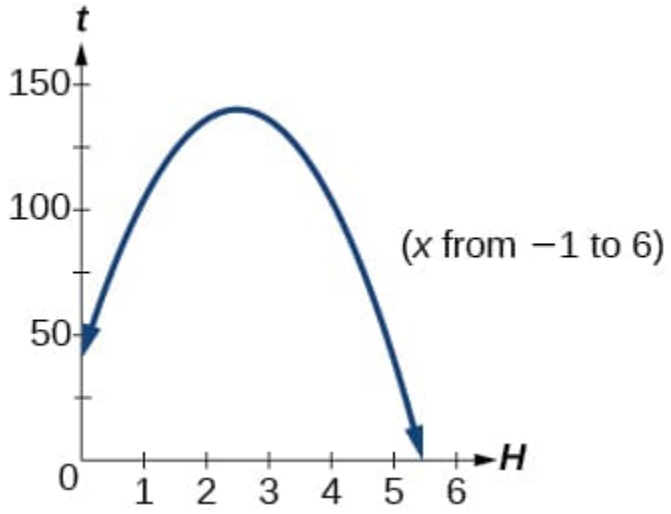


Figure 2.2.21

d. This question is akin to asking us to solve $-16t^2 + 80t + 40 > 104$. We start by getting 0 on one side:

$$-16t^2 + 80t - 64 > 0.$$

Again, it would be better to get that lead coefficient to be positive, so we multiply both sides by -1 (remember that the inequality flips):

$$16t^2 - 80t + 64 < 0.$$

We then divide both sides by the GCF of each of the coefficients (which is 16):

$$t^2 - 5t + 4 < 0.$$

We want to know when the outputs of this parabola are below the t-axis. We know this parabola is opening upward and the left side factors to $(t-4)(t-1)$. This implies the roots are $t = 1$ and $t = 4$. This means that our solution is the interval $(1, 4)$ (see Figure 2.2.22).

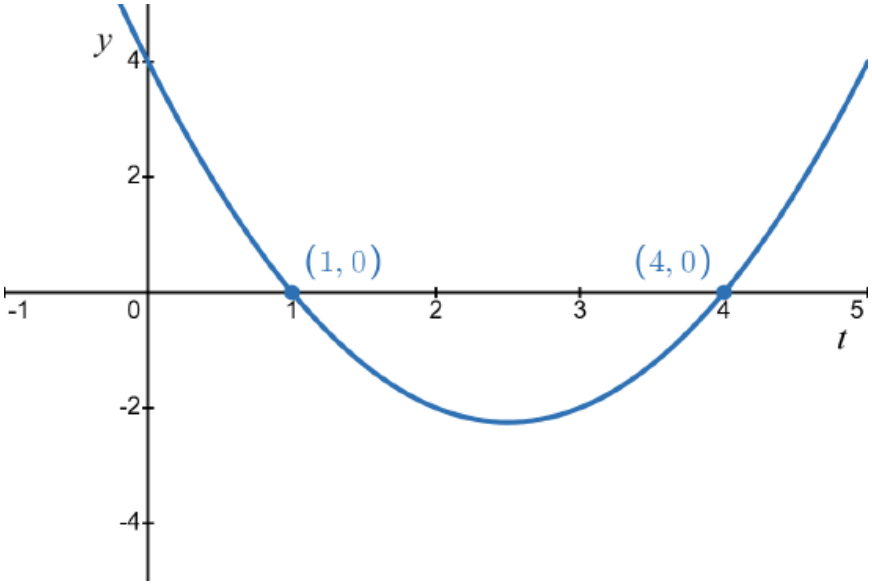


Figure 2.2.22

Example 2.2.16**: Solve Absolute Value Equations Involving Quadratic Expressions**

a. As with any synthesis question, you need to think more critically than with basic questions. In this case, you are being asked when the left side is zero. The only time the absolute value is zero is when its argument is zero. So this question boils down to solving

$$2x^2 + x - 3 = 0.$$

In Example 2.2.14, we found the zeros of this function to be $x = -\frac{3}{2}$ and $x = 1$. Thus, the solutions to this absolute value equation are $x = -\frac{3}{2}$ and $x = 1$.

b. Again, we have a synthesis question where thinking critically ahead of time saves time. We are being asked, "When is an absolute value equal to a negative number?" We know that this cannot happen. Thus, this absolute value equation does not have a solution. In Mathematics, we often say the solution **DNE** (does not exist).

c. We finally arrive at an example which requires more than superficial thinking. Using the definition of the absolute value, we see that the left side has the equivalent form

$$|x^2 - 5x| = \begin{cases} x^2 - 5x, & \text{if } x^2 - 5x \geq 0 \\ -(x^2 - 5x), & \text{if } x^2 - 5x < 0 \end{cases}$$

So, we need to determine when $x^2 - 5x = 0$ because this is where the "pivot point" of the piecewise definition occurs. We factor the equation to $x(x - 5) = 0$ and note that the solutions are $x = 0$ and $x = 5$. Since $y = x^2 - 5x$ is a parabola opening upward, and we just found out that the zeros of this function are 0 and 5, this function is at or above the horizontal axis on the interval $(-\infty, 0] \cup [5, \infty)$. Thus, it is below the horizontal axis on $(0, 5)$. We modify our piecewise definition accordingly.

$$|x^2 - 5x| = \begin{cases} x^2 - 5x, & \text{if } x \in (-\infty, 0] \cup [5, \infty) \\ -(x^2 - 5x), & \text{if } x \in (0, 5) \end{cases}$$

Thus, our original equation becomes the following:

	Case #1	**Case #2**
	$x \in (-\infty, 0] \cup [5, \infty)$	$x \in (0, 5)$
	$x^2 - 5x = 6$	$-(x^2 - 5x) = 6$
$\Longrightarrow$	$x^2 - 5x - 6 = 0$	$-x^2 + 5x - 6 = 0$
$\Longrightarrow$	$(x - 6)(x + 1) = 0$	$x^2 - 5x + 6 = 0$
$\Longrightarrow$	$x = 6$ or $x = -1$	$(x - 2)(x - 3) = 0$
$\Longrightarrow$		$x = 2$ or $x = 3$

Hence, our solutions to the original absolute value equation are $x = -1, 2, 3$ and 6.

Example 2.2.17: Graph the Absolute Value of a Quadratic Function

Using the definition of absolute value, we have

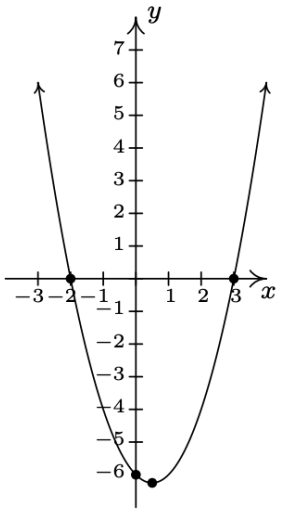
$$f(x) = \begin{cases} (x^2 - x - 6), & \text{if } x^2 - x - 6 \geq 0 \\ -(x^2 - x - 6), & \text{if } x^2 - x - 6 < 0 \end{cases}$$

We begin by graphing $y = g(x) = x^2 - x - 6$ using the intercepts and the vertex. To find the x-intercepts, we solve $x^2 - x - 6 = 0$. Factoring gives $(x - 3)(x + 2) = 0$ so $x = -2$ or $x = 3$. Hence, $(-2, 0)$ and $(3, 0)$ are x-intercepts. The y-intercept $(0, -6)$ is found by setting $x = 0$. To plot the vertex, we find

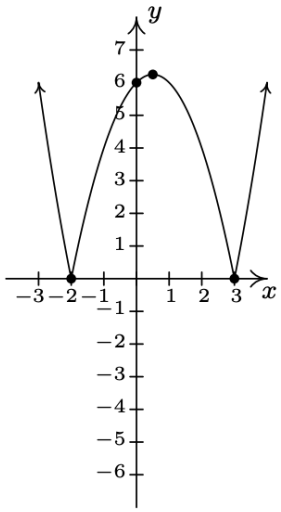
$$x = -\frac{b}{2a} = -\frac{-1}{2(1)} = \frac{1}{2}, \text{ and } y = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 6 = -\frac{25}{4} = -6.25.$$

Plotting, we get the parabola on the left side of Figure 2.2.23.

To obtain points on the graph of $y = f(x) = |x^2 - x - 6|$, we can take points on the graph of $g(x) = x^2 - x - 6$ and apply the absolute value to each of the y values on the parabola. We see from the graph of g that for $x \leq -2$ or $x \geq 3$, the y-values on the parabola are greater than or equal to zero (since the graph is on or above the x-axis), so the absolute value leaves these portions of the graph alone. For x between -2 and 3, however, the y-values on the parabola are negative. For example, the point $(0, -6)$ on $y = x^2 - x - 6$ would result in the point $(0, |-6|) = (0, -(-6)) = (0, 6)$ on the graph of $f(x) = |x^2 - x - 6|$. Proceeding in this manner for all points with x-coordinates between -2 and 3 results in the graph on the right side of Figure 2.2.23.



$y = g(x) = x^2 - x - 6$



$y = f(x) = |x^2 - x - 6|$

Figure 2.2.23

Example 2.2.18**: Solving an Equation that is Quadratic-in-Form**

Many students would immediately distribute things out to get

$$27x^2 + 21x - 20 = 0$$

and try to factor; however, this is not taking advantage of the form of the original equation.

Notice that there are some common expressions within the original equation — namely $3x - 1$. If we let $u = 3x - 1$, we could rewrite the equation as

$$3u^2 + 13u - 10 = 0.$$

Hopefully, you agree that this equation looks much easier to solve. This type of substitution is formally called a u**-substitution**. The choice of the variable u is arbitrary, and you are encouraged to use whatever variable you want (other than the variable in the original equation).

We can factor the left side of this new equation to get

$$(3u - 2)(u + 5) = 0.$$

Therefore, $u = \frac{2}{3}$ or $u = -5$.

Since the original problem involved the variable x, we need to **resubstitute** $u = 3x - 1$ to arrive at

$$\begin{aligned}
u &= \frac{2}{3} \quad \text{or} \quad u = -5 \\
\implies 3x - 1 &= \frac{2}{3} \quad \text{or} \quad 3x - 1 = -5 && \text{(resubstituting)} \\
\implies 3x &= \frac{2}{3} + 1 \quad \text{or} \quad 3x = -5 + 1 && \text{(adding 1 to both sides)} \\
\implies 3x &= \frac{5}{3} \quad \text{or} \quad 3x = -4 && \text{(simplifying)} \\
\implies x &= \frac{5}{9} \quad \text{or} \quad x = -\frac{4}{3} && \text{(dividing both sides by 3)}
\end{aligned}$$

Thus, the solutions to the original equation are $x = \frac{5}{9}$ and $x = -\frac{4}{3}$.

Example 2.2.19**: Solving an Equation that is Quadratic-in-Form**

Since this equation has several terms involving powers of x, it is best to collect all information on one side (this tends to "reveal" the type of equation we are working with). Subtracting 45 and x^2 from both sides, we get

$$x^4 - 4x^2 - 45 = 0.$$

The key observation here is that the exponent on x in the first term is *twice* that of the exponent on x in the second term. That is, this has the same form as a quadratic equation. Letting $u = x^2$, we get

$$\begin{aligned}
x^4 - 4x^2 - 45 &= \left(x^2\right)^2 - 4x^2 - 45 && \text{(Laws of Exponents: Power Law)} \\
&= u^2 - 4u - 45 && \text{(substituting } u = x^2\text{)}
\end{aligned}$$

Thus, we instead focus on solving

$$\begin{aligned}
u^2 - 4u - 45 &= 0 \\
\implies (u - 9)(u + 5) &= 0 && \text{(factoring)}
\end{aligned}$$

Hence, $u = 9$ or $u = -5$. However, we are supposed to be solving for x, so we resubstitute $u = x^2$.

$$\implies x^2 = 9 \quad \text{or} \quad x^2 = -5.$$

While we can easily solve the first equation to get $x = \pm 3$, the second equation doesn't have a solution (over the real number system) because there isn't a real number whose square becomes a negative number. Thus, the solutions to the original equation are $x = -3$ and $x = 3$.

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2.2.1: Resources and Key Concepts

Resources

Videos

- Completing the square
  - Intermediate Algebra - Completing the Square (Part A)
  - Intermediate Algebra - Completing the Square (Part B)
- Graphing quadratic functions in standard/vertex form
  - Intermediate Algebra - Everything About Graphing Parabolas
  - Intermediate Algebra - Graphing Parabolas in Vertex Form
- Graphing quadratic functions in general form
  - Quadratic Functions - Understanding How Graphs of Parabolas are Related to Their Quadratic Functions
  - Intermediate Algebra - Graphing Quadratic Equations (Part A)
  - Intermediate Algebra - Graphing Quadratic Equations (Part B)
- Solving quadratic equations
  - Intermediate Algebra - Introduction to Quadratic Equations
  - Algebra in Review: Equations - Quadratic (Zero Factor Principle)
  - Intermediate Algebra - Solving Quadratic Equations by Factoring (Part A)
  - Intermediate Algebra - Solving Quadratic Equations by Factoring (Part B)
  - Intermediate Algebra - Quadratic Formula (Part A)
  - Intermediate Algebra - Quadratic Formula (Part B)
  - Intermediate Algebra - Quadratic Formula (Part C)
  - Intermediate Algebra - Quadratic Formula (Part D)
- Solving equations that are quadratic-in-form
  - Equations - Overview - A Review for College Algebra: Solving Equations That Are Quadratic-in-Form
- Solving quadratic inequalities
  - Inequalities - Nonlinear - A Review for College Algebra: Solving Quadratic Inequalities
  - Intermediate Algebra - Quadratic Inequalities (Part A)
  - Intermediate Algebra - Quadratic Inequalities (Part B)
  - Intermediate Algebra - Quadratic Inequalities (Part C)
  - Intermediate Algebra - Quadratic Inequalities (Part D)
- Applications involving quadratic functions
  - Quadratic Functions - Determining the Maximum and Minimum Values of Quadratic Functions
  - Intermediate Algebra - Quadratic Applications
  - Optimization Problems - Maximizing Revenue the Non-Calculus Way
- CAPSTONE!!!
  - Algebra in Review: Capstone - Quadratics

Key Concepts

Definitions

- **Quadratic Function:** A function of degree two, which can be written in the general form $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and $a \neq 0$.
- **Parabola:** The U-shaped curve that is the graph of a quadratic function.
- **Vertex (of a parabola):** The point on the parabola where the function reaches its maximum or minimum value. It is also the turning point of the graph.
- **Axis of Symmetry (of a parabola):** A vertical line passing through the vertex of a parabola, dividing it into two mirror images. Its equation is $x = h$, where h is the x-coordinate of the vertex.
- **General Form (of a quadratic function):** $f(x) = ax^2 + bx + c$.

- **Standard Form (or Vertex Form of a quadratic function):** $f(x) = a(x-h)^2 + k$, where (h, k) is the vertex of the parabola.
- **Zeros (or Roots or x-intercepts of a quadratic function):** The values of x for which $f(x) = 0$. These are the points where the parabola intersects the x-axis.
- **y-intercept (of a quadratic function):** The point where the parabola intersects the y-axis, found by evaluating $f(0)$. In general form, this is $(0, c)$.
- **Discriminant:** For a quadratic equation $ax^2 + bx + c = 0$ (or function $f(x) = ax^2 + bx + c$), the discriminant is the value $D = b^2 - 4ac$. It helps determine the nature and number of real roots (x-intercepts).
- **Prime Quadratic:** A quadratic expression that cannot be factored into linear factors with real coefficients. This occurs when its discriminant is negative.
- **u-substitution:** A technique used to solve equations that are "quadratic-in-form" by substituting a variable (often u) for an expression, transforming the equation into a standard quadratic form.

Theorems

- **Axis of Symmetry and Vertex (from General Form):** For a quadratic function $f(x) = ax^2 + bx + c$, the axis of symmetry is the line $x = -\frac{b}{2a}$. The vertex is at $(-\frac{b}{2a}, f(-\frac{b}{2a}))$.
- **Vertex (from Standard Form):** For a quadratic function $f(x) = a(x-h)^2 + k$, the vertex is (h, k).
- **Relationship between General and Standard Forms:**
  - Given $f(x) = ax^2 + bx + c$, then $h = -\frac{b}{2a}$ and $k = c - ah^2 = c - a(-\frac{b}{2a})^2 = c - \frac{b^2}{4a}$. More simply, $k = f(h)$.
- **Domain and Range of a Quadratic Function:**
  - The domain of any quadratic function is all real numbers, $(-\infty, \infty)$.
  - If $a > 0$, the range is $[k, \infty)$, where k is the y-coordinate of the vertex.
  - If $a < 0$, the range is $(-\infty, k]$, where k is the y-coordinate of the vertex.
- **Zeros and the Discriminant:** For a quadratic function $f(x) = ax^2 + bx + c$, let $D = b^2 - 4ac$ be the discriminant.
  - If $D > 0$, the function has two distinct real zeros (two x-intercepts).
  - If $D = 0$, the function has one real zero (one x-intercept; the vertex is on the x-axis).
  - If $D < 0$, the function has no real zeros (no x-intercepts); the zeros are complex conjugates.
- **Quadratic Formula:** The solutions to the quadratic equation $ax^2 + bx + c = 0$ (where $a \neq 0$) are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Common Mistakes to Avoid

- **Confusing h in Standard Form:** In $f(x) = a(x-h)^2 + k$, if the form is, for example, $f(x) = a(x+3)^2 + k$, misidentifying h as 3 instead of -3.
- **Errors in Completing the Square:** Forgetting to factor out the leading coefficient a from the x^2 and x terms before completing the square.
- **Factoring Errors:** Incorrectly factoring a quadratic trinomial when trying to find x-intercepts or solve inequalities.
- **Solving Quadratic Inequalities:** Forgetting to flip the inequality sign if multiplying/dividing the entire inequality by a negative number.
- **Equations Quadratic-in-Form:** After solving for the substituted variable (e.g., u), forgetting to substitute back to solve for the original variable (e.g., x).
- **Absolute Value Equations with Quadratics:** Not considering both positive and negative cases for the expression inside the absolute value (e.g., for $|Q(x)| = c$, solving only $Q(x) = c$).
- **Graphing $|Q(x)|$:** Reflecting the entire graph of $Q(x)$ instead of only the parts below the x-axis.

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2.2.2: Homework

Reading Questions

1. What is the shape of the graph of a quadratic function called?
2. What are the key features of a parabola? (List at least three)
3. What is the general form of a quadratic function? What is the standard (or vertex) form?
4. How can you determine from the general form $f(x) = ax^2 + bx + c$ whether the parabola opens upward or downward?
5. What is the formula for the x-coordinate of the vertex (and the axis of symmetry) when a quadratic function is in general form?
6. If a quadratic function is in standard form $f(x) = a(x - h)^2 + k$, what are the coordinates of its vertex?
7. How does the value of a in $f(x) = a(x - h)^2 + k$ affect the width and direction of opening of the parabola compared to $y = x^2$?
8. What is the domain of any quadratic function? How is the range determined?
9. What is the discriminant of a quadratic function, and what does it tell you about the x-intercepts of the parabola?
10. When would you use the Quadratic Formula to find the x-intercepts of a parabola?
11. Describe the process of u-substitution for solving an equation that is quadratic-in-form, like $x^4 - 4x^2 - 45 = 0$.
12. Explain the graphical method for solving a quadratic inequality like $x^2 - x - 6 < 0$.
13. Explain the sign chart method for solving a quadratic inequality.
14. When solving an absolute value equation involving a quadratic, such as $|x^2 - 5x| = 6$, what are the two main cases you need to consider?
15. How do you graph a function of the form $f(x) = |Q(x)|$, where $Q(x)$ is a quadratic function? What happens to the parts of the graph of $Q(x)$ that are below the x-axis?
16. Explain the advantage of writing a quadratic function in standard form.
17. How can the vertex of a parabola be used in solving real world problems?
18. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.
19. What is another name for the standard form of a quadratic function?
20. What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

Homework

Algebraic

For the following exercises, rewrite the quadratic functions in standard form and give the vertex.

21. $f(x) = x^2 - 12x + 32$
22. $g(x) = x^2 + 2x - 3$
23. $f(x) = x^2 - x$
24. $f(x) = x^2 + 5x - 2$
25. $h(x) = 2x^2 + 8x - 10$
26. $k(x) = 3x^2 - 6x - 9$
27. $f(x) = 2x^2 - 6x$
28. $f(x) = 3x^2 - 5x - 1$

For the following exercises, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

29. $y(x) = 2x^2 + 10x + 12$
30. $f(x) = 2x^2 - 10x + 4$
31. $f(x) = -x^2 + 4x + 3$
32. $f(x) = 4x^2 + x - 1$
33. $h(t) = -4t^2 + 6t - 1$
34. $f(x) = \frac{1}{2}x^2 + 3x + 1$
35. $f(x) = -\frac{1}{3}x^2 - 2x + 3$

For the following exercises, determine the domain and range of the quadratic function.

36. $f(x) = (x-3)^2 + 2$
37. $f(x) = -2(x+3)^2 - 6$
38. $f(x) = x^2 + 6x + 4$
39. $f(x) = 2x^2 - 4x + 2$
40. $k(x) = 3x^2 - 6x - 9$

For the following exercises, solve the equations over the complex numbers.

41. $x^2 = -25$
42. $x^2 = -8$
43. $x^2 + 36 = 0$
44. $x^2 + 27 = 0$
45. $x^2 + 2x + 5 = 0$
46. $x^2 - 4x + 5 = 0$
47. $x^2 + 8x + 25 = 0$
48. $x^2 - 4x + 13 = 0$
49. $x^2 + 6x + 25 = 0$
50. $x^2 - 10x + 26 = 0$
51. $x^2 - 6x + 10 = 0$
52. $x(x-4) = 20$
53. $x(x-2) = 10$
54. $2x^2 + 2x + 5 = 0$
55. $5x^2 - 8x + 5 = 0$
56. $5x^2 + 6x + 2 = 0$
57. $2x^2 - 6x + 5 = 0$
58. $x^2 + x + 2 = 0$
59. $x^2 - 2x + 4 = 0$

For the following exercises, use the vertex (h,k) and a point on the graph (x,y) to find the general form of the equation of the quadratic function.

60. $(h,k) = (2,0)$, $(x,y) = (4,4)$
61. $(h,k) = (-2,-1)$, $(x,y) = (-4,3)$
62. $(h,k) = (0,1)$, $(x,y) = (2,5)$
63. $(h,k) = (2,3)$, $(x,y) = (5,12)$
64. $(h,k) = (-5,3)$, $(x,y) = (2,9)$
65. $(h,k) = (3,2)$, $(x,y) = (10,1)$
66. $(h,k) = (0,1)$, $(x,y) = (1,0)$
67. $(h,k) = (1,0)$, $(x,y) = (0,1)$

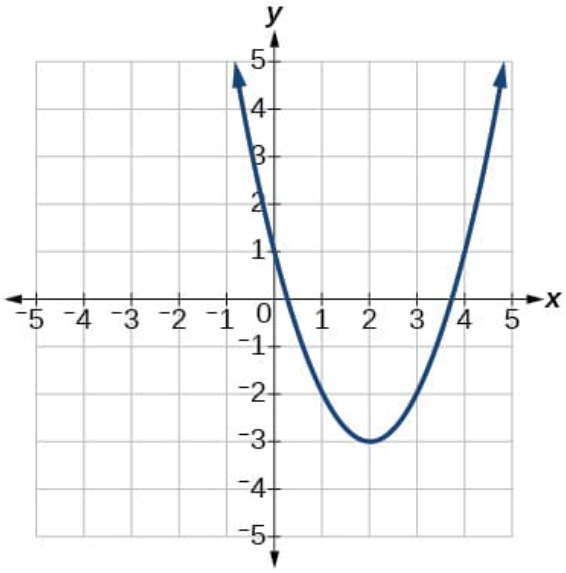
Graphical

For the following exercises, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

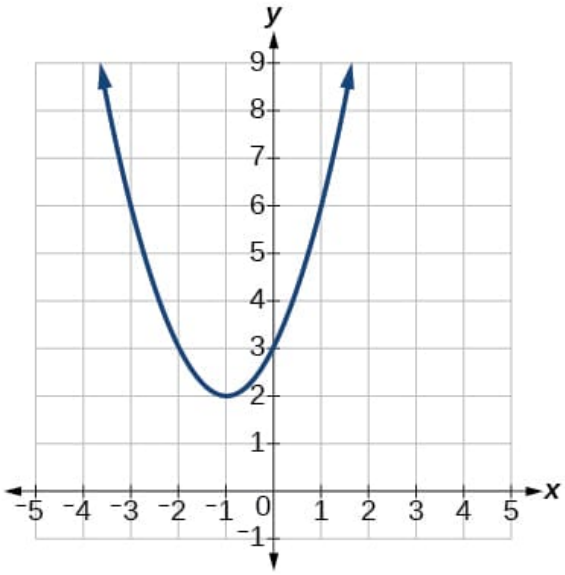
68. $f(x) = x^2 - 2x$
69. $f(x) = x^2 - 6x - 1$
70. $f(x) = x^2 - 5x - 6$
71. $f(x) = x^2 - 7x + 3$
72. $f(x) = -2x^2 + 5x - 8$
73. $f(x) = 4x^2 - 12x - 3$

For the following exercises, write the equation for the graphed function.

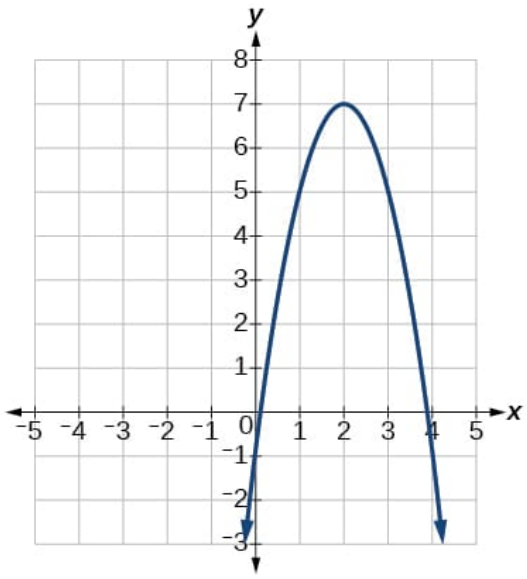
74.



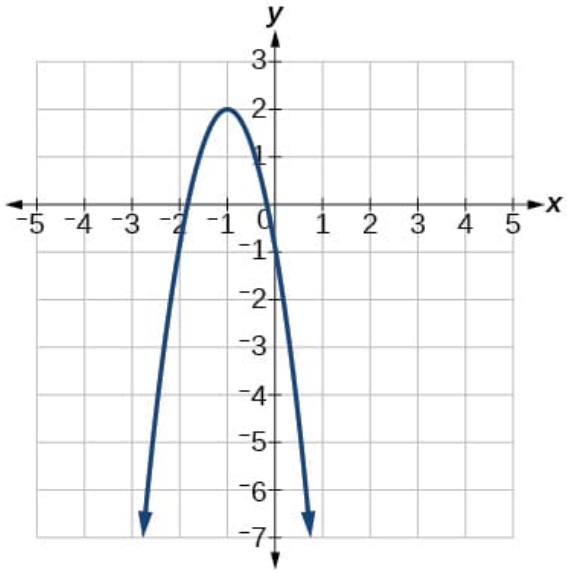
75.



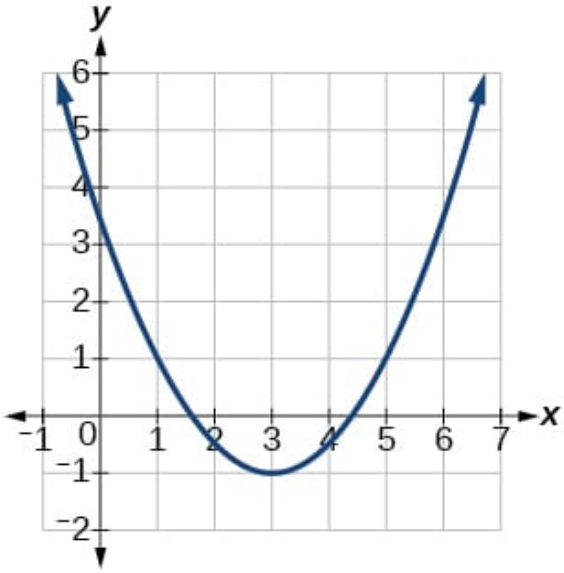
76.



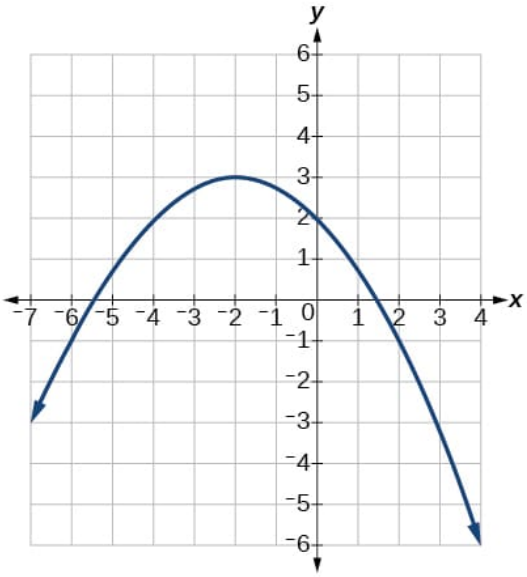
77.



78.



79.



Numeric

For the following exercises, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

80.

x	–2	–1	0	1	2
y	5	2	1	2	5

81.

x	–2	–1	0	1	2
y	1	0	1	4	9

82.

x	–2	–1	0	1	2
y	–2	1	2	1	–2

33.

x	–2	–1	0	1	2
y	–8	–3	0	1	0

34.

x	–2	–1	0	1	2
y	8	2	0	2	8

Technology

For the following exercises, use a calculator to find the answer.

35. Graph on the same set of axes the functions $f(x) = x^2$, $f(x) = 2x^2$, and $f(x) = \frac{1}{3}x^2$. What appears to be the effect of changing the coefficient?
36. Graph on the same set of axes $f(x) = x^2$, $f(x) = x^2 + 2$, $f(x) = x^2 + 5$, and $f(x) = x^2 - 3$. What appears to be the effect of adding a constant?
37. Graph on the same set of axes $f(x) = x^2$, $f(x) = (x-2)^2$, $f(x) = (x-3)^2$, and $f(x) = (x+4)^2$. What appears to be the effect of adding or subtracting those numbers?
38. The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function $h(x) = -\frac{32}{(80)^2}x^2 + x$ where x is the horizontal distance traveled and $h(x)$ is the height in feet. Use graphing technology to determine the height of the object when it has traveled 100 feet away horizontally.
39. A suspension bridge can be modeled by the quadratic function $h(x) = .0001x^2$ with $-2000 \leq x \leq 2000$ where $|x|$ is the number of feet from the center and $h(x)$ is height in feet. Use graphing technology to estimate how far from the center does the bridge have a height of 100 feet.

Extensions

For the following exercises, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

90. Vertex $(1, -2)$, opens up.
91. Vertex $(-1, 2)$ opens down.
92. Vertex $(-5, 11)$, opens down.
93. Vertex $(-100, 100)$, opens up.

For the following exercises, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

94. Contains $(1, 1)$ and has shape of $f(x) = 2x^2$. Vertex is on the y-axis.
95. Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$. Vertex is on the y-axis.
96. Contains $(2, 3)$ and has the shape of $f(x) = 3x^2$. Vertex is on the y-axis.
97. Contains $(1, -3)$ and has the shape of $f(x) = -x^2$. Vertex is on the y-axis.
98. Contains $(4, 3)$ and has the shape of $f(x) = 5x^2$. Vertex is on the y-axis.
99. Contains $(1, -6)$ has the shape of $f(x) = 3x^2$. Vertex has x-coordinate of -1.

Real-World Applications

100. Find the dimensions of the rectangular corral producing the greatest enclosed area given 200 feet of fencing.
101. Find the dimensions of the rectangular corral split into 2 pens of the same size producing the greatest possible enclosed

area given 300 feet of fencing.

92. Find the dimensions of the rectangular corral producing the greatest enclosed area split into 3 pens of the same size given 500 feet of fencing.
93. Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?
94. Among all of the pairs of numbers whose difference is 12, find the pair with the smallest product. What is the product?
95. Suppose that the price per unit in dollars of a cell phone production is modeled by $p = 45 - 0.0125x$, where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue.
96. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.
97. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$. How long does it take to reach maximum height?
98. A soccer stadium holds 62,000 spectators. With a ticket price of $11, the average attendance has been 26,000. When the price dropped to $9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?
99. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

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2.3: Power Functions and Polynomial Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure 2.3.1**:** *credit: Jason Bay, Flickr*

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in Table 2.3.1.

Table 2.3.1

Year	2009	2010	2011	2012	2013
Bird Population	800	897	992	1,083	1,169

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800$, where $P(t)$ represents the bird population on the island t years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.

Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function called a **power function**.

Definition: Power Function

A **power function** is a function of the form

$$f(x) = ax^n,$$

where a and n are real numbers and $a \neq 0$. a is known as the **coefficient** of the power function, and n is called its **exponent**.

As an example, consider functions for area or volume. The function for the area of a circle with radius r is $A(r) = \pi r^2$ and the function for the volume of a sphere with radius r is $V(r) = \frac{4}{3}\pi r^3$. Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a power.

Q&A

Is $f(x) = 2^x$ a power function?

No. A power function contains a variable base raised to a fixed power. This function has a constant base raised to a variable power. This is called an **exponential function**, not a power function. We will deal with these types of functions later.

Example 2.3.1: Identifying Power Functions

Which of the following functions are power functions?

$f(x) = 1$ Constant function

$f(x) = x$ Identify function

$f(x) = x^2$ Quadratic function

$f(x) = x^3$ Cubic function

$f(x) = \dfrac{1}{x}$ Reciprocal function

$f(x) = \dfrac{1}{x^2}$ Reciprocal squared function

$f(x) = \sqrt{x}$ Square root function

$f(x) = \sqrt[3]{x}$ Cube root function

Solution available at the end of this section.

Checkpoint 2.3.1

Which functions are power functions?

$$f(x) = 2x^2 \cdot 4x^3$$

$$g(x) = -x^5 + 5x^3 - 4x$$

$$h(x) = \frac{2x^5 - 1}{3x^2 + 4}$$

The ever-important Difference Quotient is often used in Calculus when working with power function.

Example 2.3.2: Revisiting the Difference Quotient

Compute the Difference Quotient for the given function.

$$f(x) = x^4$$

Solution available at the end of this section.

In Example 2.3.2, we could have taken the alternative approach of using factoring techniques to simplify the numerator.

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^4 - x^4}{h} && \text{(substituting)} \\
&= \frac{\left((x+h)^2\right)^2 - \left(x^2\right)^2}{h} && \text{(Laws of Exponents)} \\
&= \frac{\left((x+h)^2 - x^2\right)\left((x+h)^2 + x^2\right)}{h} && \text{(Difference of Squares)} \\
&= \frac{\left((x+h) - x\right)\left((x+h) + x\right)\left((x+h)^2 + x^2\right)}{h} && \text{(Difference of Squares)} \\
&= \frac{(h)\,(2x+h)\left(x^2 + 2hx + h^2 + x^2\right)}{h} && \text{(simplifying)} \\
&= \frac{\cancel{h}^{1}\,(2x+h)\left(x^2 + 2hx + h^2 + x^2\right)}{\cancel{h}^{1}} && \text{(canceling like factors)} \\
&= (2x+h)\left(2x^2 + 2hx + h^2\right) && \text{(simplifying)} \\
&= 4x^3 + 6x^2h + 4xh^2 + h^3 && \text{(distributing)}
\end{aligned}$$

While both methods are valid and have their place, a student well-prepared for Calculus will inevitably want to try the second method.

Identifying End Behavior of Power Functions

Figure 2.3.2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$, and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

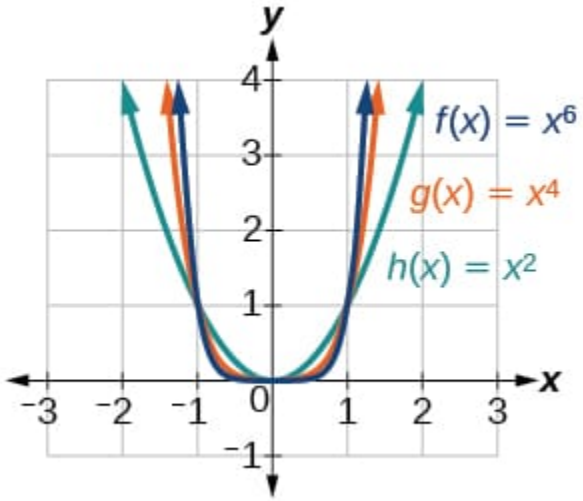


Figure 2.3.2: *Even-power functions*

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty, \quad f(x) \to \infty.$$

Figure $2.3.3$ shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

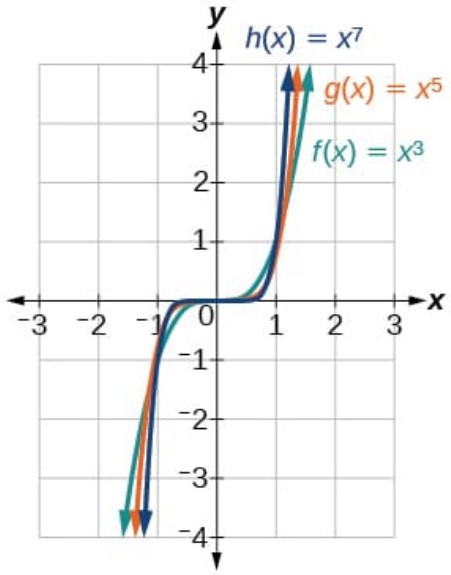


Figure $2.3.3$**:** *Odd-power functions*

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure $2.3.2$ we see that functions of the form $f(x) = x^n$, where n is even, are symmetric about the y-axis. In Figure $2.3.3$, we see that functions of the form $f(x) = x^n$, where n is odd, are symmetric about the origin.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form, we write

$$\text{as } x \to -\infty, \quad f(x) \to -\infty \quad \text{and as } x \to \infty, \quad f(x) \to \infty.$$

The behavior of the graph of a function as the input values get very negative ($x \to -\infty$) and get very large ($x \to \infty$) is referred to as the **end behavior** of the function. We can use words or symbols to describe end behavior.

Definition: End Behavior of a Function

The **end behavior** of a function refers to the way the graph of the function behaves as the input (typically the x-value) approaches positive or negative infinity (denoted $x \to \infty$ or $x \to -\infty$).

Figure $2.3.4$ shows the end behavior of power functions in the form $f(x) = kx^n$, where n is a non-negative integer depending on the power and the constant.

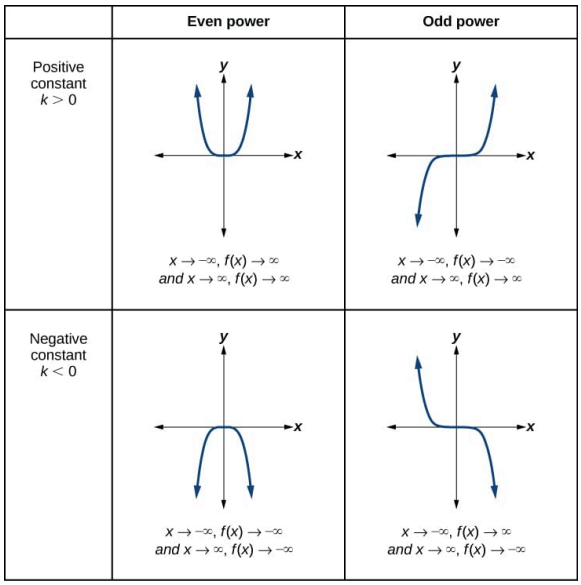


Figure 2.3.4

Interactive Element: End Behavior of a Power Function

The following Interactive Element might help you visualize how the exponent and the coefficient of a power function $f(x) = k x^n$ control its end behavior.

Interact: Drag the red point along the horizontal axis to change the exponent n (whole numbers), and drag the orange point up or down to change the coefficient k.

Observation: When n is even, the graph is symmetric about the y-axis and both ends point the same way; when n is odd, it is symmetric about the origin and the ends point in opposite directions. Making k negative reflects the graph vertically, reversing both ends.

Example 2.3.3: Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution available at the end of this section.

Example 2.3.4: Identifying the End Behavior of a Power Function.

Describe the end behavior of the graph of $f(x) = -x^9$.

Solution available at the end of this section.

Checkpoint 2.3.4

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius,

but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius, r, of the spill depends on the number of weeks, w, that have passed. This relationship is linear.

$$r(w) = 24 + 8w.$$

We can combine this with the formula for the area A of a circle, $A(r) = \pi r^2$. Composing these functions gives a formula for the area in terms of weeks.

$$A(w) = A(r(w)) = A(24 + 8w) = \pi(24 + 8w)^2.$$

Distributing gives the formula

$$A(w) = 576\pi + 384\pi w + 64\pi w^2.$$

This formula is an example of a **polynomial function**.

Definition: Polynomial Function

A **polynomial function** is a function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$

where $n \in \mathbb{N}$. This is called the **general form of a polynomial function**. Each a_i is a **coefficient** and can be any real number. Each product $a_i x_i$ is called a **term** of the function. The largest n for which $a_n \neq 0$ is called the **degree** of the polynomial function. The **leading term** is the term containing the highest power of the variable, or the term with the highest degree. The **leading coefficient** is the coefficient of the leading term.

Example 2.3.5: Identifying Polynomial Functions

Which of the following are polynomial functions?

$$\begin{aligned} f(x) &= 2x^3 \cdot 3x + 4 \\ g(x) &= -x(x^2 - 4) \\ h(x) &= 5\sqrt{x} + 2 \end{aligned}$$

Solution available at the end of this section.

Example 2.3.6: Revisiting the Difference Quotient

Find and simplify the Difference Quotient for the following function.

$$P(x) = 2x^3 + x - 4$$

Solution available at the end of this section.

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power (this is what we call, in the definition of the polynomial function, **general form**).

Example 2.3.7: Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$\begin{aligned} f(x) &= 3 + 2x^2 - 4x^3 \\ g(t) &= 5t^5 - 2t^3 + 7t \\ h(p) &= 6p - p^3 - 2 \end{aligned}$$

Solution available at the end of this section.

Checkpoint 2.3.7

Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 6$.

Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its **end behavior**. Recall the following definition.

Definition: End Behavior of a Function

The **end behavior** of a function refers to the way the graph of the function behaves as the input (typically the x-value) approaches positive or negative infinity (denoted $x \to \infty$ or $x \to -\infty$).

To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph. We state this in the following theorem along with a formal proof of the result.

Theorem: End Behavior of a Polynomial Function

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,$$

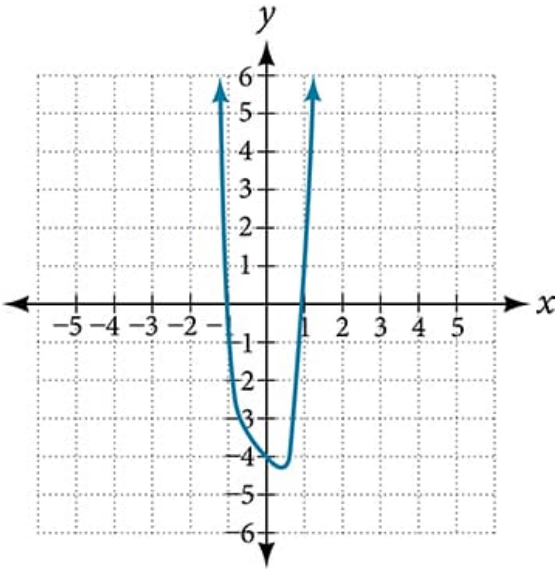
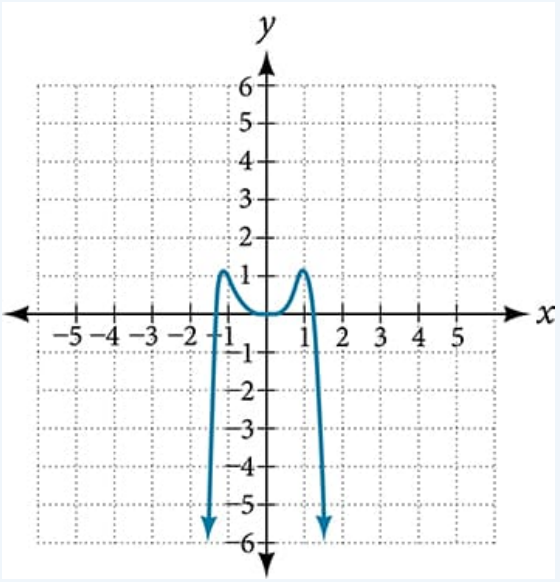
where $a_n \neq 0$ and $n \geq 1$. The end behavior of the graph of $P(x)$ matches the end behavior of the graph of $y = a_n x^n$. Hence,

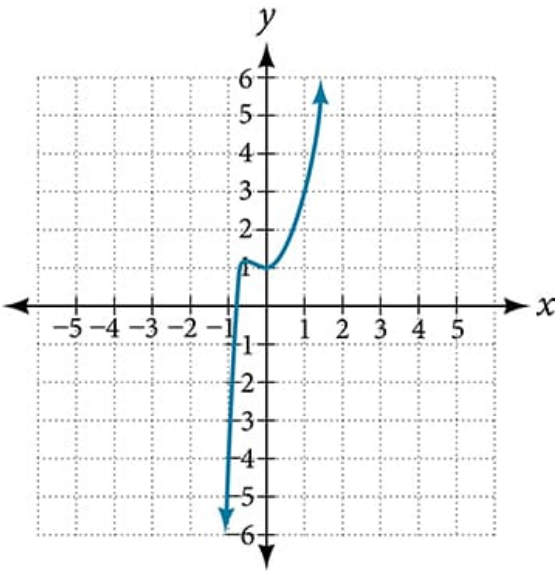
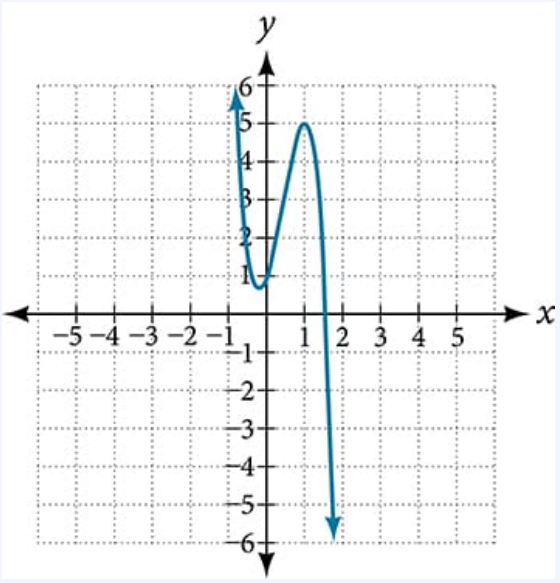
a. If n is even and $a_n > 0$, then $P(x) \to \infty$ as $x \to -\infty$ and as $x \to \infty$.
b. If n is even and $a_n < 0$, then $P(x) \to -\infty$ as $x \to -\infty$ and as $x \to \infty$.
c. If n is odd and $a_n > 0$, then $P(x) \to -\infty$ as $x \to -\infty$, and $P(x) \to \infty$ as $x \to \infty$.
d. If n is odd and $a_n < 0$, then $P(x) \to \infty$ as $x \to -\infty$, and $P(x) \to -\infty$ as $x \to \infty$.

Proof available at the end of this section.

Table 2.3.3 illustrates some examples.

Table 2.3.3

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$	
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$	

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$	$3x^5$	
$f(x) = -6x^3 + 7x^2 + 3x + 1$	$-6x^3$	

Example 2.3.8: Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure 2.3.7.

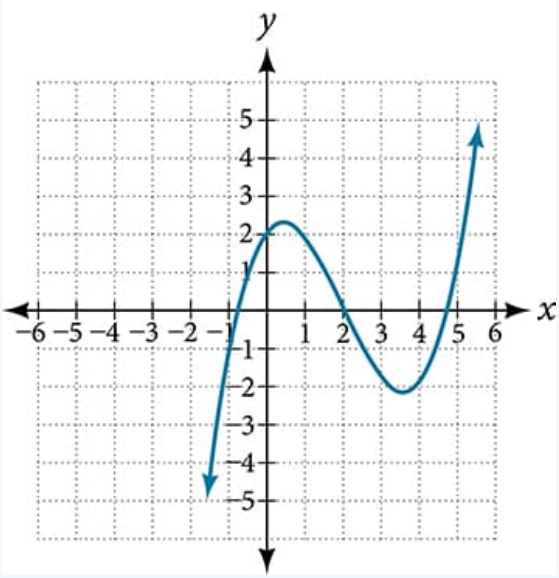


Figure 2.3.7

Solution available at the end of this section.

Checkpoint 2.3.8

Describe the end behavior, and determine a possible degree of the polynomial function in Figure 2.3.8.

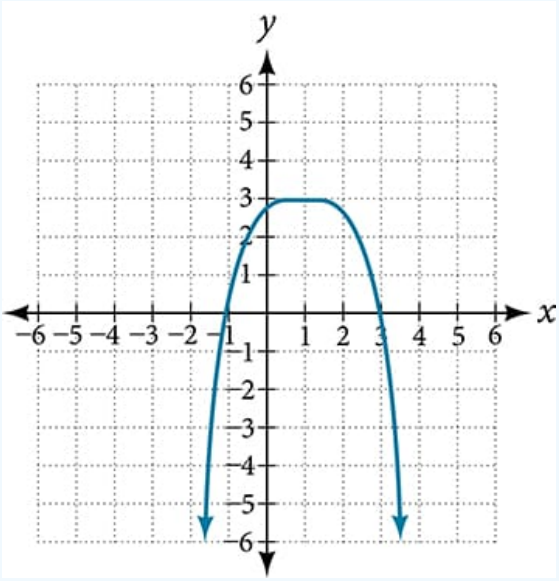


Figure 2.3.8

Example 2.3.9: Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Solution available at the end of this section.

Checkpoint 2.3.9

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. At first glance, most people think we are immediately going to find the **turning points** of the function.

Definition: Turning Point

A **turning point** of a graph is a point at which the graph changes from increasing to decreasing, or from decreasing to increasing.

Unfortunately, determining the exact locations of the turning points requires Calculus (however, we will soon be able to determine the maximum number of turning points for any given polynomial!).

While any further discussion of turning points must be placed on hold, we can (at the very least) concern ourselves with the intercepts. As with all functions, the y-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one y-intercept, $(0, a_0)$. The x-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one x-intercept. See Figure 2.3.9.

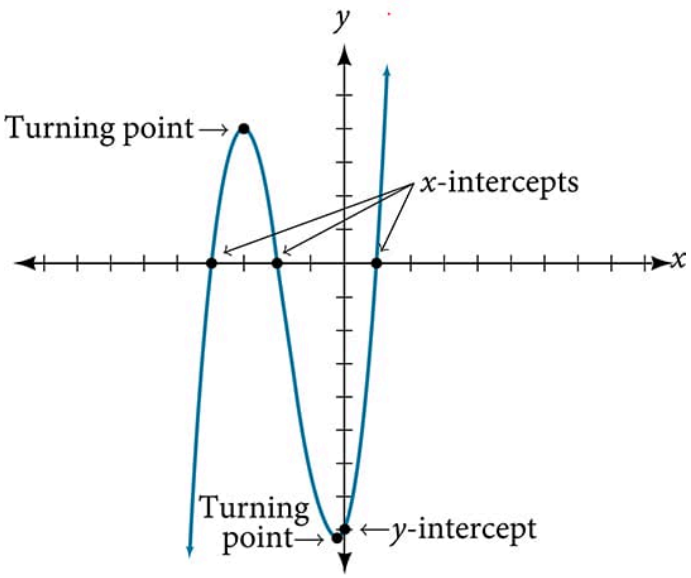


Figure 2.3.9

Example 2.3.10: Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the y- and x-intercepts.

Solution available at the end of this section.

Example 2.3.11: Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the y- and x-intercepts.

Solution available at the end of this section.

Checkpoint 2.3.11

Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the y- and x-intercepts.

Comparing Smooth and Continuous Graphs

We already know that the degree of a polynomial function helps us to determine its end behavior; however, the degree also helps us to determine the number of x-intercepts and, as such, a rough approximation to the number of turning points. A polynomial function of nth degree is the product of n factors, so it will have at most n zeros, or x-intercepts. Therefore, the graph of the polynomial function of degree n must have at most $n - 1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

Theorem: Number of Turning Points for a Polynomial Function

The graph of a polynomial function of degree $n \geq 1$ will have, at most, $n - 1$ turning points and n horizontal intercepts.

Proof available at the end of this section.

The proof of this theorem requires more mathematical "technology" than we currently have access to, so we hold off until Calculus.

While not technically true, we will, for the sake of simplicity, define a **continuous function** to be one that has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A **smooth curve** is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Example 2.3.12: Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of x-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

Solution available at the end of this section.

Checkpoint 2.3.12

Without graphing the function, determine the maximum number of x-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$.

Example 2.3.13: Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure 2.3.12 based on its intercepts and turning points?

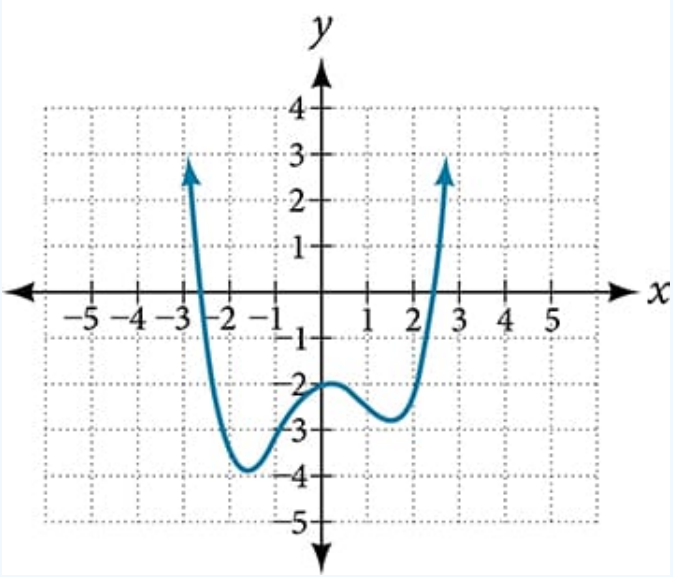


Figure 2.3.12

Solution available at the end of this section.

Checkpoint 2.3.13

What can we conclude about the polynomial represented by the graph shown in Figure 2.3.14 based on its intercepts and turning points?

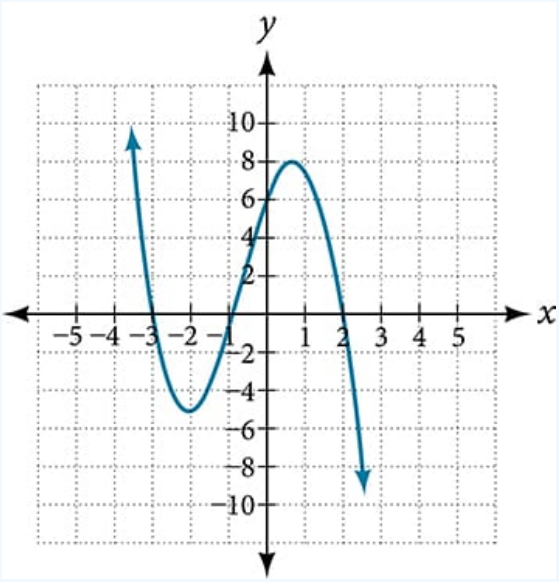


Figure 2.3.14

Example 2.3.14: Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

Solution available at the end of this section.

Checkpoint 2.3.14

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

Hidden Resources: CalFresh

CalFresh (formerly known as Food Stamps) is a nutrition program that helps students and their families buy healthy foods.

How Does CalFresh Work?

CalFresh benefits are issued on an electronic benefit transfer (EBT) card that works like a debit card and can be used at most grocery stores and many farmers' markets to buy food. You can even use CalFresh to purchase food online through Amazon and Walmart.com.

If you are on a limited income and are struggling to buy the healthy foods that you need, then CalFresh may be able to help! Plus, CalFresh benefits do not impact your financial aid, and you do not need your parents' tax return information to apply.

For more information, please see this page on the CRC website.

Success in Mathematics: Why Study Mathematics? (Part 1)

Until modern times, only a tiny percent of people had the opportunity to be "liberally educated." These people were not studying to learn a job or trade - they wanted to learn what other humans had thought and created. They were also trying to develop new knowledge. Menaechmus (350 B.C.E) was this kind of student. As he studied geometry, he discovered the properties of conic sections. Others built on his knowledge. Apollonius (350 B.C.E) wrote an eight-chapter book about conic sections.

Today we use our knowledge of conic sections to predict orbits, build telescopes, locate ships at sea, make "eyes" for robots, and collect solar energy. But we also study conic sections as an accomplishment of our human ancestors. In developing their equations, we also develop logic and creative problem-solving skills. Think about the football coach who has his team doing agility drills and running through tires. Why? When has a football team ever had to run through tires during a game? Similarly, with the right attitude, you can develop your ability to reason and think in your life outside of college by doing mathematics. You are responsible for recognizing that learning and studying may be valuable even when it is not immediately practical.

> One problem with teaching mathematics in the K-12 system - and I see it as a major difficulty - is that there is virtually nothing the pupils learn that has a non-trivial application in today's world. The most a teacher can tell a student who inquires, entirely reasonably, "How is this useful?" is that almost all mathematics finds uses, in many cases important ones, and that what they learn in school leads on to mathematics that definitely is used.
>
> - Keith Devlin, Standford University

By "non-trivial," Dr. Devlin means that many real-life applications require mathematics beyond what is studied in high school.

1. Younger students often ask, " When will I use this math?" Their teachers often reply, "You need to learn this math to go on to the next course. In college, you can take math that is very powerful for doing all kinds of practical things." Do you think this is the best reply to give to the student? Explain.
2. I've often pondered the best mixture of application and theory in a mathematics course. My love of mathematics lies in theory; however, my research in mathematics is in applications. As a student, which do you prefer? Explain.

Answers

Section Preview

The purpose of the material in this section is to introduce the vocabulary needed to work with polynomial functions. Students are introduced to power functions, polynomials, and end behavior **but** are not asked to graph at this time.

Section Preview

- **Factoring Techniques**
  - Factoring Trinomials: Finding the x-intercepts in Example 11 ($x^4 - 4x^2 - 45$ factored as a quadratic-in-form) and Checkpoint 11 ($x^2 - 3x - 10$) requires trinomial factoring.
- **Simplifying Expressions**
  - Laws of Exponents—Power Rules: Example 2 uses the power-of-a-power rule to rewrite $(x+h)^4$ as $\left((x+h)^2\right)^2$ and x^4 as $\left(x^2\right)^2$.
  - Laws of Exponents—Zero and Negative Exponent Rules: Example 1 classifies the reciprocal functions as power functions by rewriting them as x^{-1} and x^{-2}, and the constant/identity functions as x^0 and x^1.
  - Simplifying Expressions Involving Positive Rational Exponents: Example 1 identifies the square- and cube-root functions as power functions by converting $\sqrt{x}$ and $\sqrt[3]{x}$ into $x^{1/2}$ and $x^{1/3}$.

Section Preview

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.
- Compute the Difference Quotient for power and polynomial functions.

Example 2.3.1: Identifying Power Functions

All of the listed functions are power functions. The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$, respectively. The quadratic and cubic functions are power functions with whole number powers. The reciprocal and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$. The square and cube root functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Example 2.3.2: Revisiting the Difference Quotient

Recall that the Difference Quotient for the function f is defined to be

$$\frac{f(x+h) - f(x)}{h}.$$

Therefore, in this case, we have

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^4 - x^4}{h} && \text{(substituting)} \\
&= \frac{\left((x+h)^2\right)^2 - x^4}{h} && \text{(Laws of Exponents)} \\
&= \frac{\left((x+h)(x+h)\right)^2 - x^4}{h} && \\
&= \frac{\left(x^2+2hx+h^2\right)^2 - x^4}{h} && \text{(distributing)} \\
&= \frac{\left(x^2+2hx+h^2\right)\left(x^2+2hx+h^2\right) - x^4}{h} && \\
&= \frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4 - x^4}{h} && \text{(distributing)} \\
&= \frac{4x^3h+6x^2h^2+4xh^3+h^4}{h} && \text{(combining like terms)} \\
&= \frac{h\left(4x^3+6x^2h+4xh^2+h^3\right)}{h} && \text{(factoring out the GCF from the numerator)} \\
&= \frac{\cancel{h}^{1}\left(4x^3+6x^2h+4xh^2+h^3\right)}{\cancel{h}^{1}} && \text{(canceling like factors)} \\
&= 4x^3+6x^2h+4xh^2+h^3 && \text{(simplifying)}
\end{aligned}$$

Example 2.3.3: Identifying the End Behavior of a Power Function

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. Therefore, as $x \to \infty$, $f(x) \to \infty$. As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty$, $f(x) \to \infty$. We can graphically represent the function as shown in Figure 2.3.5.

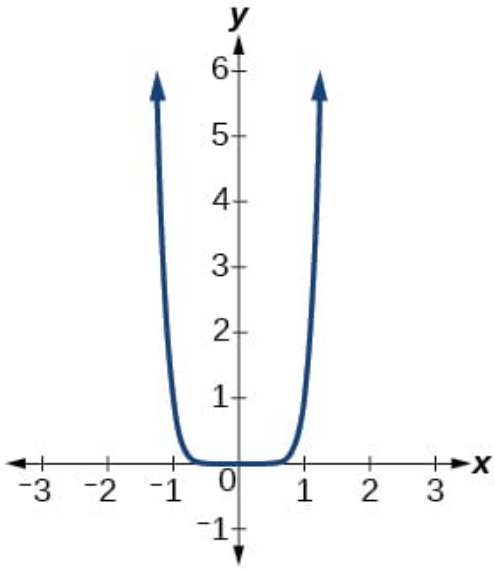


Figure 2.3.5

Example 2.3.4**: Identifying the End Behavior of a Power Function.**

The exponent of the power function is 9 (an odd number). Because the coefficient is -1 (negative), the graph is the reflection about the x-axis of the graph of $f(x) = x^9$. Figure 2.3.6 shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty,\ f(x) \to \infty; \text{ as } x \to \infty,\ f(x) \to -\infty.$$

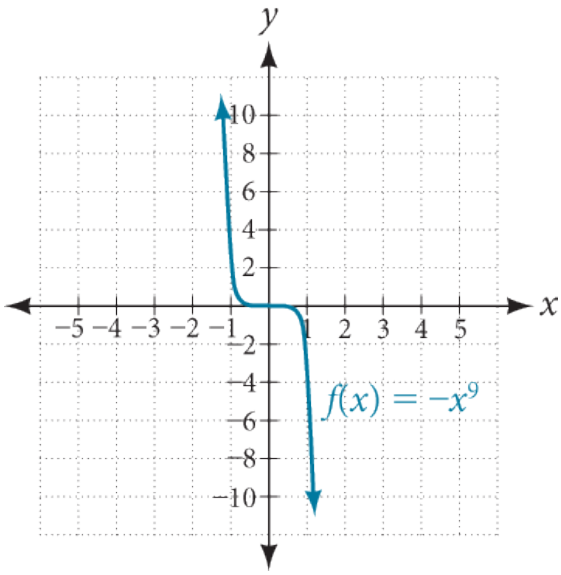


Figure 2.3.6

We can check our work by using the table feature on a graphing utility.

Table 2.3.2

x	$f(x)$
-10	1,000,000,000
-5	1,953,125
0	0
5	-1,953,125
10	-1,000,000,000

We can see from Table 2.3.2 that, when we substitute very small values for x, the output is very large, and when we substitute very large values for x, the output is

very small (meaning that it is a very large negative value).

Example 2.3.5: Identifying Polynomial Functions

The first two functions are examples of polynomial functions because they can be written in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Example 2.3.6: Revisiting the Difference Quotient

$$\begin{aligned}
\frac{P(x+h)-P(x)}{h} &= \frac{2(x+h)^3+(x+h)-4-(2x^3+x-4)}{h} && \text{(substituting)}\\
&= \frac{2(x+h)^3+(x+h)-4-2x^3-x+4}{h} && \text{(distributing the negative)}\\
&= \frac{2(x^3+3x^2h+3xh^2+h^3)+x+h-4-2x^3-x+4}{h} && \text{(distributing)}\\
&= \frac{2x^3+6x^2h+6xh^2+2h^3+x+h-4-2x^3-x+4}{h} && \text{(distributing)}\\
&= \frac{6x^2h+6xh^2+2h^3+h}{h} && \text{(combining like terms)}\\
&= \frac{h\left(6x^2+6xh+2h^2+1\right)}{h} && \text{(factoring out the GCF from the numerator)}\\
&= \frac{\cancel{h}^{1}\left(6x^2+6xh+2h^2+1\right)}{\cancel{h}^{1}} && \text{(canceling like factors)}\\
&= 6x^2+6xh+2h^2+1 && \text{(simplifying)}
\end{aligned}$$

Example 2.3.7: Identifying the Degree and Leading Coefficient of a Polynomial Function

For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, -1.

Theorem: End Behavior of a Polynomial Function

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_n \neq 0$. Then

$$\begin{aligned}
P(x) &= a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0\\
&= a_n x^n\left(1 + \frac{a_{n-1}x^{n-1}}{a_n x^n} + \frac{a_{n-2}x^{n-2}}{a_n x^n} + \cdots + \frac{a_2 x^2}{a_n x^n} + \frac{a_1 x}{a_n x^n} + \frac{a_0}{a_n x^n}\right) && \text{(factoring out } a_n x^n\text{)}\\
&= a_n x^n\left(1 + \frac{a_{n-1}}{a_n x} + \frac{a_{n-2}}{a_n x^2} + \cdots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right) && \text{(Laws of Exponents)}
\end{aligned}$$

Since $a_n \neq 0$, the fractions $\frac{a_k}{a_n}$ are finite constants. For notational efficiency, let's denote $\frac{a_k}{a_n}$ as b_k. Therefore,

$$P(x) = a_n x^n\left(1 + b_{n-1}\cdot\frac{1}{x} + b_{n-2}\cdot\frac{1}{x^2} + \cdots + b_2\cdot\frac{1}{x^{n-2}} + b_1\cdot\frac{1}{x^{n-1}} + b_0\frac{1}{x^n}\right).$$

(The next step is proved in Calculus, but can be understood here as a strong possibility.) As long as the power on x is not negative, as $x \to \pm\infty$, $\frac{1}{x^p} \to 0$. Since all the powers on x in

$$P(x) = a_n x^n\left(1 + b_{n-1}\cdot\frac{1}{x} + b_{n-2}\cdot\frac{1}{x^2} + \cdots + b_2\cdot\frac{1}{x^{n-2}} + b_1\cdot\frac{1}{x^{n-1}} + b_0\frac{1}{x^n}\right)$$

are not negative, all terms involving x in the denominator must tend to 0 as $x \to \pm\infty$. That is, as $x \to \pm\infty$, $P(x) \to a_n x^n(1+0+0+\cdots+0+0+0) = a_n x^n$. Hence, the end behavior of $P(x)$ is $y = a_n x^n$.

Example 2.3.8: Identifying End Behavior and Degree of a Polynomial Function

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very negative, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to -\infty,\ f(x) \to -\infty;\ \text{as } x \to \infty,\ f(x) \to \infty.$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

Example 2.3.9: Identifying End Behavior and Degree of a Polynomial Function

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2+3x-4) \\ &= -3x^4-9x^3+12x^2 \end{aligned}$$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (-3), so the end behavior is

$$\text{as } x \to -\infty,\ f(x) \to -\infty;\ \text{as } x \to \infty,\ f(x) \to -\infty.$$

Example $2.3.10$**: Determining the Intercepts of a Polynomial Function**

The y-intercept occurs when the input is zero so substitute 0 for x.

$$\begin{aligned} f(0) &= (0-2)(0+1)(0-4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The y-intercept is $(0, 8)$.

The x-intercepts occur when the output is zero.

$$\begin{aligned} & 0 = (x-2)(x+1)(x-4) \\ \Longrightarrow \quad & x-2=0 \quad \text{or} \quad x+1=0 \quad \text{or} \quad x-4=0 \\ \Longrightarrow \quad & x=2 \quad \text{or} \quad x=-1 \quad \text{or} \quad x=4 \end{aligned}$$

The x-intercepts are $(2, 0)$, $(-1, 0)$, and $(4, 0)$.

We can see these intercepts on the graph of the function shown in Figure $2.3.10$.

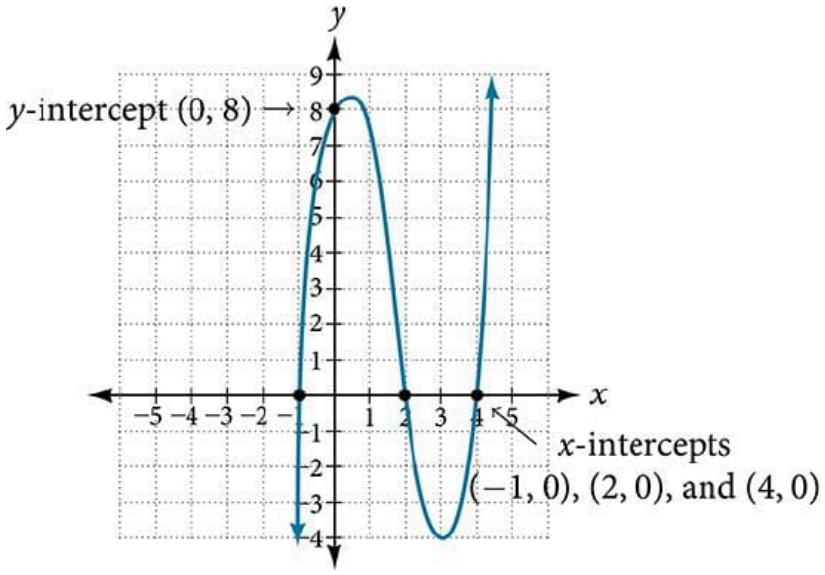


Figure $2.3.10$

Example $2.3.11$**: Determining the Intercepts of a Polynomial Function with Factoring**

The y-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^2 - 45 \\ &= -45 \end{aligned}$$

The y-intercept is $(0, -45)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$f(x) = x^4 - 4x^2 - 45 = (x^2-9)(x^2+5) = (x-3)(x+3)(x^2+5).$$

Therefore, we solve

$$0 = (x-3)(x+3)(x^2+5).$$

This mean $x-3=0$ or $x+3=0$ ($x^2+5=0$ would require $x^2=-5$, which has no real solution). Hence, $x=3$ or $x=-3$. The x-intercepts are $(3, 0)$ and $(-3, 0)$.

We can see these intercepts on the graph of the function shown in Figure $2.3.11$. We can see that the function is even because $f(x) = f(-x)$.

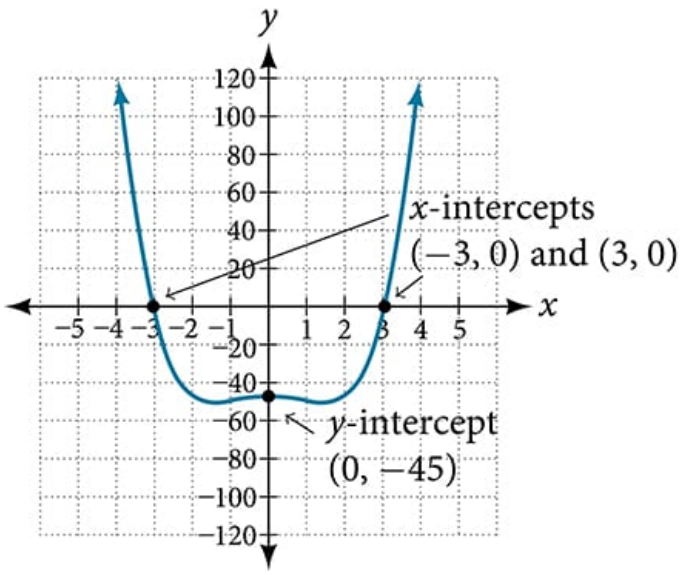


Figure 2.3.11

Theorem: Number of Turning Points for a Polynomial Function

This proof must wait until Calculus.

Example 2.3.12**: Determining the Number of Intercepts and Turning Points of a Polynomial**

The polynomial has a degree of 10, so there are at most 10 x-intercepts and at most $10-1=9$ turning points.

Example 2.3.13**: Drawing Conclusions about a Polynomial Function from the Graph**

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See Figure 2.3.13.

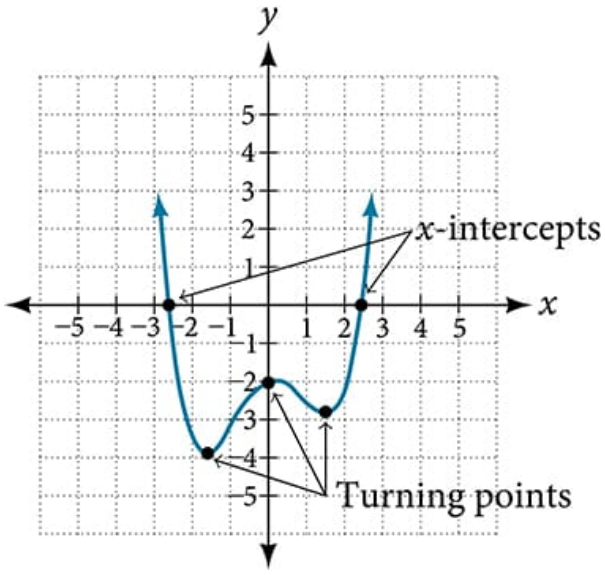


Figure 2.3.13

The graph has 2 x-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

Example 2.3.14**: Drawing Conclusions about a Polynomial Function from the Factors**

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The y-intercept is $(0,0)$.

The x-intercepts are found by determining the zeros of the function.

$$0=-4x(x+3)(x-4) \implies x=0 \text{ or } x+3=0 \text{ or } x-4=0 \implies x=0 \text{ or } x=-3 \text{ or } x=4.$$

The x-intercepts are $(0,0),(-3,0)$, and $(4,0)$.

The degree is 3 so the graph has at most 2 turning points.

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2.3.1: Resources and Key Concepts

Resources

Videos

- Power function
  - Intermediate Algebra - Integer Exponents: Power Functions
- End behavior
  - Power Functions and Polynomial Functions - Identifying End Behavior of Polynomial Functions
  - Polynomial Functions and Their Graphs - End Behavior of Polynomials
- Polynomial function
  - Polynomial Functions - The Concept
- Zeros (of a polynomial)
  - Graphs of Polynomial Functions - Using Factoring to Find Zeros of Polynomial Functions

Key Concepts

Definitions

- **Power Function**: A function that can be represented in the form $f(x) = kx^p$, where k and p are real numbers, and k is known as the coefficient.
- **End Behavior of a Function**: The way the graph of the function behaves as the input (typically the x-value) approaches positive or negative infinity (denoted $x \to \infty$ or $x \to -\infty$).
- **Polynomial Function**: A function that can be written in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where $n \in \mathbb{N}$ (a non-negative integer) and the coefficients a_i are real numbers.
- **General Form of a Polynomial Function**: The form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, typically with terms arranged in descending order of power.
- **Coefficient (of a term in a polynomial)**: The real number factor multiplying the variable part of a term (e.g., in $a_i x^i$, a_i is the coefficient).
- **Term (of a polynomial function)**: Each product $a_i x^i$ in the polynomial function.
- **Degree (of a polynomial function)**: The largest exponent n for which the coefficient a_n is not zero.
- **Leading Term (of a polynomial function)**: The term containing the highest power of the variable (the term with the highest degree).
- **Leading Coefficient (of a polynomial function)**: The coefficient of the leading term.
- **Turning Point (of a polynomial graph)**: A point at which the function values change from increasing to decreasing or decreasing to increasing.
- **Continuous Function (informal definition)**: A function whose graph has no breaks; it can be drawn without lifting the pen from the paper.
- **Smooth Curve (informal definition)**: A graph that has no sharp corners.

Theorems

- **End Behavior of a Polynomial Function**: Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_n \neq 0$. Then the end behavior of the graph of $P(x)$ will match the end behavior of the graph of its leading term $y = a_n x^n$.
- **Number of Turning Points for a Polynomial Function**: A polynomial of degree n will have, at most, $n - 1$ turning points and n horizontal intercepts (zeros).

Common Mistakes

- **Confusing Power Functions with Exponential Functions**: A power function has a variable base and a fixed power (e.g., x^2),

while an exponential function has a fixed base and a variable power (e.g., 2^x). The function $f(x) = 2^x$ is not a power function.

- **Incorrectly Identifying the Degree or Leading Coefficient**: When a polynomial is not written in standard general form (descending powers), care must be taken to identify the term with the highest power to determine the degree and leading coefficient.
- **Errors in Determining End Behavior**: Misinterpreting the combined effect of the sign of the leading coefficient and the parity (even/odd) of the degree of the polynomial on its end behavior.
- **Assuming Maximum Number of Turning Points/Intercepts is Always Achieved**: A polynomial of degree n has *at most* $n - 1$ turning points and *at most* n x-intercepts. It does not necessarily have exactly that many.

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2.3.2: Homework

Reading Questions

1. What is the general form of a power function? Give an example of a toolkit function that is a power function.
2. Is $f(x) = 2^x$ a power function? Why or why not?
3. How does the end behavior of an even-degree power function $f(x) = kx^n$ (where n is even) differ based on the sign of k?
4. How does the end behavior of an odd-degree power function $f(x) = kx^n$ (where n is odd) differ based on the sign of k?
5. What is the general form of a polynomial function? What restriction is placed on the exponents of the variable?
6. Define the "degree" of a polynomial.
7. Define the "leading term" and "leading coefficient" of a polynomial.
8. How does the leading term of a polynomial function relate to its end behavior?
9. What is a "turning point" of a polynomial graph?
10. What is the maximum number of x-intercepts a polynomial of degree n can have?
11. What is the maximum number of turning points a polynomial of degree n can have?
12. How do you find the y-intercept of a polynomial function given in general form?
13. If a polynomial function is given in factored form, like $f(x) = (x-2)(x+1)(x-4)$, how can you find its x-intercepts?
14. Are the graphs of polynomial functions smooth and continuous? What do these terms mean informally?
15. When computing the difference quotient for $f(x) = x^4$, what algebraic techniques might be useful for expanding $(x+h)^4$?
16. Explain the difference between the coefficient of a power function and its degree.
17. If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?
18. In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.
19. What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph?
20. What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$.

Homework

Algebraic

For the following exercises, identify the function as a power function, a polynomial function, or neither.

21. $f(x) = x^5$
22. $f(x) = (x^2)^3$
23. $f(x) = x - x^4$
24. $f(x) = \frac{x^2}{x^2-1}$
25. $f(x) = 2x(x+2)(x-1)^2$
26. $f(x) = 3^{x+1}$

For the following exercises, find the degree and leading coefficient for the given polynomial.

27. $-3x^4$
28. $7 - 2x^2$
29. $-2x^2 - 3x^5 + x - 6$
30. $x(4-x^2)(2x+1)$
31. $x^2(2x-3)^2$

For the following exercises, determine the end behavior of the functions.

32. $f(x) = x^4$
33. $f(x) = x^3$
34. $f(x) = -x^4$

35. $f(x) = -x^9$
36. $f(x) = -2x^4 - 3x^2 + x - 1$
37. $f(x) = 3x^2 + x - 2$
38. $f(x) = x^2(2x^3 - x + 1)$
39. $f(x) = (2 - x)^7$

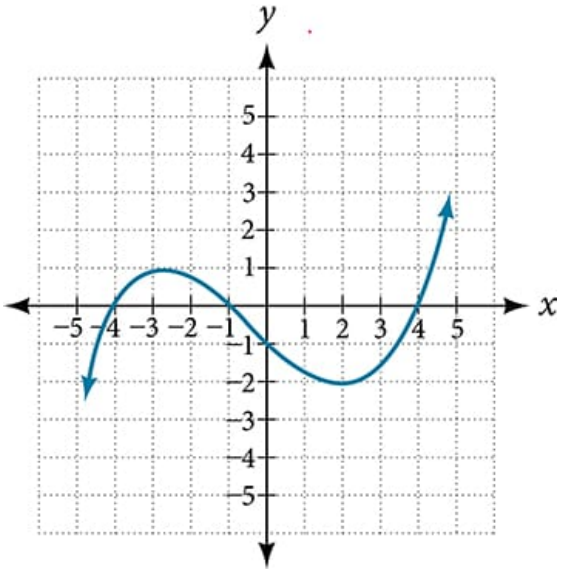
For the following exercises, find the intercepts of the functions.

40. $f(t) = 2(t - 1)(t + 2)(t - 3)$
41. $g(n) = -2(3n - 1)(2n + 1)$
42. $f(x) = x^4 - 16$
43. $f(x) = x^3 + 27$
44. $f(x) = x(x^2 - 2x - 8)$
45. $f(x) = (x + 3)(4x^2 - 1)$

Graphical

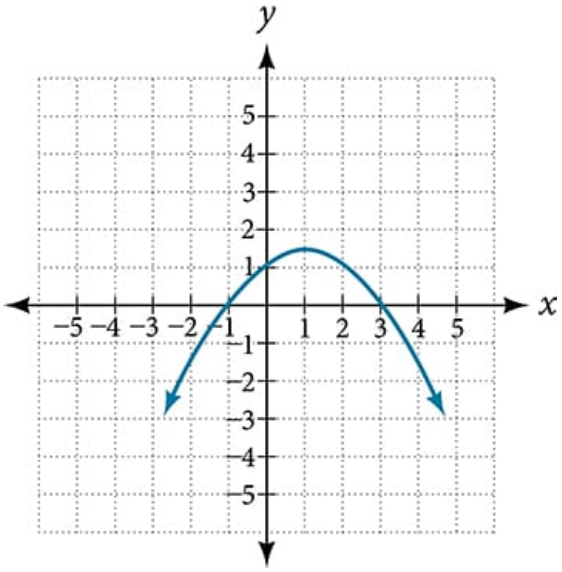
For the following exercises, determine the least possible degree of the polynomial function shown.

46.

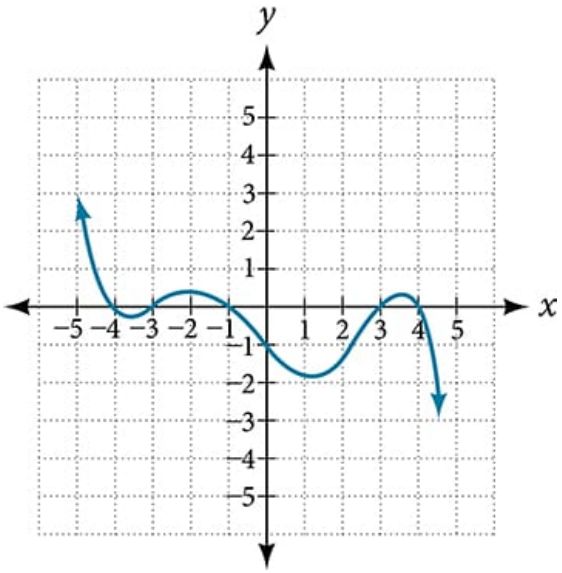


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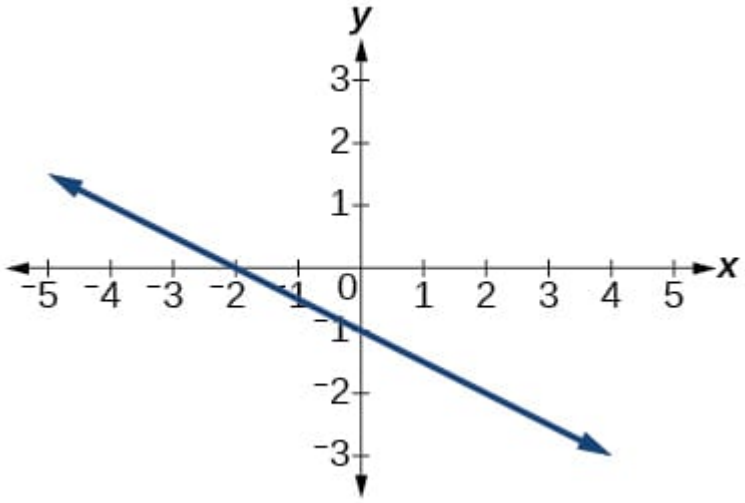
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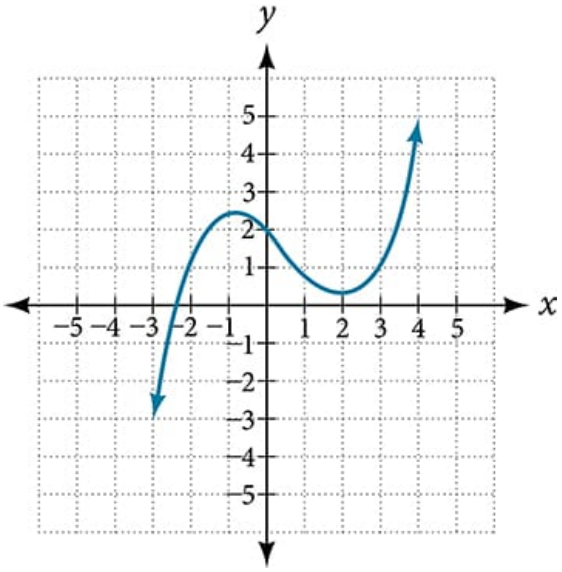

y
x
5
4
3
2
1
−5 −4 −3 −2 −1
1 2 3 4 5
−1
−2
−3
−4
−5

19.

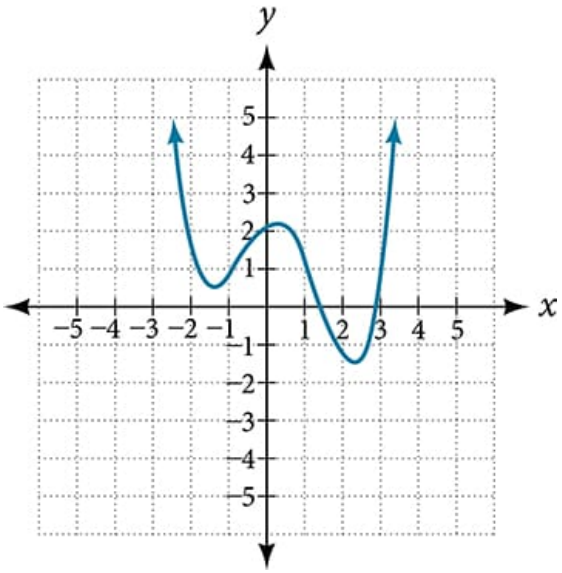

y
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1 2 3 4 5
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−2
−3
−4
−5

20.

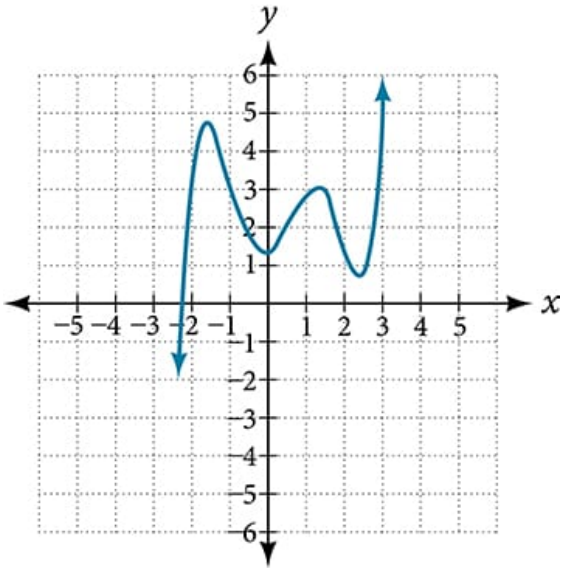

y
x
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1
−5 −4 −3 −2 −1 0
1 2 3 4 5
−1
−2
−3



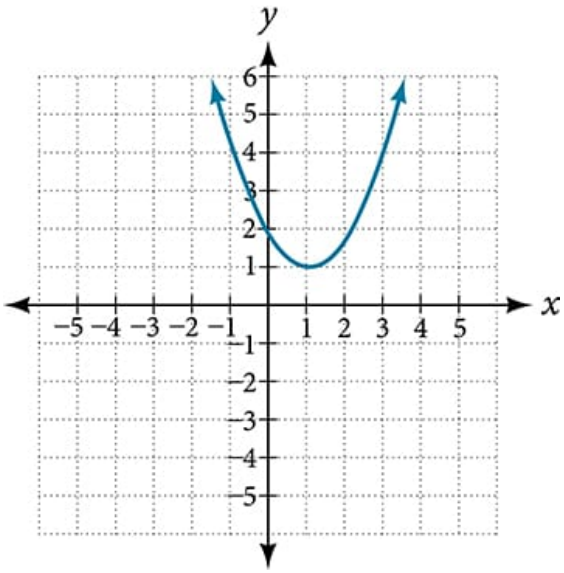
51.



52.



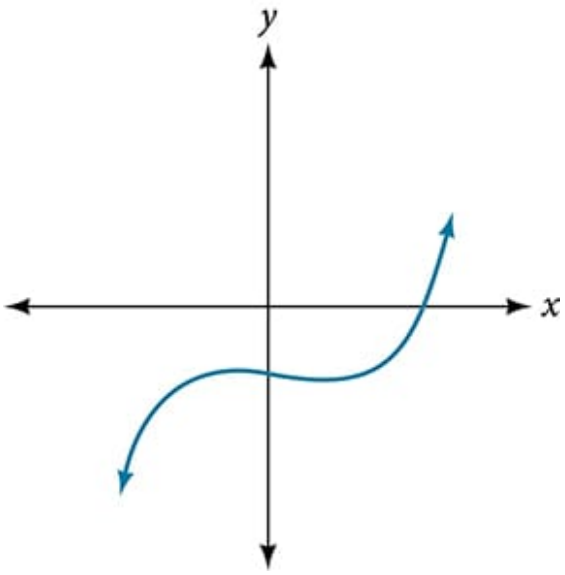
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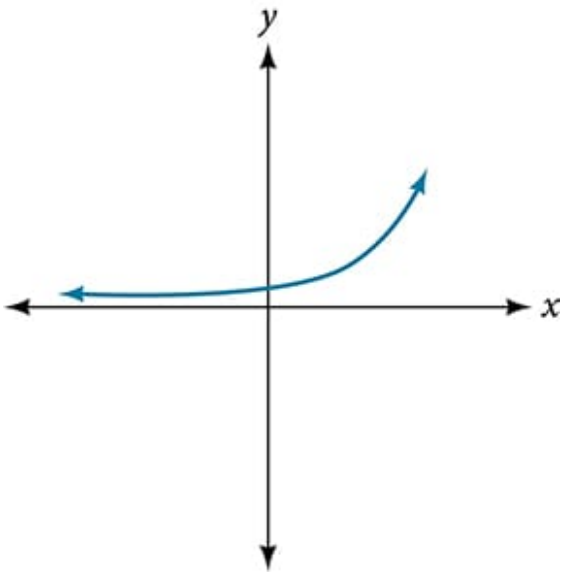
For the following exercises, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.

54.

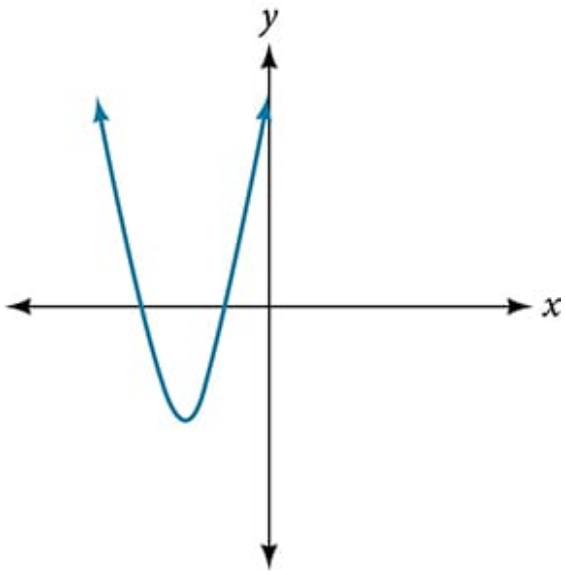
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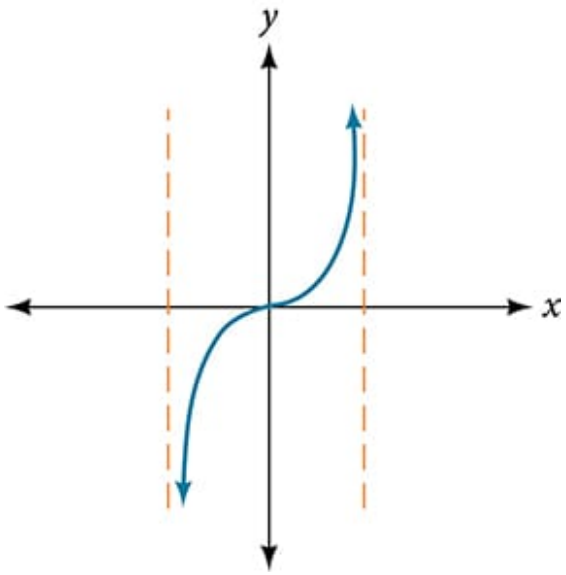
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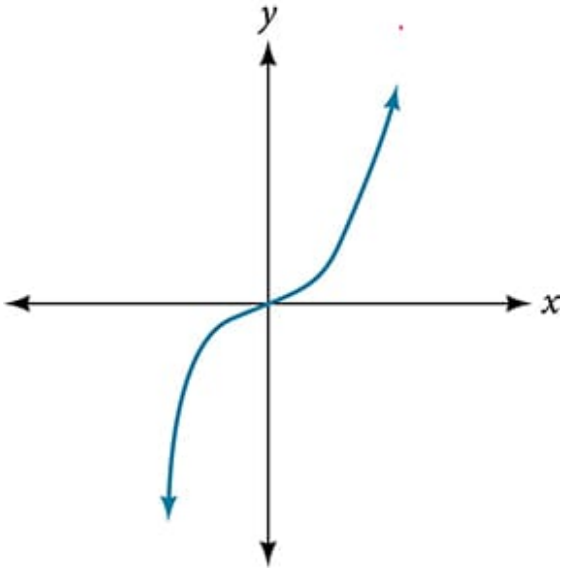
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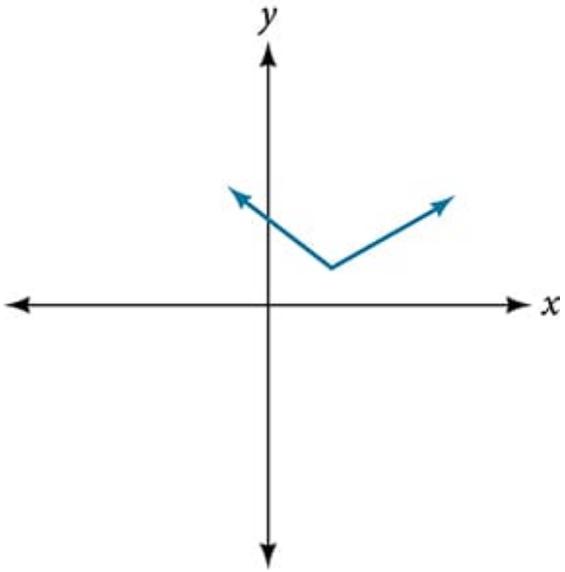
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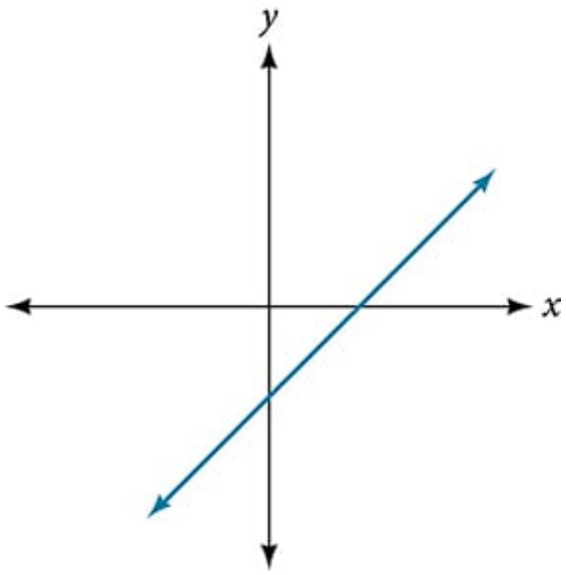


59.



60.





Numeric

For the following exercises, make a table to confirm the end behavior of the function.

61. $f(x) = -x^3$
62. $f(x) = x^4 - 5x^2$
63. $f(x) = x^2(1-x)^2$
64. $f(x) = (x-1)(x-2)(3-x)$
65. $f(x) = \frac{x^5}{10} - x^4$

Technology

For the following exercises, graph the polynomial functions using graphing technology. Based on the graph, determine the intercepts and the end behavior.

66. $f(x) = x^3(x-2)$
67. $f(x) = x(x-3)(x+3)$
68. $f(x) = x(14-2x)(10-2x)$
69. $f(x) = x(14-2x)(10-2x)^2$
70. $f(x) = x^3 - 16x$
71. $f(x) = x^3 - 27$
72. $f(x) = x^4 - 81$
73. $f(x) = -x^3 + x^2 + 2x$
74. $f(x) = x^3 - 2x^2 - 15x$
75. $f(x) = x^3 - 0.01x$

Extensions

For the following exercises, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is 1 or –1. There may be more than one correct answer.

76. The y-intercept is $(0,-4)$. The x-intercepts are $(-2,0)$, $(2,0)$. Degree is 2. End behavior: as $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to \infty$.
77. The y-intercept is $(0,9)$. The x-intercepts are $(-3,0)$, $(3,0)$. Degree is 2. End behavior: as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to -\infty$.
78. The y-intercept is $(0,0)$. The x-intercepts are $(0,0)$, $(2,0)$. Degree is 3. End behavior: as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.

79. The y-intercept is $(0, 1)$. The x-intercept is $(1, 0)$. Degree is 3. End behavior: as $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to -\infty$.
80. The y-intercept is $(0, 1)$. There is no x-intercept. Degree is 4. End behavior: as $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to \infty$.

Real-World Applications

For the following exercises, use the written statements to construct a polynomial function that represents the required information.

81. An oil slick is expanding as a circle. The radius of the circle is increasing at the rate of 20 meters per day. Express the area of the circle as a function of d, the number of days elapsed.
82. A cube has an edge of 3 feet. The edge is increasing at the rate of 2 feet per minute. Express the volume of the cube as a function of m, the number of minutes elapsed.
83. A rectangle has a length of 10 inches and a width of 6 inches. If the length is increased by x inches and the width increased by twice that amount, express the area of the rectangle as a function of x.
84. An open box is to be constructed by cutting out square corners of x-inch sides from a piece of cardboard 8 inches by 8 inches and then folding up the sides. Express the volume of the box as a function of x.
85. A rectangle is twice as long as it is wide. Squares of side 2 feet are cut out from each corner. Then the sides are folded up to make an open box. Express the volume of the box as a function of the width (x).

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2.4: Graphs of Polynomial Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in Table 2.4.1.

Table 2.4.1

Year	2006	2007	2008	2009	2010	2011	2012	2013
Revenues	52.4	52.8	51.2	49.5	48.6	48.6	48.7	47.1

The revenue can be modeled by the polynomial function $R(t) = -0.037t^4 + 1.414t^3 - 19.777t^2 + 118.696t - 205.332$, where R represents the revenue in millions of dollars and t represents the year, with $t = 6$ corresponding to 2006. Over which intervals is the revenue for the company increasing? Over which intervals is the revenue for the company decreasing?

These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous (again, as mentioned in the previous section, this is not technically true; however, we will use this flawed definition of continuity until we reach Calculus). Figure 2.4.1 shows a graph that represents a polynomial function and a graph that represents a function that is not a polynomial.

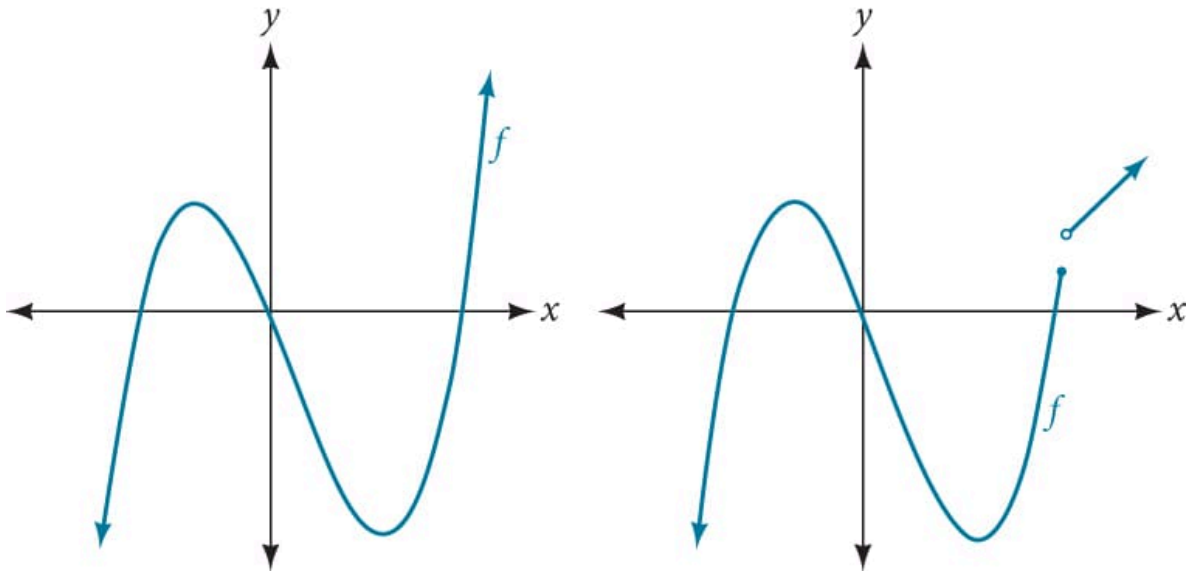


Figure 2.4.1

Example 2.4.1: Recognizing Polynomial Functions

Which of the graphs in Figure 2.4.2 represents a polynomial function?

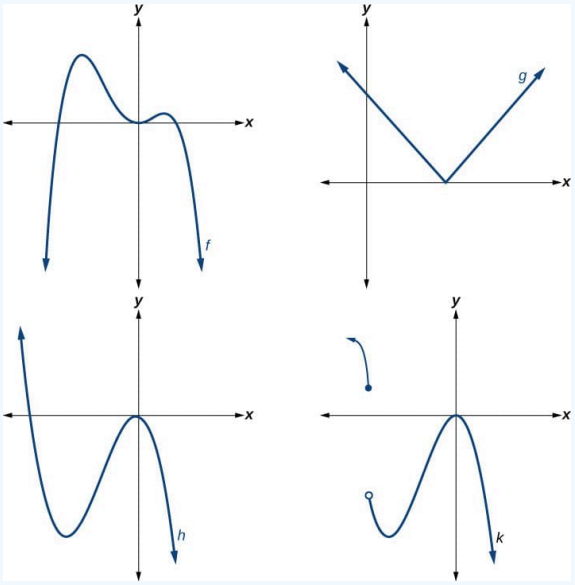


Figure 2.4.2

Solution available at the end of this section.

Q&A

Do all polynomial functions have as their domain all real numbers?

Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f(x) = 0$ are called zeros of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple Quadratic Formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

1. The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.
3. Technology is used to determine the intercepts.

Example 2.4.2: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Solution available at the end of this section.

Example 2.4.3: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

Solution available at the end of this section.

Example 2.4.4: Finding the y- and x-Intercepts of a Polynomial in Factored Form

Find the y- and x-intercepts of $g(x) = (x - 2)^2(2x + 3)$.

Solution available at the end of this section.

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial. For example, Figure 2.4.5 shows that our answers for Example 2.4.4 are correct.

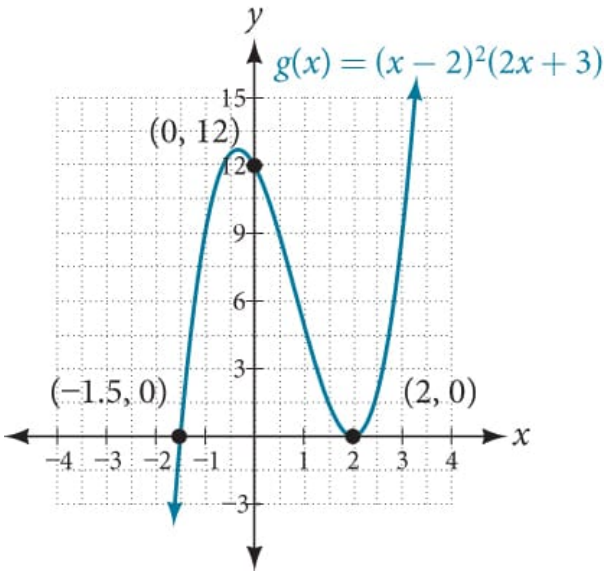


Figure 2.4.5

Example 2.4.5: Finding the x-Intercepts of a Polynomial Function Using a Graph

Find the x-intercepts of $h(x) = x^3 + 4x^2 + x - 6$.

Solution available at the end of this section.

Checkpoint 2.4.5

Find the y- and x-intercepts of the function $f(x) = x^4 - 19x^2 + 30x$.

Identifying Zeros and Their Multiplicities

Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and bounce off.

Suppose, for example, we graph the function $f(x) = (x + 3)(x - 2)^2(x + 1)^3$. Notice in Figure 2.4.7 that the behavior of the function at each of the x-intercepts is different.

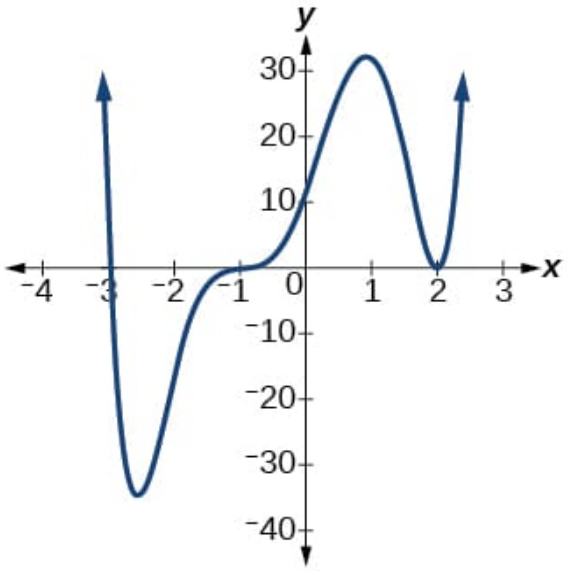


Figure 2.4.7**:** *Identifying the behavior of the graph at an x-intercept by examining the multiplicity of the zero.*

The x-intercept at $x=-3$ is the solution of equation $(x+3)=0$. The graph passes directly through the x-intercept at $x=-3$. We say the factor is **linear** (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a **single zero** because the zero corresponds to a single factor of the function.

The x-intercept at $x=2$ is the repeated solution of equation $(x-2)^2=0$. The graph touches the axis at the intercept and changes direction. We say the factor is **quadratic** (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept. Since $(x-2)^2=(x-2)(x-2)$, the factor is repeated. This leads us to an important definition that will be useful when it comes to polynomial functions.

Definition: Multiplicity of a Zero

Let h be a zero of a nonzero polynomial function f. The **multiplicity** of h is the largest positive integer p for which $(x-h)^p$ is a factor of $f(x)$. We say that $x=h$ is a **zero of multiplicity** p. A zero of multiplicity 1 is called a **simple zero**; a zero of multiplicity greater than 1 is called a **repeated zero** (or **multiple zero**).

Hence, the zero associated with $x=2$ has multiplicity 2 because the factor $(x-2)$ occurs twice.

The x-intercept at $x=-1$ is the repeated solution of factor $(x+1)^3=0$. The graph passes through the axis at the intercept, but flattens out a bit first. We say this factor is **cubic** (degree 3), so the behavior near the intercept is like that of a cubic—with the same "S"-shape near the intercept as the toolkit function $f(x)=x^3$. By definition, this is a zero of multiplicity 3.

Now that we know what it means for a zero to have multiplicity p, it is natural to ask if there is any ramification of the multiplicity of a zero.

Theorem: Local Behavior Near a Zero

If $x=h$ is a zero of multiplicity p for a given polynomial, then the local behavior of the graph of the polynomial near $x=h$ matches the behavior of $y=\pm(x-h)^p$, where the sign of the coefficient is determined by other characteristics of the given polynomial.

In truth, most people (myself included) would say that the behavior of the polynomial near to a zero of multiplicity p matches the graph of $y=x^p$, but possibly reflected about the x-axis. What we *mean* is that the behavior of the graph near to the zero matches the look of $y=x^p$ near the origin.

For zeros with even multiplicities, the graphs *touch* or are **tangent** to the x-axis. For zeros with odd multiplicities, the graphs

cross or intersect the x-axis. See Figure $2.4.8$ for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.

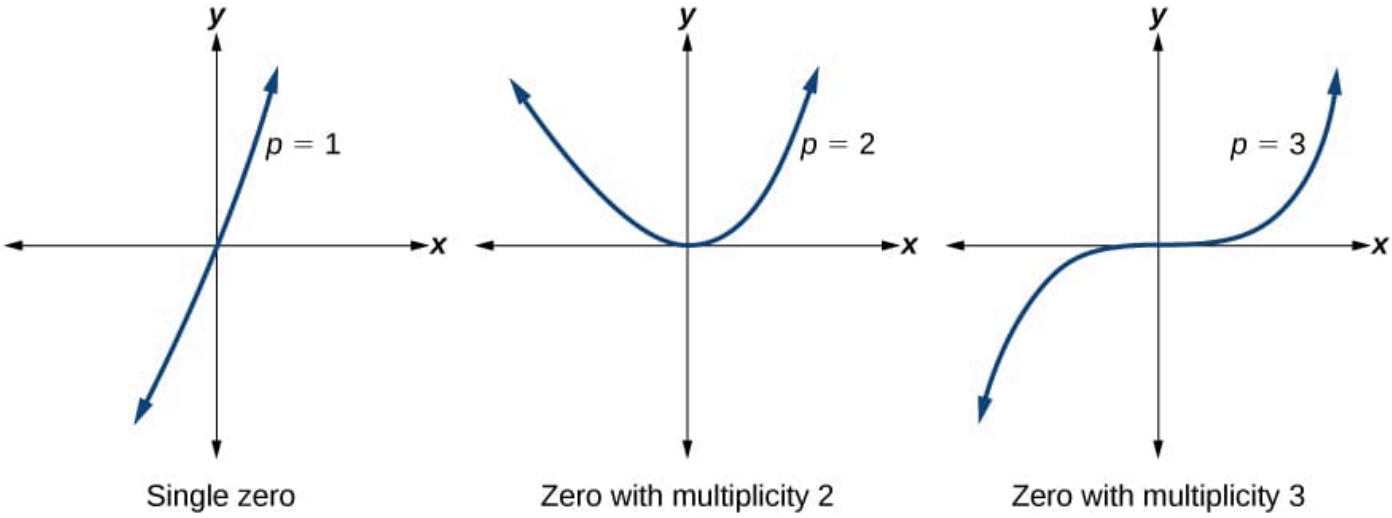


Figure $2.4.8$

For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis. For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

Interactive Element: Multiplicity and the Shape of a Graph Near a Zero

The following Interactive Element might help you visualize how the multiplicity of a zero controls whether the graph crosses or bounces, and how it flattens near the zero.

Interact: Drag the red handle left and right to change the multiplicity n of the zero at the origin.

Observation: Odd n makes the graph *cross* the x-axis, while even n makes it *touch and bounce*; and as n grows, the graph flattens more near the zero — just like $y = x^n$.

Caution: Graph Accurately Near Zeros!

You will eventually be asked to make accurate sketches of polynomial functions. Part of this skill is to show that you know the impact of high multiplicity on a zero. The higher the multiplicity, the flatter the polynomial graph is near to the zero—be prepared to show this.

Finally, you will inevitably notice the following fact.

Theorem: Sum of Multiplicities and the Degree of a Polynomial

The degree of a polynomial factored into the product of linear and irreducible quadratic factors is sum of the multiplicities of the linear factors and the twice the sum of the multiplicities of the irreducible quadratic factors. That is, if

$$P(x) = (a_1x + b_1)^{m_1}(a_2x + b_2)^{m_2} \cdots (a_kx + b_k)^{m_k}(c_1x^2 + d_1)^{p_1}(c_2x^2 + d_2)^{p_2} \cdots (c_nx^2 + d_n)^{p_n},$$

then the degree of $P(x)$ is

$$m_1 + m_2 + \cdots + m_k + 2(p_1 + p_2 + \cdots p_n).$$

Example 2.4.6: Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in Figure 2.4.9 to identify the zeros of the function and their possible multiplicities.

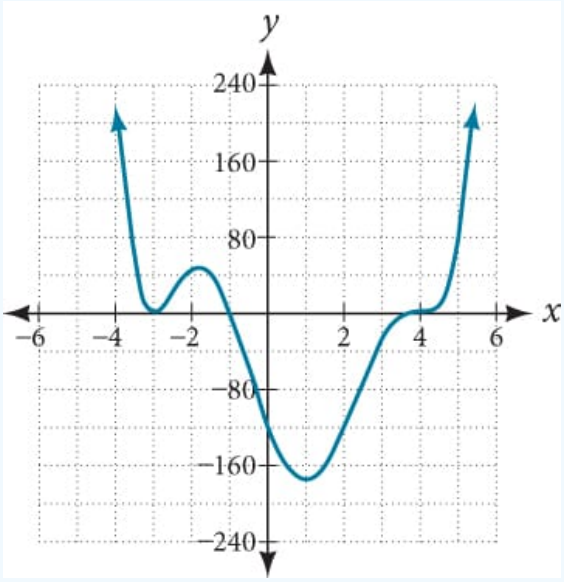


Figure 2.4.9

Solution available at the end of this section.

Checkpoint 2.4.6

Use the graph of the function of degree 7 in Figure 2.4.10 to identify the zeros of the function and their multiplicities.

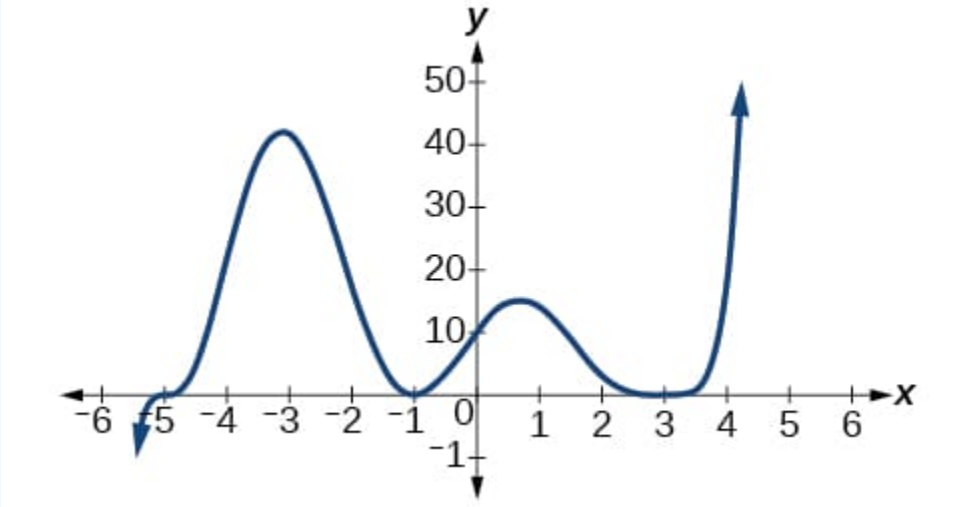


Figure 2.4.10

Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

will either ultimately rise or fall as x increases or decreases without bound. This is because, for very large inputs, say 100 or 1,000,

the leading term dominates the size of the output. The same is true for very negative inputs, say –100 or -1,000.

Recall that we call this behavior the end behavior of a function. As we pointed out when discussing power functions, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure $2.4.11$ summarizes all four cases.

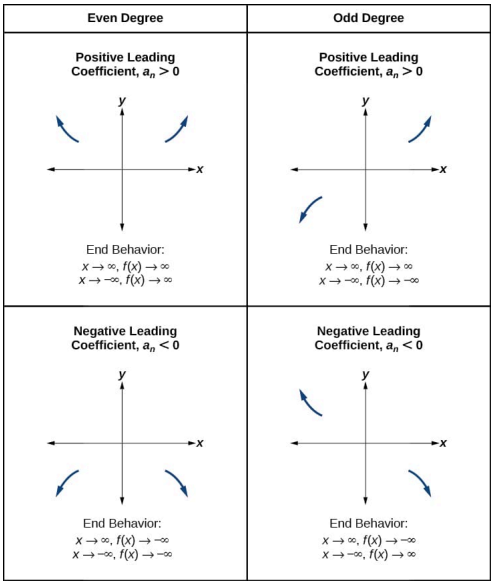


Figure $2.4.11$

Understanding the Relationship Between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's *local behavior*. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure $2.4.12$. The graph has three turning points.

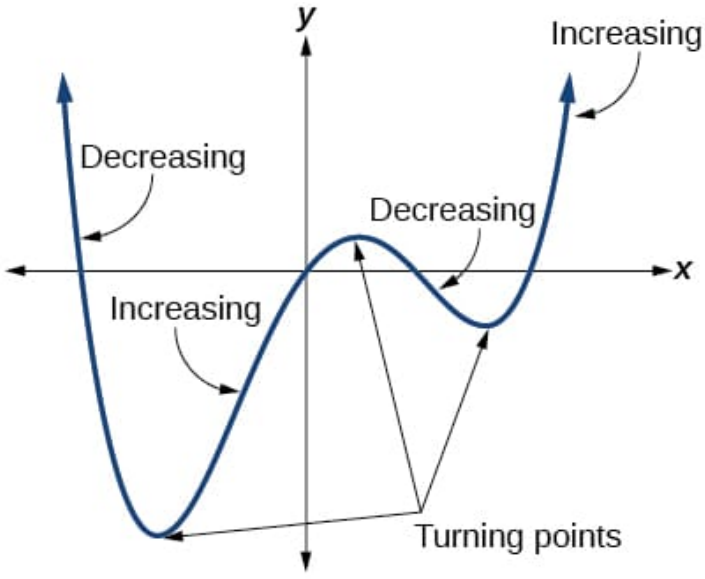


Figure $2.4.12$

This function is a 4th degree polynomial function and has 3 turning points. This should be familiar because, as mentioned in the previous section, the maximum number of turning points of a polynomial function is always one less than the degree of the

function.

Example 2.4.7: Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$
b. $f(x) = -(x-1)^2(1+2x^2)$

Solutions available at the end of this section.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

How To: Graph a polynomial function

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetric about the y-axis. That is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetric about the origin. That is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

Example 2.4.8: Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution available at the end of this section.

Checkpoint 2.4.8

Sketch a graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$.

While you will not be graphing the absolute value of polynomial functions in Calculus, you *will* be graphing the absolute value of other functions. Therefore, it's a good idea to keep that skill fresh (it's a simple skill once you know how to graph the non-absolute value function).

Example 2.4.9: Revisiting the Graphs of Absolute Value Functions

Graph

$$f(x) = \left|-0.9(x+2)^3(x-1)^6(x-5)^5\right|.$$

Solution available at the end of this section.

Solving Polynomial Inequalities

We finish this section with a topic that might not seem related to graphing polynomial functions—solving polynomial inequalities.

There are those who insist on students solving inequalities using sign charts, and then there are those who want to reinforce a student's graphing skills. If you cannot tell, I am in the latter camp. To solve *any* inequality, you could move all terms to one side of the inequality, graph the resulting expression on the non-zero side, and use that graph to solve the inequality. The key to being able to do this is know how to graph that non-zero expression.

Let's see this in an example.

Example 2.4.10

Solve

$$3x^3 - 4x \geq 6x^2 - 8.$$

Solution available at the end of this section.

Using the Intermediate Value Theorem (optional topic)

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the x-axis, we can confirm that there is a zero between them. Consider a polynomial function f whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$. We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function f at $x = a$ lies above the x-axis and another point at $x = b$ lies below the x-axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x-axis. Call this point $(c, f(c))$. This means that we are assured there is a solution c where $f(c) = 0$.

In other words, the Intermediate Value Theorem tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis. Figure 2.4.20 shows that there is a zero between a and b.

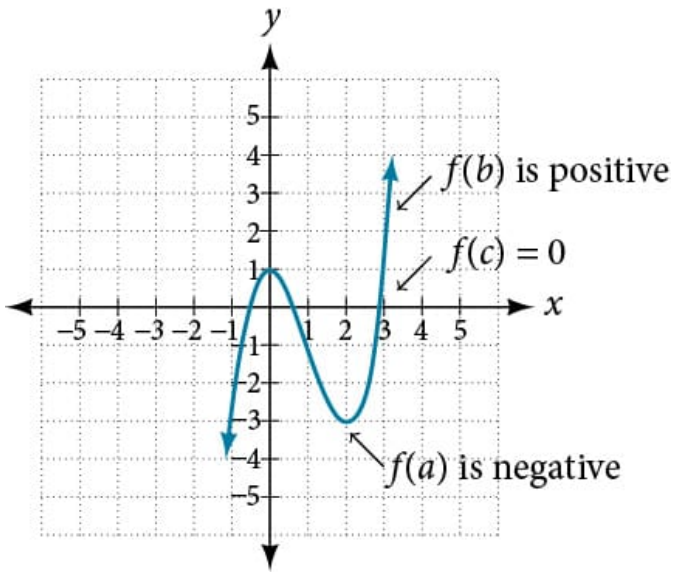


Figure 2.4.20: *Using the Intermediate Value Theorem to show there exists a zero.*

Theorem: Intermediate Value Theorem (for Polynomials)

Let f be a polynomial function. If $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

Proof available at the end of this section.

To be honest, the Intermediate Value Theorem applies to any continuous function (not just polynomials); however, that extra bit of power to the theorem is reserved for Calculus (where the Intermediate Value Theorem becomes incredibly useful).

Example 2.4.11: Using the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

Solution available at the end of this section.

Checkpoint 2.4.11

Show that the function $f(x) = 7x^5 - 9x^4 - x^2$ has at least one real zero between $x = 1$ and $x = 2$.

Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from Calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. We restate the definition from earlier in this textbook below for completeness.

Definition: Local Extrema

A function f has a **local maximum** at $x = b$ in an open interval (a, c) if $f(b) \geq f(x)$ for all $x \neq b$ in the interval. Likewise, f has a **local minimum** at $x = b$ in (a, c) if $f(b) \leq f(x)$ for all $x \neq b$ in the interval.

If f has more than one local maximum on a given interval, we call the collection of these maximums the **local maxima**. Likewise, if f has more than one local minimum on a given interval, we call the collection of these minimums the **local minima**.

Together, local maxima and minima are called **local extrema** (or **local extreme values**).

Some texts use the word "relative" instead of "local." We will use the latter throughout this text.

Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a global maximum or a global minimum. Again, we restate the definition from earlier in this text below for completeness.

Definition: Absolute Extrema

A function f has an **absolute maximum** at $x = b$ if $f(b) \geq f(x)$ for all x in the domain of f. Likewise, a function f has an **absolute minimum** at $x = b$ if $f(b) \leq f(x)$ for all x in the domain of f.

If f has more than one local absolute maximum, we call the collection of these maximums the **absolute maxima**. Likewise, if f has more than one absolute minimum, we call the collection of these minimums the **absolute minima**.

Together, absolute maxima and minima are called **absolute extrema**.

Some texts use the word "global" instead of "absolute." We will use the latter throughout this text.

We can see the difference between local and global extrema in Figure 2.4.22.

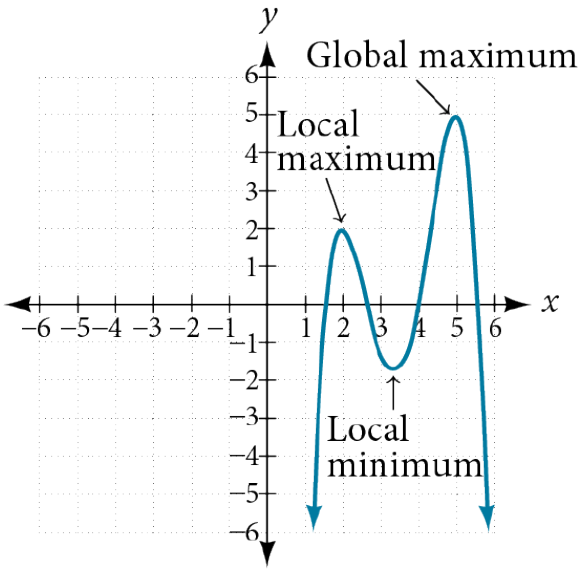


Figure 2.4.22

Q&A

Do all polynomial functions have a global minimum or maximum?

No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ has neither a global maximum nor a global minimum.

Example 2.4.12: Using Local Extrema to Solve Applications

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

Solution available at the end of this section.

Checkpoint 2.4.12

Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

Success in College: Identifying Your Learning Style(s)

Visual/Verbal Learners

These people learn best when information is presented visually and in a written form. In a classroom setting, they prefer professors who use visual aids (i.e., whiteboard or PowerPoint presentation) to list the essential points of a lecture to provide them with an outline to follow. They benefit from information obtained from textbooks and class notes. These learners like to study by themselves in quiet environments. They visualize information in their "minds' eyes" to remember something. The online environment is especially appropriate for visual/verbal learners because most of the information for a course is presented in written form.

Visual/Nonverbal Learners

These people learn best when information is presented visually and in a picture or design format. In a classroom setting, they benefit from professors who supplement their lectures with materials such as film, video, maps, and diagrams. They relate well to information obtained from the images and charts in textbooks. They tend to prefer to work alone in quiet environments. They visualize an image of something in their mind when trying to remember it. These learners may also be artistic and enjoy visual art and design. The online environment is well suited for this learner because graphical representations of information can help them remember concepts and ideas. Graphical information can be presented using charts, tables, graphs, and images.

Auditory/Verbal Learners

These people learn best when information is presented aurally. They benefit from listening to lectures and participating in group discussions in a classroom setting. They also benefit from obtaining information from audio recordings. When trying to remember something, they often repeat it aloud and mentally "hear" how the professor explained the information to them. They learn best when interacting with others in a listening/speaking activity. Online learning environments can complement these learners' styles. Although most information is presented visually (either written or graphically), group participation and collaborative activities are accomplished well online. In addition, streaming audio and computer conferencing can be incorporated into an online course to best meet the learning style of these students.

Tactile/Kinesthetic Learners

These people learn best when doing a physical "hands-on" activity. In the classroom, they prefer to learn new materials in a lab setting where they can touch and manipulate materials. They learn best in physically active learning situations. They benefit from professors who use in-class demonstrations, hands-on learning experiences, and fieldwork outside the classroom. Online environments can provide learning opportunities for tactile/kinesthetic learners. Simulations with 3-Dimensional graphics can replicate physical demonstrations. Lab sessions can be conducted at predetermined locations or homes and then discussed online. Also, outside fieldwork can be incorporated into the coursework, with ample online discussion preceding and following the experience. Finally, the online environment is well suited for the presentation and discussion of either group or individual projects and activities.

Take the learning style survey at Learning Styles.

1. What was your most extreme score, and what was it about?
2. Of all the results, which surprised you the most?

Success in Mathematics: Math Anxiety (Part 1)

The following is an excerpt of an article written at MathAcademy.com.

> A famous stage actress was once asked if she had ever suffered from stage fright and, if so, how she had gotten over it. She laughed at the interviewer's naive assumption that she must not feel that kind of anxiety since she was an accomplished actress. She assured him that she had always had stage fright and had never gotten over it. Instead, despite it, she had learned to walk on stage and perform.

Like stage fright, math anxiety can be a disabling condition, causing humiliation, resentment, and even panic.

Even though we all suffer from math anxiety to some degree – just as anyone feels at least a little nervous when speaking to an audience – for some of us, it is a severe problem. This burden interferes with our lives, preventing us from achieving our goals. The first step, without which no further progress is possible, is to recognize that math anxiety is an emotional response. (In fact, severe math anxiety is a learned emotional response.) As with any strong emotional reaction, there are constructive and unconstructive ways to manage math anxiety. Unconstructive (and damaging) ways include rationalization, suppression, and denial.

By "rationalization," we mean finding reasons why it is okay and perhaps even inevitable – and therefore justified – for you to have this reaction. The myths of "men are better at math than women" or "math is for logical people and not creative people" are examples of rationalizations. While they may make you feel better (or at least less bad) about having math anxiety, they will do nothing to lessen it or to help you get it under control. Therefore, rationalization is unconstructive.

"Suppression" means being aware of the anxiety – but trying very hard not to feel it. I have found that students very commonly attempt this, and some severe self-criticism accompanies it. Students feel that they shouldn't feel this anxiety, that it's a weakness they should overcome by brute force if necessary. When this effort doesn't succeed (as invariably it doesn't), the self-criticism becomes ever harsher, leading to a deep sense of frustration and often a severe loss of self-esteem – mainly if the stakes for a student are high, as when their career or personal goals are riding on a successful outcome in a math class, or when parental disapproval is a factor. Consequently, suppression of math anxiety is not only unconstructive but can be damaging.

Finally, there is denial. People using this approach probably aren't likely to see this essay, much less read it, for they carefully construct their lives to avoid all mathematics as much as possible. They choose college majors and careers that don't require math and let the bank or their spouse balance the checkbook. This approach has the advantage that feelings of frustration and anxiety about math are mostly avoided. However, their lives are drastically constrained, for, in our society, fewer than 25% of all careers are, so to speak, "math-free." Thus, their choices of personal and professional goals are severely limited. (Incidentally, most math-free jobs are low-status and low-pay.)

People in denial about mathematics miss out on something else, too, for the student of mathematics learns to see aspects of the structure and beauty of our world that can be seen in no other way and to which the "innumerate" necessarily remain forever blind. It would be a lot like never hearing music or never seeing colors. (I understand that some people have these disabilities but didn't choose to have them.)

Okay, so what is the constructive way to manage math anxiety? I call it "taking possession." It involves identifying the sources of math anxiety in one's life, accepting those feelings without self-criticism, and then learning strategies for disarming math anxiety's influence on one's future study of mathematics.

Begin by understanding that your feelings of math anxiety are not uncommon and do not indicate that there is anything wrong with you or inferior about your ability to learn math. For some, this can be hard to accept, but it is worth trying to accept – since, after all, it happens to be true. This can be made more accessible by exploring your own "math history." Think back across your career as a math student, and identify those experiences which have contributed most to your feelings of frustration about math. For some, this will be a memory of a humiliating experience in school, such as being made to stand at the blackboard and embarrassed in front of one's peers. For others, it may involve interaction with a parent. Whatever the principle episodes are, recall them as vividly as you can. Then, write them down. This is important. After you have written the episode on a sheet(s) of paper, write down your reaction to the episode, both at the time and how it makes you feel to recall it now.

After completing this exercise, take a fresh sheet of paper and try to sum up in a few words what your feelings about math are at this point in your life, together with the reason or reasons you wish to succeed at math. This, too, is important. Not until after we lay out for ourselves consciously and deliberately what our feelings and desires are towards mathematics will it become possible to take possession of our feelings of math anxiety and become free to implement strategies for coping with those feelings.

At this point, sharing your memories, feelings, and goals with others can be enormously helpful. In a math class I teach for art majors, I hand out a questionnaire early in the semester asking students to do precisely what is described above. After spending about twenty minutes writing down their recollections and goals, I lead them in a classroom discussion on math anxiety. This process of dialogue and sharing – though it may seem just a bit on the goopy side – invariably brings out of each student their barriers to math, often helping them become completely conscious of these barriers for the first time.

Just as important, it helps all my students understand that the negative experiences they have had, and their reactions to them, are shared one way or another by almost everyone else in the room.

If you do not have the opportunity to engage in a group discussion in a classroom setting, find friends or relatives you trust to respect your feelings and induce them to talk about their own math anxiety experiences and listen to yours.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Are there counselors at your school that are available to help students who are struggling with math anxiety? If so, where are they located, and how to make an appointment to see one?
2. Estimate the percent of students in your class you think are negatively affected by math anxiety.

If you are unsure about the answers to either of these questions, feel free to ask your instructor - that's what they are there for!

Answers

Section Preview

This section introduces multiplicities of zeros for polynomials and goes over the process of *accurately* graphing a polynomial. Make sure you focus on the local behavior of the function near a zero with multiplicities greater than 2 (discussing the "flattening" as the multiplicity increases). Solving polynomial inequalities using graphs and graphing absolute values of polynomial functions are also required topics. I have included the IVT as an ***optional topic*** for those who want to discuss it.

Section Preview

- **Factoring Techniques**
  - Factoring by Grouping: Example 3 (x^3-5x^2-x+5) and Example 10 $(3x^3-6x^2-4x+8)$ both require grouping a four-term polynomial before solving.
  - Factoring Expressions that are Quadratic-in-Form: Example 2 recognizes x^4-3x^2+2 as quadratic-in-form and uses the substitution $u=x^2$ to factor it.

Section Preview

- Recognize characteristics of graphs of polynomial functions.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Determine end behavior.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.
- Use the Intermediate Value Theorem.
- Graph the absolute value of a polynomial function.
- Solve a polynomial inequality (using a graph).

Example $2.4.1$: Recognizing Polynomial Functions

The graphs of f and h are graphs of polynomial functions. They are smooth and continuous.

The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

Example $2.4.2$: Finding the x-Intercepts of a Polynomial Function by Factoring

We can attempt to factor this polynomial to find solutions for $f(x)=0$.

$$\begin{aligned}
& x^6-3x^4+2x^2 = 0 \\
\Longrightarrow\quad & x^2(x^4-3x^2+2) = 0 && \text{(factoring out the GCF)} \\
\Longrightarrow\quad & x^2(u^2-3u+2) = 0 && \text{(substituting } u=x^2 \text{ in the trinomial)} \\
\Longrightarrow\quad & x^2(u-2)(u-1) = 0 && \text{(factoring the trinomial)} \\
\Longrightarrow\quad & x^2(x^2-2)(x^2-1) = 0 && \text{(resubstituting } u=x^2) \\
\Longrightarrow\quad & x^2(x^2-2)(x-1)(x+1) = 0 && \text{(factoring a Difference of Squares)}
\end{aligned}$$

Thus,

$$\begin{array}{lccccccc}
 & x^2=0 & \text{or} & x^2-2=0 & \text{or} & x-1=0 & \text{or} & x+1=0 \\
\Longrightarrow & x=0 & & x^2=2 & & x=1 & & x=-1 \\
\Longrightarrow & x=0 & & x=\pm\sqrt{2} & & x=1 & & x=-1
\end{array}$$

This gives us five x-intercepts: $(0,0)$, $(1,0)$, $(-1,0)$, $(\sqrt{2},0)$, and $(-\sqrt{2},0)$ (see Figure $2.4.3$). We can see that this is an even function.

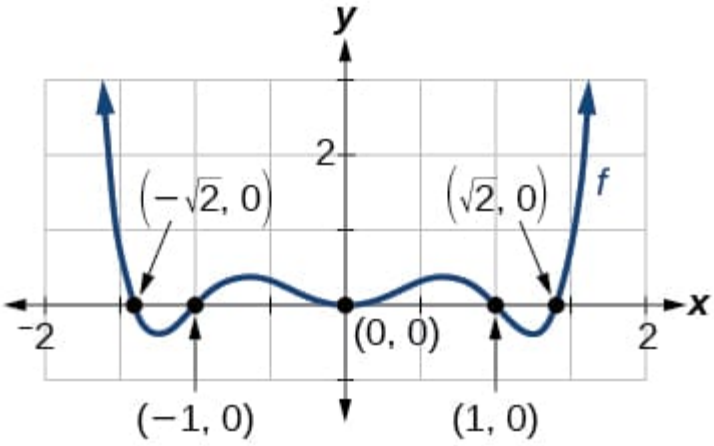


Figure 2.4.3

Example 2.4.3**: Finding the x-Intercepts of a Polynomial Function by Factoring**

Find solutions for $f(x) = 0$ by factoring.

$$\begin{aligned} & x^3 - 5x^2 - x + 5 &&= 0 && \\ \implies\ & x^2(x-5) - 1(x-5) &&= 0 && \text{(factoring by grouping)} \\ \implies\ & (x-5)(x^2-1) &&= 0 && \text{(factor out the GCF)} \\ \implies\ & (x-5)(x-1)(x+1) &&= 0 && \text{(factoring the Difference of Squares)} \end{aligned}$$

Hence, $x = 5$, or $x = 1$, or $x = -1$. There are three x-intercepts: $(-1, 0)$, $(1, 0)$, and $(5, 0)$ (see Figure 2.4.4).

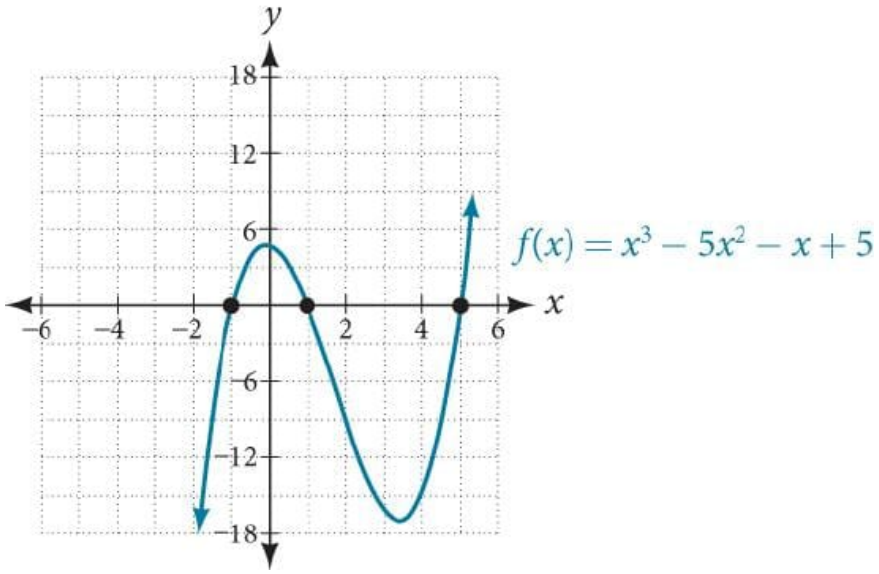


Figure 2.4.4

Example 2.4.4**: Finding the y- and x-Intercepts of a Polynomial in Factored Form**

The y-intercept can be found by evaluating $g(0)$.

$$g(0) = (0-2)^2(2(0)+3) = 12.$$

So the y-intercept is $(0, 12)$.

The x-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3) = 0 \implies x = 2 \text{ or } x = -\frac{3}{2}.$$

So the x-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

Example 2.4.5**: Finding the x-Intercepts of a Polynomial Function Using a Graph**

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in Figure 2.4.6, it appears that there are x-intercepts at $x = -3$, -2, and 1.

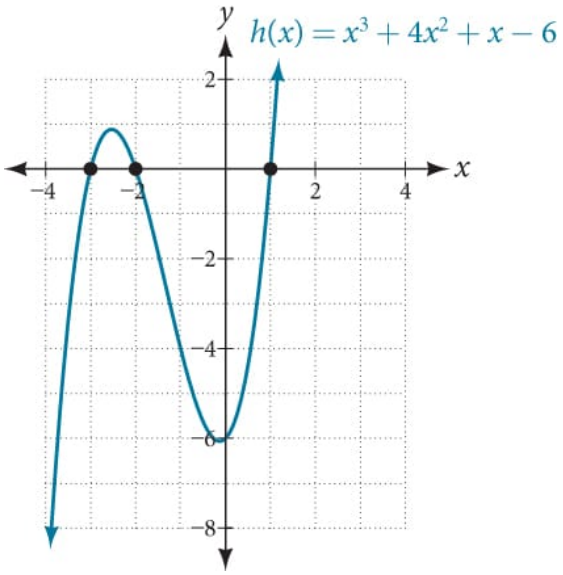


Figure 2.4.6

We can check whether these are correct by substituting these values for x and verifying that

$$h(-3) = h(-2) = h(1) = 0.$$

Since $h(x) = x^3 + 4x^2 + x - 6$, we have:

$$\begin{aligned} h(-3) &= (-3)^3 + 4(-3)^2 + (-3) - 6 = -27 + 36 - 3 - 6 = 0 \\ h(-2) &= (-2)^3 + 4(-2)^2 + (-2) - 6 = -8 + 16 - 2 - 6 = 0 \\ h(1) &= (1)^3 + 4(1)^2 + (1) - 6 = 1 + 4 + 1 - 6 = 0 \end{aligned}$$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$h(x) = x^3 + 4x^2 + x - 6 = (x+3)(x+2)(x-1).$$

Example 2.4.6**: Identifying Zeros and Their Multiplicities**

The polynomial function is of degree n. The sum of the multiplicities must be n.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of -3 could have multiplicity 2, 4, or 6 (remember, we were told this polynomial has degree 6, so the sum of the multiplicities of the zeros must be 6).

The next zero occurs at $x = -1$. The graph looks almost linear at this point, so this is likely a zero of multiplicity 1.

The last zero occurs at $x = 4$. The graph crosses the x-axis, so the multiplicity of the zero must be odd. From the shape of the curve near to $x = 4$, we know the multiplicity cannot be 1 (the graph is not *locally linear* near $x = 4$). Hence, $x = 4$ must be a zero of multiplicity either 3 or 5.

Now that we have an idea of the *possible* multiplicities, let's build a table and use a little deductive logic to identify the exact multiplicities.

Zero	Possible Multiplicities
-3	$2, 4, 6$
-1	1
4	$3, 5$

Since the sum of the multiplicities must be 6, and we have three zeros, the maximum multiplicity for any given zero is 3. This cleans up our table quite a bit:

Zero	Possible Multiplicities
-3	2
-1	1
4	3

Thus, we have identified the zeros and their exact multiplicities.

Example 2.4.7**: Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function**

a. First, rewrite the polynomial function in descending order: $f(x) = 4x^5 - x^3 - 3x^2 + 1$. Identify the degree of the polynomial function. This polynomial function is of degree 5. The maximum number of turning points is $5 - 1 = 4$.

b. First, identify the leading term of the polynomial function if the function were expanded. Looking at the given function, $f(x) = -(x-1)^2(1+2x^2)$, we can see that the factor $(x-1)^2$ will contribute x^2 (if we multiplied it out) and the factor $1 + 2x^2$ contributes $2x^2$. Therefore, the leading term is

$$a_n x^n = -(x^2)(2x^2) = -2x^4.$$

Therefore, this polynomial function is of degree 4. As such, the maximum number of turning points is $4 - 1 = 3$.

Example 2.4.8**: Sketching the Graph of a Polynomial Function**

This graph has two x-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x = 5$, the function has a multiplicity of one, indicating the graph will cross through the axis like a line at this intercept.

The y-intercept is found by evaluating $f(0)$.

$$f(0) = -2(0+3)^2(0-5) = -2 \cdot 9 \cdot (-5) = 90.$$

The y- intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See Figure $2.4.13$.

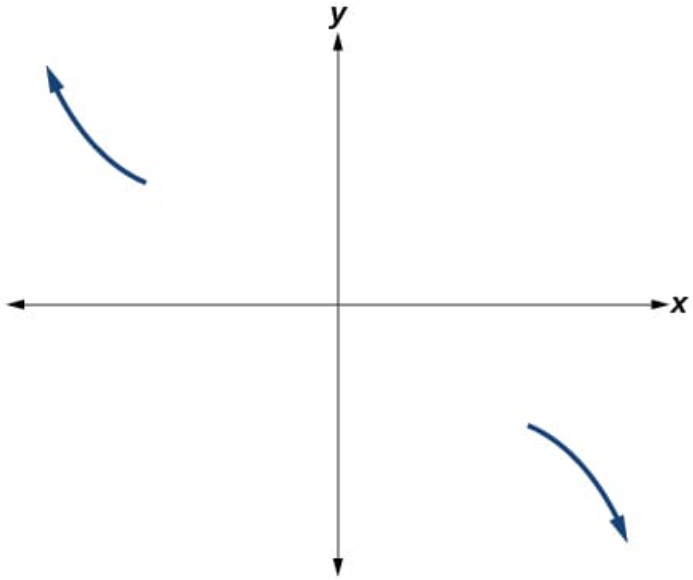


Figure $2.4.13$

To sketch this, we consider that:

- As $x \to -\infty$, $f(x) \to \infty$. So we know the graph starts in the second quadrant and is decreasing toward the x-axis.
- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $\pm f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of the x-axis, so the function must start increasing.

  At $(0, 90)$, the graph crosses the y-axis at the y-intercept. See Figure $2.4.14$.

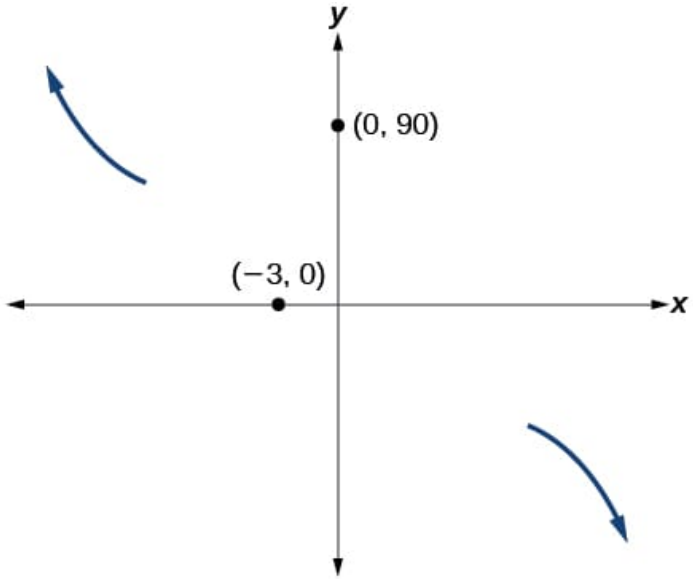


Figure $2.4.14$

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5, 0)$. See Figure $2.4.15$.

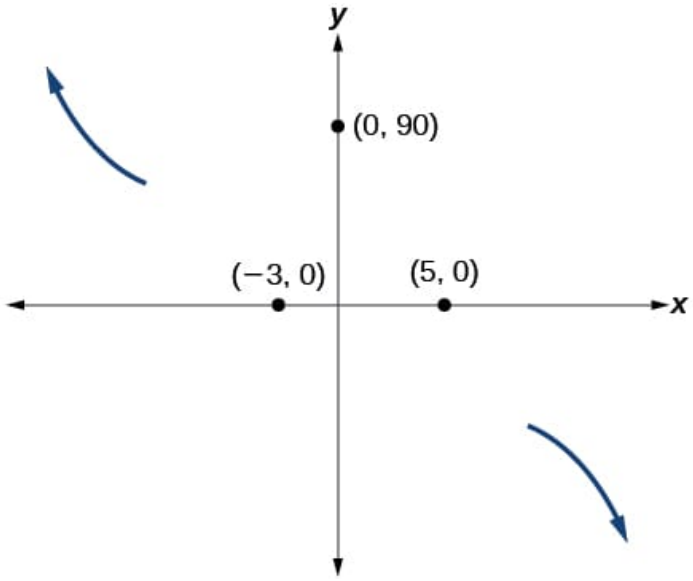


Figure 2.4.15

As $x \to \infty$, the function $f(x) \to -\infty$, so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure 2.4.16, and verify that the resulting graph looks like our sketch in Figure 2.4.15.

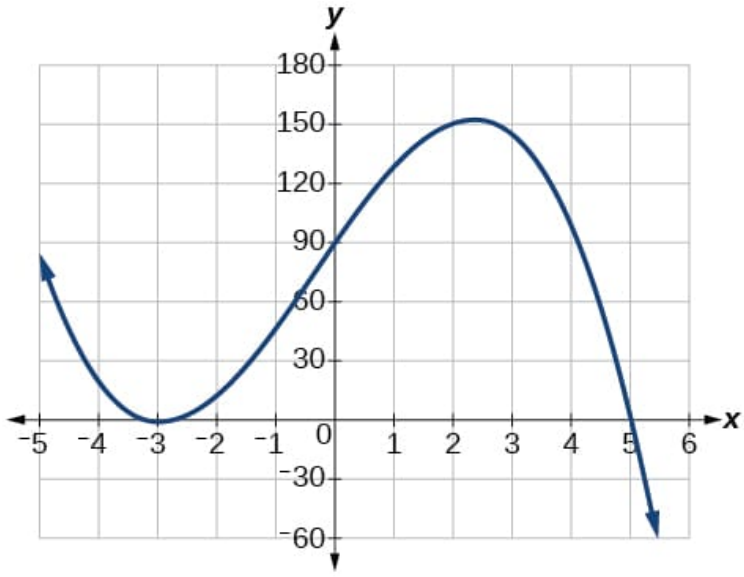


Figure 2.4.16**:** *The complete graph of the polynomial function* $f(x) = -2(x+3)^2(x-5)$

Example 2.4.9**: Revisiting the Graphs of Absolute Value Functions**

As we have done in previous sections, we initially concern ourselves with the *non-absolute value* function

$$f(x) = -0.9(x+2)^3(x-1)^6(x-5)^5.$$

We list the roots and their multiplicities, as well as the behavior of the graph near that root:

Root	Multiplicity	Behavior
-2	3	like x^3
1	6	like x^6 (a very flat parabola)
5	5	like x^5 (a flat cubic)

Next, we note that the degree of the polynomial is $3+6+5=14$, so the end behavior is like that of x^2 *except* the leading coefficient is negative. Therefore, the end behavior is more like $-x^2$. That is, as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to -\infty$. Reading from left to right, the graph of f should

- start low
- increase until it crosses the x-axis at $x = -2$ like a cubic
- climb for a bit, but then turn to fall back to the x-axis to "bounce" like a very flat parabola at $x = 1$
- climb for a bit, but then turn to fall back to the x-axis to cross at $x = 5$ like a flat cubic
- it then continues to fall forever.

Figure 2.4.17 shows the graph of the non-absolute value function.

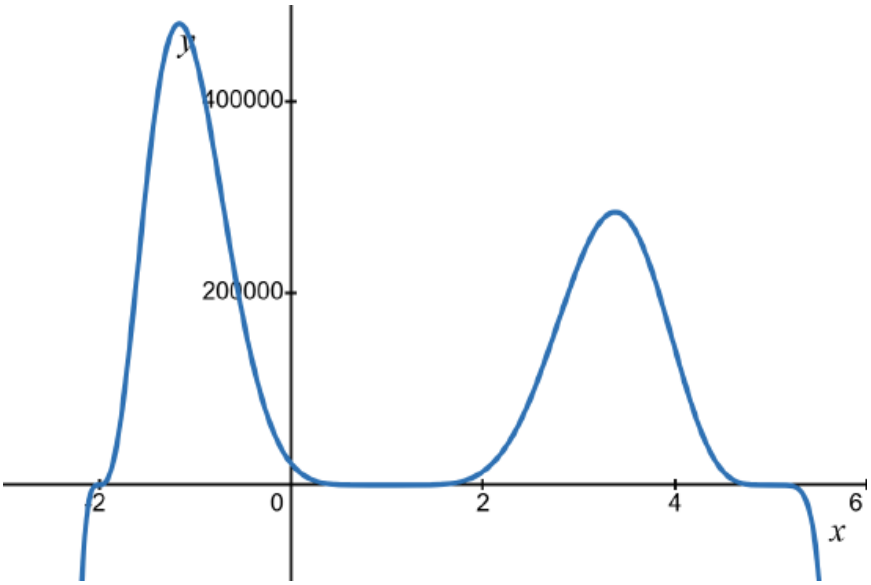


Figure 2.4.17

To graph the absolute value of this function, we take all the function outputs (y-values) that are negative and we force them to be positive. In effect, this keeps the part of the graph from Figure 2.4.17 that is already above the x-axis in place; however, all the parts that are below the x-axis get reflected about the x-axis to become positive (above the x-axis). See Figure 2.4.18.

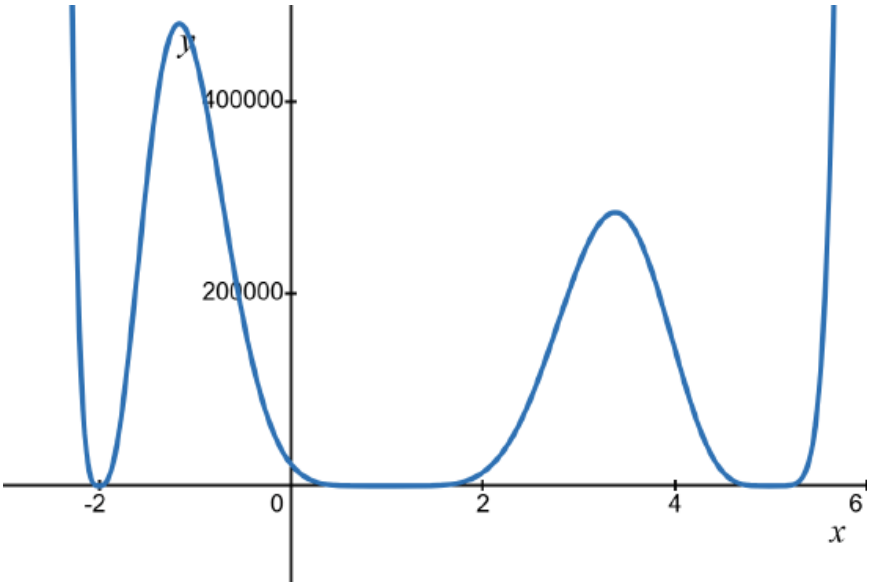


Figure 2.4.18

Example 2.4.10

Let's start by getting all terms to one side.

$$\begin{aligned} 3x^3 - 4x &\geq 6x^2 - 8 \\ \implies \quad 3x^3 - 6x^2 - 4x + 8 &\geq 0 \end{aligned}$$

Now we want to graph the expression on the left side. That is, let's graph

$$g(x) = 3x^3 - 6x^2 - 4x + 8.$$

Since this is a polynomial, we would like to find the roots and their multiplicities. Let's try factoring by grouping:

$$\begin{aligned} g(x) &= 3x^3 - 6x^2 - 4x + 8 \\ &= 3x^2(x-2) - 4(x-2) \\ &= (x-2)(3x^2 - 4) \end{aligned}$$

Hence, this polynomial has the following roots (and multiplicities):

Root	Multiplicity
2	1
$\frac{2}{\sqrt{3}}$	1
$-\frac{2}{\sqrt{3}}$	1

Since $g(x)$ is a cubic, we know it "starts low and ends high." A quick sketch yields the graph in Figure $2.4.19$.

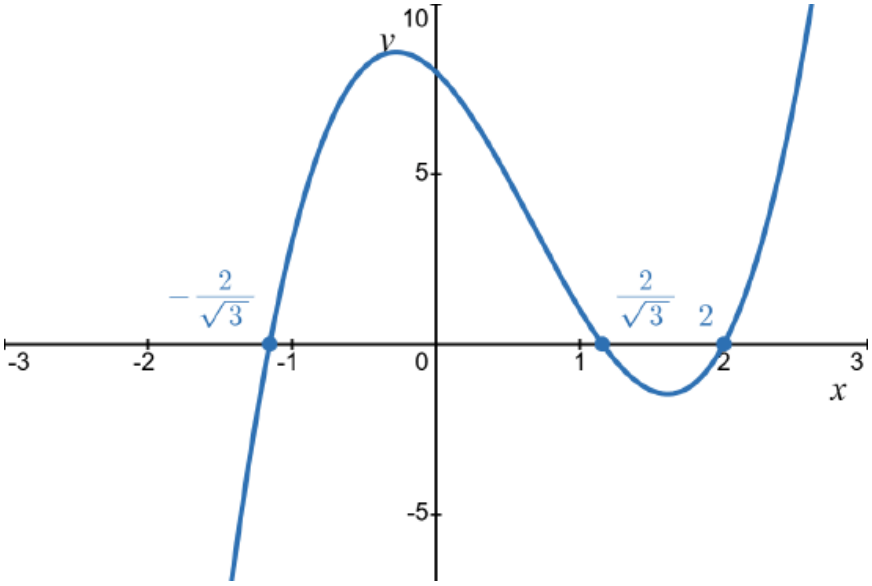


Figure $2.4.19$

Remember, we are being asked to solve $3x^3 - 6x^2 - 4x + 8 \geq 0$. This is equivalent to asking, "When is the graph of $g(x) = 3x^3 - 6x^2 - 4x + 8$ at or above the x -axis?" The answer to this question is simple once you have the graph. We can see from Figure $2.4.19$, that $g(x) \geq 0$ when

$$x \in \left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right] \cup [2, \infty) .$$

Theorem: Intermediate Value Theorem (for Polynomials)

This proof must wait until Calculus.

Example $2.4.11$**: Using the Intermediate Value Theorem**

As a start, evaluate $f(x)$ at the integer values $x = 1$, 2, 3, and 4. See Table $2.4.2$.

Table $2.4.2$

x	1	2	3	4
$f(x)$	5	0	-3	2

We see that one zero occurs at $x = 2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4. Thus, we have shown that there are *at least* two real zeros between $x = 1$ and $x = 4$.

We can verify this result by looking at the graph of the function in Figure $2.4.21$.

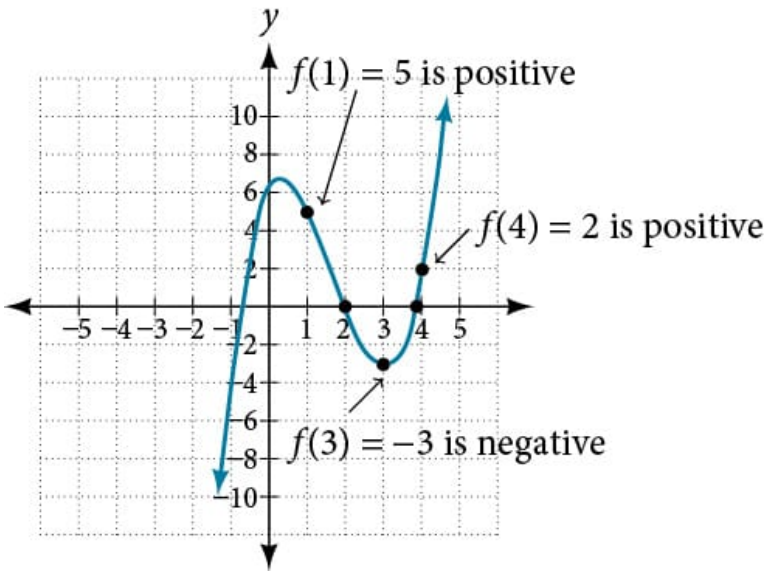


Figure 2.4.21

Example 2.4.12**: Using Local Extrema to Solve Applications**

We will start this problem by drawing a picture like that in Figure 2.4.23, labeling the width of the cut-out squares with a variable, w.

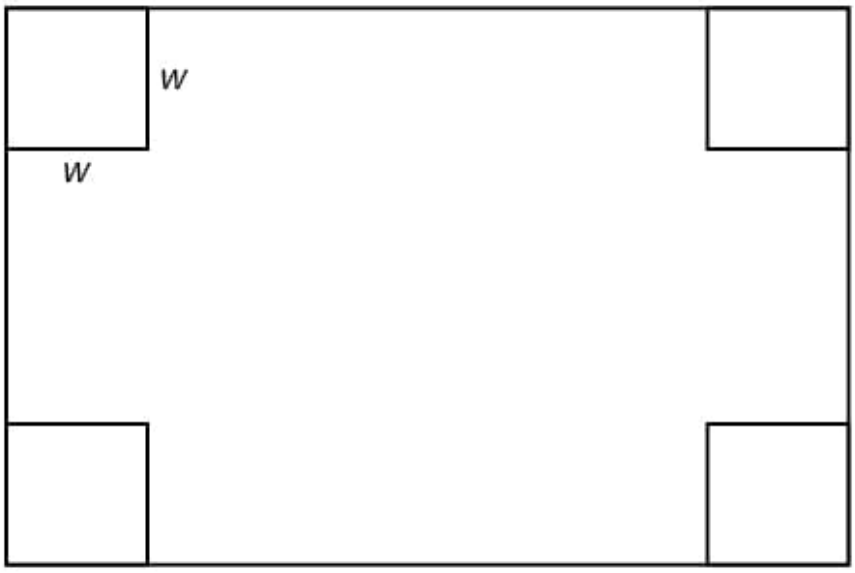


Figure 2.4.23

Notice that after a square is cut out from each end, it leaves a $(14-2w)$ cm by $(20-2w)$ cm rectangle for the base of the box, and the box will be w cm tall. This gives the volume

$$V(w)=(20-2w)(14-2w)w=280w-68w^2+4w^3.$$

Notice, since the factors are w, $20-2w$, and $14-2w$, the three zeros are 0, 10, and 7, respectively. Because a height of 0 cm is not reasonable, we consider only the zeros 10 and 7. The shortest side is 14 and we are cutting off two squares, so values w may take on are greater than zero or less than 7. This means we will restrict the domain of this function to \(0

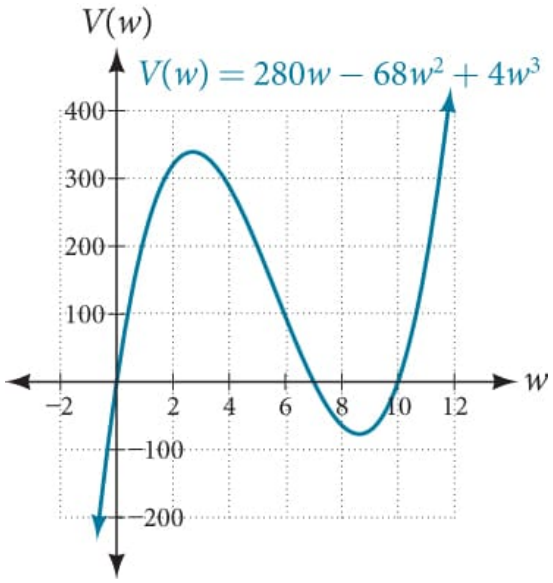


Figure 2.4.24

From this graph, we turn our focus to only the portion on the reasonable domain, $[0, 7]$. We can estimate the maximum value to be around 340 cubic cm, which occurs when the squares are about 2.75 cm on each side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce Figure 2.4.25.

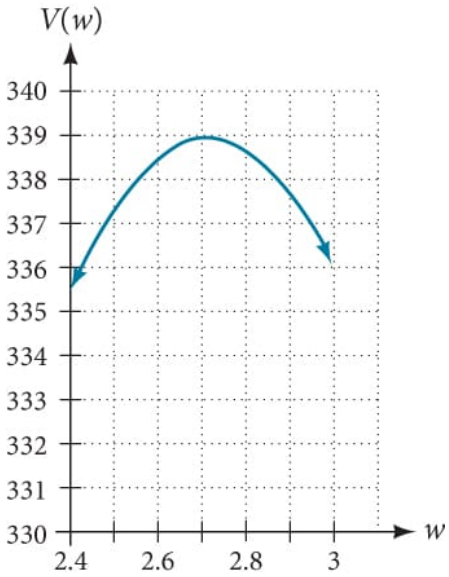


Figure 2.4.25

From this zoomed-in view, we can refine our estimate for the maximum volume to about 339 cubic cm, when the squares measure approximately 2.7 cm on each side.

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2.4.1: Resources and Key Concepts

Resources

Videos

- Graphs of polynomial functions
  - Graphs of Polynomial Functions - Graphing Polynomial Functions
  - Polynomial Functions and Their Graphs - Graphing Polynomials Using End Behavior and Zeros
- Zeros (and their multiplicities) of polynomial functions
  - Graphs of Polynomial Functions - Identifying Zeros and Their Multiplicities
  - Polynomial Functions and Their Graphs - Zeros of Functions and Their Multiplicities
- Factoring polynomials
  - Graphs of Polynomial Functions - Using Factoring to Find Zeros of Polynomial Functions
- Polynomial modeling
  - Polynomial Functions - Applications

Key Concepts

Definitions

- **Zeros of a Polynomial Function**: The values of x for which $f(x) = 0$. These correspond to the x-intercepts of the graph.
- **Multiplicity of a Zero**: If a polynomial contains a factor of the form $(x - h)^p$, and if this is the greatest such factor of $(x - p)$, we say that $x = h$ is a zero of multiplicity p.
- **Single Zero**: A zero of multiplicity 1. The graph passes directly through the x-axis at a single zero.

Theorems

- **Local Behavior Near a Zero**: If $x = h$ is a zero of multiplicity p for a given polynomial, then the local behavior of the graph of the polynomial near $x = h$ matches the behavior of $y = \pm(x - h)^p$, where the sign of the coefficient is determined by other characteristics of the given polynomial.
- **Sum of Multiplicities and the Degree of a Polynomial**: The sum of the multiplicities of the factors for a polynomial is the degree of the polynomial function.
- **Intermediate Value Theorem (for Polynomials)**: Let f be a polynomial function. If $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

Common Mistakes

- **Incorrectly Identifying Multiplicity**: Miscounting the multiplicity of a zero, especially when a factor is repeated or when a zero comes from a factor like $(x^2 - k)$ which might yield two distinct zeros each of multiplicity 1.
- **Misinterpreting Behavior at Zeros based on Multiplicity**: Confusing whether the graph crosses or touches/bounces at an x-intercept based on the multiplicity of the zero (odd multiplicity means cross, even multiplicity means touch/bounce).
- **Errors in Determining End Behavior**: Incorrectly identifying the leading term (especially if the polynomial is not in standard form) or misapplying the rules for end behavior based on the degree and sign of the leading coefficient.
- **Exceeding Maximum Number of Turning Points**: Sketching a graph with more turning points than $n - 1$, where n is the degree of the polynomial.

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2.4.2: Homework

Reading Questions

1. What are two key visual characteristics of the graphs of polynomial functions of degree 2 or more?
2. What are the "zeros" of a polynomial function $f(x)$, and how do they relate to the graph's x-intercepts?
3. Describe three methods mentioned for finding the zeros of a polynomial function.
4. What does the "multiplicity" of a zero tell us about the behavior of the graph of a polynomial function at that x-intercept?
   - What happens if the multiplicity is odd?
   - What happens if the multiplicity is even?
5. How does the graph's behavior near an x-intercept change for higher even or odd multiplicities?
6. What determines the end behavior of a polynomial function's graph?
7. What is the maximum number of turning points a polynomial function of degree n can have?
8. List the general steps suggested for sketching the graph of a polynomial function.
9. What does the Intermediate Value Theorem state about a polynomial function f if $f(a)$ and $f(b)$ have opposite signs?
10. If you know the x-intercepts and their multiplicities for a polynomial, how can you write a general formula for that polynomial function? What additional piece of information is needed to find the specific stretch factor a?
11. What is the difference between a local maximum/minimum and a global maximum/minimum for a polynomial function?
12. Do all polynomial functions have a global minimum or maximum? Explain.
13. How do you graph the function $g(x) = |f(x)|$ if you know the graph of $f(x)$, where $f(x)$ is a polynomial function?
14. Describe a graphical approach to solving a polynomial inequality like $P(x) > 0$.
15. What is the difference between an x-intercept and a zero of a polynomial function f?
16. If a polynomial function of degree n has n distinct zeros, what do you know about the graph of the function?
17. Explain how the Intermediate Value Theorem can assist us in finding a zero of a function.
18. Explain how the factored form of the polynomial helps us in graphing it.
19. If the graph of a polynomial just touches the x-axis and then changes direction, what can we conclude about the factored form of the polynomial?

Homework

Algebraic

For the following exercises, find the x- or t-intercepts of the polynomial functions.

20. $C(t) = 2(t-4)(t+1)(t-6)$
21. $C(t) = 3(t+2)(t-3)(t+5)$
22. $C(t) = 4t(t-2)^2(t+1)$
23. $C(t) = 2t(t-3)(t+1)^2$
24. $C(t) = 2t^4 - 8t^3 + 6t^2$
25. $C(t) = 4t^4 + 12t^3 - 40t^2$
26. $f(x) = x^4 - x^2$
27. $f(x) = x^3 + x^2 - 20x$
28. $f(x) = x^3 + 6x^2 - 7x$
29. $f(x) = x^3 + x^2 - 4x - 4$
30. $f(x) = x^3 + 2x^2 - 9x - 18$
31. $f(x) = 2x^3 - x^2 - 8x + 4$
32. $f(x) = x^6 - 7x^3 - 8$
33. $f(x) = 2x^4 + 6x^2 - 8$
34. $f(x) = x^3 - 3x^2 - x + 3$
35. $f(x) = x^6 - 2x^4 - 3x^2$
36. $f(x) = x^6 - 3x^4 - 4x^2$
37. $f(x) = x^5 - 5x^3 + 4x$

For the following exercises, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.

38. $f(x) = x^3 - 9x$, between $x = -4$ and $x = -2$.
39. $f(x) = x^3 - 9x$, between $x = 2$ and $x = 4$.
40. $f(x) = x^5 - 2x$, between $x = 1$ and $x = 2$.
41. $f(x) = -x^4 + 4$, between $x = 1$ and $x = 3$.
42. $f(x) = -2x^3 - x$, between $x = -1$ and $x = 1$.
43. $f(x) = x^3 - 100x + 2$, between $x = 0.01$ and $x = 0.1$

For the following exercises, find the zeros and give the multiplicity of each.

44. $f(x) = (x + 2)^3(x - 3)^2$
45. $f(x) = x^2(2x + 3)^5(x - 4)^2$
46. $f(x) = x^3(x - 1)^3(x + 2)$
47. $f(x) = x^2(x^2 + 4x + 4)$
48. $f(x) = (2x + 1)^3(9x^2 - 6x + 1)$
49. $f(x) = (3x + 2)^5(x^2 - 10x + 25)$
50. $f(x) = x(4x^2 - 12x + 9)(x^2 + 8x + 16)$
51. $f(x) = x^6 - x^5 - 2x^4$
52. $f(x) = 3x^4 + 6x^3 + 3x^2$
53. $f(x) = 4x^5 - 12x^4 + 9x^3$
54. $f(x) = 2x^4(x^3 - 4x^2 + 4x)$
55. $f(x) = 4x^4(9x^4 - 12x^3 + 4x^2)$

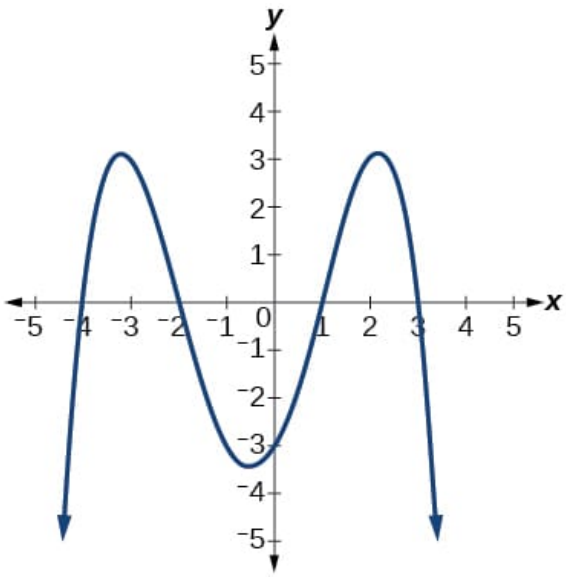
Graphical

For the following exercises, graph the polynomial functions. Note x- and y-intercepts, multiplicity, and end behavior.

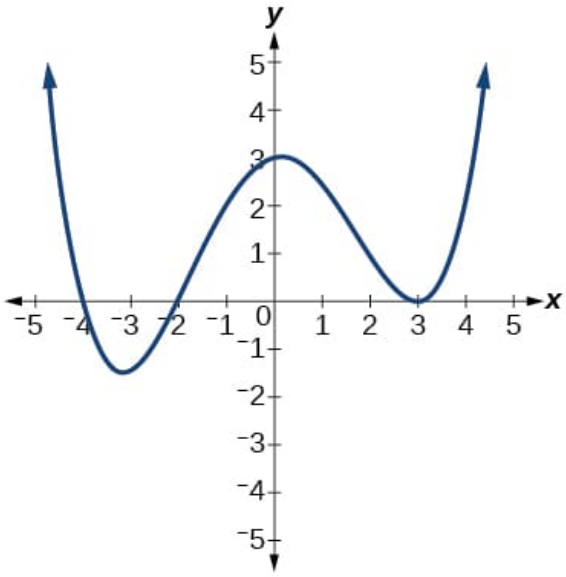
56. $f(x) = (x + 3)^2(x - 2)$
57. $g(x) = (x + 4)(x - 1)^2$
58. $h(x) = (x - 1)^3(x + 3)^2$
59. $k(x) = (x - 3)^3(x - 2)^2$
60. $m(x) = -2x(x - 1)(x + 3)$
61. $n(x) = -3x(x + 2)(x - 4)$

For the following exercises, use the graph to identify zeros and multiplicity.

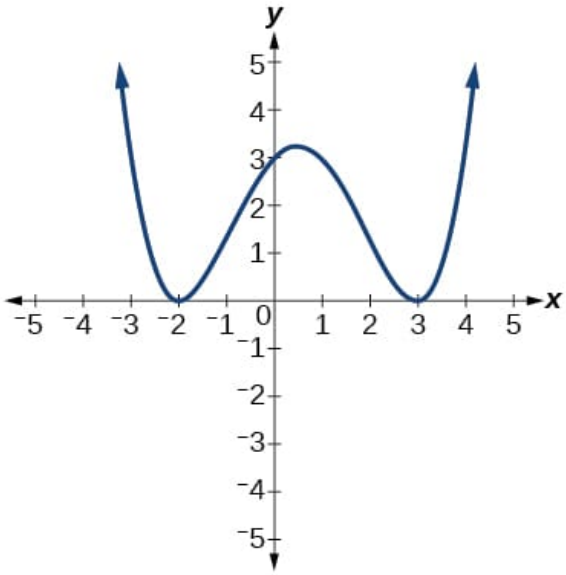
62.



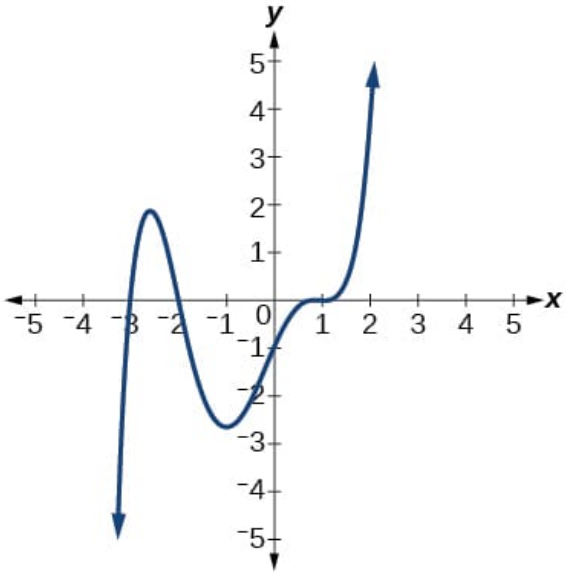
63.



64.



65.



Technology

For the following exercises, use a calculator to approximate local minima and maxima or the global minimum and maximum.

66. $f(x) = x^3 - x - 1$
67. $f(x) = 2x^3 - 3x - 1$
68. $f(x) = x^4 + x$
69. $f(x) = -x^4 + 3x - 2$
70. $f(x) = x^4 - x^3 + 1$

Real-World Applications

For the following exercises, write the polynomial function that models the given situation.

71. A rectangle has a length of 10 units and a width of 8 units. Squares of x by x units are cut out of each corner, and then the

sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of x.

72. Consider the same rectangle of the preceding problem. Squares of $2x$ by $2x$ units are cut out of each corner. Express the volume of the box as a polynomial in terms of x.
73. A square has sides of 12 units. Squares $x+1$ by $x+1$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of x.
74. A cylinder has a radius of $x+2$ units and a height of 3 units greater. Express the volume of the cylinder as a polynomial function.
75. A right circular cone has a radius of $3x+6$ and a height 3 units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V=\frac{1}{3}\pi r^2 h$ for radius r and height h.

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2.5: Dividing Polynomials

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure 2.5.1**:** *Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)*

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m, and height 30 m. We can easily find the volume using elementary Geometry.

$$V = l \cdot w \cdot h = 61.5 \cdot 40 \cdot 30 = 73,800.$$

So the volume is 73,800 cubic meters (m^3). Suppose we knew the volume, length, and width. We could divide to find the height.

$$h = \frac{V}{l \cdot w} = \frac{73,800}{61.5 \cdot 40} = 30.$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any or all of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$ and the width is given by $x - 2$. To find the height of the solid, we can use polynomial division, which is the focus of this section.

Using Long Division to Divide Polynomials

Suppose we wish to find the zeros of $f(x) = x^3 + 4x^2 - 5x - 14$. Setting $f(x) = 0$ results in the polynomial equation $x^3 + 4x^2 - 5x - 14 = 0$. Despite all the factoring techniques we learned in Intermediate Algebra, this equation foils us at every turn. If we graph f using Desmos, we get the graph in Figure 2.5.2.

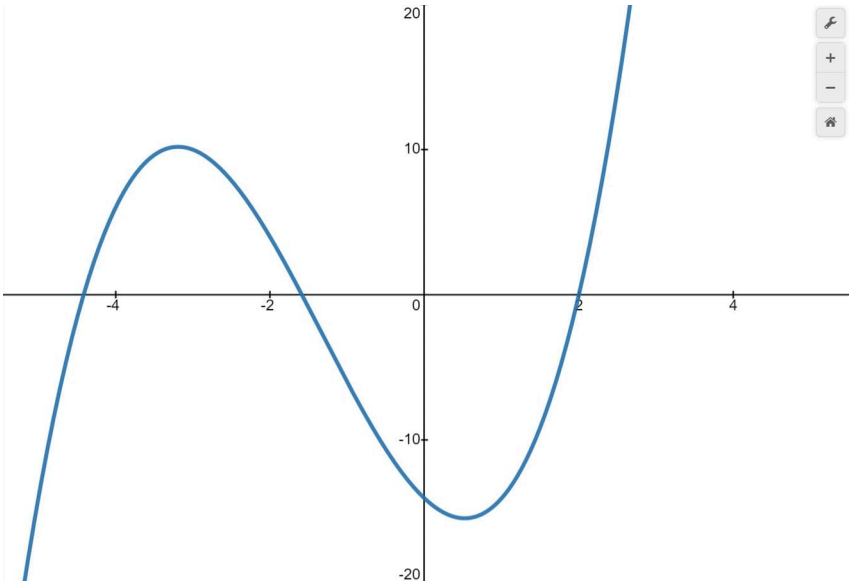


Figure 2.5.2**:** $f(x) = x^3 + 4x^2 - 5x - 14$

The graph suggests that the function has three zeros, one of which is $x = 2$. It's easy to show that $f(2) = 0$, but the other two zeros seem less friendly. Even though we could click on the curve in Desmos to find decimal approximations for these, we seek an analytical method (a method by hand) to find the remaining zeros exactly.

Since f is a polynomial function and $f(2) = 0$, it should seem realistic that $(x - 2)$ is a factor of f. That is, if $x = 2$ is a zero, it seems that there should be a factor of $(x - 2)$ lurking around in the factorization of $f(x)$. Therefore, we should expect that $x^3 + 4x^2 - 5x - 14 = (x - 2)q(x)$, where $q(x)$ is some other polynomial. If it even exists, how could we find such a $q(x)$? The answer comes from our old friend, polynomial division. Dividing $x^3 + 4x^2 - 5x - 14$ by $x - 2$ gives

$$
\begin{array}{rr|l}
 & & x^2 + 6x + 7 \\
\cline{2-3}
x - 2 & \big) & x^3 + 4x^2 - 5x - 14 \\
 & & -(x^3 - 2x^2) \\
\cline{3-3}
 & & 6x^2 - 5x \\
 & & -(6x^2 - 12x) \\
\cline{3-3}
 & & 7x - 14 \\
 & & -(7x - 14) \\
\cline{3-3}
 & & 0
\end{array}
$$

Thus, $\frac{x^3+4x^2-5x-14}{x-2} = x^2 + 6x + 7$. This means $x^3 + 4x^2 - 5x - 14 = (x - 2)(x^2 + 6x + 7)$, so to find the zeros of f, we now solve $(x - 2)(x^2 + 6x + 7) = 0$. We get $x - 2 = 0$ (which gives us our known zero, $x = 2$) as well as $x^2 + 6x + 7 = 0$. The latter doesn't factor nicely, so we apply the Quadratic Formula to get $x = -3 \pm \sqrt{2}$.

Interactive Element: When Does $x - k$ Divide Evenly?

The following Interactive Element might help you visualize why dividing $f(x) = x^3 + 4x^2 - 5x - 14$ by $x - k$ leaves no remainder exactly when k is a zero of f.

Interact: Drag the red point along the x-axis to choose the divisor $x - k$; the green bar's height is the remainder $f(k)$.

Observation: The remainder bar collapses to 0 exactly when the red point sits on an x-intercept — for example at $k = 2$, where $(x - 2)$ is a factor and the division comes out even.

The point of this section is to prepare us to be able to find as many zeros of a polynomial as possible by using division. We begin

with a friendly reminder of what we can expect when we divide polynomials.

Theorem: Division Algorithm

Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. Then there exist two unique polynomials, $q(x)$ and $r(x)$, such that

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)},$$

where either $r(x) = 0$ or the degree of r is strictly less than the degree of d. By multiplying both sides of this equation by $d(x)$, we get the equivalent form

$$p(x) = q(x)d(x) + r(x).$$

Each of these polynomials is given a special name:

- $d(x)$ is called the **divisor**
- $p(x)$ is called the **dividend**
- $q(x)$ is called the **quotient**
- $r(x)$ is called the **remainder**

For example, we just performed long division and discovered that $\frac{x^3+4x^2-5x-14}{x-2} = x^2 + 6x + 7$, where the remainder is 0. In this equation, the dividend is $x^3 + 4x^2 - 5x - 14$, the divisor is $x - 2$, the quotient is $x^2 + 6x + 7$, and (as was just mentioned) the remainder is 0.

Example 2.5.1: Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$.

