MATH 384: FOUNDATIONS FOR CALCULUS



Roy Simpson
Cosumnes River College

LibreTexts™

Math 384: Foundations for Calculus

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (https://LibreTexts.org) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide an accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden borne from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by NICE CXone Expert and were supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptions contact info@LibreTexts.org or visit our main website at https://LibreTexts.org.

The LibreTexts name, logo, and book covers are trademarked by LibreTexts, Inc. (a 501(c)(3) not-for-profit organization) and protected against unauthorized use.

This text was compiled on 09/12/2026

TABLE OF CONTENTS

Licensing

1: Functions

1.1: Functions and Function Notation

1.1.1: Resources and Key Concepts

1.1.2: Homework

1.2: Domain and Range

1.2.1: Resources and Key Concepts

1.2.2: Homework

1.3: The Arithmetic and Composition of Functions

1.3.1: Resources and Key Concepts

1.3.2: Homework

1.4: Transformations of Functions

1.4.1: Resources and Key Concepts

1.4.2: Homework

1.5: Absolute Value Functions

1.5.1: Resources and Key Concepts

1.5.2: Homework

1.6: Inverse Functions

1.6.1: Resources and Key Concepts

1.6.2: Homework

2: Polynomial and Rational Functions

2.1: Quadratic Functions

2.1.1: Resources and Key Concepts

2.1.2: Homework

2.2: Power Functions and Polynomial Functions

2.2.1: Resources and Key Concepts

2.2.2: Homework

2.3: Graphs of Polynomial Functions

2.3.1: Resources and Key Concepts

2.3.2: Homework

2.4: Dividing Polynomials

2.4.1: Resources and Key Concepts

2.4.2: Homework

2.5: Rational Functions

2.5.1: Resources and Key Concepts

2.5.2: Homework

3: Inverse and Radical Functions

3.1: Inverse Functions

3.1.1: Resources and Key Concepts

3.1.2: Homework

3.2: Radical Functions

3.2.1: Resources and Key Concepts

3.2.2: Homework

3.3: Radical Equations

3.3.1: Resources and Key Concepts

3.3.2: Homework

3.4: Rational Exponent Expressions Encountered in Calculus

3.4.1: Resources and Key Concepts

3.4.2: Homework

4: Triangles and Circles

4.1: Angles and Basic Geometry

4.1.1: Resources and Key Concepts

4.1.2: Homework

4.2: Triangles

4.2.1: Resources and Key Concepts

4.2.2: Homework

4.3: Similar Triangles

4.3.1: Resources and Key Concepts

4.3.2: Homework

4.4: Circles and Angles in the Rectangular Coordinate System

4.4.1: Resources and Key Concepts

4.4.2: Homework

5: Coordinate Trigonometry

5.1: The Trigonometric Functions - Coordinate Definition

5.1.1: Resources and Key Concepts

5.1.2: Homework

5.2: Algebra with Trigonometric Functions

5.2.1: Resources and Key Concepts

5.2.2: Homework

5.3: Introduction to Trigonometric Identities

5.3.1: Resources and Key Concepts

5.3.2: Homework

5.4: Introduction to Proofs in Trigonometry

5.4.1: Resources and Key Concepts

5.4.2: Homework

6: Right Triangle Trigonometry

6.1: The Trigonometric Functions - Right Triangle Definition

6.1.1: Resources and Key Concepts

6.1.2: Homework

6.2: Technology and Right Triangle Trigonometry

6.2.1: Resources and Key Concepts

6.2.2: Homework

6.3: Solving Right Triangles

6.3.1: Resources and Key Concepts

6.3.2: Homework

6.4: Applications Involving Right Triangles

6.4.1: Resources and Key Concepts

6.4.2: Homework

7: Radian Measure and the Circular Functions

7.1: Reference Angles

7.1.1: Resources and Key Concepts

7.1.2: Homework

7.2: Radian Measure, Arc Length, and Sector Area

7.2.1: Resources and Key Concepts

7.2.2: Homework

7.3: The Trigonometric Functions - Unit Circle Definition

7.3.1: Resources and Key Concepts

7.3.2: Homework

8: Graphs of the Trigonometric Functions

8.1: Base Graphs of the Fundamental Trigonometric Functions

8.1.1: Resources and Key Concepts

8.1.2: Homework

8.2: Non-Rigid Transformations

8.2.1: Resources and Key Concepts

8.2.2: Homework

8.3: Rigid Transformations

8.3.1: Resources and Key Concepts

8.3.2: Homework

8.4: Graphs of the Remaining Trigonometric Functions

8.4.1: Resources and Key Concepts
8.4.2: Homework

9: Analytic Trigonometry
9.1: Basic Trigonometric Identities and Proof Techniques
9.1.1: Resources and Key Concepts
9.1.2: Homework
9.2: Sum and Difference Identities
9.2.1: Resources and Key Concepts
9.2.2: Homework
9.3: Double-Angle Identities
9.3.1: Resources and Key Concepts
9.3.2: Homework

10: Trigonometric Equations
10.1: The Fundamental Inverse Trigonometric Functions
10.1.1: Resources and Key Concepts
10.1.2: Homework
10.2: The Remaining Inverse Trigonometric Functions
10.2.1: Resources and Key Concepts
10.2.2: Homework
10.3: Solving Trigonometric Equations - Algebraic Techniques
10.3.1: Resources and Key Concepts
10.3.2: Homework
10.4: Solving Trigonometric Equations Using Identities
10.4.1: Resources and Key Concepts
10.4.2: Homework

11: Non-Right Triangle Trigonometry
11.1: The Law of Sines
11.1.1: Resources and Key Concepts
11.1.2: Homework
11.2: The Law of Cosines
11.2.1: Resources and Key Concepts
11.2.2: Homework

12: Exponential and Logarithmic Functions
12.1: Exponential Functions
12.1.1: Resources and Key Concepts
12.1.2: Homework
12.2: Graphs of Exponential Functions

12.2.1: Resources and Key Concepts
12.2.2: Homework

12.3: Logarithmic Functions
12.3.1: Resources and Key Concepts
12.3.2: Homework

12.4: Graphs of Logarithmic Functions
12.4.1: Resources and Key Concepts
12.4.2: Homework

12.5: Laws of Logarithms
12.5.1: Resources and Key Concepts
12.5.2: Homework

12.6: Exponential and Logarithmic Equations
12.6.1: Resources and Key Concepts
12.6.2: Homework

12.7: Exponential and Logarithmic Models
12.7.1: Resources and Key Concepts
12.7.2: Homework

Index

Glossary

Detailed Licensing

Detailed Licensing

This page titled Math 384: Foundations for Calculus is shared under a license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College.

Licensing

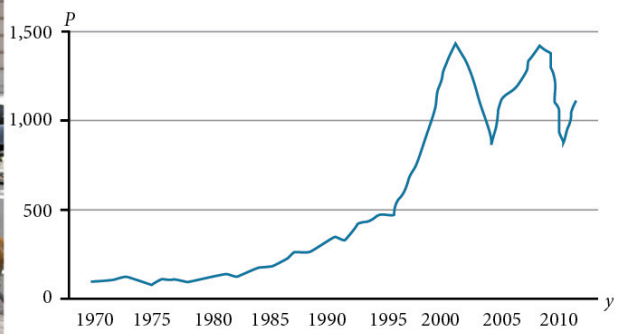
A detailed breakdown of this resource's licensing can be found in ***Back Matter/Detailed Licensing***.

This page titled Licensing is shared under a license and was authored, remixed, and/or curated by LibreTexts.

1: FUNCTIONS

Figure 1: *Standard and Poor's Index with dividends reinvested (credit "bull": modification of work by Prayitno Hadinata; credit "graph": modification of work by MeasuringWorth)*





Toward the end of the twentieth century, the values of stocks of internet and technology companies rose dramatically. As a result, the Standard and Poor's stock market average rose as well. The chart on the right tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1,100 by the beginning of 2000. That five-year period became known as the "dot-com bubble" because so many internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000.

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

1.1: Functions and Function Notation

1.1.1: Resources and Key Concepts

1.1.2: Homework

1.2: Domain and Range

1.2.1: Resources and Key Concepts

1.2.2: Homework

1.3: The Arithmetic and Composition of Functions

1.3.1: Resources and Key Concepts

1.3.2: Homework

1.4: Transformations of Functions

1.4.1: Resources and Key Concepts

1.4.2: Homework

1.5: Absolute Value Functions

1.5.1: Resources and Key Concepts
1.5.2: Homework

1.6: Inverse Functions

1.6.1: Resources and Key Concepts
1.6.2: Homework

This page titled 1: Functions is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

1.1: Functions and Function Notation

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

We begin our journey with a definition (and from this, progress to more definitions, then theorems, then more theorems, and so on until we will have derived all of College Algebra) and an exploration.

Definition: Ordered Pair

An **ordered pair** (a, b) is a listing of two objects in which the order is significant; a is called the **first coordinate** and b the **second coordinate**. Two ordered pairs, (a, b) and (c, d), are equal (written as $(a, b) = (c, d)$) if and only if $a = c$ and $b = d$.

Consider the following set of ordered pairs:

$$D = \{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)\}. \tag{1.1.1}$$

The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first. It is logical to want to describe this link between the first and second values of these ordered pairs. We do so by defining this link as a **relation**.

Definition: Relation

A **relation** from a set A to a set B is a subset of all possible ordered pairs (a, b) with $a \in A$ and $b \in B$. When $A = B$, we call it a relation **on** A.

We should spend a moment unpacking the language in that last definition.

- "A relation *from* a set A *to* a set B," implies there is a direction—we are going *from* somewhere *to* somewhere else.
- "...is a *subset* of *all possible* ordered pairs (a, b)," means that the relation might not contain every possible combination of elements from A and B—it only needs to contain a *subset* of these combinations. It is possible that it contains all

combinations of ordered pairs between these two sets, but it is not necessary.

For example, let A be the set of names of states in the United States, and let B be the set of flavors of Ben & Jerry's ice cream. We could create a (non-sensical) relation *from* A *to* B containing only the following ordered pairs: {(California, Chunky Monkey), (New York, Americone Dream), (Texas, Cherry Garcia)}. Notice that this relation definitely didn't use *all* of the states, nor did it use *all* of the Ben & Jerry's ice cream flavors—that's fine! By definition, it is still a relation between A and B.

Relations *might* be able to be described in terms of formulas (like, $y = 2x$ in the set 1.1.1), but this property need not exist. That is, we might be given a set of ordered pairs where there is no formula to the relationship between x and y.

Note

Before we go too far, it is important to mention that relations can involve more than two variables, but until you reach Calculus III, we will try our best to restrict our exploration to relations between two unknowns.

Now that we have a name for the "machine" linking the first and second values of these ordered pairs, let's introduce more language so that we can stop saying, "the first and second values of the ordered pairs."

Definition: Domain and Range (of a relation)

Let R be a relation. The **domain** of R is the set of all first coordinates that appear in R, and the **range** of R is the set of all second coordinates that appear in R:

$$\text{domain} = \{\, a \mid (a,b)^{\text{int}} R \text{ for some } b \,\}, \qquad \text{range} = \{\, b \mid (a,b)^{\text{int}} R \text{ for some } a \,\}.$$

Therefore, the domain of the relation 1.1.1 is $\{1, 2, 3, 4, 5\}$ and the range is $\{2, 4, 6, 8, 10\}$.

Definition: Input and Output Values

Each value in the domain is also known as an **input value**, and each value in the range is also known as an **output value**.

Input and output values are traditionally considered constants. That is, in 1.1.1 the input value of 3 mapped to an output value of 6. However, when referring to an *arbitrary* element from the set of input or output values, we use a slightly modified language (because, in Mathematics, the word "arbitrary" means "variable"—or non-constant).

Definition: Independent and Dependent Variables

If we are treating the input value as a variable, x, then we call x the **independent variable**. Likewise, the variable y is often used when referencing an arbitrary element from the set of output values, and y is called the **dependent variable**.

A Variable By Any Other Name...

While x and y are traditionally used in lower-level math courses as the independent and dependent variables, respectively, this need not always be the case. We can use any letter (or picture, for that matter) to represent these variables. The choice is literally limitless.

Relations are incredibly flexible—as long as there is some link between two variables, we can form a relation between them. For example, there is a relation between social security number and name. That is, the social security number 123-45-6789 (which, in truth, is not a real social security number) is related to a person's name. A natural question that arises when we do some thinking is whether the relation is unique. That is, if we grab a random social security number, will it be related to one name? This type of special relation is called a **function**.

Definition: Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say the output is a function of the input. The set of input values makes up the **domain** of the function, and the corresponding set of output values makes up the **range** of the function.

In a very real sense, functions are a special subclass of relations. While a relation can relate the input of 5 to the output 7 and then the input of 5 to the output of 10, functions are not allowed such silliness.

The relation from 1.1.1 is a function because each element in the domain, $\{1, 2, 3, 4, 5\}$, is paired with exactly one element in the range, $\{2, 4, 6, 8, 10\}$. For the social security numbers, the relation in one direction is a function—each social security number leads to a single name. However, if we reversed the direction of the relation so that the domain is the set of all names and the range is the set of all social security numbers, the relation would *not* be a function. This is because the name, Roy Simpson, leads to more than one social security number (there are more Roy Simpsons in the United States than just me!).

Let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd}, 1), (\text{even}, 2), (\text{odd}, 3), (\text{even}, 4), (\text{odd}, 5)\}.$$

Notice that each element in the domain, $\{\text{even}, \text{odd}\}$, is not paired with exactly one element in the range, $\{1, 2, 3, 4, 5\}$. For example, the term "odd" corresponds to three values from the range, $\{1, 3, 5\}$ and the term "even" corresponds to two values from the range, $\{2, 4\}$. This violates the definition of a function, so this relation is not a function. Figure $1.1.1$ compares relations that are functions and not functions.

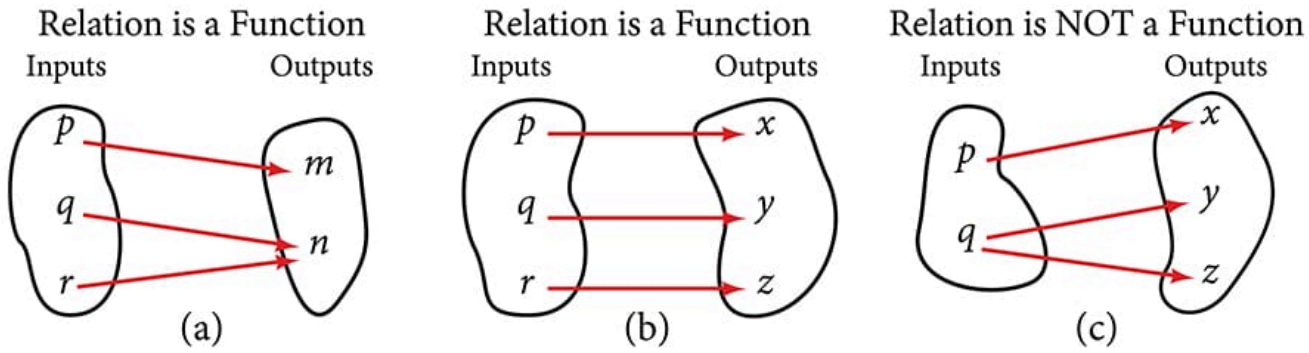


Figure $1.1.1$

Figure $1.1.1(a)$ is a function because each input is associated with a single output. Note that inputs q and r both give output n. This is perfectly fine. The definition of a function only restricts a single input from leading to two distinctly different outputs. Figure $1.1.1(b)$ is also a function. In this case, each input is associated with a single output. Finally, Figure $1.1.1(c)$ is *not* a function because input q is associated with two different outputs (y and z).

Example 1.1.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in Figure $1.1.2$, consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

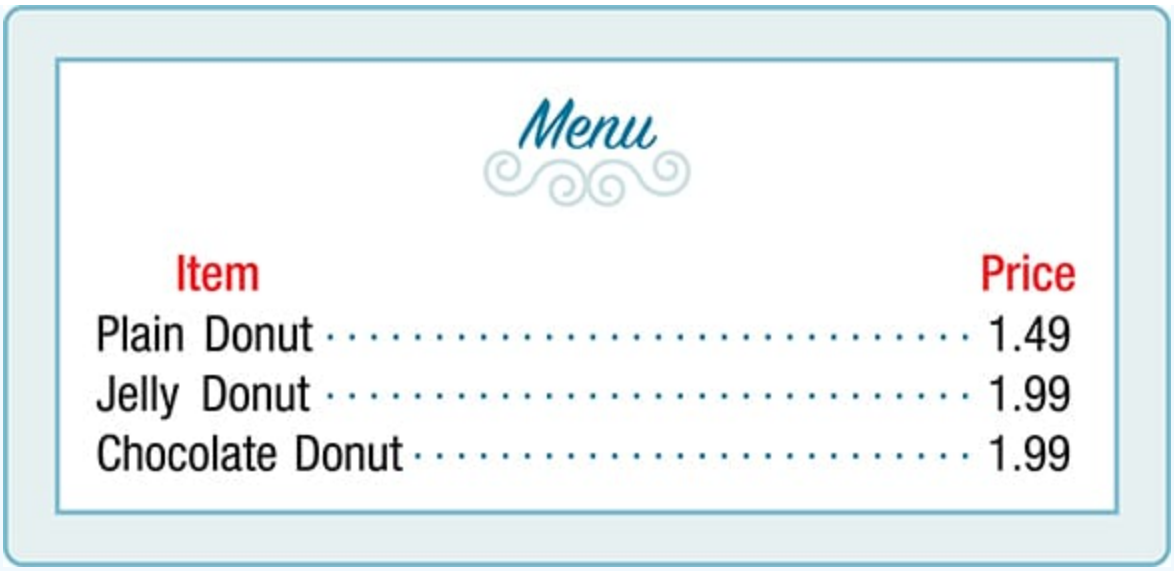


Figure 1.1.2

Solutions available at the end of this section.

Example 1.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 1.1.1 shows a possible rule for assigning grade points.

Table 1.1.1

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution available at the end of this section.

Checkpoint 1.1.2

Table 1.1.2 lists the five greatest baseball players of all time in order of rank.

Table 1.1.2

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables: h for height and a for age. The letters f, g, and h are often used to represent functions just as we use x, y, and z to represent numbers and A, B, and C to represent sets.

We then build a structure for the function notation. Since height is a function of age, we would like our notation to match.

$$\begin{array}{rccc} & \text{height} & \text{is} & \text{a function of age} \\ \Longrightarrow & h & = & f(a) \end{array}$$

Remember, we can use any letter to name the function. The notation $f(a)$ shows us that f depends on a. The value a must be put into the function f to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example, $f(a+b)$ means, "first add a and b, and then input the result into the function f." The operations must be performed in this order to obtain the correct result.

Definition: Function Notation

For the notation $y = f(x)$, f is the **name** of the function, x is the **domain value**, also known as the **argument** or the **independent variable**, and $f(x)$ is the **range value** of the function f at x. Since $y = f(x)$, we can also say that the range value of the function f at x is y. In this case, we say that y is the **dependent variable** (as its value *depends* on x). We read $f(x)$ as "f of x" or the value of f at x.

Example 1.1.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month. Assume that the domain does not include leap years.

Solution available at the end of this section.

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 1.1.4: Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Solution available at the end of this section.

Checkpoint 1.1.4

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done, especially in applied subjects that use higher math, such as Physics and Engineering. However, in exploring Mathematics itself, we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table $1.1.3$ lists the input number of each month (January = 1, February = 2, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Table $1.1.3$

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table $1.1.4$ defines a function $Q = g(n)$. Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

Table $1.1.4$

n	1	2	3	4	5
Q	8	6	7	6	8

Table $1.1.5$ displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Table $1.1.5$

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

Example $1.1.5$: Identifying Tables that Represent Functions

Which table, Table $1.1.6A$, Table $1.1.6B$, or Table $1.1.6C$, represents a function (if any)?

Table $1.1.6A$

Input	Output
2	1
5	3
8	6

Table $1.1.6B$

Input	Output
-3	5
0	1
4	5

Table $1.1.6C$

Input	Output
1	0
5	2
5	4

Solution available at the end of this section.

Checkpoint $1.1.5$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we *evaluate* the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and *solve* for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

Preview of Future Material

Each of the *types* of functions in the following examples will be studied in detail as we move through the textbook. The functions in Examples $1.1.6$, $1.1.7$, and $1.1.8$ are called **quadratic functions**, and are the focus of an entire section later on. The functions in Checkpoints $1.1.7$ and $1.1.8$ are called **square root functions** (a subset of the **radical functions**). These also get their own sections later in the text.

Example $1.1.6$: Evaluating Functions at Specific Values

Evaluate $f(x) = x^2 + 3x - 4$ at:

a. 2
b. a
c. $a+h$
d. Now evaluate $\frac{f(a+h)-f(a)}{h}$

Solutions available at the end of this section.

Example 1.1.7: Evaluating Functions

Given the function $h(p)=p^2+2p$, evaluate $h(4)$.

Solution available at the end of this section.

Checkpoint 1.1.7

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Example 1.1.8: Solving Functions

Given the function $h(p)=p^2+2p$, solve $h(p)=3$.

Solution available at the end of this section.

Checkpoint 1.1.8

Given the function $g(m)=\sqrt{m-4}$, solve $g(m)=2$.

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n+6p=12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

Example 1.1.9: Finding the Algebraic Form of a Function

Express the relationship $2n+6p=12$ as a function $p=f(n)$, if possible.

Solution available at the end of this section.

Caution: Not all relations are functions

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Example 1.1.10: Expressing the Equation of a Circle as a Function

Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution available at the end of this section.

Checkpoint 1.1.10

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Q&A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table.

Pet	Memory span in hours
Puppy	0.008
Adult dog	0.083
Cat	16
Goldfish	2160
Beta fish	3600

At times, evaluating a function in table form may be more useful than using equations. For the pet-memory span information, let's call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(\text{goldfish}) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

Example 1.1.11: Evaluating and Solving a Tabular Function

Use the provided table to answer the following questions.

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

n	1	2	3	4	5
g	8	6	7	6	8

Solutions available at the end of this section.

Checkpoint 1.1.11

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 1.1.12: Reading Function Values from a Graph

Use the graph below to answer the following questions.

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

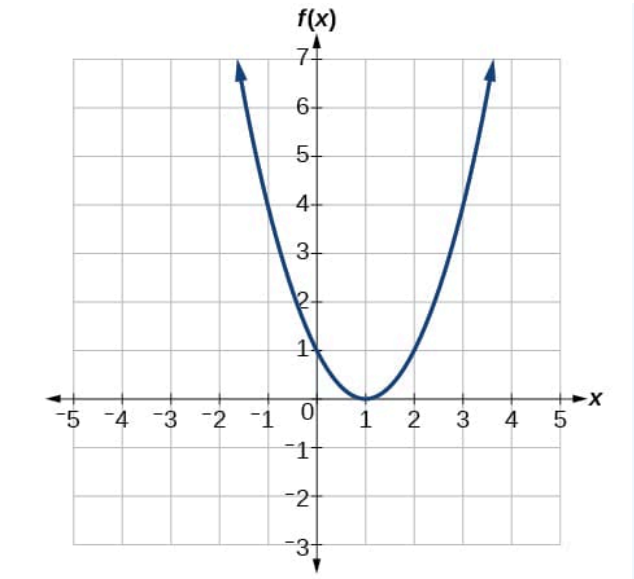


Figure 1.1.6

Solutions available at the end of this section.

Checkpoint 1.1.12

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input.

Definition: One-to-One Function

A function f is **one-to-one** (or **injective**) if distinct inputs always produce distinct outputs; that is, whenever $a \neq b$ in the domain of f, then $f(a) \neq f(b)$. Equivalently, $f(a) = f(b)$ forces $a = b$.

As an example, consider a school that uses only letter grades and decimal equivalents, as listed in the following table.

Letter grade	Grade point average
A	4.0
B	3.0
C	2.0

Letter grade	Grade point average
D	1.0

This grading system represents a one-to-one function because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figure $1.1.1(a)$ and Figure $1.1.1(b)$. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

Example $1.1.13$: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution available at the end of this section.

Checkpoint $1.1.13A$

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Checkpoint $1.1.13B$

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output value y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in the following figure tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points.

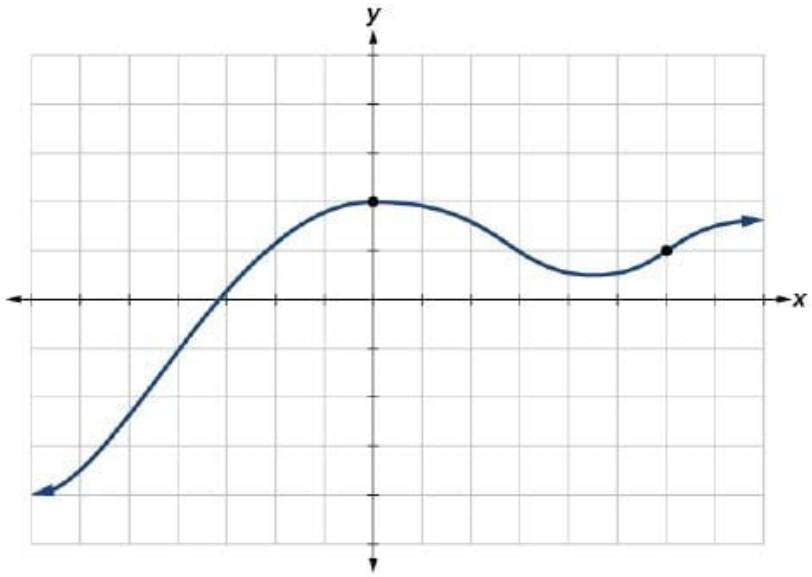


Figure 1.1.9

If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value (see the following figure).

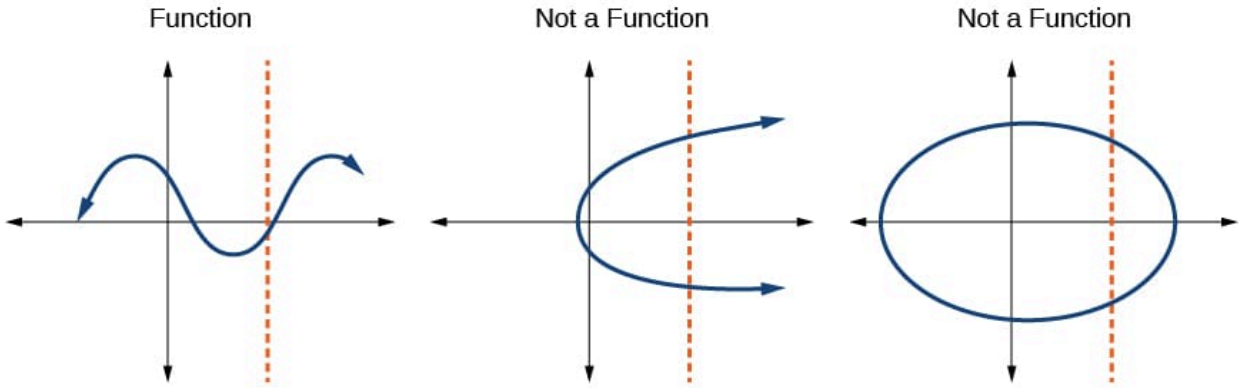


Figure 1.1.10

This leads us to the **vertical line test**.

Theorem: Vertical Line Test

A set of points in a rectangular coordinate system is the graph of a function if and only if every vertical line intersects the graph in at most one point.

Interactive Element: The Vertical Line Test

The following Interactive Element might help you visualize why a circle is not a function.

Interact: Drag the red point along the horizontal axis to sweep the vertical line across the circle.

Observation: Whenever the line passes through the interior of the circle it meets the circle at two points — one input x with two outputs — so the circle fails the Vertical Line Test and is not a function $y = f(x)$. A single vertical line with two crossings is enough to disqualify the graph.

Example 1.1.14: Applying the Vertical Line Test

Which of the graphs in the following figure represent(s) a function $y = f(x)$?