Solution available at the end of this section.

The division problem in Example 2.5.1 had a remainder of 0 and the result was that our original polynomial (the dividend) had a factor of $(x + 1)$ (our divisor). This is a preview of an important theorem introduced in the next section.

Example 2.5.2: Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$.

Solution available at the end of this section.

Checkpoint 2.5.2

Divide $16x^3 - 12x^2 + 20x - 3$ by $4x + 5$.

Focus on Calculus - Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and can be quite cumbersome. As such, we hunt for a better method. Fortunately, people like Ruffini and Horner have already blazed this trail. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1 (although, it can be adjusted to divide by linear factors like $3x - 2$).

Let's look at the long division we performed at the beginning of the section and try to streamline it. First off, let's change all of the subtractions into additions by distributing through the negative signs.

$$
\begin{array}{rl}
 & x^2 + 6x + 7 \\
\hline
x-2 \;\big)\; & x^3 + 4x^2 - 5x - 14 \\
 & -x^3 + 2x^2 \\
\hline
 & 6x^2 - 5x \\
 & -6x^2 + 12x \\
\hline
 & 7x - 14 \\
 & -7x + 14 \\
\hline
 & 0
\end{array}
$$

Next, observe that the terms $-x^3$, $-6x^2$ and $-7x$ are the exact opposite of the terms above them.

$$
\begin{array}{rl}
 & x^2 + 6x + 7 \\
\hline
x-2 \;\big)\; & x^3 + 4x^2 - 5x - 14 \\
 & \boxed{-x^3} + 2x^2 \\
\hline
 & 6x^2 - 5x \\
 & \boxed{-6x^2} + 12x \\
\hline
 & 7x - 14 \\
 & \boxed{-7x} + 14 \\
\hline
 & 0
\end{array}
$$

Our algorithm ensures this is always the case, so we can omit them without losing any information. Also, note that the terms we "bring down" (namely the $-5x$ and -14) aren't essential to recopy, so we omit them, too.

$$
\begin{array}{rl}
 & x^2 + 6x + 7 \\
\hline
x-2 \;\big)\; & x^3 + 4x^2 - 5x - 14 \\
 & 2x^2 \\
\hline
 & 6x^2 \\
 & 12x \\
\hline
 & 7x \\
 & 14 \\
\hline
 & 0
\end{array}
$$

Now, let's move things up a bit and, for reasons that will become clear in a moment, copy the x^3 into the last row.

$$
\begin{array}{rl}
 & x^2 + 6x + 7 \\
\hline
x-2 \;\big)\; & x^3 + 4x^2 - 5x - 14 \\
\hline
 & x^3 \quad 6x^2 \quad 7x \quad 0
\end{array}
\qquad
\begin{array}{r|l}
 & x^3 + 4x^2 - 5x - 14 \\
\hline
-2 & 2x^2 \quad 12x \quad 14 \\
\hline
 & x^3 \quad 6x^2 \quad 7x \quad 0
\end{array}
$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. Take the time to work back through the original division problem. You will find that this is exactly the way we determined the quotient polynomial. This means we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

We've streamlined things quite a bit, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, $6x$ and 7 , respectively, by the -2 in $x-2$, then by -1 when we changed the subtraction to addition. Multiplying by -2 then by -1 is the same as multiplying by 2, so we replace the -2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$\begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & 0 \end{array}$$

We have constructed a synthetic division tableau for this polynomial division problem. Let's rework our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3+4x^2-5x-14$ by $x-2$, we write 2 in the place of the divisor and the coefficients of $x^3+4x^2-5x-14$ in for the dividend. Then, "bring down" the first coefficient of the dividend.

$$\begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \end{array} \qquad \begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was "brought down" to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ & 1 & 6 & 7 & \boxed{0} \end{array}$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third-degree polynomial and divided by a first-degree polynomial, so the quotient is a second-degree polynomial. Hence the quotient is x^2+6x+7. The number in the box is the remainder.

Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x-c$. It is important to note that it works only for these kinds of divisors (you'll need to use good old-fashioned polynomial long division for divisors of degree larger than 1). Also, take note that when a polynomial (of degree at least 1) is divided by $x-c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division. While the authors have done their best to indicate where the algorithm comes from, there is no substitute for working through it yourself.

How To: Use synthetic division to divide a polynomial by a binomial of the form $x-k$.

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder, the next number from the right has degree 0, the next number from the right has degree 1, and so on.

Example 2.5.3: Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

Solution available at the end of this section.

Example 2.5.4: Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

Solution available at the end of this section.

Example 2.5.5: Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

Solution available at the end of this section.

Checkpoint 2.5.5

Use synthetic division to divide $3x^4 + 18x^3 - 3x + 40$ by $x + 7$.

Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

Example 2.5.6: Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$ and the width is given by $x - 2$. Find the height of the solid.

Solution available at the end of this section.

Checkpoint 2.5.6

The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6$. The width of the rectangle is given by $x + 6$. Find an expression for the length of the rectangle.

Miscellaneous Topics: Thinking About Mathematics (Part 1)

1. What is it about math that you feel is difficult?
2. Would you think it an odd statement if someone told you that they suck at reading and could barely read a comic book? Why do you think people are more apt to proclaim that they suck at math?

Hidden Resources: Microsoft Office 365

Los Rios students have free access to Microsoft Office 365 for as long as they remain enrolled! This is an incredible deal! A warning is necessary - do not use the included OneDrive storage as it gets deleted each semester. Other than that minor issue, this is a fantastic resource. Check out the details on this CRC webpage.

Answers

Section Preview

The entire purpose of this section is to introduce the student to synthetic division, which is preferable to know prior to moving forward with polynomial theory.

Section Preview

- **Arithmetic**
  - Mathematical Operations — Addition, Subtraction, Multiplication, and Division: Every synthetic-division step multiplies by k and adds signed coefficients; fluency with signed-number arithmetic is exercised throughout Examples $2.5.3$–$2.5.6$.
- **Geometry**
  - Volumes and Surface Areas of Common Shapes: The volume formula $V = l \cdot w \cdot h$ for a rectangular solid drives both the section opener (Lincoln Memorial) and the application in Example $2.5.6$.

Section Preview

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

Example $2.5.1$: Using Long Division to Divide a Second-Degree Polynomial

$$\begin{array}{rrrrrr} & & & 5x & - & 2 \\ \cline{2-6} x+1 & \big) & 5x^2 & + \; 3x & - & 2 \\ & - & (5x^2 & + \; 5x) & & \\ \cline{2-6} & & & -2x & - & 2 \\ & - & & (-2x & - & 2) \\ \cline{2-6} & & & & & 0 \end{array}$$

The quotient is $5x-2$. The remainder is 0. We write the result as

$$\frac{5x^2+3x-2}{x+1} = 5x-2$$

or

$$5x^2+3x-2 = (x+1)(5x-2).$$

Example $2.5.2$: Using Long Division to Divide a Third-Degree Polynomial

$$\begin{array}{rrrrrrrrr} & & & 2x^2 & + & 5x & - & 7 & \\ \cline{2-9} 3x-2 & \big) & 6x^3 & + & 11x^2 & - & 31x & + & 15 \\ & - & (6x^3 & - & 4x^2) & & & & \\ \cline{2-9} & & & & 15x^2 & - & 31x & & \\ & & & - & (15x^2 & - & 10x) & & \\ \cline{2-9} & & & & & & -21x & + & 15 \\ & & & & & - & (-21x & + & 14) \\ \cline{2-9} & & & & & & & & 1 \end{array}$$

There is a remainder of 1. We can express the result as:

$$\frac{6x^3+11x^2-31x+15}{3x-2} = 2x^2+5x-7+\frac{1}{3x-2}$$

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x-2)(2x^2+5x-7)+1 = 6x^3+11x^2-31x+15.$$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is 1

Example $2.5.3$: Using Synthetic Division to Divide a Second-Degree Polynomial

Begin by setting up the synthetic division. Write k and the coefficients.

$$\underline{3}| \quad 5 \quad -3 \quad -36$$

Bring down the lead coefficient. Multiply the lead coefficient by k.

$$\begin{array}{r|rrr} \underline{3} & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by k. Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} \underline{3} & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x+12$. The remainder is 0. So $x-3$ is a factor of the original polynomial.

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x-3)(5x+12)+0=5x^2-3x-36.$$

Example $2.5.4$**: Using Synthetic Division to Divide a Third-Degree Polynomial**

The binomial divisor is $x+2$ so $k=-2$. Add each column, multiply the result by -2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} \underline{-2} & 4 & 10 & -6 & -20 \\ & & -8 & -4 & 20 \\ \hline & 4 & 2 & -10 & 0 \end{array}$$

The result is $4x^2+2x-10$. The remainder is 0. Thus, $x+2$ is a factor of $4x^3+10x^2-6x-20$.

The graph of the polynomial function $f(x)=4x^3+10x^2-6x-20$ in Figure $2.5.3$ shows a zero at $x=k=-2$. This confirms that $x+2$ is a factor of $4x^3+10x^2-6x-20$.

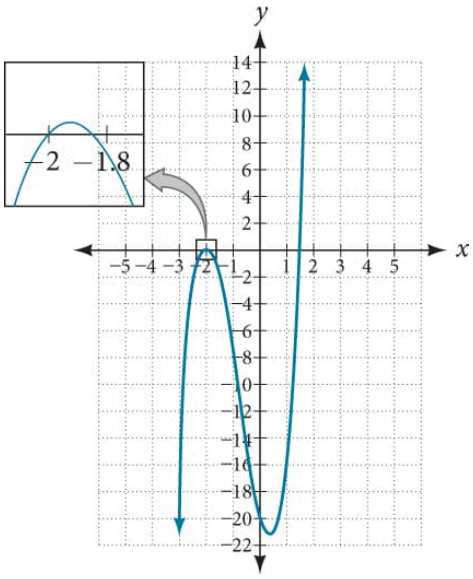


Figure $2.5.3$

Example $2.5.5$**: Using Synthetic Division to Divide a Fourth-Degree Polynomial**

Notice there is no x-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} \underline{1} & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3+x^2+8x+8+\frac{2}{x-1}$.

Example $2.5.6$**: Using Polynomial Division in an Application Problem**

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in Figure $2.5.4$.

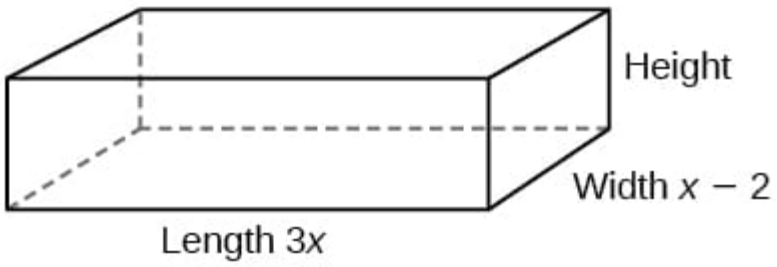


Figure 2.5.4

We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$\begin{aligned} V &= l \cdot w \cdot h \\ \implies 3x^4 - 3x^3 - 33x^2 + 54x &= 3x \cdot (x-2) \cdot h \end{aligned}$$

To solve for h, first divide both sides by $3x$.

$$\begin{aligned} \frac{3x \cdot (x-2) \cdot h}{3x} &= \frac{3x^4 - 3x^3 - 33x^2 + 54x}{3x} \\ \implies (x-2)h &= x^3 - x^2 - 11x + 18 \end{aligned}$$

Now solve for h using synthetic division.

$$h = \frac{x^3 - x^2 - 11x + 18}{x - 2}$$

$$\begin{array}{r|rrrr} \underline{2} & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The quotient is $x^2 + x - 9$ and the remainder is 0. The height of the solid is $x^2 + x - 9$.

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2.5.1: Resources and Key Concepts

Resources

Videos

- Synthetic division
  - Dividing Polynomials - Synthetic Division

Key Concepts

Definitions

- **Dividend**: In polynomial division $p(x) = d(x)q(x) + r(x)$, $p(x)$ is the dividend (the polynomial being divided).
- **Divisor**: In polynomial division $p(x) = d(x)q(x) + r(x)$, $d(x)$ is the divisor (the polynomial by which another polynomial is divided).
- **Quotient**: In polynomial division $p(x) = d(x)q(x) + r(x)$, $q(x)$ is the quotient (the result of the division, excluding the remainder).
- **Remainder**: In polynomial division $p(x) = d(x)q(x) + r(x)$, $r(x)$ is the remainder (the part left over after division if the divisor does not divide the dividend exactly).
- **Synthetic Division**: A shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1 (e.g., $x - k$).

Theorems

- **Division Algorithm**: Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. Then there exist two unique polynomials, $q(x)$ and $r(x)$, such that $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$ where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

Common Mistakes

- **Sign Errors in Long Division**: When subtracting partial products in polynomial long division, errors often occur if the signs of all terms in the polynomial being subtracted are not correctly changed.
- **Misalignment of Terms in Long Division**: Not aligning like terms vertically during the long division process, which can lead to errors in subtraction or in bringing down the next term.
- **Incorrectly Handling Missing Terms**: When setting up long division or synthetic division, failing to include a zero coefficient as a placeholder for any missing terms (e.g., if a cubic polynomial has no x^2 term, a $0x^2$ should be used).
- **Errors in Synthetic Division Setup**:
  - Using the wrong value for k when dividing by $x - k$. For example, if dividing by $x + 2$, then $k = -2$, not 2.
  - Incorrectly listing the coefficients of the dividend.
- **Arithmetic Errors in Synthetic Division**: Making mistakes in the multiply-add steps of synthetic division.
- **Misinterpreting the Result of Synthetic Division**: Incorrectly determining the degree of the quotient (it is one less than the degree of the dividend when dividing by a linear factor).
- **Using Synthetic Division for Non-Linear Divisors**: Attempting to use synthetic division when the divisor is not a linear factor of the form $x - k$ (or adaptable to it). Synthetic division is a specialized method.

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2.5.2: Homework

Reading Questions

1. What are the four main components in the Division Algorithm for polynomials: $f(x) = d(x)q(x) + r(x)$? Define each term.
2. When performing long division of polynomials, how do you know when the division process is complete? What condition must the remainder satisfy?
3. If the remainder of a polynomial division is 0, what does this imply about the relationship between the divisor, the dividend, and the quotient?
4. Describe the first step in setting up polynomial long division (e.g., dividing $2x^3 - 3x^2 + 4x + 5$ by $x + 2$).
5. In polynomial long division, after multiplying the current term of the quotient by the entire divisor, what is the next arithmetic operation performed?
6. What is synthetic division, and for what special case of divisor is it typically used?
7. In synthetic division, if you are dividing by $x - k$, what value representing the divisor do you write outside the division symbol? What if you are dividing by $x + k$?
8. Describe the process of bringing down the leading coefficient and the subsequent multiply-add steps in synthetic division.
9. How do you interpret the bottom row of numbers in a completed synthetic division problem to find the quotient and remainder?
10. If a polynomial dividend is missing a term (e.g., no x^2 term in a cubic polynomial), how should this be handled when setting up either long division or synthetic division?
11. How can you check your work after performing polynomial division?
12. Can polynomial division be used to find unknown dimensions of a geometric solid if its volume and some other dimensions are given as polynomials?
13. If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?
14. If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

Homework

Algebraic

For the following exercises, use long division to divide. Specify the quotient and the remainder.

15. $(x^2 + 5x - 1) \div (x - 1)$
16. $(2x^2 - 9x - 5) \div (x - 5)$
17. $(3x^2 + 23x + 14) \div (x + 7)$
18. $(4x^2 - 10x + 6) \div (4x + 2)$
19. $(6x^2 - 25x - 25) \div (6x + 5)$
20. $(-x^2 - 1) \div (x + 1)$
21. $(2x^2 - 3x + 2) \div (x + 2)$
22. $(x^3 - 126) \div (x - 5)$
23. $(3x^2 - 5x + 4) \div (3x + 1)$
24. $(x^3 - 3x^2 + 5x - 6) \div (x - 2)$
25. $(2x^3 + 3x^2 - 4x + 15) \div (x + 3)$

For the following exercises, use synthetic division to find the quotient.

26. $(3x^3 - 2x^2 + x - 4) \div (x + 3)$
27. $(2x^3 - 6x^2 - 7x + 6) \div (x - 4)$
28. $(6x^3 - 10x^2 - 7x - 15) \div (x + 1)$
29. $(4x^3 - 12x^2 - 5x - 1) \div (2x + 1)$
30. $(9x^3 - 9x^2 + 18x + 5) \div (3x - 1)$
31. $(3x^3 - 2x^2 + x - 4) \div (x + 3)$
32. $(-6x^3 + x^2 - 4) \div (2x - 3)$

33. $(2x^3 + 7x^2 - 13x - 3) \div (2x - 3)$
34. $(3x^3 - 5x^2 + 2x + 3) \div (x + 2)$
35. $(4x^3 - 5x^2 + 13) \div (x + 4)$
36. $(x^3 - 3x + 2) \div (x + 2)$
37. $(x^3 - 21x^2 + 147x - 343) \div (x - 7)$
38. $(x^3 - 15x^2 + 75x - 125) \div (x - 5)$
39. $(9x^3 - x + 2) \div (3x - 1)$
40. $(6x^3 - x^2 + 5x + 2) \div (3x + 1)$
41. $(x^4 + x^3 - 3x^2 - 2x + 1) \div (x + 1)$
42. $(x^4 - 3x^2 + 1) \div (x - 1)$
43. $(x^4 + 2x^3 - 3x^2 + 2x + 6) \div (x + 3)$
44. $(x^4 - 10x^3 + 37x^2 - 60x + 36) \div (x - 2)$
45. $(x^4 - 8x^3 + 24x^2 - 32x + 16) \div (x - 2)$
46. $(x^4 + 5x^3 - 3x^2 - 13x + 10) \div (x + 5)$
47. $(x^4 - 12x^3 + 54x^2 - 108x + 81) \div (x - 3)$
48. $(4x^4 - 2x^3 - 4x + 2) \div (2x - 1)$
49. $(4x^4 + 2x^3 - 4x^2 + 2x + 2) \div (2x + 1)$

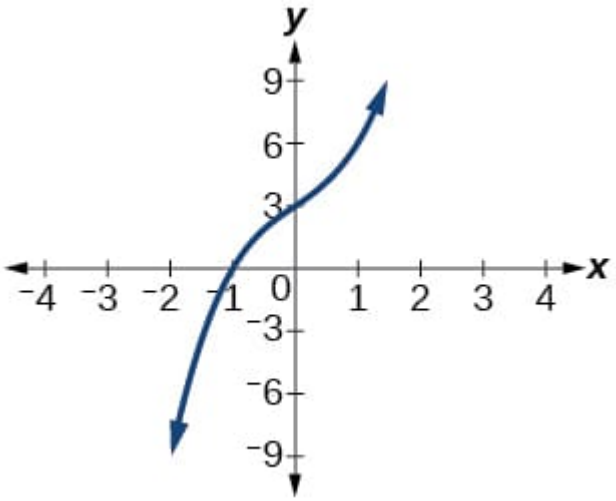
For the following exercises, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.

50. $x - 2$, $4x^3 - 3x^2 - 8x + 4$
51. $x - 2$, $3x^4 - 6x^3 - 5x + 10$
52. $x + 3$, $-4x^3 + 5x^2 + 8$
53. $x - 2$, $4x^4 - 15x^2 - 4$
54. $x - \frac{1}{2}$, $2x^4 - x^3 + 2x - 1$
55. $x + \frac{1}{3}$, $3x^4 + x^3 - 3x + 1$

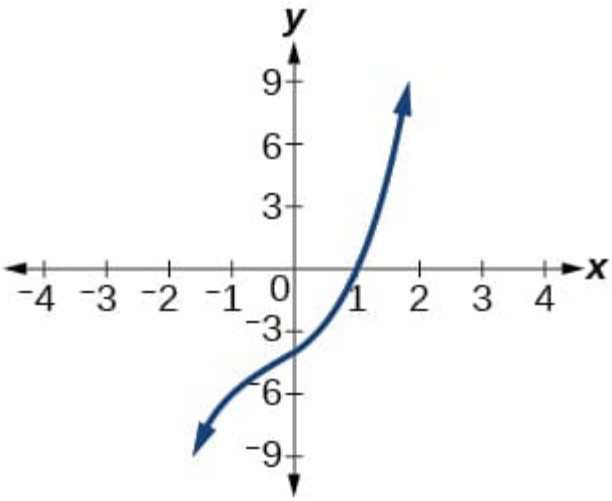
Graphical

For the following exercises, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

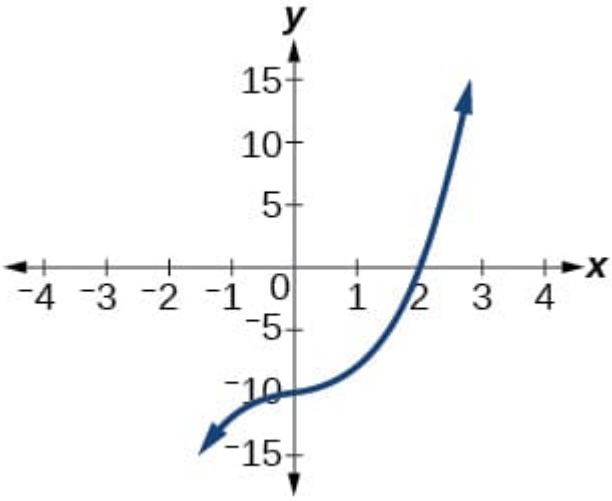
56. Factor is $x^2 - x + 3$



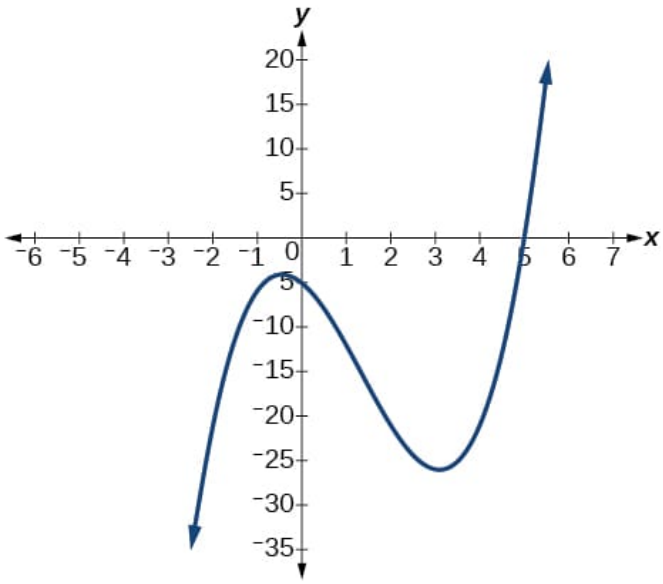
57. Factor is $x^2 + 2x + 4$



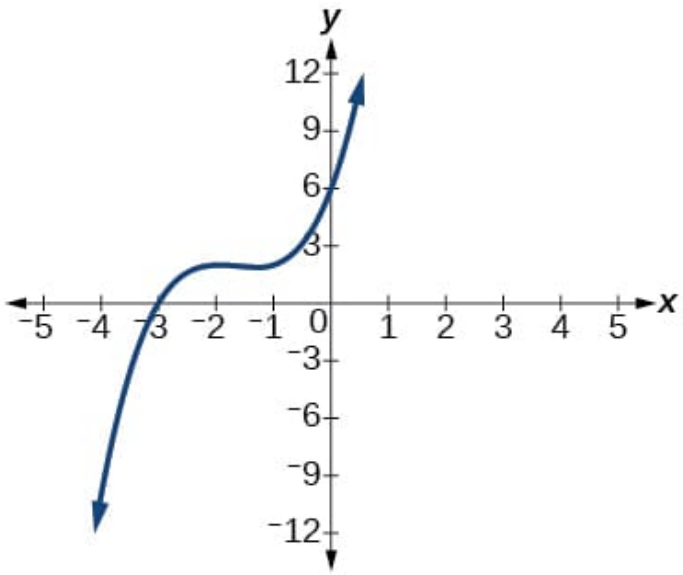
58. Factor is $x^2 + 2x + 5$



59. Factor is $x^2 + x + 1$



60. Factor is $x^2 + 2x + 2$



For the following exercises, use synthetic division to find the quotient and remainder.

61. $\frac{4x^3-33}{x-2}$
62. $\frac{2x^3+25}{x+3}$
63. $\frac{3x^3+2x-5}{x-1}$
64. $\frac{-4x^3-x^2-12}{x+4}$
65. $\frac{x^4-22}{x+2}$

Technology

For the following exercises, use technology to answer the questions.

66. Consider $\frac{x^k-1}{x-1}$ with $k = 1$, 2, and 3. What do you expect the result to be if $k = 4$?
67. Consider $\frac{x^k+1}{x+1}$ for $k = 1$, 3, and 5. What do you expect the result to be if $k = 7$?
68. Consider $\frac{x^4-k^4}{x-k}$ for $k = 1$, 2, and 3. What do you expect the result to be if $k = 4$?
69. Consider $\frac{x^k}{x+1}$ with $k = 1$, 2, and 3. What do you expect the result to be if $k = 4$?
70. Consider $\frac{x^k}{x-1}$ with $k = 1$, 2, and 3. What do you expect the result to be if $k = 4$?

Extensions

For the following exercises, use synthetic division to determine the quotient involving a complex number.

71. $\frac{x+1}{x-i}$
72. $\frac{x^2+1}{x-i}$
73. $\frac{x+1}{x+i}$
74. $\frac{x^2+1}{x+i}$
75. $\frac{x^3+1}{x-i}$

Real-World Applications

For the following exercises, use the given length and area of a rectangle to express the width algebraically.

76. Length is $x + 5$, area is $2x^2 + 9x - 5$.
77. Length is $2x + 5$, area is $4x^3 + 10x^2 + 6x + 15$

78. Length is $3x–4$, area is $6x^4 - 8x^3 + 9x^2 - 9x - 4$

For the following exercises, use the given volume of a box and its length and width to express the height of the box algebraically.

79. Volume is $12x^3 + 20x^2 - 21x - 36$, length is $2x + 3$, width is $3x - 4$.
80. Volume is $18x^3 - 21x^2 - 40x + 48$, length is $3x–4$, width is $3x–4$.
81. Volume is $10x^3 + 27x^2 + 2x - 24$, length is $5x–4$, width is $2x + 3$.
82. Volume is $10x^3 + 30x^2 - 8x - 24$, length is 2, width is $x + 3$.

For the following exercises, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.

83. Volume is $\pi(25x^3 - 65x^2 - 29x - 3)$, radius is $5x + 1$.
84. Volume is $\pi(4x^3 + 12x^2 - 15x - 50)$, radius is $2x + 5$.
85. Volume is $\pi(3x^4 + 24x^3 + 46x^2 - 16x - 32)$, radius is $x + 4$.

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2.6: Zeros of Polynomial Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

A new bakery offers decorated, multi-tiered cakes for display and cutting at Quinceañera and wedding celebrations, as well as sheet cakes to serve most of the guests. The bakery wants the volume of a small sheet cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

This problem can be solved by writing a cubic function and solving a cubic equation for the volume of the cake. In this section, we will discuss a variety of tools for writing polynomial functions and solving polynomial equations.

Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**.

Theorem: Remainder Theorem

If a polynomial $f(x)$, where $\deg(f(x)) \geq 1$, is divided by $x - c$, the remainder is $f(c)$. That is,

$$f(x) = (x - c) \cdot Q(x) + f(c)$$

for some polynomial $Q(x)$ whose degree is one less than the degree of $f(x)$.

Proof available at the end of this section.

The Remainder Theorem gives us an alternative (and sometimes faster) way to evaluate polynomial functions. Let's see how this works.

Example 2.6.1: Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate

$$f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7 \text{ at } x = 2.$$

Solution available at the end of this section.

If you pay attention to the amount of work required to evaluate $f(2)$ in Example 2.6.1 using traditional methods, you should notice that it's actually longer than using the Remainder Theorem. Moreover, synthetic division only requires the operations of addition and multiplication, whereas our traditional function evaluation method requires addition, multiplication, and powers.

Checkpoint 2.6.1

Use the Remainder Theorem to evaluate

$$f(x) = 2x^5 - 3x^4 - 9x^3 + 8x^2 + 2 \text{ at } x = -3.$$

Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors.

Theorem: Factor Theorem

Let $f(x)$ be a nonzero polynomial function having degree greater than zero. c is a zero of $f(x)$ if and only if $(x - c)$ is a factor of $f(x)$.

Proof available at the end of this section.

We have been "dancing around" the Factor Theorem for a couple of sections now; however, it wasn't until now that we formally stated the result. As we will soon see, a polynomial of degree n with complex-numbered coefficients will have n complex-numbered zeros. We can use the Factor Theorem to completely factor a polynomial into the product of n factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

Example 2.6.2: Using the Factor Theorem to Solve a Polynomial Equation

Show that $(x + 2)$ is a factor of $x^3 - 6x^2 - x + 30$. Find the remaining factors. Use the factors to determine the zeros of the polynomial.

Solution available at the end of this section.

Checkpoint 2.6.2

Use the Factor Theorem to find the zeros of $f(x) = x^3 + 4x^2 - 4x - 16$ given that $(x - 2)$ is a factor of the polynomial.

Example 2.6.3: Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in Figure 2.6.1.

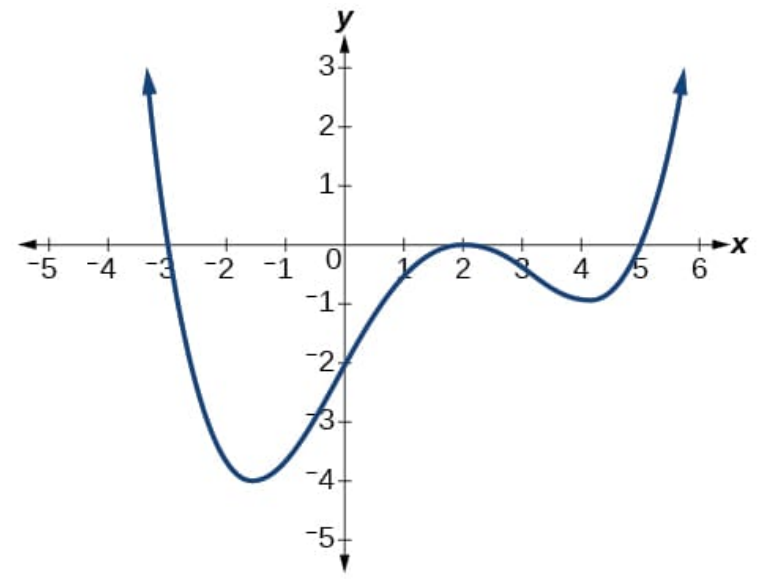


Figure 2.6.1

Solution available at the end of this section.

Interactive Element: The Stretch Factor and the Fixed Zeros

The following Interactive Element might help you visualize how the stretch factor a reshapes a polynomial's graph without moving its zeros.

Interact: Drag the red point along the y-axis to change the stretch factor a.

Observation: Stretching the graph vertically leaves the x-intercepts at $x = -3,\ 2,\ 5$ fixed; only one value of a sends the graph through a given point off the axis.

Checkpoint 2.6.3

Given the graph shown in Figure 2.6.2, write a formula for the function shown.

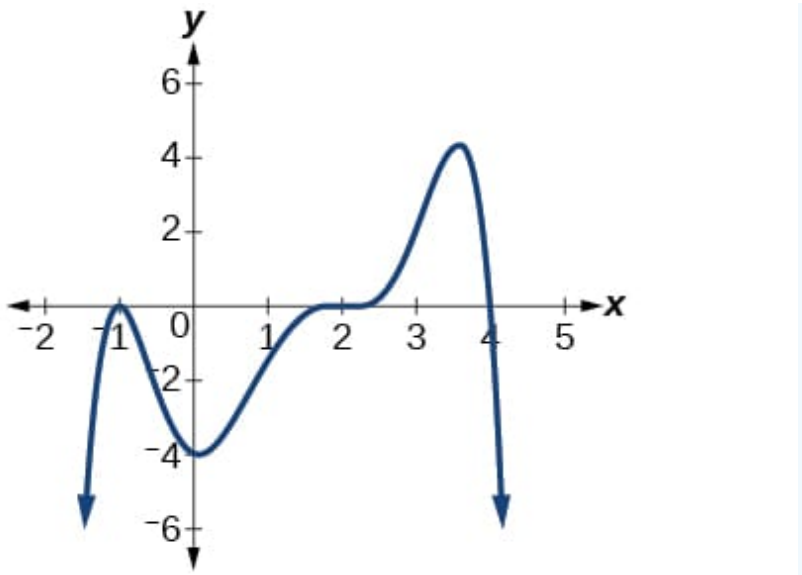


Figure 2.6.2

Using the Rational Zeros Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zeros Theorem** (also called the **Rational Roots Theorem**) helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading coefficient of the polynomial. Before stating the Rational Zeros Theorem, let's motivate *why* it works with a simple example.

Consider a quadratic function with two zeros, $x = \frac{2}{5}$ and $x = \frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0 and then construct the original quadratic function absent its stretching factor.

$$\begin{aligned}
& x - \frac{2}{5} = 0 \quad \text{or} \quad x - \frac{3}{4} = 0 && \text{(setting each factor equal to 0)} \\
\Longrightarrow \quad & 5x - 2 = 0 \quad \text{or} \quad 4x - 3 = 0 && \text{(multiplying both sides of the equation to eliminate fractions)} \\
\Longrightarrow \quad & f(x) = (5x - 2)(4x - 3) && \text{(creating the quadratic function, multiplying the factors)} \\
\Longrightarrow \quad & f(x) = 20x^2 - 23x + 6 && \text{(expanding the polynomial)} \\
\Longrightarrow \quad & f(x) = (5 \cdot 4)x^2 - 23x + (2 \cdot 3)
\end{aligned}$$

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

If we used this single example to infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient, we would be (somewhat) citing the Rational Zeros Theorem. However, as we know, examples are not proofs!

Theorem: Rational Zeros Theorem

Suppose

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

is a polynomial of degree n with $n \geq 1$, and a_0, a_1, ..., a_n are *integers*. If r is a *rational* zero of f, then r must be of the form $\pm \frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

Proof available at the end of this section.

When the leading coefficient is 1, the Rational Zeros Theorem simplifies to stating that the possible rational zeros are the factors of the constant term.

Caution: The Rational Zeros Theorem Requires Integer Coefficients

An often overlooked requirement to using the Rational Zeros Theorem is that the coefficients of the polynomial must be integers. Do not make the mistake of trying to use the Rational Zeros Theorem on polynomials with coefficients of the form $\frac{3}{7}$, $\sqrt{2}$, or $3+2i$.

Example 2.6.4: Listing All Possible Rational Zeros

List all possible rational zeros of $f(x)=2x^4-5x^3+x^2-4$

Solution available at the end of this section.

Example 2.6.5: Using the Rational Zeros Theorem to Find Rational Zeros

Use the Rational Zeros Theorem to find the rational zeros of $f(x)=2x^3+x^2-4x+1$.

Solution available at the end of this section.

Checkpoint 2.6.5

Use the Rational Zeros Theorem to find the rational zeros of

$$f(x)=x^3-5x^2+2x+1.$$

Finding the Zeros of Polynomial Functions

The Rational Zeros Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use synthetic division repeatedly to determine all of the zeros of a polynomial function.

Example 2.6.6: Finding the Zeros of a Polynomial Function with Repeated Real Zeros

Find the zeros of $f(x)=4x^3-3x-1$.

Solution available at the end of this section.

Investigating Example 2.6.6 a little more, we look at the graph of the function f in Figure 2.6.3. Notice, at $x=-0.5$, the graph bounces off the x-axis, indicating the even multiplicity $(2,4,6,...)$ for the zero -0.5. At $x=1$, the graph crosses the x-axis, indicating the odd multiplicity $(1,3,5,...)$ for the zero $x=1$.

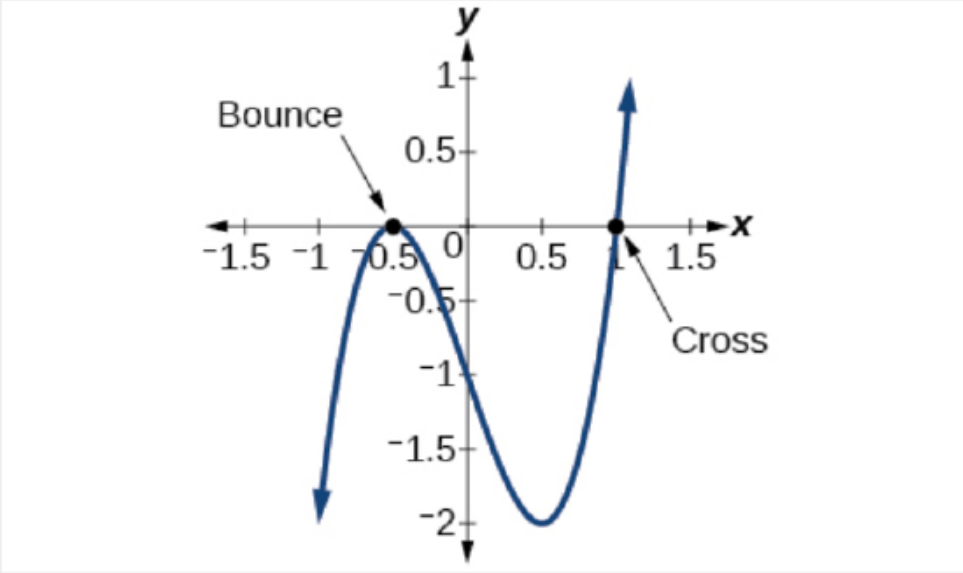


Figure 2.6.3

Interactive Element: Multiplicity and the Shape of a Graph at a Zero

The following Interactive Element might help you visualize how the multiplicity of a zero controls the behavior of a polynomial's graph at that zero.

Interact: Drag the red point up and down to change the multiplicity m of the zero at $x = 0$.

Observation: When m is odd the graph crosses the x-axis; when m is even it bounces off it. The larger m becomes, the flatter the graph lies near the zero.

Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Theorem: Fundamental Theorem of Algebra

Every nonconstant polynomial with complex coefficients has at least one complex zero.

Proof available at the end of this section.

The Fundamental Theorem of Algebra is an example of an "existence theorem" in Mathematics. It guarantees the existence of at least one complex-valued zero but gives us no algorithm to use in finding it. It took mathematicians hundreds of years to prove the theorem in its full generality, and some of that history is recorded here. Note that the Fundamental Theorem of Algebra applies to not only polynomial functions with real coefficients but to those with complex number coefficients as well.

Suppose f is a polynomial function of degree four, and $f(x) = 0$. The Fundamental Theorem of Algebra states that there is at least one complex-valued solution, call it z_1. By the Factor Theorem, we can write $f(x)$ as a product of $x - z_1$ and a polynomial quotient. Since $x - z_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it z_2. So we can write the polynomial quotient as a product of $x - z_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them, and each one will yield a factor of $f(x)$.

In truth, "a polynomial with complex-valued coefficients" means that the coefficients can be any number we can imagine—from 2 to $\sqrt{2}$ to $\frac{1}{2}$ to $2+2i$. Therefore, most people write the Fundamental Theorem of Algebra as, "If $f(x)$ is a polynomial of degree $n \geq 1$, then $f(x)$ has at least one complex zero," thereby omitting the needless "with complex-valued coefficients."

Although the Fundamental Theorem of Algebra is a major theorem in Mathematics, it is the following corollary that makes the theorem useful.

Corollary: Complete Factorization Theorem

Let $f(x)$ be a polynomial of degree $n \geq 1$ with complex coefficients and leading coefficient a. Then $f(x)$ has exactly n linear factors over the complex number system. That is, there exists complex numbers $z_1, z_2, \ldots, z_n$ such that

$$f(x) = a\,(x - z_1)\,(x - z_2)\ldots(x - z_n)\,.$$

Proof available at the end of this section.

As with the Fundamental Theorem of Algebra, the Complete Factorization Theorem can be shortened to, "If $f(x)$ is a polynomial of degree $n \geq 1$, then $f(x)$ has exactly n linear factors over the complex number system."

Q & A

Does every polynomial have at least one imaginary zero?

No. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

Example $2.6.7$: Finding the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of

$$f(x) = 3x^3 + 9x^2 + x + 3.$$

Solution available at the end of this section.

Figure $2.6.4$ shows the graph of the function f from Example $2.6.7$. Notice that, at $x = -3$, the graph crosses the x-axis, indicating an odd multiplicity (1) for the zero $x = -3$. Also note the presence of the two turning points. This means that, since there is a 3rd degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the x-intercepts for the function are shown. Either the multiplicity of $x = -3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x = -3$ is three. Either way, our result is correct.

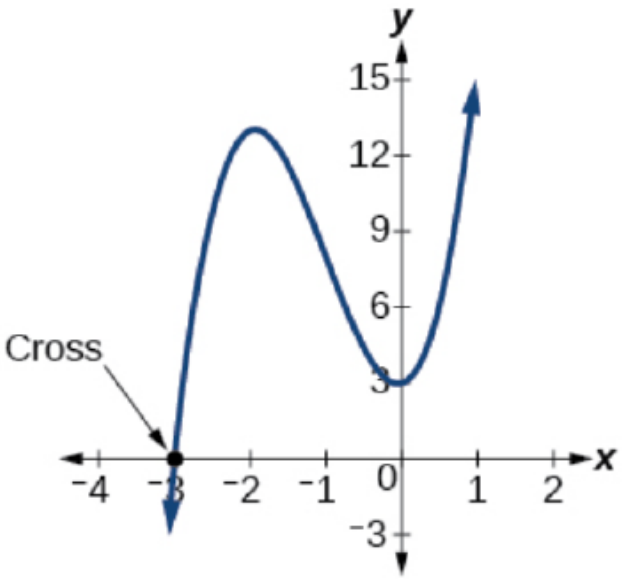


Figure 2.6.4

Checkpoint 2.6.7

Find the zeros of

$$f(x) = 2x^3 + 5x^2 - 11x + 4.$$

Using the Complete Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the Fundamental Theorem of Algebra, as we stated above, is that a polynomial function of degree n will have n zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into n factors. The Complete Factorization Theorem tells us that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x - c)$, where c is a complex number.

Theorem: Conjugate Zeros Theorem

Let $f(x)$ be a polynomial with *real* coefficients. If the complex number $z = a + bi$ is a zero of $f(x)$, then $\overline{z} = a - bi$ is also a zero of $f(x)$, and z and $\overline{z}$ have the same multiplicity.

Proof available at the end of this section.

Example 2.6.8: Finding a Polynomial with Given Zeros

Find a fourth degree polynomial with real coefficients that has zeros of -3, 2, and i, such that $f(-2) = 100$.

Solution available at the end of this section.

In Example 2.6.8, we found that both i and $-i$ were zeros, but only one of these zeros needed to be given. If i is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of i.

Q & A

If $2+3i$ were given as a zero of a polynomial with real coefficients, would $2-3i$ also need to be a zero?

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

Checkpoint 2.6.8

Find a third degree polynomial with real coefficients that has zeros of 5 and $-2i$ such that $f(1)=10$.

It's probably good to summarize the theorems and corollaries we have encountered so far in this section, along with their requirements.

Table 2.6.1

Theorem or Corollary	Requirement	Result
Rational Zeros Theorem	Coefficients must be integers	Rational zeros of the polynomial are always ratios of factors of the constant term and the leading coefficient.
Fundamental Theorem of Algebra	Coefficients can be any number	The polynomial has at least one complex zero.
Complete Factorization Theorem	Coefficients can be any number	The polynomial can be factored into n linear factors over the complex number system.
Conjugate Zeros Theorem	Coefficients must be real numbers	If $a+bi$ is a zero, then $a-bi$ is also a zero.

Solving Real-World Applications

We have now introduced a variety of tools for solving polynomial equations. Let's use these tools to solve the bakery problem from the beginning of the section.

Example 2.6.9

A new bakery offers decorated, multi-tiered cakes for display and cutting at Quinceañera and wedding celebrations, as well as sheet cakes to serve most of the guests. The bakery wants the volume of a small sheet cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

Solution available at the end of this section.

Checkpoint 2.6.9

A shipping container in the shape of a rectangular solid must have a volume of 84 cubic meters. The client tells the manufacturer that, because of the contents, the length of the container must be one meter longer than the width, and the height must be one meter greater than twice the width. What should the dimensions of the container be?

Success in College: Relying On Others and Fooling Yourself

It is a common mistake that students make:

- You do homework with your friend, and she constantly is the one with the correct answers. Instead of talking about the mistakes, you copy the answer and move on.
- Your classmates have formed a Discord, and you copy the solutions from one of them.
- You only have a couple hours before the homework is due and you use your preferred A.I. model to do most of the work, copying frantically to submit the work before it's too late.

Is this helping you learn? Do you truly think it is helping your grade?

If you answered yes, you're in the majority of students who have been fooled by this method of "get the right answer by any means and move on."

> When I was a student at UC Davis, I concentrated on understanding every concept that came my way. I studied with other students, but more often than not, they relied on my answers and explanations to succeed in the course. The problem is that they were just "parroting" my methods and not understanding them. In the end, most of them did poorly due to their lack of understanding of the material.
>
> - Professor Simpson, Cosumnes River College

This sort of "copying" also leads to another damaging process - a lack of self-confidence. Because they never solve anything by themselves, the students who copy other students' work end up freaking out more before exams and, as a result of their heightened state of panic and their fundamental lack of understanding of the material, they fail the exams.

Think about what kind of student you are, and, without fear of judgement, consider the following questions:

1. A student was having trouble with his homework. For the next few weeks, a friend helped him do his homework. The friend answered his questions and told him what to do when he was stuck on a problem. Why do you think this student may have trouble doing well on a test?
2. A student can do about 75% of the problems on most of her homework assignments without help. She thinks that this is good enough. She thinks she should be able to score about 75% on a test and pass the test. Do you think she is right?

Success in Mathematics: Why Study Mathematics? (Part 2)

Tracy Moody graduated in 2006 with a bachelor's degree in mathematics. She then went to graduate school, studying to become a physical therapist. She writes,

> "Many people cannot seem to fathom how math and physical therapy overlap. Well, one of the obvious things is that physical therapists use statistics to see if new treatments work. However, the main thing I bring as a mathematician to my graduate studies is my ability to problem solve. A degree in mathematics taught me how to look at a problem from several angles. Furthermore, one of the main classes that carried over into my career choice is, believe it or not, real analysis. I can remember working with my fellow classmates on problems for hours. However, through this process, I learned the importance of perseverance, and I learned how to communicate the problem and solutions in several ways."
>
> *Source: www.maa.org, Career Profiles.*

1. Do you think studying mathematics has helped you become a better problem solver? Explain.
2. Do you think studying mathematics has helped you learn to "persevere," to keep working on a difficult problem even though you are struggling? Explain.

Answers

Section Preview

This very dense section contains the Factor and Remainder theorems along with the Rational Roots Theorem (which I called the Rational Zero Theorem and was too lazy to change back to the Rational Roots Theorem). You also cover the Fundamental Theorem of Algebra, Complete Factorization Theorem, and the Conjugate

Zeros Theorem. Like I said, this is a dense section. Expect to spend at least two class sessions here. This is necessary theory for Calculus.

Section Preview

- **Arithmetic**
  - Factors, Prime Factorizations, and Least Common Multiples: Applying the Rational Zeros Theorem requires listing the integer factors of the constant term and leading coefficient (Examples 4–7 and Example 9).

Section Preview

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.
- Use the Rational Zeros Theorem to find rational zeros.
- Find zeros of a polynomial function.
- Use the Complete Factorization Theorem to find polynomials with given zeros.
- Solve real-world applications of polynomial equations.

Theorem: Remainder Theorem

Let $f(x)$ be a polynomial of degree one or greater. The Division Algorithm states there exists a quotient, $Q(x)$, and remainder, $R(x)$, such that

$$\frac{f(x)}{x-c} = Q(x) + \frac{R(x)}{x-c}.$$

Multiplying both sides by $x-c$, we get

$$f(x) = Q(x)(x-c) + R(x).$$

We now use this to evaluate $f(c)$.

$$f(c) = Q(c)(c-c) + R(c) = Q(c)\cdot 0 + R(c) = R(c).$$

Hence, $f(c) = R(c)$. However, the remainder must be of a degree strictly less than that of the divisor. Since the divisor, $x-c$, is of degree one, the remainder must be of degree zero. That is, $R(c) = r$ is constant. Thus,

$$f(c) = r = \text{the remainder when } f(x) \text{ is divided by } x-c.$$

Example $2.6.1$: Using the Remainder Theorem to Evaluate a Polynomial

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x-2$.

$$\begin{array}{r|rrrrr} \underline{2} & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array}$$

The remainder is 25. Therefore, $f(2) = 25$.

We can check our answer by evaluating $f(2)$.

$$\begin{aligned} f(x) &= 6x^4 - x^3 - 15x^2 + 2x - 7 \\ \Longrightarrow \quad f(2) &= 6(2)^4 - (2)^3 - 15(2)^2 + 2(2) - 7 \\ &= 6(16) - (8) - 15(4) + 2(2) - 7 \\ &= 96 - 8 - 60 + 4 - 7 \\ &= 25 \end{aligned}$$

Theorem: Factor Theorem

Let $f(x)$ be a polynomial function of degree greater than zero.

( ): Suppose c is a zero of $f(x)$. Then $f(c) = 0$. By the Remainder Theorem, if $f(x)$ is divided by $x-c$, then

$$\frac{f(x)}{x-c} = Q(x) + \frac{R(x)}{x-c} = Q(x) + \frac{f(c)}{x-c} = Q(x) + \frac{0}{x-c} = Q(x).$$

Multiplying both sides by $x-c$, we get

$$f(x) = q(x)(x-c).$$

Hence, $x-c$ is a factor of $f(x)$.

( ): Suppose $x-c$ is a factor of $f(x)$. Then

$$f(x) = (x-c)g(x),$$

for some polynomial $g(x)$. Therefore,

$$f(c) = (c-c)g(c) = 0\cdot g(c) = 0.$$

Thus, $f(c) = 0$, which means c is a zero of $f(x)$.

Example $2.6.2$: Using the Factor Theorem to Solve a Polynomial Equation

We can use synthetic division to show that $(x+2)$ is a factor of the polynomial.

$$\begin{array}{r|rrrr} \underline{-2} & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array}$$

The remainder is zero, so $(x+2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$$(x+2)\left(x^2 - 8x + 15\right).$$

We can factor the quadratic factor to write the polynomial as

$$(x+2)(x-3)(x-5).$$

By the Factor Theorem, the zeros of $x^3 - 6x^2 - x + 30$ are -2, 3, and 5.

Example 2.6.3: Writing a Formula for a Polynomial Function from the Graph

This graph has three x-intercepts: $x = -3$, 2, and 5. The y-intercept is located at $(0, -2)$. At $x = -3$ and $x = 5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

$$f(x) = a(x+3)(x-2)^2(x-5).$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, -2)$, to solve for a.

$$\begin{aligned} f(0) &= a(0+3)(0-2)^2(0-5) && \text{(substituting)} \\ \implies -2 &= a(0+3)(0-2)^2(0-5) && \text{(substituting)} \\ \implies -2 &= -60a && \text{(simplifying)} \\ \implies \frac{1}{30} &= a && \text{(dividing both sides by } -60) \end{aligned}$$

The graphed polynomial appears to represent the function $f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$.

Theorem: Rational Zeros Theorem

Suppose c is a zero of f and $c = \frac{p}{q}$ in lowest terms. This means p and q have no common factors. Since $f(c) = 0$, we have

$$f(c) = f\left(\frac{p}{q}\right) = a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_1\left(\frac{p}{q}\right) + a_0 = 0.$$

That is,

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_1\left(\frac{p}{q}\right) + a_0 = 0.$$

Multiplying both sides of this equation by q^n, we clear the denominators to get

$$a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_1 pq^{n-1} + a_0 q^n = 0.$$

Keeping the term $a_n p^n$ on the left side and moving everything else to the right side, we get

$$a_n p^n = -a_{n-1}p^{n-1}q - \ldots - a_1 pq^{n-1} - a_0 q^n.$$

However, each term on the right side has a factor of q in common. Therefore,

$$a_n p^n = \left(-a_{n-1}p^{n-1} - \ldots - a_1 pq^{n-2} - a_0 q^{n-1}\right) q.$$

Recall that p, q, and a_k are integers (for all k). Moreover, $n \geq 1$. This shows that $-a_{n-1}p^{n-1} - \ldots - a_1 pq^{n-2} - a_0 q^{n-1}$ must be an integer (can you see why?). Hence, $a_n p^n$ is an integer multiple of q. Moreover, since $a_n p^n = (a_n p^{n-1})\, p$, $a_n p^n$ is also an integer multiple of p. However, p and q have no common factors. Thus, a_n must be the integer multiple of q. This shows that q is a factor of the leading coefficient a_n.

If we rearrange the equation

$$a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_1 pq^{n-1} + a_0 q^n = 0$$

as

$$a_0 q^n = -a_n p^n - a_{n-1}p^{n-1}q - \ldots - a_1 pq^{n-1},$$

we can play the same game and conclude a_0 is a multiple of p, and we have the result we desire.

Example 2.6.4: Listing All Possible Rational Zeros

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, -4, and the factors of the leading coefficient, 2.

The factors of -4 are $p = \pm 1, \pm 2, \pm 4$.

The factors of 2 are $q = \pm 1, \pm 2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of -4 divided by one of the factors of 2.

$$\frac{p}{q} = \pm\frac{1}{1}, \pm\frac{1}{2} \quad \frac{p}{q} = \pm\frac{2}{1}, \pm\frac{2}{2} \quad \frac{p}{q} = \pm\frac{4}{1}, \pm\frac{4}{2}.$$

Note that $\frac{2}{2} = 1$ and $\frac{4}{2} = 2$, which have already been listed. So we can shorten our list.

$$\frac{p}{q} = \frac{\text{Factors of the last}}{\text{Factors of the first}} = \pm 1, \pm 2, \pm 4, \pm\frac{1}{2}.$$

Example 2.6.5: Using the Rational Zeros Theorem to Find Rational Zeros

The Rational Zeros Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 1 and q is a factor of 2.

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of } 1}{\text{factor of } 2} \end{aligned}$$

The factors of 1 are ± 1 and the factors of 2 are ± 1 and ± 2. The possible values for $\frac{p}{q}$ are ± 1 and $\pm\frac{1}{2}$. These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by substituting these values for x in $f(x)$.

$$\begin{aligned} f(-1) &= 2(-1)^3+(-1)^2-4(-1)+1=4 \\ f(1) &= 2(1)^3+(1)^2-4(1)+1=0 \\ f\left(-\frac{1}{2}\right) &= 2\left(-\frac{1}{2}\right)^3+\left(-\frac{1}{2}\right)^2-4\left(-\frac{1}{2}\right)+1=3 \\ f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^2-4\left(\frac{1}{2}\right)+1=-\frac{1}{2} \end{aligned}$$

Of those, -1, $-\frac{1}{2}$, and $\frac{1}{2}$ are not zeros of $f(x)$. 1 is the only rational zero of $f(x)$.

Example $2.6.6$**: Finding the Zeros of a Polynomial Function with Repeated Real Zeros**

First, note that the coefficients of the given polynomial function are all integers—remember, we cannot use the Rational Zeros Theorem if this is not the case. The Rational Zeros Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of -1 and q is a factor of 4.

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of } -1}{\text{factor of } 4} \end{aligned}$$

The factors of -1 are ± 1 and the factors of 4 are ± 1,± 2, and ± 4. The possible values for $\frac{p}{q}$ are ± 1, $\pm\frac{1}{2}$, and $\pm\frac{1}{4}$. These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with 1.

$$\begin{array}{r|rrrr} \underline{1} & 4 & 0 & -3 & -1 \\ & & 4 & 4 & 1 \\ \hline & 4 & 4 & 1 & 0 \end{array}$$

Dividing by $(x-1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

$$(x-1)\left(4x^2+4x+1\right).$$

The quadratic is a perfect square. Therefore, $f(x)$ can be written as

$$(x-1)(2x+1)^2.$$

We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

$$2x+1=0 \implies x=-\frac{1}{2}.$$

The zeros of the function are 1 and $-\frac{1}{2}$ with multiplicity 2.

Theorem: Fundamental Theorem of Algebra

The proof of this theorem is beyond the scope of lower-division mathematics.

Corollary: Complete Factorization Theorem

The proof of this theorem is done using mathematical induction and the Fundamental Theorem of Algebra. Since this topic is usually covered *prior* to learning proofs by mathematical induction, the proof is omitted.

Example $2.6.7$**: Finding the Zeros of a Polynomial Function with Complex Zeros**

The Rational Zeros Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 3 and q is a factor of 3.

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of } 3}{\text{factor of } 3} \end{aligned}$$

The factors of 3 are ± 1 and ± 3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are ± 3, ± 1, and $\pm\frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with -3.

$$\begin{array}{r|rrrr} \underline{-3} & 3 & 9 & 1 & 3 \\ & & -9 & 0 & -3 \\ \hline & 3 & 0 & 1 & 0 \end{array}$$

Dividing by $(x+3)$ gives a remainder of 0, so -3 is a zero of the function. The polynomial can be written as

$$(x+3)\left(3x^2+1\right).$$

We can then set the quadratic equal to 0 and solve to find the other zeros of the function.

$$\begin{aligned} 3x^2+1 &= 0 \\ \implies x^2 &= -\frac{1}{3} \\ \implies x &= \pm\sqrt{-\frac{1}{3}} \\ &= \pm\frac{i\sqrt{3}}{3} \end{aligned}$$

The zeros of $f(x)$ are -3 and $\pm\frac{i\sqrt{3}}{3}$.

Theorem: Conjugate Zeros Theorem

The proof requires several theorems from complex numbers. These are mentioned within the proof. If $z=a+bi$ is a zero of a polynomial function $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, then

$$\begin{aligned}
f(\overline{z}) &= a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \cdots + a_1\overline{z} + a_0 && \text{(substituting)}\\
&= a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + \cdots + a_1\overline{z} + a_0 && \left(\overline{z^n} = (\overline{z})^n\right)\\
&= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} && (\text{if } a_i \in \mathbb{R},\ \text{then } a_i = \overline{a_i})\\
&= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} && (\overline{z_1}\cdot\overline{z_2} = \overline{z_1\cdot z_2})\\
&= \overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} && \left(\overline{z_1}+\overline{z_2}+\cdots+\overline{z_k} = \overline{z_1+z_2+\cdots+z_k}\right)\\
&= \overline{f(z)}\\
&= \overline{0} && (z \text{ is a zero of } f(x))\\
&= 0 && \left(\overline{k} = k \text{ for real numbers } k\right)
\end{aligned}$$

Hence, $\overline{z}$ (the conjugate) is also a zero of $f(x)$.

To prove that multiplicities must be the same, one can argue via contradiction. This part of the proof is left for the reader.

Example 2.6.8**: Finding a Polynomial with Given Zeros**

Because $x = i$ is a zero, by the Conjugate Zeros Theorem $x = -i$ is also a zero. The polynomial must have factors of $(x+3)$, $(x-2)$, $(x-i)$, and $(x+i)$. Since we are looking for a degree 4 polynomial, and now have four zeros, we have all four factors. Let's begin by multiplying these factors.

$$\begin{aligned}
f(x) &= a(x+3)(x-2)(x-i)(x+i)\\
&= a\left(x^2+x-6\right)\left(x^2+1\right)\\
&= a\left(x^4+x^3-5x^2+x-6\right)
\end{aligned}$$

We need to find a to ensure $f(-2) = 100$. Substitute $x = -2$ and $f(-2) = 100$ into $f(x)$.

$$\begin{aligned}
100 &= a\left((-2)^4+(-2)^3-5(-2)^2+(-2)-6\right)\\
\Longrightarrow\quad 100 &= a(-20)\\
\Longrightarrow\quad -5 &= a
\end{aligned}$$

So the polynomial function is

$$f(x) = -5\left(x^4+x^3-5x^2+x-6\right)$$

or

$$f(x) = -5x^4-5x^3+25x^2-5x+30.$$

Example 2.6.9

Begin by writing an equation for the volume of the cake. The volume of a rectangular solid is given by $V = lwh$. We were given that the length must be four inches longer than the width, so we can express the length of the cake as $l = w+4$. We were given that the height of the cake is one-third of the width, so we can express the height of the cake as $h = \frac{1}{3}w$. Let's write the volume of the cake in terms of width of the cake.

$$\begin{aligned}
V &= (w+4)(w)\left(\frac{1}{3}w\right)\\
&= \frac{1}{3}w^3+\frac{4}{3}w^2
\end{aligned}$$

Substitute the given volume into this equation.

$$\begin{aligned}
351 &= \frac{1}{3}w^3+\frac{4}{3}w^2 && \text{(substituting)}\\
\Longrightarrow\quad 1053 &= w^3+4w^2 && \text{(multiplying both sides by 3)}\\
\Longrightarrow\quad 0 &= w^3+4w^2-1053 && \text{(subtracting 1053 from both sides)}
\end{aligned}$$

The Rational Zeros Theorem tells us that the possible rational zeros are ±1, ±3, ±9, ±13, ±27, ±39, ±81, ±117, ±351, and ±1053. We can use synthetic division to test these possible zeros. Only positive numbers make sense as dimensions for a cake, so we need not test any negative values. Let's begin by testing values that make the most sense as dimensions for a small sheet cake. Use synthetic division to check $x = 1$.

$$\begin{array}{r|rrrr}
\underline{1} & 1 & 4 & 0 & -1053\\
 & & 1 & 5 & 5\\
 & 1 & 5 & 5 & -1048
\end{array}$$

Since 1 is not a solution, we will check $x = 3$.

$$\begin{array}{r|rrrr}
\underline{3} & 1 & 4 & 0 & -1053\\
 & & 3 & 21 & 63\\
 & 1 & 7 & 21 & -990
\end{array}$$

Since 3 is not a solution either, we will test $x = 9$.

$$\begin{array}{r|rrrr}
\underline{9} & 1 & 4 & 0 & -1053\\
 & & 9 & 117 & 1053\\
 & 1 & 13 & 117 & 0
\end{array}$$

Synthetic division gives a remainder of 0, so 9 is a solution to the equation. We can use the relationships between the width and the other dimensions to determine the length and height of the sheet cake pan.

$$l = w + 4 = 9 + 4 = 13 \text{ and } h = \frac{1}{3}w = \frac{1}{3}(9) = 3.$$

The sheet cake pan should have dimensions 13 inches by 9 inches by 3 inches.

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2.6.1: Resources and Key Concepts

Resources

Videos

- Remainder and Factor Theorems
  - Dividing Polynomials - The Remainder and Factor Theorems
- Rational Zeros Theorem (Rational Roots Theorem)
  - Real Zeros of Polynomials - Motivating the Rational Roots Theorem
  - Real Zeros of Polynomials - Proof of the Rational Roots Theorem
  - College Algebra - Polynomial Functions - Rational Roots Theorem - Example
  - Real Zeros of Polynomials - Examples Using the Rational Roots Theorem
- Fundamental Theorem of Algebra
  - Complex Zeros and the Fundamental Theorem of Algebra - Intro to the Fundamental Theorem of Algebra
  - Complex Zeros and the Fundamental Theorem of Algebra - Using the Fundamental Theorem of Algebra
- Complete Factorization Theorem
  - Complex Zeros and the Fundamental Theorem of Algebra - The Complete Factorization Theorem
- Complex Conjugate Theorem
  - Complex Zeros and the Fundamental Theorem of Algebra - The Zeros and Conjugate Zeros Theorems

Key Concepts

Theorems

- **Remainder Theorem**: If a polynomial $f(x)$ of degree greater than or equal to one is divided by $x-k$, the remainder is the value $f(k)$.
- **Factor Theorem**: Let $f(x)$ be a polynomial function of degree greater than or equal to one. k is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$.
- **Rational Zeros Theorem (Rational Roots Theorem)**: Suppose $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is a polynomial of degree $n\geq 1$, and $a_0,a_1,\ldots,a_n$ are integers. If $r=\pm\frac{p}{q}$ is a rational zero of f, in lowest terms, then p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.
- **Fundamental Theorem of Algebra**: If $f(x)$ is a polynomial of degree $n\geq 1$ with complex-valued coefficients, then $f(x)$ has at least one complex-valued zero.
- **Complete Factorization Theorem (Corollary to Fundamental Theorem of Algebra)**: If $f(x)$ is a polynomial of degree $n\geq 1$ with complex-valued coefficients, then $f(x)$ has exactly n linear factors over the complex number system. That is, $f(x)=a(x-z_1)(x-z_2)\ldots(x-z_n)$, where $z_1,z_2,\ldots,z_n$ are complex numbers (the zeros) and $a\in\mathbb{R}$. Therefore, $f(x)$ has n complex-valued zeros if we allow for multiplicities.
- **Conjugate Zeros Theorem (Complex Conjugate Theorem)**: If the polynomial function f has real coefficients and a complex zero of the form $a+bi$, then the complex conjugate of the zero, $a-bi$, is also a zero of f.

Common Mistakes

- **Misapplying the Rational Zeros Theorem**: Attempting to use the Rational Zeros Theorem when the polynomial coefficients are not integers.
- **Stopping After Finding One Zero**: When a zero is found using the Rational Zeros Theorem and synthetic division, forgetting to use the quotient (depressed polynomial) to find the remaining zeros.
- **Not Accounting for Multiplicity of Zeros**: When factoring or listing zeros, not indicating the multiplicity if a zero is repeated.
- **Forgetting Complex Conjugate Zeros**: If a polynomial has real coefficients and a complex zero $a+bi$ is found or given, forgetting that its conjugate $a-bi$ must also be a zero.
- **Incorrectly Constructing a Polynomial from Zeros**: Forgetting the leading coefficient a when constructing a polynomial from its zeros.

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2.6.2: Homework

Reading Questions

1. What does the Remainder Theorem state about dividing a polynomial $f(x)$ by $x-k$? How can it be used to evaluate $f(k)$?
2. Explain the Factor Theorem. How does it relate the zeros of a polynomial to its factors?
3. What is the Rational Zeros Theorem used for? What conditions must the coefficients of the polynomial satisfy to use this theorem?
4. If $\frac{p}{q}$ is a possible rational zero of $f(x)=a_nx^n+\cdots+a_0$, what do p and q represent?
5. According to the Fundamental Theorem of Algebra, how many complex zeros does a polynomial of degree $n>0$ have (counting multiplicities)?
6. What does the Complete Factorization Theorem say about factoring a polynomial of degree $n>0$?
7. What is the Complex Conjugate Theorem? If $2+3i$ is a zero of a polynomial with real coefficients, what other zero must it have?
8. If you are asked to find a polynomial of a certain degree with given zeros, what is the general approach using its factors? What role does the leading coefficient a play?
9. When using synthetic division to test for zeros from the list generated by the Rational Zeros Theorem, what remainder value indicates that a number is indeed a zero?
10. If you find a zero k of a polynomial $f(x)$, how can you use this information to find other zeros? (Hint: consider the quotient polynomial).
11. Describe a use for the Remainder Theorem.
12. Explain why the Rational Zero Theorem does not guarantee finding zeros of a polynomial function.
13. What is the difference between rational and real zeros?
14. If synthetic division reveals a zero, why should we try that value again as a possible solution?

Homework

Algebraic

For the following exercises, use the Remainder Theorem to find the remainder.

15. $(x^4-9x^2+14)\div(x-2)$
16. $(3x^3-2x^2+x-4)\div(x+3)$
17. $(x^4+5x^3-4x-17)\div(x+1)$
18. $(-3x^2+6x+24)\div(x-4)$
19. $(5x^5-4x^4+3x^3-2x^2+x-1)\div(x+6)$
20. $(x^4-1)\div(x-4)$
21. $(3x^3+4x^2-8x+2)\div(x-3)$
22. $(4x^3+5x^2-2x+7)\div(x+2)$

For the following exercises, use the given factor and the Factor Theorem to find all real zeros for the given polynomial function.

23. $f(x)=2x^3-9x^2+13x-6;\quad x-1$
24. $f(x)=2x^3+x^2-5x+2;\quad x+2$
25. $f(x)=3x^3+x^2-20x+12;\quad x+3$
26. $f(x)=2x^3+3x^2+x+6;\quad x+2$
27. $f(x)=-5x^3+16x^2-9;\quad x-3$
28. $x^3+3x^2+4x+12;\quad x+3$
29. $4x^3-7x+3;\quad x-1$
30. $2x^3+5x^2-12x-30;\quad 2x+5$

For the following exercises, use the Rational Zeros Theorem to find all real zeros.

31. $x^3 - 3x^2 - 10x + 24 = 0$
32. $2x^3 + 7x^2 - 10x - 24 = 0$
33. $x^3 + 2x^2 - 9x - 18 = 0$
34. $x^3 + 5x^2 - 16x - 80 = 0$
35. $x^3 - 3x^2 - 25x + 75 = 0$
36. $2x^3 - 3x^2 - 32x - 15 = 0$
37. $2x^3 + x^2 - 7x - 6 = 0$
38. $2x^3 - 3x^2 - x + 1 = 0$
39. $3x^3 - x^2 - 11x - 6 = 0$
40. $2x^3 - 5x^2 + 9x - 9 = 0$
41. $2x^3 - 3x^2 + 4x + 3 = 0$
42. $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$
43. $x^4 + 2x^3 - 9x^2 - 2x + 8 = 0$
44. $4x^4 + 4x^3 - 25x^2 - x + 6 = 0$
45. $2x^4 - 3x^3 - 15x^2 + 32x - 12 = 0$
46. $x^4 + 2x^3 - 4x^2 - 10x - 5 = 0$
47. $4x^3 - 3x + 1 = 0$
48. $8x^4 + 26x^3 + 39x^2 + 26x + 6$

For the following exercises, find all complex solutions (real and non-real).

49. $x^3 + x^2 + x + 1 = 0$
50. $x^3 - 8x^2 + 25x - 26 = 0$
51. $x^3 + 13x^2 + 57x + 85 = 0$
52. $3x^3 - 4x^2 + 11x + 10 = 0$
53. $x^4 + 2x^3 + 22x^2 + 50x - 75 = 0$
54. $2x^3 - 3x^2 + 32x + 17 = 0$

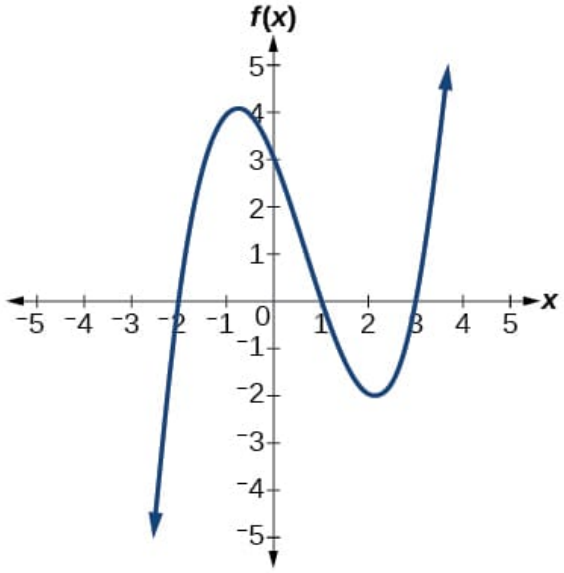
Numeric

For the following exercises, list all possible rational zeros for the functions.

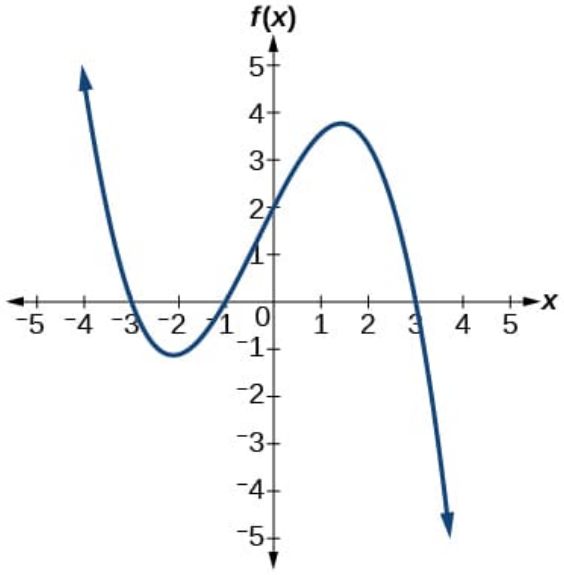
55. $f(x) = x^4 + 3x^3 - 4x + 4$
56. $f(x) = 2x^3 + 3x^2 - 8x + 5$
57. $f(x) = 3x^3 + 5x^2 - 5x + 4$
58. $f(x) = 6x^4 - 10x^2 + 13x + 1$
59. $f(x) = 4x^5 - 10x^4 + 8x^3 + x^2 - 8$

For the following exercises, use the graphs to write the formula for a polynomial function of least degree.

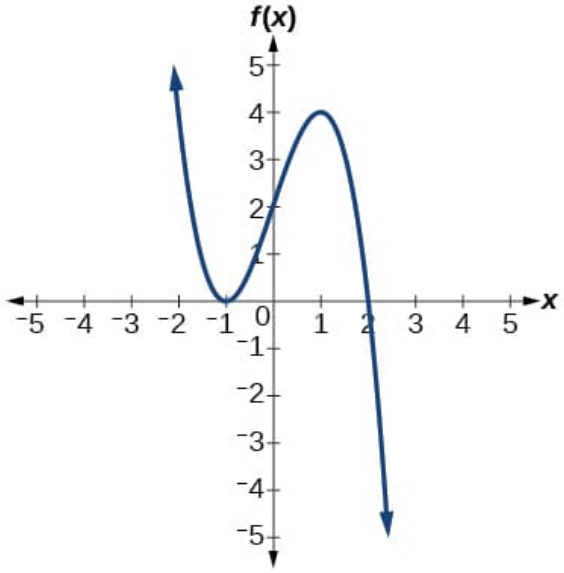
60.



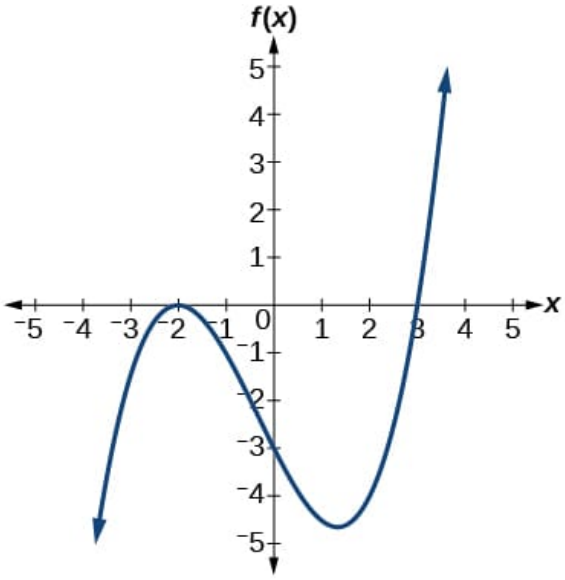
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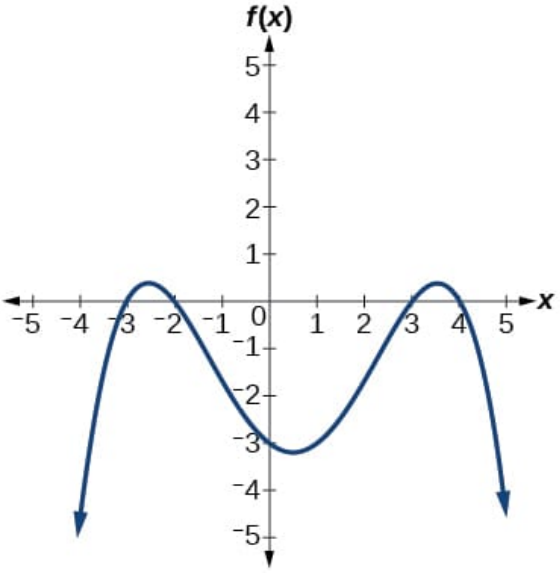
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53.

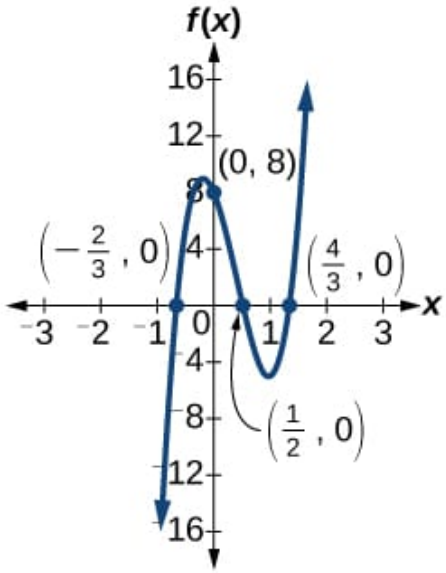


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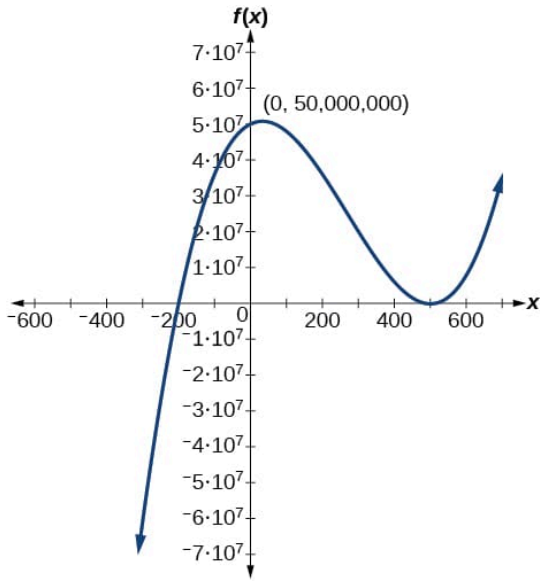


For the following exercises, use the graphs to write a polynomial function of least degree.

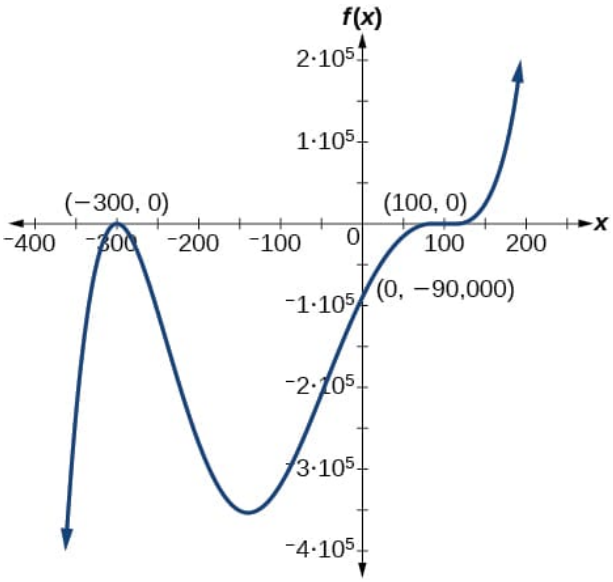
55.



56.



67.



Technology

For the following exercises, use your calculator to graph the polynomial function. Based on the graph, find the rational zeros. All real solutions are rational.

68. $f(x) = 6x^3 - 7x^2 + 1$
69. $f(x) = 4x^3 - 4x^2 - 13x - 5$
70. $f(x) = 8x^3 - 6x^2 - 23x + 6$
71. $f(x) = 12x^4 + 55x^3 + 12x^2 - 117x + 54$
72. $f(x) = 16x^4 - 24x^3 + x^2 - 15x + 25$

Extensions

For the following exercises, use the given information about the polynomial graph to write the equation.

73. Degree 3. Zeros at $x = -2$, $x = 1$, and $x = 3$. y-intercept at $(0, -4)$.
74. Degree 3. Zeros at $x = -5$, $x = -2$, and $x = 1$. y-intercept at $(0, 6)$.
75. Degree 5. Roots of multiplicity 2 at $x = 3$ and $x = 1$, and a root of multiplicity 1 at $x = -3$. y-intercept at $(0, 9)$.
76. Degree 4. Root of multiplicity 2 at $x = 4$, and a roots of multiplicity 1 at $x = 1$ and $x = -2$. y-intercept at $(0, -3)$.
77. Degree 5. Double zero at $x = 1$, and triple zero at $x = 3$. Passes through the point $(2, 15)$.
78. Degree 3. Zeros at $x = 4$, $x = 3$, and $x = 2$. y-intercept at $(0, -24)$.
79. Degree 3. Zeros at $x = -3$, $x = -2$ and $x = 1$. y-intercept at $(0, 12)$.
80. Degree 5. Roots of multiplicity 2 at $x = -3$ and $x = 2$ and a root of multiplicity 1 at $x = -2$. y-intercept at $(0, 4)$.
81. Degree 4. Roots of multiplicity 2 at $x = \frac{1}{2}$ and roots of multiplicity 1 at $x = 6$ and $x = -2$. y-intercept at $(0, 18)$.
82. Double zero at $x = -3$ and triple zero at $x = 0$. Passes through the point $(1, 32)$.

For the following exercises, construct a polynomial function of least degree possible using the given information.

83. Real roots: –1, 1, 3 and $(2, f(2)) = (2, 4)$
84. Real roots: –1, 1 (with multiplicity 2 and 1) and $(2, f(2)) = (2, 4)$
85. Real roots: $-2, \frac{1}{2}$ (with multiplicity 2) and $(-3, f(-3)) = (-3, 5)$
86. Real roots: $-\frac{1}{2}, 0, \frac{1}{2}$ and $(-2, f(-2)) = (-2, 6)$
87. Real roots: –4, –1, 1, 4 and $(-2, f(-2)) = (-2, 10)$

Real-World Applications

For the following exercises, find the dimensions of the box described.

88. The length is twice as long as the width. The height is 2 inches greater than the width. The volume is 192 cubic inches.
89. The length, width, and height are consecutive whole numbers. The volume is 120 cubic inches.
90. The length is one inch more than the width, which is one inch more than the height. The volume is 86.625 cubic inches.
91. The length is three times the height and the height is one inch less than the width. The volume is 108 cubic inches.
92. The length is 3 inches more than the width. The width is 2 inches more than the height. The volume is 120 cubic inches.

For the following exercises, find the dimensions of the right circular cylinder described.

93. The radius is 3 inches more than the height. The volume is 16π cubic inches.
94. The height is one less than one half the radius. The volume is 72π cubic meters.
95. The radius and height differ by one meter. The radius is larger and the volume is 48π cubic meters.
96. The radius and height differ by two meters. The height is greater and the volume is 28.125π cubic meters.
97. The radius is $\frac{1}{3}$ meter greater than the height. The volume is $\frac{98}{9}\pi$ cubic meters.

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2.7: Rational Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Suppose we know that the cost of making a product is dependent on the number of items, x, produced. This is given by the equation $C(x) = 15,000x - 0.1x^2 + 1000$. If we want to know the average cost for producing x items, we would divide the cost function by the number of items, x.

The average cost function, which yields the average cost per item for x items produced, is $f(x) = \frac{15,000x-0.1x^2+1000}{x}$. Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore **rational functions**, which have variables in the denominator.

Focus on Calculus - Using Arrow Notation

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our study of toolkit functions. Let's examine these graphs and notice some of their features.

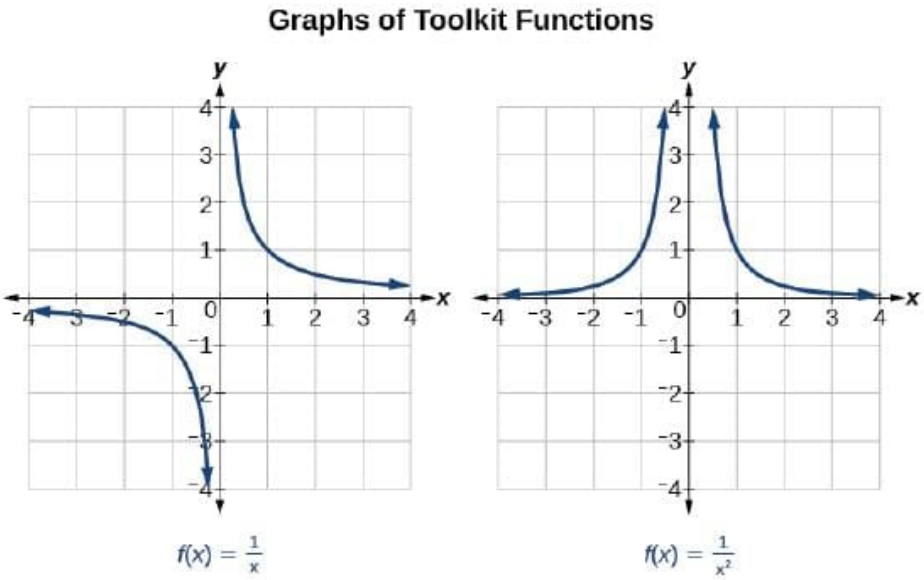


Figure 2.7.1

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

1. On the left branch of the graph, the curve approaches the x-axis ($y = 0$) as $x \to -\infty$.
2. As the graph approaches $x = 0$ from the left ($x \to 0^-$), the curve drops ($f(x) \to -\infty$), but as we approach zero from the right ($x \to 0^+$), the curve rises ($f(x) \to \infty$).
3. Finally, on the right branch of the graph, the curve approaches the x-axis ($y = 0$) as $x \to \infty$.

To summarize, we use arrow notation to show that x or $f(x)$ is approaching a particular value. See Table 2.7.1.

***Table** 2.7.1: Defining Arrow Notation*

Symbol	Meaning
$x \to a^-$	x approaches a from the left (\(x
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table 2.7.2.

***Table** 2.7.2: Numerical Investigation of $f(x) = \frac{1}{x}$ as $x \to 0^-$.*

x	-0.1	-0.01	-0.001	-0.0001
$f(x) = \frac{1}{x}$	-10	-100	-1000	$-10,000$

We would indicate this behavior using the arrow notation: "as $x \to 0^-$, $f(x) \to -\infty$."

On the other hand, as the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in Table 2.7.3.

***Table** 2.7.3: Numerical Investigation of $f(x) = \frac{1}{x}$ as $x \to 0^+$.*

x	0.1	0.01	0.001	0.0001
$f(x) = \frac{1}{x}$	10	100	1000	$10,000$

This is interpreted using the following arrow notation: "as $x \to 0^+$, $f(x) \to \infty$." See Figure 2.7.2.

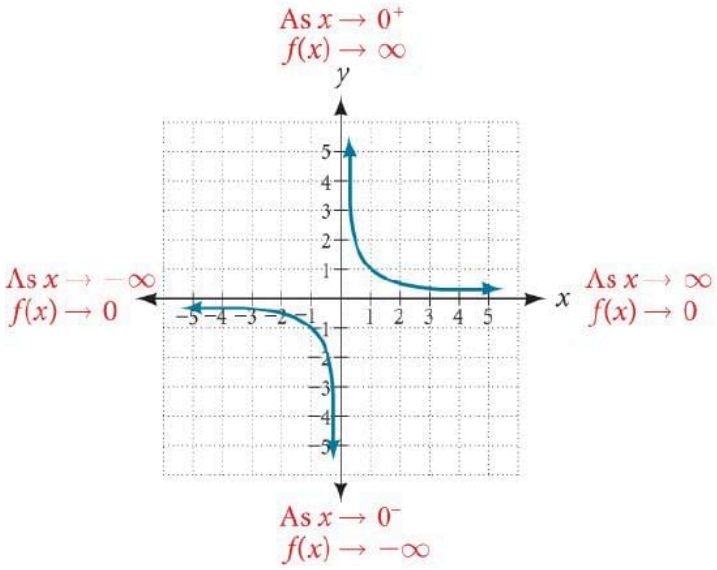


Figure 2.7.2**:** *Graphical Investigation of* $f(x) = \frac{1}{x}$.

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero. See Figure 2.7.3.

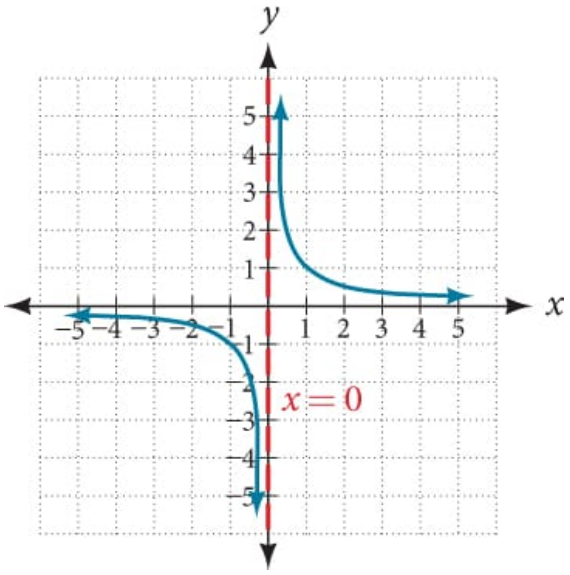


Figure 2.7.3**:** *Showing the vertical asymptote,* $x = 0$*, for the function* $f(x) = \frac{1}{x}$.

Definition: Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of f if the values of $f(x)$ increase or decrease without bound as x approaches a from the left, from the right, or from both sides. We write

$f(x) \to \infty$ as $x \to a^-$ if the graph increases without bound as x approaches a from the left,

$f(x) \to -\infty$ as $x \to a^-$ if the graph decreases without bound as x approaches a from the left,

$f(x) \to \infty$ as $x \to a^+$ if the graph increases without bound as x approaches a from the right,

and

$f(x) \to -\infty$ as $x \to a^+$ if the graph decreases without bound as x approaches a from the right.

End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. Likewise, as the values of x approach negative infinity, the function values approach 0. See Figure $2.7.4$. Symbolically, using arrow notation, we write: "as $x \to \infty$, $f(x) \to 0$," and "as $x \to -\infty$, $f(x) \to 0$."

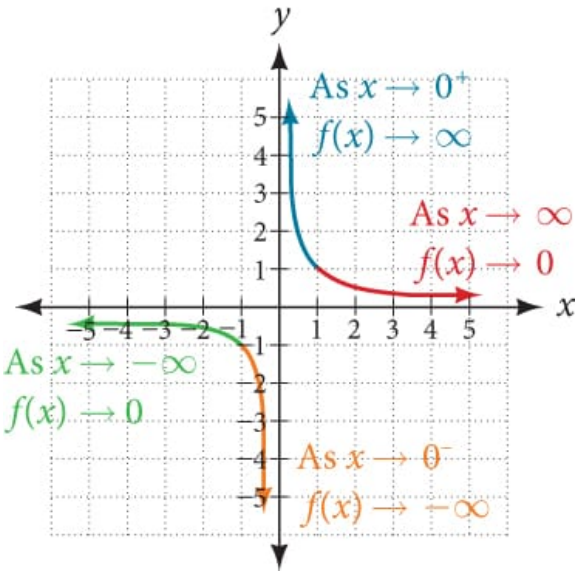


Figure $2.7.4$**:** *Graphical Investigation of* $f(x) = \frac{1}{x}$.

Based on this overall behavior (and the graph), we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure $2.7.5$.

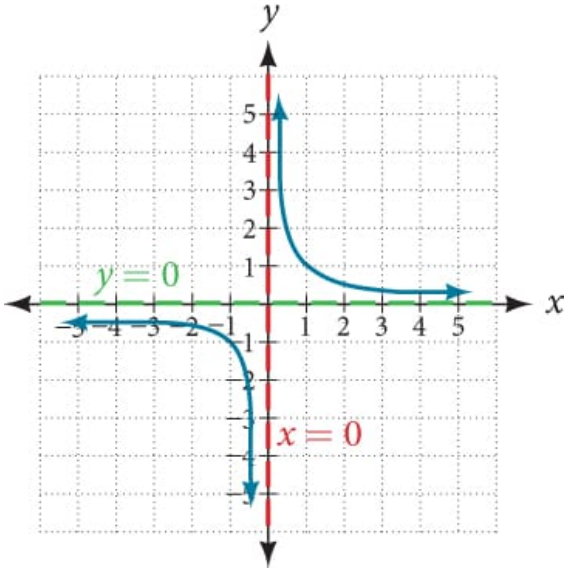


Figure $2.7.5$**:** *Showing the vertical and horizontal asymptotes for the function* $f(x) = \frac{1}{x}$.

Interactive Element: Arrow Notation on $f(x) = \frac{1}{x}$

The following Interactive Element might help you visualize the local and end behavior of the reciprocal function described by arrow notation.

Interact: Drag the orange point along the curve toward $x = 0$ and then far out to the left and right.

Observation: As $x \to 0^-$, $f(x) \to -\infty$ and as $x \to 0^+$, $f(x) \to \infty$; as $x \to \pm\infty$, $f(x) \to 0$.

Definition: Horizontal Asymptote

The line $y = c$ is a **horizontal asymptote** of the graph of f if the values of $f(x)$ approach b as x increases without bound, as x decreases without bound, or both. We write

$$f(x) \to b \quad \text{as} \quad x \to \infty \quad \text{if the graph approaches the line } y = b \text{ as } x \text{ increases without bound}$$

and

$$f(x) \to b \quad \text{as} \quad x \to -\infty \quad \text{if the graph approaches the line } y = b \text{ as } x \text{ decreases without bound.}$$

Example 2.7.1: Using Arrow Notation

Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure 2.7.6.

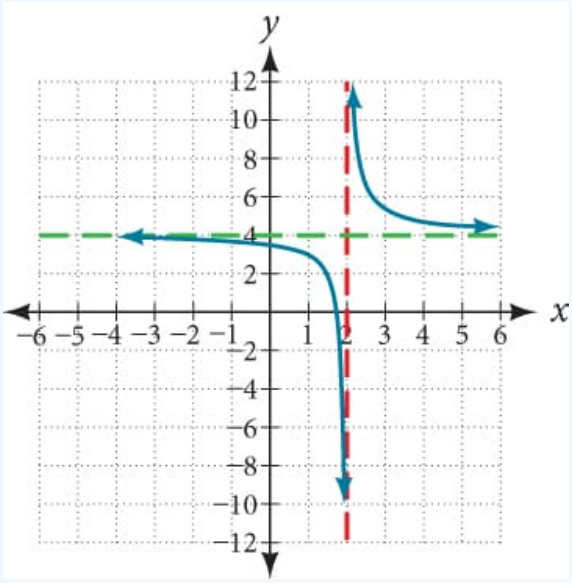


Figure 2.7.6: *A transformation of* $y = \frac{1}{x}$.

Solution available at the end of this section.

Checkpoint 2.7.1

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

Example 2.7.2: Using Transformations to Graph a Rational Function

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution available at the end of this section.

Interactive Element: Translating the Reciprocal Function

The following Interactive Element might help you visualize how the asymptotes of a shifted reciprocal function move along with the curve.

Interact: Drag the purple center point (h, k) around the plane.

Observation: The curve carries its asymptotes with it — the vertical asymptote stays at $x = h$ and the horizontal asymptote stays at $y = k$.

Checkpoint 2.7.2

Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Solving Applied Problems Involving Rational Functions

In Example 2.7.2, we shifted a toolkit function in a way that resulted in the function $f(x) = \frac{3x+7}{x+2}$. This is an example of a rational function. A **rational function** is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

Definition: Rational Function

A **rational function** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0}, \text{ where } Q(x) \neq 0.$$

Example 2.7.3: Solving an Applied Problem Involving a Rational Function

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

Solution available at the end of this section.

In Example 2.7.3, we found the horizontal asymptote by dividing the leading coefficient in the numerator by the leading coefficient in the denominator, which led to $y = \frac{1}{10} = 0.1$. This means the concentration, C, the ratio of pounds of sugar to gallons of water, will approach 0.1 in the long term.

Checkpoint 2.7.3

There are $1,200$ freshmen and $1,500$ sophomores at a pep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

While not a true "application," finding the Difference Quotient (as we have been stating over and over again) is something you will repeatedly need to do in Calculus. Let's revisit that topic right now.

Example 2.7.4: Revisiting the Difference Quotient

Find the Difference Quotient of the following function.

$$R(t) = \frac{2t-1}{3t+7}$$

Solution available at the end of this section.

Caution: Know When Not to Distribute

With difference quotients, you will commonly need to multiply the numerator **and** denominator of an expression by the LCD of all fractions within a compound fraction (a.k.a. simplifying compound fractions). When doing so, knowing what to distribute and what not to distribute is imperative.

Simplifying Compound Fractions

The entire point of multiplying the numerator **and** denominator of your compound fraction by the LCD of all the minor fractions is to "get rid of" denominators in those minor fractions; however, with difference quotients, it is common that the denominator of the entire (major) fraction is only h. In this case, **do not** distribute out the denominator! Doing so will complicate the mathematics, and you will lose visibility of factors that cancel.

Finding the Domains of Rational Functions

A vertical asymptote represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Theorem: Domain of a Rational Function

The domain of a rational function includes all real numbers except those that cause the denominator of the unsimplified rational function to equal zero.

Caution: Domain is Always Considered Prior to Simplifications

A common mistake is for students to simplify or otherwise perform Algebra on an expression and *then* state the domain. In this simplification process, you might "remove" a domain issue that originally existed. Therefore, you need to state the domain *prior* to simplifying the function.

For example, if $f(x) = \frac{x+2}{(x+2)(x+3)}$, you would need to state the domain of the function *prior* to simplifying. In this case, $x = -2$ and $x = -3$ cause division by zero. Therefore, the domain is $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$.

Had you simplified the function first, you would have arrived at

$$f(x) = \frac{x+2}{(x+2)(x+3)} = \frac{1}{x+3},$$

and you would have (incorrectly) stated the domain as $(-\infty, -3) \cup (-3, \infty)$. In truth, $f(x)$ is *not* equal to $\frac{1}{x+3}$ because, if it were, we would be able to evaluate $f(-2)$. However, looking at the original function, evaluating f at $x = -2$ causes division by zero!

Example 2.7.5: Finding the Domain of a Rational Function

Find the domain of $f(x) = \frac{x+3}{x^2-9}$.

Solution available at the end of this section.

A graph of the function from Example 2.7.5 is shown in Figure 2.7.8. It confirms that the function is not defined when $x = \pm 3$.

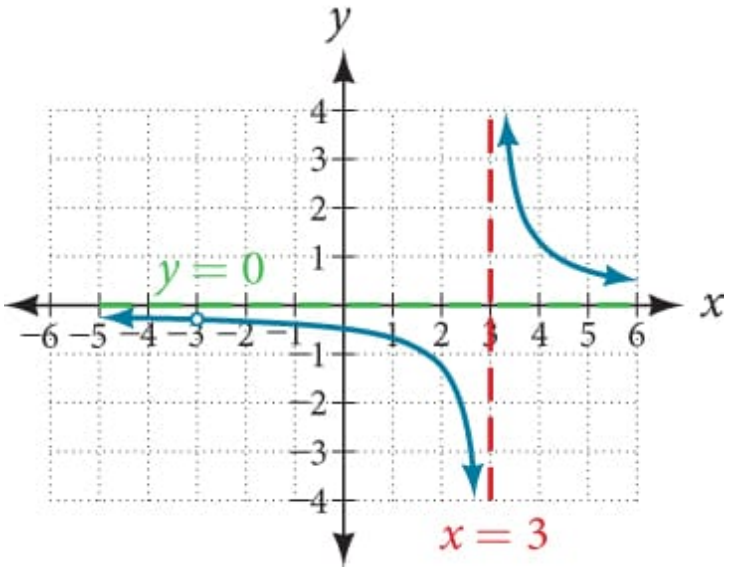


Figure 2.7.8: *Graph of* $f(x) = \frac{x+3}{x^2-9}$.

There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

Checkpoint 2.7.5

Find the domain of $f(x) = \frac{4x}{5(x-1)(x-5)}$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

Example 2.7.6: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \frac{5+2x^2}{2-x-x^2}$.

Solution available at the end of this section.

Removable Discontinuities

Occasionally, a graph will contain a hole.

Definition: Removable Discontinuity

A single point where the graph of a function is not defined, indicated by an open circle on the graph, is called a **removable discontinuity**.

For example, the function $f(x) = \frac{x^2-1}{x^2-2x-3}$ may be rewritten by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}.$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x = -1$, is the location of the removable discontinuity. Notice also that $x-3$ is *not* a factor in both the numerator and denominator. The zero of this factor, $x = 3$, is the vertical asymptote. See Figure 2.7.10.

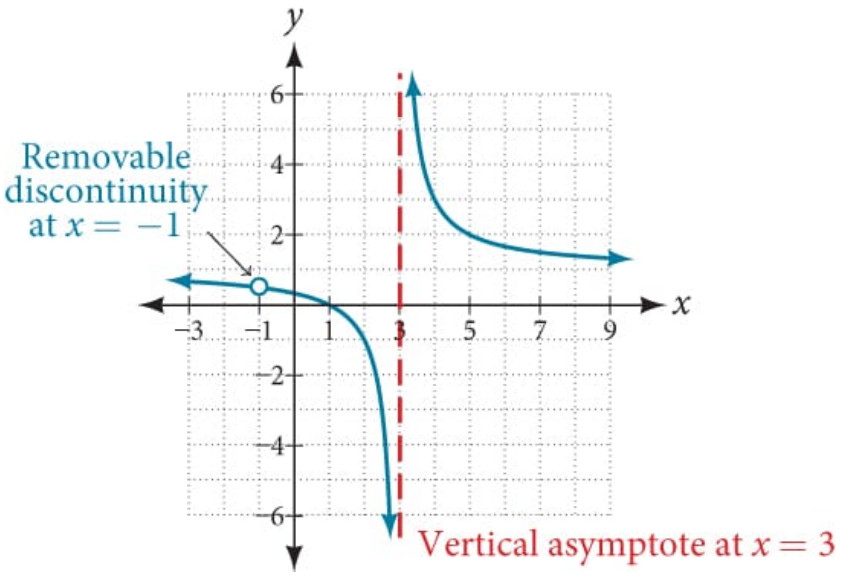


Figure 2.7.10: *Showcasing the removable discontinuity for the graph of the function* $f(x) = \frac{x^2-1}{x^2-2x-3}$.

Theorem: Removable Discontinuity of a Rational Function

A removable discontinuity for the graph of a rational function occurs at $x = a$ if

1. a is a zero for a factor in the denominator that is common with a factor in the numerator, and
2. after simplification, a is no longer a zero of the denominator.

Proof available at the end of this section.

A clean way to summarize this theorem is to first factor the numerator and denominator and check for common factors. If we find

any, we set the common factor equal to 0 and solve. This x-value leads to a removable discontinuity if the multiplicity of the factor in the numerator is greater than or equal to that in the denominator. If the multiplicity of the factor is greater in the denominator, then there is still an asymptote at that value.

Example 2.7.7: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \frac{x-2}{x^2-4}$.

Solution available at the end of this section.

Checkpoint 2.7.7

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \frac{x^2-25}{x^3-6x^2+5x}$.

Identifying End Behavior of Rational Functions

As we have discussed, horizontal asymptotes help describe the behavior of a graph as the input gets infinitely large (either positive or negative); however, it is possible that the graph of a function doesn't tend to a horizontal line as $x \to \pm\infty$. In fact, there are an infinite number of cases in which graphs tend to non-horizontal lines or, in more advanced cases, other higher-degreed functions. Therefore, rather than speaking of identifying "horizontal asymptotes" of graphs, it's best to keep things general and speak of the **end behavior** of a graph.

Definition: End Behavior

The **end behavior** of a graph is a description, usually as a function, of what the function values of the graph tend to approach as the inputs increase or decrease without bound.

For example, the function $f(x) = 3x + 1 + \frac{1}{x}$ is the solid curve graphed in Figure 2.7.12.

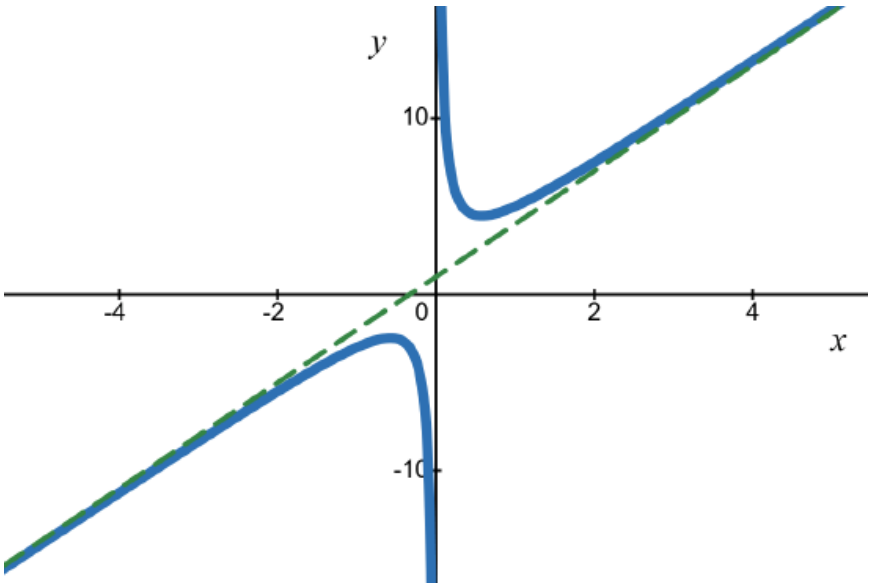


Figure 2.7.12: *Graph of $f(x) = 3x + 1 + \frac{1}{x}$ showcasing a slant asymptote.*

As $x \to \pm\infty$, the graph quickly begins to "mimic" the dashed line, which happens to have equation $y = 3x + 1$. Why does this happen? The third term of the $f(x)$, which is $\frac{1}{x}$, tends to 0 (it tends to disappear) as $x \to \pm\infty$. This forces $f(x)$ to eventually "mimic" what *doesn't* disappear (specifically, $y = 3x + 1$). Hence, we would say that the end behavior of $f(x) = 3x + 1 + \frac{1}{x}$ is $y = 3x$

$+1$. This type of end behavior is called a **slant asymptote** because it is a line.

With this stated, we can see that a horizontal asymptote is a *type* of end behavior, but it is not the *only* type. Our goal at this point is to discover the types of end behaviors a rational function might have.

Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. There are three distinct outcomes when checking for the end behavior of a rational function; however, let's define a new notation that will simplify some of our discussion.

Definition: Polynomial Degree Notation

The degree of a polynomial $f(x)$ is denoted $\deg(f(x))$.

For example, $\deg(1-x^{17}+x)=17$.

With that definition, we are ready to explore the end behavior possibilities of rational functions. We begin by letting $f(x)=\frac{N(x)}{D(x)}$, where N and D are polynomials and $D(x)\neq 0$.

Case 1 ($\deg(D(x))>\deg(N(x))$**)**

Consider the function $f(x)=\frac{4x+2}{x^2+4x-5}$. A little Algebra goes a long way:

$$\begin{aligned}\frac{4x+2}{x^2+4x-5} &= \frac{\frac{1}{x^2}}{\frac{1}{x^2}}\cdot\frac{4x+2}{x^2+4x-5} && \left(\text{multiplying numerator and denominator by } \tfrac{1}{x^2}\right)\\ &= \frac{\frac{4}{x}+\frac{2}{x^2}}{1+\frac{4}{x}-\frac{5}{x^2}} && (\text{distributing})\end{aligned}$$

As $x\to\pm\infty$, all expressions of the form $\frac{\text{constant}}{x}$ and $\frac{\text{constant}}{x^2}$ will tend to 0. Thus, as $x\to\pm\infty$, $f(x)\to\frac{0+0}{1+0+0}=\frac{0}{1}=0$. That is, $y=0$ is a horizontal asymptote for $f(x)$. See Figure 2.7.13. Note that this graph crosses the horizontal asymptote.

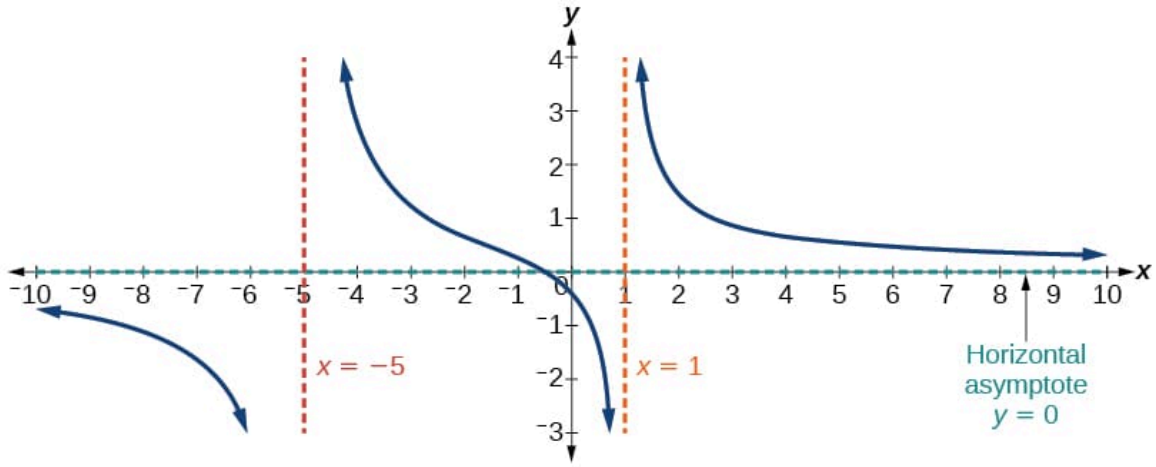


Figure 2.7.13**:** *Horizontal Asymptote* $y=0$ *when* $f(x)=\frac{N(x)}{D(x)}$*, where* $D(x)\neq 0$ *and* $\deg(N(x))<\deg(D(x))$.

Case 2 $\deg(D(x))<\deg(N(x))$ **(by** 1**)**

In this textbook, we will only consider the possibility of the numerator being one degree more than the denominator. Your instructor might spend a few minutes of class time to discuss what happens when the difference of degrees between numerator and denominator is greater than 1, but that is not a major topic in Calculus.

Consider the function $f(x)=\frac{3x^2-2x+1}{x-1}$. Since we can think of this as an improper fraction (where the numerator might be somewhat divisible by the denominator), we perform synthetic division to arrive at

$$\frac{3x^2-2x+1}{x-1} = 3x+1+\frac{2}{x-1}.$$

As $x \to \pm\infty$, the expression $\frac{2}{x-1} \to 0$. Therefore, our function will "mimic" $y = 3x+1$. That is, $f(x) \to 3x+1$ as $x \to \pm\infty$. Thus, the line $y = 3x+1$ is a slant asymptote of $f(x)$ (see Figure 2.7.14).

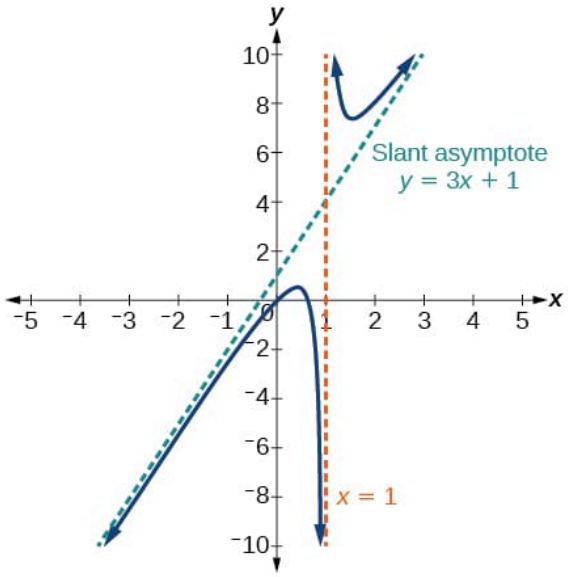


Figure 2.7.14: *Slant Asymptote when* $f(x) = \frac{N(x)}{D(x)}$, *where* $D(x) \neq 0$ *and* $\deg(N(x)) > \deg(D(x))$ *by* 1.

Case 3 $\deg(D(x)) = \deg(N(x))$

Consider the function $f(x) = \frac{3x^2+2}{x^2+4x-5}$. Performing some Algebra, we get

$$\begin{aligned} \frac{3x^2+2}{x^2+4x-5} &= \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \cdot \frac{3x^2+2}{x^2+4x-5} && \left(\text{multiplying numerator and denominator by } \tfrac{1}{x^2}\right) \\ &= \frac{3+\frac{2}{x^2}}{1+\frac{4}{x}-\frac{5}{x^2}} && (\text{distributing}) \end{aligned}$$

As $x \to \pm\infty$, all expressions of the form $\frac{\text{constant}}{x}$ and $\frac{\text{constant}}{x^2}$ will tend to 0. Thus, as $x \to \pm\infty$, $f(x) \to \frac{3+0}{1+0-0} = \frac{3}{1} = 3$. That is, $y = 3$ is a horizontal asymptote for $f(x)$. See Figure 2.7.15. Again, note that this graph crosses the horizontal asymptote.

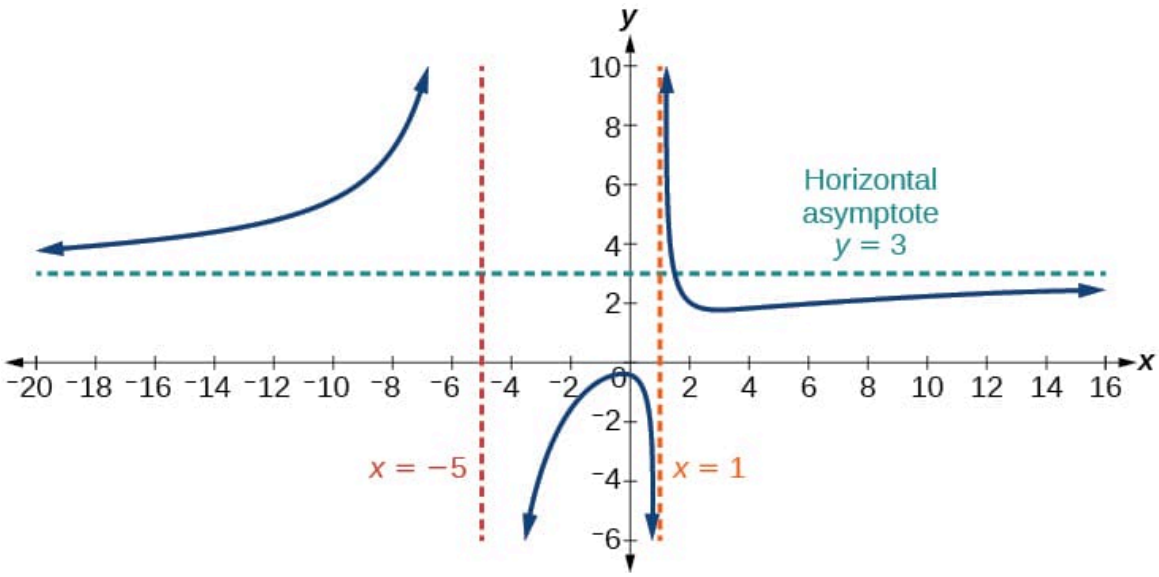


Figure 2.7.15**:** *Horizontal Asymptote when* $f(x) = \frac{N(x)}{D(x)}$*, where* $D(x) \neq 0$ *and* $\deg(N(x)) = \deg(D(x))$.

Notice that, while the graph of a rational function will never cross a *vertical* asymptote, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph of a *rational function* will have at most one horizontal (or slant) asymptote. As an aside, there exist *non-rational* functions that have two horizontal or slant asymptotes (and there are some that have both a horizontal *and* a slant asymptote); however, we are only focusing our attention on rational functions for this discussion.

Theorem: End Behavior of a Rational Function

Let $f(x) = \frac{N(x)}{D(x)}$ be a rational function. Then

- $f(x)$ has a horizontal asymptote of $y = 0$ if $\deg(D(x)) > \deg(N(x))$
- $f(x)$ has a horizontal asymptote of $y = \frac{a}{b}$ if $\deg(D(x)) = \deg(N(x))$, where a is the lead coefficient of $N(x)$ and b is the lead coefficient of $D(x)$
- $f(x)$ has a slant or oblique asymptote if $\deg(D(x)) < \deg(N(x))$. The asymptote is a slant asymptote if $\deg(N(x)) = \deg(D(x)) + 1$; otherwise, it is an oblique asymptote. In either case, the equation of the asymptote is the quotient after dividing $N(x)$ by $D(x)$.

Example 2.7.8: Identifying Horizontal and Slant Asymptotes

For the functions below, identify the horizontal or slant asymptote.

a. $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$
b. $h(x) = \frac{x^2 - 4x + 1}{x + 2}$
c. $k(x) = \frac{x^2 + 4x}{x^3 - 8}$

Solution available at the end of this section.

Example 2.7.9: Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \frac{5 + t}{100 + 10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

Solution available at the end of this section.

Example 2.7.10: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

Solution available at the end of this section.

Checkpoint 2.7.10

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Like all functions, a rational function will have a y-intercept when the input is zero, if the function is defined at zero. A rational function will not have a y-intercept if the function is not defined at zero. Likewise, a rational function will have x-intercepts at the inputs that cause the output to be zero; however, since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

Example 2.7.11: Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution available at the end of this section.

Checkpoint 2.7.11

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

Graphing Rational Functions

In Example 2.7.10, we see that the numerator of a rational function reveals the x-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure 2.7.18.

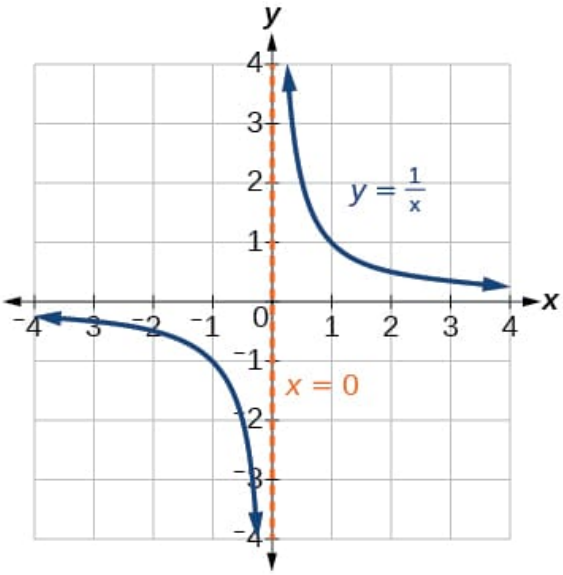


Figure 2.7.18**:** *Graph of the toolkit function* $y = \frac{1}{x}$.

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure 2.7.19.

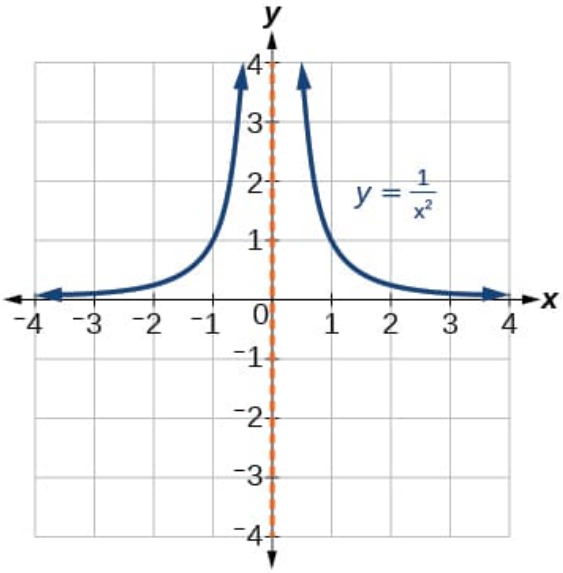


Figure 2.7.19**:** *Graph of the toolkit function* $y = \frac{1}{x^2}$.

For example, the graph of $f(x) = \frac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure 2.7.20.

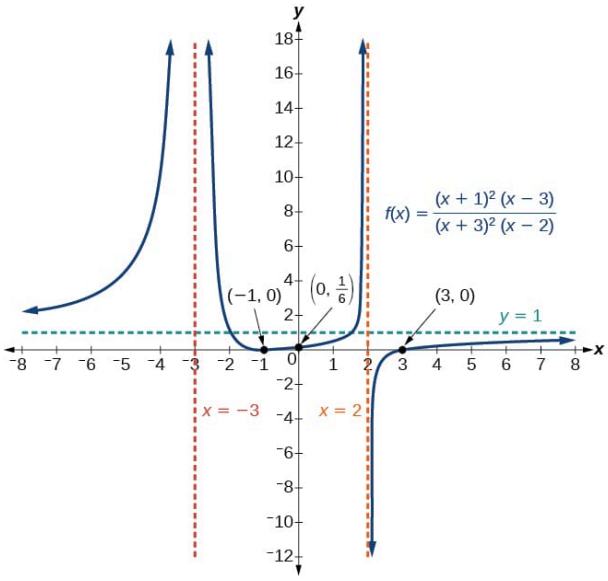


Figure 2.7.20**:** *Graph of* $f(x) = \frac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$.

- At the x-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph bounces (like a parabola), consistent with the quadratic nature of the factor.
- At the x-intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis (like a line) as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \frac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of $f(x) = \frac{1}{x}$.

How To: Graph a rational function

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Example 2.7.12: Graphing a Rational Function

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution available at the end of this section.

Checkpoint 2.7.12

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an x-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of x-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

Writing Rational Functions from Intercepts and Asymptotes

If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and $x_i \neq v_j$ for all i and j, then the function can be written in the form:

$$f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_m}},$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the x-intercept or by the horizontal asymptote if it is nonzero.

Example 2.7.13: Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in Figure 2.7.23.

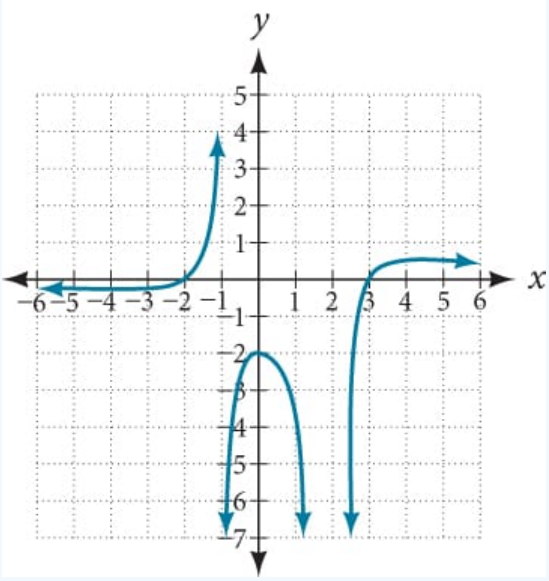


Figure 2.7.23

Solution available at the end of this section.

Revisiting Solving Inequalities

As we did with polynomials, we can solve inequalities involving rational expressions using our graphing skills. Again, many instructors would have you solve rational inequalities using sign charts; however, I find solving via graphing (1) is faster, (2) leads to a more natural understanding of what the solution to a rational inequality means, and (3) reinforces graphing skills. Let's see how this is done via an example.

Example 2.7.14: Solving a Rational Inequality

Find all solutions to the inequality.

$$\frac{6x^2+14x-34}{x+4} > 5x-9$$

Solution available at the end of this section.

Success in College: Cheating and Consequences

The following is an excerpt from an article at WorldNetDaily.com by Dennis Prager.

> A report by Donald McCabe, professor of organization management at Rutgers University and founding president of the Center for Academic Integrity at Duke University, describes a pretty bad ethical situation in America's high schools. Seventy-five percent of students engage in "serious cheating" (such as repeated cheating and significant plagiarism).
>
> It gets worse.
>
> First, according to the professor, this epidemic is the culmination of three decades of increased cheating in high schools.
>
> Second, about half of all students (not just half of all cheaters) see nothing wrong with cheating.
>
> As one who has written and lectured on ethics for 30 years, during which time I, too, have spoken at high schools across America, I am certain that these statistics are accurate. I also believe that I know the major reasons.
>
> One reason is that parents emphasize many things over character.
>
> A second reason is the absurd emphasis on getting into the best college. This preoccupation on the part of parents (which often begins in preschool) has utterly distorted our children's (and high schools') priorities. Because students are taught that getting into the right college is the most important thing in their lives, they readily compromise their character to achieve this goal. And they could not care less about learning. Since the purpose of school is to get good grades, learning is incidental, and loving learning is a non-issue.
>
> A third and final reason is the death of the concept of sin, a product of the thoroughly secular education American high schools offer. As noted, far worse than the number of students who cheat is the number of students who think that there is nothing wrong with cheating. I can testify from my talks at high schools that the vast majority of students believe what they have been taught - that right and wrong - are entirely subjective concepts. There is no morality higher than one's own; therefore, right and wrong are defined by the individual, who can rationalize any behavior.

When students do not understand a concept, they need help. Instead, some students in this situation choose to cheat. In college, any form of cheating is "academic dishonesty."

1. Has your professor discussed or posted their cheating policy?
2. What happens if you cheat in this math class?

> To be clear, cheating - in any form - results in more than just a "zero" on that assignment. This includes, but is not limited to, using notes on exams, using a device that is not allowed during an exam, or using unapproved technology to help during an exam. If I suspect a student of cheating, I will have them meet me outside of class time in my office

to discuss what happened and why it is considered cheating. They will receive a zero on the assignment/exam and will be reported to the Student Discipline Officer for further discussions. The Student Discipline Officer is involved for two reasons: to give the student tools to help them break the "cheating habit," and to create an official record within the Los Rios Community College District to track that student's cheating offenses - if multiple offenses have occurred, the student could like be placed on academic probation or, in severe cases, completely expelled from the District.

- Professor Simpson, Cosumnes River College

Success in College: Getting Help

Learning math can be a struggle. Sometimes your professor seems to be talking in a foreign language. Other students look like they get it. The professor keeps going. The stuff you don't understand keeps piling up.

Struggling in a college math class is expected. You must learn to work through your confusion. Your professor expects you to ask questions in class. If needed, you must also get help outside of class. Visit your professor during office hours. Ask if appointments are available outside of office hours. Use the math tutors on campus. It is your responsibility to get the help you need. You must work, ask questions, practice, and think until you get it. You have to take the time to get extra help.

Students who let frustration get to them fail. They put off getting help. They sometimes skip class. They forget that college is different than many high schools. In college, you can fail. If you do not learn enough, you cannot go on.

When successful students are confused, they get help. They work until they understand. You should expect to be confused at times. It would be best if you planned on needing extra time and energy to deal with it.

Here are some questions that may help you succeed in this class (and possibly many other classes as well):

1. Where is your professor's office located?
2. Your professor might keep regular office hours when they are available for help. When are your professor's office hours?

If you are unsure about the answers to either of these questions, feel free to ask your professor - that's what they are there for!

Answers

Section Preview

This section covers rational functions and their graphs. You should use the Calculus "arrow" notation as often as possible so that it is not foreign to students once they get into Calculus. Solving rational inequalities by using graphs is also covered (and is a great reinforcement of the need to graph).

Section Preview

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the Difference Quotient of rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.
- Solve a rational inequality by graphing.

Example $2.7.1$: Using Arrow Notation

Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$. We can see that as $x \to 2^-$, $f(x) \to -\infty$, and as $x \to 2^+$, $f(x) \to \infty$. Moreover, as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y = 4$. Likewise, as the inputs increase without bound, the graph levels off at 4. That is, $f(x) \to 4$ as $x \to \pm\infty$.

Example $2.7.2$: Using Transformations to Graph a Rational Function

Shifting the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3$$

or equivalently, by giving the terms a common denominator,

$$f(x) = \frac{3x+7}{x+2}.$$

The graph of the shifted function is displayed in Figure $2.7.7$.

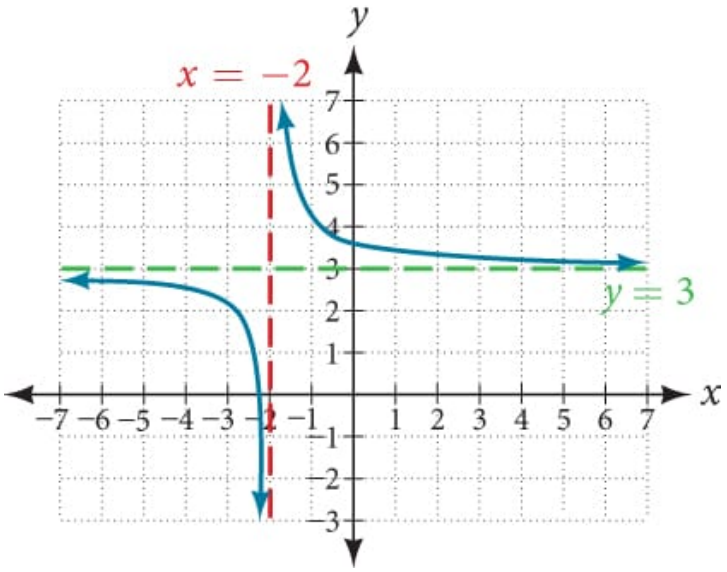


Figure 2.7.7**:** *The graph of* $f(x)=\frac{1}{x+2}+3$.

As $x \to -2^-$, $f(x) \to -\infty$. Therefore, the graph has a vertical asymptote at $x=-2$. We could have arrived at this same conclusion by considering that $f(x) \to \infty$ as $x \to -2^+$.

As $x \to \pm\infty$, $f(x) \to 3$. Thus, the graph has a horizontal asymptote at $y=3$.

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

Example 2.7.3**: Solving an Applied Problem Involving a Rational Function**

Let t be the number of minutes since the tap opened. Since the water flows in at 10 gallons per minute, and the sugar is poured in at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\text{water:} \quad W(t)=100+10t \text{ in gallons}$$

$$\text{sugar:} \quad S(t)=5+1t \text{ in pounds}$$

The concentration, C, will be the ratio of pounds of sugar to gallons of water

$$C(t)=\frac{5+t}{100+10t}.$$

The concentration after 12 minutes is given by evaluating $C(t)$ at $t=12$.

$$C(12)=\frac{5+12}{100+10(12)}=\frac{17}{220}.$$

This means the concentration is 17 pounds of sugar to 220 gallons of water.

At the beginning, the concentration is

$$C(0)=\frac{5+0}{100+10(0)}=\frac{1}{20}.$$

Since $\frac{17}{220} \approx 0.08 > \frac{1}{20}=0.05$, the concentration is greater after 12 minutes than at the beginning.

Example 2.7.4**: Revisiting the Difference Quotient**

$$
\begin{aligned}
\frac{R(t+h)-R(t)}{h} &= \frac{\frac{2(t+h)-1}{3(t+h)+7}-\frac{2t-1}{3t+7}}{h} && \text{(substituting)}\\
&= \frac{\frac{2t+2h-1}{3t+3h+7}-\frac{2t-1}{3t+7}}{h} && \text{(distributing)}\\
&= \frac{\frac{2t+2h-1}{3t+3h+7}-\frac{2t-1}{3t+7}}{h}\cdot\frac{(3t+3h+7)(3t+7)}{(3t+3h+7)(3t+7)} && \text{(simplifying the compound rational expression)}\\
&= \frac{(2t+2h-1)(3t+7)-(2t-1)(3t+3h+7)}{h(3t+3h+7)(3t+7)} && \text{(distributing)}\\
&= \frac{6t^2+14t+6ht+14h-3t-7-(6t^2+6ht+14t-3t-3h-7)}{h(3t+3h+7)(3t+7)} && \text{(distributing (numerator only))}\\
&= \frac{6t^2+14t+6ht+14h-3t-7-6t^2-6ht-14t+3t+3h+7}{h(3t+3h+7)(3t+7)} && \text{(distributing the negative (numerator only))}\\
&= \frac{17h}{h(3t+3h+7)(3t+7)} && \text{(combining like terms)}\\
&= \frac{17\cancel{h}^{1}}{\cancel{h}^{1}(3t+3h+7)(3t+7)} && \text{(canceling like factors)}\\
&= \frac{17}{(3t+3h+7)(3t+7)} && \text{(simplifying)}
\end{aligned}
$$

Example 2.7.5**: Finding the Domain of a Rational Function**

Begin by setting the denominator equal to zero and solving.

$$
\begin{aligned}
x^2-9 &= 0\\
\Rightarrow\quad x^2 &= 9\\
\Rightarrow\quad x &= \pm 3
\end{aligned}
$$

The denominator is equal to zero when $x=\pm 3$. The domain of the function is all real numbers except $x=\pm 3$. Writing this in interval notation, we say the domain of f is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$.

Example 2.7.6**: Identifying Vertical Asymptotes**

First, factor the numerator and denominator.

$$
\begin{aligned}
k(x) &= \frac{5+2x^2}{2-x-x^2}\\
&= \frac{5+2x^2}{(2+x)(1-x)}
\end{aligned}
$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$
\begin{aligned}
(2+x)(1-x) &= 0\\
\Rightarrow\quad x &= -2, 1
\end{aligned}
$$

Neither $x=-2$ nor $x=1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 2.7.9 confirms the location of the two vertical asymptotes.

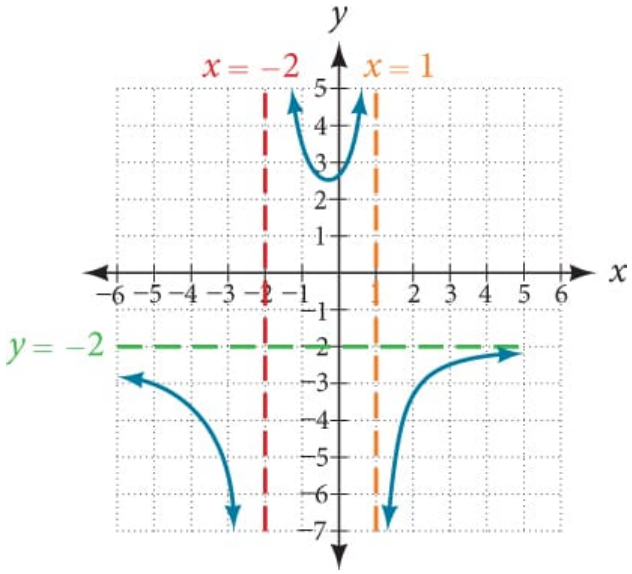


Figure 2.7.9**:** *Graph of* $k(x)=\frac{5+2x^2}{2-x-x^2}$.

Theorem: Removable Discontinuity of a Rational Function

The most elegant proof of this theorem occurs in Calculus.

Example 2.7.7: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)}.$$

Notice that there is a common factor in the numerator and the denominator, $x-2$. The zero for this factor is $x=2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x+2$. The zero for this factor is $x=-2$. Therefore, the vertical asymptote is $x=-2$. See Figure 2.7.11.

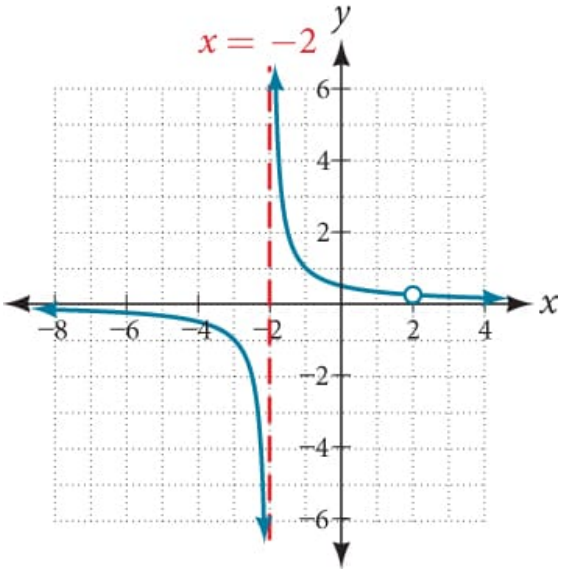


Figure 2.7.11: *Graph of $k(x) = \frac{x-2}{x^2-4}$ showing the vertical asymptote and removable discontinuity.*

The graph of this function will have the vertical asymptote at $x=-2$, but at $x=2$ the graph will have a hole.

Example 2.7.8: Identifying Horizontal and Slant Asymptotes

For these solutions, we will let $p(x)$ represent the numerator and $q(x)$ the denominator.

a. The degree of p is equal to the degree of q (3) so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y=\frac{6}{2}$ or $y=3$.

b. The degree of p is greater than the degree of q by 1. Therefore, this rational function has a slant asymptote. To find the equation of the slant asymptote, we divide $\frac{x^2-4x+1}{x+2}$. This is easily done using synthetic division.

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

The quotient is $x-6$ and the remainder is 13. Thus, the equation of the slant asymptote is $y=x-6$.

c. The degree of p is less than the degree of q, so there is a horizontal asymptote at $y=0$.

Example 2.7.9: Identifying Horizontal Asymptotes

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty \implies C(t) \to \frac{1}{10}.$$

This function will have a horizontal asymptote at $y=\frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Example 2.7.10: Identifying Horizontal and Vertical Asymptotes

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x=1$, -2, and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y=0$. See Figure 2.7.16.

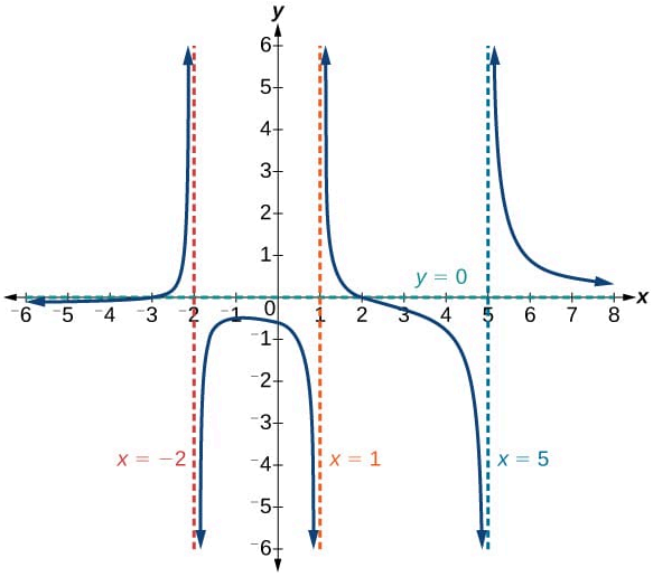


Figure 2.7.16**:** *Graph of* $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Example 2.7.11**: Finding the Intercepts of a Rational Function**

We can find the y-intercept by evaluating the function at zero

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} = \frac{-6}{10} = -\frac{3}{5} = -0.6.$$

The x-intercepts will occur when the function is equal to zero:

$$\begin{aligned} 0 &= \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \\ \implies \quad 0 &= (x-2)(x+3) \qquad \text{(a fraction is zero only when the numerator is zero)} \end{aligned}$$

This happens when $x = 2$ or when $x = -3$. Therefore, the y-intercept is $(0, -0.6)$, and the x-intercepts are $(2, 0)$ and $(-3, 0)$. See Figure 2.7.17.

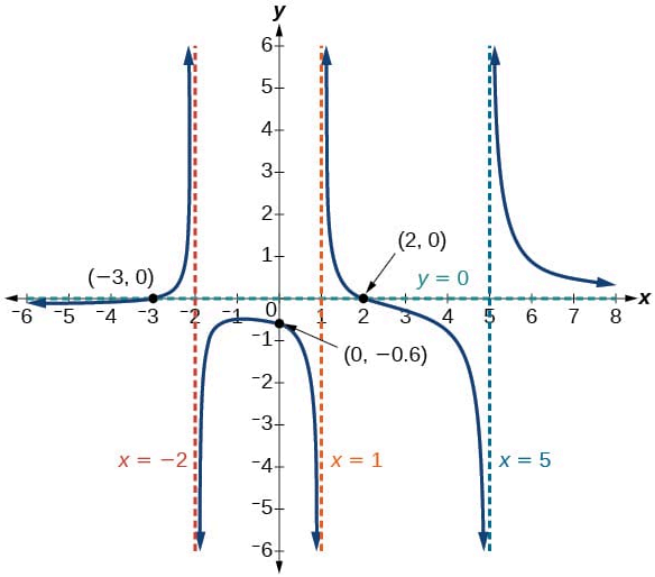


Figure 2.7.17**:** *Graph of* $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Example 2.7.12**: Graphing a Rational Function**

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the y-intercept:

$$f(0) = \frac{(0+2)(0-3)}{(0+1)^2(0-2)} = 3.$$

To find the x-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find x-intercepts at $x = -2$ and $x = 3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a y-intercept at $(0, 3)$ and x-intercepts at $(-2, 0)$ and $(3, 0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x + 1 = 0$ and when $x - 2 = 0$, giving us vertical asymptotes

at $x = -1$ and $x = 2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no x-intercepts between the vertical asymptotes, and the y-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure 2.7.21.

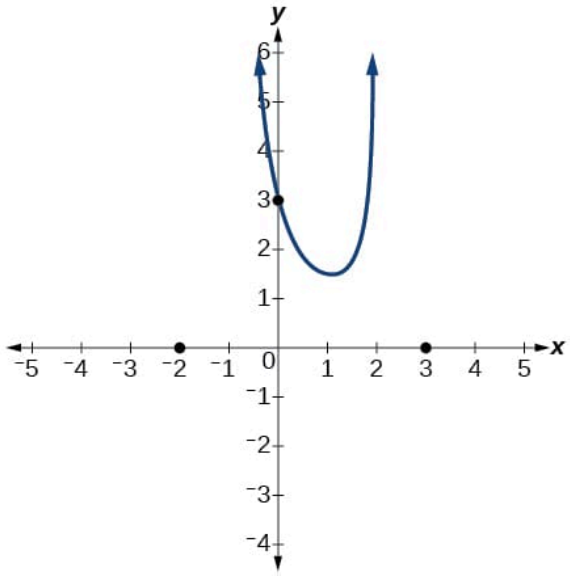


Figure 2.7.21: *An unfinished graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$ (the middle portion).*

The factor associated with the vertical asymptote at $x = -1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x = 2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure 2.7.22. After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

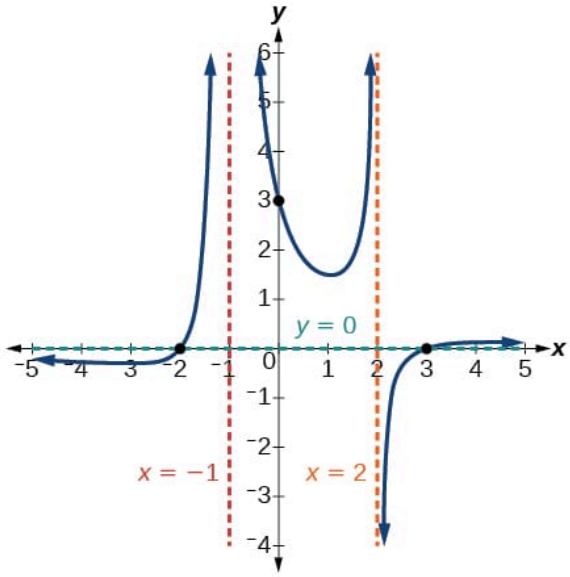


Figure 2.7.22: *Graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.*

Example 2.7.13: Writing a Rational Function from Intercepts and Asymptotes

The graph appears to have x-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See Figure 2.7.24.

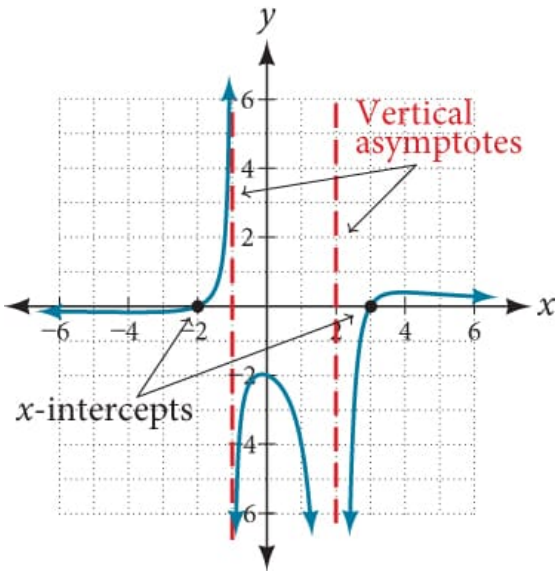


Figure 2.7.24**:** *The original graph with asymptotes drawn in for guidance.*

We can use this information to write a function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}.$$

To find the stretch factor, we can use another clear point on the graph, such as the y-intercept $(0,-2)$.

$$\begin{aligned} -2 &= a\frac{(0+2)(0-3)}{(0+1)(0-2)^2} \\ \Rightarrow \quad -2 &= a\frac{-6}{4} \\ \Rightarrow \quad a &= \frac{-8}{-6} \\ \Rightarrow \quad a &= \frac{4}{3} \end{aligned}$$

This gives us a final function of

$$f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}.$$

Example 2.7.14**: Solving a Rational Inequality**

We begin as we did when solving polynomial inequalities—we move all terms to one side and simplify as much as possible.

$$\begin{aligned} & \frac{6x^2+14x-34}{x+4} > 5x-9 \\ \Rightarrow \quad & \frac{6x^2+14x-34}{x+4} - (5x-9) > 0 && \text{(subtracting } 5x-9 \text{ from both sides)} \\ \Rightarrow \quad & \frac{6x^2+14x-34}{x+4} - \frac{(5x-9)(x+4)}{x+4} > 0 && \text{(rewriting with a common denominator)} \\ \Rightarrow \quad & \frac{6x^2+14x-34}{x+4} - \frac{5x^2+11x-36}{x+4} > 0 && \text{(distributing)} \\ \Rightarrow \quad & \frac{6x^2+14x-34-(5x^2+11x-36)}{x+4} > 0 && \text{(combining fractions)} \\ \Rightarrow \quad & \frac{6x^2+14x-34-5x^2-11x+36}{x+4} > 0 && \text{(distributing)} \\ \Rightarrow \quad & \frac{x^2+3x+2}{x+4} > 0 && \text{(combining like terms)} \\ \Rightarrow \quad & \frac{(x+2)(x+1)}{x+4} > 0 && \text{(factoring the trinomial)} \end{aligned}$$

It's important to note that you would have done all of this work if you were going to solve the inequality using sign charts or graphing—there is no difference in "grunt work" up to this point.

As we did with polynomial inequalities, we graph the expression on the left side. That is, we want to sketch $f(x) = \frac{(x+2)(x+1)}{x+4}$. Our goal is to identify when the graph of f is *above* the x-axis (because we are solving $f(x) > 0$).

- **Horizontal Intercepts:** The numerator of f is zero at $x = -2$ and $x = -1$. Therefore, the graph has x-intercepts at $(-2,0)$ and $(-1,0)$. Both of these zeros have multiplicity 1, so the graph will pass through each like a line.
- **Vertical Intercept:** $f(0) = \frac{2}{4} = \frac{1}{2}$, so the y-intercept is $\left(0, \frac{1}{2}\right)$.

- **Vertical Asymptote:** The denominator is zero when $x = -4$, which is not a zero shared with the numerator. Therefore, f has a vertical asymptote at $x = -4$. Additionally, the degree of the factor, $x + 4$, in the denominator is 1, so the vertical asymptote shares the behavior of $\frac{1}{x}$.
- **Horizontal/Slant Asymptote:** Since the numerator of f has degree 2 and the denominator has degree 1, there will be a slant asymptote. To find the equation of the slant asymptote, it's best to use the unfactored form, $f(x) = \frac{x^2+3x+2}{x+4}$, and divide.

$$\begin{array}{r|rrr} -4 & 1 & 3 & 2 \\ & & -4 & 4 \\ \hline & 1 & -1 & 6 \end{array}$$

Hence, the quotient is $x - 1$ and the remainder is 6. Therefore, the equation of the slant asymptote is $y = x - 1$.

We now have enough information to sketch the graph. Starting at the y-intercept (which is slightly above the x-axis), we have a choice, start sketching to the left or right. For no better reason than random choice, I will choose to sketch to the right. Does the graph rise or fall? If we started drawing to the right and had the graph fall, it would eventually intersect the x-axis; however, we do not have any x-intercepts greater than 0. Thus, the graph climbs and eventually tends to the line $y = x - 1$ (see Figure $2.7.25$).

Going back to the y-intercept, we now start sketching the curve to the left. Since there is an x-intercept at $x = -1$, our curve falls (as we draw to the left), goes through $(-1, 0)$, and (since there is another x-intercept) climbs to strike the x-axis at $x = -2$. It goes through this intercept and we are now above the x-axis. If we continue to draw to the left (again, see Figure $2.7.25$), our curve would approach the vertical asymptote, $x = -4$. We have a choice—have the graph rise to infinity as $x \to -4^+$, or have the graph fall to negative infinity as $x \to -4^+$. Since we do not have any more x-intercepts, the choice is out of our hands. The function grows without bound (to infinity) as $x \to -4^+$.

As stated in the list above, the behavior of the function near to the vertical asymptote mimics that of $\frac{1}{x}$, so we know $f(x) \to -\infty$ as $x \to -4^-$. That is, since the graph of f is rising on one side of $x = -4$, it must be falling on the other side. Finally, as we continue drawing to the left, the function tends to the slant asymptote.

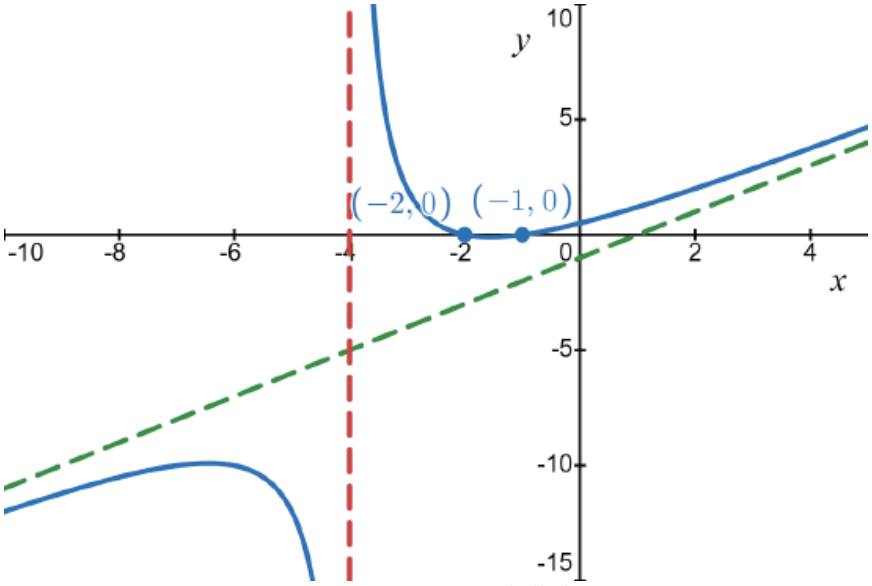


Figure $2.7.25$**:** *Graph of* $y = \frac{(x+2)(x+1)}{x+4}$.

Trust me when I say that you will be able to sketch a function like this in a couple of minutes once you practice these types of problems. Do not freak out.

Our ultimate goal was to find out when $f(x) > 0$. Now that we have the graph, we can see this occurs when $x \in (-4, -2) \cup (-1, \infty)$.

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2.7.1: Resources and Key Concepts

Resources

Videos

- Rational function
  - Rational Functions - A Brief Review of Rational Functions
- Vertical asymptote
  - Rational Functions - Identifying Vertical Asymptotes of Rational Functions
  - Rational Functions - Asymptotic Behavior of Rational Functions
- Horizontal asymptote
  - Rational Functions - Identifying Horizontal Asymptotes of Rational Functions
- Slant asymptote (oblique asymptote)
  - Rational Functions - Identifying Slant Asymptotes of Rational Functions
- Domain of a rational function
  - Rational Functions - Finding the Domain of Rational Functions
- Graphing rational functions
  - Rational Functions - Graphing Rational Functions
  - Rational Functions - Graphing Rational Functions
  - Algebra Lecture: Graphing Rational Functions
- Solving rational inequalities
  - Rational Functions - Solving Rational Inequalities
  - Inequalities - Nonlinear - A Review for College Algebra: Solving Rational Inequalities

Key Concepts

Definitions

- **Arrow Notation**: A symbolic way to describe the behavior of a function as its input x approaches a certain value or infinity. Examples: $x \to a^-$ (approaches a from the left), $x \to \infty$ (approaches infinity), $f(x) \to L$ (the output approaches L).
- **Vertical Asymptote**: A vertical line $x = a$ where the graph of a function tends toward positive or negative infinity as the inputs approach a from either the left or the right.
- **Horizontal Asymptote**: A horizontal line $y = b$ where the graph of a function approaches the line as the inputs increase or decrease without bound ($x \to \infty$ or $x \to -\infty$).
- **Rational Function**: A function that can be written as the quotient of two polynomial functions, $f(x) = \frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$.
- **Removable Discontinuity (Hole)**: A single point where the graph of a function is not defined, indicated by an open circle on the graph. It occurs in a rational function at $x = a$ if $(x - a)$ is a factor of both the numerator and the denominator, and after simplification, a is no longer a zero of the denominator.
- **End Behavior (of a rational function's graph)**: A description, usually as a function (e.g., horizontal or slant asymptote), of what the function values of the graph tend to approach as the inputs increase or decrease without bound.
- **Polynomial Degree Notation ($deg(f(x))$)**: Denotes the degree of a polynomial $f(x)$.
- **Slant Asymptote (Oblique Asymptote)**: A non-horizontal, non-vertical line that the graph of a rational function approaches as $x \to \infty$ or $x \to -\infty$. It occurs when the degree of the numerator is exactly one more than the degree of the denominator.

Theorems

- **Domain of a Rational Function**: The domain of a rational function includes all real numbers except those that cause the denominator of the unsimplified rational function to equal zero.
- **Vertical Asymptotes of Rational Functions**: Vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

- **Removable Discontinuity of a Rational Function**: A removable discontinuity for the graph of a rational function occurs at $x = a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator, and after simplification, a is no longer a zero of the denominator.
- **End Behavior of a Rational Function** $f(x) = \frac{N(x)}{D(x)}$:
  - If $deg(D(x)) > deg(N(x))$, then $f(x)$ has a horizontal asymptote at $y = 0$.
  - If $deg(D(x)) = deg(N(x))$, then $f(x)$ has a horizontal asymptote at $y = \frac{a}{b}$, where a is the lead coefficient of $N(x)$ and b is the lead coefficient of $D(x)$.
  - If $deg(N(x)) = deg(D(x)) + 1$, then $f(x)$ has a slant (oblique) asymptote whose equation is the quotient after dividing $N(x)$ by $D(x)$ (ignoring the remainder).

Common Mistakes

- **Incorrectly Identifying Domain Before Simplification**: Finding the domain of a rational function *after* simplifying it by canceling common factors. The domain must be determined from the original, unsimplified function.
- **Confusing Holes with Vertical Asymptotes**: If a factor $(x - a)$ cancels from the numerator and denominator, there is a hole at $x = a$, not a vertical asymptote, provided a is not still a zero of the simplified denominator.
- **Errors in Determining Horizontal/Slant Asymptotes**:
  - Making errors in calculating the ratio of leading coefficients for horizontal asymptotes when degrees are equal.
  - Errors in polynomial division when finding slant asymptotes.
- **Assuming Graphs Cannot Cross Horizontal/Slant Asymptotes**: While graphs of rational functions never cross their vertical asymptotes, they can cross their horizontal or slant asymptotes.
- **Errors in Simplifying Difference Quotients for Rational Functions**: Mistakes in finding a common denominator or in distributing terms when simplifying the complex fraction that arises.
- **Solving Rational Inequalities Incorrectly**: Making errors when finding critical points (zeros of numerator and denominator) or when testing intervals, or misinterpreting the graph if using the graphical method.

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2.7.2: Homework

Reading Questions

1. What is a rational function? Give its general form.
2. Explain what arrow notation like $x \to a^+$ and $f(x) \to -\infty$ means.
3. What is a vertical asymptote of a rational function? How do you typically find the location(s) of vertical asymptotes from the function's equation?
4. What is a horizontal asymptote? Describe the three cases for determining the horizontal asymptote of a rational function based on the degrees of the numerator and denominator.
5. Under what condition does a rational function have a slant asymptote instead of a horizontal asymptote? How is its equation found?
6. What is a removable discontinuity (or hole) in the graph of a rational function? How is it identified from the function's equation?
7. How do you find the domain of a rational function?
8. How are the x-intercepts of a rational function typically found? What about the y-intercept?
9. Describe how the multiplicity of a factor in the denominator affects the behavior of the graph near a vertical asymptote (odd vs. even multiplicity).
10. When writing the equation of a rational function from its graph, what information do the x-intercepts provide? What about the vertical asymptotes? How can the stretch factor a be determined?
11. When solving a rational inequality graphically, like $\frac{P(x)}{Q(x)} > 0$, what are you looking for on the graph of $f(x) = \frac{P(x)}{Q(x)}$?
12. What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?
13. What is the fundamental difference in the graphs of polynomial functions and rational functions?
14. If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?
15. Can a graph of a rational function have no vertical asymptote? If so, how?
16. Can a graph of a rational function have no x-intercepts? If so, how?

Homework

Algebraic

For the following exercises, find the domain of the rational functions.

17. $f(x) = \frac{x-1}{x+2}$
18. $f(x) = \frac{x+1}{x^2-1}$
19. $f(x) = \frac{x^2+4}{x^2-2x-8}$
20. $f(x) = \frac{x^2+4x-3}{x^4-5x^2+4}$

For the following exercises, find the domain, vertical asymptotes, and horizontal asymptotes of the functions.

21. $f(x) = \frac{4}{x-1}$
22. $f(x) = \frac{2}{5x+2}$
23. $f(x) = \frac{x}{x^2-9}$
24. $f(x) = \frac{x}{x^2+5x-36}$
25. $f(x) = \frac{3+x}{x^3-27}$
26. $f(x) = \frac{3x-4}{x^3-16x}$
27. $f(x) = \frac{x^2-1}{x^3+9x^2+14x}$
28. $f(x) = \frac{x+5}{x^2-25}$
29. $f(x) = \frac{x-4}{x-6}$
30. $f(x) = \frac{4-2x}{3x-1}$

For the following exercises, find the x-and y-intercepts for the functions.

31. $f(x) = \frac{x+5}{x^2+4}$
32. $f(x) = \frac{x}{x^2-x}$
33. $f(x) = \frac{x^2+8x+7}{x^2+11x+30}$
34. $f(x) = \frac{x^2+x+6}{x^2-10x+24}$
35. $f(x) = \frac{94-2x^2}{3x^2-12}$

For the following exercises, describe the local and end behavior of the functions.

36. $f(x) = \frac{x}{2x+1}$
37. $f(x) = \frac{2x}{x-6}$
38. $f(x) = \frac{-2x}{x-6}$
39. $f(x) = \frac{x^2-4x+3}{x^2-4x-5}$
40. $f(x) = \frac{2x^2-32}{6x^2+13x-5}$

For the following exercises, find the slant asymptote of the functions.

41. $f(x) = \frac{24x^2+6x}{2x+1}$
42. $f(x) = \frac{4x^2-10}{2x-4}$
43. $f(x) = \frac{81x^2-18}{3x-2}$
44. $f(x) = \frac{6x^3-5x}{3x^2+4}$
45. $f(x) = \frac{x^2+5x+4}{x-1}$

Graphical

For the following exercises, use the given transformation to graph the function. Note the vertical and horizontal asymptotes.

46. The reciprocal function shifted up two units.
47. The reciprocal function shifted down one unit and left three units.
48. The reciprocal squared function shifted to the right 2 units.
49. The reciprocal squared function shifted down 2 units and right 1 unit.

For the following exercises, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal or slant asymptote of the functions. Use that information to sketch a graph.

50. $p(x) = \frac{2x-3}{x+4}$
51. $q(x) = \frac{x-5}{3x-1}$
52. $s(x) = \frac{4}{(x-2)^2}$
53. $r(x) = \frac{5}{(x+1)^2}$
54. $f(x) = \frac{3x^2-14x-5}{3x^2+8x-16}$
55. $g(x) = \frac{2x^2+7x-15}{3x^2-14x+15}$
56. $a(x) = \frac{x^2+2x-3}{x^2-1}$
57. $b(x) = \frac{x^2-x-6}{x^2-4}$
58. $h(x) = \frac{2x^2+x-1}{x-4}$
59. $k(x) = \frac{2x^2-3x-20}{x-5}$
60. $w(x) = \frac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$
61. $z(x) = \frac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

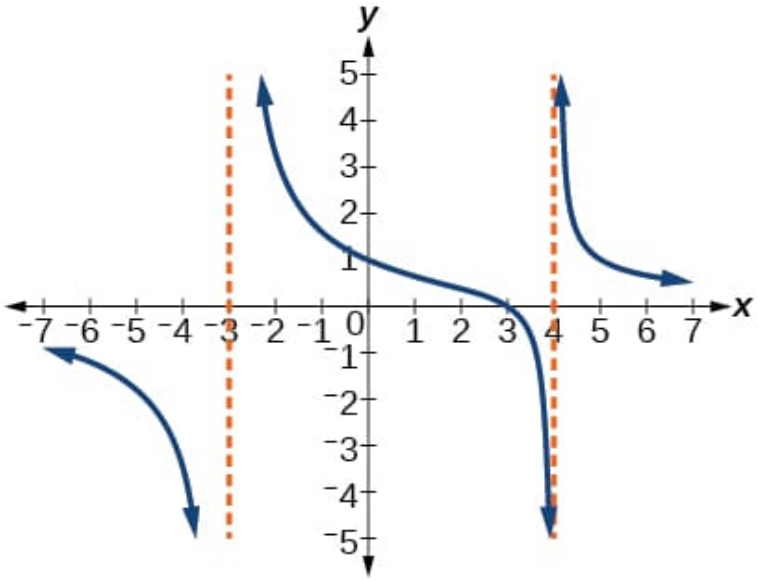
For the following exercises, write an equation for a rational function with the given characteristics.

62. Vertical asymptotes at $x = 5$ and $x = -5$, x-intercepts at $(2, 0)$ and $(-1, 0)$, y-intercept at $(0, 4)$
63. Vertical asymptotes at $x = -4$ and $x = -1$, x-intercepts at $(1, 0)$ and $(5, 0)$, y-intercept at $(0, 7)$
64. Vertical asymptotes at $x = -4$ and $x = -5$, x-intercepts at $(4, 0)$ and $(-6, 0)$, Horizontal asymptote at $y = 7$
65. Vertical asymptotes at $x = -3$ and $x = 6$, x-intercepts at $(-2, 0)$ and $(1, 0)$, Horizontal asymptote at $y = -2$
66. Vertical asymptote at $x = -1$, Double zero at $x = 2$, y-intercept at $(0, 2)$

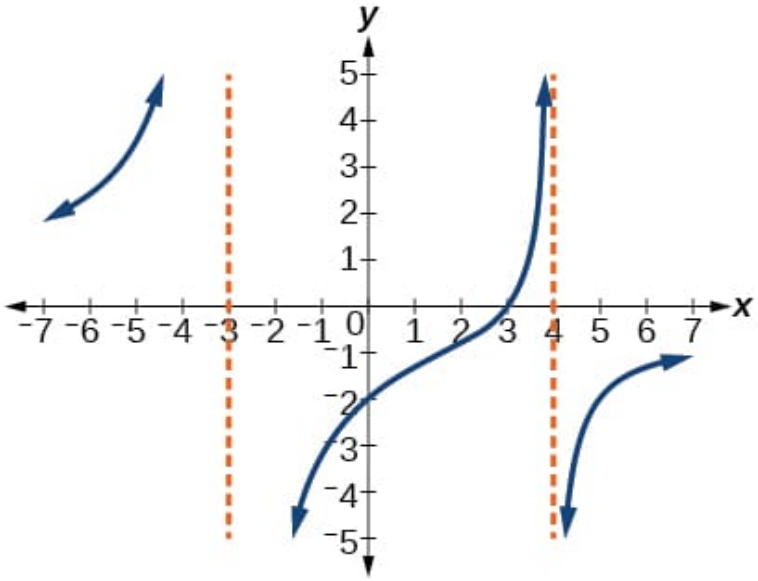
67. Vertical asymptote at $x = 3$, Double zero at $x = 1$, y-intercept at $(0, 4)$

For the following exercises, use the graphs to write an equation for the function.

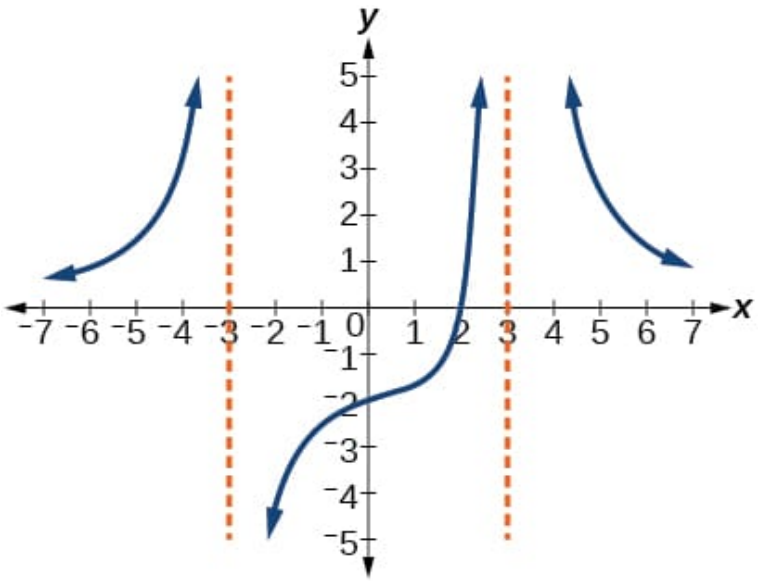
68.



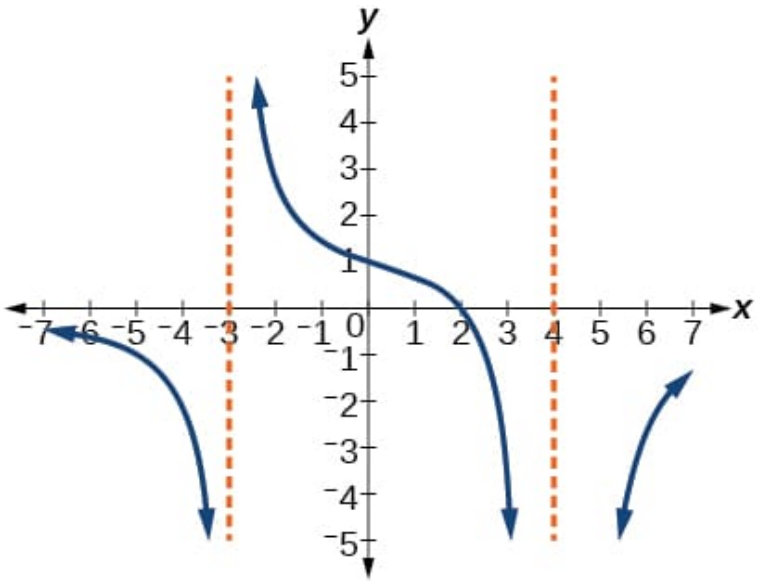
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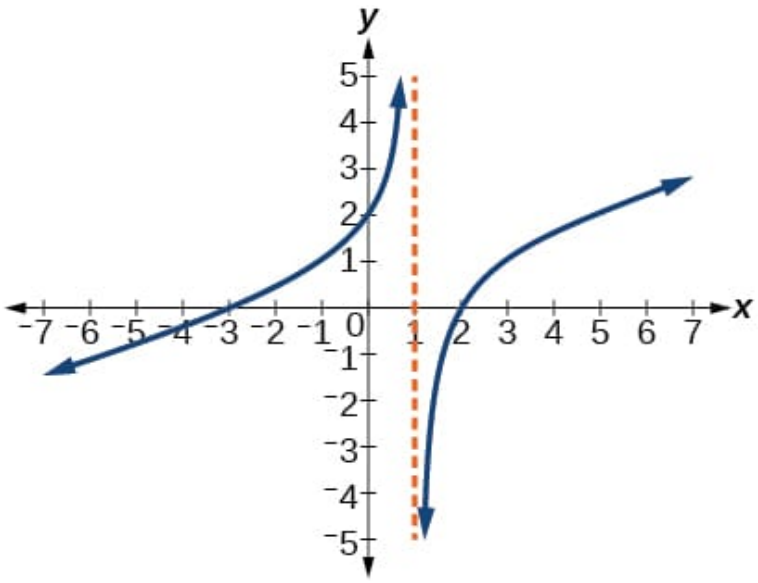
70.


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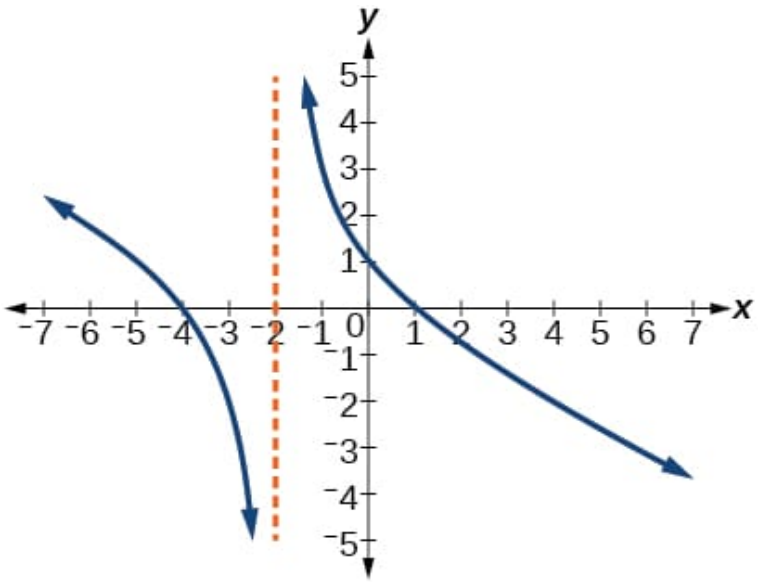
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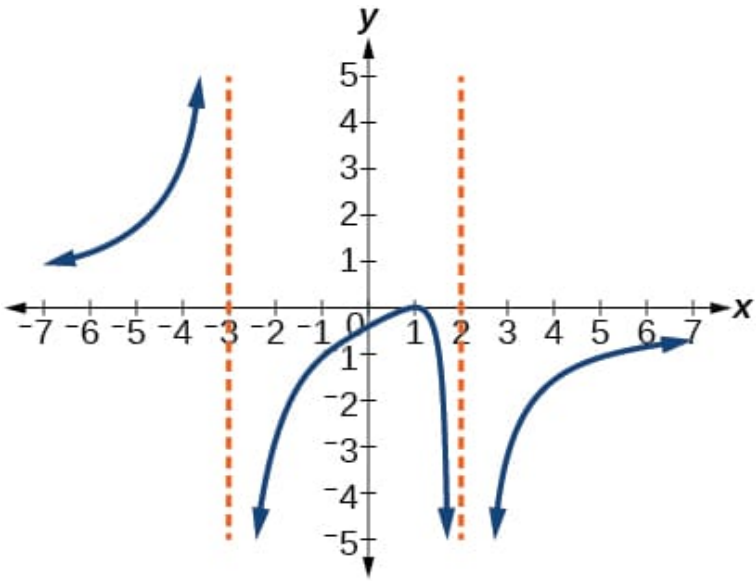
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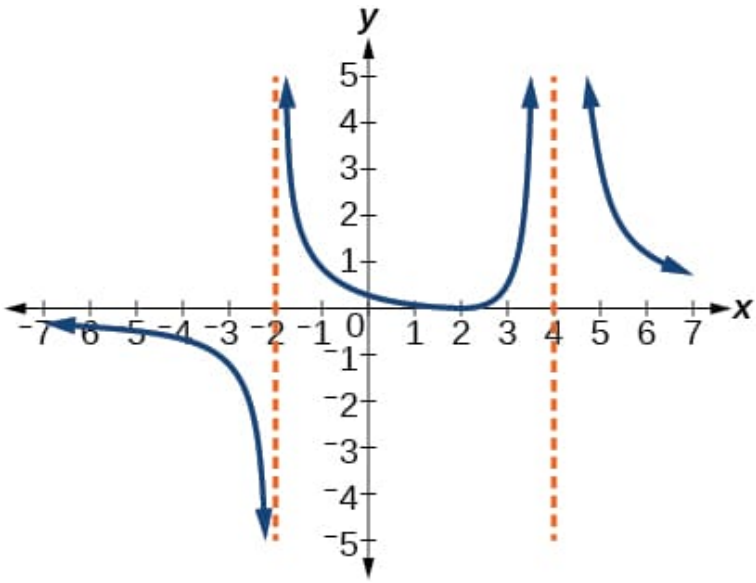
73.


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74.



75.



Numeric

For the following exercises, make tables to show the behavior of the function near the vertical asymptote and reflecting the horizontal asymptote

76. $f(x) = \frac{1}{x-2}$
77. $f(x) = \frac{x}{x-3}$
78. $f(x) = \frac{2x}{x+4}$
79. $f(x) = \frac{2x}{(x-3)^2}$
80. $f(x) = \frac{x^2}{x^2+2x+1}$

Technology

For the following exercises, use a calculator to graph $f(x)$. Use the graph to solve $f(x) > 0$.

81. $f(x) = \frac{2}{x+1}$

32. $f(x) = \frac{4}{2x-3}$
33. $f(x) = \frac{2}{(x-1)(x+2)}$
34. $f(x) = \frac{x+2}{(x-1)(x-4)}$
35. $f(x) = \frac{(x+3)^2}{(x-1)^2(x+1)}$

Extensions

For the following exercises, identify the removable discontinuity.

36. $f(x) = \frac{x^2-4}{x-2}$
37. $f(x) = \frac{x^3+1}{x+1}$
38. $f(x) = \frac{x^2+x-6}{x-2}$
39. $f(x) = \frac{2x^2+5x-3}{x+3}$
40. $f(x) = \frac{x^3+x^2}{x+1}$

Real-World Applications

For the following exercises, express a rational function that describes the situation.

91. A large mixing tank currently contains 200 gallons of water, into which 10 pounds of sugar have been mixed. A tap will open, pouring 10 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 3 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.
92. A large mixing tank currently contains 300 gallons of water, into which 8 pounds of sugar have been mixed. A tap will open, pouring 20 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 2 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

For the following exercises, use the given rational function to answer the question.

93. The concentration C of a drug in a patient's bloodstream t hours after injection in given by $C(t) = \frac{2t}{3+t^2}$. What happens to the concentration of the drug as t increases?
94. The concentration C of a drug in a patient's bloodstream t hours after injection is given by $C(t) = \frac{100t}{2t^2+75}$. Use a calculator to approximate the time when the concentration is highest.

For the following exercises, construct a rational function that will help solve the problem. Then, use a calculator to answer the question.

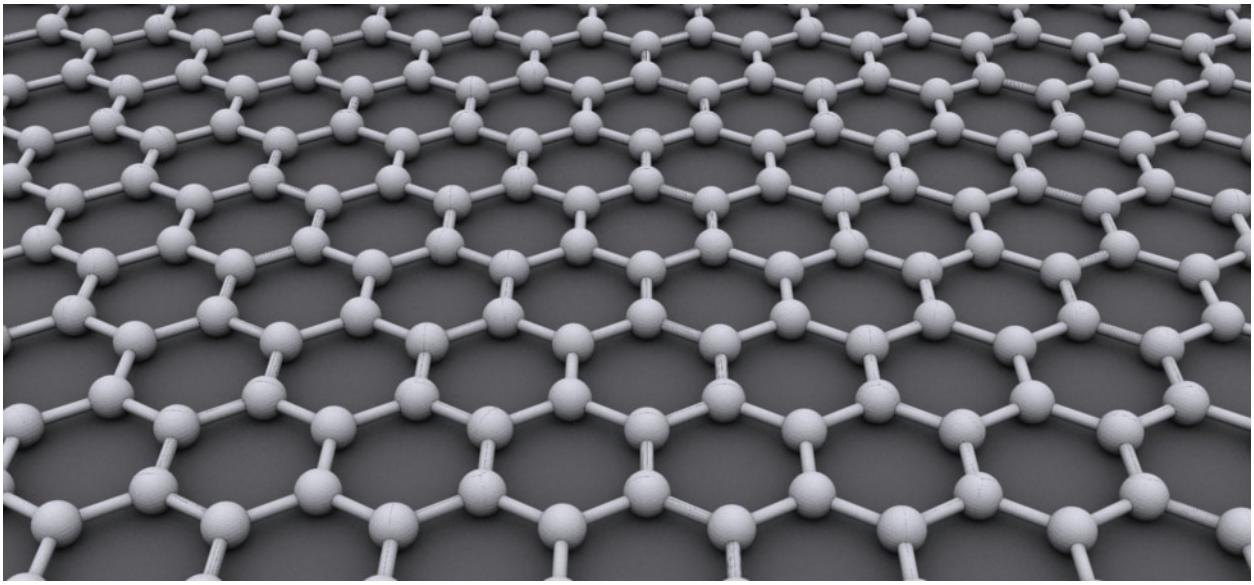
95. An open box with a square base is to have a volume of 108 cubic inches. Find the dimensions of the box that will have minimum surface area. Let x = length of the side of the base.
96. A rectangular box with a square base is to have a volume of 20 cubic feet. The material for the base costs 30 cents/ square foot. The material for the sides costs 10 cents/square foot. The material for the top costs 20 cents/square foot. Determine the dimensions that will yield minimum cost. Let x = length of the side of the base.
97. A right circular cylinder has volume of 100 cubic inches. Find the radius and height that will yield minimum surface area. Let x = radius.
98. A right circular cylinder with no top has a volume of 50 cubic meters. Find the radius that will yield minimum surface area. Let x = radius.
99. A right circular cylinder is to have a volume of 40 cubic inches. It costs 4 cents/square inch to construct the top and bottom and 1 cent/square inch to construct the rest of the cylinder. Find the radius to yield minimum cost. Let x = radius.

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3: INVERSE AND RADICAL FUNCTIONS

Imagine charging your cell phone is less than five seconds. Consider cleaning radioactive waste from contaminated water. Think about filtering salt from ocean water to make an endless supply of drinking water. Ponder the idea of bionic devices that can repair spinal injuries. These are just of few of the many possible uses of a material called graphene. Materials scientists are developing a material made up of a single layer of carbon atoms that is stronger than any other material, completely flexible, and conducts electricity better than most metals. Research into this type of material requires a solid background in mathematics, including understanding roots and radicals. In this chapter, you will learn to simplify expressions containing roots and radicals, perform operations on radical expressions and equations, and evaluate radical functions.

Graphene is an incredible strong and flexible material made from carbon. It can also conduct electricity. Notice the hexagonal grid pattern. (credit: "AlexanderAIUS" / Wikimedia Commons)



3.1: Inverse Functions

3.1.1: Resources and Key Concepts

3.1.2: Homework

3.2: Radical Functions

3.2.1: Resources and Key Concepts

3.2.2: Homework

3.3: Radical Equations

3.3.1: Resources and Key Concepts

3.3.2: Homework

3.4: Rational Exponent Expressions Encountered in Calculus

3.4.1: Resources and Key Concepts

3.4.2: Homework

3.5: Radical Expressions Encountered in Calculus

3.5.1: Resources and Key Concepts

3.5.2: Homework

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3.1: Inverse Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure $3.1.1$ provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

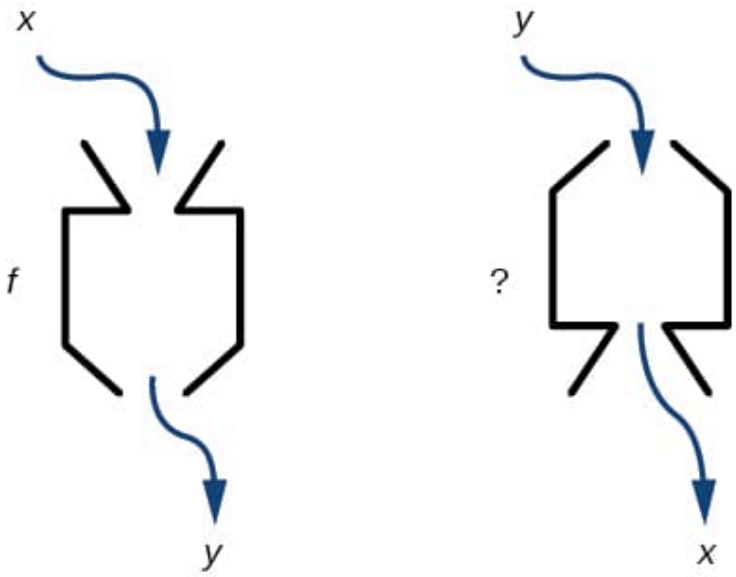


Figure $3.1.1$**:** *Can a function "machine" operate in reverse?*

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the Celsius scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees Fahrenheit to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32)$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24^\circ \text{ C}.$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure $3.1.2$ for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

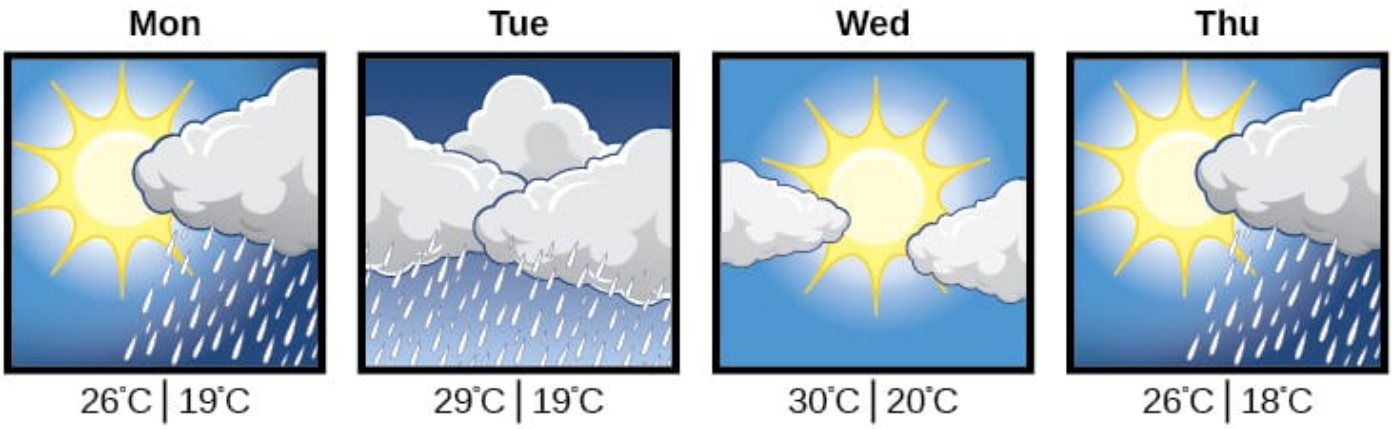


Figure 3.1.2

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her Algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$\begin{aligned} 26 &= \frac{5}{9}(F-32) \\ \implies 26\cdot\frac{9}{5} &= F-32 \\ \implies F &= 26\cdot\frac{9}{5}+32\approx 79 \end{aligned}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Definition: Inverse Function

Let f be a one-to-one function. The **inverse** of f, denoted f^{-1}, is the function that reverses the pairing produced by f: for each y in the range of f,

$$f^{-1}(y)=x \quad \text{precisely when} \quad f(x)=y.$$

f^{-1} is called the **inverse function** of f and the notation, f^{-1}, is read "f-inverse".