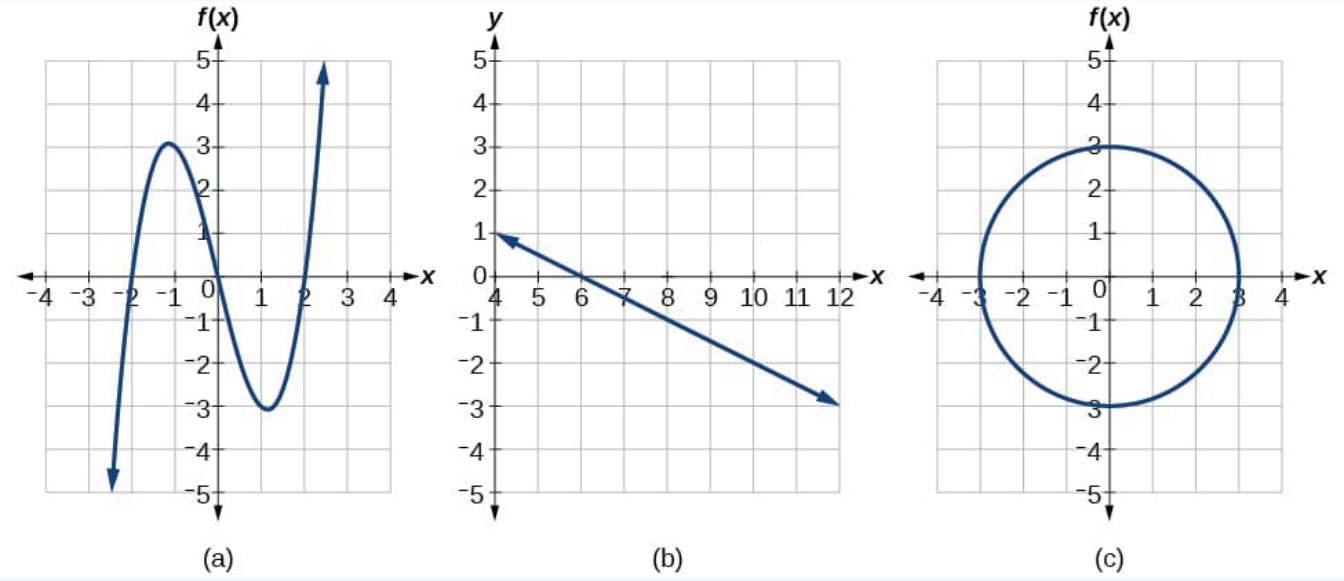


Figure 1.1.11

Solution available at the end of this section.

Checkpoint 1.1.14

Does the graph in the following figure represent a function?

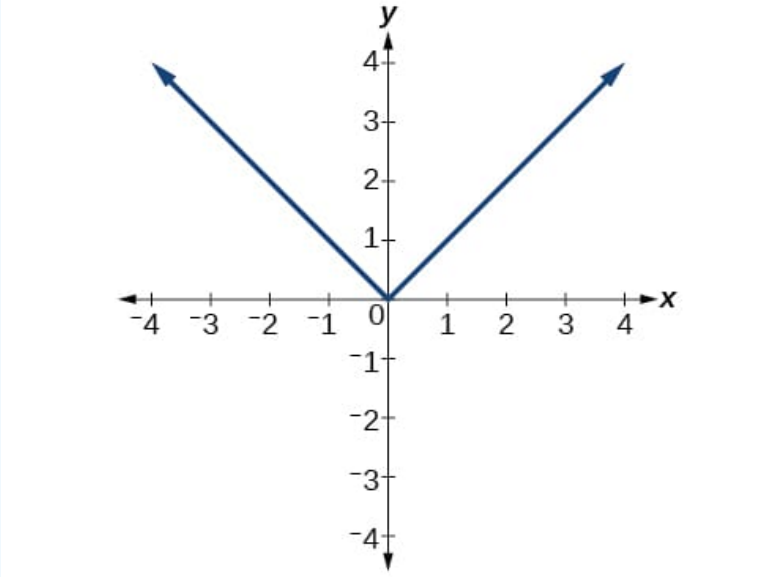


Figure 1.1.13

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the

Horizontal Line Test.

Theorem: Horizontal Line Test

A set of points representing a function in a rectangular coordinate system is the graph of a one-to-one function if and only if every horizontal line intersects the graph in at most one point.

To use the Horizontal Line Test, draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Example 1.1.15: Applying the Horizontal Line Test

Consider the functions shown in parts (a) and (b) of Example 1.1.14. Are either of the functions one-to-one?

Solution available at the end of this section.

Checkpoint 1.1.15

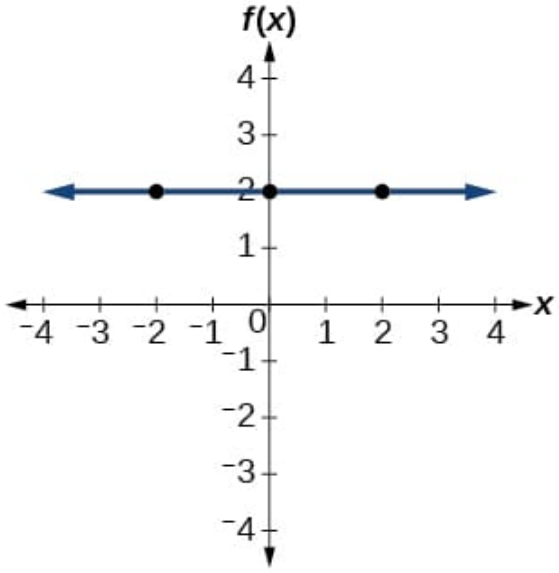
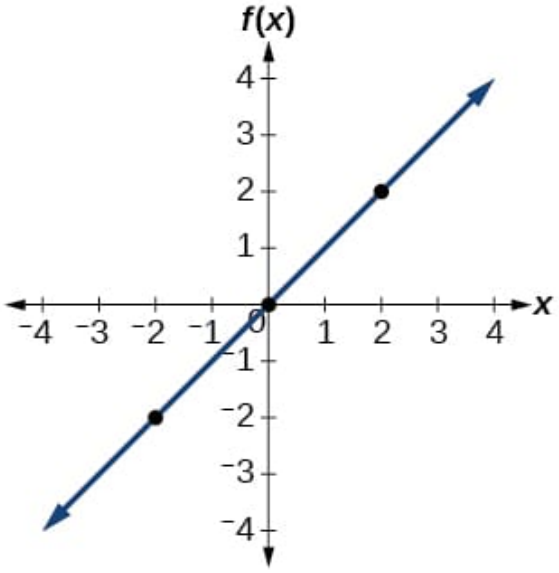
Is the graph shown in part (c) of Example 1.1.14 one-to-one?

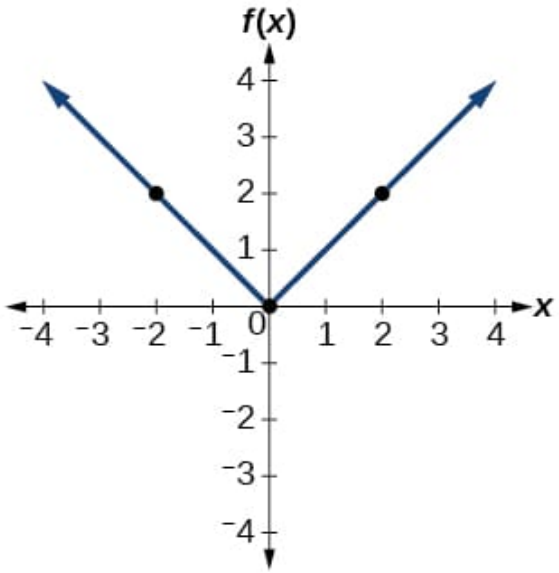
Identifying Basic Toolkit Functions

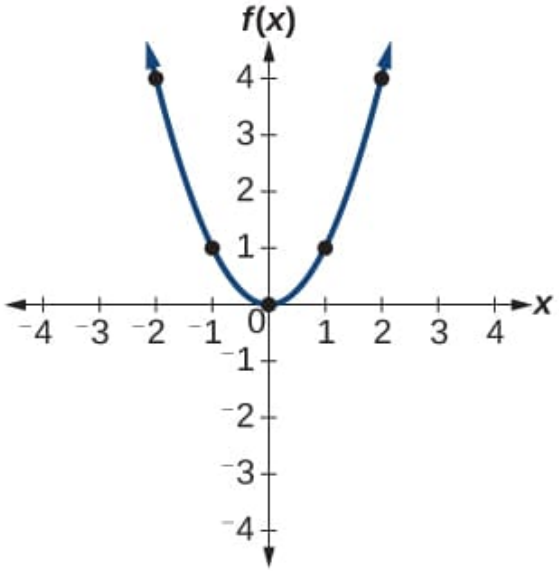
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do Arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "**toolkit functions**," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in the following table.

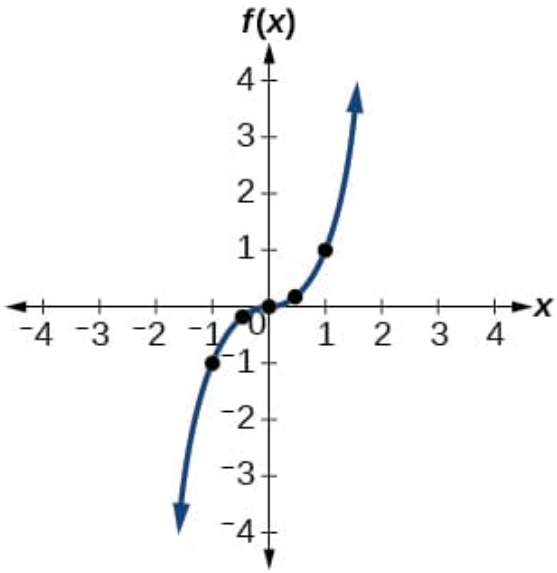
Toolkit Functions

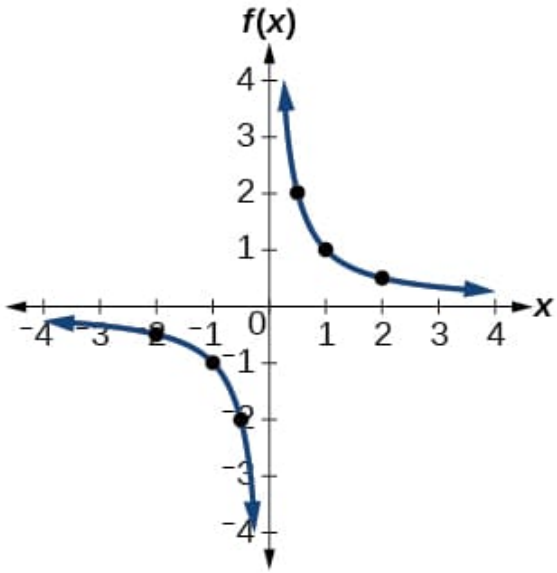
Name	Function	Graph
Constant	$f(x) = c$, where c is a constant	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>2</td></tr></table>
Identity	$f(x) = x$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table>

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Absolute value</td><td>$f(x) = |x|$</td><td> 

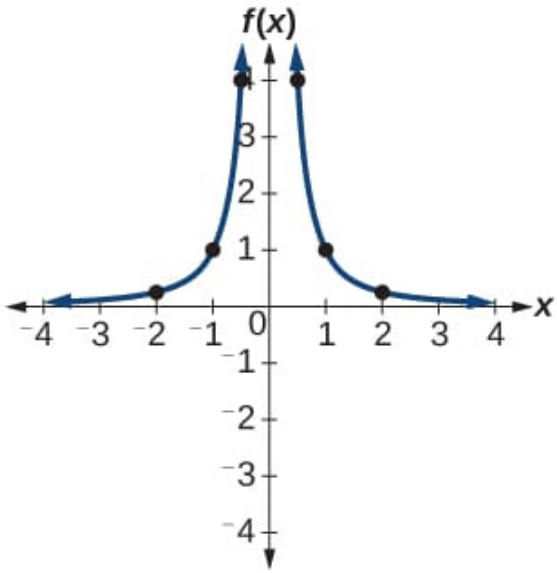
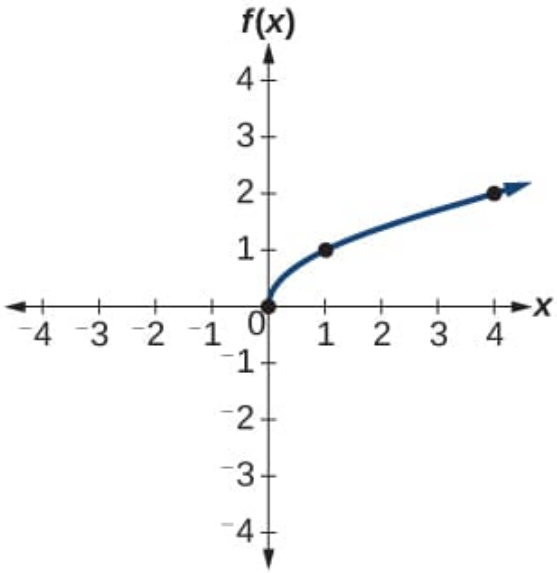
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>2</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
<tr><td>Quadratic</td><td>$f(x) = x^2$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>4</td></tr>
<tr><td>−1</td><td>1</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>4</td></tr>
</table></td></tr>
</table>

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Cubic</td><td>$f(x) = x^3$</td><td> 

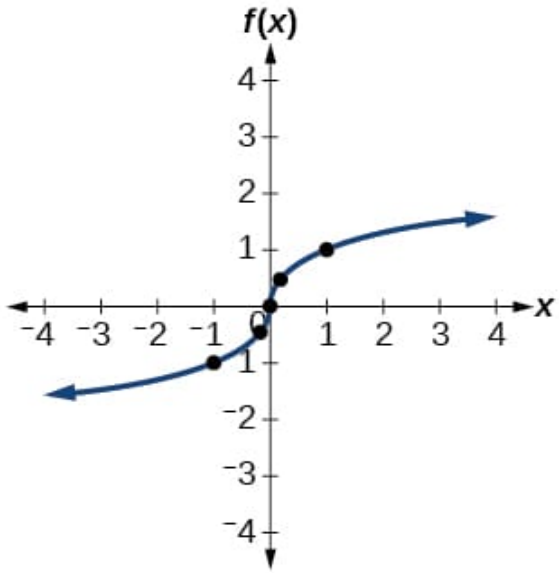
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−0.125</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.5</td><td>0.125</td></tr>
<tr><td>1</td><td>1</td></tr>
</table></td></tr>
<tr><td>Reciprocal</td><td>$f(x) = \frac{1}{x}$</td><td> 

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−0.5</td></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.5</td><td>−2</td></tr>
<tr><td>0.5</td><td>2</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.5</td></tr>
</table></td></tr>
</table>

Name	Function	Graph
Reciprocal squared	$f(x) = \frac{1}{x^2}$	
Square root	$f(x) = \sqrt{x}$	

x	f(x)
−2	0.25
−1	1
−0.5	4
0.5	4
1	1
2	0.25

x	f(x)
0	0
1	1
4	2

Name	Function	Graph
Cube root	$f(x) = \sqrt[3]{x}$	

x	f(x)
−1	−1
−0.125	−0.5
0	0
0.125	0.5
1	1

Hidden Resources: Borrowing Technology

Having a computer is a requirement for any college class these days; however, what do you do if you don't have a computer? Cosumnes River College (and the Los Rios Community College District) has pulled together a great list of resources for students needing a computer. Whether you want to borrow a Chromebook from the CRC library for the semester or you prefer to purchase a discounted computer, the CRC Library has all the information you need!

MyOpenMath Advice: "Message Instructor" Within Questions

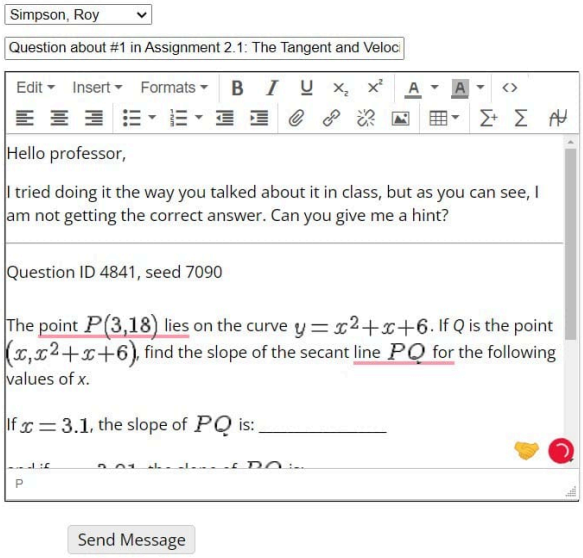
What is it?

If you are stuck on a homework question, and if your professor has set up a discussion area on Canvas, post your question to that area first; however, not all professors have enabled a discussions area. In this case, you can ask your professor for help directly by clicking on the "Message instructor" link at the bottom of the question.



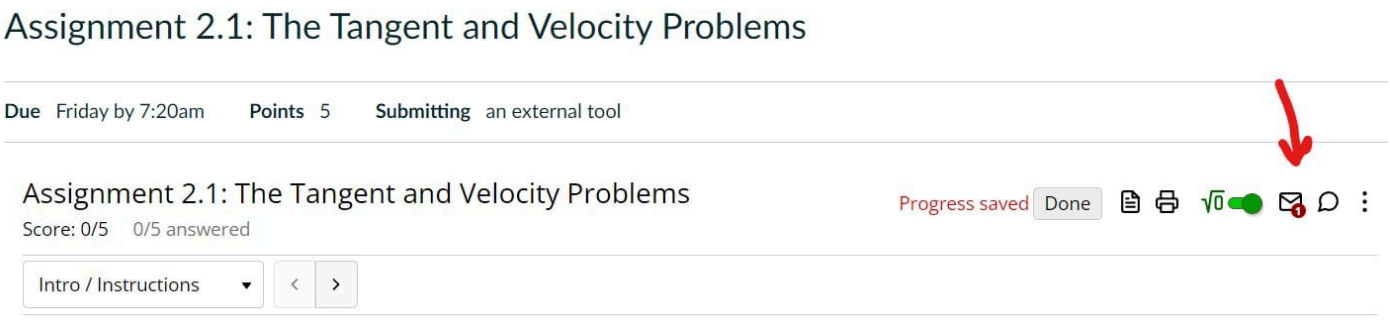
Why use it?

The nice thing about using the "Message instructor" link is that it sends *your version* of the question you are asking about directly to your professor. This helps them better understand your question. Be sure to include some information as to what might be causing your confusion.



How do you know when your professor has responded?

This is the only tricky part. When your professor responds, you will *not* get an email notification. This is a weakness of the MyOpenMath embed into Canvas (the messaging systems do not work well together); however, when your professor responds, you will see a message flag within *any* homework assignment you open.



Just click on the envelope and you will see your professor's response.

Answers

Section Preview

Treat this section as a review of functions (represented as tables, algebraic functions, word problems, and graphs). As far as computation is concerned, the algebraic functions used within this section are restricted to linear, quadratic, radical, and very simple rational functions. Finally, the nine Toolkit functions are introduced, but do not cover transformations (as this is held off for later).

Section Preview

- **Sets and Numbers**
  - Sets of Numbers: The opening relation $D=\{(1,2),(2,4),...\}$ is built from the natural numbers, and domains/ranges throughout the section are described as sets of numbers; students must be able to recognize and classify these number types to read the examples.
- **Graphs**
  - The Cartesian Coordinate System and the Quadrants: All graphical representations of functions, the Vertical Line Test, and the Horizontal Line Test presuppose a working understanding of the coordinate plane and how points live on it.
  - Ordered Pairs and Graphing Equations by Point-Plotting: The section introduces relations and functions as sets of ordered pairs (x,y); a student must already understand what an ordered pair represents and how it corresponds to a plotted point to follow the development.

Section Preview

- **Arithmetic**
  - Mathematical Operations — Powers and Square Roots: Evaluating the quadratic $f(x)=x^2$ and square-root $f(x)=\sqrt{x}$ toolkit functions, and computing values such as $h(4)=4^2+2(4)$ in Example 7, requires fluency with powers and roots.
  - The Order of Operations: Correctly evaluating expressions like $f(2)=2^2+3(2)-4$ in Example 6a depends on applying PEMDAS in the right sequence.
- **Geometry**

- Circles — Properties, Area, and Circumference: Example 13 relies on the area formula $A = \pi r^2$ to argue that area is a one-to-one function of radius; students must already know and apply this formula.
- **Factoring Techniques**
  - Greatest Common Factor: Example 6d factors the common h out of $2ah + h^2 + 3h$ in order to cancel and simplify the difference-quotient result.
- **Simplifying Expressions**
  - Algebraic Vocabulary and Evaluating Expressions: Nearly every example (6, 7, 8) hinges on substituting a value or expression for the input variable and simplifying the result, the central operational skill of the section.
  - Multiplying Polynomials and the Distributive Property: Evaluating $f(a + h)$ in Example 6c requires expanding $(a + h)^2$ and distributing $3(a + h)$.
- **Solving Equations**
  - Solving Literal Equations: Example 9 rewrites $2n + 6p = 12$ as $p = f(n)$, which requires isolating one variable while treating the other as a constant.
  - Solving Quadratic Equations by Factoring: Example 8 solves $h(p) = 3$ by rewriting it as $p^2 + 2p - 3 = 0$, factoring, and applying the Zero Product Property.
  - Solving Quadratic Equations Using Extraction of Roots: Example 10 isolates y^2 in $x^2 + y^2 = 1$ and extracts roots to obtain $y = \pm\sqrt{1 - x^2}$.
- **Graphs**
  - Reading Values from Graphs: Example 12 asks students to evaluate $f(2)$ and solve $f(x) = 4$ by reading input and output coordinates directly off a plotted curve.

Section Preview

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the Vertical Line Test to identify functions.
- Graph the functions listed in the library of functions.

Example 1.1.1: Determining If Menu Price Lists Are Functions

a. Let's begin by considering the input as the items on the menu. The output values are then the prices (see Figure 1.1.2). Each item on the menu has only one price, so the price is a function of the item.

b. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it (see the figure below).

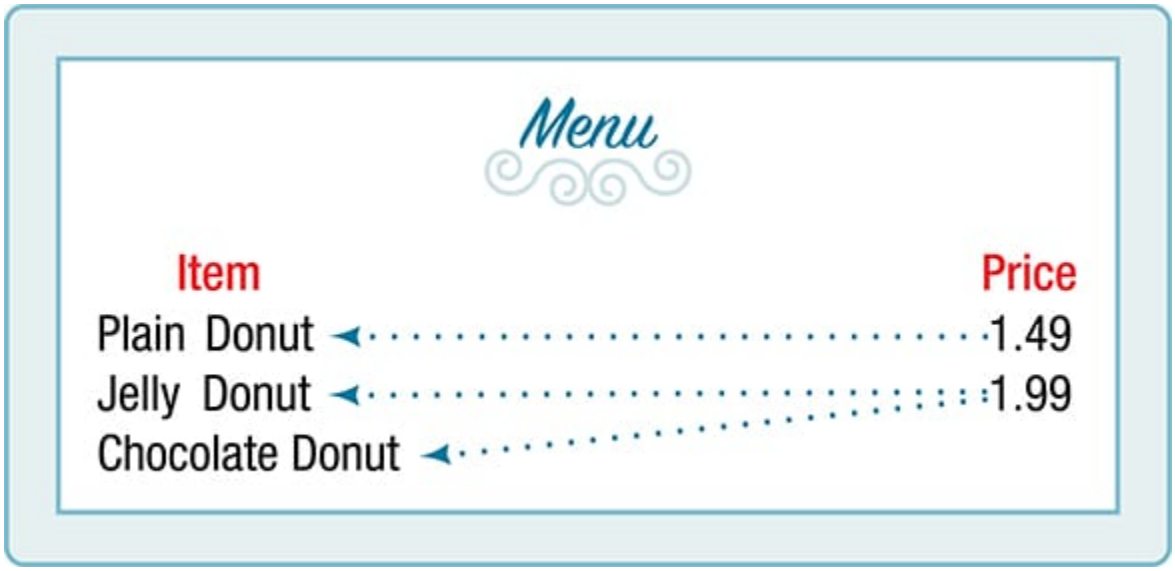


Figure 1.1.3

Example 1.1.2: Determining If Class Grade Rules Are Functions

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Example 1.1.3: Using Function Notation for Days in a Month

The number of days in a month is a function of the name of the month, so if we name the function f, we write

$$\text{days} = f(\text{month})$$

or

$$d = f(m).$$

The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

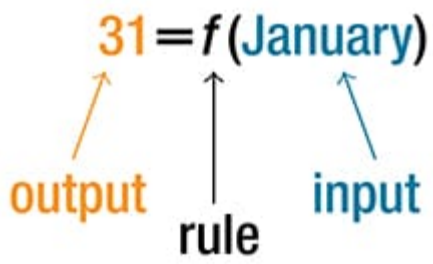


Figure 1.1.4

For example, $f(\text{March}) = 31$ because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Example 1.1.4**: Interpreting Function Notation**

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Example 1.1.5**: Identifying Tables that Represent Functions**

Table $1.1.6A$ and Table $1.1.6B$ define functions. In both, each input value corresponds to exactly one output value. Table $1.1.6C$ does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table $1.1.6A$ can be represented by writing

$$f(2) = 1, f(5) = 3, \text{ and } f(8) = 6.$$

Similarly, the statements

$$g(-3) = 5, g(0) = 1, \text{ and } g(4) = 5$$

represent the function in Table $1.1.6B$.

Table $1.1.6C$ cannot be expressed in a similar way because it does not represent a function.

Example 1.1.6**: Evaluating Functions at Specific Values**

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple Algebra to simplify.

$$\begin{aligned} f(2) &= (2)^2 + 3(2) - 4 && \text{(substituting)} \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 && \text{(substituting)} \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 && \text{(distributing)} \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4.$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} && \text{(substituting)} \\ &= \frac{a^2 + 2ah + h^2 + 3a + 3h - 4 - a^2 - 3a + 4}{h} && \text{(distributing)} \\ &= \frac{2ah + h^2 + 3h}{h} && \text{(combining like terms)} \\ &= \frac{h\,(2a + h + 3)}{h} && \text{(factoring out the GCF)} \\ &= \frac{\cancel{h}^{1}\,(2a + h + 3)}{\cancel{h}^{1}} && \text{(canceling like factors)} \\ &= 2a + h + 3 \end{aligned}$$

Example 1.1.7**: Evaluating Functions**

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(4) &= (4)^2 + 2(4) && \text{(substituting)} \\ &= 16 + 8 && \text{(simplifying)} \\ &= 24 && \text{(simplifying)} \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

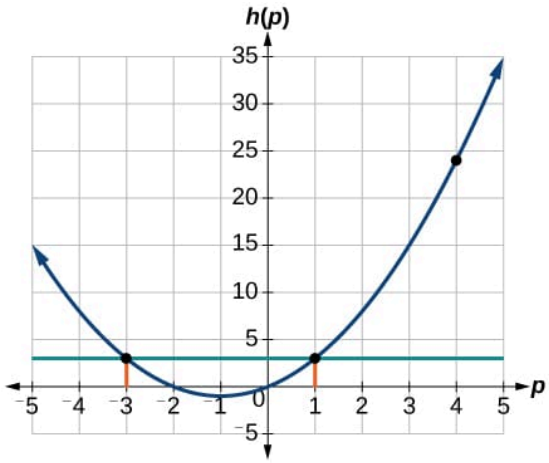
Example $1.1.8$**: Solving Functions**

$$\begin{aligned} & h(p) = 3 \\ \Longrightarrow \quad & p^2 + 2p = 3 && \text{(substituting)} \\ \Longrightarrow \quad & p^2 + 2p - 3 = 0 && \text{(subtracting 3 from both sides)} \\ \Longrightarrow \quad & (p+3)(p-1) = 0 && \text{(factoring)} \end{aligned}$$

If $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case (recall from Elementary Algebra, this is called the **Zero Factor Property**).

$$\begin{aligned} (p+3) = 0 \quad &\Longrightarrow \quad p = -3 \\ (p-1) = 0 \quad &\Longrightarrow \quad p = 1 \end{aligned}$$

This gives us two solutions, so we know that $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing (see the figure below). The graph verifies that $h(1) = h(-3) = 3$ and (for Example $1.1.7$) $h(4) = 24$.



p	-3	-2	0	1	4
$h(p)$	3	0	0	3	24

Figure $1.1.5$

Example $1.1.9$**: Finding the Algebraic Form of a Function**

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as

$$p = [\text{expression involving } n].$$

$$\begin{aligned} 2n + 6p &= 12 \\ \Longrightarrow \quad 6p &= 12 - 2n && \text{(subtracting } 2n \text{ from both sides)} \\ \Longrightarrow \quad p &= \frac{12 - 2n}{6} && \text{(dividing both sides by 6)} \\ \Longrightarrow \quad p &= \frac{12}{6} - \frac{2n}{6} \\ \Longrightarrow \quad p &= 2 - \frac{n}{3} && \text{(canceling like factors)} \end{aligned}$$

Therefore, p as a function of n is written as

$$p = f(n) = 2 - \frac{1}{3}n.$$

Example $1.1.10$**: Expressing the Equation of a Circle as a Function**

First we subtract x^2 from both sides, to get

$$y^2 = 1 - x^2.$$

We now try to solve for y in this equation.

$$\begin{aligned} y^2 &= 1 - x^2 \\ \Longrightarrow \quad y &= \pm\sqrt{1 - x^2} && \text{(Extraction of Roots)} \end{aligned}$$

Thus, either $y = \sqrt{1 - x^2}$ or $y = -\sqrt{1 - x^2}$. That is, we get two outputs corresponding to the same input. Therefore, this relationship cannot be represented as a

single function $y = f(x)$.