Caution

The symbol f^{-1} denotes the inverse function, not a reciprocal. In general, $f^{-1}(x)\neq\frac{1}{f(x)}$. The reciprocal $\frac{1}{f(x)}$ is written $\big(f(x)\big)^{-1}$.

Additionally, not all functions have inverses.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a=1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1}\circ f$ equals the identity function, that is,

$$(f^{-1}\circ f)(x)=f^{-1}\left(f(x)\right)=x.$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a reciprocal, some functions do not have inverses.

Inverse Function Notation

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x."

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

Theorem

If f^{-1} is the inverse of f, then $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}\right) = x.$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Example 3.1.1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution available at the end of this section.

If we show the coordinate pairs from Example 3.1.1 in a table form, the input and output are clearly reversed. See Table 3.1.1.

Table 3.1.1

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Checkpoint 3.1.1

Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

Example 3.1.2: Testing Inverse Relationships Algebraically

If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$, is $g = f^{-1}$?

Solution available at the end of this section.

Checkpoint 3.1.2

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

Example 3.1.3: Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

Solution available at the end of this section.

Checkpoint 3.1.3

If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure 3.1.3.

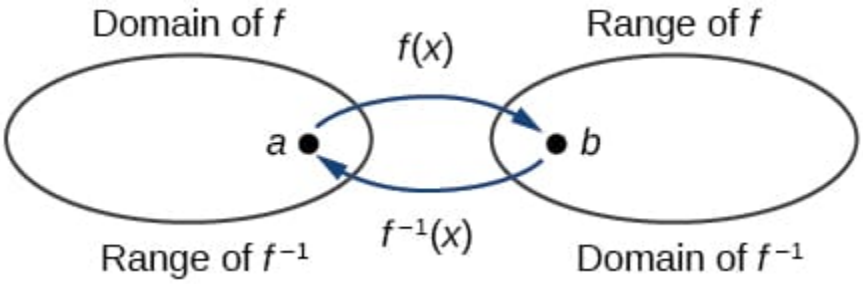


Figure 3.1.3: *Domain and range of a function and its inverse*

Theorem: Domain and Range of Inverse Functions

If f is one-to-one, then the domain of f^{-1} equals the range of f, and the range of f^{-1} equals the domain of f.

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and -3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all!

To put it differently, the quadratic function is not a one-to-one function; it fails the Horizontal Line Test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

Interactive Element: The Horizontal Line Test

The following Interactive Element might help you visualize why the quadratic function is not one-to-one.

Interact: Drag the red point up and down to move the horizontal line.

Observation: Above the vertex the line meets $y = x^2$ at two points, so one output has two inputs — the function fails the Horizontal Line Test.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its domain limited to $[0, \infty)$, which is a one-to-one function (it passes the Horizontal Line Test) and which has an inverse (the square-root function).

If $f(x) = (x-1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of f^{-1} = $[1, \infty)$.
- The domain of f^{-1} = range of f = $[0, \infty)$.

Q&A

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Example 3.1.4: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 3.1.2. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Table 3.1.2

Constant	**Identity**	**Quadratic**	**Cubic**	**Reciprocal**
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	**Cube root**	**Square root**	**Absolute value**	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Solution available at the end of this section.

We can see that the absolute value and reciprocal-squared functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 3.1.4. They both would fail the Horizontal Line Test. However, if a function is restricted to a certain domain so that it passes the Horizontal Line Test, then in that restricted domain, it can have an inverse.

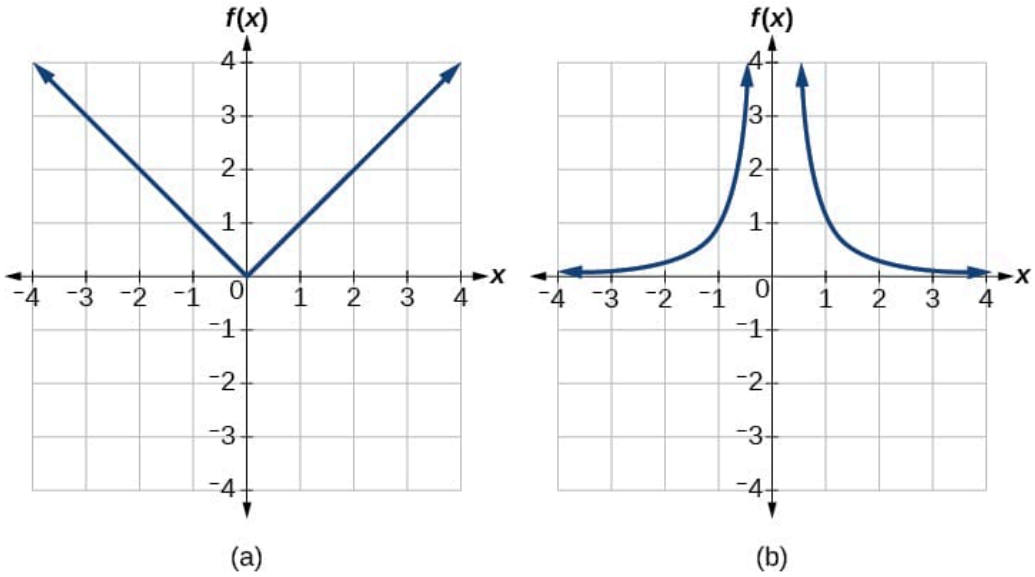


Figure 3.1.4**:** *(a) Absolute value; (b) Reciprocal squared*

Checkpoint 3.1.4

The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example 3.1.5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table 3.1.3, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$.

Table 3.1.3

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Solution available at the end of this section.

Checkpoint 3.1.5

Using Table 3.1.4, find and interpret each of the following:

a. $f(60)$
b. $f^{-1}(60)$

Table 3.1.4

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We know that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

Example 3.1.6: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 3.1.5. Find $g(3)$ and $g^{-1}(3)$.

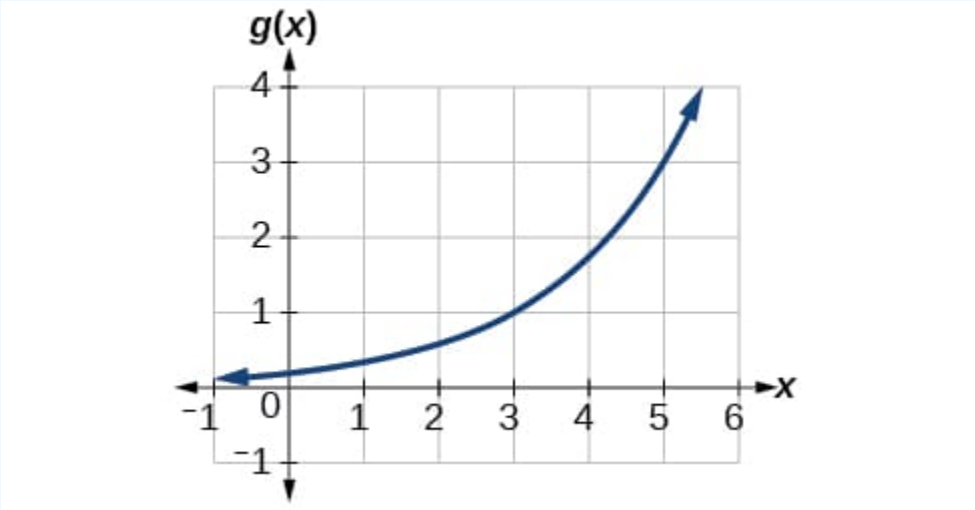


Figure 3.1.5

Solution available at the end of this section.

Checkpoint 3.1.6

Using the graph in Figure 3.1.6 to answer the following questions.

a. Find $g^{-1}(1)$.
b. Estimate $g^{-1}(4)$.

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula—for example, y as a function of x—we can often find the inverse function by solving to obtain x as a function of y.

How To: Find the Equation of an Invertible Function

Given a function, $f(x)$, we find the inverse (if it exists) by doing the following:

1. Let $y = f(x)$.
2. Solve for x.
3. Interchange x and y.

The result of these operations, if it is a function, is $f^{-1}(x)$.

Example 3.1.7: Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature,

$$C = \frac{5}{9}(F - 32).$$

Solution available at the end of this section.

Checkpoint 3.1.7

Solve for x in terms of y given $y = \frac{1}{3}(x - 5)$.

Example 3.1.8: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

Solution available at the end of this section.

Note that the domain and range of f in Example 3.1.8 exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table 3.1.5.

Table 3.1.5

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example 3.1.9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x - 4}$.

Solution available at the end of this section.

The formula we found for $f^{-1}(x)$ in Example 3.1.9 looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

Checkpoint 3.1.9

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in Figure 3.1.7.

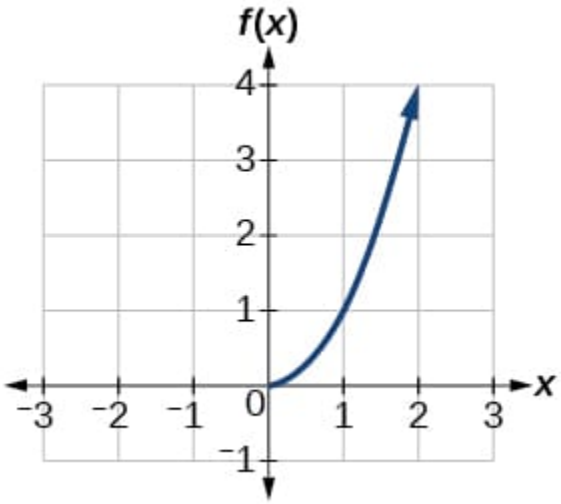


Figure 3.1.7**:** *Quadratic function with domain restricted to* $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the Horizontal Line Test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure 3.1.8.

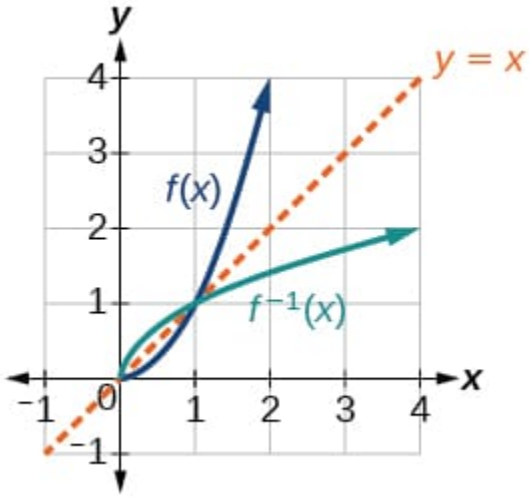


Figure 3.1.8**:** *Square and square-root functions on the non-negative domain*

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Interactive Element: Reflecting a Function Across the Line y = x

The following Interactive Element might help you visualize how the graph of an inverse function is the mirror image of the original across the line $y = x$.

Interact: Drag the large red point along the blue curve $y = x^2$.

Observation: The orange point always lands on $y = \sqrt{x}$, the mirror image across $y = x$; a pair (a, b) on f becomes (b, a) on f^{-1}.

Example 3.1.10: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure 3.1.9, sketch a graph of $f^{-1}(x)$.

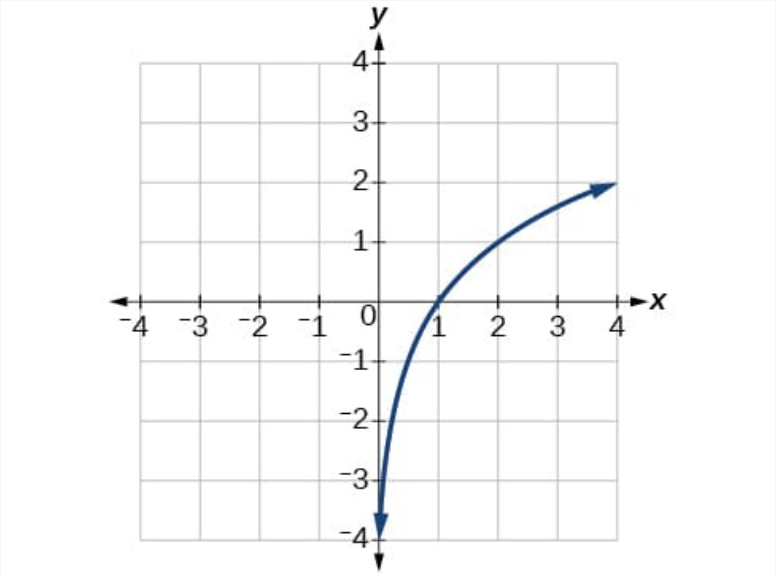


Figure 3.1.9

Solution available at the end of this section.

Checkpoint 3.1.10

Draw graphs of the functions f and f^{-1} from Example 3.1.8.

Q&A

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x.$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Restricting the Domain to Find the Inverse of a Polynomial Function

So far, we have been able to find the inverse functions of cubic functions without having to restrict their domains. However, as we know, not all cubic polynomials are one-to-one. Some functions that are not one-to-one may have their domain restricted so that they are one-to-one, but only over that domain. The function over the restricted domain would then have an inverse function. Since quadratic functions are not one-to-one, we must restrict their domain in order to find their inverses.

Theorem: Forcing Invertibility by Restricting Domain

If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

How To: Restrict Domain to Force Invertibility

Given a function that is not one-to-one,

1. Restrict the domain by determining a domain on which the original function is one-to-one (i.e., a domain where the function passes the Horizontal Line Test).
2. Replace $f(x)$ with y.
3. Solve for x interchange x and y.
4. Rename the function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

Example 3.1.11: Restricting the Domain to Find the Inverse of a Polynomial Function

Find the inverse function of f :

a. $f(x)=(x-4)^2,\ x\geq 4$
b. $f(x)=(x-4)^2,\ x\leq 4$

Solution available at the end of this section.

On the graphs in Figure 3.1.12, we see the original function from Example 3.1.11 graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line $y=x$. The coordinate pair $(4,0)$ is on the graph of f and the coordinate pair $(0,4)$ is on the graph of f^{-1}. For any coordinate pair, if (a,b) is on the graph of f, then (b,a) is on the graph of f^{-1}. Finally, observe that the graph of f intersects the graph of f^{-1} on the line $y=x$. Points of intersection for the graphs of f and f^{-1} will always lie on the line $y=x$.

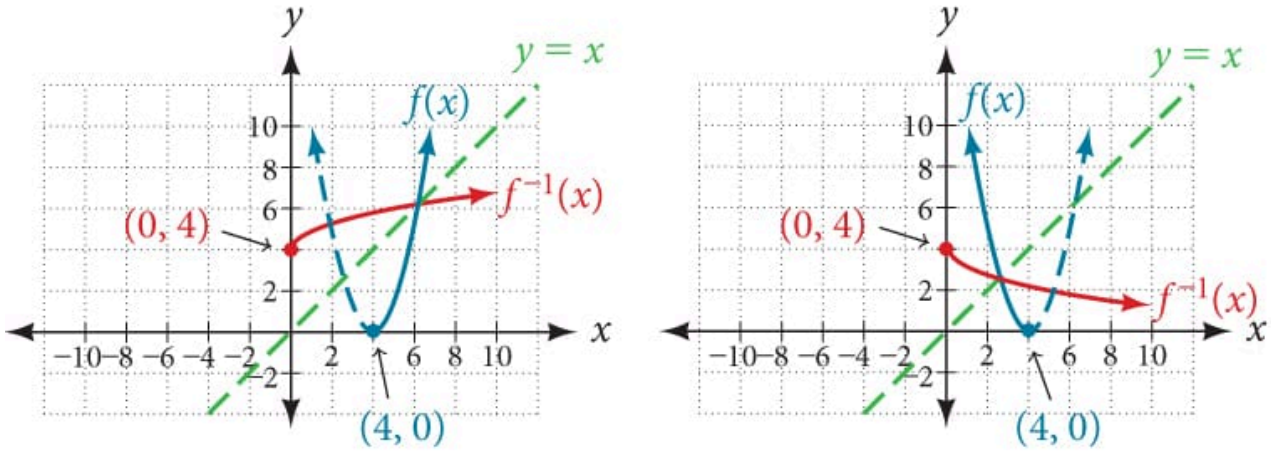


Figure 3.1.12

Example 3.1.12: Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified

Restrict the domain and then find the inverse of

$$f(x) = (x-2)^2 - 3.$$

Solution available at the end of this section.

Notice that we arbitrarily decided to restrict the domain on $x \geq 2$. We could just have easily opted to restrict the domain on $x \leq 2$, in which case $f^{-1}(x) = 2 - \sqrt{x+3}$. Observe the original function graphed on the same set of axes as its inverse function in Figure 3.1.13. Notice that both graphs show symmetry about the line $y = x$. The coordinate pair $(2, -3)$ is on the graph of f and the coordinate pair $(-3, 2)$ is on the graph of f^{-1}. Observe from the graph of both functions on the same set of axes that

$$\text{domain of } f = \text{range of } f^{-1} = [2, \infty)$$

and

$$\text{domain of } f^{-1} = \text{range of } f = [-3, \infty).$$

Finally, observe that the graph of f intersects the graph of f^{-1} along the line $y = x$.

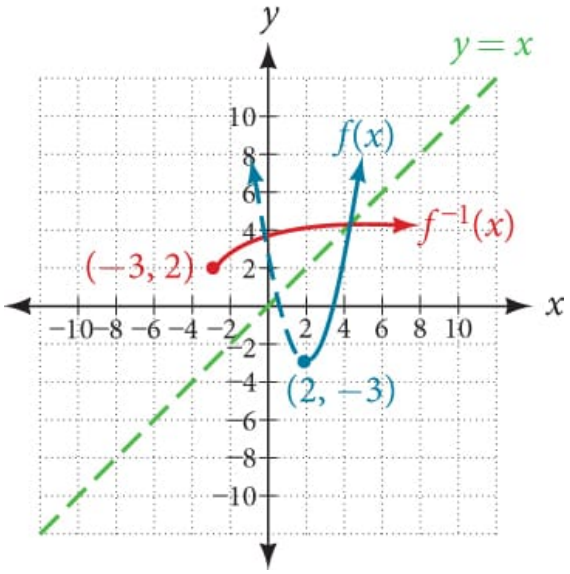


Figure 3.1.13

Checkpoint 3.1.12

Find the inverse of the function $f(x) = x^2 + 1$, on the domain $x \geq 0$.

Solving Applications Involving the Inverse of Polynomial Functions

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

Example 3.1.13: Solving an Application with a Cubic Function

A mound of gravel is in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

$$V = \frac{2}{3}\pi r^3.$$

Find the inverse of this function. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi \approx 3.14$.

Solution available at the end of this section.

Finding Inverses of Rational Functions

As with finding inverses of quadratic functions, it is sometimes desirable to find the inverse of a rational function, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

Example 3.1.14: Finding the Inverse of a Rational Function

The function $C = \frac{20+0.4n}{100+n}$ represents the concentration C of an acid solution after n mL of 40% solution has been added to 100 mL of a 20% solution. First, find the inverse of the function; that is, find an expression for n in terms of C. Then use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

Solution available at the end of this section.

Checkpoint 3.1.14

Find the inverse of the function $f(x) = \dfrac{x+3}{x-2}$.

Success in Mathematics: Mathematics in Other Courses

In other courses, we use mathematics to understand and explore the big ideas of the subject. Professors may use different vocabulary, variables, and notations than you learned in your math classes. The mathematics is the same. The way of talking about it may be different.

Also, some courses will use formulas based on mathematics that you have not yet learned. You may not be ready to understand why these formulas work. Sometimes you have to accept that the formula is correct.

Making connections between different courses takes work. It would be best if you were willing to ask questions. You may need to ask your math instructor about topics in your other classes. You may need to ask your other professors why they use different

words or notations. It is your responsibility to make these connections. Your ability to do critical thinking in other classes depends on it.

1. There are many formulas that we hear about in daily news or history. Of all these famous formulas, name one.
2. Can you describe the use of the formula you mentioned and what each variable stands for?

Hidden Resources: CRC Hub

As a college student, you have access to so many resources that it can be overwhelming. Luckily, the CRC Hub is your "one-stop shop" for all local resources covering various topics (from healthcare to legal fees). Check out the CRC Hub here.

Answers

Section Preview

We begin this chapter with defining and finding the inverse of a function. This is a traditional treatment of inverses, including how to graph them and how to find their equations given the original function. We also include domain restriction (as we are about to encounter radical functions).

Section Preview

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.
- Restrict the domain to find the inverse of a polynomial function.
- Find the inverse of a rational function.

Example 3.1.1: Identifying an Inverse Function for a Given Input-Output Pair

The inverse function reverses the input and output quantities, so if $f(2)=4$, then $f^{-1}(4)=2$; If $f(5)=12$, then $f^{-1}(12)=5$. Alternatively, if we want to name the inverse function g, then $g(4)=2$ and $g(12)=5$.

Example 3.1.2: Testing Inverse Relationships Algebraically

To determine if g is the inverse function of f, we just need to check if g "undoes" f.

$$\begin{aligned} g(f(x)) &= g\left(\frac{1}{x+2}\right) && \text{(substituting)} \\ &= \frac{1}{\frac{1}{x+2}}-2 && \text{(substituting)} \\ &= x+2-2 && \left(\text{recognizing that } \frac{1}{1/a}=a\right) \\ &= x && \text{(simplifying)} \end{aligned}$$

so

$$g=f^{-1} \text{ and } f=g^{-1}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$\begin{aligned} f(g(x)) &= f\left(\frac{1}{x}-2\right) && \text{(substituting)} \\ &= \frac{1}{\left(\frac{1}{x}+2\right)-2} && \text{(substituting)} \\ &= \frac{1}{\frac{1}{x}} && \text{(simplifying)} \\ &= x && \left(\text{recognizing that } \frac{1}{1/a}=a\right) \end{aligned}$$

Example 3.1.3: Determining Inverse Relationships for Power Functions

$$f(g(x))=f\left(\frac{x}{3}\right)=\left(\frac{x}{3}\right)^3=\frac{x^3}{27}\neq x.$$

No, the functions are not inverses. The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x}=x^{1/3}$, that is, the one-third is an exponent, not a multiplier.

Example 3.1.4: Finding the Inverses of Toolkit Functions

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0,\infty)$, where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0,\infty)$.

Example 3.1.5: Interpreting the Inverse of a Tabular Function

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70

miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Example 3.1.6**: Evaluating a Function and Its Inverse from a Graph at Specific Points**

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 3.1.6.

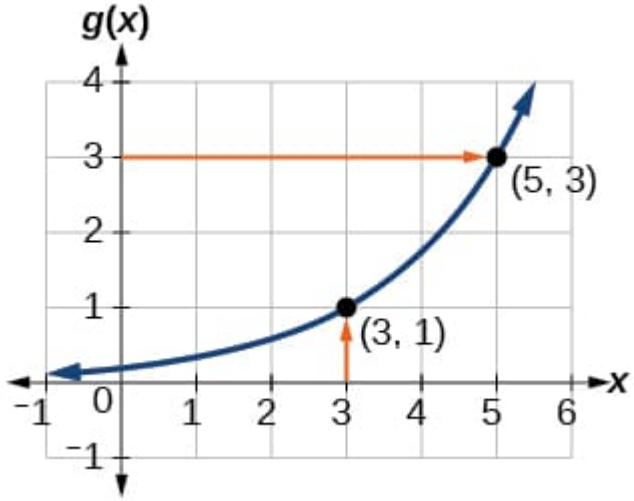


Figure 3.1.6

Example 3.1.7**: Inverting the Fahrenheit-to-Celsius Function**

$$\begin{aligned} C &= \frac{5}{9}(F-32) \\ \Longrightarrow \quad \frac{9}{5}C &= F-32 && \left(\text{multiplying both sides by } \frac{9}{5}\right) \\ \Longrightarrow \quad \frac{9}{5}C+32 &= F && (\text{adding 32 to both sides}) \end{aligned}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F-32),$$

then

$$F = h^{-1}(C) = \frac{9}{5}C+32.$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Example 3.1.8**: Solving to Find an Inverse Function**

$$\begin{aligned} y &= \frac{2}{x-3}+4 && (\text{starting with the original equation}) \\ \Longrightarrow \quad y-4 &= \frac{2}{x-3} && (\text{subtracting 4 from both sides}) \\ \Longrightarrow \quad (x-3)(y-4) &= 2 && (\text{multiplying both sides by } (x-3)) \\ \Longrightarrow \quad x-3 &= \frac{2}{y-4} && (\text{dividing both sides by } (y-4)) \\ \Longrightarrow \quad x &= \frac{2}{y-4}+3 && (\text{adding 3 to both sides}) \end{aligned}$$

So $f^{-1}(y) = \frac{2}{y-4} + 3$ or $f^{-1}(x) = \frac{2}{x-4} + 3$.

Example 3.1.9**: Solving to Find an Inverse with Radicals**

$$\begin{aligned} y &= 2+\sqrt{x-4} && (\text{starting with the original equation}) \\ \Longrightarrow \quad y-2 &= \sqrt{x-4} && (\text{subtracting 2 from both sides}) \\ \Longrightarrow \quad (y-2)^2 &= x-4 && (\text{squaring both sides}) \\ \Longrightarrow \quad (y-2)^2+4 &= x && (\text{adding 4 to both sides}) \end{aligned}$$

So $f^{-1}(x) = (x-2)^2 + 4$.

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$.

Example 3.1.10**: Finding the Inverse of a Function Using Reflection about the Identity Line**

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 3.1.10.

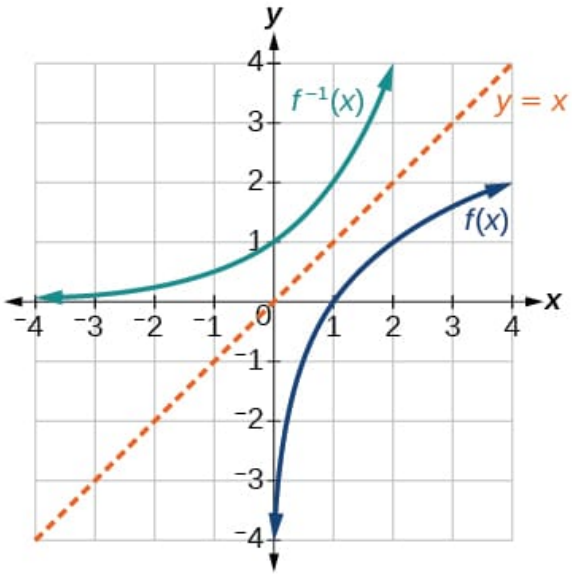


Figure 3.1.10**:** *The function and its inverse, showing reflection about the identity line*

Example 3.1.11**: Restricting the Domain to Find the Inverse of a Polynomial Function**

The original function $f(x) = (x - 4)^2$ is not one-to-one, but the function is restricted to a domain of $x \geq 4$ or $x \leq 4$ on which it is one-to-one. See Figure 3.1.11.

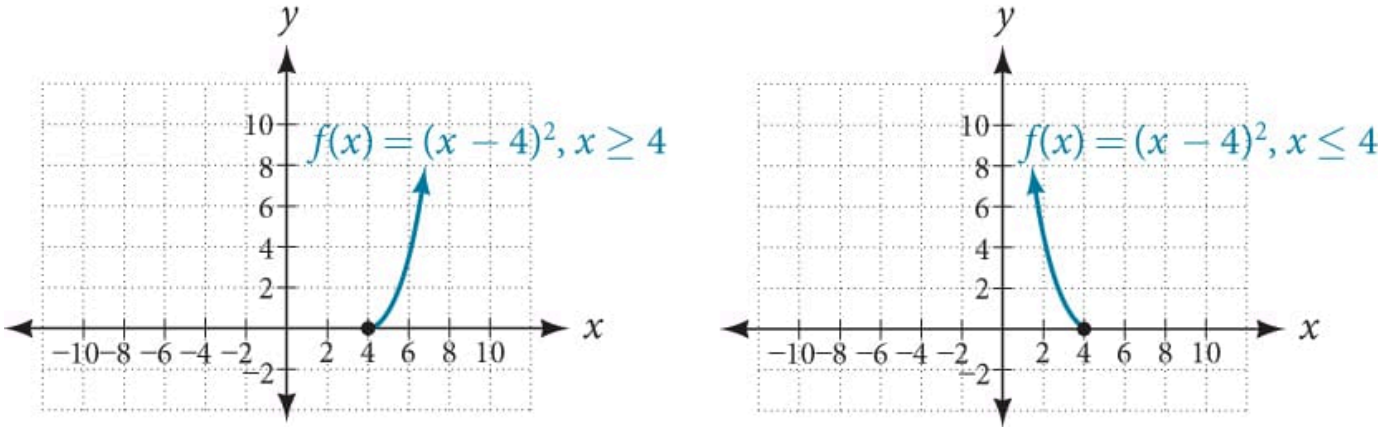


Figure 3.1.11

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$$\begin{aligned} y &= (x-4)^2 && \text{(letting } y = f(x)\text{)} \\ \Rightarrow \quad \pm\sqrt{y} &= x-4 && \text{(Extraction of Roots)} \\ \Rightarrow \quad 4 \pm \sqrt{y} &= x && \text{(adding 4 to both sides)} \\ \Rightarrow \quad 4 \pm \sqrt{x} &= y && \text{(exchanging } y \text{ and } x\text{)} \end{aligned}$$

This is not a function as written. We need to examine the restrictions on the domain of the original function to determine the inverse. Since we reversed the roles of x and y for the original $f(x)$, we need to look at the domain of the original function. When we reversed the roles of x and y, this gave us the values y could assume. For this function, $x \geq 4$, so for the inverse, we should have $y \geq 4$, which is what our inverse function gives.

a. The domain of the original function was restricted to $x \geq 4$, so the outputs of the inverse need to be the same, $f(x) \geq 4$, and we must use the "+" case:

$$f^{-1}(x) = 4 + \sqrt{x}.$$

b. The domain of the original function was restricted to $x \leq 4$, so the outputs of the inverse need to be the same, $f(x) \leq 4$, and we must use the "-" case:

$$f^{-1}(x) = 4 - \sqrt{x}.$$

Example 3.1.12**: Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified**

We can see this is a parabola with vertex at $(2, -3)$ that opens upward. Because the graph will be decreasing on one side of the vertex and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we repeat what we did in Example 3.1.11.

$$\begin{aligned} y &= (x-2)^2-3 && \text{(substituting } y=f(x)\text{)} \\ \Rightarrow \quad y+3 &= (x-2)^2 && \text{(adding 3 to both sides)} \\ \Rightarrow \quad \pm\sqrt{y+3} &= x-2 && \text{(Extraction of Roots)} \\ \Rightarrow \quad 2\pm\sqrt{y+3} &= x && \text{(adding 2 to both sides)} \\ \Rightarrow \quad 2\pm\sqrt{x+3} &= y && \text{(interchanging } x \text{ and } y\text{)} \end{aligned}$$

Now we need to determine which case to use. Because we restricted our original function to a domain of $x \geq 2$, the outputs of the inverse should be the same, telling us to utilize the "+" case. Therefore,

$$f^{-1}(x) = 2+\sqrt{x+3}.$$

If the quadratic had not been given in vertex form, rewriting it into vertex form would be the first step. This way we may easily observe the coordinates of the vertex to help us restrict the domain.

Example $3.1.13$**: Solving an Application with a Cubic Function**

Start with the given function for V. Notice that the meaningful domain for the function is $r \geq 0$ since negative radii would not make sense in this context. Also note the range of the function (hence, the domain of the inverse function) is $V \geq 0$. Solve for r in terms of V, using the method outlined previously.

$$\begin{aligned} V &= \frac{2}{3}\pi r^3 \\ \Rightarrow \quad \frac{3}{2}V &= \pi r^3 \\ \Rightarrow \quad \frac{3}{2\pi}V &= r^3 \\ \Rightarrow \quad \sqrt[3]{\frac{3}{2\pi}V} &= r \end{aligned}$$

This is the result stated in the section opener. Now evaluate this for $V = 100$ and $\pi \approx 3.14$.

$$\begin{aligned} r &= \sqrt[3]{\frac{3V}{2\pi}} \\ &\approx \sqrt[3]{\frac{3(100)}{2(3.14)}} \\ &\approx \sqrt[3]{47.7707} \\ &\approx 3.63 \end{aligned}$$

Therefore, the radius is about 3.63 ft.

Example $3.1.14$**: Finding the Inverse of a Rational Function**

We first want the inverse of the function.

$$\begin{aligned} C &= \frac{20+0.4n}{100+n} \\ \Rightarrow \quad C(100+n) &= 20+0.4n \\ \Rightarrow \quad 100C+Cn &= 20+0.4n \\ \Rightarrow \quad 100C-20 &= 0.4n-Cn \\ \Rightarrow \quad 100C-20 &= (0.4-C)n \\ \Rightarrow \quad \frac{100C-20}{0.4-C} &= n \end{aligned}$$

Now evaluate this function for $C = 0.35$ (35%).

$$n = \frac{100(0.35)-20}{0.4-0.35} = \frac{15}{0.05} = 300.$$

We can conclude that 300 mL of the 40% solution should be added.

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3.1.1: Resources and Key Concepts

Resources

Videos

- Inverse function (concept)
  - Intermediate Algebra - Functions: The Concept of Inverse Functions
- $f^{-1}(x)$ notation
  - Intermediate Algebra - Functions: Inverse Function Notation
- Verifying inverse functions
  - Inverse Functions - Verifying That Two Functions Are Inverse Functions
- One-to-one function
  - One-to-One Functions and Their Inverses - A Brief Review for Precalculus
- Domain and Range of inverse functions
  - Inverse Functions - Finding the Domain and Range of Inverse Functions
- Finding inverse from a formula
  - Inverse Functions - Finding Inverse Functions and Their Graphs
  - Inverse Functions - Finding and Evaluating Inverse Functions
  - Intermediate Algebra - Functions: Finding an Equation for the Inverse Function
- Graph of inverse function
  - Intermediate Algebra - Functions: Graphing the Inverse Function

Key Concepts

Definitions

- **Inverse Function:** For a one-to-one function $y = f(x)$, a function $f^{-1}(x)$ is an inverse function of f if $f^{-1}(f(x)) = x$ for all x in the domain of f, and $f(f^{-1}(y)) = y$ for all y in the domain of f^{-1} (which is the range of f). The input of the original function becomes the output of the inverse function, and the output of the original function becomes the input of the inverse function.
- **Inverse Function Notation (f^{-1}):** The notation used to represent the inverse of a function f. It is read as "f inverse."
- **Identity Line:** The line $y = x$. The graph of a function and its inverse are reflections of each other across this line.

Theorems

- **Inverse Function Property:** If f^{-1} is the inverse of f, then $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}. (And $f^{-1}(f(x)) = x$ for all x in the domain of f).
- **Domain and Range of Inverse Functions:** The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$, and the domain of $f(x)$ is the range of $f^{-1}(x)$.
- **Condition for Existence of an Inverse Function:** In order for a function to have an inverse, it must be a one-to-one function.
- **Forcing Invertibility by Restricting Domain**: If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

Common Mistakes

- **Confusing Inverse Function Notation with Reciprocal:** The notation $f^{-1}(x)$ for an inverse function does not mean $\frac{1}{f(x)}$. These are generally different concepts.
- **Assuming All Functions Have Inverses:** Not all functions have inverses. A function must be one-to-one to have an inverse function.
- **Forgetting to Restrict the Domain for Non-One-to-One Functions:** When finding an inverse for a function that is not naturally one-to-one (like $f(x) = x^2$), failing to restrict the domain of the original function appropriately will result in an inverse

relation that is not a function.

- **Incorrectly Determining the Domain of the Inverse Function:** The domain of f^{-1} is the range of f. Sometimes, after algebraically finding a formula for $f^{-1}(x)$, the apparent domain of this new formula might be larger than the actual range of f. The domain of f^{-1} must be restricted to match the range of f.
- **Forgetting to Restrict the Domain for Non-One-to-One Functions**: Attempting to find an inverse for a function like $f(x) = x^2$ without first restricting its domain to an interval where it is one-to-one. This leads to an inverse relation that is not a function (e.g., $y = \pm\sqrt{x}$).
- **Incorrectly Choosing the Branch of the Inverse**: After restricting the domain of a function (e.g., a quadratic) and finding an inverse involving a square root, selecting the wrong sign (positive or negative) for the square root, which would not correspond to the original restricted domain.
- **Not Verifying if the Original Function is One-to-One**: Proceeding to find an inverse without first confirming that the function (or its restricted version) is one-to-one, for example, by using the Horizontal Line Test.
- **Domain/Range Confusion for Inverse Functions**: Incorrectly stating the domain or range of the inverse function. Remember, the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

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3.1.2: Homework

Reading Questions

1. What does it mean for a function f^{-1} to be the inverse of a function f?
2. How can you algebraically verify if two functions, $f(x)$ and $g(x)$, are inverses of each other?
3. Is the notation $f^{-1}(x)$ the same as $\frac{1}{f(x)}$? Explain.
4. What condition must a function satisfy to have an inverse function? How can this be checked graphically?
5. If a function is not one-to-one, what can sometimes be done to allow an inverse function to be found?
6. How are the domain and range of a function f related to the domain and range of its inverse f^{-1}?
7. If the point (a, b) is on the graph of a one-to-one function f, what corresponding point must be on the graph of f^{-1}?
8. Describe the graphical relationship between a function and its inverse. What line is involved?
9. Outline the algebraic steps to find the formula for an inverse function $f^{-1}(x)$ if you are given the formula for $f(x)$.
10. If $f(x) = \sqrt{x}$ has the domain $[0, \infty)$, its inverse is $f^{-1}(x) = x^2$. What must be the domain of this inverse function $f^{-1}(x)$, and why?
11. Can a function have more than one inverse function over the same domain?
12. If you are given a table of values for a one-to-one function $f(t)$ showing distance traveled in t minutes, how would you find $f^{-1}(d)$ for a given distance d? What would $f^{-1}(d)$ represent?
13. Name a toolkit function (other than the identity function) that is its own inverse.
14. Describe why the Horizontal Line Test is an effective way to determine whether a function is one-to-one?
15. Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?
16. Can a function be its own inverse? Explain.
17. Are one-to-one functions either always increasing or always decreasing? Why or why not?
18. How do you find the inverse of a function algebraically?
19. Why is it necessary to restrict the domain of a function like $f(x) = x^2$ before finding its inverse?
20. If you restrict the domain of $f(x) = (x-2)^2 - 3$ to $x \geq 2$, what will be the formula for its inverse $f^{-1}(x)$? What if you restrict the domain to $x \leq 2$?
21. When finding the inverse of a quadratic function, such as $y = ax^2$, you might arrive at $\pm\sqrt{\frac{y}{a}} = x$. How do you decide whether to use the positive or negative root for the inverse function?
22. What is the relationship between the graph of a function and the graph of its inverse with respect to the line $y = x$?
23. Outline the general steps to find the inverse of a polynomial function whose domain has been appropriately restricted.
24. Outline the general steps to find the inverse of a rational function of the form $f(x) = \frac{ax+b}{cx+d}$.
25. If $f(x) = 5x^3 + 1$, and you find its inverse $f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$, how can you verify that these functions are indeed inverses?
26. In the gravel mound problem (Example 5), why is the domain of the volume function $V(r) = \frac{2}{3}\pi r^3$ considered to be $r \geq 0$? What does this imply for the domain of its inverse function?
27. Can all polynomial functions have an inverse if their domain is restricted? Explain why or why not, considering a constant function as an example.
28. When solving for n in terms of C in Example 6 for the function $C = \frac{20+0.4n}{100+n}$, what algebraic steps are involved in isolating n?
29. Explain why we cannot find inverse functions for all polynomial functions.
30. Why must we restrict the domain of a quadratic function when finding its inverse?
31. When finding the inverse of a radical function, what restriction will we need to make?
32. The inverse of a quadratic function will always take what form?

Homework

Algebraic

33. Show that the function $f(x) = a - x$ is its own inverse for all real numbers a.

For the following exercises, find $f^{-1}(x)$ for each function.

34. $f(x) = x + 3$

35. $f(x) = x + 5$
36. $f(x) = 2 - x$
37. $f(x) = 3 - x$
38. $f(x) = \frac{x}{x+2}$
39. $f(x) = \frac{2x+3}{5x+4}$

For the following exercises, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

40. $f(x) = (x + 7)^2$
41. $f(x) = (x - 6)^2$
42. $f(x) = x^2 - 5$
43. Given $f(x) = \frac{x}{2+x}$ and $g(x) = \frac{2x}{1-x}$:
    a. Find $f(g(x))$ and $g(f(x))$.
    b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

For the following exercises, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

44. $f(x) = \sqrt[3]{x - 1}$ and $g(x) = x^3 + 1$
45. $f(x) = -3x + 5$ and $g(x) = \frac{x-5}{-3}$

For the following exercises, find the inverse of the function on the given domain.

46. $f(x) = (x - 4)^2$, $[4, \infty)$
47. $f(x) = (x + 2)^2$, $[-2, \infty)$
48. $f(x) = (x + 1)^2 - 3$, $[-1, \infty)$
49. $f(x) = 2 - \sqrt{3 + x}$, $[-3, \infty)$
50. $f(x) = 3x^2 + 5$, $(-\infty, 0]$
51. $f(x) = 12 - x^2$, $[0, \infty)$
52. $f(x) = 9 - x^2$, $[0, \infty)$
53. $f(x) = 2x^2 + 4$, $[0, \infty)$

For the following exercises, find the inverse of the functions.

54. $f(x) = x^3 + 5$
55. $f(x) = 3x^3 + 1$
56. $f(x) = 4 - x^3$
57. $f(x) = 4 - 2x^3$

For the following exercises, find the inverse of the functions.

58. $f(x) = \sqrt{2x + 1}$
59. $f(x) = \sqrt{3 - 4x}$
60. $f(x) = 9 + \sqrt{4x - 4}$
61. $f(x) = \sqrt{6x - 8} + 5$
62. $f(x) = 9 + 2\sqrt[3]{x}$
63. $f(x) = 3 - \sqrt[3]{x}$
64. $f(x) = \frac{2}{x+8}$
65. $f(x) = \frac{3}{x-4}$
66. $f(x) = \frac{x+3}{x+7}$
67. $f(x) = \frac{x-2}{x+7}$
68. $f(x) = \frac{3x+4}{5-4x}$
69. $f(x) = \frac{5x+1}{2-5x}$
70. $f(x) = x^2 + 2x$, $[-1, \infty)$
71. $f(x) = x^2 + 4x + 1$, $[-2, \infty)$
72. $f(x) = x^2 - 6x + 3$, $[3, \infty)$

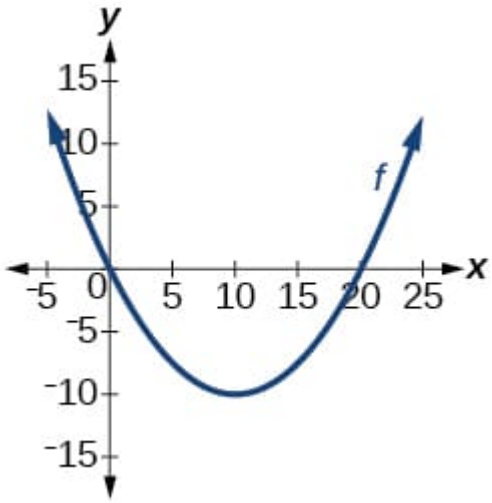
Graphical

For the following exercises, use graphing technology to determine whether each function is one-to-one.

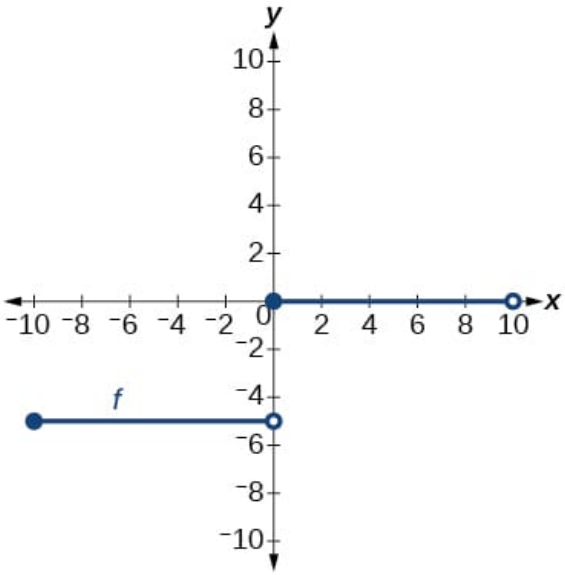
73. $f(x) = \sqrt{x}$
74. $f(x) = \sqrt[3]{3x+1}$
75. $f(x) = -5x + 1$
76. $f(x) = x^3 - 27$

For the following exercises, determine whether the graph represents a one-to-one function.

77.

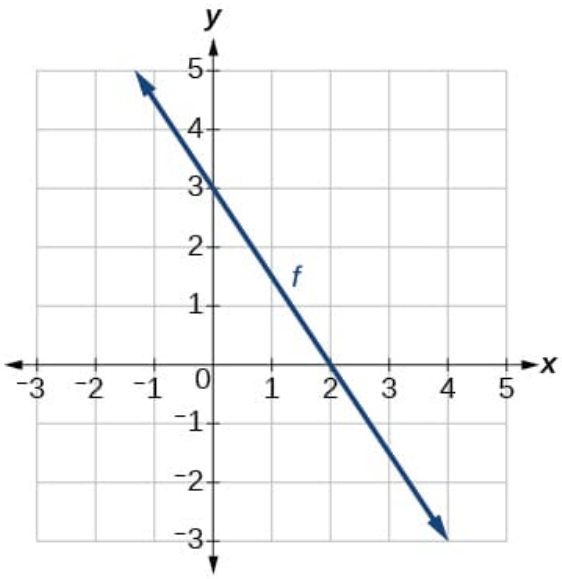


78.



For the following exercises, use the graph of f shown in Figure 3.1.2.1.

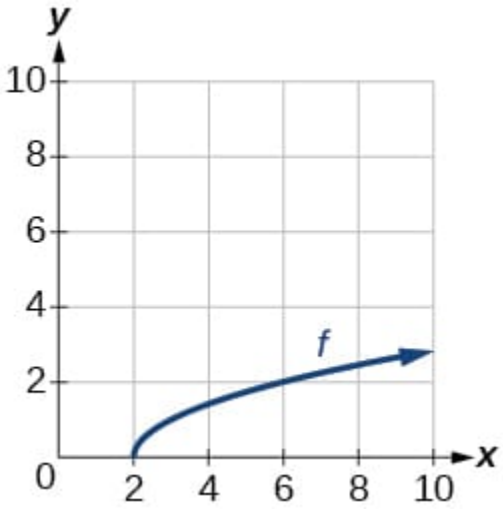
Figure 3.1.2.1



79. Find $f(0)$.
80. Solve $f(x) = 0$.
81. Find $f^{-1}(0)$.
82. Solve $f^{-1}(x) = 0$.

For the following exercises, use the graph of the one-to-one function shown in Figure 3.1.2.2**.**

Figure 3.1.2.2



83. Sketch the graph of f^{-1}.
84. Find $f(6)$ and $f^{-1}(2)$.
85. If the complete graph of f is shown, find the domain of f.
86. If the complete graph of f is shown, find the range of f.

For the following exercises, find the inverse of the function and graph both the function and its inverse.

87. $f(x) = x^2 + 2, x \geq 0$
88. $f(x) = 4 - x^2, x \geq 0$
89. $f(x) = (x + 3)^2, x \geq -3$

90. $f(x) = (x-4)^2$, $x \geq 4$
91. $f(x) = x^3 + 3$
92. $f(x) = 1 - x^3$
93. $f(x) = x^2 + 4x$, $x \geq -2$
94. $f(x) = x^2 - 6x + 1$, $x \geq 3$
95. $f(x) = \frac{2}{x}$
96. $f(x) = \frac{1}{x^2}$, $x \geq 0$

For the following exercises, use a graph to help determine the domain of the functions.

97. $f(x) = \sqrt{\frac{(x+1)(x-1)}{x}}$
98. $f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$
99. $f(x) = \sqrt{\frac{x(x+3)}{x-4}}$
100. $f(x) = \sqrt{\frac{x^2-x-20}{x-2}}$
101. $f(x) = \sqrt{\frac{9-x^2}{x+4}}$

Numeric

For the following exercises, evaluate or solve, assuming that the function f is one-to-one.

102. If $f(6) = 7$, find $f^{-1}(7)$.
103. If $f(3) = 2$, find $f^{-1}(2)$.
104. If $f^{-1}(-4) = -8$, find $f(-8)$.
105. If $f^{-1}(-2) = -1$, find $f(-1)$.

For the following exercises, use the values listed in Table 3.1.2.1 to evaluate or solve.

Table 3.1.2.1

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	8	0	7	4	2	6	5	3	9	1

106. Find $f(1)$.
107. Solve $f(x) = 3$.
108. Find $f^{-1}(0)$.
109. Solve $f^{-1}(x) = 7$.
110. Use the tabular representation of f in Table 3.1.2.2 to create a table for $f^{-1}(x)$.

Table 3.1.2.2

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

Technology

For the following exercises, find the inverse function. Then, graph the function and its inverse.

111. $f(x) = \frac{3}{x-2}$
112. $f(x) = x^3 - 1$
113. Find the inverse function of $f(x) = \frac{1}{x-1}$. Use graphing technology to find its domain and range. Write the domain and range in interval notation.

For the following exercises, use a calculator to graph the function. Then, using the graph, give three points on the graph

of the inverse with y-coordinates given.

14. $f(x) = x^3 - x - 2$, $y = 1, 2, 3$
15. $f(x) = x^3 + x - 2$, $y = 0, 1, 2$
16. $f(x) = x^3 + 3x - 4$, $y = 0, 1, 2$
17. $f(x) = x^3 + 8x - 4$, $y = -1, 0, 1$
18. $f(x) = x^4 + 5x + 1$, $y = -1, 0, 1$

Extensions

For the following exercises, find the inverse of the functions with a, b, c positive real numbers.

19. $f(x) = ax^3 + b$
20. $f(x) = x^2 + bx$
21. $f(x) = \sqrt{ax^2 + b}$
22. $f(x) = \sqrt[3]{ax + b}$
23. $f(x) = \frac{ax+b}{x+c}$

Real-World Applications

24. To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x) = \frac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.
25. The circumference C of a circle is a function of its radius given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.
26. A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t, in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

For the following exercises, determine the function described and then use it to answer the question.

27. An object dropped from a height of 200 meters has a height, $h(t)$, in meters after t seconds have lapsed, such that $h(t) = 200 - 4.9t^2$. Express t as a function of height, h, and find the time to reach a height of 50 meters.
28. An object dropped from a height of 600 feet has a height, $h(t)$, in feet after t seconds have elapsed, such that $h(t) = 600 - 16t^2$. Express t as a function of height h, and find the time to reach a height of 400 feet.
29. The volume, V, of a sphere in terms of its radius, r, is given by $V(r) = \frac{4}{3}\pi r^3$. Express r as a function of V, and find the radius of a sphere with volume of 200 cubic feet.
30. The surface area, A, of a sphere in terms of its radius, r, is given by $A(r) = 4\pi r^2$. Express r as a function of A, and find the radius of a sphere with a surface area of 1000 square inches.
31. A container holds 100 ml of a solution that is 25 ml acid. If n ml of a solution that is 60% acid is added, the function $C(n) = \frac{25+.6n}{100+n}$ gives the concentration, C, as a function of the number of ml added, n. Express n as a function of C and determine the number of mL that need to be added to have a solution that is 50% acid.
32. The period T, in seconds, of a simple pendulum as a function of its length l, in feet, is given by $T(l) = 2\pi\sqrt{\frac{l}{32.2}}$. Express l as a function of T and determine the length of a pendulum with period of 2 seconds.
33. The volume of a cylinder , V, in terms of radius, r, and height, h, is given by $V = \pi r^2 h$. If a cylinder has a height of 6 meters, express the radius as a function of V and find the radius of a cylinder with volume of 300 cubic meters.
34. The surface area, A, of a cylinder in terms of its radius, r, and height, h, is given by $A = 2\pi r^2 + 2\pi rh$. If the height of the cylinder is 4 feet, express the radius as a function of A and find the radius if the surface area is 200 square feet.
35. The volume of a right circular cone, V, in terms of its radius, r, and its height, h, is given by $V = \frac{1}{3}\pi r^2 h$. Express r in terms of V if the height of the cone is 12 inches and find the radius of a cone with volume of 50 cubic inches.
36. Consider a cone with height of 30 feet. Express the radius, r, in terms of the volume, V, and find the radius of a cone with volume of 1000 cubic feet.

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3.2: Radical Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Evaluate a Radical Function

In this section we will extend our previous work with functions to *officially* include radicals. A lot of the material in this section will feel like a review of material from early in the course; however, after a few brief definitions, theorems, and examples at this "review" level, we will increase the level to be a bit more challenging.

Recall from Algebra that if a function is defined by a radical expression, we call it a **radical function**. In particular,

- the square root function is $f(x) = \sqrt{x}$.
- the cube root function is $f(x) = \sqrt[3]{x}$.

For completeness, we include the official definition of a radical function here.

Definition: Radical Function

A **radical function** is a function that is defined by a radical expression.

Let's take a moment to review how to evaluate radical functions.

Example 3.2.1

For the function $f(x) = \sqrt{2x-1}$, find

a. $f(5)$
b. $f(-2)$

Solutions available at the end of this section.

Example $3.2.1b$ illustrates an important point—when working with functions (radical or not), we are assuming that the number system we're interested in is the real number system. Therefore, we do not want imaginary numbers as function values.

Checkpoint 3.2.1

For the function $f(x) = \sqrt{3x-2}$, find

a. $f(6)$
b. $f(0)$

Answers available at the end of this section.

Given a cube root function, we follow the same procedure.

Example 3.2.2

For the function $g(x) = \sqrt[3]{x-6}$, find

a. $g(14)$
b. $g(-2)$

Solutions available at the end of this section.

As Example $3.2.2b$ shows, one of the major differences between a square root function and a cube root function is that we can evaluate the cube root of a negative number. In fact, as you (hopefully) remember from Algebra, the domain of a radical function with an odd index is all real numbers, while the domain of an even-indexed radical function is restricted so that the radicand (the argument of the radical function) is non-negative. We will write this as a theorem in a little bit.

Checkpoint 3.2.2

For the function $g(x) = \sqrt[3]{3x-4}$, find

a. $g(4)$
b. $g(1)$

Answers available at the end of this section.

Again, as you should recall from Algebra, radical functions need not be restricted to square roots and cube roots.

Example 3.2.3

For the function $f(x) = \sqrt[4]{5x-4}$, find

a. $f(4)$
b. $f(-12)$

Solutions available at the end of this section.

Checkpoint 3.2.3

For the function $f(x) = \sqrt[4]{3x+4}$, find

a. $f(4)$
b. $f(-1)$

Answers available at the end of this section.

The Domain of a Radical Function

To find the domain of a radical function, we use our properties of radicals from Algebra. For a radical with an even index, we said the radicand had to be greater than or equal to zero. This is because even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

Theorem: Properties of $\sqrt[n]{a}$

When n is an even number and:

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.

Therefore, to find the domain of a radical function with an even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number. Again, for completeness, we include this statement as a theorem.

Theorem: Domain of a Radical Function

When the index of the radical is even, the domain of the radical function consists of all real numbers for which the radicand is greater than or equal to zero. When the index of the radical is odd, the domain of a radical function is all real numbers.

Example 3.2.4

Find the domain of the function, $f(x) = \sqrt{3x - 4}$. Write the domain in interval notation.

Solution available at the end of this section.

Checkpoint 3.2.4

Find the domain of the function, $f(x) = \sqrt{6x - 5}$. Write the domain in interval notation.

Answer available at the end of this section.

Up to this point in this section, we really have not stated anything new. Everything we have covered should feel like a repeat from your previous Algebra classes. Now we begin to improve upon these older skills to introduce examples more relevant to Calculus.

Example 3.2.5

Find the domain of the function, $g(x) = \sqrt{\frac{6}{x-1}}$. Write the domain in interval notation.

Solution available at the end of this section.

Checkpoint 3.2.5

Find the domain of the function, $f(x) = \sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

Answer available at the end of this section.

Example 3.2.6

Find the domain of the function, $h(x) = \sqrt[6]{-21x^2 + 2x + 8}$. Write the domain in interval notation.

Solution available at the end of this section.

Checkpoint 3.2.6

Find the domain of the function, $f(x) = \sqrt{x^3 + x^2 - 4x - 4}$. Write the domain in interval notation.

Answer available at the end of this section.

Example 3.2.6 and Checkpoint 3.2.6 illustrate the need to keep our skills with the material we previously covered in this course sharp. This is the way of Calculus—once you have learned a skill, it is likely going to come into play at some point in the future.

Example 3.2.7: Finding the Domain of a Radical Function Composed with a Rational Function

Find the domain of the function $f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$.

Solution available at the end of this section.

Domains of Non-Radical Functions Containing Radicals

In Calculus, you often work with functions that *contain* radical expressions *in addition* to containing non-radical expressions. Technically speaking, these are not called radical functions—they are simply referred to as compositions of functions (a catchall phrase encompassing most functions). Examples of these styles of functions are:

- $f(x) = 3x\sqrt[3]{2 - x}$
- $g(x) = \sqrt{2 - \sqrt[4]{x + 3}}$
- $k(x) = \frac{2x}{\sqrt{x^2 - 1}}$

Being able to identify the domains of functions like these is going to be a necessary skill for success in Calculus.

To find the domains of functions like these, we use a method akin to the Order of Operations—working from the innermost function to the outermost. It is often easier to use the method of sign charts when finding the domains of composite functions.

When creating a sign chart for a given function, we are looking for when a denominator becomes zero, when an even-indexed radical has a negative as its radicand, or when the argument of a logarithm (which we haven't encountered yet) is zero or negative. There are other, more rare, cases where domain issues occur, but we will approach those only as needed.

Let's spend some time investigating how such domains can be determined.

Example 3.2.8

State the domains of the following functions.

a. $f(x) = 3x\sqrt[3]{2 - x}$
b. $g(x) = \sqrt{2 - \sqrt[4]{x + 3}}$
c. $k(x) = \frac{2x}{\sqrt{x^2 - 1}}$

Solutions available at the end of this section.

Graphing Higher-Indexed Radical Functions

We already learned the graphs of the square root and cube root functions, however, we have yet to graph higher-indexed radical functions.

It is worth remarking that we could define $f(x) = \sqrt[n]{x}$ functionally as the inverse of $g(x) = x^n$ with the stipulation that when n is even, the domain of g is restricted to $[0, \infty)$. From what we know about $g(x) = x^n$ from our investigations of polynomials and their inverses, we can produce the graphs of $f(x) = \sqrt[n]{x}$ by reflecting the graphs of $g(x) = x^n$ across the line $y = x$. Below are the graphs of $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$. The point $(0, 0)$ is indicated as a reference. The axes are hidden so we can see the vertical steepening near $x = 0$ and the horizontal flattening as $x \to \infty$.

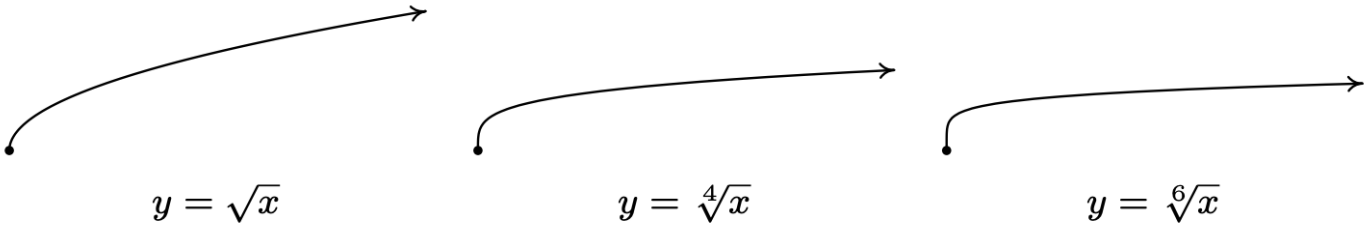


Figure 3.2.5

The odd-indexed radical functions also follow a predictable trend—steepening near $x = 0$ and flattening as $x \to \pm\infty$. In the exercises, you'll have a chance to graph some basic radical functions using transformations.

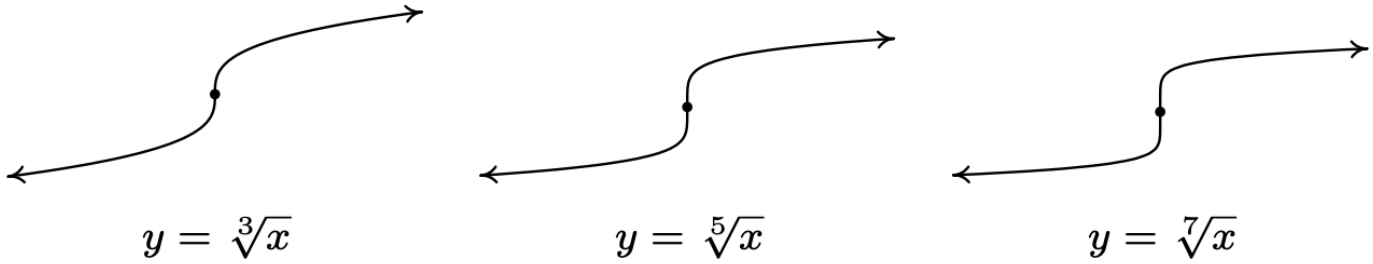


Figure 3.2.6

Interactive Element: Shapes of nth-Root Functions

The following Interactive Element might help you visualize how the shape of $y = \sqrt[n]{x}$ changes as the index n increases.

Interact: Drag the red handle left and right to change the index n in $y = \sqrt[n]{x}$ (shown for $x \geq 0$).

Observation: Every n^{th}-root curve passes through $(0, 0)$ and $(1, 1)$; as n grows the curve steepens near $x = 0$ and flattens as x increases.

Example 3.2.9

Sketch a graph of

$$f(x) = -\frac{1}{2}\sqrt[6]{2x + 8} + 1.$$

Solution available at the end of this section.

Applications Involving Radicals

Radical functions often arise in Calculus as a result of a process called **optimization**. The following example illustrates one such optimization problem (but we are allowing ourselves technology because we don't have Calculus yet).

Example 3.2.10

Carl wishes to get high speed internet service installed in his remote Sasquatch observation post located 30 miles from Route 117. The nearest junction box is located 50 miles down the road from the post, as indicated in the diagram below. Suppose it costs $\$15$ per mile to run cable along the road and $\$20$ per mile to run cable off of the road.

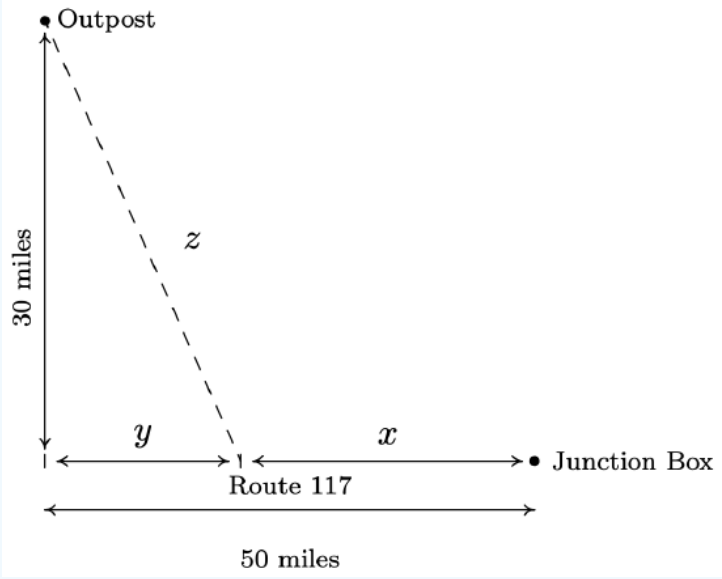


Figure 3.2.8

a. Express the total cost C of connecting the junction box to the post as a function of x, the number of miles the cable is run along Route 117 before heading off road directly towards the post. Determine a reasonable applied domain for the problem.
b. Use graphing technology to graph $y = C(x)$ on its domain. What is the minimum cost? How far along Route 117 should the cable be run before turning off the road?

Solutions available at the end of this section.

Success in College: Dealing With Failure (Part 1)

People sometimes do not achieve their goals. Athletes lose games. Students fail tests or classes. A job interview does not go well. It is your responsibility to learn how to cope with failure.

Former Chicago Bears football coach, Mike Ditka, said, "Success isn't permanent, and failure isn't fatal." This idea applies beyond the game of football. Failure on one test or even failing a class does not mean you cannot earn your degree. However, a continuing pattern of failures can prevent success.

When failure happens, you need to take action. Change the way you study. Redo the problems you missed on a test. Make an appointment with your professor to learn what you don't understand. If you are failing a course, meet with your advisor. Perhaps you should consider withdrawing. Make sure you know the effect of withdrawing or failing on any financial aid you receive.

Failure is not fun. It is tempting to skip class the day after a nasty test. It is easy to avoid talking to your professor about it. However, it is your responsibility to be honest about failure. Ignoring failure does not make it go away.

1. A class has four tests. A student taking this class fails one test with a score of 45%. The student's other test scores were 72%, 72%, and 79%. Find the student's test average for this class.
2. The effect of a bad test score depends on the number of tests in the class. It also depends on how much test scores contribute to the final course grade. If you had the same test average in this class as the student in the last question, could you still pass? Explain.

Success in College: Memorization Learning Preferences

Students may be inclined to approach their courses in one of three ways. Those with a reproducing orientation tend to take a surface approach to learning, relying on rote memorization and mechanical formula substitution and making little or no effort to understand the material being taught. Those with a meaning orientation tend to adopt a deep approach, probing and questioning and exploring the limits of applicability of new material. Those with an achieving orientation tend to use a strategic approach, doing whatever is necessary to get the highest grade they can, taking a surface approach if that suffices and a deep approach when necessary.

Source: R. Felder, R. Brent , Journal of Engineering Education, 94 (1), 57-72 (2005).

1. Students have different motivations for going to math class. What do you think the main motivation is for students who have a reproducing orientation and who take a surface approach?
2. What do you think is the main motivation for students who have a meaning orientation and who take a deep approach?

Answers

Section Preview

Typical textbooks do not have a full chapter dedicated to radical functions and equations involving radicals; however, this topic just seemed disjoint enough to warrant its own chapter. In this section, we officially introduce the concept of the radical function. While they have seen most of this before, it's best to spend some extra time here on domains of radical functions as well as difference quotients. Finally, getting a basic idea of the shapes of the nth root functions is useful for Calculus.

Section Preview

- **Geometry**
  - The Pythagorean Theorem, Distance Formula, and Midpoint Formula: Setting up the cost model in the optimization application (Example 10) requires $y^2 + 30^2 = z^2$; used only in that example.

Section Preview

- Evaluate a radical function
- Find the domain of a radical function
- Graph radical functions

Example 3.2.1

a. To evaluate $f(5)$, substitute 5 for x.

$$\begin{aligned} f(5) &= \sqrt{2 \cdot 5 - 1} && \text{(substituting)} \\ &= \sqrt{10 - 1} && \text{(simplifying)} \\ &= \sqrt{9} && \text{(simplifying)} \\ &= 3 && \text{(simplifying)} \end{aligned}$$

b. We follow the same process as in part a.

$$\begin{aligned} f(-2) &= \sqrt{2 \cdot (-2) - 1} && \text{(substituting)} \\ &= \sqrt{-4 - 1} && \text{(simplifying)} \\ &= \sqrt{-5} && \text{(simplifying)} \end{aligned}$$

Since the square root of a negative number is not a real number, the function does not have a (real) value at $x = -2$.

Checkpoint 3.2.1

a. $f(6) = 4$
b. no value at $x = 0$

Example 3.2.2

a. We substitute 14 into $g(x)$.

$$\begin{aligned} g(14) &= \sqrt[3]{14 - 6} && \text{(substituting)} \\ &= \sqrt[3]{8} && \text{(simplifying)} \\ &= 2 && \text{(simplifying)} \end{aligned}$$

b. As we did with part a, we substitute.

$$\begin{aligned} g(-2) &= \sqrt[3]{-2-6} && \text{(substituting)} \\ &= \sqrt[3]{-8} && \text{(simplifying)} \\ &= -2 && \text{(simplifying)} \end{aligned}$$

Checkpoint 3.2.2

a. $g(4) = 2$
b. $g(1) = -1$

Example 3.2.3

a.

$$\begin{aligned} f(4) &= \sqrt[4]{5 \cdot 4 - 4} && \text{(substituting)} \\ &= \sqrt[4]{20 - 4} && \text{(simplifying)} \\ &= \sqrt[4]{16} && \text{(simplifying)} \\ &= 2 && \text{(simplifying)} \end{aligned}$$

b.

$$\begin{aligned} f(-12) &= \sqrt[4]{5 \cdot (-12) - 4} && \text{(substituting)} \\ &= \sqrt[4]{-60 - 4} && \text{(simplifying)} \\ &= \sqrt[4]{-64} && \text{(simplifying)} \end{aligned}$$

Since the fourth root of a negative number is not a real number, the function does not have a (real) value at $x = -12$.

Checkpoint 3.2.3

a. $f(4) = 2$
b. $f(-1) = 1$

Example 3.2.4

Since the function, $f(x) = \sqrt{3x - 4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0.

$$\begin{aligned} 3x - 4 &\geq 0 && \text{(setting the radicand greater than or equal to 0)} \\ \implies 3x &\geq 4 && \text{(adding 4 to both sides)} \\ \implies x &\geq \frac{4}{3} && \text{(dividing both sides by 3)} \end{aligned}$$

The domain of $f(x) = \sqrt{3x - 4}$ is all values $x \geq \frac{4}{3}$, and we write it in interval notation as $\left[\frac{4}{3}, \infty\right)$.

Checkpoint 3.2.4

$\left[\frac{5}{6}, \infty\right)$

Example 3.2.5

Since the function, $g(x) = \sqrt{\frac{6}{x-1}}$, has a radical with an even index, we know the radicand must be greater than or equal to 0. Moreover, since the numerator is a nonzero constant, the radicand cannot be zero.

For $\frac{6}{x-1}$ to be greater than zero, the denominator must be positive (since the numerator is positive). Therefore, we set $x - 1 > 0$ and solve. This yields $x > 1$. So far, our *candidate* domain is $(1, \infty)$.

An additional consideration when dealing with functions involving fractions is that the denominator cannot be zero. Therefore, we know $x \neq 1$; however, since 1 is not in the interval $(1, \infty)$, we don't have to worry about this extra restriction.

Putting this together we get the domain is $(1, \infty)$.

Checkpoint 3.2.5

$(-3, \infty)$

Example 3.2.6

The even index on this radical function tells us that the radicand must be greater than or equal to zero. Therefore, we need to solve

$$-21x^2 + 2x + 8 \geq 0.$$

Proceeding as we did when solving quadratic inequalities, we factor this quadratic to get

$$(-3x + 2)(7x + 4) \geq 0.$$

We then sketch a graph of the quadratic expression on the left side of the inequality to find when the curve is at or above the x-axis.

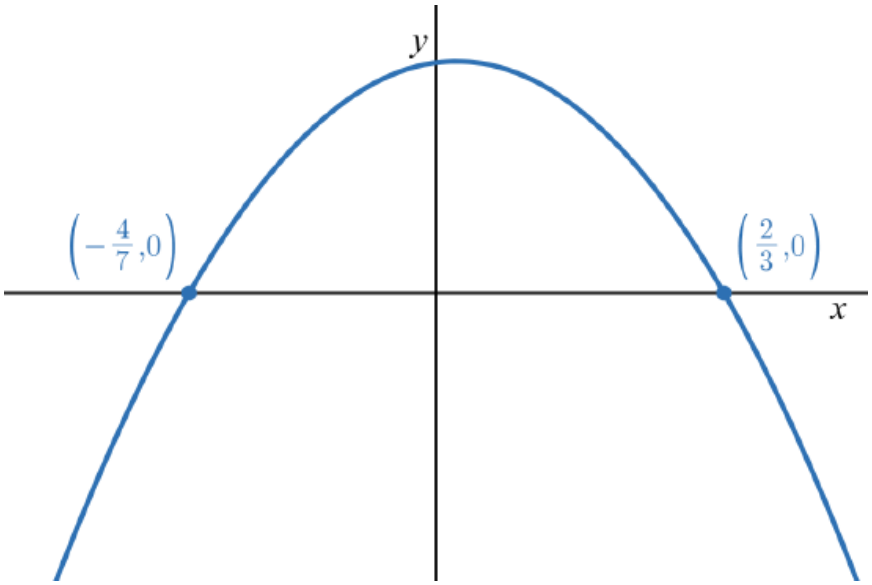


Figure 3.2.1

We can see from the graph that the inequality is satisfied when $-\frac{4}{7} \leq x \leq \frac{2}{3}$. Hence, the domain of the radical function is $\left[-\frac{4}{7}, \frac{2}{3}\right]$.

Checkpoint 3.2.6

$[-2,-1] \cup [2,\infty)$

Example 3.2.7**: Finding the Domain of a Radical Function Composed with a Rational Function**

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where $\frac{(x+2)(x-3)}{x-1} \geq 0$. To determine the intervals on which the rational expression is positive, we could test some values in the expression or sketch a graph. While both approaches work equally well, we will continue to reinforce the graphing skills we learned previously and choose to sketch the rational function (see Figure 3.2.2).

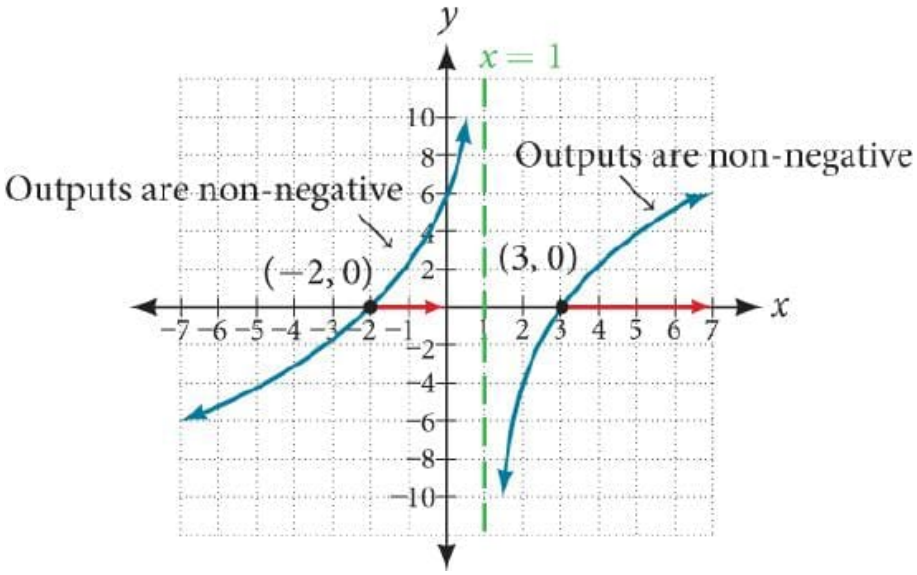


Figure 3.2.2

From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f(x)$ will be defined. $f(x)$ has domain $-2 \leq x < 1$ or $x \geq 3$, or in interval notation, $[-2,1) \cup [3,\infty)$.

Example 3.2.8

a. As far as domain is concerned, $f(x)$ has no denominators and no even roots, which means there are no domain issues. Hence, the domain of this function is $(-\infty,\infty)$.

b. In $g(x) = \sqrt{2-\sqrt[4]{x+3}}$, we have two radicals both of which are even-indexed. To satisfy the innermost radical, $\sqrt[4]{x+3}$, we require $x+3 \geq 0$ or $x \geq -3$. Thus, our *candidate* domain of $g(x)$ is $[-3,\infty)$. This is the largest possible domain for the function g; however, we have more to consider, and this domain might shrink. "Zooming out" in our function, we see we need $2-\sqrt[4]{x+3} \geq 0$. While it may be tempting to write this as $2 \geq \sqrt[4]{x+3}$ and take both sides to the fourth power, there are times when this technique will produce erroneous results. Instead, we solve $2-\sqrt[4]{x+3} \geq 0$ using a sign chart. We take a moment to find out when $2-\sqrt[4]{x+3} = 0$.

$$\begin{aligned} 2-\sqrt[4]{x+3} &= 0 \\ \Longrightarrow \quad 2 &= \sqrt[4]{x+3} && \left(\text{adding } \sqrt[4]{x+3} \text{ to both sides}\right) \\ \Longrightarrow \quad 2^4 &= \left(\sqrt[4]{x+3}\right)^4 && \text{(raising both sides to the fourth power)} \\ \Longrightarrow \quad 16 &= x+3 && \text{(simplifying)} \\ \Longrightarrow \quad 13 &= x && \text{(subtracting 3 from both sides)} \end{aligned}$$

One thing to be careful of throughout all of Mathematics is raising both sides of an equation to an even power—this *sometimes* introduces extraneous solutions. Therefore, we need to check to see if $x=13$ is an extraneous solution. Substituting $x=13$ into the expression $\sqrt{2-\sqrt[4]{x+3}}$ does check since

$$2-\sqrt[4]{x+3} = 2-\sqrt[4]{13+3} = 2-\sqrt[4]{16} = 2-2 = 0.$$

Remember, we are currently trying to figure out when $2-\sqrt[4]{x+3} \geq 0$. We make a sign chart with marked with $x=13$ (where the expression on the left side of the inequality is zero) and, since the *candidate* domain is $[-3,\infty)$, we start our number line at $x=-3$.

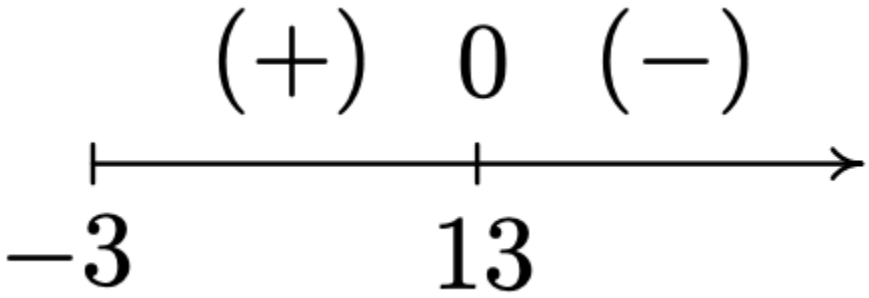


Figure 3.2.3

We find $2-\sqrt[4]{x+3} \geq 0$ on $[-3,13]$ so this is the domain of g.

c. To find the domain of k, we have both an even root and a denominator to concern ourselves with. To satisfy the square root, $x^2-1 \geq 0$. Setting $r(x)=x^2-1$, we find the zeros of r to be $x=\pm 1$, and we find the sign chart of r to be

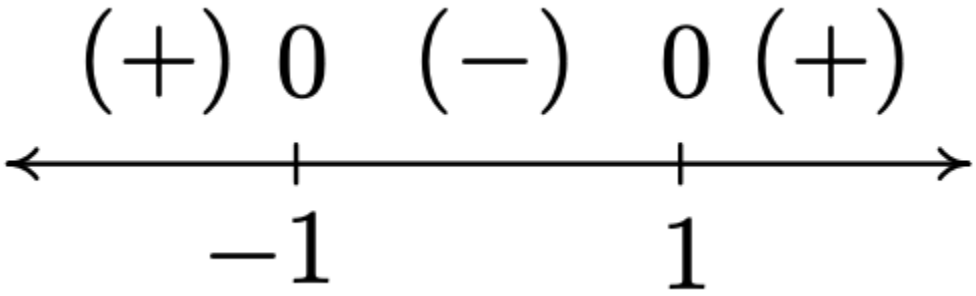


Figure 3.2.4

We find $x^2-1 \geq 0$ for $(-\infty,-1] \cup [1,\infty)$. This is our *candidate* domain, which might shrink as we look further into the problem.

To keep the denominator of $k(x)$ away from zero, we set $\sqrt{x^2-1}=0$. We leave it to the reader to verify the solutions are $x=\pm 1$, both of which must be excluded from the domain. Hence, the domain of k is $(-\infty,-1) \cup (1,\infty)$.

Example 3.2.9

We know the base graph of this function (it is in the images above), so we just need to apply the transformations. We begin by modifying the function to get it into an appropriate form.

$$f(x) = -\frac{1}{2}\sqrt[6]{2x+8}+1 = -\frac{1}{2}\sqrt[6]{2\,(x+4)}+1.$$

Reading from left to right, the base graph of $y=\sqrt[6]{x}$:

- gets reflected about the x-axis ($y=-\sqrt[6]{x}$)
- gets compressed vertically by a factor of $\frac{1}{2}$ ($y=-\frac{1}{2}\sqrt[6]{x}$)
- gets compressed horizontally by a factor of $\frac{1}{2}$ ($y=-\frac{1}{2}\sqrt[6]{2x}$)
- gets shifted 4 units to the left ($y=-\frac{1}{2}\sqrt[6]{2\,(x+4)}$)
- gets shifted 1 unit up ($y=-\frac{1}{2}\sqrt[6]{2\,(x+4)}+1$)

The graph of $f(x)$ is shown below.

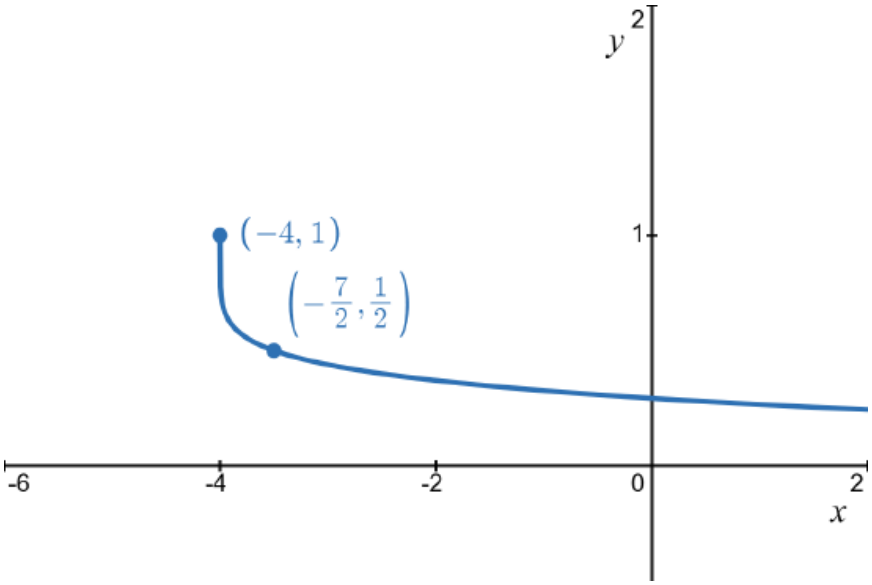


Figure 3.2.7

Example 3.2.10

a. The cost is broken into two parts: the cost to run cable along Route 117 at $\$15$ per mile, and the cost to run it off road at $\$20$ per mile. Since x represents the miles of cable run along Route 117, the cost for that portion is $15x$. From the diagram, we see that the number of miles the cable is run off road is z, so the cost of that portion is $20z$. Hence, the total cost is $C = 15x + 20z$.

Our next goal is to determine z as a function of x. The diagram suggests we can use the Pythagorean Theorem to get

$$y^2 + 30^2 = z^2.$$

But we also see $x + y = 50$, so that $y = 50 - x$. Hence,

$$z^2 = (50 - x)^2 + 900.$$

Solving for z, we obtain

$$z = \pm\sqrt{(50 - x)^2 + 900}.$$

Since z represents a distance, we choose $z = \sqrt{(50 - x)^2 + 900}$ so that our cost as a function of x only is given by

$$C(x) = 15x + 20\sqrt{(50 - x)^2 + 900}.$$

From the context of the problem, we have the applied domain of $x \in [0, 50]$.

b. Graphing $y = C(x)$ using Desmos, we find the relative minimum (which is also the absolute minimum in this case) to two decimal places to be $(15.98, 1146.86)$.

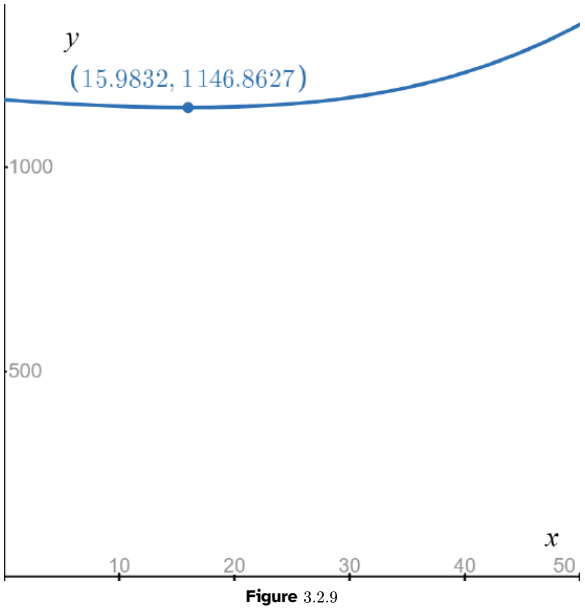


Figure 3.2.9

Here the x-coordinate tells us that in order to minimize cost, we should run 15.98 miles of cable along Route 117 and then turn off of the road and head towards the outpost. The y-coordinate tells us that the minimum cost, in dollars, to do so is $\$1146.86$. The ability to stream live SasquatchCasts? Priceless. The following Interactive Element shows the actual physical situation.

Interactive Element: Minimizing Cable Cost

The following Interactive Element might help you visualize how the total cost changes with where the cable leaves the road and heads for the post.

Interact: Drag the blue point along Route 117 to change x, the number of miles of cable run along the road before turning off toward the post.

Observation: The total cost falls to a minimum near $x \approx 15.98$ miles (about $\$1146.86$); turning off too early forces a long expensive diagonal, while running too far along the road overshoots.

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3.2.1: Resources and Key Concepts

Resources

Key Concepts

Definitions

- **Radical Function**: A function that is defined by a radical expression.

Theorems

- **Properties of $\sqrt[n]{a}$ (Regarding Real Numbers)**:
  - When n is an even number and $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
  - When n is an even number and $a < 0$, then $\sqrt[n]{a}$ is not a real number.
  - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Domain of a Radical Function**:
  - When the index of the radical is even, the domain of the radical function consists of all real numbers for which the radicand is greater than or equal to zero.
  - When the index of the radical is odd, the domain of a radical function consists of all real numbers (assuming the radicand itself is defined for all real numbers).

Common Mistakes

- **Ignoring Domain Restrictions for Even-Indexed Radicals**: Attempting to evaluate an even-indexed radical function for an input that makes the radicand negative, or failing to correctly identify the domain by setting the radicand ≥ 0.
- **Forgetting Denominator Restrictions within Radicands**: When the radicand of an even-indexed root is a rational expression, not only must the entire rational expression be ≥ 0, but the denominator of that rational expression also cannot be zero.

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3.2.2: Homework

Reading Questions

1. What is a radical function? Give an example of a square root function and a cube root function.
2. When evaluating radical functions, why is it important to consider the index of the radical (even vs. odd) if the radicand might be negative?
3. What is the primary condition that the radicand of an even-indexed radical function must satisfy for the function to yield a real number?
4. What is the domain of an odd-indexed radical function like $f(x) = \sqrt[3]{x-2}$?
5. How do you find the domain of an even-indexed radical function like $f(x) = \sqrt{3x-4}$?
6. If the radicand of an even-indexed radical function is a rational expression, such as $f(x) = \sqrt{\frac{x+2}{x-1}}$, what conditions must be met to determine the domain?
7. Describe the general shape of the graph of $y = \sqrt{x}$ and $y = \sqrt[3]{x}$.
8. How do transformations (shifts, stretches, reflections) affect the graph of a basic radical function like $y = \sqrt{x}$?
9. When finding the domain of a function like $g(x) = \sqrt[4]{2 - \sqrt[4]{x+3}}$, which radical's domain requirements should you address first?
10. What does it mean if, when evaluating a radical function for a specific input, you find the radicand of an even root is negative (e.g., $f(-2)$ for $f(x) = \sqrt{2x-1}$)?

Homework

In the following exercises, evaluate each function.

1. $f(x) = \sqrt{4x-4}$, find
   a. $f(5)$
   b. $f(0)$
2. $f(x) = \sqrt{6x-5}$, find
   a. $f(5)$
   b. $f(-1)$
3. $g(x) = \sqrt{6x+1}$, find
   a. $g(4)$
   b. $g(8)$
4. $g(x) = \sqrt{3x+1}$, find
   a. $g(8)$
   b. $g(5)$
5. $F(x) = \sqrt{3-2x}$, find
   a. $F(1)$
   b. $F(-11)$
6. $F(x) = \sqrt{8-4x}$, find
   a. $F(1)$
   b. $F(-2)$
7. $G(x) = \sqrt{5x-1}$, find
   a. $G(5)$
   b. $G(2)$
8. $G(x) = \sqrt{4x+1}$, find
   a. $G(11)$
   b. $G(2)$
9. $g(x) = \sqrt[3]{2x-4}$, find
   a. $g(6)$
   b. $g(-2)$
10. $g(x) = \sqrt[3]{7x-1}$, find
    a. $g(4)$

b. $g(-1)$

11. $h(x)=\sqrt[3]{x^2-4}$, find
    a. $h(-2)$
    b. $h(6)$
12. $h(x)=\sqrt[3]{x^2+4}$, find
    a. $h(-2)$
    b. $h(6)$
13. For the function $f(x)=\sqrt[4]{2x^3}$, find
    a. $f(0)$
    b. $f(2)$
14. For the function $f(x)=\sqrt[4]{3x^3}$, find
    a. $f(0)$
    b. $f(3)$
15. For the function $g(x)=\sqrt[4]{4-4x}$, find
    a. $g(1)$
    b. $g(-3)$
16. For the function $g(x)=\sqrt[4]{8-4x}$, find
    a. $g(-6)$
    b. $g(2)$

In the following exercises, find the domain of the function and write the domain in interval notation.

17. $f(x)=\sqrt{3x-1}$
18. $f(x)=\sqrt{4x-2}$
19. $g(x)=\sqrt{2-3x}$
20. $g(x)=\sqrt{8-x}$
21. $h(x)=\sqrt{\frac{5}{x-2}}$
22. $h(x)=\sqrt{\frac{6}{x+3}}$
23. $f(x)=\sqrt{\frac{x+3}{x-2}}$
24. $f(x)=\sqrt{\frac{x-1}{x+4}}$
25. $g(x)=\sqrt[3]{8x-1}$
26. $g(x)=\sqrt[3]{6x+5}$
27. $f(x)=\sqrt[3]{4x^2-16}$
28. $f(x)=\sqrt[3]{6x^2-25}$
29. $F(x)=\sqrt[4]{8x+3}$
30. $F(x)=\sqrt[4]{10-7x}$
31. $G(x)=\sqrt[5]{2x-1}$
32. $G(x)=\sqrt[5]{6x-3}$

In the following exercises, find the domain of the function, graph the function, and use the graph to determine the range.

33. $f(x)=\sqrt{x+1}$
34. $f(x)=\sqrt{x-1}$
35. $g(x)=\sqrt{x+4}$
36. $g(x)=\sqrt{x-4}$
37. $f(x)=\sqrt{x}+2$
38. $f(x)=\sqrt{x}-2$
39. $g(x)=2\sqrt{x}$
40. $g(x)=3\sqrt{x}$
41. $f(x)=\sqrt{3-x}$
42. $f(x)=\sqrt{4-x}$
43. $g(x)=-\sqrt{x}$
44. $g(x)=-\sqrt{x}+1$
45. $f(x)=\sqrt[3]{x+1}$
46. $f(x)=\sqrt[3]{x-1}$
47. $g(x)=\sqrt[3]{x+2}$

48. $g(x) = \sqrt[3]{x-2}$
49. $f(x) = \sqrt[3]{x}+3$
50. $f(x) = \sqrt[3]{x}-3$
51. $g(x) = \sqrt[3]{x}$
52. $g(x) = -\sqrt[3]{x}$
53. $f(x) = 2\sqrt[3]{x}$
54. $f(x) = -2\sqrt[3]{x}$

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3.3: Radical Equations

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

As is the theme in College Algebra, we begin this section with the assumption that you already have exposure to solving simple radical equations. We still spend a few moments recalling old techniques, but quickly jump into solving more complex radical equations.

Solving Radical Equations Containing One Radical

We start our trip down memory lane by recalling an old definition.

Definition: Radical Equation

An equation in which variables are part of a radicand is called a **radical equation**.

When solving these equations, our goal is to isolate the radical. At that point, our strategy will be to raise both sides of the equation to the power of the index. This will, in turn, eliminate the radical; however, we have something to watch out for when solving radical equations!

Caution: Solving Equations Containing Even-Indexed Radicals

Care must be taken when working with expressions and equations involving even-indexed radicals. This is due to the nature of such beasts. Remember, the argument of an even-indexed radical cannot be negative, nor can an even-indexed radical return a negative value.

If, while solving a radical equation, we obtain a "solution" that would otherwise cause an even-indexed radical to be negative or the argument of the radical to be negative, then we know this "solution" will not work. In such cases we call this an **extraneous solution** as we did when we solved rational equations.

Definition: Extraneous Solution

An **extraneous solution**, also known as a **spurious solution**, is the solution of a transformed version of the original equation that is not an actual solution of the original equation because it was excluded from the domain (of the original equation).

As the definition suggests, extraneous solutions are always a concern when functions within an equation have implicit domain restrictions (e.g., rational and radical functions).

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index n to the nth power. This will eliminate the radical.

Example 3.3.1: Solving a radical equation involving a single radical expression

Solve: $\sqrt{5n-4}-9=0$.

Solution available at the end of this section.

Checkpoint 3.3.1

Solve: $\sqrt{3m+2}-5=0$.

Answer available at the end of this section.

It is important to remember the following advice:

> Whenever you raise both sides of an equation to an even power, you must check your final solutions.

In truth, the reason why we (as mathematicians) ask you to do this is to catch any inconsistencies that might have slipped past you. For example, suppose we were asked to solve $\sqrt{3x-1}-7=-9$. We would start by isolating the radical to get $\sqrt{3x-1}=-2$. If you are paying attention (and not asleep at the "mathematical wheel"), you would notice that you have an even-indexed radical equaling a negative number—**this cannot happen!** At this point, you would stop the solution process and state that the solution doesn't exist (some people use the notation "DNE" for "does not exist").

However, if you didn't notice the negative on the right side, you would continue to solve by squaring both sides to get

$$3x-1=4 \implies 3x=5 \implies x=\frac{5}{3}.$$

As long as we develop the habit to check our solutions whenever we square both sides of an equation, we have a second safety net.

$$\sqrt{3\left(\frac{5}{3}\right)-1}-7=\sqrt{5-7}-7=\sqrt{-2}-7$$

which is not a real number. As such, it cannot become -9.

I often try to get my students to stop at each step of the solution process to think about whether or not the statement before them makes sense, but checking your work at the end is your second opportunity to catch the logical flaw.

Interactive Element: Why Squaring Can Create Extra Solutions

The following Interactive Element might help you visualize why squaring both sides of a radical equation can introduce "solutions" that don't actually work. The purpose is to "visually" solve $\sqrt{x}=x-b$. That is, we want to know when the blue curve meets the green line. This obviously happens at the green dot (unless you drag the purple point too far up), so we expect only one solution (or none, again, if the purple point is dragged too far up).

Interact: Drag the purple point along the y-axis to slide the line $y=x-b$ by changing the value of b.

Observation: The equation $\sqrt{x}=x-b$ is solved where the blue curve meets the green line (green point). Squaring both sides also allows $\sqrt{x}=-(x-b)$, the gray dashed mirror line. This introduces the false solution at the red point on that dashed line. That is, an *extraneous* solution. Drag the purple point far enough up and the curve and line no longer meet: no solution.

Example 3.3.2

Solve: $\sqrt{9k-2}+1=0$.

Solution available at the end of this section.

Having recognized the square root equaling a negative number in the last step of Example 3.3.2 saved us time. Had we not seen that issue, we would have continued on to get

$$\begin{aligned} & \sqrt{9k-2}+1 &=& \ 0 & \\ \implies & \sqrt{9k-2} &=& -1 & \text{(isolating the radical by subtracting 1 from both sides)} \\ \implies & 9k-2 &=& \ 1 & \text{(squaring both sides)} \\ \implies & 9k &=& \ 3 & \text{(adding 2 to both sides)} \\ \implies & k &=& \ \frac{1}{3} & \text{(dividing both sides by 9)} \end{aligned}$$

Because we squared both sides, we go back to the original equation and check our answer.

$$\sqrt{9\left(\frac{1}{3}\right)-2}+1=\sqrt{3-2}+1=\sqrt{1}+1=1+1=2\neq 0.$$

This is a **contradiction**. Hence, $k=\frac{1}{3}$ is an extraneous solution.

Checkpoint 3.3.2

Solve: $\sqrt{2r-3}+5=0$.

Answer available at the end of this section.

Example 3.3.3

Solve: $\sqrt{p-1}+1=p$.

Solution available at the end of this section.

Checkpoint 3.3.3

Solve: $\sqrt{x-2}+2=x$.

Answer available at the end of this section.

It should be obvious that when the index of the radical is n, we eventually raise both sides of the radical equation to the nth power.

Example 3.3.4

Solve: $\sqrt[3]{5x+1}+8=4$.

Solution available at the end of this section.

It's always a good idea to check your answers when solving equations in Mathematics; however, it's only truly required (at least for now) when we have raised both sides of an equation to an even power (e.g., when solving radical equations) or when we have

multiplied both sides of an equation by an expression containing a variable (e.g., when solving a rational equation). This is why we did not bother checking our solution in Example 3.3.4. We leave it to the reader to check that $x = -13$ works.

Checkpoint 3.3.4

Solve: $\sqrt[3]{4x-3}+8=5$

Answer available at the end of this section.

Solving Equations Containing Rational Exponents

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical equation. Since $a^{m/n} = \sqrt[n]{a^m}$, we first raise each side of the equation to the power of the denominator of the rational exponent. Once that is done, we take the mth root of both sides of the equation (or raise both sides to the $\frac{1}{m}{}^{\text{th}}$ power).

Example 3.3.5

Solve: $(3x-2)^{1/4}+3=5$.

Solution available at the end of this section.

Checkpoint 3.3.5

Solve: $(9x+9)^{1/4}-2=1$

Answer available at the end of this section.

As with radical equations, care must be taken when solving equations containing rational exponents. We must remember all of the techniques we previously learned. For example, in the equation $a^{4/7}=1$, we start by raising both sides to the 7th power to get

$$a^4 = 1^7 = 1.$$

However, we now have the variable a raised to an even power. Extraction of Roots tells us that $a = \pm 1$. This is important!

Example 3.3.6

Solve: $3\left(\frac{1}{2}x+1\right)^{2/3}-5=7$

Solution available at the end of this section.

Checkpoint 3.3.6

Solve: $6\left(3x-5\right)^{3/2}-4=2$

Answer available at the end of this section.

Solving Radical Equations with Two Radicals

If we have a radical equation containing two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first. In the next example, when one radical is isolated, the second radical is also isolated.

Example 3.3.7

Solve: $\sqrt[3]{4x-3}-\sqrt[3]{3x+2}=0$.

Solution available at the end of this section.

Checkpoint 3.3.7

Solve: $\sqrt[3]{7x+1}=\sqrt[3]{2x-5}$.

Answer available at the end of this section.

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we "rinse and repeat" by isolating the remaining radical and raising both sides of the equation to the power of the index again.

Example 3.3.8

Solve: $\sqrt{m}+1=\sqrt{m+9}$.

Solution available at the end of this section.

Checkpoint 3.3.8

Solve: $3-\sqrt{x}=\sqrt{x-3}$.

Answer available at the end of this section.

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation. This procedure will now work for any radical equation.

How To: Solve a Radical Equation

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
   If yes, repeat Step 1 and Step 2 again.
   If no, solve the new equation.
4. Check the answer in the original equation.

Be careful as you square binomials (as in the next example). Remember the pattern in $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Example 3.3.9

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution available at the end of this section.

Checkpoint 3.3.9

Solve: $\sqrt{x-1}+2=\sqrt{2x+6}$

Answer available at the end of this section.

Equations involving rational exponents are fertile territory to run into equations that are quadratic-in-form. Not only must you recognize their form, but you also must watch out for all the pitfalls mentioned throughout this section concerning checking your solutions.

Example 3.3.10: Revisiting Solving Equations Quadratic-in-Form

Solve: $x-2\sqrt{x}-8=0$.

Solution available at the end of this section.

Checkpoint 3.3.10

Solve: $\sqrt{x}-\sqrt[4]{x}-2=0$

Answer available at the end of this section.

While Example 3.3.10 is interesting, the following example showcases a more frequent occurrence.

Example 3.3.11: Revisiting Solving Equations Quadratic-in-Form

Solve: $x^{-2/3}-3x^{-1/3}-10=0$.

Solution available at the end of this section.

Applications Involving Radicals

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. The following Problem Solving Strategy is incredibly helpful as you progress through College Algebra and into Calculus.

Problem Solving Strategy

1. **Read** the problem carefully and get clarification on words or concepts that are foreign to you.
2. **Understand** what is being stated and asked by rephrasing statements into language you connect with. Draw a figure to represent the situation, if possible.
3. **Label** the given and wanted information. The wanted information should have a variable representing it.
4. **Translate** into an equation relating the given and wanted information by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good Algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **State** the answer to the question with a complete sentence.

One application of radicals has to do with the effect of gravity on falling objects. The following (non-memorizable) formula allows us to determine how long it will take a fallen object to hit the ground.

Theorem: Time for a Dropped Object to Strike the Ground

On Earth, if an object is dropped from a height of h feet (where h is not too large) and if we ignore air resistance, the time (in seconds) it will take to reach the ground is approximated by using the formula $t = \frac{\sqrt{h}}{4}$.

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h = 64$ into the formula.

$$\begin{aligned} & t = \frac{\sqrt{h}}{4} & \\ \Longrightarrow \quad & t = \frac{\sqrt{64}}{4} & (\text{substituting}) \\ \Longrightarrow \quad & t = \frac{8}{4} & (\text{simplifying}) \\ \Longrightarrow \quad & t = 2 & (\text{simplifying}) \end{aligned}$$

Therefore, it would take 2 seconds for an object dropped from a height of 64 feet to reach the ground (ignoring deceleration from air resistance).

Don't Memorize!

The previous theorem is (for most instructors) a non-memorizable theorem. It's just a shortcut to approximate the time it takes for an object to fall a certain distance (and, in truth, it is only partially accurate for short falls).

Example 3.3.12

Kaitlyn dropped her sunglasses from a bridge 400 feet above a river. How many seconds does it take for the sunglasses to reach the river?

Solution available at the end of this section.

Checkpoint 3.3.12

A window washer dropped a squeegee from a platform 196 feet above the sidewalk. How many seconds does it take for the squeegee to reach the sidewalk?

Answer available at the end of this section.

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Theorem: Skid Marks and Speed of a Car

If the length of the skid marks is d feet, then the speed, s, of the car (in miles per hour) before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

Again, this formula is not memorizable *and* it is only accurate under special circumstances (e.g., it is not accurate at all if the roadway was covered in ice).

Example 3.3.13

After a car accident, the skid marks for one car measured 190 feet. Find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution available at the end of this section.

Checkpoint 3.3.13

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer available at the end of this section.

Success in College: Learning From Tests (Part 1)

1. A student scored 75% on their last math test. They tried to redo the questions and problems that they missed. Now they need to know if their new answers are correct. For your current math class, how could they find this out?
2. Look over your last math test. Did you miss any points because you did not show enough work? If so, explain why you didn't show the work on the test.

Success in Mathematics: Math Anxiety (Part 2)

I'm going to keep this short (for a change). Please take this math anxiety test to help you better understand your math anxiety. While at that website, check out the hints on overcoming math anxiety.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. A suggestion for reducing math anxiety is to come to class prepared. If you are prepared, then you are ready to be successful. When you are successful, you start learning that you do not have to be anxious. What do you think it means to be prepared for your math class?
2. A suggestion for reducing math anxiety is to practice math every day. How many days a week do you do math? How long, on average, do you spend doing math on those days?

Answers

Section Preview

This entire section is focused on solving equations involving radicals and rational exponents. While not terribly common in Calculus, these occur enough that exposure is critical.

Section Preview

- **Arithmetic**
  - Measurement, Approximation, Rounding, and Significant Figures: Rounding to the nearest tenth is required in the skid-mark application (Example 13 and Checkpoint 13).
- **Solving Equations**
  - Solving Linear Equations — Multistep with Variables on Both Sides: Once radicals are removed, the resulting linear equation must be solved, including the variables-on-both-sides case in Example 7.

Section Preview

- Solve radical equations involving one or two radicals.
- Solve equations involving rational exponents.
- Solve equations that are quadratic-in-form involving radicals or rational exponents.
- Use radicals in applications.

Example 3.3.1: Solving a radical equation involving a single radical expression

$$\begin{aligned}
& \sqrt{5n-4}-9 &&= 0 && \\
\Longrightarrow\quad & \sqrt{5n-4} &&= 9 && \text{(isolating the radical by adding 9 to both sides)} \\
\Longrightarrow\quad & \left(\sqrt{5n-4}\right)^2 &&= 9^2 && \text{(squaring both sides. } \textbf{Warning:} \text{ This forces us to check our final solution)} \\
\Longrightarrow\quad & 5n-4 &&= 81 && \text{(simplifying)} \\
\Longrightarrow\quad & 5n &&= 85 && \text{(adding 4 to both sides)} \\
\Longrightarrow\quad & n &&= 17 && \text{(dividing both sides by 5)}
\end{aligned}$$

Now we **must** check the solution. When doing so, you need to go back to the original equation to check your answer.

$$\sqrt{5(17)-4}-9=\sqrt{85-4}-9=\sqrt{81}-9=9-9=0.\checkmark$$

Thus, $n=17$ is the solution to the equation.

Checkpoint 3.3.1

$m=\frac{23}{3}$

Example 3.3.2

$$\begin{aligned}
& \sqrt{9k-2}+1 &&= 0 && \\
\Longrightarrow\quad & \sqrt{9k-2} &&= -1 && \text{(isolating the radical by subtracting 1 from both sides)}
\end{aligned}$$

At this point, if we were paying attention, we could stop because a square root is never equal to -1 (at least, not over the real numbers).

Checkpoint 3.3.2

no solution

Example 3.3.3

$$\begin{aligned}
& \sqrt{p-1}+1 &&= p && \\
\Longrightarrow\quad & \sqrt{p-1} &&= p-1 && \text{(isolating the radical by subtracting 1 from both sides)} \\
\Longrightarrow\quad & \left(\sqrt{p-1}\right)^2 &&= (p-1)^2 && \text{(squaring both sides)} \\
\Longrightarrow\quad & p-1 &&= p^2-2p+1 && \text{(simplifying and distributing)} \\
\Longrightarrow\quad & 0 &&= p^2-3p+2 && \text{(subtracting } p \text{ and adding 1 to both sides)} \\
\Longrightarrow\quad & 0 &&= (p-2)(p-1) && \text{(factoring)}
\end{aligned}$$

Therefore, our *candidate*solutions are $p=2$ or $p=1$. However, because we squared both sides, we need to check these results (using the original equation).

$$\begin{array}{lll}
p=2 & & p=1 \\
\Longrightarrow\quad \sqrt{2-1}+1 \stackrel{?}{=} 2 & & \Longrightarrow\quad \sqrt{1-1}+1 \stackrel{?}{=} 1 \\
\Longrightarrow\quad \sqrt{1}+1 \stackrel{?}{=} 2 & & \Longrightarrow\quad \sqrt{0}+1 \stackrel{?}{=} 1 \\
\Longrightarrow\quad 1+1 \stackrel{?}{=} 2 & & \Longrightarrow\quad 0+1 \stackrel{?}{=} 1 \\
\Longrightarrow\quad 2=2\checkmark & & \Longrightarrow\quad 1=1\checkmark
\end{array}$$

Therefore, the solutions are $p=1$ and $p=2$.

Checkpoint 3.3.3

$x=2, x=3$

Example 3.3.4

$$\begin{aligned}
& \sqrt[3]{5x+1}+8 &&= 4 && \\
\Longrightarrow\quad & \sqrt[3]{5x+1} &&= -4 && \text{(isolating the radical by subtracting 4 from both sides)} \\
\Longrightarrow\quad & \left(\sqrt[3]{5x+1}\right)^3 &&= (-4)^3 && \text{(raising both sides to the third power)} \\
\Longrightarrow\quad & 5x+1 &&= -64 && \text{(simplifying)} \\
\Longrightarrow\quad & 5x &&= -65 && \text{(subtracting 1 from both sides)} \\
\Longrightarrow\quad & x &&= -13 && \text{(dividing both sides by 5)}
\end{aligned}$$

Checkpoint 3.3.4

$x=-6$

Example 3.3.5

$$
\begin{aligned}
(3x-2)^{1/4}+3 &= 5 \\
\Longrightarrow \quad (3x-2)^{1/4} &= -2 && \text{(subtracting 3 from both sides)} \\
\Longrightarrow \quad \left((3x-2)^{1/4}\right)^4 &= (-2)^4 && \text{(raising both sides to the fourth power)} \\
\Longrightarrow \quad (3x-2)^{4/4} &= 16 && \text{(Laws of Exponents)} \\
\Longrightarrow \quad (3x-2)^{1} &= 16 && \text{(simplifying)} \\
\Longrightarrow \quad 3x-2 &= 16 && \text{(simplifying)} \\
\Longrightarrow \quad 3x &= 18 && \text{(adding 2 to both sides)} \\
\Longrightarrow \quad x &= 6 && \text{(dividing both sides by 3)}
\end{aligned}
$$

Since we raised both sides to an even power, we need to check for extraneous solutions. Going back to the original equation and substituting $x=6$, we get

$$(3(6)-2)^{1/4}+3=(18-2)^{1/4}+3=(16)^{1/4}+3=\sqrt[4]{16}+3=2+3=5\checkmark.$$

Checkpoint 3.3.5

$x=8$

Example 3.3.6

$$
\begin{aligned}
3\left(\frac{1}{2}x+1\right)^{2/3}-5 &= 7 \\
\Longrightarrow \quad 3\left(\frac{1}{2}x+1\right)^{2/3} &= 12 && \text{(begin isolating the rational exponent expression by adding 5 to both sides)} \\
\Longrightarrow \quad \left(\frac{1}{2}x+1\right)^{2/3} &= 4 && \text{(continue isolating the rational exponent expression by dividing both sides by 3)} \\
\Longrightarrow \quad \left(\left(\frac{1}{2}x+1\right)^{2/3}\right)^3 &= (4)^3 && \text{(raise both sides to the third power)} \\
\Longrightarrow \quad \left(\frac{1}{2}x+1\right)^{6/3} &= 64 && \text{(Laws of Exponents)} \\
\Longrightarrow \quad \left(\frac{1}{2}x+1\right)^{2} &= 64 && \text{(simplifying)} \\
\Longrightarrow \quad \frac{1}{2}x+1 &= \pm 8 && \text{(Extraction of Roots)} \\
\Longrightarrow \quad \frac{1}{2}x &= -1\pm 8 && \text{(subtracting 1 from both sides)} \\
\Longrightarrow \quad x &= -2\pm 16 && \text{(multiplying both sides by 2)}
\end{aligned}
$$

Thus, the solutions are $x=-2-16=-18$ and $x=-2+16=14$.

Checkpoint 3.3.6

$x=2$

Example 3.3.7

$$
\begin{aligned}
\sqrt[3]{4x-3}-\sqrt[3]{3x+2} &= 0 \\
\Longrightarrow \quad \sqrt[3]{4x-3} &= \sqrt[3]{3x+2} && \text{(isolating one of the radicals (which actually isolates the other in this example))} \\
\Longrightarrow \quad \left(\sqrt[3]{4x-3}\right)^3 &= \left(\sqrt[3]{3x+2}\right)^3 && \text{(raising both sides to the third power)} \\
\Longrightarrow \quad 4x-3 &= 3x+2 && \text{(simplifying)} \\
\Longrightarrow \quad x &= 5 && \text{(subtracting } 3x \text{ and adding 3 to both sides)}
\end{aligned}
$$

The solution is $x=5$. We leave it to you to show that 5 checks.

Checkpoint 3.3.7

$x=-\frac{6}{5}$

Example 3.3.8

$$\begin{aligned}
\sqrt{m}+1 &= \sqrt{m+9} \\
\implies (\sqrt{m}+1)^2 &= \left(\sqrt{m+9}\right)^2 && \text{(since a radical was already isolated, square both sides)} \\
\implies (\sqrt{m}+1)(\sqrt{m}+1) &= m+9 && \text{(simplifying)} \\
\implies m+2\sqrt{m}+1 &= m+9 && \text{(distributing)} \\
\implies 2\sqrt{m} &= 8 && \text{(begin isolating the remaining radical by subtracting both } m \text{ and 1 from both sides)} \\
\implies \sqrt{m} &= 4 && \text{(dividing both sides by 2)} \\
\implies (\sqrt{m})^2 &= 4^2 && \text{(squaring both sides)} \\
\implies m &= 16 && \text{(simplifying)}
\end{aligned}$$

Since we squared both sides during our solution process, we need to check that our solution works in the original equation.

$$\begin{aligned}
\sqrt{16}+1 &\stackrel{?}{=} \sqrt{16+9} \\
\implies 4+1 &\stackrel{?}{=} \sqrt{25} \\
\implies 5 &= 5 \checkmark
\end{aligned}$$

Therefore, the solution is $m = 16$.

Checkpoint 3.3.8

$x = 4$

Example 3.3.9

$$\begin{aligned}
\sqrt{q-2}+3 &= \sqrt{4q+1} \\
\implies \left(\sqrt{q-2}+3\right)^2 &= \left(\sqrt{4q+1}\right)^2 && \text{(squaring both sides since one radical is already isolated)} \\
\implies \left(\sqrt{q-2}+3\right)\left(\sqrt{q-2}+3\right) &= 4q+1 && \text{(simplifying)} \\
\implies q-2+6\sqrt{q-2}+9 &= 4q+1 && \text{(distributing)} \\
\implies q+7+6\sqrt{q-2} &= 4q+1 && \text{(combining like terms)} \\
\implies 6\sqrt{q-2} &= 3q-6 && \text{(begin isolating the remaining radical by subtracting } q \text{ and 7 from both sides)} \\
\implies 2\sqrt{q-2} &= q-2 && \text{(noticing that all coefficients share a common divisor, we divide both sides by 3)} \\
\implies \left(2\sqrt{q-2}\right)^2 &= (q-2)^2 && \text{(instead of dividing by 2 (which would introduce fractions), we square both sides)} \\
\implies 2^2\left(\sqrt{q-2}\right)^2 &= (q-2)(q-2) && \text{(Laws of Exponents)} \\
\implies 4(q-2) &= q^2-4q+4 && \text{(simplifying and distributing)} \\
\implies 4q-8 &= q^2-4q+4 && \text{(distributing)} \\
\implies 0 &= q^2-8q+12 && \text{(subtracting } 4q \text{ and adding 8 to both sides)} \\
\implies 0 &= (q-6)(q-2) && \text{(factoring the trinomial)}
\end{aligned}$$

Therefore, our *candidate* solutions are $q = 6$ or $q = 2$. We leave it to the reader to check that neither is extraneous.

Checkpoint 3.3.9

$x = 5$

Example 3.3.10**: Revisiting Solving Equations Quadratic-in-Form**

This is a radical equation that can be written in quadratic form. If we let $u = \sqrt{x}$, then $u^2 = \left(\sqrt{x}\right)^2 = x$ and we can write

$$\begin{aligned}
x-2\sqrt{x}-8 &= 0 \\
\implies u^2-2u-8 &= 0 && \text{(substituting)} \\
\implies (u-4)(u+2) &= 0 && \text{(factoring the trinomial)}
\end{aligned}$$

Therefore, $u = 4$ or $u = -2$. Since $u = \sqrt{x}$, we resubstitute to get $\sqrt{x} = 4$ or $\sqrt{x} = -2$; however, square roots are never negative. Hence, we throw out that possibility and focus on $\sqrt{x} = 4 \implies x = 16$. Checking this solution in the *original* equation, we get

$$16-2\sqrt{16}-8 = 16-2(4)-8 = 16-8-8 = 0. \checkmark$$

Thus, the solution to the equation is $x = 16$.

Checkpoint 3.3.10

$x = 16$

Example 3.3.11**: Revisiting Solving Equations Quadratic-in-Form**

If we let $u = x^{-1/3}$, then $u^2 = \left(x^{-1/3}\right)^2 = x^{-2/3}$ and we can write

$$\begin{aligned} x^{-2/3} - 3x^{-1/3} - 10 &= 0 \\ \implies u^2 - 3u - 10 &= 0 && \text{(substituting)} \\ \implies (u-5)(u+2) &= 0 && \text{(factoring the trinomial)} \\ \implies \left(x^{-1/3} - 5\right)\left(x^{-1/3} + 2\right) &= 0 && \text{(resubstituting)} \end{aligned}$$

Therefore, $x^{-1/3} = 5$ or $x^{-1/3} = -2$.

$$\begin{aligned} x^{-1/3} &= 5 & x^{-1/3} &= -2 \\ \implies \frac{1}{x^{1/3}} &= 5 & \frac{1}{x^{1/3}} &= -2 && \text{(Laws of Exponents)} \\ \implies x^{1/3} &= \frac{1}{5} & x^{1/3} &= -\frac{1}{2} && \left(\text{if } a = b, \text{ where neither are zero, then } \tfrac{1}{a} = \tfrac{1}{b}\right) \\ \implies \left(x^{1/3}\right)^3 &= \left(\frac{1}{5}\right)^3 & \left(x^{1/3}\right)^3 &= \left(-\frac{1}{2}\right)^3 && \text{(raising both sides to the third power)} \\ \implies x^{3/3} &= \frac{1^3}{5^3} & x^{3/3} &= -\frac{1^3}{2^3} && \text{(Laws of Exponents)} \\ \implies x &= \frac{1}{125} & x &= -\frac{1}{8} && \text{(simplifying)} \end{aligned}$$

The domain of the original equation only has the restriction that $x \neq 0$ (can you see why?). Furthermore, the solution process involved cubing, not squaring. Squaring can introduce extraneous solutions, but cubing does not, so a check is not necessary. We leave it to the reader to check that these solutions work in the original equation.

Example 3.3.12

1. **Read** the problem.
2. **Understand** the situation. We are looking for the time it takes for Kaitlyn's sunglasses to reach the river.
3. **Label** the given and wanted information. Let t represent the unknown time it will take the sunglasses to fall to the river.

Given	Want
$h = 400$	t

4. **Translate** into an equation. The recent theorem allows us to relate h and t by the formula

$$t = \frac{\sqrt{h}}{4}.$$

Substituting in the given information, we get

$$t = \frac{\sqrt{400}}{4}.$$

5. **Solve** the equation.

$$\begin{aligned} t &= \frac{\sqrt{400}}{4} \\ &= \frac{20}{4} && \text{(simplifying)} \\ &= 5 && \text{(simplifying)} \end{aligned}$$

6. **Check** the solution.

$$\begin{aligned} 5 &\stackrel{?}{=} \frac{\sqrt{400}}{4} \\ \implies 5 &\stackrel{?}{=} \frac{20}{4} \\ \implies 5 &= 5 \checkmark \end{aligned}$$

Thus, 5 is correct *mathematically*. It also makes sense logically (for example, if the answer was negative, that would not make sense).

7. **State** the answer. It takes 5 seconds for Kaitlyn's sunglasses to reach the river.

Checkpoint 3.3.12

3.5 seconds

Example 3.3.13

1. **Read!**
2. **Understand:** We are looking for the initial speed of the car before the brakes were applied.
3. **Label:** Let s represent the unknown speed of the car and d represent the length of the skid.

Given	Want
$d = 190$	s

4. **Translate:**

$$s = \sqrt{24d} = \sqrt{24(190)}.$$

5. **Solve:**

$$\begin{aligned} s &= \sqrt{24(190)} \\ &= \sqrt{4560} && \text{(simplifying)} \\ &\approx 67.5277\ldots && \text{(approximating)} \\ &\approx 67.5 && \text{(rounding to the nearest tenth)} \end{aligned}$$

6. **Check:**

$$\begin{aligned} 67.5 &\overset{?}{\approx} \sqrt{24(190)} \\ \implies 67.5 &\overset{?}{\approx} \sqrt{4560} \\ \implies 67.5 &\approx 67.5277\ldots \checkmark \end{aligned}$$

   Thus, 67.5 is correct *mathematically*. It also makes sense logically that the car could have been going 67.5 miles per hour.
7. **State:** The speed of the car before the brakes were applied was roughly 67.5 miles per hour.

Checkpoint $3.3.13$

42.7 mph

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3.3.1: Resources and Key Concepts

Resources

Videos

- Solving radical equations
  - Equations - Overview - A Review for College Algebra: Solving Radical Equations
  - Intermediate Algebra - Radical Equations: Solving

Key Concepts

Definitions

- **Radical Equation**: An equation in which unknowns (variables) are part of a radicand.
- **Extraneous Solution (Spurious Solution)**: A root of a transformed (manipulated) equation that is not a root of the original equation because it was excluded from the domain of the original equation or introduced by operations like squaring both sides.
- **Contradiction**: A statement that is both true and false simultaneously (e.g., arriving at $2 \neq 0$ after assuming a solution exists and checking it).

Common Mistakes

- **Forgetting to Check for Extraneous Solutions**: When raising both sides of an equation to an even power (e.g., squaring), it is crucial to check all candidate solutions in the original equation, as this process can introduce extraneous solutions.
- **Incorrectly Isolating the Radical**: Failing to isolate the radical term on one side of the equation before raising both sides to a power. This leads to more complicated expressions that still contain radicals.
- **Errors in Raising Both Sides to a Power**: Not raising the entire side to the power, especially if it's a binomial (e.g., if $\sqrt{A} = B + C$, then $A = (B + C)^2$, not $B^2 + C^2$).
- **Errors with Rational Exponents**: Incorrectly applying powers to "undo" rational exponents (e.g., for $x^{m/n} = k$, raise both sides to n/m).
- **Not Recognizing Equations Quadratic-in-Form**: Failing to see that a radical or rational exponent equation can be solved using u-substitution if it's quadratic-in-form.

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3.3.2: Homework

Reading Questions

1. What is a radical equation?
2. What is the primary strategy for solving a radical equation containing one radical?
3. Why is it crucial to check your solutions when solving radical equations, especially if you've raised both sides to an even power? What kind of solutions might arise?
4. If, after isolating an even-indexed radical, you find it is equal to a negative number (e.g., $\sqrt{x} = -2$), what can you conclude about the real solutions to the equation?
5. When solving an equation with a rational exponent, like $(3x-2)^{1/4}+3=5$, what is the first step after isolating the term with the rational exponent?
6. If an equation contains two radicals, what is the general approach to solving it?
7. When solving an equation like $x - 2\sqrt{x} - 8 = 0$, what substitution could you make to transform it into a quadratic equation?
8. What does the term "extraneous solution" mean in the context of solving radical equations?
9. If you are solving an equation and raise both sides to an odd power (e.g., cubing both sides), is it still mandatory to check for extraneous solutions for the same reasons as with even powers?
10. Describe the general steps for solving a radical equation as outlined in the "How To: Solve a Radical Equation" box.

Homework

In the following exercises, solve.

11. $\sqrt{5x-6}=8$
12. $\sqrt{4x-3}=7$
13. $\sqrt{5x+1}=-3$
14. $\sqrt{3y-4}=-2$
15. $\sqrt[3]{2x}=-2$
16. $\sqrt[3]{4x-1}=3$
17. $\sqrt{2m-3}-5=0$
18. $\sqrt{2n-1}-3=0$
19. $\sqrt{6v-2}-10=0$
20. $\sqrt{12u+1}-11=0$
21. $\sqrt{4m+2}+2=6$
22. $\sqrt{6n+1}+4=8$
23. $\sqrt{2u-3}+2=0$
24. $\sqrt{5v-2}+5=0$
25. $\sqrt{u-3}+3=u$
26. $\sqrt{v-10}+10=v$
27. $\sqrt{r-1}=r-1$
28. $\sqrt{s-8}=s-8$
29. $\sqrt[3]{6x+4}=4$
30. $\sqrt[3]{11x+4}=5$
31. $\sqrt[3]{4x+5}-2=-5$
32. $\sqrt[3]{9x-1}-1=-5$
33. $(6x+1)^{1/2}-3=4$
34. $(3x-2)^{1/2}+1=6$
35. $(8x+5)^{1/3}+2=-1$
36. $(12x-5)^{1/3}+8=3$
37. $(12x-3)^{1/4}-5=-2$
38. $(5x-4)^{1/4}+7=9$
39. $\sqrt{x+1}-x+1=0$
40. $\sqrt{y+4}-y+2=0$
41. $\sqrt{z+100}-z=-10$

42. $\sqrt{w+25}-w=-5$
43. $3\sqrt{2x-3}-20=7$
44. $2\sqrt{5x+1}-8=0$
45. $2\sqrt{8r+1}-8=2$
46. $3\sqrt{7y+1}-10=8$

In the following exercises, solve.

47. $\sqrt{3u+7}=\sqrt{5u+1}$
48. $\sqrt{4v+1}=\sqrt{3v+3}$
49. $\sqrt{8+2r}=\sqrt{3r+10}$
50. $\sqrt{10+2c}=\sqrt{4c+16}$
51. $\sqrt[3]{5x-1}=\sqrt[3]{x+3}$
52. $\sqrt[3]{8x-5}=\sqrt[3]{3x+5}$
53. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$
54. $\sqrt[3]{x^2-x+18}=\sqrt[3]{2x^2-3x-6}$
55. $\sqrt{a}+2=\sqrt{a+4}$
56. $\sqrt{r}+6=\sqrt{r+8}$
57. $\sqrt{u}+1=\sqrt{u+4}$
58. $\sqrt{x}+1=\sqrt{x+2}$
59. $\sqrt{a+5}-\sqrt{a}=1$
60. $-2=\sqrt{d-20}-\sqrt{d}$
61. $\sqrt{2x+1}=1+\sqrt{x}$
62. $\sqrt{3x+1}=1+\sqrt{2x-1}$
63. $\sqrt{2x-1}-\sqrt{x-1}=1$
64. $\sqrt{x+1}-\sqrt{x-2}=1$
65. $\sqrt{x+7}-\sqrt{x-5}=2$
66. $\sqrt{x+5}-\sqrt{x-3}=2$

In the following exercises, solve. Round approximations to one decimal place.

67. **Landscaping.** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.
68. **Landscaping.** Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.
69. **Gravity.** A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.
70. **Gravity.** A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.
71. **Accident investigation.** The skid marks for a car involved in an accident measured 216 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.
72. **Accident investigation.** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

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3.4: Rational Exponent Expressions Encountered in Calculus

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

While you will be presented with radical expressions in Calculus, more often than not these will be written with rational exponents. In fact, it is advantageous in Calculus to use rational exponents instead of radical notation.

When working with expressions in Calculus involving rational exponents, there are three skills we need to review (or introduce). In order of frequency and importance, these are:

1. understanding the meaning and impact of a negative exponent,
2. factoring out the GCF from an expression, and
3. knowing how to multiply the numerator and denominator of an expression by a unit fraction to simplify.

The first of these is addressed with a quick reminder:

$$a^{-n} = \frac{1}{a^n} \quad \text{and} \quad \frac{1}{a^{-n}} = a^n$$

We will be using this Law of Exponents frequently in this course and in Calculus.

The second might *seem* silly to mention (because you likely have been factoring out the GCF from algebraic expressions since elementary school); however, a simple change in the *look* of an expression throws many students into a downward spiral. The critical concept to remember when factoring out the GCF from an algebraic expression is that factoring is a division process. That is, when we factor $3x - 6$, we *divide out* the 3 from both $3x$ and -6. Hence,

$$3x - 6 = 3\left(\frac{3x}{3} - \frac{6}{3}\right) = 3(x - 2).$$

Of course, most people skip that middle step. With this being said, we can focus on factoring more complex algebraic expressions.

When factoring the GCF out of an algebraic expression, we were taught to factor out the smallest-powered versions of the shared factors. For example, given $a^M b^n + a^m b^N$, where $m < M$ and $n < N$, we identify the common factors as a (to a power) and b (to a power). Since the smallest power on a is m and the smallest power on b is n, we factor out $a^m b^n$.

$$a^M b^n + a^m b^N = a^m b^n \left(\frac{a^M b^n}{a^m b^n} + \frac{a^m b^N}{a^m b^n}\right) = a^m b^n \left(a^{M-m} + b^{N-n}\right).$$

To memorize this formula is preposterous, so we will focus on the concept rather than the rote memorization.

Example 3.4.1

Simplify the expression by factoring:

$$3(1+x)^{1/3} - x(1+x)^{-2/3}$$

Solution available at the end of this section.

If you pay close attention, Example 3.4.1 reveals a shortcut to factoring out the smallest-powered version of each shared factor: factoring out $(1+x)^{-2/3}$ ends up subtracting $-\frac{2}{3}$ from the remaining exponents. That is, it *adds* the $\frac{2}{3}$ to those exponents. If you pay attention and do a few examples of your own, this trick speeds up factoring problems like this significantly.

Another thing to note from Example 3.4.1 is that we left our answer with a negative exponent—this is perfectly fine (and actually preferred in Calculus).

The next example demonstrates that we always try to factor completely, if possible.

Example 3.4.2

Factor:

$$x\left(2x+7\right)^{1/3}+5\left(2x+7\right)^{-2/3}$$

Solution available at the end of this section.

Checkpoint 3.4.2

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online



Expressions like those in Examples 3.4.1 and 3.4.2 commonly occur as the result of the Product Rule in Differential Calculus (Calculus I). You will have need to simplify those without any prompting from your instructor.

Another style of algebraic expression that frequently occurs in Calculus is in the following example.

Example 3.4.3

Simplify the expression.

$$\frac{2\left(18+x\right)^{1/2}-x\left(18+x\right)^{-1/2}}{x+18}$$

Solution available at the end of this section.

While both approaches in Example 3.4.3 look like they take the same amount of work, the second (Method #2) is often a little faster. Expressions of the form in Example 3.4.3 are the result of the Quotient Rule in Differential Calculus. As before, you will have need to simplify these without any prompting from your instructor.

Checkpoint 3.4.3

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online



Success in College: Learning From Tests (Part 2)

1. A student earned 75% of the points on her last math test. On the first class day after the test, the class started a new chapter

with new material. She does not understand several vital procedures from the last test. What do you think she should do about this situation?

2. A student usually finishes tests right as class ends. He doesn't have time to check for careless mistakes. He looks carefully at his corrected tests. He realizes that many of his errors are arithmetic or algebra mistakes. He knew how to do the problems; he didn't do them well enough. What do you think he should do?

Miscellaneous Topics: Thinking About Mathematics (Part 2)

A significant concern in mathematics education is the trend that students rely on calculators more. An argument for calculator usage is that, in "real life," a person will always have access to a computer or a calculator. Therefore, forcing students to do problems "by hand" is useless.

1. Give an argument against high calculator usage in math classes.
2. Do you think that your professor relies too much on calculator usage in class? What about their use of technology? Would you like more or less?

Answers

Section Preview

The purpose of this *very short* section is to reintroduce students to rational exponents and to acquaint them with factoring expressions having rational exponents. These occur somewhat frequently in Calculus.

Section Preview

- Factor and simplify common expressions found in Calculus involving rational powers

Example 3.4.1

Both terms in the expression share the factor $(1+x)$; however, the second term has the smallest-powered version of this factor. Therefore, we factor out $(1+x)^{-2/3}$.

$$\begin{aligned}
3(1+x)^{1/3} - x(1+x)^{-2/3} &= (1+x)^{-2/3}\left(\frac{3(1+x)^{1/3}}{(1+x)^{-2/3}} - \frac{x(1+x)^{-2/3}}{(1+x)^{-2/3}}\right) && \text{(factoring out the GCF (which is just division))}\\
&= (1+x)^{-2/3}\left(3(1+x)^{1/3+2/3} - x(1+x)^{-2/3+2/3}\right) && \text{(Laws of Exponents)}\\
&= (1+x)^{-2/3}\left(3(1+x) - x(1+x)^{0}\right) && \text{(simplifying)}\\
&= (1+x)^{-2/3}\left(3(1+x) - x\right) && \text{(Laws of Exponents)}\\
&= (1+x)^{-2/3}\left(3+2x\right) && \text{(distributing and combining like terms)}
\end{aligned}$$

Example 3.4.2

$$\begin{aligned}
x(2x+7)^{1/3} + 5(2x+7)^{-2/3} &= (2x+7)^{-2/3}\left(x(2x+7)^{3/3} + 5(2x+7)^{0}\right) && \text{(factoring out the GCF)}\\
&= (2x+7)^{-2/3}\left(x(2x+7) + 5\right) && \text{(simplifying)}\\
&= (2x+7)^{-2/3}\left(2x^2 + 7x + 5\right) && \text{(distributing)}\\
&= (2x+7)^{-2/3}(2x+5)(x+1) && \text{(factoring the trinomial)}
\end{aligned}$$

Example 3.4.3

There are two approaches to simplifying expressions like these. The first is to mimic what we did in Examples 3.4.1 and 3.4.2.

Method #1

$$\begin{aligned}
\frac{2(18+x)^{1/2} - x(18+x)^{-1/2}}{x+18} &= \frac{(18+x)^{-1/2}\left(2(18+x)^{2/2} - x(18+x)^{0}\right)}{x+18} && \text{(factoring the GCF from the numerator)}\\
&= \frac{(18+x)^{-1/2}\left(2(18+x) - x\right)}{x+18} && \text{(simplifying)}\\
&= \frac{(18+x)^{-1/2}(36+2x-x)}{x+18} && \text{(distributing)}\\
&= \frac{(18+x)^{-1/2}(36+x)}{x+18} && \text{(combining like terms)}\\
&= \frac{(36+x)}{(18+x)^{1/2}(x+18)} && \text{(Laws of Exponents)}\\
&= \frac{36+x}{(18+x)^{3/2}} && \text{(Laws of Exponents)}
\end{aligned}$$

While this method works, and there is nothing wrong with it, it is helpful to show another approach.

Method #2

This approach will use the fact that a^{-n} is the fraction $\frac{1}{a^n}$.

$$\begin{aligned}
\frac{2(18+x)^{1/2} - x(18+x)^{-1/2}}{x+18} &= \frac{2(18+x)^{1/2} - x\frac{1}{(18+x)^{1/2}}}{x+18} && \text{(Laws of Exponents)} \\
&= \frac{\left(2(18+x)^{1/2} - x\frac{1}{(18+x)^{1/2}}\right)}{(x+18)} \cdot \frac{(18+x)^{1/2}}{(18+x)^{1/2}} && \text{(simplifying the compound fraction)} \\
&= \frac{2(18+x)^{2/2} - x}{(x+18)^{3/2}} && \text{(distributing)} \\
&= \frac{2(18+x) - x}{(x+18)^{3/2}} && \text{(simplifying)} \\
&= \frac{36+2x-x}{(x+18)^{3/2}} && \text{(distributing)} \\
&= \frac{36+x}{(x+18)^{3/2}} && \text{(combining like terms)}
\end{aligned}$$

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3.4.1: Resources and Key Concepts

Resources

Videos

- Factoring with rational exponents
  - Factoring Polynomials - Factoring Expressions with Rational or Negative Exponents
  - Algebraic Expressions - Factoring Expressions with Negative Rational Exponents
  - Intermediate Algebra - Rational Exponents: Factoring/Distributing Expressions

Key Concepts

Common Mistakes

- **Errors with Negative Exponents**: Incorrectly interpreting or manipulating negative exponents, such as $a^{-n} \neq -a^n$ or misapplying rules when factoring.
- **Incorrectly Identifying the GCF with Rational/Negative Exponents**: When factoring expressions like $3(1+x)^{1/3} - x(1+x)^{-2/3}$, choosing the wrong power for the GCF $(1+x)$. The rule is to factor out the term with the smallest (most negative or least positive) exponent.
- **Errors in Subtracting Exponents During Factoring**: When factoring out a GCF like A^m from A^M, the remaining exponent is $M-m$. Mistakes often occur with signs, especially if m is negative (e.g., $1/3-(-2/3)=1/3+2/3$).
- **Distributing Unnecessarily in Denominators**: When simplifying complex fractions or expressions resulting from the quotient rule, distributing terms in the overall denominator can make further simplification or cancellation harder. It's often better to keep denominators factored.
- **Algebraic Errors in Simplifying Compound Fractions**: When multiplying by LCD to clear minor denominators, making errors in distribution or combining terms.
- **Not Factoring Completely**: After an initial factoring step, failing to see if any resulting polynomial factors (like a quadratic) can be factored further.

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3.4.2: Homework

Reading Questions

1. What does a negative exponent, such as a^{-n}, signify? How can it be rewritten with a positive exponent?
2. When factoring out the Greatest Common Factor (GCF) from an algebraic expression where terms have different powers of a common base (e.g., $x^{1/3}$ and $x^{-2/3}$), which power of the common base should you factor out?
3. If you factor $(1+x)^{-2/3}$ from $3(1+x)^{1/3}$, what is the exponent of $(1+x)$ in the remaining factor inside the parentheses? Show the subtraction of exponents.
4. When simplifying an expression like $\frac{2(18+x)^{1/2}-x(18+x)^{-1/2}}{x+18}$ by first factoring out the GCF from the numerator, what is a common GCF to choose?
5. Describe an alternative method for simplifying an expression like $\frac{2(18+x)^{1/2}-x(18+x)^{-1/2}}{x+18}$ that involves treating negative exponents as fractions.
6. Why is it often preferred in Calculus to leave answers with negative exponents rather than converting them to fractions?
7. When simplifying expressions that arise from the Quotient Rule in Calculus, such as the one in Example 3, why is it generally advisable not to distribute terms in the final overall denominator?
8. What is the critical concept to remember when factoring out a GCF from an algebraic expression? (Hint: "factoring is a _____ process")

Homework

The textbook homework has not been written at this time.

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3.5: Radical Expressions Encountered in Calculus

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We dealt with inverses of polynomial functions (over restricted domains) at the end of the previous chapter. At that time, we discovered that such inverses are often **radical functions**. Other than the square root and cube root functions, however, radical functions (as a class of functions) are not very common in Calculus. On the other hand, manipulating functions and expressions involving radicals (and, alternatively, expressions involving rational exponents) occurs so frequently in Calculus that we would benefit from improving on our base skills.

Simplify Variable Expressions of the Form $\sqrt[n]{a^n}$

There are many subtleties in Mathematics when dealing with radical expressions and, unfortunately, almost all of these occur in Calculus. The first such subtlety involves the nth root of an expression raised to the nth power. The following theorem reminds us of the complication.

Theorem: Roots of Powers

For any natural numer $n \geq 2$,

$$\sqrt[n]{a^n} = \begin{cases} a, & \text{if } n \text{ is odd} \\ |a|, & \text{if } n \text{ is even} \end{cases}$$

An easy way to remember this theorem is to know that even-indexed radicals only return non-negative values. Therefore, $\sqrt[10]{a^{10}}$, for example, must always return $|a|$ and $\sqrt[10]{a^{20}} = \sqrt[10]{(a^2)^{10}}$ always returns $|a^2|$. Of course, after forcing the absolute values on the result, we need to use some logic to simplify. In the latter case, we know that $a^2 \geq 0$. Therefore, $|a^2| = a^2$.

Interactive Element: Why $\sqrt{x^2} = |x|$

The following Interactive Element might help you visualize what the even-indexed root of a square actually returns. The blue curve is $y = |x|$ and the dashed grey line is $y = x$.

Interact: Drag the large red point left and right along the curve, crossing the corner at the origin.

Observation: For $x \geq 0$ the value of $\sqrt{x^2}$ sits right on the line $y = x$; for $x < 0$ they split, because the root stays non-negative. So $\sqrt{x^2} = |x|$, which equals x only when x is non-negative.

Let's try some examples to solidify this concept.

Example 3.5.1

Simplify each expression.

a. $\sqrt{x^2}$
b. $\sqrt[3]{n^3}$
c. $\sqrt[4]{p^4}$
d. $\sqrt[5]{y^5}$

Solutions available at the end of this section.

Checkpoint 3.5.1

Simplify:

a. $\sqrt{b^2}$
b. $\sqrt[3]{w^3}$
c. $\sqrt[4]{m^4}$
d. $\sqrt[5]{q^5}$

Answers available at the end of this section.

What about square roots of higher powers of variables? The Power Property of the Laws of Exponents says $(a^m)^n = a^{m \cdot n}$. So if we square a^m, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}.$$

Looking now at the square root, we get

$$\sqrt{a^{2m}}.$$

Since $(a^m)^2 = a^{2m}$,

$$\sqrt{(a^m)^2}.$$

Since n is even, $\sqrt[n]{a^n} = |a|$. Therefore,

$$|a^m| \implies \sqrt{a^{2m}} = |a^m|.$$

We apply this concept in the next example.

Example 3.5.2

Simplify:

a. $\sqrt{x^6}$
b. $\sqrt{y^{16}}$
c. $\sqrt[3]{y^{18}}$
d. $\sqrt[4]{z^8}$
e. $\sqrt{16n^2}$
f. $-\sqrt{81c^2}$
g. $\sqrt[3]{64p^6}$
h. $\sqrt[4]{16q^{12}}$

Solutions available at the end of this section.

Checkpoint 3.5.2

Simplify:

a. $\sqrt{y^{18}}$

b. $\sqrt{z^{12}}$
c. $\sqrt[4]{u^{12}}$
d. $\sqrt[3]{v^{15}}$
e. $\sqrt{64x^2}$
f. $-\sqrt{100p^2}$
g. $\sqrt[3]{27x^{27}}$
h. $\sqrt[4]{81q^{28}}$

Answers available at the end of this section.

Rationalizing Numerators and Denominators

By now, you have probably figured out that finding the Difference Quotient of a function is a big deal in Calculus. After all, we have covered the Difference Quotient in multiple sections up to this point. It turns out that in order to find the Difference Quotient of a radical function, we need to be able to rationalize denominators (and, sometimes, numerators) of fractions involving radicals. To do this, we need to generalize a definition we gave previously.

Definition: Conjugate

The **conjugate** of a binomial $a+b$ is $a-b$, and the two expressions are called **conjugates** of each other. Likewise, the conjugate of $a-b$ is $a+b$.

When we encountered this concept in the previous chapter, it was in reference to complex conjugates; however, we need not restrict the definition to complex-valued numbers.

Recall from Algebra, when the denominator of a fraction is a sum or difference involving at least one square root, we use **conjugate multiplication** to **rationalize the denominator**.

$$\begin{aligned}(a-b)(a+b) \quad & (2-\sqrt{5})(2+\sqrt{5}) \\ =a^2-b^2 \quad & =2^2-(\sqrt{5})^2 \\ & =4-5 \\ & =-1\end{aligned}$$

As seen above, when we multiply a binomial that includes a square root by its conjugate, the product has no square roots. This process is so important and common in Calculus, we should codify it with a theorem.

Theorem: Conjugate Multiplication

Given algebraic expressions a and b, the product of $a+b$ with its conjugate, $a-b$, is always a^2-b^2.

Proof available at the end of this section.

Of course, this is simply the Difference of Squares factorization theorem stated in reverse, so it should not be a surprising result.

Example 3.5.3

Simplify: $\frac{5}{2-\sqrt{3}}$

Solution available at the end of this section.

Notice we did not distribute the 5 in the answer of Example $3.5.3$. By leaving the result factored we can see if there are any factors

that may be common to both the numerator and denominator.

Checkpoint 3.5.3

Simplify: $\frac{3}{1-\sqrt{5}}$.

Answer available at the end of this section.

Example 3.5.4

Simplify: $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution available at the end of this section.

Checkpoint 3.5.4

Simplify: $\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

Answer available at the end of this section.

As with simplifying compound fractions, you must instinctively know when to rationalize a numerator or denominator throughout Calculus (without explicit instruction from your instructor).

Example 3.5.5: Revisiting the Difference Quotient

Find the Difference Quotient of the following function.

$$f(x) = \frac{1}{\sqrt{x}}$$

Simplify this expression as much as possible, and make sure your final simplification does not have a factor of h in the denominator.

Solution available at the end of this section.

In the solution to Example 3.5.5, there were a couple of times when we did *not* (and, technically, could not) distribute the denominator. We were forced to multiply by the LCD because of the fractions in the *numerator*. Remember: only use distribution on the offending piece (the numerator in both cases).

Checkpoint 3.5.5

Find and simplify the Difference Quotient for the following function.

$$r(x) = \sqrt{x}$$

Answer available at the end of this section.

Interactive Element: The Difference Quotient as a Secant Slope

The following Interactive Element might help you visualize why we bother clearing h from the denominator of a difference quotient.

Interact: Drag the red point along the curve $y = \sqrt{x}$ toward the blue anchor point.

Observation: The line through the two points is a secant, and its slope is exactly the difference quotient $\frac{f(x+h)-f(x)}{h}$. As the points come together (as $h \to 0$), the secant settles onto the dashed line and its slope approaches $\frac{1}{4}$ — the value the simplified difference quotient gives at $h = 0$. That limit exists only because we removed the h from the denominator.

Caution: To Distribute or Not To Distribute... That is the Question

With difference quotients, you will commonly need to multiply the numerator **and** denominator of an expression by either

a. the LCD of all fractions within a compound fraction (a.k.a. simplifying compound fractions), or
b. the conjugate of either the numerator *or* the denominator (a.k.a. rationalizing).

When doing so, knowing what to distribute and what not to distribute is imperative.

Simplifying Compound Fractions

The entire point of multiplying the numerator **and** denominator of your compound fraction by the LCD of all the minor fractions is to "get rid of" denominators in those minor fractions; however, with difference quotients, it is common that the denominator of the entire (major) fraction is only h. In this case, **do not** distribute out the denominator! Doing so will complicate the mathematics, and you will lose visibility of factors that cancel.

Rationalizing

In a difference quotient, the entire point of multiplying the numerator **and** denominator by the conjugate of either the numerator *or* the denominator is to clear radicals. This will only happen with the conjugate pairs. So, definitely distribute the conjugate pairs, but **do not** distribute the non-conjugates.

For example, in Example $3.5.5$, we multiplied the numerator **and** denominator by the conjugate of the numerator. We then distributed the *numerator* but did not bother with the distribution in the denominator. This is because distribution in the numerator cleared the radicals, but distribution in the denominator would do nothing other than make a mess.

Hidden Resources: Adobe Creative Cloud

The Adobe Creative Cloud is a collection of Adobe software products. In addition to offering the products, the Creative Cloud also offers cloud storage.

Apps Included in the Adobe Creative Suite are:

- Design Apps: Photoshop, InCopy, Spark, Illustrator, Adobe XD, Dimension, InDesign
- Web & UX Apps: Adobe XD, Spark, Dreamweaver, Animate, InCopy, InDesign, Photoshop
- Photography Apps: Photoshop, Lightroom
- Video & Audio Apps: Premiere Pro, Premiere Rush, After Effects, Spark, Audition

Typically, the Adobe Creative Suite costs $55/month, but Los Rios students have access to Adobe Creative Cloud for $39.99 per semester or $79.99 per year. Check out this page for more details.

Success in College: Negative Talk

Sometimes math students use “negative self-talk” about math. They say, “I can’t do math” instead of, “I can do some math and, with hard work, I’ll be able to do more.” They say, “I should be able to work faster” instead of “I can work fast enough to pass the course.”

It has been proven that positive thinking is part of success. Being an optimist about math, but more importantly, about life, is a crucial building block of success.

A favorite quote of mine comes from Thomas Edison. When asked how he felt about so many failed tests before finally inventing the lightbulb, he responded, “I didn’t fail the tests; I just found 100 ways to do it wrong.”

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. What effect(s) do you think negative self-talk can have on a student’s attitude about math? Why?
2. What effect do you think negative self-talk can have on a student’s performance in math? Why?

Answers

Section Preview

This is kind of a "catch-all" section for simplifying radical expressions (at least the kinds of simplifications needed for Calculus). The focus is on roots of powers and rationalizing (both topics, of which, are incredibly important in Calculus). It is critical that you spend time to do some difference quotient problems here involving radical functions.

Section Preview

- Simplify expressions of the form $\sqrt[q]{a^n}$.
- Rationalize denominators and numerators.
- Compute the Difference Quotient of radical functions.

Example $3.5.1$

a. Since the index, 2, is even, $\sqrt{x^2} = \sqrt[2]{x^2} = |x|$.
b. This is an odd-indexed root so there is no need for an absolute value sign.

$$\sqrt[3]{n^3} = n.$$

c. Since the index is even $\sqrt[4]{p^4} = |p|$.
d. Since the index is odd, $\sqrt[5]{y^5} = y$.

Checkpoint $3.5.1$

a. $|b|$
b. w
c. $|m|$
d. q

Example $3.5.2$

a. Since $(x^3)^2 = x^6$,

$$\sqrt{(x^3)^2}.$$

Since the index is even, this becomes $|x^3|$. Hence,

$$\sqrt{x^6} = |x^3|.$$

b. Since $(y^8)^2 = y^{16}$,

$$\sqrt{(y^8)^2}.$$

Since the index is even, this becomes $|y^8|$. In this case the absolute value sign is not needed as y^8 is not negative. Therefore,

$$\sqrt{y^{16}} = y^8.$$

c. Since $(y^6)^3 = y^{18}$,

$$\sqrt[3]{(y^6)^3}.$$

Since 3 is odd, this becomes y^6.
d. Since $(z^2)^4 = z^8$,

$$\sqrt[4]{(z^2)^4}.$$

Since z^2 is positive, we do not need an absolute value sign. Therefore, this simplifies to z^2.
e. Since $(4n)^2 = 16n^2$,

$$\sqrt{(4n)^2}.$$

Since the index is even, this reduces to $4|n|$.
f. Since $(9c)^2 = 81c^2$,

$$-\sqrt{(9c)^2}.$$

Since the index is even, this becomes $-9|c|$.
g. Rewrite $64p^6$ as $(4p^2)^3$. Therefore, we have

$$\sqrt[3]{(4p^2)^3}.$$

Taking the cube root, we get

$$4p^2.$$

h. Rewrite the radicand as a fourth power.

$$\sqrt[4]{(2q^3)^4}.$$

Take the fourth root to get

$$2|q^3|.$$

Checkpoint 3.5.2

a. $|y^9|$
b. z^6
c. $|u^3|$
d. v^5
e. $8|x|$
f. $-10|p|$
g. $3x^9$
h. $3|q^7|$

Theorem: Conjugate Multiplication

$$\begin{aligned}(a+b)(a-b) &= a^2-ab+ab-b^2 && \text{(distributing)}\\ &= a^2-b^2 && \text{(simplifying)}\end{aligned}$$

Example 3.5.3

$$\begin{aligned}\frac{5}{2-\sqrt{3}} &= \frac{5}{\left(2-\sqrt{3}\right)}\cdot\frac{\left(2+\sqrt{3}\right)}{\left(2+\sqrt{3}\right)} && \text{(multiplying numerator and denominator by the conjugate of the denominator)}\\ &= \frac{5\left(2+\sqrt{3}\right)}{2^2-\left(\sqrt{3}\right)^2} && \text{(distributing (denominator only))}\\ &= \frac{5\left(2+\sqrt{3}\right)}{4-3} && \text{(simplifying)}\\ &= \frac{5\left(2+\sqrt{3}\right)}{1} && \text{(simplifying)}\\ &= 5\left(2+\sqrt{3}\right) && \text{(simplifying)}\end{aligned}$$

Checkpoint 3.5.3

$-\frac{3(1+\sqrt{5})}{4}$

Example 3.5.4

$$\begin{aligned}\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}} &= \frac{\sqrt{3}}{\left(\sqrt{u}-\sqrt{6}\right)}\cdot\frac{\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}+\sqrt{6}\right)} && \text{(multiplying numerator and denominator by the conjugate of the denominator)}\\ &= \frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}\right)^2-\left(\sqrt{6}\right)^2} && \text{(distributing (denominator only))}\\ &= \frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{u-6} && \text{(simplifying)}\end{aligned}$$

Checkpoint 3.5.4

$\frac{\sqrt{5}(\sqrt{x}-\sqrt{2})}{x-2}$

Example 3.5.5**: Revisiting the Difference Quotient**

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h} && \text{(substituting)}\\ &= \frac{\left(\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right)}{h}\cdot\frac{\sqrt{x}\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} && \text{(multiplying numerator and denominator by the LCD)}\\ &= \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} && \text{(distributing (numerator only))}\end{aligned}$$

Now that the compound fraction has been simplified, we are faced with an expression that *still* has the h in the denominator. Remember, a hidden, underlying reason will compel us to somehow remove that through creative uses of Algebra. One such use is conjugate multiplication!

$$\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \frac{\left(\sqrt{x}-\sqrt{x+h}\right)}{h\sqrt{x}\sqrt{x+h}}\cdot\frac{\left(\sqrt{x}+\sqrt{x+h}\right)}{\left(\sqrt{x}+\sqrt{x+h}\right)} \quad \text{(multiplying numerator and denominator by the conjugate of the numerator)}$$

$$= \frac{\left(\sqrt{x}\right)^2-\left(\sqrt{x+h}\right)^2}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} \quad \text{(distributing (numerator only))}$$

$$= \frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} \quad \text{(simplifying)}$$

$$= \frac{x-x-h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} \quad \text{(distributing)}$$

$$= \frac{-h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} \quad \text{(simplifying)}$$

$$= \frac{-\cancel{h}^{1}}{\cancel{h}^{1}\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} \quad \text{(canceling like factors)}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)} \quad \text{(simplifying)}$$

Our final expression is no longer a compound fraction, nor does it have that pesky h in the denominator!

Checkpoint 3.5.5

$\frac{1}{\sqrt{x+h}+\sqrt{x}}$

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3.5.1: Resources and Key Concepts

Resources

Videos

- Rationalizing denominators and numerators
  - Expressions - Radicals - Simplifying by Rationalizing Denominators and Numerators: A Review
  - Radicals and Rational Exponents - A Review of Rationalizing Denominators
  - Intermediate Algebra - Radical Expressions: Rationalizing Denominators

Key Concepts

Definitions

- **Conjugate (of an expression involving square roots)**: The conjugate of $a+b$ is $a-b$. Likewise, the conjugate of $a-b$ is $a+b$. This definition is extended from complex conjugates to expressions involving square roots (e.g., the conjugate of $2-\sqrt{3}$ is $2+\sqrt{3}$).
- **Radical Functions**: Functions involving roots (e.g., square root, cube root).

Theorems

- **Roots of Powers**: For any natural number $n \geq 2$, $\sqrt[n]{a^n} = \begin{cases} a, & \text{if } n \text{ is odd} \\ |a|, & \text{if } n \text{ is even} \end{cases}$.
- **Conjugate Multiplication**: Given algebraic expressions a and b, the product of $a+b$ with its conjugate, $a-b$, is always a^2-b^2. (This is essentially the difference of squares factorization.)

Common Mistakes

- **Forgetting Absolute Value with Even Roots**: When simplifying an even-indexed root of a variable raised to an even power, forgetting to use absolute value signs. For example, $\sqrt{x^6} = |x^3|$, not just x^3, unless it's known that $x \geq 0$.
- **Incorrectly Applying Absolute Value with Odd Roots**: Using absolute value signs when simplifying odd-indexed roots. For odd roots, $\sqrt[n]{a^n} = a$.
- **Errors in Rationalizing**:
  - Multiplying only the numerator or only the denominator by the conjugate, instead of both.
  - Incorrectly multiplying out the conjugate pair (e.g., not getting a^2-b^2).
  - Distributing unnecessarily in the non-conjugate part, which might obscure common factors that could cancel.
- **Errors in Simplifying Compound Fractions (in Difference Quotients)**:
  - Not multiplying by the correct LCD of all minor fractions.
  - Incorrectly distributing or canceling terms when simplifying the compound fraction.
- **Not Simplifying Radicals Completely**: Leaving perfect n^{th} powers inside an nth root (e.g., $\sqrt{16n^2}$ should be simplified to $4|n|$).

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3.5.2: Homework

Reading Questions

1. What is the result of simplifying $\sqrt[n]{a^n}$ when n is an even natural number? What if n is an odd natural number?
2. When simplifying $\sqrt{x^6}$, why is the result $|x^3|$ and not just x^3 (unless $x \geq 0$)?
3. What is the conjugate of the expression $\sqrt{u} - \sqrt{6}$?
4. When rationalizing a denominator (or numerator) that is a binomial involving a square root, what do you multiply the numerator and denominator by?
5. What is the product of an expression like $(a - b)$ and its conjugate $(a + b)$?
6. When simplifying a difference quotient for a radical function like $f(x) = \frac{1}{\sqrt{x}}$, what is the primary algebraic technique used to eliminate the radicals from the numerator after initial simplification?
7. In the context of simplifying difference quotients involving compound fractions with radicals, when should you distribute terms in the denominator, and when should you avoid it?
8. Simplify $\sqrt[4]{(2q^3)^4}$, assuming q can be any real number.
9. Why is it important to rationalize the numerator when working with difference quotients of radical functions in Calculus?
10. If you are simplifying $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ by multiplying the numerator and denominator by $\sqrt{x+h}+\sqrt{x}$, which part (numerator or denominator) should you fully multiply out (distribute), and which part should you generally leave in factored form? Why?

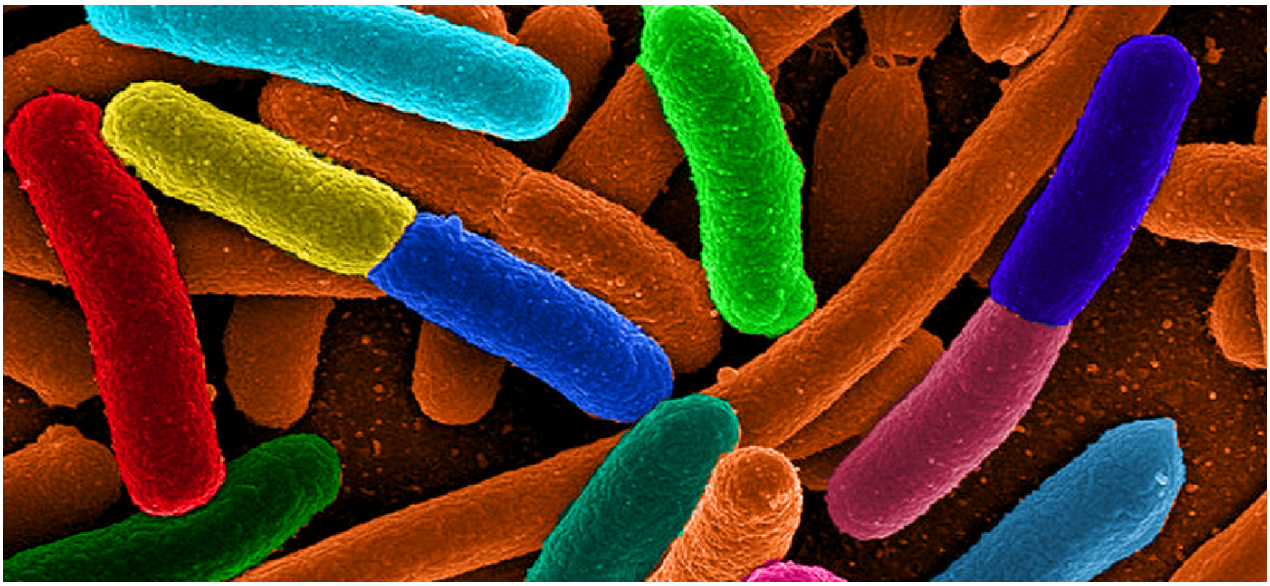
Homework

A textbook homework assignment needs to be created for this section.

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4: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Electron micrograph of E.Coli bacteria (credit: "Mattosaurus," Wikimedia Commons)



Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years.

For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. The table below shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

4.1: Exponential Functions

4.1.1: Resources and Key Concepts

4.1.2: Homework

4.2: Graphs of Exponential Functions

4.2.1: Resources and Key Concepts

4.2.2: Homework

4.3: Logarithmic Functions

4.3.1: Resources and Key Concepts

4.3.2: Homework

4.4: Graphs of Logarithmic Functions

4.4.1: Resources and Key Concepts

4.4.2: Homework

4.5: Laws of Logarithms

4.5.1: Resources and Key Concepts

4.5.2: Homework

4.6: Exponential and Logarithmic Equations

4.6.1: Resources and Key Concepts

4.6.2: Homework

4.7: Exponential and Logarithmic Models

4.7.1: Resources and Key Concepts

4.7.2: Homework

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4.1: Exponential Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

When the original iteration of this textbook was written (2013), India was the second most populous country in the world with a population of about 1.25 billion people. At that time, the population was growing at a rate of about 1.2% each year. If that rate continued, the population of India would have exceeded China's population by the year 2031. Interestingly, the rate of growth for India's population actually sped up! In 2025, India's population was 1.42 billion while China's was 1.41 billion. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at **exponential functions**, which model this kind of rapid growth.

Identifying Exponential Functions

When exploring linear growth back in Algebra, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x) = 3x + 4$, the slope tells us the output increases by 3 each time the input increases by 1. We call this the **Slope Addition Property** and it is unique to linear functions. If you never knew of this property from your old Algebra class, then it's best we codify it here.

Theorem: Slope Addition Property

For a linear function with slope m, every increase of 1 in input results in adding m to the function value. This is called the **slope addition property**.

The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, 2.5% of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continued, vegans would have made up 10% of the U.S. population in 2015, 40% in 2019, and 80% in 2021.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of exponential growth, let us contrast exponential growth with linear growth. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs. See Table $4.1.1$.

Table $4.1.1$

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12
7	128	14
8	256	16
9	512	18
10	1024	20

From Table $4.1.1$ we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** means the original value from the range increases by the *same percentage* over equal increments in the domain.
- **Linear growth** means the original value from the range increases by the *same amount* over equal increments in the domain (again, this is called the Slope Addition Property).

Apparently, the difference between "the same percentage" and "the same amount" is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. If we agree to call this multiplier the **base**, then this aspect that is unique to exponential functions is called the **Base Multiplier Property**.

Theorem: Base Multiplier Property

For an exponential function with base b, every increase of 1 in input results in multiplying the previous function value by b. This is called the **base multiplier property**.

So, how do we define exponential functions?

Definition: Exponential Function

An **exponential function** is a function with the form

$$f(x) = ab^x,$$

where a is a non-zero real number called the **initial value** and b is any positive real number, called the **base**, such that $b \neq 1$. This function is also referred to as the **exponential function base** b.

Let's look at the function $f(x) = 2^x$ from our example. We will create a table (Table $4.1.2$) to determine the corresponding outputs over an interval in the domain from -3 to 3.

Table 4.1.2

x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	$2^{-3}=\frac{1}{8}$	$2^{-2}=\frac{1}{4}$	$2^{-1}=\frac{1}{2}$	$2^0=1$	$2^1=2$	$2^2=4$	$2^3=8$

Let us examine the graph of f by plotting the ordered pairs we observe on the table in Figure 4.1.1, and then make a few observations.

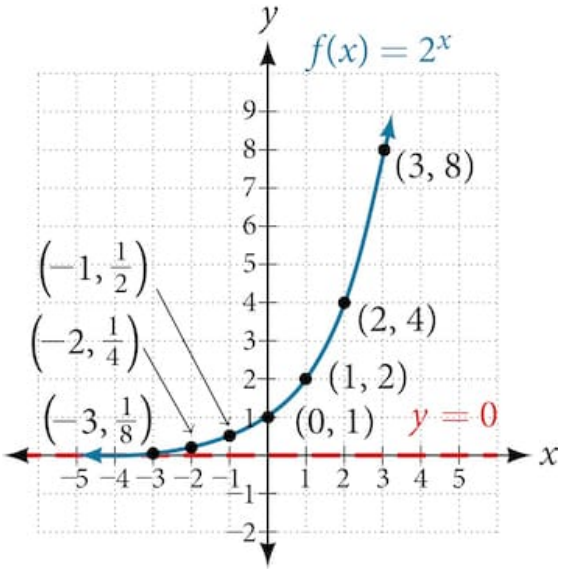


Figure 4.1.1

From the graph, it appears that the following are some key characteristics of $f(x)=2^x$:

- the domain is $(-\infty,\infty)$,
- the range is $(0,\infty)$,
- as $x\to\infty$, $f(x)\to\infty$,
- as $x\to-\infty$, $f(x)\to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the x-axis because base two raised to any exponent never has the result of zero,
- $y=0$ appears to be the horizontal asymptote, and
- the y-intercept is 1.

In fact, these characteristics are not unique to the exponential function base 2.

Theorem: Properties of the Exponential Function

Let $f(x)=ab^x$, with $a>0$, $b>0$, and $b\neq 1$:

- The domain of f is all real numbers.
- The range of f is all positive real numbers.
- The y-intercept is $(0,a)$.
- The graph of f includes the point $(1,ab)$.
- The horizontal asymptote is $y=0$.
- If $b>1$,
  - f is increasing,
  - as $x\to-\infty$, $f(x)\to 0$, and
  - as $x\to\infty$, $f(x)\to\infty$.
- If $0<b<1$,
  - $f(x)$ is decreasing,

- as $x \to -\infty$, $f(x) \to \infty$, and
- as $x \to \infty$, $f(x) \to 0$.

We state this theorem assuming $a > 0$ for sake of simplicity. If $a < 0$, then we can visualize the graphical change in f as a reflection about the x-axis. Moreover, we require $a \neq 0$. Otherwise, $f(x) = 0 \cdot b^x = 0$, and this is the horizontal line along the x-axis. Hence, it is not exponentially increasing nor decreasing.

The domain of an exponential function is all real numbers because we can evaluate b^{power} for any power we desire. The range is all positive real numbers because, as long as $b > 0$, $b^{\text{power}} > 0$.

If $f(x) = ab^x$, where $b \neq 0$, then $f(0) = a \cdot b^0 = a \cdot 1 = a$ (remember, we cannot determine the value of 0^0, so we cannot let $b = 0$). Furthermore, $f(1) = a \cdot b^1 = ab$. Thus, the points $(0, 1)$ and $(1, ab)$ are on the graph of the exponential function.

Example 4.1.1: Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution available at the end of this section.

Checkpoint 4.1.1

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1. Why do we limit the base b to positive values?

Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{1/2} = \sqrt{-9}$, which is not a real number. This should convince us that allowing b to be negative would cause too much of a headache. Therefore, it is convention to restrict the base of exponential functions to positive values only.

Why do we limit the base to positive values other than 1?

Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x. Since this is a constant, it is neither exponentially increasing nor exponentially decreasing. As such, we further restrict the base of exponential functions to not include 1.

To evaluate an exponential function with the form $f(x) = b^x$, we simply substitute x with the given value, and calculate the resulting power. For example, let $f(x) = 2^x$. What is $f(3)$?

$$\begin{aligned} & f(x) = 2^x \\ \implies & f(3) = 2^3 && \text{(substituting)} \\ & = 8 && \text{(simplifying)} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the Order of Operations. For example, let $f(x) = 30(2)^x$. What is $f(3)$?

$$\begin{aligned} & f(x) = 30(2)^x & \\ \Longrightarrow \quad & f(3) = 30(2)^3 & \text{(substituting)} \\ & = 30(8) & \text{(exponents before multiplication)} \\ & = 240 & \text{(multiplication)} \end{aligned}$$

Note that if the Order of Operations were not followed, the result would be incorrect:

$$f(3) = 30(2)^3 \neq 60^3 = 216,000.$$

Example 4.1.2: Evaluating Exponential Functions

Let $f(x) = 5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

Solution available at the end of this section.

Checkpoint 4.1.2

Let $f(x) = 8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term "exponential growth" is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

Definition: Exponential Growth and Decay

A function that models **exponential growth** or **exponential decay** grows or decays by a rate proportional to the amount present. For any real number t and any positive real numbers P_0 and b such that $b \neq 1$, an exponential growth or decay function has the form

$$P(t) = P_0\, b^t,$$

where

- P_0 is the **initial** or **starting value** of the function, and
- b is the **growth** or **decay factor**
  - if $b > 1$, it is a growth factor or **growth multiplier** per unit t
  - if $0 < b < 1$, it is a decay factor or **decay multiplier** per unit t.

Note that this definition mirrors the definition of an exponential function, but with f replaced by P, a replaced by P_0, and x replaced by t. The traditional notation introduced with the original definition (using $f(x)$ and a) only differs from this new notation in letters. Most applications involving exponential functions will use time (t) as the independent variable and are modeling populations (thus, the choice of function name, $P(t)$). Moreover, P_0 (read as, "P naught") stands for the initial population size (or the time-zero population size).

To differentiate between linear and exponential functions, let's consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A(x) = 100 + 50x$. Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x) = 100(1 + 0.5)^x$.

A few years of growth for these companies are illustrated in Table 4.1.3.

Table 4.1.3

Year, x	Stores, Company A	Stores, Company B
0	$100 + 50(0) = 100$	$100(1 + 0.5)^0 = 100$
1	$100 + 50(1) = 150$	$100(1 + 0.5)^1 = 150$
2	$100 + 50(2) = 200$	$100(1 + 0.5)^2 = 225$
3	$100 + 50(3) = 250$	$100(1 + 0.5)^3 = 337.5$
x	$A(x) = 100 + 50x$	$B(x) = 100(1 + 0.5)^x$

The graphs comparing the number of stores for each company over a five-year period are shown in Figure 4.1.2. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.

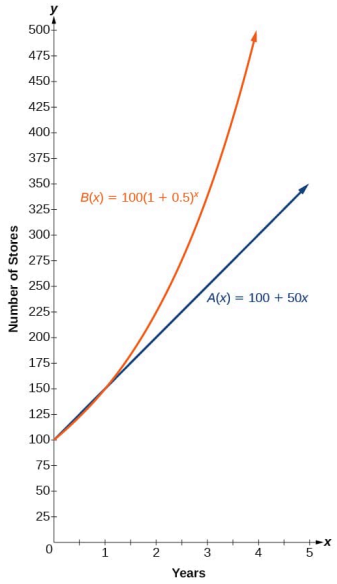


Figure 4.1.2**:** *The graph shows the numbers of stores Companies A and B opened over a five-year period.*

Notice that the *realistic* domain for both functions is $[0, \infty)$, and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Interactive Element: Exponential vs. Linear Growth

The following Interactive Element might help you visualize how exponential growth overtakes linear growth.

Interact: Drag the green point, which is Company B's number of stores at year 1, up and down.

Observation: Both companies start at 100 stores, but once Company B's growth factor rises above 1, its curve bends upward and overtakes Company A's straight line — pulling further ahead every year after.

Now we will turn our attention to the function representing the number of stores for Company B, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the **growth rate**, and $1 + 0.5 = 1.5$ represents the **growth factor**. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$, where 100 is the initial value, 1.5 is the base, and x is the exponent.

Example 4.1.3: Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, and using this model, what will the population of India be in 2031?

Solution available at the end of this section.

Checkpoint 4.1.3

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about 0.6%. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in Example 4.1.3?

Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a and b, and evaluate the function.

How to... find the equation of an exponential function given two points

1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

Example 4.1.4: Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an exponential function $N(t)$ representing the population (N) of deer over time t.

Solution available at the end of this section.

Checkpoint 4.1.4

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

Example 4.1.5: Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

Solution available at the end of this section.

Checkpoint 4.1.5

Given the two points $(1, 3)$ and $(2, 4.5)$, find the equation of the exponential function that passes through these two points.

Q&A

Do two points always determine a unique exponential function?

Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in x, which in many real world cases involves time.

Example 4.1.6: Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in Figure 4.1.5.

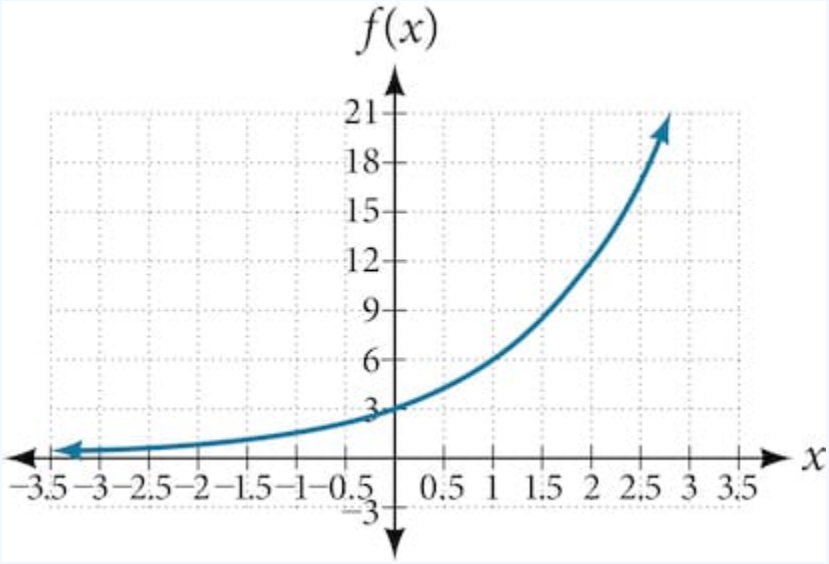


Figure 4.1.5

Solution available at the end of this section.

Checkpoint 4.1.6

Find an equation for the exponential function graphed in Figure 4.1.6.

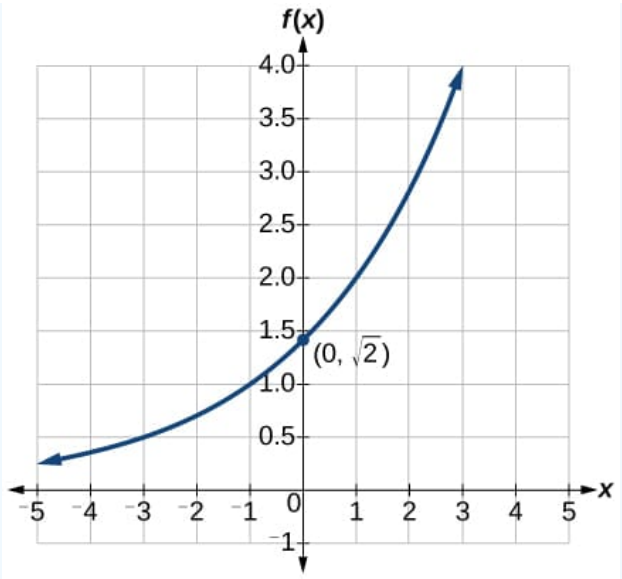


Figure 4.1.6

Compound Interest

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, earn interest that is compounded.

Definition: Compounding

Compounding refers to the process of earning interest not only on the initial investment (**principal**) but also on the accumulated interest itself.

To convey to the customer how much their money is earning annually (per year), investors cite the **annual percentage rate** (**APR**).

Definition: Annual Percentage Rate

The **annual percentage rate** (**APR**) of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account.

The term nominal is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the **effective interest rate** ends up being *greater* than the nominal rate! This is a powerful tool for investing.

Definition: Effective Rate

The **effective interest rate** (**EIR**), **effective annual interest rate**, **annual equivalent rate** (**AER**) or simply **effective rate** is the percentage of interest on a loan or financial product if compound interest accumulates in periods different than a year. Another, very popular, name for the effective rate is the **annual percentage yield** (**APY**).

Typical compounding periods (other than once per year) are:

- semiannual (twice per year, at six-month intervals)
- quarterly (once per three months)
- monthly

- daily

Other compounding periods exist (e.g., weekly), but they are not as common.

Developing a formula to predict the effects of compound interest on an investment takes a little bit of patience and insight. Let's assume we have $1,000 and invest it for one year at a bank that pays 10% interest.

If the bank is only paying **simple interest** (which means it is compounding annually), then at the end of the year we would have

$$1000 + 10\% \text{ of } 1000 = 1000 + 0.10 \times 1000 = 1000 + 100 = 1100.$$

Our money has grown from $1,000 to $1,100, earning us $100 in interest!

What would happen if the bank was paying 10% APR, but compounding semiannually? Since they are compounding twice per year, at the end of six months they pay 5% on what we invested. Then, at the end of the second six months, they pay another 5% *of what is now in our account*! That is, the second compounding period earns interest not only on what we invested, but also on the interest we received during the first six months.

$$\begin{aligned} \text{End of First Six Months:} \quad & 1000 + 5\% \text{ of } 1000 = 1000 + 0.05 \times 1000 = 1000 + 50 = 1050 \\ \text{End of Second Six Months:} \quad & 1050 + 5\% \text{ of } 1050 = 1050 + 0.05 \times 1050 = 1050 + 52.50 = 1102.50 \end{aligned}$$

We end the year with $1,102.50. We earned more interest than before!

Let's show one more case before stating the observation. In this case, let's assume that the bank compounds interest quarterly. Moreover, instead of simplifying our computations, we will use some Algebra to showcase a pattern.

$$
\begin{aligned}
\text{End of First Quarter:} \quad 1000 + \frac{10\%}{4} \text{ of } 1000 &= 1000 + \frac{0.10}{4} \times 1000 \\
&= 1000\left(1 + \frac{0.10}{4}\right) && \text{(factoring out 1000)} \\
&= P_1 && \text{(labeling this amount as } P_1\text{)} \\
\text{End of Second Quarter:} \quad P_1 + \frac{10\%}{4} \text{ of } P_1 &= P_1 + \frac{0.10}{4} \times P_1 && \text{(earning interest on the previous quarte} \\
&= P_1\left(1 + \frac{0.10}{4}\right) && \text{(factoring out } P_1\text{)} \\
&= 1000\left(1 + \frac{0.10}{4}\right)\left(1 + \frac{0.10}{4}\right) && \text{(substituting)} \\
&= 1000\left(1 + \frac{0.10}{4}\right)^2 && \text{(simplifying)} \\
&= P_2 && \text{(labeling this amount as } P_2\text{)} \\
\text{End of ThirdQuarter:} \quad P_2 + \frac{10\%}{4} \text{ of } P_2 &= P_2 + \frac{0.10}{4} \times P_2 && \text{(earning interest on the previous quarte} \\
&= P_2\left(1 + \frac{0.10}{4}\right) && \text{(factoring out } P_2\text{)} \\
&= 1000\left(1 + \frac{0.10}{4}\right)^2\left(1 + \frac{0.10}{4}\right) && \text{(substituting)} \\
&= 1000\left(1 + \frac{0.10}{4}\right)^3 && \text{(simplifying)} \\
&= P_3 && \text{(labeling this amount as } P_3\text{)} \\
\text{End of FourthQuarter:} \quad P_3 + \frac{10\%}{4} \text{ of } P_3 &= P_3 + \frac{0.10}{4} \times P_3 && \text{(earning interest on the previous quarte} \\
&= P_3\left(1 + \frac{0.10}{4}\right) && \text{(factoring out } P_3\text{)} \\
&= 1000\left(1 + \frac{0.10}{4}\right)^3\left(1 + \frac{0.10}{4}\right) && \text{(substituting)} \\
&= 1000\left(1 + \frac{0.10}{4}\right)^4 && \text{(simplifying)}
\end{aligned}
$$

We can grab a calculator to compute this final value to be approximately \$1,103.81. This is even more than our previous amount!

Now that we have taken our time, hopefully we can see a pattern. If we replace the interest rate (10%) with r, the number of compounding periods (4) with n, and the principal invested (\$1000) with P_0, we have the amount of money at the end of a single year as

$$A = P_0\left(1 + \frac{r}{n}\right)^n.$$

Allowing our money to sit over t years gives us the **compound interest formula**.

Theorem: Compound Interest Formula

The amount an investment of P_0 dollars is worth after t years earning an annual percentage rate of $r \times 100\%$ compounding n times per year is given by

$$A(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}.$$

P_0 is called the **principal**, or, more generally, the **present value**.

For example, observe Table 4.1.4, which shows the result of investing $1,000 at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Table 4.1.4

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

Example 4.1.7: Calculating Compound Interest

If we invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution available at the end of this section.

Checkpoint 4.1.7

An initial investment of $100,000 at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

Example 4.1.8: Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $40,000 over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Solution available at the end of this section.

Checkpoint 4.1.8

Refer to Example 4.1.8. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

Evaluating Functions with Base e

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. Table 4.1.5 shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of $1 invested at 100% interest for 1 year, compounded at various frequencies, listed in Table 4.1.5.

Table 4.1.5

Frequency	$A(n) = \left(1 + \frac{1}{n}\right)^n$	Value
Annually	$\left(1 + \frac{1}{1}\right)^1$	$2

Frequency	$A(n) = \left(1 + \frac{1}{n}\right)^n$	Value
Semiannually	$\left(1 + \frac{1}{2}\right)^2$	\$2.25
Quarterly	$\left(1 + \frac{1}{4}\right)^4$	\$2.441406
Monthly	$\left(1 + \frac{1}{12}\right)^{12}$	\$2.613035
Daily	$\left(1 + \frac{1}{365}\right)^{365}$	\$2.714567
Hourly	$\left(1 + \frac{1}{8760}\right)^{8760}$	\$2.718127
Once per minute	$\left(1 + \frac{1}{525600}\right)^{525600}$	\$2.718279
Once per second	$\left(1 + \frac{1}{31536000}\right)^{31536000}$	\$2.718282

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1 + \frac{1}{n}\right)^n$ approaches a number used so frequently in Mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating.

Definition: e

The letter e represents the irrational number

$$\left(1 + \frac{1}{n}\right)^n, \text{ as } n \text{ increases without bound.}$$

It is approximately 2.718281828459045. This is also known as **Euler's number**, **Euler's constant**, and the **natural exponential base**.

The letter e is used as a base for many real-world exponential models. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Example 4.1.9: Using a Calculator to Find Powers of e

Calculate $e^{3.14}$. Round to five decimal places.

Solution available at the end of this section.

Checkpoint 4.1.9

Use a calculator to find $e^{-0.5}$. Round to five decimal places.

Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, e is used as the base for exponential functions. Exponential models that use e as the base are called **continuous growth or decay models**. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

Definition: Continuous Growth/Decay

For all real numbers t, and all positive numbers a and r, **continuous exponential growth** (or, simply **continuous growth**) or **continuous exponential decay** (**continuous decay**) is represented by the formula

$$P(t) = P_0 e^{rt},$$

where r is called the **continuous growth** (or **decay**) **rate** per unit time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

Example 4.1.10: Calculating Continuous Growth

A person invested $1,000 in an account earning a nominal 10% per year compounded continuously. How much was in the account at the end of one year?

Solution available at the end of this section.

Checkpoint 4.1.10

A person invests $100,000 at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

Example 4.1.11: Calculating Continuous Decay

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Solution available at the end of this section.

Checkpoint 4.1.11

Using the data in Example 4.1.11, how much radon-222 will remain after one year?

Time-Sensitive Advice: Drop Deadline

Within the Los Rios Community College District, the last day you can drop a class is the Monday (or, sometimes, the Tuesday) of the thirteenth week of the semester. After this date, you will no longer be able to drop a class for that semester and the only grades you can receive in the classes you remain in are A, B, C, D, or F.

Will you still get a "W?"

Dropping any class between the third week and this final drop deadline will result in a "W" notation on your record for that class.

What does this mean?

Once you withdraw (W) or fail (D or F) a course three times within the Los Rios Community College District (CRC, SCC, FLC, or ARC), you are never allowed to take that course again at any of the colleges within the district. Thus, if you wanted to continue through the math sequence, you would need to take this course outside the Los Rios Community College District.

Success in Mathematics: Why Study Mathematics? (Part 3)

Matthew Jannsen graduated with a degree in mathematics in 2000. He went to law school and is now an attorney. He writes,

> "At this stage of my career I spend a majority of my time conducting legal research and drafting legal memoranda. Most people assume that mathematics would not be the most appropriate preparation for a person who makes their living

engaging in these activities. I would disagree. In fact, I would suggest that my mathematics background has been one of my greatest assets throughout my law school and legal careers and will continue to help me develop as a lawyer throughout my career. In order to succeed as a math major, I always had to engage in logical thought... I constantly rely on the logical thinking skills that I developed as a math major in executing my tasks as an attorney and I am grateful for my mathematics education."

Source: http://www.maa.org/careers/janssen.html

1. Describe, in detail, what you think it means to "think logically."
2. Is logical thought a vital skill in the career that you hope to have? Explain.

Answers

Section Preview

After all this time, we finally get to a chapter where we can assume that students have either never seen the material or have so little experience with it that it essentially needs to be built from the ground up. In this section, we cover the basics of exponential functions—essentially everything excpetion for graphing them (that is reserved for the next section).

Section Preview

- **Arithmetic**
  - Simplifying Fractions and Mixed Numbers: Example 4 reduces $\frac{180}{80}$ to $\frac{9}{4}$ before solving for the base.
  - Percents: Converting a stated rate to decimal form (e.g., $3\% \to 0.03$; a 50% increase $\to$ factor 1.5) is required throughout the growth and compound-interest examples (Examples 3, 7, 8).

Section Preview

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base e.

Example $4.1.1$: Identifying Exponential Functions

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base b of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

Example $4.1.2$: Evaluating Exponential Functions

Follow the Order of Operations. Be sure to pay attention to the parentheses.

$$\begin{aligned} & f(x) = 5(3)^{x+1} \\ \implies & f(2) = 5(3)^{2+1} && \text{(substituting)} \\ & = 5(3)^3 && \text{(simplifying)} \\ & = 5(27) && \text{(simplifying)} \\ & = 135 && \text{(simplifying)} \end{aligned}$$

Example $4.1.3$: Evaluating a Real-World Exponential Model

To estimate the population in 2031, we evaluate the model for $t = 18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549.$$

There will be about 1.549 billion people in India in the year 2031.

Example $4.1.4$: Writing an Exponential Model When the Initial Value Is Known

We let our independent variable t be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: $(0, 80)$ and $(6, 180)$. Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find b:

$$\begin{aligned} & N(t) = 80b^t \\ \implies & 180 = 80b^6 && \text{(substituting)} \\ \implies & \frac{9}{4} = b^6 && \text{(dividing both sides by 80)} \\ \implies & \left(\frac{9}{4}\right)^{1/6} = b && \left(\text{raising both sides to the } \tfrac{1}{6} \text{ power}\right) \\ \implies & b \approx 1.1447 && \text{(rounding to four decimal places)} \end{aligned}$$

NOTES: When we took the sixth root of both sides, we did not bother writing $b = \pm\left(\frac{9}{4}\right)^{1/6}$. This is because the base of an exponential function is never negative. Moreover, *do not round any intermediate calculations. Doing so leads to error propagation.*

The exponential model for the population of deer is $N(t) \approx 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs get large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in Figure $4.1.3$ passes through the initial points

given in the problem, $(0, 80)$ and $(6, 180)$. We can also see that the domain for the function is $[0, \infty)$, and the range for the function is $[80, \infty)$.

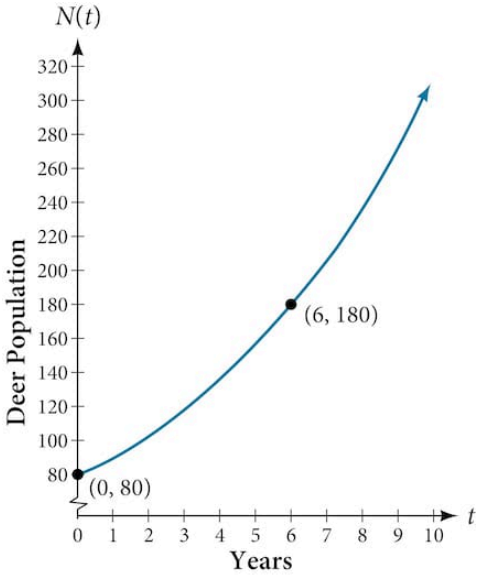


Figure 4.1.3**:** *Graph showing the population of deer over time,* $N(t) \approx 80(1.1447)^t$*, t years after 2006*

Example 4.1.5**: Writing an Exponential Model When the Initial Value is Not Known**

Because we don't have the initial value, we substitute both points into an equation of the form $f(x) = a\ b^x$, and then solve the system for a and b.

- Substituting $(-2, 6)$ gives $6 = a\ b^{-2}$
- Substituting $(2, 1)$ gives $1 = a\ b^2$

Use the first equation to solve for a in terms of b:

$$\begin{aligned} & 6 = a\,b^{-2} \\ \Longrightarrow\quad & 6b^2 = a \qquad \text{(multiplying both sides by } b^2\text{)} \end{aligned}$$

Substitute a in the second equation, and solve for b:

$$\begin{aligned} & 1 = a\,b^2 \\ \Longrightarrow\quad & 1 = 6b^2b^2 \qquad \text{(substituting } a = 6b^2\text{)} \\ \Longrightarrow\quad & 1 = 6b^4 \qquad \text{(Laws of Exponents)} \\ \Longrightarrow\quad & \frac{1}{6} = b^4 \qquad \text{(dividing both sides by 6)} \\ \Longrightarrow\quad & \left(\frac{1}{6}\right)^{1/4} = b \qquad \text{(raise both sides to the } \tfrac{1}{4} \text{ power)} \end{aligned}$$

Now that we have b, we go back to the first equation to find a:

$$a = 6b^2 = 6\left(\left(\frac{1}{6}\right)^{1/4}\right)^2 = 6\left(\frac{1}{6}\right)^{2/4} = 6\left(\frac{1}{6}\right)^{1/2} = \frac{6}{6^{1/2}} = 6^{1-1/2} = 6^{1/2} = \sqrt{6}.$$

Thus, the equation is

$$f(x) = \sqrt{6}\left(\left(\frac{1}{6}\right)^{1/4}\right)^x = \sqrt{6}\left(\frac{1}{6}\right)^{x/4}.$$

We can graph our model to check our work. Notice that the graph in Figure 4.1.4 passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

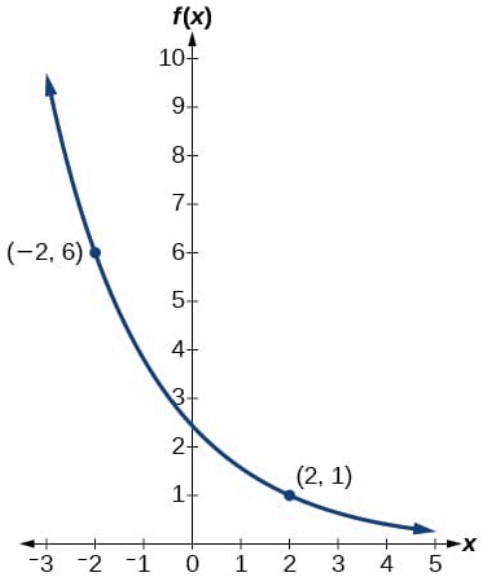


Figure 4.1.4**:** *The graph of* $f(x) = \sqrt{6}\left(\frac{1}{6}\right)^{x/4}$ *models exponential decay.*

Example 4.1.6**: Writing an Exponential Function Given Its Graph**

We can choose the y-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

$$\begin{aligned} & y = ab^x && \text{(writing the general form)} \\ \Longrightarrow\ & y = 3b^x && \text{(substituting } a = 3\text{)} \\ \Longrightarrow\ & 12 = 3b^2 && \text{(substituting } x = 2 \text{ and } y = 12\text{)} \\ \Longrightarrow\ & 4 = b^2 && \text{(dividing both sides by 3)} \\ \Longrightarrow\ & \pm 2 = b && \text{(Extraction of Roots)} \\ \Longrightarrow\ & 2 = b && \text{(exponential bases are positive)} \end{aligned}$$

Substitute a and b into the standard form to yield the equation $f(x) = 3(2)^x$.

Example 4.1.7**: Calculating Compound Interest**

Because we are starting with $3,000, $P = 3000$. Our interest rate is 3%, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$.

$$\begin{aligned} A(t) &= P\left(1 + \frac{r}{n}\right)^{nt} \\ \Longrightarrow\ A(10) &= 3000\left(1 + \frac{0.03}{4}\right)^{4 \cdot 10} && \text{(substituting known values)} \\ &\approx \$4,045.05 && \text{(rounding to two decimal places)} \end{aligned}$$

The account will be worth about $4,045.05 in 10 years.

Example 4.1.8**: Using the Compound Interest Formula to Solve for the Principal**

The nominal interest rate is 6%, so $r = 0.06$. Interest is compounded twice a year, so $n = 2$.

We want to find the initial investment, P, needed so that the value of the account will be worth $40,000 in 18 years. Substitute the given values into the compound interest formula, and solve for P.

$$\begin{aligned} A(t) &= P\left(1 + \frac{r}{n}\right)^{nt} \\ \Longrightarrow\ 40,000 &= P\left(1 + \frac{0.06}{2}\right)^{2 \cdot 18} && \text{(substituting known values)} \\ \Longrightarrow\ 40,000 &= P\,(1.03)^{36} && \text{(simplifying)} \\ \Longrightarrow\ \frac{40,000}{1.03^{36}} &= P && \text{(isolating } P\text{)} \\ \Longrightarrow\ P &\approx \$13,801 && \text{(rounding to the nearest dollar)} \end{aligned}$$

Lily will need to invest $13,801 to have $40,000 in 18 years.

Example 4.1.9**: Using a Calculator to Find Powers of** e

On a calculator, press the button labeled $[e^x]$. The window shows $e^($. Type 3.14 and then close parenthesis,). Press [ENTER]. Rounding to five decimal places, $e^{3.14} \approx 23.10387$.

Caution: Many scientific calculators have an Exp button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

Example 4.1.10**: Calculating Continuous Growth**

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was \$1,000, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

$$\begin{aligned} A(t) &= Pe^{rt} \\ &= 1000e^{0.1} && \text{(substituting known values)} \\ &\approx 1105.17 && \text{(rounding to the nearest hundredth)} \end{aligned}$$

The account is worth \$1,105.17 after one year.

Example $4.1.11$**: Calculating Continuous Decay**

Since the substance is decaying, the rate, 17.3%, is negative. So, $r = -0.173$. The initial amount of radon-222 was 100 mg, so $a = 100$. We use the continuous decay formula to find the value after $t = 3$ days:

$$\begin{aligned} A(t) &= ae^{rt} \\ &= 100e^{-0.173(3)} && \text{(substituting known values)} \\ &\approx 59.5115 && \text{(rounding to four decimal places)} \end{aligned}$$

So approximately 59.5115 mg of radon-222 will remain.

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4.1.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Arithmetic**
  - Percents: Understanding how to convert percentages to decimals (e.g., 9% to 0.09) is crucial for working with growth/decay rates in exponential functions and compound interest formulas.
- **Systems**
  - Solving 2x2 Systems of Linear Equations: This skill is needed to find equations of exponential functions going through two given points.

Videos

- Exponential functions
  - Intermediate Algebra - Exponential Functions: The Concepts
  - Exponential Functions - A Brief Review of Exponential Functions
- Exponential growth and decay
  - Exponential Functions - Types of Exponential Growth and Decay
- Finding equation of exponential function
  - Exponential Functions - Finding Equations of Exponential Functions
- Compound interest formula
  - Exponential Functions - Applying the Compound Interest Formula
- Natural exponential base e
  - Exponential Functions - Investigating Continuous Growth
  - Exponential Functions - Calculus and the Natural Base
- Applications of exponential functions
  - Exponential Functions - Applications Involving Exponential Functions
  - Intermediate Algebra - Exponential Functions: Creating from Applications

Key Concepts

Definitions

- **Percent Change**: A change based on a percent of the original amount.
- **Exponential Growth**: An increase based on a constant multiplicative rate of change over equal increments of time, that is, a percent increase of the original amount over time.
- **Exponential Decay**: A decrease based on a constant multiplicative rate of change over equal increments of time, that is, a percent decrease of the original amount over time.
- **Exponential Function**: A function with the form $f(x) = ab^x$, where a is a non-zero real number called the initial value and b is any positive real number, called the base, such that $b \neq 1$.
- **Initial Value (of an exponential function)**: In the form $f(x) = ab^x$, a is the initial value (the value of the function when $x = 0$).
- **Base (of an exponential function)**: In the form $f(x) = ab^x$, b is the base. It must be a positive real number not equal to 1.
- **Exponential Growth/Decay Function (alternative form)**: A function that models exponential growth or decay can be written as $P(t) = P_0 b^t$, where P_0 is the initial value, and b is the growth/decay factor. If $b > 1$, it's a growth factor. If $0 < b < 1$, it's a decay factor.

- **Compounding**: The process of earning interest not only on the initial investment (principal) but also on the accumulated interest itself.
- **Annual Percentage Rate (APR) / Nominal Rate**: The yearly interest rate earned by an investment account before considering the effect of compounding frequency.
- **Effective Rate / Annual Percentage Yield (APY)**: The percentage of interest on a loan or financial product if compound interest accumulates in periods different than a year; it reflects the actual annual rate earned after compounding.
- **The number e (Euler's number / natural exponential base)**: An irrational number approximately equal to 2.71828, which arises as the limit of $(1+\frac{1}{n})^n$ as n increases without bound. It is used as the base for continuous growth/decay models.
- **Continuous Growth/Decay**: Exponential growth or decay represented by the formula $P(t) = P_0 e^{rt}$, where P_0 is the initial value, r is the continuous growth rate (if $r > 0$) or continuous decay rate (if $r < 0$) per unit time, and t is time.
- **Continuous Growth/Decay Rate**: The parameter r in the continuous growth/decay formula $P(t) = P_0 e^{rt}$.

Theorems

- **Slope Addition Property**: For a linear function with slope m, every increase of 1 in input results in adding m to the function value.
- **Base Multiplier Property**: For an exponential function with base b, every increase of 1 in input results in multiplying the previous function value by b.
- **Properties of the Exponential Function $f(x) = ab^x$, $a > 0$**:
  - Domain: All real numbers.
  - Range: All positive real numbers $(0, \infty)$.
  - y-intercept: $(0, a)$.
  - Includes point $(1, ab)$.
  - Horizontal asymptote: $y = 0$.
  - If $b > 1$, f is increasing; as $x \to -\infty, f(x) \to 0$; as $x \to \infty, f(x) \to \infty$.
  - If $0 < b < 1$, f is decreasing; as $x \to -\infty, f(x) \to \infty$; as $x \to \infty, f(x) \to 0$.
- **Compound Interest Formula**: The amount $A(t)$ an investment of P_0 dollars is worth after t years earning an annual percentage rate of $r \times 100\%$ compounding n times per year is given by $A(t) = P_0\left(1+\frac{r}{n}\right)^{nt}$.

Common Mistakes

- **Confusing Exponential Growth with Linear Growth**: Mistaking a situation with a constant additive change for exponential growth, or vice-versa. Exponential growth involves a constant multiplicative factor (percent change).
- **Incorrect Base for Exponential Functions**: Using a base $b \leq 0$ or $b = 1$ for an exponential function $f(x) = ab^x$. The base must be positive and not equal to 1.
- **Order of Operations Errors in Evaluation**: When evaluating exponential functions like $f(x) = ab^{x+c} + d$, performing operations in the wrong order (e.g., multiplying by a before evaluating the exponent).
- **Incorrectly Using the Continuous Compounding Formula**: Applying $A = P_0 e^{rt}$ when interest is compounded discretely (e.g., monthly, quarterly), or using the discrete compounding formula for continuous compounding.
- **Sign Errors with Decay Rates**: For continuous decay $P(t) = P_0 e^{rt}$, forgetting that the rate r must be negative.

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4.1.2: Homework

Reading Questions

1. Explain the difference between linear growth and exponential growth.
2. What is the general form of an exponential function? Define the "initial value" and the "base," including any restrictions on the base.
3. Describe the key characteristics of the graph of $f(x) = ab^x$ when $a > 0$ and $b > 1$ (domain, range, intercepts, asymptotes, increasing/decreasing behavior).
4. How does the graph of $f(x) = ab^x$ change if $0 < b < 1$ compared to when $b > 1$?
5. If you are given two points, how can you find the formula for an exponential function $f(x) = ab^x$ that passes through them? What if one of the points is the y-intercept $(0, a)$?
6. What is the compound interest formula? Define each variable in the formula.
7. What is the number e? Why is it significant in the context of exponential functions?
8. What is the formula for continuous exponential growth or decay? Define each variable.
9. If a population is decaying continuously at a rate of 5% per year, what would be the value of r in the formula $P(t) = P_0e^{rt}$?
10. What is the difference between the annual percentage rate (APR) and the annual percentage yield (APY)?

Homework

Verbal

11. Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.
12. Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.
13. The Oxford Dictionary defines the word **nominal** as a value that is "stated or expressed but not necessarily corresponding exactly to the real value." Develop a reasonable argument for why the term **nominal rate** is used to describe the annual percentage rate of an investment account that compounds interest.

Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.

14. The average annual population increase of a pack of wolves is 25.
15. A population of bacteria decreases by a factor of $\frac{1}{8}$ every 24 hours.
16. The value of a coin collection has increased by 3.25% annually over the last 20 years.
17. For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.
18. The height of a projectile at time t is represented by the function $h(t) = -4.9t^2 + 18t + 40$.

For the following exercises, consider this scenario: For each year t, the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)

19. Which forest's population is growing at a faster rate?
20. Which forest had a greater number of trees initially? By how many?
21. Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?
22. Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100 years? By how many?
23. Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.

24. $y = 300(1-t)^5$
25. $y = 220(1.06)^x$
26. $y = 16.5(1.025)^{1/x}$
27. $y = 11,701(0.97)^t$

For the following exercises, find the formula for an exponential function that passes through the two points given.

28. $(0, 6)$ and $(3, 750)$
29. $(0, 2000)$ and $(2, 20)$
30. $\left(-1, \frac{3}{2}\right)$ and $(3, 24)$
31. $(-2, 6)$ and $(3, 1)$
32. $(3, 1)$ and $(5, 4)$

For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

33.

x	1	2	3	4
$f(x)$	70	40	10	-20

34.

x	1	2	3	4
$h(x)$	70	49	34.3	24.01

35.

x	1	2	3	4
$m(x)$	80	61	42.9	25.61

36.

x	1	2	3	4
$f(x)$	10	20	40	80

37.

x	1	2	3	4
$g(x)$	-3.25	2	7.25	12.5

For the following exercises, use the compound interest formula, $A(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$.

38. After a certain number of years, the value of an investment account is represented by the equation $A = 10,250\left(1 + \frac{0.04}{12}\right)^{120}$. What is the value of the account?
39. What was the initial deposit made to the account in the previous exercise?
40. How many years had the account from the previous exercise been accumulating interest?
41. An account is opened with an initial deposit of \$6,500 and earns 3.6% interest compounded semi-annually. What will the account be worth in 20 years?
42. How much more would the account in the previous exercise have been worth if the interest were compounding weekly?
43. Solve the compound interest formula for the principal, P_0.

44. Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $\$14,472.74$ after earning 5.5% interest compounded monthly for 5 years. (Round to the nearest dollar.)
45. How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?
46. Use properties of rational exponents to solve the compound interest formula for the interest rate, r.
47. Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $9,000 and was worth $13,373.53 after 10 years.
48. Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $5,500, and was worth $38,455 after 30 years.

For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.

49. $y = 3742e^{0.75t}$
50. $y = 150e^{3.25/t}$
51. $y = 2.25e^{-2t}$
52. Suppose an investment account is opened with an initial deposit of $\$12,000$ earning 7.2% interest compounded continuously. How much will the account be worth after 30 years?
53. How much less would the account from the previous exercise be worth after 30 years if it were compounded monthly instead?

Numeric

For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.

54. $f(x) = 2(5)^x$, for $f(-3)$
55. $f(x) = -4^{2x+3}$, for $f(-1)$
56. $f(x) = e^x$, for $f(3)$
57. $f(x) = -2e^{x-1}$, for $f(-1)$
58. $f(x) = 2.7(4)^{-x+1} + 1.5$, for $f(-2)$
59. $f(x) = 1.2e^{2x} - 0.3$, for $f(3)$
60. $f(x) = -\frac{3}{2}(3)^{-x} + \frac{3}{2}$, for $f(2)$

Technology

For the following exercises, use graphing technology to find the equation of an exponential function given the points on the curve.

61. $(0, 3)$ and $(3, 375)$
62. $(3, 222.62)$ and $(10, 77.456)$
63. $(20, 29.495)$ and $(150, 730.89)$
64. $(5, 2.909)$ and $(13, 0.005)$
65. $(11, 310.035)$ and $(25, 356365.2)$

Extensions

66. The annual percentage yield (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $\text{APY} = \left(1 + \frac{r}{12}\right)^{12} - 1$.
67. Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds n times per year.
68. Recall that an exponential function is any equation written in the form $f(x) = a \cdot b^x$ such that a and b are positive numbers and $b \neq 1$. Any positive number b can be written as $b = e^n$ for some value of n. Use this fact to rewrite the formula for an exponential function that uses the number e as a base.
69. In an exponential decay function, the base of the exponent is a value between 0 and 1. Thus, for some number $b > 1$, the exponential decay function can be written as $f(x) = a \cdot \left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b = e^n$, to show that an exponential decay function takes the form $f(x) = ae^{-nx}$ for some positive number n.
70. The formula for the amount A in an investment account with a nominal interest rate r at any time t is given by $A(t) = P_0e^{rt}$, where P_0 is the amount of principal initially deposited into an account that compounds continuously. Prove that the

percentage of interest earned to principal at any time t can be calculated with the formula $I(t) = e^{rt} - 1$.

Real-World Applications

71. The fox population in a certain region has an annual growth rate of 9% per year. In the year 2012, there were 23,900 fox counted in the area. What is the fox population predicted to be in the year 2020?
72. A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?
73. In the year 1985, a house was valued at $110,000. By the year 2005, the value had appreciated to $145,000. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?
74. A car was valued at $38,000 in the year 2007. By 2013, the value had depreciated to $11,000. If the car's value continues to drop by the same percentage, what will it be worth by 2017?
75. Jamal wants to save $54,000 for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?
76. Kyoko has $10,000 that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $15,000 by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (Hint: solve the compound interest formula for the interest rate.)
77. Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was $13,500. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?
78. An investment account with an annual interest rate of 7% was opened with an initial deposit of $4,000. Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

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4.2: Graphs of Exponential Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events.

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table $4.2.1$ change as the input increases by 1.

Table $4.2.1$

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the **constant ratio**. In fact, for any exponential function with the form $f(x) = ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a. If this sounds familiar, it's because we gave this property a name in the previous section—the Base Multiplier Property.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

Figure $4.2.1$ shows the exponential growth function $f(x) = 2^x$.

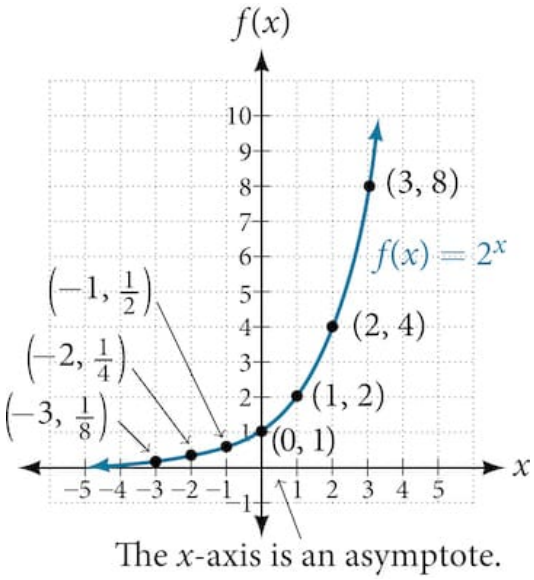


Figure 4.2.1**:** *Notice that the graph gets close to the x-axis, but never touches it.*

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table 4.2.2 change as the input increases by 1.

Table 4.2.2

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

Figure 4.2.2 shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

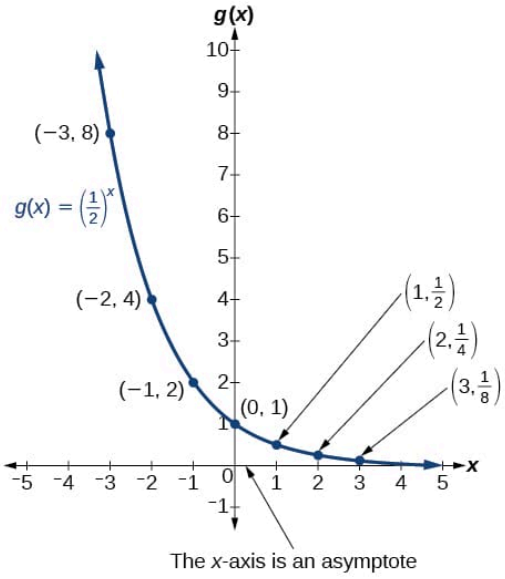


Figure 4.2.2

The domain of $g(x) = \left(\frac{1}{2}\right)^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

Interactive Element: Growth, Decay, and the Base

The following Interactive Element might help you visualize how the base b alone decides whether $f(x) = b^x$ grows or decays.

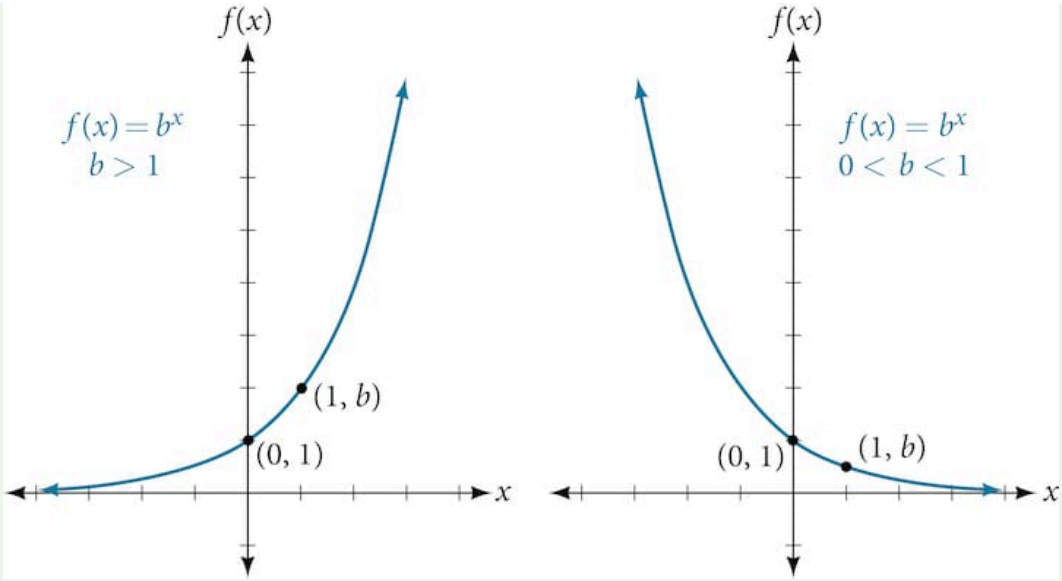
Interact: Drag the red point up and down — its height is the base b.

Observation: When $b > 1$ the curve grows; when $0 < b < 1$ it decays; at $b = 1$ it flattens to $y = 1$. The point $(0, 1)$ never moves.

Theorem: Properties of the Graphs of Exponential Functions

An exponential function with the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$, has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x-intercept: none
- y-intercept: $(0, 1)$
- Secondary key point: $(1, b)$
- increasing if $b > 1$
- decreasing if $0 < b < 1$



Example 4.2.1: Sketching the Graph of an Exponential Function of the Form $f(x) = b^x$

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 4.2.1

Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other toolkit functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the function $f(x) = b^x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in Figure 4.2.4.

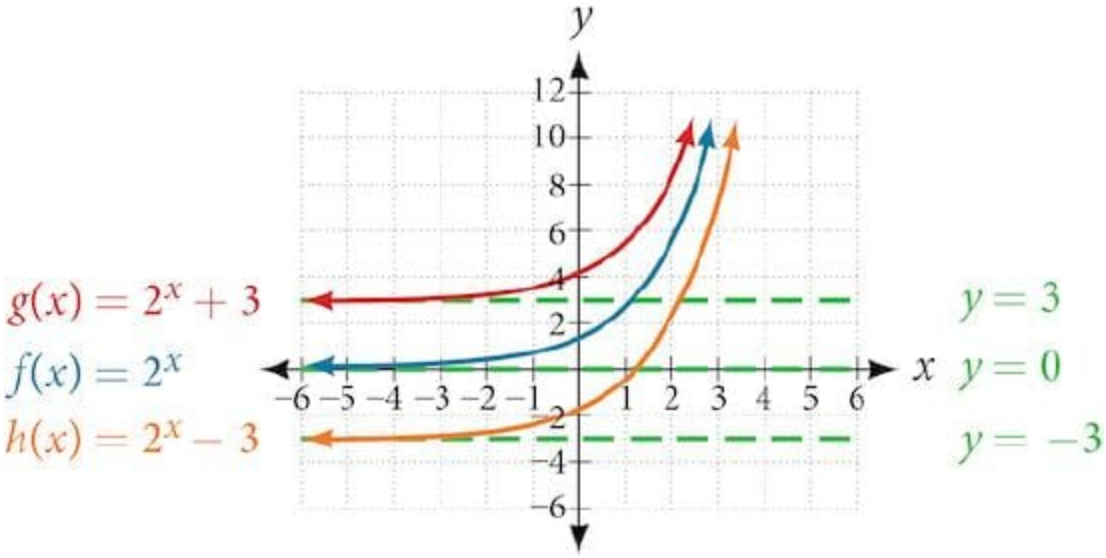


Figure 4.2.4

Interactive Element: Vertical Shift and the Asymptote

The following Interactive Element might help you visualize how a vertical shift d moves both the curve and its horizontal asymptote.

Interact: Drag the red point on the dashed line up and down to change d.

Observation: The curve and its horizontal asymptote $y = d$ always move together, so the range is (d, ∞).

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The y-intercept shifts up 3 units to $(0, 4)$.
  - The asymptote shifts up 3 units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The y-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. Both horizontal shifts are shown in Figure 4.2.5.

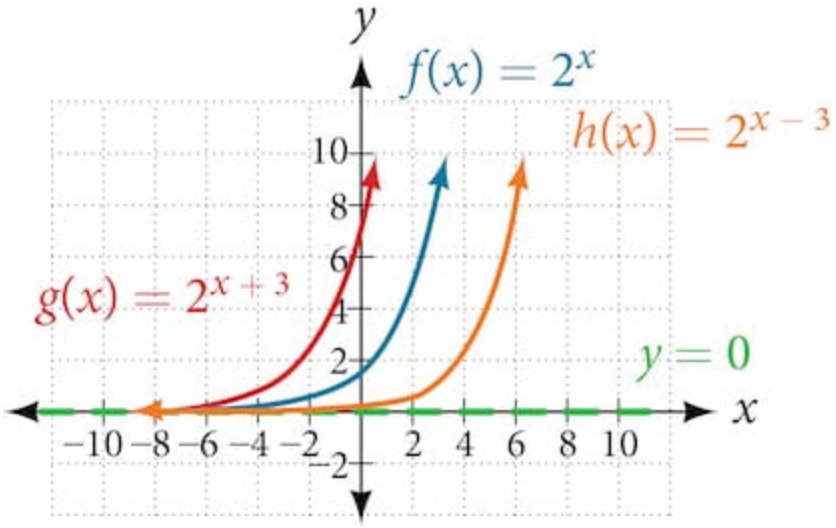


Figure 4.2.5

Interactive Element: Horizontal Shift

The following Interactive Element might help you visualize why shifting an exponential sideways also changes its y-intercept.

Interact: Drag the red point (it rides the curve at height 1) left and right.

Observation: The curve slides sideways and the asymptote stays at $y = 0$, but the orange y-intercept moves — a shift of c acts like multiplying by 2^c.

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$, remains unchanged.
- The asymptote, $y = 0$, remains unchanged.
- The y-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$, the y-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)2^x$, so the initial value of the function is 8.
  - When the function is shifted right 3 units to $h(x) = 2^{x-3}$, the y-intercept becomes $(0, \frac{1}{8})$. Again, see that $2^{x-3} = \left(\frac{1}{8}\right)2^x$, so the initial value of the function is $\frac{1}{8}$.

Example 4.2.2: Graphing a Shift of an Exponential Function

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 4.2.2

Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

Example 4.2.3: Approximating the Solution of an Exponential Equation

Solve $42 = 1.2(5)^x + 2.8$ using graphing technology. Round to the nearest thousandth.

Solution available at the end of this section.

Checkpoint 4.2.3

Solve $4 = 7.85(1.15)^x - 2.27$ graphically. Round to the nearest thousandth.

Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a stretch or compression occurs when we multiply the function $f(x) = b^x$ by a constant $|a| > 0$. For example, if we begin by graphing the function $f(x) = 2^x$, we can then graph the stretch, using $a = 3$, to get $g(x) = 3(2)^x$ as shown on the left in Figure 4.2.7, and the compression, using $a = \frac{1}{3}$, to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure 4.2.7.

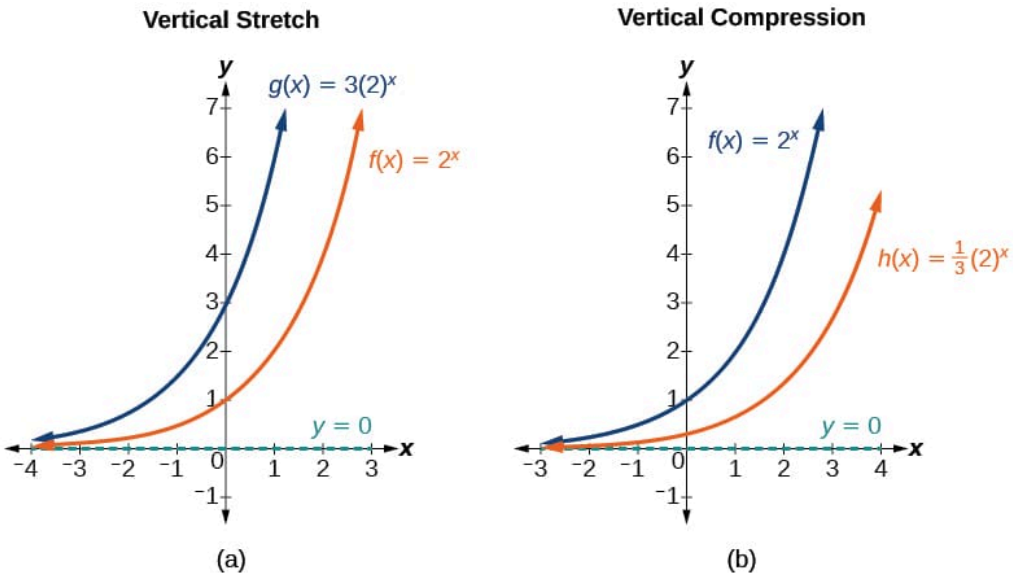


Figure 4.2.7: *(a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.*

Interactive Element: The Leading Coefficient a

The following Interactive Element might help you visualize how the coefficient a in $f(x) = a\ b^x$ stretches, compresses, or flips the graph.

Interact: Drag the red y-intercept point up and down — its height is a.

Observation: The curve always crosses at $(0, a)$; $|a| > 1$ stretches, $0 < |a| < 1$ compresses, and $a < 0$ reflects it about the x -axis.

Example 4.2.4: Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 4.2.4

Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the x-axis or the y-axis. When we multiply the function $f(x) = b^x$ by -1, we get a reflection about the x-axis. When we multiply the input by -1, we get a reflection about the y-axis. For example, if we begin by graphing the function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection about the x-axis, $g(x) = -2^x$, is shown on the left side of Figure 4.2.9, and the reflection about the y-axis $h(x) = 2^{-x}$, is shown on the right side of Figure 4.2.9.

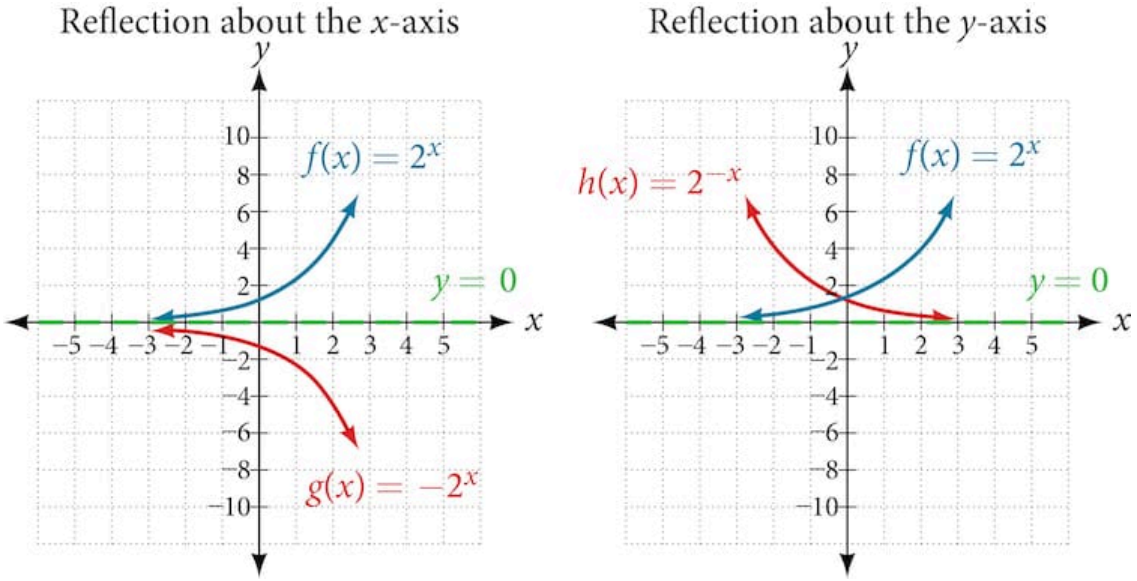


Figure 4.2.9: *(a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $h(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the y-axis.*

Example 4.2.5: Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = \left(\frac{1}{4}\right)^x$ about the x-axis. State its domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 4.2.5

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the y-axis. State its domain, range, and asymptote.

Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in Table 4.2.6 to arrive at the general equation for translating exponential functions.

Table $4.2.6$***:*** *Transformations of the Parent Function* $f(x) = b^x$

Transformation	Form
Shift • Horizontally c units to the left (if $c > 0$) • Vertically d units up (if $d > 0$)	$f(x) = b^{x+c} + d$
Stretch and Compress • Stretch if $\|a\| > 1$ • Compression if $0 < \|a\| < 1$	$f(x) = ab^x$
Reflect about the x-axis	$f(x) = -b^x$
Reflect about the y-axis	$f(x) = b^{-x} = \left(\frac{1}{b}\right)^x$
General equation for all transformations	$f(x) = ab^{x+c} + d$

Example $4.2.6$: Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the y-axis, and then shifted up 4 units.

Solution available at the end of this section.

Checkpoint $4.2.6$

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the x-axis and then shifted down 2 units.

Success in College: Dealing With Failure (Part 2)

A student gets a test back from the professor. The test score is 55%. The professor has written at the top of the exam, "Make an appointment to talk to me about this."

1. Why do you think that the professor wants to talk to the student?
2. Making an appointment to talk to your professor is usually an option in college. You can choose not to do this. Describe the reasons someone may use for making the appointment. Describe the reasons someone may use for not making the appointment.

Miscellaneous Topics: Thinking About Mathematics (Part 3)

When I would get frustrated in school, I often found it good to reflect upon some of my favorite teachers and think about how they would approach the subject.

- Professor Simpson, Cosumnes River College

1. Who was your favorite teacher, and what subject did they teach?
2. Can you make any suggestions I can incorporate into the classroom that worked for that teacher?

Answers

Section Preview

This is a standard treatment of the graphs of exponential functions. Give some extra time for the base e exponential as it is incredibly important in Calculus.

Section Preview

- Graph exponential functions.
- Graph exponential functions using transformations.

Example $4.2.1$**: Sketching the Graph of an Exponential Function of the Form** $f(x) = b^x$

Before graphing, identify the behavior and create a table of points for the graph.

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in Table $4.2.3$.

Table $4.2.3$

x	-3	-2	-1	0	1	2	3
$f(x) = 0.25^x$	64	16	4	1	0.25	0.0625	0.015625

- Plot the y-intercept, $(0, 1)$, along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in Figure $4.2.3$.

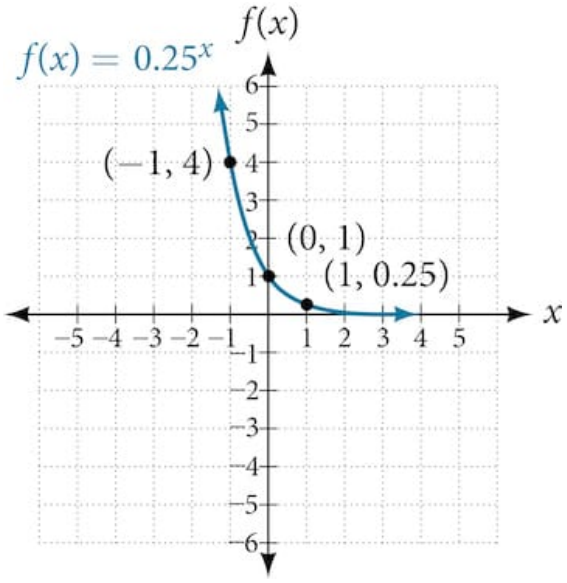


Figure $4.2.3$

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Example $4.2.2$**: Graphing a Shift of an Exponential Function**

We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d = -3$.

- Draw the horizontal asymptote $y = d$, so draw $y = -3$.
- Identify the shift as $(-c, d)$, so the shift is $(-1, -3)$.
- Shift the graph of $f(x) = b^x$ left 1 unit and down 3 units.

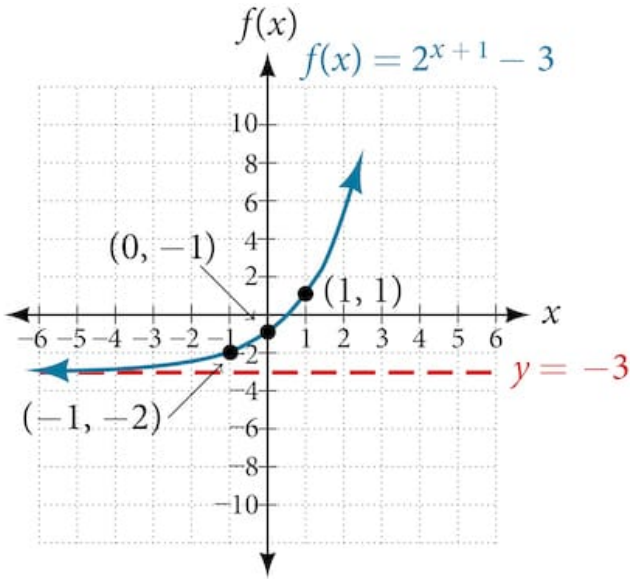


Figure 4.2.6

The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

Example 4.2.3**: Approximating the Solution of an Exponential Equation**

If you use Desmos, then you would go to the graphing calculator and type in $f(x) = 1.2(5)^x + 2.8$. On the next line, type $y = 42$. For a window, use the values -3 to 3 for x and -5 to 55 for y. If you click where the two graphs intersect, Desmos will display the point $(2.1662, 42)$. Therefore, the solution, which is the value of x where these graphs intersect, is approximately 2.166.

Example 4.2.4**: Graphing the Stretch of an Exponential Function**

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the x-axis as x increases.
- Since $a = 4$, the graph of $f(x) = \left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.
- Create a table of points as shown in Table 4.2.4.

Table 4.2.4

x	-3	-2	-1	0	1	2	3
$f(x) = 4\left(\frac{1}{2}\right)^x$	32	16	8	4	2	1	0.5

- Plot the y-intercept, $(0, 4)$, along with two other points. We can use $(-1, 8)$ and $(1, 2)$.

Draw a smooth curve connecting the points, as shown in Figure 4.2.8.

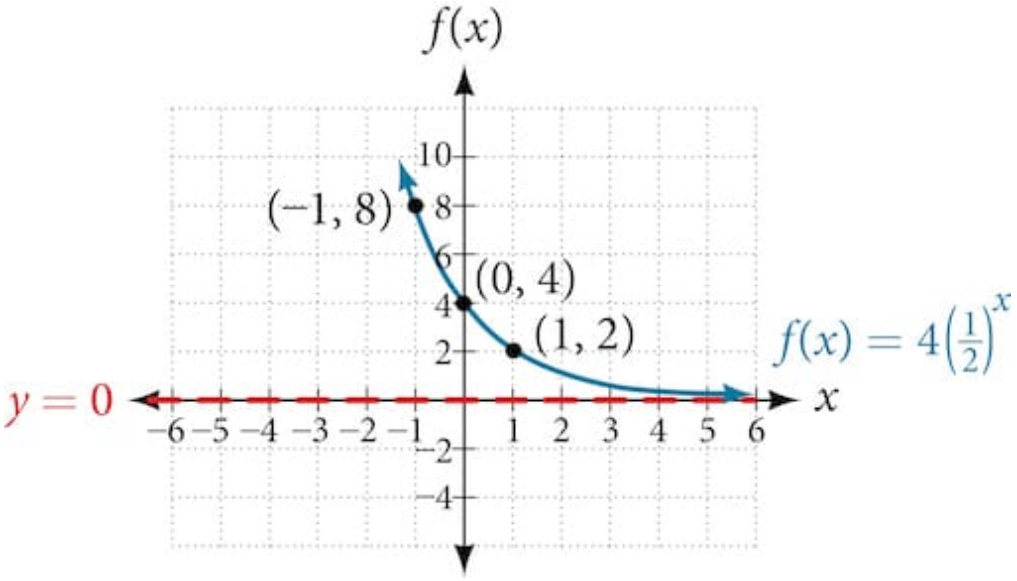


Figure 4.2.8

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Example $4.2.5$**: Writing and Graphing the Reflection of an Exponential Function**

Since we want to reflect the parent function $f(x) = \left(\frac{1}{4}\right)^x$ about the x-axis, we multiply $f(x)$ by -1 to get, $g(x) = -\left(\frac{1}{4}\right)^x$. Next we create a table of points as in Table $4.2.5$.

Table $4.2.5$

x	-3	-2	-1	0	1	2	3
$g(x) = -\left(\frac{1}{4}\right)^x$	-64	-16	-4	-1	-0.25	-0.0625	-0.0156

Plot the y-intercept, $(0, -1)$, along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

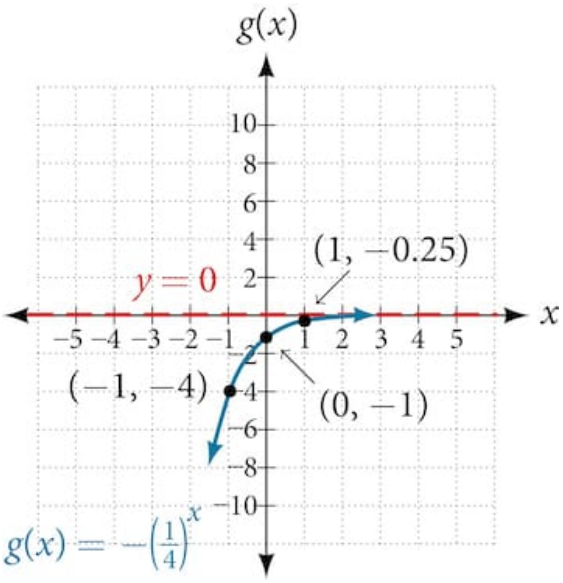


Figure $4.2.10$

The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

Example $4.2.6$**: Writing a Function from a Description**

We want to find an equation of the general form $f(x) = ab^{x+c} + d$. We use the description provided to find a, b, c, and d.

- We are given the parent function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of 2, so $a = 2$.
- The function is reflected about the y-axis. We replace x with $-x$ to get: e^{-x}.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,

$$f(x) = ab^{x+c} + d = 2e^{-x+0} + 4 = 2e^{-x} + 4.$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

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4.2.1: Resources and Key Concepts

Resources

Videos

- Graphs of exponential functions
  - Graphs of Exponential Functions - Graphing Exponential Functions
  - Exponential Functions - Advanced Topics in Exponential Functions
  - Intermediate Algebra - Exponential Functions: Graphs of Exponential Functions

Key Concepts

Definitions

- **Constant Ratio (for exponential functions)**: For an exponential function $f(x) = ab^x$, the base b is the constant ratio by which the output is multiplied when the input increases by 1.

Theorems

- **Properties of the Graphs of Exponential Functions** $f(x) = b^x$ **(where** $b > 0, b \neq 1$**)**:
  - One-to-one function.
  - Horizontal asymptote: $y = 0$.
  - Domain: $(-\infty, \infty)$.
  - Range: $(0, \infty)$.
  - x-intercept: none.
  - y-intercept: $(0, 1)$.
  - Increasing if $b > 1$.
  - Decreasing if $0 < b < 1$.

Common Mistakes

- **Incorrect Order of Transformations**: Applying shifts, reflections, and stretches/compressions in the wrong order, which can lead to an incorrect final graph. For example, applying a vertical shift before a vertical stretch will result in the shift also being stretched.
- **Errors in Horizontal Shifts**: Confusing the direction of horizontal shifts (e.g., $f(x + c)$ shifts left, $f(x - c)$ shifts right for $c > 0$).
- **Misinterpreting Stretches vs. Shifts**: Confusing the effect of multiplying the function/input (stretch/compression) with adding to the function/input (shift).
- **Incorrectly Identifying the Asymptote after Transformations**: Forgetting that vertical shifts change the horizontal asymptote, while horizontal shifts, reflections, and vertical stretches/compressions (by a non-zero factor) do not change the location of the horizontal asymptote $y = 0$ unless there's also a vertical shift.
- **Sign Errors with Reflections**: Incorrectly applying reflections; $-f(x)$ reflects across the x-axis, and $f(-x)$ reflects across the y-axis.
- **Errors in Graphing Key Points after Transformation**: Incorrectly transforming key points like the y-intercept or the point $(1, b)$ from the parent function.

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4.2.2: Homework

Reading Questions

1. What is the "constant ratio" for an exponential function of the form $f(x) = ab^x$, and what does it signify about the function's values as the input increases by 1?
2. Describe the domain, range, and horizontal asymptote of the basic exponential function $f(x) = b^x$ where $b > 0$ and $b \neq 1$.
3. How does a vertical shift by a constant d (i.e., $g(x) = b^x + d$) affect the domain, range, and horizontal asymptote of $f(x) = b^x$?
4. How does a horizontal shift by a constant c (i.e., $g(x) = b^{x+c}$) affect the domain, range, and horizontal asymptote of $f(x) = b^x$?
5. Explain how the graph of $g(x) = a \cdot b^x$ is related to the graph of $f(x) = b^x$ if $|a| > 1$. What if $0 < |a| < 1$?
6. How is the graph of $g(x) = -b^x$ related to the graph of $f(x) = b^x$?
7. How is the graph of $g(x) = b^{-x}$ related to the graph of $f(x) = b^x$?
8. When graphing $f(x) = ab^{x+c} + d$, what is a general order in which transformations can be applied to the parent graph $y = b^x$?
9. If the graph of $f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the y-axis, and then shifted up 4 units, what is the equation of the new function?
10. What is the y-intercept of the function $f(x) = 4(\frac{1}{2})^x$?
11. What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?
12. What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

Homework

Algebraic

13. The graph of $f(x) = 3^x$ is reflected about the y-axis and stretched vertically by a factor of 4. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.
14. The graph of $f(x) = \left(\frac{1}{2}\right)^{-x}$ is reflected about the y-axis and compressed vertically by a factor of $\frac{1}{5}$. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.
15. The graph of $f(x) = 10^x$ is reflected about the x-axis and shifted upward 7 units. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.
16. The graph of $f(x) = (1.68)^x$ is shifted right 3 units, stretched vertically by a factor of 2, reflected about the x-axis, and then shifted downward 3 units. What is the equation of the new function, $g(x)$? State its y-intercept (to the nearest thousandth), domain, and range.
17. The graph of $f(x) = -\frac{1}{2}\left(\frac{1}{4}\right)^{x-2} + 4$ is shifted downward 4 units, and then shifted left 2 units, stretched vertically by a factor of 4, and reflected about the x-axis. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

Graphical

For the following exercises, graph the function and its reflection about the y-axis on the same axes, and give the y-intercept.

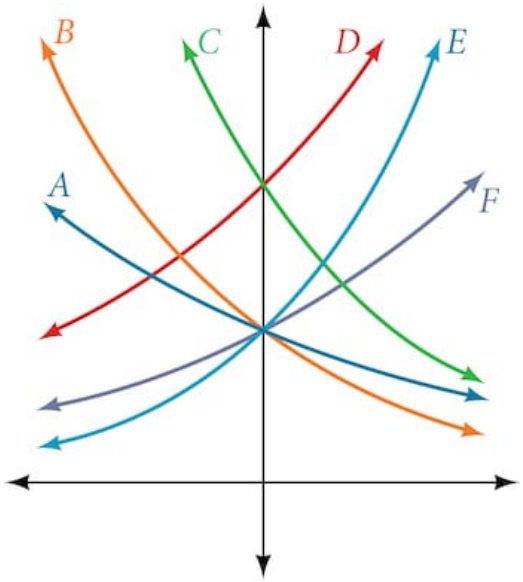
18. $f(x) = 3\left(\frac{1}{2}\right)^x$
19. $g(x) = -2(0.25)^x$
20. $h(x) = 6(1.75)^{-x}$

For the following exercises, graph each set of functions on the same axes.

21. $f(x) = 3\left(\frac{1}{4}\right)^x$, $g(x) = 3(2)^x$, and $h(x) = 3(4)^x$
22. $f(x) = \frac{1}{4}(3)^x$, $g(x) = 2(3)^x$, and $h(x) = 4(3)^x$

For the following exercises, match each function with one of the graphs in Figure $4.2.2.1$.

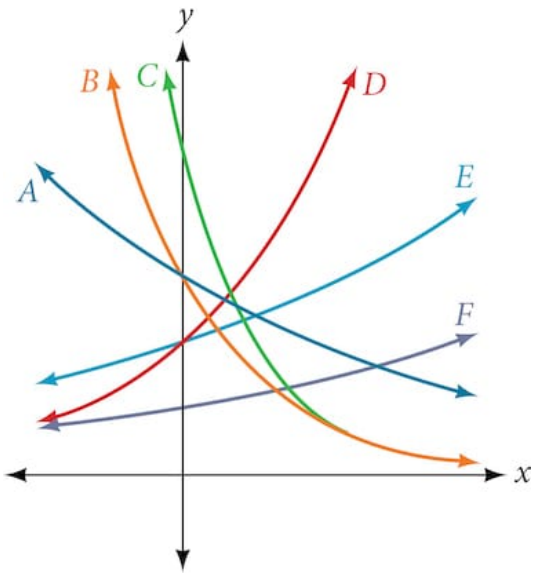
Figure 4.2.2.1



23. $f(x) = 2(0.69)^x$
24. $f(x) = 2(1.28)^x$
25. $f(x) = 2(0.81)^x$
26. $f(x) = 4(1.28)^x$
27. $f(x) = 2(1.59)^x$
28. $f(x) = 4(0.69)^x$

For the following exercises, use the graphs shown in Figure 4.2.2.2**. All have the form** $f(x) = ab^x$**.**

Figure 4.2.2.2



29. Which graph has the largest value for b?
30. Which graph has the smallest value for b?
31. Which graph has the largest value for a?
32. Which graph has the smallest value for a?

For the following exercises, graph the function and its reflection about the x**-axis on the same axes.**

33. $f(x) = \frac{1}{2}(4)^x$
34. $f(x) = 3(0.75)^x - 1$
35. $f(x) = -4(2)^x + 2$

For the following exercises, graph the transformation of $f(x) = 2^x$. Give the horizontal asymptote, the domain, and the range.

36. $f(x) = 2^{-x}$
37. $h(x) = 2^x + 3$
38. $f(x) = 2^{x-2}$

For the following exercises, describe the end behavior of the graphs of the functions.

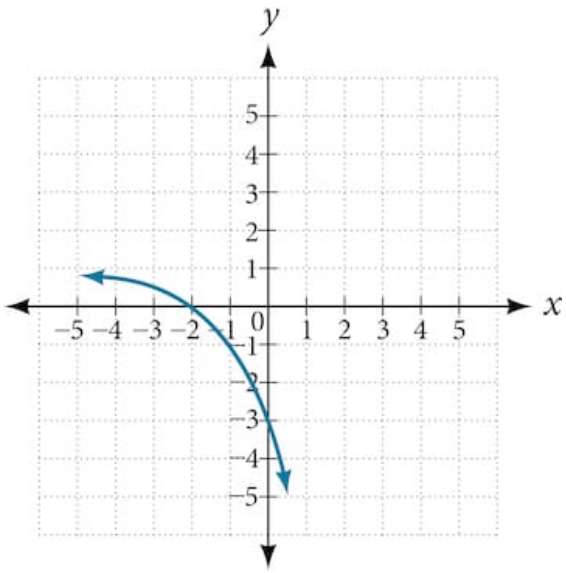
39. $f(x) = -5(4)^x - 1$
40. $f(x) = 3\left(\frac{1}{2}\right)^x - 2$
41. $f(x) = 3(4)^{-x} + 2$

For the following exercises, start with the graph of $f(x) = 4^x$. Then write a function that results from the given transformation.

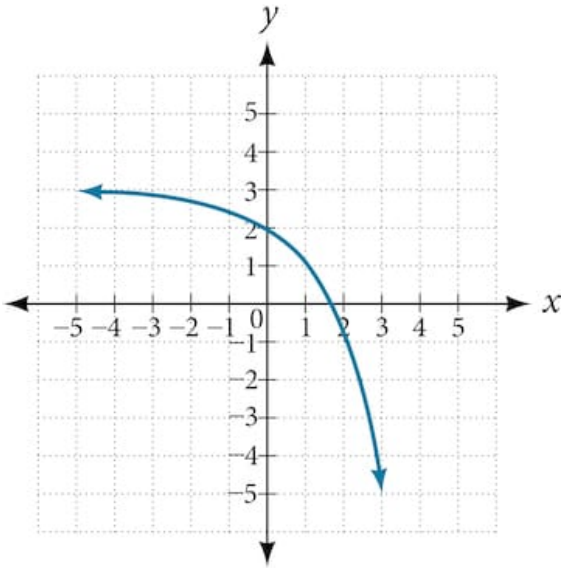
42. Shift $f(x)$ 4 units upward
43. Shift $f(x)$ 3 units downward
44. Shift $f(x)$ 2 units left
45. Shift $f(x)$ 5 units right
46. Reflect $f(x)$ about the x-axis
47. Reflect $f(x)$ about the y-axis

For the following exercises, each graph is a transformation of $y = 2^x$. Write an equation describing the transformation.

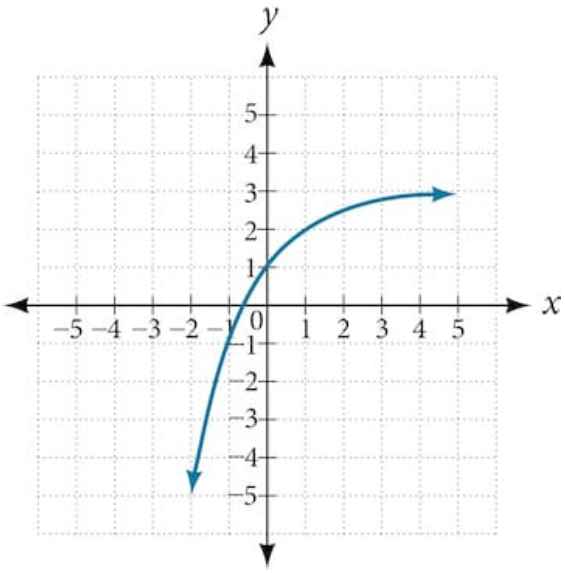
48.



49.



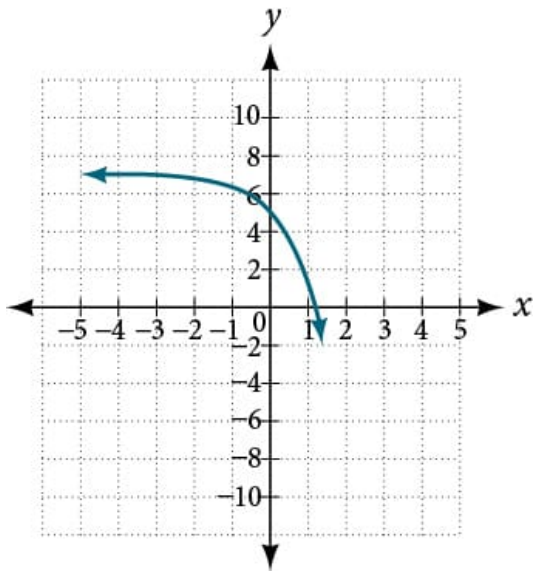
50.

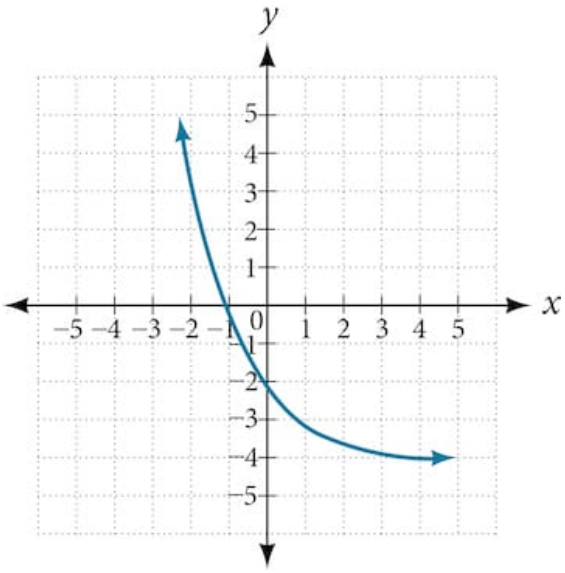


For the following exercises, find an exponential equation for the graph.

51.

52.





Numeric

For the following exercises, evaluate the exponential functions for the indicated value of x.

53. $g(x) = \frac{1}{3}(7)^{x-2}$ for $g(6)$.
54. $f(x) = 4(2)^{x-1} - 2$ for $f(5)$.
55. $h(x) = -\frac{1}{2}\left(\frac{1}{2}\right)^{x} + 6$ for $h(-7)$.

Technology

For the following exercises, use graphing technology to approximate the solutions of the equation. Round to the nearest thousandth.

56. $-50 = -\left(\frac{1}{2}\right)^{-x}$
57. $116 = \frac{1}{4}\left(\frac{1}{8}\right)^{x}$
58. $12 = 2(3)^{x} + 1$
59. $5 = 3\left(\frac{1}{2}\right)^{x-1} - 2$

60. $-30 = -4(2)^{x+2} + 2$

Extensions

61. Explore and discuss the graphs of $F(x) = b^x$ and $G(x) = \left(\frac{1}{b}\right)^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b}\right)^x$ for any real number $b > 0$.
62. Prove the conjecture made in the previous exercise.
63. Explore and discuss the graphs of $f(x) = 4^x$, $g(x) = 4^{x-2}$, and $h(x) = \left(\frac{1}{16}\right)4^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b^n}\right)b^x$ for any real number n and real number $b > 0$.
64. Prove the conjecture made in the previous exercise.

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4.3: Logarithmic Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure $4.3.1$**:** *Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce)*

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings, like those shown in Figure $4.3.1$. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the **Richter Scale**. The Haitian earthquake registered a 7.0 on the Richter Scale whereas the Japanese earthquake registered a 9.0.

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is $10^{8-4} = 10^4 = 10,000$ times as great! In this section, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in Figure $4.3.2$, to better estimate the solution.

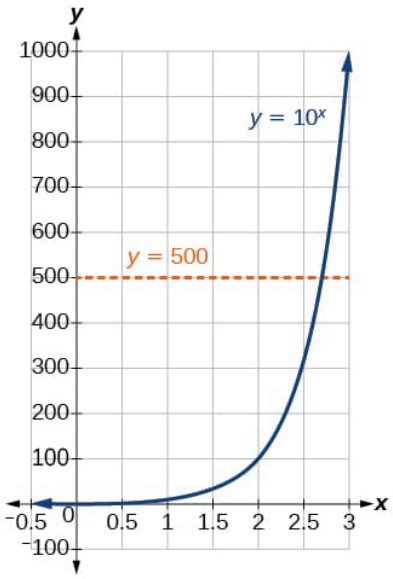


Figure 4.3.2

Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure 4.3.2 passes the Horizontal Line Test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a **logarithmic function** of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

Interactive Element: A Logarithm Is an Exponent

The following Interactive Element might help you see that a common logarithm is simply the exponent you put on 10 to reach a given number.

Interact: Drag the red point along the curve $y = 10^x$; its x-coordinate is the exponent and its height is the output. Drag it up until the height reaches the line $y = 500$.

Observation: At every position the x-coordinate equals $\log$ of the height. When the height is 500, the exponent is about 2.7, so $\log(500) \approx 2.7$ — a logarithm is just an exponent.

Definition: Logarithmic Function

The **base** b **logarithmic function** is defined as follows:

$$y = \log_b(x) \text{ is equivalent to } b^y = x,$$

where, $x > 0$, $b > 0$, and $b \neq 1$. We read $\log_b(x)$ as, "the logarithm with base b of x," or the "log base b of x."

It is critical to recognize that logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2(32) = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \leftrightarrow b^y = x, b > 0, b \neq 1.$$

The symbol, $\leftrightarrow$, means "**if and only if**." Since b is the base of an exponential function, b is always positive.



Figure 4.3.3

Note: Lax Notation

Because the logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. While this "lax" notation has been adopted as acceptable by most mathematicians and scientists, we will stick with parentheses as often as possible to reinforce the concept that the logarithm is a function.

Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:

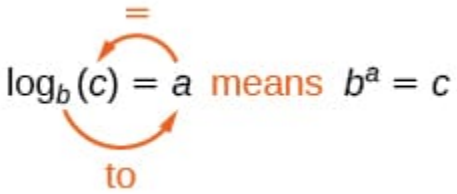


Figure 4.3.4

Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log_b(x)$ and $y = b^x$ are inverse functions. This leads us to quickly develop some fundamental theory about logarithmic functions.

Theorem: Domain and Range of the Logarithmic Function

The domain of the logarithmic function $f(x) = \log_b(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

Proof available at the end of this section.

In fact, it is due to the inverse nature of the logarithmic and exponential functions that $x > 0$, $b > 0$, and $b \neq 1$ are restrictions for the logarithmic function. Since $y = \log_b(x)$ is the same as $b^y = x$, and we know the base of an exponential function must be positive, $b > 0$. Moreover, when defining the exponential function, we took great care to state that the base could not be 1. Thus, $b \neq 1$. Finally, exponential functions only return positive values. Hence, $x > 0$.

Q&A

Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

Example 4.3.1: Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$
b. $\log_3(9) = 2$

Solutions available at the end of this section.

Checkpoint 4.3.1

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1,000,000) = 6$
b. $\log_5(25) = 2$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b, exponent x, and output y. Then we write $x = \log_b(y)$.

Example 4.3.2: Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \frac{1}{10,000}$

Solutions available at the end of this section.

Checkpoint 4.3.2

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \frac{1}{2}$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2(8)$. We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2(8) = 3$.

Now consider simplifying $\log_7(49)$ and $\log_3(27)$ mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know $7^2 = 49$. Therefore, $\log_7(49) = 2$
- We ask, "To what exponent must 3 be raised in order to get 27?" We know $3^3 = 27$. Therefore, $\log_3(27) = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{2/3}\left(\frac{4}{9}\right)$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$?" We know $2^2 = 4$ and $3^2 = 9$, so $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Therefore, $\log_{2/3}\left(\frac{4}{9}\right) = 2$.

Example 4.3.3: Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

Solution available at the end of this section.

Checkpoint 4.3.3

Solve $y = \log_{121}(11)$ without using a calculator.

Example 4.3.4: Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

Solution available at the end of this section.

Checkpoint 4.3.4

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base-10 logarithm a **common logarithm**.

Definition: Common Logarithm

A **common logarithm** is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$.

The common logarithm otherwise follows the standard definition of a logarithmic function: For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x.$$

Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars, decibel levels of sounds, and the pH of acids and bases also use common logarithms.

Example 4.3.5: Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution available at the end of this section.

Checkpoint 4.3.5

Evaluate $y = \log(1,000,000)$.

Example 4.3.6: Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution available at the end of this section.

We can perform a quick "logic check" on our answer to Example 4.3.6. Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100) = 2$ and $\log(1000) = 3$.

Checkpoint 4.3.6

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Example 4.3.7: Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution available at the end of this section.

Checkpoint 4.3.7

The amount of energy released from one earthquake was 8,500 times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in Calculus and some scientific applications; they are called **natural logarithms**.

Definition: Natural Logarithm

A **natural logarithm** is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$.

The natural logarithm otherwise follows the standard definition of a logarithmic function: For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x.$$

Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln(1) = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

Example 4.3.8: Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Solution available at the end of this section.

Checkpoint 4.3.8

Evaluate $\ln(-500)$.

Success in College: Self-Motivation Learning Preferences

Choose the statements that best describe you.

I come to class because I don't want to miss anything.	or	I come to class because it is required.
I would probably do less work in this class if I were graded pass/fail instead of a letter grade.	or	I would probably do the same amount of work in this class if I was graded pass/fail instead of with a letter grade.
I need frequent homework assignments to keep me working.	or	I do homework frequently to know what is going on in class.

1. Do you consider the choices above to be good or bad for your education?
2. Some professors provide detailed schedules with assignments and test dates at the beginning of the term. Others announce the homework assignments at each class meeting and test days during the term. Which do you prefer?

Success in Mathematics: Math Anxiety (Part 3)

For some math students, anxiety changes their ability to show what they know. They have physical and emotional symptoms. You are responsible for recognizing if math anxiety is interfering with your success. Do you need to make some changes to reduce your math anxiety? Do you need help to do so?

Researchers have identified various events that seem to result in math anxiety. Perhaps you had negative experiences learning math. For you, learning math meant feeling dumb or embarrassed. Perhaps a teacher or parent who struggled with math has passed on their fears to you. Or you believe that your gender or ethnic heritage means you cannot do the math.

Some students learn to cope on their own with their math anxiety. They use relaxation techniques. They practice problems over and over. At test time, their anxiety does not block them from doing the work. Others find help from counselors at their college.

Math anxiety is common among college students. It is your responsibility to think about it and, if needed, get help so that you can cope with your fears.

Math anxiety affects students in physical ways such as increasing blood pressure and heart rate, breathing faster, excess sweating, dry mouth, restlessness, or nausea. Emotionally, it can cause fear, anger, nervousness, worry, and feelings of hopelessness. Students may have trouble concentrating, sitting still, reading, or keeping their legs from bouncing. They may feel desperate - like they need to run away or cry.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Using a scale of 1 to 10, with 10 describing overwhelming anxiety that makes it extremely difficult to do math, rate your math anxiety. How does it affect you?
2. Most students have some anxiety about their classes. They worry about giving speeches, taking tests, writing a good paper, or completing all their work. Worrying can be motivating. It can help us get going and to get things done. Or, worrying can be a bad thing. It paralyzes us. In general, how does worrying affect you?

Answers

Section Preview

The structure of this section mimics that of the introduction to exponential functions. It is important to note that the properties of logarithms are held off for a later section.

Section Preview

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

Theorem: Domain and Range of the Logarithmic Function

Let $y = \log_b(x)$. Since the logarithm base b is the inverse of the exponential function base b, the domain of the logarithm is the range of the exponential function. This is precisely $(0, \infty)$. Moreover, the range of the logarithm is the domain of the exponential function, which is $(-\infty, \infty)$.

Example 4.3.1: Converting from Logarithmic Form to Exponential Form

First, identify the values of b, y, and x. Then, write the equation in the form $b^y = x$.

a. Here, $b = 6$, $y = \frac{1}{2}$, and $x = \sqrt{6}$. Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to
$$6^{1/2} = \sqrt{6}.$$
b. Here, $b = 3$, $y = 2$, and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9$.

Example 4.3.2: Converting from Exponential Form to Logarithmic Form

First, identify the values of b, y, and x. Then, write the equation in the form $x = \log_b(y)$.

a. Here, $b = 2$, $x = 3$, and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.
b. Here, $b = 5$, $x = 2$, and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.
c. Here, $b = 10$, $x = -4$, and $y = \frac{1}{10,000}$. Therefore, the equation $10^{-4} = \frac{1}{10,000}$ is equivalent to $\log_{10}\left(\frac{1}{10,000}\right) = -4$.

Example 4.3.3: Solving Logarithms Mentally

First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know
$$4^3 = 64.$$
Therefore,
$$\log_4(64) = 3.$$

Example 4.3.4: Evaluating the Logarithm of a Reciprocal

First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

We know $3^3 = 27$, but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write
$$3^{-3} = \frac{1}{3^3} = \frac{1}{27}.$$
Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

Example 4.3.5: Finding the Value of a Common Logarithm Mentally

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know
$$10^3 = 1000.$$
Therefore, $\log(1000) = 3$.

Example 4.3.6: Finding the Value of a Common Logarithm Using a Calculator

- Press **[LOG]**.
- Enter 321, followed by **)**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

Example 4.3.7: Rewriting and Solving a Real-World Exponential Model

We begin by rewriting the exponential equation in logarithmic form.

$$\begin{aligned} & 10^x = 500 \\ \Longrightarrow\ & \log(500) = x && \text{(definition of the common logarithm)} \\ \Longrightarrow\ & x \approx 2.699 && \text{(using a calculator)} \end{aligned}$$

The difference in Richter Scale magnitudes was about 2.699.

Example 4.3.8: Evaluating a Natural Logarithm Using a Calculator

- Press **[LN]**.
- Enter 500, followed by **)**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$.

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4.3.1: Resources and Key Concepts

Resources

Videos

- Logarithmic functions
  - Intermediate Algebra - Logarithmic Functions: Concepts and Basic Use
  - Intermediate Algebra - Logarithms: Logarithmic Functions
  - Logarithmic Functions - A Focus on Logarithmic Functions
- Converting between logarithmic to exponential forms
  - Logarithmic Functions - Converting Between Logarithmic and Exponential Forms
- Evaluating logarithms
  - Intermediate Algebra - Logarithmic Functions: Evaluating Logarithms

Key Concepts

Definitions

- **Logarithmic Function (base b)**: The function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. It is defined as: $y = \log_b(x)$ is equivalent to $b^y = x$, where $x > 0$, $b > 0$, and $b \neq 1$. The base b logarithm of a number x is the exponent by which we must raise b to get x.
- **Common Logarithm**: A logarithm with base 10. It is written as $\log(x)$, which is shorthand for $\log_{10}(x)$.
- **Natural Logarithm**: A logarithm with base e (Euler's number). It is written as $\ln(x)$, which is shorthand for $\log_e(x)$.

Theorems

- **Domain and Range of the Logarithmic Function**: The domain of the logarithmic function $f(x) = \log_b(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

Common Mistakes

- **Incorrect Conversion Between Logarithmic and Exponential Form**: Misplacing the base, argument, or exponent when converting between $y = \log_b(x)$ and $b^y = x$.
- **Taking the Logarithm of a Non-Positive Number**: Attempting to evaluate the logarithm of zero or a negative number (e.g., $\ln(-500)$ in Checkpoint 4.3.8). The argument of a logarithm must be positive.
- **Errors in Evaluating Basic Logarithms**: Difficulty in mentally evaluating logarithms by not recognizing them as exponents (e.g., for $\log_2(8)$, thinking "2 to what power is 8?").
- **Calculator Errors**: Incorrectly inputting expressions, especially with parentheses, when using a calculator.
- **Assuming $\log_b(0) = 1$ or similar misconceptions**: The logarithm of 0 is undefined. $\log_b(1) = 0$ because $b^0 = 1$.

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4.3.2: Homework

Reading Questions

1. What is a base b logarithm? Explain its relationship to an exponential function.
2. How do you convert the logarithmic equation $\log_b(x) = y$ into exponential form?
3. How do you convert the exponential equation $b^y = x$ into logarithmic form?
4. What is the domain and range of the logarithmic function $f(x) = \log_b(x)$?
5. Can you take the logarithm of a negative number or zero? Explain why or why not based on the properties of exponential functions.
6. What is a "common logarithm"? What is its base, and how is it typically notated?
7. What is a "natural logarithm"? What is its base, and how is it typically notated?
8. How would you mentally evaluate $\log_5(25)$?
9. If $10^x = 500$, how can you rewrite this equation using a common logarithm to solve for x?
10. What is the value of $\ln(1)$? What is the value of $\log(1)$? Why?
11. What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b(x) = y$ for $b > 0$, $b \neq 1$.
12. How is the logarithmic function $f(x) = \log_b(x)$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?
13. How can the logarithmic equation $\log_b(x) = y$ be solved for x using the properties of exponents?
14. Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?
15. Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Homework

Algebraic

For the following exercises, rewrite each equation in exponential form.

16. $\log_4(q) = m$
17. $\log_a(b) = c$
18. $\log_{16}(y) = x$
19. $\log_x(64) = y$
20. $\log_y(x) = -11$
21. $\log_{15}(a) = b$
22. $\log_y(137) = x$
23. $\log_{13}(142) = a$
24. $\log(v) = t$
25. $\ln(w) = n$

For the following exercises, rewrite each equation in logarithmic form.

26. $4^x = y$
27. $c^d = k$
28. $m^{-7} = n$
29. $19^x = y$
30. $x^{-10/13} = y$
31. $n^4 = 103$
32. $\left(\frac{7}{5}\right)^m = n$
33. $y^x = \frac{39}{100}$
34. $10^a = b$

35. $e^k = h$

For the following exercises, solve for x by converting the logarithmic equation to exponential form.

36. $\log_3(x) = 2$
37. $\log_2(x) = -3$
38. $\log_5(x) = 2$
39. $\log_3(x) = 3$
40. $\log_2(x) = 6$
41. $\log_9(x) = \frac{1}{2}$
42. $\log_{18}(x) = 2$
43. $\log_6(x) = -3$
44. $\log(x) = 3$
45. $\ln(x) = 2$

For the following exercises, use the definition of common and natural logarithms to simplify.

46. $\log(100^8)$
47. $10^{\log(32)}$
48. $2\log(0.0001)$
49. $e^{\ln(1.06)}$
50. $\ln(e^{-5.03})$
51. $e^{\ln(10.125)} + 4$

Numeric

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

52. $\log_3\left(\frac{1}{27}\right)$
53. $\log_6(\sqrt{6})$
54. $\log_2\left(\frac{1}{8}\right) + 4$
55. $6\log_8(4)$

For the following exercises, evaluate the common logarithmic expression without using a calculator.

56. $\log(10,000)$
57. $\log(0.001)$
58. $\log(1) + 7$
59. $2\log(100^{-3})$

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

60. $\ln(e^{1/3})$
61. $\ln(1)$
62. $\ln(e^{-0.225}) - 3$
63. $25\ln(e^{2/5})$

Technology

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

64. $\log(0.04)$
65. $\ln(15)$
66. $\ln\left(\frac{4}{5}\right)$
67. $\log(\sqrt{2})$
68. $\ln(\sqrt{2})$

Extensions

69. Is $x = 0$ in the domain of the function $f(x) = \log(x)$? If so, what is the value of the function when $x = 0$? Verify the result.
70. Is $f(x) = 0$ in the range of the function $f(x) = \log(x)$? If so, for what value of x? Verify the result.
71. Is there a number x such that $\ln(x) = 2$? If so, what is that number? Verify the result.
72. Is the following true: $\frac{\log_3(27)}{\log_4(\frac{1}{64})} = -1$? Verify the result.
73. Is the following true: $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$? Verify the result.

Real-World Applications

74. The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where f is the "f-stop" setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?
75. Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?
76. The intensity levels I of two earthquakes measured on a seismograph can be compared by the formula

$$\log\left(\frac{I_1}{I_2}\right) = M_1 - M_2,$$

where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0. How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.

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4.4: Graphs of Logarithmic Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

When we investigated graphs of exponential functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest \$2500 in an account that offers an annual interest rate of 5%, compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? Figure $4.4.1$ shows this point on the logarithmic graph.

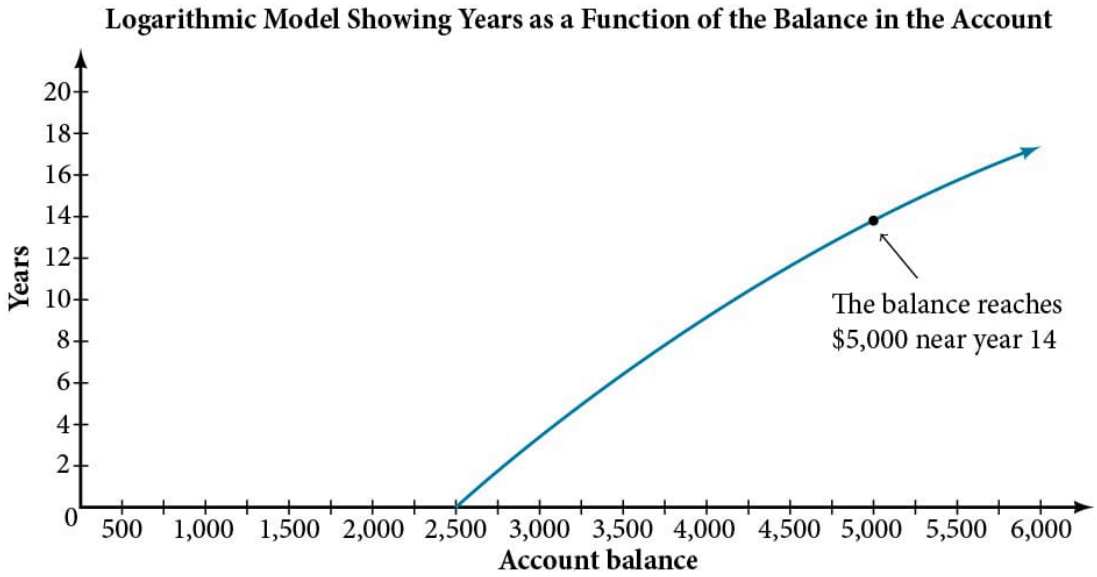


Figure $4.4.1$

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where the domain of y is $(-\infty, \infty)$ and the range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Theorem: Domain and Range of the Logarithmic Function

The domain of the logarithmic function $f(x) = \log_b(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

Proof available at the end of this section.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

We have seen that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be strictly greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$, is greater than zero. To find the domain, we set up an inequality and solve for x:

$$\begin{aligned} & 2x - 3 > 0 \\ \implies \quad & 2x > 3 && \text{(adding 3 to both sides)} \\ \implies \quad & x > 1.5 && \text{(dividing both sides by 2)} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

Example 4.4.1: Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x) = \log_2(x + 3)$?

Solution available at the end of this section.

Checkpoint 4.4.1

What is the domain of $f(x) = \log_5(x - 2) + 1$?

Example 4.4.2: Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x) = \log(5 - 2x)$?

Solution available at the end of this section.

Checkpoint 4.4.2

What is the domain of $f(x) = \log(x - 5) + 2$?

Of course, there is no need to restrict the argument of a logarithm to linear expressions. We can rely on the graphing skills we built in previous chapters to find the domain of the following example.

Example 4.4.3: Identifying the Domain of a Logarithm Composed with a Nonlinear Function

State the domain of the function.

$$g(x) = -2\log_5\left(\frac{2x^3 - 3x^2 - 3x + 2}{3x^3 + 375}\right)$$

Solution available at the end of this section.

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the parent function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in Table 4.4.1.

Table 4.4.1

x	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from Table 4.4.1, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See Table 4.4.2.

Table 4.4.2

$f(x) = 2^x$	$\left(-3, \frac{1}{8}\right)$	$\left(-2, \frac{1}{4}\right)$	$\left(-1, \frac{1}{2}\right)$	$(0, 1)$	$(1, 2)$	$(2, 4)$	$(3, 8)$
$g(x) = \log_2(x)$	$\left(\frac{1}{8}, -3\right)$	$\left(\frac{1}{4}, -2\right)$	$\left(\frac{1}{2}, -1\right)$	$(1, 0)$	$(2, 1)$	$(4, 2)$	$(8, 3)$

As we'd expect, the x- and y-coordinates are reversed for the inverse functions. Figure 4.4.3 shows the graph of f and g.

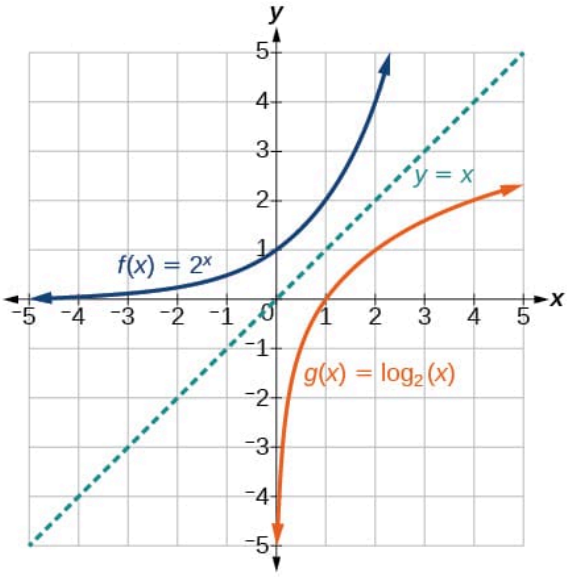


Figure 4.4.3**:** *Notice that the graphs of* $f(x) = 2^x$ *and* $g(x) = \log_2(x)$ *are reflections about the line* $y = x$.

Observe the following from the graph:

- $f(x) = 2^x$ has a y-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an x- intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

Interactive Element: A Logarithm Is the Mirror of Its Exponential

The following Interactive Element might help you visualize why $y = \log_2(x)$ is the reflection of $y = 2^x$ across the line $y = x$.

Interact: Drag the point along the curve $y = 2^x$.

Observation: Its mirror image across $y = x$ always lands on $y = \log_2(x)$, and the coordinates swap: $(t,\ 2^t)$ becomes $(2^t,\ t)$.

This leads to the following theorem.

Theorem: Properties of the Graphs of Logarithmic Functions

A logarithmic function with the form $f(x) = \log_b(x)$, where $b > 0$ and $b \neq 1$, has these characteristics:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- x-intercept: $(1, 0)$
- Secondary key point: $(b, 1)$
- y-intercept: none
- increasing if $b > 1$
- decreasing if \(0

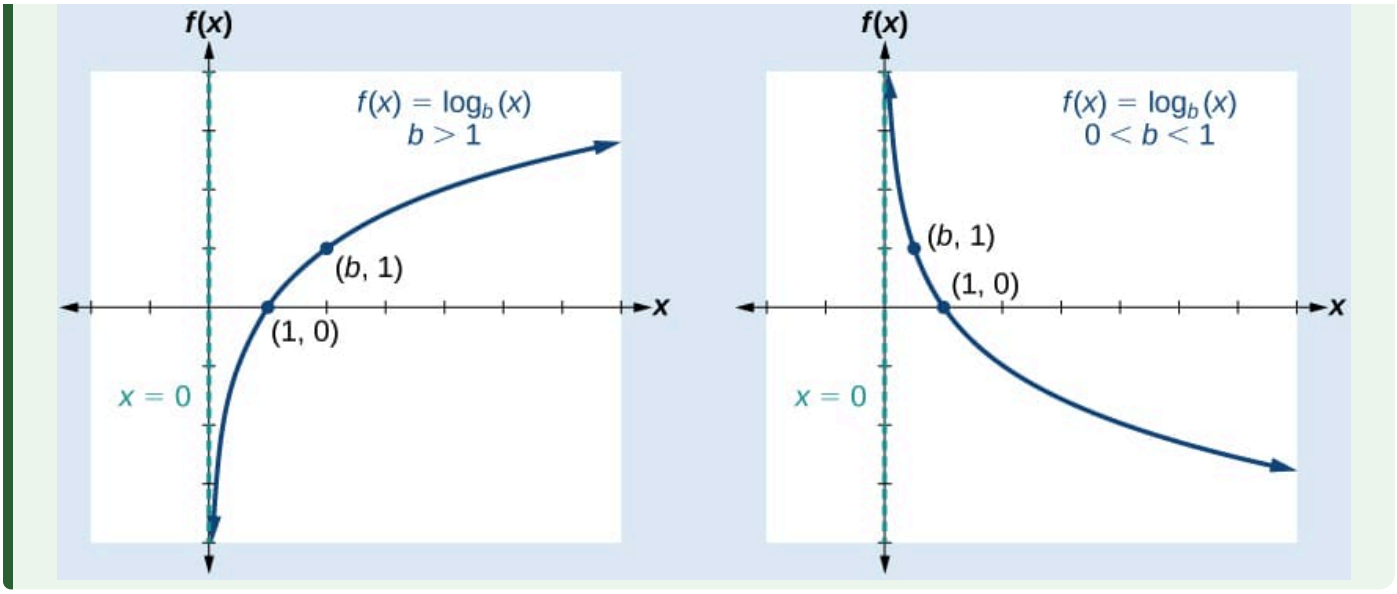


Figure 4.4.4 shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (*Note:* recall that the function $\ln(x)$ has base $e \approx 2.718$).

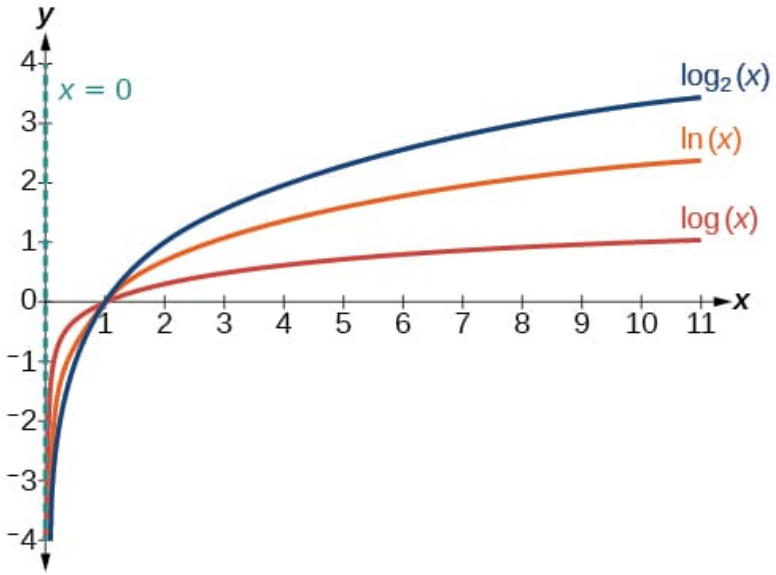


Figure 4.4.4**:** *The graphs of three logarithmic functions with different bases, all greater than 1.*

Example 4.4.4: Graphing a Logarithmic Function with the Form $f(x) = \log_b(x)$.

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 4.4.4

Graph $f(x) = \log_{1/5}(x)$. State the domain, range, and asymptote.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift of $f(x) = \log_b(x)$

When a constant c is added to the input of the parent function $f(x) = \log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 4.4.6.

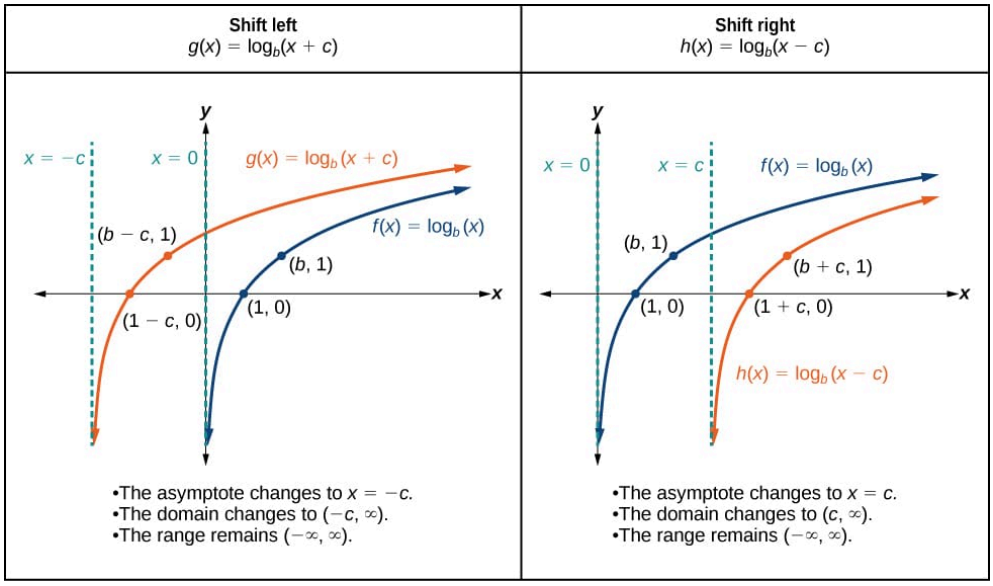


Figure 4.4.6

Example 4.4.5: Graphing a Horizontal Shift of the Parent Function $y = \log_b(x)$

Sketch the horizontal shift $f(x) = \log_3(x - 2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 4.4.5

Sketch a graph of $f(x) = \log_3(x + 4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Graphing a Vertical Shift of $y = \log_b(x)$

When a constant d is added to the parent function $f(x) = \log_b(x)$, the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$. See Figure 4.4.8.

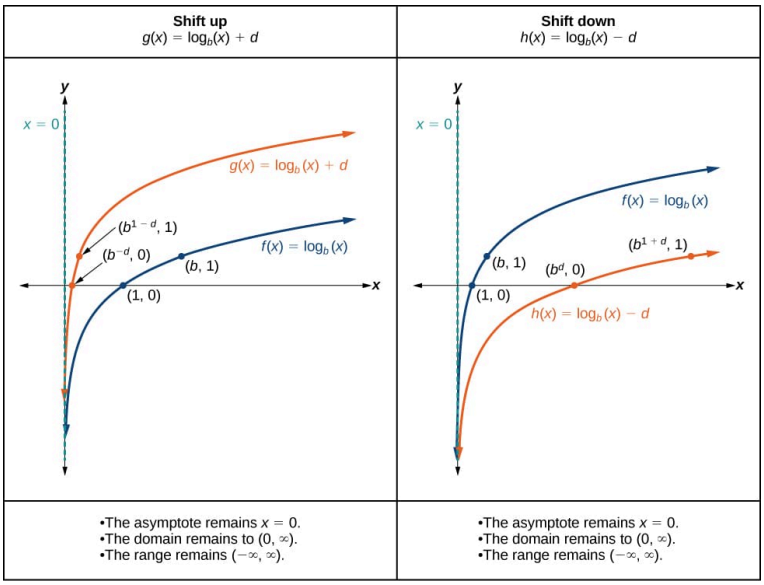


Figure 4.4.8

Example 4.4.6: Graphing a Vertical Shift of the Parent Function $y = \log_b(x)$

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 4.4.6

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Graphing Stretches and Compressions of $y = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a > 1$ and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$. See Figure 4.4.10.

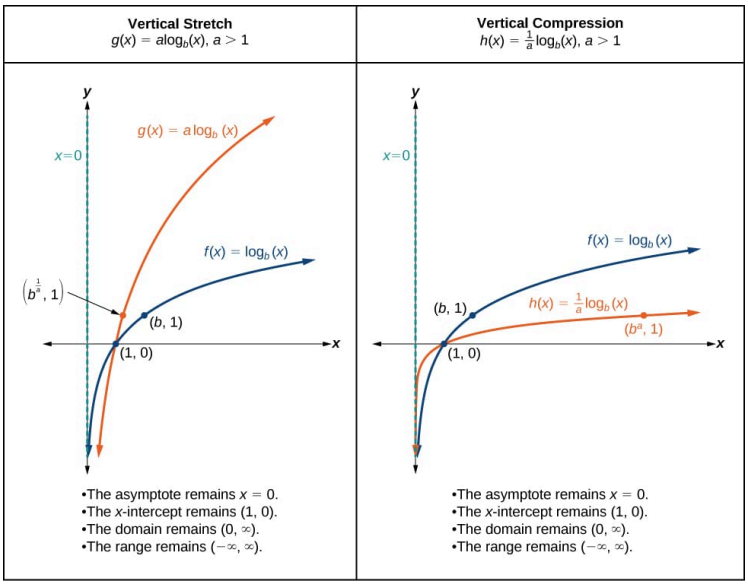


Figure 4.4.10

Example 4.4.7: Graphing a Stretch or Compression of the Parent Function $y = \log_b(x)$

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 4.4.7

Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Example 4.4.8: Combining a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x + 2)$. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 4.4.8

Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.

Graphing Reflections of $f(x) = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by -1, the result is a reflection about the x-axis. When the *input* is multiplied by -1, the result is a reflection about the y-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the x-axis, $g(x) = -\log_b(x)$ and the reflection about the y-axis, $h(x) = \log_b(-x)$.

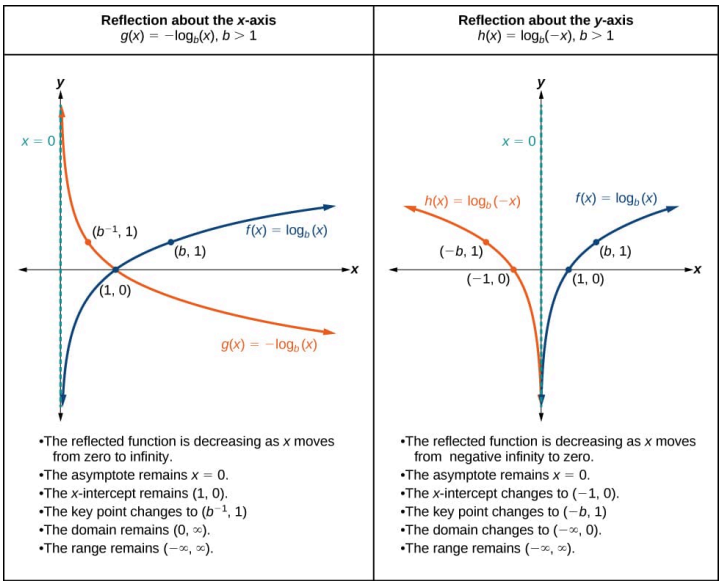


Figure 4.4.13

Example 4.4.9: Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 4.4.9

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

Example 4.4.10: Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x) + 1 = -2\ln(x - 1)$ using graphing technology. Round to the nearest thousandth.

Solution available at the end of this section.

Checkpoint 4.4.10

Solve $5\log(x + 2) = 4 - \log(x)$ using graphing technology. Round to the nearest thousandth.

Summarizing Transformations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in Table 4.4.3 to arrive at the general equation for translating exponential functions.

Table 4.4.3

Transformations of the Parent Function $y = \log_b(x)$

Transformation	Form
Shift • Horizontally c units to the left (if $c > 0$) • Vertically d units up (if $d > 0$)	$y = \log_b(x + c) + d$
Stretch and Compress • Stretch if $\lvert a\rvert > 1$ • Compression if $\lvert a\rvert < 1$	$y = a\log_b(x)$
Reflect about the x-axis	$y = -\log_b(x)$
Reflect about the y-axis	$y = \log_b(-x)$
General equation for all transformations	$y = a\log_b(x + c) + d$

Translations of Logarithmic Functions

All translations of the parent logarithmic function, $y = \log_b(x)$, have the form

$$f(x) = a\log_b(x + c) + d$$

where the parent function, $y = \log_b(x)$, $b > 1$, is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $\lvert a\rvert$ if $\lvert a\rvert > 0$.
- compressed vertically by a factor of $\lvert a\rvert$ if $0 < \lvert a\rvert < 1$.
- reflected about the x-axis when $a < 0$.

For $f(x) = \log_b(-x)$, the graph of the parent function is reflected about the y-axis.

Interactive Element: Transforming the Logarithmic Parent Function

The following Interactive Element might help you visualize how the constants in $a\log_2(x + c) + d$ reshape the graph, and which one moves the vertical asymptote.

Interact: Drag the orange anchor point to shift the graph, and drag the green handle up or down to change a.

Observation: Moving the anchor left or right slides the vertical asymptote along with the curve, but moving it up or down—or changing a—leaves the asymptote where it is. Only the horizontal shift relocates the asymptote.

Example 4.4.11: Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2\log_3(x + 4) + 5$?

Solution available at the end of this section.

In Example 4.4.11, the coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to $x = -4$.

Checkpoint 4.4.11

What is the vertical asymptote of $f(x) = 3 + \ln(x - 1)$?

Example 4.4.12: Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in Figure 4.4.15.

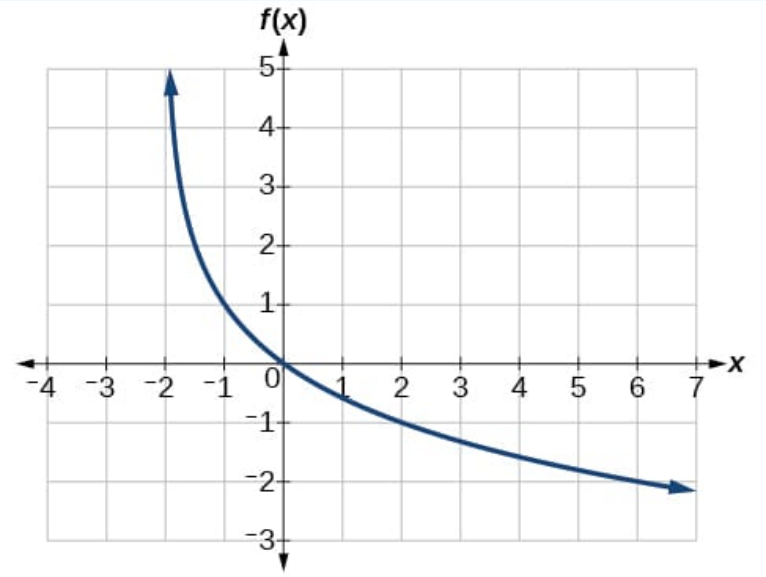


Figure 4.4.15

Solution available at the end of this section.

Checkpoint 4.4.12

Give the equation of the natural logarithm graphed in Figure 4.4.16.

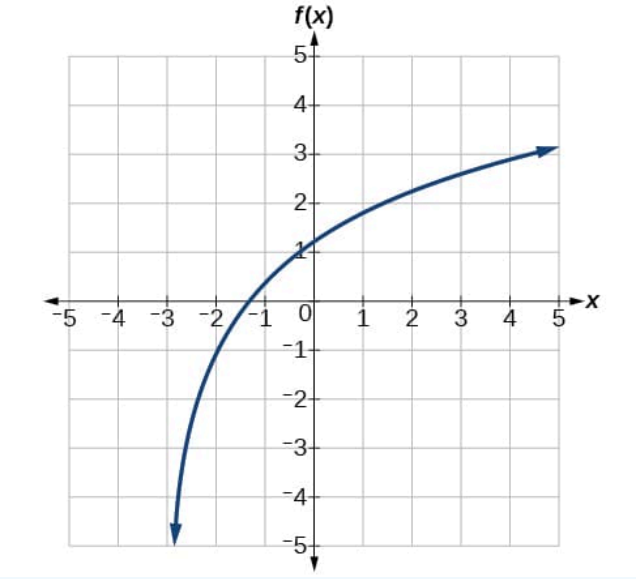


Figure 4.4.16

Q&A

Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?

Yes, if we know the function is a general logarithmic function. For example, look at the graph in Figure $4.4.16$. The graph approaches $x = -3$ (or thereabouts) more and more closely, so $x = -3$ is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right, $\{x \mid x > -3\}$. The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is that as $x \to -3^+$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.

More Complex Transformations

Just as we encountered early in this course, logarithmic functions can also have horizontal stretches or compressions. Since these more robust transformations occur in Calculus, we spend a few moments reminding ourselves how to deal with graphing such types of functions.

Example $4.4.13$: Graphing a More Complex Logarithmic Function

Sketch a graph of the function

$$h(x) = 2\ln\left(-\frac{1}{4}x + 1\right) - 1.$$

Solution available at the end of this section.

Revisiting Graphs of Absolute Value Functions

Finally, while you might not know it yet, absolute values and natural logarithms will appear together quite often in Calculus. It's best that we spend an example reminding ourselves of the effect of the absolute value on a function.

Example $4.4.14$

Graph

$$k(x) = \left|2\ln\left(-\frac{1}{4}x + 1\right) - 1\right|.$$

Solution available at the end of this section.

Applications of Mathematics: Poker

So, where does mathematics apply in poker? Poker players can use mathematics in many different situations; however, it is most commonly used when a player is on a draw, such as a flush or straight draw. If an opponent makes a bet and you are on a draw, you are faced with the decision of whether or not you should call to try and complete the draw by the next card or fold and let your opponent take the pot. In this situation, a player with good knowledge of poker mathematics will always know whether or not to call, while a player who does not know mathematics will be unsure.

Players unfamiliar with poker odds will make an educated guess on whether or not they should call. If the bet is large, they may feel it is too expensive to try and catch the right card, but if the bet is small, they will be more inclined to call. On the other hand, a player that can use mathematics correctly will be able to work out the pot odds they are getting on the hand and act accordingly. Pot odds consider the amount your opponent has bet in relation to the pot size and the likelihood of completing your draw to inform you about whether or not you should call or fold.

1. Can you think of three other games that rely explicitly or implicitly on mathematics? Explain.
2. Oddly enough, most of us are superstitious about numbers. Some people even play their "favorite" numbers when they play the lottery. Suppose you pick your favorite numbers, and I pick the numbers 1, 2, 3, 4, 5, and 6. Who has better odds of winning? Why are you still likely to win more if the odds are the same?

Hidden Resources: Kanopy

Did you know that as a Los Rios Community College student, you have free access to Kanopy?

What is Kanopy?

According to Wikipedia, Kanopy "is an on-demand streaming video platform for public libraries and universities that offers films and documentaries."

Why would you be interested?

There are a lot of great movies, documentaries, and educational videos on this service - and it's free for you because you are a student of the Los Rios Community College District.

Is it as good as Hulu or Netflix?

No, but it does have a lot of the same titles.

Why aren't any of the Great Courses available to us?

Not being able to access the Great Courses is a disappointment. That was one of the options Los Rios chose not to purchase; however, drown your sorrows in What We Do in the Shadows instead!

How do I sign up?

All you need to do is go to https://losrios.kanopy.com and start typing Los Rios when asked for your institution. That should be it.

Answers

Section Preview

This section covers graphing logarithmic functions—**DO NOT SKIP THIS MATERIAL!** Knowing how to graph a logarithmic function is critical in Calculus.

Section Preview

- **Factoring Techniques**
  - Factoring Sums and Differences of Cubes: Example 3 factors $x^3 + 125$ as a sum of cubes to locate the denominator's real zero and vertical asymptote at $x = -5$.
- **Absolute Value Functions**
  - Graphing the Composition of a Function and the Absolute Value Function: Example 14 graphs $|h(x)|$ by reflecting the negative-y portion of $h(x)$ about the x-axis while leaving the rest unchanged.

Section Preview

- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

Theorem: Domain and Range of the Logarithmic Function

Let $y = \log_b(x)$. Since the logarithm base b is the inverse of the exponential function base b, the domain of the logarithm is the range of the exponential function. This is precisely $(0, \infty)$. Moreover, the range of the logarithm is the domain of the exponential function, which is $(-\infty, \infty)$.

Example $4.4.1$: Identifying the Domain of a Logarithmic Shift

The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

$$x + 3 > 0 \implies x > -3.$$

The domain of $f(x) = \log_2(x + 3)$ is $(-3, \infty)$.

Example $4.4.2$: Identifying the Domain of a Logarithmic Shift and Reflection

The logarithmic function is defined only when the input is positive, so this function is defined when $5-2x>0$. Solving this inequality,

$$\begin{aligned} & 5-2x > 0 \\ \Longrightarrow \quad & -2x > -5 && \text{(subtracting 5 from both sides)} \\ \Longrightarrow \quad & x < \frac{5}{2} && \text{(dividing both sides by } -2 \text{ and switching the inequality)} \end{aligned}$$

The domain of $f(x)=\log(5-2x)$ is $\left(-\infty, \frac{5}{2}\right)$.

Example 4.4.3**: Identifying the Domain of a Logarithm Composed with a Nonlinear Function**

Since the argument of a logarithm must be positive, answering this question requires us to solve

$$\frac{2x^3-3x^2-3x+2}{3x^3+375}>0.$$

From our previous work, we have two ways to solve this rational inequality—sign charts or a quick graph. As you probably will guess, I am in favor of a quick sketch. In either case, we will need to factor the numerator and denominator of the rational expression.

Factoring the denominator is the easier task, so we start there.

$$\begin{aligned} 3x^3+375 &= 3\left(x^3+125\right) && \text{(factoring out the GCF)} \\ &= 3(x+5)\left(x^2-5x+25\right) && \text{(factoring the sum of cubes)} \end{aligned}$$

Hence, the rational inequality is

$$\frac{2x^3-3x^2-3x+2}{3(x+5)\left(x^2-5x+25\right)}>0,$$

where the trinomial in the denominator is prime (i.e., it does not have real zeros).

Turning our attention to the numerator, we are forced to use the Rational Zeros Theorem. The candidate roots are ± 1, ± 2, and $\pm\frac{1}{2}$. Starting with 1, we get

$$\begin{array}{r|cccc} \underline{1} & 2 & -3 & -3 & 2 \\ & & 2 & -1 & -4 \\ \hline & 2 & -1 & -4 & -2 \end{array}$$

Since the remainder is nonzero, $(x-1)$ is not a factor. Let's move to -1.

$$\begin{array}{r|cccc} \underline{-1} & 2 & -3 & -3 & 2 \\ & & -2 & 5 & -2 \\ \hline & 2 & -5 & 2 & 0 \end{array}$$

The remainder is zero. Therefore, $(x-(-1))=(x+1)$ is a factor. Moreover, the remaining factor is $2x^2-5x+2=(2x-1)(x-2)$. Thus, the rational inequality we need to solve can be rewritten as

$$\frac{(x+1)(2x-1)(x-2)}{3(x+5)(x^2-5x+25)}>0.$$

The intercepts of the function on the left side are $(-1,0)$, $\left(\frac{1}{2},0\right)$, $(2,0)$, and $\left(0,\frac{2}{375}\right)$. The vertical asymptote is $x=-5$ and the horizontal asymptote is $y=\frac{2}{3}$. Noting that the multiplicities of the x-intercepts are all 1 and the multiplicity of the vertical asymptote is odd, we quickly sketch the following graph.

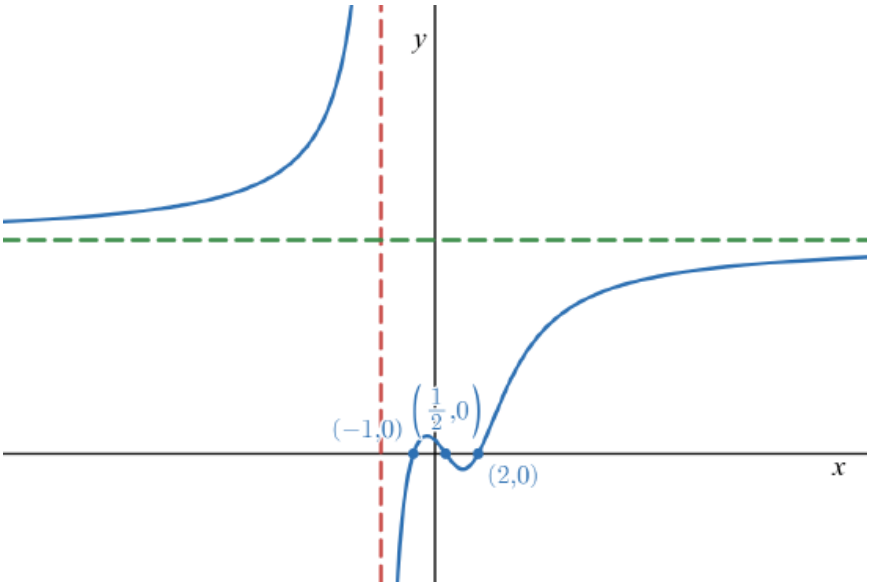


Figure 4.4.2

We can see this graph is above the x-axis on the interval $(-\infty,-5)\cup\left(-1,\frac{1}{2}\right)\cup(2,\infty)$. Hence, this is the domain of the logarithmic function.

Example 4.4.4**: Graphing a Logarithmic Function with the Form** $f(x)=\log_b(x)$**.**

Before graphing, identify the behavior and key points for the graph.

- Since $b=5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x=0$, and the right tail will increase slowly without bound.

- The x-intercept is $(-1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see Figure $4.4.5$).

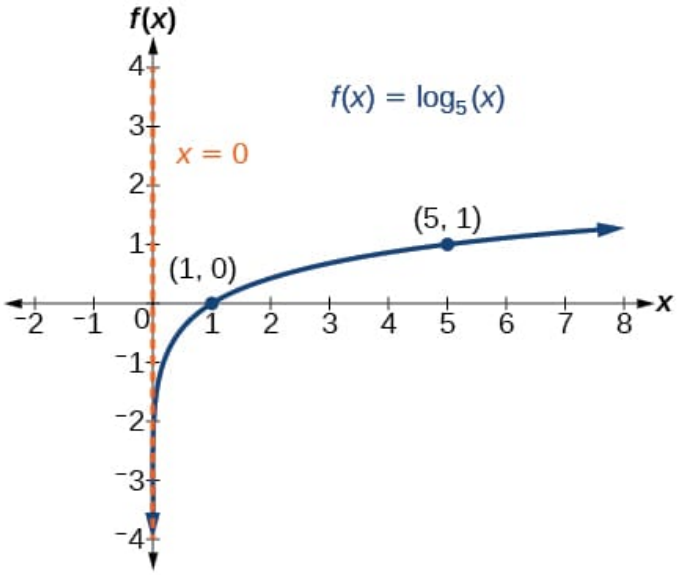


Figure $4.4.5$

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Example $4.4.5$**: Graphing a Horizontal Shift of the Parent Function** $y = \log_b(x)$

Since the function is $f(x) = \log_3(x - 2)$, we notice $x + (-2) = x - 2$.

Thus $c = -2$, so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the parent function, $(\frac{1}{3}, -1)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by adding 2 to the x coordinates.

Label the points $(\frac{7}{3}, -1)$, $(3, 0)$, and $(5, 1)$.

The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.

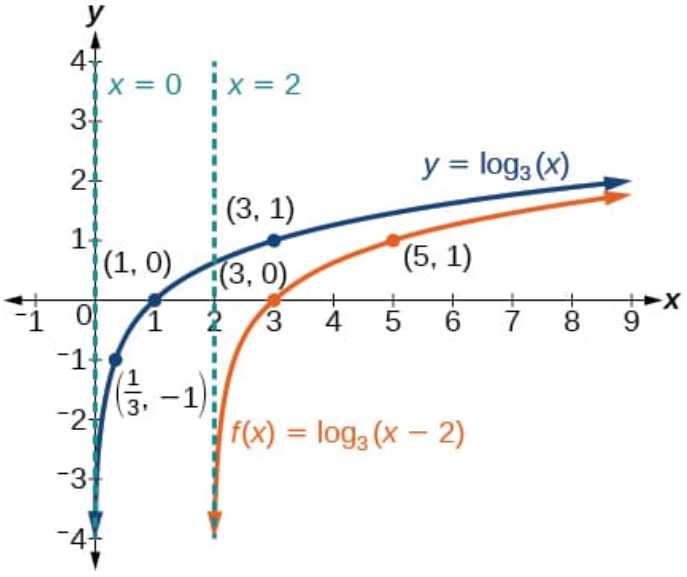


Figure $4.4.7$

Example $4.4.6$**: Graphing a Vertical Shift of the Parent Function** $y = \log_b(x)$

Since the function is $f(x) = \log_3(x) - 2$, we will notice $d = -2$. Thus $d < 0$. This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $(\frac{1}{3}, -1)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by subtracting 2 from the y coordinates.

Label the points $(\frac{1}{3}, -3)$, $(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

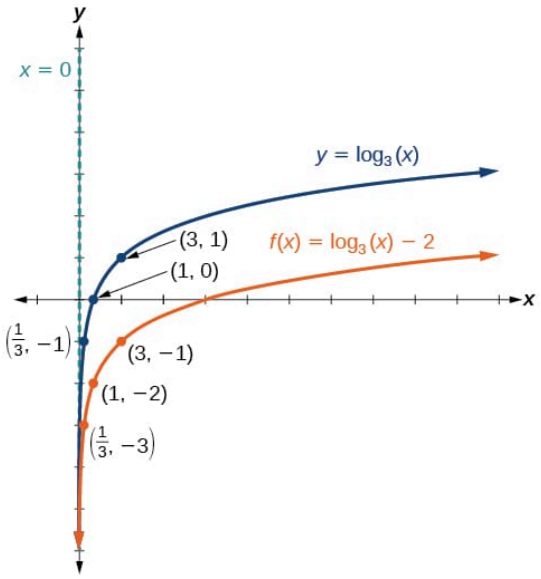


Figure 4.4.9

Example 4.4.7**: Graphing a Stretch or Compression of the Parent Function** $y = \log_b(x)$

Since the function is $f(x) = 2\log_4(x)$, we will notice $a = 2$. This means we will stretch the function $f(x) = \log_4(x)$ by a factor of 2.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $(\frac{1}{4}, -1)$, $(-1, 0)$, and $(4, 1)$.

The new coordinates are found by multiplying the y coordinates by 2.

Label the points $(\frac{1}{4}, -2)$, $(-1, 0)$, and $(4, 2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$. See Figure 4.4.11.

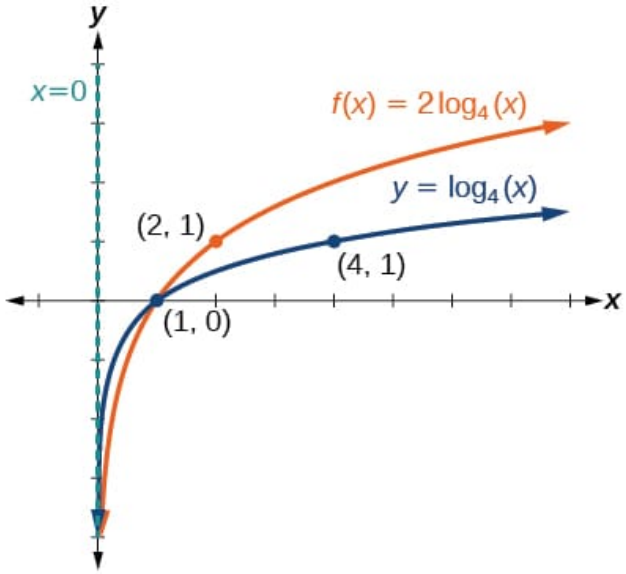


Figure 4.4.11

Example 4.4.8**: Combining a Shift and a Stretch**

Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in Figure 4.4.12. The vertical asymptote will be shifted to $x = -2$. The x-intercept will be $(-1, 0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5)$. We chose $x = 8$ as the x-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.

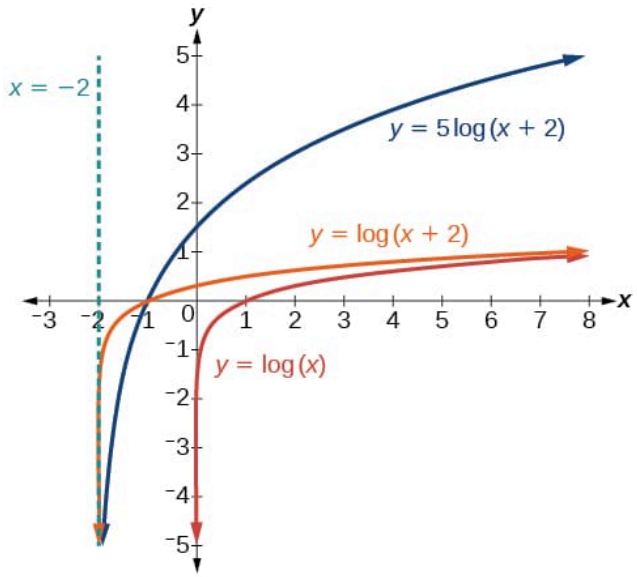


Figure 4.4.12

The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -2$.

Example 4.4.9**: Graphing a Reflection of a Logarithmic Function**

Before graphing $f(x) = \log(-x)$, identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by -1, f is a reflection of the parent graph about the y-axis. Thus, $f(x) = \log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The x-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

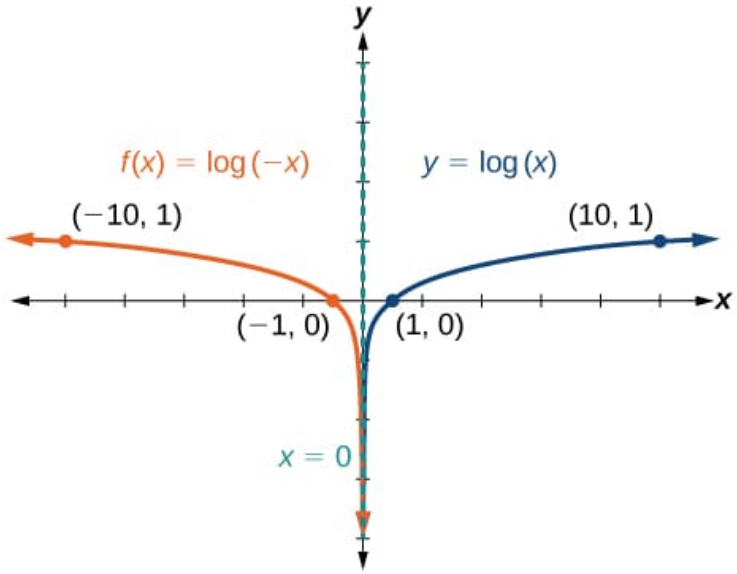


Figure 4.4.14

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Example 4.4.10**: Approximating the Solution of a Logarithmic Equation**

In the graphing calculator portion of Desmos, we enter $y = 4\ln(x) + 1$ on one line and $y = -2\ln(x - 1)$ on the next line. For a window, use the values 0 to 5 for x and -10 to 10 for y. The graphs should intersect somewhere a little to right of $x = 1$. Clicking on their intersection point, we see the x-coordinate of the point of intersection is displayed as 1.3385. So, to the nearest thousandth, $x \approx 1.339$.

Example 4.4.11**: Finding the Vertical Asymptote of a Logarithm Graph**

The vertical asymptote is at $x = -4$.

Example 4.4.12**: Finding the Equation from a Graph**

This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

$$f(x) = -a\log(x + 2) + k$$

It appears the graph passes through the points $(-1, 1)$ and $(2, -1)$. Substituting $(-1, 1)$,

$$\begin{aligned} 1 &= -a\log(-1+2)+k && \text{(substituting)} \\ \Rightarrow\ 1 &= -a\log(1)+k && \text{(simplifying)} \\ \Rightarrow\ 1 &= -a(0)+k && (\log(1)=0) \\ \Rightarrow\ 1 &= k && (\log(1)=0) \end{aligned}$$

Next, substituting in $(2,-1)$,

$$\begin{aligned} -1 &= -a\log(2+2)+1 && \text{(substituting)} \\ \Rightarrow\ -1 &= -a\log(4)+1 && \text{(simplifying)} \\ \Rightarrow\ -2 &= -a\log(4) && \text{(subtracting 1 from both sides)} \\ \Rightarrow\ -\frac{2}{\log(4)} &= a && \text{(dividing both sides by } -\log(4)) \end{aligned}$$

This gives us the equation $f(x)=-\frac{2}{\log(4)}\log(x+2)+1$.

We can verify this answer by comparing the function values in Table 4.4.4 with the points on the graph in Figure 4.4.15.

Table 4.4.4

x	-1	0	1	2	3
$f(x)$	1	0	-0.58496	-1	-1.3219
x	4	5	6	7	8
$f(x)$	-1.5850	-1.8074	-2	-2.1699	-2.3219

Example 4.4.13**: Graphing a More Complex Logarithmic Function**

As we have done previously, we begin by factoring the argument of the function.

$$h(x)=2\ln\left(-\frac{1}{4}(x-4)\right)-1.$$

Starting with the parent graph (which contains the anchor points $(1,0)$ and $(e,1)$), we read our function from left to right:

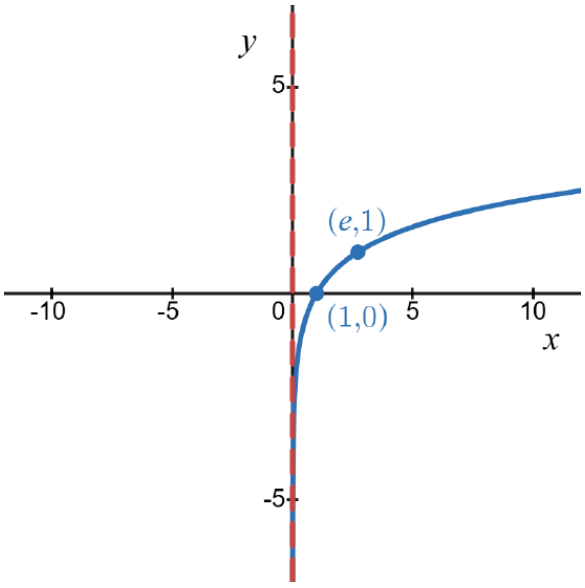


Figure 4.4.17

- The parent graph, $y=\ln(x)$, has been **stretched vertically** by a factor of 2. Thus, we multiply the y-coordinates of our anchor points by 2 to get $(1,0)$ and $(e,2)$ to get the graph of $y=2\ln(x)$.

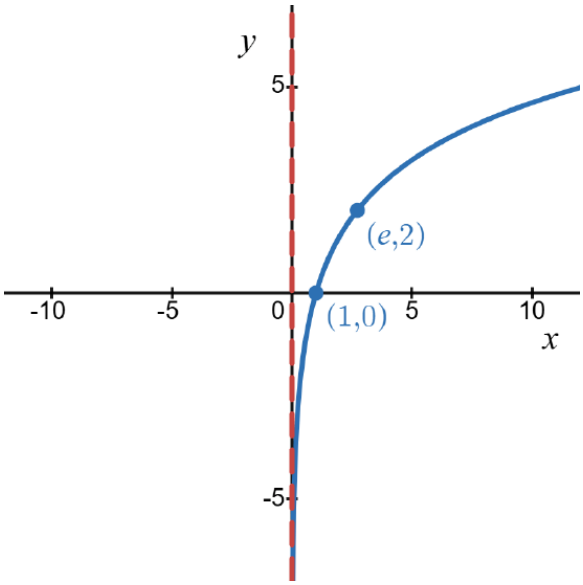


Figure 4.4.18

- The graph of $y = 2\ln(x)$ has been **reflected horizontally** about the y-axis and **stretched horizontally** by a factor of 4. Therefore, we multiply the x-coordinates of our anchor points by -4 to get $(-4, 0)$ and $(-4e, 2)$ to get the graph of $y = 2\ln\left(-\frac{1}{4}x\right)$.

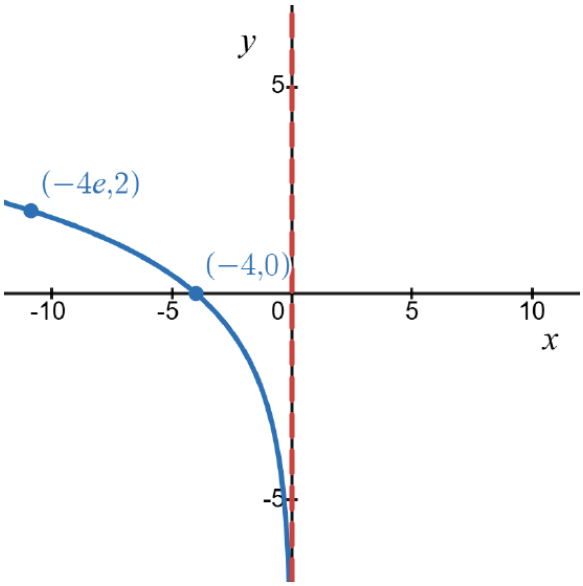


Figure 4.4.19

- The graph of $y = 2\ln\left(-\frac{1}{4}x\right)$ gets **shifted horizontally** 4 units to the right. In this horizontal shift, the vertical asymptote also shifts. Hence, we add 4 to all the x-coordinates of our anchor points and to the vertical asymptote. Thus, the graph of $y = 2\ln\left(-\frac{1}{4}(x-4)\right)$ contains the points $(0, 0)$ and $(-4e + 4, 2)$, and the vertical asymptote is at $x = 4$.

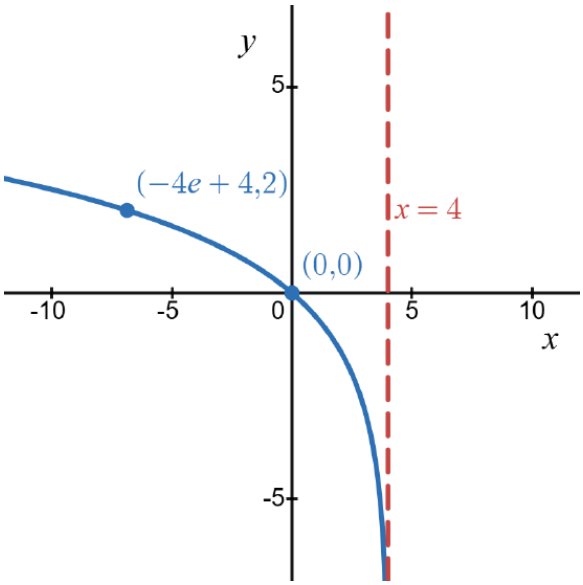


Figure 4.4.20

- The graph of $y = 2\ln\left(-\frac{1}{4}(x-4)\right)$ gets **shifted vertically** 1 unit down to arrive at our final graph of the function $h(x) = 2\ln\left(-\frac{1}{4}(x-4)-1\right)$. The anchor points get moved to $(0,-1)$ and $(-4e+4,1)$. The final graph is shown below.

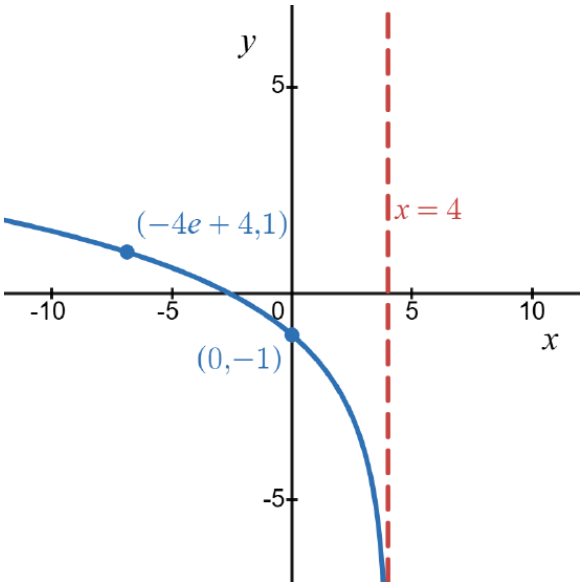


Figure 4.4.21

Example 4.4.14

If this function looks familiar, it is because it is the absolute value of the function $h(x)$ from Example 4.4.13. The easiest way to graph the absolute value of a function is to recall that the absolute value keeps non-negative values the same, but forces negative values to become positive. Graphically, this means that any coordinate on the graph of $h(x)$ with negative y-values will be reflected about the x-axis to their positive values. All other coordinates remain unchanged.

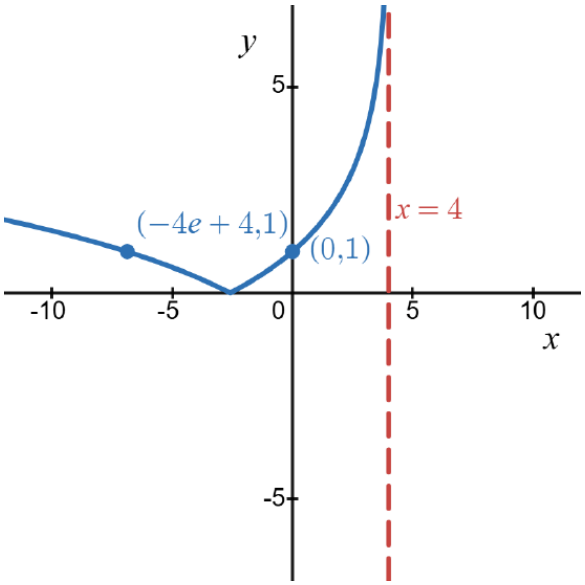


Figure 4.4.22

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4.4.1: Resources and Key Concepts

Resources

Videos

- Graphs of logarithmic functions
  - Graphs of Logarithmic Functions - Graphing Logarithmic Functions
  - Logarithmic Functions - Graphs of Logarithmic Functions
  - Intermediate Algebra - Logarithms: Graphing Logarithmic Functions
  - Intermediate Algebra - Logarithms: Graphing Logarithmic Functions Using Transformations
- Domain of logarithmic function
  - Graphs of Logarithmic Functions - Finding the Domain of a Logarithmic Function
- Logarithms and You
  - Logarithmic Functions - Logarithms and You

Key Concepts

Theorems

- **Domain and Range of the Logarithmic Function** $f(x) = \log_b(x)$: The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
- **Properties of the Graphs of Logarithmic Functions** $f(x) = \log_b(x)$ **(where** $b > 0, b \neq 1$**)**:
  - One-to-one function.
  - Vertical asymptote: $x = 0$.
  - Domain: $(0, \infty)$.
  - Range: $(-\infty, \infty)$.
  - x-intercept: $(1, 0)$.
  - Secondary key point: $(b, 1)$.
  - y-intercept: none.
  - Increasing if $b > 1$.
  - Decreasing if $0 < b < 1$.

Common Mistakes

- **Incorrectly Identifying the Domain of Transformed Logarithmic Functions**: The argument of the logarithm must always be positive. For $f(x) = \log_b(expression)$, one must solve $expression > 0$. Errors occur if this inequality is set up or solved incorrectly, especially with reflections or multiple transformations.
- **Errors in Horizontal Shifts and Vertical Asymptote Location**: Confusing the direction of horizontal shifts (e.g., $\log_b(x + c)$ shifts left, $\log_b(x - c)$ shifts right for $c > 0$) and consequently misplacing the vertical asymptote. The vertical asymptote shifts with the horizontal shift.
- **Order of Transformations**: Applying shifts, reflections, and stretches/compressions in the wrong order, particularly when combining horizontal transformations (e.g., in $h(x) = 2\ln(-\frac{1}{4}x + 1) - 1$, factoring the argument is key: $-\frac{1}{4}(x - 4)$).
- **Misinterpreting Stretches vs. Shifts**: Confusing the effect of multiplying the function/input (stretch/compression) with adding to the function/input (shift).
- **Incorrectly Identifying the Range of Logarithmic Functions**: The range of any untransformed or vertically stretched/compressed/shifted logarithmic function $y = a\log_b(expression) + d$ is always $(-\infty, \infty)$. Horizontal transformations do not affect the range.
- **Sign Errors with Reflections**: Incorrectly applying reflections; $-f(x)$ reflects across the x-axis, and $f(-x)$ reflects across the y-axis. This also impacts the domain for $f(-x)$.
- **Graphing Absolute Value of Logarithmic Function**: Incorrectly reflecting parts of the graph. Only the portions of the original logarithmic graph that are *below* the x-axis should be reflected *above* the x-axis. Portions already above remain unchanged.

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4.4.2: Homework

Reading Questions

1. What is the domain of the basic logarithmic function $f(x) = \log_b(x)$? What is its range?
2. What is the equation of the vertical asymptote for the basic logarithmic function $f(x) = \log_b(x)$?
3. How does a horizontal shift by a constant c (i.e., $g(x) = \log_b(x + c)$) affect the domain and vertical asymptote of $f(x) = \log_b(x)$?
4. How does a vertical shift by a constant d (i.e., $g(x) = \log_b(x) + d$) affect the domain, range, and vertical asymptote of $f(x) = \log_b(x)$?
5. Explain how the graph of $g(x) = a \log_b(x)$ is related to the graph of $f(x) = \log_b(x)$ if $|a| > 1$. What if $0 < |a| < 1$?
6. How is the graph of $g(x) = -\log_b(x)$ related to the graph of $f(x) = \log_b(x)$?
7. How is the graph of $g(x) = \log_b(-x)$ related to the graph of $f(x) = \log_b(x)$? How does this affect the domain?
8. When graphing $f(x) = a \log_b(B(x - h)) + k$, what is a general order in which transformations can be applied to the parent graph $y = \log_b(x)$? Why is factoring the argument $B(x - h)$ important?
9. What are the two "anchor" points typically used when graphing transformations of $y = \log_b(x)$?
10. How do you graph the function $k(x) = |h(x)|$ if you know the graph of $h(x) = 2\ln(-\frac{1}{4}x + 1) - 1$?
11. The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?
12. What type(s) of transformation(s), if any, affect the range of a logarithmic function?
13. What type(s) of transformation(s), if any, affect the domain of a logarithmic function?
14. Consider the general logarithmic function $f(x) = \log_b(x)$. Why can't x be zero?
15. Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

Homework

Algebraic

For the following exercises, state the domain and range of the function.

16. $f(x) = \log_3(x + 4)$
17. $h(x) = \ln\left(\frac{1}{2} - x\right)$
18. $g(x) = \log_5(2x + 9) - 2$
19. $h(x) = \ln(4x + 17) - 5$
20. $f(x) = \log_2(12 - 3x) - 3$

For the following exercises, state the domain and the vertical asymptote of the function.

21. $f(x) = \log_b(x - 5)$
22. $g(x) = \ln(3 - x)$
23. $f(x) = \log(3x + 1)$
24. $f(x) = 3\log(-x) + 2$
25. $g(x) = -\ln(3x + 9) - 7$

For the following exercises, state the domain, vertical asymptote, and end behavior of the function.

26. $f(x) = \ln(2 - x)$
27. $f(x) = \log\left(x - \frac{3}{7}\right)$
28. $h(x) = -\log(3x - 4) + 3$
29. $g(x) = \ln(2x + 6) - 5$
30. $f(x) = \log_3(15 - 5x) + 6$

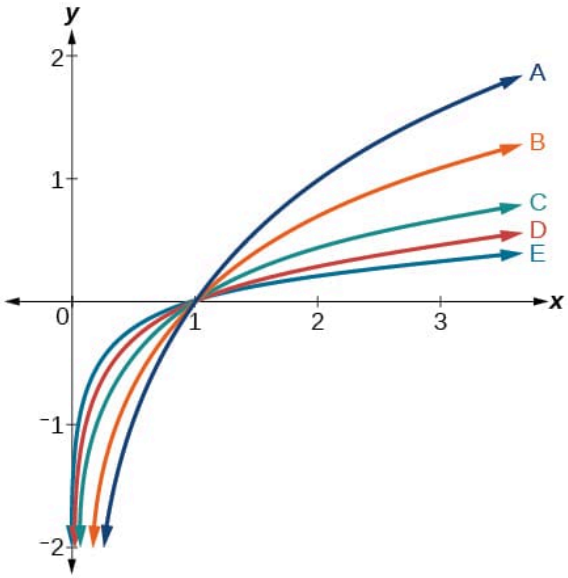
For the following exercises, state the domain, range, and x- and y-intercepts, if they exist. If they do not exist, write DNE.

31. $h(x) = \log_4(x-1)+1$
32. $f(x) = \log(5x+10)+3$
33. $g(x) = \ln(-x)-2$
34. $f(x) = \log_2(x+2)-5$
35. $h(x) = 3\ln(x)-9$

Graphical

For the following exercises, match each function in Figure 4.4.2.1 with the letter corresponding to its graph.

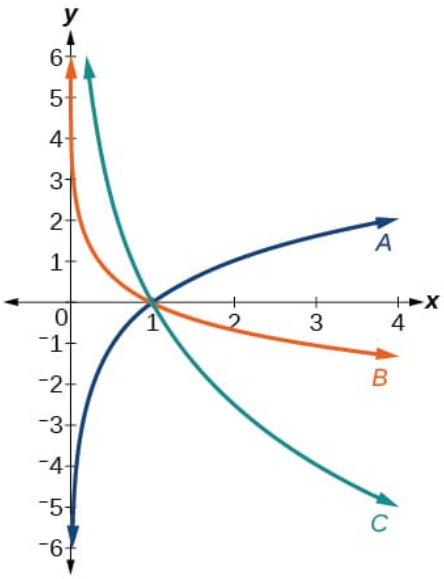
Figure 4.4.2.1



36. $d(x) = \log(x)$
37. $f(x) = \ln(x)$
38. $g(x) = \log_2(x)$
39. $h(x) = \log_5(x)$
40. $j(x) = \log_{25}(x)$

For the following exercises, match each function in Figure 4.4.2.2 with the letter corresponding to its graph.

Figure 4.4.2.2



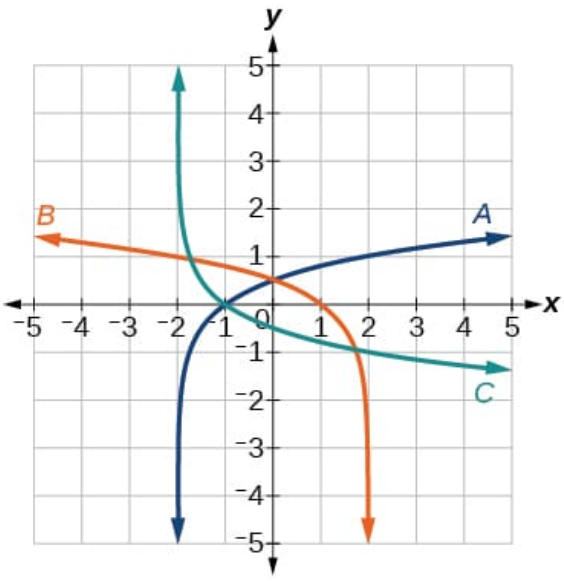
41. $f(x) = \log_{1/3}(x)$
42. $g(x) = \log_2(x)$
43. $h(x) = \log_{3/4}(x)$

For the following exercises, sketch the graphs of each pair of functions on the same axis.

44. $f(x) = \log(x)$ and $g(x) = 10^x$
45. $f(x) = \log(x)$ and $g(x) = \log_{1/2}(x)$
46. $f(x) = \log_4(x)$ and $g(x) = \ln(x)$
47. $f(x) = e^x$ and $g(x) = \ln(x)$

For the following exercises, match each function in Figure 4.4.2.3 **with the letter corresponding to its graph.**

Figure 4.4.2.3



48. $f(x) = \log_4(-x + 2)$
49. $g(x) = -\log_4(x + 2)$

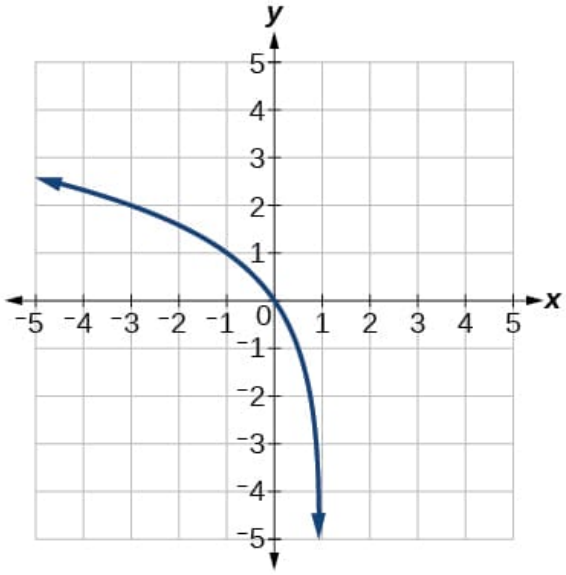
50. $h(x) = \log_4(x+2)$

For the following exercises, sketch the graph of the indicated function.

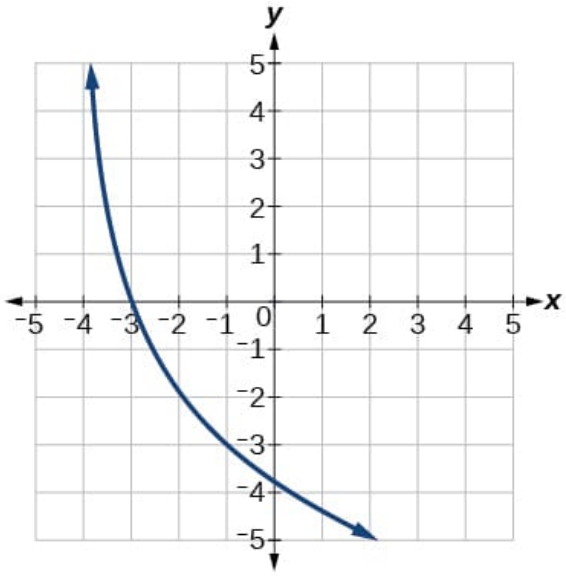
51. $f(x) = \log_2(x+2)$
52. $f(x) = 2\log(x)$
53. $f(x) = \ln(-x)$
54. $g(x) = \log(4x+16)+4$
55. $g(x) = \log(6-3x)+1$
56. $h(x) = -\frac{1}{2}\ln(x+1)-3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

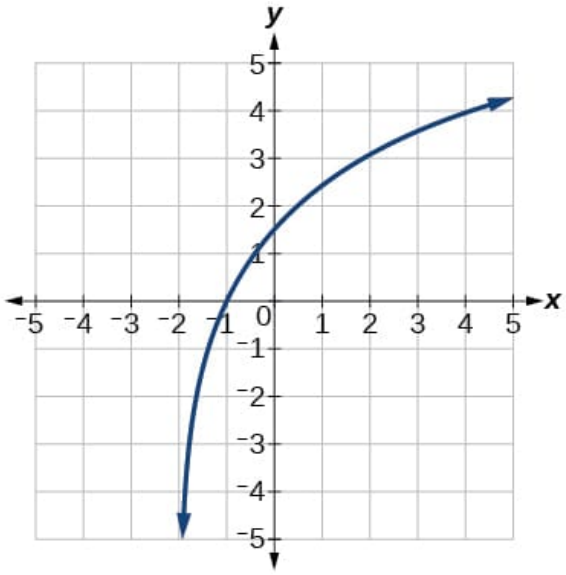
57. Use $y = \log_2(x)$ as the parent function.



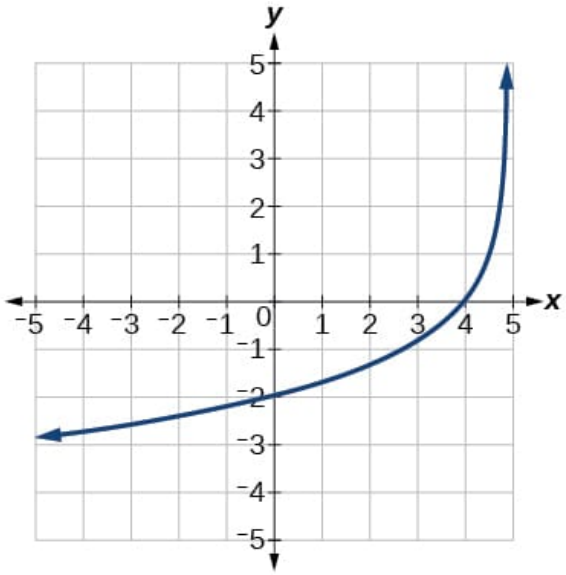
58. Use $f(x) = \log_3(x)$ as the parent function.



59. Use $f(x) = \log_4(x)$ as the parent function.



60. Use $f(x) = \log_5(x)$ as the parent function.



Technology

For the following exercises, use graphing technology to find approximate solutions to each equation.

61. $\log(x-1)+2 = \ln(x-1)+2$
62. $\log(2x-3)+2 = -\log(2x-3)+5$
63. $\ln(x-2) = -\ln(x+1)$
64. $2\ln(5x+1) = \frac{1}{2}\ln(-5x)+1$
65. $\frac{1}{3}\log(1-x) = \log(x+1)+\frac{1}{3}$

Extensions

66. Let b be any positive real number such that $b \neq 1$. What must $\log_b 1$ be equal to? Verify the result.
67. Explore and discuss the graphs of $f(x) = \log_{1/2}(x)$ and $g(x) = -\log_2(x)$. Make a conjecture based on the result.
68. Prove the conjecture made in the previous exercise.
69. What is the domain of the function $f(x) = \ln\left(\frac{x+2}{x-4}\right)$? Discuss the result.
70. Use properties of exponents to find the x-intercepts of the function $f(x) = \log(x^2+4x+4)$ algebraically. Show the steps for

solving, and then verify the result by graphing the function.

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4.5: Laws of Logarithms

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure $4.5.1$**:** *The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan)*

In chemistry, pH is used as a measure of the acidity or alkalinity of a substance. The pH scale runs from 0 to 14. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 (at 25° C)
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where H^+, is the concentration of hydrogen ion in the solution

$$\text{pH} = -\log([H^+]) = \log\left(\frac{1}{[H^+]}\right).$$

The equivalence of $-\log([H^+])$ and $\log\left(\frac{1}{[H^+]}\right)$, is one of the Laws of Logarithms we will examine in this section.

Using the Product Law for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar laws to

exponents. Some important laws of logarithms are given here. First, the following "Base Laws" are easy to prove.

Theorem: Laws of Logarithms - Base Laws

Let b be a positive real number with $b \neq 1$. Then

$$\log_b(1) = 0 \quad \text{and} \quad \log_b(b) = 1.$$

Proofs available at the end of this section.

For example, $\log_5 1 = 0$, since $5^0 = 1$, and $\log_5 5 = 1$, since $5^1 = 5$.

Next, we have the Inverse Laws.

Theorem: Laws of Logarithms - Inverse Laws

$$\log_b(b^x) = x \quad \text{and} \quad b^{\log_b(x)} = x, \text{ where } x > 0.$$

Proofs available at the end of this section.

To illustrate these laws, we imagine evaluating $\log(100)$. We can rewrite the logarithm as $\log_{10}(10^2)$, and then apply the Inverse Law $\log_b(b^x) = x$, to get $\log_{10}(10^2) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the Inverse Law $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the One-to-One Law.

Theorem: Laws of Logarithms - One-to-One Law

$$\log_b(M) = \log_b(N) \text{ if and only if } M = N.$$

Proof available at the end of this section.

We can use the One-to-One Law to solve the equation $\log_3(3x) = \log_3(2x + 5)$, for x. Since the bases are the same, we can apply the One-to-One Law by setting the arguments equal and solving for x:

$$\begin{aligned} 3x &= 2x + 5 && \text{(setting the arguments equal to each other)} \\ \implies x &= 5 && \text{(subtracting } 2x \text{ from both sides)} \end{aligned}$$

But what about the equation $\log_3(3x) + \log_3(2x + 5) = 2$? The One-to-One Law does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *Product Law of Exponents* to combine the product of powers by adding exponents: $x^a x^b = x^{a+b}$. We have a similar law for logarithms, called the **Product Law for Logarithms**.

Theorem: Laws of Logarithms - Product Law

Let b, M, and N be positive real numbers with $b \neq 1$. Then

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

That is, the logarithm of a product is the sum of the logarithms of its factors.

Proof available at the end of this section.

Note that repeated applications of the Product Law for Logarithms allow us to simplify the logarithm of the product of any number

of factors. For example, consider $\log_b(wxyz)$. Using the Product Law for Logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b(w) + \log_b(x) + \log_b(y) + \log_b(z).$$

Example 4.5.1: Using the Product Law for Logarithms

Expand $\log_3(30x(3x+4))$.

Solution available at the end of this section.

Checkpoint 4.5.1

Expand $\log_b(8k)$.

Using the Quotient Law for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *Quotient Law of Exponents* to combine the quotient of exponents by subtracting: $\frac{x^a}{x^b} = x^{a-b}$. The **Quotient Law for Logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the Product Law, we can use the Inverse Law to derive the Quotient Law.

Theorem: Laws of Logarithms - Quotient Law

Let b, M, and N be positive real numbers with $b \neq 1$. Then

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

That is, the logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator.

Proof available at the end of this section.

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{aligned}
\log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{(factoring out the GCF from numerator and denominator)} \\
&= \log\left(\frac{2x\ \cancel{(x+3)}}{3\ \cancel{(x+3)}}\right) && \text{(canceling like factors)} \\
&= \log\left(\frac{2x}{3}\right) && \text{(simplifying)}
\end{aligned}$$

Next we apply the Quotient Law by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the Product Law.

$$\log\left(\frac{2x}{3}\right) = \log(2x) - \log(3) = \log(2) + \log(x) - \log(3).$$

Caution: Assumptions Abound

When using the Product and Quotient Laws of Logarithms, we are *assuming* (often falsely) that the component logarithms are well-defined. That is, when expanding $\log\left(\frac{2x^2+6x}{3x+9}\right)$ to arrive at $\log(2) + \log(x) - \log(3)$, we are assuming

1. $x > 0$, and

2. $x+3 \neq 0$ (remember, this is the factor we canceled in the process of simplifying the logarithm).

These assumptions are likely *not correct*! However, when *practicing* the Laws of Logarithms, we allow ourselves the "training wheels" of assuming everything is going to work out. When we start to solve equations involving logarithms, on the other hand, we remove those "training wheels" and become concerned with the possibility that the component logarithms are not well-defined.

Example 4.5.2: Using the Quotient Law for Logarithms

Expand

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right).$$

Solution available at the end of this section.

Let's restate our "Caution" from above. There are exceptions to consider in Example 4.5.2. First, because denominators must never be zero, the expression in Example 4.5.2 is not defined for $x=-\frac{4}{3}$ and $x=2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x>0$, $x>1$, $x>-\frac{4}{3}$, and $x<2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

Checkpoint 4.5.2

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

Using the Power Law for Logarithms

We've explored the Product Law and the Quotient Law, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\begin{aligned}
\log_b(x^2) &= \log_b(x\cdot x) && \text{(Laws of Exponents)}\\
&= \log_b x+\log_b x && \text{(Laws of Logarithms: Product Law)}\\
&= 2\log_b x && \text{(combining like terms)}
\end{aligned}$$

Notice that we used the Product Law for Logarithms to find a solution for the example above. By doing so, we have derived the **Power Law for Logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100=10^2 \quad \sqrt{3}=3^{1/2} \quad \frac{1}{e}=e^{-1}.$$

Theorem: Laws of Logarithms - Power Law

Let b and M be positive real numbers with $b\neq 1$, and let p be any real number. Then

$$\log_b(M^p)=p\log_b(M).$$

Proof available at the end of this section.

Example 4.5.3: Expanding a Logarithm with Powers

Expand $\log_2(x^5)$.

Solution available at the end of this section.

Checkpoint 4.5.3

Expand $\ln(x^2)$.

Example 4.5.4: Rewriting an Expression as a Power before Using the Power Law

Expand $\log_3(25)$, using the Power Law for Logs.

Solution available at the end of this section.

Checkpoint 4.5.4

Expand $\ln\left(\frac{1}{x^2}\right)$.

Example 4.5.5: Using the Power Law in Reverse

Rewrite $4\ln(x)$, using the Power Law for logs to a single logarithm with a leading coefficient of 1.

Solution available at the end of this section.

Checkpoint 4.5.5

Rewrite $2\log_3 4$, using the Power Law for logs to a single logarithm with a leading coefficient of 1.

Expanding Logarithmic Expressions

Taken together, the laws introduced in this section so far are collectively called the "Laws of Logarithms." Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the Power Law to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the Quotient Law for Logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b(AC^{-1}) \\ &= \log_b(A) + \log_b(C^{-1}) \\ &= \log_b A + (-1)\log_b C \\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the Product Law to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Example 4.5.6: Expanding Logarithms Using Product, Quotient, and Power Laws

Rewrite $\ln\left(\frac{x^4y}{7}\right)$, as a sum or difference of logs.

Solution available at the end of this section.

Checkpoint 4.5.6

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

Example 4.5.7: Using the Power Law for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log(\sqrt{x})$.

Solution available at the end of this section.

Checkpoint 4.5.7

Expand $\ln(\sqrt[3]{x^2})$.

Q&A

Can we expand $\ln(x^2+y^2)$?

No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

Example 4.5.8: Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{2x-1}\right)$.

Solution available at the end of this section.

Checkpoint 4.5.8

Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{x^2-9}\right)$.

Condensing Logarithmic Expressions

We can use the Laws of Logarithms to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that *the logarithms must have the same base to be combined*. We will learn later how to change the base of any logarithm before condensing. Because students sometimes make mistakes on the order with which they apply the Laws of Logarithms, we provide the following "How To" as a beginner's reference guide.

How To

Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1. Apply the Power Law first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the Product Law. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the Quotient Law last. Rewrite differences of logarithms as the logarithm of a quotient.

Example 4.5.9: Using the Product and Quotient Laws to Combine Logarithms

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

Solution available at the end of this section.

Checkpoint 4.5.9

Condense $\log 3 - \log 4 + \log 5 - \log 6$.

Example 4.5.10: Condensing Complex Logarithmic Expressions

Condense $\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2)$.

Solution available at the end of this section.

Checkpoint 4.5.10

Rewrite $\log(5) + 0.5\log(x) - \log(7x-1) + 3\log(x-1)$ as a single logarithm.

Example 4.5.11: Rewriting as a Single Logarithm

Rewrite $2\log x - 4\log(x+5) + \frac{1}{x}\log(3x+5)$ as a single logarithm.

Solution available at the end of this section.

Checkpoint 4.5.11

Condense $4(3\log(x) + \log(x+5) - \log(2x+3))$.

Example 4.5.12: Applying of the Laws of Logs

Recall that, in chemistry, $\mathrm{pH} = -\log[H^+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution available at the end of this section.

Checkpoint 4.5.12

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e, we use the **Change-of-Base Formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

Theorem: Change-of-Base Formula

Given any positive real numbers M, b, and n, where $n \neq 1$, and $b \neq 1$,

$$\log_b(M) = \frac{\log_n(M)}{\log_n(b)}.$$

Proof available at the end of this section.

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\log_5 36 = \frac{\log(36)}{\log(5)} \approx 2.2266.$$

Example 4.5.13: Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5(3)$ to a quotient of natural logarithms.

Solution available at the end of this section.

Checkpoint 4.5.13

Change $\log_{0.5}(8)$ to a quotient of natural logarithms.

Q&A

Can we change common logarithms to natural logarithms?

Yes. Remember that $\log 9$, means $\log_{10} 9$. So, $\log 9 = \frac{\ln 9}{\ln 10}$.

Example 4.5.14: Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$, using the Change-of-Base Formula with a calculator.

Solution available at the end of this section.

Checkpoint 4.5.14

Evaluate $\log_5(100)$, using the Change-of-Base Formula.

Success in College: Environment and Learning Preferences

Throughout your academic career, you have completed many exams. While you might not have thought about it, you should be able to identify a strong correlation between the environment you study in and your performance on exams.

Learning preferences describe the situations in which you best learn math. Are you a visual, auditory, or kinesthetic learner? Should you take notes as you listen? Should you record your class? Should you trace the graphs in your book as you study?

Think about whether you learn better by working by yourself or with others. Do you prefer to work in quiet or with music in the background? Do you like to sit at a desk or recline in an easy chair? Do you learn best at a particular time of the day? When you study, does it help if you can move around? All of these are part of your learning preferences.

If you know your learning preferences, you can try to adjust your studying and class behavior to improve your work. It is your responsibility to think about and, if needed, make changes in how you are learning and studying.

Think about the following list. When you study, do you prefer...

- a quiet room, or background noise (music or TV)
- bright light shining on your work, or dim lighting
- sitting at a desk or table, or being relaxed on a couch or a bed
- eating or drinking while you learn, or only eating or drinking during breaks
- working hard for a reasonably long time or taking frequent breaks

1. Do you work in conditions that match these preferences when you do your math homework?
2. Do you work in conditions that match these preferences when you study for math tests?

Success in Mathematics: Sequential Versus Global Learners

> Sequential learners tend to think in a linear manner and are able to function with only partial understanding of material they have been taught. Global learners think in a systems-oriented manner, and may have trouble applying new material until they fully understand it and see how it relates to material they already know about and understand. Once they grasp the big picture, however, their holistic perspective enables them to see innovative solutions to problems that sequential learners might take much longer to reach.
>
> Source: R. Felder, R. Brent , Journal of Engineering Education, 94 (1), 57-72 (2005).

1. When learning math, do you think you are more of a sequential learner or a global learner?
2. Students often have different learning preferences for different subjects. When learning history or another subject in the humanities, do you think you are more a sequential learner or a global learner?

Answers

Section Preview

This is the typical "Laws of Logarithms" material that you would find in any College Algebra textbook.

Section Preview

- Use the Product Law for logarithms.
- Use the Quotient Law for logarithms.
- Use the Power Law for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the Change-of-Base Formula for logarithms.

Theorem: Laws of Logarithms - Base Laws

By definition of a logarithm, $\log_b(1)$ is the exponent, p, for which $b^p = 1$. Therefore, $p = 0$. As such,

$$\log_b(1) = 0.$$

Similarly, $\log_b(b)$ is the exponent, q, such that $b^q = b$. Since $b = b^1$, this means that $b^q = b^1$. In other words, $q = 1$. Hence,

$$\log_b(b) = 1.$$

Theorem: Laws of Logarithms - Inverse Laws

Consider $\log_b(b^x)$. By the definition of a logarithm, this is the exponent, p, such that $b^p = b^x$. That is, $x = p$. Thus, $\log_b(b^x) = x$.

Now consider $b^{\log_b(x)}$, where $x > 0$. Recall that $\log_b(x)$ is the exponent, q, such that $b^q = x$. That is,

$$b^{\log_b(x)} = b^q = x.$$

Theorem: Laws of Logarithms - One-to-One Law

From the definition of a logarithm, if we let $\log_b(M) = m$ and $\log_b(N) = n$, then $b^m = M$ and $b^n = N$.

( ): If $\log_b(M) = \log_b(N)$, then $m = n$. However, if $m = n$, then $b^m = b^n$. Therefore, $M = N$.

( ): If $M = N$, then $b^m = b^n$. However, by laws of exponents, this means $m = n$. Hence, $\log_b(M) = \log_b(N)$.

Theorem: Laws of Logarithms - Product Law

Let $m = \log_b(M)$, and $n = \log_b(N)$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{(substituting)} \\ &= \log_b(b^{m+n}) && \text{(Laws of Exponents: Product Rule)} \\ &= m + n && \text{(Laws of Logarithms: Inverse Law)} \\ &= \log_b(M) + \log_b(N) && \text{(substituting)} \end{aligned}$$

Example 4.5.1: Using the Product Law for Logarithms

We begin by recognizing the argument as a product of three factors.

$$\log_3(30x(3x+4)) = \log_3(30 \cdot x \cdot (3x+4)).$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30 \cdot x \cdot (3x+4)) = \log_3(30) + \log_3(x) + \log_3(3x+4).$$

Finally, we need to be aware of the base and the possibility that the 30 can be factored into a power of the base and some other constant. Specifically, $30 = 3 \cdot 10$. Therefore,

$$\begin{aligned} \log_3(30x(3x+4)) &= \log_3(3 \cdot 10) + \log_3(x) + \log_3(3x+4) \\ &= \log_3(3) + \log_3(10) + \log_3(x) + \log_3(3x+4) && \text{(Laws of Logarithms: Product Law)} \\ &= 1 + \log_3(10) + \log_3(x) + \log_3(3x+4) && \text{(Laws of Logarithms: } \log_b b = 1) \end{aligned}$$

Theorem: Laws of Logarithms - Quotient Law

Let $m = \log_b(M)$, and $n = \log_b(N)$. In exponential form, these equations are $b^m = M$, and $b^n = N$. It follows that

$$\begin{aligned} \log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{(substituting)} \\ &= \log_b\left(b^{m-n}\right) && \text{(Laws of Exponents: Quotient Law)} \\ &= m - n && \text{(Laws of Logarithms: Inverse Law)} \\ &= \log_b(M) - \log_b(N) && \text{(substituting)} \end{aligned}$$

Example 4.5.2: Using the Quotient Law for Logarithms

First we note that the quotient is factored and in lowest terms, so we apply the Quotient Law.

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2(15x(x-1)) - \log_2((3x+4)(2-x)).$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the Product Law, noting that 15 is not divisible by a power of the base (2).

$$\begin{aligned} \log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= \log_2(15) + \log_2(x) + \log_2(x-1) - [\log_2(3x+4) + \log_2(2-x)] && \text{(Product Law)} \\ &= \log_2(15) + \log_2(x) + \log_2(x-1) - \log_2(3x+4) - \log_2(2-x) && \text{(distributing)} \end{aligned}$$

Theorem: Laws of Logarithms - Power Law

Let $\log_b(M) = m$.

$$
\begin{aligned}
\implies \quad b^m &= M && \text{(definition of the logarithm)} \\
\implies \quad (b^m)^p &= M^p && \text{(Laws of Exponents)} \\
\implies \quad b^{mp} &= M^p && \text{(Laws of Exponents)} \\
\implies \quad \log_b\left(b^{mp}\right) &= \log_b\left(M^p\right) && \text{(One-to-One Law of Logarithms)} \\
\implies \quad \log_b\left(b^m \cdot b^p\right) &= \log_b\left(M^p\right) && \text{(Laws of Exponents)} \\
\implies \quad \log_b\left(b^m\right) \cdot \log_b\left(b^p\right) &= \log_b\left(M^p\right) && \text{(Product Law of Logarithms)} \\
\implies \quad m \cdot p &= \log_b\left(M^p\right) && \text{(Base Laws of Logarithms)} \\
\implies \quad \log_b(M) \cdot p &= \log_b\left(M^p\right) && \text{(our original substitution } \log_b(M) = m\text{)} \\
\implies \quad p\log_b(M) &= \log_b\left(M^p\right) && \text{(Commutative Property of Multiplication)}
\end{aligned}
$$

Example 4.5.3: Expanding a Logarithm with Powers

The argument is already written as a power, so we identify the exponent, 5, and the base, x, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2(x^5) = 5\log_2 x$$

Example 4.5.4: Rewriting an Expression as a Power before Using the Power Law

Expressing the argument as a power, we get $\log_3(25) = \log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(5^2) = 2\log_3(5)$$

Example 4.5.5: Using the Power Law in Reverse

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, x, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

Example 4.5.6: Expanding Logarithms Using Product, Quotient, and Power Laws

First, because we have a quotient of two expressions, we can use the Quotient Law:

$$\ln\left(\frac{x^4y}{7}\right) = \ln(x^4y) - \ln(7).$$

Then seeing the product in the first term, we use the Product Law:

$$\ln(x^4y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7).$$

Finally, we use the Power Law on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7).$$

Example 4.5.7: Using the Power Law for Logarithms to Simplify the Logarithm of a Radical Expression

$$\log(\sqrt{x}) = \log x^{1/2} = \frac{1}{2}\log x$$

Example 4.5.8: Expanding Complex Logarithmic Expressions

We can expand by applying the Product and Quotient Laws.

$$
\begin{aligned}
\log_6\left(\frac{64x^3(4x+1)}{2x-1}\right) &= \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) \\
&= \log_6 64 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1)
\end{aligned}
$$

Note that we *could* have rewritten $\log_6 64$ as $\log_6 2^6 = 6\log_6 2$, however, this is not any better than $\log_6 64$. When it comes to logarithms of constants, we are only concerned with the possibility that a power of the base is a factor of the constant.

Example 4.5.9: Using the Product and Quotient Laws to Combine Logarithms

Using the product and Quotient Laws

$$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40).$$

This reduces our original expression to

$$\log_3(40) - \log_3(2).$$

Then, using the Quotient Law

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20).$$

Example 4.5.10: Condensing Complex Logarithmic Expressions

We apply the Power Law first:

$$\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2) = \log_2(x^2) + \log_2(\sqrt{x-1}) - \log_2((x+3)^6).$$

Next we apply the Product Law to the sum:

$$\log_2(x^2) + \log_2(\sqrt{x-1}) - \log_2((x+3)^6) = \log_2(x^2\sqrt{x-1}) - \log_2((x+3)^6).$$

Finally, we apply the Quotient Law to the difference:

$$\log_2(x^2\sqrt{x-1}) - \log_2((x+3)^6) = \log_2\left(\frac{x^2\sqrt{x-1}}{(x+3)^6}\right).$$

Example 4.5.11**: Rewriting as a Single Logarithm**

We apply the Power Law first:

$$2\log x - 4\log(x+5) + \frac{1}{x}\log(3x+5) = \log(x^2) - \log\left[(x+5)^4\right] + \log\left[(3x+5)^{x^{-1}}\right].$$

Next, we rearrange and apply the Product Law to the sum:

$$\begin{aligned}\log(x^2) - \log\left[(x+5)^4\right] + \log\left[(3x+5)^{x^{-1}}\right] &= \log(x^2) + \log\left[(3x+5)^{x^{-1}}\right] - \log\left[(x+5)^4\right] \\ &= \log\left[x^2(3x+5)^{x^{-1}}\right] - \log\left[(x+5)^4\right]\end{aligned}$$

Finally, we apply the Quotient Law to the difference:

$$\log\left[x^2(3x+5)^{x^{-1}}\right] - \log\left[(x+5)^4\right] = \log\left[\frac{x^2(3x+5)^{x^{-1}}}{(x+5)^4}\right].$$

Example 4.5.12**: Applying of the Laws of Logs**

Suppose C, is the original concentration of hydrogen ions, and P, is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$\text{pH} = -\log(2C).$$

Using the Product Law of logs

$$\text{pH} = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C).$$

Since $P = -\log(C)$, the new pH is

$$\text{pH} = P - \log(2) \approx P - 0.301.$$

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

Theorem: Change-of-Base Formula

Let $y = \log_b(M)$. By the definition of a logarithm, $b^y = M$. It follows that

$$\begin{aligned}\log_n(b^y) &= \log_n(M) && \text{(One-to-One Law of Logarithms)} \\ \Longrightarrow\quad y\log_n(b) &= \log_n(M) && \text{(Power Law of Logarithms)} \\ \Longrightarrow\quad y &= \frac{\log_n(M)}{\log_n(b)} && \text{(dividing both sides by } \log_n(b)) \\ \Longrightarrow\quad \log_b(M) &= \frac{\log_n(M)}{\log_n(b)} && \text{(substituting)}\end{aligned}$$

Example 4.5.13**: Changing Logarithmic Expressions to Expressions Involving Only Natural Logs**

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the Change-of-Base Formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_5 3 = \frac{\ln 3}{\ln 5}.$$

Example 4.5.14**: Using the Change-of-Base Formula with a Calculator**

According to the Change-of-Base Formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

$$\log_2 10 = \frac{\ln 10}{\ln 2} \approx 3.3219.$$

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4.5.1: Resources and Key Concepts

Resources

Videos

- Laws of Logarithms
  - Intermediate Algebra - Logarithms: Properties of Logarithms
  - Logarithmic Properties - A Brief Review of the Properties of Logarithms
  - Logarithmic Functions - A Brief Review of Properties of Logarithms
  - Logarithmic Properties - Using Properties of Logarithms
- Expanding and condensing logarithmic expressions
  - Logarithmic Functions - Expanding and Combining Logarithms

Key Concepts

Theorems

- **Laws of Logarithms - Base Laws**:
  - $\log_b(1) = 0$
  - $\log_b(b) = 1$
- **Laws of Logarithms - Inverse Laws**:
  - $\log_b(b^x) = x$
  - $b^{\log_b(x)} = x$, where $x > 0$
- **Laws of Logarithms - One-to-One Law**: $\log_b(M) = \log_b(N)$ if and only if $M = N$.
- **Laws of Logarithms - Product Law**: Given any real number x, and positive real numbers M and N, $\log_b(MN) = \log_b(M) + \log_b(N)$.
- **Laws of Logarithms - Quotient Law**: Given any real number x, and positive real numbers M and N, $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$.
- **Laws of Logarithms - Power Law**: For any real number p and positive real number M, $\log_b(M^p) = p\log_b(M)$.
- **Change-of-Base Formula**: Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$, $\log_b(M) = \frac{\log_n(M)}{\log_n(b)}$.

Common Mistakes

- **Incorrectly Applying Logarithm Laws**:
  - $\log_b(M+N) \neq \log_b(M) + \log_b(N)$ (Log of a sum is not sum of logs).
  - $\log_b(M-N) \neq \log_b(M) - \log_b(N)$ (Log of a difference is not difference of logs).
  - $\frac{\log_b(M)}{\log_b(N)} \neq \log_b(M) - \log_b(N)$ (Quotient of logs is not log of a quotient).
  - $(\log_b(M))^p \neq p\log_b(M)$ (Power of a log is not product of exponent and log, unless the power is on the argument).
- **Errors in Expanding Logarithms**: Not fully expanding an expression, or incorrectly applying laws (e.g., distributing a log over a sum/difference inside the argument).
- **Errors in Condensing Logarithms**:
  - Incorrectly combining terms, especially with subtraction (order matters for division).
  - Not applying the power law first to make coefficients into exponents before combining with product/quotient laws.
  - Combining logarithms with different bases.
- **Ignoring Domain Restrictions (Conceptual vs. Practical Application)**: While the "Caution: Assumptions Abound" notes that for practice with laws, well-definedness is assumed, in solving equations or other applications, the domain of each logarithmic expression must be considered.
- **Errors with Change-of-Base Formula**: Incorrectly setting up the ratio, e.g., $\frac{\log_n(b)}{\log_n(M)}$ instead of $\frac{\log_n(M)}{\log_n(b)}$.
- **Calculator Errors with Change-of-Base**: Using the wrong base on the calculator or inputting the fraction incorrectly.

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4.5.2: Homework

Reading Questions

1. What is the Product Law for Logarithms? State the formula.
2. What is the Quotient Law for Logarithms? State the formula.
3. What is the Power Law for Logarithms? State the formula.
4. How does the Power Law for Logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$?
5. When expanding a complex logarithmic expression, what is the general order in which you might apply the Product, Quotient, and Power Laws?
6. When condensing multiple logarithmic terms into a single logarithm, what must be true about the bases of the logarithms?
7. When condensing an expression like $2\log(x) + 3\log(y) - \log(z)$, which law should generally be applied first to the terms $2\log(x)$ and $3\log(y)$?
8. What is the Change-of-Base Formula for logarithms? Why is it particularly useful when using a calculator?
9. Rewrite $\log_5(36)$ as a quotient of common logarithms (base 10).
10. Rewrite $\log_5(3)$ as a quotient of natural logarithms (base e).
11. Can $\log_b(A + C)$ be expanded using the Product Law for Logarithms? Why or why not?
12. How does the Power Law for Logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$?
13. What does the change-of-base formula do? Why is it useful when using a calculator?

Homework

Algebraic

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

14. $\log_b(7x \cdot 2y)$
15. $\ln(3ab \cdot 5c)$
16. $\log_b\left(\frac{13}{17}\right)$
17. $\log_4\left(\frac{x/z}{w}\right)$
18. $\ln\left(\frac{1}{4^k}\right)$
19. $\log_2(y^x)$

For the following exercises, condense to a single logarithm if possible.

20. $\ln(7) + \ln(x) + \ln(y)$
21. $\log_3(2) + \log_3(a) + \log_3(11) + \log_3(b)$
22. $\log_b(28) - \log_b(7)$
23. $\ln(a) - \ln(d) - \ln(c)$
24. $-\log_b\left(\frac{1}{7}\right)$
25. $\frac{1}{3}\ln(8)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

26. $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$
27. $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$
28. $\log(\sqrt{x^3y^{-4}})$
29. $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$
30. $\log(x^2y^3\sqrt[3]{x^2y^5})$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

31. $\log(2x^4) + \log(3x^5)$
32. $\ln(6x^9) - \ln(3x^2)$
33. $2\log(x) + 3\log(x+1)$
34. $\log(x) - \frac{1}{2}\log(y) + 3\log(z)$
35. $4\log_7(c) + \frac{\log_7(a)}{3} + \frac{\log_7(b)}{3}$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

36. $\log_7(15)$ to base e
37. $\log_{14}(55.875)$ to base 10

For the following exercises, suppose $\log_5(6) = a$ and $\log_5(11) = b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b. Show the steps for solving.

38. $\log_{11}(5)$
39. $\log_6(55)$
40. $\log_{11}\left(\frac{6}{11}\right)$

Numeric

For the following exercises, use properties of logarithms to evaluate without using a calculator.

41. $\log_3\left(\frac{1}{9}\right) - 3\log_3(3)$
42. $6\log_8(2) + \frac{\log_8(64)}{3\log_8(4)}$
43. $2\log_9(3) - 4\log_9(3) + \log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

44. $\log_3(22)$
45. $\log_8(65)$
46. $\log_6(5.38)$
47. $\log_4\left(\frac{15}{2}\right)$
48. $\log_{1/2}(4.7)$

Extensions

49. Use the Product Law of Logarithms to find all x values such that $\log_{12}(2x+6) + \log_{12}(x+2) = 2$. Show the steps for solving.
50. Use the Quotient Law of Logarithms to find all x values such that $\log_6(x+2) - \log_6(x-3) = 1$. Show the steps for solving.
51. Can the Power Law of Logarithms be derived from the Power Law of Exponents using the equation $b^x = m$? If not, explain why. If so, show the derivation.
52. Prove that $\log_b(n) = \frac{1}{\log_n(b)}$ for any positive integers $b > 1$ and $n > 1$.
53. Does $\log_{81}(2401) = \log_3(7)$? Verify the claim algebraically.

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4.6: Exponential and Logarithmic Equations

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure 4.6.1**:** *Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the "rabbit plague." (credit: Richard Taylor, Flickr)*

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential equations.

Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall the following theorem from Algebra (the proof of which requires recalling that logarithms "undo" exponential functions).

Theorem: One-to-One Property of Exponents

For any algebraic expressions S and T, and any positive real number $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T.$$

Proof available at the end of this section.

In other words, when an exponential equation has the same base on each side, the exponents must be equal. Therefore, we can solve many exponential equations by using the Laws of Exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7} = \frac{3^{2x}}{3}$. To solve for x, we use the Quotient Rule for Exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the One-to-One Property of exponents by setting the exponents equal to one another and solving for x:

$$\begin{aligned}
& 3^{4x-7} = \frac{3^{2x}}{3} && \\
\Rightarrow\ & 3^{4x-7} = \frac{3^{2x}}{3^1} && (3 = 3^1) \\
\Rightarrow\ & 3^{4x-7} = 3^{2x-1} && (\text{Laws of Exponents: Quotient Rule}) \\
\Rightarrow\ & 4x - 7 = 2x - 1 && (\text{One-to-One Property of Exponents}) \\
\Rightarrow\ & 2x = 6 && (\text{subtracting } 2x \text{ and adding } 7 \text{ to both sides}) \\
\Rightarrow\ & x = 3 && (\text{dividing both sides by } 2)
\end{aligned}$$

Example 4.6.1: Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

Solution available at the end of this section.

Checkpoint 4.6.1

Solve $5^{2x} = 5^{3x+2}$.

Rewriting Equations So All Powers Have the Same Base

Sometimes the common base for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the One-to-One Property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the Laws of Exponents, along with the One-to-One Property, to solve for x:

$$\begin{aligned}
& 256 = 4^{x-5} && \\
\Rightarrow\ & 4^4 = 4^{x-5} && (\text{rewriting both sides in the same base}) \\
\Rightarrow\ & 4 = x - 5 && (\text{One-to-One Property}) \\
\Rightarrow\ & 9 = x && (\text{adding } 5 \text{ to both sides})
\end{aligned}$$

Example 4.6.2: Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2} = 16^{x+1}$.

Solution available at the end of this section.

Checkpoint 4.6.2

Solve $5^{2x} = 25^{3x+2}$.

Example 4.6.3: Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve $2^{5x} = \sqrt{2}$.

Solution available at the end of this section.

Checkpoint 4.6.3

Solve $5^{x} = \sqrt{5}$.

Q&A

Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?

No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.

Example 4.6.4: Solving an Equation with Positive and Negative Powers

Solve $3^{x+1} = -2$.

Solution available at the end of this section.

Checkpoint 4.6.4

Solve $2^{x} = -100$.

Solving Exponential Equations Using Logarithms

Most of the time the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

Example 4.6.5: Solving an Equation Containing Powers of Different Bases

Solve $5^{x+2} = 4^{x}$.

Solution available at the end of this section.

Checkpoint 4.6.5

Solve $2^x = 3^{x+1}$.

Q&A

Is there any way to solve $2^x = 3^x$ without logarithms?

Yes. The solution is 0.

Equations Containing e

Exponential equations involving the base e are very common in Science. This constant occurs again and again in nature, in Mathematics, in Science, in Engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

Example 4.6.6: Solve an Equation of the Form $y = Ae^{kt}$

Solve $100 = 20e^{2t}$.