Example 1.1.11**: Evaluating and Solving a Tabular Function**

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.

b. Solving $g(n) = 6$ means identifying the input values, n, that produce an output value of 6. The table shows two solutions: 2 and 4. That is, when we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

Example 1.1.12**: Reading Function Values from a Graph**

a. To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$ (see the following figure).

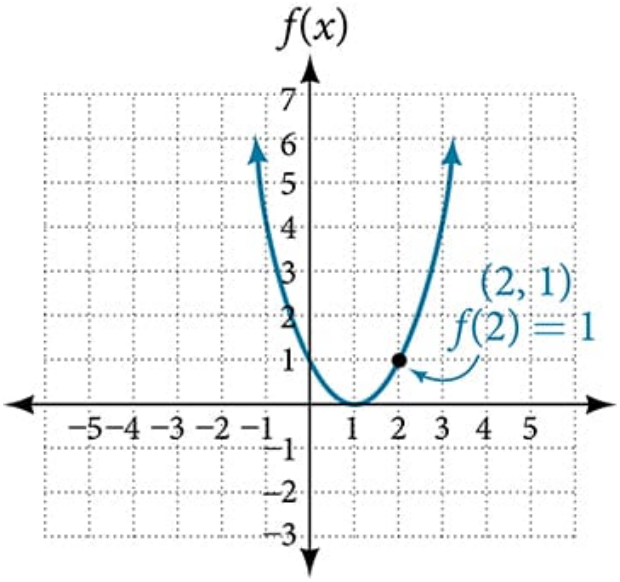


Figure 1.1.7

b. To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: -1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4 (see the figure below).

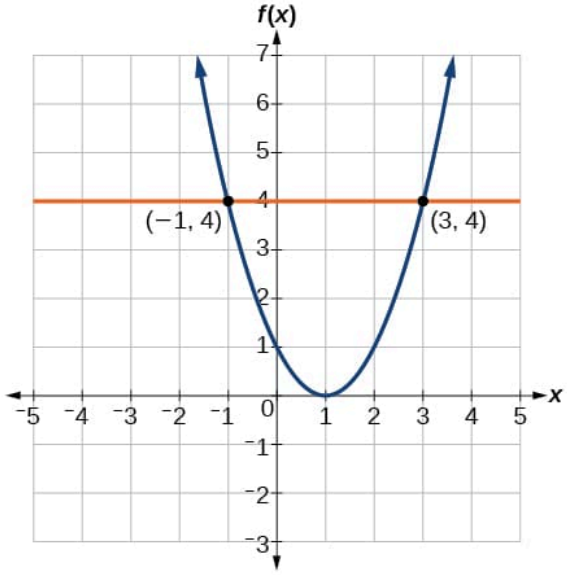


Figure 1.1.8

Example 1.1.13**: Determining Whether a Relationship Is a One-to-One Function**

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius r.

If the function is one-to-one, the output value (the area) must correspond to a unique input value (the radius). Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Example 1.1.14**: Applying the Vertical Line Test**

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of the figure. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in the figure below.

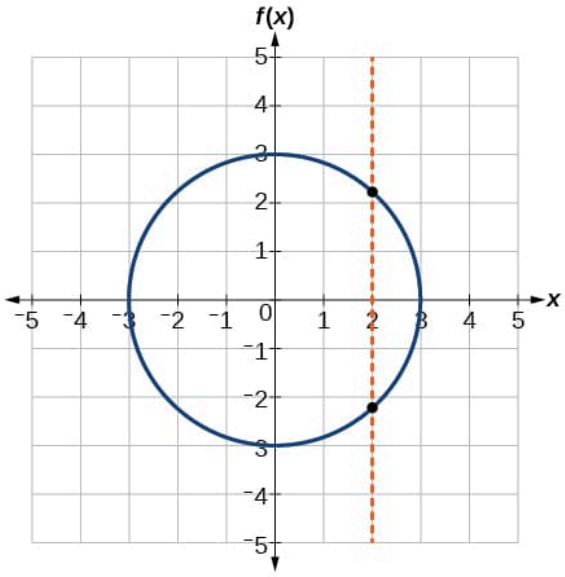


Figure 1.1.12

Example 1.1.15**: Applying the Horizontal Line Test**

The function in part (a) is not one-to-one. The horizontal line shown in the following figure intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

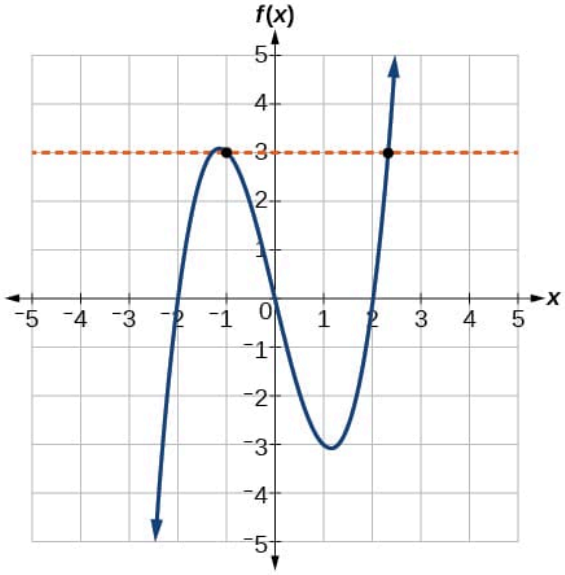


Figure 1.1.14

The function in part (b) is one-to-one. Any horizontal line will intersect a diagonal line at most once.

This page titled 1.1: Functions and Function Notation is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

1.1.1: Resources and Key Concepts

Resources

Videos

- Relations
  - Functions - Overview - A Review for College Algebra: Relations and Functions
- Functions
  - Algebra in Review: Functions - Overview
  - Intermediate Algebra - Functions: An Introduction
- Function notation
  - Algebra in Review: Functions - Notation
- Evaluating functions
  - Functions - Overview - A Review for College Algebra: Evaluating Functions
  - Functions - Overview - A Review for Precalculus: Evaluating Functions
  - Algebra in Review: Functions - Evaluation
- Representation of functions
  - Intermediate Algebra - Functions: The Four Representations
- One-to-one function
  - Functions - Overview - A Review for College Algebra: One-to-One Functions
- Vertical Line Test
  - Functions - Overview - A Review for College Algebra: The Vertical Line Test
- Horizontal Line Test
  - Functions - Overview - A Review for College Algebra: The Horizontal Line Test
- Toolkit functions
  - Intermediate Algebra - Functions: Base Graphs

Key Concepts

Definitions

- **Relation:** Any set of ordered pairs, denoted as (x, y).
- **Domain (of a relation/function):** The set of all first components (x-values) from the ordered pairs in a relation; it represents all possible input values.
- **Range (of a relation/function):** The set of all second components (y-values) from the ordered pairs in a relation; it represents all possible output values.
- **Input Value:** A value from the domain of a relation or function.
- **Output Value:** A value from the range of a relation or function, resulting from a given input value.
- **Independent Variable:** The variable (commonly x) that represents the input values of a function.
- **Dependent Variable:** The variable (commonly y) that represents the output values of a function, as its value depends on the input.
- **Function:** A specific type of relation where each possible input value is associated with exactly one output value.
- **Function Notation:** A convention, typically written as $y = f(x)$, used to denote a function, where f is the name of the function, x is the input (independent variable), and $f(x)$ is the output (dependent variable) corresponding to the input x.
- **One-to-One Function:** A function for which every distinct input value produces a distinct output value, meaning each output value corresponds to exactly one input value.

Theorems

- **Vertical Line Test:** A graph in a rectangular coordinate system represents a function if and only if no vertical line intersects the graph at more than one point. If any vertical line intersects the graph at more than one point, the graph does not represent

a function.
- **Horizontal Line Test:** The graph of a function represents a one-to-one function if and only if no horizontal line intersects the graph at more than one point. If any horizontal line intersects the graph at more than one point, the function is not one-to-one.

Common Mistakes

- **Misinterpreting Parentheses in Function Notation:** The parentheses in function notation, such as in $f(a)$, signify that a is the input to the function f. They do not indicate multiplication of f by a.
- **Assuming All Equations Define Functions:** Not every relationship expressed by an equation can be written as a single function where one variable is a function of the other. For example, the equation of a circle $x^2 + y^2 = 1$ does not define y as a single function of x because for a given x-value (other than 1 or -1), there are two corresponding y-values.

This page titled 1.1.1: Resources and Key Concepts is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

1.1.2: Homework

Reading Questions

1. What is a relation?
2. What are the domain and range of a relation?
3. How does a function differ from a general relation?
4. In the function notation $y = f(x)$, what do x, f, and $f(x)$ represent?
5. What does it mean to "evaluate" a function at a specific value, for example, $f(2)$?
6. Describe the Vertical Line Test. What is it used for?
7. What is a one-to-one function?
8. How can the Horizontal Line Test be used?
9. Does the equation $x^2 + y^2 = 1$ represent y as a function of x? Why or why not?
10. Name three of the "toolkit functions" mentioned in the section.

Homework

1. What is the difference between a relation and a function?
2. What is the difference between the input and the output of a function?
3. Why does the Vertical Line Test tell us whether the graph of a relation represents a function?
4. How can you determine if a relation is a one-to-one function?
5. Why does the Horizontal Line Test tell us whether the graph of a function is one-to-one?

For the following exercises, determine whether the relation represents a function.

6. $\{(a, b), (c, d), (a, c)\}$
7. $\{(a, b), (b, c), (c, c)\}$

For the following exercises, determine whether the relation represents y as a function of x.

8. $5x + 2y = 10$
9. $y = x^2$
10. $x = y^2$
11. $3x^2 + y = 14$
12. $2x + y^2 = 6$
13. $y = -2x^2 + 40x$
14. $y = \frac{1}{x}$
15. $x = \frac{3y+5}{7y-1}$
16. $x = \sqrt{1 - y^2}$
17. $y = \frac{3x+5}{7x-1}$
18. $x^2 + y^2 = 9$
19. $2xy = 1$
20. $x = y^3$
21. $y = x^3$
22. $y = \sqrt{1 - x^2}$
23. $x = \pm\sqrt{1 - y}$
24. $y = \pm\sqrt{1 - x}$
25. $y^2 = x^2$
26. $y^3 = x^2$

For the following exercises, evaluate $f(-3)$, $f(2)$, $f(-a)$, $-f(a)$, and $f(a + h)$.

27. $f(x) = 2x - 5$
28. $f(x) = -5x^2 + 2x - 1$

29. $f(x) = \sqrt{2-x} + 5$
30. $f(x) = \frac{6x-1}{5x+2}$
31. $f(x) = |x-1| - |x+1|$
32. Given the function $g(x) = 5 - x^2$, evaluate

$$\frac{g(x+h) - g(x)}{h}, \text{ where } h \neq 0.$$

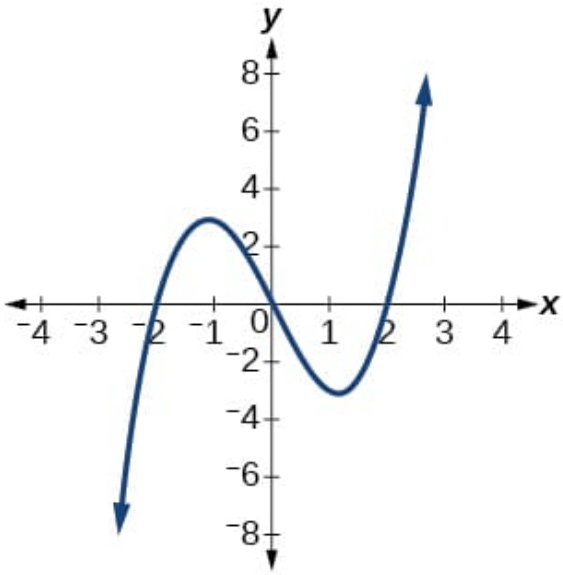
33. Given the function $g(x) = x^2 + 2x$, evaluate

$$\frac{g(x) - g(a)}{x - a}, \text{ where } x \neq a.$$

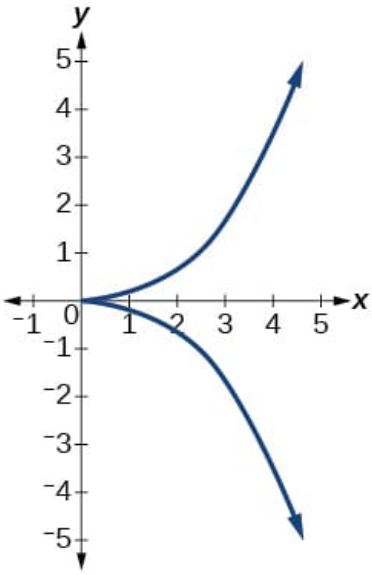
34. Given the function $k(t) = 2t - 1$:
    a. Evaluate $k(2)$.
    b. Solve $k(t) = 7$.
35. Given the function $f(x) = 8 - 3x$:
    a. Evaluate $f(-2)$.
    b. Solve $f(x) = -1$.
36. Given the function $p(c) = c^2 + c$:
    a. Evaluate $p(-3)$.
    b. Solve $p(c) = 2$.
37. Given the function $f(x) = x^2 - 3x$:
    a. Evaluate $f(5)$.
    b. Solve $f(x) = 4$.
38. Given the function $f(x) = \sqrt{x+2}$:
    a. Evaluate $f(7)$.
    b. Solve $f(x) = 4$.
39. Consider the relationship $3r + 2t = 18$.
    a. Write the relationship as a function $r = f(t)$.
    b. Evaluate $f(-3)$.
    c. Solve $f(t) = 2$.

For the following exercises, use the Vertical Line Test to determine which graphs show relations that are functions.

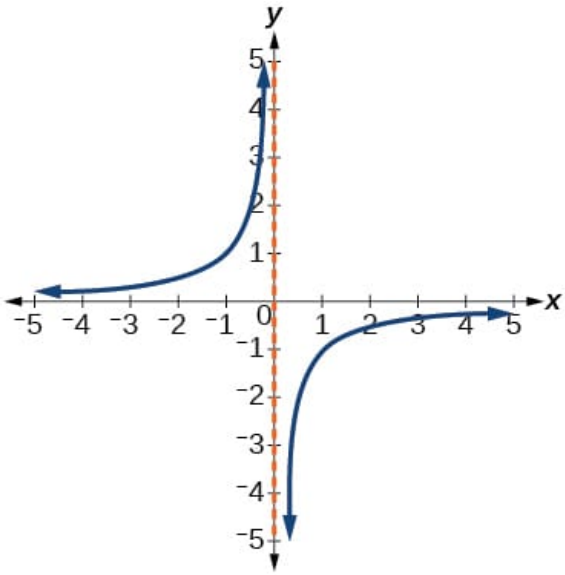
40.



41.

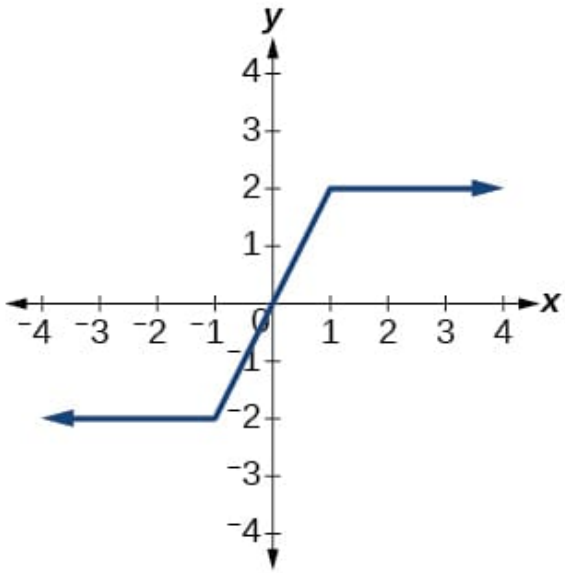


12.

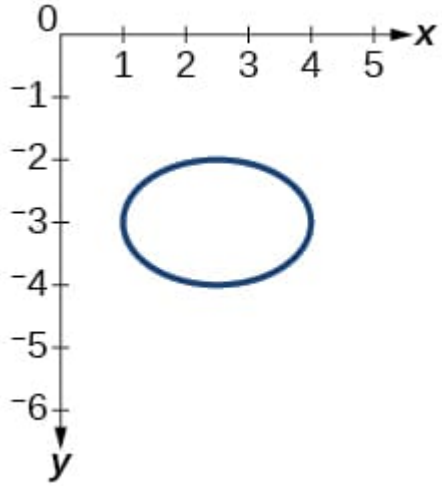


13.

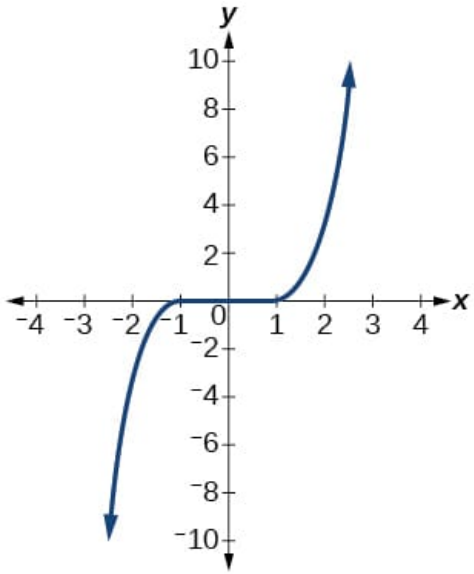
14.

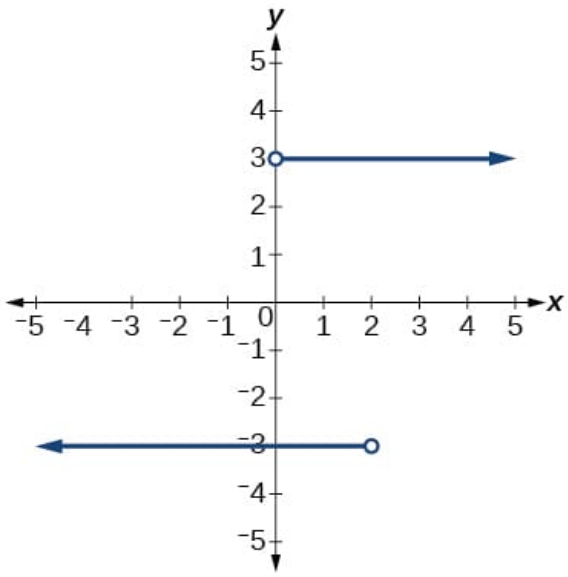

y
x
4
3
2
1
0
−1
−2
−3
−4
−4
−3
−2
−1
1
2
3
4

15.

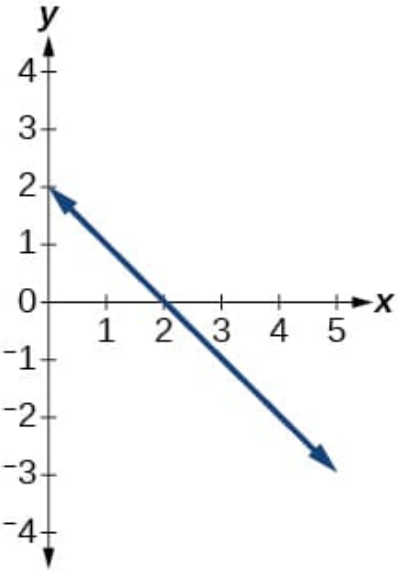

0
x
1
2
3
4
5
−1
−2
−3
−4
−5
−6
y

16.

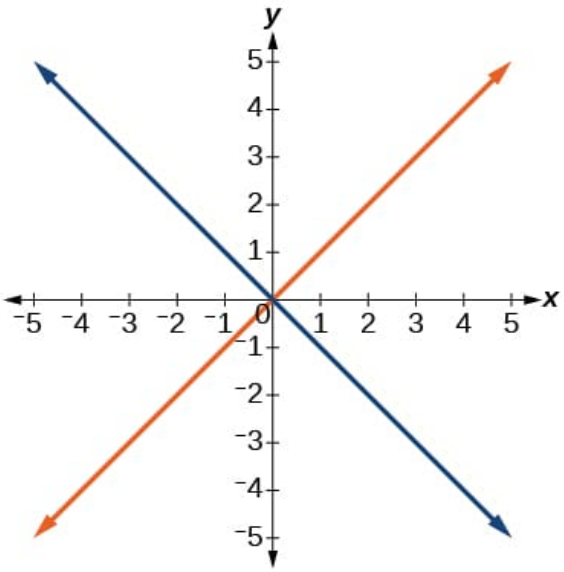

y
x
10
8
6
4
2
0
−2
−4
−6
−8
−10
−4
−3
−2
−1
1
2
3
4



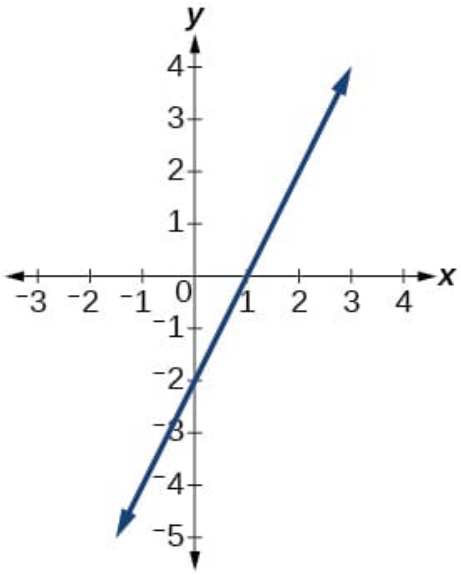
47.



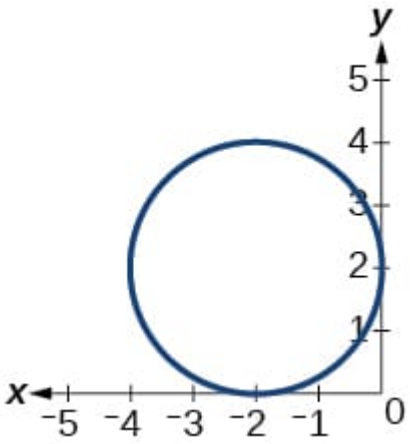
48.


y
x
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5 -4 -3 -2 -1 1 2 3 4 5

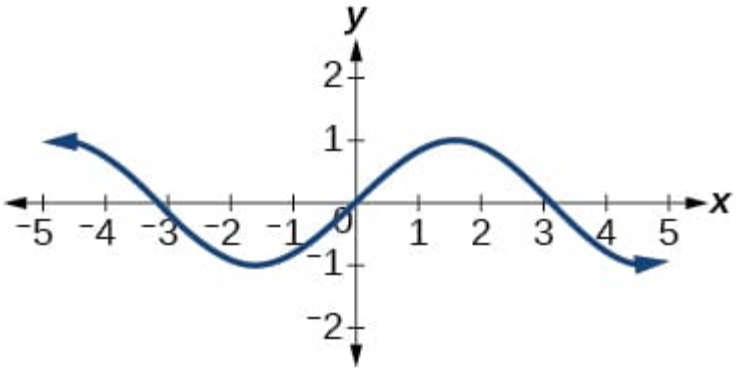
49.


y
x
4
3
2
1
0
-1
-2
-3
-4
-5
-3 -2 -1 1 2 3 4

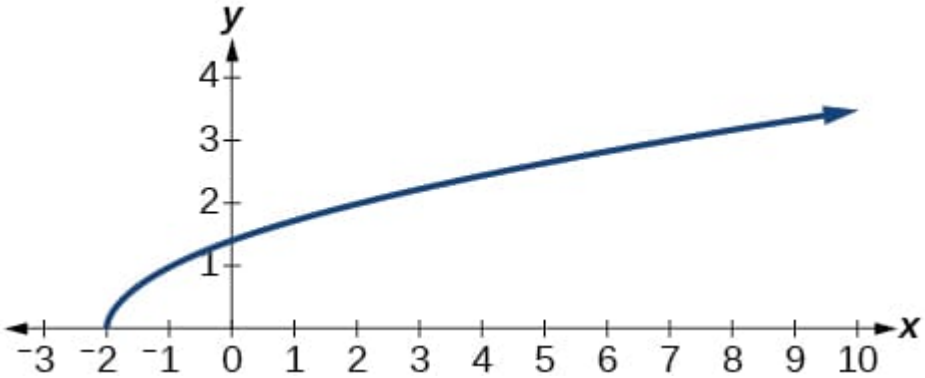
50.


y
x
5
4
3
2
1
0
-5 -4 -3 -2 -1

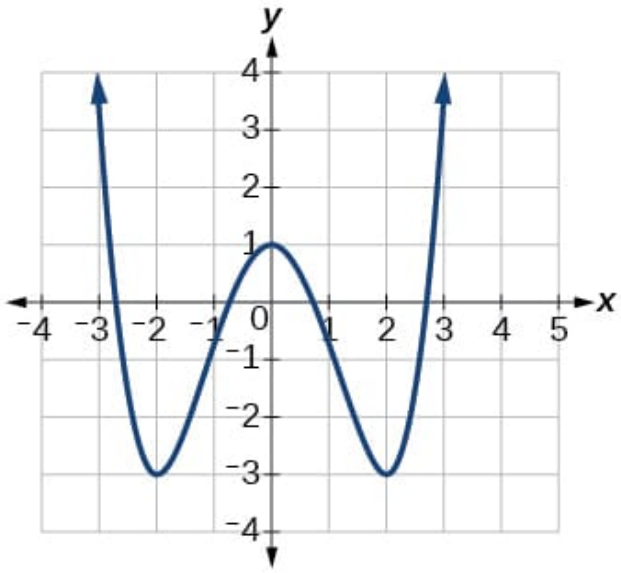
51.



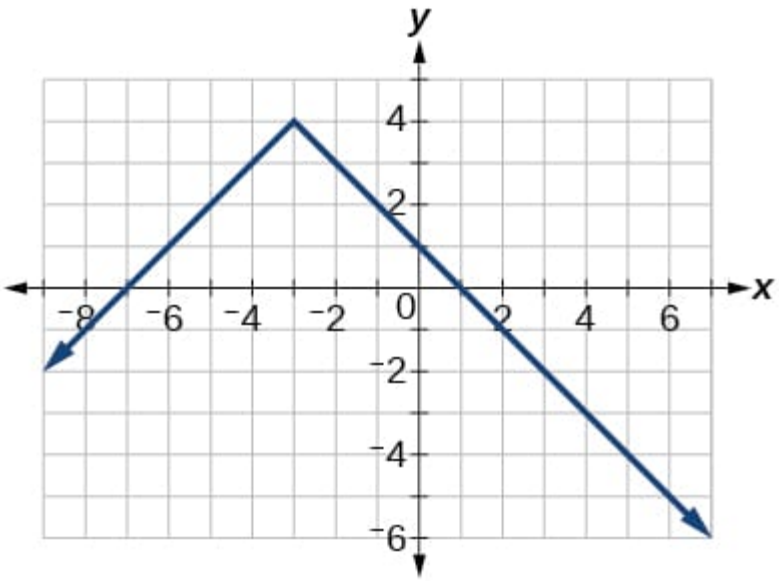
52. Given the following graph,
    a. Evaluate $f(-1)$.
    b. Solve for $f(x) = 3$.