Solution available at the end of this section.

Checkpoint 4.6.6

Solve $3e^{0.5t} = 11$.

Q&A

Does every equation of the form $y = Ae^{kt}$ have a solution?

No. There is a solution when $k \neq 0$, and when y and A are either both 0 or neither 0, and they have the same sign. An example of an equation with this form that has no solution is $2 = -3e^t$.

Example 4.6.7: Solving an Equation That Can Be Simplified to the Form $y = Ae^{kt}$

Solve $4e^{2x} + 5 = 12$.

Solution available at the end of this section.

Checkpoint 4.6.7

Solve $3 + e^{2t} = 7e^{2t}$.

Extraneous Solutions

Sometimes the methods used to solve an equation introduce an extraneous solution, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the

number we are evaluating in a logarithm function is negative, there is no output.

Example 4.6.8: Solving Exponential Equations in Quadratic Form

Solve $e^{2x} - e^x = 56$.

Solution available at the end of this section.

When we plan to use factoring to solve a problem, as we did in Example 4.6.8, we always get zero on one side of the equation. This is because zero has the unique property that when a product is zero, one or both of the factors must be zero. We rejected the equation $e^x = -7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

Checkpoint 4.6.8

Solve $e^{2x} = e^x + 2$.

Q&A

Does every logarithmic equation have a solution?

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the Laws of Logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use Laws of Logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x:

$$\begin{aligned}
& \log_2(2) + \log_2(3x-5) = 3 && \\
\Longrightarrow\ & \log_2(2(3x-5)) = 3 && \text{(Laws of Logarithms)} \\
\Longrightarrow\ & \log_2(6x-10) = 3 && \text{(distributing)} \\
\Longrightarrow\ & 6x-10 = 2^3 && \text{(definition of a logarithm)} \\
\Longrightarrow\ & 6x-10 = 8 && \text{(simplifying)} \\
\Longrightarrow\ & 6x = 18 && \text{(adding 10 to both sides)} \\
\Longrightarrow\ & x = 3 && \text{(dividing both sides by 6)}
\end{aligned}$$

As you can see, when solving a logarithmic equation, we rely on the following definition of logarithms:

$$\text{For any algebraic expression } S \text{ and real numbers } b \text{ and } c, \text{ where } b > 0 \text{ and } b \neq 1, \log_b(S) = c \text{ if and only if } b^c = S.$$

Example 4.6.9: Using Algebra to Solve a Logarithmic Equation

Solve $2\ln(x) + 3 = 7$.

Solution available at the end of this section.

Checkpoint 4.6.9

Solve $6 + \ln x = 10$.

Example 4.6.10: Using Algebra Before and After Using the Definition of the Natural Logarithm

Solve $2\ln(6x) = 7$.

Solution available at the end of this section.

Checkpoint 4.6.10

Solve $2\ln(x + 1) = 10$.

Example 4.6.11: Showcasing the Need to Check Solutions

Solve $\log_7(1 - 2x) = 1 - \log_7(3 - x)$.

Solution available at the end of this section.

Example 4.6.12: Using a Graph to Understand the Solution to a Logarithmic Equation

Solve $\ln x = 3$ and use a graph to interpret the solution.

Solution available at the end of this section.

Checkpoint 4.6.12

Use graphing technology to estimate the approximate solution to the exponential equation $2^x = 1000$ to two decimal places.

Interactive Element: Solving $2^x = c$ as an Intersection

The following Interactive Element might help you visualize how solving an exponential equation corresponds to finding where two graphs meet.

Interact: Drag the red point up and down the vertical axis to change the target height c of the line $y = c$.

Observation: The green point sits where $y = 2^x$ meets $y = c$, and its x-coordinate is the solution of $2^x = c$. Because 2^x is always positive, dragging the line to $c \leq 0$ leaves the graphs never touching — the equation has no solution.

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the One-to-One Property to solve logarithmic equations. The One-to-One Property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{ then } x-1=8.$$

So, if $x-1=8$, then we can solve for x, and we get $x=9$. To check, we can substitute $x=9$ into the original equation: $\log_2(9-1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use the Laws of Logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the One-to-One Property to solve for x:

$$\begin{aligned}
& \log(3x-2) - \log(2) = \log(x+4) && \\
\Longrightarrow \quad & \log\left(\frac{3x-2}{2}\right) = \log(x+4) && \text{(Laws of Logarithms)} \\
\Longrightarrow \quad & \frac{3x-2}{2} = x+4 && \text{(One-to-One Property)} \\
\Longrightarrow \quad & 3x-2 = 2x+8 && \text{(multiplying both sides by 2)} \\
\Longrightarrow \quad & x = 10 && \text{(subtracting } 2x \text{ and adding 2 to both sides)}
\end{aligned}$$

To check the result, substitute $x=10$ into the original equation.

$$\begin{aligned}
& \log(3(10)-2) - \log(2) \stackrel{?}{=} \log(10+4) \\
\Longrightarrow \quad & \log(28) - \log(2) \stackrel{?}{=} \log(14) \\
\Longrightarrow \quad & \log\left(\frac{28}{2}\right) \stackrel{?}{=} \log(14) \\
\Longrightarrow \quad & \log(14) \stackrel{\checkmark}{=} \log(14)
\end{aligned}$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

Example 4.6.13: Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2) = \ln(2x+3)$.

Solution available at the end of this section.

Checkpoint 4.6.13

Solve $\ln(x^2) = \ln 1$.

Miscellaneous Topics: What Are Your Top Degree Choices?

I have seen all the statistics regarding the correlation between income and education. In general, the higher your education, the higher your income.

1. Name three benefits, besides money, that a higher degree can give you.
2. What are your top three choices for degrees (e.g., English, Mathematics, Psychology)?

Success in College: Affirmations

Do you ever tell yourself (or others) that you're not good at math? This is called negative self-talk. The more you talk that way, the more likely your subconscious will believe it - and you will do poorly in math.

You can counteract years of negative self-talk by telling yourself with conviction that you're good at math. It might seem strange to state that something that hasn't happened yet is true, but it works! Such statements are called affirmations.

There are four guiding principles for getting the most out of affirmations:

I. Say affirmations that imply that the desired event is currently happening. For example, say, "I am good at algebra," not "I will be good at algebra."
II. Say affirmations that you are making progress toward the desired result. For example, say, "I am good at trigonometry, and I continue to improve at it."
II. Say affirmations in the positive rather than the negative. For example, say "I attend each class," not "I don't cut classes."
IV. Say affirmations with conviction.

1. Do you practice affirmations? If so, how?
2. After reading this, will you try to practice affirmations? Where else in life could affirmations be put to work?

Answers

Section Preview

Take care to cover the domain issues that arise when solving logarithmic equations.

Section Preview

- **Solving Equations**
  - Solving Equations Quadratic-in-Form: Example 8 (and Checkpoint 8) uses the substitution $u = e^x$ to convert $e^{2x} - e^x = 56$ into a solvable quadratic before back-substituting.

Section Preview

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the One-to-One Property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

Theorem: One-to-One Property of Exponents

Let S and T be algebraic expressions, and b be a positive real number not equal to 1.

$$\begin{aligned} & b^S = b^T \\ \leftrightarrow \quad & \ln(b^S) = \ln(b^T) && \text{(taking the natural log of both sides)} \\ \leftrightarrow \quad & S\ln(b) = T\ln(b) && \text{(Laws of Logarithms)} \\ \leftrightarrow \quad & S = T && \text{(dividing both sides by } \ln(b)\text{)} \end{aligned}$$

Example 4.6.1**: Solving an Exponential Equation with a Common Base**

$$\begin{aligned} & 2^{x-1} = 2^{2x-4} \\ \Longrightarrow \quad & x-1 = 2x-4 && \text{(One-to-One Property)} \\ \Longrightarrow \quad & 3 = x && \text{(subtracting } x \text{ and adding 4 to both sides)} \end{aligned}$$

Example 4.6.2**: Solving Equations by Rewriting Them to Have a Common Base**

$$\begin{aligned} & 8^{x+2} = 16^{x+1} \\ \Longrightarrow \quad & (2^3)^{x+2} = (2^4)^{x+1} && \text{(rewriting both sides with the same base)} \\ \Longrightarrow \quad & 2^{3(x+2)} = 2^{4(x+1)} && \text{(Laws of Exponents)} \\ \Longrightarrow \quad & 2^{3x+6} = 2^{4x+4} && \text{(distributing)} \\ \Longrightarrow \quad & 3x+6 = 4x+4 && \text{(One-to-One Property)} \\ \Longrightarrow \quad & 2 = x && \text{(subtracting } 3x \text{ and 4 from both sides)} \end{aligned}$$

Example 4.6.3**: Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base**

$$\begin{aligned} & 2^{5x} = \sqrt{2} \\ \Longrightarrow \quad & 2^{5x} = 2^{1/2} && \text{(rewriting the radical as a rational exponent)} \\ \Longrightarrow \quad & 5x = \frac{1}{2} && \text{(One-to-One Property)} \\ \Longrightarrow \quad & x = \frac{1}{10} && \text{(dividing both sides by 5)} \end{aligned}$$

Example 4.6.4**: Solving an Equation with Positive and Negative Powers**

This equation has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive.

Figure 4.6.2 shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

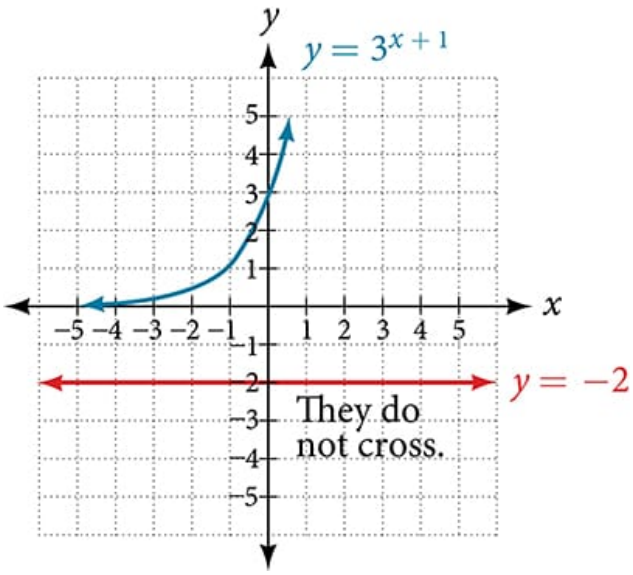


Figure 4.6.2

Example 4.6.5**: Solving an Equation Containing Powers of Different Bases**

$$\begin{aligned}
& 5^{x+2} = 4^x && \text{(notice there is no common base between 5 and 4)} \\
\Longrightarrow\ & \ln\left(5^{x+2}\right) = \ln\left(4^x\right) && \text{(take the natural log of both sides)} \\
\Longrightarrow\ & (x+2)\ln(5) = x\ln(4) && \text{(Laws of Logarithms)} \\
\Longrightarrow\ & x\ln(5) + 2\ln(5) = x\ln(4) && \text{(distributing)} \\
\Longrightarrow\ & x\ln(5) - x\ln(4) = -2\ln(5) && \text{(moving all terms containing } x \text{ to one side)} \\
\Longrightarrow\ & x(\ln(5) - \ln(4)) = -2\ln(5) && \text{(factoring out } x\text{)} \\
\Longrightarrow\ & x = \frac{-2\ln(5)}{\ln(5) - \ln(4)} && \text{(dividing both sides by } \ln(5) - \ln(4)\text{)}
\end{aligned}$$

Some could correctly argue that we could further simplify the final expression using the Laws of Logarithms.

$$x = \frac{-2\ln(5)}{\ln(5) - \ln(4)} = \frac{\ln\left(5^{-2}\right)}{\ln\left(\frac{5}{4}\right)}.$$

However, this is as correct as $\frac{-2\ln(5)}{\ln(5)-\ln(4)}$. In fact, you could go even further in your simplification to

$$x = \frac{\ln\left(5^{-2}\right)}{\ln\left(\frac{5}{4}\right)} = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}.$$

Be sure to check with your instructor as to their preference, but my guess is that any of these is acceptable as equivalent.

Example 4.6.6**: Solve an Equation of the Form** $y = Ae^{kt}$

$$\begin{aligned}
& 100 = 20e^{2t} && \\
\Longrightarrow\ & 5 = e^{2t} && \text{(dividing both sides by 20)} \\
\Longrightarrow\ & \ln(5) = \ln\left(e^{2t}\right) && \text{(taking the natural log of both sides)} \\
\Longrightarrow\ & \ln(5) = 2t && \text{(Laws of Logarithms)} \\
\Longrightarrow\ & \frac{\ln(5)}{2} = t && \text{(dividing both sides by 2)}
\end{aligned}$$

Using Laws of Logarithms, we can also write this answer in the form $t = \ln\sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

Example 4.6.7**: Solving an Equation That Can Be Simplified to the Form** $y = Ae^{kt}$

$$\begin{aligned} & 4e^{2x}+5 &=& 12 & \\ \Longrightarrow\quad & 4e^{2x} &=& 7 & \text{(subtracting 5 from both sides)} \\ \Longrightarrow\quad & e^{2x} &=& \frac{7}{4} & \text{(dividing both sides by 4)} \\ \Longrightarrow\quad & \ln\left(e^{2x}\right) &=& \ln\left(\frac{7}{4}\right) & \text{(taking the natural log of both sides)} \\ \Longrightarrow\quad & 2x &=& \ln\left(\frac{7}{4}\right) & \text{(Laws of Logarithms)} \\ \Longrightarrow\quad & x &=& \frac{\ln(7/4)}{2} & \text{(dividing both sides by 2)} \end{aligned}$$

Example 4.6.8**: Solving Exponential Equations in Quadratic Form**

$$\begin{aligned} & e^{2x}-e^{x} &=& 56 & \\ \Longrightarrow\quad & e^{2x}-e^{x}-56 &=& 0 & \text{(subtracting 56 from both sides)} \\ \Longrightarrow\quad & \left(e^{x}\right)^{2}-e^{x}-56 &=& 0 & \text{(Laws of Exponents)} \\ \Longrightarrow\quad & u^{2}-u-56 &=& 0 & \text{(substituting } u=e^{x}\text{)} \\ \Longrightarrow\quad & (u-8)(u+7) &=& 0 & \text{(factoring the trinomial)} \end{aligned}$$

Therefore, $u=8$ or $u=-7$. Resubstituting $u=e^{x}$, we get

$$e^{x}=8 \quad \text{or} \quad e^{x}=-7.$$

Since $e^{x}>0$, we throw out the possibility that $e^{x}=-7$. To solve $e^{x}=8$, we take the natural log of both sides to get $x=\ln(8)$.

Example 4.6.9**: Using Algebra to Solve a Logarithmic Equation**

$$\begin{aligned} & 2\ln(x)+3 &=& 7 & \\ \Longrightarrow\quad & 2\ln(x) &=& 4 & \text{(subtracting 3 from both sides)} \\ \Longrightarrow\quad & \ln(x) &=& 2 & \text{(dividing both sides by 2)} \\ \Longrightarrow\quad & x &=& e^{2} & \text{(rewriting in exponential form)} \end{aligned}$$

Example 4.6.10**: Using Algebra Before and After Using the Definition of the Natural Logarithm**

$$\begin{aligned} & 2\ln(6x) &=& 7 & \\ \Longrightarrow\quad & \ln(6x) &=& \frac{7}{2} & \text{(dividing both sides by 2)} \\ \Longrightarrow\quad & 6x &=& e^{7/2} & \text{(rewriting in exponential form)} \\ \Longrightarrow\quad & x &=& \frac{1}{6}e^{7/2} & \text{(dividing both sides by 6)} \end{aligned}$$

Example 4.6.11**: Showcasing the Need to Check Solutions**

$$\begin{aligned} & \log_{7}(1-2x) &=& 1-\log_{7}(3-x) & \\ \Longrightarrow\quad & \log_{7}(1-2x)+\log_{7}(3-x) &=& 1 & \text{(adding } \log_{7}(3-x) \text{ to both sides)} \\ \Longrightarrow\quad & \log_{7}\left((1-2x)(3-x)\right) &=& 1 & \text{(Laws of Logarithms)} \\ \Longrightarrow\quad & (1-2x)(3-x) &=& 7^{1} & \text{(rewriting in exponential form)} \\ \Longrightarrow\quad & 2x^{2}-7x+3 &=& 7 & \text{(distributing)} \\ \Longrightarrow\quad & 2x^{2}-7x-4 &=& 0 & \text{(subtracting 7 from both sides)} \\ \Longrightarrow\quad & (2x+1)(x-4) &=& 0 & \text{(factoring the trinomial)} \end{aligned}$$

Therefore, either $x=-\frac{1}{2}$ or $x=4$. We now check our candidate solutions (as we always must do when solving logarithmic equations).

Substituting $x=-\frac{1}{2}$ into the original equation, we get

$$\begin{aligned} & \log_{7}\left(1-2\left(-\frac{1}{2}\right)\right) &\stackrel{?}{=}& 1-\log_{7}\left(3-\left(-\frac{1}{2}\right)\right) & \text{(substituting)} \\ \Longrightarrow\quad & \log_{7}(1+1) &\stackrel{?}{=}& 1-\log_{7}\left(\frac{7}{2}\right) & \text{(simplifying)} \\ \Longrightarrow\quad & \log_{7}(2) &\stackrel{?}{=}& 1-\left(\log_{7}(7)-\log_{7}(2)\right) & \text{(simplifying and Laws of Logarithms)} \\ \Longrightarrow\quad & \log_{7}(2) &\stackrel{?}{=}& 1-\log_{7}(7)+\log_{7}(2) & \text{(distributing)} \\ \Longrightarrow\quad & \log_{7}(2) &\stackrel{?}{=}& 1-1+\log_{7}(2) & \text{(Laws of Logarithms)} \\ \Longrightarrow\quad & \log_{7}(2) &\stackrel{\checkmark}{=}& \log_{7}(2) & \text{(simplifying)} \end{aligned}$$

Thus, $x=-\frac{1}{2}$ is a solution.

Substituting $x=4$ into the original equation, we get the following.

$$\log_7(1-2(4)) \stackrel{?}{=} 1-\log_7(3-4)$$

$$\implies \log_7(-7) \stackrel{?}{=} \log_7(-1)$$

We can stop at this point and confidently state that $x=4$ is an extraneous solution.

WARNING: Our determination that $x=4$ is extraneous is *not* because $-7 \neq -1$. Instead, it is due to the fact that the argument of a logarithm can never be negative!

Hence, the only solution to the equation is $-\frac{1}{2}$.

Example $4.6.12$**: Using a Graph to Understand the Solution to a Logarithmic Equation**

$$\ln x = 3 \implies x = e^3.$$

Figure $4.6.3$ represents the graph of the equation. On the graph, the x-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3 \approx 20$. A calculator gives a better approximation: $e^3 \approx 20.0855$.

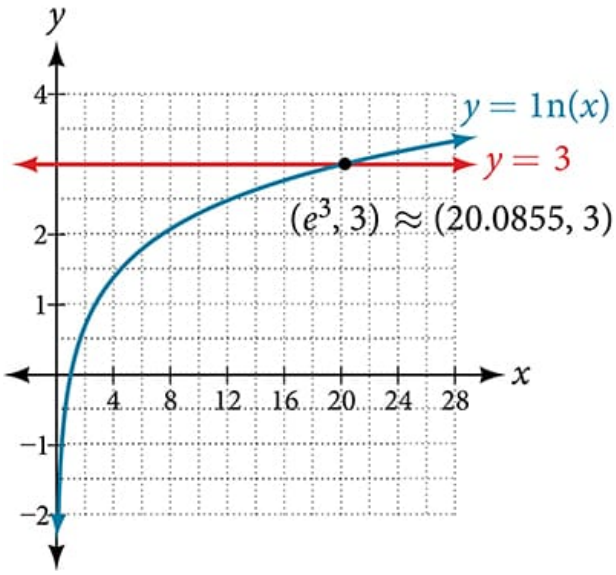


Figure $4.6.3$**:** *The graphs of* $y=\ln x$ *and* $y=3$ *cross at the point* $(e^3,3)$*, which is approximately* $(20.0855,3)$.

Example $4.6.13$**: Solving an Equation Using the One-to-One Property of Logarithms**

$$\begin{aligned} & \ln(x^2) = \ln(2x+3) & \\ \implies & x^2 = 2x+3 & \text{(One-to-One Property)} \\ \implies & x^2-2x-3 = 0 & \text{(subtracting } 2x \text{ and } 3 \text{ from both sides)} \\ \implies & (x-3)(x+1) = 0 & \text{(factoring the trinomial)} \end{aligned}$$

Therefore, $x=3$ or $x=-1$.

We leave it to the reader to check the solutions; however, it is important to note that, while the solution -1 is negative, it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

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4.6.1: Resources and Key Concepts

Resources

Videos

- Solving exponential equations
  - Exponential and Logarithmic Equations - Solving Exponential Equations and Inequalities
- Like bases property exponents
  - Intermediate Algebra - Exponential Functions: Solving Exponential Equations (no logs)
- Using logarithms to solve exponential equations
  - Exponential and Logarithmic Equations - Solving Exponential Equations Using Logarithms
  - Intermediate Algebra - Logarithms: Solving Any Exponential Equation
  - Intermediate Algebra - Exponential and Logarithmic Functions: Solving Equations
- Solving logarithmic equations
  - Exponential and Logarithmic Equations - Solving Logarithmic Equations
  - Exponential and Logarithmic Equations - Solving Logarithmic Equations and Inequalities

Key Concepts

Theorems

- **One-to-One Property of Exponents**: For any algebraic expressions S and T, and any positive real number $b \neq 1$, $b^S = b^T$ if and only if $S = T$.

Common Mistakes

- **Incorrectly Applying the One-to-One Property of Exponents**: Attempting to equate exponents when the bases are not the same, without first rewriting them to have a common base.
- **Errors in Using Logarithms to Solve Exponential Equations**: Incorrectly applying logarithm laws, especially the power rule (e.g., $\ln(5^{x+2}) = (x+2)\ln(5)$, not $x\ln(5) + 2$).
- **Forgetting to Check for Extraneous Solutions in Logarithmic Equations**: Failing to verify that all solutions obtained make the arguments of the original logarithms positive.
- **Errors in Using the One-to-One Property of Logarithms**: Not first condensing logarithmic expressions on each side of the equation to a single logarithm before equating arguments.
- **Treating $\ln(A - B)$ as $\ln(A) - \ln(B)$ or $\frac{\ln(A)}{\ln(B)}$**: Incorrectly splitting logarithms of differences or quotients of logarithms. (Common algebraic errors with log laws)
- **Solving Equations Quadratic-in-Form Incorrectly**: Forgetting to substitute back after solving for u in an equation like $e^{2x} - e^x - 56 = 0$, or incorrectly solving the resulting exponential/logarithmic equations after substitution.

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4.6.2: Homework

Reading Questions

1. What is the One-to-One Property of Exponents, and how is it used to solve exponential equations?
2. If an exponential equation cannot be easily rewritten with like bases, what is the general strategy for solving it using logarithms?
3. When solving an exponential equation by taking the logarithm of both sides, does it matter which base of logarithm you use (e.g., common log vs. natural log)?
4. What is an extraneous solution in the context of solving exponential or logarithmic equations? Why do they sometimes occur?
5. How is the definition of a logarithm used to solve logarithmic equations of the form $\log_b(S) = c$?
6. What is the One-to-One Property of Logarithms, and how can it be used to solve logarithmic equations? What must be true about the bases of the logarithms?
7. Before applying the One-to-One Property of Logarithms to an equation like $\log(3x-2) - \log(2) = \log(x+4)$, what must you do first?
8. Why is it essential to check solutions when solving logarithmic equations?
9. Describe how to solve an equation like $e^{2x} - e^x = 56$ which is quadratic in form.
10. If you solve a logarithmic equation and obtain a candidate solution that makes an argument of a logarithm in the original equation negative or zero, what do you conclude about that candidate solution?
11. How can an exponential equation be solved?
12. When does an extraneous solution occur? How can an extraneous solution be recognized?
13. When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

Homework

Algebraic

For the following exercises, use like bases to solve the exponential equation.

14. $4^{-3v-2} = 4^{-v}$
15. $64 \cdot 4^{3x} = 16$
16. $3^{2x+1} \cdot 3^x = 243$
17. $2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$
18. $625 \cdot 5^{3x+3} = 125$
19. $\frac{36^{3b}}{36^{2b}} = 216^{2-b}$
20. $\left(\frac{1}{64}\right)^{3n} \cdot 8 = 2^6$

For the following exercises, use logarithms to solve.

21. $9^{x-10} = 1$
22. $2e^{6x} = 13$
23. $e^{r+10} - 10 = -42$
24. $2 \cdot 10^{9a} = 29$
25. $-8 \cdot 10^{p+7} - 7 = -24$
26. $7e^{3n-5} + 5 = -89$
27. $e^{-3k} + 6 = 44$
28. $-5e^{9x-8} - 8 = -62$
29. $-6e^{9x+8} + 2 = -74$
30. $2^{x+1} = 5^{2x-1}$
31. $e^{2x} - e^x - 132 = 0$
32. $7e^{8x+8} - 5 = -95$
33. $10e^{8x+3} + 2 = 8$
34. $4e^{3x+3} - 7 = 53$

35. $8e^{-5x-2} - 4 = -90$
36. $3^{2x+1} = 7^{x-2}$
37. $e^{2x} - e^x - 6 = 0$
38. $3e^{3-3x} + 6 = -31$

For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.

39. $\log\left(\frac{1}{100}\right) = -2$
40. $\log_{324}(18) = \frac{1}{2}$

For the following exercises, use the definition of a logarithm to solve the equation.

41. $5\log_7(n) = 10$
42. $-8\log_9(x) = 16$
43. $4 + \log_2(9k) = 2$
44. $2\log(8n + 4) + 6 = 10$
45. $10 - 4\ln(9 - 8x) = 6$

For the following exercises, use the one-to-one property of logarithms to solve.

46. $\ln(10 - 3x) = \ln(-4x)$
47. $\log_{13}(5n - 2) = \log_{13}(8 - 5n)$
48. $\log(x + 3) - \log(x) = \log(74)$
49. $\ln(-3x) = \ln(x^2 - 6x)$
50. $\log_4(6 - m) = \log_4(3m)$
51. $\ln(x - 2) - \ln(x) = \ln(54)$
52. $\log_9(2n^2 - 14n) = \log_9(-45 + n^2)$
53. $\ln(x^2 - 10) + \ln(9) = \ln(10)$

For the following exercises, solve each equation for x.

54. $\log(x + 12) = \log(x) + \log(12)$
55. $\ln(x) + \ln(x - 3) = \ln(7x)$
56. $\log_2(7x + 6) = 3$
57. $\ln(7) + \ln(2 - 4x^2) = \ln(14)$
58. $\log_8(x + 6) - \log_8(x) = \log_8(58)$
59. $\ln(3) - \ln(3 - 3x) = \ln(4)$
60. $\log_3(3x) - \log_3(6) = \log_3(77)$

Graphical

For the following exercises, solve the equation for x, if there is a solution. Then graph both sides of the equation, and observe the point of intersection (if it exists) to verify the solution.

61. $\log_9(x) - 5 = -4$
62. $\log_3(x) + 3 = 2$
63. $\ln(3x) = 2$
64. $\ln(x - 5) = 1$
65. $\log(4) + \log(-5x) = 2$
66. $-7 + \log_3(4 - x) = -6$
67. $\ln(4x - 10) - 6 = -5$
68. $\log(4 - 2x) = \log(-4x)$
69. $\log_{11}(-2x^2 - 7x) = \log_{11}(x - 2)$
70. $\ln(2x + 9) = \ln(-5x)$
71. $\log_9(3 - x) = \log_9(4x - 8)$
72. $\log(x^2 + 13) = \log(7x + 3)$
73. $\frac{3}{\log_2(10)} - \log(x - 9) = \log(44)$
74. $\ln(x) - \ln(x + 3) = \ln(6)$

For the following exercises, solve for the indicated value, and graph the situation showing the solution point.

75. An account with an initial deposit of $6,500 earns 7.25% annual interest, compounded continuously. How much will the account be worth after 20 years?
76. The formula for measuring sound intensity in decibels D is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8.3 \cdot 10^2$ watts per square meter?
77. The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach 20,000?

Technology

For the following exercises, solve each equation by rewriting the exponential expression using the indicated logarithm. Then use a calculator to approximate the variable to 3 decimal places.

78. $1000(1.03)^t = 5000$ using the common log
79. $e^{5x} = 17$ using the natural log
80. $3(1.04)^{3t} = 8$ using the common log
81. $3^{4x-5} = 38$ using the common log
82. $50e^{-0.12t} = 10$ using the natural log

For the following exercises, use a calculator to solve the equation. Unless indicated otherwise, round all answers to the nearest ten-thousandth.

83. $7e^{3x-5} + 7.9 = 47$
84. $\ln(3) + \ln(4.4x + 6.8) = 2$
85. $\log(-0.7x - 9) = 1 + 5\log(5)$
86. Atmospheric pressure P in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where x is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of 8.369 pounds per square inch? (*Hint*: there are 5,280 feet in a mile)
87. The magnitude M of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where E is the amount of energy released by the earthquake in joules and $E_0 = 10^{4.4}$ is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4 \cdot 10^{13}$ joules of energy?

Extensions

88. Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b(x)} = x$.
89. Recall the formula for continually compounding interest, $y = Ae^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.
90. Recall the compound interest formula $A = a\left(1 + \frac{r}{k}\right)^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t.
91. Newton's Law of Cooling states that the temperature T of an object at any time t can be described by the equation $T = T_s + (T_0 - T_s)e^{-kt}$, where T_s is the temperature of the surrounding environment, T_0 is the initial temperature of the object, and k is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

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4.7: Exponential and Logarithmic Models

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

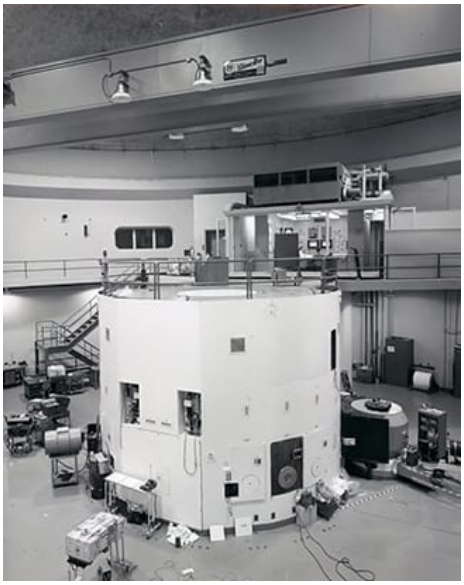


Figure 4.7.1**:** *A nuclear research reactor inside the Neely Nuclear Research Center on the Georgia Institute of Technology campus (credit: Georgia Tech Research Institute)*

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and **Newton's Law of Cooling**.

Modeling Exponential Growth and Decay

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function $P(t) = P_0 e^{kt}$, where P_0 is equal to the value at time zero, e is Euler's constant, and k is a positive constant that determines the rate (percentage) of growth. We may use the exponential growth function in applications involving **doubling time**, the time it takes for a quantity to double.

Definition: Doubling Time

The time it takes for a quantity undergoing exponential growth to double is called the quantity's **doubling time**.

Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit growth

based on a doubling time.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the exponential decay model. Again, we have the form $P(t) = P_0 e^{kt}$ where P_0 is the starting value, and e is Euler's constant. Now k is a negative constant that determines the rate of decay. We may use the exponential decay model when we are calculating **half-life**, or the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

Definition: Half-Life

The time it takes for a quantity undergoing exponential decay to decay by half is called the quantity's **half-life**.

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape, as we can see in Figure $4.7.2$ and Figure $4.7.3$. It is important to remember that, although parts of each of the two graphs seem to lie on the x-axis, they are really a tiny distance above the x-axis.

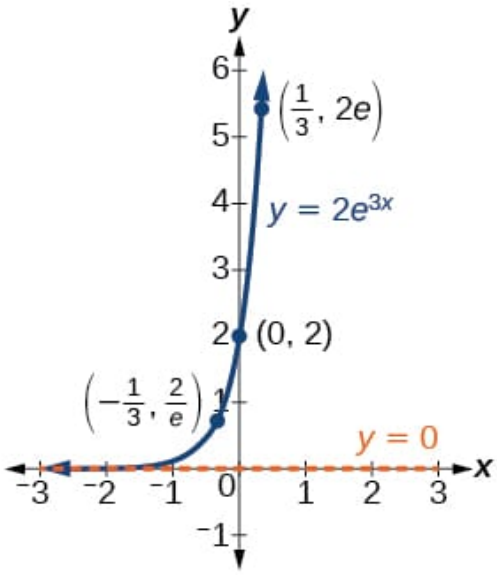


Figure $4.7.2$**:** *A graph showing exponential growth. The equation is* $y = 2e^{3x}$.

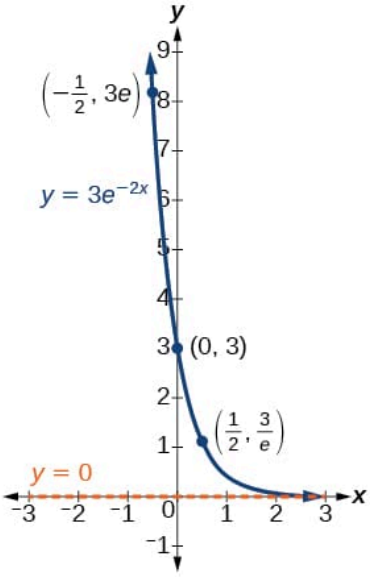


Figure $4.7.3$**:** *A graph showing exponential decay. The equation is* $y = 3e^{-2x}$.

Exponential growth and decay often involve very large or very small numbers. To describe these numbers, we often use orders of magnitude.

Definition: Order of Magnitude

The **order of magnitude** of a number is the exponent of 10 when the number is written in scientific notation.

For example, the distance to the nearest star, Proxima Centauri, measured in kilometers, is $40,113,497,200,000$ kilometers. Expressed in scientific notation, this is $4.01134972 \times 10^{13}$. So, we could describe this number as having order of magnitude 13.

Example 4.7.1: Graphing Exponential Growth

A population of bacteria doubles every hour. If the culture started with 10 bacteria, graph the population as a function of time.

Solution available at the end of this section.

Half-Life

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its half-life. Table $4.7.1$ lists the half-life for several of the more common radioactive substances.

Table $4.7.1$

Substance	Use	Half-life
gallium-67	nuclear medicine	80 hours
cobalt-60	manufacturing	5.3 years
technetium-99m	nuclear medicine	6 hours
americium-241	construction	432 years
carbon-14	archeological dating	5,715 years
uranium-235	atomic power	703,800,000 years

To find the half-life of a function describing exponential decay, we need to know how long it takes for the initial population, P_0, to decay by half to $\frac{1}{2}P_0$. This requires us to solve the following equation:

$$\frac{1}{2}P_0 = P_0 e^{kt}.$$

We find that the half-life depends only on the constant k and not on the starting quantity P_0.

The formula is derived as follows:

$$\begin{aligned}
\frac{1}{2}P_0 &= P_0 e^{kt} \\
\Longrightarrow \quad \frac{1}{2} &= e^{kt} && \text{(dividing both sides by } P_0\text{)} \\
\Longrightarrow \quad \ln\left(\frac{1}{2}\right) &= kt && \text{(taking the natural logarithm of both sides)} \\
\Longrightarrow \quad \frac{\ln(1/2)}{k} &= t && \text{(dividing both sides by } k\text{)} \\
\Longrightarrow \quad \frac{\ln(2^{-1})}{k} &= t && \text{(Laws of Exponents)} \\
\Longrightarrow \quad -\frac{\ln(2)}{k} &= t && \text{(Laws of Logarithms)}
\end{aligned}$$

Since t, the time, is positive, k must, as expected, be negative. This gives us the half-life formula.

Theorem: Half-Life Formula

If an exponentially decaying population is modeled by the function $P(t) = P_0 e^{kt}$, then the time it takes for the population to decay by half is

$$t = -\frac{\ln(2)}{k}.$$

While most students tend to memorize this formula, you are expected to understand it and derive it, if needed.

Example 4.7.2: Finding the Function that Describes Radioactive Decay

The half-life of carbon-14 is 5,730 years. Express the amount of carbon-14 remaining as a function of time, t.

Solution available at the end of this section.

Checkpoint 4.7.2

The half-life of plutonium-244 is 80,000,000 years. Find a function that gives the amount of plutonium-244 remaining as a function of time, measured in years.

Radiocarbon Dating

The formula for exponential decay is important in radiocarbon dating, which is used to calculate the approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby, who won a Nobel Prize for his discovery. It compares the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about 1% error for plants or animals that died within the last 60,000 years.

Carbon-14 is a radioactive isotope of carbon that has a half-life of 5,730 years. It occurs in small quantities in the carbon dioxide in the air we breathe. Most of the carbon on Earth is carbon-12, which has an atomic weight of 12 and is not radioactive. Scientists have determined the ratio of carbon-14 to carbon-12 in the air for the last 60,000 years, using tree rings and other organic samples of known dates—although the ratio has changed slightly over the centuries.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Since the half-life of carbon-14 is 5,730 years, we derive a formula for the amount present at any time t as follows:

$$
\begin{aligned}
& P(t) = P_0 e^{kt} \\
\Longrightarrow \quad & \frac{1}{2} P_0 = P_0 e^{k \cdot 5730} && \text{(after 5,730 years, half the initial amount remains)} \\
\Longrightarrow \quad & \frac{1}{2} = e^{5730k} && \text{(dividing both sides by } P_0\text{)} \\
\Longrightarrow \quad & \ln\left(\tfrac{1}{2}\right) = 5730k && \text{(taking the natural logarithm of both sides)} \\
\Longrightarrow \quad & \frac{\ln(0.5)}{5730} = k && \text{(dividing both sides by 5730)}
\end{aligned}
$$

Thus, the formula for the amount of carbon-14 remaining after t years is

$$P(t) \approx P_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t},$$

where

- $P(t)$ is the amount of carbon-14 remaining at time t, and
- P_0 is the amount of carbon-14 when the plant or animal began decaying.

Again, I *highly* discourage you from memorizing this formula—it's fast to derive, and the half-life changes based on substance given. It is much preferred to derive the formula each time.

Example 4.7.3: Finding the Age of a Bone

A bone fragment is found that contains 20% of its original carbon-14. To the nearest year, how old is the bone?

Solution available at the end of this section.

In reality, the instruments that measure the percentage of carbon-14 are extremely sensitive and a scientist would need to do much more work than we did in order to be satisfied. Even so, carbon dating is only accurate to about 1%, so the age of the sample from Example 4.7.3 should be given as

$$13,301 \text{ years } \pm \text{ } 113,301 \text{ years } = 13,301 \pm 133 \text{ years.}$$

Checkpoint 4.7.3

Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

Calculating Doubling Time

For decaying quantities, we determined how long it took for half of a substance to decay. For growing quantities, we might want to find out how long it takes for a quantity to double. As we mentioned above, the time it takes for a quantity to double is called the doubling time.

Given the basic exponential growth equation $P(t) = P_0 e^{kt}$, doubling time can be found by solving for when the original quantity has doubled—that is, by solving $2P_0 = P_0 e^{kt}$. Rather than attracting you to memorize a formula, let's show how to derive it.

Example 4.7.4: Finding a Function That Describes Exponential Growth

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

Solution available at the end of this section.

Checkpoint 4.7.4

Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.

Newton's Law of Cooling

Exponential decay can also be applied to temperature. When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature. On a graph of the temperature function, the leveling off will correspond to a horizontal asymptote at the temperature of the surrounding air. Unless the room temperature is zero, this will correspond to a vertical shift of the generic exponential decay function. This translation leads to **Newton's Law of Cooling**, the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature.

Theorem: Newton's Law of Cooling

The temperature of an object, T, in a surrounding medium having temperature T_s will behave according to the formula

$$T(t) = T_s + (T_0 - T_s)e^{kt},$$

where t is the time the object has been immersed in the medium, T_0 is the temperature of the object at time 0 (usually before being immersed in the medium), and k is a constant called the **continuous rate of cooling** of the object.

Example 4.7.5: Using Newton's Law of Cooling

A cheesecake is taken out of the oven with an ideal internal temperature of 165°F and is placed into a 35°F refrigerator. After 10 minutes, the cheesecake has cooled to 150°F. If we must wait until the cheesecake has cooled to 70°F before we eat it, how long will we have to wait?

Solution available at the end of this section.

Checkpoint 4.7.5

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Using Logistic Growth Models

Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Consider an aspiring writer who writes a single line on day one and plans to double the number of lines she writes each day for a month. By the end of the month, she must write over 17 billion lines, or one-half-billion pages. It is impractical, if not impossible, for anyone to write that much in such a short period of time. Eventually, an exponential model must begin to approach some limiting value, and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an exponential growth model, though the exponential growth model is still useful over a short term, before approaching the limiting value.

Definition: Logistic Growth Model

A **logistic growth model** is any model that can be written in the form

$$P(t) = \frac{c}{1 + ae^{-bt}},$$

where a, b, and c are constants. c is called the **carrying capacity** (also known as the **limiting value**) of the population, b is a constant determined by the rate of growth, and the ratio $\frac{c}{1+a}$ is the initial value.

The logistic growth model is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, or carrying capacity. The graph in Figure $4.7.5$ shows how the growth rate changes over time. The graph increases from left to right, but the growth rate only increases until it reaches its point of maximum growth rate, at which point the rate of increase decreases.

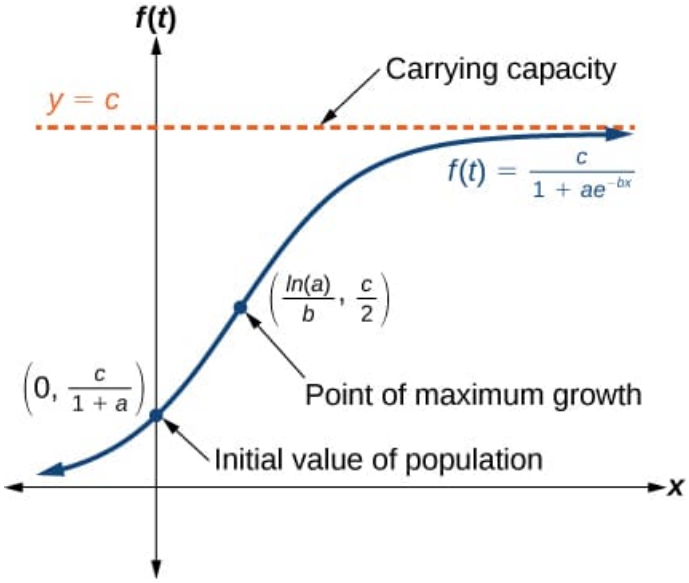


Figure $4.7.5$

Example $4.7.6$: Using the Logistic-Growth Model

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors: The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of 1,000 people who has the flu. So, in that community, at most 1,000 people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is $b = 0.6030$. Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.

Solution available at the end of this section.

Checkpoint $4.7.6$

Using the model in Example $4.7.6$, estimate the number of cases of flu on day 15.

Expressing an Exponential Model in Base e

While powers and logarithms of any base can be used in modeling, the two most common bases are 10 and e. In Science and Mathematics, the base e is often preferred. We can use Laws of Exponents and Laws of Logarithms to change any base to base e.

Given a model with the form $y = ab^x$, we change it to the form $y = A_0e^{kx}$ using the following steps.

1. Rewrite $y = ab^x$ as $y = ae^{\ln(b^x)}$.
2. Use the Power Rule of Logarithms to rewrite y as $y = ae^{x\ln(b)} = ae^{\ln(b)x}$.
3. Note that $a = A_0$ and $k = \ln(b)$ in the equation $y = A_0e^{kx}$.

Example 4.7.7: Changing to base e

Change the function $y = 2.5(3.1)^x$ so that this same function is written in the form $y = A_0e^{kx}$.

Solution available at the end of this section.

Checkpoint 4.7.7

Change the function $y = 3(0.5)^x$ to one having e as the base.

Logarithmic Scales

For quantities that vary greatly in magnitude, a standard scale of measurement is not always effective, and utilizing logarithms can make the values more manageable. For example, if the average distances from the sun to the major bodies in our solar system are listed, you see they vary greatly.

Table 4.7.2

Planet	Distance (millions of km)
Mercury	58
Venus	108
Earth	150
Mars	228
Jupiter	779
Saturn	1430
Uranus	2880
Neptune	4500

Placed on a linear scale—one with equally spaced values—these values get bunched up (see Figure 4.7.7).

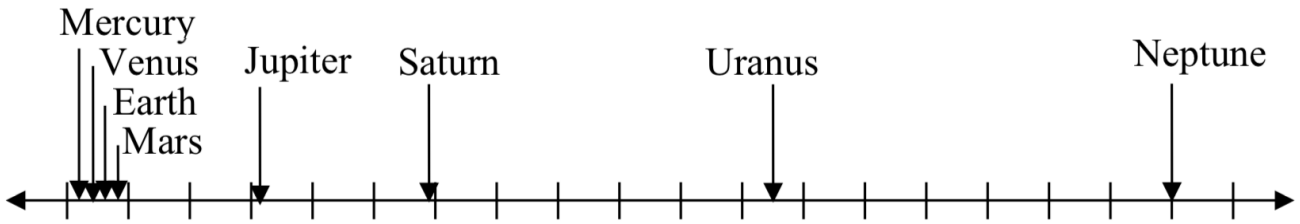


Figure 4.7.7: *Linear scale of astronomical distances in our solar system.*

However, computing the logarithm of each value and plotting these new values on a number line results in a more manageable graph, and makes the relative distances more apparent, as seen in Figure 4.7.8 below Table 4.7.3. (It is interesting to note the large gap between Mars and Jupiter on the log number line. The asteroid belt is located there, which scientists believe is a planet that never formed because of the effects of the gravity of Jupiter.)

Table 4.7.3

Planet	Distance (millions of km)	log(distance)
Mercury	58	1.76
Venus	108	2.03
Earth	150	2.18
Mars	228	2.36
Jupiter	779	2.89
Saturn	1430	3.16
Uranus	2880	3.46
Neptune	4500	3.65

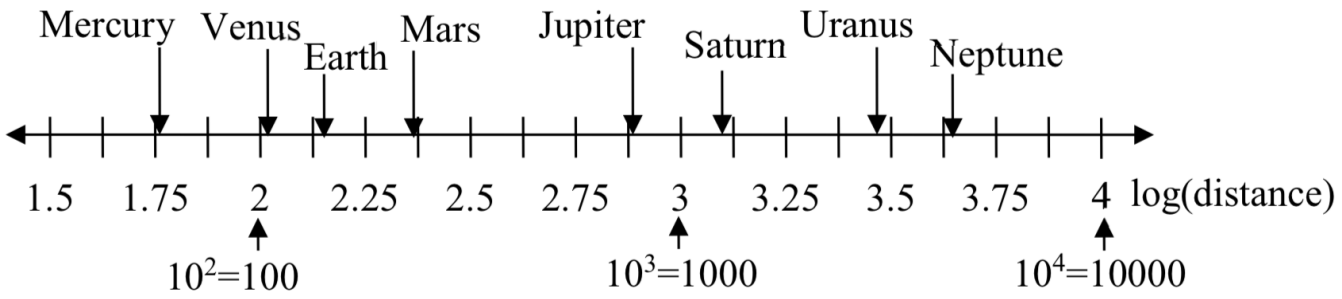


Figure 4.7.8**:** *Logarithmic scale of astronomical distances in our solar system.*

Sometimes, as shown above in Figure 4.7.8, the scale on a logarithmic number line will show the log values, but more commonly the original values are listed as powers of 10, as shown below in Figure 4.7.9.

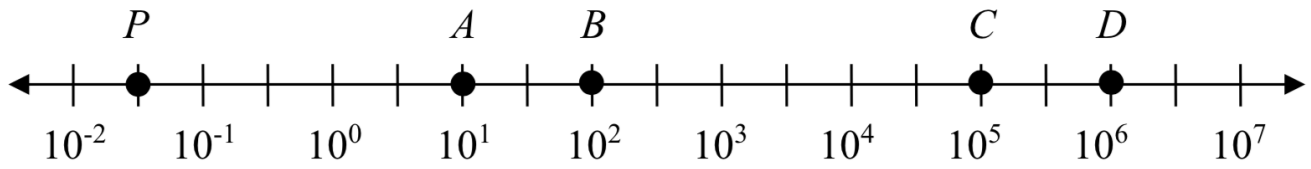


Figure 4.7.9

Example 4.7.8

Estimate the value of point P on the log scale above

Solution available at the end of this section.

Example 4.7.9

Place the number 6000 on a logarithmic scale.

Solution available at the end of this section.

Checkpoint 4.7.9

Plot the data in the table below on a logarithmic scale (From http://www.epd.gov.hk/epd/noise_educ...1/intro_5.html, retrieved Oct 2, 2010).

Table 4.7.4

Source of Sound/Noise	Approximate Sound Pressure in μ Pa (micro Pascals)
Launching of the Space Shuttle	2000,000,000
Full Symphony Orchestra	2000,000
Diesel Freight Train at High Speed at 25 m	200,000
Normal Conversation	20,000
Soft Whispering at 2 m in Library	2,000
Unoccupied Broadcast Studio	200
Softest Sound a human can hear	20

Answer available at the end of this section.

Notice that on the log scale above Figure 4.7.9, the visual distance on the scale between points A and B and between C and D is the same. When looking at the values these points correspond to, notice B is ten times the value of A, and D is ten times the value of C. A visual (linear) difference between points corresponds to a relative (ratio) change between the corresponding values.

Logarithms are useful for showing these relative changes. For example, comparing \$1,000,000 to \$10,000, the first is 100 times larger than the second.

$$\frac{1,000,000}{10,000} = 100 = 10^2.$$

Likewise, comparing \$1000 to \$10, the first is 100 times larger than the second.

$$\frac{1,000}{10} = 100 = 10^2.$$

When one quantity is roughly ten times larger than another, its exponent when written in scientific notation increases by 1. Thus, we say it is one order of magnitude larger. In both cases described above, the first number was two orders of magnitude larger than the second.

Notice that the order of magnitude can be found as the common logarithm of the ratio of the quantities. On the log scale above, B is one order of magnitude larger than A, and D is one order of magnitude larger than C.

Theorem: Determining the Order of Magnitude Between Two Numbers

Given two values A and B, the number of orders of magnitude A is greater than B is

$$\log\left(\frac{A}{B}\right).$$

Example 4.7.10

On the log scale in Figure 4.7.9, how many orders of magnitude larger is C than B?

Solution available at the end of this section.

Checkpoint 4.7.10

Using the table from Checkpoint 4.7.9, what is the difference of order of magnitude between the softest sound a human can hear and the launching of the space shuttle?

Answer available at the end of this section.

Earthquakes

An example of a logarithmic scale is the **Richter Scale** used for earthquakes. While not the only scale for measuring earthquakes, it is the most commonly known. The Richter scale compares the amplitude, A, of the earthquake's seismographic trace with the amplitude, A_0, of the smallest detectable earthquake.

Definition: Richter Scale

The **Richter scale** is a logarithmic scale used to express the magnitude of an earthquake, specifically the energy released during it. For an earthquake with **seismic trace amplitude** A (a measurement from the baseline of the seismograph), the **Richter value**, or **magnitude** of the earthquake, is defined to be

$$M = \log\left(\frac{A}{A_0}\right),$$

where A_0 is a baseline measure for the seismograph.

Example 4.7.11

If one earthquake has a Richter value of 6.0, and another has a magnitude of 8.0, how much more powerful (in terms of earth movement) is the second earthquake?

Solution available at the end of this section.

Example 4.7.12

One earthquake has magnitude of 3.0. If a second earthquake has twice as much earth movement as the first earthquake, find the magnitude of the second quake.

Solution available at the end of this section.

In fact, using the Laws of Logarithms, we could show that whenever the earth movement doubles, the Richter magnitude will increase by about 0.301:

$$\begin{aligned} M &= \log\left(\frac{2A}{A_0}\right) \\ &= \log\left(2 \cdot \frac{A}{A_0}\right) \\ &= \log(2) + \log\left(\frac{A}{A_0}\right) \\ &\approx 0.3010 + \log\left(\frac{A}{A_0}\right) \end{aligned}$$

This illustrates the most important feature of a log scale: that multiplying the quantity being considered will add to the scale value, and vice versa.

Logarithms are useful in measuring the intensities of many quantities based on senses. Earthquakes (the Richter scale), sound (decibels) and acids and bases (pH) are common examples. We now present yet a different use of the basic logarithm function, password strength.

Example 4.7.13

The information entropy H, in bits, of a randomly generated password consisting of L characters is given by

$$H = L\log_2(N),$$

where N is the number of possible symbols for each character in the password. In general, the higher the entropy, the stronger the password.

a. If a 7 character case-sensitive password is comprised of letters and numbers only, find the associated information entropy.
b. How many possible symbol options per character are required to produce a 7 character password with an information entropy of 50 bits?

Solutions available at the end of this section.

Chemical systems known as buffer solutions can adjust to small changes in acidity to maintain a range of pH values. Buffer solutions have a wide variety of applications, from maintaining a healthy fish tank to regulating the pH levels in blood.

Example 4.7.14

Blood is a buffer solution. When carbon dioxide is absorbed into the bloodstream, it produces carbonic acid and lowers the pH. The body compensates by producing bicarbonate, a weak base, to partially neutralize the acid. The equation which models blood pH in this situation is

$$\text{pH} = 6.1 + \log\left(\frac{800}{x}\right),$$

where x is the partial pressure of carbon dioxide in arterial blood, measured in torr. Find the partial pressure of carbon dioxide in arterial blood if the pH is 7.4.

Solution available at the end of this section.

Success in Mathematics: Process-Oriented Learning Preferences

Choose the statements that best describe you.

I learn better if I know why I'm learning something.	or	I don't need to know why I'm doing something before learning how to do it.
I find it annoying to learn something I don't think I will use anywhere else.	or	I believe that learning new things is always worth it because it exercises my brain.
I naturally think in steps.	or	It is hard for me to learn something step by step.

1. Do you consider your choices above good or bad for your education?
2. In specific textbooks, many procedures are summarized as a list of steps. This list is shown in a separate box. Do you refer to these boxes when you do your homework? Why or why not?

Miscellaneous Topics: How Does Math Relate to Your Field?

1. Please tell me what degree you are currently pursuing.
2. Name at least five things in your chosen field with which mathematics can help.

Answers

Section Preview

This is the culmination of the exponential and logarithmic function material—skipping this would be criminal. If any area in Mathematics deserves extra time for applications, it's this chapter.

Section Preview

- **Sets and Numbers**
  - Scientific Notation: Example 1 expresses populations such as $10{,}240$ and $10{,}485{,}760$ as 1.024×10^4 and 1.049×10^7; converting between standard and scientific form is needed to track orders of magnitude.

Section Preview

- Model exponential growth and decay.
- Use Newton's Law of Cooling.
- Use logistic-growth models.
- Express an exponential model in base e.
- Use logarithmic models.

Example $4.7.1$: Graphing Exponential Growth

When an amount grows at a fixed percent per unit time, the growth is exponential. To find P_0 we use the fact that P_0 is the amount at time zero, so $P_0 = 10$. To find k, use the fact that after one hour ($t = 1$) the population doubles from 10 to 20. The formula is derived as follows

$$\begin{aligned} 20 &= 10e^{k \cdot 1} && \text{(substituting)} \\ \Longrightarrow \quad 2 &= e^k && \text{(dividing both sides by 10)} \\ \Longrightarrow \quad \ln 2 &= k && \text{(applying the natural logarithm to both sides)} \end{aligned}$$

So $k = \ln(2)$. Thus, the equation we want to graph is

$$y = 10e^{(\ln 2)t} = 10\left(e^{\ln 2}\right)^t = 10 \cdot 2^t.$$

The graph is shown in Figure $4.7.4$.

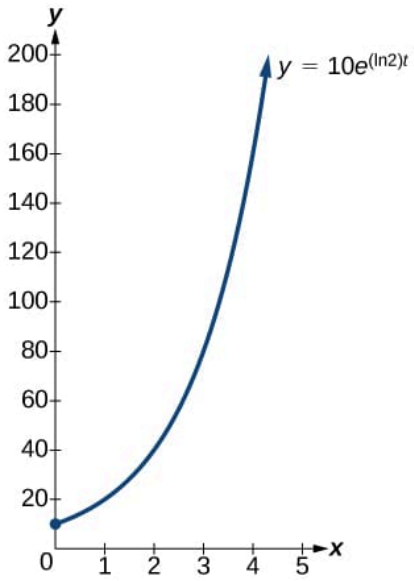


Figure $4.7.4$: *The graph of $P(t) = 10e^{(\ln 2)t}$*

The population of bacteria after ten hours is $10,240$. We could describe this amount as being an order of magnitude 4 because it can be written as 1.024×10^4. The population of bacteria after twenty hours is $10,485,760 \approx 1.049 \times 10^7$, which is an order of magnitude of 7, so we could say that the population has increased by three orders of magnitude in ten hours.

Example $4.7.2$: Finding the Function that Describes Radioactive Decay

This formula is derived as follows:

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ \implies \frac{1}{2}P_0 &= P_0 e^{k\cdot 5730} && \text{(substituting)} \\ \implies \frac{1}{2} &= e^{k\cdot 5730} && \text{(dividing both sides by } P_0\text{)} \\ \implies \ln\left(\frac{1}{2}\right) &= 5730k && \text{(taking the natural logarithm of both sides)} \\ \implies \frac{\ln(1/2)}{5730} &= k && \text{(dividing both sides by 5730)} \end{aligned}$$

Substituting this back into the original function, we get

$$P(t) = P_0 e^{\left(\frac{\ln(1/2)}{5730}\right)t}.$$

We observe that the coefficient of t, $\frac{\ln(0.5)}{5730} \approx -1.2097 \times 10^{-4}$, is negative, as expected in the case of exponential decay.

Example $4.7.3$**: Finding the Age of a Bone**

We are told that 20% of an initial sample remains. That is, the current amount of carbon-14 in the sample is $0.20P_0$. Therefore, we want to solve $0.2P_0 = P_0 e^{kt}$. This gives

$$0.2 = e^{kt} \implies \ln 0.2 = kt \implies t = \frac{\ln 0.2}{k}.$$

Since we already have computed k for carbon-14 earlier, we substitute that value into this formula:

$$t = \frac{\ln 0.2}{k} = \frac{\ln 0.2}{\frac{\ln 0.5}{5730}} = \frac{5730\ln(0.2)}{\ln 0.5} \approx 13305$$

The bone fragment is about 13,305 years old.

Example $4.7.4$**: Finding a Function That Describes Exponential Growth**

The formula is derived as follows:

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ \implies 2P_0 &= P_0 e^{k\cdot 2} && \text{(substituting } t = 2 \text{ and } P(2) = 2P_0\text{)} \\ \implies 2 &= e^{2k} && \text{(dividing both sides by } P_0\text{)} \\ \implies \ln 2 &= 2k && \text{(taking the natural log of both sides)} \\ \implies \frac{\ln 2}{2} &= k && \text{(dividing both sides by 2)} \end{aligned}$$

Thus, the function is

$$P(t) = P_0 e^{(\ln(2)/2)t}.$$

Example $4.7.5$**: Using Newton's Law of Cooling**

Because the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially toward 35, following the equation

$$T(t) = 35 + (T_0 - 35)e^{kt}.$$

We know the initial temperature was 165, so $T(0) = T_0 = 165$. Therefore,

$$T(t) = 35 + (165 - 35)e^{k\cdot t} = 35 + 130e^{kt}.$$

We were given another data point, $T(10) = 150$, which we can use to solve for k.

$$\begin{aligned} 150 &= 35 + 130e^{k\cdot 10} && \text{(subsituting the point } (10, 150)\text{)} \\ \implies 115 &= 130e^{10k} && \text{(subtracting 35 from both sides)} \\ \implies \frac{115}{130} &= e^{10k} && \text{(dividing both sides by 130)} \\ \implies \ln\left(\tfrac{115}{130}\right) &= 10k && \text{(taking the natural log of both sides)} \\ \implies \frac{\ln\left(\frac{115}{130}\right)}{10} &= k && \text{(dividing both sides by 10)} \end{aligned}$$

Hence, $k \approx -0.0123$. This gives us the equation for the cooling of the cheesecake:

$$T(t) \approx 35 + 130e^{-0.0123t}.$$

Now we can solve for the time it will take for the temperature to cool to 70 degrees. To keep numbers as exact as possible, we wait until the end to replace k with our found value.

$$
\begin{aligned}
70 &= 35 + 130e^{kt} && \text{(substituting 70 for } T(t)\text{)} \\
\Longrightarrow \quad 35 &= 130e^{kt} && \text{(subtracting 35 from both sides)} \\
\Longrightarrow \quad \frac{35}{130} &= e^{kt} && \text{(dividing both sides by 130)} \\
\Longrightarrow \quad \ln\left(\tfrac{35}{130}\right) &= kt && \text{(taking the natural log of both sides)} \\
\Longrightarrow \quad \frac{\ln(35/130)}{k} &= t && \text{(dividing both sides by } k\text{)} \\
\Longrightarrow \quad \frac{\ln(35/130)}{\frac{\ln(115/130)}{10}} &= t && \left(\text{substituting } k = \frac{\ln\left(\frac{115}{130}\right)}{10}\right)
\end{aligned}
$$

Computing this value, we get $t \approx 106.68$. Thus, it will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool to 70°F.

Example $4.7.6$**: Using the Logistic-Growth Model**

We substitute the given data into the logistic growth model

$$f(t) = \frac{c}{1 + ae^{-bt}}.$$

Because at most 1,000 people, the entire population of the community, can get the flu, we know the limiting value is $c = 1000$. To find a, we use the formula that the number of cases at time $t = 0$ is $\frac{c}{1+a} = 1$, from which it follows that $a = 999$. This model predicts that, after ten days, the number of people who have had the flu is $f(10) = \frac{1000}{1+999e^{-0.6030(10)}} \approx 293.9$. Because the actual number must be a whole number (a person has either had the flu or not) we round *down* to 293. In the long term, the number of people who will contract the flu is the limiting value, $c = 1000$.

Remember that, because we are dealing with a virus, we cannot predict with certainty the number of people infected. The model only approximates the number of people infected and will not give us exact or actual values.

The graph in Figure $4.7.6$ gives a good picture of how this model fits the data.

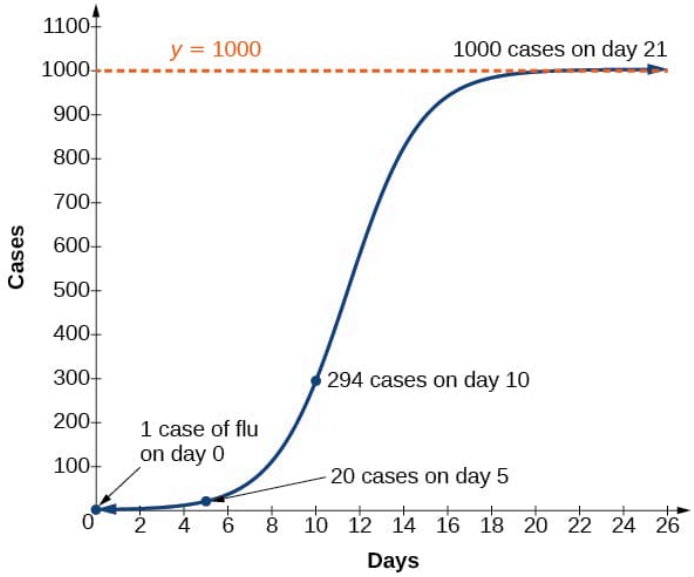


Figure $4.7.6$**:** *The graph of* $f(t) = \frac{1000}{1+999e^{-0.6030t}}$.

Example $4.7.7$**: Changing to base** e

The formula is derived as follows:

$$
\begin{aligned}
y &= 2.5(3.1)^x \\
&= 2.5e^{\ln(3.1^x)} && \text{(Inverse Property of Exponentials and Logarithms)} \\
&= 2.5e^{x\ln(3.1)} && \text{(Laws of Logarithms)} \\
&= 2.5e^{\ln(3.1)x} && \text{(Commutative Property of Multiplication)}
\end{aligned}
$$

Example $4.7.8$

The point P appears to be half way between -2 and -1 in log value. So if V is the value of this point,

$$\log(V) \approx -1.5.$$

Rewriting in exponential form,

$$V \approx 10^{-1.5} \approx 0.0316.$$

Example $4.7.9$

Since $\log(6000) \approx 3.8$, this point would belong on the log scale about here:

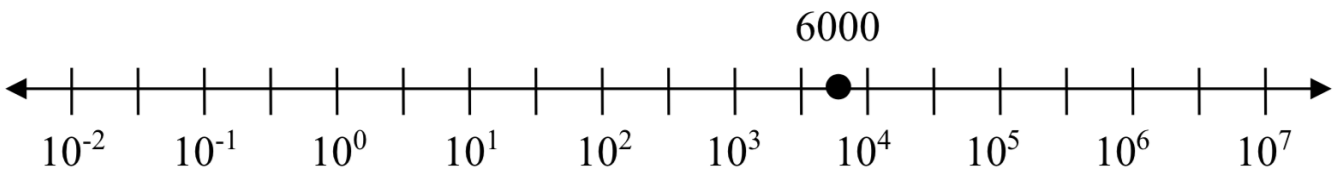


Figure 4.7.10

Checkpoint 4.7.9

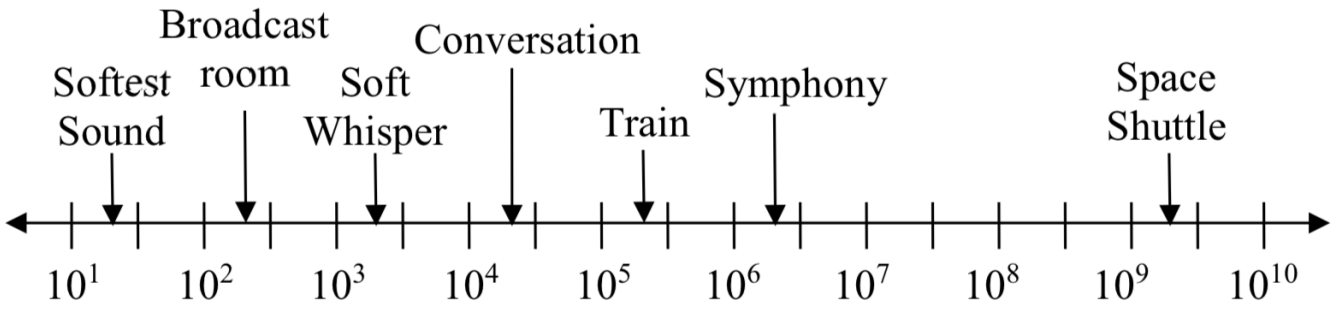


Figure 4.7.11

Example 4.7.10

The value B corresponds to $10^2 = 100$ and the value C corresponds to $10^5 = 100,000$. The relative change is $\frac{100,000}{100} = 1000 = \frac{10^5}{10^2} = 10^3$. The log of this value is 3. Therefore, C is three orders of magnitude greater than B, which can be seen on the log scale by the visual difference between the points on the scale.

Checkpoint 4.7.10

$\frac{2\times10^9}{2\times10^1} = 10^8$. The sound pressure in μ Pa created by launching the space shuttle is 8 orders of magnitude greater than the sound pressure in μ Pa created by the softest sound a human ear can hear.

Example 4.7.11

Since the first earthquake has Richter value of 6.0, we can find the amount of earth movement for that quake, which we'll denote A_1. The value of A_0 is not particularly relevant, so we will not replace it with its value.

$$\begin{aligned} 6.0 &= \log\left(\frac{A_1}{A_0}\right) \\ \Longrightarrow \quad 10^{6.0} &= \frac{A_1}{A_0} && \text{(rewriting the logarithm in exponential form)} \\ \Longrightarrow \quad 10^{6.0}A_0 &= A_1 && \text{(multiplying both sides by } A_0\text{)} \end{aligned}$$

This tells us the first earthquake has about 10^6 times more earth movement than the baseline measure.

Doing the same with the second earthquake, A_2, with a magnitude of 8.0,

$$\begin{aligned} 8.0 &= \log\left(\frac{A_2}{A_0}\right) \\ \Longrightarrow \quad 10^{8.0} &= \frac{A_2}{A_0} && \text{(rewriting the logarithm in exponential form)} \\ \Longrightarrow \quad 10^{8.0}A_0 &= A_2 && \text{(multiplying both sides by } A_0\text{)} \end{aligned}$$

Comparing the earth movement of the second earthquake to the first,

$$\frac{A_2}{A_1} = \frac{10^8 A_0}{10^6 A_0} = 10^2 = 100.$$

The second value's earth movement is 100 times as large as the first earthquake.

Example 4.7.12

Since the first quake has magnitude 3.0,

$$3.0 = \log\left(\frac{A}{A_0}\right).$$

Solving for A,

$$\begin{aligned} 3.0 &= \log\left(\frac{A}{A_0}\right) \\ \Longrightarrow \quad 10^{3.0} &= \frac{A}{A_0} && \text{(rewriting in exponential form)} \\ \Longrightarrow \quad 10^{3.0}A_0 &= A && \text{(multiplying both sides by } A_0\text{)} \end{aligned}$$

Since the second earthquake has twice as much earth movement, for the second quake we get

$$A = 2 \cdot 10^{3.0} A_0.$$

Finding the magnitude,

$$\begin{aligned} M &= \log\left(\frac{2 \cdot 10^{3.0} A_0}{A_0}\right) \\ &= \log\left(2 \cdot 10^{3.0}\right) \\ &\approx 3.3010 \end{aligned}$$

The second earthquake with twice as much earth movement will have a magnitude of about 3.3.

Example $4.7.13$

a. There are 26 letters in the alphabet, 52 if upper and lower case letters are counted as different. There are 10 digits (0 through 9) for a total of $N = 62$ symbols. Since the password is to be 7 characters long, $L = 7$. Thus, $H = 7\log_2(62) = \frac{7\ln(62)}{\ln(2)} \approx 41.68$.

b. We have $L = 7$ and $H = 50$ and we need to find N. Solving the equation $50 = 7\log_2(N)$ gives $N = 2^{50/7} \approx 141.323$, so we would need 142 different symbols to choose from.

Example $4.7.14$

We set $\mathrm{pH} = 7.4$ and get $7.4 = 6.1 + \log\left(\frac{800}{x}\right)$, or $\log\left(\frac{800}{x}\right) = 1.3$. Solving, we find $x = \frac{800}{10^{1.3}} \approx 40.09$. Hence, the partial pressure of carbon dioxide in the blood is about 40 torr.

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4.7.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Sets and Numbers**
  - Scientific Notation: Scientific notation is often used when working with exponential models.

Videos

- Exponential growth and decay models
  - Exponential Functions - Applications Involving Exponential Functions
- Half-life and doubling time models
  - Exponential and Logarithmic Models - Doubling Time and Half-Life
  - Modeling with Exponential and Logarithmic Equations - Applications Involving Half Life
- Newton's Law of Cooling
  - Modeling with Exponential and Logarithmic Equations - Applications Involving Newton's Law of Cooling
- Logarithmic models
  - Modeling with Exponential and Logarithmic Equations - Logarithmic Applications

Key Concepts

Definitions

- **Doubling Time**: The time it takes for a quantity undergoing exponential growth to double its initial amount.
- **Half-Life**: The time it takes for a quantity undergoing exponential decay to decay to half of its original quantity.
- **Order of Magnitude**: The power of ten of a number when written in scientific notation. It describes how large or small a number is relative to powers of ten.
- **Radiocarbon Dating**: A method used to calculate the approximate date a plant or animal died by comparing the ratio of carbon-14 to carbon-12 in a sample to the known ratio in the atmosphere.
- **Continuous Rate of Cooling**: The constant k in Newton's Law of Cooling formula.
- **Logistic Growth Model**: A model for growth that is approximately exponential at first but has a reduced rate of growth as the output approaches the model's upper bound (carrying capacity). The formula is $P(t) = \frac{c}{1+ae^{-bt}}$.
- **Carrying Capacity (Limiting Value)**: The constant c in the logistic growth model, representing the upper bound or maximum sustainable population/value.
- **Richter Scale**: A logarithmic scale used to express the magnitude of an earthquake, based on the seismic trace amplitude. Defined as $M = \log\left(\frac{A}{A_0}\right)$.

Theorems

- **Half-Life Formula**: If an exponentially decaying population is modeled by $P(t) = P_0e^{kt}$, then the time it takes for the population to decay by half (half-life) is $t = -\frac{\ln(2)}{k}$ (or $t = \frac{\ln(1/2)}{k}$).
- **Newton's Law of Cooling**: The temperature of an object, T, in a surrounding medium having temperature T_s will behave according to the formula $T(t) = T_s + (T_0 - T_s)e^{kt}$, where t is time, T_0 is the initial temperature, and k is the continuous rate of cooling.
- **Determining the Order of Magnitude Between Two Numbers**: Given two values A and B, the number of orders of

magnitude A is greater than B is $\log\left(\frac{A}{B}\right)$.

Common Mistakes

- **Incorrect Sign for Rate Constant k**: Using a positive k for decay models (like half-life or Newton's cooling) or a negative k for growth models (like doubling time) in the $P(t) = P_0 e^{kt}$ formula.
- **Confusing Half-Life/Doubling Time with Rate Constant**: Mistaking the half-life or doubling time itself for the rate constant k, or vice-versa, instead of using the derived formulas (e.g., $k = -\frac{\ln(2)}{T_{half}}$) to relate them.
- **Errors in Setting Up Newton's Law of Cooling**: Incorrectly identifying T_s (ambient temperature), T_0 (initial temperature), or the temperature at a given time $T(t)$.
- **Algebraic Errors When Solving for t or k**: Making mistakes in isolating the exponential term, taking logarithms, or using logarithm properties when solving for time or the rate constant in various models.
- **Misinterpreting Logistic Growth Parameters**: Incorrectly identifying the carrying capacity c or the initial value $\frac{c}{1+a}$ from the logistic model formula.
- **Using Linear Models for Exponential Situations**: Attempting to model situations like population growth or radioactive decay with linear functions when an exponential model is appropriate.
- **Rounding Too Early**: Rounding intermediate calculations, which can lead to significant errors in the final answer, especially in exponential models.
- **Incorrectly Applying Logarithmic Scales**: Misunderstanding that a linear difference on a logarithmic scale corresponds to a multiplicative (ratio) difference in actual values (e.g., Richter scale).

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4.7.2: Homework

Reading Questions

1. What is the difference between "doubling time" and "half-life" in the context of exponential models? Which applies to growth and which to decay?
2. What is the general formula for exponential growth/decay using base e? Define each variable.
3. How is the half-life T related to the decay constant k in the formula $P(t) = P_0e^{kt}$?
4. Describe Newton's Law of Cooling. What does each variable in the formula $T(t) = T_s + (T_0 - T_s)e^{kt}$ represent?
5. What is a logistic growth model, and how does its behavior differ from a simple exponential growth model, especially in the long term?
6. What is the "carrying capacity" in a logistic growth model?
7. If an exponential function is given as $y = ab^x$, how can you convert it to the form $y = A_0e^{kx}$? What are A_0 and k in terms of a and b?
8. Explain what an "order of magnitude" means. If one quantity is 3 orders of magnitude larger than another, approximately how many times larger is it?
9. What is the Richter scale used to measure, and is it a linear or logarithmic scale?
10. In the context of radiocarbon dating, what does the half-life of Carbon-14 (5,730 years) signify?
11. With what kind of exponential model would half-life be associated? What role does half-life play in these models?
12. What is carbon dating? Why does it work? Give an example in which carbon dating would be useful.
13. With what kind of exponential model would *doubling time* be associated? What role does doubling time play in these models?
14. Define Newton's Law of Cooling. Then name at least three real-world situations where Newton's Law of Cooling would be applied.
15. What is an order of magnitude? Why are orders of magnitude useful? Give an example to explain.

Homework

Numeric

16. The temperature of an object in degrees Fahrenheit after t minutes is represented by the equation $T(t) = 72 + 68e^{-0.0174t}$. To the nearest degree, what is the temperature of the object after one and a half hours?

For the following exercises, use the logistic growth model $f(x) = \frac{150}{1+8e^{-2x}}$.

17. Find and interpret $f(0)$. Round to the nearest tenth.
18. Find and interpret $f(4)$. Round to the nearest tenth.
19. Find the carrying capacity.
20. Graph the model.
21. Determine whether the data from the table could best be represented as a function that is linear, exponential, or logarithmic. Then write a formula for a model that represents the data.

x	$f(x)$
–2	0.694
–1	0.833
0	1
1	1.2
2	1.44
3	1.728
4	2.074

x	$f(x)$
5	2.488

22. Rewrite $f(x) = 1.68(0.65)^x$ as an exponential equation with base e to five decimal places.

Technology

For the following exercises, enter the data from each table into a graphing calculator and graph the resulting scatter plots. Determine whether the data from the table could represent a function that is linear, exponential, or logarithmic.

23.

x	$f(x)$
1	2
2	4.079
3	5.296
4	6.159
5	6.828
6	7.375
7	7.838
8	8.238
9	8.592
10	8.908

24.

x	$f(x)$
1	2.4
2	2.88
3	3.456
4	4.147
5	4.977
6	5.972
7	7.166
8	8.6
9	10.32
10	12.383

25.

x	$f(x)$
4	9.429
5	9.972

x	$f(x)$
6	10.415
7	10.79
8	11.115
9	11.401
10	11.657
11	11.889
12	12.101
13	12.295

26.

x	$f(x)$
1.25	5.75
2.25	8.75
3.56	12.68
4.2	14.6
5.65	18.95
6.75	22.25
7.25	23.75
8.6	27.8
9.25	29.75
10.5	33.5

For the following exercises, use a graphing calculator and this scenario: the population of a fish farm in t years is modeled by the equation $P(t) = \frac{1000}{1+9e^{-0.6t}}$.

27. Graph the function.
28. What is the initial population of fish?
29. To the nearest tenth, what is the doubling time for the fish population?
30. To the nearest whole number, what will the fish population be after 2 years?
31. To the nearest tenth, how long will it take for the population to reach 900?
32. What is the carrying capacity for the fish population? Justify your answer using the graph of P.

Extensions

33. A substance has a half-life of 2.045 minutes. If the initial amount of the substance was 132.8 grams, how many half-lives will have passed before the substance decays to 8.3 grams? What is the total time of decay?
34. The formula for an increasing population is given by $P(t) = P_0e^{rt}$ where P_0 is the initial population and $r > 0$. Derive a general formula for the time t it takes for the population to increase by a factor of M.
35. Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. Show each step for solving this equation algebraically for the seismic moment S.
36. What is the y-intercept of the logistic growth model $y = \frac{c}{1+ae^{-rx}}$? Show the steps for calculation. What does this point tell us about the population?
37. Prove that $b^x = e^{xln(b)}$ for positive $b \neq 1$.

Real-World Applications

For the following exercises, use this scenario: A doctor prescribes 125 milligrams of a therapeutic drug that decays by about 30% each hour.

38. To the nearest hour, what is the half-life of the drug?
39. Write an exponential model representing the amount of the drug remaining in the patient's system after t hours. Then use the formula to find the amount of the drug that would remain in the patient's system after 3 hours. Round to the nearest milligram.
40. Using the model found in the previous exercise, find $f(10)$ and interpret the result. Round to the nearest hundredth.

For the following exercises, use this scenario: A tumor is injected with 0.5 grams of Iodine-125, which has a decay rate of 1.15 per day.

41. To the nearest day, how long will it take for half of the Iodine-125 to decay?
42. Write an exponential model representing the amount of Iodine-125 remaining in the tumor after t days. Then use the formula to find the amount of Iodine-125 that would remain in the tumor after 60 days. Round to the nearest tenth of a gram.
43. A scientist begins with 250 grams of a radioactive substance. After 250 minutes, the sample has decayed to 32 grams. Rounding to five decimal places, write an exponential equation representing this situation. To the nearest minute, what is the half-life of this substance?
44. The half-life of Radium-226 is 1590 years. What is the annual decay rate? Express the decimal result to four decimal places and the percentage to two decimal places.
45. The half-life of Erbium-165 is 10.4 hours. What is the hourly decay rate? Express the decimal result to four decimal places and the percentage to two decimal places.
46. A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. To the nearest year, about how many years old is the artifact? (The half-life of carbon-14 is 5730 years.)
47. A research student is working with a culture of bacteria that doubles in size every twenty minutes. The initial population count was 1350 bacteria. Rounding to five decimal places, write an exponential equation representing this situation. To the nearest whole number, what is the population size after 3 hours?

For the following exercises, use this scenario: A biologist recorded a count of 360 bacteria present in a culture after 5 minutes and 1000 bacteria present after 20 minutes.

48. To the nearest whole number, what was the initial population in the culture?
49. Rounding to six decimal places, write an exponential equation representing this situation. To the nearest minute, how long did it take the population to double?

For the following exercises, use this scenario: A pot of warm soup with an internal temperature of 100° Fahrenheit was taken off the stove to cool in a 69° F room. After fifteen minutes, the internal temperature of the soup was 95° F.

50. Use Newton's Law of Cooling to write a formula that models this situation.
51. To the nearest minute, how long will it take the soup to cool to 80° F?
52. To the nearest degree, what will the temperature be after 2 and a half hours?

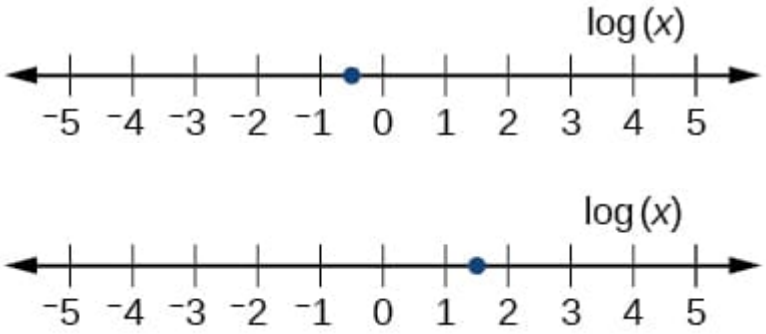
For the following exercises, use this scenario: A turkey is taken out of the oven with an internal temperature of 165° F and is allowed to cool in a 75° F room. After half an hour, the internal temperature of the turkey is 145° F.

53. Write a formula that models this situation.
54. To the nearest degree, what will the temperature be after 50 minutes?
55. To the nearest minute, how long will it take the turkey to cool to 110° F?

For the following exercises, find the value of the number shown on each logarithmic scale. Round all answers to the nearest thousandth.

56.

57.



58. Plot each set of approximate values of intensity of sounds on a logarithmic scale: Whisper: $10^{-10}\frac{W}{m^2}$, Vacuum: $10^{-4}\frac{W}{m^2}$, Jet: $10^{2}\frac{W}{m^2}$.
59. Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. One earthquake has magnitude 3.9 on the MMS scale. If a second earthquake has 750 times as much energy as the first, find the magnitude of the second quake. Round to the nearest hundredth.

For the following exercises, use this scenario: The equation $N(t) = \frac{500}{1+49e^{-0.7t}}$ models the number of people in a town who have heard a rumor after t days.

60. How many people started the rumor?
61. To the nearest whole number, how many people will have heard the rumor after 3 days?
62. As t increases without bound, what value does $N(t)$ approach? Interpret your answer.

For the following exercise, choose the correct answer choice.

63. A doctor injects a patient with 13 milligrams of radioactive dye that decays exponentially. After 12 minutes, there are 4.75 milligrams of dye remaining in the patient's system. Which is an appropriate model for this situation?
    a. $f(t) = 13(0.0805)^t$
    b. $f(t) = 13e^{0.9195t}$
    c. $f(t) = 13e^{(}-0.0839t)$
    d. $f(t) = \frac{4.75}{1+13e^{-0.83925t}}$

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5: SEQUENCES, SUMMATIONS, AND LOGIC

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A lottery winner has some big decisions to make regarding what to do with the winnings. Buy a villa in Saint Barthélemy? A luxury convertible? A cruise around the world?

The likelihood of winning the lottery is slim, but we all love to fantasize about what we could buy with the winnings. One of the first things a lottery winner has to decide is whether to take the winnings in the form of a lump sum or as a series of regular payments, called an annuity, over the next 30 years or so.

This decision is often based on many factors, such as tax implications, interest rates, and investment strategies. There are also personal reasons to consider when making the choice, and one can make many arguments for either decision. However, most lottery winners opt for the lump sum.

In this chapter, we will explore the mathematics behind situations such as these. We will take an in-depth look at annuities. We will also look at the branch of mathematics that would allow us to calculate the number of ways to choose lottery numbers and the probability of winning.

5.1: Sequences and Their Notations

5.1.1: Resources and Key Concepts

5.1.2: Homework

5.2: Arithmetic Sequences

5.2.1: Resources and Key Concepts

5.2.2: Homework

5.3: Geometric Sequences

5.3.1: Resources and Key Concepts

5.3.2: Homework

5.4: Series and Their Notations

5.4.1: Resources and Key Concepts
5.4.2: Homework

5.5: Mathematical Induction

5.5.1: Resources and Key Concepts
5.5.2: Homework

5.6: Binomial Theorem

5.6.1: Resources and Key Concepts
5.6.2: Homework

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5.1: Sequences and Their Notations

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

A video game company launches an exciting new advertising campaign. They predict the number of online visits to their website, or hits, will double each day. The model they are using shows 2 hits the first day, 4 hits the second day, 8 hits the third day, and so on. See Table $5.1.1$.

Table $5.1.1$

Day	1	2	3	4	5	...
Hits	2	4	8	16	32	...

If their model continues, how many hits will there be at the end of the month? To answer this question, we'll first need to know how to determine a list of numbers written in a specific order. In this section, we will explore these kinds of ordered lists.

Writing the Terms of a Sequence Defined by an Explicit Formula

One way to describe an ordered list of numbers is as a **sequence**.

Definition: Sequence

A **sequence** is a function whose domain is the set of natural numbers. The numbers in a sequence are called **terms**.

For example, the sequence established by the number of hits on the website from Table $5.1.1$ is

$$\{2, 4, 8, 16, 32, ...\}.$$

The first five terms of this sequence are 2, 4, 8, 16, and 32. The ellipsis (...) indicates that the sequence continues indefinitely.

Definition: Infinite Sequence

A sequence that continues indefinitely is an **infinite sequence**.

Therefore, the sequence of website hits is considered an infinite sequence.

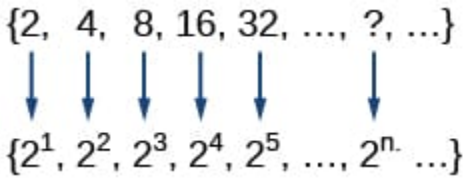
Listing all of the terms for a sequence can be cumbersome. For example, finding the number of hits on the website at the end of the month would require listing out as many as 31 terms. A more efficient way to determine a specific term is by writing a formula to define the sequence.

One type of formula is an **explicit formula**, which defines the terms of a sequence using their position in the sequence.

Definition: Explicit Formula

Given the sequence $\{a_1, a_2, a_3, ...\}$, where a_1 is the first term of the sequence, a_2 is the second term of the sequence, a_3 the third term of the sequence, and so on, an **explicit formula** defines a_n, the nth term of the sequence, in relation to the position of the term, n. That is, if a function formula, $f(n)$, can be found to represent the given sequence, the explicit formula would be $a_n = f(n)$.

Explicit formulas are helpful if we want to find a specific term of a sequence without finding all of the previous terms. We can use the formula to find the nth term of the sequence, where n is any positive integer. In our example, each number in the sequence is double the previous number, so we can use powers of 2 to write a formula for the nth term.



The first term of the sequence is $2^1 = 2$, the second term is $2^2 = 4$, the third term is $2^3 = 8$, and so on. The nth term of the sequence can be found by raising 2 to the nth power. An explicit formula for a sequence is named by a lower case letter a, b, c, ... with the subscript n. The explicit formula for this sequence is

$$a_n = 2^n.$$

Now that we have a formula for the nth term of the sequence, we can answer the question posed at the beginning of this section. We were asked to find the number of hits at the end of the month, which we will take to be 31 days. To find the number of hits on the last day of the month, we need to find the 31st term of the sequence. We will substitute 31 for n in the formula.

$$a_{31} = 2^{31} = 2,147,483,648.$$

If the doubling trend continues, the company will get $2,147,483,648$ hits on the last day of the month. That is over 2.1 billion hits! The huge number is probably a little unrealistic because it does not take consumer interest and competition into account. It does, however, give the company a starting point from which to consider business decisions.

Another way to represent the sequence is by using a table. The first five terms of the sequence and the nth term of the sequence are shown in Table $5.1.2$.

Table $5.1.2$

n	1	2	3	4	5	n
a_n	2	4	8	16	32	2^n

Graphing provides a visual representation of the sequence as a set of distinct points. We can see from the graph in Figure $5.1.1$ that the number of hits is rising at an exponential rate. This particular sequence forms an exponential function.

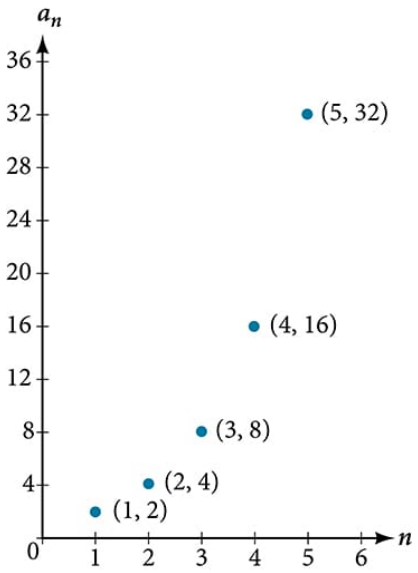


Figure 5.1.1

Lastly, we can write this particular sequence as

$$\{2, 4, 8, 16, 32, \ldots, 2n, \ldots\}.$$

If we consider only the first 10 terms of the sequence, we could write

$$\{2, 4, 8, 16, 32, \ldots, 2n, \ldots, 1024\}.$$

This sequence is called a **finite sequence** because it does not continue indefinitely.

Definition: Finite Sequence

A **finite sequence** is a sequence whose domain consists of only the first n natural numbers.

Q&A

Does a sequence always have to begin with a_1?

No. In certain problems, it may be useful to define the initial term as a_0 instead of a_1. In these problems, the domain of the function includes 0.

Example 5.1.1: Writing the Terms of a Sequence Defined by an Explicit Formula

Write the first five terms of the sequence defined by the explicit formula $a_n = -3n + 8$.

Solution available at the end of this section.

The sequence values in Example 5.1.1 can be listed in a table. A table, such as Table 5.1.3, is a convenient way to input the function into graphing technology like Desmos.

Table 5.1.3

n	1	2	3	4	5
a_n	5	2	-1	-4	-7

A graph can be made from this table of values. From the graph in Figure 5.1.2, we can see that this sequence represents a linear

function, but notice the graph is not continuous because the domain is over the natural numbers only.

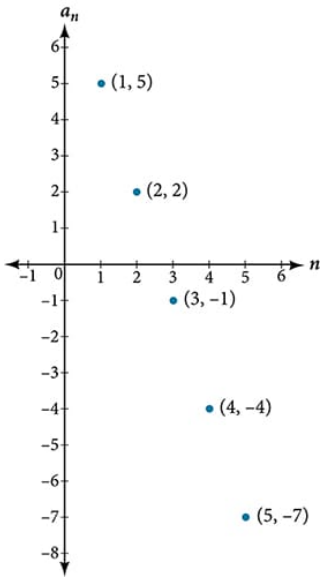


Figure 5.1.2

Checkpoint 5.1.1

Write the first five terms of the sequence defined by the explicit formula $t_n = 5n - 4$.

Investigating Alternating Sequences

Sometimes sequences have terms that alternate in sign.

Definition: Alternating Sequence

A sequence in which consecutive terms alternate in sign, or which can be simplified to a sequence where consecutive terms alternate in sign, is called an **alternating sequence**.

The steps to finding terms of such sequences are the same as if the signs did not alternate. However, the resulting terms will not show increasing or decreasing behavior as n increases. Let's take a look at the following sequence.

$$\{2, -4, 6, -8\}$$

Notice the first term is greater than the second term, the second term is less than the third term, and the third term is greater than the fourth term. This trend continues forever. *Do not* rearrange the terms in numerical order to interpret the sequence.

Example 5.1.2: Writing the Terms of an Alternating Sequence Defined by an Explicit Formula

Write the first five terms of the sequence.

$$a_n = \frac{(-1)^n n^2}{n+1}$$

Solution available at the end of this section.

Q&A

In Example 5.1.2, does the (-1) to the power of n account for the oscillations of signs?

Yes, the power might be n, $n+1$, $n-1$, and so on, but any odd powers will result in a negative term, and any even power will result in a positive term.

Checkpoint 5.1.2

Write the first five terms of the sequence.

$$a_n = \frac{4n}{(-2)^n}$$

Investigating Piecewise Explicit Formulas

We've learned that sequences are functions whose domain is over the natural numbers. This is true for other types of functions, including some piecewise functions. Recall that a piecewise function is a function defined by multiple subsections. A different formula might represent each individual subsection.

Example 5.1.3: Writing the Terms of a Sequence Defined by a Piecewise Explicit Formula

Write the first six terms of the sequence.

$$a_n = \begin{cases} n^2, & \text{if } n \text{ is not divisible by } 3 \\ \frac{n}{3}, & \text{if } n \text{ is divisible by } 3 \end{cases}$$

Solution available at the end of this section.

Checkpoint 5.1.3

Write the first six terms of the sequence.

$$a_n = \begin{cases} 2n^3, & \text{if } n \text{ is odd} \\ \frac{5n}{2}, & \text{if } n \text{ is even} \end{cases}$$

Finding an Explicit Formula

Thus far, we have been given the explicit formula and asked to find a number of terms of the sequence. Sometimes, the explicit formula for the nth term of a sequence is not given. Instead, we are given several terms from the sequence. When this happens, we can work in reverse to find an explicit formula from the first few terms of a sequence. The key to finding an explicit formula is to look for a pattern in the terms. Keep in mind that the pattern may involve alternating terms, formulas for numerators, formulas for denominators, exponents, or bases.

Example 5.1.4: Writing an Explicit Formula for the nth Term of a Sequence

Write an explicit formula for the nth term of each sequence.

a. $\left\{-\frac{2}{11}, \frac{3}{13}, -\frac{4}{15}, \frac{5}{17}, -\frac{6}{19}, \ldots\right\}$
b. $\left\{-\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, -\frac{2}{3{,}125}, -\frac{2}{15{,}625}, \ldots\right\}$
c. $\left\{e^4, e^5, e^6, e^7, e^8, \ldots\right\}$

Solutions available at the end of this section.

Checkpoint 5.1.4

Write an explicit formula for the nth term of each sequence.

a. $\{9, -81, 729, -6561, 59049, ...\}$
b. $\left\{-\frac{3}{4}, -\frac{9}{8}, -\frac{27}{12}, -\frac{81}{16}, -\frac{243}{20}, ...\right\}$
c. $\left\{\frac{1}{e^2}, \frac{1}{e}, 1, e, e^2, ...\right\}$

Writing the Terms of a Sequence Defined by a Recursive Formula

Sequences occur naturally in the growth patterns of nautilus shells, pinecones, tree branches, and many other natural structures. We may see the sequence in the leaf or branch arrangement, the number of petals of a flower, or the pattern of the chambers in a nautilus shell. Their growth follows the **Fibonacci sequence**, a famous sequence in which each term can be found by adding the preceding two terms. The numbers in the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Other examples from the natural world that exhibit the Fibonacci sequence are the Calla Lily, which has just one petal, the Black-Eyed Susan with 13 petals, and different varieties of daisies that may have 21 or 34 petals.

Each term of the Fibonacci sequence depends on the terms that come before it. The Fibonacci sequence cannot easily be written using an explicit formula. Instead, we describe the sequence using a **recursive formula**.

Definition: Recursive Formula

A **recursive formula** is a formula that defines each term of a sequence using a preceding term (or multiple preceding terms). Recursive formulas must always state the **initial term**, or terms, of the sequence.

For example, suppose we know the following:

$$\begin{aligned} a_1 &= 3 \\ a_n &= 2a_{n-1} - 1 \qquad \text{for } n \geq 2 \end{aligned}$$

We can find the subsequent terms of the sequence using the first term.

$$\begin{aligned} a_1 &= 3 \\ a_2 &= 2a_1 - 1 = 2(3) - 1 = 5 \\ a_3 &= 2a_2 - 1 = 2(5) - 1 = 9 \\ a_4 &= 2a_3 - 1 = 2(9) - 1 = 17 \end{aligned}$$

So the first four terms of the sequence are $\{3, 5, 9, 17\}$.

The recursive formula for the Fibonacci sequence states the first two terms and defines each successive term as the sum of the preceding two terms.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3 \end{aligned}$$

To find the tenth term of the sequence, for example, we would need to add the eighth and ninth terms. We were told previously that the eighth and ninth terms are 21 and 34, so

$$a_{10} = a_9 + a_8 = 34 + 21 = 55.$$

Q&A

Must the first two terms always be given in a recursive formula?

No. The Fibonacci sequence defines each term using the two preceding terms, but many recursive formulas define each term using only one preceding term. These sequences need only the first term to be defined.

Example 5.1.5: Writing the Terms of a Sequence Defined by a Recursive Formula

Write the first five terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 9 \\ a_n &= 3a_{n-1} - 20 \qquad \text{for } n \geq 2 \end{aligned}$$

Solution available at the end of this section.

Checkpoint 5.1.5

Write the first five terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 2 \\ a_n &= 2a_{n-1} + 1 \qquad \text{for } n \geq 2 \end{aligned}$$

Example 5.1.6: Writing the Terms of a Sequence Defined by a Recursive Formula

Write the first six terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2 \\ a_n &= 3a_{n-1} + 4a_{n-2} \qquad \text{for } n \geq 3 \end{aligned}$$

Solution available at the end of this section.

Checkpoint 5.1.6

Write the first eight terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 1 \\ a_3 &= 1 \\ a_n &= \frac{a_{n-1}}{a_{n-2}} + a_{n-3} \qquad \text{for } n \geq 4 \end{aligned}$$

Using Factorial Notation

The formulas for some sequences include products of consecutive natural numbers—these are known as **factorials** and there are two (somewhat equivalent) definitions. Since we just learned about recursive formulas, let's start with the recursive formula definition of a factorial.

Definition: Factorial (Recursive Definition)

The **factorial** of a whole number n, denoted $n!$ and reads as "n **factorial**," is defined by the recursive formula

$$\begin{aligned} a_0 &= 0! = 1 \\ a_n &= n! = n \cdot a_{n-1} \qquad \text{for } n \geq 1 \end{aligned}$$

Therefore,

$$\begin{aligned} 0! &= a_0 = 1 \\ 1! &= a_1 = 1 \cdot a_0 = 1 \cdot 1 = 1 \\ 2! &= a_2 = 2 \cdot a_1 = 2 \cdot 1 = 2 \\ 3! &= a_3 = 3 \cdot a_2 = 3 \cdot 2 = 6 \\ 4! &= a_4 = 4 \cdot a_3 = 4 \cdot 6 = 24 \\ 5! &= a_5 = 5 \cdot a_4 = 5 \cdot 24 = 120 \end{aligned}$$

We could continue forever, but there is a pattern to notice here. A habit we need to introduce at this point is to *not* simplify when trying to find patterns. Ignoring $0!$ for the moment,

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \cdot 1 \\ 3! &= 3 \cdot 2 \cdot 1 \\ 4! &= 4 \cdot 3 \cdot 2 \cdot 1 \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

In general, $n!$ is the product of the natural numbers from 1 to n. This gives us our second (somewhat equivalent) definition.

Definition: Factorial (Non-Recursive Definition)

The **factorial** of a whole number n, denoted $n!$ and reads as "n **factorial**," is defined to be following piecewise function.

$$n! = \begin{cases} 1, & \text{if } n = 0 \\ n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1, & \text{if } n \geq 1 \end{cases}$$

For example,

$$\begin{aligned} 4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \end{aligned}$$

An example of formula containing a factorial is $a_n = (n+1)!$. The sixth term of the sequence can be found by substituting 6 for n.

$$a_6 = (6+1)! = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

The factorial of any natural number n is $n(n-1)!$ We can therefore also think of $5!$ as $5 \cdot 4!$. The following theorem codifies this statement.

Theorem: Factorial Recurrence Relation

For $n \geq 1$,

$$n! = n(n-1)!$$

Q&A

Can factorials always be found using a calculator?

No. Factorials get large very quickly—faster than exponential functions! When the output gets too large for the calculator, it will not be able to calculate the factorial.

Example 5.1.7: Writing the Terms of a Sequence Using Factorials

Write the first five terms of the sequence defined by the explicit formula $a_n = \frac{5n}{(n+2)!}$.

Solution available at the end of this section.

Checkpoint 5.1.7

Write the first five terms of the sequence defined by the explicit formula $a_n = \frac{(n+1)!}{2n}$.

Success in College: To Be Smart or Not To Be Smart... Is This Your Question?

> When I attended CRC, I had a computer science professor who had some great sayings. One of my favorites occurred when a student raised her hand and said, in a sheepish voice, "This may be a stupid question, but..."
>
> The brilliantly opinionated professor immediately interrupted the student and stated, in a voice beaming with inspiring confidence, "There are no stupid questions... only stupid people who don't ask them."
>
> "That is so true," I thought to myself. Here we are, learning, experiencing, and becoming better citizens of this great wide world. Still, most students refuse to involve themselves in their education by remaining silent when they should be speaking up. This single abrasive comment by the professor summed up the majority of my fellow students!
>
> I immediately swore to myself not to be shy anymore about asking questions. If I annoy my neighbor by slowing the lecture, so be it! If I look like I am lost in front of others, that's a small price to pay to gain understanding.
>
> And so it went... I carried this habit throughout the universities I attended, in every course, with every professor. As time went on, and I was in classes with more capable students because those earlier students fell by the wayside, I found I was surrounded by those with the same habit. It turned out that one of the (many) keys to success on the road to becoming a critical thinker was being engaged and asking questions.
>
> - Professor Simpson, Cosumnes River College

So what about you? Are you willing to let go of that social stigma that asking questions makes you look "dumb?" Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. In some high schools, it is not "socially acceptable" to act as if you care about classwork. Students do not ask for help in front of others. This attitude may or may not continue in beginning college classes, even for older students. Is this a typical attitude at your college?
2. Some professors check for understanding by calling on students in class. This way, the professor knows if more examples or discussions are needed. Some students have no problem answering questions in front of the class. Other students struggle to answer in front of the class even if they think they know the correct answer. How do you react when a professor asks you a question in front of the class? Why?

Success in Mathematics: Check Your Work

Checking your work is essential, but most students want to complete it and walk away from it. The fastest way to check your work in algebra is to plug your answers into the original problem. In trigonometry and calculus, this method may be a bit difficult. However, you can always check your solutions with graphs on the graphing calculator! It's a great idea.

1. Do you usually check your work?
2. Have you used the graphing calculator to check your work, or do you usually check "by hand?"

Answers

Section Preview

The material in this section should be considered as completely new to the student. You should expect to spend more time with the concepts than with previous sections because the students are learning this without previous scaffolding.

Section Preview

- Write the terms of a sequence defined by an explicit formula.
- Write the terms of a sequence defined by a recursive formula.
- Use factorial notation.

Example $5.1.1$: Writing the Terms of a Sequence Defined by an Explicit Formula

Substitute $n = 1$ into the formula. Repeat with values 2 through 5 for n.

$$\begin{aligned} n &= 1 \qquad a_1 = -3(1) + 8 = 5 \\ n &= 2 \qquad a_2 = -3(2) + 8 = 2 \\ n &= 3 \qquad a_3 = -3(3) + 8 = -1 \\ n &= 4 \qquad a_4 = -3(4) + 8 = -4 \\ n &= 5 \qquad a_5 = -3(5) + 8 = -7 \end{aligned}$$

The first five terms are $\{5, 2, -1, -4, -7\}$.

Example $5.1.2$: Writing the Terms of an Alternating Sequence Defined by an Explicit Formula

Substitute $n = 1$, $n = 2$, and so on into the formula.

$$\begin{aligned} n &= 1 \qquad a_1 = \frac{(-1)^1 1^2}{1+1} = -\frac{1}{2} \\ n &= 2 \qquad a_2 = \frac{(-1)^2 2^2}{2+1} = \frac{4}{3} \\ n &= 3 \qquad a_3 = \frac{(-1)^3 3^2}{3+1} = -\frac{9}{4} \\ n &= 4 \qquad a_4 = \frac{(-1)^4 4^2}{4+1} = \frac{16}{5} \\ n &= 5 \qquad a_5 = \frac{(-1)^5 5^2}{5+1} = -\frac{25}{6} \end{aligned}$$

The first five terms are $\left\{-\frac{1}{2}, \frac{4}{3}, -\frac{9}{4}, \frac{16}{5}, -\frac{25}{6}\right\}$.

The graph of this function, shown in Figure $5.1.3$, looks different from the ones we have seen previously in this section because the terms of the sequence alternate between positive and negative values.

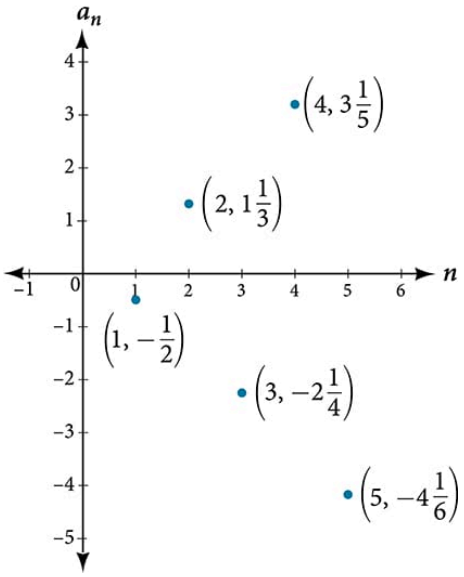


Figure 5.1.3

Example 5.1.3**: Writing the Terms of a Sequence Defined by a Piecewise Explicit Formula**

Substitute $n = 1$, $n = 2$, and so on into the appropriate formula. Use n^2 when n is not a multiple of 3. Use $\frac{n}{3}$ when n is a multiple of 3.

$$\begin{aligned}
n = 1 \qquad & a_1 = 1^2 = 1 && (1 \text{ is not a multiple of } 3 \implies \text{Use } n^2)\\
n = 2 \qquad & a_2 = 2^2 = 4 && (2 \text{ is not a multiple of } 3 \implies \text{Use } n^2)\\
n = 3 \qquad & a_3 = \frac{3}{3} = 1 && \left(3 \text{ is a multiple of } 3 \implies \text{Use } \frac{n}{3}\right)\\
n = 4 \qquad & a_4 = 4^2 = 16 && (4 \text{ is not a multiple of } 3 \implies \text{Use } n^2)\\
n = 5 \qquad & a_5 = 5^2 = 25 && (5 \text{ is not a multiple of } 3 \implies \text{Use } n^2)\\
n = 6 \qquad & a_6 = \frac{6}{3} = 2 && \left(6 \text{ is a multiple of } 3 \implies \text{Use } \frac{n}{3}\right)
\end{aligned}$$

The first six terms are $\{1, 4, 1, 16, 25, 2\}$.

Every third point on the graph shown in Figure 5.1.4 stands out from the two nearby points. This occurs because the sequence was defined by a piecewise function.

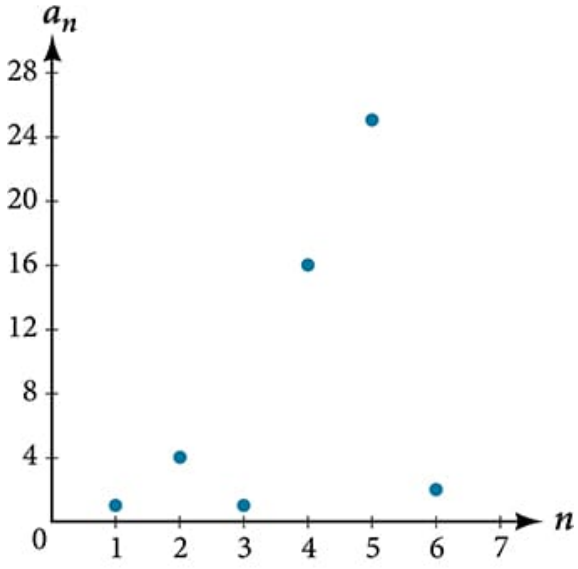


Figure 5.1.4

Example 5.1.4**: Writing an Explicit Formula for the nth Term of a Sequence**

Look for the pattern in each sequence.

a. The terms alternate between positive and negative. We can use $(-1)^n$ to make the terms alternate. The numerator can be represented by $n + 1$. The denominator can be represented by $2n + 9$. Therefore,

$$a_n = \frac{(-1)^n(n+1)}{2n+9}$$

b. The terms are all negative.

$$\left\{-\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, -\frac{2}{3,125}, -\frac{2}{15,625}, \ldots\right\}$$

The numerator is always 2.

$$\left\{-\frac{2}{5^2}, -\frac{2}{5^3}, -\frac{2}{5^4}, -\frac{2}{5^5}, -\frac{2}{5^6}, \ldots\right\}$$

The denominators are increasing powers of 5, starting with the second power when $n = 1$. So we know that the denominator can be represented by 5^{n+1}. Therefore,

$$a_n = -\frac{2}{5^{n+1}}.$$

c. The terms are powers of e. For $n = 1$, we get e^4. Therefore, the exponent must be $n + 3$. That is,

$$a_n = e^{n+3}.$$

Example $5.1.5$: Writing the Terms of a Sequence Defined by a Recursive Formula

The first term is given in the formula. For each subsequent term, we replace a_{n-1} with the value of the preceding term.

$$\begin{aligned}
n &= 1 \qquad a_1 = 9 \\
n &= 2 \qquad a_2 = 3a_1 - 20 = 3(9) - 20 = 27 - 20 = 7 \\
n &= 3 \qquad a_3 = 3a_2 - 20 = 3(7) - 20 = 21 - 20 = 1 \\
n &= 4 \qquad a_4 = 3a_3 - 20 = 3(1) - 20 = 3 - 20 = -17 \\
n &= 5 \qquad a_5 = 3a_4 - 20 = 3(-17) - 20 = -51 - 20 = -71
\end{aligned}$$

The first five terms are $\{9, 7, 1, -17, -71\}$. See Figure $5.1.5$

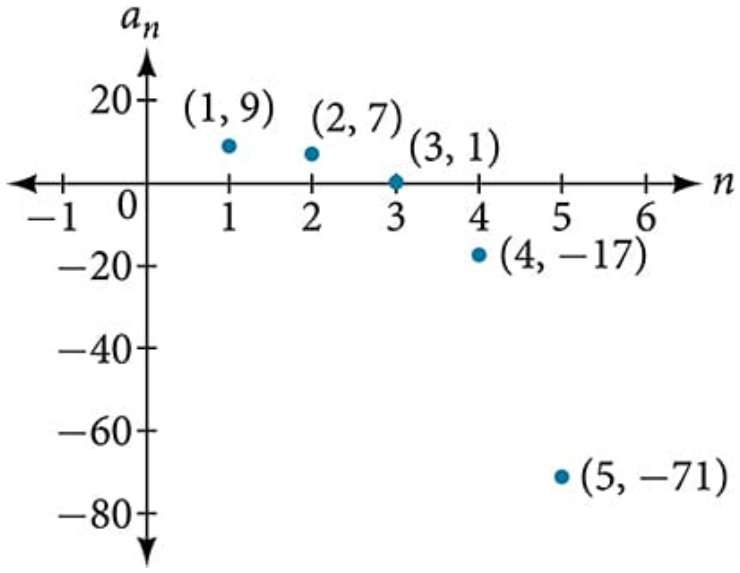


Figure $5.1.5$

Example $5.1.6$: Writing the Terms of a Sequence Defined by a Recursive Formula

The first two terms are given. For each subsequent term, we replace a_{n-1} and a_{n-2} with the values of the two preceding terms.

$$\begin{aligned}
n &= 3 \qquad a_3 = 3a_2 + 4a_1 = 3(2) + 4(1) = 10 \\
n &= 4 \qquad a_4 = 3a_3 + 4a_2 = 3(10) + 4(2) = 38 \\
n &= 5 \qquad a_5 = 3a_4 + 4a_3 = 3(38) + 4(10) = 154 \\
n &= 6 \qquad a_6 = 3a_5 + 4a_4 = 3(154) + 4(38) = 614
\end{aligned}$$

The first six terms are $\{1, 2, 10, 38, 154, 614\}$. See Figure $5.1.6$.

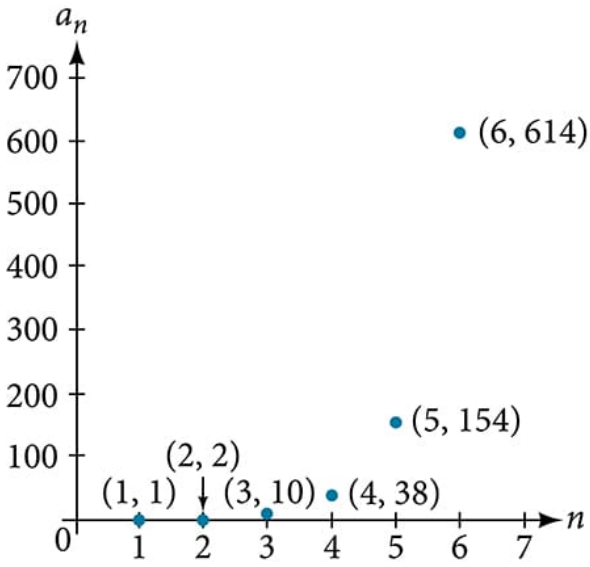


Figure 5.1.6

Example 5.1.7**: Writing the Terms of a Sequence Using Factorials**

Substitute $n = 1$, $n = 2$, and so on in the formula.

$$n = 1 \qquad a_1 = \frac{5(1)}{(1+2)!} = \frac{5}{3!} = \frac{5}{3 \cdot 2 \cdot 1} = \frac{5}{6}$$

$$n = 2 \qquad a_2 = \frac{5(2)}{(2+2)!} = \frac{10}{4!} = \frac{10}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{5}{12}$$

$$n = 3 \qquad a_3 = \frac{5(3)}{(3+2)!} = \frac{15}{5!} = \frac{15}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{8}$$

$$n = 4 \qquad a_4 = \frac{5(4)}{(4+2)!} = \frac{20}{6!} = \frac{20}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{36}$$

$$n = 5 \qquad a_5 = \frac{5(5)}{(5+2)!} = \frac{25}{7!} = \frac{25}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{5}{1008}$$

The first five terms are $\left\{\frac{5}{6}, \frac{5}{12}, \frac{1}{8}, \frac{1}{36}, \frac{5}{1008}\right\}$.

Figure 5.1.7 shows the graph of the sequence. Notice that, since factorials grow very quickly, the presence of the factorial term in the denominator results in the denominator becoming much larger than the numerator as n increases. This means the quotient gets smaller and, as the plot of the terms shows, the terms are decreasing and nearing zero.

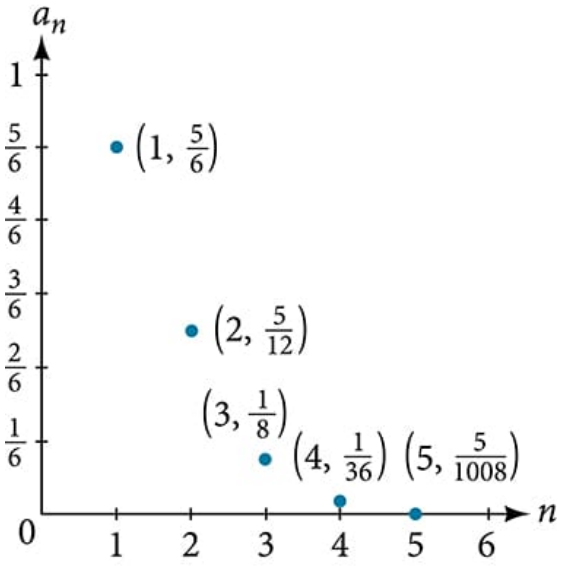


Figure 5.1.7

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5.1.1: Resources and Key Concepts

Resources

Videos

- Sequences
  - Sequences and Summation Notation - Sequences
- Recursive formula for sequence
  - Sequences and Summation Notation - Recursively-Defined Sequences

Key Concepts

Definitions

- **Sequence**: A function whose domain is the set of natural numbers.
- **Terms (of a sequence)**: The numbers in a sequence; the output values of the sequence function.
- **Infinite Sequence**: A sequence that continues indefinitely.
- **Finite Sequence**: A sequence whose domain consists of only the first n natural numbers.
- **Explicit Formula (for a sequence)**: A formula that defines the nth term of a sequence, a_n, in relation to its position n.
- **Alternating Sequence**: A sequence whose terms alternate in sign (positive, negative, positive, negative, ... or vice-versa).
- **Recursive Formula (for a sequence)**: A formula that defines each term of a sequence using a preceding term (or multiple preceding terms). Recursive formulas must always state the initial term(s).
- **Initial Term(s) (of a recursive sequence)**: The first term (or first few terms) of a sequence that are given directly in a recursive definition, before the recursive part of the formula applies.
- **Factorial (Recursive Definition)**: The factorial of a natural number n, denoted $n!$, is defined by $0! = 1$ and $n! = n \cdot (n-1)!$ for $n \geq 1$.
- **Factorial (Product of Natural Numbers Definition)**: The factorial of a natural number n, denoted $n!$, is defined as the product of all natural numbers less than or equal to n. That is, $n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1$.

Common Mistakes

- **Incorrectly Identifying Patterns for Explicit Formulas**: When given terms and asked to find an explicit formula, misinterpreting the pattern of numerators, denominators, signs, or exponents.
- **Errors in Evaluating Recursive Formulas**: Not starting with the given initial term(s).
- **Misinterpreting Alternating Sign Notation**: Incorrectly using or interpreting powers of (-1) (e.g., $(-1)^n$, $(-1)^{n+1}$) to generate alternating signs.
- **Errors in Factorial Calculations**: Incorrectly expanding or simplifying expressions involving factorials (e.g., $\frac{n!}{(n-2)!} \neq \frac{1}{-2!}$).
- **Domain Issues with Piecewise Explicit Formulas**: Applying the wrong piece of the formula for a given term number n.
- **Assuming a Sequence is Continuous**: Graphing a sequence as a continuous curve rather than a set of distinct points, as the domain is the natural numbers.

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5.1.2: Homework

Reading Questions

1. What is a sequence? What are the numbers in a sequence called?
2. What is the difference between an infinite sequence and a finite sequence?
3. What is an explicit formula for a sequence? How does it define the terms?
4. If a sequence alternates in sign, what common factor is often used in its explicit formula to achieve this alternation?
5. What is a recursive formula for a sequence? What must always be stated along with a recursive formula?
6. Give an example of a sequence that is easily defined by a recursive formula but might be more difficult to define with an explicit formula.
7. What is factorial notation (e.g., $n!$)? Provide both the recursive and product definitions.
8. What is the value of $0!$?
9. How would you find the 5th term of the sequence defined by $a_1 = 2$ and $a_n = 3a_{n-1} + 1$ for $n \geq 2$?
10. If the terms of a sequence are $\{2, 4, 8, 16, ...\}$, what is a possible explicit formula for a_n?
11. When graphing a sequence, why are the points distinct and not connected by a continuous curve?
12. Discuss the meaning of a sequence. If a finite sequence is defined by a formula, what is its domain? What about an infinite sequence?
13. Describe three ways that a sequence can be defined.
14. Is the ordered set of even numbers an infinite sequence? What about the ordered set of odd numbers? Explain why or why not.
15. What happens to the terms, a_n, of a sequence when there is a negative factor in the formula that is raised to a power that includes n? What is the term used to describe this phenomenon?
16. What is a factorial, and how is it denoted? Use an example to illustrate how factorial notation can be beneficial.

Homework

Algebraic

For the following exercises, write the first four terms of the sequence.

17. $a_n = 2^n - 2$
18. $a_n = -\frac{16}{n+1}$
19. $a_n = -(-5)^{n-1}$
20. $a_n = \frac{2^n}{n^3}$
21. $a_n = \frac{2n+1}{n^3}$
22. $a_n = 1.25 \cdot (-4)^{n-1}$
23. $a_n = -4 \cdot (-6)^{n-1}$
24. $a_n = \frac{n^2}{2n+1}$
25. $a_n = (-10)^n + 1$
26. $a_n = -\left(\frac{4 \cdot (-5)^{n-1}}{5}\right)$

For the following exercises, write the first eight terms of the piecewise sequence.

27. $a_n = \begin{cases} (-2)^n - 2, & \text{if } n \text{ is even} \\ (3)^{n-1}, & \text{if } n \text{ is odd} \end{cases}$
28. $a_n = \begin{cases} \frac{n^2}{2n+1}, & \text{if } n \leq 5 \\ n^2 - 5, & \text{if } n > 5 \end{cases}$
29. $a_n = \begin{cases} (2n+1)^2, & \text{if } n \text{ is divisible by } 4 \\ \frac{2}{n}, & \text{if } n \text{ is not dividible by } 4 \end{cases}$

30. $a_n = \begin{cases} -0.6 \cdot 5^{n-1}, & \text{if } n \text{ is prime or } 1 \\ 2.5 \cdot (-2)^{n-1}, & \text{if } n \text{ is composite} \end{cases}$
31. $a_n = \begin{cases} 4(n^2-2), & \text{if } n \leq 3 \text{ or } n > 6 \\ \frac{n^2-2}{4}, & \text{if } 3 < n \leq 6 \end{cases}$

For the following exercises, write an explicit formula for each sequence.

32. $4, 7, 12, 19, 28, \ldots$
33. $-4, 2, -10, 14, -34, \ldots$
34. $1, 1, \frac{4}{3}, 2, \frac{16}{5}, \ldots$
35. $0, \frac{1-e^1}{1+e^2}, \frac{1-e^2}{1+e^3}, \frac{1-e^3}{1+e^4}, \frac{1-e^4}{1+e^5}, \ldots$
36. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$

For the following exercises, write the first five terms of the sequence.

37. $a_1 = 9$, $a_n = a_{n-1} + n$
38. $a_1 = 3$, $a_n = (-3)a_{n-1}$
39. $a_1 = -4$, $a_n = \frac{a_{n-1}+2n}{a_{n-1}-1}$
40. $a_1 = -1$, $a_n = \frac{(-3)^{n-1}}{a_{n-1}-2}$
41. $a_1 = -30$, $a_n = (2 + a_{n-1})\left(\frac{1}{2}\right)^n$

For the following exercises, write the first eight terms of the sequence.

42. $a_1 = \frac{1}{24}$, $a_2 = 1$, $a_n = (2a_{n-2})(3a_{n-1})$
43. $a_1 = -1$, $a_2 = 5$, $a_n = a_{n-2}(3 - a_{n-1})$
44. $a_1 = 2$, $a_2 = 10$, $a_n = \frac{2(a_{n-1}+2)}{a_{n-2}}$

For the following exercises, write a recursive formula for each sequence.

45. $-2.5, -5, -10, -20, -40, \ldots$
46. $-8, -6, -3, 1, 6, \ldots$
47. $2, 4, 12, 48, 240, \ldots$
48. $35, 38, 41, 44, 47, \ldots$
49. $15, 3, \frac{3}{5}, \frac{3}{25}, \frac{3}{125}, \cdots$

For the following exercises, evaluate the factorial.

50. $6!$
51. $(\frac{12}{6})!$
52. $\frac{12!}{6!}$
53. $\frac{100!}{99!}$

For the following exercises, write the first four terms of the sequence.

54. $a_n = \frac{n!}{n^2}$
55. $a_n = \frac{3 \cdot n!}{4 \cdot n!}$
56. $a_n = \frac{n!}{n^2-n-1}$
57. $a_n = \frac{100n}{n(n-1)!}$

Graphical

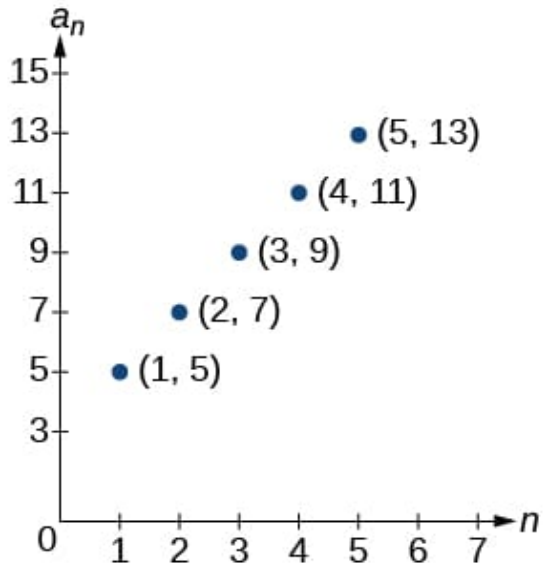
For the following exercises, graph the first five terms of the indicated sequence

58. $a_n = \frac{(-1)^n}{n} + n$
59. $a_n = \begin{cases} \frac{4+n}{2n}, & \text{if } n \text{ is even} \\ 3 + n, & \text{if } n \text{ is odd} \end{cases}$

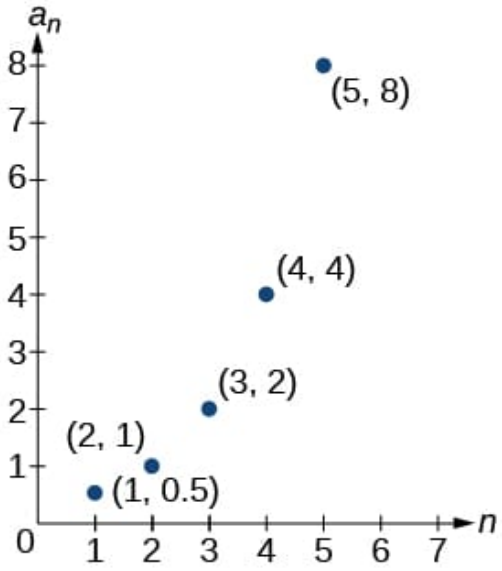
60. $a_1 = 2, a_n = (-a_{n-1} + 1)^2$
61. $a_1 = 1, a_n = a_{n-1} + 8$
62. $a_n = \frac{(n+1)!}{(n-1)!}$

For the following exercises, write an explicit formula for the sequence using the first five points shown on the graph.

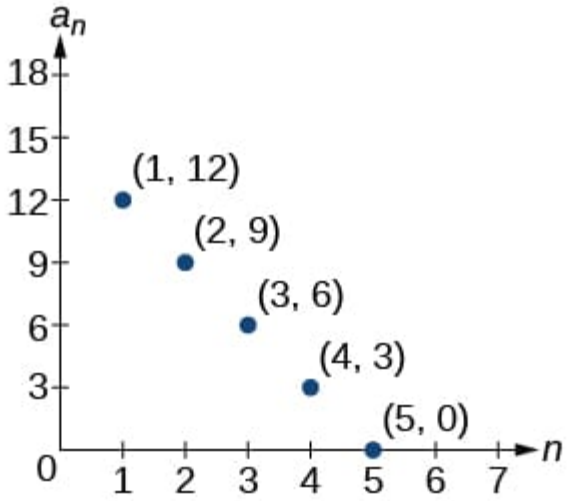
63.



64.

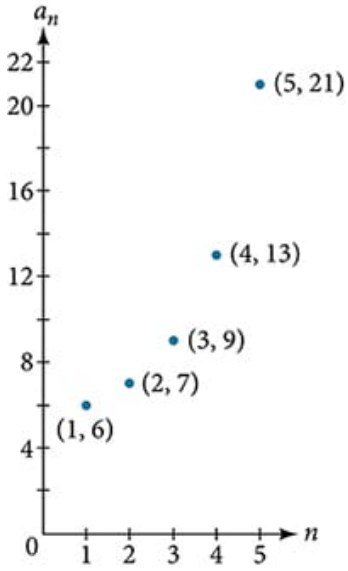


65.

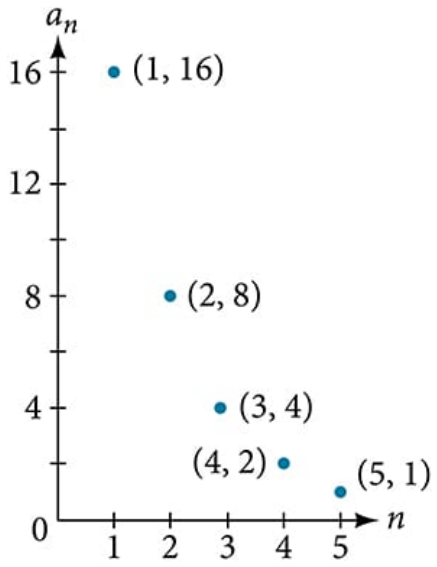


For the following exercises, write a recursive formula for the sequence using the first five points shown on the graph.

56.



57.



Technology

Follow these steps to evaluate a sequence defined recursively using a graphing calculator:

- On the home screen, key in the value for the initial term a_1 and press $[\text{ENTER}]$.
- Enter the recursive formula by keying in all numerical values given in the formula, along with the key strokes $[2\text{ND}]\text{ANS}$ for the previous term a_{n-1}. Press $[\text{ENTER}]$.
- Continue pressing $[\text{ENTER}]$ to calculate the values for each successive term.

For the following exercises, use the steps above to find the indicated term or terms for the sequence.

68. Find the first five terms of the sequence $a_1 = \frac{87}{111}$, $a_n = \frac{4}{3}a_{n-1} + \frac{12}{37}$. Use the $>\text{Frac}$ feature to give fractional results.
69. Find the 15th term of the sequence $a_1 = 625$, $a_n = 0.8a_{n-1} + 18$.
70. Find the first five terms of the sequence $a_1 = 2$, $a_n = 2^{(a_{n-1}-1)} + 1$.
71. Find the first ten terms of the sequence $a_1 = 8$, $a_n = \frac{(a_{n-1}+1)!}{a_{n-1}!}$.
72. Find the tenth term of the sequence $a_1 = 2$, $a_n = na_{n-1}$

Extensions

73. Consider the sequence defined by $a_n = -6 - 8n$. Is $a_n = -421$ a term in the sequence? Verify the result.
74. What term in the sequence $a_n = \frac{n^2+4n+4}{2(n+2)}$ has the value 41? Verify the result.
75. Find a recursive formula for the sequence $1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, \ldots$. (Hint: find a pattern for a_n based on the first two terms.)
76. Calculate the first eight terms of the sequences $a_n = \frac{(n+2)!}{(n-1)!}$ and $b_n = n^3 + 3n^2 + 2n$, and then make a conjecture about the relationship between these two sequences.
77. Prove the conjecture made in the preceding exercise.

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5.2: Arithmetic Sequences

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Companies often make large purchases, such as computers and vehicles, for business use. The book-value of these supplies decreases each year for tax purposes. This decrease in value is called **depreciation**. One method of calculating depreciation is **straight-line depreciation**, in which the value of the asset decreases by the same amount each year.

As an example, consider a woman who starts a small contracting business. She purchases a new truck for $25,000. After five years, she estimates that she will be able to sell the truck for $8,000. The loss in value of the truck will therefore be $17,000, which is $3,400 per year for five years. The truck will be worth $21,600 after the first year; $18,200 after two years; $14,800 after three years; $11,400 after four years; and $8,000 at the end of five years. In this section, we will consider specific kinds of sequences that will allow us to calculate depreciation, such as the truck's value.

An Introduction to Arithmetic Sequences

The values of the truck in the example are said to form an **arithmetic sequence** because they change by a constant amount each year. Each term increases or decreases by the same constant value called the **common difference** of the sequence.

Definition: Arithmetic Sequence

An **arithmetic sequence** is a sequence having the property that the difference between any two consecutive terms is constant. This constant is called the **common difference**. Therefore, if a_1 is the first term of an arithmetic sequence and d is the common difference, the sequence will be:

$$\{a_n\} = \{a_1, a_1 + d, a_1 + 2d, a_1 + 3d, ...\}.$$

For this sequence representing the value of the truck, the common difference is -3400.

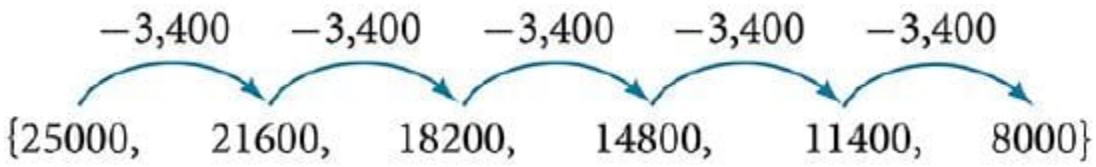


Figure 5.2.1

The sequence below is another example of an arithmetic sequence. In this case, the constant difference is 3. You can choose any term of the sequence, and add 3 to find the subsequent term.

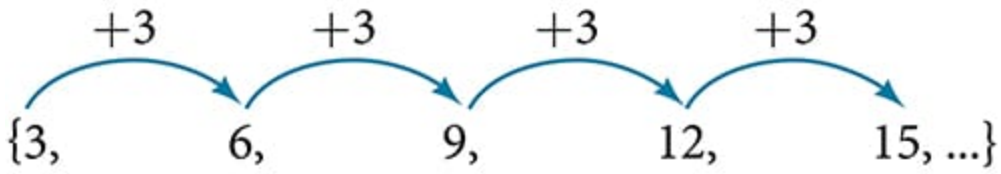


Figure 5.2.2

Example 5.2.1: Finding Common Differences

Is each sequence arithmetic? If so, find the common difference.

a. $\{a_n\} = \{1, 2, 4, 8, 16, ...\}$
b. $\{b_n\} = \{-3, 1, 5, 9, 13, ...\}$

Solutions available at the end of this section.

Q&A

If we are told that a sequence is arithmetic, do we have to subtract every term from the following term to find the common difference?

No. If we know that the sequence is arithmetic, we can choose any one term in the sequence, and subtract it from the subsequent term to find the common difference.

Checkpoint 5.2.1

Is the given sequence arithmetic? If so, find the common difference.

a. $\{c_n\} = \{18, 16, 14, 12, 10, ...\}$
b. $\{d_n\} = \{1, 3, 6, 10, 15, ...\}$

The Recursive Formula for Arithmetic Sequences

Now that we can recognize an arithmetic sequence, we will find the terms using a recursive formula. For example, if the common difference is 5, then each term is the previous term plus 5. As with any recursive formula, the first term must be given.

$$a_n = a_{n-1} + d, \quad n \geq 2$$

Theorem: Recursive Formula of an Arithmetic Sequence

The recursive formula for the arithmetic sequence with first term a_1 and common difference d is

$$a_n = a_{n-1} + d.$$

Proof available at the end of this section.

Example 5.2.2: Writing a Recursive Formula for an Arithmetic Sequence

Write a recursive formula for the arithmetic sequence.

$$\{-18, -7, 4, 15, 26, ...\}$$

Solution available at the end of this section.

Q&A

Do we have to subtract the first term from the second term to find the common difference?

No. We can subtract any term in the sequence from the subsequent term. It is, however, most common to subtract the first term from the second term because it is often the easiest method of finding the common difference.

Checkpoint 5.2.2

Write a recursive formula for the arithmetic sequence.

$$\{25, 37, 49, 61, ...\}$$

The Explicit Formula for Arithmetic Sequences

We now find the terms if we are given the first term and the common difference. The terms can be found by beginning with the first term and adding the common difference repeatedly. Doing so results in the explicit formula for an arithmetic sequence.

Theorem: Explicit Formula of an Arithmetic Sequence

The explicit formula for the arithmetic sequence with first term a_1 and common difference d is

$$a_n = a_1 + (n-1)d.$$

Proof (by Mathematical Induction) available at the end of this section.

Example 5.2.3: Writing Terms of Arithmetic Sequences

Write the first five terms of the arithmetic sequence with $a_1 = 17$ and $d = -3$.

Solution available at the end of this section.

Checkpoint 5.2.3

List the first five terms of the arithmetic sequence with $a_1 = 1$ and $d = 5$.

Example 5.2.4: Writing Terms of Arithmetic Sequences

Given $a_1 = 8$ and $a_4 = 14$, find a_5.

Solution available at the end of this section.

Checkpoint 5.2.4

Given $a_3 = 7$ and $a_5 = 17$, find a_2.

Using Explicit Formulas for Arithmetic Sequences

We can think of an arithmetic sequence as a "linear-like" function on the domain of the natural numbers—it is a "linear-like" function because it has a constant rate of change. The common difference is the constant rate of change, or the slope of the function. We can construct the linear function if we know the slope and the vertical intercept.

$$a_n = a_1 + d(n-1)$$

To find the y-intercept of the function, we can subtract the common difference from the first term of the sequence. Consider the following sequence.

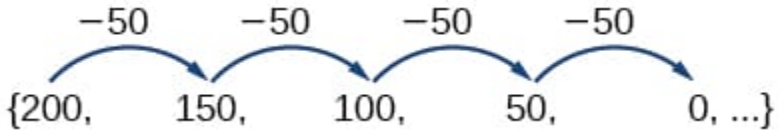


Figure 5.2.8

The common difference is -50, so the sequence represents a linear function with a slope of -50. To find the y-intercept, we subtract -50 from 200:

$$200 - (-50) = 200 + 50 = 250.$$

The graph is shown in Figure 5.2.9.

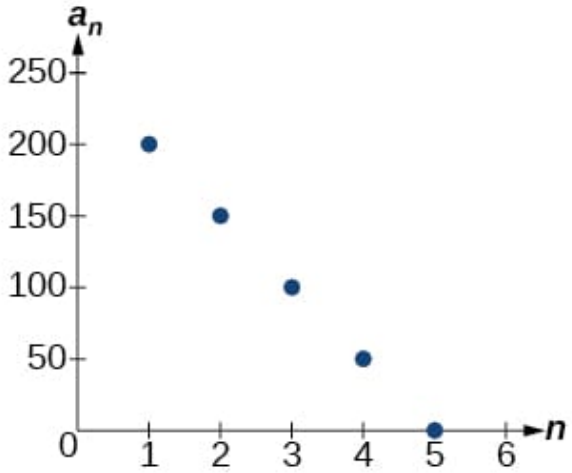


Figure 5.2.9

Recall the slope-intercept form of a line is $y = mx + b$. When dealing with sequences, we use a_n in place of y and n in place of x. If we know the slope and vertical intercept of the function, we can substitute them for m and b in the slope-intercept form of a line. Substituting -50 for the slope and 250 for the vertical intercept, we get the following equation:

$$a_n = -50n + 250.$$

We do not need to find the vertical intercept to write an explicit formula for an arithmetic sequence. Another explicit formula for this sequence is $a_n = 200 - 50(n-1)$, which simplifies to $a_n = -50n + 250$.

Example 5.2.5: Writing the nth Term Explicit Formula for an Arithmetic Sequence

Write an explicit formula for the arithmetic sequence.

$$\{2, 12, 22, 32, 42, ...\}$$

Solution available at the end of this section.

Checkpoint 5.2.5

Write an explicit formula for the following arithmetic sequence.

$$\{50, 47, 44, 41, ...\}$$

Finding the Number of Terms in a Finite Arithmetic Sequence

Explicit formulas can be used to determine the number of terms in a finite arithmetic sequence. We need to find the common difference, and then determine how many times the common difference must be added to the first term to obtain the final term of the sequence.

Example 5.2.6: Finding the Number of Terms in a Finite Arithmetic Sequence

Find the number of terms in the finite arithmetic sequence.

$$\{8, 1, -6, \ldots, -41\}$$

Solution available at the end of this section.

Checkpoint 5.2.6

Find the number of terms in the finite arithmetic sequence.

$$\{6, 11, 16, \ldots, 56\}$$

Solving Application Problems with Arithmetic Sequences

In many application problems, it often makes sense to use an initial term of a_0 instead of a_1. In these problems, we alter the explicit formula slightly to account for the difference in initial terms. We use the following formula:

$$a_n = a_0 + dn.$$

Example 5.2.7: Solving Application Problems with Arithmetic Sequences

A five-year-old child receives an allowance of $1 each week. His parents promise him an annual increase of $2 per week.

a. Write a formula for the child's weekly allowance in a given year.
b. What will the child's allowance be when he is 16 years old?

Solutions available at the end of this section.

Checkpoint 5.2.7

A woman decides to go for a 10-minute run every day this week and plans to increase the time of her daily run by four minutes each week. Write a formula for the time of her run after n weeks. How long will her daily run be eight weeks from today?

Success in College: Dealing With Failure (Part 3)

Sometimes unexpected events happen during a term. You get sick and are in bed for a week. Someone that you need to help

care for gets sick. You are in a car accident and lose your transportation. A daycare provider quits. A relationship falls apart. Your boss at work needs you to increase your hours. Because this happens, you don't have time to keep up with your courses. You miss class. You fail one or more tests.

1. A student in this situation decides to withdraw from the math class. She reasons that it is better to concentrate on passing her other classes. She will re-enroll in math the next term. Are there any adverse consequences of making this choice? Explain.
2. A student in this situation decides to keep coming to class and trying to do the homework. He plans to take the course over the next term but wants to learn as much as he can now. This should make the course easier next term. Are there any adverse consequences of making this choice? Explain.

Miscellaneous Topics: Teachers and Role Models

Parents, teachers, and other role models influence our attitudes about math. What they say and do can change our view of the importance of math, how hard math is, and whether we are good at math. Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Describe the attitudes towards math of your parent, teacher (not me, of course), or another personal role model.
2. Describe how the opinion of this role model has influenced your attitudes about math.

Answers

Section Preview

The link between arithmetic sequences and linear functions should be stressed throughout the instruction of this material. This creates a cognitive link to past material for the student.

Section Preview

- **Linear Functions**
  - Equations of Lines — Slope-Intercept Form: The derivation of the explicit formula explicitly says "Recall the slope-intercept form of a line is $y = mx + b$" and builds $a_n = -50n + 250$ by analogy to $y = mx + b$.

Section Preview

- Find the common difference for an arithmetic sequence.
- Write terms of an arithmetic sequence.
- Use a recursive formula for an arithmetic sequence.
- Use an explicit formula for an arithmetic sequence.

Example 5.2.1: Finding Common Differences

Subtract each term from the subsequent term to determine whether a common difference exists.

a. The sequence is not arithmetic because there is no common difference.

$2 - 1 = 1 \qquad 4 - 2 = 2 \qquad 8 - 4 = 4 \qquad 16 - 8 = 8$

Figure 5.2.3

b. The sequence is arithmetic because there is a common difference. The common difference is 4.

$1 - (-3) = 4 \qquad 5 - 1 = 4 \qquad 9 - 5 = 4 \qquad 13 - 9 = 4$

Figure 5.2.4

The graph of each of these sequences is shown in Figure 5.2.5. We can see from the graphs that, although both sequences show growth, $\{a_n\}$ is not linear whereas $\{b_n\}$ is linear. Arithmetic sequences have a constant rate of change so their graphs will always be points on a line.

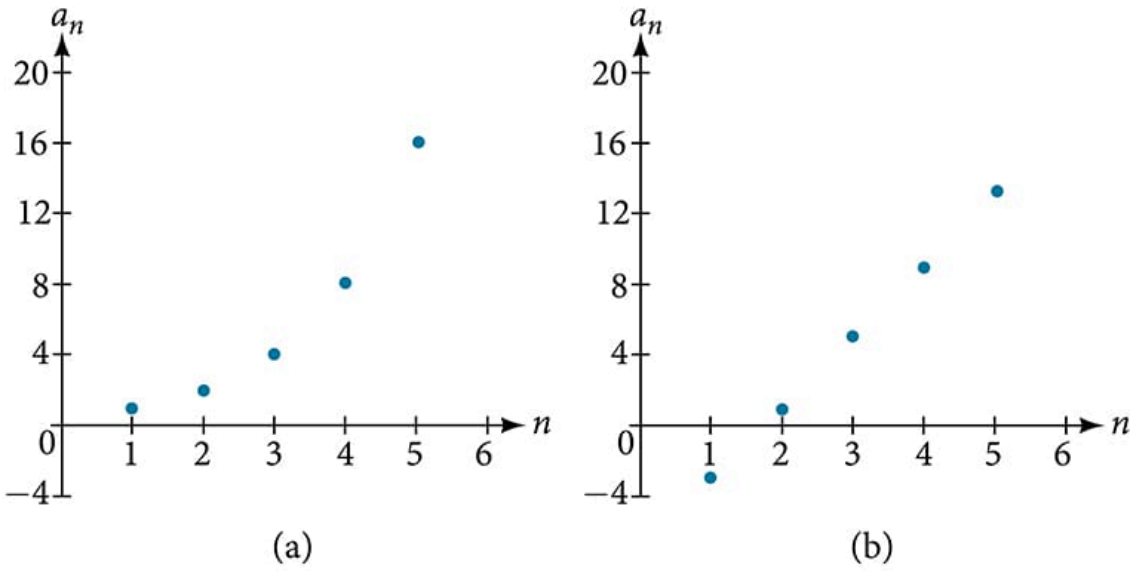


Figure 5.2.5

Theorem: Recursive Formula of an Arithmetic Sequence

This is a direct result of the definition of an arithmetic sequence.

Example 5.2.2**: Writing a Recursive Formula for an Arithmetic Sequence**

The first term is given as -18. The common difference can be found by subtracting the first term from the second term.

$$d = -7 - (-18) = 11$$

Substitute the initial term and the common difference into the recursive formula for arithmetic sequences.

$$\begin{aligned} a_1 &= -18 \\ a_n &= a_{n-1} + 11, \text{ for } n \geq 2 \end{aligned}$$

We see that the common difference is the slope of the line formed when we graph the terms of the sequence, as shown in Figure 5.2.6. The growth pattern of the sequence shows the constant difference of 11 units.

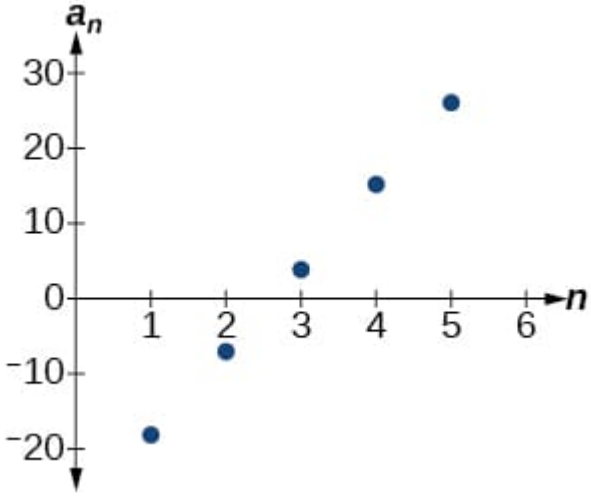


Figure 5.2.6

Theorem: Explicit Formula of an Arithmetic Sequence

NOTE: The method of proof for this theorem is Mathematical Induction, which is a topic generally covered *after* learning about sequences and series. If you do not know about proofs by Mathematical Induction, feel free to come back to this proof later.

Let a_n be the nth term of an arithmetic sequence with common difference d. Our claim is that $a_n = a_1 + (n-1)d$.

Base Case ($n = 1$): For $n = 1$, we get $a_1 = a_1 + (1-1)d = a_1 + 0 = a_1$, which is correct. Therefore, the claim is true for $n = 1$.

Induction Hypothesis: Assume the claim is true for $n = k$. That is, assume $a_k = a_1 + (k-1)d$.

Inductive Step ($n = k+1$): For $n = k+1$, we would like to show that $a_{k+1} = a_1 + ((k+1)-1)d$.

$$\begin{aligned} a_{k+1} &= a_k + d && \text{(since the sequence is arithmetic, the next term is } a_k + d) \\ &= a_1 + (k-1)d + d && \text{(from the Induction Hypothesis)} \\ &= a_1 + (k-1+1)d && \text{(factoring out } d) \\ &= a_1 + (k+1-1)d && \text{(Commutative Property of Addition)} \\ &= a_1 + ((k+1)-1)d && \text{(Associative Property of Addition)} \end{aligned}$$

Thus, $a_{k+1} = a_1 + ((k+1)-1)d$, and so the claim must be true for $n = k+1$.

Hence, the claim is true for all natural numbers by Mathematical Induction.

Example $5.2.3$**: Writing Terms of Arithmetic Sequences**

Adding -3 is the same as subtracting 3. Beginning with the first term, subtract 3 from each term to find the next term. The first five terms are $\{17, 14, 11, 8, 5\}$. As expected, the graph of the sequence consists of points on a line as shown in Figure $5.2.7$.

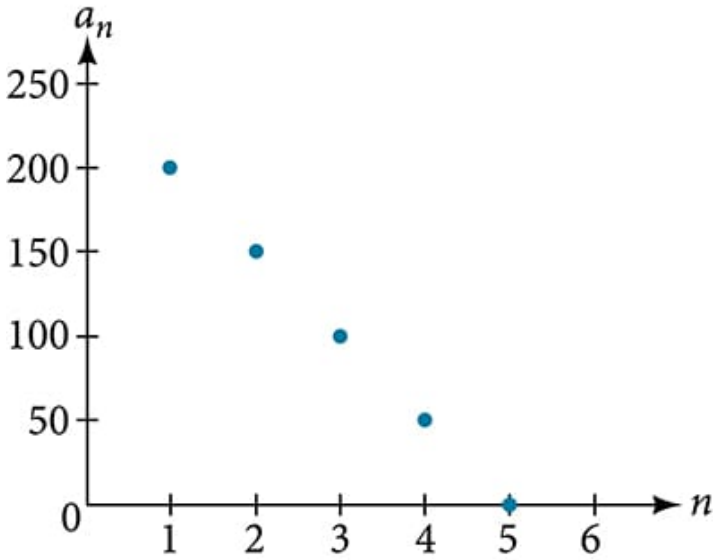


Figure $5.2.7$

Example $5.2.4$**: Writing Terms of Arithmetic Sequences**

The sequence can be written in terms of the initial term, 8, and the common difference d.

$$\{8, 8+d, 8+2d, 8+3d\}.$$

We know the fourth term equals 14; we know the fourth term has the form $a_1 + 3d = 8 + 3d$.

We can find the common difference d.

$$\begin{aligned} & a_n = a_1 + (n-1)d \\ \Longrightarrow \quad & 14 = 8 + 3d && \text{(substituting } n = 4, a_1 = 8, \text{ and } a_4 = 14) \\ \Longrightarrow \quad & 6 = 3d && \text{(subtracting 8 from both sides)} \\ \Longrightarrow \quad & 2 = d && \text{(dividing both sides by 3)} \end{aligned}$$

Find the fifth term by adding the common difference to the fourth term.

$$a_5 = a_4 + 2 = 16.$$

Notice that the common difference is added to the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by adding the common difference to the first term nine times or by using the equation $a_n = a_1 + (n-1)d$.

Example $5.2.5$**: Writing the** n^{th} **Term Explicit Formula for an Arithmetic Sequence**

The common difference can be found by subtracting the first term from the second term.

$$\begin{aligned} d &= a_2 - a_1 \\ &= 12 - 2 \\ &= 10 \end{aligned}$$

The common difference is 10. Substitute the common difference and the first term of the sequence into the formula and simplify.

$$\begin{aligned} & a_n = 2 + 10(n-1) \\ \Longrightarrow \quad & a_n = 10n - 8 \end{aligned}$$

The graph of this sequence, represented in Figure $5.2.10$, shows a slope of 10 and a vertical intercept of -8.

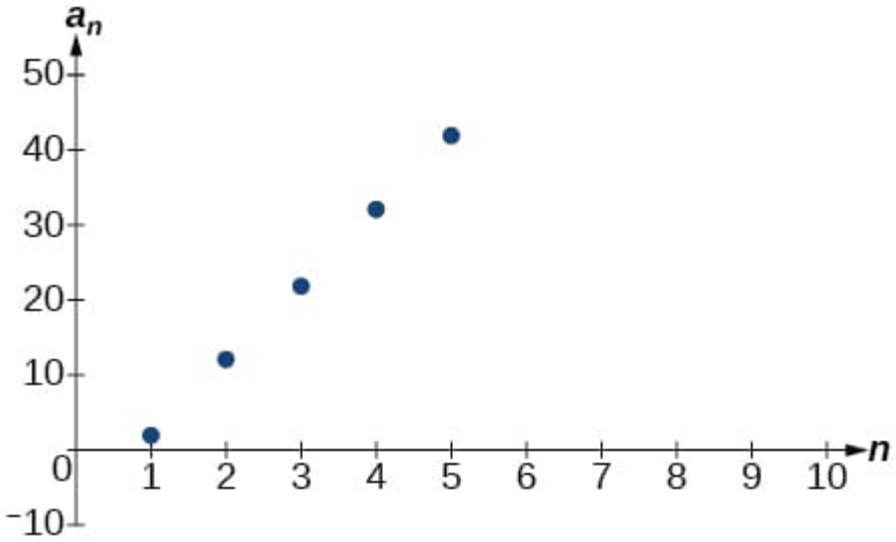


Figure 5.2.10

Example 5.2.6**: Finding the Number of Terms in a Finite Arithmetic Sequence**

The common difference can be found by subtracting the first term from the second term.

$$1 - 8 = -7$$

The common difference is -7. Substitute the common difference and the initial term of the sequence into the nth term formula and simplify.

$$\begin{aligned} & a_n = a_1 + d(n-1) \\ \Longrightarrow \quad & a_n = 8 + (-7)(n-1) && \text{(substituting)} \\ \Longrightarrow \quad & a_n = 15 - 7n && \text{(simplifying)} \end{aligned}$$

Substitute -41 for a_n and solve for n

$$\begin{aligned} & -41 = 15 - 7n \\ \Longrightarrow \quad & -56 = -7n && \text{(subtracting 15 from both sides)} \\ \Longrightarrow \quad & 8 = n && \text{(dividing both sides by } -7) \end{aligned}$$

There are eight terms in the sequence.

Example 5.2.7**: Solving Application Problems with Arithmetic Sequences**

a. The situation can be modeled by an arithmetic sequence with an initial term of 1 and a common difference of 2. Let A be the amount of the allowance and n be the number of years after age 5. Using the altered explicit formula for an arithmetic sequence we get:

$$A_n = 1 + 2n.$$

b. We can find the number of years since age 5 by subtracting.

$$16 - 5 = 11$$

We are looking for the child's allowance after 11 years. Substitute 11 into the formula to find the child's allowance at age 16.

$$A_{11} = 1 + 2(11) = 23$$

The child's allowance at age 16 will be $23 per week.

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5.2.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Linear Functions**
  - Equations of Lines: The explicit formula for an arithmetic sequence, $a_n = a_1 + (n-1)d$, resembles the point-slope form of a linear equation.

Videos

- Arithmetic sequence
  - Arithmetic Sequences - Arithmetic Sequences

Key Concepts

Definitions

- **Arithmetic Sequence**: A sequence that has the property that the difference between any two consecutive terms is a constant.
- **Common Difference (d)**: The constant difference between any two consecutive terms in an arithmetic sequence.

Theorems

- **Recursive Formula of an Arithmetic Sequence**: The recursive formula for the arithmetic sequence with first term a_1 and common difference d is $a_n = a_{n-1} + d$.
- **Explicit Formula of an Arithmetic Sequence**: The explicit formula for the arithmetic sequence with first term a_1 and common difference d is $a_n = a_1 + (n-1)d$.

Common Mistakes

- **Incorrectly Calculating the Common Difference**: Subtracting terms in the wrong order (e.g., $a_n - a_{n+1}$ instead of $a_{n+1} - a_n$) or assuming a sequence is arithmetic without checking if the difference is constant between all consecutive terms.
- **Errors in Applying the Recursive Formula**: Using the wrong previous term or an incorrect common difference when generating terms recursively.
- **Errors in Applying the Explicit Formula**: Using n instead of $(n-1)$ in the term $(n-1)d$, i.e., writing $a_n = a_1 + nd$.
- **Confusing Arithmetic Sequences with Other Types of Sequences**: Assuming a sequence is arithmetic when it might be geometric or neither, especially if only looking at the first few terms.
- **Off-by-One Errors with a_0 vs. a_1**: When using an explicit formula based on an initial term a_0 (like $a_n = a_0 + dn$), confusing it with the formula based on a_1 ($a_n = a_1 + (n-1)d$), or misinterpreting which term is considered the "starting" term in an application.
- **Misinterpreting "Number of Terms" in a Finite Sequence**: Errors in setting up or solving the explicit formula to find the total number of terms in a finite arithmetic sequence.

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5.2.2: Homework

Reading Questions

1. What defines an arithmetic sequence?
2. How is the common difference of an arithmetic sequence found?
3. If you are told a sequence is arithmetic, do you need to check the difference between every pair of consecutive terms to find the common difference? Explain.
4. What is the recursive formula for an arithmetic sequence with first term a_1 and common difference d?
5. What is the explicit formula for an arithmetic sequence with first term a_1 and common difference d?
6. How can an arithmetic sequence be thought of as a "linear-like" function? What does the common difference correspond to in this analogy?
7. If the first term of an arithmetic sequence is $a_1 = 8$ and the fourth term is $a_4 = 14$, how can you find the common difference d?
8. How can you use an explicit formula to find the number of terms in a finite arithmetic sequence if you know the first term, the last term, and the common difference?
9. What is the difference between the explicit formula $a_n = a_1 + (n-1)d$ and $a_n = a_0 + dn$? When might the second formula be more convenient?
10. When graphing an arithmetic sequence, why are the points collinear (lie on a straight line)?
11. What is an arithmetic sequence?
12. How is the common difference of an arithmetic sequence found?
13. How do we determine whether a sequence is arithmetic?
14. What are the main differences between using a recursive formula and using an explicit formula to describe an arithmetic sequence?
15. Describe how linear functions and arithmetic sequences are similar. How are they different?

Homework

Algebraic

For the following exercises, find the common difference for the arithmetic sequence provided.

16. $\{5, 11, 17, 23, 29, ...\}$
17. $\left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\right\}$

For the following exercises, determine whether the sequence is arithmetic. If so find the common difference.

18. $\{11.4, 9.3, 7.2, 5.1, 3, ...\}$
19. $\{4, 16, 64, 256, 1024, ...\}$

For the following exercises, write the first five terms of the arithmetic sequence given the first term and common difference.

20. $a_1 = -25$, $d = -9$
21. $a_1 = 0$, $d = \frac{2}{3}$

For the following exercises, write the first five terms of the arithmetic series given two terms.

22. $a_1 = 17$, $a_7 = -31$
23. $a_{13} = -60$, $a_{33} = -160$

For the following exercises, find the specified term for the arithmetic sequence given the first term and common difference.

24. First term is 3, common difference is 4, find the 5th term.
25. First term is 4, common difference is 5, find the 4th term.
26. First term is 5, common difference is 6, find the 8th term.
27. First term is 6, common difference is 7, find the 6th term.
28. First term is 7, common difference is 8, find the 7th term.

For the following exercises, find the first term given two terms from an arithmetic sequence.

29. Find the first term or a_1 of an arithmetic sequence if $a_6 = 12$ and $a_{14} = 28$.
30. Find the first term or a_1 of an arithmetic sequence if $a_7 = 21$ and $a_{15} = 42$.
31. Find the first term or a_1 of an arithmetic sequence if $a_8 = 40$ and $a_{23} = 115$.
32. Find the first term or a_1 of an arithmetic sequence if $a_9 = 54$ and $a_{17} = 102$.
33. Find the first term or a_1 of an arithmetic sequence if $a_{11} = 11$ and $a_{21} = 16$.

For the following exercises, find the specified term given two terms from an arithmetic sequence.

34. $a_1 = 33$ and $a_7 = -15$. Find a_4.
35. $a_3 = -17.1$ and $a_{10} = -15.7$. Find a_{21}.

For the following exercises, use the recursive formula to write the first five terms of the arithmetic sequence.

36. $a_1 = 39$; $a_n = a_{n-1} - 3$
37. $a_1 = -19$; $a_n = a_{n-1} - 1.4$

For the following exercises, write a recursive formula for each arithmetic sequence.

38. $a = \{40, 60, 80, ...\}$
39. $a = \{17, 26, 35, ...\}$
40. $a = \{-1, 2, 5, ...\}$
41. $a = \{12, 17, 22, ...\}$
42. $a = \{-15, -7, 1, ...\}$
43. $a = \{8.9, 10.3, 11.7, ...\}$
44. $a = \{-0.52, -1.02, -1.52, ...\}$
45. $a = \left\{\frac{1}{5}, \frac{9}{20}, \frac{7}{10}, ...\right\}$
46. $a = \left\{-\frac{1}{2}, -\frac{5}{4}, -2, ...\right\}$
47. $a = \left\{\frac{1}{6}, -\frac{11}{12}, -2, ...\right\}$

For the following exercises, write a recursive formula for the given arithmetic sequence, and then find the specified term.

48. $a = \{7, 4, 1, ...\}$; Find the 17th term.
49. $a = \{4, 11, 18, ...\}$; Find the 14th term.
50. $a = \{2, 6, 10, ...\}$; Find the 12th term.

For the following exercises, use the explicit formula to write the first five terms of the arithmetic sequence.

51. $a_n = 24 - 4n$
52. $a_n = \frac{1}{2}n - \frac{1}{2}$

For the following exercises, write an explicit formula for each arithmetic sequence.

53. $a = \{3, 5, 7, ...\}$
54. $a = \{32, 24, 16, ...\}$
55. $a = \{-5, 95, 195, ...\}$
56. $a = \{-17, -217, -417, ...\}$
57. $a = \{1.8, 3.6, 5.4, ...\}$
58. $a = \{-18.1, -16.2, -14.3, ...\}$
59. $a = \{15.8, 18.5, 21.2, ...\}$
60. $a = \left\{\frac{1}{3}, -\frac{4}{3}, -3, ...\right\}$

61. $a = \left\{0, \frac{1}{3}, \frac{2}{3}, \ldots\right\}$
62. $a = \left\{-5, -\frac{10}{3}, -\frac{5}{3}, \ldots\right\}$

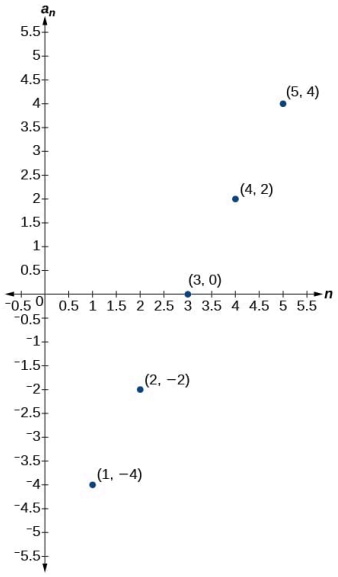
For the following exercises, find the number of terms in the given finite arithmetic sequence.

63. $a = \{3, -4, -11, \ldots, -60\}$
64. $a = \{1.2, 1.4, 1.6, \ldots, 3.8\}$
65. $a = \left\{\frac{1}{2}, 2, \frac{7}{2}, \ldots, 8\right\}$

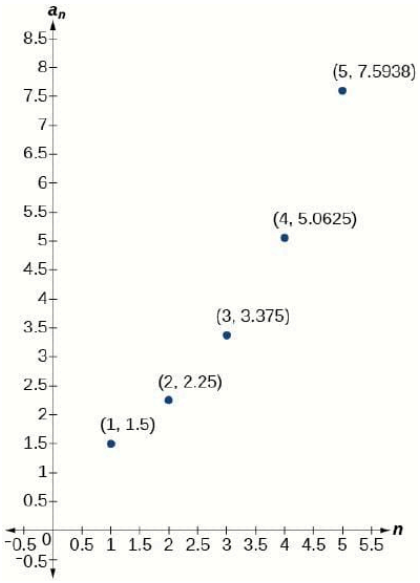
Graphical

For the following exercises, determine whether the graph shown represents an arithmetic sequence.

66.



67.



For the following exercises, use the information provided to graph the first 5 terms of the arithmetic sequence.

68. $a_1 = 0, d = 4$
69. $a_1 = 9; a_n = a_{n-1} - 10$

70. $a_n = -12 + 5n$

Extensions

71. Give two examples of arithmetic sequences whose 4th terms are 9.
72. Give two examples of arithmetic sequences whose 10th terms are 206.
73. Find the 5th term of the arithmetic sequence $\{9b, 5b, b, ...\}$.
74. Find the 11th term of the arithmetic sequence $\{3a - 2b, a + 2b, -a + 6b ...\}$.
75. At which term does the sequence $\{5.4, 14.5, 23.6, ...\}$ exceed 151?
76. At which term does the sequence $\left\{\frac{17}{3}, \frac{31}{6}, \frac{14}{3}, ...\right\}$ begin to have negative values?
77. For which terms does the finite arithmetic sequence $\left\{\frac{5}{2}, \frac{19}{8}, \frac{9}{4}, ..., \frac{1}{8}\right\}$ have integer values?
78. Write an arithmetic sequence using a recursive formula. Show the first 4 terms, and then find the 31st term.
79. Write an arithmetic sequence using an explicit formula. Show the first 4 terms, and then find the 28th term.

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5.3: Geometric Sequences

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Many jobs offer an annual cost-of-living increase to keep salaries consistent with inflation. Suppose, for example, a recent college graduate finds a position as a sales manager earning an annual salary of $26,000. He is promised a 2% cost-of-living increase each year. His annual salary in any given year can be found by multiplying his salary from the previous year by 102%. His salary will be $26,520 after one year; $27,050.40 after two years; $27,591.41 after three years; and so on. When a salary increases by a constant rate each year, the salary grows by a constant factor. In this section, we will review sequences that grow in this way.

An Introduction to Geometric Sequences

The yearly salary values described form a **geometric sequence** because they change by a constant factor each year. Each term of a geometric sequence increases or decreases by a constant factor called the **common ratio**.

Definition: Geometric Sequence

A **geometric sequence** is a sequence having the property that the ratio between any two consecutive terms is constant. This constant is called the **common ratio**. Therefore, if a_1 is the first term of a geometric sequence and r is the common ratio, the sequence will be:

$$\{a_n\} = \{a_1, a_1 r, a_1 r^2, a_1 r^3, ...\}.$$

The sequence below is an example of a geometric sequence because each term increases by a constant factor of 6. Multiplying any term of the sequence by the common ratio 6 generates the subsequent term.



Figure $5.3.1$

Example $5.3.1$: Finding Common Ratios

Is the sequence geometric? If so, find the common ratio.

a. $1, 2, 4, 8, 16, ...$
b. $48, 12, 4, 2, ...$

Solution available at the end of this section.

Q&A

If you are told that a sequence is geometric, do you have to divide every term by the previous term to find the common ratio?

No. If you know that the sequence is geometric, you can choose any one term in the sequence and divide it by the previous term to find the common ratio.

Checkpoint $5.3.1$

Is the sequence geometric? If so, find the common ratio.

a. $5, 10, 15, 20, \ldots$
b. $100, 20, 4, \frac{4}{5}, \ldots$

The Recursive Formula for Geometric Sequences

We begin finding a formula for a geometric sequence as we did in the previous section—by developing a recursive formula. This allows us to find any term of a geometric sequence by using the previous term. Each term is the product of the common ratio and the previous term. For example, suppose the common ratio is 9. Then each term is nine times the previous term. As with any recursive formula, the initial term must be given.

Theorem: Recursive Formula of a Geometric Sequence

The recursive formula for the geometric sequence with first term a_1 and common ratio r is

$$a_n = a_{n-1}\, r.$$

Proof available at the end of this section.

Example $5.3.2$: Using Recursive Formulas for Geometric Sequences

Write a recursive formula for the following geometric sequence.

$$\{6, 9, 13.5, 20.25, \ldots\}$$

Solution available at the end of this section.

Q&A

Do we have to divide the second term by the first term to find the common ratio?

No. We can divide any term in the sequence by the previous term. It is, however, most common to divide the second term by the first term because it is often the easiest method of finding the common ratio.

Checkpoint $5.3.2$

Write a recursive formula for the following geometric sequence.

$$\left\{2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \ldots\right\}$$

The Explicit Formula for Geometric Sequences

Now that we can identify a geometric sequence, we will learn how to find the terms of a geometric sequence if we are given the first term and the common ratio. The terms of a geometric sequence can be found by beginning with the first term and multiplying by the common ratio repeatedly. Doing so results in the following theorem.

Theorem: Explicit Formula of a Geometric Sequence

The explicit formula for the geometric sequence with first term a_1 and common ratio r is

$$a_n = a_1 r^{n-1}.$$

Proof (by Mathematical Induction) available at the end of this section.

Example 5.3.3: Writing the Terms of a Geometric Sequence

List the first four terms of the geometric sequence with $a_1 = 5$ and $r = -2$.

Solution available at the end of this section.

Checkpoint 5.3.3

List the first five terms of the geometric sequence with $a_1 = 18$ and $r = \frac{1}{3}$.

Using Explicit Formulas for Geometric Sequences

Let's take a look at the sequence $\{18, 36, 72, 144, 288, ...\}$. This is a geometric sequence with a common ratio of 2 and an "exponential-like" function with a base of 2. An explicit formula for this sequence is

$$a_n = 18 \cdot 2^{n-1}.$$

The graph of the sequence is shown in Figure 5.3.4.

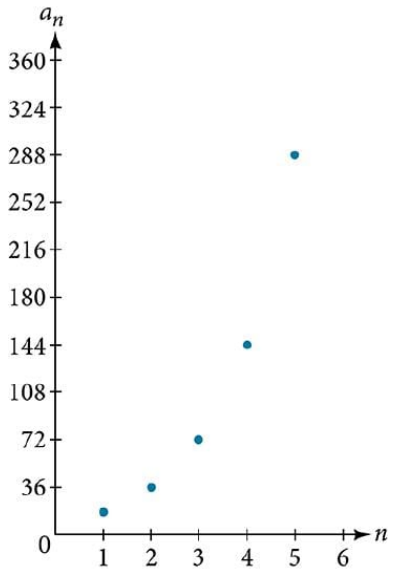


Figure 5.3.4

Example 5.3.4: Writing Terms of Geometric Sequences Using the Explicit Formula

Given a geometric sequence with $a_1 = 3$ and $a_4 = 24$, find a_2.

Solution available at the end of this section.

Checkpoint 5.3.4

Given a geometric sequence with $a_2 = 4$ and $a_3 = 32$, find a_6.

Example 5.3.5: Writing an Explicit Formula for the nth Term of a Geometric Sequence

Write an explicit formula for the nth term of the following geometric sequence.

$$\{2, 10, 50, 250, ...\}$$

Solution available at the end of this section.

Checkpoint 5.3.5

Write an explicit formula for the following geometric sequence.

$$\{-1, 3, -9, 27, ...\}$$

Solving Application Problems with Geometric Sequences

In real-world scenarios involving geometric sequences, we may need to use an initial term of a_0 instead of a_1. In these problems, we can alter the explicit formula slightly by using the following formula:

$$a_n = a_0 r^n.$$

Example 5.3.6: Solving Application Problems with Geometric Sequences

In 2013, the number of students in a small school is 284. It is estimated that the student population will increase by 4% each year.

a. Write a formula for the student population.
b. Estimate the student population in 2030.

Solution available at the end of this section.

Checkpoint 5.3.6

A business starts a new website. Initially the number of hits is 293 due to the curiosity factor. The business estimates the number of hits will increase by 2.6% per week.

a. Write a formula for the number of hits.
b. Estimate the number of hits in 5 weeks.

Success in College: Learning From Tests (Part 3)

1. Look at your last test from this class. How many points did you miss because you could not remember something or did not know how to do the problem?
2. Knowing what you know now, do you think you could do better on this exam?

Success in College: The Classroom and Learning Preferences

Read the strategies suggested for your learning preferences below.

Visual Learners

Strategies for visual learners include the following:

- take detailed notes during class
- reorganize and rewrite notes after class
- rework important examples
- watch the videos linked to your textbook
- use flashcards to study vocabulary

Auditory Learners

Strategies for auditory learners include the following:

- record your class, take brief notes of the big ideas and examples, and listen to the recording after class
- restate the main ideas from the lecture aloud to yourself
- talk to yourself as you do your homework, explaining each step to yourself
- study vocabulary by repeating definitions (listen to a recording of the words and definitions)

Kinesthetic Learners

Strategies for kinesthetic learners include the following:

- record the lectures, take brief notes of the big ideas and examples, and listen to the recording after class
- draw pictures, when possible
- with your finger, trace diagrams and graphs - don't just look at them
- imagine symbols as three-dimensional objects, or even cartoon characters
- study vocabulary as you exercise or walk to your car
- walk around your room as you repeat definitions

1. Do you use any of these strategies currently? If not, should you be?
2. Can you suggest any other ideas that I should add to this list?

Answers

Section Preview

Of the material so far in this chapter, this section is the most important—geometric sequences and series becomes incredibly important in Calculus. Make sure to create the connection between these sequences and exponential functions.

Section Preview

- **Arithmetic**
  - Ratios, Proportions, and Proportional Reasoning: The defining idea of a geometric sequence is the *common ratio*, so students must understand a ratio as the multiplicative comparison (quotient) between two consecutive terms in order to grasp why each term is a constant factor of the one before it (Examples 1–2).

Section Preview

- Find the common ratio for a geometric sequence.

- List the terms of a geometric sequence.
- Use a recursive formula for a geometric sequence.
- Use an explicit formula for a geometric sequence.

Example 5.3.1: Finding Common Ratios

Divide each term by the previous term to determine whether a common ratio exists.

a. The sequence is geometric because there is a common ratio. The common ratio is 2.
b. The sequence is not geometric because there is not a common ratio.

The graph of each sequence is shown in Figure 5.3.2. It seems from the graphs that both (a) and (b) appear to have the form of the graph of an exponential function in this viewing window. However, we know that (a) is geometric and so this interpretation holds, but (b) is not.

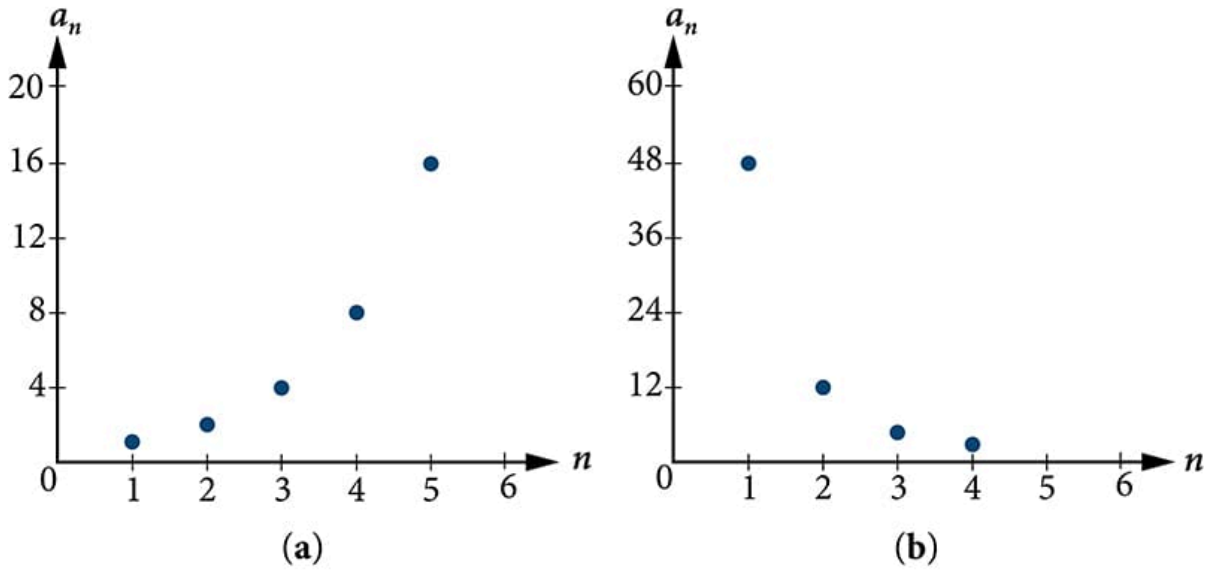


Figure 5.3.2

Theorem: Recursive Formula of a Geometric Sequence

This is a direct result of the definition of a geometric sequence.

Example 5.3.2: Using Recursive Formulas for Geometric Sequences

The first term is given as 6. The common ratio can be found by dividing the second term by the first term.

$$r = \frac{9}{6} = 1.5$$

Substitute the common ratio into the recursive formula for geometric sequences and define a_1.

$$\begin{aligned} a_n &= 1.5a_{n-1}, \text{ for } n \geq 2 \\ a_1 &= 6 \end{aligned}$$

The sequence of data points follows an exponential pattern. The common ratio is also the base of an exponential function as shown in Figure 5.3.3.

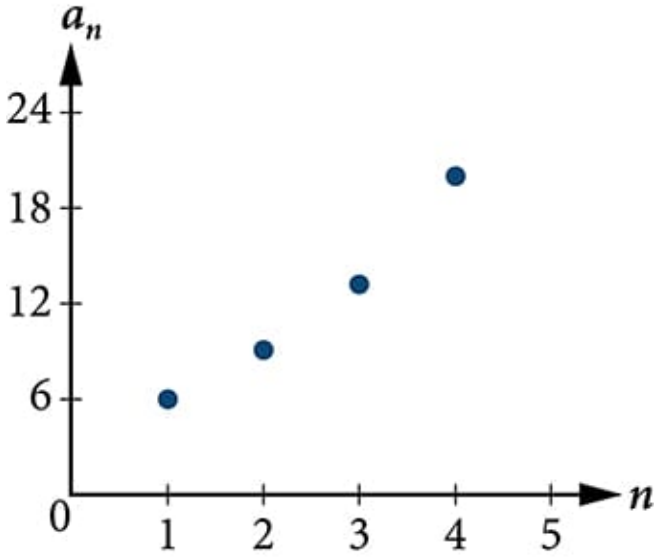


Figure 5.3.3

Theorem: Explicit Formula of a Geometric Sequence

NOTE: The method of proof for this theorem is Mathematical Induction, which is a topic generally covered *after* learning about sequences and series. If you do not know about proofs by Mathematical Induction, feel free to come back to this proof later.

Let a_n be the nth term of a geometric sequence with common ratio r. Our claim is that $a_n = a_1 r^{n-1}$.

Base Case ($n = 1$): For $n = 1$, we get $a_1 = a_1 r^{1-1} = a_1 r^0 = a_1 \cdot 1 = a_1$, which is correct. Therefore, the claim is true for $n = 1$.

Induction Hypothesis: Assume the claim is true for $n = k$. That is, assume $a_k = a_1 r^{k-1}$.

Inductive Step ($n = k + 1$): For $n = k + 1$, we would like to show that $a_{k+1} = a_1 r^{(k+1)-1)}$.

$$\begin{aligned} a_{k+1} &= a_k\, r && \text{(since the sequence is geometric, the next term is } a_k\, r) \\ &= a_1 r^{k-1} \cdot r && \text{(from the Induction Hypothesis)} \\ &= a_1 r^{k-1+1} && \text{(Laws of Exponents)} \\ &= a_1 r^{k+1-1} && \text{(Commutative Property of Addition)} \\ &= a_1 r^{(k+1)-1} && \text{(Associative Property of Addition)} \end{aligned}$$

Thus, $a_{k+1} = a_1 r^{(k+1)-1}$, and so the claim must be true for $n = k + 1$.

Hence, the claim is true for all natural numbers by Mathematical Induction.

Example 5.3.3: Writing the Terms of a Geometric Sequence

Multiply a_1 by -2 to find a_2. Repeat the process, using a_2 to find a_3, and so on.

$$\begin{aligned} a_1 &= 5 \\ a_2 &= -2a_1 = (-2) \cdot 5 = -10 \\ a_3 &= -2a_2 = (-2) \cdot (-10) = 20 \\ a_4 &= -2a_3 = (-2) \cdot 20 = -40 \end{aligned}$$

The first four terms are $\{5, -10, 20, -40\}$.

Example 5.3.4: Writing Terms of Geometric Sequences Using the Explicit Formula

The sequence can be written in terms of the initial term and the common ratio r.

$$3, 3r, 3r^2, 3r^3, \ldots$$

Find the common ratio using the given fourth term.

$$\begin{aligned} & a_n = a_1 r^{n-1} \\ \Longrightarrow\ & a_4 = 3r^3 && \text{(substituting } n = 4) \\ \Longrightarrow\ & 24 = 3r^3 && \text{(substituting } a_4 = 24) \\ \Longrightarrow\ & 8 = r^3 && \text{(dividing both sides by 3)} \\ \Longrightarrow\ & 2 = r && \text{(taking the cube root of both sides)} \end{aligned}$$

Find the second term by multiplying the first term by the common ratio.

$$a_2 = 2a_1 = 2(3) = 6$$

Note that the common ratio is multiplied by the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by multiplying the first term by the common ratio nine times or by multiplying by the common ratio raised to the ninth power.

Example 5.3.5: Writing an Explicit Formula for the nth Term of a Geometric Sequence

The first term is 2. The common ratio can be found by dividing the second term by the first term.

$$\frac{10}{2} = 5$$

The common ratio is 5. Substitute the common ratio and the first term of the sequence into the formula.

$$a_n = 2 \cdot 5^{n-1}$$

The graph of this sequence in Figure 5.3.5 shows an exponential pattern.

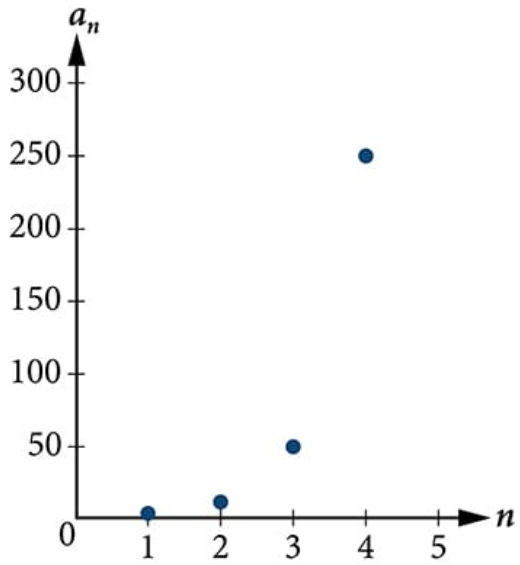


Figure 5.3.5

Example 5.3.6**: Solving Application Problems with Geometric Sequences**

a. The situation can be modeled by a geometric sequence with an initial term of 284. The student population will be 104% of the prior year, so the common ratio is 1.04.