53. Given the following graph,
    a. Evaluate $f(0)$.
    b. Solve for $f(x) = -3$.

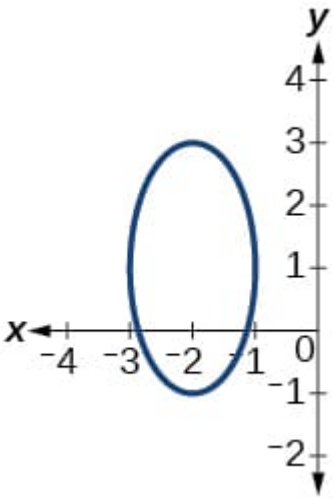


54. Given the following graph,
    a. Evaluate $f(4)$.
    b. Solve for $f(x) = 1$.

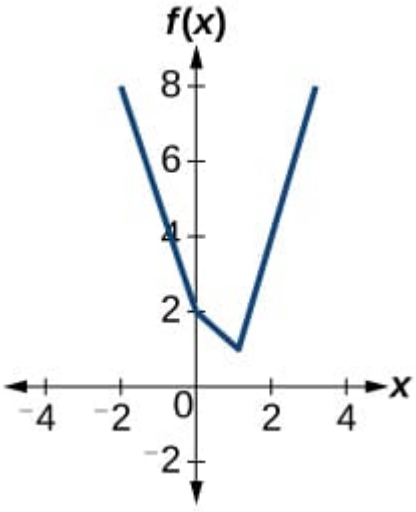


For the following exercises, determine if the given graph is a one-to-one function.

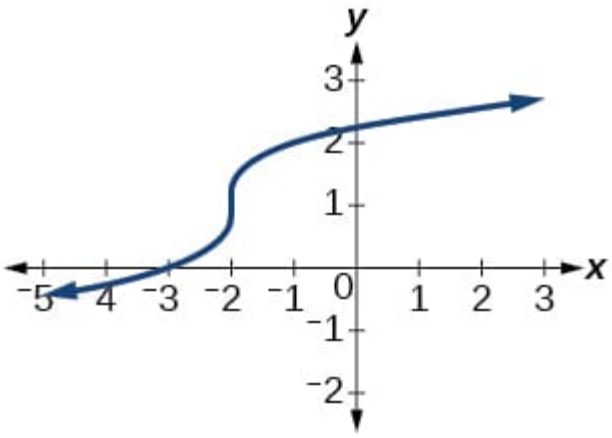
55.



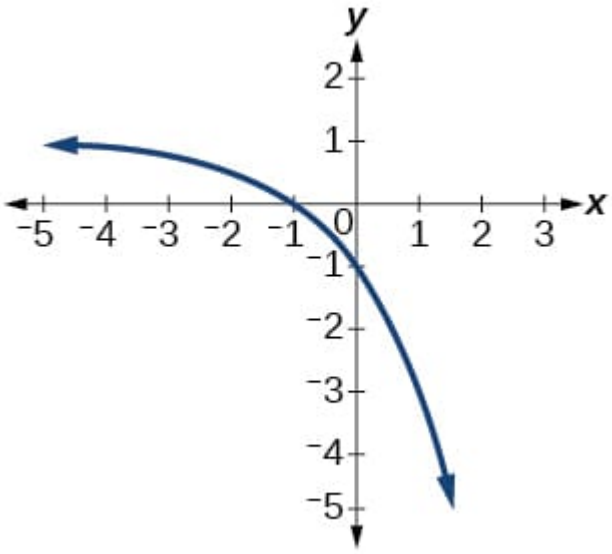
56.



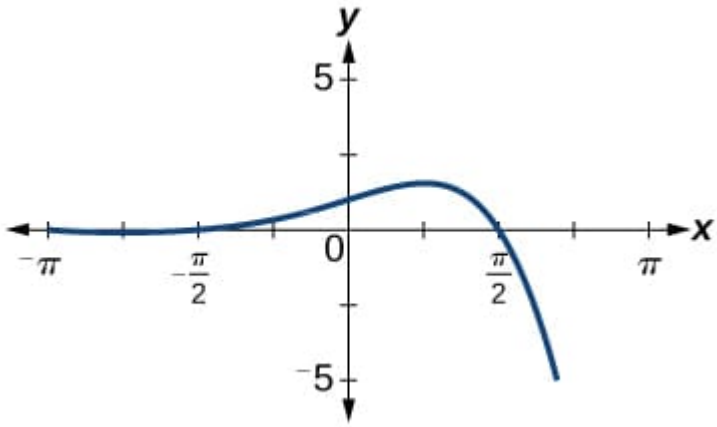
57.



58.



59.



For the following exercises, determine whether the relation represents a function.

60. $\{(-1,-1),(-2,-2),(-3,-3)\}$
61. $\{(3,4),(4,5),(5,6)\}$
62. $\{(2,5),(7,11),(15,8),(7,9)\}$

For the following exercises, determine if the relation represented in table form represents y as a function of x.

63.

x	5	10	15

y	3	8	14

54.

x	5	10	15
y	3	8	8

55.

x	5	10	10
y	3	8	14

For the following exercises, use the function f represented in the following table.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

56. Evaluate $f(3)$.
57. Solve $f(x) = 1$.

For the following exercises, evaluate the function f at the values -2, -1, 0, 1, and 2.

58. $f(x) = 4 - 2x$
59. $f(x) = 8 - 3x$
60. $f(x) = 8x^2 - 7x + 3$
61. $f(x) = 3 + \sqrt{x+3}$
62. $f(x) = \frac{x-2}{x+3}$
63. $f(x) = 3^x$

For the following exercises, evaluate the expressions, given functions f, g, and h:

- $f(x) = 3x - 2$
- $g(x) = 5 - x^2$
- $h(x) = -2x^2 + 3x - 1$

64. $3f(1) - 4g(-2)$
65. $f\left(\frac{7}{3}\right) - h(-2)$

For the following exercises, graph $y = x^2$ on the given domain. Determine the corresponding range. Show each graph.

66. $[-0.1, 0.1]$
67. $[-10, 10]$
68. $[-100, 100]$

For the following exercises, graph $y = x^3$ on the given domain. Determine the corresponding range. Show each graph.

69. $[-0.1, 0.1]$
70. $[-10, 10]$
71. $[-100, 100]$

For the following exercises, graph $y = \sqrt{x}$ on the given domain. Determine the corresponding range. Show each graph.

72. $[0, 0.01]$
73. $[0, 100]$
74. $[0, 10000]$

For the following exercises, graph $y = \sqrt[3]{x}$ on the given domain. Determine the corresponding range. Show each graph.

75. $[-0.001, 0.001]$
76. $[-1000, 1000]$

37. $[-1000000, 1000000]$
38. The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.
    a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
    b. Explain the meaning of the statement $f(5) = 2$.
39. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.
    a. A garden with area 5000 ft^2 requires 50 yd^3 of dirt. Express this information in terms of the function g.
    b. Explain the meaning of the statement $g(100) = 1$.
40. Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:
    a. $f(5) = 30$
    b. $f(10) = 40$
41. Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:
    a. $h(1) = 200$
    b. $h(2) = 350$
42. Show that the function $f(x) = 3(x - 5)^2 + 7$ is not one-to-one.

This page titled 1.1.2: Homework is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by OpenStax.

1.2: Domain and Range

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—*I am Legend*, *Hannibal*, *The Ring*, *The Grudge*, and *The Conjuring*. Figure 1.2.1 shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

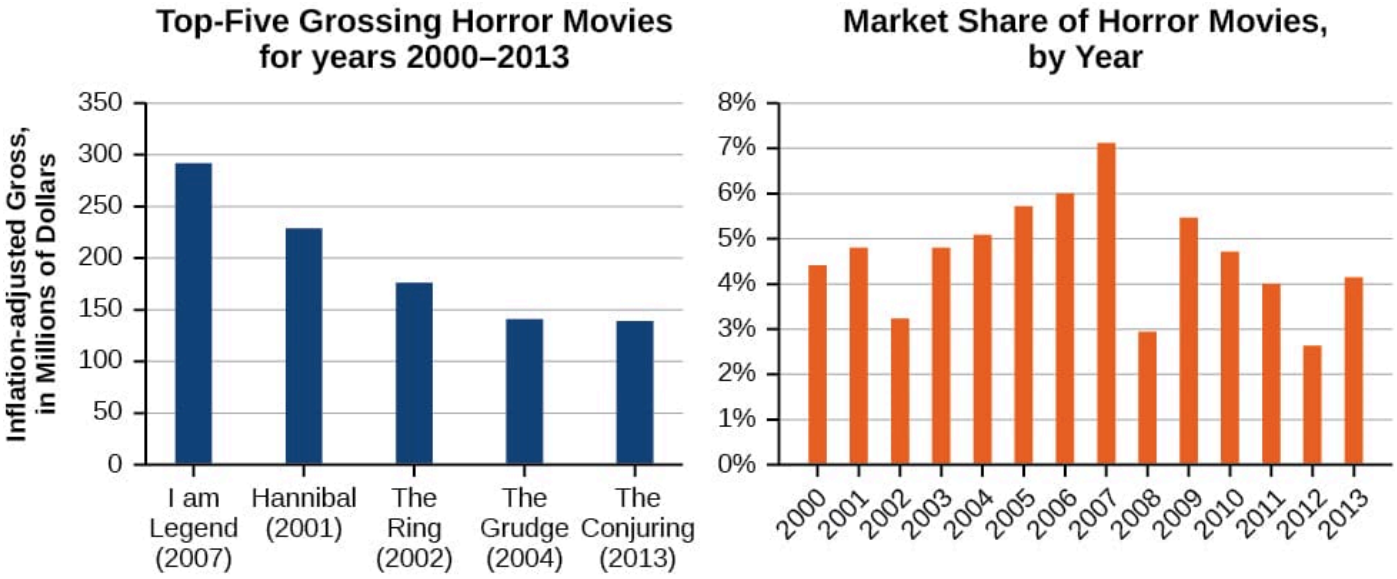


Figure 1.2.1

Finding the Domain of a Function Defined by an Equation

In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as ticket sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include

any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products (see Figure 1.2.2).

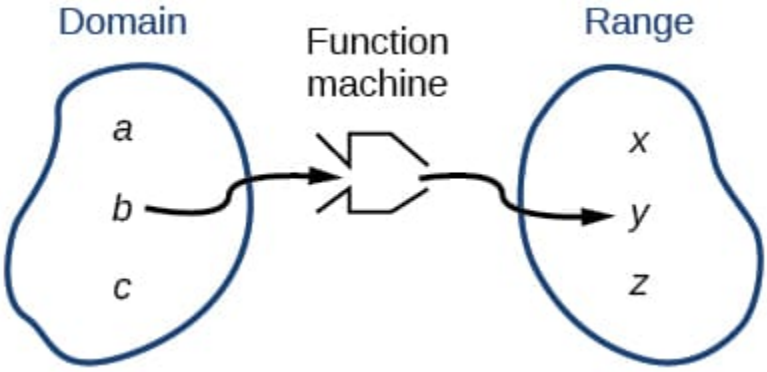


Figure 1.2.2

We can write the domain and range in interval notation, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket, [, when the set includes the endpoint and a parenthesis, (, to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. We will review interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an odd root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called **exclusive**.
- Brackets, [or], are used to indicate that an endpoint is included, called **inclusive**.

Figure 1.2.3 provides a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Figure 1.2.3

Example 1.2.1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2, 10), (3, 10), (4, 20), (5, 30), (6, 40)\}$.

Solution available at the end of this section.

Checkpoint 1.2.1

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Example 1.2.2: Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$.

Solution available at the end of this section.

Checkpoint 1.2.2

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Example 1.2.3: Finding the Domain of a Function Involving a Denominator

Find the domain of the function

$$f(x) = \frac{x+1}{2-x}.$$

Solution available at the end of this section.

Checkpoint 1.2.3

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Example 1.2.4: Finding the Domain of a Function with an Even Root

Find the domain of the function

$$f(x) = \sqrt{7-x}.$$

Solution available at the end of this section.

Checkpoint 1.2.4

Find the domain of the function

$$f(x) = \sqrt{5+2x}.$$

Preview of Future Material

Functions of the types found in Examples 1.2.2 through 1.2.4 will be explored in more detail later in the textbook. For now, any domain questions involving these functions will be fairly straightforward; however, later we will see them "composed" with other functions.

Q&A

Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function

$$f(x) = -\frac{1}{\sqrt{x}}$$

has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in **set-builder notation**.

Definition: Set-Builder Notation

Set-builder notation is a notation for specifying a set by a property that characterizes its members.

Notation: Set-Builder Notation

Set-builder notation commonly uses the following notation:

$$\{x \mid x \text{ satisfies some condition}\}.$$

The braces, $\{\ \}$, are read literally as "the set of." The vertical bar is read as, "such that." Hence, set-builder notation is read as "the set of x such that x satisfies some given condition."

For example, $\{x \mid 10 \leq x < 30\}$ describes the behavior of x in set-builder notation. So we would read $\{x \mid 10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

The following figure compares inequality notation, set-builder notation, and interval notation.

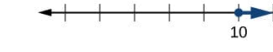
	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure 1.2.5

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$, to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one *or* the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x \mid |x| \geq 3\} = (-\infty, -3] \cup [3, \infty).$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x \mid \text{statement about } x\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x \mid 4 < x \leq 12\}.$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12].$$

Example 1.2.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in the following figure using inequality notation, set-builder notation, and interval notation.

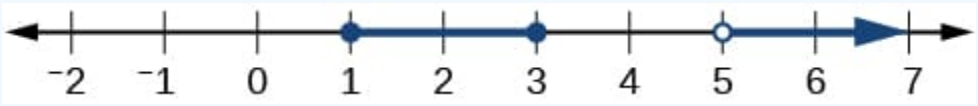


Figure 1.2.6

Solution available at the end of this section.