Let P be the student population and n be the number of years after 2013. Using the explicit formula for a geometric sequence we get

$$P_n = 284 \cdot 1.04^n.$$

b. We can find the number of years since 2013 by subtracting.

$$2030 - 2013 = 17$$

We are looking for the population after 17 years. We can substitute 17 for n to estimate the population in 2030.

$$P_{17} = 284 \cdot 1.04^{17} \approx 553.$$

The student population will be about 553 in 2030.

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5.3.1: Resources and Key Concepts

Resources

Videos

- Geometric sequence
  - Geometric Sequences - Geometric Sequences

Key Concepts

Definitions

- **Geometric Sequence**: A sequence having the property that the ratio between any two consecutive terms is constant.
- **Common Ratio (r)**: The constant ratio between any two consecutive terms in a geometric sequence. It is found by dividing any term by its preceding term ($r = \frac{a_n}{a_{n-1}}$).

Theorems

- **Recursive Formula of a Geometric Sequence**: The recursive formula for the geometric sequence with first term a_1 and common ratio r is $a_n = a_{n-1} \cdot r$.
- **Explicit Formula of a Geometric Sequence**: The explicit formula for the geometric sequence with first term a_1 and common ratio r is $a_n = a_1 r^{n-1}$.

Common Mistakes

- **Incorrectly Calculating the Common Ratio**: Dividing terms in the wrong order (e.g., a_n/a_{n+1} instead of a_{n+1}/a_n) or assuming a sequence is geometric without checking if the ratio is constant between all consecutive terms.
- **Errors in Applying the Recursive Formula**: Using the wrong previous term or an incorrect common ratio when generating terms recursively.
- **Errors in Applying the Explicit Formula**: Using r^n instead of r^{n-1} in the formula $a_n = a_1 r^{n-1}$.
- **Confusing Geometric Sequences with Arithmetic Sequences**: Applying concepts of common difference to geometric sequences or common ratio to arithmetic sequences.
- **Off-by-One Errors with a_0 vs. a_1**: When using an explicit formula based on an initial term a_0 (like $a_n = a_0 r^n$), confusing it with the formula based on a_1 ($a_n = a_1 r^{n-1}$), or misinterpreting which term is considered the "starting" term in an application.
- **Sign Errors with Negative Common Ratios**: Forgetting that terms will alternate in sign if the common ratio r is negative.

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5.3.2: Homework

Reading Questions

1. What defines a geometric sequence?
2. How is the common ratio of a geometric sequence found?
3. If you are told a sequence is geometric, do you need to divide every term by its preceding term to find the common ratio? Explain.
4. What is the recursive formula for a geometric sequence with first term a_1 and common ratio r?
5. What is the explicit formula for a geometric sequence with first term a_1 and common ratio r?
6. How are geometric sequences similar to exponential functions? What corresponds to the base of the exponential function?
7. If the first term of a geometric sequence is $a_1 = 3$ and the fourth term is $a_4 = 24$, how can you find the common ratio r?
8. What is the difference between the explicit formula $a_n = a_1 r^{n-1}$ and $a_n = a_0 r^n$? When might the second formula be more convenient?
9. If the common ratio r of a geometric sequence is negative, what happens to the signs of the terms?
10. When graphing a geometric sequence, what kind of pattern do the points typically form if all terms are positive and $r > 1$?
11. What is a geometric sequence?
12. How is the common ratio of a geometric sequence found?
13. What is the procedure for determining whether a sequence is geometric?
14. What is the difference between an arithmetic sequence and a geometric sequence?
15. Describe how exponential functions and geometric sequences are similar. How are they different?

Homework

Algebraic

For the following exercises, find the common ratio for the geometric sequence.

16. $1, 3, 9, 27, 81, \ldots$
17. $-0.125, 0.25, -0.5, 1, -2, \ldots$
18. $-2, -\frac{1}{2}, -\frac{1}{8}, -\frac{1}{32}, -\frac{1}{128}, \ldots$

For the following exercises, determine whether the sequence is geometric. If so, find the common ratio.

19. $-6, -12, -24, -48, -96, \ldots$
20. $5, 5.2, 5.4, 5.6, 5.8, \ldots$
21. $-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots$
22. $6, 8, 11, 15, 20, \ldots$
23. $0.8, 4, 20, 100, 500, \ldots$

For the following exercises, write the first five terms of the geometric sequence, given the first term and common ratio.

24. $a_1 = 8$, $r = 0.3$
25. $a_1 = 5$, $r = \frac{1}{5}$

For the following exercises, write the first five terms of the geometric sequence, given any two terms.

26. $a_7 = 64$, $a_{10} = 512$
27. $a_6 = 25$, $a_8 = 6.25$

For the following exercises, find the specified term for the geometric sequence, given the first term and common ratio.

28. The first term is 2, and the common ratio is 3. Find the 5th term.
29. The first term is 16 and the common ratio is $-\frac{1}{3}$. Find the 4th term.

For the following exercises, find the specified term for the geometric sequence, given the first four terms.

30. $a_n = \{-1, 2, -4, 8, ...\}$. Find a_{12}.
31. $a_n = \left\{-2, \frac{2}{3}, -\frac{2}{9}, \frac{2}{27}, ...\right\}$. Find a_7.

For the following exercises, write the first five terms of the geometric sequence.

32. $a_1 = -486$, $a_n = -\frac{1}{3}a_{n-1}$
33. $a_1 = 7$, $a_n = 0.2a_{n-1}$

For the following exercises, write a recursive formula for each geometric sequence.

34. $a_n = \{-1, 5, -25, 125, ...\}$
35. $a_n = \{-32, -16, -8, -4, ...\}$
36. $a_n = \{14, 56, 224, 896, ...\}$
37. $a_n = \{10, -3, 0.9, -0.27, ...\}$
38. $a_n = \{0.61, 1.83, 5.49, 16.47, ...\}$
39. $a_n = \left\{\frac{3}{5}, \frac{1}{10}, \frac{1}{60}, \frac{1}{360}, ...\right\}$
40. $a_n = \left\{-2, \frac{4}{3}, -\frac{8}{9}, \frac{16}{27}, ...\right\}$
41. $a_n = \left\{\frac{1}{512}, -\frac{1}{128}, \frac{1}{32}, -\frac{1}{8}, ...\right\}$

For the following exercises, write the first five terms of the geometric sequence.

42. $a_n = -4 \cdot 5^{n-1}$
43. $a_n = 12 \cdot \left(-\frac{1}{2}\right)^{n-1}$

For the following exercises, write an explicit formula for each geometric sequence.

44. $a_n = \{-2, -4, -8, -16, ...\}$
45. $a_n = \{1, 3, 9, 27, ...\}$
46. $a_n = \{-4, -12, -36, -108, ...\}$
47. $a_n = \{0.8, -4, 20, -100, ...\}$
48. $a_n = \{-1.25, -5, -20, -80, ...\}$
49. $a_n = \left\{-1, -\frac{4}{5}, -\frac{16}{25}, -\frac{64}{125}, ...\right\}$
50. $a_n = \left\{2, \frac{1}{3}, \frac{1}{18}, \frac{1}{108}, ...\right\}$
51. $a_n = \left\{3, -1, \frac{1}{3}, -\frac{1}{9}, ...\right\}$

For the following exercises, find the specified term for the geometric sequence given.

52. Let $a_1 = 4$, $a_n = -3a_{n-1}$. Find a_8.
53. Let $a_n = -\left(-\frac{1}{3}\right)^{n-1}$. Find a_{12}.

For the following exercises, find the number of terms in the given finite geometric sequence.

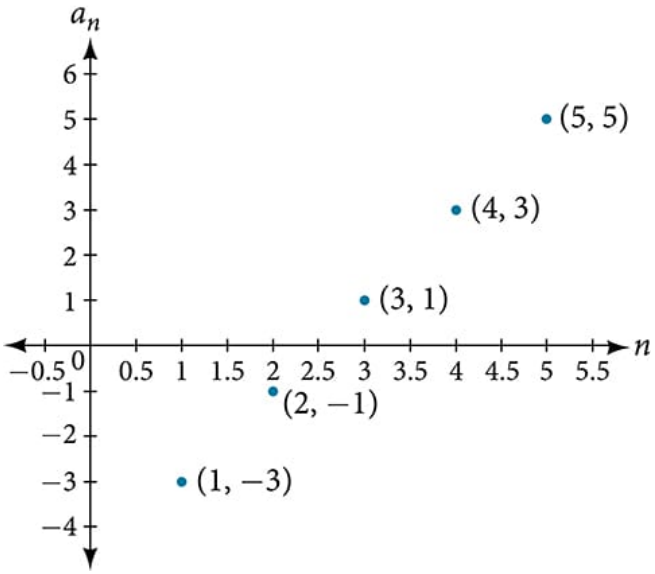
54. $a_n = \{-1, 3, -9, ..., 2187\}$
55. $a_n = \left\{2, 1, \frac{1}{2}, ..., \frac{1}{1024}\right\}$

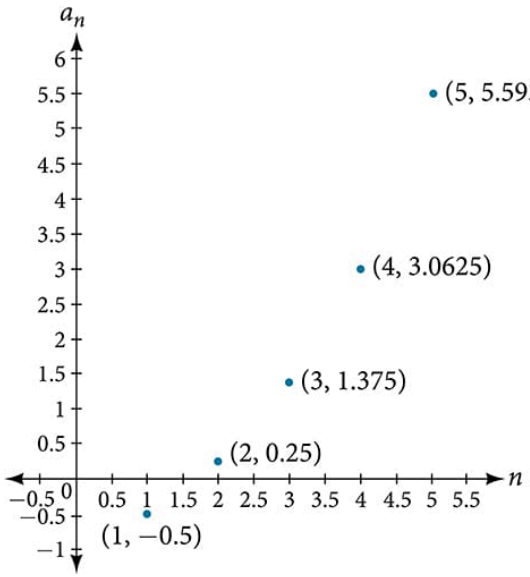
Graphical

For the following exercises, determine whether the graph shown represents a geometric sequence.

56.

57.





For the following exercises, use the information provided to graph the first five terms of the geometric sequence.

58. $a_1 = 1, r = \frac{1}{2}$
59. $a_1 = 3, a_n = 2a_{n-1}$
60. $a_n = 27 \cdot 0.3^{n-1}$

Extensions

61. Use recursive formulas to give two examples of geometric sequences whose 3rd terms are 200.
62. Use explicit formulas to give two examples of geometric sequences whose 7th terms are 1024.
63. Find the 5th term of the geometric sequence $\{b, 4b, 16b, ...\}$.
64. Find the 7th term of the geometric sequence $\{64a(-b), 32a(-3b), 16a(-9b), ...\}$.
65. At which term does the sequence $\{10, 12, 14.4, 17.28, ...\}$ exceed 100?
66. At which term does the sequence $\left\{\frac{1}{2187}, \frac{1}{729}, \frac{1}{243}, \frac{1}{81}, ...\right\}$ begin to have integer values?
67. For which term does the geometric sequence $a_n = -36\left(\frac{2}{3}\right)^{n-1}$ first have a non-integer value?
68. Use the recursive formula to write a geometric sequence whose common ratio is an integer. Show the first four terms, and then find the 10th term.
69. Use the explicit formula to write a geometric sequence whose common ratio is a decimal number between 0 and 1. Show the

first 4 terms, and then find the 8th term.

70. Is it possible for a sequence to be both arithmetic and geometric? If so, give an example.

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5.4: Series and Their Notations

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

A couple decides to start a college fund for their daughter. They plan to invest $50 in the fund each month. The fund pays 6% annual interest, compounded monthly. How much money will they have saved when their daughter is ready to start college in 6 years? In this section, we will learn how to answer this question. To do so, we need to consider the amount of money invested and the amount of interest earned.

Using Summation Notation

To find the total amount of money in the college fund and the sum of the amounts deposited, we need to add the amounts deposited each month and the amounts earned in interest monthly. The sum of the terms of a sequence is called a **series**.

Definition: Series

The sum of the terms of a (finite or infinite) sequence is called a **series**.

Consider, for example, the following sequence.

$$\{a_n\} = \{3, 7, 11, 15, 19, ...\}$$

The series for this sequence is the infinite summation

$$3 + 7 + 11 + 15 + 19 + ...$$

Obviously, adding terms of a sequence forever is impossible, so we begin our excursion into summations of sequences by asking ourselves what would happen if we only added up, say, the first n terms of the sequence. This is called the nth **partial sum**.

Definition: nth Partial Sum of a Series

The nth **partial sum** of a series is the sum of the first n terms of that series, and it is denoted S_n.

For example, with the series above, the first few partial sums are as follows:

$$\begin{aligned} S_1 &= 3 \\ S_2 &= 3 + 7 = 10 \\ S_3 &= 3 + 7 + 11 = 21 \\ S_4 &= 3 + 7 + 11 + 15 = 36 \end{aligned}$$

Since partial sums can get pretty bulky to write, we introduce a new notation. **Summation notation** is used to represent series.

Definition: Summation Notation

Summation notation, also known as **Sigma notation**, is a concise way to express the sum of a series of terms. Given a function f, we write

$$\sum_{k=a}^{b} f(k)$$

to represent the operation

$$f(a) + f(a+1) + f(a+2) + \cdots + f(n).$$

Summation notation uses the Greek capital letter Sigma (Σ) to indicate operation of summation, along with an **index variable** (k in the notation above, but this choice is arbitrary), a **lower limit** of summation (a in the notation above), and an **upper limit** of summation (n in the notation above). We read the notation $\sum_{k=a}^{b} f(k)$ as "the sum from $k = a$ to $k = b$ of $f(k)$."

The index variable is also called the **index of summation**.

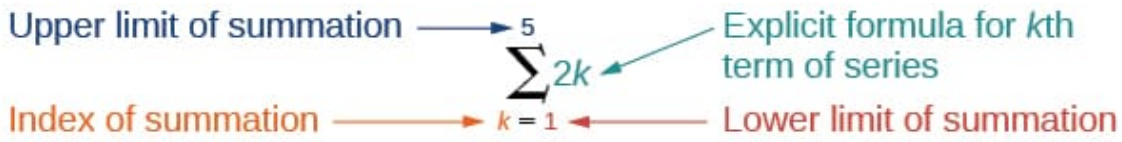


Figure 5.4.1

If we interpret the given notation, we see that it asks us to find the sum of the terms in the series $a_k = 2k$ for $k = 1$ through $k = 5$. We can begin by substituting the terms for k and listing out the terms of this series.

$$\begin{aligned} a_1 &= 2(1) = 2 \\ a_2 &= 2(2) = 4 \\ a_3 &= 2(3) = 6 \\ a_4 &= 2(4) = 8 \\ a_5 &= 2(5) = 10 \end{aligned}$$

We can find the sum of the series by adding the terms:

$$\sum_{k=1}^{5} 2k = 2 + 4 + 6 + 8 + 10 = 30.$$

Q&A

Does the lower limit of summation have to be 1?

No (and this is why our definition didn't start with the lower limit set as 1). The lower limit of summation can be any number, but 1 is frequently used. We will look at examples with lower limits of summation other than 1.

Example 5.4.1: Using Summation Notation

Evaluate $\sum_{k=3}^{7} k^2$.

Solution available at the end of this section.

Checkpoint 5.4.1

Evaluate $\sum_{k=2}^{5}(3k-1)$.

Arithmetic Series

Just as we studied special types of sequences, we will look at special types of series. Recall that an arithmetic sequence is a sequence in which the difference between any two consecutive terms is the common difference, d. That is, if d is the common difference and a_1 is the first term for the arithmetic sequence, then $a_2 - a_1 = d$. Moreover, in general, $a_n - a_{n-1} = d$. Since this difference is constant between any two terms in the arithmetic sequence, we could write the sequence terms as follows:

$$\begin{aligned} a_2 &= a_1 + d \\ a_3 &= a_1 + 2d \\ a_4 &= a_1 + 3d \\ \vdots &= \vdots \\ a_n &= a_1 + (n-1)d \end{aligned}$$

Definition: Arithmetic Series

The sum of the terms of a (finite or infinite) arithmetic sequence is called an **arithmetic series**.

Arithmetic series are special in that we have a closed formula for their n^{th} partial sum.

Theorem: nth Partial Sum of an Arithmetic Series

Given the arithmetic sequence $a_n = a_1 + (n-1)d$, the sum of the first n terms of the corresponding arithmetic series is

$$S_n = n \cdot \frac{a_1 + a_n}{2}.$$

Proof available at the end of this section.

That is, the sum of the first n terms of an arithmetic sequence is the product of n and the average of the first and n^{th} terms of the sequence.

Example 5.4.2: Finding the First n Terms of an Arithmetic Series

Find the sum of each arithmetic series.

a. $5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32$
b. $20 + 15 + 10 + \ldots + (-50)$
c. $\sum_{k=1}^{12} 3k - 8$

Solutions available at the end of this section.

Checkpoint 5.4.2

Find the sum of each arithmetic series.

a. $1.4 + 1.6 + 1.8 + 2.0 + 2.2 + 2.4 + 2.6 + 2.8 + 3.0 + 3.2 + 3.4$
b. $13 + 21 + 29 + ... + 69$
c. $\sum_{k=1}^{10} 5 - 6k$

Example 5.4.3: Solving Application Problems with Arithmetic Series

On the Sunday after a minor surgery, a woman is able to walk a half-mile. Each Sunday, she walks an additional quarter-mile. After 8 weeks, what will be the total number of miles she has walked?

Solution available at the end of this section.

Checkpoint 5.4.3

A man earns $100 in the first week of June. Each week, he earns $12.50 more than the previous week. After 12 weeks, how much has he earned?

Geometric Series

Just as the sum of the terms of an arithmetic sequence is called an arithmetic series, the sum of the terms in a geometric sequence is called a **geometric series**.

Definition: Geometric Series

The sum of the terms of a (finite or infinite) geometric sequence is called an **geometric series**.

Likewise, we can derive a closed formula for the nth partial sum of a geometric series.

Theorem: nth Partial Sum of a Geometric Series

Given the geometric sequence $a_n = a_1 r^{n-1}$, the sum of the first n terms of the corresponding geometric series is

$$S_n = \frac{a_1(1 - r^n)}{1 - r}.$$

Proof available at the end of this section.

Notice that this formula does not exist when $r = 1$. This is not an issue because geometric sequences (which geometric series are based upon) are required to have common ratios not equal to 1.

Example 5.4.4: Finding the First n Terms of a Geometric Series

Use the formula to find the indicated partial sum of each geometric series.

a. S_{11} for the series $8 + (-4) + 2 + ...$

b. $\sum_{k=1}^{6} 3 \cdot 2^k$

Solutions available at the end of this section.

Checkpoint 5.4.4

Use the formula to find the indicated partial sum of each geometric series.

a. S_{20} for the series $1,000 + 500 + 250 + ...$

b. $\sum_{k=1}^{8} 3^k$

Example 5.4.5: Solving an Application Problem with a Geometric Series

At a new job, an employee's starting salary is $26,750. He receives a 1.6% annual raise. Find his total earnings at the end of 5 years.

Solution available at the end of this section.

Checkpoint 5.4.5

At a new job, an employee's starting salary is $32,100. She receives a 2% annual raise. How much will she have earned by the end of 8 years?

Infinite Series

Thus far, we have only considered partial sums of series. These are also known as **finite series**. Sometimes, however, we are interested in the sum of the terms of an infinite sequence rather than the sum of only the first n terms.

Definition: Infinite Series

An **infinite series** is the sum of the terms of an infinite sequence.

An example of an infinite series is $2 + 4 + 6 + 8 + ...$

This series can also be written in summation notation as $\sum_{k=1}^{\infty} 2k$, where the upper limit of summation is infinity. Because the terms are not tending to zero, the sum of the series increases without bound as we add more terms. Therefore, the sum of this infinite series is not defined; however, some infinite series tend to a finite value. A major focus of Calculus is determining which infinite series tend to finite values. As such, we need to introduce new language.

Definition: Convergent/Divergent Infinite Series

When the sum of an infinite series is not a real number, we say the series **diverges** and we call the series a **divergent series**. On the other hand, if the sum of an infinite series becomes finite, we say the series **converges** and we call the series a **convergent series**.

Caution: Convergence is Not Intuitive (at first)

The study of Calculus will present you with many concepts that seem counterintuitive. For example, it seems reasonable to state that an infinite series where the terms are tending to zero must become finite.

For example, consider the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

The terms are tending to zero as we keep writing out the series. In fact, by the time we get to the thousandth term, we are only adding $\frac{1}{1000}$. Therefore, it *seems* logical to state that this summation would approach a finite value. After all, we are essentially adding zero as time goes on.

Unfortunately, this is *not true*!

In general, if the terms of a series tend to zero, we cannot guarantee that the sum of those terms will become finite. The series

$$\sum_{n=1}^{\infty}\frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

have terms that tend to zero as n tends to get large; however, the first of these series is divergent, while the second is convergent (in fact, eerily, it converges to $\ln(2)$).

In general, finding the sum of a convergent (infinite) series is a task left for Calculus.

Infinite Geometric Series

Among infinite series, infinite *geometric* series claim a special place. This is because, when an infinite geometric series converges, we can calculate its sum—this is often not true for other types of convergent infinite series. The formula for the sum of an infinite geometric series is related to the formula for the sum of the first n terms of the geometric series:

$$S_n=\frac{a_1(1-r^n)}{1-r}.$$

We will examine an infinite series with $r=\frac{1}{2}$. What happens to r^n as n increases?

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$\left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

The value of r^n decreases rapidly. What happens for greater values of n?

$$\left(\frac{1}{2}\right)^{10} = \frac{1}{1,024}$$

$$\left(\frac{1}{2}\right)^{20} = \frac{1}{1,048,576}$$

$$\left(\frac{1}{2}\right)^{30} = \frac{1}{1,073,741,824}$$

As n gets very large, r^n gets very small. We say that, as n increases without bound, r^n approaches 0. Borrowing notation from our exploration of rational functions, as $n\to\infty$, $r^n\to 0$. As r^n approaches 0, $1-r^n$ approaches 1. When this happens, the numerator approaches a_1. This gives us a formula for the sum of a convergent infinite geometric series.

Theorem: Sum of a Convergent Infinite Geometric Series

The infinite geometric series

$$\sum_{n=1}^{\infty} a_1 r^{n-1}$$

converges to $\frac{a_1}{1-r}$ as long as $-1 < r < 1$. That is,

$$\sum_{n=1}^{\infty} a_1 r^{n-1} = \frac{a_1}{1-r}, \quad \text{if } |r| < 1.$$

The infinite geometric series diverges otherwise.

Proof available at the end of this section.

Example 5.4.6: Determining Whether the Sum of an Infinite Series is Defined

Determine whether the sum of each infinite series is defined.

a. $\frac{3}{4} + \frac{1}{2} + \frac{1}{3} + \ldots$
b. $\sum_{k=1}^{\infty} 27 \cdot \left(\frac{1}{3}\right)^k$

Solutions available at the end of this section.

Checkpoint 5.4.6

Determine whether the sum of the infinite series is defined.

a. $24 + (-12) + 6 + (-3) + \ldots$
b. $\frac{1}{3} + \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \ldots$
c. $\sum_{k=1}^{\infty} 15 \cdot (-0.3)^k$

Example 5.4.7: Finding the Sum of an Infinite Geometric Series

Find the sum, if it exists, for the following:

a. $10 + 9 + 8 + 7 + \ldots$
b. $248.6 + 99.44 + 39.776 + \ldots$
c. $\sum_{k=1}^{\infty} 4,374 \cdot \left(-\frac{1}{3}\right)^{k-1}$
d. $\sum_{k=1}^{\infty} \frac{1}{9} \cdot \left(\frac{4}{3}\right)^k$

Solutions available at the end of this section.

Checkpoint 5.4.7

Find the sum, if it exists.

a. $2 + \frac{2}{3} + \frac{2}{9} + \ldots$

b. $\sum_{k=1}^{\infty} 0.76k + 1$

c. $\sum_{k=1}^{\infty} \left(-\frac{3}{8}\right)^k$

A common "classroom" use of convergent infinite geometric series is to find fractional equivalents of repeating decimal numbers.

Example 5.4.8: Finding an Equivalent Fraction for a Repeating Decimal

Find an equivalent fraction for the repeating decimal $0.\overline{3}$.

Solution available at the end of this section.

Solving Annuity Problems

At the beginning of the section, we looked at a problem in which a couple invested a set amount of money each month into a college fund for six years. An annuity is an investment in which the purchaser makes a sequence of periodic, equal payments. To find the amount of an annuity, we need to find the sum of all the payments and the interest earned. In the example, the couple invests $50 each month. This is the value of the initial deposit. The account paid 6% annual interest, compounded monthly. To find the interest rate per payment period, we need to divide the 6% annual percentage rate (APR) by 12. So the monthly interest rate is 0.5%. We can multiply the amount in the account each month by 100.5% to find the value of the account after interest has been added.

We can find the value of the annuity right after the last deposit by using a geometric series with $a_1 = 50$ and $r = 100.5\% = 1.005$. After the first deposit, the value of the annuity will be $50. Let us see if we can determine the amount in the college fund and the interest earned.

We can find the value of the annuity after n deposits using the formula for the sum of the first n terms of a geometric series. In 6 years, there are 72 months, so $n = 72$. We can substitute $a_1 = 50$, $r = 1.005$, and $n = 72$ into the formula, and simplify to find the value of the annuity after 6 years.

$$S_{72} = \frac{50(1 - 1.005^{72})}{1 - 1.005} \approx 4,320.44.$$

After the last deposit, the couple will have a total of $4,320.44 in the account. Notice, the couple made 72 payments of $50 each for a total of $72(50) = \$3,600$. This means that because of the annuity, the couple earned $720.44 interest in their college fund.

Example 5.4.9: Solving an Annuity Problem

A deposit of $100 is placed into a college fund at the beginning of every month for 10 years. The fund earns 9% annual interest, compounded monthly, and paid at the end of the month. How much is in the account right after the last deposit?

Solution available at the end of this section.

Checkpoint 5.4.9

At the beginning of each month, $200 is deposited into a retirement fund. The fund earns 6% annual interest, compounded monthly, and paid into the account at the end of the month. How much is in the account if deposits are made for 10 years?

Success in Mathematics: Why Study Mathematics? (Part 4)

Dr. Robert Cabin teaches biology in New York State. He writes,

"... but many students are disillusioned by and uninterested in academics. They are on our campuses because they think they need a degree to get a good job, because their parents expect them to go to college, or because they just don't know what else to do."

Source: Chronicle of Higher Education, October 2003.

1. Dr. Cabin states that many students are uninterested in academics. What do you think he means by academics? Explain.
2. Are you interested in academics? Explain.

Applications of Mathematics: Game Theory Could Save the World!

The following is from The Telegraph (U.K. newspaper):

A team has been using "game theory" - where mathematics is used to capture how people deal with each other - to show that a classic problem that undermines the ability of individuals to cooperate can be overcome, if people are diverse enough, as is the case when it comes to the 6.5 billion citizens of planet Earth.

Working together for the common good is crucial for progress in any society - not least for effectively addressing big issues such as recycling and tackling climate change. But there is a basic problem with how to make the public share responsibility for common problems, such as climate change.

This was most vividly illustrated by Prof Garrett Hardin, of the University of California, Santa Barbara, in his highly influential 1968 paper 'The tragedy of the commons'.

He used the example of a public pasture. Each herdsman will keep adding cows to a common field, because the benefit of an additional cow goes exclusively to the herdsman. Because the cost of overgrazing is shared by all, the pasture will end up ruined.

Mathematical models have confirmed that this is a profound problem, whether for coming together to fight climate change or pay taxes, but today, in the journal Nature, hope for cooperation is raised by Prof Jorge Pacheco of the University of Lisbon, who did the work there with Ms Marta Santos and Dr Francisco Santos from the Free University of Brussels.

Existing mathematical models treat individuals as equivalent, ignoring real-life diversity and population structure. So the team made the mathematics more realistic in this respect.

The team shows that, contrary to expectations, the temptation to cheat ("Why shouldn't I let others pay more for being green?") declines as society becomes more diverse.

Another discovery is that diversity also plays an important role in wealth distribution.

"Cooperation blooms whenever the act of giving is more important than the amount given," said Prof Pacheco.

The most important factor, he said, seems to be that people actually contribute to the public good.

"In particular, the fact that the wealthy minority contributes stimulates the contribution from the rest, who take them as role models," he said.

Prof Pacheco recalled the words of Anna Dreber and Prof Martin Nowak from Harvard University, leading game theorists, who stated that "preserving the global climate is the biggest public good dilemma ever, the one we cannot afford to lose."

1. The basis of this article, game theory, is used to predict human behavior in many areas. Name one possible application of game theory.
2. It turns out that we, as humans, are reasonably predictable. We do things in patterns, and our behaviors can be modeled using complex functions. Do you believe there is a formula for how many heartbeats you will have (that may depend on how many cigarettes you smoke, steps you take, etc.)?

Answers

Section Preview

Series are necessary for both Differential and Integral Calculus. Be as thorough as possible with this material and drive home the importance of the material.

Section Preview

- Use summation notation.
- Use the formula for the sum of the first n terms of an arithmetic series.
- Use the formula for the sum of the first n terms of a geometric series.
- Use the formula for the sum of an infinite geometric series.
- Solve annuity problems.

Example $5.4.1$: Using Summation Notation

According to the notation, the lower limit of summation is 3 and the upper limit is 7. So we need to find the sum of k^2 from $k=3$ to $k=7$. We find the terms of the series by substituting $k=3,4,5,6,$ and 7 into the function k^2. We add the terms to find the sum.

$$\begin{aligned}\sum_{k=3}^{7} k^2 &= 3^2+4^2+5^2+6^2+7^2\\ &= 9+16+25+36+49\\ &= 135\end{aligned}$$

Theorem: nth Partial Sum of an Arithmetic Series

We can write the sum of the first n terms of the arithmetic series as:

$$S_n = a_1+(a_1+d)+(a_1+2d)+\ldots+(a_1+(n-3)d)+(a_1+(n-2)d)+(a_1+(n-1)d).$$

We can also reverse the order of the terms and write the sum as

$$S_n = (a_1+(n–1)d)+(a_1+(n-2)d)+(a_1+(n-3)d)+\ldots+(a_1+2d)+(a_1+d)+a_1.$$

If we add these two expressions for the sum of the first n terms of the arithmetic series, we can derive a formula for the sum of the first n terms of any arithmetic series.

$$\begin{array}{rrcccccccccccc} & S_n &=& a_1 &+& a_1+d &+& a_1+2d &+\cdots+& a_1+(n-3)d &+& a_1+(n-2)d &+& a_1+(n-1)d\\ + & S_n &=& a_1+(n-1)d &+& a_1+(n-2)d &+& a_1+(n-3)d &+\cdots+& a_1+2d &+& a_1+d &+& a_1\\ & 2S_n &=& 2a_1+(n-1)d &+& 2a_1+(n-1)d &+& 2a_1+(n-1)d &+\cdots+& 2a_1+(n-1)d &+& 2a_1+(n-1)d &+& 2a_1+(n-1)d \end{array}$$

Because there are n terms in the series, we can simplify this sum to

$$2S_n = n(2a_1+(n-1)d).$$

However, we know that $a_1+(n-1)d=a_n$. Therefore, we can rewrite the relation as

$$2S_n = n(a_1+a_1+(n-1)d) = n(a_1+a_2).$$

We divide by 2 to find the formula for the sum of the first n terms of an arithmetic series.

$$S_n = n\cdot\frac{a_1+a_n}{2}.$$

Example $5.4.2$: Finding the First n Terms of an Arithmetic Series

a. We are given $a_1=5$ and $a_n=32$. A quick count shows the number of terms to be 10. Thus, $n=10$.
Substitute values for a_1, a_n, and n into the formula and simplify.

$$S_{10} = 10\cdot\frac{5+32}{2} = 5\cdot(37) = 185.$$

b. We are given $a_1=20$ and $a_n=-50$. The sequence given suggests that $d=-5$. Use the formula for the general term of an arithmetic sequence to find n.

$$\begin{aligned} & a_n &=& a_1+(n-1)d & \\ \Longrightarrow\ & -50 &=& 20+(n-1)(-5) & \text{(substituting)}\\ \Longrightarrow\ & -70 &=& (n-1)(-5) & \text{(subtracting 20 from both sides)}\\ \Longrightarrow\ & 14 &=& n-1 & \text{(dividing both sides by } -5)\\ \Longrightarrow\ & 15 &=& n & \text{(adding 1 to both sides)}\end{aligned}$$

Substitute values for a_1, a_n, and n into the formula and simplify.

$$S_{15} = 15\cdot\frac{20-50}{2} = 15\cdot(-15) = -225.$$

c. To find a_1, substitute $k=1$ into the given explicit formula.

$$\begin{aligned} & a_k &=& 3k-8\\ \Longrightarrow\ & a_1 &=& 3(1)-8=-5\end{aligned}$$

We are given that $n=12$. To find a_{12}, substitute $k=12$ into the given explicit formula.

$$\begin{aligned} & a_k &=& 3k-8\\ \Longrightarrow\ & a_{12} &=& 3(12)-8=28\end{aligned}$$

Substitute values for a_1, a_n, and n into the formula and simplify.

$$S_n = n \cdot \frac{a_1 + a_n}{2}$$

$$\Longrightarrow \quad S_{12} = 12 \cdot \frac{-5 + 28}{2} = 6 \cdot 23 = 138$$

Example 5.4.3: Solving Application Problems with Arithmetic Series

This problem can be modeled by an arithmetic series with $a_1 = \frac{1}{2}$ and $d = \frac{1}{4}$. We are looking for the total number of miles walked after 8 weeks, so we know that $n = 8$, and we are looking for S_8. To find a_8, we can use the explicit formula for an arithmetic sequence.

$$a_n = a_1 + d(n-1)$$

$$\Longrightarrow \quad a_8 = \frac{1}{2} + \frac{1}{4}(8-1) = \frac{9}{4}$$

We can now use the formula for arithmetic series.

$$S_n = n \cdot \frac{a_1 + a_n}{2}$$

$$\Longrightarrow \quad S_8 = 8 \cdot \frac{\frac{1}{2} + \frac{9}{4}}{2} = 4 \cdot \left(\frac{11}{4}\right) = 11$$

She will have walked a total of 11 miles.

Theorem: nth Partial Sum of a Geometric Series

We can write the sum of the first n terms of a geometric series as

$$S_n = a_1 + ra_1 + r^2a_1 + \ldots + r^{n-1}a_1.$$

Multiplying both sides of the equation by r, we get the following:

$$rS_n = ra_1 + r^2a_1 + r^3a_1 + \ldots + r^na_1.$$

Next, we subtract this equation from the original equation.

$$\begin{array}{rcccccccccccccc} S_n & = & a_1 & + & ra_1 & + & r^2a_1 & + & \cdots & + & r^{n-3}a_1 & + & r^{n-2}a_1 & + & r^{n-1}a_1 \\ - \quad rS_n & = - (& & & ra_1 & + & r^2a_1 & + & \cdots & + & r^{n-3}a_1 & + & r^{n-2}a_1 & + & r^{n-1}a_1 \quad + \quad r^na_1 \ ) \\ (1-r)S_n & = & a_1 & + & 0 & + & 0 & + & \cdots & + & 0 & + & 0 & + & 0 \quad - \quad r^na_1 \end{array}$$

Notice that when we subtract, all but the first term of the top equation and the last term of the bottom equation cancel out. To obtain a formula for S_n, divide both sides by $1 - r$.

$$S_n = \frac{a_1(1-r^n)}{1-r}, \text{ where } r \neq 1.$$

Example 5.4.4: Finding the First n Terms of a Geometric Series

a. $a_1 = 8$, and we are given that $n = 11$. We can find r by dividing the second term of the series by the first.

$$r = \frac{-4}{8} = -\frac{1}{2}.$$

Substitute values for a_1, r, and n into the formula and simplify.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$\Longrightarrow \quad S_{11} = \frac{8\left(1 - \left(-\frac{1}{2}\right)^{11}\right)}{1 - \left(-\frac{1}{2}\right)} \approx 5.336$$

b. Find a_1 by substituting $k = 1$ into the given explicit formula.

$$a_1 = 3 \cdot 2^1 = 6.$$

We can see from the given explicit formula that $r = 2$. The upper limit of summation is 6, so $n = 6$.

Substitute values for a_1, r, and n into the formula, and simplify.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$\Longrightarrow \quad S_6 = \frac{6\left(1 - (2)^6\right)}{1-2} = 378$$

Example 5.4.5: Solving an Application Problem with a Geometric Series

The problem can be represented by a geometric series with $a_1 = 26,750$, $n = 5$, and $r = 1.016$. Substitute values for a_1, r, and n into the formula and simplify to find the total amount earned at the end of 5 years.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$\Longrightarrow \quad S_5 = \frac{26,750\left(1 - 1.016^5\right)}{1 - 1.016} \approx 138,099.03$$

He will have earned a total of $138,099.03 by the end of 5 years.

Theorem: Sum of a Convergent Infinite Geometric Series

A true proof of this theorem cannot be done without Calculus; however, your instructor can *justify* this theorem in class using a little bit of "hand-waving."

Example 5.4.6: Determining Whether the Sum of an Infinite Series is Defined

a. The ratio of the second term to the first is the same as the ratio of the third term to the second. The series is geometric with a common ratio of $\frac{2}{3}$. Since $-1 < \frac{2}{3} < 1$, this series must converge.
b. The given formula is exponential with a base of $\frac{1}{3}$. Therefore, the series is geometric with a common ratio of $\frac{1}{3}$, and so the infinite series is convergent.

Example 5.4.7: Finding the Sum of an Infinite Geometric Series

a. Since there is not a constant ratio between consecutive terms of the series, the series is not geometric. As such, even if it did converge (which it doesn't), we don't have a theorem to help us find its value.
b. There is a constant ratio; the series is geometric. $a_1 = 248.6$ and $r = \frac{99.44}{248.6} = 0.4$, which is between -1 and 1. Therefore, the geometric series must converge. Substitute $a_1 = 248.6$ and $r = 0.4$ into the formula and simplify to find the sum:

$$\begin{aligned} S &= \frac{a_1}{1-r} \\ \Longrightarrow \quad S &= \frac{248.6}{1-0.4} = 414.\overline{3} \end{aligned}$$

c. The formula is exponential, so the series is geometric with $r = -\frac{1}{3}$. Find a_1 by substituting $k = 1$ into the given explicit formula:

$$a_1 = 4,374 \cdot \left(-\frac{1}{3}\right)^{1-1} = 4,374$$

Substitute $a_1 = 4,374$ and $r = -\frac{1}{3}$ into the formula, and simplify to find the sum:

$$\begin{aligned} S &= \frac{a_1}{1-r} \\ \Longrightarrow \quad S &= \frac{4,374}{1-\left(-\frac{1}{3}\right)} = 3,280.5 \end{aligned}$$

d. The formula is exponential, so the series is geometric, but $r \geq 1$. The series must diverge.

Example 5.4.8: Finding an Equivalent Fraction for a Repeating Decimal

We notice the repeating decimal $0.\overline{3} = 0.333\ldots$, so we can rewrite the repeating decimal as a sum of terms.

$$0.\overline{3} = 0.3 + 0.03 + 0.003 + \ldots$$

Looking for a pattern, we rewrite the sum, noticing that $0.3 = 0.3 \cdot 1 = 0.3 \cdot (0.1)^0$, $0.03 = 0.3 \cdot 0.1 = 0.3 \cdot (0.1)^1$, $0.003 = 0.3 \cdot 0.01 = 0.3 \cdot (0.1)^2$, and so on.

$$0.\overline{3} = 0.3 + 0.3(0.1)^1 + 0.3(0.1)^2 + \cdots$$

Notice the pattern? We multiply each consecutive term by a common ratio of 0.1 starting with the first term of 0.3. So, substituting into our formula for an infinite geometric sum, we have

$$S_n = \frac{a_1}{1-r} = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}.$$

Example 5.4.9: Solving an Annuity Problem

The value of the initial deposit is $100, so $a_1 = 100$. A total of 120 monthly deposits are made in the 10 years, so $n = 120$. To find r, divide the annual interest rate by 12 to find the monthly interest rate and add 1 to represent the new monthly deposit.

$$r = 1 + \frac{0.09}{12} = 1.0075.$$

Substitute $a_1 = 100$, $r = 1.0075$, and $n = 120$ into the formula for the sum of the first n terms of a geometric series, and simplify to find the value of the annuity.

$$S_{120} = \frac{100(1 - 1.0075^{120})}{1 - 1.0075} \approx 19,351.43$$

So the account has $19,351.43 after the last deposit is made.

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5.4.1: Resources and Key Concepts

Resources

Videos

- Partial sums
  - Sequences and Summation Notation - Partial Sums of a Sequence
  - Arithmetic Sequences - Partial Sums of Arithmetic Sequences
  - Geometric Sequences - Partial Sums of Geometric Sequences
- Summation notation
  - Sequences and Summation Notation - Using Sigma Notation
- Infinite series
  - Geometric Series - Infinite Series
- Infinite geometric series
  - Geometric Series - Infinite Geometric Series
  - Geometric Series - Applications of Infinite Geometric Series

Key Concepts

Definitions

- **Series**: The sum of the terms of a (finite or infinite) sequence.
- n^{th} **Partial Sum (S_n)**: The sum of the first n terms of a sequence.
- **Summation Notation (Sigma Notation)**: A concise way to express the sum of a series of terms using the Greek capital letter Sigma (Σ). It includes an index variable (index of summation), a lower limit of summation, an upper limit of summation, and an explicit formula for the kth term. For example, $\sum_{k=a}^{b} f(k)$.
- **Index Variable (Index of Summation)**: The variable used in summation notation (e.g., k in $\sum_{k=a}^{b} f(k)$) that takes on integer values from the lower limit to the upper limit.
- **Lower Limit of Summation**: The starting value for the index variable in summation notation.
- **Upper Limit of Summation**: The ending value for the index variable in summation notation.
- **Arithmetic Series**: The sum of the terms of a (finite or infinite) arithmetic sequence.
- **Geometric Series**: The sum of the terms of a (finite or infinite) geometric sequence.
- **Infinite Series**: The sum of the terms of an infinite sequence.
- **Convergent Series**: An infinite series whose sum approaches a finite real number.
- **Divergent Series**: An infinite series whose sum is not a real number (e.g., tends to infinity or oscillates without approaching a limit).
- **Annuity**: An investment in which the purchaser makes a sequence of periodic, equal payments.

Theorems

- **Formula for the nth Partial Sum of an Arithmetic Series**: Given the arithmetic sequence $a_n = a_1 + (n-1)d$, the sum of the first n terms is $S_n = n\frac{a_1+a_n}{2}$ or $S_n = \frac{n}{2}(2a_1 + (n-1)d)$. (The first form is explicitly derived and used in the text, the second is a common alternative.)
- **Formula for the nth Partial Sum of a Geometric Series**: Given the geometric sequence $a_n = a_1 r^{n-1}$, the sum of the first n terms is $S_n = \frac{a_1(1-r^n)}{1-r}$, for $r \neq 1$.
- **Formula for the Sum of a Convergent Infinite Geometric Series**: The infinite geometric series $\sum_{n=1}^{\infty} a_1 r^{n-1}$ converges to the sum $S = \frac{a_1}{1-r}$ if and only if $|r| < 1$. If $|r| \geq 1$, the series diverges.

Common Mistakes

- **Confusing Sequences with Series**: A sequence is a list of terms, while a series is the sum of those terms.
- **Errors in Summation Notation**: Incorrectly identifying the lower or upper limits of summation.
- **Using the Wrong Sum Formula**: Applying the arithmetic series sum formula to a geometric series, or vice-versa. (Implicit from having distinct formulas)
- **Errors in Arithmetic Series Sum Formula**: Incorrectly identifying a_1, a_n, or n.
- **Errors in Geometric Series Sum Formula (S_n)**: Incorrectly identifying a_1, r, or n.
- **Errors with Infinite Geometric Series**: Attempting to find the sum of a divergent infinite geometric series (i.e., when $|r| \geq 1$).
- **Assuming All Series with Terms Tending to Zero Converge**: Believing that if the terms of an infinite series approach zero, the series must converge. This is not true (e.g., the harmonic series $\sum \frac{1}{n}$ diverges).

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5.4.2: Homework

Reading Questions

1. What is an nth partial sum of a series?
2. What is the difference between an arithmetic sequence and an arithmetic series?
3. What is a geometric series?
4. How is finding the sum of an infinite geometric series different from finding the nth partial sum of a geometric series? What condition must be met for an infinite geometric series to have a sum?
5. What is an annuity in the context of finance?
6. Explain what each part of the summation notation $\sum_{k=1}^{N} a_k$ represents.
7. What is the formula for the sum of the first n terms of an arithmetic series?
8. What is the formula for the sum of the first n terms of a geometric series?
9. Under what condition does an infinite geometric series converge (have a finite sum)? What is the formula for this sum?
10. If the terms of an infinite series tend to zero, does the series necessarily converge? Explain.
11. What is an nth partial sum?
12. What is the difference between an arithmetic sequence and an arithmetic series?
13. What is a geometric series?
14. How is finding the sum of an infinite geometric series different from finding the nth partial sum?
15. What is an annuity?

Homework

Algebraic

For the following exercises, express each description of a sum using summation notation.

16. The sum of terms $m^2 + 3m$ from $m = 1$ to $m = 5$
17. The sum from of $n = 0$ to $n = 4$ of $5n$
18. The sum of $6k - 5$ from $k = -2$ to $k = 1$
19. The sum that results from adding the number 4 five times

For the following exercises, express each arithmetic sum using summation notation.

20. $5 + 10 + 15 + 20 + 25 + 30 + 35 + 40 + 45 + 50$
21. $10 + 18 + 26 + ... + 162$
22. $\frac{1}{2} + 1 + \frac{3}{2} + 2 + ... + 4$

For the following exercises, use the formula for the sum of the first n terms of each arithmetic sequence.

23. $\frac{3}{2} + 2 + \frac{5}{2} + 3 + \frac{7}{2}$
24. $19 + 25 + 31 + ... + 73$
25. $3.2 + 3.4 + 3.6 + ... + 5.6$

For the following exercises, express each geometric sum using summation notation.

26. $1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187$
27. $8 + 4 + 2 + ... + 0.125$
28. $-\frac{1}{6} + \frac{1}{12} - \frac{1}{24} + ... + \frac{1}{768}$

For the following exercises, use the formula for the sum of the first n terms of each geometric sequence, and then state the indicated sum.

29. $9 + 3 + 1 + \frac{1}{3} + \frac{1}{9}$

30. $\sum_{n=1}^{9} 5 \cdot 2^{n-1}$
31. $\sum_{a=1}^{11} 64 \cdot 0.2^{a-1}$

For the following exercises, determine whether the infinite series has a sum. If so, write the formula for the sum. If not, state the reason.

32. $12 + 18 + 24 + 30 + ...$
33. $2 + 1.6 + 1.28 + 1.024 + ...$
34. $\sum_{m=1}^{\infty} 4^{m-1}$
35. $\sum_{k=1}^{\infty} -\left(-\frac{1}{2}\right)^{k-1}$

Graphical

For the following exercises, use the following scenario. Javier makes monthly deposits into a savings account. He opened the account with an initial deposit of $50. Each month thereafter he increased the previous deposit amount by $20.

36. Graph the arithmetic sequence showing one year of Javier's deposits.
37. Graph the arithmetic series showing the monthly sums of one year of Javier's deposits.

For the following exercises, use the geometric series $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$.

38. Graph the first 7 partial sums of the series.
39. What number does S_n seem to be approaching in the graph? Find the sum to explain why this makes sense.

Numeric

For the following exercises, find the indicated sum.

40. $\sum_{a=1}^{14} a$
41. $\sum_{n=1}^{6} n(n-2)$
42. $\sum_{k=1}^{17} k^2$
43. $\sum_{k=1}^{7} 2^k$

For the following exercises, use the formula for the sum of the first n terms of an arithmetic series to find the sum.

44. $-1.7 + -0.4 + 0.9 + 2.2 + 3.5 + 4.8$
45. $6 + \frac{15}{2} + 9 + \frac{21}{2} + 12 + \frac{27}{2} + 15$
46. $-1 + 3 + 7 + ... + 31$
47. $\sum_{k=1}^{11} \left(\frac{k}{2} - \frac{1}{2}\right)$

For the following exercises, use the formula for the sum of the first n terms of a geometric series to find the partial sum.

48. S_6 for the series $-2 - 10 - 50 - 250 - ...$
49. S_7 for the series $0.4 - 2 + 10 - 50 + ...$

50. $\sum_{k=1}^{9} 2^{k-1}$
51. $\sum_{n=1}^{10} -2 \cdot \left(\frac{1}{2}\right)^{n-1}$

For the following exercises, find the sum of the infinite geometric series.

52. $4 + 2 + 1 + \frac{1}{2} + \ldots$
53. $-1 - \frac{1}{4} - \frac{1}{16} - \frac{1}{64} - \ldots$
54. $\sum_{k=1}^{\infty} 3 \cdot \left(\frac{1}{4}\right)^{k-1}$
55. $\sum_{n=1}^{\infty} 4.6 \cdot 0.5^{n-1}$

For the following exercises, determine the value of the annuity for the indicated monthly deposit amount, the number of deposits, and the interest rate.

56. Deposit amount: \$50; total deposits: 60; interest rate: 5%, compounded monthly
57. Deposit amount: \$150; total deposits: 24; interest rate: 3%, compounded monthly
58. Deposit amount: \$450; total deposits: 60; interest rate: 4.5%, compounded quarterly
59. Deposit amount: \$100; total deposits: 120; interest rate: 10%, compounded semi-annually

Extensions

60. The sum of terms $50 - k^2$ from $k = x$ through 7 is 115. What is x?
61. Write an explicit formula for a_k such that $\sum_{k=0}^{6} a_k = 189$. Assume this is an arithmetic series.
62. Find the smallest value of n such that $\sum_{k=1}^{n} (3k - 5) > 100$.
63. How many terms must be added before the series $-1 - 3 - 5 - 7 - \ldots$ has a sum less than -75?
64. Write $0.\overline{65}$ as an infinite geometric series using summation notation. Then use the formula for finding the sum of an infinite geometric series to convert $0.\overline{65}$ to a fraction.
65. The sum of an infinite geometric series is five times the value of the first term. What is the common ratio of the series?
66. To get the best loan rates available, the Riches want to save enough money to place 20% down on a \$160,000 home. They plan to make monthly deposits of \$125 in an investment account that offers 8.5% annual interest compounded semi-annually. Will the Riches have enough for a 20% down payment after five years of saving? How much money will they have saved?
67. Karl has two years to save \$10,000 to buy a used car when he graduates. To the nearest dollar, what would his monthly deposits need to be if he invests in an account offering a 4.2% annual interest rate that compounds monthly?

Real-World Applications

68. Keisha devised a week-long study plan to prepare for finals. On the first day, she plans to study for 1 hour, and each successive day she will increase her study time by 30 minutes. How many hours will Keisha have studied after one week?
69. A boulder rolled down a mountain, traveling 6 feet in the first second. Each successive second, its distance increased by 8 feet. How far did the boulder travel after 10 seconds?
70. A scientist places 50 cells in a petri dish. Every hour, the population increases by 1.5%. What will the cell count be after 1 day?
71. A pendulum travels a distance of 3 feet on its first swing. On each successive swing, it travels $\frac{3}{4}$ the distance of the previous swing. What is the total distance traveled by the pendulum when it stops swinging?
72. Rachael deposits \$1,500 into a retirement fund each year. The fund earns 8.2% annual interest, compounded monthly. If she opened her account when she was 19 years old, how much will she have by the time she is 55? How much of that amount will be interest earned?

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5.5: Mathematical Induction

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The Chinese philosopher Laozi is credited with the saying, "A journey of a thousand miles begins with a single step." In many ways, this is the central theme of this section. Here, we introduce a method of proof, **Mathematical Induction**, which allows us to prove many of the formulas we have merely motivated in previous sections. A good example is the formula for arithmetic sequences.

As mentioned previously, arithmetic sequences can be defined recursively, starting with $a_1 = a$ and then $a_{n+1} = a_n + d$ for $n \geq 1$. This tells us that we start the sequence with a and go from one term to the next by successively adding d. In symbols,

$$a, a + d, a + 2d, a + 3d, a + 4d, ...$$

The pattern suggested here is that to reach the nth term, we start with a and add d to it exactly $n - 1$ times, which leads us to our formula $a_n = a + (n - 1)d$ for $n \geq 1$. But how do we prove this to be the case?

The Principle of Mathematical Induction (PMI)

Suppose $P(n)$ is a sentence involving the natural number n.

IF

$P(1)$ is true and whenever $P(k)$ is true, it follows that $P(k + 1)$ is also true,

THEN

the sentence $P(n)$ is true for all natural numbers n.

The **Principle of Mathematical Induction**, or **PMI** for short, is exactly that—a principle (also known as an **axiom**). It is a property of the natural numbers we either choose to accept or reject. In English, it says that if we want to prove a formula works for all natural numbers n, we start by showing it is true for $n = 1$ (the "**base case**") and then show that if it is true for an arbitrary natural number k, it must be true for the next natural number, $k + 1$ (the "**inductive step**").

The notation $P(n)$ acts just like function notation. For example, if $P(n)$ is the claim that "$n^2 + 1 = 3$," then $P(1)$ would be "$1^2 + 1 = 3$," which is false. The construction $P(k + 1)$ would be "$(k + 1)^2 + 1 = 3$." As usual, this new concept is best illustrated with an example.

Returning to our quest to prove the formula for an arithmetic sequence, we first identify $P(n)$ as the claim $a_n = a + (n - 1)d$. We need to do two things to prove this claim is true for all natural numbers n. First, we need to establish that $P(1)$ is true. In other words, is it true that $a_1 = a + (1 - 1)d$? The answer is yes since this simplifies to $a_1 = a$, which is part of the definition of the arithmetic sequence. The second thing we need to show is that whenever $P(k)$ is true, it follows that $P(k + 1)$ is true. In other words, we *assume* $P(k)$ is true (this is called the "**induction hypothesis**") and *deduce* (through logical arguments) that $P(k + 1)$ is also true.

Assuming $P(k)$ to be true seems to invite disaster—after all, isn't this essentially what we're trying to prove in the first place? To help explain this step better, we show how this works for specific values of n. We've already established that $P(1)$ is true, and we now want to show that $P(2)$ is true. Thus, we need to show that $a_2 = a + (2-1)d$. Since $P(1)$ is true, we have $a_1 = a$, and by the definition of an arithmetic sequence, $a_2 = a_1 + d = a + d = a + (2-1)d$. So $P(2)$ is true.

We now use the fact that $P(2)$ is true to show that $P(3)$ is true. Using the fact that $a_2 = a + (2-1)d$, we show $a_3 = a + (3-1)d$. Since $a_3 = a_2 + d$, we get $a_3 = (a + (2-1)d) + d = a + 2d = a + (3-1)d$, so we have shown $P(3)$ is true. Similarly, we can use the fact that $P(3)$ is true to show that $P(4)$ is true, and so forth.

In general, if $P(k)$ is true (i.e., $a_k = a + (k-1)d$) we set out to show that $P(k+1)$ is true (i.e., $a_{k+1} = a + ((k+1)-1)d$). Assuming $a_k = a + (k-1)d$, we have by the definition of an arithmetic sequence that $a_{k+1} = a_k + d$ so we get $a_{k+1} = (a + (k-1)d) + d = a + kd = a + ((k+1)-1)d$. Hence, $P(k+1)$ is true.

In essence, by showing that $P(k+1)$ must always be true when $P(k)$ is true, we are showing that the formula $P(1)$ can be used to get the formula $P(2)$, which in turn can be used to derive the formula $P(3)$, which in turn can be used to establish the formula $P(4)$, and so on. Thus, as long as $P(k)$ is true for some natural number k, $P(n)$ is true for all of the natural numbers n which follow k. Coupling this with the fact that $P(1)$ is true, we have established $P(k)$ is true for all natural numbers that follow $n = 1$; in other words, all natural numbers n. One might liken Mathematical Induction to a repetitive process like climbing stairs. If you are sure that (1) you can get on the stairs (the base case) and (2) you can climb from any one step to the next step (the inductive step), then presumably you can climb the entire staircase (or falling dominoes, which is the most widely used metaphor in mainstream College Algebra books). We get some more practice with induction in the following example.

Example 5.5.1: Proving a Summation Formula

Prove the following, where $n \in \mathbb{N}$.

$$\sum_{j=1}^{n}(a + (j-1)d) = \frac{n}{2}(2a + (n-1)d)$$

Solution available at the end of this section.

Caution: Unknown Equality

Pay close attention to the fact that we used the symbol $\stackrel{?}{=}$, called an **unknown equality**, when comparing both sides of the equations in Example 5.5.1 for $P(1)$ and $P(k+1)$. This symbol means that we are not certain if those two sides are actually equal. As such, we cannot treat it like an equality. That is, we are not allowed to add, subtract, multiply, or divide between sides (as you can only do that if you have an equality or inequality). In fact, we cannot perform *any* operations *between* sides!

To be clear, it would be mathematically *illegal* to do the following when comparing both sides of $P(n)$ for $n = 1$:

$$\begin{aligned}
\sum_{j=1}^{1}(a + (j-1)d) &\stackrel{?}{=} \frac{1}{2}(2a + (1-1)d) \\
a + (1-1)d &\stackrel{?}{=} \frac{1}{2}(2a) \\
a &\stackrel{?}{=} \frac{1}{2}(2a) \\
0 &\stackrel{?}{=} \frac{1}{2}(2a) - a \quad \text{(subtracting } a \text{ from both sides)} \\
0 &\stackrel{?}{=} a - a \\
0 &= 0\checkmark
\end{aligned}$$

Once you have performed an operation between the two sides of the $\stackrel{?}{=}$ symbol, you have *assumed* that the equation is already true.

Example 5.5.2: Proving an Algebraic Claim

Prove that, for a complex number z, $(\overline{z})^n = \overline{z^n}$, where $n \geq 1$ and $n \in \mathbb{N}$.

Solution available at the end of this section.

Example 5.5.3: Proving a Summation Formula

Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$, where $n \in \mathbb{N}$.

Solution available at the end of this section.

Advice

Notice in the Induction Hypothesis step of Example 5.5.3 that we clearly stated, "Assume $P(k)$ is true. That is, assume $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$." This statement is intentional. It tells us that, from this point forward, whenever we see $1^3 + 2^3 + 3^3 + \cdots + k^3$, we can replace it with $\frac{k^2(k+1)^2}{4}$. This replacement always happens in the next step of the proof.

Additionally, when starting our work on $P(k+1)$, we clearly stated that, "we need to show that $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$." Clearly stating what we need to show, and simplifying that down to $\frac{(k+1)^2(k+2)^2}{4}$, provides us with a goal—we need to manipulate $P(k+1)$ (using the Induction Hypothesis) to eventually become $\frac{(k+1)^2(k+2)^2}{4}$.

Example 5.5.4: Proving an Inequality Claim

Prove that $3^n > 100n$ for $n > 5$, where $n \in \mathbb{N}$.

Solution available at the end of this section.

Example 5.5.4 showcases a challenge when working with inequalities and Mathematical Induction—you need to be very creative and truly understand your prerequisite mathematics to do well with these types of induction proofs. Read through Example 5.5.4 a few times to truly understand what we did.

Example 5.5.5: Proving a Divisibility Claim

Prove $n^2 + n$ is divisible by 2, where $n \in \mathbb{N}$.

Solution available at the end of this section.

While we have used the Principle of Mathematical Induction to prove some of the formulas we have merely motivated in the text, our primary use of this result is to prove the celebrated Binomial Theorem. The ardent Mathematics student will undoubtedly see the PMI in many courses that have yet to be taught. Sometimes, it is explicitly stated, and sometimes, it remains hidden in the background. If ever you see a property expressed as being true "for all natural numbers n," it's a solid bet that the formal proof requires the Principle of Mathematical Induction.

Success in College: Learning From Tests (Part 4)

Tests and quizzes have two purposes. First, they assess your performance. Your test scores are part of a professor's information to assign you a grade for the course. Tests also allow you to see what you do not know. It is your responsibility to use tests to improve your skills and knowledge.

Often, a passing score on a test is 70%. However, the missing 30% is still critically important. When preparing for the next test, you must study your mistakes from the last test. Perhaps you will realize that you still do not understand an important concept. This concept will not disappear. Now is the time to get extra help. When it reappears, you will be ready. Perhaps you will realize that you are making many careless mistakes. Now is the time to plan strategies for checking your work on the next test. Perhaps you will realize that you made most of your mistakes on the last few questions of the exam. Now is the time to ponder how to pace yourself as you work on a test.

Learning from tests takes self-discipline. It is tempting to throw the test away or hide it in a folder. It is your responsibility to review your tests critically. Your ability to do well in the future depends on it.

1. Describe what you usually do with a returned math test.
2. Does what you do with a returned math test depend on whether you earned a good or poor grade on the test? Explain.

Success in College: Groupwork Learning Preferences

Most students are very intimidated by working in groups. It usually takes quite a while for students to realize that studying together can be very rewarding.

Think about the following questions regarding learning or practicing complicated math.

- Do you prefer to do homework by yourself, or do you like to do homework with others?
- Do you like to do homework at school, or do you like to do homework at home?
- If you need help, do you prefer to get it from friends or a spouse, or do you prefer to get it from your professor or a tutor?
- Do you concentrate on learning what you don't know by yourself, or do you prefer to help others because it helps you learn it better?

1. Do you work in conditions that match these preferences when you do your math homework? Why or why not?
2. Spending time with old and new friends is an integral part of college. However, studying with friends is not always practical. Do you ever feel peer pressure to study in ways that are not the best for you?

Answers

Section Preview

Mathematical Induction is the first dedicated proof section in Algebra that the students are likely to have seen. This is a foundational moment—and one not to be shrugged off. Spend a good amount of time discussing proof styles and *why* we start with learning proofs by Mathematical Induction (it's because it's the most algorithmic of the proof styles). Also, make sure students understand that most proofs cannot be done using this niche technique.

Section Preview

- **Sets and Numbers**
  - The Language of Algebra: The PMI is introduced explicitly as an axiom—a property of the natural numbers we choose to accept—so students must understand what an axiom is (versus a definition or theorem) to grasp why induction is a legitimate method of proof rather than something to be derived.

Section Preview

- Use the Principle of Mathematical Induction to prove algebraic identities.

Example $5.5.1$: Proving a Summation Formula

We set $P(n)$ to be the equation we are asked to prove. For $n = 1$, we compare both sides of the equation given in $P(n)$:

$$\sum_{j=1}^{1}(a+(j-1)d) \stackrel{?}{=} \frac{1}{2}(2a+(1-1)d)$$

$$a+(1-1)d \stackrel{?}{=} \frac{1}{2}(2a)$$

$$a = a\checkmark$$

This shows the base case $P(1)$ is true. Next we assume $P(k)$ is true, that is, we assume

$$\sum_{j=1}^{k}(a+(j-1)d) = \frac{k}{2}(2a+(k-1)d)$$

and attempt to use this to show $P(k+1)$ is true. Namely, we must show

$$\sum_{j=1}^{k+1}(a+(j-1)d) = \frac{k+1}{2}(2a+(k+1-1)d).$$

To see how we can use $P(k)$ in this case to prove $P(k+1)$, we note that the sum in $P(k+1)$ is the sum of the first $k+1$ terms of the sequence $a_k = a+(k-1)d$

for $k \geq 1$ while the sum in $P(k)$ is the sum of the first k terms. We compare both sides of the equation in $P(k+1)$.

$$\underbrace{\sum_{j=1}^{k+1}(a+(j-1)d)}_{\text{summing the first } k+1 \text{ terms}} \stackrel{?}{=} \frac{k+1}{2}(2a+(k+1-1)d)$$

$$\underbrace{\sum_{j=1}^{k}(a+(j-1)d)}_{\text{summing the first } k \text{ terms}} + \underbrace{(a+(k+1-1)d)}_{\text{adding the } (k+1)\text{st term}} \stackrel{?}{=} \frac{k+1}{2}(2a+kd)$$

$$\underbrace{\frac{k}{2}(2a+(k-1)d)}_{\text{Using } P(k)} + (a+kd) \stackrel{?}{=} \frac{(k+1)(2a+kd)}{2}$$

$$\frac{k(2a+(k-1)d)+2(a+kd)}{2} \stackrel{?}{=} \frac{2ka+k^2d+2a+kd}{2}$$

$$\frac{2ka+2a+k^2d+kd}{2} = \frac{2ka+2a+k^2d+kd}{2} \checkmark$$

Since all of our steps on both sides of the string of equations are reversible, we conclude that the two sides of the equation are equivalent, and hence, $P(k+1)$ is true. By the Principle of Mathematical Induction, we have that $P(n)$ is true for all natural numbers n.

Example 5.5.2: Proving an Algebraic Claim

Let $P(n)$ be the formula $(\overline{z})^n = \overline{z^n}$.

For $n=1$, we have $(\overline{z})^1 = \overline{z} = \overline{z^1}$. Thus, $P(1)$ is true.

Induction Hypothesis: Assume $P(k)$ is true. That is, assume $(\overline{z})^k = \overline{z^k}$.

To show $P(k+1)$ is true, we need to show that $(\overline{z})^{k+1} = \overline{z^{k+1}}$.

$$\begin{aligned}(\overline{z})^{k+1} &= (\overline{z})^k\,\overline{z} \\ &= \overline{z^k}\,\overline{z} && \text{(Induction Hypothesis)} \\ &= \overline{z^k z} && \text{(Product Rule for Conjugates)} \\ &= \overline{z^{k+1}}\end{aligned}$$

This establishes $(\overline{z})^{k+1} = \overline{z^{k+1}}$, so that $P(k+1)$ is true. Hence, by the Principle of Mathematical Induction, $(\overline{z})^n = \overline{z^n}$ for all $n \geq 1$.

Example 5.5.3: Proving a Summation Formula

Let $P(n)$ be the formula $1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$.

For $n=1$, we have $1^3 = \frac{1^2(1+1)^2}{4}$. Thus, $P(1)$ is true.

Induction Hypothesis: Assume $P(k)$ is true. That is, assume $1^3+2^3+3^3+\cdots+k^3 = \frac{k^2(k+1)^2}{4}$.

To show $P(k+1)$ is true, we need to show that $1^3+2^3+3^3+\cdots+k^3+(k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$.

$$\begin{aligned}1^3+2^3+3^3+\ldots+k^3+(k+1)^3 &= (1^3+2^3+3^3+\ldots+k^3)+(k+1)^3 \\ &= \frac{k^2(k+1)^2}{4}+(k+1)^3 && \text{(Induction Hypothesis)} \\ &= \frac{k^2(k+1)^2+4(k+1)^3}{4} \\ &= \frac{(k+1)^2\left(k^2+4(k+1)\right)}{4} \\ &= \frac{(k+1)^2\left(k^2+4k+4\right)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4}\end{aligned}$$

Therefore, $1^3+2^3+3^3+\cdots+k^3+(k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$, and so $P(k+1)$ is true. Hence, by the Principle of Mathematical Induction, $1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$ for all natural numbers n.

Example 5.5.4: Proving an Inequality Claim

The first wrinkle we encounter in this problem is that we are asked to prove this formula for $n > 5$ instead of $n \geq 1$. Since n is a natural number, our base step occurs at $n=6$. We can still use the PMI in this case, but we will conclude that the formula is valid for all $n \geq 6$.

Let $P(n)$ be the inequality $3^n > 100n$.

For $n=6$, we have $3^6 = 729 > 600 = 100(6)$. Thus, $P(6)$ is true.

Induction Hypothesis: Assume $P(k)$ is true. That is, assume $3^k > 100k$.

To show $P(k+1)$ is true, we need to show $3^{k+1} > 100(k+1) = 100k+100$.

$$\begin{aligned}3^{k+1} &= 3^k \cdot 3 \\ &> 100k \cdot 3 && \text{(Induction Hypothesis)} \\ &= 100k+200k && \text{(our goal is to get to } 100k+100\text{, so isolating } 100k \text{ is helpful)} \\ &> 100k+100 && \left(200k > 100 \text{ when } k > \tfrac{1}{2}\text{. Since } k \geq 6,\ 200k > 100\right)\end{aligned}$$

Hence $P(k+1)$ is true. By induction, $3^n > 100n$ for all $n \geq 6$.

Example $5.5.5$: Proving a Divisibility Claim

Let $P(n)$ be the statement that $n^2 + n$ is divisible by 2.

For $n = 1$, we have $1^2 + 1 = 2$. This is definitely divisible by 2 (since $\frac{2}{2} = 1$), so $P(1)$ is true.

Induction Hypothesis: Assume $P(k)$ is true. That is, assume $k^2 + k$ is divisible by 2.

To show $P(k+1)$ is true, we need to show that $(k+1)^2 + (k+1)$ is divisible by 2.

$$\begin{aligned}(k+1)^2 + (k+1) &= k^2 + 2k + 1 + k + 1 \\ &= (k^2 + k) + 2k + 1 + 1 \quad \text{(Associative and Commutative Properties of Addition)} \\ &= (k^2 + k) + 2k + 2 \\ &= (k^2 + k) + 2(k+1)\end{aligned}$$

We know that $k^2 + k$ is divisible by 2, by the Induction Hypothesis. We also know that $2(k+1)$ is divisible by 2. The sum of numbers divisible by 2 is also divisible by 2, so $(k+1)^2 + (k+1)$ must be divisible by 2. Hence, $P(k+1)$ is true. Thus, the claim is true for all natural numbers n by Mathematical Induction.

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5.5.1: Resources and Key Concepts

Resources

Videos

- Principle of Mathematical Induction (PMI)
  - Mathematical Induction - What is Mathematical Induction, and How Does it Work?
- Proving summation formulas
  - Mathematical Induction - Proving Summation Conjectures Using Mathematical Induction
- Proving divisibility and inequalities by induction
  - Mathematical Induction - Proving Non-Summation Conjectures Using Mathematical Induction

Key Concepts

Definitions

- **Mathematical Induction**: A method of proof used to establish that a given statement is true for all natural numbers (or for all natural numbers from a certain point onwards).
- **Base Case (in Mathematical Induction)**: The first step in an inductive proof, where the statement $P(n)$ is shown to be true for the initial value (usually $n = 1$, or the starting value specified in the claim).
- **Induction Hypothesis (Inductive Hypothesis)**: The second step in an inductive proof, where it is assumed that the statement $P(k)$ is true for an arbitrary natural number k (or k greater than or equal to the base case value).
- **Inductive Step (in Mathematical Induction)**: The part of an inductive proof where, using the induction hypothesis (assuming $P(k)$ is true), it is shown that $P(k + 1)$ must also be true.

Theorems

- **The Principle of Mathematical Induction (PMI)**: Suppose $P(n)$ is a claim involving the natural number n. If 1. $P(1)$ is true (base case) and 2. assuming $P(k)$ is true (Induction Hypothesis), it follows that $P(k + 1)$ is also true (inductive step), then the claim $P(n)$ is true for all natural numbers n.

Common Mistakes

- **Skipping or Incorrectly Proving the Base Case**: The base case is a crucial first step. Failing to establish its truth, or choosing the wrong starting value for n, invalidates the entire proof.
- **Incorrectly Stating or Using the Inductive Hypothesis**: The inductive hypothesis must be clearly stated as an assumption ($P(k)$ is true). In the inductive step, this assumption must be explicitly used to prove $P(k + 1)$.
- **Proving $P(k)$ instead of $P(k + 1)$ in the Inductive Step**: The goal of the inductive step is to show that *if* $P(k)$ is true, *then* $P(k + 1)$ is true. Simply restating or manipulating $P(k)$ is not sufficient.
- **Assuming $P(k + 1)$ is True**: Circular reasoning where the statement to be proven ($P(k + 1)$) is assumed as part of its own proof.
- **Algebraic Errors in the Inductive Step**: Making mistakes in algebraic manipulation when trying to transform the expression for $P(k + 1)$ to show it follows from $P(k)$. This is common in summation, inequality, and divisibility proofs.
- **Not Clearly Showing How $P(k)$ Leads to $P(k + 1)$**: The logical connection between the assumption of $P(k)$ and the conclusion of $P(k + 1)$ must be explicit and clear.
- **Incorrectly Handling Inequalities in Inductive Step**: When proving inequalities, not carefully maintaining the direction of the inequality or making invalid steps (e.g., multiplying by a negative without reversing the sign).

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5.5.2: Homework

Reading Questions

1. What is the Principle of Mathematical Induction (PMI)? Describe its two main parts.
2. What is the "base case" in a proof by mathematical induction, and why is it important?
3. What is the "inductive hypothesis" in a proof by mathematical induction?
4. What is the "inductive step" in a proof by mathematical induction? What are you trying to show in this step?
5. Explain the "domino effect" analogy for mathematical induction.
6. When proving a summation formula like $\sum_{j=1}^{n}(a+(j-1)d) = \frac{n}{2}(2a+(n-1)d)$ using induction, what does $P(k)$ represent, and what does $P(k+1)$ represent?
7. In the inductive step of proving a summation formula, how is the sum for $P(k+1)$ typically related to the sum for $P(k)$?
8. If you are asked to prove a statement for $n > 5$, what would be your base case?
9. When proving a divisibility claim like "n^2+n is divisible by 2," what does it mean to assume $P(k)$ is true (i.e., k^2+k is divisible by 2)? How might you express this algebraically?
10. Why is it important not to perform operations *between* the two sides of an equation when you are in the process of verifying if $P(k+1)$ is true (using the $\stackrel{?}{=}$ notation)?

Homework

For the following exercises, prove the claim using the Principle of Mathematical Induction.

11. $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$
12. $\displaystyle\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$
13. $2^n > 500n$ for $n > 12$
14. $3^n \geq n^3$ for $n \geq 4$
15. Use the Product Rule for Absolute Value to show $|x^n| = |x|^n$ for all real numbers x and all natural numbers $n \geq 1$
16. Use the Product Rule for Logarithms to show $\log(x^n) = n\log(x)$ for all real numbers $x > 0$ and all natural numbers $n \geq 1$.
17. For the sequence $a_1 = a$, $a_{n+1} = ra_n$, for $n \geq 1$, prove $a_n = ar^{n-1}$.
18. Discuss the classic 'paradox' All Horses are the Same Color problem with your classmates.

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5.6: Binomial Theorem

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Recall that a polynomial with two terms is called a binomial. We have already learned how to multiply binomials and to raise binomials to powers, but raising a binomial to a high power can be tedious and time-consuming. In this section, we will discuss a shortcut that will allow us to find $(x+y)^n$ without multiplying the binomial by itself n times.

In this section, we aim to prove the celebrated **Binomial Theorem**. Simply stated, the Binomial Theorem is a formula for the expansion of quantities $(a+b)^n$ for natural numbers n. In Elementary and Intermediate Algebra, you should have seen specific instances of the formula, namely

$$\begin{aligned}(a+b)^1 &= a+b\\ (a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

If we wanted the expansion for $(a+b)^4$, we would write $(a+b)^4=(a+b)(a+b)^3$ and use the formula that we have for $(a+b)^3$ to get

$$(a+b)^4=(a+b)\left(a^3+3a^2b+3ab^2+b^3\right)=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

Generalizing this a bit, we see that if we have a formula for $(a+b)^k$, we can obtain a formula for $(a+b)^{k+1}$ by rewriting the latter as

$$(a+b)^{k+1}=(a+b)(a+b)^k.$$

This means Mathematical Induction plays a significant role in the proof of the Binomial Theorem.

Factorials

Before we can state the theorem, we need to revisit the sequence of factorials that were introduced earlier in this chapter.

Definition: Factorial (Recursive Definition)

The **factorial** of a whole number n, denoted $n!$ and reads as "n **factorial**," is defined by the recursive formula

$$\begin{aligned}a_0 &= 0! = 1\\ a_n &= n! = n\cdot a_{n-1} \qquad \text{for } n\geq 1\end{aligned}$$

Recall this means $0!=1$ and $n!=n(n-1)!$ for $n\geq 1$. Using the recursive definition, we get: $1!=1\cdot 0!=1\cdot 1=1$, $2!=2\cdot 1!=2\cdot 1=2$, $3!=3\cdot 2!=3\cdot 2\cdot 1=6$, and $4!=4\cdot 3!=4\cdot 3\cdot 2\cdot 1=24$. Informally, $n!=n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$ with $0!=1$ as our "base case." Our first example refamiliarizes us with some of the basic computations involving factorials.

Example 5.6.1: Computing Factorials

a. Simplify the following expressions.
   i. $\frac{3!\,2!}{0!}$

ii. $\frac{7!}{5!}$
iii. $\frac{1000!}{998!\,2!}$
iv. $\frac{(k+2)!}{(k-1)!}$, $k \geq 1$

b. Prove $n! > 3^n$ for all $n \geq 7$.

Solutions available at the end of this section.

Of all the mathematical animals we discussed in the text, factorials grow most quickly. In Example $5.6.1b$, we proved that $n!$ overtakes 3^n at $n = 7$. "Overtakes" may be too polite a word since $n!$ thoroughly trounces 3^n for $n \geq 7$, as any reasonable set of data will show. It can be shown that for any real number $x > 0$, not only does $n!$ eventually overtake x^n, but the ratio $\frac{x^n}{n!} \to 0$ as $n \to \infty$. This fact is far more important than you might realize right now, but wait until Calculus and you *might* begin to understand the significance of this statement.

Binomial Coefficients

Applications of factorials in the wild often involve counting arrangements. For example, suppose you have fifty songs on your phone and wish to arrange these songs in a playlist in which the order of the songs matters. There are $50!$ different possible playlists in that case. That equates to $30,414,093,201,713,378,043,612,608,166,064,768,844,377,641,568,960,512,000,000,000,000$ possible playlists. If you wish to select only ten of the songs to create a playlist, then there are $\frac{50!}{40!} = 37,276,043,023,296,000$ such playlists. If, on the other hand, you want to select ten song files out of the fifty to play in *any* order, there are $\frac{50!}{40!10!} = 10,272,278,170$ ways to achieve this. The authors encourage you to take courses such as Finite Mathematics, Discrete Mathematics, and Statistics if you are interested in concepts like this. We introduce these concepts here because this is how the factorials make their way into the Binomial Theorem, as the following definition indicates.

Definition: Binomial Coefficient

For integers n and j satisfying $0 \leq j \leq n$, the **binomial coefficient** $\binom{n}{j}$ (read, "n choose j") is the whole number given by

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}.$$

The name "binomial coefficient" will be justified shortly. For now, we can physically interpret $\binom{n}{j}$ as the number of ways to select j items from n items where the order of the items selected is unimportant. For example, suppose you win two free tickets to a special screening of the latest Hollywood blockbuster and have five friends, each of whom would love to accompany you to the movies. There are $\binom{5}{2}$ ways to choose who goes with you. Applying the definition of the binomial coefficient, we get

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2} = 10.$$

So there are 10 different ways to distribute those two tickets among five friends. (Some will see it as 10 ways to decide which three friends must stay home.) The reader is encouraged to verify this by listing all the possibilities.

Example $5.6.2$: Finding the Number of Combinations Using the Formula

A fast food restaurant offers five side dish options. Your meal comes with two side dishes.

a. How many ways can you select your side dishes?
b. How many ways can you select three side dishes?

Solutions available at the end of this section.

The equivalence of the answers to Example $5.6.2a$ and $5.6.2b$ is not a coincidence. Asking how many ways to choose two items from five should be the same as asking how to *exclude* three items from five.

Checkpoint 5.6.2

An ice cream shop offers 10 flavors of ice cream. How many ways are there to choose three flavors for a banana split?

Q&A

Is a binomial coefficient always a whole number?

Yes. Just as the number of combinations must always be a whole number, a binomial coefficient will always be a whole number.

Example 5.6.3: Finding Binomial Coefficients

Find each binomial coefficient.

a. $\binom{5}{3}$
b. $\binom{9}{2}$
c. $\binom{9}{7}$

Solutions available at the end of this section.

Checkpoint 5.6.3

Find each binomial coefficient.

a. $\binom{7}{3}$
b. $\binom{11}{4}$

The Binomial Theorem

We now state and prove a theorem, named after the celebrated French mathematician Blaise Pascal, which is crucial to the proof of the Binomial Theorem.

Theorem: Pascal's Rule

For integers n and j with $1 \leq j \leq n$,

$$\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}.$$

Proof available at the end of this section.

We can now state and prove the Binomial Theorem. This theorem states that binomial coefficients are just that—coefficients in the binomial expansion.

Theorem: Binomial Theorem

For nonzero real numbers a and b,

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$$

for all natural numbers n.

Proof available at the end of this section.

To get a feel of what this theorem is saying and how it is easier to remember than it may first appear, let's consider the specific case of $n = 4$. According to the theorem, we have

$$\begin{aligned}(a+b)^4 &= \sum_{j=0}^{4} \binom{4}{j} a^{4-j} b^j \\ &= \binom{4}{0} a^{4-0} b^0 + \binom{4}{1} a^{4-1} b^1 + \binom{4}{2} a^{4-2} b^2 + \binom{4}{3} a^{4-3} b^3 + \binom{4}{4} a^{4-4} b^4 \\ &= \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} a b^3 + \binom{4}{4} b^4\end{aligned}$$

We forgo the simplification of the coefficients to note the pattern in the expansion. First, note that in each term, the total of the exponents is 4, which matches the exponent of the binomial $(a+b)^4$. The exponent on a begins at 4 and decreases by one as we move from one term to the next, while the exponent on b starts at 0 and increases by one each time. Also, note that the binomial coefficients themselves have a pattern. The upper number, 4, matches the exponent on the binomial $(a+b)^4$, whereas the lower number changes from term to term and matches the exponent of b in that term. This is no coincidence and corresponds to the kind of counting we discussed earlier. If we think of obtaining $(a+b)^4$ by multiplying $(a+b)^2(a+b)^2$, our answer is the sum of all possible products with exactly four factors—some a, some b. If we wish to count, for instance, the number of ways we obtain 1 factor of b out of a total of 4 possible factors, thereby forcing the remaining 3 factors to be a, the answer is $\binom{4}{1}$. Hence, the term $\binom{4}{1} a^3 b$ is in the expansion. The other terms which appear cover the remaining cases.

Example 5.6.4: Evaluating Binomials Using the Binomial Theorem

Use the Binomial Theorem to find the following.

a. $(x-2)^4$
b. 2.1^3
c. The term containing x^3 in the expansion $(2x+y)^5$

Solutions available at the end of this section.

Checkpoint 5.6.4

Find the sixth term of $(3x-y)^9$ without fully expanding the binomial.

Pascal's Triangle

We close this section with Pascal's Triangle. Pascal's Triangle is obtained by arranging the binomial coefficients in the triangular fashion below.

$$\begin{array}{ccccccccc}
 & & & \binom{0}{0} & & & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \searrow\swarrow & & & & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \searrow\swarrow & & \searrow\swarrow & & & & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & \vdots & & & & &
\end{array}$$

Since $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for all whole numbers n, we get that each row of Pascal's Triangle begins and ends with 1. We use the additive relationship expressed in our first theorem in this section to generate the numbers in the middle of the rows (from the third row onwards). For instance, $\binom{1}{0} + \binom{1}{1} = \binom{2}{1}$, $\binom{2}{0} + \binom{2}{1} = \binom{3}{1}$, and so forth. The arrows in the array above indicate this relationship. With these two facts, we can quickly generate Pascal's Triangle. We start with the first two rows, 1 and $1 \quad 1$. From that point on, each successive row begins and ends with 1, and the middle numbers are generated using the first theorem in this section. Below we attempt to demonstrate this building process to generate the first five rows of Pascal's Triangle.

$$\begin{array}{ccccccc}
 & & 1 & & & & \\
 & 1 & & 1 & & & \\
 & & \searrow\swarrow & & & & \\
\boxed{1} & & \underline{1+1} & & \boxed{1} & & \\
\end{array}
\quad\longrightarrow\quad
\begin{array}{ccccc}
 & & 1 & & \\
 & 1 & & 1 & \\
1 & & 2 & & 1
\end{array}$$

$$\begin{array}{ccccccc}
 & & 1 & & & & \\
 & 1 & & 1 & & & \\
1 & & 2 & & 1 & & \\
 & \searrow\swarrow & & \searrow\swarrow & & & \\
\boxed{1} & & \underline{1+2} & & \underline{2+1} & & \boxed{1}
\end{array}
\quad\longrightarrow\quad
\begin{array}{ccccc}
 & & 1 & & \\
 & 1 & & 1 & \\
1 & & 2 & & 1 \\
1 & & 3 & & 3 & & 1
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & 1 & & & & & \\
 & & 1 & & 1 & & & & \\
 & 1 & & 2 & & 1 & & & \\
1 & & 3 & & 3 & & 1 & & \\
 & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & & \\
\boxed{1} & & \underline{1+3} & & \underline{3+3} & & \underline{3+1} & & \boxed{1}
\end{array}
\quad\longrightarrow\quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1 \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

To see how we can use Pascal's Triangle to expedite the Binomial Theorem, suppose we wish to expand $(3x - y)^4$. The coefficients we need are $\binom{4}{j}$ for $j = 0, 1, 2, 3, 4$ and are the numbers that form the fifth row of Pascal's Triangle. Since we know that the exponent of $3x$ in the first term is 4 and then decreases by one as we go from left to right while the exponent of $-y$ starts at 0 in the first term and then increases by one as we move from left to right, we quickly obtain

$$\begin{aligned}(3x-y)^4 &= (1)(3x)^4+(4)(3x)^3(-y)+(6)(3x)^2(-y)^2+4(3x)(-y)^3+1(-y)^4\\ &= 81x^4-108x^3y+54x^2y^2-12xy^3+y^4\end{aligned}$$

We would like to stress that Pascal's Triangle is a rapid method to expand an *entire* binomial. If only a term (or two or three) is required, then the Binomial Theorem is the way to go.

Checkpoint 5.6.5

Write in expanded form.

a. $(x-y)^5$
b. $(2x+5y)^3$

Success in Mathematics: Why Study Mathematics? (Part 5)

Mathematics reveals hidden patterns that help us understand the world around us. Now, much more than arithmetic and geometry, mathematics today is a diverse discipline that deals with data, measurements, and observations from science; with inference, deduction, and proof; and with mathematical models of natural phenomena, human behavior, and social systems.

The process of "doing" mathematics is far more than just calculation or deduction; it involves observing patterns, testing conjectures, and estimating results. As a practical matter, mathematics is a science of pattern and order. Its domain is not molecules or cells but numbers, chance, form, algorithms, and change. As a science of abstract objects, mathematics relies on logic rather than observation as its standard of truth yet employs observation, simulation, and even experimentation to discover the truth.

The unique role of mathematics in education is a consequence of its universal applicability. The mathematics results - theorems and theories - are significant and valuable; the best results are also elegant and profound. Through its theorems, mathematics offers science both a foundation of truth and a standard of certainty. In addition to theorems and theories, mathematics offers distinctive modes of thought that are versatile and powerful, including modeling, abstraction, optimization, logical analysis, inference from data, and the use of symbols.

As a major intellectual tradition, mathematics is a subject appreciated as much for its beauty as for its power. The enduring qualities of such abstract concepts as symmetry, proof, and change have been developed through more than 3,000 years of intellectual effort. Like language, religion, and music, mathematics is a universal part of human culture.

1. What was your favorite moment or lecture in this course? Explain.
2. Has this course helped your appreciation for mathematics? Explain.

Answers

Section Preview

Aside from proofs by Mathematical Induction, this is the most abstract material in the course; however, a lot can be gained here—not least of which is Pascal's Triangle. This material *can* be used in Differential Calculus, but really steps into the spotlight in Integral Calculus and Differential Equations. Exposure at that time is assumed.

Section Preview

- Simplify expressions involving factorials.
- Understand and use binomial coefficients and the Binomial Theorem to expand the powers of binomials quickly.

Example 5.6.1: Computing Factorials

a. We keep in mind the mantra, "When in doubt, write it out!" as we simplify the following.
   i. We have been programmed to react with alarm to the presence of a 0 in the denominator, but in this case, $0!=1$, so the fraction is defined after all. As for the numerator, $3!=3\cdot2\cdot1=6$ and $2!=2\cdot1=2$, so we have $\frac{3!\,2!}{0!}=\frac{(6)(2)}{1}=12$.
   ii. We have $7!=7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1=5040$ while $5!=5\cdot4\cdot3\cdot2\cdot1=120$. Dividing, we get $\frac{7!}{5!}=\frac{5040}{120}=42$. While this is correct, we note that we could have saved ourselves some time had we proceeded as follows

$$\frac{7!}{5!}=\frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1}=\frac{7\cdot6\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}=7\cdot6=42.$$

   In fact, should we want to exploit the recursive nature of the factorial fully, we can write

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5!}{5!} = \frac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} = 42.$$

iii. Keeping in mind the lesson we learned from the previous problem, we have

$$\frac{1000!}{998!\,2!} = \frac{1000 \cdot 999 \cdot 998!}{998! \cdot 2!} = \frac{1000 \cdot 999 \cdot \cancel{998!}}{\cancel{998!} \cdot 2!} = \frac{999000}{2} = 499500$$

iv. This problem continues the theme we have seen in the previous two problems. We first note that since $k+2$ is larger than $k-1$, $(k+2)!$ contains all of the factors of $(k-1)!$ and as a result we can get the $(k-1)!$ to cancel from the denominator. To see this, we begin by writing out $(k+2)!$ starting with $(k+2)$ and multiplying it by the numbers which precede it until we reach $(k-1)$: $(k+2)! = (k+2)(k+1)(k)(k-1)!$. As a result, we have

$$\frac{(k+2)!}{(k-1)!} = \frac{(k+2)(k+1)(k)(k-1)!}{(k-1)!} = \frac{(k+2)(k+1)(k)\,\cancel{(k-1)!}}{\cancel{(k-1)!}} = k(k+1)(k+2)$$

The stipulation $k \geq 1$ is there to ensure that all of the factorials involved are defined.

b. This will require a proof by Mathematical Induction. In this case, the claim should only hold for $n \geq 7$. Let $P(n)$ be the statement $n! > 3^n$.

For $n = 7$, we have $7! = 5040 > 2187 = 3^7$. Thus, $P(7)$ is true.

Induction Hypothesis: Assume $P(k)$ is true. That is, assume $k! > 3^k$.

We want to show that $P(k+1)$ is true. That is, we want to show $(k+1)! > 3^{k+1} = 3 \cdot 3^k$.

$$\begin{aligned}
(k+1)! &= (k+1)k! && \text{(Definition of a factorial)} \\
&> (k+1)3^k && \text{(Induction Hypothesis)} \\
&> (3)3^k && (k+1 > 3 \text{ when } k > 2. \text{ Luckily, we know that } k \geq 7, \text{ so } k+1 > 3) \\
&= 3^{k+1}
\end{aligned}$$

Hence, we have shown $(k+1)! > 3^{k+1}$. Thus, $P(k+1)$ is true. By the Principle of Mathematical Induction, we have $n! > 3^n$ for all $n \geq 7$.

Example 5.6.2: Finding the Number of Combinations Using the Formula

a. We want to choose two side dishes from five options.

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{2! \cdot \cancel{3!}} = \frac{20}{2} = 10.$$

There are ten ways to choose two side dishes from five options.

b. We want to choose three side dishes from five options.

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!} = \frac{20}{2} = 10.$$

Therefore, there are also ten ways to choose three side dishes from five options.

Example 5.6.3: Finding Binomial Coefficients

Use the formula to calculate each binomial coefficient:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

a. $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!} = \frac{20}{2} = 10$

b. $\binom{9}{2} = \frac{9!}{2!(9-2)!} = \frac{9 \cdot 8 \cdot \cancel{7!}}{2! \cdot \cancel{7!}} = \frac{72}{2} = 36$

c. $\binom{9}{7} = \frac{9!}{7!(9-7)!} = \frac{9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!} \cdot 2!} = \frac{72}{2} = 36$

Theorem: Pascal's Rule

$$\begin{aligned}
\binom{n}{j-1}+\binom{n}{j} &= \frac{n!}{(j-1)!(n-(j-1))!}+\frac{n!}{j!(n-j)!} \\
&= \frac{n!}{(j-1)!(n-j+1)!}+\frac{n!}{j!(n-j)!} \\
&= \frac{n!}{(j-1)!(n-j+1)(n-j)!}+\frac{n!}{j(j-1)!(n-j)!} \\
&= \frac{n!\,j}{j(j-1)!(n-j+1)(n-j)!}+\frac{n!(n-j+1)}{j(j-1)!(n-j+1)(n-j)!} \\
&= \frac{n!\,j}{j!(n-j+1)!}+\frac{n!(n-j+1)}{j!(n-j+1)!} \\
&= \frac{n!\,j+n!(n-j+1)}{j!(n-j+1)!} \\
&= \frac{n!\,(j+(n-j+1))}{j!(n-j+1)!} \\
&= \frac{(n+1)n!}{j!(n+1-j))!} \\
&= \frac{(n+1)!}{j!((n+1)-j))!} \\
&= \binom{n+1}{j}\checkmark
\end{aligned}$$

Theorem: Binomial Theorem

To prove the Binomial Theorem, we let $P(n)$ be the expansion formula given in the statement of the theorem, and we note that $P(1)$ is true since

$$\begin{aligned}
(a+b)^1 &\stackrel{?}{=} \sum_{j=0}^{1}\binom{1}{j}a^{1-j}b^j \\
a+b &\stackrel{?}{=} \binom{1}{0}a^{1-0}b^0+\binom{1}{1}a^{1-1}b^1 \\
a+b &= a+b\checkmark
\end{aligned}$$

Now we assume that $P(k)$ is true. That is, we assume that we can expand $(a+b)^k$ using the formula given in the Binomial Theorem and attempt to show that $P(k+1)$ is true.

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k \\
&= (a+b)\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\
&= a\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j+b\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\
&= \sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j+\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1}
\end{aligned}$$

We aim to combine as many terms as possible within the two summations. As the counter j in the first summation runs from 0 through k, we get terms involving a^{k+1}, a^kb, $a^{k-1}b^2$, \ldots , ab^k. In the second summation, we get terms involving a^kb, $a^{k-1}b^2$, \ldots , ab^k, b^{k+1}. In other words, they have common terms apart from the first term in the first summation and the last term in the second summation. Our next move is to "kick out" the terms that we cannot combine and rewrite the summations so that we can combine them. To that end, we note

$$\sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j = a^{k+1}+\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j$$

and

$$\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1} = \sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1}+b^{k+1}$$

so that

$$(a+b)^{k+1} = a^{k+1}+\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j+\sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1}+b^{k+1}.$$

We now wish to write

$$\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^j+\sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1}$$

as a single summation. The wrinkle is that the first summation starts with $j=1$, while the second starts with $j=0$. Even though the sums produce terms with the same powers of a and b, they do so for different values of j. To resolve this, we need to shift the index on the second summation so that the index j starts at $j=1$ instead of $j=0$.

$$\begin{aligned}\sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} &= \sum_{j=0+1}^{k-1+1}\binom{k}{j-1}a^{k-(j-1)}b^{(j-1)+1}\\ &= \sum_{j=1}^{k}\binom{k}{j-1}a^{k+1-j}b^{j}\end{aligned}$$

We can now combine our two sums and simplify.

$$\begin{aligned}\sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^{j}+\sum_{j=0}^{k-1}\binom{k}{j}a^{k-j}b^{j+1} &= \sum_{j=1}^{k}\binom{k}{j}a^{k+1-j}b^{j}+\sum_{j=1}^{k}\binom{k}{j-1}a^{k+1-j}b^{j}\\ &= \sum_{j=1}^{k}\left[\binom{k}{j}+\binom{k}{j-1}\right]a^{k+1-j}b^{j}\\ &= \sum_{j=1}^{k}\binom{k+1}{j}a^{k+1-j}b^{j}\end{aligned}$$

Using this and the fact that $\binom{k+1}{0}=1$ and $\binom{k+1}{k+1}=1$, we get

$$\begin{aligned}(a+b)^{k+1} &= a^{k+1}+\sum_{j=1}^{k}\binom{k+1}{j}a^{k+1-j}b^{j}+b^{k+1}\\ &= \binom{k+1}{0}a^{k+1}b^{0}+\sum_{j=1}^{k}\binom{k+1}{j}a^{k+1-j}b^{j}+\binom{k+1}{k+1}a^{0}b^{k+1}\\ &= \sum_{j=0}^{k+1}\binom{k+1}{j}a^{(k+1)-j}b^{j}\end{aligned}$$

which shows that $P(k+1)$ is true. Hence, by induction, we have established that the Binomial Theorem holds for all natural numbers n.

Example $5.6.4$**: Evaluating Binomials Using the Binomial Theorem**

a. Since $(x-2)^4=(x+(-2))^4$, we identify $a=x$, $b=-2$ and $n=4$ and obtain

$$\begin{aligned}(x-2)^4 &= \sum_{j=0}^{4}\binom{4}{j}x^{4-j}(-2)^{j}\\ &= \binom{4}{0}x^{4-0}(-2)^{0}+\binom{4}{1}x^{4-1}(-2)^{1}+\binom{4}{2}x^{4-2}(-2)^{2}+\binom{4}{3}x^{4-3}(-2)^{3}+\binom{4}{4}x^{4-4}(-2)^{4}\\ &= x^4-8x^3+24x^2-32x+16\end{aligned}$$

b. At first, this problem seems misplaced, but we can write $2.1^3=(2+0.1)^3$. Identifying $a=2$, $b=0.1=\frac{1}{10}$ and $n=3$, we get

$$\begin{aligned}\left(2+\frac{1}{10}\right)^3 &= \sum_{j=0}^{3}\binom{3}{j}2^{3-j}\left(\frac{1}{10}\right)^{j}\\ &= \binom{3}{0}2^{3-0}\left(\frac{1}{10}\right)^{0}+\binom{3}{1}2^{3-1}\left(\frac{1}{10}\right)^{1}+\binom{3}{2}2^{3-2}\left(\frac{1}{10}\right)^{2}+\binom{3}{3}2^{3-3}\left(\frac{1}{10}\right)^{3}\\ &= 8+\frac{12}{10}+\frac{6}{100}+\frac{1}{1000}\\ &= 8+1.2+0.06+0.001\\ &= 9.261\end{aligned}$$

c. Identifying $a=2x$, $b=y$ and $n=5$, the Binomial Theorem gives

$$(2x+y)^5=\sum_{j=0}^{5}\binom{5}{j}(2x)^{5-j}y^{j}$$

Since we are concerned with only the term containing x^3, there is no need to expand the entire sum. The exponents on each term must add to 5 and if the exponent on x is 3, the exponent on y must be 2. Plucking out the term $j=2$, we get

$$\binom{5}{2}(2x)^{5-2}y^2=10(2x)^3y^2=80x^3y^2$$

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5.6.1: Resources and Key Concepts

Resources

Videos

- Binomial Theorem
  - The Binomial Theorem - The Binomial Theorem
- Binomial coefficients
  - The Binomial Theorem - Factorials and Combinations
- Pascal's Triangle
  - The Binomial Theorem - Pascal's Triangle and Powers of Binomials

Key Concepts

Definitions

- **Binomial Coefficient**: Given two whole numbers n and j with $n \geq j$, the binomial coefficient $\binom{n}{j}$ (read "n choose j") is the whole number given by $\binom{n}{j} = \frac{n!}{j!(n-j)!}$. It represents the number of ways to select j items from n items where order is unimportant.
- **Pascal's Triangle**: A triangular arrangement of binomial coefficients, where each number is the sum of the two numbers directly above it. The kth number (starting $k = 0$) in row n (starting $n = 0$) is $\binom{n}{k}$.

Theorems

- **Pascal's Rule**: For natural numbers n and j with $n \geq j$, $\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$.
- **Binomial Theorem**: For nonzero real numbers a and b, and any natural number n, $(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$. This can be expanded as: $(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$.

Common Mistakes

- **Miscalculating Binomial Coefficients**: Errors in applying the formula $\binom{n}{j} = \frac{n!}{j!(n-j)!}$.
- **Errors in Applying the Binomial Theorem**:
  - Using incorrect binomial coefficients for terms.
  - Errors in the exponents of a or b for a given term (they should sum to n; the exponent of b matches j, the lower index of the coefficient).
  - Forgetting to apply exponents to the entire term if a or b are themselves products or have coefficients (e.g., in $(2x+y)^5$, a term involves $(2x)^k$, not just $2x^k$).
  - Sign errors when one of the terms in the binomial is negative (e.g., $(x-2)^4$).
- **Incorrectly Identifying Terms in Pascal's Triangle**: Using the wrong row or miscounting positions when using Pascal's Triangle for binomial coefficients. Remember row n gives coefficients for $(a+b)^n$, and rows/positions start counting from 0.
- **Using Pascal's Triangle for Single Term vs. Full Expansion**: Pascal's Triangle is efficient for full expansion. For finding a single term, the Binomial Theorem formula with binomial coefficients is generally better.
- **Assuming $\binom{n}{j} = \binom{n}{n-j}$ is Pascal's Rule**: While $\binom{n}{j} = \binom{n}{n-j}$ is a true property (symmetry of binomial coefficients), Pascal's Rule is the additive relationship $\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$.

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5.6.2: Homework

Reading Questions

1. What is a factorial, and how is $n!$ defined recursively? What is the value of $0!$?
2. What is a binomial coefficient $\binom{n}{j}$, and how is it calculated using factorials? What does it represent in terms of combinations?
3. State the Binomial Theorem. What does it provide a formula for?
4. In the expansion of $(a+b)^n$, what is the sum of the exponents of a and b in each term?
5. In the term $\binom{n}{j}a^{n-j}b^j$ from the Binomial Theorem, how does the value of j relate to the exponent of b?
6. Describe how Pascal's Triangle is constructed. How do its entries relate to binomial coefficients?
7. When is it more advantageous to use Pascal's Triangle for binomial expansion compared to using the Binomial Theorem formula directly?
8. What is Pascal's Rule for binomial coefficients?
9. How many terms are there in the expansion of $(a+b)^n$?
10. What are the first three terms in the expansion of $(x+y)^5$ using the Binomial Theorem?
11. What is a binomial coefficient, and how it is calculated?
12. What role do binomial coefficients play in a binomial expansion? Are they restricted to any type of number?
13. What is the Binomial Theorem and what is its use?
14. When is it an advantage to use the Binomial Theorem? Explain.

Homework

Algebraic

For the following exercises, evaluate the binomial coefficient.

15. $\binom{6}{2}$
16. $\binom{5}{3}$
17. $\binom{7}{4}$
18. $\binom{9}{7}$
19. $\binom{10}{9}$
20. $\binom{25}{11}$
21. $\binom{17}{6}$
22. $\binom{200}{199}$

For the following exercises, use the Binomial Theorem to expand each binomial.

23. $(4a-b)^3$
24. $(5a+2)^3$
25. $(3a+2b)^3$
26. $(2x+3y)^4$
27. $(4x+2y)^5$
28. $(3x-2y)^4$
29. $(4x-3y)^5$
30. $\left(\frac{1}{x}+3y\right)^5$
31. $(x^{-1}+2y^{-1})^4$
32. $(\sqrt{x}-\sqrt{y})^5$

For the following exercises, use the Binomial Theorem to write the first three terms of each binomial.

33. $(a+b)^{17}$

34. $(x-1)^{18}$
35. $(a-2b)^{15}$
36. $(x-2y)^{8}$
37. $(3a+b)^{20}$
38. $(2a+4b)^{7}$
39. $(x^3-\sqrt{y})^{8}$

For the following exercises, find the indicated term of each binomial without fully expanding the binomial.

40. The fourth term of $(2x-3y)^4$
41. The fourth term of $(3x-2y)^5$
42. The third term of $(6x-3y)^7$
43. The eighth term of $(7+5y)^{14}$
44. The seventh term of $(a+b)^{11}$
45. The fifth term of $(x-y)^7$
46. The tenth term of $(x-1)^{12}$
47. The ninth term of $(a-3b^2)^{11}$
48. The fourth term of $\left(x^3-\frac{1}{2}\right)^{10}$
49. The eighth term of $\left(\frac{y}{2}+\frac{2}{x}\right)^9$

Graphical

For the following exercises, use the Binomial Theorem to expand the binomial $f(x)=(x+3)^4$. Then find and graph each indicated sum on one set of axes.

50. Find and graph $f_1(x)$, such that $f_1(x)$ is the first term of the expansion.
51. Find and graph $f_2(x)$, such that $f_2(x)$ is the sum of the first two terms of the expansion.
52. Find and graph $f_3(x)$, such that $f_3(x)$ is the sum of the first three terms of the expansion.
53. Find and graph $f_4(x)$, such that $f_4(x)$ is the sum of the first four terms of the expansion.
54. Find and graph $f_5(x)$, such that $f_5(x)$ is the sum of the first five terms of the expansion.

Extensions

55. In the expansion of $(5x+3y)^n$, each term has the form $\binom{n}{k}a^{n-k}b^k$, where k successively takes on the value 0, 1, 2, ..., n. If $\binom{n}{k}=\binom{7}{2}$, what is the corresponding term?
56. In the expansion of $(a+b)^n$, the coefficient of $a^{n-k}b^k$ is the same as the coefficient of which other term?
57. Consider the expansion of $(x+b)^{40}$. What is the exponent of b in the kth term?
58. Find $\binom{n}{k-1}+\binom{n}{k}$ and write the answer as a binomial coefficient in the form $\binom{n}{k}$. Prove it. *Hint:* Use the fact that, for any integer p, such that $p\geq 1$, $p!=p(p-1)!$.
59. Which expression cannot be expanded using the Binomial Theorem? Explain.
    a. (x^2-2x+1)
    b. $(\sqrt{a}+4\sqrt{a}-5)^8$
    c. $(x^3+2y^2-z)^5$
    d. $(3x^2-\sqrt{2y^3})^{12}$

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6: ANALYTIC GEOMETRY

(a) Greek philosopher Aristotle (384–322 BCE) (b) German mathematician and astronomer Johannes Kepler (1571–1630)



Videos

- A Visual Introduction to Conic Sections

The Greek mathematician Menaechmus (c. 380 - c. 320 BCE) is generally credited with discovering the shapes formed by the intersection of a plane and a right circular cone. Depending on how he tilted the plane when it intersected the cone, he formed different shapes at the intersection–beautiful shapes with near-perfect symmetry.

It was also said that Aristotle may have had an intuitive understanding of these shapes, as he observed the orbit of the planet to be circular. He presumed that the planets moved in circular orbits around Earth, and for nearly 2000 years this was the commonly held belief.

It was not until the Renaissance movement that Johannes Kepler noticed that the orbits of the planet were not circular in nature. His published law of planetary motion in the 1600s changed our view of the solar system forever. He claimed that the sun was at one end of the orbits, and the planets revolved around the sun in an oval-shaped path.

In this chapter, we will investigate the two-dimensional figures that are formed when a right circular cone is intersected by a plane. We will begin by studying each of three figures created in this manner. We will develop defining equations for each figure and then learn how to use these equations to solve a variety of problems.

6.1: The Ellipse

6.1.1: Resources and Key Concepts

6.1.2: Homework

6.2: The Hyperbola

6.2.1: Resources and Key Concepts

6.2.2: Homework

6.3: The Parabola

6.3.1: Resources and Key Concepts

6.3.2: Homework

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6.1: The Ellipse

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure 6.1.1**:** *The National Statuary Hall in Washington, D.C. (credit: Greg Palmer, Flickr)*

Can you imagine standing at one end of a large room and still being able to hear a whisper from a person standing at the other end? The National Statuary Hall in Washington, D.C., shown in Figure 6.1.1, is such a room. It is an oval-shaped room called a **whispering chamber** because the shape makes it possible for sound to travel along the walls. In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.

Ellipses as Conic Sections

We begin this material with the general concept of a **conic section**.

Definition: Conic Section

A **conic section**, or **conic**, is a shape resulting from intersecting a right circular cone with a plane.

The angle at which the plane intersects the cone determines the shape, as shown in Figure 6.1.2.

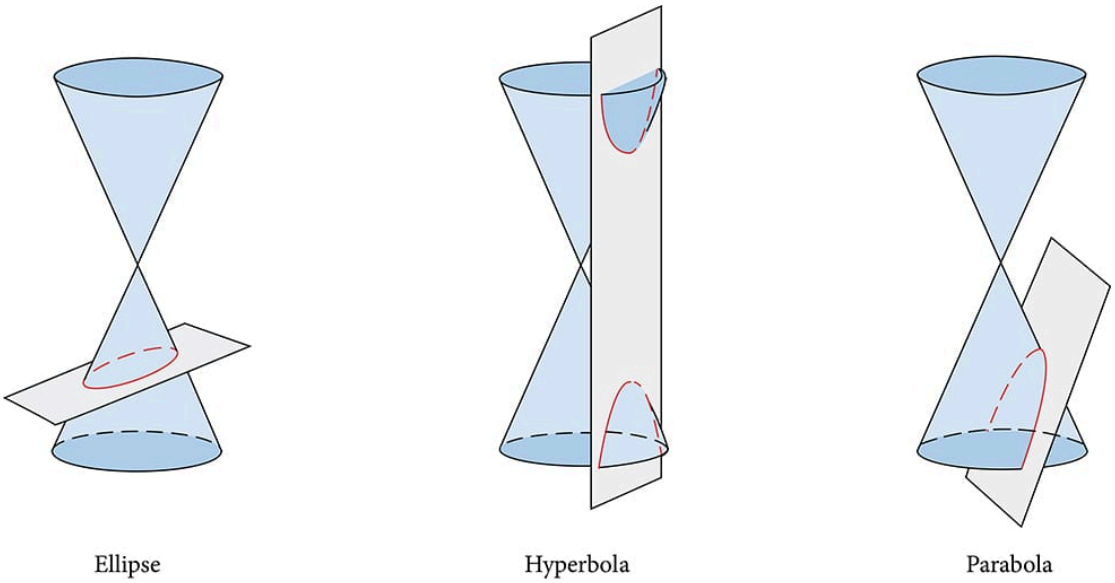


Figure 6.1.2

Conic sections can also be described by a set of points in the coordinate plane. Later in this chapter, we will see that the graph of any quadratic equation in two variables is a conic section. The signs within the equations and the coefficients of the variable terms determine the shape. This section focuses on the four variations of the standard form of the equation for the **ellipse**.

Definition: Ellipse

An **ellipse** is the set of all points in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a **focus** (plural: **foci**).

We can draw an ellipse using a piece of cardboard, two thumbtacks, a pencil, and string. Place the thumbtacks in the cardboard to form the foci of the ellipse. Cut a piece of string longer than the distance between the two thumbtacks (the length of the string represents the constant in the definition). Tack each end of the string to the cardboard, and trace a curve with a pencil held taut against the string. The result is an ellipse. See Figure 6.1.3.

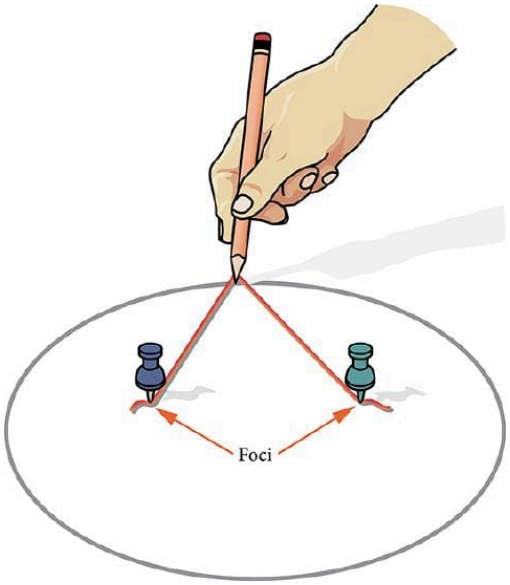


Figure 6.1.3

Every ellipse has two axes of symmetry and several key points.

Definition: Major and Minor Axes, Vertex and Co-Vertex, and Center of an Ellipse

The longer axis of an ellipse is called the **major axis**, and the shorter axis is called the **minor axis**. Each endpoint of the major axis is called a **vertex** of the ellipse (plural: **vertices**), and each endpoint of the minor axis is called a **co-vertex** of the ellipse. The **center** of the ellipse is the midpoint of both the major and minor axes.

The axes are perpendicular at the center. The foci always lie on the major axis, and the sum of the distances from the foci to any point on the ellipse (the constant sum) is greater than the distance between the foci. See Figure $6.1.4$.

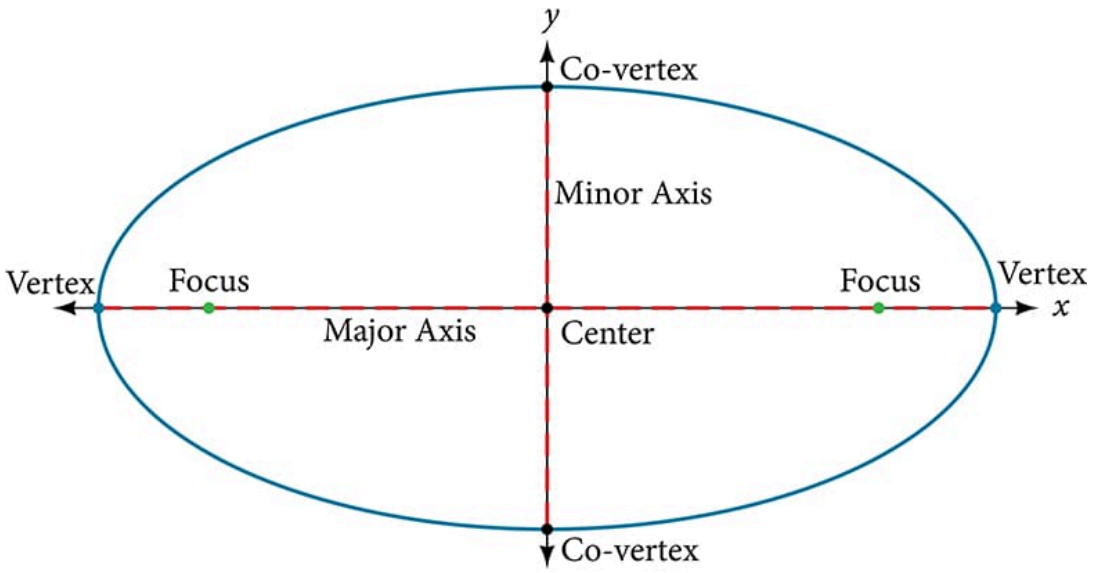


Figure $6.1.4$

In this course, we restrict ellipses to those that are positioned vertically or horizontally in the coordinate plane. That is, the axes will either lie on or be parallel to the x- and y-axes.

To work with horizontal and vertical ellipses in the coordinate plane, we consider two cases: those that are centered at the origin and those that are centered at a point other than the origin. First, we will learn to derive the equations of ellipses, and then we will learn how to write the equations of ellipses in standard form. Later we will use what we learn to draw the graphs.

Standard Form of an Ellipse Centered at the Origin

Since there are two ways we are going to orient our ellipses (horizontally and vertically), there are two equations for the standard form of an ellipse.

Theorem: Standard Form of the Equation of an Ellipse Centered at the Origin

The standard form of the equation of an ellipse with center $(0,0)$ and major axis on the x-axis is

$$\frac{x^2}{v_x^2}+\frac{y^2}{v_y^2}=1,$$

where

- v_x and v_y are positive
- $v_x > v_y$
- the length of the major axis is $2v_x$
- the coordinates of the vertices are $(\pm v_x, 0)$
- the length of the minor axis is $2v_y$

- the coordinates of the co-vertices are $(0, \pm v_y)$
- the coordinates of the foci are $(\pm c, 0)$, where $c^2 = |v_x^2 - v_y^2|$.

The standard form of the equation of an ellipse with center $(0, 0)$ and major axis on the y-axis is

$$\frac{x^2}{v_x^2} + \frac{y^2}{v_y^2} = 1,$$

where

- v_x and v_y are positive
- $v_y > v_x$
- the length of the major axis is $2v_y$
- the coordinates of the vertices are $(0, \pm v_y)$
- the length of the minor axis is $2v_x$
- the coordinates of the co-vertices are $(\pm v_x, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = |v_x^2 - v_y^2|$.

Proof available at the end of this section.

You should spend time reading through the proof of this theorem (or, better yet, you should ask your instructor to go over this theorem in class!).

This theorem can be shortened quite a bit by saying that the larger of v_x and v_y dictates the major axis (and the smaller dictates the minor axis). In any case, the vertices in the horizontal direction are located v_x units from the center (at $(\pm v_x, 0)$) and the vertices in the vertical direction are v_y units from the center (at $(0, \pm v_y)$). The foci are located c units from the center along the major axis, where $c^2 = |v_x^2 - v_y^2|$. The absolute values in this final equation force positivity of the right side.

Example 6.1.1: Writing the Equation of an Ellipse Centered at the Origin in Standard Form

What is the standard form equation of the ellipse that has vertices $(\pm 8, 0)$ and foci $(\pm 5, 0)$?

Solution available at the end of this section.

Checkpoint 6.1.1

What is the standard form equation of the ellipse that has vertices $(0, \pm 4)$ and foci $(0, \pm\sqrt{15})$?

Q&A

Can we write the equation of an ellipse centered at the origin given coordinates of just one focus and vertex?

Yes. Ellipses are symmetrical, so the coordinates of the vertices of an ellipse centered around the origin will always have the form $(\pm v_x, 0)$ or $(0, \pm v_y)$. Similarly, the coordinates of the foci will always have the form $(\pm c, 0)$ or $(0, \pm c)$. Knowing this, we can use v_x (or v_y) and c from the given points, along with the equation $c^2 = |v_x^2 - v_y^2|$, to find the missing denominator.

Standard Form of an Ellipse Centered Off the Origin

Like the graphs of other equations, the graph of an ellipse can be transformed—specifically, in this conversation, shifted. If an ellipse is shifted h units horizontally and k units vertically, the center of the ellipse will be (h, k). This transformation results in the standard form of the equation we saw previously, with x replaced by $(x - h)$ and y replaced by $(y - k)$.

Theorem: The Standard Form of the Equation of an Ellipse with Center (h, k)

The standard form of the equation of an ellipse with center (h, k) and major axis parallel to the x-axis is

$$\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1,$$

where

- v_x and v_y are positive
- $v_x^2 > v_y^2$
- the length of the major axis is $2v_x$
- the coordinates of the vertices are $(h \pm v_x, k)$
- the length of the minor axis is $2v_y$
- the coordinates of the co-vertices are $(h, k \pm v_y)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^2 = |v_x^2 - v_y^2|$.

The standard form of the equation of an ellipse with center (h, k) and major axis parallel to the y-axis is

$$\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1,$$

where

- v_x and v_y are positive
- $v_y^2 > v_x^2$
- the length of the major axis is $2v_y$
- the coordinates of the vertices are $(h, k \pm v_y)$
- the length of the minor axis is $2v_x$
- the coordinates of the co-vertices are $(h \pm v_x, k)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^2 = |v_x^2 - v_y^2|$.

Again, this theorem can be shortened to saying that the larger of v_x and v_y dictates the major axis. In any case, the vertices in the horizontal direction are located v_x units from the center (at $(h \pm v_x, k)$) and the vertices in the vertical direction are v_y units from the center (at $(h, k \pm v_y)$). The foci are located c units from the center along the major axis, where $c^2 = |v_x^2 - v_y^2|$.

Example 6.1.2: Writing the Equation of an Ellipse Centered at a Point Other Than the Origin

What is the standard form equation of the ellipse that has vertices $(-2, -8)$ and $(-2, 2)$ and foci $(-2, -7)$ and $(-2, 1)$?

Solution available at the end of this section.

Checkpoint 6.1.2

What is the standard form equation of the ellipse that has vertices $(-3, 3)$ and $(5, 3)$ and foci $(1 - 2\sqrt{3}, 3)$ and $(1 + 2\sqrt{3}, 3)$?

Graphing Ellipses Centered at the Origin

Just as we can write the equation for an ellipse given information about its graph, we can graph an ellipse given its equation. To graph ellipses centered at the origin, we use the standard form $\frac{x^2}{v_x^2} + \frac{y^2}{v_y^2} = 1$.

Example 6.1.3: Graphing an Ellipse Given an Equation

Graph the ellipse given by the equation, $\frac{x^2}{9}+\frac{y^2}{25}=1$. Identify and label the center, vertices, co-vertices, and foci.

Solution available at the end of this section.

Checkpoint 6.1.3

Graph the ellipse given by the equation $\frac{x^2}{36}+\frac{y^2}{4}=1$. Identify and label the center, vertices, co-vertices, and foci.

Example 6.1.4: Graphing an Ellipse Centered at the Origin from an Equation Not in Standard Form

Graph the ellipse given by the equation $4x^2+25y^2=100$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

Solution available at the end of this section.

Checkpoint 6.1.4

Graph the ellipse given by the equation $49x^2+16y^2=784$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

Graphing Ellipses Not Centered at the Origin

When an ellipse is not centered at the origin, we can still use the standard form to find the key features of the graph. When the ellipse is centered at some point, (h,k), we use the standard form $\frac{(x-h)^2}{v_x^2}+\frac{(y-k)^2}{v_y^2}=1$. From this standard equation, we can easily determine the center, vertices, co-vertices, foci, and positions of the major and minor axes.

Example 6.1.5: Graphing an Ellipse Centered at (h,k)

Graph the ellipse given by the equation, $\frac{(x+2)^2}{4}+\frac{(y-5)^2}{9}=1$. Identify and label the center, vertices, co-vertices, and foci.

Solution available at the end of this section.

Checkpoint 6.1.5

Graph the ellipse given by the equation $\frac{(x-4)^2}{36}+\frac{(y-2)^2}{20}=1$. Identify and label the center, vertices, co-vertices, and foci.

Ellipses in General Form

We are sometimes given the equation of a conic in **general form**.

Definition: General Form of the Equation of a Conic Section

The **general form** of a conic is the unfactored form of the conic's equation, which is generally written in the form

$$ax^2+by^2+cx+dy+e=0,$$

where a and b are not both zero.

To get equations like these into the celebrated standard form, we use our old friend—Completing the Square.

Example 6.1.6: Graphing an Ellipse Centered at (h, k) by First Writing It in Standard Form

Graph the ellipse given by the equation $4x^2 + 9y^2 - 40x + 36y + 100 = 0$. Identify and label the center, vertices, co-vertices, and foci.

Solution available at the end of this section.

Solving Applied Problems Involving Ellipses

Many real-world situations can be represented by ellipses, including orbits of planets, satellites, moons and comets, and shapes of boat keels, rudders, and some airplane wings. A medical device called a lithotripter uses elliptical reflectors to break up kidney stones by generating sound waves. Some buildings, called whispering chambers, are designed with elliptical domes so that a person whispering at one focus can easily be heard by someone standing at the other focus. This occurs because of the acoustic properties of an ellipse. When a sound wave originates at one focus of a whispering chamber, the sound wave will be reflected off the elliptical dome and back to the other focus. See Figure 6.1.9. In the whisper chamber at the Museum of Science and Industry in Chicago, two people standing at the foci—about 43 feet apart—can hear each other whisper.

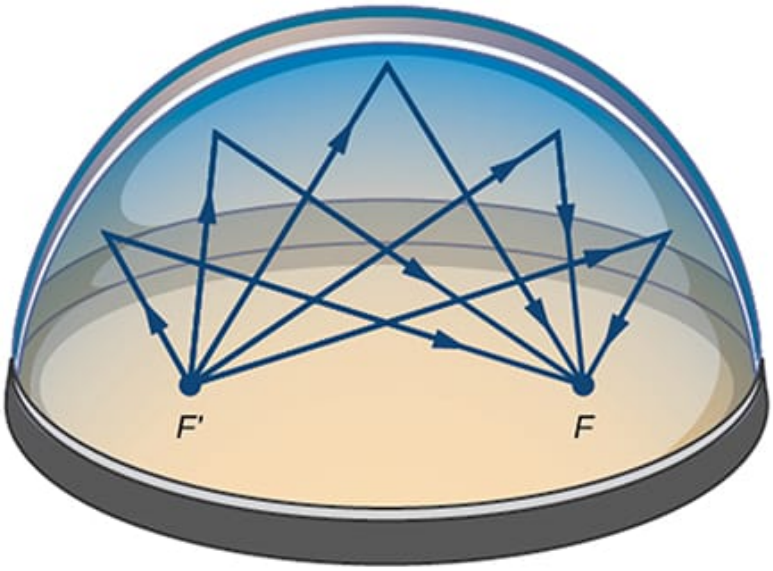


Figure 6.1.9: *Sound waves are reflected between foci in an elliptical room, called a whispering chamber.*

Example 6.1.7: Locating the Foci of a Whispering Chamber

The Statuary Hall in the Capitol Building in Washington, D.C. is a whispering chamber. Its dimensions are 46 feet wide by 96 feet long as shown in Figure 6.1.10.

a. What is the standard form of the equation of the ellipse representing the outline of the room? Hint: assume a horizontal ellipse, and let the center of the room be the point $(0, 0)$.
b. If two senators standing at the foci of this room can hear each other whisper, how far apart are the senators? Round to the nearest foot.

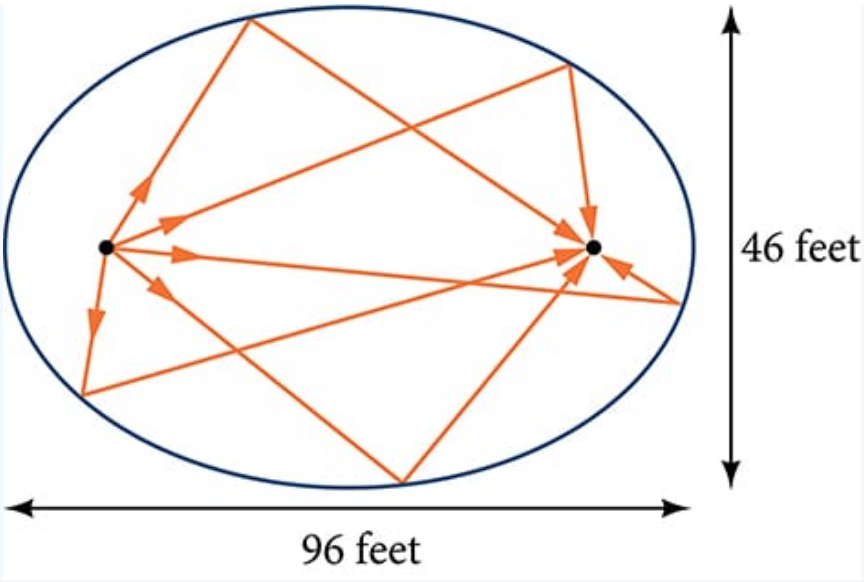


Figure 6.1.10

Solutions available at the end of this section.

Checkpoint 6.1.7

Suppose a whispering chamber is 480 feet long and 320 feet wide.

a. What is the standard form of the equation of the ellipse representing the room? Hint: assume a horizontal ellipse, and let the center of the room be the point $(0, 0)$.
b. If two people are standing at the foci of this room and can hear each other whisper, how far apart are the people? Round to the nearest foot

Success in College: Groupwork Learning Preferences

Most students are very intimidated by working in groups. It usually takes quite a while for students to realize that studying together can be very rewarding.

Think about the following questions regarding learning or practicing complicated math.

- Do you prefer to do homework by yourself, or do you like to do homework with others?
- Do you like to do homework at school, or do you like to do homework at home?
- If you need help, do you prefer to get it from friends or a spouse, or do you prefer to get it from your professor or a tutor?
- Do you concentrate on learning what you don't know by yourself, or do you prefer to help others because it helps you learn it better?

1. Do you work in conditions that match these preferences when you do your math homework? Why or why not?
2. Spending time with old and new friends is an integral part of college. However, studying with friends is not always practical. Do you ever feel peer pressure to study in ways that are not the best for you?

Answers

Section Preview

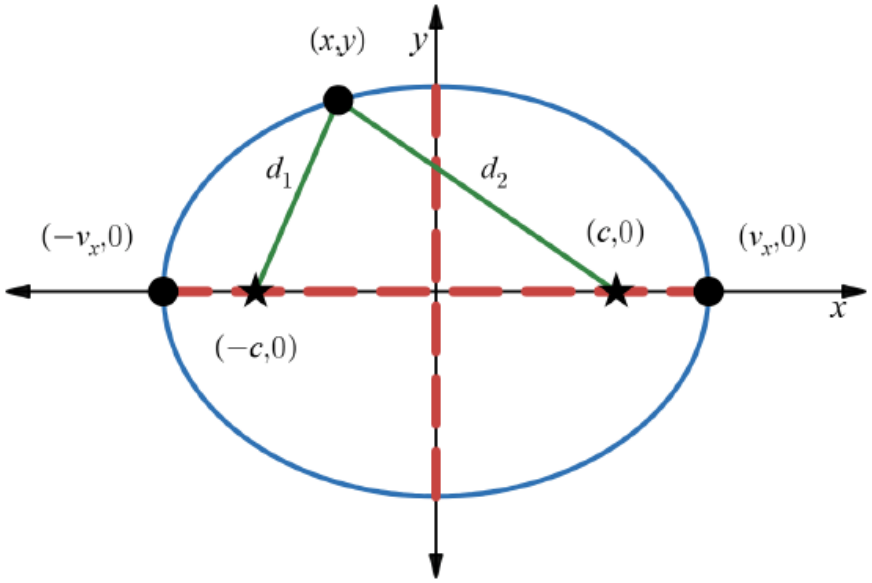
Ellipses are peripherally used in Differential Calculus, but become more important in Integral Calculus and beyond. Students must have some pre-exposure to this material prior to those courses.

Section Preview

- Write equations of ellipses in standard form.
- Graph ellipses centered at the origin.
- Graph ellipses not centered at the origin.
- Solve applied problems involving ellipses.

Theorem: Standard Form of the Equation of an Ellipse Centered at the Origin

To derive the equation of an ellipse centered at the origin, we begin with the foci $(-c, 0)$ and $(c, 0)$. The ellipse is the set of all points (x, y) such that the sum of the distances from (x, y) to the foci is constant, as shown in the figure below.



If $(v_x, 0)$ is a vertex of the ellipse (on the major axis), the distance from $(-c, 0)$ to $(v_x, 0)$ is $v_x - (-c) = v_x + c$. The distance from $(c, 0)$ to $(v_x, 0)$ is $v_x - c$. The sum of the distances from the foci to the vertex is

$$(v_x + c) + (v_x - c) = 2v_x$$

If (x, y) is a point on the ellipse, then we can define the following variables:

$$\begin{aligned} d_1 &= \text{the distance from } (-c, 0) \text{ to } (x, y) \\ d_2 &= \text{the distance from } (c, 0) \text{ to } (x, y) \end{aligned}$$

By the definition of an ellipse, $d_1 + d_2$ is constant for any point (x, y) on the ellipse. We know that the sum of these distances is $2v_x$ for the vertex $(v_x, 0)$. It follows that $d_1 + d_2 = 2v_x$ for any point on the ellipse. We will begin the derivation by applying the Distance Formula. The rest of the derivation is algebraic.

$$
\begin{aligned}
d_1 + d_2 &= 2v_x \\
\implies \sqrt{(x-(-c))^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} &= 2v_x && \text{(substituting)} \\
\implies \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} &= 2v_x && \text{(simplifying)} \\
\implies \sqrt{(x+c)^2 + y^2} &= 2v_x - \sqrt{(x-c)^2 + y^2} && \text{(isolating a radical)} \\
\implies (x+c)^2 + y^2 &= \left[2v_x - \sqrt{(x-c)^2 + y^2}\right]^2 && \text{(squaring both sides)} \\
\implies (x+c)^2 + y^2 &= 4v_x^2 - 4v_x\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 && \text{(distributing)} \\
\implies x^2 + 2cx + c^2 + y^2 &= 4v_x^2 - 4v_x\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2 && \text{(distributing)} \\
\implies 2cx &= 4v_x^2 - 4v_x\sqrt{(x-c)^2 + y^2} - 2cx && \text{(subtracting } x^2,\ c^2,\ \text{and } y^2 \\
& && \text{from both sides)} \\
\implies 4cx - 4v_x^2 &= -4v_x\sqrt{(x-c)^2 + y^2} && \text{(isolating the radical)} \\
\implies cx - v_x^2 &= -v_x\sqrt{(x-c)^2 + y^2} && \text{(dividing both sides by 4)} \\
\implies (cx - v_x^2)^2 &= v_x^2\left((x-c)^2 + y^2\right) && \text{(squaring both sides)} \\
\implies c^2x^2 - 2v_x^2cx + v_x^4 &= v_x^2\left(x^2 - 2cx + c^2 + y^2\right) && \text{(distributing)} \\
\implies c^2x^2 - 2v_x^2cx + v_x^4 &= v_x^2x^2 - 2v_x^2cx + v_x^2c^2 + v_x^2y^2 && \text{(distributing)} \\
\implies c^2x^2 + v_x^4 &= v_x^2x^2 + v_x^2c^2 + v_x^2y^2 && \text{(adding } 2v_x^2cx \text{ to both sides)} \\
\implies v_x^4 - v_x^2c^2 &= v_x^2x^2 - c^2x^2 + v_x^2y^2 && \text{(subtracting } v_x^2c^2 \text{ and } c^2x^2 \\
& && \text{from both sides)} \\
\implies v_x^2(v_x^2 - c^2) &= x^2(v_x^2 - c^2) + v_x^2y^2 && \text{(factor)} \\
\implies v_x^2b^2 &= x^2b^2 + v_x^2y^2 && \text{(let } b^2 = v_x^2 - c^2\text{)} \\
\implies \frac{v_x^2b^2}{v_x^2b^2} &= \frac{x^2b^2}{v_x^2b^2} + \frac{v_x^2y^2}{v_x^2b^2} && \text{(divide both sides by } v_x^2b^2\text{)} \\
\implies 1 &= \frac{x^2}{v_x^2} + \frac{y^2}{b^2} && \text{(simplify)}
\end{aligned}
$$

If we let $x = 0$ in our final result, we get $y^2 = b^2$, which implies $y = \pm b$. Therefore, the point $(0, b)$ must be the co-vertex of the ellipse. As such, we relabel $b = v_y$ to arrive at the standard equation of an ellipse:

$$\frac{x^2}{v_x^2} + \frac{y^2}{v_y^2} = 1.$$

This equation defines an ellipse centered at the origin. If $v_x > v_y$, the ellipse is stretched further in the horizontal direction, and if $v_y > v_x$, the ellipse is stretched further in the vertical direction.

Example 6.1.1**: Writing the Equation of an Ellipse Centered at the Origin in Standard Form**

The foci are on the x-axis, so the major axis is the x-axis. Thus, the equation will have the form

$$\frac{x^2}{v_x^2} + \frac{y^2}{v_y^2} = 1$$

The vertices are $(\pm 8, 0)$, so $v_x = 8$ and $v_x^2 = 64$.

The foci are $(\pm 5, 0)$, so $c = 5$ and $c^2 = 25$.

We know that the vertices and foci are related by the equation $c^2 = |v_x^2 - v_y^2|$. Solving for v_y^2, we have:

$$
\begin{aligned}
c^2 &= \left|v_x^2 - v_y^2\right| \\
\implies 25 &= \left|64 - v_y^2\right| && \text{(substituting)} \\
\implies 25 &= 64 - v_y^2 && \left(\text{since the major axis is horizontal, } v_x^2 > v_y^2\right) \\
\implies v_y^2 &= 39 && \left(\text{solving for } v_y^2\right)
\end{aligned}
$$

Now we need only substitute $v_x^2 = 64$ and $v_y^2 = 39$ into the standard form of the equation. The equation of the ellipse is $\frac{x^2}{64} + \frac{y^2}{39} = 1$.

Example 6.1.2**: Writing the Equation of an Ellipse Centered at a Point Other Than the Origin**

The x-coordinates of the vertices and foci are the same, so the major axis is parallel to the y-axis. Thus, the equation of the ellipse will have the form

$$\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1$$

First, we identify the center, (h, k). The center is halfway between the vertices, $(-2, -8)$ and $(-2, 2)$. Applying the Midpoint Formula, we have:

$$\begin{aligned}(h,k) &= \left(\frac{-2+(-2)}{2}, \frac{-8+2}{2}\right)\\ &= (-2,-3)\end{aligned}$$

Next, we find v_y^2. The length of the major axis, $2v_y$, is bounded by the vertices. We solve for v_y by finding the distance between the y-coordinates of the vertices.

$$\begin{aligned}2v_y &= 2-(-8)\\ \implies 2v_y &= 10\\ \implies v_y &= 5\end{aligned}$$

So $v_y^2 = 25$.

Now we find c^2. The foci are given by $(h, k \pm c)$. So, $(h, k-c) = (-2,-7)$ and $(h, k+c) = (-2, 1)$. We substitute $k = -3$ using either of these points to solve for c.

$$\begin{aligned}k+c &= 1\\ \implies -3+c &= 1\\ \implies c &= 4\end{aligned}$$

So $c^2 = 16$.

Next, we solve for v_x^2 using the equation $c^2 = |v_x^2 - v_y^2|$.

$$\begin{aligned}c^2 &= |v_x^2 - v_y^2|\\ \implies 16 &= |v_x^2 - 25| && \text{(substituting)}\\ \implies 16 &= 25 - v_x^2 && \text{(since } v_y^2 > v_x^2\text{)}\\ \implies v_x^2 &= 9 && \text{(solving for } v_x^2\text{)}\end{aligned}$$

Finally, we substitute the values found for h, k, v_x^2, and v_y^2 into the standard form equation for an ellipse:

$$\frac{(x+2)^2}{9} + \frac{(y+3)^2}{25} = 1$$

Example $6.1.3$**: Graphing an Ellipse Given an Equation**

From the equation, $v_x^2 = 9$ and $v_y^2 = 25$. Moreover, $h = 0$ and $k = 0$. It follows that:

- the major axis is vertical ($v_y^2 > v_x^2$)
- the center of the ellipse is $(0, 0)$
- the coordinates of the vertices are $(0, \pm v_y) = (0, \pm\sqrt{25}) = (0, \pm 5)$
- the coordinates of the co-vertices are $(\pm v_x, 0) = (\pm\sqrt{9}, 0) = (\pm 3, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = |v_x^2 - v_y^2|$. Solving for c, we have:

$$\begin{aligned}c^2 &= |9-25|\\ &= |-16|\\ &= 16\\ \implies c &= 4\end{aligned}$$

Therefore, the coordinates of the foci are $(0, \pm 4)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse. See Figure $6.1.5$

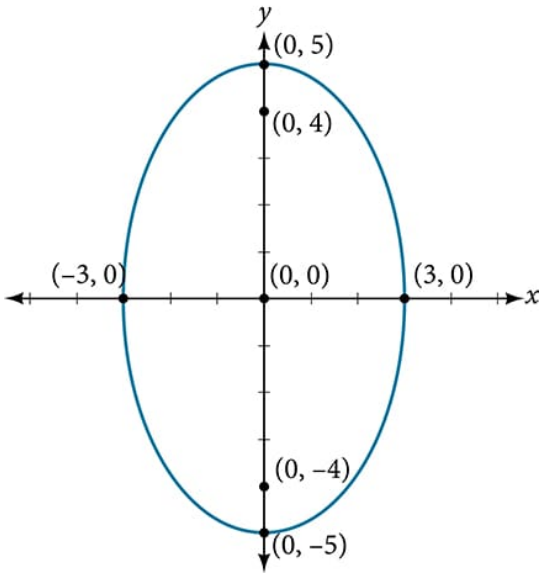


Figure $6.1.5$

Example $6.1.4$**: Graphing an Ellipse Centered at the Origin from an Equation Not in Standard Form**

First, use algebra to rewrite the equation in standard form.

$$\begin{aligned} 4x^2+25y^2 &= 100 \\ \Rightarrow \quad \frac{4x^2}{100}+\frac{25y^2}{100} &= \frac{100}{100} \\ \Rightarrow \quad \frac{x^2}{25}+\frac{y^2}{4} &= 1 \end{aligned}$$

From the equation, $v_x^2=25$ and $v_y^2=4$. Moreover, $h=0$ and $k=0$. It follows that:

- the major axis is horizontal ($v_x^2>v_y^2$)
- the center of the ellipse is $(0,0)$
- the coordinates of the vertices are $(\pm v_x,0)=(\pm\sqrt{25},0)=(\pm 5,0)$
- the coordinates of the co-vertices are $(0,\pm v_y)=(0,\pm\sqrt{4})=(0,\pm 2)$
- the coordinates of the foci are $(\pm c,0)$, where $c^2=|v_x^2-v_y^2|$. Solving for c, we have:

$$\begin{aligned} c^2 &= |25-4| \\ &= |21| \\ &= 21 \\ \Rightarrow \quad c &= \sqrt{21} \end{aligned}$$

Therefore the coordinates of the foci are $(\pm\sqrt{21},0)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

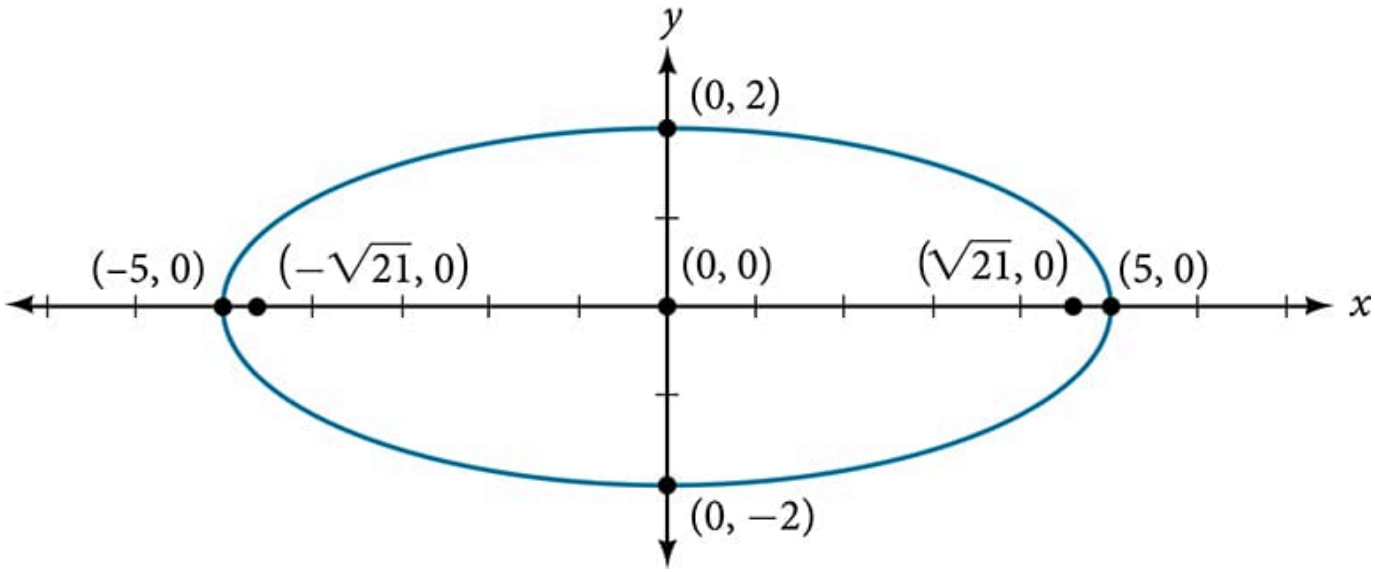


Figure 6.1.6

Example 6.1.5**: Graphing an Ellipse Centered at** (h,k)

From the equation, $v_x^2=4$ and $v_y^2=9$. Moreover, $h=-2$ and $k=5$. It follows that:

- the major axis is vertical ($v_y^2>v_x^2$)
- the center of the ellipse is $(-2,5)$
- the coordinates of the vertices are $(-2,5\pm v_y)=(-2,5\pm\sqrt{9})=(-2,5\pm 3)$. That is, the vertices are at $(-2,8)$ and $(-2,2)$.
- the coordinates of the co-vertices are $(-2\pm v_x,5)=(-2\pm\sqrt{4},5)=(-2\pm 2,5)$. Hence, they are at $(-4,5)$ and $(0,5)$.
- the coordinates of the foci are $(-2,5\pm c)$, where $c^2=|v_x^2-v_y^2|$. Solving for c, we have:

$$\begin{aligned} c^2 &= |4-9| \\ &= |-5| \\ &= 5 \\ \Rightarrow \quad c &= \sqrt{5} \end{aligned}$$

Therefore, the coordinates of the foci are $(-2,5-\sqrt{5})$ and $(-2,5+\sqrt{5})$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

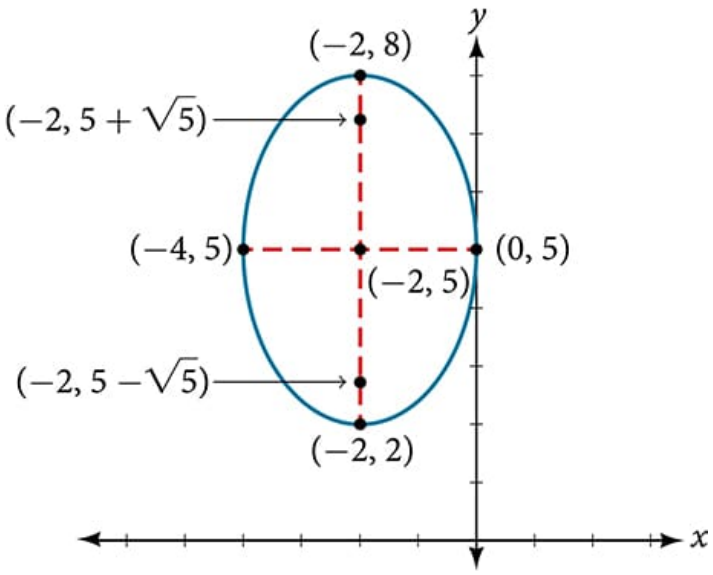


Figure 6.1.7

Example 6.1.6**: Graphing an Ellipse Centered at** (h, k) **by First Writing It in Standard Form**

We must begin by rewriting the equation in standard form. Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$\left(4x^2 - 40x\right) + \left(9y^2 + 36y\right) = -100$$

Factor out the coefficients of the squared terms.

$$4\left(x^2 - 10x\right) + 9\left(y^2 + 4y\right) = -100$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$4\left(x^2 - 10x + 25\right) + 9\left(y^2 + 4y + 4\right) = -100 + 100 + 36$$

Rewrite as perfect squares.

$$4(x-5)^2 + 9(y+2)^2 = 36$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x-5)^2}{9} + \frac{(y+2)^2}{4} = 1$$

Now that the equation is in standard form, we can determine the position of the major axis. From the equation, $v_x^2 = 9$ and $v_y^2 = 4$. Moreover, $h = 5$ and $k = -2$. It follows that:

- the major axis is horizontal ($v_x^2 > v_y^2$)
- the center of the ellipse is $(5, -2)$
- the coordinates of the vertices are $(5 \pm v_x, -2) = (5 \pm \sqrt{9}, -2) = (5 \pm 3, -2)$. That is, $(2, -2)$ and $(8, -2)$
- the coordinates of the co-vertices are $(5, -2 \pm v_y) = (5, -2 \pm \sqrt{4}) = (5, -2 \pm 2)$. That is, $(5, -4)$ and $(5, 0)$
- the coordinates of the foci are $(5 \pm c, -2)$, where $c^2 = \left|v_x^2 - v_y^2\right|$. Solving for c, we have:

$$\begin{aligned} c^2 &= |9 - 4| \\ &= |5| \\ &= 5 \\ \Longrightarrow \quad c &= \sqrt{5} \end{aligned}$$

Therefore, the coordinates of the foci are $(5 - \sqrt{5}, -2)$ and $(5 + \sqrt{5}, -2)$.

Next we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse as shown in Figure 6.1.8.

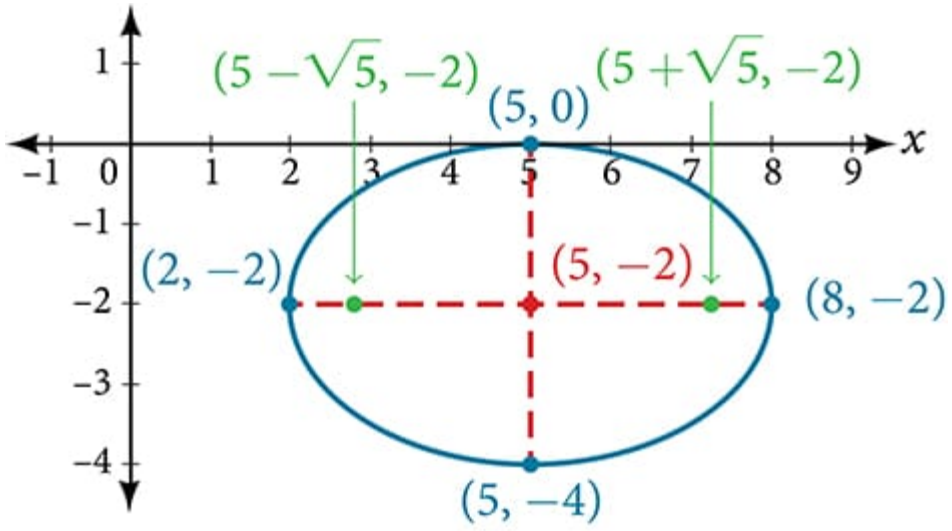


Figure 6.1.8

Example 6.1.7**: Locating the Foci of a Whispering Chamber**

a. We are assuming a horizontal ellipse with center $(0,0)$, so we need to find an equation of the form $\frac{x^2}{v_x^2}+\frac{y^2}{v_y^2}=1$, where $v_x^2>v_y^2$. We know that the length of the major axis, $2v_x$, is longer than the length of the minor axis, $2v_y$. So the length of the room, 96 , is represented by the major axis, and the width of the room, 46 , is represented by the minor axis.

Solving for v_x, we have $2v_x=96$, so $v_x=48$, and $v_x^2=2304$. Additionally, solving for v_y, we have $2v_y=46$, so $v_y=23$, and $v_y^2=529$.

Therefore, the equation of the ellipse is

$$\frac{x^2}{2304}+\frac{y^2}{529}=1.$$

b. To find the distance between the senators, we must find the distance between the foci, $(\pm c,0)$, where $c^2=|v_x^2-v_y^2|$. Solving for c, we have:

$$\begin{aligned} c^2 &= |2304-529| \\ &= |1775| \\ &= 1775 \\ \Longrightarrow \quad c &= \sqrt{1775} \\ &\approx 42 \end{aligned}$$

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6.1.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Geometry**
  - The Pythagorean Theorem, Distance Formula, and Midpoint Formula: The distance formula is used to derive the standard equation of an ellipse from its definition. The midpoint formula is used to find the center of an ellipse given its vertices or foci. (LibreTexts 6.1.3, Example 6.1.2)

Videos

- Ellipse
  - The Ellipse - Definitions and Concept
- Equation of an ellipse
  - The Ellipse - Building Equations for Ellipses
- Graphing ellipses
  - The Ellipse - Graphing Ellipses

Key Concepts

Definitions

- **Conic Section (Conic)**: A curve formed by the intersection of a plane and a double-napped right circular cone.
- **Ellipse**: The set of all points in a plane such that the sum of their distances from two fixed points is a constant.
- **Focus (plural: foci)**: One of the two fixed points used to define an ellipse. The sum of the distances from any point on the ellipse to the two foci is constant.
- **Major Axis**: The longer axis of an ellipse; it passes through the center and the vertices.
- **Minor Axis**: The shorter axis of an ellipse; it passes through the center and the co-vertices and is perpendicular to the major axis.
- **Vertex (of an ellipse)**: An endpoint of the major axis.
- **Co-vertex (of an ellipse)**: An endpoint of the minor axis.
- **Center (of an ellipse)**: The midpoint of both the major and minor axes.
- **General Form of the Equation of a Conic Section**: An equation of the form $ax^2 + by^2 + cx + dy + e = 0$, where a and b are not both zero.

Theorems

- **Standard Form of the Equation of an Ellipse with Center at the Origin**:
  - The standard form for an ellipse with its major axis on the x-axis is $\frac{x^2}{v_x^2} + \frac{y^2}{v_y^2} = 1$, where $v_x > v_y$. The vertices are $(\pm v_x, 0)$ and the foci are $(\pm c, 0)$ where $c^2 = v_x^2 - v_y^2$.
  - The standard form for an ellipse with its major axis on the y-axis is $\frac{x^2}{v_x^2} + \frac{y^2}{v_y^2} = 1$, where $v_y > v_x$. The vertices are $(0, \pm v_y)$ and the foci are $(0, \pm c)$ where $c^2 = v_y^2 - v_x^2$.
- **Standard Form of the Equation of an Ellipse with Center at** (h, k):
  - The standard form for an ellipse with its major axis parallel to the x-axis is $\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1$, where $v_x > v_y$. The vertices

are $(h \pm v_x, k)$ and the foci are $(h \pm c, k)$ where $c^2 = v_x^2 - v_y^2$.
- The standard form for an ellipse with its major axis parallel to the y-axis is $\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1$, where $v_y > v_x$. The vertices are $(h, k \pm v_y)$ and the foci are $(h, k \pm c)$ where $c^2 = v_y^2 - v_x^2$.

Common Mistakes

- **Confusing Major and Minor Axes**: Incorrectly identifying the major axis. The major axis corresponds to the larger denominator in the standard form equation.
- **Mixing up Vertices and Co-vertices**: Vertices are the endpoints of the major axis, while co-vertices are the endpoints of the minor axis.
- **Errors in Finding Foci**: Using the wrong formula for c^2. For an ellipse, c^2 is the difference of the squared semi-axis lengths ($c^2 = |v_x^2 - v_y^2|$), not the sum.
- **Forgetting to Take Square Roots**: The denominators in the standard form are v_x^2 and v_y^2. Forgetting to take the square root to find the values of v_x and v_y, which represent the distances from the center to the vertices and co-vertices.
- **Errors in Completing the Square**: When converting from general form to standard form, making algebraic mistakes during the completing the square process, especially forgetting to properly balance the equation when a coefficient has been factored out (e.g., adding $4(25)$ to the right side, not just 25).
- **Sign Errors with Center Coordinates**: When identifying the center (h, k) from the standard form $\frac{(x-h)^2}{v_x^2} + \frac{(y-k)^2}{v_y^2} = 1$, incorrectly determining the signs of h and k. For example, in $(x+2)^2$, $h = -2$.

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6.1.2: Homework

Reading Questions

1. Define an ellipse in terms of its foci.
2. What is the difference between the major axis and the minor axis of an ellipse?
3. What are the vertices and co-vertices of an ellipse?
4. What is the standard form of the equation for an ellipse centered at the origin with its major axis on the x-axis?
5. How can you determine if the major axis of an ellipse is horizontal or vertical by looking at its standard form equation?
6. What is the relationship between the values v_x, v_y, and c (the distance from the center to a focus) for an ellipse?
7. How does the standard form equation of an ellipse change if its center is at (h, k) instead of the origin?
8. What algebraic process is used to convert the general form of an ellipse's equation into standard form?
9. What is a whispering chamber and what property of an ellipse does it utilize?
10. If an ellipse has vertices at $(\pm 5, 0)$ and co-vertices at $(0, \pm 3)$, what are the values of v_x and v_y?
11. Define an ellipse in terms of its foci.
12. Where must the foci of an ellipse lie?
13. What special case of the ellipse do we have when the major and minor axis are of the same length?
14. For the special case mentioned in the previous question, what would be true about the foci of that ellipse?
15. What can be said about the symmetry of the graph of an ellipse with center at the origin and foci along the y-axis?

Homework

Algebraic

For the following exercises, determine whether the given equations represent ellipses. If yes, write in standard form.

16. $2x^2 + y = 4$
17. $4x^2 + 9y^2 = 36$
18. $4x^2 - y^2 = 4$
19. $4x^2 + 9y^2 = 1$
20. $4x^2 - 8x + 9y^2 - 72y + 112 = 0$

For the following exercises, write the equation of an ellipse in standard form, and identify the end points of the major and minor axes as well as the foci.

21. $\frac{x^2}{4} + \frac{y^2}{49} = 1$
22. $\frac{x^2}{100} + \frac{y^2}{64} = 1$
23. $x^2 + 9y^2 = 1$
24. $4x^2 + 16y^2 = 1$
25. $\frac{(x-2)^2}{49} + \frac{(y-4)^2}{25} = 1$
26. $\frac{(x-2)^2}{81} + \frac{(y+1)^2}{16} = 1$
27. $\frac{(x+5)^2}{4} + \frac{(y-7)^2}{9} = 1$
28. $\frac{(x-7)^2}{49} + \frac{(y-7)^2}{49} = 1$
29. $4x^2 - 8x + 9y^2 - 72y + 112 = 0$
30. $9x^2 - 54x + 9y^2 - 54y + 81 = 0$
31. $4x^2 - 24x + 36y^2 - 360y + 864 = 0$
32. $4x^2 + 24x + 16y^2 - 128y + 228 = 0$
33. $4x^2 + 40x + 25y^2 - 100y + 100 = 0$
34. $x^2 + 2x + 100y^2 - 1000y + 2401 = 0$
35. $4x^2 + 24x + 25y^2 + 200y + 336 = 0$
36. $9x^2 + 72x + 16y^2 + 16y + 4 = 0$

For the following exercises, find the foci for the given ellipses.

37. $\frac{(x+3)^2}{25}+\frac{(y+1)^2}{36}=1$
38. $\frac{(x+1)^2}{100}+\frac{(y-2)^2}{4}=1$
39. $x^2+y^2=1$
40. $x^2+4y^2+4x+8y=1$
41. $10x^2+y^2+200x=0$

Graphical

For the following exercises, graph the given ellipses, noting center, vertices, and foci.

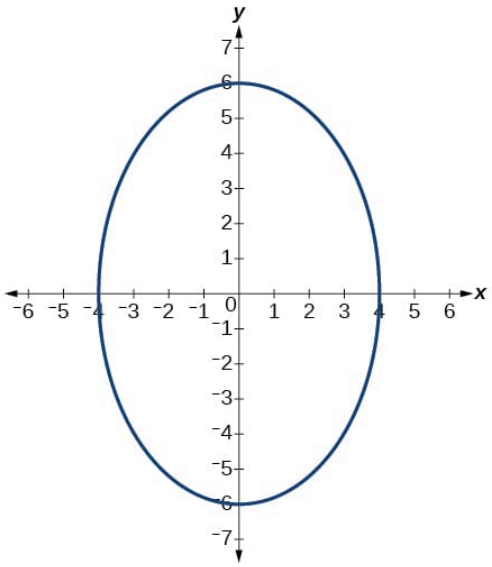
42. $\frac{x^2}{25}+\frac{y^2}{36}=1$
43. $\frac{x^2}{16}+\frac{y^2}{9}=1$
44. $4x^2+9y^2=1$
45. $81x^2+49y^2=1$
46. $\frac{(x-2)^2}{64}+\frac{(y-4)^2}{16}=1$
47. $\frac{(x+3)^2}{9}+\frac{(y-3)^2}{9}=1$
48. $\frac{x^2}{2}+\frac{(y+1)^2}{5}=1$
49. $4x^2-8x+16y^2-32y-44=0$
50. $x^2-8x+25y^2-100y+91=0$
51. $x^2+8x+4y^2-40y+112=0$
52. $64x^2+128x+9y^2-72y-368=0$
53. $16x^2+64x+4y^2-8y+4=0$
54. $100x^2+1000x+y^2-10y+2425=0$
55. $4x^2+16x+4y^2+16y+16=0$

For the following exercises, use the given information about the graph of each ellipse to determine its equation.

56. Center at the origin, symmetric with respect to the x- and y-axes, focus at $(4,0)$, and point on graph $(0,3)$.
57. Center at the origin, symmetric with respect to the x- and y-axes, focus at $(0,-2)$, and point on graph $(5,0)$.
58. Center at the origin, symmetric with respect to the x- and y-axes, focus at $(3,0)$, and major axis is twice as long as minor axis.
59. Center $(4,2)$; vertex $(9,2)$; one focus: $(4+2\sqrt{6},2)$.
60. Center $(3,5)$; vertex $(3,11)$; one focus: $(3,5+4\sqrt{2})$
61. Center $(-3,4)$; vertex $(1,4)$; one focus: $(-3+2\sqrt{3},4)$

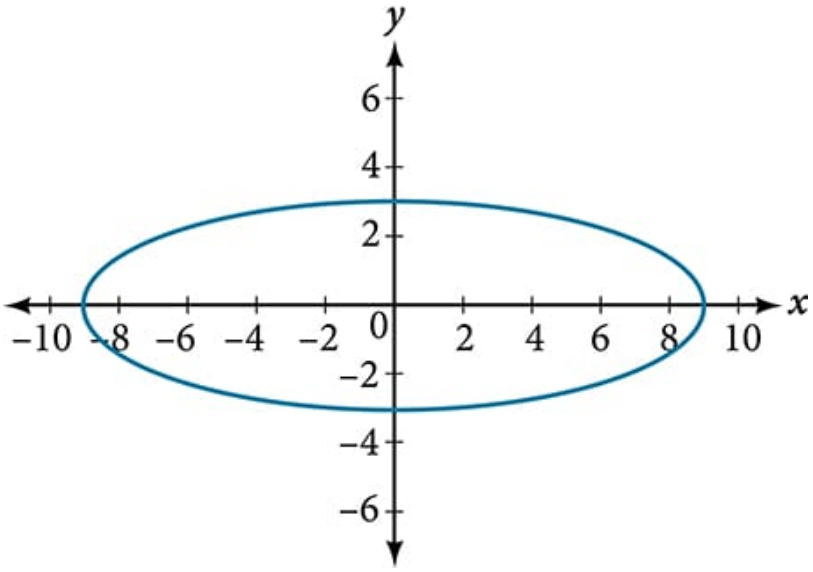
For the following exercises, given the graph of the ellipse, determine its equation.

62.

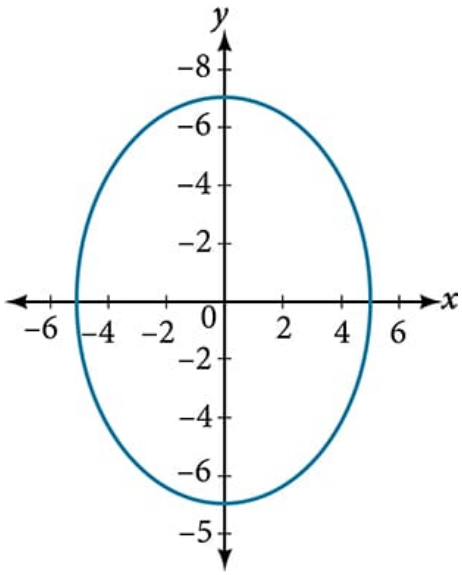


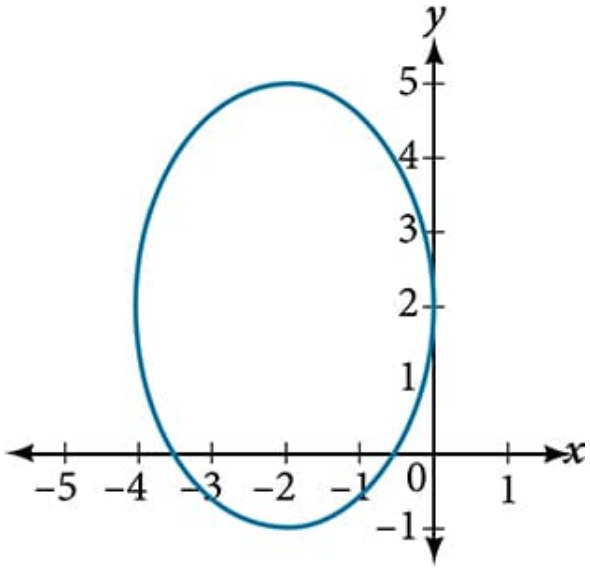
63.

54.

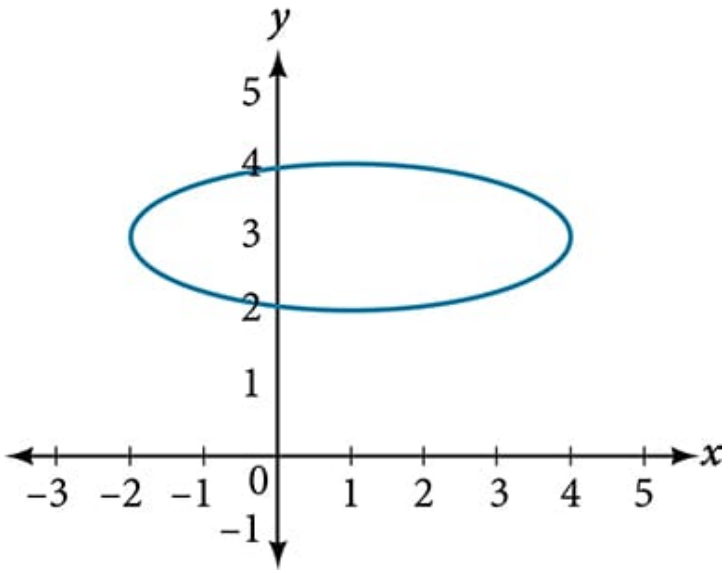


55.





56.



Extensions

For the following exercises, find the area of the ellipse. The area of an ellipse is given by the formula Area $= v_x \cdot v_y \cdot \pi$.

57. $\frac{(x-3)^2}{9} + \frac{(y-3)^2}{16} = 1$
58. $\frac{(x+6)^2}{16} + \frac{(y-6)^2}{36} = 1$
59. $\frac{(x+1)^2}{4} + \frac{(y-2)^2}{5} = 1$
70. $4x^2 - 8x + 9y^2 - 72y + 112 = 0$
71. $9x^2 - 54x + 9y^2 - 54y + 81 = 0$

Real-World Applications

72. Find the equation of the ellipse that will just fit inside a box that is 8 units wide and 4 units high.
73. Find the equation of the ellipse that will just fit inside a box that is four times as wide as it is high. Express in terms of h, the height.
74. An arch has the shape of a semi-ellipse (the top half of an ellipse). The arch has a height of 8 feet and a span of 20 feet. Find an equation for the ellipse, and use that to find the height to the nearest 0.01 foot of the arch at a distance of 4 feet from the

center.

75. An arch has the shape of a semi-ellipse. The arch has a height of 12 feet and a span of 40 feet. Find an equation for the ellipse, and use that to find the distance from the center to a point at which the height is 6 feet. Round to the nearest hundredth.
76. A bridge is to be built in the shape of a semi-elliptical arch and is to have a span of 120 feet. The height of the arch at a distance of 40 feet from the center is to be 8 feet. Find the height of the arch at its center.
77. A person in a whispering gallery standing at one focus of the ellipse can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet, and the foci are located 30 feet from the center, find the height of the ceiling at the center.
78. A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus, and the other focus is 80 feet away, what is the length and height at the center of the gallery?

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6.2: The Hyperbola

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

What do paths of comets, supersonic booms, ancient Grecian pillars, and natural draft cooling towers have in common? They can all be modeled by the same type of conic. For instance, when something moves faster than the speed of sound, a shock wave in the form of a cone is created. A portion of a conic is formed when the wave intersects the ground, resulting in a sonic boom. See Figure 6.2.1.

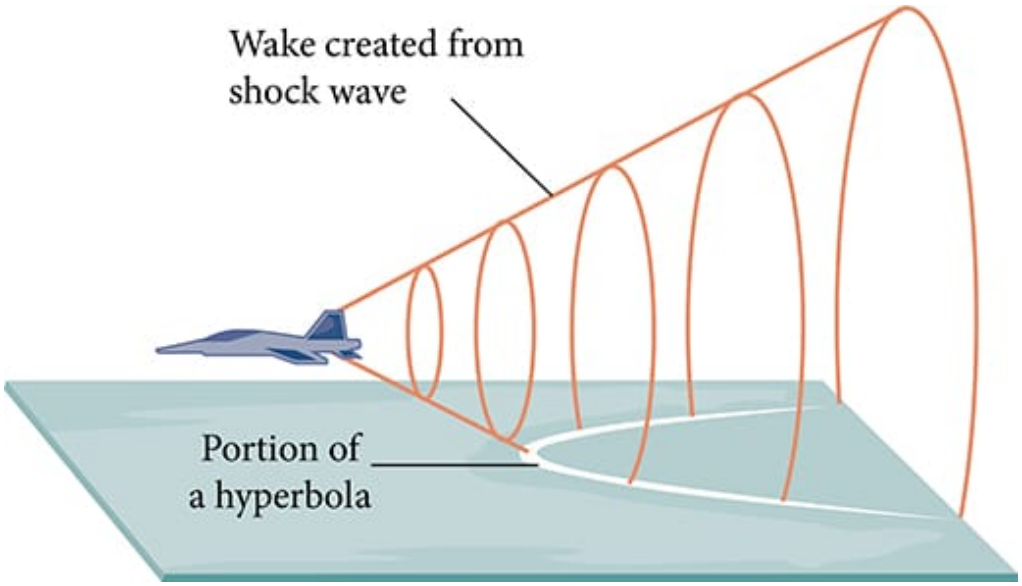


Figure 6.2.1**:** *A shock wave intersecting the ground forms a portion of a conic and results in a sonic boom.*

Most people are familiar with the sonic boom created by supersonic aircraft, but humans were breaking the sound barrier long before the first supersonic flight. The crack of a whip occurs because the tip is exceeding the speed of sound. The bullets shot from many firearms also break the sound barrier, although the bang of the gun usually supersedes the sound of the sonic boom.

Hyperbolas as Conic Sections

In analytic Geometry, a **hyperbola** is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other. See Figure 6.2.2.

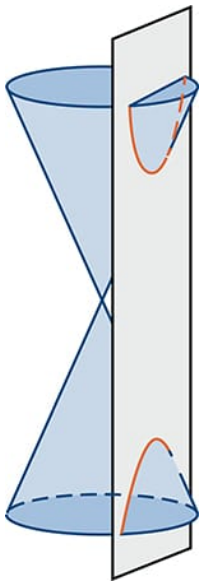


Figure 6.2.2**:** *A hyperbola*

Like the ellipse, the hyperbola can also be defined as a set of points in the coordinate plane.

Definition: Hyperbola

A **hyperbola** is the set of all points in a plane such that the difference of the distances between the point and two fixed points is a positive constant. Each fixed point is called a **focus** (plural: **foci**).

Notice that the definition of a hyperbola is very similar to that of an ellipse. The distinction is that the hyperbola is defined in terms of the *difference* of two distances, whereas the ellipse is defined in terms of the *sum* of two distances.

As with the ellipse, every hyperbola has two axes of symmetry and several key points.

Definition: Transverse and Conjugate Axes, Vertex and Co-Vertex, and Center of a Hyperbola

The line segment that passes through the center of the hyperbola and has **vertices** as its endpoints is called **transverse axis**. The **conjugate axis** is perpendicular to the transverse axis and has the **co-vertices** as its endpoints. The **center** of the hyperbola is the midpoint of both the transverse and conjugate axes.

Students often think of the transverse axis of a hyperbola as analogous to the major axis for an ellipse. Likewise, they consider conjugate axis of a hyperbola as analogous to the minor axis.

Every hyperbola also has two asymptotes that pass through its center. As a hyperbola recedes from the center, its branches approach these asymptotes. The **central rectangle** of the hyperbola is centered at the center of the hyperbola with sides that pass through each vertex and co-vertex; it is a useful tool for graphing the hyperbola and its asymptotes. To sketch the asymptotes of the hyperbola, simply sketch and extend the diagonals of the central rectangle. See Figure 6.2.3.

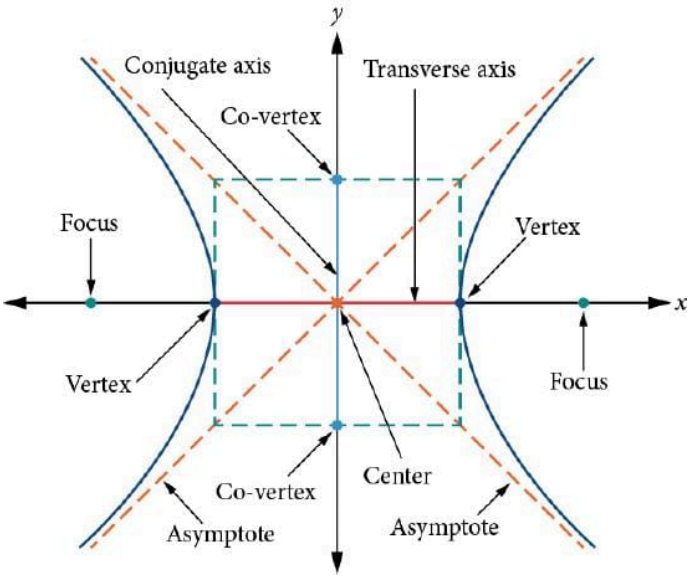


Figure 6.2.3**:** *Key features of the hyperbola*

As we did with ellipses, we limit our discussion to hyperbolas that are positioned vertically or horizontally in the coordinate plane; the axes will either lie on or be parallel to the x- and y-axes. We will consider two cases: those that are centered at the origin, and those that are centered at a point other than the origin.

Standard Form of a Hyperbola Centered at the Origin

Theorem: Standard Form of the Equation of a Hyperbola Centered at the Origin

The standard form of the equation of a hyperbola with center $(0,0)$ and transverse axis on the x-axis is

$$\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1,$$

where

- v_x and v_y are positive
- the length of the transverse axis is $2v_x$
- the coordinates of the vertices are $(\pm v_x, 0)$
- the length of the conjugate axis is $2v_y$
- the coordinates of the co-vertices are $(0, \pm v_y)$
- the coordinates of the foci are $(\pm c, 0)$, where $c^2 = |v_x^2 + v_y^2|$
- the equations of the asymptotes are $y = \pm \frac{v_y}{v_x} x$.

The standard form of the equation of a hyperbola with center $(0,0)$ and transverse axis on the y-axis is

$$\frac{y^2}{v_y^2} - \frac{x^2}{v_x^2} = 1,$$

where

- v_x and v_y are positive
- the length of the transverse axis is $2v_y$
- the coordinates of the vertices are $(0, \pm v_y)$
- the length of the conjugate axis is $2v_x$
- the coordinates of the co-vertices are $(\pm v_x, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = |v_x^2 + v_y^2|$
- the equations of the asymptotes are $y = \pm \frac{v_y}{v_x} x$.

Proof available at the end of this section.

Figure 6.2.4 shows the graphs of the horizontal and vertical hyperbolas centered at the origin.

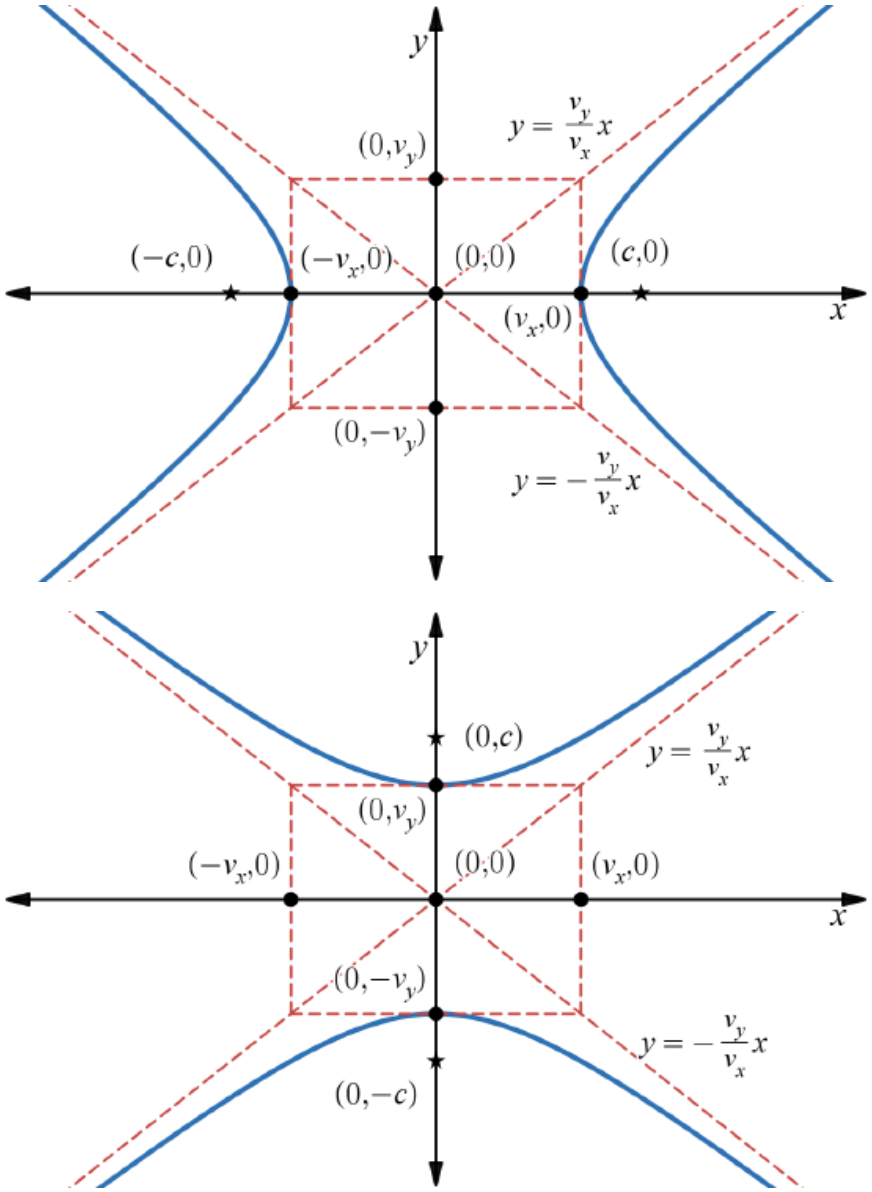


Figure 6.2.4: *Top:* $\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1$
Bottom: $\frac{y^2}{v_y^2} - \frac{x^2}{v_x^2} = 1$

One of the major differences between the ellipse and the hyperbola is the fact that one of the variable terms involving either x^2 or y^2 is negative in the equation of the hyperbola. This means that the order of the terms matters. That is,

$$\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1 \text{ is not the same as } \frac{y^2}{v_y^2} - \frac{x^2}{v_x^2} = 1.$$

The former results in a graph with branches opening left and right (this is what we derived above). The latter results in a graph with branches opening up and down. In either case, $c^2 = |v_x^2 + v_y^2|$.

This theorem can be shortened quite a bit by saying that the variable associated with the positive term dictates the transverse axis (and the variable associated with the negative terms dictates the conjugate axis). In any case, $c^2 = |v_x^2 + v_y^2|$. While there is no need for the absolute values in this equation, it doesn't hurt to have them and it aligns nicely with our method for ellipses.

Example 6.2.1: Locating a Hyperbola's Vertices and Foci

Identify the vertices and foci of the hyperbola with equation $\frac{y^2}{49} - \frac{x^2}{32} = 1$.

Solution available at the end of this section.

Checkpoint 6.2.1

Identify the vertices and foci of the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{25} = 1$.

Just as with ellipses, writing the equation for a hyperbola in standard form allows us to calculate the key features: its center, vertices, co-vertices, foci, asymptotes, and the lengths and positions of the transverse and conjugate axes. Conversely, an equation for a hyperbola can be found given its key features. We begin by finding standard equations for hyperbolas centered at the origin. Then we will turn our attention to finding standard equations for hyperbolas centered at some point other than the origin.

Reviewing the standard forms given for hyperbolas centered at $(0, 0)$, we see that the vertices, co-vertices, and foci are related by the equation $c^2 = |v_x^2 + v_y^2|$. Note that this equation can also be rewritten as $c^2 = v_x^2 + v_y^2$, which can be rewritten as $v_y^2 = c^2 - v_x^2$. This relationship is used to write the equation for a hyperbola when given the coordinates of its foci and vertices.

Example 6.2.2: Finding the Equation of a Hyperbola Centered at $(0, 0)$ Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices $(\pm 6, 0)$ and foci $\left(\pm 2\sqrt{10}, 0\right)$?

Solution available at the end of this section.

Checkpoint 6.2.2

What is the standard form equation of the hyperbola that has vertices $(0, \pm 2)$ and foci $\left(0, \pm 2\sqrt{5}\right)$?

When we have an equation in standard form for a hyperbola centered at the origin, we can interpret its parts to identify the key features of its graph: the center, vertices, co-vertices, asymptotes, foci, and lengths and positions of the transverse and conjugate axes. To graph hyperbolas centered at the origin, we use the standard form

$$\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1$$

for horizontal hyperbolas and the standard form

$$\frac{y^2}{v_y^2} - \frac{x^2}{v_x^2} = 1$$

for vertical hyperbolas.

Example 6.2.3: Graphing a Hyperbola Centered at $(0, 0)$ Given an Equation in Standard Form

Graph the hyperbola given by the equation $\frac{y^2}{64} - \frac{x^2}{36} = 1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

Solution available at the end of this section.

Checkpoint 6.2.3

Graph the hyperbola given by the equation $\frac{x^2}{144} - \frac{y^2}{81} = 1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

Standard Form of a Hyperbola Centered Off the Origin

Like the graphs for other equations, the graph of a hyperbola can be translated. If a hyperbola is translated h units horizontally and k units vertically, the center of the hyperbola will be (h, k). This translation results in the standard form of the equation we saw previously, with x replaced by $(x - h)$ and y replaced by $(y - k)$.

Theorem: Standard Form of the Equation of a Hyperbola Centered Off-Origin

The standard form of the equation of a hyperbola with center (h, k) and transverse axis parallel to the x-axis is

$$\frac{(x-h)^2}{v_x^2} - \frac{(y-k)^2}{v_y^2} = 1,$$

where

- v_x and v_y are positive
- the length of the transverse axis is $2v_x$
- the coordinates of the vertices are $(h \pm v_x, k)$
- the length of the conjugate axis is $2v_y$
- the coordinates of the co-vertices are $(h, k \pm v_y)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^2 = |v_x^2 + v_y^2|$
- the equations of the asymptotes are $y - k = \pm \frac{v_y}{v_x}(x - h)$.

The standard form of the equation of a hyperbola with center (h, k) and transverse axis parallel to the y-axis is

$$\frac{(y-k)^2}{v_y^2} - \frac{(x-h)^2}{v_x^2} = 1,$$

where

- v_x and v_y are positive
- the length of the transverse axis is $2v_y$
- the coordinates of the vertices are $(h, k \pm v_y)$
- the length of the conjugate axis is $2v_x$
- the coordinates of the co-vertices are $(h \pm v_x, k)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^2 = |v_x^2 + v_y^2|$
- the equations of the asymptotes are $y - k = \pm \frac{v_y}{v_x}(x - h)$.

Figure 6.2.7 shows the graphs of the horizontal and vertical hyperbolas centered off-origin.

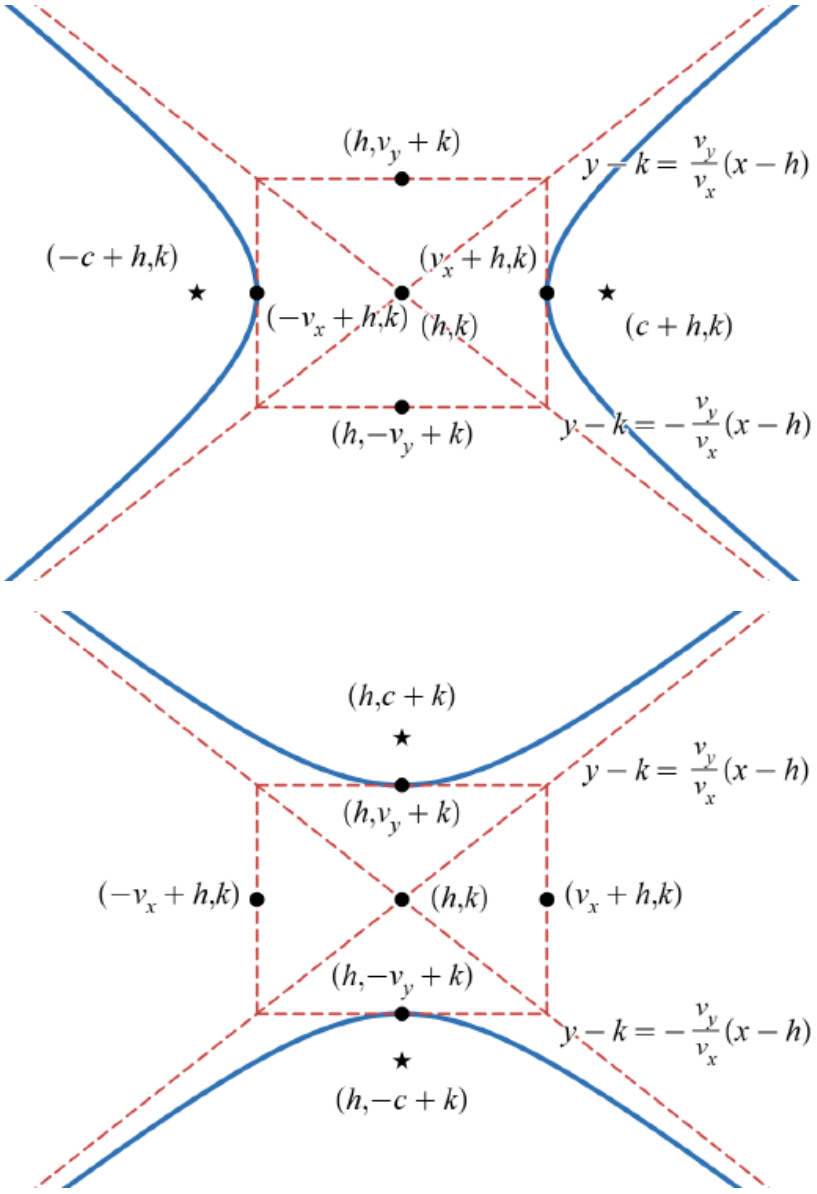


Figure 6.2.7**:** *Top:* $\frac{(x-h)^2}{v_x^2} - \frac{(y-k)^2}{v_y^2} = 1$
Bottom: $\frac{(y-k)^2}{v_y^2} - \frac{(x-h)^2}{v_x^2} = 1$

Like hyperbolas centered at the origin, hyperbolas centered at a point (h, k) have vertices, co-vertices, and foci that are related by the equation $c^2 = |v_x^2 + v_y^2|$. Again, there is no need for the absolute values; however, including them while introducing the concepts allows a firmer link between the equations for ellipses and equations for hyperbolas. We can use this relationship along with the Midpoint and Distance Formulas to find the standard equation of a hyperbola when the vertices and foci are given.

Example 6.2.4: Finding the Equation of a Hyperbola Centered at (h, k) Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices at $(0, -2)$ and $(6, -2)$ and foci at $(-2, -2)$ and $(8, -2)$?

Solution available at the end of this section.

Checkpoint 6.2.4

What is the standard form equation of the hyperbola that has vertices $(1,-2)$ and $(1,8)$ and foci $(1,-10)$ and $(1,16)$?

Graphing hyperbolas centered at a point (h,k) other than the origin is similar to graphing ellipses centered at a point other than the origin. We use the standard forms

$$\frac{(x-h)^2}{v_x^2}-\frac{(y-k)^2}{v_y^2}=1$$

for horizontal hyperbolas, and

$$\frac{(y-k)^2}{v_y^2}-\frac{(x-h)^2}{v_x^2}=1$$

for vertical hyperbolas. From these standard form equations we can easily calculate and plot key features of the graph: the coordinates of its center, vertices, co-vertices, and foci; the equations of its asymptotes; and the positions of the transverse and conjugate axes.

Example 6.2.5: Graphing a Hyperbola Centered at (h,k) Given an Equation in General Form

Graph the hyperbola given by the equation $9x^2-4y^2-36x-40y-388=0$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Solution available at the end of this section.

Checkpoint 6.2.5

Graph the hyperbola given by the standard form of an equation $\frac{(y+4)^2}{100}-\frac{(x-3)^2}{64}=1$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Solving Applied Problems Involving Hyperbolas

As we discussed at the beginning of this section, hyperbolas have real-world applications in many fields, such as Astronomy, Physics, Engineering, and Architecture. The design efficiency of hyperbolic cooling towers is particularly interesting. Cooling towers are used to transfer waste heat to the atmosphere and are often touted for their ability to generate power efficiently. Because of their hyperbolic form, these structures are able to withstand extreme winds while requiring less material than any other forms of their size and strength. See Figure 6.2.9. For example, a 500-foot tower can be made of a reinforced concrete shell only 6 or 8 inches wide!



Figure 6.2.9**:** *Cooling towers at the Drax power station in North Yorkshire, United Kingdom (credit: Les Haines, Flickr)*

The first hyperbolic towers were designed in 1914 and were 35 meters high. Today, the tallest cooling towers are in France, standing a remarkable 170 meters tall. In Example 6.2.6 we will use the design layout of a cooling tower to find a hyperbolic equation that models its sides.

Example 6.2.6: Solving Applied Problems Involving Hyperbolas

The design layout of a cooling tower is shown in Figure 6.2.10. The tower stands 179.6 meters tall. The diameter of the top is 72 meters. At their closest, the sides of the tower are 60 meters apart.

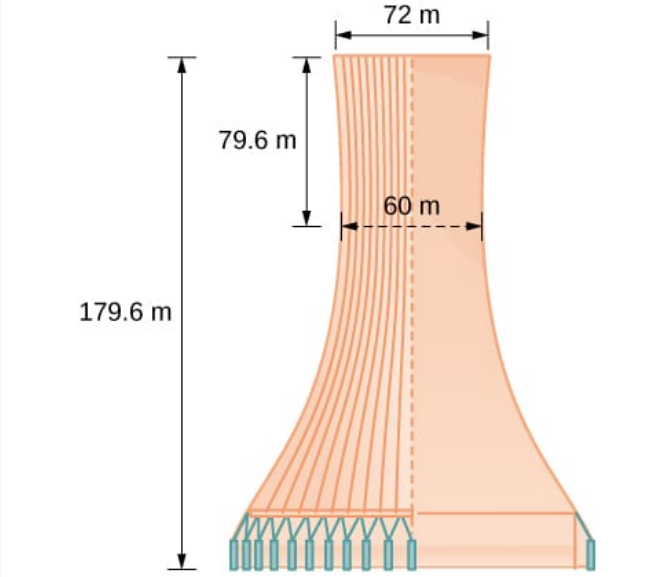


Figure 6.2.10**:** *Project design for a natural draft cooling tower*

Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

Solution available at the end of this section.

Checkpoint 6.2.6

A design for a cooling tower project is shown in Figure 6.2.11. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

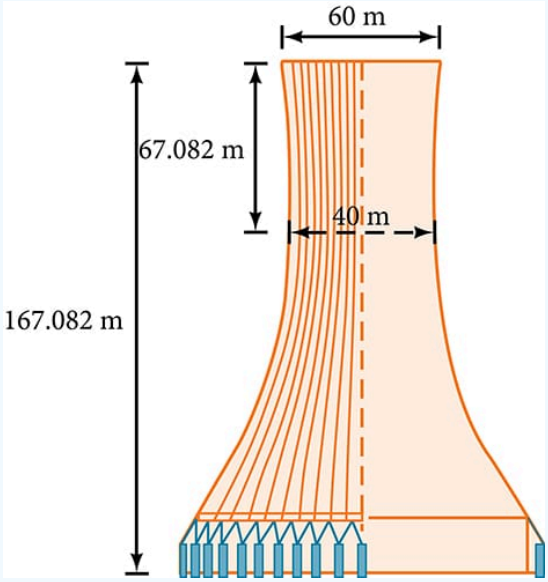


Figure 6.2.11

Success in College: Dealing With Failure (Part 4)

1. A first-year student fails her math class. She hates math. Her advisor wants her to retake the course the next term. The student wants to wait a year or so before trying again. Discuss the advantages and disadvantages of waiting to retake this math course.
2. Some colleges restrict the number of times a student can repeat a course. Describe any restrictions on repeating a course at your school.

Success in College: Preparing for the Final Exam (Part 1)

1. Describe what you will do to prepare for the final exam in this course.
2. Describe how well you think you are prepared right now to take the final exam.

Answers

Section Preview

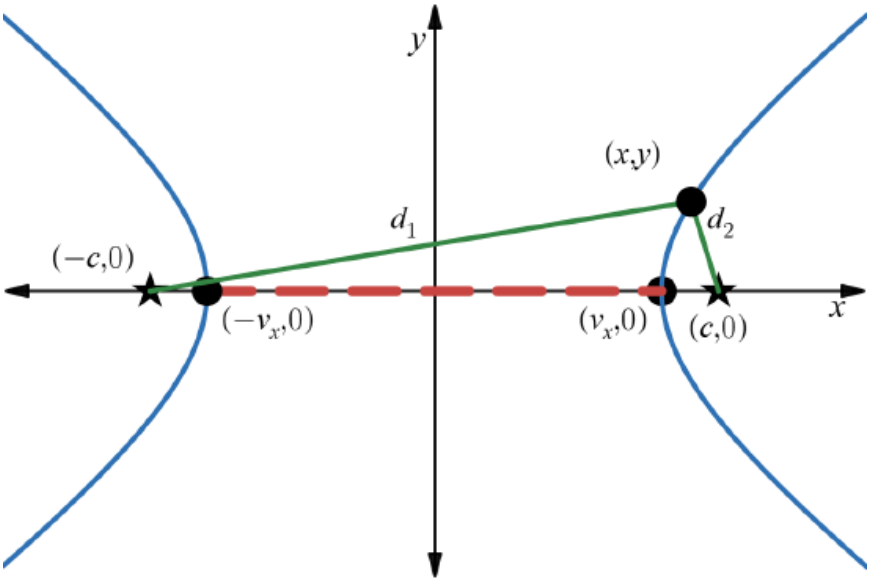
The material in this section is arguably the most important when it comes to a student's preparation for Calculus. In Differential Calculus, specifically, a good chunk of the material requires understanding hyperbolic functions (which are derived from hyperbolas).

Section Preview

- Locate a hyperbola's vertices and foci.
- Write equations of hyperbolas in standard form.
- Graph hyperbolas centered at the origin.
- Graph hyperbolas not centered at the origin.
- Solve applied problems involving hyperbolas.

Theorem: Standard Form of the Equation of a Hyperbola Centered at the Origin

To derive the equation of a hyperbola centered at the origin, we begin with the foci $(-c, 0)$ and $(c, 0)$. The hyperbola is the set of all points (x, y) such that the difference of the distances from (x, y) to the foci is constant, as shown in the figure below.



If $(v_x, 0)$ is a vertex of the hyperbola (on the transverse axis), the distance from $(-c, 0)$ to $(v_x, 0)$ is $v_x - (-c) = v_x + c$. The distance from $(c, 0)$ to $(v_x, 0)$ is $c - v_x$ (this is where the difference in the proof between the ellipse and the hyperbola begin to differ - notice from the figure above that $c > v_x$). The difference of the distances from the foci to the vertex is

$$(v_x + c) - (c - v_x) = 2v_x.$$

If (x, y) is a point on the hyperbola, then we can define the following variables:

$$\begin{aligned} d_1 &= \text{the distance from } (-c, 0) \text{ to } (x, y) \\ d_2 &= \text{the distance from } (c, 0) \text{ to } (x, y) \end{aligned}$$

By the definition of a hyperbola, $d_1 - d_2$ is constant for any point (x, y) on the hyperbola. We know that the difference of these distances is $2v_x$ for the vertex $(v_x, 0)$. It follows that $d_1 - d_2 = 2v_x$ for any point on the hyperbola. We will begin the derivation by applying the Distance Formula. The rest of the derivation is algebraic.

$$
\begin{aligned}
& & d_1 - d_2 &= 2v_x & & \\
&\implies & \sqrt{(x-(-c))^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} &= 2v_x & & \text{(substituting)} \\
&\implies & \sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} &= 2v_x & & \text{(simplifying)} \\
&\implies & \sqrt{(x+c)^2+y^2} &= 2v_x + \sqrt{(x-c)^2+y^2} & & \text{(isolating a radical)} \\
&\implies & (x+c)^2+y^2 &= \left[2v_x + \sqrt{(x-c)^2+y^2}\right]^2 & & \text{(squaring both sides)} \\
&\implies & (x+c)^2+y^2 &= 4v_x^2 + 4v_x\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2 & & \text{(distributing)} \\
&\implies & x^2+2cx+c^2+y^2 &= 4v_x^2 + 4v_x\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2 & & \text{(distributing)} \\
&\implies & 2cx &= 4v_x^2 + 4v_x\sqrt{(x-c)^2+y^2} - 2cx & & \text{(subtracting } x^2,\ c^2,\ \text{and } y^2 \\
& & & & & \text{ from both sides)} \\
&\implies & 4cx - 4v_x^2 &= 4v_x\sqrt{(x-c)^2+y^2} & & \text{(isolating the radical)} \\
&\implies & cx - v_x^2 &= v_x\sqrt{(x-c)^2+y^2} & & \text{(dividing both sides by 4)} \\
&\implies & (cx-v_x^2)^2 &= v_x^2\left((x-c)^2+y^2\right) & & \text{(squaring both sides)} \\
&\implies & c^2x^2 - 2v_x^2cx + v_x^4 &= v_x^2\left(x^2-2cx+c^2+y^2\right) & & \text{(distributing)} \\
&\implies & c^2x^2 - 2v_x^2cx + v_x^4 &= v_x^2x^2 - 2v_x^2cx + v_x^2c^2 + v_x^2y^2 & & \text{(distributing)} \\
&\implies & c^2x^2 + v_x^4 &= v_x^2x^2 + v_x^2c^2 + v_x^2y^2 & & \text{(adding } 2v_x^2cx \text{ to both sides)} \\
&\implies & v_x^4 - v_x^2c^2 &= v_x^2x^2 - c^2x^2 + v_x^2y^2 & & \text{(subtracting } v_x^2c^2 \text{ and } c^2x^2 \\
& & & & & \text{ from both sides)} \\
&\implies & v_x^2(v_x^2-c^2) &= x^2(v_x^2-c^2) + v_x^2y^2 & & \text{(factor)} \\
&\implies & -v_x^2(v_x^2-c^2) &= -x^2(v_x^2-c^2) - v_x^2y^2 & & \text{(multiplying both sides by } -1) \\
&\implies & v_x^2(c^2-v_x^2) &= x^2(c^2-v_x^2) - v_x^2y^2 & & \text{(distributing)} \\
&\implies & v_x^2b^2 &= x^2b^2 - v_x^2y^2 & & \text{(let } b^2 = c^2 - v_x^2) \\
&\implies & \frac{v_x^2b^2}{v_x^2b^2} &= \frac{x^2b^2}{v_x^2b^2} - \frac{v_x^2y^2}{v_x^2b^2} & & \text{(divide both sides by } v_x^2b^2) \\
&\implies & 1 &= \frac{x^2}{v_x^2} - \frac{y^2}{b^2} & & \text{(simplify)}
\end{aligned}
$$

At this point in the proof of the equation for the *ellipse*, we were able to let $x = 0$ to determine the meaning of b; however, the hyperbola opening horizontally in the figure above does not exist at $x = 0$. Therefore, such a simple maneuver is not possible here. Instead, let's consider our result,

$$\frac{x^2}{v_x^2} - \frac{y^2}{b^2} = 1,$$

and solve for y.

$$
\begin{aligned}
& \frac{x^2}{v_x^2} - \frac{y^2}{b^2} &= 1 & \\
\implies & \frac{x^2}{v_x^2} &= 1 + \frac{y^2}{b^2} & \left(\text{adding } \tfrac{y^2}{b^2} \text{ to both sides}\right) \\
\implies & \frac{x^2}{v_x^2} - 1 &= \frac{y^2}{b^2} & \text{(subtracting 1 from both sides)} \\
\implies & \pm\sqrt{\frac{x^2}{v_x^2} - 1} &= \frac{y}{b} & \text{(taking the square root of both sides)} \\
\implies & \pm b\sqrt{\frac{x^2}{v_x^2} - 1} &= y & \text{(multiplying both sides by } b) \\
\implies & \pm b\sqrt{\frac{x^2}{v_x^2}\left(1 - \frac{v_x^2}{x^2}\right)} &= y & \left(\text{factoring } \tfrac{x^2}{v_x^2} \text{ within the radicand}\right) \\
\implies & \pm b\left|\frac{x}{v_x}\right|\sqrt{1 - \frac{v_x^2}{x^2}} &= y & \left(\text{simplifying using } \sqrt{\blacksquare^2} = |\blacksquare|\right)
\end{aligned}
$$

Since we are requiring $v_x > 0$, $|v_x| = v_x$. Moreover, for simplicity (and without loss of generality), let's only consider points on the right branch of the hyperbola. That is, let's assume $x > 0$. Then our final result becomes

$$y = \pm\frac{b}{v_x}x\sqrt{1-\frac{v_x^2}{x^2}}.$$

Using the same logic we developed when discussing slant asymptotes (see the section on graphing rational functions), we consider what happens as $x \to \infty$. To truly grasp this concept, you must keep in mind that v_x is a constant.

$$\text{As } x \to \infty, \qquad \frac{v_x^2}{x^2} \to 0$$

$$\text{As } \frac{v_x^2}{x^2} \to 0, \qquad 1-\frac{v_x^2}{x^2} \to 1$$

$$\text{As } 1-\frac{v_x^2}{x^2} \to 1, \qquad \sqrt{1-\frac{v_x^2}{x^2}} \to 1$$

Thus, as $x \to \infty$, the y-values for the points on our hyperbola approach

$$y = \pm\frac{b}{v_x}x\sqrt{1-\frac{v_x^2}{x^2}} \to \pm\frac{b}{v_x}x \cdot 1 = \pm\frac{b}{v_x}x.$$

That is, the lines $y = \frac{b}{v_x}x$ and $y = -\frac{b}{v_x}x$ are slant asymptotes (guidelines for graphing) as $x \to \infty$. Both of these are linear equations where the slopes are $\pm\frac{b}{v_x}$. Since slope is "rise over run," or "increase in y over increase in x," we can interpret the numerator of these slopes as the corresponding change in y when x-values change by v_x. Making the notational change, $b = v_y$, we arrive at our desired result: the equation of the hyperbola centered at the origin and opening horizontally is

$$\frac{x^2}{v_x^2}-\frac{y^2}{v_y^2} = 1,$$

where the slant asymptotes are $y = \pm\frac{v_y}{v_x}x$.

The proof for the equation of the hyperbola opening vertically is similar.

Example 6.2.1**: Locating a Hyperbola's Vertices and Foci**

The equation has the form $\frac{y^2}{v_y^2}-\frac{x^2}{v_x^2} = 1$, so the transverse axis lies on the y-axis. The hyperbola is centered at the origin, so the vertices serve as the y-intercepts of the graph. To find the vertices, set $x = 0$ and solve for y

$$\begin{aligned} & 1 = \frac{y^2}{49}-\frac{x^2}{32} & \\ \Rightarrow \quad & 1 = \frac{y^2}{49}-\frac{0^2}{32} & \text{(substituting)} \\ \Rightarrow \quad & 1 = \frac{y^2}{49} & \text{(simplifying)} \\ \Rightarrow \quad & 49 = y^2 & \text{(multiplying both sides by 49)} \\ \Rightarrow \quad & \pm 7 = y & \text{(Extraction of Roots)} \end{aligned}$$

The foci are located at $(0, \pm c)$. Solving for c,

$$\begin{aligned} & c^2 = |v_x^2 + v_y^2| & \\ \Rightarrow \quad & c^2 = |49 + 32| & \text{(substituting)} \\ \Rightarrow \quad & c^2 = 81 & \text{(simplifying)} \\ \Rightarrow \quad & c = 9 & \text{(Extraction of Roots)} \end{aligned}$$

Note that we stick with the convention that v_x, v_y, and c are all positive. Thus, we have no need to worry about negative radicals when performing Extraction of Roots.

We have found that the vertices are located at $(0, \pm 7)$, and the foci are at $(0, \pm 9)$.

Example 6.2.2**: Finding the Equation of a Hyperbola Centered at** $(0, 0)$ **Given its Foci and Vertices**

The vertices and foci are on the x-axis. Thus, the equation for the hyperbola will have the form $\frac{x^2}{v_x^2}-\frac{y^2}{v_y^2} = 1$.

The vertices are $(\pm 6, 0)$, so $v_x = 6$ and $v_x^2 = 36$.

The foci are $\left(\pm 2\sqrt{10}, 0\right)$, so $c = 2\sqrt{10}$ and $c^2 = 40$.

Solving for v_y^2, we have

$$\begin{aligned} & c^2 = |v_x^2 + v_y^2| & \\ \Rightarrow \quad & 40 = |36 + v_y^2| & \text{(substituting)} \\ \Rightarrow \quad & 40 = 36 + v_y^2 & \text{(removing the unnecessary absolute values)} \\ \Rightarrow \quad & 4 = v_y^2 & \text{(subtracting 36 from both sides)} \end{aligned}$$

Therefore, the equation of the hyperbola is

$$\frac{x^2}{36}-\frac{y^2}{4} = 1,$$

as shown in Figure 6.2.5.

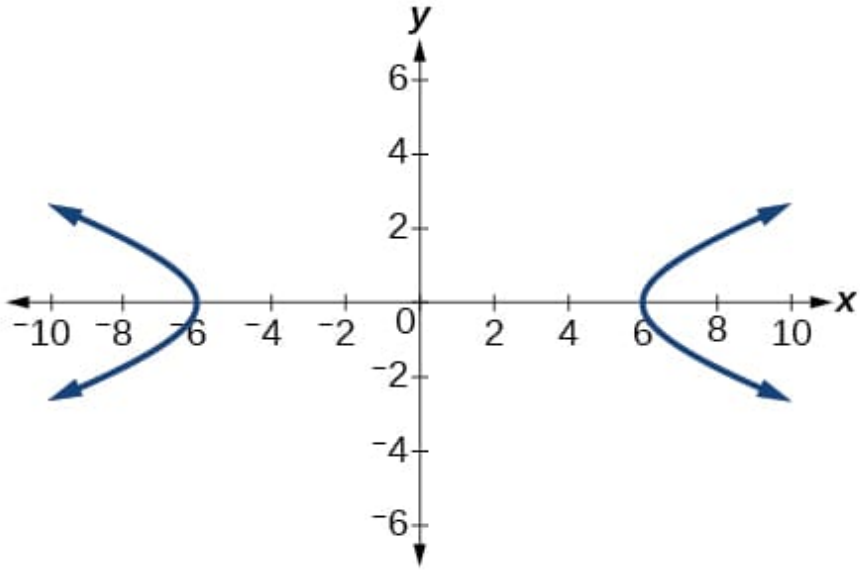


Figure 6.2.5

Example 6.2.3**: Graphing a Hyperbola Centered at** $(0,0)$ **Given an Equation in Standard Form**

The standard form that applies to the given equation is $\frac{y^2}{v_y^2}-\frac{x^2}{v_x^2}=1$. Thus, the transverse axis is on the y-axis.

The coordinates of the vertices are $(0,\pm v_y)=(0,\pm\sqrt{64})=(0,\pm 8)$.

The coordinates of the co-vertices are $(\pm v_x,0)=(\pm\sqrt{36},0)=(\pm 6,0)$.

The coordinates of the foci are $(0,\pm c)$, where $c=\sqrt{v_x^2+v_y^2}$. Solving for c, we have

$$c=\sqrt{64+36}=\sqrt{100}=10.$$

Therefore, the coordinates of the foci are $(0,\pm 10)$.

The equations of the asymptotes are $y=\pm\frac{v_y}{v_x}x=\pm\frac{8}{6}x=\pm\frac{4}{3}x$.

Plot and label the vertices and co-vertices, and then sketch the central rectangle. Sides of the rectangle are parallel to the axes and pass through the vertices and co-vertices. Sketch and extend the diagonals of the central rectangle to show the asymptotes. The central rectangle and asymptotes provide the framework needed to sketch an accurate graph of the hyperbola. Label the foci and asymptotes, and draw a smooth curve to form the hyperbola, as shown in Figure 6.2.6.

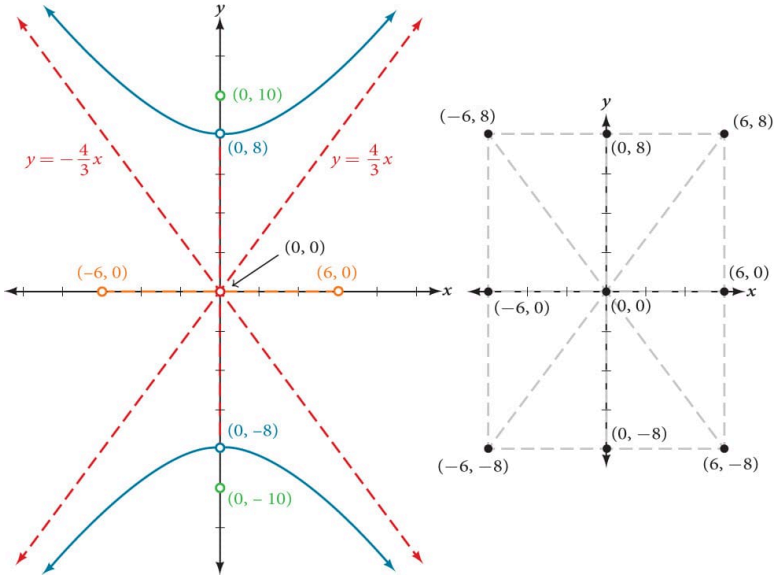


Figure 6.2.6

Example 6.2.4**: Finding the Equation of a Hyperbola Centered at** (h,k) **Given its Foci and Vertices**

The y-coordinates of the vertices and foci are the same, so the transverse axis is parallel to the x-axis. Thus, the equation of the hyperbola will have the form

$$\frac{(x-h)^2}{v_x^2}-\frac{(y-k)^2}{v_y^2}=1.$$

First, we identify the center, (h,k). The center is halfway between the vertices $(0,-2)$ and $(6,-2)$. Applying the Midpoint Formula, we have

$$(h,k)=\left(\frac{0+6}{2},\frac{-2+(-2)}{2}\right)=(3,-2).$$

Next, we find v_x^2. The length of the transverse axis, $2v_x$, is bounded by the vertices. So, we can find v_x^2 by finding the distance between the x-coordinates of the vertices.

$$\begin{aligned} 2v_x &= |0-6| \\ \implies 2v_x &= 6 && \text{(simplifying)} \\ \implies v_x &= 3 && \text{(dividing both sides by 2)} \\ \implies v_x^2 &= 9 && \text{(squaring both sides)} \end{aligned}$$

Now we need to find c^2. The coordinates of the foci are $(h \pm c, k)$. So $(h-c,k) = (-2,-2)$ and $(h+c,k) = (8,-2)$. We can use the x-coordinate from either of these points to solve for c. Using the point $(8,-2)$ and substituting $h = 3$, we get

$$\begin{aligned} h+c &= 8 \\ \implies 3+c &= 8 && \text{(substituting)} \\ \implies c &= 5 && \text{(subtracting 3 from both sides)} \\ \implies c^2 &= 25 && \text{(squaring both sides)} \end{aligned}$$

Next, solve for v_y^2 using the equation $c^2 = |v_x^2 + v_y^2| = v_x^2 + v_y^2$.

$$\begin{aligned} c^2 &= v_x^2 + v_y^2 \\ \implies 25 &= 9 + v_y^2 && \text{(substituting)} \\ \implies 16 &= v_y^2 && \text{(subtracting 9 from both sides)} \end{aligned}$$

Finally, substitute the values found for h, k, v_x^2, and v_y^2 into the standard form of the equation.

$$\frac{(x-3)^2}{9} - \frac{(y+2)^2}{16} = 1$$

Example 6.2.5: Graphing a Hyperbola Centered at (h,k) Given an Equation in General Form

Start by expressing the equation in standard form. Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$(9x^2 - 36x) - (4y^2 + 40y) = 388$$

Factor the leading coefficient of each expression.

$$9(x^2 - 4x) - 4(y^2 + 10y) = 388$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$9(x^2 - 4x + 4) - 4(y^2 + 10y + 25) = 388 + 36 - 100$$

Rewrite as perfect squares.

$$9(x-2)^2 - 4(y+5)^2 = 324$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x-2)^2}{36} - \frac{(y+5)^2}{81} = 1$$

The standard form that applies to the given equation is $\frac{(x-h)^2}{v_x^2} - \frac{(y-k)^2}{v_y^2} = 1$, where $v_x^2 = 36$ and $v_y^2 = 81$, or $v_x = 6$ and $v_y = 9$. Thus, the transverse axis is parallel to the x-axis. It follows that:

- the center of the hyperbola is $(h,k) = (2,-5)$
- the coordinates of the vertices are $(h \pm v_x, k) = (2 \pm 6, -5)$, or $(-4,-5)$ and $(8,-5)$
- the coordinates of the co-vertices are $(h, k \pm v_y) = (2, -5 \pm 9)$, or $(2,-14)$ and $(2,4)$
- the coordinates of the foci are $(h \pm c, k)$, where $c = \sqrt{v_x^2 + v_y^2}$. Solving for c, we have

  $$c = \sqrt{36+81} = \sqrt{117} = 3\sqrt{13}$$

  Therefore, the coordinates of the foci are $\left(2 - 3\sqrt{13}, -5\right)$ and $\left(2 + 3\sqrt{13}, -5\right)$.
- The equations of the asymptotes are

  $$y - k = \pm\frac{v_y}{v_x}(x-h) \implies y - (-5) = \pm\frac{9}{6}(x-2) \implies y = \pm\frac{3}{2}(x-2) - 5.$$

Next, we plot and label the center, vertices, co-vertices, foci, and asymptotes and draw smooth curves to form the hyperbola, as shown in Figure 6.2.8.

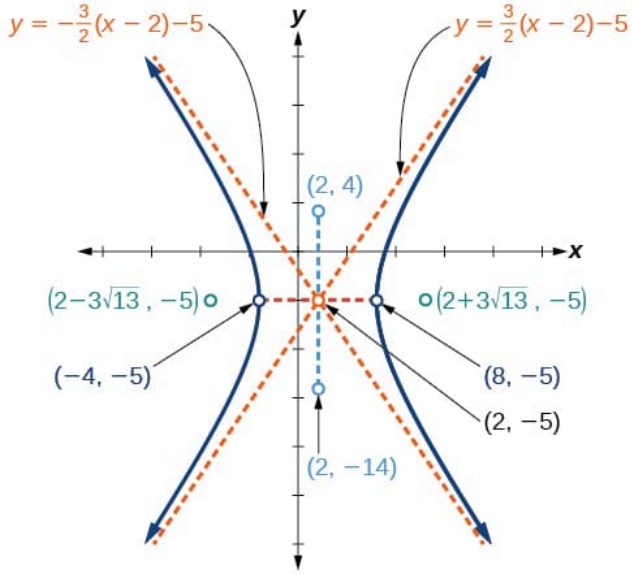


Figure 6.2.8

Example 6.2.6**: Solving Applied Problems Involving Hyperbolas**

We are assuming the center of the tower is at the origin, so we can use the standard form of a horizontal hyperbola centered at the origin: $\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1$, where the branches of the hyperbola form the sides of the cooling tower. We must find the values of v_x^2 and v_y^2 to complete the model.

First, we find v_x^2. Recall that the length of the transverse axis of a hyperbola is $2v_x$. This length is represented by the distance where the sides are closest, which is given as 60 meters. So, $2v_x = 60$. Therefore, $v_x = 30$ and $v_x^2 = 900$.

To solve for v_y^2, we need to substitute for x and y in our equation using a known point. To do this, we can use the dimensions of the tower to find some point (x, y) that lies on the hyperbola. We will use the top right corner of the tower to represent that point. Since the y-axis bisects the tower, our x-value can be represented by the radius of the top, or 36 meters. The y-value is represented by the distance from the origin to the top, which is given as 79.6 meters. Therefore,

$$\begin{aligned}
& \frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1 \\
\Rightarrow\quad & \frac{36^2}{900} - \frac{79.6^2}{v_y^2} = 1 && \text{(substituting)} \\
\Rightarrow\quad & \frac{36^2}{900} - 1 = \frac{79.6^2}{v_y^2} && \left(\text{subtracting 1 and adding } \tfrac{79.6^2}{v_y^2} \text{ to both sides}\right) \\
\Rightarrow\quad & \frac{1}{\frac{36^2}{900} - 1} = \frac{v_y^2}{79.6^2} && \text{(taking the reciprocal of both sides)} \\
\Rightarrow\quad & \frac{79.6^2}{\frac{36^2}{900} - 1} = v_y^2 && \left(\text{multiplying both sides by } 79.6^2\right) \\
\Rightarrow\quad & v_y^2 \approx 14400.3636 && \text{(Extraction of Roots)}
\end{aligned}$$

The sides of the tower can be modeled by the hyperbolic equation

$$\frac{x^2}{900} - \frac{y^2}{14400.3636} = 1.$$

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6.2.1: Resources and Key Concepts

Resources

Videos

- Hyperbola
  - The Hyperbola - Definitions and Concept
- Equation of a hyperbola
  - The Hyperbola - Building Equations for Hyperbolas
- Graphing hyperbolas
  - The Hyperbola - Graphing Hyperbolas

Key Concepts

Definitions

- **Hyperbola**: The set of all points in a plane such that the absolute value of the difference of the distances between the point and two fixed points is a positive constant.
- **Focus (plural: foci)**: One of the two fixed points used to define a hyperbola.
- **Transverse Axis**: The line segment that passes through the center of the hyperbola and has the vertices as its endpoints.
- **Conjugate Axis**: The line segment that is perpendicular to the transverse axis, passes through the center, and has the co-vertices as its endpoints.
- **Center (of a hyperbola)**: The midpoint of both the transverse and conjugate axes.
- **Vertices (of a hyperbola)**: The endpoints of the transverse axis.
- **Co-vertices (of a hyperbola)**: The endpoints of the conjugate axis.
- **Central Rectangle**: A rectangle centered at the center of the hyperbola with sides that pass through each vertex and co-vertex. The diagonals of this rectangle form the asymptotes of the hyperbola.

Theorems

- **Standard Form of the Equation of a Hyperbola with Center at the Origin**:
  - The standard form for a hyperbola with its transverse axis on the x-axis is $\frac{x^2}{v_x^2} - \frac{y^2}{v_y^2} = 1$. The vertices are $(\pm v_x, 0)$, the foci are $(\pm c, 0)$ where $c^2 = v_x^2 + v_y^2$, and the asymptotes are $y = \pm \frac{v_y}{v_x} x$.
  - The standard form for a hyperbola with its transverse axis on the y-axis is $\frac{y^2}{v_y^2} - \frac{x^2}{v_x^2} = 1$. The vertices are $(0, \pm v_y)$, the foci are $(0, \pm c)$ where $c^2 = v_x^2 + v_y^2$, and the asymptotes are $y = \pm \frac{v_y}{v_x} x$.
- **Standard Form of the Equation of a Hyperbola with Center at** (h, k):
  - The standard form for a hyperbola with its transverse axis parallel to the x-axis is $\frac{(x-h)^2}{v_x^2} - \frac{(y-k)^2}{v_y^2} = 1$. The equations of the asymptotes are $y - k = \pm \frac{v_y}{v_x}(x - h)$.
  - The standard form for a hyperbola with its transverse axis parallel to the y-axis is $\frac{(y-k)^2}{v_y^2} - \frac{(x-h)^2}{v_x^2} = 1$. The equations of the asymptotes are $y - k = \pm \frac{v_y}{v_x}(x - h)$.

Common Mistakes

- **Confusing the Foci Equation with that of an Ellipse**: For a hyperbola, the relationship is $c^2 = v_x^2 + v_y^2$. This is different from the ellipse, which uses subtraction.
- **Incorrectly Identifying the Transverse Axis**: For a hyperbola, the transverse axis corresponds to the positive term in the standard form equation, not necessarily the term with the larger denominator.
- **Errors in Asymptote Equations**: Mixing up the numerator and denominator in the slope of the asymptotes. The slope is

always a variation of $\pm\frac{\text{rise}}{\text{run}}$ or $\pm\frac{v_y}{v_x}$.

- **Errors in Completing the Square**: When converting from general to standard form, failing to correctly account for factored coefficients when balancing the equation. For example, when changing $9x^2 - 36x$ to $9(x^2 - 4x + 4)$, you have added $9 \times 4 = 36$ to that side of the equation.
- **Sign Errors with Center Coordinates**: Incorrectly identifying the center (h, k) from the standard form. For example, in $\frac{(x-2)^2}{36}$, $h = 2$, and in $\frac{(y+5)^2}{81}$, $k = -5$.

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6.2.2: Homework

Reading Questions

1. Define a hyperbola in terms of its foci. How does this definition differ from the definition of an ellipse?
2. What is the transverse axis of a hyperbola? How can you identify which axis is the transverse axis from the standard form equation?
3. What is the role of the "central rectangle" in graphing a hyperbola?
4. What is the equation relating the distances from the center to the vertices (v_x, v_y) and the distance from the center to the foci (c)? How does this differ from the corresponding equation for an ellipse?
5. What is the standard form of the equation for a hyperbola with a horizontal transverse axis, centered at (h, k)?
6. What is the formula for the slopes of the asymptotes of a hyperbola centered at (h, k)?
7. If the asymptotes of a hyperbola intersect at the origin, what can you conclude about the hyperbola?
8. Do the foci of a hyperbola lie on the transverse axis or the conjugate axis?
9. If the transverse axis of a hyperbola is vertical, what does this tell you about its graph and its standard form equation?
10. Can the graph of a hyperbola ever cross its asymptotes? Can it cross its transverse axis? Can it cross its conjugate axis?
11. Define a hyperbola in terms of its foci.
12. What can we conclude about a hyperbola if its asymptotes intersect at the origin?
13. What must be true of the foci of a hyperbola?
14. If the transverse axis of a hyperbola is vertical, what do we know about the graph?
15. Where must the center of hyperbola be relative to its foci?

Homework

Algebraic

For the following exercises, determine whether the following equations represent hyperbolas. If so, write in standard form.

16. $3y^2 + 2x = 6$
17. $\frac{x^2}{36} - \frac{y^2}{9} = 1$
18. $5y^2 + 4x^2 = 6x$
19. $25x^2 - 16y^2 = 400$
20. $-9x^2 + 18x + y^2 + 4y - 14 = 0$

For the following exercises, write the equation for the hyperbola in standard form if it is not already, and identify the vertices and foci, and write equations of asymptotes.

21. $\frac{x^2}{25} - \frac{y^2}{36} = 1$
22. $\frac{x^2}{100} - \frac{y^2}{9} = 1$
23. $\frac{y^2}{4} - \frac{x^2}{81} = 1$
24. $9y^2 - 4x^2 = 1$
25. $\frac{(x-1)^2}{9} - \frac{(y-2)^2}{16} = 1$
26. $\frac{(y-6)^2}{36} - \frac{(x+1)^2}{16} = 1$
27. $\frac{(x-2)^2}{49} - \frac{(y+7)^2}{49} = 1$
28. $4x^2 - 8x - 9y^2 - 72y + 112 = 0$
29. $-9x^2 - 54x + 9y^2 - 54y + 81 = 0$
30. $4x^2 - 24x - 36y^2 - 360y + 864 = 0$
31. $-4x^2 + 24x + 16y^2 - 128y + 156 = 0$
32. $-4x^2 + 40x + 25y^2 - 100y + 100 = 0$
33. $x^2 + 2x - 100y^2 - 1000y + 2401 = 0$
34. $-9x^2 + 72x + 16y^2 + 16y + 4 = 0$
35. $4x^2 + 24x - 25y^2 + 200y - 464 = 0$

For the following exercises, find the equations of the asymptotes for each hyperbola.

36. $\frac{y^2}{3^2} - \frac{x^2}{3^2} = 1$
37. $\frac{(x-3)^2}{5^2} - \frac{(y+4)^2}{2^2} = 1$
38. $\frac{(y-3)^2}{3^2} - \frac{(x+5)^2}{6^2} = 1$
39. $9x^2 - 18x - 16y^2 + 32y - 151 = 0$
40. $16y^2 + 96y - 4x^2 + 16x + 112 = 0$

Graphical

For the following exercises, sketch a graph of the hyperbola, labeling vertices and foci.

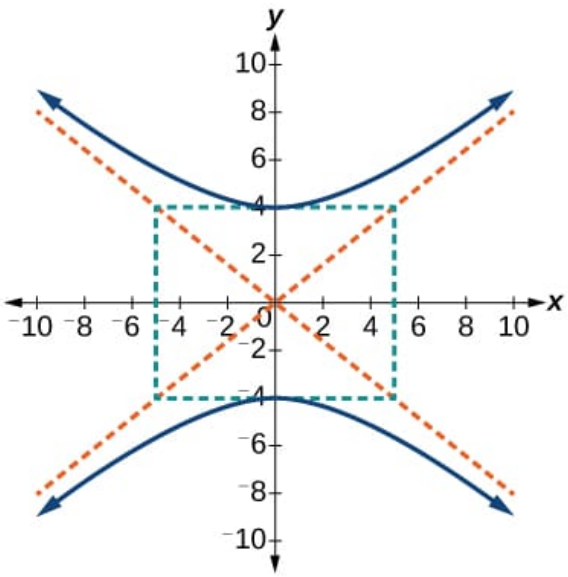
41. $\frac{x^2}{49} - \frac{y^2}{16} = 1$
42. $\frac{x^2}{64} - \frac{y^2}{4} = 1$
43. $\frac{y^2}{9} - \frac{x^2}{25} = 1$
44. $81x^2 - 9y^2 = 1$
45. $\frac{(y+5)^2}{9} - \frac{(x-4)^2}{25} = 1$
46. $\frac{(x-2)^2}{8} - \frac{(y+3)^2}{27} = 1$
47. $\frac{(y-3)^2}{9} - \frac{(x-3)^2}{9} = 1$
48. $-4x^2 - 8x + 16y^2 - 32y - 52 = 0$
49. $x^2 - 8x - 25y^2 - 100y - 109 = 0$
50. $-x^2 + 8x + 4y^2 - 40y + 88 = 0$
51. $64x^2 + 128x - 9y^2 - 72y - 656 = 0$
52. $16x^2 + 64x - 4y^2 - 8y - 4 = 0$
53. $-100x^2 + 1000x + y^2 - 10y - 2575 = 0$
54. $4x^2 + 16x - 4y^2 + 16y + 16 = 0$

For the following exercises, given information about the graph of the hyperbola, find its equation.

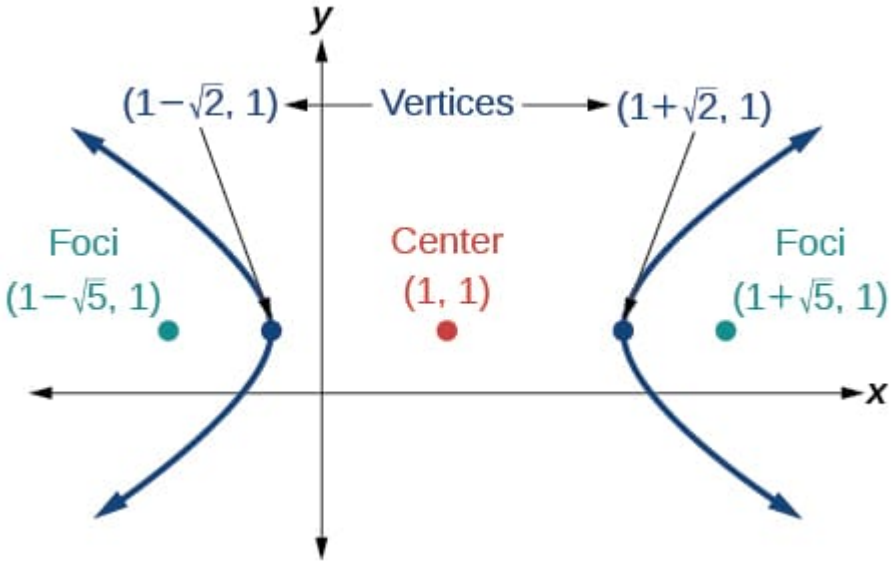
55. Vertices at $(3, 0)$ and $(-3, 0)$ and one focus at $(5, 0)$.
56. Vertices at $(0, 6)$ and $(0, -6)$ and one focus at $(0, -8)$.
57. Vertices at $(1, 1)$ and $(11, 1)$ and one focus at $(12, 1)$.
58. Center: $(0, 0)$; vertex: $(0, -13)$; one focus: $(0, \sqrt{313})$.
59. Center: $(4, 2)$; vertex: $(9, 2)$; one focus: $(4 + \sqrt{26}, 2)$.
60. Center: $(3, 5)$; vertex: $(3, 11)$; one focus: $(3, 5 + 2\sqrt{10})$.

For the following exercises, given the graph of the hyperbola, find its equation.

61.

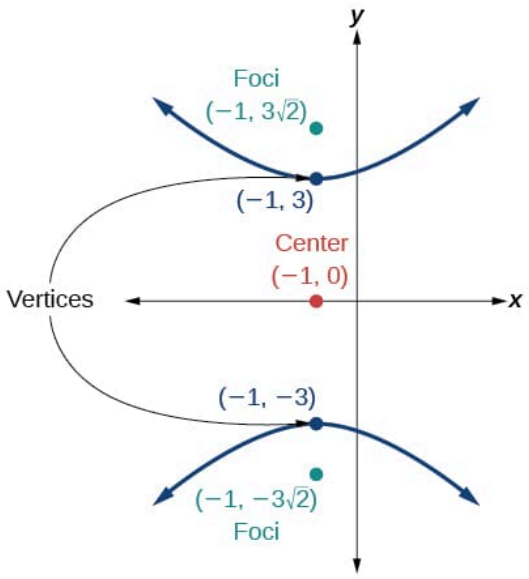

y
x
10
8
6
4
2
−10
−8
−6
−4
−2
2
4
6
8
10
−2
−4
−6
−8
−10

62.

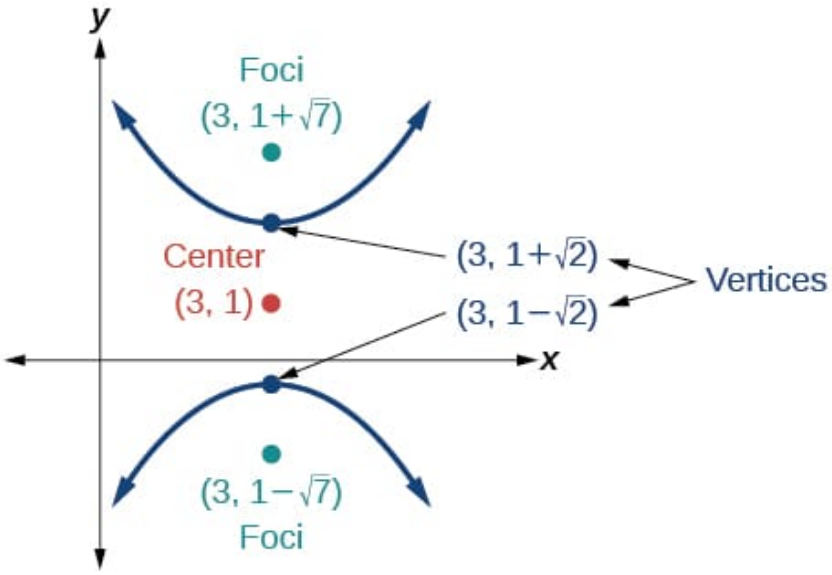

y
x
(1−√2, 1)
Vertices
(1+√2, 1)
Foci
(1−√5, 1)
Center
(1, 1)
Foci
(1+√5, 1)

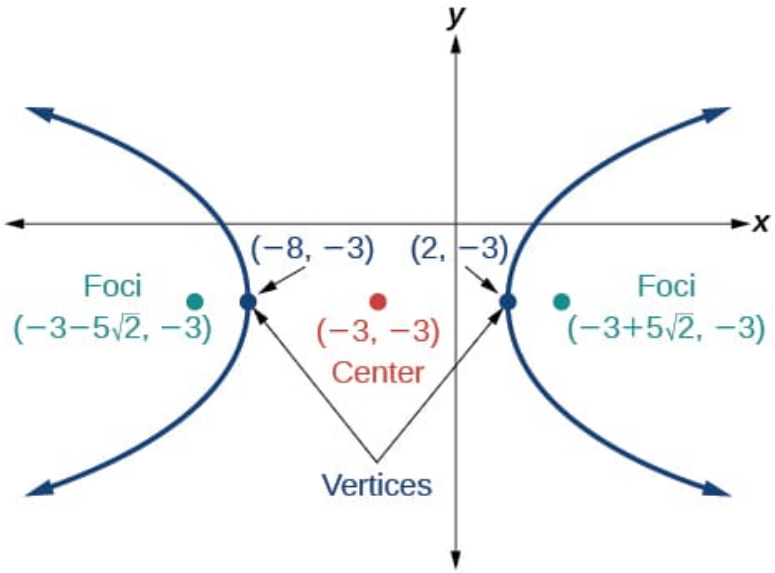
63.

54.


y
Foci
(−1, 3√2)
(−1, 3)
Center
(−1, 0)
Vertices
x
(−1, −3)
(−1, −3√2)
Foci

55.


y
Foci
(3, 1+√7)
Center
(3, 1)
(3, 1+√2)
(3, 1−√2)
Vertices
x
(3, 1−√7)
Foci



Extensions

For the following exercises, express the equation for the hyperbola as two functions, with y as a function of x. Express as simply as possible. Use graphing technology to sketch the graph of the two functions on the same axes.

56. $\frac{x^2}{4} - \frac{y^2}{9} = 1$
57. $\frac{y^2}{9} - x^2 = 1$
58. $\frac{(x-2)^2}{16} - \frac{(y+3)^2}{25} = 1$
59. $-4x^2 - 16x + y^2 - 2y - 19 = 0$
70. $4x^2 - 24x - y^2 - 4y + 16 = 0$

Real-World Applications

For the following exercises, a hedge is to be constructed in the shape of a hyperbola near a fountain at the center of the yard. Find the equation of the hyperbola and sketch the graph.

71. The hedge will follow the asymptotes $y = x$ and $y = -x$, and its closest distance to the center fountain is 5 yards.
72. The hedge will follow the asymptotes $y = 2x$ and $y = -2x$, and its closest distance to the center fountain is 6 yards.
73. The hedge will follow the asymptotes $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$, and its closest distance to the center fountain is 10 yards.
74. The hedge will follow the asymptotes $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$, and its closest distance to the center fountain is 12 yards.
75. The hedge will follow the asymptotes $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$, and its closest distance to the center fountain is 20 yards.

For the following exercises, assume an object enters our solar system and we want to graph its path on a coordinate system with the sun at the origin and the x-axis as the axis of symmetry for the object's path. Give the equation of the flight path of each object using the given information.

76. The object enters along a path approximated by the line $y = x - 2$ and passes within 1 au (astronomical unit) of the sun at its closest approach, so that the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -x + 2$.
77. The object enters along a path approximated by the line $y = 2x - 2$ and passes within 0.5 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -2x + 2$.
78. The object enters along a path approximated by the line $y = 0.5x + 2$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -0.5x - 2$.
79. The object enters along a path approximated by the line $y = \frac{1}{3}x - 1$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -\frac{1}{3}x + 1$

.

30. The object enters along a path approximated by the line $y = 3x - 9$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -3x + 9$.

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6.3: The Parabola

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure 6.3.1**:** *The Olympic torch concludes its journey around the world when it is used to light the Olympic cauldron during the opening ceremony. (credit: Ken Hackman, U.S. Air Force)*

Did you know that the Olympic torch is lit several months before the start of the games? The ceremonial method for lighting the flame is the same as in ancient times. The ceremony takes place at the Temple of Hera in Olympia, Greece, and is rooted in Greek mythology, paying tribute to Prometheus, who stole fire from Zeus to give to all humans. One of eleven acting priestesses places the torch at the focus of a parabolic mirror (see Figure 6.3.1), which focuses light rays from the sun to ignite the flame.

Parabolic mirrors (or reflectors) are able to capture energy and focus it to a single point. The advantages of this property are evidenced by the vast list of parabolic objects we use every day: satellite dishes, suspension bridges, telescopes, microphones, spotlights, and car headlights, to name a few. Parabolic reflectors are also used in alternative energy devices, such as solar cookers and water heaters, because they are inexpensive to manufacture and need little maintenance. In this section we will explore the parabola and its uses, including low-cost, energy-efficient solar designs.

Graphing Parabolas with Vertices at the Origin

We previously saw that an ellipse is formed when a plane cuts through a right circular cone. If the plane is parallel to the edge of the cone, an unbounded curve is formed. This curve is a **parabola**. See Figure 6.3.2.

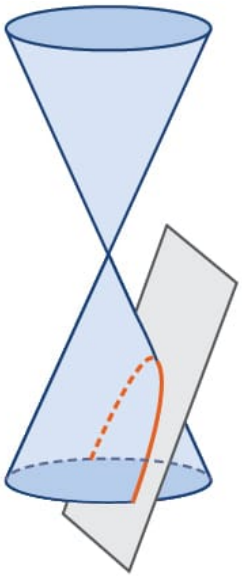


Figure 6.3.2**:** *Parabola*

Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane.

Definition: Parabola

A **parabola** is the set of all points in a plane that are the same distance from a fixed line, called the **directrix**, and a fixed point called the **focus** (where the focus is not on the directrix).

When studying quadratic functions, we learned about a parabola's vertex and axis of symmetry. Now we extend the discussion to include other key features of the parabola. See Figure 6.3.3. Notice that the axis of symmetry passes through the focus and vertex and is perpendicular to the directrix. The vertex is the midpoint between the directrix and the focus.

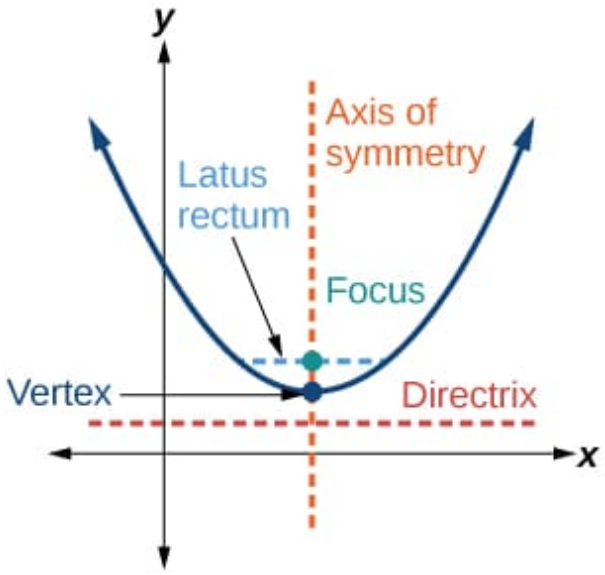


Figure 6.3.3**:** *Key features of the parabola*

Definition: Latus Rectum

The line segment that passes through the focus and is parallel to the directrix is called the **latus rectum**.

The endpoints of the latus rectum lie on the curve. By definition, the distance d from the focus to any point P on the parabola is equal to the distance from P to the directrix.

To work with parabolas in the coordinate plane, we consider two cases: those with a vertex at the origin and those with a vertex at a point other than the origin. We begin with the former.

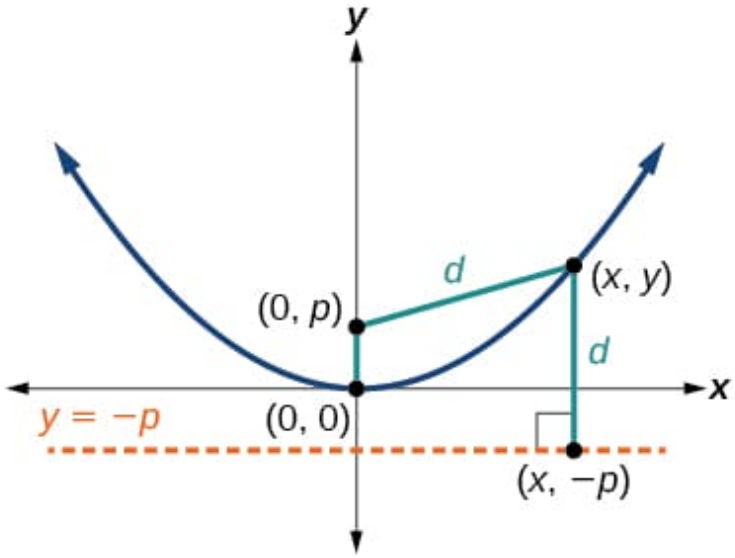


Figure 6.3.4

Let (x, y) be a point on the parabola with vertex $(0, 0)$, focus $(0, p)$, and directrix $y = -p$ as shown in Figure 6.3.4. The distance d from point (x, y) to point $(x, -p)$ on the directrix is the difference of the y-values: $d = y + p$. The distance from the focus $(0, p)$ to the point (x, y) is also equal to d and can be expressed using the Distance Formula.

$$d = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}.$$

Set the two expressions for d equal to each other and solve for y to derive the equation of the parabola. We do this because the distance from (x, y) to $(0, p)$ equals the distance from (x, y) to $(x, -p)$.

$$\sqrt{x^2 + (y-p)^2} = y + p$$

We then square both sides of the equation, expand the squared terms, and simplify by combining like terms.

$$\begin{aligned}
& x^2 + (y-p)^2 = (y+p)^2 && \text{(squaring both sides)} \\
\Longrightarrow\ & x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 && \text{(distributing)} \\
\Longrightarrow\ & x^2 - 2py = 2py && \text{(subtracting } y^2 \text{ and } p^2 \text{ from both sides)} \\
\Longrightarrow\ & x^2 = 4py && \text{(adding } 2py \text{ to both sides)}
\end{aligned}$$

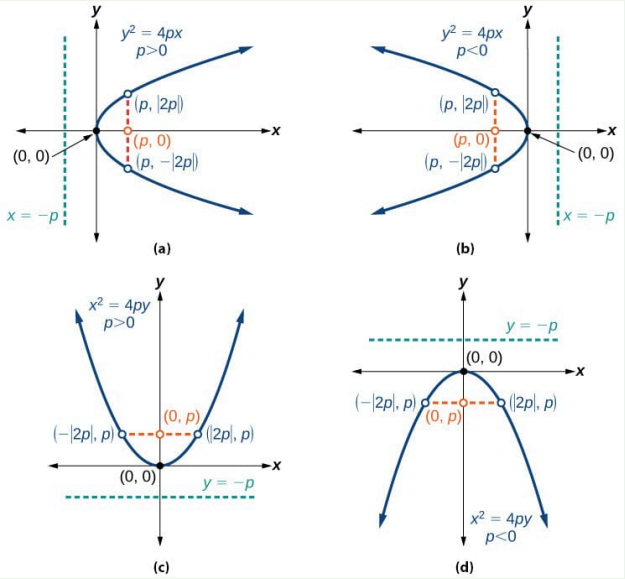
The equations of parabolas with vertex $(0, 0)$ are $y^2 = 4px$ when the x-axis is the axis of symmetry and $x^2 = 4py$ when the y-axis is the axis of symmetry. These standard forms are given below, along with their general graphs and key features.

Theorem: Standard Form of the Equation of a Parabola with Vertex at the Origin

The table below and the following figure summarize the standard features of parabolas with a vertex at the origin.

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
x-axis	$y^2 = 4px$	$(p, 0)$	$x = -p$	$(p, \pm 2p)$
y-axis	$x^2 = 4py$	$(0, p)$	$y = -p$	$(\pm 2p, p)$

(a) When $p > 0$ and the axis of symmetry is the x-axis, the parabola opens right.
(b) When $p < 0$ and the axis of symmetry is the x-axis, the parabola opens left.
(c) When $p > 0$ and the axis of symmetry is the y-axis, the parabola opens up.
(d) When $p < 0$ and the axis of symmetry is the y-axis, the parabola opens down.



The key features of a parabola are its vertex, axis of symmetry, focus, directrix, and latus rectum. When given a standard equation for a parabola centered at the origin, we can easily identify the key features to graph the parabola.

A line is said to be **tangent** to a curve if it intersects the curve at exactly one point. If we sketch lines tangent to the parabola at the endpoints of the latus rectum, these lines intersect on the axis of symmetry, as shown in Figure $6.3.5$.

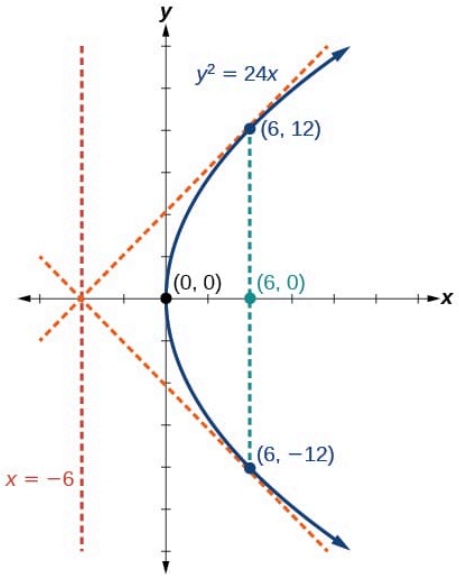


Figure $6.3.5$

Example $6.3.1$: Graphing a Parabola with Vertex $(0, 0)$ and the x-axis as the Axis of Symmetry

Graph $y^2 = 24x$. Identify and label the focus, directrix, and endpoints of the latus rectum.

Solution available at the end of this section.

Checkpoint 6.3.1

Graph $y^2 = -16x$. Identify and label the focus, directrix, and endpoints of the latus rectum.

Example 6.3.2: Graphing a Parabola with Vertex $(0, 0)$ and the y-axis as the Axis of Symmetry

Graph $x^2 = -6y$. Identify and label the focus, directrix, and endpoints of the latus rectum.

Solution available at the end of this section.

Checkpoint 6.3.2

Graph $x^2 = 8y$. Identify and label the focus, directrix, and endpoints of the latus rectum.

Writing Equations of Parabolas in Standard Form

In the previous examples, we used the standard form equation of a parabola to calculate the locations of its key features. We can also use the calculations in reverse to write an equation for a parabola when given its key features.

Example 6.3.3: Writing the Equation of a Parabola in Standard Form Given its Focus and Directrix

What is the equation for the parabola with focus $\left(-\frac{1}{2}, 0\right)$ and directrix $x = \frac{1}{2}$?

Solution available at the end of this section.

Exercise 6.3.3

What is the equation for the parabola with focus $\left(0, \frac{7}{2}\right)$ and directrix $y = -\frac{7}{2}$?

Graphing Parabolas with Vertices Not at the Origin

Like other graphs we've worked with, the graph of a parabola can be translated. If a parabola is translated h units horizontally and k units vertically, the vertex will be (h, k). This translation results in the standard form of the equation we saw previously with x replaced by $(x - h)$ and y replaced by $(y - k)$.

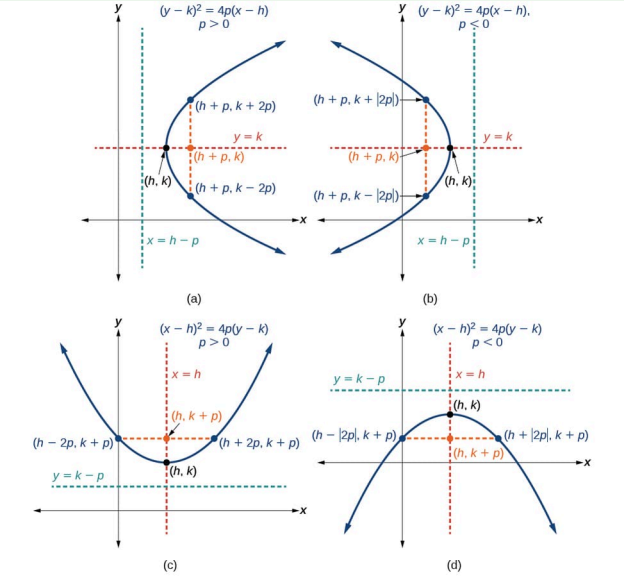
To graph parabolas with a vertex (h, k) other than the origin, we use the standard form $(y - k)^2 = 4p(x - h)$ for parabolas that have an axis of symmetry parallel to the x-axis, and $(x - h)^2 = 4p(y - k)$ for parabolas that have an axis of symmetry parallel to the y-axis. These standard forms are given below, along with their general graphs and key features.

Theorem: Standard Form of the Equation of a Parabola with Vertex Off-Origin

The table below and the following figure summarize the standard features of parabolas with a vertex at the origin.

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
$y = k$	$(y - k)^2 = 4p(x - h)$	$(h + p, k)$	$x = h - p$	$(h + p, k \pm 2p)$
$x = h$	$(x - h)^2 = 4p(y - k)$	$(h, k + p)$	$y = k - p$	$(h \pm 2p, k + p)$

(a) When $p > 0$, the parabola opens right.
(b) When $p < 0$, the parabola opens left.
(c) When $p > 0$, the parabola opens up.
(d) When $p < 0$, the parabola opens down.



Example 6.3.4: Graphing a Parabola with Vertex (h, k) and Axis of Symmetry Parallel to the x-axis

Graph $(y-1)^2 = -16(x+3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Solution available at the end of this section.

Checkpoint 6.3.4

Graph $(y+1)^2 = 4(x-8)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Example 6.3.5: Graphing a Parabola from an Equation Given in General Form

Graph $x^2 - 8x - 28y - 208 = 0$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Solution available at the end of this section.

Checkpoint 6.3.5

Graph $(x+2)^2 = -20(y-3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Solving Applied Problems Involving Parabolas

As we mentioned at the beginning of the section, parabolas are used to design many objects we use every day, such as telescopes, suspension bridges, microphones, and radar equipment. Parabolic mirrors, such as the one used to light the Olympic torch, have a very unique reflecting property. When rays of light parallel to the parabola's axis of symmetry are directed toward any surface of the mirror, the light is reflected directly to the focus. See Figure 6.3.10. This is why the Olympic torch is ignited when it is held at

the focus of the parabolic mirror.

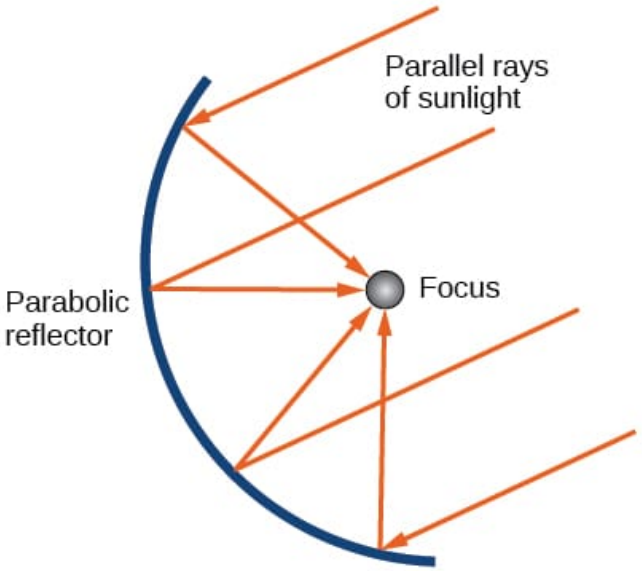


Figure 6.3.10**:** *Reflecting property of parabolas*

Parabolic mirrors have the ability to focus the sun's energy to a single point, raising the temperature hundreds of degrees in a matter of seconds. Thus, parabolic mirrors are featured in many low-cost, energy efficient solar products, such as solar cookers, solar heaters, and even travel-sized fire starters.

Example 6.3.6: Solving Applied Problems Involving Parabolas

A cross-section of a design for a travel-sized solar fire starter is shown in Figure 6.3.11. The sun's rays reflect off the parabolic mirror toward an object attached to the igniter. Because the igniter is located at the focus of the parabola, the reflected rays cause the object to burn in just seconds.

a. Find the equation of the parabola that models the fire starter. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane.
b. Use the equation found in part (a) to find the depth of the fire starter.

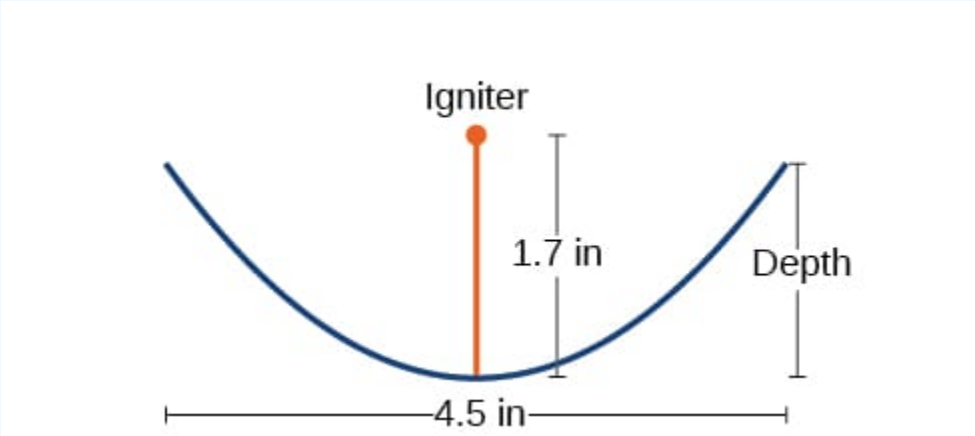


Figure 6.3.11**:** *Cross-section of a travel-sized solar fire starter*

Solution available at the end of this section.

Checkpoint 6.3.6

Balcony-sized solar cookers have been designed for families living in India. The top of a dish has a diameter of 1600 mm. The sun's rays reflect off the parabolic mirror toward the "cooker," which is placed 320 mm from the base.

a. Find an equation that models a cross-section of the solar cooker. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane, and that the parabola opens to the right (i.e., has the x-axis as its axis of symmetry).
b. Use the equation found in part (a) to find the depth of the cooker.

Success in College: Learning From Tests (Part 5)

1. A math class includes one or two quizzes for each major test. A student is averaging about 85% on these quizzes. When he gets them back, he looks them over and drops them in the wastebasket as he leaves class. Explain why he should keep the quizzes.
2. A math professor tells her class to go over their tests so that they can learn from their mistakes. She does not require that students do this. Most do not. Do you think the professor should require the students to make corrections on the test and turn them back in? (Assume that the corrections will not change their original test grade.) Explain.

Answers

Section Preview

I hate to say it, but this material is likely never going to be used outside of manufactured examples specifically targeting paraboloid, satellite dishes, and telescopes. It's great material—don't get me wrong—and it is theroetically beautiful (from a Mathematics perspective), but it is the protien bar of conic sections. Sure, it is delicious and fills the mathematical soul with nourishment, but there are better and cleaner topics to digest.

Section Preview

- Graph parabolas with vertices at the origin.
- Write equations of parabolas in standard form.
- Graph parabolas with vertices not at the origin.
- Solve applied problems involving parabolas.

Example $6.3.1$: Graphing a Parabola with Vertex $(0,0)$ and the x-axis as the Axis of Symmetry

The standard form that applies to the given equation is $y^2 = 4px$. Thus, the axis of symmetry is the x-axis. It follows that:

- $24 = 4p$, so $p = 6$. Since $p > 0$, the parabola opens right
- the coordinates of the focus are $(p, 0) = (6, 0)$
- the equation of the directrix is $x = -p = -6$
- the endpoints of the latus rectum have the same x-coordinate at the focus. To find the endpoints, substitute $x = 6$ into the original equation: $(6, \pm 12)$

Next we plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

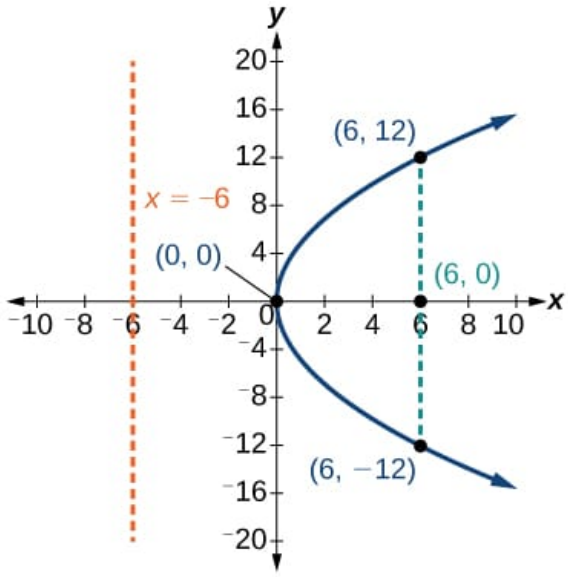


Figure $6.3.6$

Example $6.3.2$**: Graphing a Parabola with Vertex** $(0,0)$ **and the** y**-axis as the Axis of Symmetry**

The standard form that applies to the given equation is $x^2 = 4py$. Thus, the axis of symmetry is the y-axis. It follows that:

- $-6 = 4p$, so $p = -\frac{3}{2}$. Since $p < 0$, the parabola opens down
- the coordinates of the focus are $(0, p) = \left(0, -\frac{3}{2}\right)$
- the equation of the directrix is $y = -p = \frac{3}{2}$
- the endpoints of the latus rectum can be found by substituting $y = \frac{3}{2}$ into the original equation: $\left(\pm 3, -\frac{3}{2}\right)$

Next we plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

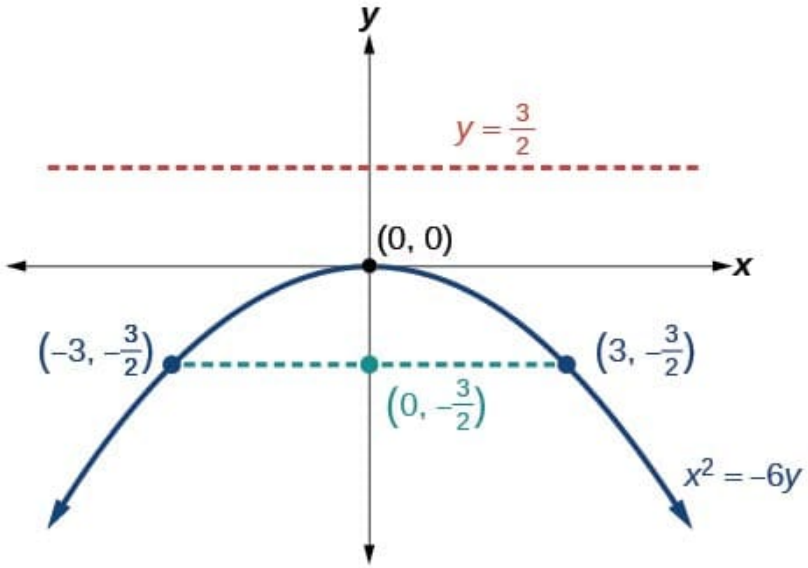


Figure $6.3.7$

Example $6.3.3$**: Writing the Equation of a Parabola in Standard Form Given its Focus and Directrix**

The focus has the form $(p, 0)$, so the equation will have the form $y^2 = 4px$. Multiplying $4p$, we have $4p = 4\left(-\frac{1}{2}\right) = -2$. Substituting for $4p$, we have $y^2 = 4px = -2x$. Therefore, the equation for the parabola is $y^2 = -2x$.

Example $6.3.4$**: Graphing a Parabola with Vertex** (h,k) **and Axis of Symmetry Parallel to the** x**-axis**

The standard form that applies to the given equation is $(y-k)^2 = 4p(x-h)$. Thus, the axis of symmetry is parallel to the x-axis. It follows that:

- the vertex is $(h, k) = (-3, 1)$
- the axis of symmetry is $y = k = 1$
- $-16 = 4p$, so $p = -4$. Since $p < 0$, the parabola opens left.
- the coordinates of the focus are $(h + p, k) = (-3 + (-4), 1) = (-7, 1)$
- the equation of the directrix is $x = h - p = -3 - (-4) = 1$
- the endpoints of the latus rectum are $(h + p, k \pm 2p) = (-3 + (-4), 1 \pm 2(-4))$, or $(-7, -7)$ and $(-7, 9)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola. See Figure $6.3.8$.

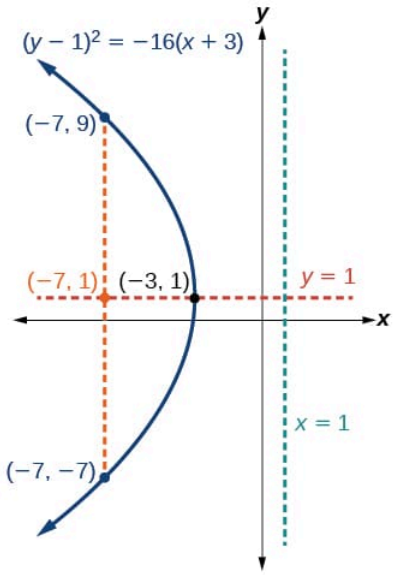


Figure $6.3.8$

Example $6.3.5$**: Graphing a Parabola from an Equation Given in General Form**

Start by writing the equation of the parabola in standard form. The standard form that applies to the given equation is $(x-h)^2 = 4p(y-k)$. Thus, the axis of

symmetry is parallel to the y-axis. To express the equation of the parabola in this form, we begin by isolating the terms that contain the variable x in order to complete the square.

$$\begin{aligned} x^2-8x-28y-208 &= 0 \\ \Longrightarrow \quad x^2-8x &= 28y+208 \\ \Longrightarrow \quad x2-8x+16 &= 28y+208+16 \\ \Longrightarrow \quad (x-4)^2 &= 28y+224 \\ \Longrightarrow \quad (x-4)^2 &= 28(y+8) \\ \Longrightarrow \quad (x-4)^2 &= 4\cdot 7(y+8) \end{aligned}$$

It follows that:

- the vertex is $(h,k)=(4,-8)$
- the axis of symmetry is $x=h=4$
- since $p=7$, $p>0$ and so the parabola opens up
- the coordinates of the focus are $(h,k+p)=(4,-8+7)=(4,-1)$
- the equation of the directrix is $y=k-p=-8-7=-15$
- the endpoints of the latus rectum are $(h\pm 2p,k+p)=(4\pm 2(7),-8+7)$, or $(-10,-1)$ and $(18,-1)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola. See Figure 6.3.9.

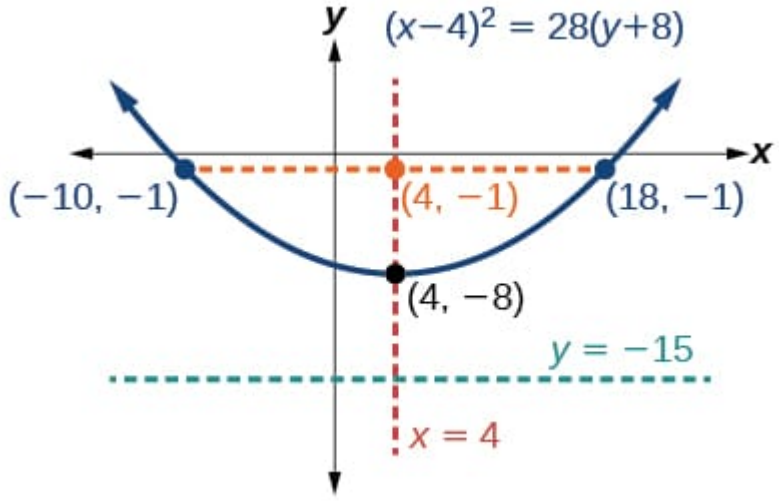


Figure 6.3.9

Example 6.3.6**: Solving Applied Problems Involving Parabolas**

a. The vertex of the dish is the origin of the coordinate plane, so the parabola will take the standard form $x^2=4py$, where $p>0$. The igniter, which is the focus, is 1.7 inches above the vertex of the dish. Thus we have $p=1.7$.

$$\begin{aligned} x^2 &= 4py && \text{(Standard form of upward-facing parabola with vertex } (0,0)) \\ \Longrightarrow \quad x^2 &= 4(1.7)y && \text{(substituting)} \\ \Longrightarrow \quad x^2 &= 6.8y && \text{(multiplying)} \end{aligned}$$

b. The dish extends $\frac{4.5}{2}=2.25$ inches on either side of the origin. We can substitute 2.25 for x in the equation from part (a) to find the depth of the dish.

$$\begin{aligned} x^2 &= 6.8y \\ \Longrightarrow \quad (2.25)^2 &= 6.8y && \text{(substituting)} \\ \Longrightarrow \quad 0.74 &\approx y && \text{(solving for } y) \end{aligned}$$

The dish is about 0.74 inches deep.

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6.3.1: Resources and Key Concepts

Resources

Videos

- Parabola
  - Algebra Mini-Lecture: Conic Sections - The Parabola

Key Concepts

Definitions

- **Parabola**: The set of all points in a plane that are the same distance from a fixed line, called the directrix, and a fixed point called the focus (where the focus is not on the directrix).
- **Directrix**: A fixed line used to define a parabola. Every point on the parabola is equidistant from the directrix and the focus.
- **Focus (of a parabola)**: A fixed point used to define a parabola. Every point on the parabola is equidistant from the focus and the directrix.
- **Axis of Symmetry (of a parabola)**: A line that passes through the focus and vertex and is perpendicular to the directrix.
- **Vertex (of a parabola)**: The point where the axis of symmetry intersects the parabola; it is the midpoint between the focus and the directrix.
- **Latus Rectum**: The line segment that passes through the focus and is parallel to the directrix; its endpoints lie on the parabola.
- **Tangent (to a curve)**: A line that intersects a curve at exactly one point.

Theorems

- **Standard Form of the Equation of a Parabola with Vertex at the Origin**:
  - If the axis of symmetry is the y-axis, the standard form is $x^2 = 4py$. The focus is $(0, p)$, and the directrix is the line $y = -p$.
  - If the axis of symmetry is the x-axis, the standard form is $y^2 = 4px$. The focus is $(p, 0)$, and the directrix is the line $x = -p$.
- **Standard Form of the Equation of a Parabola with Vertex at** (h, k):
  - If the axis of symmetry is parallel to the y-axis, the standard form is $(x - h)^2 = 4p(y - k)$.
  - If the axis of symmetry is parallel to the x-axis, the standard form is $(y - k)^2 = 4p(x - h)$.

Common Mistakes

- **Confusing the Parabola's Orientation**: Mistaking the standard form $y^2 = 4px$ (opens left/right) for $x^2 = 4py$ (opens up/down). The variable that is squared determines the axis of symmetry.
- **Incorrect Sign or Value for** p: The standard form has a $4p$ term. A common error is to mistake the coefficient for p instead of $4p$ (e.g., in $y^2 = 24x$, $p = 6$, not 24). Also, the sign of p determines the direction the parabola opens.
- **Incorrect Formulas for Focus and Directrix**: Applying the focus/directrix formula for a vertical parabola to a horizontal one, or vice-versa. For example, for $y^2 = 4px$, the focus is $(p, 0)$ and directrix is $x = -p$.
- **Errors in Completing the Square**: When converting from general form to standard form, making algebraic mistakes, especially when factoring out a leading coefficient before completing the square and forgetting to balance the other side of the equation accordingly.
- **Sign Errors with Center Coordinates** (h, k): Incorrectly identifying the center (h, k) from the standard form. For example, in $(y - 1)^2 = -16(x + 3)$, the vertex is $(-3, 1)$.

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6.3.2: Homework

Reading Questions

1. Define a parabola in terms of its focus and directrix.
2. If the equation of a parabola is written in standard form and p is positive and the directrix is a vertical line, then what can we conclude about its graph? (Which way does it open?)
3. If the equation of a parabola is written in standard form and p is negative and the directrix is a horizontal line, then what can we conclude about its graph? (Which way does it open?)
4. What is the effect on the graph of a parabola if its equation in standard form has increasing values of $|p|$?
5. What are the two standard forms for the equation of a parabola with its vertex at the origin? What determines which way the parabola opens?
6. How are the focus and directrix related to the value of p?
7. How does the standard form equation of a parabola change if its vertex is at (h, k) instead of the origin?
8. What algebraic process is used to convert the general form of a parabola's equation into standard form?
9. What is the reflecting property of a parabola and how is it used in real-world objects like car headlights or satellite dishes?
10. What is the latus rectum of a parabola?
11. Define a parabola in terms of its focus and directrix.
12. If the equation of a parabola is written in standard form and p is positive and the directrix is a vertical line, then what can we conclude about its graph?
13. If the equation of a parabola is written in standard form and p is negative and the directrix is a horizontal line, then what can we conclude about its graph?
14. What is the effect on the graph of a parabola if its equation in standard form has increasing values of p?
15. As the graph of a parabola becomes wider, what will happen to the distance between the focus and directrix?

Homework

Algebraic

For the following exercises, determine whether the given equation is a parabola. If so, rewrite the equation in standard form.

16. $y^2 = 4 - x^2$
17. $y = 4x^2$
18. $3x^2 - 6y^2 = 12$
19. $(y-3)^2 = 8(x-2)$
20. $y^2 + 12x - 6y - 51 = 0$

For the following exercises, rewrite the given equation in standard form, and then determine the vertex (V), focus (F), and directrix (d) of the parabola.

21. $x = 8y^2$
22. $y = \frac{1}{4}x^2$
23. $y = -4x^2$
24. $x = \frac{1}{8}y^2$
25. $x = 36y^2$
26. $x = \frac{1}{36}y^2$
27. $(x-1)^2 = 4(y-1)$
28. $(y-2)^2 = \frac{4}{5}(x+4)$
29. $(y-4)^2 = 2(x+3)$
30. $(x+1)^2 = 2(y+4)$
31. $(x+4)^2 = 24(y+1)$
32. $(y+4)^2 = 16(x+4)$

33. $y^2 + 12x - 6y + 21 = 0$
34. $x^2 - 4x - 24y + 28 = 0$
35. $5x^2 - 50x - 4y + 113 = 0$
36. $y^2 - 24x + 4y - 68 = 0$
37. $x^2 - 4x + 2y - 6 = 0$
38. $y^2 - 6y + 12x - 3 = 0$
39. $3y^2 - 4x - 6y + 23 = 0$
40. $x^2 + 4x + 8y - 4 = 0$

Graphical

For the following exercises, graph the parabola, labeling the focus and the directrix.

41. $x = \frac{1}{8}18y^2$
42. $y = 36x^2$
43. $y = \frac{1}{36}x^2$
44. $y = -9x^2$
45. $(y-2)^2 = -\frac{4}{3}(x+2)$
46. $-5(x+5)^2 = 4(y+5)$
47. $-6(y+5)^2 = 4(x-4)$
48. $y^2 - 6y - 8x + 1 = 0$
49. $x^2 + 8x + 4y + 20 = 0$
50. $3x^2 + 30x - 4y + 95 = 0$
51. $y^2 - 8x + 10y + 9 = 0$
52. $x^2 + 4x + 2y + 2 = 0$
53. $y^2 + 2y - 12x + 61 = 0$
54. $-2x^2 + 8x - 4y - 24 = 0$

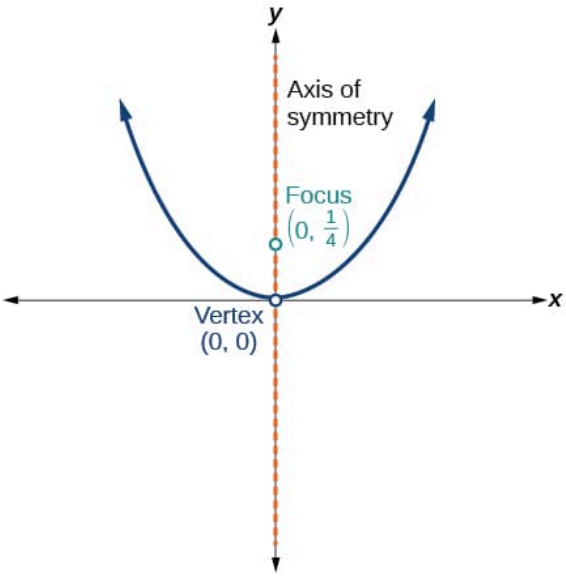
For the following exercises, find the equation of the parabola given information about its graph.

55. Vertex is $(0, 0)$; directrix is $y = 4$, focus is $(0, -4)$.
56. Vertex is $(0, 0)$; directrix is $x = 4$, focus is $(-4, 0)$.
57. Vertex is $(2, 2)$; directrix is $x = 2 - \sqrt{2}$, focus is $(2 + \sqrt{2}, 2)$.
58. Vertex is $(-2, 3)$; directrix is $x = -\frac{7}{2}$, focus is $\left(-\frac{1}{2}, 3\right)$.
59. Vertex is $(\sqrt{2}, -\sqrt{3})$; directrix is $x = 2\sqrt{2}$, focus is $(0, -\sqrt{3})$.
60. Vertex is $(1, 2)$; directrix is $y = \frac{11}{3}$, focus is $\left(1, \frac{1}{3}\right)$.

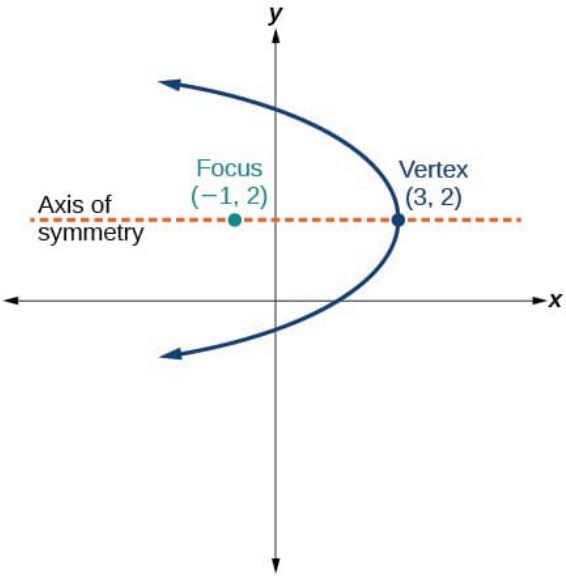
For the following exercises, determine the equation for the parabola from its graph.

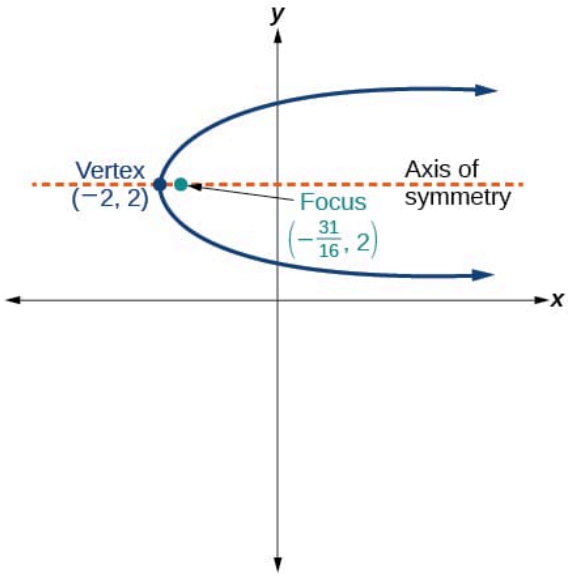
61.

52.

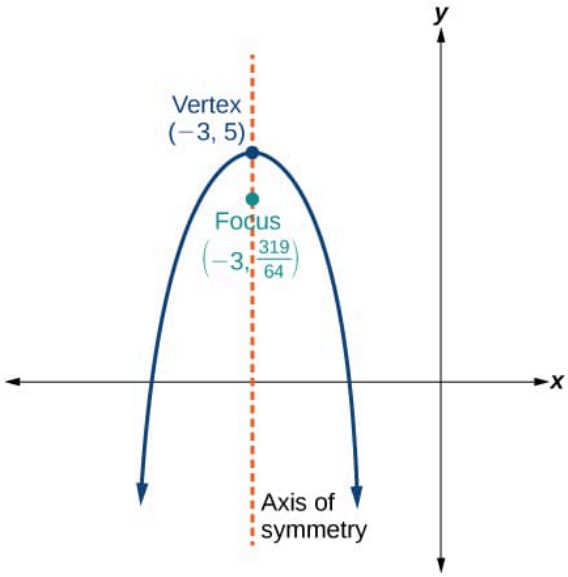

y
Axis of
symmetry
Focus
$\left(0, \frac{1}{4}\right)$
x
Vertex
(0, 0)

53.

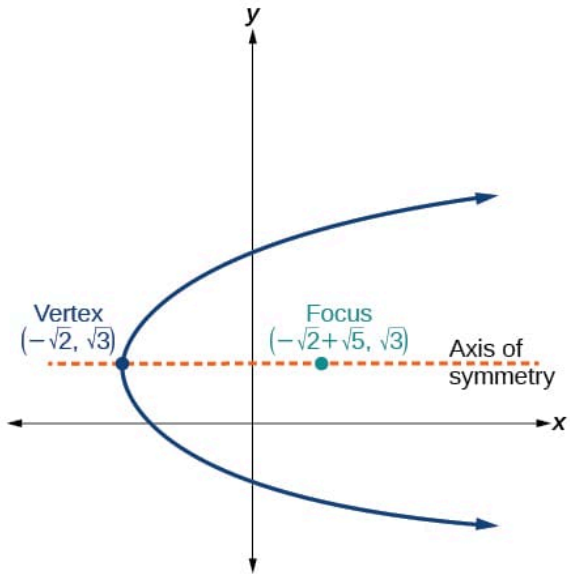

y
Focus
(−1, 2)
Vertex
(3, 2)
Axis of
symmetry
x


y
Vertex
(−2, 2)
Axis of
symmetry
Focus
$\left(-\frac{31}{16}, 2\right)$
x

64.


y
Vertex
(−3, 5)
Focus
$\left(-3, \frac{319}{64}\right)$
x
Axis of
symmetry

65.



Extensions

For the following exercises, the vertex and endpoints of the latus rectum of a parabola are given. Find the equation.

56. $V(0,0)$, Endpoints $(2,1)$, $(-2,1)$
57. $V(0,0)$, Endpoints $(-2,4)$, $(-2,-4)$
58. $V(1,2)$, Endpoints $(-5,5)$, $(7,5)$
59. $V(-3,-1)$, Endpoints $(0,5)$, $(0,-7)$
70. $V(4,-3)$, Endpoints $\left(5,-\frac{7}{2}\right)$, $\left(3,-\frac{7}{2}\right)$

Real-World Applications

71. The mirror in an automobile headlight has a parabolic cross-section with the light bulb at the focus. On a schematic, the equation of the parabola is given as $x^2 = 4y$. At what coordinates should you place the light bulb?
72. If we want to construct the mirror from the previous exercise such that the focus is located at $(0, 0.25)$, what should the equation of the parabola be?
73. A satellite dish is shaped like a paraboloid of revolution. This means that it can be formed by rotating a parabola around its axis of symmetry. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?
74. Consider the satellite dish from the previous exercise. If the dish is 8 feet across at the opening and 2 feet deep, where should we place the receiver?
75. The reflector in a searchlight is shaped like a paraboloid of revolution. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.
76. If the reflector in the searchlight from the previous exercise has the light source located 6 inches from the base along the axis of symmetry and the opening is 4 feet, find the depth.
77. An arch is in the shape of a parabola. It has a span of 100 feet and a maximum height of 20 feet. Find the equation of the parabola, and determine the height of the arch 40 feet from the center.
78. If the arch from the previous exercise has a span of 160 feet and a maximum height of 40 feet, find the equation of the parabola, and determine the distance from the center at which the height is 20 feet.
79. An object is projected so as to follow a parabolic path given by $y = -x^2 + 96x$, where x is the horizontal distance traveled in feet and y is the height. Determine the maximum height the object reaches.
80. For the object from the previous exercise, assume the path followed is given by $y = -0.5x^2 + 80x$. Determine how far along the horizontal the object traveled to reach maximum height.

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7: SYSTEMS OF EQUATIONS AND INEQUALITIES

Enigma machines like this one, once owned by Italian dictator Benito Mussolini, were used by government and military officials for enciphering and deciphering top-secret communications during World War II. (credit: Dave Addey, Flickr)



At the start of the Second World War, British military and intelligence officers recognized that defeating Nazi Germany would require the Allies to know what the enemy was planning. This task was complicated by the fact that the German military transmitted all of its communications through a presumably uncrackable code created by a machine called Enigma. The Germans had been encoding their messages with this machine since the early 1930s. Not long after the war started, the British recruited a team of brilliant codebreakers to crack the Enigma code. The codebreakers used what they knew about the Enigma machine to build a mechanical computer that could crack the code. The Germans were so confident that the code could not be cracked, that they felt comfortable transmitting all manner of battlefield intelligence encoded with the machine. But the Allies had cracked it. And that knowledge of what the Germans were planning proved to be a key part of the ultimate Allied victory of Nazi Germany in 1945.

The Enigma is perhaps the most famous cryptographic device ever known. It stands as an example of the pivotal role cryptography has played in society. Now, technology has moved cryptanalysis to the digital world.

Many ciphers are designed using invertible matrices as the method of message transference, as finding the inverse of a matrix is generally part of the process of decoding. In addition to knowing the matrix and its inverse, the receiver must also know the key that, when used with the matrix inverse, will allow the message to be read.

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions. We will not be breaking any secret codes here, but we will lay the foundation for future courses.

7.1: Systems of Linear Equations - Two Variables

7.1.1: Resources and Key Concepts

7.1.2: Homework

7.2: Systems of Linear Equations - Three Variables

7.2.1: Resources and Key Concepts

7.2.2: Homework

7.3: Systems of Nonlinear Equations and Inequalities - Two Variables

7.3.1: Resources and Key Concepts

7.3.2: Homework

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7.1: Systems of Linear Equations - Two Variables

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure 7.1.1**:** *(credit: Thomas Sørenes)*

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. This is where **systems of equations** come into play.

Definition: System of Equations

A **system of equations** , sometimes simply called a **system**, consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.

It would not be an overstatement to say that systems of equations are one of the most important mathematical structures in modern society. From cell phone operation to web search to generative artificial intelligence, the ability to work with and solve systems of equations is a fundamental necessity in technology.

The simplest of systems involves only linear equations.

Definition: System of Linear Equations

A **system of linear equations** (or **linear system**) consists of two or more linear equations involving the same variables such that all equations in the linear system are considered simultaneously.

For the remainder of this section (and the next), we will focus on systems of linear equations.

Since a system involves two or more variables, we can think of the values of the variables as dials on a safe. We are looking for the very special combination of those dials that makes all equations within the system true simultaneously.

Definition: Solution and Solution Set of a System

A combination of values for the variables that causes each equation in a system to simultaneously become true is called a **solution** to the system. The **solution set** is the set of all solutions of the system.

For example, we might have a system of three equations, each containing the variables x, y, and z. If the combination $x = 0$, $y = -1$, and $z = \frac{1}{2}$ makes all three equations in the system true, we say that the solution to the system is $x = 0$, $y = -1$, and $z = \frac{1}{2}$; however, we have a much more succinct way of writing a solution to a system.

Definition: n-tuple

An n**-tuple** is an ordered collection of n elements of the form

$$(x_1, x_2, x_3 \ldots, x_n),$$

where n is a non-negative integer. It's a generalization of ordered pairs ($n = 2$), triples ($n = 3$), etc.

Therefore, in our case mentioned above, we could say the solution to the system is $\left(0, -1, \frac{1}{2}\right)$ (where we assume the entries in the n-tuple are written in alphabetical order).

Caution: Ambiguous Notation

When dealing with ordered pairs, an ambiguity appears in our notation. For example, without context, $(1, 5)$ could mean the ordered pair (the point on the plane) where $x = 1$ and $y = 5$, or it could mean the interval $1 < x < 5$.

You need to use the context of the given situation to determine which interpretation should be used.

As we shall see shortly, some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables.

Definition: Systems of Two Linear Equations in Two Variables

A **system of two linear equations in two variables** x and y is a pair of equations that can be written in the standard form

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are real constants, and in each equation the coefficients of x and y are not both zero.

For example, consider the following system of linear equations in two variables.

$$\begin{cases} 2x + y = 15 \\ 3x - y = 5 \end{cases}$$

The ordered pair $(4, 7)$ is the solution to this system (for now, don't worry how we arrived at that). We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. We will investigate methods of finding such a solution, if it exists.

$$\begin{cases} 2(4) + (7) = 15 & \text{True} \\ 3(4) - (7) = 5 & \text{True} \end{cases}$$

In addition to considering the number of equations and variables, we can categorize systems (not just linear systems, but systems in general) by the number of solutions.

Systems can be defined in terms of whether or not they have solutions, and they can be further categorized based upon the number of solutions (finite or infinite).

Definition: Types of Systems

A system having one or more solutions is called a **consistent system**. A system having no solutions is called an **inconsistent system**.

Consistent systems are further categorized into **independent systems**, which are consistent systems with a finite number of solutions, and **dependent systems**, which are consistent systems with an infinite number of solutions.

Table 7.1.1 summarizes these possibilities.

Table 7.1.1***:*** *Types of Systems*

Consistent Systems		Inconsistent Systems
Independent Systems	**Dependent Systems**	
Finite Solutions	Infinite Solutions	No Solutions

Consider a system of linear equations involving two equations with two unknowns. Recall that the graph of an equation is a visual representation of the solution set for that specific equation. Thus, when we graph a system of two linear equations, we are seeing the solution sets to each component equation separately, and we are trying to find out if these solution sets ever overlap. Figure 7.1.2 shows us the possibilities when considering a system of two linear equations.

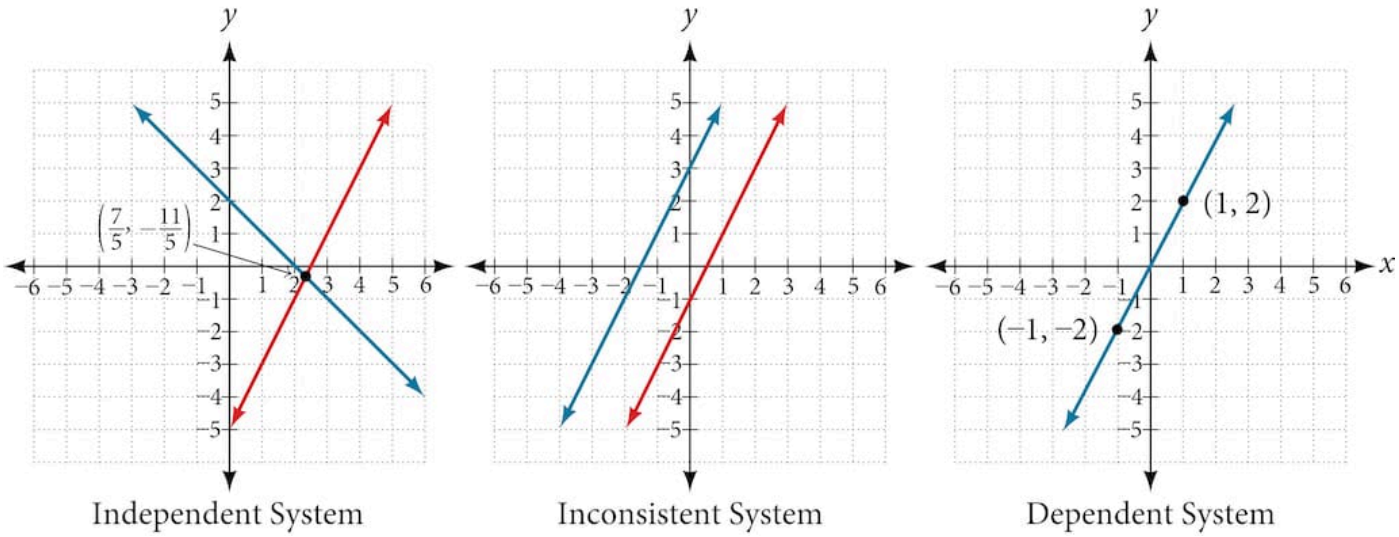


Figure 7.1.2

The independent and dependent systems are consistent because they have at least one solution. For the independent system, the two lines have different slopes and intersect at one point in the plane. For the dependent system, the equations have the same slope and the same y-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions. Finally, the inconsistent system is one in which the equations represent two *different* parallel lines. The lines have the same slope and different y-intercepts. There are no points common to both lines; hence, there is no solution to the system.

This should convince us of the following theorem.

Theorem: Classification of Two Linear Equations in Two Variables

Given a system of two linear equations in two variables, each equation graphs as a line, and exactly one of the following holds.

a. The two lines intersect at exactly one point. The system has exactly one solution and is consistent and independent.
b. The two lines are parallel and distinct. The system has no solution and is inconsistent.
c. The two lines coincide. The system has infinitely many solutions and is consistent and dependent.

Example 7.1.1: Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

$$\begin{cases} x + 3y = 8 \\ 2x - 9 = y \end{cases}$$

Solution available at the end of this section.

Checkpoint 7.1.1

Determine whether the ordered pair $(8, 5)$ is a solution to the following system.

$$\begin{cases} 5x - 4y = 20 \\ 2x + 1 = 3y \end{cases}$$

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

Example 7.1.2: Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$$\begin{cases} 2x + y = -8 \\ x - y = -1 \end{cases}$$

Solution available at the end of this section.

Checkpoint 7.1.2

Solve the following system of equations by graphing.

$$\begin{cases} 2x - 5y = -25 \\ -4x + 5y = 35 \end{cases}$$

Q&A

Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a system of linear equations that are more precise than graphing. One such method is solving a system of equations by the **Substitution Method**.

Definition: The Substitution Method (for solving a system of equations)

The **Substitution Method** is a heuristic process in which we solve one equation in a system for one variable and then substitute that result into another equation in the system. This process is then repeated until the value of a single variable can be determined (thereby, through a series of previous relations, revealing the values of the remaining variables in the system).

The word "heuristic" used within this definition is purposeful. A process is heuristic if it requires the user to investigate using trial and error. In this regard, solving systems using the Substitution Method is not ideal—especially for larger systems (systems involving more than two equations and two variables). However, we can learn much from the Substitution Method, and its use cannot be avoided.

How To... Solve a 2x2 System Using Substitution

1. Solve one of the equations for one of its variables. Choose the variable that is easiest to isolate—typically one whose coefficient is 1 or -1.

2. Substitute the resulting expression into the *other* equation. This produces a single equation in one variable.
3. Solve that equation. If it reduces to a false statement (a contradiction), the system has no solution. If it reduces to a true statement (an identity), the system has infinitely many solutions.
4. Back-substitute the value found into either original equation, or into the isolated expression from Step 1, to find the remaining variable.
5. Write the solution as an ordered pair (x, y) and verify it in both original equations.

Before we continue, I should clarify what I mean by "2x2." A system of m equations involving n unknowns is called an $m \times n$ **system**. Therefore, a system of 2 equations with 2 unknowns is also called a 2×2 **system**.

Example 7.1.3: Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$\begin{cases} -x + y = -5 \\ 2x - 5y = 1 \end{cases}$$

Solution available at the end of this section.

Checkpoint 7.1.3

Solve the following system of equations by substitution.

$$\begin{cases} x = y + 3 \\ 4 = 3x - 2y \end{cases}$$

Q&A

Can the Substitution Method be used to solve any linear system in two variables?

Yes, but the method works best if one of the equations contains a coefficient of 1 or -1 so that we do not have to deal with fractions.

Solving Systems of Equations in Two Variables by the Elimination Method

A third method of solving systems of linear equations is the **Elimination Method**. Before introducing this method, we need to define what it means for two systems to be **equivalent**.

Definition: Equivalent Systems

Two systems of linear equations in the same variables are **equivalent** if they have the same solution set.

How is it that two systems can have the same solution set? Well, we have seen such beasts throughout our mathematical careers already! Consider the two equations (not a system, but two separate equations) $3x = 6$ and $2x - 1 = 3$. These equations are called equivalent because they share the same solution set (namely, $x = 2$). In a similar fashion, the systems

$$\begin{cases} x + y = 2 \\ -x + 4y = 3 \end{cases} \quad \text{and} \quad \begin{cases} x + 7y = 8 \\ 2x + 5y = 7 \end{cases}$$

are equivalent because they both have the same solution set (namely, $(1, 1)$).

Now that we have defined what it means for two systems to be equivalent, we can consider the Elimination Method.

Definition: The Elimination Method

The **Elimination Method** (also called the **addition method**) is an algorithm to solve a system by replacing it with an equivalent system in which the coefficients of one variable are additive inverses. This, in turn, allows adding the two equations to remove that variable and produce a single equation in one less variable.

As with the Substitution Method, our choice of wording here is purposeful. Specifically, the use of the word "algorithm." An algorithm is a prescribed set of rules to achieve a specific goal. As such, the Elimination Method tends to be the better method to solve systems in the long run.

The theory behind the use of the Elimination Method is the initial focus of a course called **Linear Algebra**. For our purposes, it's best to reduce the complexity and say that we are allowed three options when creating equivalent systems using the Elimination Method—row interchanges, row scaling, and row replacement.

Theorem: Operations That Produce an Equivalent System

Each of the following operations transforms a system of linear equations into an equivalent system:

a. interchanging two equations;
b. multiplying both sides of an equation by a nonzero constant;
c. replacing one equation by the sum of itself and a constant multiple of another equation.

To motivate why these operations don't change the solution set of a system, let's look at a trivial example—specifically, the system

$$\begin{cases} 3x &= 6 \\ x &= 2 \end{cases}.$$

We can all agree that the solution of this system is $x = 2$ (it's the only value of x that simultaneously solves both equations). If we interchange the two rows, our new system becomes

$$\begin{cases} x &= 2 \\ 3x &= 6 \end{cases}$$

and we can (again) easily see that the solution is $x = 2$. Since the solution set didn't change, these two systems are equivalent. That should convince you that interchanging (or swapping) two equations in a system doesn't change the solution to the system (we often say the solutions to the system are **invariant** under the operation of row interchanges).

What if we decided to multiply both sides of the equation $x = 2$ in the original system by -10? Then the system would become

$$\begin{cases} 3x &= 6 \\ -10x &= -20 \end{cases}.$$

Again, it is easily seen that the solution to this system is $x = 2$. Hence, the original system and this new one are equivalent. In general, this will always be true (as long as we are multiplying both sides of one of the equations in the system by a *nonzero* constant).

Finally, what if we decided to replace the top equation in

$$\begin{cases} 3x &= 6 \\ x &= 2 \end{cases}$$

with the sum of itself and 4 times the second equation? That is, what if we replaced the top equation with $(3x + 4x) = (6 + 4 \cdot 2)$. The new system would be

$$\begin{cases} 7x &= 14 \\ x &= 2 \end{cases}.$$

Again, notice that the only solution to this system is $x = 2$, and we have yet another equivalent system.

Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

How To... Solve a 2x2 System Using Elimination

1. Write both equations with x- and y-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Example 7.1.4: Solving a System by the Elimination Method

Solve the given system of equations using the Elimination Method.

$$\begin{cases} x & + & 2y & = & -1 \\ -x & + & y & = & 3 \end{cases}$$

Solution available at the end of this section.

Example 7.1.5: Using the Elimination Method When Multiplication of One Equation Is Required

Solve the given system of equations by the Elimination Method.

$$\begin{cases} 3x & + & 5y & = & -11 \\ x & - & 2y & = & 11 \end{cases}$$

Solution available at the end of this section.

Checkpoint 7.1.5

Solve the system of equations using the Elimination Method.

$$\begin{cases} 2x & - & 7y & = & 2 \\ 3x & + & y & = & -20 \end{cases}$$

Example 7.1.6: Using the Elimination Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables using the Elimination Method.

$$\begin{cases} 2x & + & 3y & = & -16 \\ 5x & - & 10y & = & 30 \end{cases}$$

Solution available at the end of this section.

Example 7.1.7: Using the Elimination Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables using the Elimination Method.

$$\begin{cases} \frac{x}{3} & + & \frac{y}{6} & = & 3 \\ \frac{x}{2} & - & \frac{y}{4} & = & 1 \end{cases}$$

Solution available at the end of this section.

Checkpoint 7.1.7

Solve the system of equations using the Elimination Method.

$$\begin{cases} 2x + 3y = 8 \\ 3x + 5y = 10 \end{cases}$$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12 = 0$.

Example 7.1.8: Solving an Inconsistent System of Equations

Solve the following system of equations.

$$\begin{cases} x = 9 - 2y \\ x + 2y = 13 \end{cases}$$

Solution available at the end of this section.

Checkpoint 7.1.8

Solve the following system of equations in two variables.

$$\begin{cases} 2y - 2x = 2 \\ 2y - 2x = 6 \end{cases}$$

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using Substitution or Elimination, the resulting equation will be an identity, such as $0 = 0$.

Example 7.1.9: Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the Elimination Method.

$$\begin{cases} x + 3y = 2 \\ 3x + 9y = 6 \end{cases}$$

Solution available at the end of this section.

Checkpoint 7.1.9

Solve the following system of equations in two variables.

$$\begin{cases} y - 2x = 5 \\ -3y + 6x = -15 \end{cases}$$

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's **revenue function** is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where $x = \text{quantity}$ and $p = \text{price}$. The revenue function is shown in orange in Figure 7.1.10.

The **cost function** is the function used to calculate the costs of doing business. It includes **fixed costs**, such as rent and salaries, and **variable costs**, such as utilities. The cost function is shown in blue in Figure 7.1.10. The x-axis represents quantity in hundreds of units. The y-axis represents either cost or revenue in hundreds of dollars.

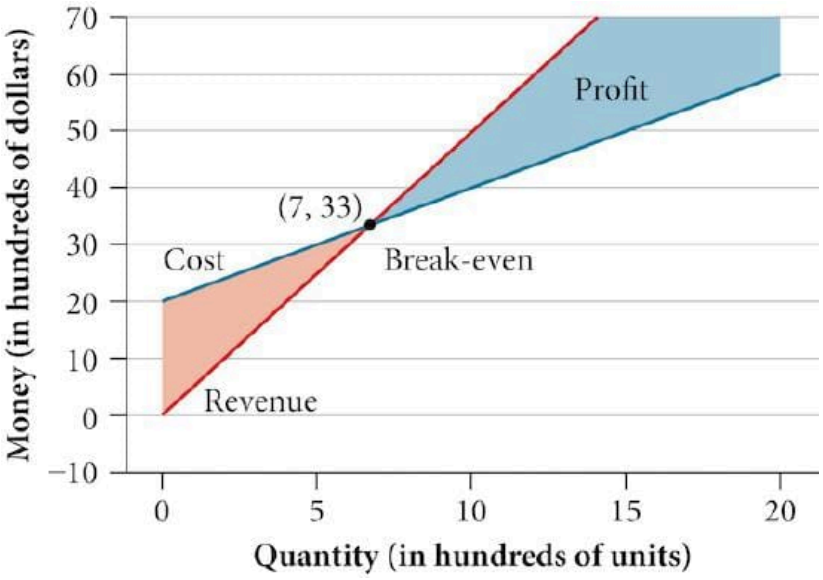


Figure 7.1.10

The point at which the two lines intersect is called the **break-even point**. We can see from the graph that if 700 units are produced, the cost is $3,300 and the revenue is also $3,300. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The **profit function** is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

Example 7.1.10: Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35,000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

Solution available at the end of this section.

Example 7.1.11: Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $25.00 for children and $50.00 for adults. On a certain day, attendance at the circus is 2,000 and the total gate revenue is $70,000. How many children and how many adults bought tickets?

Solution available at the end of this section.

Checkpoint 7.1.11

Meal tickets at the circus cost $4.00 for children and $12.00 for adults. If 1,650 meal tickets were bought for a total of $14,200, how many children and how many adults bought meal tickets?

Success in College: Learning From Tests (Part 6)

1. How complex are mathematics tests for you compared to exams in other subjects?
2. How hard do you try on the mathematics test compared to your efforts in other subjects?

Success in College: Preparing for the Final Exam (Part 2)

1. Think about the concepts and skills you have learned this semester. Describe two concepts or skills that were most difficult for you.
2. Describe what you are doing to ensure you can answer the questions about these concepts or skills on the final exam.

Answers

Section Preview

This entire section should be review for the students.

Section Preview

- **Linear Functions**
  - Parallel and Perpendicular Lines: Identifying an inconsistent system depends on recognizing that two lines with equal slopes but different y-intercepts are parallel and never intersect (Example 8; Checkpoint 8).

Section Preview

- **Solving Equations**
  - Solving Linear Equations — Multistep with Variables on One Side: After substituting or eliminating, students solve the resulting one-variable equation by distributing and combining like terms (Examples 3 and 11).
- **Linear Functions**
  - Graphing Linear Equations: The graphing method (Example 2; Checkpoint 2) requires actually plotting each line from its slope-intercept form on a shared set of axes.

Section Preview

- Solve systems of equations by graphing.
- Solve systems of equations by Substitution.
- Solve systems of equations by Elimination.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

Example 7.1.1: Determining Whether an Ordered Pair Is a Solution to a System of Equations

Substitute the ordered pair $(5, 1)$ into both equations.

$$\begin{aligned} (5)+3(1) &= 8 \\ \Longrightarrow \quad 8 &= 8 \qquad \text{True} \\ 2(5)-9 &= (1) \\ \Longrightarrow \quad 1 &= 1 \qquad \text{True} \end{aligned}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure 7.1.3.

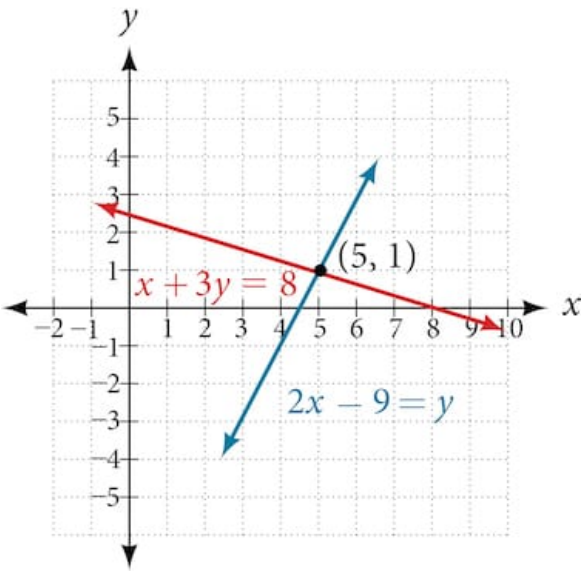


Figure 7.1.3

Example 7.1.2**: Solving a System of Equations in Two Variables by Graphing**

Solve the first equation for y.

$$\begin{aligned} 2x+y &= -8 \\ \Rightarrow \quad y &= -2x-8 \end{aligned}$$

Solve the second equation for y.

$$\begin{aligned} x-y &= -1 \\ \Rightarrow \quad y &= x+1 \end{aligned}$$

Graph both equations on the same set of axes as in Figure 7.1.4.

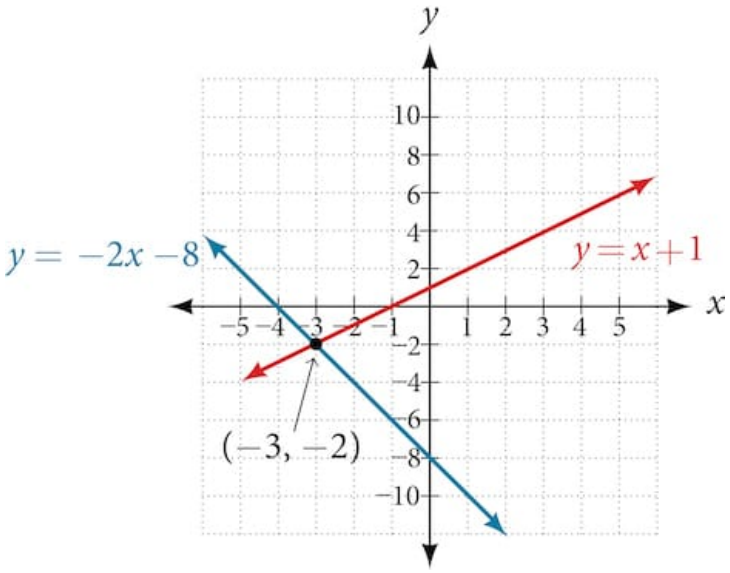


Figure 7.1.4

The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned} 2(-3)+(-2) &= -8 \\ \Rightarrow \quad -8 &= -8 \qquad \text{True} \\ (-3)-(-2) &= -1 \\ \Rightarrow \quad -1 &= -1 \qquad \text{True} \end{aligned}$$

The solution to the system is the ordered pair $(-3,-2)$, so the system is independent.

Example 7.1.3**: Solving a System of Equations in Two Variables by Substitution**

We solve the first equation for y.

$$\begin{aligned} -x+y &= -5 \\ \Longrightarrow \quad y &= x-5 \end{aligned}$$

Now we can substitute the expression $x-5$ for y in the second equation.

$$\begin{aligned} 2x-5y &= 1 \\ \Longrightarrow \quad 2x-5(x-5) &= 1 && \text{(substituting)} \\ \Longrightarrow \quad 2x-5x+25 &= 1 && \text{(distributing)} \\ \Longrightarrow \quad -3x+25 &= 1 && \text{(combining like terms)} \\ \Longrightarrow \quad -3x &= -24 && \text{(subtracting 25 from both sides)} \\ \Longrightarrow \quad x &= 8 && \text{(dividing both sides by } -3\text{)} \end{aligned}$$

Now, we substitute $x=8$ into the first equation and solve for y.

$$\begin{aligned} -x+y &= -5 \\ \Longrightarrow \quad -(8)+y &= -5 && \text{(substituting)} \\ \Longrightarrow \quad y &= 3 && \text{(adding 8 to both sides)} \end{aligned}$$

Our solution is $(8,3)$.

Check the solution by substituting $(8,3)$ into both equations.

$$\begin{aligned} -x+y &= -5 \\ \Longrightarrow \quad -(8)+(3) &= -5 && \text{True} \\ 2x-5y &= 1 \\ \Longrightarrow \quad 2(8)-5(3) &= 1 && \text{True} \end{aligned}$$

Example 7.1.4**: Solving a System by the Elimination Method**

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, -1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{aligned} x \quad &+ \quad 2y &&= -1 \\ + \quad -x \quad &+ \quad y &&= 3 \\ &\qquad 3y &&= 2 \end{aligned}$$

Now that we have eliminated x, we can solve the resulting equation for y.

$$3y=2 \implies y=\frac{2}{3}$$

Then, we substitute this value for y into one of the original equations and solve for x.

$$\begin{aligned} -x+y &= 3 \\ \Longrightarrow \quad -x+\left(\frac{2}{3}\right) &= 3 && \text{(substituting)} \\ \Longrightarrow \quad -x &= \frac{7}{3} && \left(\text{subtracting } \tfrac{7}{3} \text{ from both sides}\right) \\ \Longrightarrow \quad x &= -\frac{7}{3} && \text{(multiplying both sides by } -1\text{)} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3},\frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned} x+2y &= -1 \\ \Longrightarrow \quad \left(-\frac{7}{3}\right)+2\left(\frac{2}{3}\right) &= -1 && \text{(substituting)} \\ \Longrightarrow \quad -7+2\,(2) &= -3 && \text{(multiplying both sides by 3)} \\ \Longrightarrow \quad -7+4 &= -3 && \text{(simplifying)} \\ \Longrightarrow \quad -3 &= -3 && \text{(combining like terms)} \end{aligned}$$

Thus, the solution works in the first equation. We leave it to the reader to verify it satisfies the second equation.

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure 7.1.5 to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

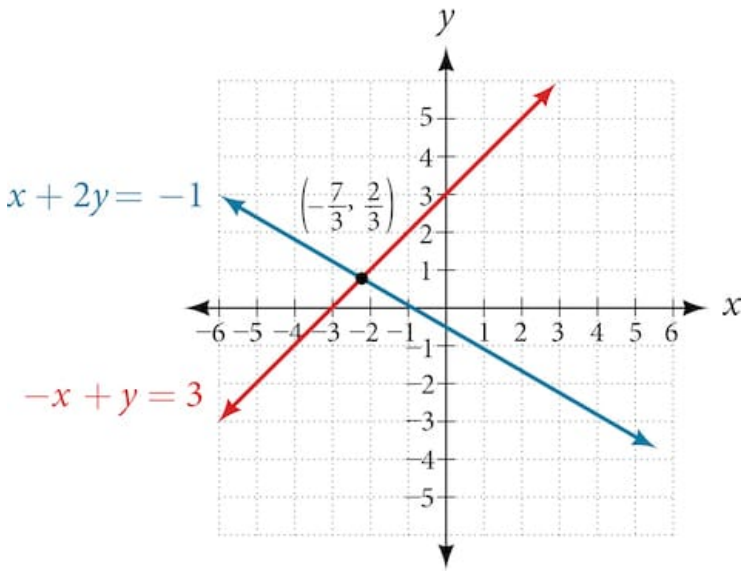


Figure 7.1.5

Example 7.1.5**: Using the Elimination Method When Multiplication of One Equation Is Required**

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by -3, the x-terms will add to zero.

$$\begin{aligned} & x - 2y = 11 \\ \Rightarrow\ & -3(x - 2y) = -3(11) \\ \Rightarrow\ & -3x + 6y = -33 \end{aligned}$$

Now, let's add them.

$$\begin{aligned} 3x + 5y &= -11 \\ +\ -3x + 6y &= -33 \\ 11y &= -44 \\ \Rightarrow\quad y &= -4 \end{aligned}$$

Substituting this into the second equation (in the original system), we get the following:

$$\begin{aligned} x - 2y &= 11 \\ \Rightarrow\ x - 2(-4) &= 11 \qquad \text{(substituting)} \\ \Rightarrow\ x + 8 &= 11 \qquad \text{(simplifying)} \\ \Rightarrow\ x &= 3 \qquad \text{(subtracting 8 from both sides)} \end{aligned}$$

Our solution is the ordered pair $(3, -4)$ (we leave it to the reader to algebraically check the solution). See Figure 7.1.6.

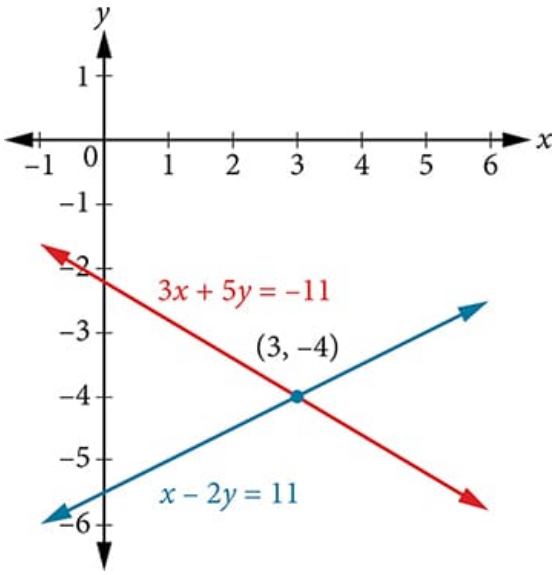


Figure 7.1.6

Example 7.1.6**: Using the Elimination Method When Multiplication of Both Equations Is Required**

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.

$$\begin{aligned} & 2x + 3y = -16 \\ \Longrightarrow\ & -5(2x + 3y) = -5(-16) \\ \Longrightarrow\ & -10x - 15y = 80 \\ & 5x - 10y = 30 \\ \Longrightarrow\ & 2(5x - 10y) = 2(30) \\ \Longrightarrow\ & 10x - 20y = 60 \end{aligned}$$

Then, we add the two equations together.

$$\begin{aligned} & -10x - 15y = 80 \\ + \ & 10x - 20y = 60 \\ & -35y = 140 \\ \Longrightarrow\ & y = -4 \end{aligned}$$

Substitute $y = -4$ into the original first equation.

$$\begin{aligned} & 2x + 3y = -16 && \\ \Longrightarrow\ & 2x + 3(-4) = -16 && \text{(substituting)} \\ \Longrightarrow\ & 2x - 12 = -16 && \text{(simplifying)} \\ \Longrightarrow\ & 2x = -4 && \text{(adding 12 to both sides)} \\ \Longrightarrow\ & x = -2 && \text{(dividing both sides by 2)} \end{aligned}$$

The solution is $(-2, -4)$ (again, we leave it to the reader to check this). See Figure 7.1.7.

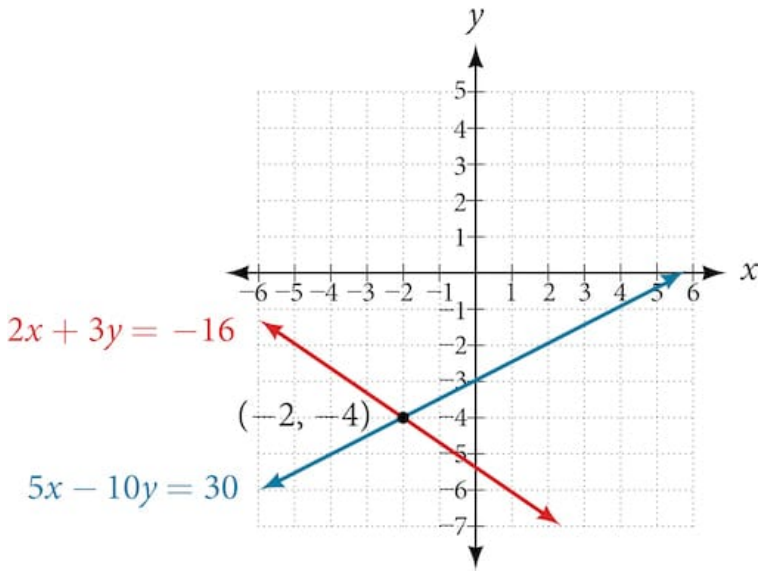


Figure 7.1.7

Example 7.1.7**: Using the Elimination Method in Systems of Equations Containing Fractions**

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$$\begin{aligned} \frac{x}{3} + \frac{y}{6} &= 3 \\ \Rightarrow \quad 6\left(\frac{x}{3} + \frac{y}{6}\right) &= 6(3) \\ \Rightarrow \quad 2x + y &= 18 \\ \frac{x}{2} - \frac{y}{4} &= 1 \\ \Rightarrow \quad 4\left(\frac{x}{2} - \frac{y}{4}\right) &= 4(1) \\ \Rightarrow \quad 2x - y &= 4 \end{aligned}$$

Add the two equations to eliminate the y-variable and solve the resulting equation.

$$\begin{aligned} 2x + y &= 18 \\ + \quad 2x - y &= 4 \\ 4x &= 22 \\ \Rightarrow \quad x &= \frac{11}{2} \end{aligned}$$

Substitute $x = \frac{11}{2}$ into the first equation.

$$\begin{aligned} 2x + y &= 18 \\ \Rightarrow \quad 2\left(\frac{11}{2}\right) + y &= 18 \qquad \text{(substituting)} \\ \Rightarrow \quad 11 + y &= 18 \qquad \text{(simplifying)} \\ \Rightarrow \quad y &= 7 \qquad \text{(subtracting 11 from both sides)} \end{aligned}$$

The solution is $\left(\frac{11}{2}, 7\right)$. We leave it to the reader to check it in the other equation.

Example 7.1.8**: Solving an Inconsistent System of Equations**

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$\begin{aligned} x + 2y &= 13 \\ \Rightarrow \quad (9 - 2y) + 2y &= 13 \qquad \text{(substituting)} \\ \Rightarrow \quad 9 &= 13 \qquad \text{(simplifying)} \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$\begin{aligned} x &= 9 - 2y \\ \Rightarrow \quad 2y &= -x + 9 \qquad \text{(subtracting } x \text{ from and adding } 2y \text{ to both sides)} \\ \Rightarrow \quad y &= -\frac{1}{2}x + \frac{9}{2} \qquad \text{(dividing both sides by 2)} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

$$\begin{aligned} x+2y &= 13 \\ \Longrightarrow \quad 2y &= -x+13 && \text{(subtracting } 2y \text{ from both sides)} \\ \Longrightarrow \quad y &= -\frac{1}{2}x+\frac{13}{2} && \text{(dividing both sides by 2)} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different y-intercepts. Therefore, the lines are parallel and do not intersect.

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines (with different y-intercepts) will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in Figure 7.1.8.

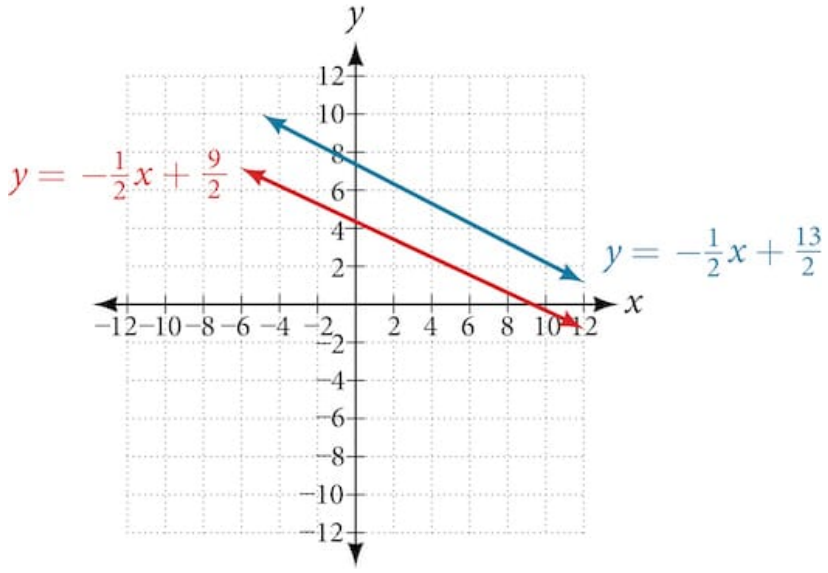


Figure 7.1.8

Example 7.1.9**: Finding a Solution to a Dependent System of Linear Equations**

With the Elimination Method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable.

$$\begin{aligned} x &+ 3y &&= 2 \\ \Longrightarrow \quad (-3)(x &+ 3y) &&= (-3)(2) \\ \Longrightarrow \quad -3x &- 9y &&= -6 \end{aligned}$$

Now add the equations.

$$\begin{aligned} -3x &- 9y &&= -6 \\ 3x &+ 9y &&= 6 \\ + \qquad &\quad 0 &&= 0 \end{aligned}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.

$$\begin{aligned} x+3y &= 2 \\ \Longrightarrow \quad 3y &= -x+2 \\ \Longrightarrow \quad y &= -\frac{1}{3}x+\frac{2}{3} \\ 3x+9y &= 6 \\ \Longrightarrow \quad 9y &= -3x+6 \\ \Longrightarrow \quad y &= -\frac{1}{3}x+\frac{2}{3} \end{aligned}$$

See Figure 7.1.9. Notice the results are the same. The general solution to the system is $\left(x, -\frac{1}{3}x+\frac{2}{3}\right)$.

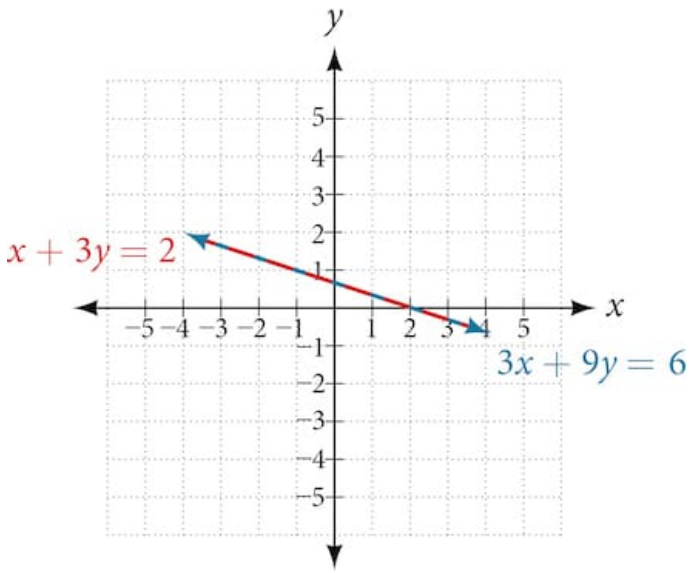


Figure 7.1.9

Example 7.1.10**: Finding the Break-Even Point and the Profit Function Using Substitution**

Write the system of equations using y to replace function notation.

$$\begin{cases} y &= 0.85x &+ &35000 \\ y &= 1.55x \end{cases}$$

Substitute the expression $0.85x + 35,000$ from the first equation into the second equation and solve for x.

$$\begin{aligned} 0.85x + 35000 &= 1.55x \\ \Longrightarrow \quad 35000 &= 0.7x \qquad \text{(subtracting } 0.85x \text{ from both sides)} \\ \Longrightarrow \quad 50000 &= x \qquad \text{(dividing both sides by } 0.7\text{)} \end{aligned}$$

Then, we substitute $x = 50,000$ into either the cost function or the revenue function.

$$1.55(50000) = 77500$$

The break-even point is $(50,000, 77,500)$.

The profit function is found using the formula $P(x) = R(x) - C(x)$.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 1.55x - (0.85x + 35000) \\ &= 0.7x - 35000 \end{aligned}$$

The profit function is $P(x) = 0.7x - 35,000$.

The cost to produce 50,000 units is $77,500, and the revenue from the sales of 50,000 units is also $77,500. To make a profit, the business must produce and sell more than 50,000 units. See Figure 7.1.11.

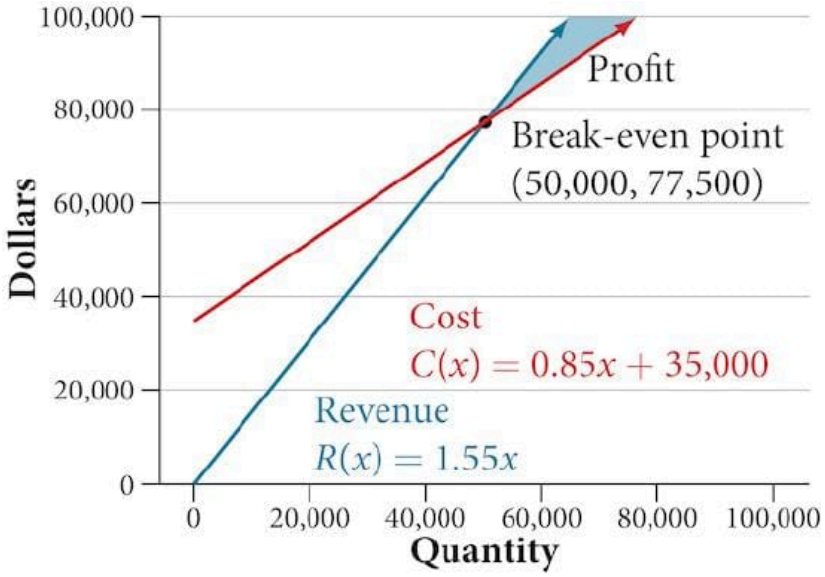


Figure 7.1.11

We see from the graph in Figure 7.1.12 that the profit function has a negative value until $x = 50,000$, when the graph crosses the x-axis. Then, the graph emerges into positive y-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.

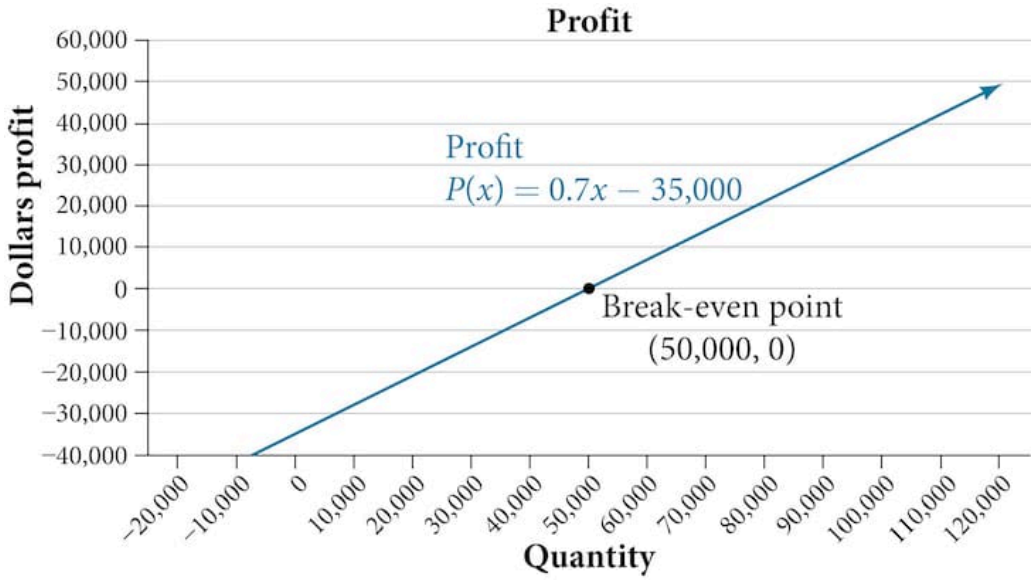


Figure 7.1.12

Example 7.1.11**: Writing and Solving a System of Equations in Two Variables**

Let c = the number of children and a = the number of adults in attendance.

The total number of people is $2,000$. We can use this to write an equation for the number of people at the circus that day.

$$c + a = 2,000$$

The revenue from all children can be found by multiplying $25.00 by the number of children, $25c$. The revenue from all adults can be found by multiplying $50.00 by the number of adults, $50a$. The total revenue is $70,000. We can use this to write an equation for the revenue.

$$25c + 50a = 70,000$$

We now have a system of linear equations in two variables.

$$\begin{cases} c & + & a & = & 2,000 \\ 25c & + & 50a & = & 70,000 \end{cases}$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.

$$c + a = 2,000 \implies a = 2,000 - c.$$

Substitute the expression $2,000 - c$ in the second equation for a and solve for c.

$$
\begin{aligned}
25c + 50a &= 70,000 \\
\implies 25c + 50(2,000 - c) &= 70,000 && \text{(substituting)} \\
\implies 25c + 100,000 - 50c &= 70,000 && \text{(distributing)} \\
\implies -25c + 100,000 &= 70,000 && \text{(combining like terms)} \\
\implies -25c &= -30,000 && \text{(subtracting } 100,000 \text{ from both sides)} \\
\implies c &= 1,200 && \text{(dividing both sides by } -25\text{)}
\end{aligned}
$$

Substitute $c = 1,200$ into the first equation to solve for a.

$$
\begin{aligned}
c + a &= 2,000 \\
\implies 1,200 + a &= 2,000 && \text{(substituting)} \\
\implies a &= 800 && \text{(subtracting } 1,200 \text{ from both sides)}
\end{aligned}
$$

We find that 1,200 children and 800 adults bought tickets to the circus that day.

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7.1.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Linear Functions**
  - Graphing Linear Functions: A direct prerequisite for solving systems of linear equations by graphing.
  - Linear Models: Required for working with applications of systems of linear equations.
  - Parallel and Perpendicular Lines: Understanding that parallel lines have the same slope is key to identifying inconsistent systems graphically.

Videos

- Systems of linear equations
  - Intermediate Algebra - An Introduction to Systems of Equations (Part A)
  - Intermediate Algebra - An Introduction to Systems of Equations (Part B)
  - Intermediate Algebra - An Introduction to Systems of Equations (Part C)
- Solving systems by substitution
  - Intermediate Algebra - Solving Small Systems of Linear Equations Algebraically (Part A)
- Solving systems by elimination
  - Intermediate Algebra - Solving Small Systems of Linear Equations Algebraically (Part B)

Key Concepts

Definitions

- **System of Equations**: Two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- **System of Linear Equations (Linear System)**: A system consisting of two or more linear equations.
- **Solution to a System**: A combination of values for the variables that causes each equation in the system to become a true statement. For a 2x2 system, this is an ordered pair (x, y).
- **n-tuple**: An ordered collection of n elements, used to represent solutions to systems with n variables.
- **Consistent System**: A system of equations that has one or more solutions.
- **Inconsistent System**: A system of equations that has no solutions. Graphically, this corresponds to parallel lines.
- **Independent System**: A consistent system with a finite number of solutions (for a 2x2 linear system, this means exactly one solution). Graphically, this corresponds to intersecting lines.
- **Dependent System**: A consistent system with an infinite number of solutions. Graphically, this corresponds to two identical lines (coincident lines).
- **Substitution Method**: A heuristic process for solving a system where one equation is solved for one variable, and that expression is then substituted into another equation in the system.
- **Elimination Method**: An algorithmic process for solving a system where two equations are added together to eliminate a variable. This often requires first multiplying one or both equations by a constant to ensure one variable has opposite coefficients.

Theorems

- **Number of Solutions of Linear Systems**: A consistent linear system has either exactly one solution or an infinite number of solutions. (This implies a linear system cannot have exactly two, three, etc., solutions).

Common Mistakes

- **Solving for Only One Variable**: After finding the value of the first variable (e.g., x), forgetting to substitute it back into an original equation to solve for the second variable.
- **Errors in Substitution**: When substituting an expression into another equation, making algebraic errors, especially with distribution if the expression is being multiplied by a coefficient.
- **Errors in Elimination**:
  - Incorrectly multiplying an entire equation by a constant (e.g., forgetting to multiply the constant term on the other side of the equals sign).
  - Making addition errors after multiplying, especially with negative numbers.
  - Multiplying by a constant that does not result in opposite coefficients for one of the variables.
- **Assuming a Solution Exists**: Concluding there is one solution when the system is actually inconsistent (parallel lines) or dependent (same line). Arriving at a false statement (e.g., $9 = 13$) indicates no solution. Arriving at a true identity (e.g., $0 = 0$) indicates infinite solutions.
- **Incorrectly Stating the Solution for a Dependent System**: For a dependent system with infinite solutions, simply writing "infinite solutions" is often insufficient. The solution should be expressed as an ordered pair in terms of one of the variables (e.g., $(x, -\frac{1}{3}x + \frac{2}{3})$).
- **Errors in Setting Up Systems from Word Problems**: Misinterpreting the problem and creating incorrect linear equations to represent the given information.

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7.1.2: Homework

Reading Questions

1. What is a system of linear equations?
2. Can a system of two linear equations in two variables have exactly two solutions? Why or why not?
3. Describe the three types of systems based on their number of solutions (independent, dependent, and inconsistent) and what their graphs look like.
4. Outline the steps of the Substitution Method for solving a 2x2 system.
5. Outline the steps of the Elimination Method for solving a 2x2 system.
6. If you are solving a system and arrive at a false statement like $9 = 13$, what can you conclude about the system?
7. If you are solving a system and arrive at an identity like $0 = 0$, what can you conclude about the system, and how should you express the solution?
8. In a business context, what is the "break-even point"? How is it found using a system of equations?
9. What is the profit function, $P(x)$, in relation to the cost function, $C(x)$, and the revenue function, $R(x)$?
10. List the three methods for solving a system of linear equations in two variables discussed in this section.
11. Can a system of linear equations have exactly two solutions? Explain why or why not.
12. If you are performing a break-even analysis for a business and their cost and revenue equations are dependent, explain what this means for the company's profit margins.
13. If you are solving a break-even analysis and get a negative break-even point, explain what this signifies for the company?
14. If you are solving a break-even analysis and there is no break-even point, explain what this means for the company. How should they ensure there is a break-even point?
15. Given a system of equations, explain at least two different methods of solving that system.

Homework

Algebraic

For the following exercises, determine whether the given ordered pair is a solution to the system of equations.

16. $\begin{cases} 5x - y = 4 \\ x + 6y = 2 \end{cases}$ and $(4, 0)$
17. $\begin{cases} -3x - 5y = 13 \\ -x + 4y = 10 \end{cases}$ and $(-6, 1)$
18. $\begin{cases} 3x + 7y = 1 \\ 2x + 4y = 0 \end{cases}$ and $(2, 3)$
19. $\begin{cases} -2x + 5y = 7 \\ 2x + 9y = 7 \end{cases}$ and $(-1, 1)$
20. $\begin{cases} x + 8y = 43 \\ 3x - 2y = -1 \end{cases}$ and $(3, 5)$

For the following exercises, solve each system by Substitution.

21. $\begin{cases} x + 3y = 5 \\ 2x + 3y = 4 \end{cases}$
22. $\begin{cases} 3x - 2y = 18 \\ 5x + 10y = -10 \end{cases}$
23. $\begin{cases} 4x + 2y = -10 \\ 3x + 9y = 0 \end{cases}$
24. $\begin{cases} 2x + 4y = -3.8 \\ 9x - 5y = 1.3 \end{cases}$
25. $\begin{cases} -2x + 3y = 1.2 \\ -3x - 6y = 1.8 \end{cases}$

26. $\begin{cases} x - 0.2y = 1 \\ -10x + 2y = 5 \end{cases}$

27. $\begin{cases} 3x + 5y = 9 \\ 30x + 50y = -90 \end{cases}$

28. $\begin{cases} -3x + y = 2 \\ 12x - 4y = -8 \end{cases}$

29. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 16 \\ \frac{1}{6}x + \frac{1}{4}y = 9 \end{cases}$

30. $\begin{cases} -\frac{1}{4}x + \frac{3}{2}y = 11 \\ -\frac{1}{8}x + \frac{1}{3}y = 3 \end{cases}$

For the following exercises, solve each system by Elimination.

31. $\begin{cases} -2x + 5y = -42 \\ 7x + 2y = 30 \end{cases}$

32. $\begin{cases} 6x - 5y = -34 \\ 2x + 6y = 4 \end{cases}$

33. $\begin{cases} 5x - y = -2.6 \\ -4x - 6y = 1.4 \end{cases}$

34. $\begin{cases} 7x - 2y = 3 \\ 4x + 5y = 3.25 \end{cases}$

35. $\begin{cases} -x + 2y = -1 \\ 5x - 10y = 6 \end{cases}$

36. $\begin{cases} 7x + 6y = 2 \\ -28x - 24y = -8 \end{cases}$

37. $\begin{cases} \frac{5}{6}x + \frac{1}{4}y = 0 \\ \frac{1}{8}x - \frac{1}{2}y = -\frac{43}{120} \end{cases}$

38. $\begin{cases} \frac{1}{3}x + \frac{1}{9}y = \frac{2}{9} \\ -\frac{1}{2}x + \frac{4}{5}y = -\frac{1}{3} \end{cases}$

39. $\begin{cases} -0.2x + 0.4y = 0.6 \\ x - 2y = -3 \end{cases}$

40. $\begin{cases} -0.1x + 0.2y = 0.6 \\ 5x - 10y = 1 \end{cases}$

For the following exercises, solve each system by any method.

41. $\begin{cases} 5x + 9y = 16 \\ x + 2y = 4 \end{cases}$

42. $\begin{cases} 6x - 8y = -0.6 \\ 3x + 2y = 0.9 \end{cases}$

43. $\begin{cases} 5x - 2y = 2.25 \\ 7x - 4y = 3 \end{cases}$

44. $\begin{cases} x - \frac{5}{12}y = -\frac{55}{12} \\ -6x + \frac{5}{2}y = \frac{55}{2} \end{cases}$

45. $\begin{cases} 7x - 4y = \frac{7}{6} \\ 2x + 4y = \frac{1}{3} \end{cases}$

46. $\begin{cases} 3x + 6y = 11 \\ 2x + 4y = 9 \end{cases}$

47. $\begin{cases} \frac{7}{3}x - \frac{1}{6}y = 2 \\ -\frac{21}{6}x + \frac{3}{12}y = -3 \end{cases}$

48. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = \frac{1}{3} \\ \frac{3}{2}x + \frac{1}{4}y = -\frac{1}{8} \end{cases}$

49. $\begin{cases} 2.2x + 1.3y = -0.1 \\ 4.2x + 4.2y = 2.1 \end{cases}$

50. $\begin{cases} 0.1x + 0.2y = 2 \\ 0.35x - 0.3y = 0 \end{cases}$

Graphical

For the following exercises, graph the system of equations and state whether the system is consistent, inconsistent, or dependent and whether the system has one solution, no solution, or infinite solutions.

51. $\begin{cases} 3x - y = 0.6 \\ x - 2y = 1.3 \end{cases}$

52. $\begin{cases} -x + 2y = 4 \\ 2x - 4y = 1 \end{cases}$

53. $\begin{cases} x + 2y = 7 \\ 2x + 6y = 12 \end{cases}$

54. $\begin{cases} 3x - 5y = 7 \\ x - 2y = 3 \end{cases}$

55. $\begin{cases} 3x - 2y = 5 \\ -9x + 6y = -15 \end{cases}$

Technology

For the following exercises, use the intersect function on a graphing device to solve each system. Round all answers to the nearest hundredth.

56. $\begin{cases} 0.1x + 0.2y = 0.3 \\ -0.3x + 0.5y = 1 \end{cases}$

57. $\begin{cases} -0.01x + 0.12y = 0.62 \\ 0.15x + 0.20y = 0.52 \end{cases}$

58. $\begin{cases} 0.5x + 0.3y = 4 \\ 0.25x - 0.9y = 0.46 \end{cases}$

59. $\begin{cases} 0.15x + 0.27y = 0.39 \\ -0.34x + 0.56y = 1.8 \end{cases}$

60. $\begin{cases} -0.71x + 0.92y = 0.13 \\ 0.83x + 0.05y = 2.1 \end{cases}$

Extensions

For the following exercises, solve each system in terms of A, B, C, D, E, and F where A–F are nonzero numbers. Note that $A \neq B$ and $A \cdot E \neq B \cdot D$.

61. $\begin{cases} x + y = A \\ x - y = B \end{cases}$

62. $\begin{cases} x + Ay = 1 \\ x + By = 1 \end{cases}$

63. $\begin{cases} Ax + y = 0 \\ Bx + y = 1 \end{cases}$

64. $\begin{cases} Ax + By = C \\ x + y = 1 \end{cases}$

65. $\begin{cases} Ax + By = C \\ Dx + Ey = F \end{cases}$

Real-World Applications

For the following exercises, solve for the desired quantity.

56. A stuffed animal business has a total cost of production $C = 12x + 30$ and a revenue function $R = 20x$. Find the break-even point.
57. A fast-food restaurant has a cost of production $C(x) = 11x + 120$ and a revenue function $R(x) = 5x$. When does the company start to turn a profit?
58. A cell phone factory has a cost of production $C(x) = 150x + 10,000$ and a revenue function $R(x) = 200x$. What is the break-even point?
59. A musician charges $C(x) = 64x + 20,000$ where x is the total number of attendees at the concert. The venue charges $80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?
70. A guitar factory has a cost of production $C(x) = 75x + 50,000$. If the company needs to break even after 150 units sold, at what price should they sell each guitar? Round up to the nearest dollar, and write the revenue function.

For the following exercises, use a system of linear equations with two variables and two equations to solve.

71. Find two numbers whose sum is 28 and difference is 13.
72. A number is 9 more than another number. Twice the sum of the two numbers is 10. Find the two numbers.
73. The startup cost for a restaurant is $120,000, and each meal costs $10 for the restaurant to make. If each meal is then sold for $15, after how many meals does the restaurant break even?
74. A moving company charges a flat rate of $150, and an additional $5 for each box. If a taxi service would charge $20 for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?
75. A total of 1,595 first- and second-year college students gathered at a pep rally. The number of freshmen exceeded the number of sophomores by 15. How many freshmen and sophomores were in attendance?
76. 276 students enrolled in a freshman-level chemistry class. By the end of the semester, 5 times the number of students passed as failed. Find the number of students who passed, and the number of students who failed.
77. There were 130 faculty at a conference. If there were 18 more women than men attending, how many of each gender attended the conference?
78. A jeep and BMW enter a highway running east-west at the same exit heading in opposite directions. The jeep entered the highway 30 minutes before the BMW did, and traveled 7 mph slower than the BMW. After 2 hours from the time the BMW entered the highway, the cars were 306.5 miles apart. Find the speed of each car, assuming they were driven on cruise control.
79. If a scientist mixed 10% saline solution with 60% saline solution to get 25 gallons of 40% saline solution, how many gallons of 10% and 60% solutions were mixed?
80. An investor earned triple the profits of what she earned last year. If she made $500,000.48 total for both years, how much did she earn in profits each year?
81. An investor who dabbles in real estate invested 1.1 million dollars into two land investments. On the first investment, Swan Peak, her return was a 110% increase on the money she invested. On the second investment, Riverside Community, she earned 50% over what she invested. If she earned $1 million in profits, how much did she invest in each of the land deals?
82. If an investor invests a total of $25,000 into two bonds, one that pays 3% simple interest, and the other that pays $2\frac{7}{8}$% interest, and the investor earns $737.50 annual interest, how much was invested in each account?
83. If an investor invests $23,000 into two bonds, one that pays 4% in simple interest, and the other paying 2% simple interest, and the investor earns $710.00 annual interest, how much was invested in each account?
84. CDs cost $5.96 more than DVDs at All Bets Are Off Electronics. How much would 6 CDs and 2 DVDs cost if 5 CDs and 2 DVDs cost $127.73?
85. A store clerk sold 60 pairs of sneakers. The high-tops sold for $98.99 and the low-tops sold for $129.99. If the receipts for the two types of sales totaled $6,404.40, how many of each type of sneaker were sold?
86. A concert manager counted 350 ticket receipts the day after a concert. The price for a student ticket was $12.50, and the price for an adult ticket was $16.00. The register confirms that $5,075 was taken in. How many student tickets and adult tickets were sold?
87. Admission into an amusement park for 4 children and 2 adults is $116.90. For 6 children and 3 adults, the admission is $175.35. Assuming a different price for children and adults, what is the price of the child's ticket and the price of the adult ticket?

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7.2: Systems of Linear Equations - Three Variables

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure 7.2.1**:** *(credit: "Elembis," Wikimedia Commons)*

John received an inheritance of $12,000 that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $4,000 more in mutual funds than in municipal bonds. He earned $670 in interest the first year. How much did John invest in each type of fund?

Understanding the correct approach to setting up problems such as the one above makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as **three-by-three systems**, the primary tool we will be using is called **Gaussian Elimination**, named after the prolific German mathematician Carl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve **upper triangular form**, the ideal form for a three-by-three system because it allows for straightforward **back-substitution** to find a solution (x, y, z), which we call an ordered triple.

Definition: Upper Triangular Form

A 3×3 system is said to be in **upper triangular form** if it is written as follows:

$$\begin{cases} Ax & + & By & + & Cz & = & D \\ & & Ey & + & Fz & = & G \\ & & & & Hz & = & K \end{cases}$$

The upper triangular form for an $n \times n$ systems is likewise defined to be a system in the form

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & c_1 \\ & & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & c_2 \\ & & & & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & c_3 \\ & & & & & \ddots & & & \vdots & & \vdots \\ & & & & & & \ddots & & \vdots & & \vdots \\ & & & & & & & & a_{nn}x_n & = & c_n \end{cases}$$

There are entire branches of Mathematics dedicated to developing faster methods for solving large systems; however, in this section, we limit ourselves to 3×3 systems. Therefore, let's focus on the upper triangular form for a 3×3 system.

The upper triangular form is useful because, in the 3×3 case, the third equation can be solved for z, and then we back-substitute to find y and x. Therefore, our task is to find a method to write a given system in upper triangular form. To achieve our goal, we introduce the following theorem.

Theorem: Invariant Solutions Under Row Operations

The solution to a system of linear equations in invariant (remains unchanged) under the following operations:

1. **Row Exchange:** Interchange the order of any two equations.
2. **Row Scaling:** Multiply both sides of an equation by a nonzero constant.
3. **Row Replacement:** Add a nonzero multiple of one equation to another equation.

As mentioned in the previous section, the solution set to a three-by-three system is an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

Theorem: Number of Solutions to 3x3 Linear Systems

A system of three linear equations in three variables has exactly one of the following outcomes, corresponding to how its three planes meet:

- **Exactly one solution.** The three planes meet at a single point, and the solution set is that point.
- **Infinitely many solutions.** The three planes share a common line, or they coincide as a single plane; the solution set is that line or plane.
- **No solution.** The three planes have no point in common, and the solution set is empty.

Figures 7.2.2 and 7.2.3 illustrate possible solution scenarios for three-by-three systems.

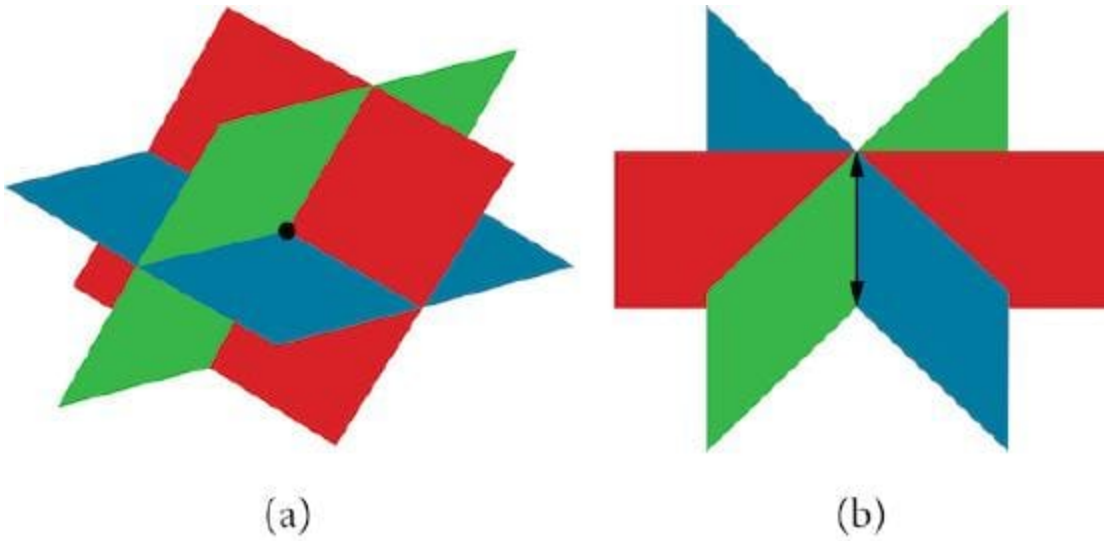


Figure 7.2.2**:** *(a): Three planes intersect at a single point, representing a three-by-three system with a single solution. (b): Three planes intersect in a line, representing a three-by-three system with infinite solutions.*

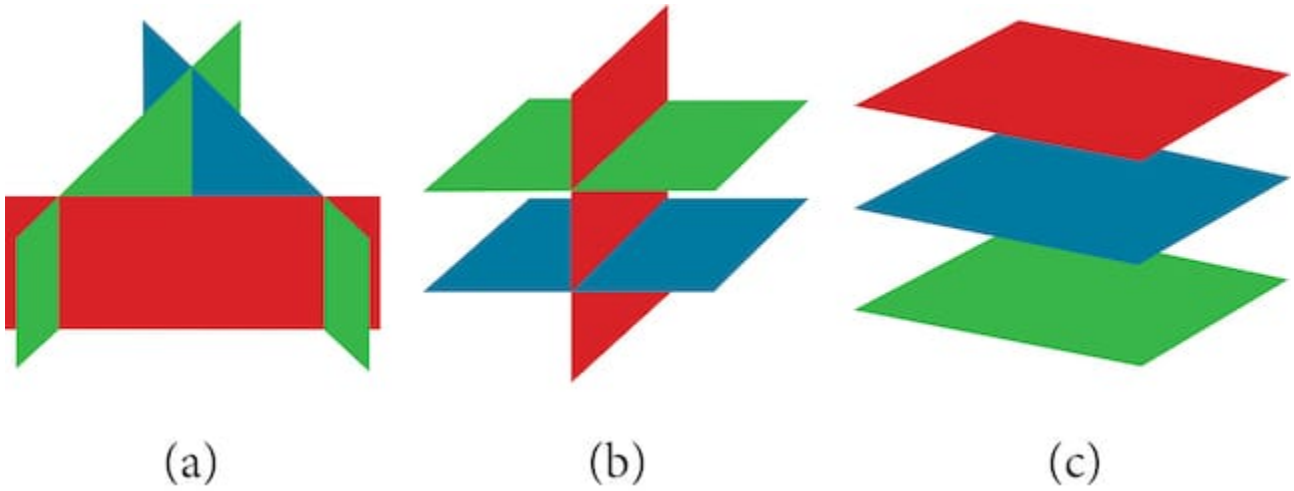


Figure 7.2.3**:** *All three figures represent three-by-three systems with no solution. (a): The three planes intersect with each other, but not at a common point. (b): Two of the planes are parallel and intersect with the third plane, but not with each other. (c): All three planes are parallel, so there is no point of intersection.*

- Systems that have a single solution are those which, after elimination, result in a solution set consisting of an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.

Example 7.2.1: Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$\begin{cases} x + y + z = 2 \\ 6x - 4y + 5z = 31 \\ 5x + 2y + 2z = 13 \end{cases}$$

Solution available at the end of this section.

The process of solving a linear system of three equations involving three unknowns can be cumbersome at times—especially if you do not have a systematic approach. We will begin by taking the following approach, which employs the Substitution Method:

1. Pick a variable and solve one of the equations for that variable.
2. Substitute this solution into the remaining two equations.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

It is important to note that the method we have written is not ideal, but it links well with previously learned methods. We will introduce a more algorithmic approach in a moment.

Example 7.2.2: Solving a System of Three Equations in Three Variables by Substitution

Find a solution to the following system:

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y - z = -6 \\ 2x - 5y + 5z = 17 \end{cases}$$

Solution available at the end of this section.

Solving linear systems using the method applied in Example 7.2.2 can get us into trouble when dealing with larger systems. Therefore, it is beneficial to showcase a method based on the Elimination Method we learned previously. The process is as follows:

1. Pick any pair of equations and use the technique discussed in the previous section to eliminate one variable.
2. Pick another pair of equations and eliminate the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Example 7.2.3: Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$\begin{cases} x - 2y + 3z = 9 & (E1) \\ -x + 3y - z = -6 & (E2) \\ 2x - 5y + 5z = 17 & (E3) \end{cases}$$

Solution available at the end of this section.

Checkpoint 7.2.3

Solve the system of equations in three variables.

$$\begin{cases} 2x + y - 2z = -1 \\ 3x - 3y - z = 5 \\ x - 2y + 3z = 6 \end{cases}$$

Example 7.2.4: Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of $12,000 in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. John invested $4,000 more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $670. How much did he invest in each type of fund?

Solution available at the end of this section.

Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The Elimination Method will result in a false statement, such as $3 = 7$ or some other contradiction.

Example 7.2.5: Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.

$$\begin{cases} x - 3y + z = 4 & (E1) \\ -x + 2y - 5z = 3 & (E2) \\ 5x - 13y + 13z = 8 & (E3) \end{cases}$$

Solution available at the end of this section.

Checkpoint 7.2.5

Solve the system of three equations in three variables.

$$\begin{cases} x + y + z = 2 \\ y - 3z = 1 \\ 2x + y + 5z = 0 \end{cases}$$

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

Example 7.2.6: Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$$\begin{cases} 2x + y - 3z = 0 & (E1) \\ 4x + 2y - 6z = 0 & (E2) \\ x - y + z = 0 & (E3) \end{cases}$$

Solution available at the end of this section.

Q&A

Does the generic solution to a dependent system always have to be written in terms of x?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x or, if needed, x and y.

Checkpoint 7.2.6

Solve the following system.

$$\begin{cases} x & + & y & + & z & = & 7 \\ 3x & - & 2y & - & z & = & 4 \\ x & + & 6y & + & 5z & = & 24 \end{cases}$$

Success in College: Preparing for the Final Exam (Part 3)

One suggestion I like to make concerning preparing for a final exam is to generate your own tests. This is pretty simple; however, you wouldn't believe how many students fail to do this. When studying, most people focus on the wrong things. You get sidetracked and work on problems that may only make up 5% of the entire exam. Try the following:

Let's assume that your final exam covers nine chapters. Let's also suppose that chapter four has the most sections, with eight. Finally, suppose the greatest number of problems within any given section is 82.

- Open Excel (or Google Sheets)
- In the first cell, type "=RANDBETWEEN(1,9)". This will generate a random chapter number.
- In the second cell, type "=RANDBETWEEN(1,8)". This will generate a random section between 1 and 8.
- In the third cell, type "=RANDBETWEEN(1,82)". This will generate a random problem number between 1 and 82.

Of course, it is entirely possible to generate problem 5.7.77 (chapter 5, section 7, problem 77) when section 7 of chapter 5 only has, for example, 50 problems. Just hit F9, and Excel will generate a new combination for you!

All you have to do is repeat this a few times, and you have a randomly generated test!

1. What do you think are the advantages of generating an exam this way? Explain.
2. How do you usually prepare for final exams?

Answers

Section Preview

Solving larger systems is useful in all levels of Mathematics; however, the students will definitely need this material to succeed in Integral Calculus (for Partial Fraction Decomposition).

Section Preview

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

Example 7.2.1: Determining Whether an Ordered Triple Is a Solution to a System

We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$$\begin{aligned} & x+y+z = 2 \\ \implies\ & (3)+(-2)+(1) = 2 \\ \implies\ & 2 = 2\checkmark \end{aligned} \qquad \begin{aligned} & 6x-4y+5z = 31 \\ \implies\ & 6(3)-4(-2)+5(1) = 31 \\ \implies\ & 18+8+5 = 31 \\ \implies\ & 31 = 31\checkmark \end{aligned} \qquad \begin{aligned} & 5x+2y+2z = 13 \\ \implies\ & 5(3)+2(-2)+2(1) = 13 \\ \implies\ & 15-4+2 = 13 \\ \implies\ & 13 = 13\checkmark \end{aligned}$$

The ordered triple $(3,-2,1)$ is indeed a solution to the system.

Example 7.2.2: Solving a System of Three Equations in Three Variables by Substitution

We begin by choosing a variable to solve for. While we could choose any of the three variables, x is easy to solve for in the first equation, so that will be our choice.

$$x-2y+3z=9 \implies x=9+2y-3z.$$

Substituting this for x in the remaining two equations, we get the following:

$$\begin{aligned} & -x+3y-z = -6 && && 2x-5y+5z = 17 \\ \Longrightarrow\ & -(9+2y-3z)+3y-z = -6 && \Longrightarrow\ & 2(9+2y-3z)-5y+5z = 17 \\ \Longrightarrow\ & -9-2y+3z+3y-z = -6 \quad \text{and} \quad && \Longrightarrow\ & 18+4y-6z-5y+5z = 17 \\ \Longrightarrow\ & -9+y+2z = -6 && \Longrightarrow\ & 18-y-z = 17 \\ \Longrightarrow\ & y+2z = 3 && \Longrightarrow\ & -y-z = -1 \end{aligned}$$

At this point, we have reduced the system of three equations in three unknowns down to the following system of two linear equations in two unknowns.

$$\begin{cases} y + 2z = 3 \\ -y - z = -1 \end{cases}$$

We can solve this 2×2 system using the Elimination Method. Adding the two equations, we get

$$\begin{aligned} y + 2z &= 3 \\ + \quad -y - z &= -1 \\ z &= 2 \end{aligned}$$

Substituting $z = 2$ into $y + 2z = 3$, we get

$$y + 2(2) = 3 \implies y = -1.$$

Finally, going back to our original equation solved for x, we let $z = 2$ and $y = -1$:

$$\begin{aligned} & x = 9+2y-3z \\ \Longrightarrow\ & x = 9+2(-1)-3(2) && \text{(substituting)} \\ \Longrightarrow\ & x = 9-2-6 && \text{(simplifying)} \\ \Longrightarrow\ & x = 1 && \text{(simplifying)} \end{aligned}$$

The solution is the ordered triple $(1, -1, 2)$. See Figure 7.2.4.

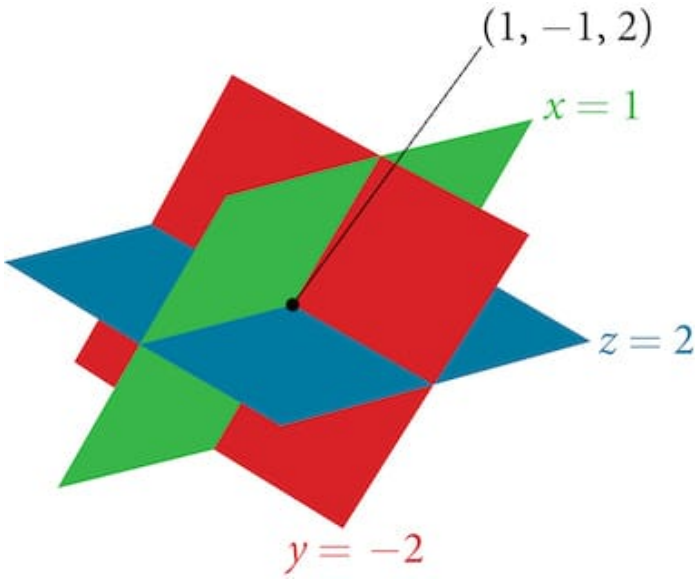


Figure 7.2.4

Example 7.2.3**: Solving a System of Three Equations in Three Variables by Elimination**

When learning the Elimination Method, it's best to label the equations (in this case, we used $(E1)$, $(E2)$, and $(E3)$). We begin by choosing a variable to eliminate. The ideal looks to be x. We will add $(E1)$ to $(E2)$ to eliminate the x from those equations.

$$\begin{aligned} (E1): \quad & x - 2y + 3z = 9 \\ + \quad (E2): \quad & -x + 3y - z = -6 \\ & y + 2z = 3 \end{aligned}$$

We now add $2 \times (E2)$ to $(E3)$ to eliminate the x from those.

$$\begin{aligned} 2 \times (E2): \quad & -2x + 6y - 2z = -12 \\ + \quad (E3): \quad & 2x - 5y + 5z = 17 \\ & y + 3z = 5 \end{aligned}$$

At this point, we have reduced the system of three equations in three unknowns down to the following system of two linear equations in two unknowns.

$$\begin{cases} y + 2z = 3 & (E4) \\ y + 3z = 5 & (E5) \end{cases}$$

We choose to continue with the Elimination Method by adding the opposite of $(E4)$ to $(E5)$.

$$\begin{array}{rrrrrrrr} -(E4): & -y & - & 2z & = & -3 \\ + \quad (E5): & y & + & 3z & = & 5 \\ & & & z & = & 2 \end{array}$$

Substituting $z = 2$ into $y + 2z = 3$, we get

$$y + 2(2) = 3 \implies y = -1.$$

Finally, we can use any of the equations involving x and we let $z = 2$ and $y = -1$. In this case, let's just use $(E1)$:

$$\begin{array}{rrcll} & x - 2y + 3z & = & 9 & \\ \implies & x - 2(-1) + 3(2) & = & 9 & \text{(substituting)} \\ \implies & x + 2 + 6 & = & 9 & \text{(simplifying)} \\ \implies & x + 8 & = & 9 & \text{(simplifying)} \\ \implies & x & = & 1 & \text{(subtracting 8 from both sides)} \end{array}$$

The solution is the ordered triple $(1, -1, 2)$.

Example 7.2.4**: Solving a Real-World Problem Using a System of Three Equations in Three Variables**

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$\begin{array}{rcl} x & = & \text{amount invested in money-market fund} \\ y & = & \text{amount invested in municipal bonds} \\ z & = & \text{amount invested in mutual funds} \end{array}$$

The first equation indicates that the sum of the three principal amounts is $12,000.

$$x + y + z = 12,000$$

We form the second equation according to the information that John invested $4,000 more in mutual funds than he invested in municipal bonds.

$$z = y + 4,000$$

The third equation shows that the total amount of interest earned from each fund equals $670.

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.

$$\begin{cases} x & + & y & + & z & = & 12,000 \\ & - & y & + & z & = & 4,000 \\ 0.03x & + & 0.04y & + & 0.07z & = & 670 \end{cases}$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,

$$\begin{cases} x & + & y & + & z & = & 12,000 & (E1) \\ & - & y & + & z & = & 4,000 & (E2) \\ 3x & + & 4y & + & 7z & = & 67,000 & (E3) \end{cases}$$

Since one of the equations is already missing a term involving x, let's eliminate the x in the other two equations. This requires us to add $-3 \times (E1)$ to $(E3)$.

$$\begin{array}{rrrrrrrr} -3(E1): & -3x & - & 3y & - & 3z & = & -36,000 \\ + \quad (E3): & 3x & + & 4y & + & 7z & = & 67,000 \\ & & & y & + & 4z & = & 31,000 \end{array}$$

We can take this result and place it in a new system involving $(E2)$—this is because $(E2)$ is already missing a term with x.

$$\begin{cases} -y & + & z & = & 4,000 & (E2) \\ y & + & 4z & = & 31,000 & (E4) \end{cases}$$

Adding these two equations together immediately eliminates y to get

$$5z = 35,000 \implies z = 7,000.$$

We now substitute this into $(E2)$ to get

$$-y + 7,000 = 4,000 \implies -y = -3,000 \implies y = 3,000.$$

Finally, we substitute these values into any equation involving x, y, and z. Let's choose $(E1)$:

$$x + 3,000 + 7,000 = 12,000 \implies x = 2,000.$$

John invested $2,000 in a money-market fund, $3,000 in municipal bonds, and $7,000 in mutual funds.

Example 7.2.5**: Solving an Inconsistent System of Three Equations in Three Variables**

Looking at the coefficients of x, we can see that we can eliminate x by adding equation $(E1)$ to equation $(E2)$.

$$\begin{array}{rrrrrrrr} (E1): & x & - & 3y & + & z & = & 4 \\ + \quad (E2): & -x & + & 2y & - & 5z & = & 3 \\ & & & -y & - & 4z & = & 7 \end{array}$$

Next, we multiply equation $(E1)$ by -5 and add it to equation $(E3)$.

$$\begin{array}{rrrrrrrr} -5(E1): & -5x & + & 15y & - & 5z & = & -20 \\ + \quad (E3): & 5x & - & 13y & + & 13z & = & 8 \\ & & & 2y & + & 8z & = & -12 \end{array}$$

We now have the simpler system

$$\begin{cases} -y - 4z = 7 & (E4) \\ 2y + 8z = -12 & (E5) \end{cases}$$

We now multiply equation $(E4)$ by 2 and add it to equation $(E5)$.

$$\begin{aligned} 2(E4): \quad -2y - 8z &= 14 \\ + \quad (E5): \quad 2y + 8z &= -12 \\ 0 &= 2 \end{aligned}$$

The final equation, $0 = 2$, is a contradiction, so we conclude that the system of equations is inconsistent and, therefore, has no solution.

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

Example $7.2.6$**: Finding the Solution to a Dependent System of Equations**

First, we can multiply equation $(E1)$ by -2 and add it to equation $(E2)$.

$$\begin{aligned} -2(E1): \quad -4x - 2y + 6z &= 0 \\ + \quad (E2): \quad 4x + 2y - 6z &= 0 \\ 0 &= 0 \end{aligned}$$

We do not need to proceed any further. The result we get is an identity, $0 = 0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation $(E3)$ by -2, and adding it to equation $(E1)$. We then perform the same steps as above and find the same result, $0 = 0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations $(E1)$ and $(E3)$, we have

$$\begin{aligned} (E1): \quad 2x + y - 3z &= 0 \\ + \quad (E3): \quad x - y + z &= 0 \\ 3x - 2z &= 0 \end{aligned}$$

We then solve the resulting equation for z.

$$3x - 2z = 0 \implies z = \frac{3}{2}x$$

We back-substitute the expression for z into one of the equations and solve for y.

$$\begin{aligned} & 2x + y - 3z = 0 \\ \implies & 2x + y - 3\left(\frac{3}{2}x\right) = 0 && \text{(substituting)} \\ \implies & 2x + y - \frac{9}{2}x = 0 && \text{(simplifying)} \\ \implies & -\frac{5}{2}x + y = 0 && \text{(combining like terms)} \\ \implies & y = \frac{5}{2}x && \left(\text{adding } \tfrac{5}{2}x \text{ to both sides}\right) \end{aligned}$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are dependent on the value selected for x.

As shown in Figure $7.2.5$, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.

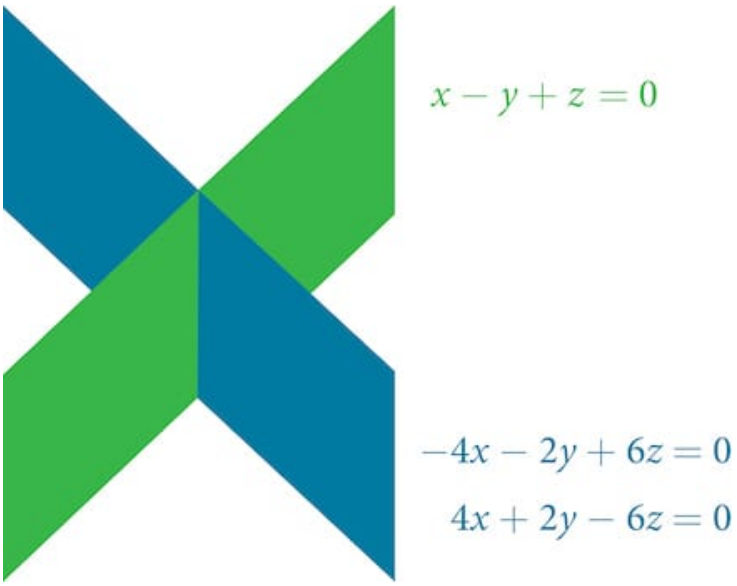


Figure $7.2.5$

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7.2.1: Resources and Key Concepts

Resources

Videos

- 3x3 systems
  - Systems of Linear Equations: Three Unknowns - Review of Solving Systems of 3 Equations in 3 Unknowns
  - Systems of Linear Equations: Three Unknowns - Creating Your Own Systems
  - Intermediate Algebra - Solving Medium Systems of Linear Equations (Part A)
- Gaussian Elimination
  - CIA: Solving 3x3 Systems Algebraically
- Dependent systems 3x3
  - Systems of Linear Equations: Three Unknowns - Parameterized Solutions to Linear Systems
- Applications of 3x3 systems
  - Intermediate Algebra - Solving Medium Systems of Linear Equations (Part B)
  - CIA: Solving Applications Using Systems
  - Systems of Linear Equations: Three Unknowns - Review of Applications Requiring Systems of Equations

Key Concepts

Definitions

- **Upper Triangular Form**: A system of linear equations is in upper triangular form if the variables are eliminated from the equations in a way that creates a "triangle" shape. For a 3x3 system, the first equation may contain x, y, z; the second contains only y, z; and the third contains only z.
- **Gaussian Elimination**: An algorithmic process used to solve systems of linear equations by applying a series of row operations to transform the system into upper triangular form.
- **Ordered Triple**: An ordered collection of three elements, (x, y, z), that represents the solution to a system of three equations in three variables. Graphically, it is the point of intersection of three planes in space.

Theorems

- **Invariant Solutions Under Row Operations**: The solution to a system of linear equations remains unchanged (is invariant) under the following operations:
  1. **Row Exchange**: Interchange the order of any two equations.
  2. **Row Scaling**: Multiply both sides of an equation by a nonzero constant.
  3. **Row Replacement**: Add a nonzero multiple of one equation to another equation.
- **Number of Solutions to 3x3 Linear Systems**: A system of three linear equations involving three unknowns has either one solution, no solution, or infinitely many solutions.

Common Mistakes

- **Not Eliminating the Same Variable**: When using elimination, a common error is to eliminate one variable from a first pair of equations (e.g., eliminate x from E1 and E2) but a different variable from a second pair (e.g., eliminate y from E1 and E3). This does not successfully reduce the system to a 2x2 system.
- **Arithmetic Errors in Row Operations**: Making simple addition or multiplication errors when performing row scaling or row replacement. For example, when multiplying an entire equation by a constant, forgetting to multiply the term on the other side of the equals sign.
- **Sign Errors During Elimination**: Forgetting to distribute a negative sign correctly when adding a multiple of one equation to another, leading to incorrect cancellation of terms.

- **Incorrectly Expressing Dependent Solutions**: When a system is found to be dependent (infinite solutions), stating "infinite solutions" is not enough. A common mistake is not expressing the general solution as an ordered triple in terms of one of the variables (e.g., $(x, \frac{5}{2}x, \frac{3}{2}x)$).
- **Misinterpreting Contradictions and Identities**:
  - Arriving at a contradiction (e.g., $0 = 2$) and not concluding that the system is inconsistent and has no solution.
  - Arriving at an identity (e.g., $0 = 0$) and not concluding that the system is dependent and has infinite solutions.
- **Errors in Back-Substitution**: After finding the value of the first variable (e.g., z) from the upper triangular form, making arithmetic errors when substituting it back into the other equations to find the remaining variables.

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7.2.2: Homework

Reading Questions

1. Can a linear system of three equations have exactly two solutions? Explain why or why not.
2. What is Gaussian Elimination and what is its goal when solving a 3x3 system?
3. Describe what "upper triangular form" means for a system of three linear equations. Why is this form useful?
4. List the three "row operations" that can be performed on a system of equations without changing its solution set.
5. From a geometric perspective, what does the solution to a 3x3 system of linear equations represent?
6. If you use elimination to solve a 3x3 system and arrive at the statement $0 = 2$, what does this tell you about the system and its solution?
7. If you use elimination to solve a 3x3 system and arrive at the statement $0 = 0$, what does this tell you about the system? How should you express the solution?
8. When using the elimination method, after you have combined one pair of equations to eliminate a variable, what must you do next?
9. What is "back-substitution"?
10. If a given ordered triple does not solve a system of equations, does that mean there is no solution to the system? Explain.
11. Can a linear system of three equations have exactly two solutions? Explain why or why not?
12. If a given ordered triple solves the system of equations, is that solution unique? If so, explain why. If not, give an example where it is not unique.
13. If a given ordered triple does not solve the system of equations, is there no solution? If so, explain why. If not, give an example.
14. Using the method of addition, is there only one way to solve the system?
15. Can you explain whether there can be only one method to solve a linear system of equations? If yes, give an example of such a system of equations. If not, explain why not.

Homework

Algebraic

For the following exercises, determine whether the ordered triple given is the solution to the system of equations.

16. $\begin{cases} 2x-6y+6z &= -12 \\ x+4y+5z &= -1 \\ -x+2y+3z &= -1 \end{cases}$ and $(0,1,-1)$

17. $\begin{cases} 6x-y+3z &= 6 \\ 3x+5y+2z &= 0 \\ x+y &= 0 \end{cases}$ and $(3,-3,-5)$

18. $\begin{cases} 6x-7y+z &= 2 \\ -x-y+3z &= 4 \\ 2x+y-z &= 1 \end{cases}$ and $(4,2,-6)$

19. $\begin{cases} x-y &= 0 \\ x-z &= 5 \\ x-y+z &= -1 \end{cases}$ and $(4,4,-1)$

20. $\begin{cases} -x-y+2z &= 3 \\ 5x+8y-3z &= 4 \\ -x+3y-5z &= -5 \end{cases}$ and $(4,1,-7)$

For the following exercises, solve each system by Elimination.

21. $\begin{cases} 3x-4y+2z &= -15 \\ 2x+4y+z &= 16 \\ 2x+3y+5z &= 20 \end{cases}$

22. $\begin{cases} 5x-2y+3z &= 20 \\ 2x-4y-3z &= -9 \\ x+6y-8z &= 21 \end{cases}$

23. $\begin{cases} 5x+2y+4z &= 9 \\ -3x+2y+z &= 10 \\ 4x-3y+5z &= -3 \end{cases}$

24. $\begin{cases} 4x-3y+5z &= 31 \\ -x+2y+4z &= 20 \\ x+5y-2z &= -29 \end{cases}$

25. $\begin{cases} 5x-2y+3z &= 4 \\ -4x+6y-7z &= -1 \\ 3x+2y-z &= 4 \end{cases}$

26. $\begin{cases} 4x+6y+9z &= 0 \\ -5x+2y-6z &= 3 \\ 7x-4y+3z &= -3 \end{cases}$

For the following exercises, solve each system by Gaussian Elimination.

27. $\begin{cases} 2x-y+3z &= 17 \\ -5x+4y-2z &= -46 \\ 2y+5z &= -7 \end{cases}$

28. $\begin{cases} 5x-6y+3z &= 50 \\ -x+4y &= 10 \\ 2x-z &= 10 \end{cases}$

29. $\begin{cases} 2x+3y-6z &= 1 \\ -4x-6y+12z &= -2 \\ x+2y+5z &= 10 \end{cases}$

30. $\begin{cases} 4x+6y-2z &= 8 \\ 6x+9y-3z &= 12 \\ -2x-3y+z &= -4 \end{cases}$

31. $\begin{cases} 2x+3y-4z &= 5 \\ -3x+2y+z &= 11 \\ -x+5y+3z &= 4 \end{cases}$

32. $\begin{cases} 10x+2y-14z &= 8 \\ -x-2y-4z &= -1 \\ -12x-6y+6z &= -12 \end{cases}$

33. $\begin{cases} x+y+z &= 14 \\ 2y+3z &= -14 \\ -16y-24z &= -112 \end{cases}$

34. $\begin{cases} 5x-3y+4z &= -1 \\ -4x+2y-3z &= 0 \\ -x+5y+7z &= -11 \end{cases}$

35. $\begin{cases} x+y+z &= 0 \\ 2x-y+3z &= 0 \\ x-z &= 0 \end{cases}$

36. $\begin{cases} 3x+2y-5z &= 6 \\ 5x-4y+3z &= -12 \\ 4x+5y-2z &= 15 \end{cases}$

37. $\begin{cases} x+y+z &= 0 \\ 2x-y+3z &= 0 \\ x-z &= 1 \end{cases}$

38. $\begin{cases} 3x-\frac{1}{2}y-z &= -\frac{1}{2} \\ 4x+z &= 3 \\ -x+\frac{3}{2}y &= \frac{5}{2} \end{cases}$

39. $\begin{cases} 6x-5y+6z &= 38 \\ \frac{1}{5}x-\frac{1}{2}y+\frac{3}{5}z &= 1 \\ -4x-\frac{3}{2}y-z &= -74 \end{cases}$

40. $\begin{cases} \frac{1}{2}x-\frac{1}{5}y+\frac{2}{5}z &= -\frac{13}{10} \\ \frac{1}{4}x-\frac{2}{5}y-\frac{1}{5}z &= -\frac{7}{20} \\ -\frac{1}{2}x-\frac{3}{4}y-\frac{1}{2}z &= -\frac{5}{4} \end{cases}$

41. $\begin{cases} -\frac{1}{3}x-\frac{1}{2}y-\frac{1}{4}z &= \frac{3}{4} \\ -\frac{1}{2}x-\frac{1}{4}y-\frac{1}{2}z &= 2 \\ -\frac{1}{4}x-\frac{3}{4}y-\frac{1}{2}z &= -\frac{1}{2} \end{cases}$

42. $\begin{cases} \frac{1}{2}x-\frac{1}{4}y+\frac{3}{4}z &= 0 \\ \frac{1}{4}x-\frac{1}{10}y+\frac{2}{5}z &= -2 \\ \frac{1}{8}x+\frac{1}{5}y-\frac{1}{8}z &= 2 \end{cases}$

43. $\begin{cases} \frac{4}{5}x-\frac{7}{8}y+\frac{1}{2}z &= 1 \\ -\frac{4}{5}x-\frac{3}{4}y+\frac{1}{3}z &= -8 \\ -\frac{2}{5}x-\frac{7}{8}y+\frac{1}{2}z &= -5 \end{cases}$

44. $\begin{cases} -\frac{1}{3}x-\frac{1}{8}y+\frac{1}{6}z &= -\frac{4}{3} \\ -\frac{2}{3}x-\frac{7}{8}y+\frac{1}{3}z &= -\frac{23}{3} \\ -\frac{1}{3}x-\frac{5}{8}y+\frac{5}{6}z &= 0 \end{cases}$

45. $\begin{cases} -\frac{1}{4}x-\frac{5}{4}y+\frac{5}{2}z &= -5 \\ -\frac{1}{2}x-\frac{5}{3}y+\frac{5}{4}z &= \frac{55}{12} \\ -\frac{1}{3}x-\frac{1}{3}y+\frac{1}{3}z &= \frac{5}{3} \end{cases}$

46. $\begin{cases} \frac{1}{40}x+\frac{1}{60}y+\frac{1}{80}z &= \frac{1}{100} \\ -\frac{1}{2}x-\frac{1}{3}y-\frac{1}{4}z &= -\frac{1}{5} \\ \frac{3}{8}x+\frac{3}{12}y+\frac{3}{16}z &= \frac{3}{20} \end{cases}$

47. $\begin{cases} 0.1x-0.2y+0.3z &= 2 \\ 0.5x-0.1y+0.4z &= 8 \\ 0.7x-0.2y+0.3z &= 8 \end{cases}$

48. $\begin{cases} 0.2x+0.1y-0.3z &= 0.2 \\ 0.8x+0.4y-1.2z &= 0.1 \\ 1.6x+0.8y-2.4z &= 0.2 \end{cases}$

49. $\begin{cases} 1.1x+0.7y-3.1z &= -1.79 \\ 2.1x+0.5y-1.6z &= -0.13 \\ 0.5x+0.4y-0.5z &= -0.07 \end{cases}$

50. $\begin{cases} 0.5x-0.5y+0.5z &= 10 \\ 0.2x-0.2y+0.2z &= 4 \\ 0.1x-0.1y+0.1z &= 2 \end{cases}$

51. $\begin{cases} 0.1x + 0.2y + 0.3z &= 0.37 \\ 0.1x - 0.2y - 0.3z &= -0.27 \\ 0.5x - 0.1y - 0.3z &= -0.03 \end{cases}$

52. $\begin{cases} 0.5x - 0.5y - 0.3z &= 0.13 \\ 0.4x - 0.1y - 0.3z &= 0.11 \\ 0.2x - 0.8y - 0.9z &= -0.32 \end{cases}$

53. $\begin{cases} 0.5x + 0.2y - 0.3z &= 1 \\ 0.4x - 0.6y + 0.7z &= 0.8 \\ 0.3x - 0.1y - 0.9z &= 0.6 \end{cases}$

54. $\begin{cases} 0.3x + 0.3y + 0.5z &= 0.6 \\ 0.4x + 0.4y + 0.4z &= 1.8 \\ 0.4x + 0.2y + 0.1z &= 1.6 \end{cases}$

55. $\begin{cases} 0.8x + 0.8y + 0.8z &= 2.4 \\ 0.3x - 0.5y + 0.2z &= 0 \\ 0.1x + 0.2y + 0.3z &= 0.6 \end{cases}$

Extensions

For the following exercises, solve the system for x, y, and z.

56. $\begin{cases} x + y + z &= 3 \\ \frac{x-1}{2} + \frac{y-3}{2} + \frac{z+1}{2} &= 0 \\ \frac{x-2}{3} + \frac{y+4}{3} + \frac{z-3}{3} &= \frac{2}{3} \end{cases}$

57. $\begin{cases} 5x - 3y - \frac{z+1}{2} &= \frac{1}{2} \\ 6x + \frac{y-9}{2} + 2z &= -3 \\ \frac{x+8}{2} - 4y + z &= 4 \end{cases}$

58. $\begin{cases} \frac{x+4}{7} - \frac{y-1}{6} + \frac{z+2}{3} &= 1 \\ \frac{x-2}{4} + \frac{y+1}{8} - \frac{z+8}{12} &= 0 \\ \frac{x+6}{3} - \frac{y+2}{3} + \frac{z+4}{2} &= 3 \end{cases}$

59. $\begin{cases} \frac{x-3}{6} + \frac{y+2}{2} - \frac{z-3}{3} &= 2 \\ \frac{x+2}{4} + \frac{y-5}{2} + \frac{z+4}{2} &= 1 \\ \frac{x+6}{2} - \frac{y-3}{2} + z + 1 &= 9 \end{cases}$

60. $\begin{cases} \frac{x-1}{3} + \frac{y+3}{4} + \frac{z+2}{6} &= 1 \\ 4x + 3y - 2z &= 11 \\ 0.02x + 0.015y - 0.01z &= 0.065 \end{cases}$

Real-World Applications

61. Three even numbers sum up to 108. The smaller is half the larger and the middle number is $\frac{3}{4}$ the larger. What are the three numbers?
62. Three numbers sum up to 147. The smallest number is half the middle number, which is half the largest number. What are the three numbers?
63. At a family reunion, there were only blood relatives, consisting of children, parents, and grandparents, in attendance. There were 400 people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents, and grandparents were in attendance?
64. An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each animal are at the shelter?
65. Your roommate, Sarah, offered to buy groceries for you and your other roommate. The total bill was $82. She forgot to save the individual receipts but remembered that your groceries were $0.05 cheaper than half of her groceries, and that your

other roommate's groceries were $2.10 more than your groceries. How much was each of your share of the groceries?

56. Your roommate, John, offered to buy household supplies for you and your other roommate. You live near the border of three states, each of which has a different sales tax. The total amount of money spent was $100.75. Your supplies were bought with 5% tax, John's with 8% tax, and your third roommate's with 9% sales tax. The total amount of money spent without taxes is $93.50. If your supplies before tax were $1 more than half of what your third roommate's supplies were before tax, how much did each of you spend? Give your answer both with and without taxes.
57. Three coworkers work for the same employer. Their jobs are warehouse manager, office manager, and truck driver. The sum of the annual salaries of the warehouse manager and office manager is $82,000. The office manager makes $4,000 more than the truck driver annually. The annual salaries of the warehouse manager and the truck driver total $78,000. What is the annual salary of each of the co-workers?
58. At a carnival, $2,914.25 in receipts were taken at the end of the day. The cost of a child's ticket was $20.50, an adult ticket was $29.75, and a senior citizen ticket was $15.25. There were twice as many senior citizens as adults in attendance, and 20 more children than senior citizens. How many children, adult, and senior citizen tickets were sold?
59. A local band sells out for their concert. They sell all 1,175 tickets for a total purse of $28,112.50. The tickets were priced at $20 for student tickets, $22.50 for children, and $29 for adult tickets. If the band sold twice as many adult as children tickets, how many of each type was sold?
70. In a bag, a child has 325 coins worth $19.50. There were three types of coins: pennies, nickels, and dimes. If the bag contained the same number of nickels as dimes, how many of each type of coin was in the bag?
71. Last year, at Haven's Pond Car Dealership, for a particular model of BMW, Jeep, and Toyota, one could purchase all three cars for a total of $140,000. This year, due to inflation, the same cars would cost $151,830. The cost of the BMW increased by 8%, the Jeep by 5%, and the Toyota by 12%. If the price of last year's Jeep was $7,000 less than the price of last year's BMW, what was the price of each of the three cars last year?
72. A recent college graduate took advantage of his business education and invested in three investments immediately after graduating. He invested $80,500 into three accounts, one that paid 4% simple interest, one that paid $3\frac{1}{8}$% simple interest, and one that paid $2\frac{1}{2}$% simple interest. He earned $2,670 interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?
73. You inherit one million dollars. You invest it all in three accounts for one year. The first account pays 3% compounded annually, the second account pays 4% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $34,000 in interest. If you invest four times the money into the account that pays 3% compared to 2%, how much did you invest in each account?
74. You inherit one hundred thousand dollars. You invest it all in three accounts for one year. The first account pays 4% compounded annually, the second account pays 3% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $3,650 in interest. If you invest five times the money in the account that pays 4% compared to 3%, how much did you invest in each account?
75. The top three countries in oil consumption in a certain year are as follows: the United States, Japan, and China. In millions of barrels per day, the three top countries consumed 39.8% of the world's consumed oil. The United States consumed 0.7% more than four times China's consumption. The United States consumed 5% more than triple Japan's consumption. What percent of the world oil consumption did the United States, Japan, and China consume?
76. The top three countries in oil production in the same year are Saudi Arabia, the United States, and Russia. In millions of barrels per day, the top three countries produced 31.4% of the world's produced oil. Saudi Arabia and the United States combined for 22.1% of the world's production, and Saudi Arabia produced 2% more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States, and Russia produce?
77. The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico, and Canada. The three top countries accounted for 47% of oil imports. The United States imported 1.8% more from Saudi Arabia than they did from Mexico, and 1.7% more from Saudi Arabia than they did from Canada. What percent of the United States oil imports were from these three countries?
78. The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas, and Alaska. The three regions were responsible for 64% of the United States oil production. The Gulf of Mexico and Texas combined for 47% of oil production. Texas produced 3% more than Alaska. What percent of United States oil production came from these regions?
79. At one time, in the United States, 398 species of animals were on the endangered species list. The top groups were mammals, birds, and fish, which comprised 55% of the endangered species. Birds accounted for 0.7% more than fish, and fish accounted for 1.5% more than mammals. What percent of the endangered species came from mammals, birds, and fish?
80. Meat consumption in the United States can be broken into three categories: red meat, poultry, and fish. If fish makes up 4% less than one-quarter of poultry consumption, and red meat consumption is 18.2% higher than poultry consumption, what are the percentages of meat consumption?

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7.3: Systems of Nonlinear Equations and Inequalities - Two Variables

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Halley's Comet (Figure 7.3.1) orbits the sun about once every 75 years. Its path can be considered to be a very elongated ellipse. Other comets follow similar paths in space. These orbital paths can be studied using systems of equations. These systems, however, are different from the ones we considered in the previous section because the equations are not linear.



Figure 7.3.1**:** *Halley's Comet (credit: "NASA Blueshift"/Flickr)*

In this section, we will consider the intersection of a parabola and a line, a circle and a line, and a circle and an ellipse. The methods for solving **systems of nonlinear equations** are similar to those for linear equations.

Solving a System of Nonlinear Equations Using Substitution

Definition: System of Nonlinear Equations

A **system of nonlinear equations** is a system of two or more equations in two or more variables containing at least one equation that is not linear.

Recall that a linear equation can take the form $Ax + By + C = 0$. Any equation that cannot be written in this form is **nonlinear**.

The Substitution Method we used for linear systems is the same method we will use for nonlinear systems. We solve one equation for one variable and then substitute the result into the second equation to solve for another variable, and so on. There is, however, a variation in the possible outcomes.

Intersection of a Parabola and a Line

There are three possible types of solutions for a system of nonlinear equations involving a parabola and a line. Figure 7.3.2 illustrates possible solution sets for a system of equations involving a parabola and a line.

- No solution. The line will never intersect the parabola.
- One solution. The line is tangent to the parabola and intersects the parabola at exactly one point.
- Two solutions. The line crosses on the inside of the parabola and intersects the parabola at two points.

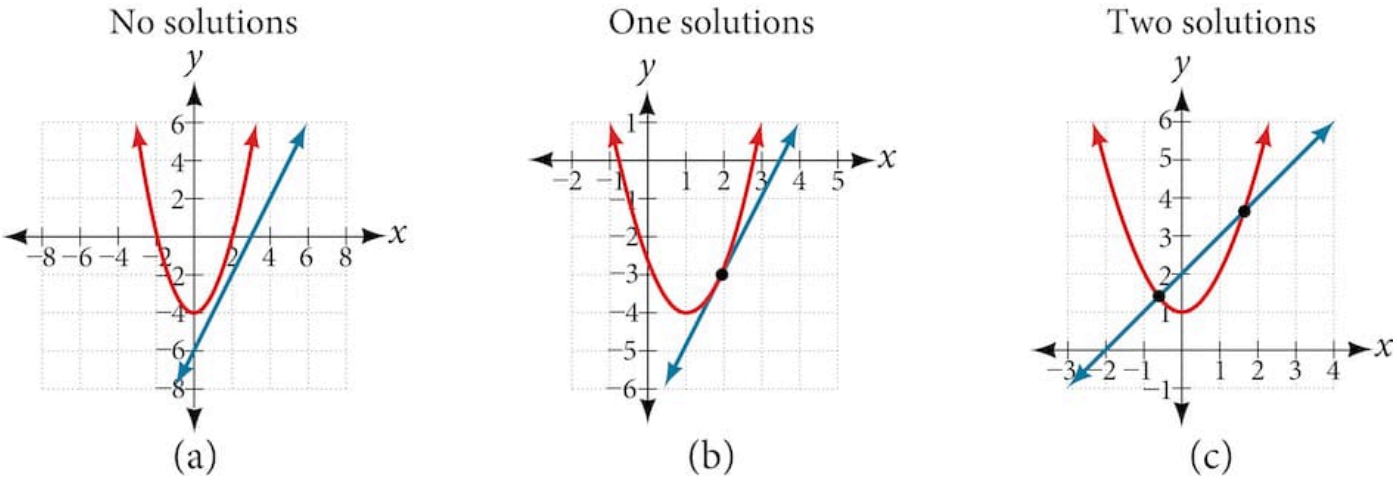


Figure 7.3.2

Example 7.3.1: Solving a System of Nonlinear Equations Representing a Parabola and a Line

Solve the system of equations.

$$\begin{cases} x - y = -1 \\ y = x^2 + 1 \end{cases}$$

Solution available at the end of this section.

Q&A

Could we have substituted values for y into the second equation to solve for x in Example 7.3.1?

Yes, but because x is squared in the second equation this could give us extraneous solutions for x.

For $y = 1$

$$\begin{aligned} & y = x^2 + 1 \\ \Longrightarrow \quad & 1 = x^2 + 1 && \text{(substituting)} \\ \Longrightarrow \quad & 0 = x^2 && \text{(subtracting 1 from both sides)} \\ \Longrightarrow \quad & \pm\sqrt{0} = x && \text{(Extraction of Roots)} \\ \Longrightarrow \quad & 0 = x && \text{(simplifying)} \end{aligned}$$

This gives us the same value as in the solution.

For $y = 2$

$$\begin{aligned} & y = x^2 + 1 & \\ \implies \quad & 2 = x^2 + 1 & \text{(substituting)} \\ \implies \quad & 1 = x^2 & \text{(subtracting 1 from both sides)} \\ \implies \quad & \pm\sqrt{1} = x & \text{(Extraction of Roots)} \\ \implies \quad & \pm 1 = x & \text{(simplifying)} \end{aligned}$$

Notice that -1 is an extraneous solution.

Checkpoint 7.3.1

Solve the given system of equations by substitution.

$$\begin{cases} 3x - y = -2 \\ 2x^2 - y = 0 \end{cases}$$

Intersection of a Circle and a Line

Just as with a parabola and a line, there are three possible outcomes when solving a system of equations representing a circle and a line.

- No solution. The line does not intersect the circle.
- One solution. The line is tangent to the circle and intersects the circle at exactly one point.
- Two solutions. The line crosses the circle and intersects it at two points.

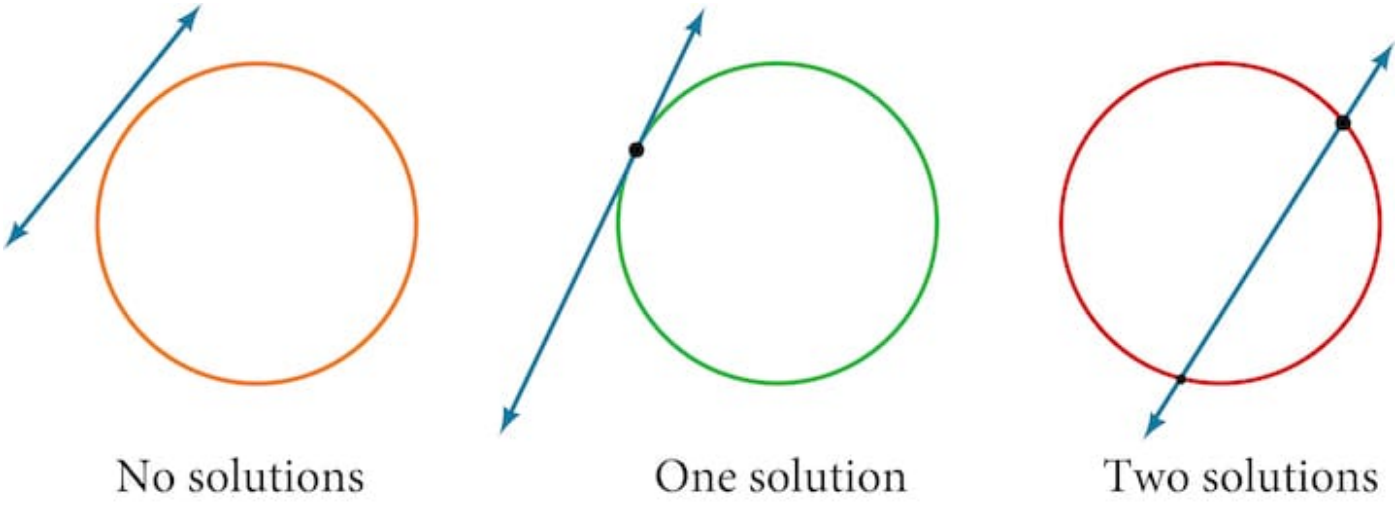


Figure 7.3.4

Example 7.3.2: Finding the Intersection of a Circle and a Line by Substitution

Find the intersection of the given circle and the given line by substitution.

$$\begin{cases} x^2 + y^2 = 5 \\ y = 3x - 5 \end{cases}$$

Solution available at the end of this section.

Checkpoint 7.3.2

Solve the system of nonlinear equations.

$$\begin{cases} x^2 + y^2 &= 10 \\ x - 3y &= -10 \end{cases}$$

Solving a System of Nonlinear Equations Using Elimination

We have seen that substitution is often the preferred method when a system of equations includes a linear equation and a nonlinear equation. However, when both equations in the system have like variables of the second degree, solving them using elimination by addition is often easier than substitution. Generally, elimination is a far simpler method when the system involves only two equations in two variables (a two-by-two system), rather than a three-by-three system, as there are fewer steps. As an example, we will investigate the possible types of solutions when solving a system of equations representing a circle and an ellipse.

Possible Types of Solutions for the Points of Intersection of a Circle and an Ellipse

Figure 7.3.6 illustrates possible solution sets for a system of equations involving a circle and an ellipse.

- No solution. The circle and ellipse do not intersect. One shape is inside the other or the circle and the ellipse are a distance away from the other.
- One solution. The circle and ellipse are tangent to each other, and intersect at exactly one point.
- Two solutions. The circle and the ellipse intersect at two points.
- Three solutions. The circle and the ellipse intersect at three points.
- Four solutions. The circle and the ellipse intersect at four points.

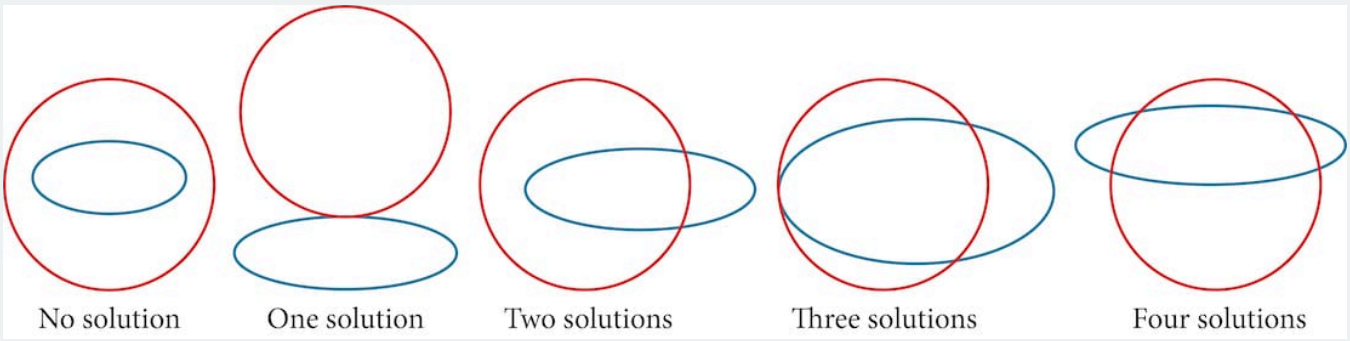


Figure 7.3.6

Example 7.3.3: Solving a System of Nonlinear Equations Representing a Circle and an Ellipse

Solve the system of nonlinear equations.

$$\begin{cases} x^2 + y^2 &= 26 \quad (1) \\ 3x^2 + 25y^2 &= 100 \quad (2) \end{cases}$$

Solution available at the end of this section.

Checkpoint 7.3.3

Find the solution set for the given system of nonlinear equations.

$$\begin{cases} 4x^2 + y^2 &= 13 \\ x^2 + y^2 &= 10 \end{cases}$$

Graphing a Nonlinear Inequality

All of the equations in the systems that we have encountered so far have involved equalities, but we may also encounter systems that involve inequalities. We have already learned to graph linear inequalities by graphing the corresponding equation, and then shading the region represented by the inequality symbol. Now, we will follow similar steps to graph a nonlinear inequality so that we can learn to solve systems of nonlinear inequalities. A nonlinear inequality is an inequality containing a nonlinear expression. Graphing a nonlinear inequality is much like graphing a linear inequality.

Recall that when the inequality is greater than, $y > a$, or less than, \(y

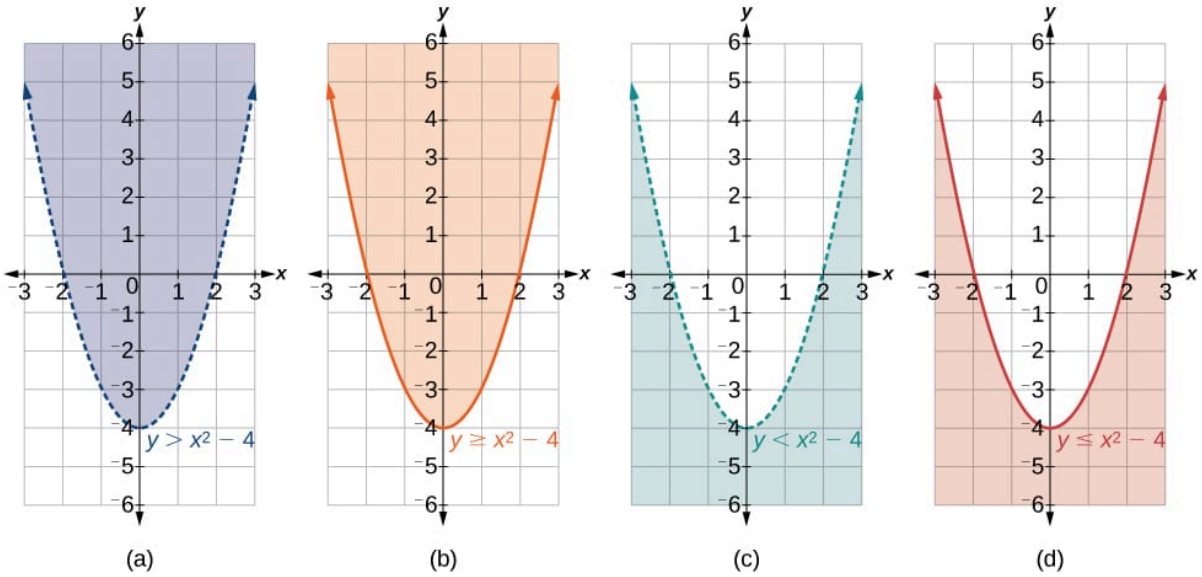


Figure 7.3.8**:** *(a) an example of* $y > a$*;*
(b) an example of $y \geq a$*;*
(c) an example of \(y *(d) an example of* $y \leq a$

How To

Given an inequality bounded by a parabola, sketch a graph.

1. Graph the parabola as if it were an equation. This is the boundary for the region that is the solution set.
2. If the boundary is included in the region (the operator is $\leq$ or $\geq$), the parabola is graphed as a solid line.
3. If the boundary is not included in the region (the operator is < or >), the parabola is graphed as a dashed line.
4. Test a point in one of the regions to determine whether it satisfies the inequality statement. If the statement is true, the solution set is the region including the point. If the statement is false, the solution set is the region on the other side of the boundary line.
5. Shade the region representing the solution set.

Example 7.3.4: Graphing an Inequality for a Parabola

Graph the inequality $y > x^2 + 1$.

Solution available at the end of this section.

Graphing a System of Nonlinear Inequalities

Now that we have learned to graph nonlinear inequalities, we can learn how to graph systems of nonlinear inequalities. A system of nonlinear inequalities is a system of two or more inequalities in two or more variables containing at least one inequality that is

not linear. Graphing a system of nonlinear inequalities is similar to graphing a system of linear inequalities. The difference is that our graph may result in more shaded regions that represent a solution than we find in a system of linear inequalities. The solution to a nonlinear system of inequalities is the region of the graph where the shaded regions of the graph of each inequality overlap, or where the regions intersect, called the feasible region.

How To... Solve a System of Nonlinear Inequalities

1. Find the intersection points by solving the corresponding system of nonlinear equations.
2. Graph the nonlinear equations.
3. Find the shaded regions of each inequality.
4. Identify the feasible region as the intersection of the shaded regions of each inequality or the set of points common to each inequality.

Example 7.3.5: Graphing a System of Inequalities

Graph the given system of inequalities.

$$\begin{cases} x^2 - y & \leq & 0 \\ 2x^2 + y & \leq & 12 \end{cases}$$

Solution available at the end of this section.

Checkpoint 7.3.4

Graph the given system of inequalities.

$$\begin{cases} y & \geq & x^2 - 1 \\ x - y & \geq & -1 \end{cases}$$

Time-Sensitive Advice: Pass/No Pass Application Deadline

Within the Los Rios Community College District, the last day you can apply to switch the grading scheme on your class to Pass/No Pass is the last day of finals week.[1]

Why should you not apply for this for your math classes?

Generally, applying for a Pass/No Pass grade in mathematics looks sloppy on a transcript. When you apply for transfer to a four-year university, you are competing with thousands of other applicants, most of whom have letter grades. Individuals who exercise the Pass/No Pass grade scheme notoriously do so for fear of failing a class, and the application reviewers at four-year institutions know this! Do yourself a favor and stick with letter grades for math courses.

[1] Strangely, this was recently changed from Friday of the fifth week of class without any notice to faculty.

Success in College: Reflecting on the Course

1. Henry Ford said, "Failure is only the opportunity to begin again more intelligently." If you could start this course over, would you do anything differently? Explain.
2. Barack Obama said, "Making your mark on the world is hard. If it were easy, everybody would do it. But it's not. It takes patience, it takes commitment, and it comes with plenty of failure along the way. The real test is not whether you avoid this failure, because you won't. It's whether you let it harden or shame you into inaction, or whether you learn from it; whether you

choose to persevere." Do you find it challenging to persevere, keep going, and get your college education? Explain.

Answers

Section Preview

The final instructional material in the text—congratulations!

Section Preview

- **Linear Functions**
  - Graphing Linear Inequalities in Two Variables: The section explicitly extends the linear-inequality model (graph the boundary, test a point, shade the satisfying region) to nonlinear boundaries; without this conceptual model the "How To" procedures and Examples 4 and 5 are opaque.

Section Preview

- Solve a system of nonlinear equations using substitution.
- Solve a system of nonlinear equations using elimination.
- Graph a nonlinear inequality.
- Graph a system of nonlinear inequalities.

Example 7.3.1**: Solving a System of Nonlinear Equations Representing a Parabola and a Line**

Solve the first equation for x.

$$\begin{aligned} & x-y = -1 \\ \Rightarrow \quad & x = y-1 \qquad \text{(adding } y \text{ to both sides)} \end{aligned}$$

We now substitute the resulting expression into the second equation.

$$\begin{aligned} & y = x^2+1 \\ \Rightarrow \quad & y = (y-1)^2+1 \qquad \text{(substituting)} \\ \Rightarrow \quad & y = y^2-2y+1+1 \qquad \text{(distributing)} \\ \Rightarrow \quad & y = y^2-2y+2 \qquad \text{(combining like terms)} \\ \Rightarrow \quad & 0 = y^2-3y+2 \qquad \text{(subtracting } y \text{ from both sides)} \\ \Rightarrow \quad & 0 = (y-2)(y-1) \qquad \text{(factoring)} \end{aligned}$$

Solving for y gives $y=2$ and $y=1$. Next, substitute each value for y into the first equation to solve for x. Always substitute the value into the linear equation to check for extraneous solutions.

$$\begin{aligned} & x-y = -1 \\ \Rightarrow \quad & x-(2) = -1 \qquad \text{(substituting)} \\ \Rightarrow \quad & x = 1 \qquad \text{(adding 2 to both sides)} \end{aligned}$$

$$\begin{aligned} & x-y = -1 \\ \Rightarrow \quad & x-(1) = -1 \qquad \text{(substituting)} \\ \Rightarrow \quad & x = 0 \qquad \text{(adding 1 to both sides)} \end{aligned}$$

The solutions are $(1,2)$ and $(0,1)$, which can be verified by substituting these (x,y) values into both of the original equations. See Figure 7.3.3.

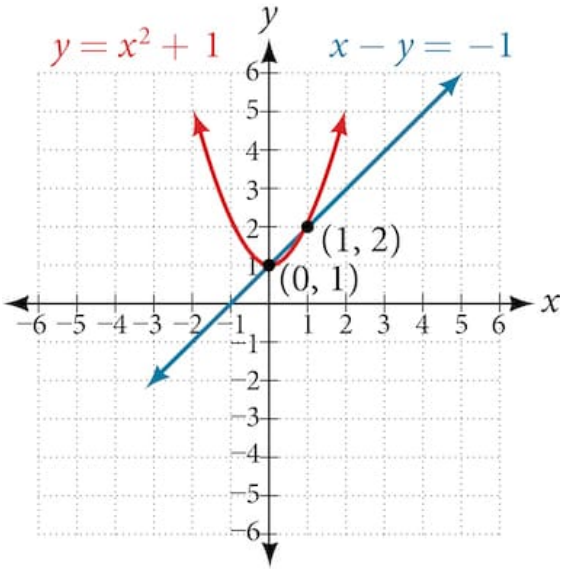


Figure 7.3.3

Example 7.3.2: Finding the Intersection of a Circle and a Line by Substitution

One of the equations has already been solved for y. We will substitute $y = 3x - 5$ into the equation for the circle.

$$\begin{aligned} & x^2 + y^2 = 5 \\ \Longrightarrow \quad & x^2 + (3x-5)^2 = 5 && \text{(substituting)} \\ \Longrightarrow \quad & x^2 + 9x^2 - 30x + 25 = 5 && \text{(distributing)} \\ \Longrightarrow \quad & 10x^2 - 30x + 25 = 5 && \text{(combining like terms)} \\ \Longrightarrow \quad & 10x^2 - 30x + 20 = 0 && \text{(subtracting 5 from both sides)} \\ \Longrightarrow \quad & x^2 - 3x + 2 = 0 && \text{(dividing both sides by 10)} \\ \Longrightarrow \quad & (x-2)(x-1) = 0 && \text{(factoring)} \end{aligned}$$

Substitute the two x-values into the original linear equation to solve for y.

$$\begin{aligned} y &= 3(2) - 5 = 1 \\ y &= 3(1) - 5 = -2 \end{aligned}$$

The line intersects the circle at $(2, 1)$ and $(1, -2)$, which can be verified by substituting these (x, y) values into both of the original equations. See Figure 7.3.5.

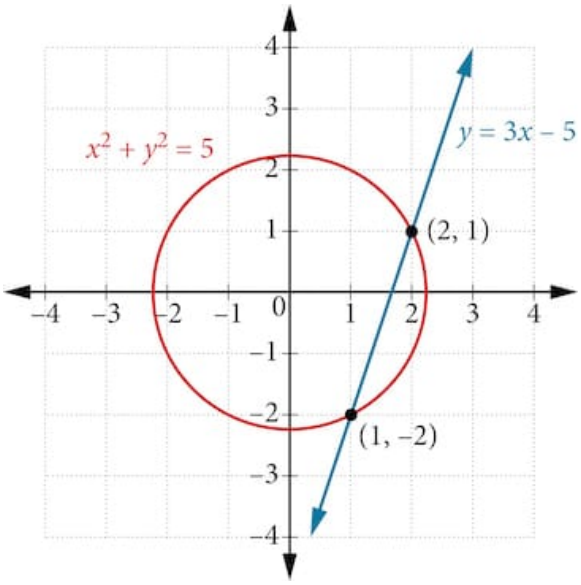


Figure 7.3.5

Example 7.3.3: Solving a System of Nonlinear Equations Representing a Circle and an Ellipse

Let's begin by multiplying equation (1) by -3. This gives $-3x^2 - 3y^2 = -78$. We now add this to equation (2).

$$\begin{array}{rcrcl} -3x^2 & - & 3y^2 & = & -78 \\ 3x^2 & + & 25y^2 & = & 100 \\ & & 22y^2 & = & 22 \end{array}$$

After we add the two equations together, we solve for y.

$$y^2 = 1 \implies y = \pm\sqrt{1} = \pm 1.$$

Substitute $y = \pm 1$ into one of the equations and solve for x.

$$\begin{aligned} & x^2 + (1)^2 = 26 \\ \Longrightarrow \quad & x^2 + 1 = 26 \\ \Longrightarrow \quad & x^2 = 25 \\ \Longrightarrow \quad & x = \pm 5 \end{aligned}$$

and

$$\begin{aligned} & x^2 + (-1)^2 = 26 \\ \Longrightarrow \quad & x^2 + 1 = 26 \\ \Longrightarrow \quad & x^2 = 25 \\ \Longrightarrow \quad & x = \pm 5 \end{aligned}$$

There are four solutions: $(5, 1)$, $(-5, 1)$, $(5, -1)$, and $(-5, -1)$. See Figure 7.3.7.

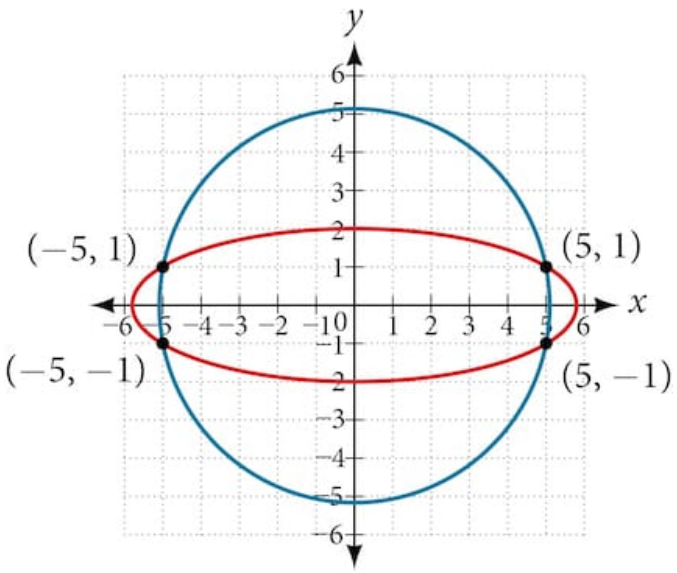


Figure 7.3.7

Example 7.3.4**: Graphing an Inequality for a Parabola**

First, graph the corresponding equation $y = x^2 + 1$. Since $y > x^2 + 1$ has a greater than symbol, we draw the graph with a dashed line. Then we choose points to test both inside and outside the parabola. Let's test the points $(0, 2)$ and $(2, 0)$. One point is clearly inside the parabola and the other point is clearly outside.

$$\begin{aligned} & y > x^2 + 1 \\ \Longrightarrow \quad & 2 > (0)^2 + 1 \qquad \text{(substituting)} \\ \Longrightarrow \quad & 2 > 1 \qquad \text{(True)} \end{aligned}$$

and

$$\begin{aligned} & y > x^2 + 1 \\ \Longrightarrow \quad & 0 > (2)^2 + 1 \qquad \text{(substituting)} \\ \Longrightarrow \quad & 0 > 5 \qquad \text{(False)} \end{aligned}$$

The graph is shown in Figure 7.3.9. We can see that the solution set consists of all points inside the parabola, but not on the graph itself.

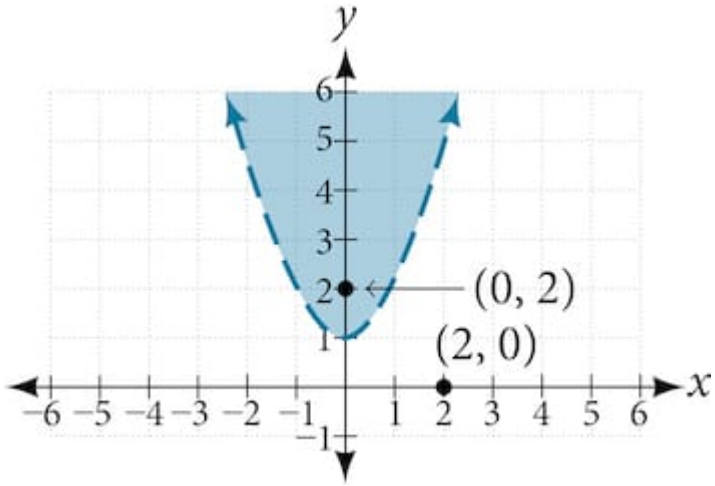


Figure 7.3.9

Example 7.3.5**: Graphing a System of Inequalities**

These two equations are clearly parabolas. We can find the points of intersection by the elimination process: Add both equations and the variable y will be eliminated. Then we solve for x.

$$\begin{aligned} x^2 - y &= 0 \\ 2x^2 + y &= 12 \\ 3x^2 &= 12 \\ \Longrightarrow \quad x^2 &= 4 \\ \Longrightarrow \quad x &= \pm 2 \end{aligned}$$

Substitute the x-values into one of the equations and solve for y.

$$\begin{aligned} x^2 - y &= 0 \\ \Rightarrow \quad (2)^2 - y &= 0 \\ \Rightarrow \quad 4 - y &= 0 \\ \Rightarrow \quad y &= 4 \end{aligned}$$

and

$$\begin{aligned} x^2 - y &= 0 \\ \Rightarrow \quad (-2)^2 - y &= 0 \\ \Rightarrow \quad 4 - y &= 0 \\ \Rightarrow \quad y &= 4 \end{aligned}$$

The two points of intersection are $(2, 4)$ and $(-2, 4)$. Notice that the equations can be rewritten as follows.

$$\begin{aligned} x^2 - y &\leq 0 \\ \Rightarrow \quad x^2 &\leq y \\ \Rightarrow \quad y &\geq x^2 \end{aligned}$$

and

$$\begin{aligned} 2x^2 + y &\leq 12 \\ \Rightarrow \quad y &\leq -2x^2 + 12 \end{aligned}$$

Graph each inequality. See Figure 7.3.10. The feasible region is the region between the two equations bounded by $2x^2 + y \leq 12$ on the top and $x^2 - y \leq 0$ on the bottom.

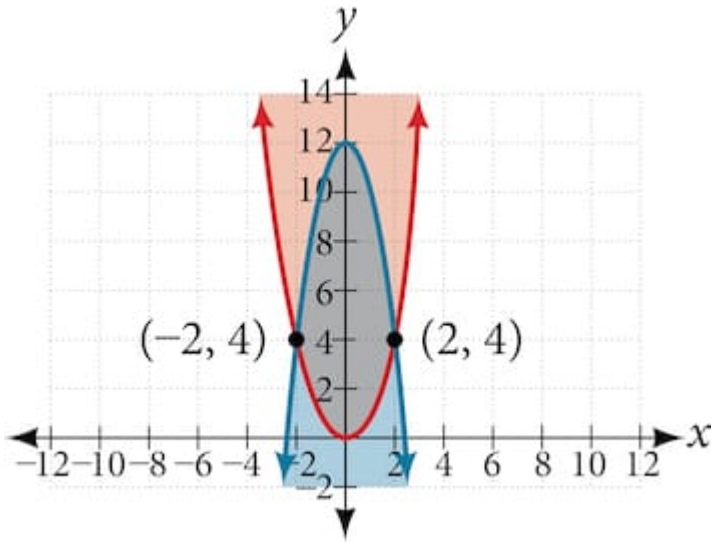


Figure 7.3.10

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7.3.1: Resources and Key Concepts

Resources

Videos

- System of nonlinear equations
  - Intermediate Algebra - Solving Non-Linear Systems of Equations (Part A)
- System of nonlinear inequalities
  - Intermediate Algebra - Graphing Systems of Linear Inequalities
  - CIA: Solving a System of Linear Inequalities

Key Concepts

Definitions

- **System of Nonlinear Equations**: A system of two or more equations in two or more variables containing at least one equation that is not linear.
- **Substitution Method (for nonlinear systems)**: A method for solving systems where one equation is solved for one variable and the result is substituted into the other equation.
- **Elimination Method (for nonlinear systems)**: A method for solving systems where two equations are added or subtracted to eliminate a variable, often used when both equations have like variables of the second degree.
- **Nonlinear Inequality**: An inequality containing a nonlinear expression.
- **System of Nonlinear Inequalities**: A system of two or more inequalities in two or more variables containing at least one inequality that is not linear.
- **Feasible Region**: The region of a graph where the shaded regions of each inequality in a system overlap; this region represents the set of all solutions to the system of inequalities.

Common Mistakes

- **Errors in Substitution**: When substituting one equation into another, especially when squaring a binomial, algebraic errors are common. For example, incorrectly expanding $(y-1)^2$ as y^2-1.
- **Forgetting All Solutions**: When solving an equation like $x^2=4$, only finding one solution ($x=2$) instead of both ($x=\pm 2$). This can lead to missing intersection points in the system.
- **Introducing Extraneous Solutions**: When solving a system, substituting a variable into the more complex (higher-degree) equation can sometimes introduce extraneous solutions that must be checked in the simpler (often linear) equation.
- **Errors in Elimination**: Making sign errors when adding or subtracting entire equations, or incorrectly multiplying an equation by a constant before adding.
- **Graphing Inequality Boundaries Incorrectly**: Using a solid line for a strict inequality ($<$ or $>$) or a dashed line for an inclusive inequality ($\leq$ or $\geq$).
- **Shading the Wrong Region for an Inequality**: After graphing the boundary curve, choosing an incorrect test point or misinterpreting the result, leading to the wrong region being shaded as the solution set.
- **Incorrectly Identifying the Feasible Region**: When graphing a system of inequalities, incorrectly identifying the final solution set, which should only be the region where *all* individual shaded areas overlap.

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7.3.2: Homework

Reading Questions

1. Explain whether a system of two nonlinear equations can have exactly two solutions. What about exactly three?
2. When graphing a nonlinear inequality, why do we only need to test one point in each region created by the boundary curve?
3. Will a system of nonlinear inequalities always have a feasible region (a solution set)? Explain why or why not.
4. What are the three possible solution outcomes for a system consisting of a parabola and a line?
5. What are the five possible solution outcomes for a system consisting of a circle and an ellipse?
6. Which method, substitution or elimination, is generally preferred for solving a system with one linear and one nonlinear equation?
7. Which method, substitution or elimination, is often easier for solving a system where both equations have like variables of the second degree (e.g., x^2 and y^2)?
8. When graphing a nonlinear inequality, how do you determine whether the boundary curve should be solid or dashed?
9. What is a "feasible region" in the context of a system of inequalities?
10. Why is it important to check for extraneous solutions when solving nonlinear systems?
11. Explain whether a system of two nonlinear equations can have exactly two solutions. What about exactly three? If not, explain why not. If so, give an example of such a system, in graph form, and explain why your choice gives two or three answers.
12. When graphing an inequality, explain why we only need to test one point to determine whether an entire region is the solution?
13. When you graph a system of inequalities, will there always be a feasible region? If so, explain why. If not, give an example of a graph of inequalities that does not have a feasible region. Why does it not have a feasible region?
14. If you graph a revenue and cost function, explain how to determine in what regions there is profit.
15. If you perform your break-even analysis and there is more than one solution, explain how you would determine which x-values are profit and which are not.

Homework

Algebraic

For the following exercises, solve the system of nonlinear equations using Substitution.

16. $\begin{cases} x+y &= 4 \\ x^2+y^2 &= 9 \end{cases}$

17. $\begin{cases} y &= x-3 \\ x^2+y^2 &= 9 \end{cases}$

18. $\begin{cases} y &= x \\ x^2+y^2 &= 9 \end{cases}$

19. $\begin{cases} y &= -x \\ x^2+y^2 &= 9 \end{cases}$

20. $\begin{cases} x &= 2 \\ x^2-y^2 &= 9 \end{cases}$

For the following exercises, solve the system of nonlinear equations using Elimination.

21. $\begin{cases} 4x^2-9y^2 &= 36 \\ 4x^2+9y^2 &= 36 \end{cases}$

22. $\begin{cases} x^2+y^2 &= 25 \\ x^2-y^2 &= 1 \end{cases}$

23. $\begin{cases} 2x^2+4y^2 &= 4 \\ 2x^2-4y^2 &= 25x-10 \end{cases}$

24. $\begin{cases} y^2-x^2 &= 9 \\ 3x^2+2y^2 &= 8 \end{cases}$

25. $\begin{cases} x^2+y^2+\frac{1}{16} &= 2500 \\ y &= 2x^2 \end{cases}$

For the following exercises, use any method to solve the system of nonlinear equations.

26. $\begin{cases} -2x^2+y &= -5 \\ 6x-y &= 9 \end{cases}$

27. $\begin{cases} -x^2+y &= 2 \\ -x+y &= 2 \end{cases}$

28. $\begin{cases} x^2+y^2 &= 1 \\ y &= 20x^2-1 \end{cases}$

29. $\begin{cases} x^2+y^2 &= 1 \\ y &= -x^2 \end{cases}$

30. $\begin{cases} 2x^3-x^2 &= y \\ y &= \frac{1}{2}-x \end{cases}$

31. $\begin{cases} 9x^2+25y^2 &= 225 \\ (x-6)^2+y^2 &= 1 \end{cases}$

32. $\begin{cases} x^4-x^2 &= y \\ x^2+y &= 0 \end{cases}$

33. $\begin{cases} 2x^3-x^2 &= y \\ x^2+y &= 0 \end{cases}$

For the following exercises, use any method to solve the nonlinear system.

34. $\begin{cases} x^2+y^2 &= 9 \\ y &= 3-x^2 \end{cases}$

35. $\begin{cases} x^2-y^2 &= 9 \\ x &= 3 \end{cases}$

36. $\begin{cases} x^2-y^2 &= 9 \\ y &= 3 \end{cases}$

37. $\begin{cases} x^2-y^2 &= 9 \\ x-y &= 0 \end{cases}$

38. $\begin{cases} -x^2+y &= 2 \\ -4x+y &= -1 \end{cases}$

39. $\begin{cases} -x^2+y &= 2 \\ 2y &= -x \end{cases}$

40. $\begin{cases} x^2+y^2 &= 25 \\ x^2-y^2 &= 36 \end{cases}$

41. $\begin{cases} x^2+y^2 &= 1 \\ y^2 &= x^2 \end{cases}$

42. $\begin{cases} 16x^2-9y^2+144 &= 0 \\ y^2+x^2 &= 16 \end{cases}$

43. $\begin{cases} 3x^2-y^2 &= 12 \\ (x-1)^2+y^2 &= 1 \end{cases}$

44. $\begin{cases} 3x^2-y^2 &= 12 \\ (x-1)^2+y^2 &= 4 \end{cases}$

45. $\begin{cases} 3x^2-y^2 &= 12 \\ x^2+y^2 &= 16 \end{cases}$

46. $\begin{cases} x^2-y^2-6x-4y-11 &= 0 \\ -x^2+y^2 &= 5 \end{cases}$

47. $\begin{cases} x^2+y^2-6y &= 7 \\ x^2+y &= 1 \end{cases}$

48. $\begin{cases} x^2+y^2 &= 6 \\ xy &= 1 \end{cases}$

Graphical

For the following exercises, graph the inequality.

49. $x^2+y<9$
50. $x^2+y^2<4$

For the following exercises, graph the system of inequalities. Label all points of intersection.

51. $\begin{cases} x^2+y &< 1 \\ y &> 2x \end{cases}$

52. $\begin{cases} x^2+y &< -5 \\ y &> 5x+10 \end{cases}$

53. $\begin{cases} x^2+y^2 &< 25 \\ 3x^2-y^2 &> 12 \end{cases}$

54. $\begin{cases} x^2-y^2 &> -4 \\ x^2+y^2 &< 12 \end{cases}$

55. $\begin{cases} x^2+3y^2 > 16 \\ 3x^2-y^2 < 1 \end{cases}$

Extensions

For the following exercises, graph the inequality.

56. $\begin{cases} y \geq e^x \\ y \leq \ln(x)+5 \end{cases}$

57. $\begin{cases} y \leq -\log(x) \\ y \leq e^x \end{cases}$

For the following exercises, find the solutions to the nonlinear equations with two variables.

58. $\begin{cases} \frac{4}{x^2}+\frac{1}{y^2} = 24 \\ \frac{5}{x^2}-\frac{2}{y^2}+4 = 0 \end{cases}$

59. $\begin{cases} \frac{6}{x^2}-\frac{1}{y^2} = 8 \\ \frac{1}{x^2}-\frac{6}{y^2} = \frac{1}{8} \end{cases}$

60. $\begin{cases} x^2-xy+y^2-2 = 0 \\ x+3y = 4 \end{cases}$

61. $\begin{cases} x^2-xy-2y^2-6 = 0 \\ x^2+y^2 = 1 \end{cases}$

62. $\begin{cases} x^2+4xy-2y^2-6 = 0 \\ x = y+2 \end{cases}$

Technology

For the following exercises, solve the system of inequalities. Use a calculator to graph the system to confirm the answer.

63. $\begin{cases} xy < 1 \\ y > \sqrt{x} \end{cases}$

64. $\begin{cases} x^2+y < 3 \\ y > 2x \end{cases}$

Real-World Applications

For the following exercises, construct a system of nonlinear equations to describe the given behavior, then solve for the requested solutions.

65. Two numbers add up to 300. One number is twice the square of the other number. What are the numbers?
66. The squares of two numbers add to 360. The second number is half the value of the first number squared. What are the numbers?
67. A laptop company has discovered their cost and revenue functions for each day: $C(x)=x^2-10x+200$ and $R(x)=-2x^2+100x+50$. If they want to make a profit, what is the range of laptops per day that they should produce? Round to the nearest number which would generate profit.

58. A cell phone company has the following cost and revenue functions: $C(x) = 8x^2 - 600x + 21,500$ and $R(x) = -3x^2 + 480x$. What is the range of cell phones they should produce each day so there is profit? Round to the nearest number that generates profit.

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8: EXTRACURRICULAR DATA MODELING

The material in this chapter is optional and only provided for those classes and/or students who wish to delve deeper into data analysis.

8.1: Fitting Linear Models to Data

8.1E: Exercises

8.2: Modeling Using Variation

8.2E: Exercises

8.3: Fitting Exponential Models to Data

8.3E: Exercises

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8.1: Fitting Linear Models to Data

Learning Objectives

In this section, you will:

- Draw and interpret scatter plots.
- Find the line of best fit.
- Distinguish between linear and nonlinear relations.
- Use a linear model to make predictions.

A professor is attempting to identify trends among final exam scores. His class has a mixture of students, so he wonders if there is any relationship between age and final exam scores. One way for him to analyze the scores is by creating a diagram that relates the age of each student to the exam score received. In this section, we will examine one such diagram known as a scatter plot.

Drawing and Interpreting Scatter Plots

A scatter plot is a graph of plotted points that may show a relationship between two sets of data. If the relationship is from a linear model, or a model that is nearly linear, the professor can draw conclusions using his knowledge of linear functions. Figure 1 shows a sample scatter plot.

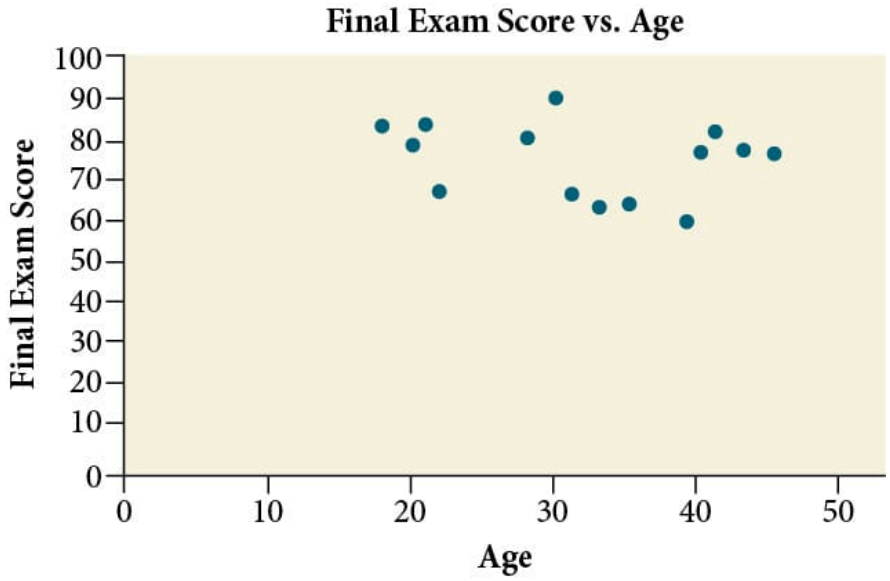


Figure 1 A scatter plot of age and final exam score variables

Notice this scatter plot does *not* indicate a linear relationship. The points do not appear to follow a trend. In other words, there does not appear to be a relationship between the age of the student and the score on the final exam.

Example 1

Using a Scatter Plot to Investigate Cricket Chirps

Table 1 shows the number of cricket chirps in 15 seconds, for several different air temperatures, in degrees Fahrenheit[5]. Plot this data, and determine whether the data appears to be linearly related.

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temperature	80.5	70.5	57	66	68	72	52	73.5	53

Table 1

Answer

Plotting this data, as depicted in Figure 2 suggests that there may be a trend. We can see from the trend in the data that the number of chirps increases as the temperature increases. The trend appears to be roughly linear, though certainly not perfectly so.

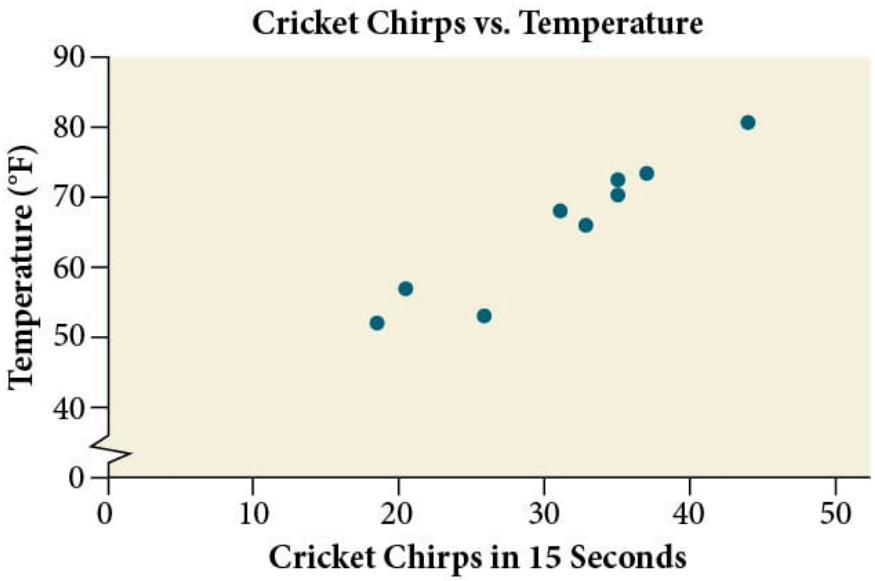


Figure 2

Finding the Line of Best Fit

Once we recognize a need for a linear function to model that data, the natural follow-up question is "what is that linear function?" One way to approximate our linear function is to sketch the line that seems to best fit the data. Then we can extend the line until we can verify the *y*-intercept. We can approximate the slope of the line by extending it until we can estimate the $\frac{\text{rise}}{\text{run}}$.

Example 2

Finding a Line of Best Fit

Find a linear function that fits the data in Table 1 by "eyeballing" a line that seems to fit.

Answer available at the end of this section.

Analysis

This linear equation can then be used to approximate answers to various questions we might ask about the trend.

Recognizing Interpolation or Extrapolation

While the data for most examples does not fall perfectly on the line, the equation is our best guess as to how the relationship will behave outside of the values for which we have data. We use a process known as **interpolation** when we predict a value inside

the domain and range of the data. The process of **extrapolation** is used when we predict a value outside the domain and range of the data.

Figure 4 compares the two processes for the cricket-chirp data addressed in Example 2. We can see that interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44. Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

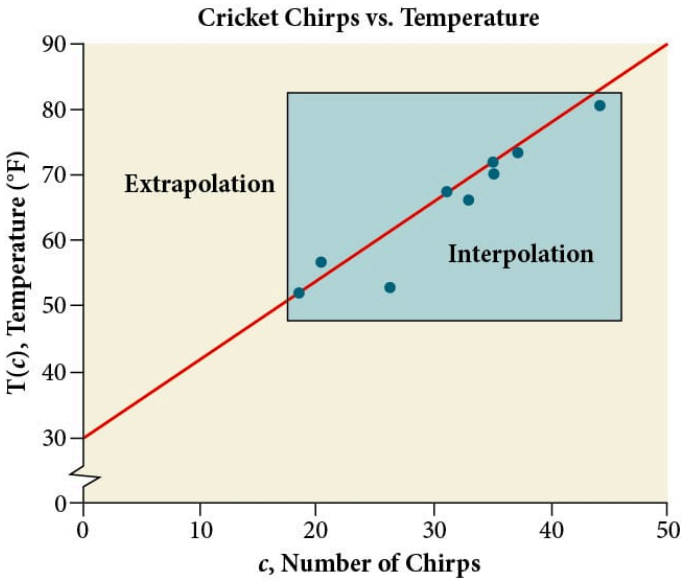


Figure 4 Interpolation occurs within the domain and range of the provided data whereas extrapolation occurs outside.

There is a difference between making predictions inside the domain and range of values for which we have data and outside that domain and range. Predicting a value outside of the domain and range has its limitations. When our model no longer applies after a certain point, it is sometimes called model breakdown. For example, predicting a cost function for a period of two years may involve examining the data where the input is the time in years and the output is the cost. But if we try to extrapolate a cost when $x = 50$, that is in 50 years, the model would not apply because we could not account for factors fifty years in the future.

Interpolation and Extrapolation

Different methods of making predictions are used to analyze data.

- The method of interpolation involves predicting a value inside the domain and/or range of the data.
- The method of extrapolation involves predicting a value outside the domain and/or range of the data.
- **Model breakdown** occurs at the point when the model no longer applies.

Example 3

Understanding Interpolation and Extrapolation

Use the cricket data from Table 1 to answer the following questions:

1. ⓐ Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.
2. ⓑ Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.

Answer available at the end of this section.

Analysis

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

Try It #1

According to the data from Table 1, what temperature can we predict it is if we counted 20 chirps in 15 seconds?

Finding the Line of Best Fit Using a Graphing Utility

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values[6]. One such technique is called least squares regression and can be computed by many graphing calculators, spreadsheet software, statistical software, and many web-based calculators[7]. Least squares regression is one means to determine the line that best fits the data, and here we will refer to this method as linear regression.

How To

Given data of input and corresponding outputs from a linear function, find the best fit line using linear regression.

1. Enter the input in List 1 (**L1**).
2. Enter the output in List 2 (**L2**).
3. On a graphing utility, select Linear Regression (**LinReg**).

Example 4

Finding a Least Squares Regression Line

Find the least squares regression line using the cricket-chirp data in Table 1.

Answer available at the end of this section.

- On a graphing utility, select Linear Regression (LinReg). Using the cricket chirp data from earlier, with technology we obtain the equation:

$$T(c) = 30.281 + 1.143c$$

Analysis

Notice that this line is quite similar to the equation we "eyeballed" but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:

$$\begin{aligned} T(30) &= 30.281 + 1.143(30) \\ &= 64.571 \\ &\approx 64.6 \text{degrees} \end{aligned}$$

The graph of the scatter plot with the least squares regression line is shown in Figure 6.

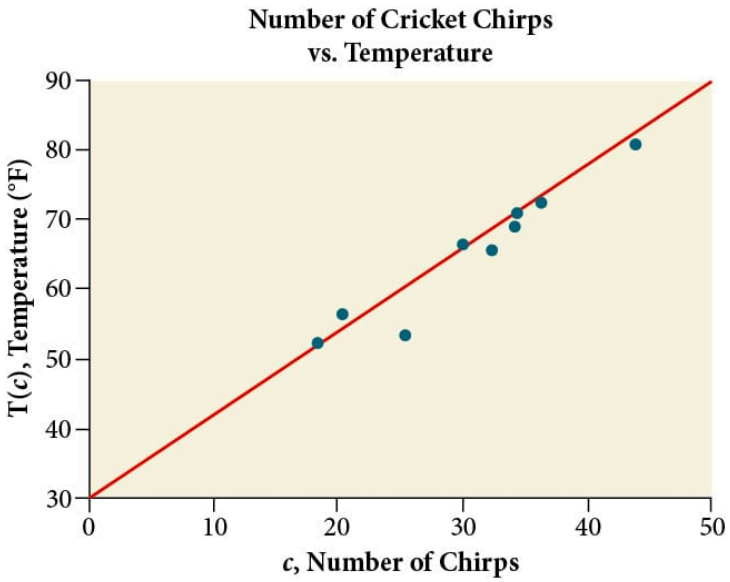


Figure 6

Q&A

Will there ever be a case where two different lines will serve as the best fit for the data?

No. There is only one best fit line.

Distinguishing Between Linear and Non-Linear Models

As we saw above with the cricket-chirp model, some data exhibit strong linear trends, but other data, like the final exam scores plotted by age, are clearly nonlinear. Most calculators and computer software can also provide us with the **correlation coefficient**, which is a measure of how closely the line fits the data. Many graphing calculators require the user to turn a "diagnostic on" selection to find the correlation coefficient, which mathematicians label as r. The correlation coefficient provides an easy way to get an idea of how close to a line the data falls.

We should compute the correlation coefficient only for data that follows a linear pattern or to determine the degree to which a data set is linear. If the data exhibits a nonlinear pattern, the correlation coefficient for a linear regression is meaningless. To get a sense for the relationship between the value of r and the graph of the data, Figure 7 shows some large data sets with their correlation coefficients. Remember, for all plots, the horizontal axis shows the input and the vertical axis shows the output.

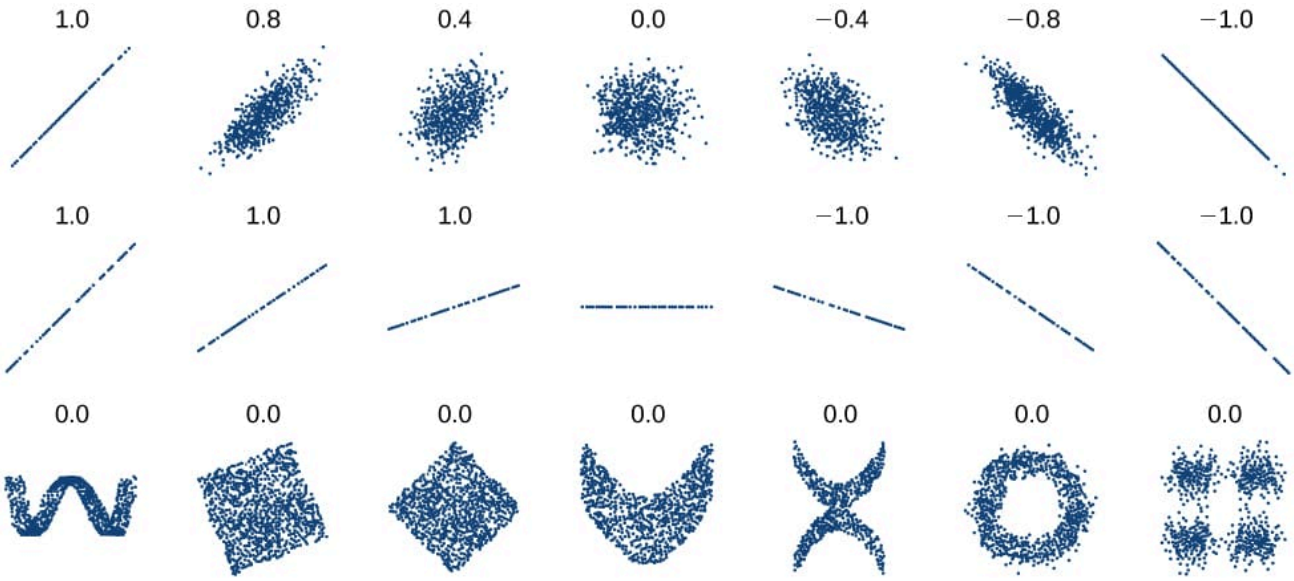


Figure 7 Plotted data and related correlation coefficients. (credit: "DenisBoigelot," Wikimedia Commons)

Correlation Coefficient

The correlation coefficient is a value, r, between –1 and 1.

- $r > 0$ suggests a positive (increasing) relationship
- $r < 0$ suggests a negative (decreasing) relationship
- The closer the value is to 0, the more scattered the data.
- The closer the value is to 1 or –1, the less scattered the data is.

Example 5

Finding a Correlation Coefficient

Calculate the correlation coefficient for cricket-chirp data in Table 1.

Answer available at the end of this section.

Predicting with a Regression Line

Once we determine that a set of data is linear using the correlation coefficient, we can use the regression line to make predictions. As we learned above, a regression line is a line that is closest to the data in the scatter plot, which means that only one such line is a best fit for the data.

Example 6

Using a Regression Line to Make Predictions

Gasoline consumption in the United States has been steadily increasing. Consumption data from 1994 to 2004 is shown in Table 3[8]. Determine whether the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billions of gallons)	113	116	118	119	123	125	126	128	131	133	136

Table 3

The scatter plot of the data, including the least squares regression line, is shown in Figure 8.

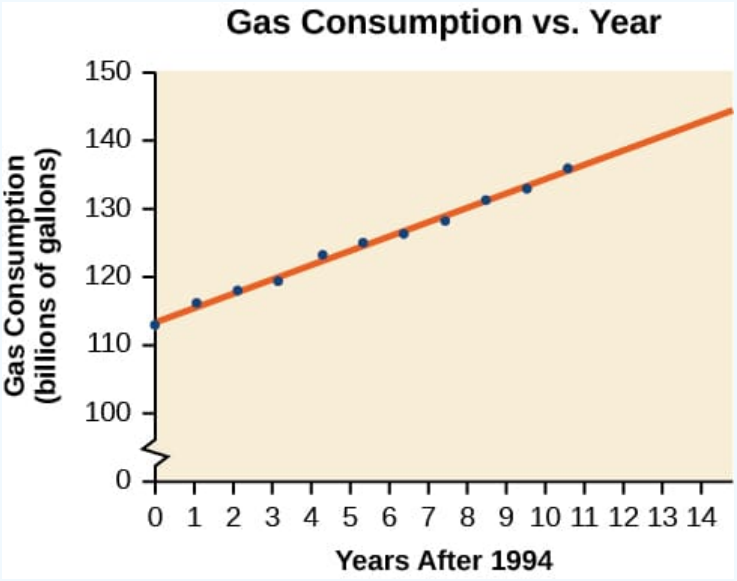


Figure 8

Answer

We can introduce new input variable, t, representing years since 1994.

The least squares regression equation is:

$$C(t) = 113.318 + 2.209t$$

Using technology, the correlation coefficient was calculated to be 0.9965, suggesting a very strong increasing linear trend.

Using this to predict consumption in 2008 $(t = 14)$,

$$\begin{aligned} C(14) &= 113.318 + 2.209(14) \\ &= 144.244 \end{aligned}$$

The model predicts 144.244 billion gallons of gasoline consumption in 2008.

Try It #2

Use the model we created using technology in Example 6 to predict the gas consumption in 2011. Is this an interpolation or an extrapolation?

Access these online resources for additional instruction and practice with fitting linear models to data.

- Introduction to Regression Analysis

- Linear Regression

Answers

On a graph, we could try sketching a line.

Using the starting and ending points of our hand drawn line, points (0, 30) and (50, 90), this graph has a slope of

$$m = \frac{60}{50} = 1.2$$

and a *y*-intercept at 30. This gives an equation of

$$T(c) = 1.2c+$$

where c is the number of chirps in 15 seconds, and $T(c)$ is the temperature in degrees Fahrenheit. The resulting equation is represented in Figure 3.

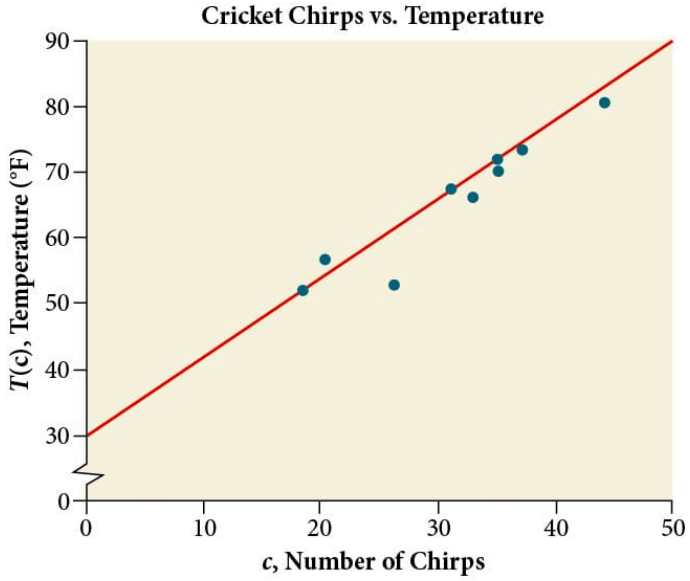


Figure 3

1. ⓐ The number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model:

$$T(30) = 30 + 1.2(30)$$
$$= 66 \quad \text{degrees}$$

   Based on the data we have, this value seems reasonable.
2. ⓑ The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation because 40 is outside the range of our data. Using our model:

$$40 = 30 + 1.2c$$
$$10 = 1.2c$$
$$c \approx 8.33$$

We can compare the regions of interpolation and extrapolation using Figure 5.

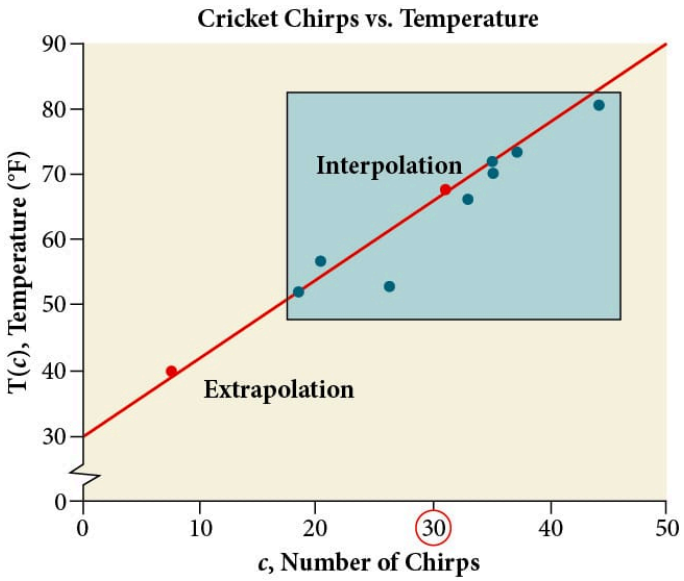


Figure 5

1. Enter the input (chirps) in List 1 (L1).
2. Enter the output (temperature) in List 2 (L2). See Table 2.

L1	44	35	20.4	33	31	35	18.5	37	26
L2	80.5	70.5	57	66	68	72	52	73.5	53

Table 2

Because the data appear to follow a linear pattern, we can use technology to calculate r. Enter the inputs and corresponding outputs and select the Linear Regression. The calculator will also provide you with the correlation coefficient, $r = 0.9509$. This value is very close to 1, which suggests a strong increasing linear relationship.

Note: For some calculators, the Diagnostics must be turned "on" in order to get the correlation coefficient when linear regression is performed: [2nd]>[0]>[alpha][x –1], then scroll to **DIAGNOSTICSON**.

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8.1E: Exercises

Exercises

Verbal

1.

Describe what it means if there is a model breakdown when using a linear model.

2.

What is interpolation when using a linear model?

3.

What is extrapolation when using a linear model?

4.

Explain the difference between a positive and a negative correlation coefficient.

5.

Explain how to interpret the absolute value of a correlation coefficient.

Algebraic

6.

A regression was run to determine whether there is a relationship between hours of TV watched per day (x) and number of sit-ups a person can do (y). The results of the regression are given below. Use this to predict the number of sit-ups a person who watches 11 hours of TV can do.

$$y = ax + b$$
$$a = -1.341$$
$$b = 32.234$$
$$r = -0.896$$

7.

A regression was run to determine whether there is a relationship between the diameter of a tree (x, in inches) and the tree's age (y, in years). The results of the regression are given below. Use this to predict the age of a tree with diameter 10 inches.

$$y = ax + b$$
$$a = 6.301$$
$$b = -1.044$$
$$r = -0.970$$

For the following exercises, draw a scatter plot for the data provided. Does the data appear to be linearly related?

8.

0	2	4	6	8	10
–22	–19	–15	–11	–6	–2

9.

1	2	3	4	5	6
46	50	59	75	100	136

10.

100	250	300	450	600	750
12	12.6	13.1	14	14.5	15.2

11.

1	3	5	7	9	11
1	9	28	65	125	216

12.

For the following data, draw a scatter plot. If we wanted to know when the population would reach 15,000, would the answer involve interpolation or extrapolation? Eyeball the line, and estimate the answer.

Year	Population
1990	11,500
1995	12,100
2000	12,700
2005	13,000
2010	13,750

13.

For the following data, draw a scatter plot. If we wanted to know when the temperature would reach 28 °F, would the answer involve interpolation or extrapolation? Eyeball the line and estimate the answer.

Temperature, °F	16	18	20	25	30
Time, seconds	46	50	54	55	62

Graphical

For the following exercises, match each scatterplot with one of the four specified correlations in Figure 9 and Figure 10.

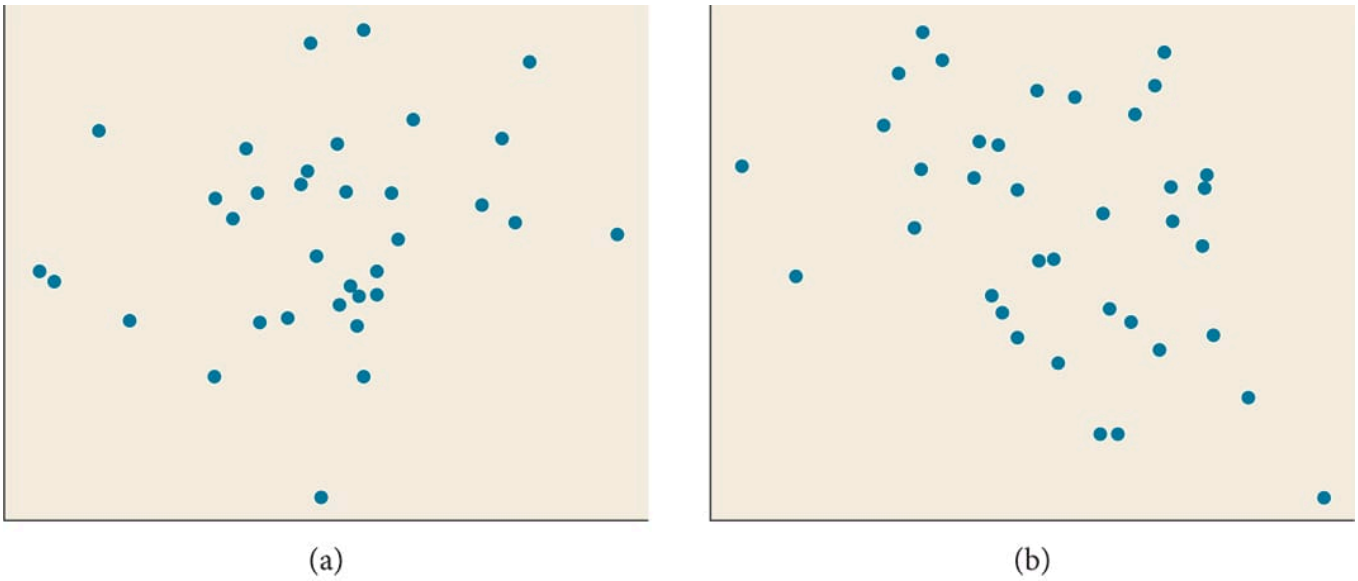


Figure 9

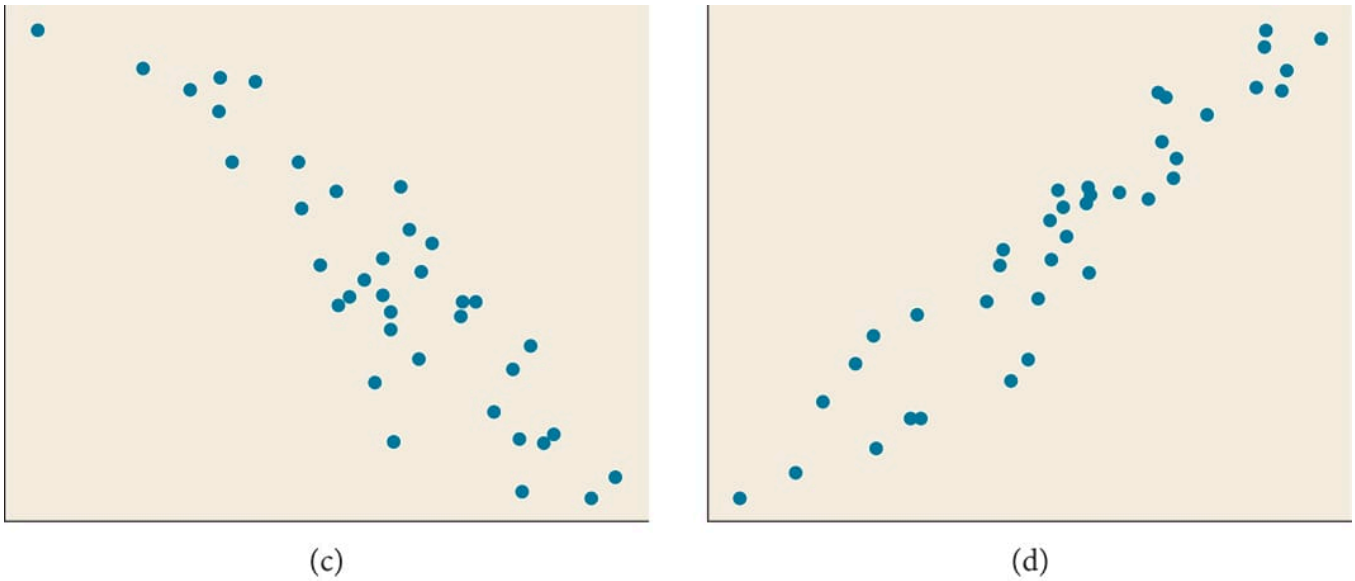


Figure 10

14.

$r = 0.95$

15.

$r = -0.89$

16.

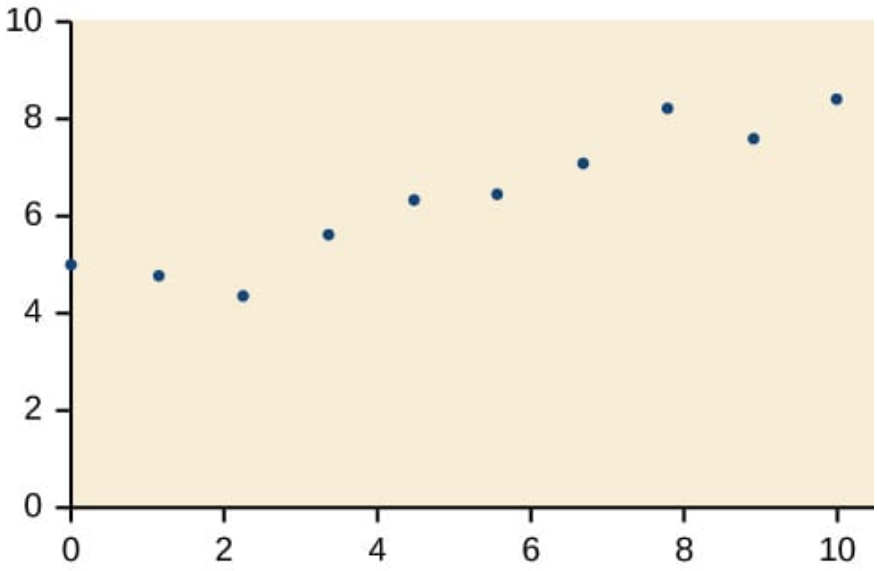
$r = 0.26$

17.

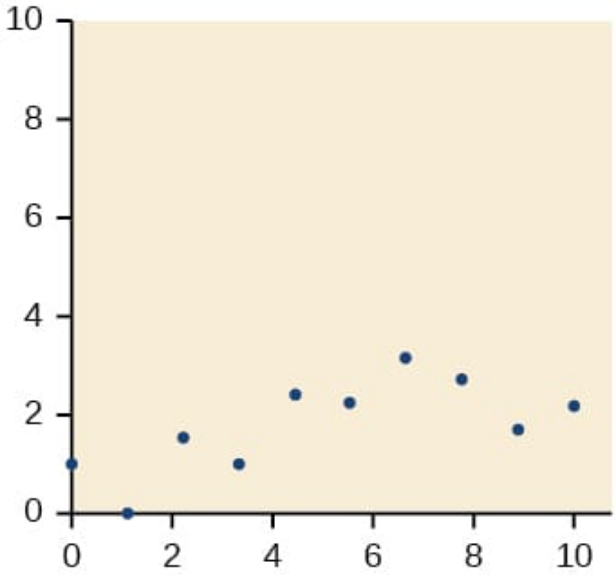
$r = -0.39$

For the following exercises, draw a best-fit line for the plotted data.

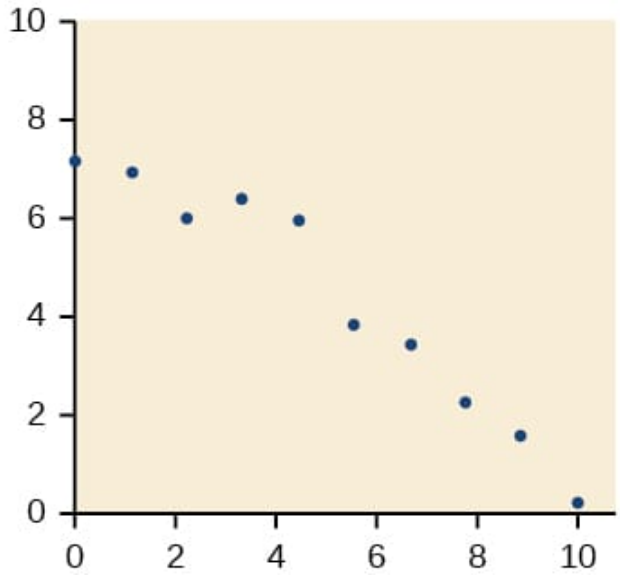
18.



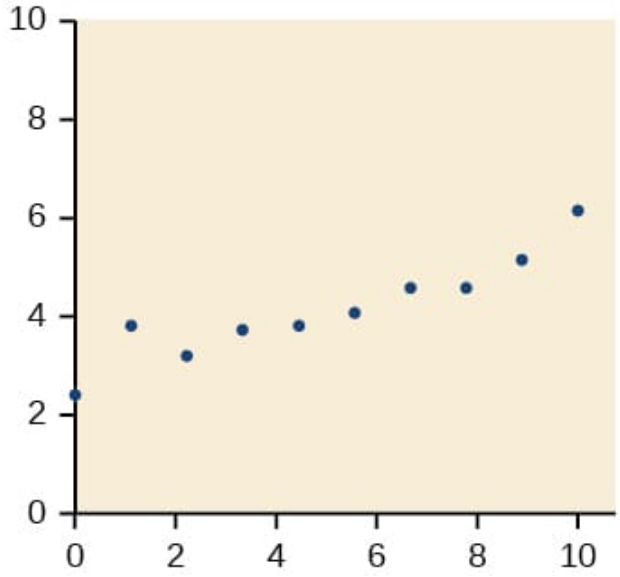
19.



20.



21.



Numeric

22.

The U.S. Census tracks the percentage of persons 25 years or older who are college graduates. That data for several years is given in Table 4[9]. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will the percentage exceed 35%?

Year	Percent Graduates
1990	21.3
1992	21.4
1994	22.2
1996	23.6
1998	24.4

Year	Percent Graduates
2000	25.6
2002	26.7
2004	27.7
2006	28
2008	29.4

Table 4

23.

The U.S. import of wine (in hectoliters) for several years is given in Table 5. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will imports exceed 12,000 hectoliters?

Year	Imports
1992	2665
1994	2688
1996	3565
1998	4129
2000	4584
2002	5655
2004	6549
2006	7950
2008	8487
2009	9462

Table 5

24.

Table 6 shows the year and the number of people unemployed in a particular city for several years. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will the number of unemployed reach 5?

Year	Number Unemployed
1990	750
1992	670
1994	650
1996	605
1998	550
2000	510
2002	460
2004	420
2006	380
2008	320

Table 6

Technology

For the following exercises, use each set of data to calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to 3 decimal places of accuracy.

25.

x	8	15	26	31	56
y	23	41	53	72	103

26.

x	5	7	10	12	15
y	4	12	17	22	24

27.

x	y	x	y
3	21.9	11	15.76
4	22.22	12	13.68
5	22.74	13	14.1
6	22.26	14	14.02
7	20.78	15	11.94
8	17.6	16	12.76
9	16.52	17	11.28
10	18.54	18	9.1

28.

x	y
4	44.8
5	43.1
6	38.8
7	39
8	38
9	32.7
10	30.1
11	29.3
12	27
13	25.8

29.

x	21	25	30	31	40	50
y	17	11	2	-1	-18	-40

30.

x	100	80	60	55	40	20
y	2000	1798	1589	1580	1390	1202

31.

x	900	988	1000	1010	1200	1205
y	70	80	82	84	105	108

Extensions

32.

Graph $f(x) = 0.5x + 10$. Pick a set of 5 ordered pairs using inputs $x = -2, 1, 5, 6, 9$ and use linear regression to verify that the function is a good fit for the data.

33.

Graph $f(x) = -2x - 10$. Pick a set of 5 ordered pairs using inputs $x = -2, 1, 5, 6, 9$ and use linear regression to verify the function.

For the following exercises, consider this scenario: The profit of a company decreased steadily over a ten-year span. The following ordered pairs shows dollars and the number of units sold in hundreds and the profit in thousands of over the ten-year span, (number of units sold, profit) for specific recorded years:

$(46, 1, 600)$, $(48, 1, 550)$, $(50, 1, 505)$, $(52, 1, 540)$, $(54, 1, 495)$.

34.

Use linear regression to determine a function P where the profit in thousands of dollars depends on the number of units sold in hundreds.

35.

Find to the nearest tenth and interpret the *x*-intercept.

36.

Find to the nearest tenth and interpret the *y*-intercept.

Real-World Applications

For the following exercises, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs shows the population and the year over the ten-year span, (population, year) for specific recorded years:

$(2500, 2000)$, $(2650, 2001)$, $(3000, 2003)$, $(3500, 2006)$, $(4200, 2010)$

37.

Use linear regression to determine a function y, where the year depends on the population. Round to three decimal places of accuracy.

38.

Predict when the population will hit 8,000.

For the following exercises, consider this scenario: The profit of a company increased steadily over a ten-year span. The following ordered pairs show the number of units sold in hundreds and the profit in thousands of over the ten year span, (number of units

sold, profit) for specific recorded years:

$(46, 250)\,, \quad (48, 305)\,, \quad (50, 350)\,, \quad (52, 390)\,, \quad (54, 410).$

39.

Use linear regression to determine a function *y*, where the profit in thousands of dollars depends on the number of units sold in hundreds .

40.

Predict when the profit will exceed one million dollars.

For the following exercises, consider this scenario: The profit of a company decreased steadily over a ten-year span. The following ordered pairs show dollars and the number of units sold in hundreds and the profit in thousands of over the ten-year span (number of units sold, profit) for specific recorded years:

$(46,\ 250),\ (48,\ 225),\ (50,\ 205),\ (52,\ 180),\ (54,\ 165).$

41.

Use linear regression to determine a function *y*, where the profit in thousands of dollars depends on the number of units sold in hundreds .

42.

Predict when the profit will dip below the $25,000 threshold.

Footnotes

- 5Selected data from http://classic.globe.gov/fsl/scientistsblog/2007/10/. Retrieved Aug 3, 2010
- 6Technically, the method minimizes the sum of the squared differences in the vertical direction between the line and the data values.
- 7For example, http://www.shodor.org/unchem/math/lls/leastsq.html
- 8http://www.bts.gov/publications/nati...ble_04_10.html
- 9http://www.census.gov/hhes/socdemo/e...cal/index.html. Accessed 5/1/2014.

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8.2: Modeling Using Variation

Learning Objectives

In this section, you will:

- Solve direct variation problems.
- Solve inverse variation problems.
- Solve problems involving joint variation.

A used-car company has just offered their best candidate, Nicole, a position in sales. The position offers 16% commission on her sales. Her earnings depend on the amount of her sales. For instance, if she sells a vehicle for $4,600, she will earn $736. She wants to evaluate the offer, but she is not sure how. In this section, we will look at relationships, such as this one, between earnings, sales, and commission rate.

Solving Direct Variation Problems

In the example above, Nicole's earnings can be found by multiplying her sales by her commission. The formula $e = 0.16s$ tells us her earnings, e, come from the product of 0.16, her commission, and the sale price of the vehicle. If we create a table, we observe that as the sales price increases, the earnings increase as well, which should be intuitive. See Table 1.

s, sales prices	$e = 0.16s$	Interpretation
$4,600	$e = 0.16\,(4{,}600) = 736$	A sale of a $4,600 vehicle results in $736 earnings.
$9,200	$e = 0.16\,(9{,}200) = 1{,}472$	A sale of a $9,200 vehicle results in $1472 earnings.
$18,400	$e = 0.16\,(18{,}400) = 2{,}944$	A sale of a $18,400 vehicle results in $2944 earnings.

Table 1

Notice that earnings are a multiple of sales. As sales increase, earnings increase in a predictable way. Double the sales of the vehicle from $4,600 to $9,200, and we double the earnings from $736 to $1,472. As the input increases, the output increases as a multiple of the input. A relationship in which one quantity is a constant multiplied by another quantity is called **direct variation**. Each variable in this type of relationship **varies directly** with the other.

Figure 1 represents the data for Nicole's potential earnings. We say that earnings vary directly with the sales price of the car. The formula $y = kx^n$ is used for direct variation. The value k is a nonzero constant greater than zero and is called the **constant of variation**. In this case, $k = 0.16$ and $n = 1$.

s

l:

ion problems.
ation problems.
volving joint variation.

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Variation Problems

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	$e = 0.16\,(18{,}400) = 2{,}944$	A sale of a $18,400 vehicle results in $2944 earnings.

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data-media-type="image/jpg"
52FPrecalculus%252FPrecalculus_2e_(OpenStax)%252F03%25253A_Polynomial_and_Rational_Functions%252F3.10%25253A_M
830c5246429f7688276ed66d" fileid="106213" />

d by an equation of the form

$$y = kx^n$$

lationship is direct variation and y varies directly with the $n\text{th}$ power of x. In direct variation relationships, there is a nonzero const
t of variation, which help defines the relationship between the variables.

of a direct variation problem, solve for an unknown.

1. Identify the input, x, and the output, y.
2. Determine the constant of variation. You may need to divide y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Example 1

Solving a Direct Variation Problem

The quantity y varies directly with the cube of x. If $y = 25$ when $x = 2$, find y when x is 6.

Answer available at the end of this section.

Analysis

The graph of this equation is a simple cubic, as shown in Figure 2.

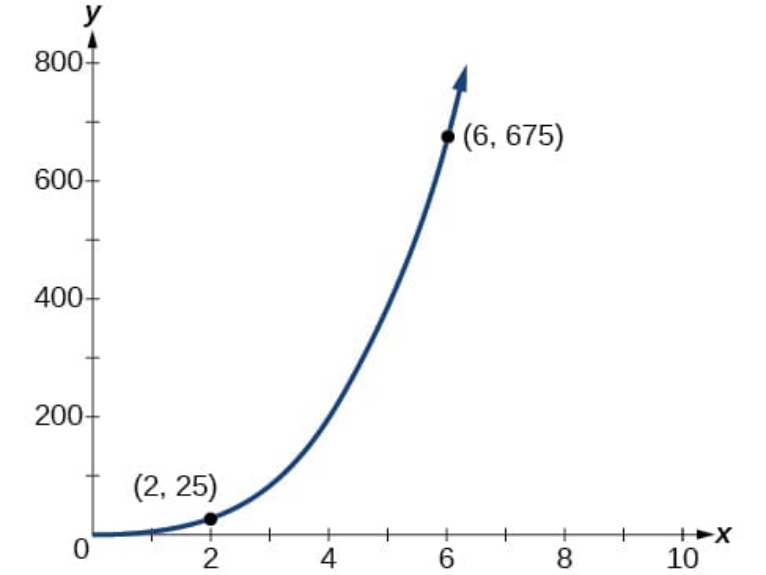


Figure 2

Q&A

Do the graphs of all direct variation equations look like Example 1?

No. Direct variation equations are power functions—they may be linear, quadratic, cubic, quartic, radical, etc. But all of the graphs pass through $(0,0)$.

Try It #1

The quantity y varies directly with the square of x. If $y = 24$ when $x = 3$, find y when x is 4.

Solving Inverse Variation Problems

Water temperature in an ocean varies inversely to the water's depth. Between the depths of 250 feet and 500 feet, the formula $T = \frac{14{,}000}{d}$ gives us the temperature in degrees Fahrenheit at a depth in feet below Earth's surface. Consider the Atlantic Ocean, which covers 22% of Earth's surface. At a certain location, at the depth of 500 feet, the temperature may be 28°F.

If we create Table 2, we observe that, as the depth increases, the water temperature decreases.

d, depth	$T = \frac{14{,}000}{d}$	Interpretation
500 ft	$\frac{14{,}000}{500} = 28$	At a depth of 500 ft, the water temperature is 28° F.
350 ft	$\frac{14{,}000}{350} = 40$	At a depth of 350 ft, the water temperature is 40° F.
250 ft	$\frac{14{,}000}{250} = 56$	At a depth of 250 ft, the water temperature is 56° F.

Table 2

We notice in the relationship between these variables that, as one quantity increases, the other decreases. The two quantities are said to be **inversely proportional** and each term **varies inversely** with the other. Inversely proportional relationships are also called **inverse variations**.

For our example, Figure 3 depicts the inverse variation. We say the water temperature varies inversely with the depth of the water because, as the depth increases, the temperature decreases. The formula $y = \frac{k}{x}$ for inverse variation in this case uses $k = 14{,}000$.

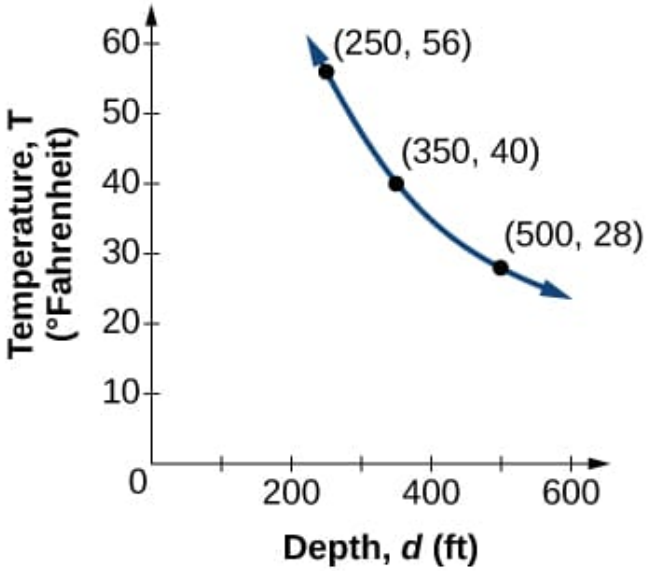


Figure 3

Inverse Variation

If x and y are related by an equation of the form

$$y = \frac{k}{x^n}$$

where k is a nonzero constant, then we say that y varies inversely with the nth power of x. In inversely proportional relationships, or inverse variations, there is a constant multiple $k = x^n y$.

Example 2

Writing a Formula for an Inversely Proportional Relationship

A tourist plans to drive 100 miles. Find a formula for the time the trip will take as a function of the speed the tourist drives.

Answer available at the end of this section.

How To

Given a description of an indirect variation problem, solve for an unknown.

1. Identify the input, x, and the output, y.
2. Determine the constant of variation. You may need to multiply y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Example 3

Solving an Inverse Variation Problem

A quantity y varies inversely with the cube of x. If $y = 25$ when $x = 2$, find y when x is 6.

Answer available at the end of this section.

Analysis

The graph of this equation is a rational function, as shown in Figure 4.

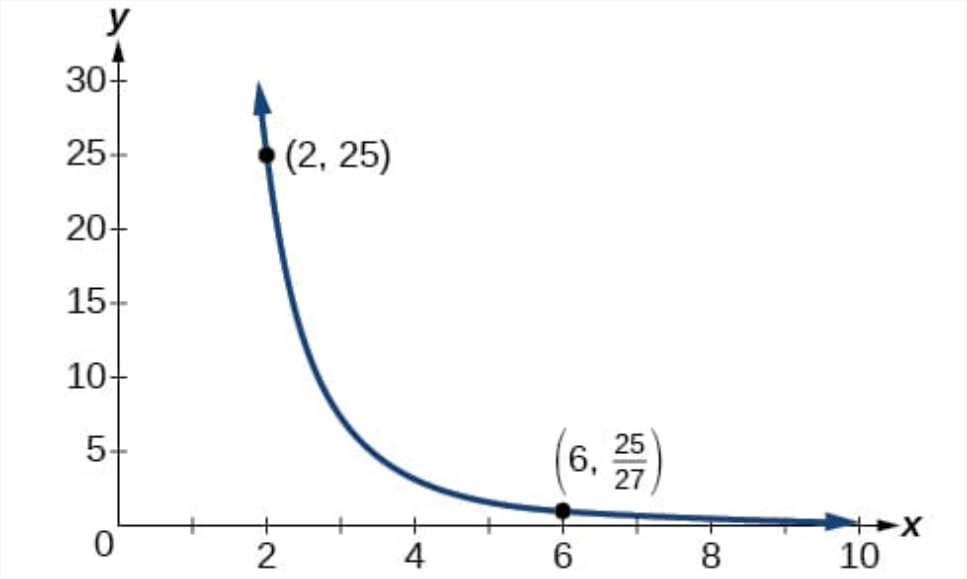


Figure 4

Try It #2

A quantity y varies inversely with the square of x. If $y = 8$ when $x = 3$, find y when x is 4.

Solving Problems Involving Joint Variation

Many situations are more complicated than a basic direct variation or inverse variation model. One variable often depends on multiple other variables. When a variable is dependent on the product or quotient of two or more variables, this is called joint variation. For example, the cost of busing students for each school trip varies with the number of students attending and the distance from the school. The variable c, cost, varies jointly with the number of students, n, and the distance, d.

Joint Variation

Joint variation occurs when a variable varies directly or inversely with multiple variables.

For instance, if x varies directly with both y and z, we have $x = kyz$. If x varies directly with y and inversely with z, we have $x = \frac{ky}{z}$. Notice that we only use one constant in a joint variation equation.

Example 4

Solving Problems Involving Joint Variation

A quantity x varies directly with the square of y and inversely with the cube root of z. If $x = 6$ when $y = 2$ and $z = 8$, find x when $y = 1$ and $z = 27$.

Answer available at the end of this section.

Try It #3

x varies directly with the square of y and inversely with z. If $x = 40$ when $y = 4$ and $z = 2$, find x when $y = 10$ and $z = 25$.

Media

Access these online resources for additional instruction and practice with direct and inverse variation.

- Direct Variation
- Inverse Variation
- Direct and Inverse Variation

3.9 Section Exercises

Verbal

1.

What is true of the appearance of graphs that reflect a direct variation between two variables?

2.

If two variables vary inversely, what will an equation representing their relationship look like?

3.

Is there a limit to the number of variables that can jointly vary? Explain.

Algebraic

For the following exercises, write an equation describing the relationship of the given variables.

4.

y varies directly as x and when $x = 6, \quad y = 12.$

5.

y varies directly as the square of x and when $x = 4, \quad y = 80.$

6.

y varies directly as the square root of x and when $x = 36, \quad y = 24.$

7.

y varies directly as the cube of x and when $x = 36, \quad y = 24.$

8.

y varies directly as the cube root of x and when $x = 27, \quad y = 15.$

9.

y varies directly as the fourth power of x and when $x = 1, \quad y = 6.$

10.

y varies inversely as x and when $x = 4, \quad y = 2.$

11.

y varies inversely as the square of x and when $x = 3, \quad y = 2.$

12.

y varies inversely as the cube of x and when $x = 2, \quad y = 5.$

13.

y varies inversely as the fourth power of x and when $x = 3, \quad y = 1.$

14.

y varies inversely as the square root of x and when $x = 25, \quad y = 3.$

15.

y varies inversely as the cube root of x and when $x = 64, \quad y = 5.$

16.

y varies jointly with x and z and when $x = 2$ and $z = 3$, $y = 36$.

17.

y varies jointly as x, z, and w and when $x = 1$, $z = 2$, $w = 5$, then $y = 100$.

18.

y varies jointly as the square of x and the square of z and when $x = 3$and $z = 4$, then $y = 72$.

19.

y varies jointly as x and the square root of z and when $x = 2$ and $z = 25$, then $y = 100$.

20.

y varies jointly as the square of x the cube of z and the square root of w. When $x = 1$,$z = 2$, and $w = 36$, then$y = 48$.

21.

y varies jointly as $x$and z and inversely as w. When $x = 3$, $z = 5$, and $w = 6$, then $y = 10$.

22.

y varies jointly as the square of x and the square root of z and inversely as the cube of w. When $x = 3, z = 4,$ and$w = 3,$ then$y = 6$.

23.

y varies jointly as x and z and inversely as the square root of w and the square of t . When $x = 3$, $z = 1, w = 25$, and $t = 2$, then $y = 6$.

Numeric

For the following exercises, use the given information to find the unknown value.

24.

y varies directly as x. When $x = 3$, then $y = 12$. Find $y$when $x = 20$.

25.

y varies directly as the square of x. When $x = 2$, then $y = 16$. Find y when $x = 8$.

26.

y varies directly as the cube of x. When $x = 3$, then $y = 5$. Find y when $x = 4$.

27.

y varies directly as the square root of x. When $x = 16$, then $y = 4$. Find $y$when$x = 36$.

28.

y varies directly as the cube root of x. When $x = 125$, then $y = 15$. Find y when $x = 1,000$.

29.

y varies inversely with x. When $x = 3$, then $y = 2$. Find y when $x = 1$.

30.

y varies inversely with the square of x. When $x = 4$, then $y = 3$. Find y when $x = 2$.

31.

y varies inversely with the cube of x. When $x = 3$, then $y = 1$. Find y when $x = 1$.

32.

y varies inversely with the square root of x. When $x = 64$, then $y = 12$. Find y when $x = 36$.

33.

y varies inversely with the cube root of x. When $x = 27$, then $y = 5$. Find y when $x = 125$.

34.

y varies jointly as $x \quad \text{and} \quad z$. When $x = 4$ and $z = 2$, then $y = 16$. Find y when $x = 3$ and $z = 3$.

35.

y varies jointly as $x, \quad z, \quad \text{and} \quad w$. When $x = 2$, $z = 1$, and $w = 12$, then $y = 72$. Find y when $x = 1$, $z = 2$, and $w = 3$.

36.

y varies jointly as x and the square of z. When $x = 2$ and $z = 4$, then $y = 144$. Find y when $x = 4$ and $z = 5$.

37.

y varies jointly as the square of x and the square root of z. When $x = 2$ and $z = 9$, then $y = 24$. Find y when $x = 3$ and $z = 25$.

38.

y varies jointly as x and z and inversely as w. When $x = 5$, $z = 2$, and $w = 20$, then $y = 4$. Find y when $x = 3$ and $z = 8$, and $w = 48$.

39.

y varies jointly as the square of x and the cube of z and inversely as the square root of w. When $x = 2$, $z = 2$, and $w = 64$, then $y = 12$. Find y when $x = 1$, $z = 3$, and $w = 4$.

40.

y varies jointly as the square of x and of z and inversely as the square root of w and of t. When $x = 2$, $z = 3$, $w = 16$, and $t = 3$, then $y = 1$. Find y when $x = 3$, $z = 2$, $w = 36$, and $t = 5$.

Technology

For the following exercises, use a calculator to graph the equation implied by the given variation.

41.

y varies directly with the square of x and when $x = 2, \quad y = 3$.

42.

y varies directly as the cube of x and when $x = 2, \quad y = 4$.

43.

y varies directly as the square root of x and when $x = 36, \quad y = 2$.

44.

y varies inversely with x and when $x = 6, \quad y = 2$.

45.

y varies inversely as the square of x and when $x = 1, \quad y = 4$.

Extensions

For the following exercises, use Kepler's Law, which states that the square of the time, T, required for a planet to orbit the Sun varies directly with the cube of the mean distance, a, that the planet is from the Sun.

46.

Using the Earth's time of 1 year and mean distance of 93 million miles, find the equation relating T and a.

47.

Use the result from the previous exercise to determine the time required for Mars to orbit the Sun if its mean distance is 142 million miles.

48.

Using Earth's distance of 150 million kilometers, find the equation relating T and a.

49.

Use the result from the previous exercise to determine the time required for Venus to orbit the Sun if its mean distance is 108 million kilometers.

50.

Using Earth's distance of 1 astronomical unit (A.U.), determine the time for Saturn to orbit the Sun if its mean distance is 9.54 A.U.

Real-World Applications

For the following exercises, use the given information to answer the questions.

51.

The distance s that an object falls varies directly with the square of the time, t, of the fall. If an object falls 16 feet in one second, how long for it to fall 144 feet?

52.

The velocity v of a falling object varies directly to the time, t, of the fall. If after 2 seconds, the velocity of the object is 64 feet per second, what is the velocity after 5 seconds?

53.

The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is 24 inches long and vibrates 128 times per second, what is the length of a string that vibrates 64 times per second?

54.

The volume of a gas held at constant temperature varies indirectly as the pressure of the gas. If the volume of a gas is 1200 cubic centimeters when the pressure is 200 millimeters of mercury, what is the volume when the pressure is 300 millimeters of mercury?

55.

The weight of an object above the surface of the Earth varies inversely with the square of the distance from the center of the Earth. If a body weighs 50 pounds when it is 3960 miles from Earth's center, what would it weigh it were 3970 miles from Earth's center?

56.

The intensity of light measured in foot-candles varies inversely with the square of the distance from the light source. Suppose the intensity of a light bulb is 0.08 foot-candles at a distance of 3 meters. Find the intensity level at 8 meters.

57.

The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is 40 amperes, the resistance is 10 ohms. Find the current if the resistance is 12 ohms.

58.

The force exerted by the wind on a plane surface varies jointly with the square of the velocity of the wind and with the area of the plane surface. If the area of the surface is 40 square feet surface and the wind velocity is 20 miles per hour, the resulting force is 15 pounds. Find the force on a surface of 65 square feet with a velocity of 30 miles per hour.

59.

The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute (rpm)) and the cube of the diameter. If the shaft of a certain material 3 inches in diameter can transmit 45 hp at 100 rpm, what must the diameter be in order to transmit 60 hp at 150 rpm?

60.

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Figure 1

Direct Variation

If x and y are related by an equation of the form

$$y = kx^n$$

then we say that the relationship is direct variation and y varies directly with the $n\text{th}$ power of x. In direct variation relationships, there is a nonzero constant ratio $k = \frac{y}{x^n}$, where k is called the constant of variation, which help defines the relationship between the variables.

How To

Given a description of a direct variation problem, solve for an unknown.

1. Identify the input, x, and the output, y.
2. Determine the constant of variation. You may need to divide y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Example 1

Solving a Direct Variation Problem

The quantity y varies directly with the cube of x. If $y = 25$ when $x = 2$, find y when x is 6.

Answer available at the end of this section.

Analysis

The graph of this equation is a simple cubic, as shown in Figure 2.

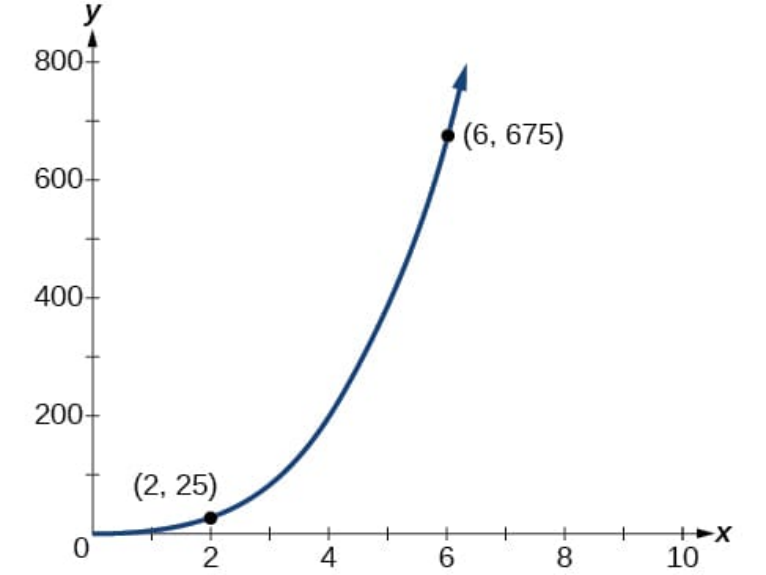


Figure 2

Q&A

Do the graphs of all direct variation equations look like Example 1?

No. Direct variation equations are power functions—they may be linear, quadratic, cubic, quartic, radical, etc. But all of the graphs pass through $(0,0)$.

Try It #1

The quantity y varies directly with the square of x. If $y = 24$ when $x = 3$, find y when x is 4.

Solving Inverse Variation Problems

Water temperature in an ocean varies inversely to the water's depth. Between the depths of 250 feet and 500 feet, the formula $T = \frac{14{,}000}{d}$ gives us the temperature in degrees Fahrenheit at a depth in feet below Earth's surface. Consider the Atlantic Ocean, which covers 22% of Earth's surface. At a certain location, at the depth of 500 feet, the temperature may be 28°F.

If we create Table 2, we observe that, as the depth increases, the water temperature decreases.

d, depth	$T = \frac{14{,}000}{d}$	Interpretation
500 ft	$\frac{14{,}000}{500} = 28$	At a depth of 500 ft, the water temperature is 28° F.
350 ft	$\frac{14{,}000}{350} = 40$	At a depth of 350 ft, the water temperature is 40° F.
250 ft	$\frac{14{,}000}{250} = 56$	At a depth of 250 ft, the water temperature is 56° F.

Table 2

We notice in the relationship between these variables that, as one quantity increases, the other decreases. The two quantities are said to be **inversely proportional** and each term **varies inversely** with the other. Inversely proportional relationships are also called **inverse variations**.

For our example, Figure 3 depicts the inverse variation. We say the water temperature varies inversely with the depth of the water because, as the depth increases, the temperature decreases. The formula $y = \frac{k}{x}$ for inverse variation in this case uses $k = 14{,}000$.

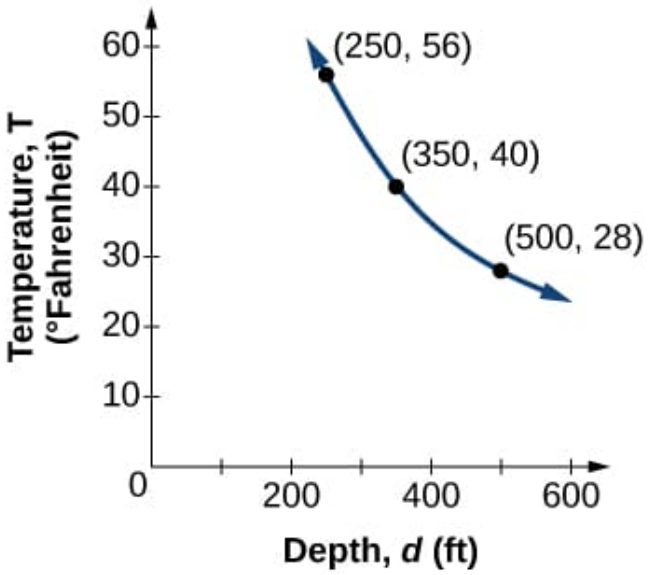


Figure 3

Inverse Variation

If x and y are related by an equation of the form

$$y = \frac{k}{x^n}$$

where k is a nonzero constant, then we say that y varies inversely with the nth power of x. In inversely proportional relationships, or inverse variations, there is a constant multiple $k = x^n y$.

Example 2

Writing a Formula for an Inversely Proportional Relationship

A tourist plans to drive 100 miles. Find a formula for the time the trip will take as a function of the speed the tourist drives.

Answer available at the end of this section.

How To

Given a description of an indirect variation problem, solve for an unknown.

1. Identify the input, x, and the output, y.
2. Determine the constant of variation. You may need to multiply y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Example 3

Solving an Inverse Variation Problem

A quantity y varies inversely with the cube of x. If $y = 25$ when $x = 2$, find y when x is 6.

Answer available at the end of this section.

Analysis

The graph of this equation is a rational function, as shown in Figure 4.

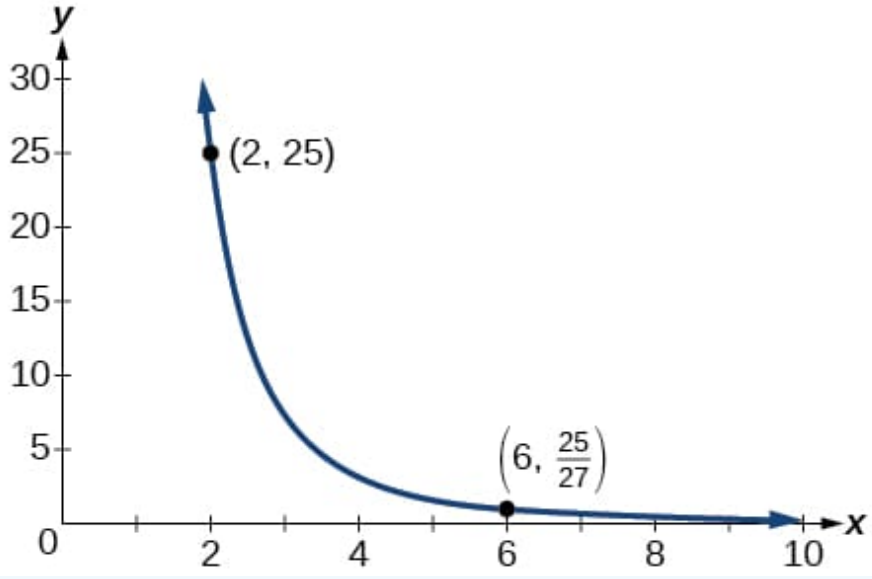


Figure 4

Try It #2

A quantity y varies inversely with the square of x. If $y = 8$ when $x = 3$, find y when x is 4.

Solving Problems Involving Joint Variation

Many situations are more complicated than a basic direct variation or inverse variation model. One variable often depends on multiple other variables. When a variable is dependent on the product or quotient of two or more variables, this is called joint variation. For example, the cost of busing students for each school trip varies with the number of students attending and the distance from the school. The variable c, cost, varies jointly with the number of students, n, and the distance, d.

Joint Variation

Joint variation occurs when a variable varies directly or inversely with multiple variables.

For instance, if x varies directly with both y and z, we have $x = kyz$. If x varies directly with y and inversely with z, we have $x = \frac{ky}{z}$. Notice that we only use one constant in a joint variation equation.

Example 4

Solving Problems Involving Joint Variation

A quantity x varies directly with the square of y and inversely with the cube root of z. If $x = 6$ when $y = 2$ and $z = 8$, find x when $y = 1$ and $z = 27$.

Answer available at the end of this section.

Try It #3

x varies directly with the square of y and inversely with z. If $x = 40$ when $y = 4$ and $z = 2$, find x when $y = 10$ and $z = 25$.

Media

Access these online resources for additional instruction and practice with direct and inverse variation.

- Direct Variation
- Inverse Variation
- Direct and Inverse Variation

Answers

The general formula for direct variation with a cube is $y = kx^3$. The constant can be found by dividing y by the cube of x.

$$\begin{aligned} k &= \tfrac{y}{x^3} \\ &= \tfrac{25}{2^3} \\ &= \tfrac{25}{8} \end{aligned}$$

Now use the constant to write an equation that represents this relationship.

$$y = \frac{25}{8}x^3$$

Substitute $x = 6$ and solve for y.

$$\begin{aligned} y &= \tfrac{25}{8}(6)^3 \\ &= 675 \end{aligned}$$

Recall that multiplying speed by time gives distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which the tourist drives, then $vt = \text{distance}$. Because the distance is fixed at 100 miles, $vt = 100$. Solving this relationship for the time gives us our function.

$$\begin{aligned} t(v) &= \tfrac{100}{v} \\ &= 100v^{-1} \end{aligned}$$

We can see that the constant of variation is 100 and, although we can write the relationship using the negative exponent, it is more common to see it written as a fraction.

The general formula for inverse variation with a cube is $y = \frac{k}{x^3}$. The constant can be found by multiplying y by the cube of x.

$$\begin{aligned} k &= x^3y \\ &= 2^3 \cdot 25 \\ &= 200 \end{aligned}$$

Now we use the constant to write an equation that represents this relationship.

$$\begin{aligned} y &= \tfrac{k}{x^3}, \quad k = 200 \\ y &= \tfrac{200}{x^3} \end{aligned}$$

Substitute $x = 6$ and solve for y.

$$\begin{aligned} y &= \tfrac{200}{6^3} \\ &= \tfrac{25}{27} \end{aligned}$$

Begin by writing an equation to show the relationship between the variables.

$$x = \frac{ky^2}{\sqrt[3]{z}}$$

Substitute $x = 6$, $y = 2$, and $z = 8$ to find the value of the constant k.

$$\begin{aligned} 6 &= \tfrac{k2^2}{\sqrt[3]{8}} \\ 6 &= \tfrac{4k}{2} \\ 3 &= k \end{aligned}$$

Now we can substitute the value of the constant into the equation for the relationship.

$$x = \frac{3y^2}{\sqrt[3]{z}}$$

To find x when $y = 1$ and $z = 27$, we will substitute values for y and z into our equation.

$$\begin{aligned} x &= \tfrac{3(1)^2}{\sqrt[3]{27}} \\ &= 1 \end{aligned}$$

The general formula for direct variation with a cube is $y = kx^3$. The constant can be found by dividing y by the cube of x.

$$\begin{aligned} k &= \tfrac{y}{x^3} \\ &= \tfrac{25}{2^3} \\ &= \tfrac{25}{8} \end{aligned}$$

Now use the constant to write an equation that represents this relationship.

$$y = \frac{25}{8}x^3$$

Substitute $x = 6$ and solve for y.

$$\begin{aligned} y &= \tfrac{25}{8}(6)^3 \\ &= 675 \end{aligned}$$

Recall that multiplying speed by time gives distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which the tourist drives, then $vt = \text{distance}$. Because the distance is fixed at 100 miles, $vt = 100$. Solving this relationship for the time gives us our function.

$$\begin{aligned} t(v) &= \tfrac{100}{v} \\ &= 100v^{-1} \end{aligned}$$

We can see that the constant of variation is 100 and, although we can write the relationship using the negative exponent, it is more common to see it written as a

fraction.

The general formula for inverse variation with a cube is $y = \frac{k}{x^3}$. The constant can be found by multiplying y by the cube of x.

$$\begin{aligned} k &= x^3 y \\ &= 2^3 \cdot 25 \\ &= 200 \end{aligned}$$

Now we use the constant to write an equation that represents this relationship.

$$\begin{aligned} y &= \frac{k}{x^3}, \quad k = 200 \\ y &= \frac{200}{x^3} \end{aligned}$$

Substitute $x = 6$ and solve for y.

$$\begin{aligned} y &= \frac{200}{6^3} \\ &= \frac{25}{27} \end{aligned}$$

Begin by writing an equation to show the relationship between the variables.

$$x = \frac{ky^2}{\sqrt[3]{z}}$$

Substitute $x = 6$, $y = 2$, and $z = 8$ to find the value of the constant k.

$$\begin{aligned} 6 &= \frac{k2^2}{\sqrt[3]{8}} \\ 6 &= \frac{4k}{2} \\ 3 &= k \end{aligned}$$

Now we can substitute the value of the constant into the equation for the relationship.

$$x = \frac{3y^2}{\sqrt[3]{z}}$$

To find x when $y = 1$ and $z = 27$, we will substitute values for y and z into our equation.

$$\begin{aligned} x &= \frac{3(1)^2}{\sqrt[3]{27}} \\ &= 1 \end{aligned}$$

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8.2E: Exercises

Exercises

Verbal

1.

What is true of the appearance of graphs that reflect a direct variation between two variables?

2.

If two variables vary inversely, what will an equation representing their relationship look like?

3.

Is there a limit to the number of variables that can jointly vary? Explain.

Algebraic

For the following exercises, write an equation describing the relationship of the given variables.

4.

y varies directly as x and when $x = 6, \quad y = 12.$

5.

y varies directly as the square of x and when $x = 4, \quad y = 80.$

6.

y varies directly as the square root of x and when $x = 36, \quad y = 24.$

7.

y varies directly as the cube of x and when $x = 36, \quad y = 24.$

8.

y varies directly as the cube root of x and when $x = 27, \quad y = 15.$

9.

y varies directly as the fourth power of x and when $x = 1, \quad y = 6.$

10.

y varies inversely as x and when $x = 4, \quad y = 2.$

11.

y varies inversely as the square of x and when $x = 3, \quad y = 2.$

12.

y varies inversely as the cube of x and when $x = 2, \quad y = 5$.

13.

y varies inversely as the fourth power of x and when $x = 3, \quad y = 1$.

14.

y varies inversely as the square root of x and when $x = 25, \quad y = 3$.

15.

y varies inversely as the cube root of x and when $x = 64, \quad y = 5$.

16.

y varies jointly with x and z and when $x = 2$ and $z = 3$, $y = 36$.

17.

y varies jointly as x, z, and w and when $x = 1$, $z = 2$, $w = 5$, then $y = 100$.

18.

y varies jointly as the square of x and the square of z and when $x = 3$and $z = 4$, then $y = 72$.

19.

y varies jointly as x and the square root of z and when $x = 2$ and $z =$, then $y = 100$.

20.

y varies jointly as the square of x the cube of z and the square root of w. When $x = 1$,$z = 2$, and $w = 36, \text{then} y = 48$.

21.

y varies jointly as $x$and z and inversely as w. When $x = 3$, $z = 5$, and $w = 6$, then $y = 10$.

22.

y varies jointly as the square of x and the square root of z and inversely as the cube of w. When $x = 3, z = 4, \text{and} w = 3, \text{then} y = 6$.

23.

y varies jointly as x and z and inversely as the square root of w and the square of t . When $x = 3$, $z = 1, w =$, and $t = 2$, then $y = 6$.

Numeric

For the following exercises, use the given information to find the unknown value.

24.

y varies directly as x. When $x = 3$, then $y =$. Find $y$when $x = 20$.

25.

y varies directly as the square of x. When $x = 2$, then $y =$. Find y when $x = 8$.

26.

y varies directly as the cube of x. When $x = 3$, then $y = 5$. Find y when $x = 4$.

27.

y varies directly as the square root of x. When $x = 16$, then $y = 4$. Find $y \text{when} x = 36$.

28.

y varies directly as the cube root of x. When $x = 125$, then $y =$. Find y when $x = 1,000$.

29.

y varies inversely with x. When $x = 3$, then $y = 2$. Find y when $x = 1$.

30.

y varies inversely with the square of x. When $x = 4$, then $y = 3$. Find y when $x = 2$.

31.

y varies inversely with the cube of x. When $x = 3$, then $y = 1$. Find y when $x = 1$.

32.

y varies inversely with the square root of x. When $x = 64$, then $y = 12$. Find y when $x = 36$.

33.

y varies inversely with the cube root of x. When $x = 27$, then $y = 5$. Find y when $x = 125$.

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y varies jointly as $x \quad \text{and} \quad z$. When $x = 4$ and $z = 2$, then $y = 16$. Find y when $x = 3$ and $z = 3$.

35.

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36.

y varies jointly as x and the square of z. When $x = 2$ and $z = 4$, then $y = 144$. Find y when $x = 4$ and $z = 5$.

37.

y varies jointly as the square of x and the square root of z. When $x = 2$ and $z = 9$, then $y = 24$. Find y when $x = 3$ and $z = 25$.

38.

y varies jointly as x and z and inversely as w. When $x = 5$, $z = 2$, and $w = 20$, then $y = 4$. Find y when $x = 3$ and $z = 8$, and $w = 48$.

39.

y varies jointly as the square of x and the cube of z and inversely as the square root of w. When $x = 2$, $z = 2$, and $w = 64$, then $y = 12$. Find y when $x = 1$, $z = 3$, and $w = 4$.

40.

y varies jointly as the square of x and of z and inversely as the square root of w and of t. When $x = 2$, $z = 3$, $w = 16$, and $t = 3$, then $y = 1$. Find y when $x = 3$, $z = 2$, $w = 36$, and $t = 5$.

Technology

For the following exercises, use a calculator to graph the equation implied by the given variation.

41.

y varies directly with the square of x and when $x = 2, \quad y = 3$.

42.

y varies directly as the cube of x and when $x = 2, \quad y = 4$.

43.

y varies directly as the square root of x and when $x = 36, \quad y = 2$.

44.

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45.

y varies inversely as the square of x and when $x = 1, \quad y = 4$.

Extensions

For the following exercises, use Kepler's Law, which states that the square of the time, T, required for a planet to orbit the Sun varies directly with the cube of the mean distance, a, that the planet is from the Sun.

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Use the result from the previous exercise to determine the time required for Mars to orbit the Sun if its mean distance is 142 million miles.

48.

Using Earth's distance of 150 million kilometers, find the equation relating T and a.

49.

Use the result from the previous exercise to determine the time required for Venus to orbit the Sun if its mean distance is 108 million kilometers.

50.

Using Earth's distance of 1 astronomical unit (A.U.), determine the time for Saturn to orbit the Sun if its mean distance is 9.54 A.U.

Real-World Applications

For the following exercises, use the given information to answer the questions.

51.

The distance s that an object falls varies directly with the square of the time, t, of the fall. If an object falls 16 feet in one second,

how long for it to fall 144 feet?

52.

The velocity v of a falling object varies directly to the time, t, of the fall. If after 2 seconds, the velocity of the object is 64 feet per second, what is the velocity after 5 seconds?

53.

The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is 24 inches long and vibrates 128 times per second, what is the length of a string that vibrates 64 times per second?

54.

The volume of a gas held at constant temperature varies indirectly as the pressure of the gas. If the volume of a gas is 1200 cubic centimeters when the pressure is 200 millimeters of mercury, what is the volume when the pressure is 300 millimeters of mercury?

55.

The weight of an object above the surface of the Earth varies inversely with the square of the distance from the center of the Earth. If a body weighs 50 pounds when it is 3960 miles from Earth's center, what would it weigh it were 3970 miles from Earth's center?

56.

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57.

The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is 40 amperes, the resistance is 10 ohms. Find the current if the resistance is 12 ohms.

58.

The force exerted by the wind on a plane surface varies jointly with the square of the velocity of the wind and with the area of the plane surface. If the area of the surface is 40 square feet surface and the wind velocity is 20 miles per hour, the resulting force is 15 pounds. Find the force on a surface of 65 square feet with a velocity of 30 miles per hour.

59.

The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute (rpm)) and the cube of the diameter. If the shaft of a certain material 3 inches in diameter can transmit 45 hp at 100 rpm, what must the diameter be in order to transmit 60 hp at 150 rpm?

60.

The kinetic energy K of a moving object varies jointly with its mass m and the square of its velocity v. If an object weighing 40 kilograms with a velocity of 15 meters per second has a kinetic energy of 1000 joules, find the kinetic energy if the velocity is increased to 20 meters per second.

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8.3: Fitting Exponential Models to Data

Learning Objectives

In this section, you will:

- Build an exponential model from data.
- Build a logarithmic model from data.
- Build a logistic model from data.

In previous sections of this chapter, we were either given a function explicitly to graph or evaluate, or we were given a set of points that were guaranteed to lie on the curve. Then we used algebra to find the equation that fit the points exactly. In this section, we use a modeling technique called *regression analysis* to find a curve that models data collected from real-world observations. With regression analysis, we don't expect all the points to lie perfectly on the curve. The idea is to find a model that best fits the data. Then we use the model to make predictions about future events.

Do not be confused by the word *model*. In mathematics, we often use the terms *function*, *equation*, and *model* interchangeably, even though they each have their own formal definition. The term *model* is typically used to indicate that the equation or function approximates a real-world situation.

We will concentrate on three types of regression models in this section: exponential, logarithmic, and logistic. Having already worked with each of these functions gives us an advantage. Knowing their formal definitions, the behavior of their graphs, and some of their real-world applications gives us the opportunity to deepen our understanding. As each regression model is presented, key features and definitions of its associated function are included for review. Take a moment to rethink each of these functions, reflect on the work we've done so far, and then explore the ways regression is used to model real-world phenomena.

Building an Exponential Model from Data

As we've learned, there are a multitude of situations that can be modeled by exponential functions, such as investment growth, radioactive decay, atmospheric pressure changes, and temperatures of a cooling object. What do these phenomena have in common? For one thing, all the models either increase or decrease as time moves forward. But that's not the whole story. It's the *way* data increase or decrease that helps us determine whether it is best modeled by an exponential equation. Knowing the behavior of exponential functions in general allows us to recognize when to use exponential regression, so let's review exponential growth and decay.

Recall that exponential functions have the form $y = ab^x$ or $y = A_0e^{kx}$. When performing regression analysis, we use the form most commonly used on graphing utilities, $y = ab^x$. Take a moment to reflect on the characteristics we've already learned about the exponential function $y = ab^x$ (assume $a > 0$) :

- b must be greater than zero and not equal to one.
- The initial value of the model is $y = a$.
  - If $b > 1$, the function models exponential growth. As x increases, the outputs of the model increase slowly at first, but then increase more and more rapidly, without bound.
  - If $0 < b < 1$, the function models exponential decay. As x increases, the outputs for the model decrease rapidly at first and then level off to become asymptotic to the *x*-axis. In other words, the outputs never become equal to or less than zero.

As part of the results, your calculator will display a number known as the *correlation coefficient*, labeled by the variable r, or r^2. (You may have to change the calculator's settings for these to be shown.) The values are an indication of the "goodness of fit" of the regression equation to the data. We more commonly use the value of r^2 instead of r, but the closer either value is to 1, the better the regression equation approximates the data.

Exponential Regression

Exponential regression is used to model situations in which growth begins slowly and then accelerates rapidly without bound, or where decay begins rapidly and then slows down to get closer and closer to zero. We use the command "**ExpReg**" on a graphing utility to fit an exponential function to a set of data points. This returns an equation of the form,

$$y = ab^x$$

Note that:

- b must be non-negative.
- when $b > 1$, we have an exponential growth model.
- when $0 < b < 1$, we have an exponential decay model.

How To

Given a set of data, perform exponential regression using a graphing utility.

1. Use the **STAT** then **EDIT** menu to enter given data.
   a. Clear any existing data from the lists.
   b. List the input values in the L1 column.
   c. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the **STATPLOT** feature.
   a. Use **ZOOM** [**9**] to adjust axes to fit the data.
   b. Verify the data follow an exponential pattern.
3. Find the equation that models the data.
   a. Select "**ExpReg**" from the **STAT** then **CALC** menu.
   b. Use the values returned for *a* and *b* to record the model, $y = ab^x$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

Example 1

Using Exponential Regression to Fit a Model to Data

In 2007, a university study was published investigating the crash risk of alcohol impaired driving. Data from 2,871 crashes were used to measure the association of a person's blood alcohol level (BAC) with the risk of being in an accident. Table 1 shows results from the study [9]. The *relative risk* is a measure of how many times more likely a person is to crash. So, for example, a person with a BAC of 0.09 is 3.54 times as likely to crash as a person who has not been drinking alcohol.

BAC	0	0.01	0.03	0.05	0.07	0.09
Relative Risk of Crashing	1	1.03	1.06	1.38	2.09	3.54
BAC	0.11	0.13	0.15	0.17	0.19	0.21
Relative Risk of Crashing	6.41	12.6	22.1	39.05	65.32	99.78

Table 1

a. Let x represent the BAC level, and let y represent the corresponding relative risk. Use exponential regression to fit a model to these data.
b. After 6 drinks, a person weighing 160 pounds will have a BAC of about 0.16. How many times more likely is a person with this weight to crash if they drive after having a 6-pack of beer? Round to the nearest hundredth.

Answer

a. Using the **STAT** then **EDIT** menu on a graphing utility, list the **BAC** values in L1 and the relative risk values in L2. Then use the **STATPLOT** feature to verify that the scatterplot follows the exponential pattern shown in Figure 1:

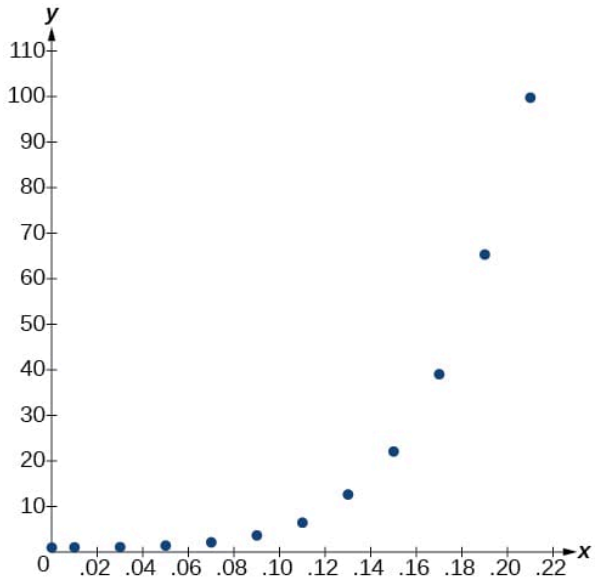


Figure 1

Use the "**ExpReg**" command from the **STAT** then **CALC** menu to obtain the exponential model,

$$y = 0.58304829(2.20720213\text{E}10)^x$$

Converting from scientific notation, we have:

$$y = 0.58304829(22{,}072{,}021{,}300)^x$$

Notice that $r^2 \approx 0.97$ which indicates the model is a good fit to the data. To see this, graph the model in the same window as the scatterplot to verify it is a good fit as shown in Figure 2:

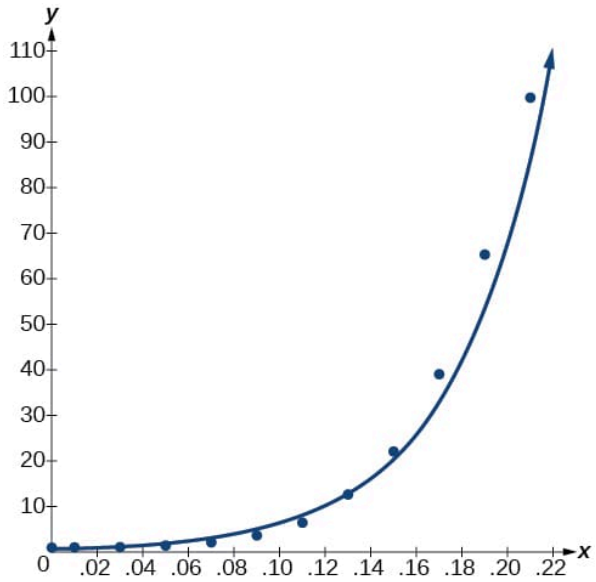


Figure 2

b. Use the model to estimate the risk associated with a BAC of 0.16. Substitute 0.16 for x in the model and solve for y.

If a 160-pound person drives after having 6 drinks, he or she is about 26.35 times more likely to crash than if driving while sober.

Try It #1

Table 2 shows a recent graduate's credit card balance each month after graduation.

Month	1	2	3	4	5	6	7	8
Debt ($)	620.00	761.88	899.80	1039.93	1270.63	1589.04	1851.31	2154.92

Table 2

ⓐ Use exponential regression to fit a model to these data.
ⓑ If spending continues at this rate, what will the graduate's credit card debt be one year after graduating?

Q&A

Is it reasonable to assume that an exponential regression model will represent a situation indefinitely?

No. Remember that models are formed by real-world data gathered for regression. It is usually reasonable to make estimates within the interval of original observation (interpolation). However, when a model is used to make predictions, it is important to use reasoning skills to determine whether the model makes sense for inputs far beyond the original observation interval (extrapolation).

Building a Logarithmic Model from Data

Just as with exponential functions, there are many real-world applications for logarithmic functions: intensity of sound, pH levels of solutions, yields of chemical reactions, production of goods, and growth of infants. As with exponential models, data modeled by logarithmic functions are either always increasing or always decreasing as time moves forward. Again, it is the *way* they increase or decrease that helps us determine whether a logarithmic model is best.

Recall that logarithmic functions increase or decrease rapidly at first, but then steadily slow as time moves on. By reflecting on the characteristics we've already learned about this function, we can better analyze real world situations that reflect this type of growth or decay. When performing logarithmic regression analysis, we use the form of the logarithmic function most commonly used on graphing utilities, $y = a + b\ln(x)$. For this function

- All input values, x, must be greater than zero.
- The point $(1, a)$ is on the graph of the model.
- If $b > 0$, the model is increasing. Growth increases rapidly at first and then steadily slows over time.
- If $b < 0$, the model is decreasing. Decay occurs rapidly at first and then steadily slows over time.

Logarithmic Regression

Logarithmic regression is used to model situations where growth or decay accelerates rapidly at first and then slows over time. We use the command "LnReg" on a graphing utility to fit a logarithmic function to a set of data points. This returns an equation of the form,

$$y = a + b\ln(x)$$

Note that

- all input values, x, must be non-negative.

- when $b > 0$, the model is increasing.
- when $b < 0$, the model is decreasing.

How To

Given a set of data, perform logarithmic regression using a graphing utility.

1. Use the **STAT** then **EDIT** menu to enter given data.
   a. Clear any existing data from the lists.
   b. List the input values in the L1 column.
   c. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the **STATPLOT** feature.
   a. Use **ZOOM** [**9**] to adjust axes to fit the data.
   b. Verify the data follow a logarithmic pattern.
3. Find the equation that models the data.
   a. Select "**LnReg**" from the **STAT** then **CALC** menu.
   b. Use the values returned for *a* and *b* to record the model, $y = a + b\ln(x)$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

Example 2

Using Logarithmic Regression to Fit a Model to Data

Due to advances in medicine and higher standards of living, life expectancy has been increasing in most developed countries since the beginning of the 20th century.

Table 3 shows the average life expectancies, in years, of Americans from 1900–2010[10].

Year	1900	1910	1920	1930	1940	1950
Life Expectancy(Years)	47.3	50.0	54.1	59.7	62.9	68.2
Year	1960	1970	1980	1990	2000	2010
Life Expectancy(Years)	69.7	70.8	73.7	75.4	76.8	78.7

Table 3

1. ⓐ Let x represent time in decades starting with $x = 1$ for the year 1900, $x = 2$ for the year 1910, and so on. Let y represent the corresponding life expectancy. Use logarithmic regression to fit a model to these data.
2. ⓑ Use the model to predict the average American life expectancy for the year 2030.

Answer

1. ⓐ Using the **STAT** then **EDIT** menu on a graphing utility, list the years using values 1–12 in L1 and the corresponding life expectancy in L2. Then use the **STATPLOT** feature to verify that the scatterplot follows a logarithmic pattern as shown in Figure 3:

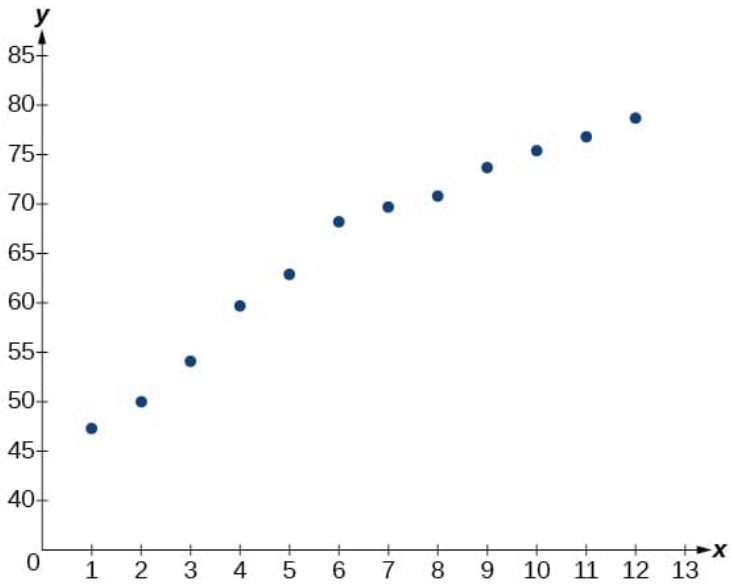


Figure 3

Use the "**LnReg**" command from the **STAT** then **CALC** menu to obtain the logarithmic model,

$$y = 42.52722583 + 13.85752327 \ln(x)$$

Next, graph the model in the same window as the scatterplot to verify it is a good fit as shown in Figure 4:

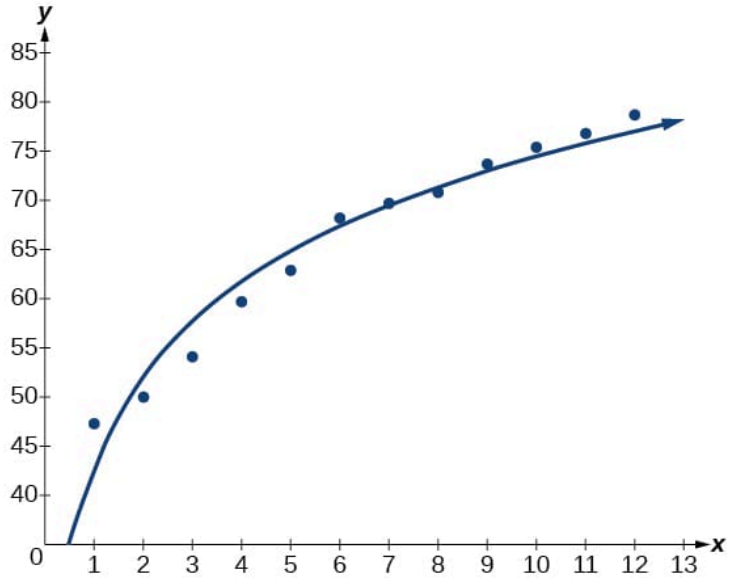


Figure 4

	y	$= 42.52722583 + 13.85752327 \ln(x)$	Use the regression model found in part (a).
2.		$= 42.52722583 + 13.85752327 \ln(14)$	Substitute 14 for x.
		≈ 79.1	Round to the nearest tenth.

Try It #2

Sales of a video game released in the year 2000 took off at first, but then steadily slowed as time moved on. Table 4 shows the number of games sold, in thousands, from the years 2000–2010.

Year	2000	2001	2002	2003	2004	2005
Number Sold (thousands)	142	149	154	155	159	161
Year	2006	2007	2008	2009	2010	-
Number Sold (thousands)	163	164	164	166	167	-

Table 4

- ⓐ Let x represent time in years starting with $x = 1$ for the year 2000. Let y represent the number of games sold in thousands. Use logarithmic regression to fit a model to these data.
- ⓑ If games continue to sell at this rate, how many games will sell in 2015? Round to the nearest thousand.

Building a Logistic Model from Data

Like exponential and logarithmic growth, logistic growth increases over time. One of the most notable differences with logistic growth models is that, at a certain point, growth steadily slows and the function approaches an upper bound, or *limiting value*. Because of this, logistic regression is best for modeling phenomena where there are limits in expansion, such as availability of living space or nutrients.

It is worth pointing out that logistic functions actually model resource-limited exponential growth. There are many examples of this type of growth in real-world situations, including population growth and spread of disease, rumors, and even stains in fabric. When performing logistic regression analysis, we use the form most commonly used on graphing utilities:

$$y = \frac{c}{1 + ae^{-bx}}$$

Recall that:

- $\frac{c}{1+a}$ is the initial value of the model.
- when $b > 0$, the model increases rapidly at first until it reaches its point of maximum growth rate, $\left(\frac{\ln(a)}{b}, \frac{c}{2}\right)$. At that point, growth steadily slows and the function becomes asymptotic to the upper bound $y = c$.
- c is the limiting value, sometimes called the *carrying capacity*, of the model.

Logistic Regression

Logistic regression is used to model situations where growth accelerates rapidly at first and then steadily slows to an upper limit. We use the command “Logistic” on a graphing utility to fit a logistic function to a set of data points. This returns an equation of the form

$$y = \frac{c}{1 + ae^{-bx}}$$

Note that

- The initial value of the model is $\frac{c}{1+a}$.
- Output values for the model grow closer and closer to $y = c$ as time increases.

How To

Given a set of data, perform logistic regression using a graphing utility.

1. Use the **STAT** then **EDIT** menu to enter given data.
   a. Clear any existing data from the lists.

b. List the input values in the L1 column.
c. List the output values in the L2 column.

2. Graph and observe a scatter plot of the data using the **STATPLOT** feature.
   a. Use **ZOOM** [**9**] to adjust axes to fit the data.
   b. Verify the data follow a logistic pattern.
3. Find the equation that models the data.
   a. Select "**Logistic**" from the **STAT** then **CALC** menu.
   b. Use the values returned for a, b, and c to record the model, $y = \frac{c}{1+ae^{-bx}}$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

Example 3

Using Logistic Regression to Fit a Model to Data

Mobile telephone service has increased rapidly in America since the mid 1990s. Today, almost all residents have cellular service. Table 5 shows the percentage of Americans with cellular service between the years 1995 and 2012 [11].

Year	Americans with Cellular Service (%)	Year	Americans with Cellular Service (%)
1995	12.69	2004	62.852
1996	16.35	2005	68.63
1997	20.29	2006	76.64
1998	25.08	2007	82.47
1999	30.81	2008	85.68
2000	38.75	2009	89.14
2001	45.00	2010	91.86
2002	49.16	2011	95.28
2003	55.15	2012	98.17

Table 5

- ⓐ Let x represent time in years starting with $x = 0$ for the year 1995. Let y represent the corresponding percentage of residents with cellular service. Use logistic regression to fit a model to these data.
- ⓑ Use the model to calculate the percentage of Americans with cell service in the year 2013. Round to the nearest tenth of a percent.
- ⓒ Discuss the value returned for the upper limit, c. What does this tell you about the model? What would the limiting value be if the model were exact?

Answer

- ⓐ Using the **STAT** then **EDIT** menu on a graphing utility, list the years using values 0–15 in L1 and the corresponding percentage in L2. Then use the **STATPLOT** feature to verify that the scatterplot follows a logistic pattern as shown in Figure 5:

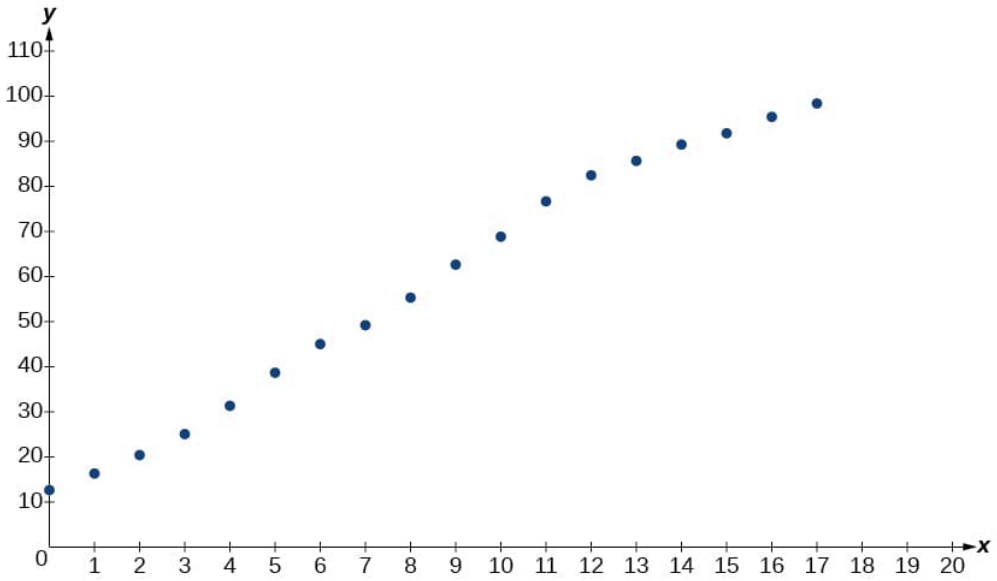


Figure 5

Use the "**Logistic**" command from the **STAT** then **CALC** menu to obtain the logistic model,

$$y = \frac{105.7379526}{1 + 6.88328979e^{-0.2595440013x}}$$

Next, graph the model in the same window as shown in Figure 6 the scatterplot to verify it is a good fit:

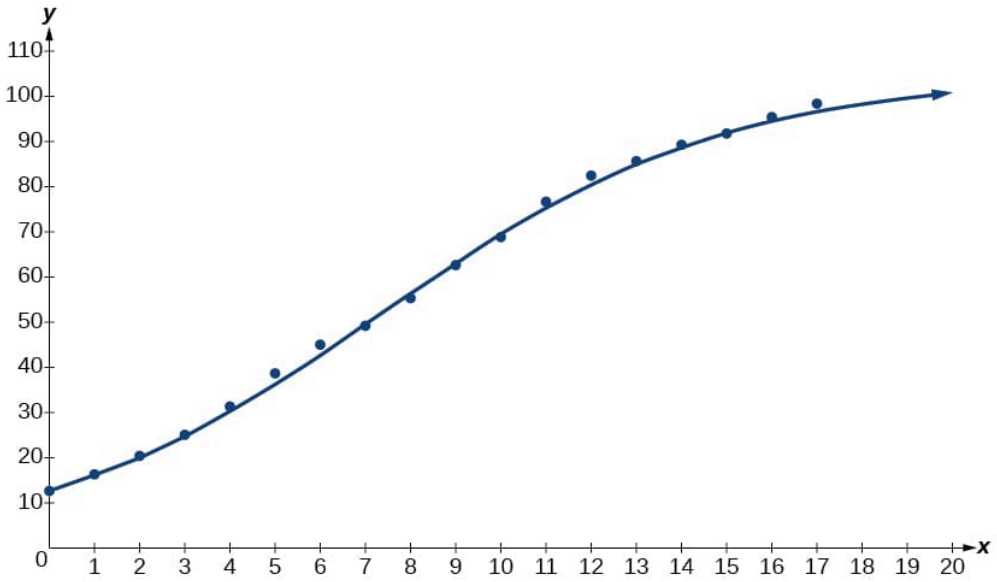


Figure 6

- To approximate the percentage of Americans with cellular service in the year 2013, substitute $x = 18$ for the in the model and solve for y :

  According to the model, about 99.3% of Americans had cellular service in 2013.

- The model gives a limiting value of about 105. This means that the maximum possible percentage of Americans with cellular service would be 105%, which is impossible. (How could over 100% of a population have cellular service?) If the model were exact, the limiting value would be $c = 100$ and the model's outputs would get very close to, but never actually reach 100%. After all, there will always be someone out there without cellular service!

Try It #3

Table 6 shows the population, in thousands, of harbor seals in the Wadden Sea over the years 1997 to 2012.

Year	Seal Population (Thousands)	Year	Seal Population (Thousands)
1997	3.493	2005	19.590
1998	5.282	2006	21.955
1999	6.357	2007	22.862
2000	9.201	2008	23.869
2001	11.224	2009	24.243
2002	12.964	2010	24.344
2003	16.226	2011	24.919
2004	18.137	2012	25.108

Table 6

- ⓐ Let x represent time in years starting with $x = 0$ for the year 1997. Let y represent the number of seals in thousands. Use logistic regression to fit a model to these data.
- ⓑ Use the model to predict the seal population for the year 2020.
- ⓒ To the nearest whole number, what is the limiting value of this model?

Media

Access this online resource for additional instruction and practice with exponential function models.

- Exponential Regression on a Calculator

Choosing an Appropriate Model for Data

Now that we have discussed various mathematical models, we need to learn how to choose the appropriate model for the raw data we have. Many factors influence the choice of a mathematical model, among which are experience, scientific laws, and patterns in the data itself. Not all data can be described by elementary functions. Sometimes, a function is chosen that approximates the data over a given interval. For instance, suppose data were gathered on the number of homes bought in the United States from the years 1960 to 2013. After plotting these data in a scatter plot, we notice that the shape of the data from the years 2000 to 2013 follow a logarithmic curve. We could restrict the interval from 2000 to 2010, apply regression analysis using a logarithmic model, and use it to predict the number of home buyers for the year 2015.

Three kinds of functions that are often useful in mathematical models are linear functions, exponential functions, and logarithmic functions. If the data lies on a straight line, or seems to lie approximately along a straight line, a linear model may be best. If the data is non-linear, we often consider an exponential or logarithmic model, though other models, such as quadratic models, may also be considered.

In choosing between an exponential model and a logarithmic model, we look at the way the data curves. This is called the concavity. If we draw a line between two data points, and all (or most) of the data between those two points lies above that line, we say the curve is concave down. We can think of it as a bowl that bends downward and therefore cannot hold water. If all (or most) of the data between those two points lies below the line, we say the curve is concave up. In this case, we can think of a bowl that bends upward and can therefore hold water. An exponential curve, whether rising or falling, whether representing growth or decay, is always concave up away from its horizontal asymptote. A logarithmic curve is always concave away from its vertical asymptote. In the case of positive data, which is the most common case, an exponential curve is always concave up, and a logarithmic curve

always concave down.

A logistic curve changes concavity. It starts out concave up and then changes to concave down beyond a certain point, called a point of inflection.

After using the graph to help us choose a type of function to use as a model, we substitute points, and solve to find the parameters. We reduce round-off error by choosing points as far apart as possible.

Example 8.3.7: Choosing a Mathematical Model

Does a linear, exponential, logarithmic, or logistic model best fit the values listed in Table 8.3.1? Find the model, and use a graph to check your choice.

x	1	2	3	4	5	6	7	8	9
y	0	1.386	2.197	2.773	3.219	3.584	3.892	4.159	4.394

Table 8.3.1

Solution

First, plot the data on a graph as in Figure 8.3.8. For the purpose of graphing, round the data to two decimal places.

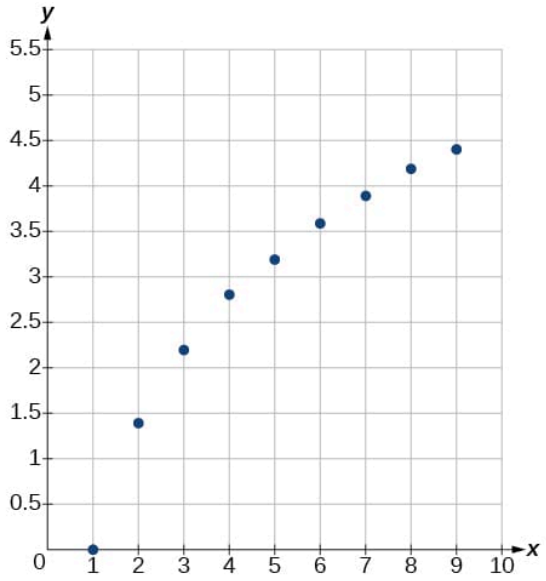


Figure 8.3.8

Clearly, the points do not lie on a straight line, so we reject a linear model. If we draw a line between any two of the points, most or all of the points between those two points lie above the line, so the graph is concave down, suggesting a logarithmic model. We can try $y = aln(bx)$. Plugging in the first point, $(1,0)$, gives $0 = alnb$. We reject the case that $a = 0$ (if it were, all outputs would be 0), so we know $ln(b) = 0$. Thus $b = 1$ and $y = aln(\mathrm{x})$. Next we can use the point $(9,4.394)$ to solve for a:

$$y = aln(x)4.394 = aln(9)a = 4.394ln(9)$$

Because $a = 4.394ln(9) \approx 2$, an appropriate model for the data is $y = 2ln(x)$.

To check the accuracy of the model, we graph the function together with the given points as in Figure 8.3.9.

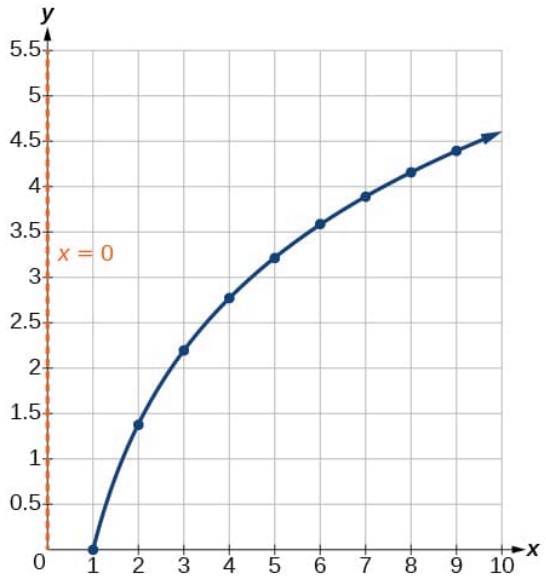


Figure $8.3.9$ *The graph of* $y = 2lnx$.

We can conclude that the model is a good fit to the data.

Compare Figure $8.3.9$ to the graph of $y = ln(x2)$ shown in Figure $8.3.10$.

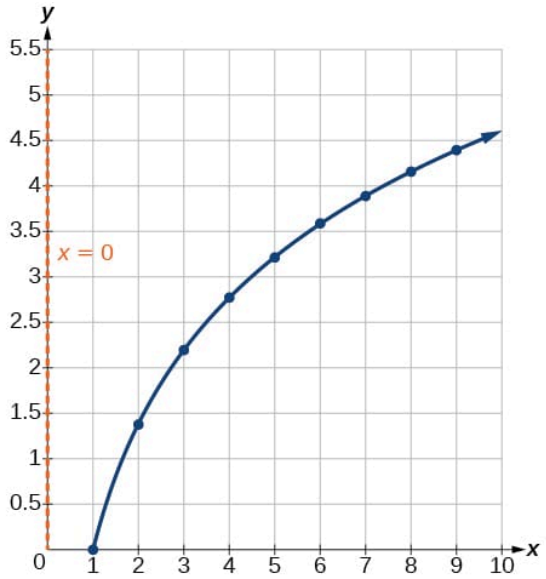


Figure $8.3.10$ *The graph of* $y = ln(x2)$

The graphs appear to be identical when $x > 0$. A quick check confirms this conclusion: $y = ln(x2) = 2ln(x)$ for $x > 0$.

However, if $x < 0$, the graph of $y = ln(x2)$ includes a "extra" branch, as shown in Figure $8.3.11$. This occurs because, while $y = 2ln(x)$ cannot have negative values in the domain (as such values would force the argument to be negative), the function $y = ln(x2)$ can have negative domain values.

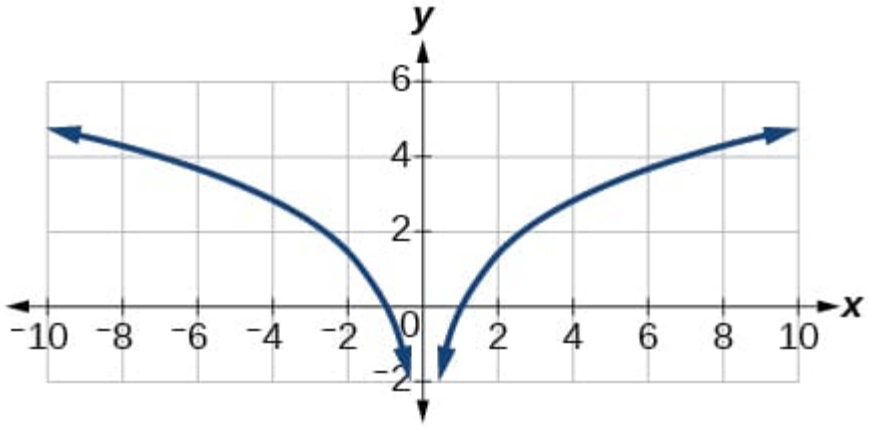


Figure 8.3.11

Checkpoint 8.3.6

Does a linear, exponential, or logarithmic model best fit the data in Table 8.3.2? Find the model.

x	1	2	3	4	5	6	7	8	9
y	3.297	5.437	8.963	14.778	24.365	40.172	66.231	109.196	180.034

Table 8.3.2

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8.3E: Exercises

Exercises

Verbal

1.

What situations are best modeled by a logistic equation? Give an example, and state a case for why the example is a good fit.

2.

What is a carrying capacity? What kind of model has a carrying capacity built into its formula? Why does this make sense?

3.

What is regression analysis? Describe the process of performing regression analysis on a graphing utility.

4.

What might a scatterplot of data points look like if it were best described by a logarithmic model?

5.

What does the *y*-intercept on the graph of a logistic equation correspond to for a population modeled by that equation?

Graphical

For the following exercises, match the given function of best fit with the appropriate scatterplot in Figure 7 through Figure 11. Answer using the letter beneath the matching graph.

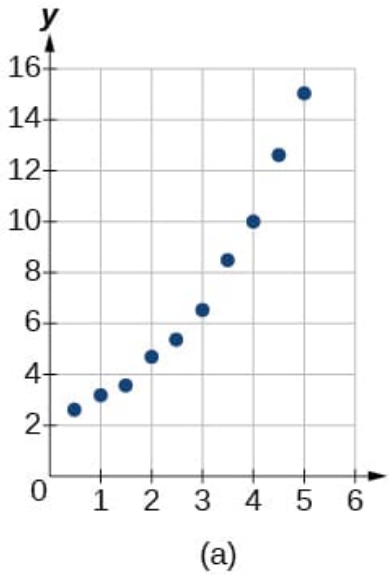
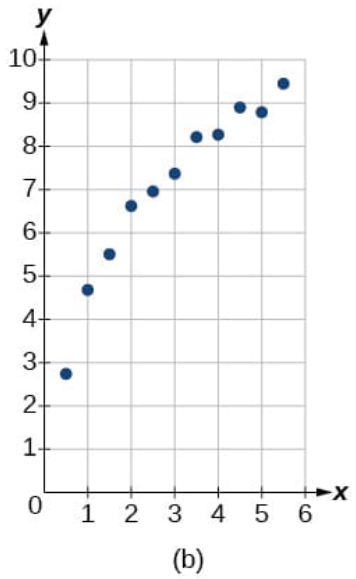
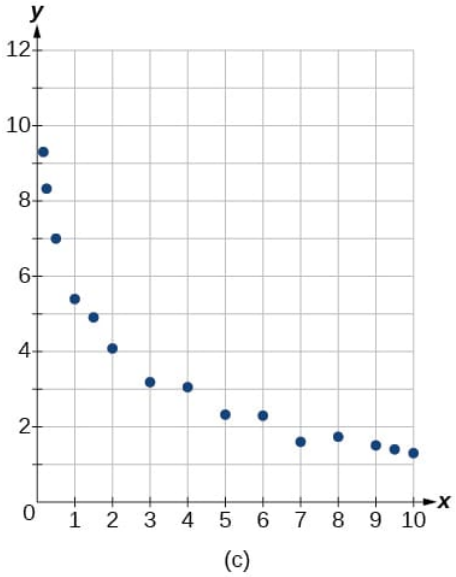


Figure 7



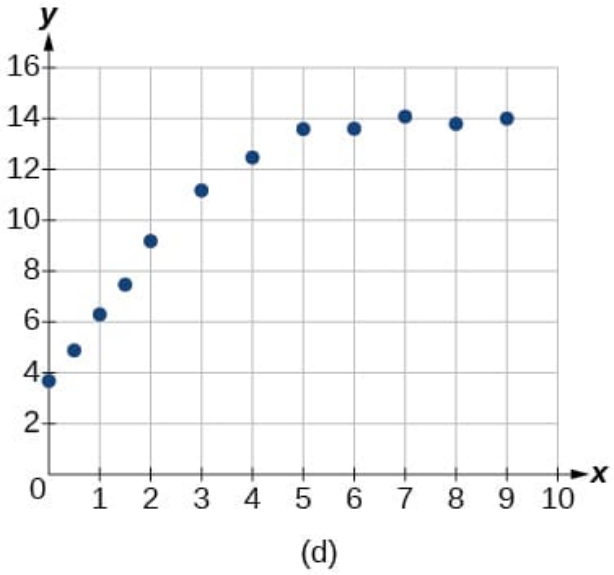
(b)

Figure 8



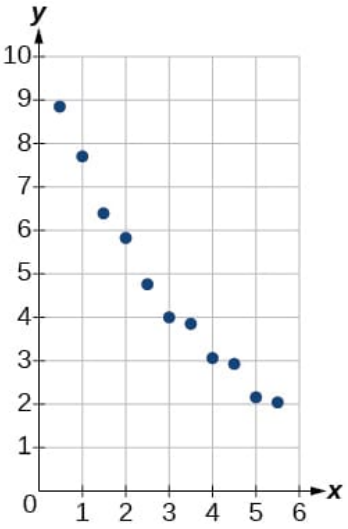
(c)

Figure 9



(d)

Figure 10



(e)

Figure 11

6.

$y = 10.209e^{-0.294x}$

7.

$y = 5.598 - 1.912\ln(x)$

8.

$y = 2.104(1.479)^x$

9.

$y = 4.607 + 2.733\ln(x)$

10.

$y = \frac{14.005}{1+2.79e^{-0.812x}}$

Numeric

11.

To the nearest whole number, what is the initial value of a population modeled by the logistic equation $P(t) = \frac{175}{1+6.995e^{-0.68t}}$? What is the carrying capacity?

12.

Rewrite the exponential model $A(t) = 1550(1.085)^x$ as an equivalent model with base e. Express the exponent to four significant digits.

13.

A logarithmic model is given by the equation $h(p) = 67.682 - 5.792 \ln(p)$. To the nearest hundredth, for what value of p does $h(p) = 62$?

14.

A logistic model is given by the equation $P(t) = \frac{90}{1+5e^{-0.42t}}$. To the nearest hundredth, for what value of *t* does $P(t) = 45$?

15.

What is the *y*-intercept on the graph of the logistic model given in the previous exercise?

Technology

For the following exercises, use this scenario: The population P of a koi pond over x months is modeled by the function $P(x) = \frac{68}{1+16e^{-0.28x}}$.

16.

Graph the population model to show the population over a span of 3 years.

17.

What was the initial population of koi?

18.

How many koi will the pond have after one and a half years?

19.

How many months will it take before there are 20 koi in the pond?

20.

Use the intersect feature to approximate the number of months it will take before the population of the pond reaches half its carrying capacity.

For the following exercises, use this scenario: The population P of an endangered species habitat for wolves is modeled by the function $P(x) = \frac{558}{1+54.8e^{-0.462x}}$, where x is given in years.

21.

Graph the population model to show the population over a span of 10 years.

22.

What was the initial population of wolves transported to the habitat?

23.

How many wolves will the habitat have after 3 years?

24.

How many years will it take before there are 100 wolves in the habitat?

25.

Use the intersect feature to approximate the number of years it will take before the population of the habitat reaches half its carrying capacity.

For the following exercises, refer to Table 7.

x	1	2	3	4	5	6
f(x)	1125	1495	2310	3294	4650	6361

Table 7

26.

Use a graphing calculator to create a scatter diagram of the data.

27.

Use the regression feature to find an exponential function that best fits the data in the table.

28.

Write the exponential function as an exponential equation with base e.

29.

Graph the exponential equation on the scatter diagram.

30.

Use the intersect feature to find the value of x for which $f(x) = 4000$.

For the following exercises, refer to Table 8.

x	1	2	3	4	5	6
f(x)	555	383	307	210	158	122

Table 8

31.

Use a graphing calculator to create a scatter diagram of the data.

32.

Use the regression feature to find an exponential function that best fits the data in the table.

33.

Write the exponential function as an exponential equation with base e.

34.

Graph the exponential equation on the scatter diagram.

35.

Use the intersect feature to find the value of x for which $f(x) = 250$.

For the following exercises, refer to Table 9.

x	1	2	3	4	5	6
f(x)	5.1	6.3	7.3	7.7	8.1	8.6

Table 9

36.

Use a graphing calculator to create a scatter diagram of the data.

37.

Use the LOGarithm option of the REGression feature to find a logarithmic function of the form $y = a + b\ln(x)$ that best fits the data in the table.

38.

Use the logarithmic function to find the value of the function when $x = 10$.

39.

Graph the logarithmic equation on the scatter diagram.

40.

Use the intersect feature to find the value of x for which $f(x) = 7$.

For the following exercises, refer to Table 10.

x	1	2	3	4	5	6	7	8
f(x)	7.5	6	5.2	4.3	3.9	3.4	3.1	2.9

Table 10

41.

Use a graphing calculator to create a scatter diagram of the data.

42.

Use the **LOG**arithm option of the **REG**ression feature to find a logarithmic function of the form $y = a + b\ln(x)$ that best fits the data in the table.

43.

Use the logarithmic function to find the value of the function when $x = 10$.

44.

Graph the logarithmic equation on the scatter diagram.

45.

Use the intersect feature to find the value of x for which $f(x) = 8$.

For the following exercises, refer to Table 11.

x	1	2	3	4	5	6	7	8	9	10
f(x)	8.7	12.3	15.4	18.5	20.7	22.5	23.3	24	24.6	24.8

Table 11

46.

Use a graphing calculator to create a scatter diagram of the data.

47.

Use the LOGISTIC regression option to find a logistic growth model of the form $y = \frac{c}{1+ae^{-bx}}$ that best fits the data in the table.

48.

Graph the logistic equation on the scatter diagram.

49.

To the nearest whole number, what is the predicted carrying capacity of the model?

50.

Use the intersect feature to find the value of x for which the model reaches half its carrying capacity.

For the following exercises, refer to Table 12.

x	0	2	4	5	7	8	10	11	15	17
$f(x)$	12	28.6	52.8	70.3	99.9	112.5	125.8	127.9	135.1	135.9

Table 12

51.

Use a graphing calculator to create a scatter diagram of the data.

52.

Use the LOGISTIC regression option to find a logistic growth model of the form $y = \frac{c}{1+ae^{-bx}}$ that best fits the data in the table.

53.

Graph the logistic equation on the scatter diagram.

54.

To the nearest whole number, what is the predicted carrying capacity of the model?

55.

Use the intersect feature to find the value of x for which the model reaches half its carrying capacity.

Extensions

56.

Recall that the general form of a logistic equation for a population is given by $P(t) = \frac{c}{1+ae^{-bt}}$, such that the initial population at time $t = 0$ is $P(0) = P_0$. Show algebraically that $\frac{c-P(t)}{P(t)} = \frac{c-P_0}{P_0}e^{-bt}$.

57.

Use a graphing utility to find an exponential regression formula $f(x)$ and a logarithmic regression formula $g(x)$ for the points $(1.5, 1.5)$ and $(8.5,\ 8.5)$. Round all numbers to 6 decimal places. Graph the points and both formulas along with the line $y = x$ on the same axis. Make a conjecture about the relationship of the regression formulas.

58.

Verify the conjecture made in the previous exercise. Round all numbers to six decimal places when necessary.

59.

Find the inverse function $f^{-1}(x)$ for the logistic function $f(x) = \frac{c}{1+ae^{-bx}}$. Show all steps.

60.

Use the result from the previous exercise to graph the logistic model $P(t) = \frac{20}{1+4e^{-0.5t}}$ along with its inverse on the same axis. What are the intercepts and asymptotes of each function?

Footnotes

- 9Source: *Indiana University Center for Studies of Law in Action, 2007*
- 10Source: *Center for Disease Control and Prevention, 2013*
- 11Source: *The World Bank, 2013*

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