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Checkpoint 1.2.5

Use the following figure to specify the graphed set in

a. words
b. set-builder notation
c. interval notation



Figure 1.2.7

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values.

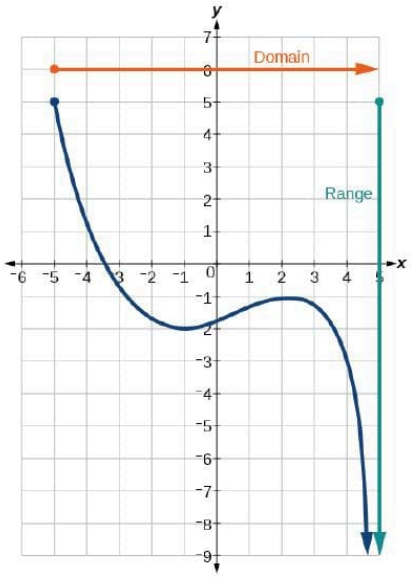


Figure 1.2.8

We can observe that the graph in the figure above extends horizontally from -5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example 1.2.6: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown below.

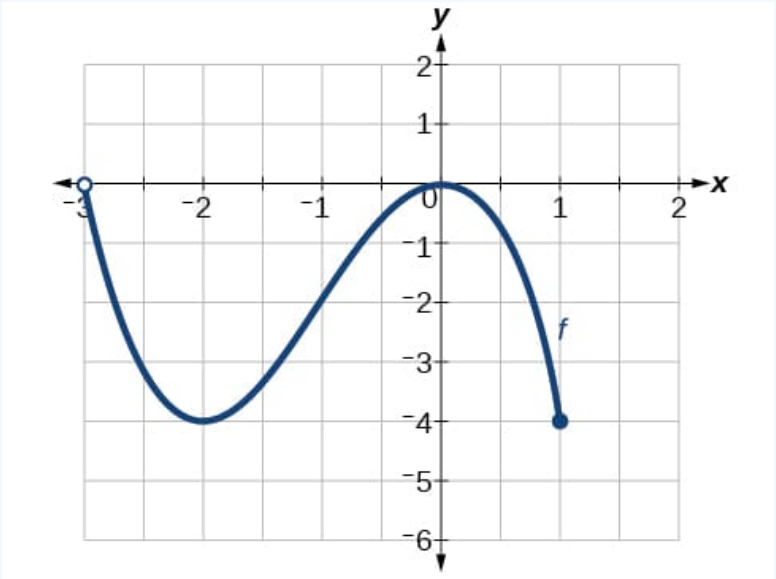


Figure 1.2.9

Solution available at the end of this section.

Example 1.2.7: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown below.

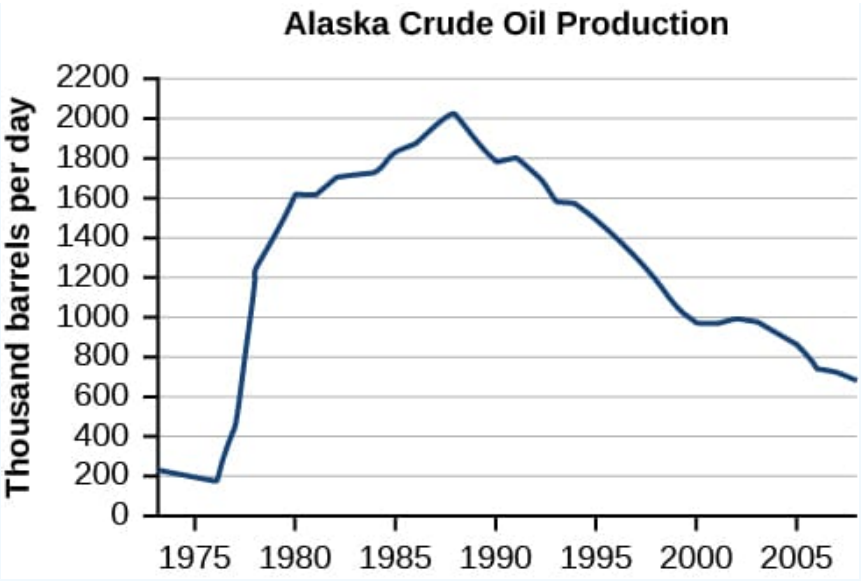


Figure 1.2.11

Solution available at the end of this section.

Checkpoint 1.2.7

Identify the domain and range of the graph in the figure below using interval notation.

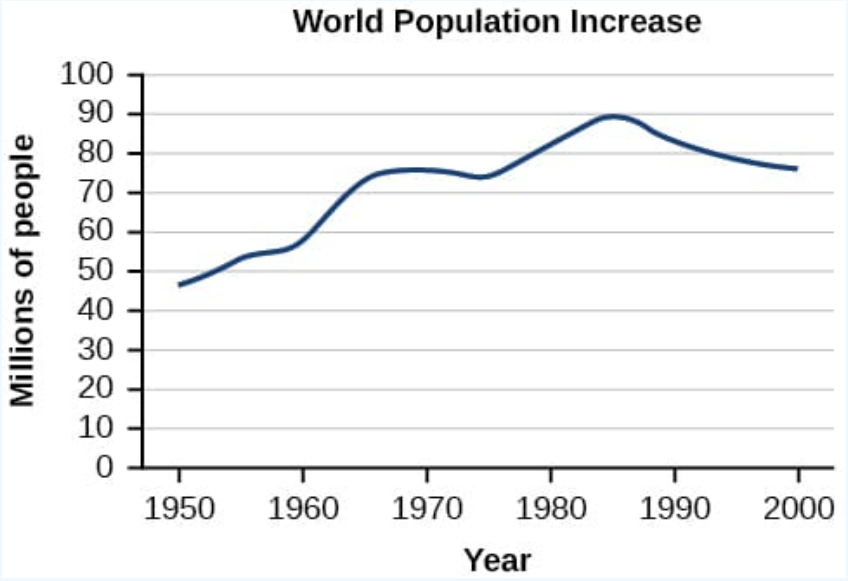


Figure 1.2.12

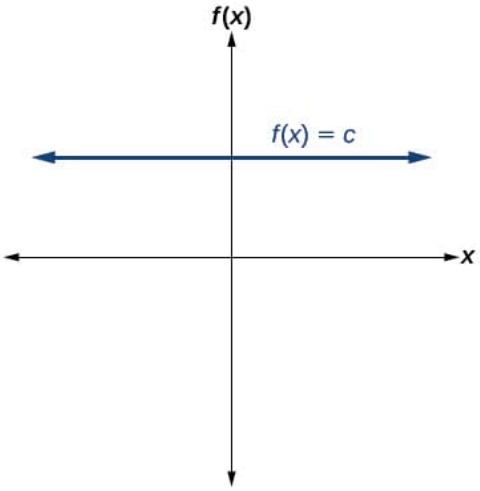
Q&A

Can a function's domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

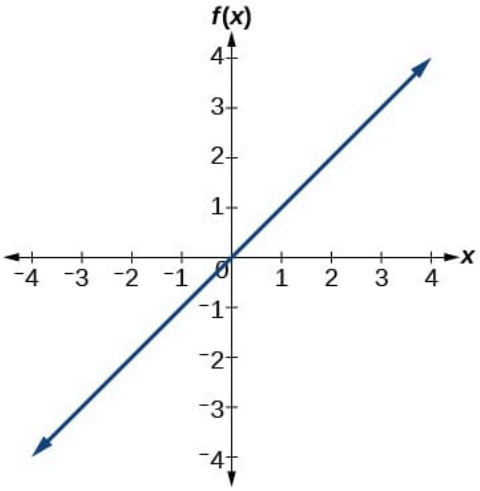
Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.



Domain: $(-\infty, \infty)$
Range: [c,c]

Figure 1.2.13**:** *For the constant function $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.*



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 1.2.14**:** *For the identity function $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.*

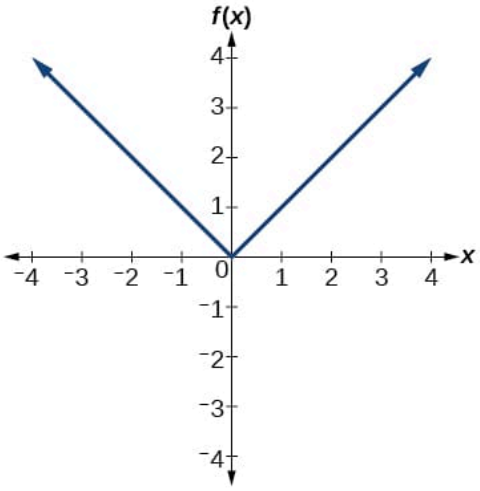


Figure 1.2.15**:** *For the absolute value function $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.*

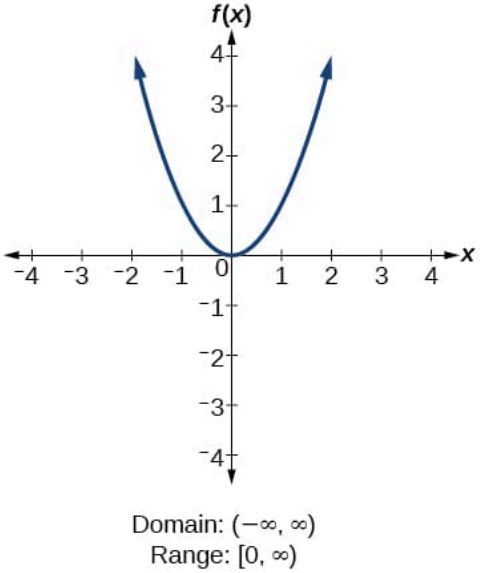


Figure 1.2.16**:** *For the quadratic function $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.*

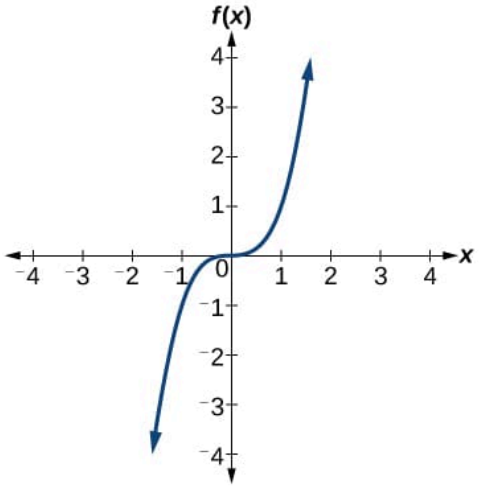


Figure 1.2.17**:** *For the cubic function $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.*

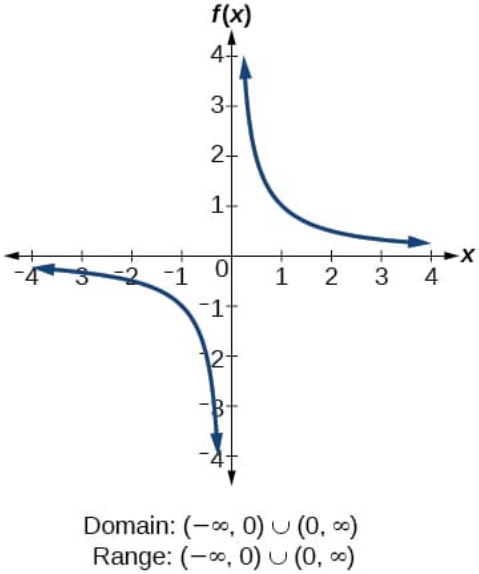
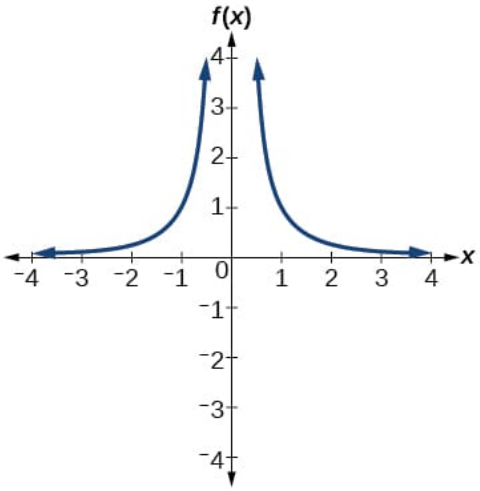
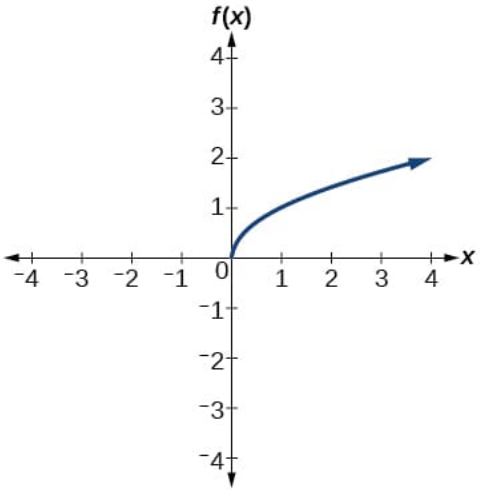


Figure 1.2.18**:** *For the reciprocal function $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x \mid x \neq 0\}$, the set of all real numbers that are not zero.*



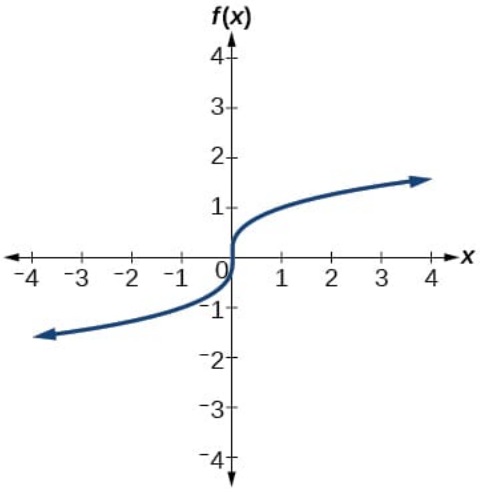
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

Figure 1.2.19**:** *For the reciprocal squared function $f(x) = \frac{1}{x^2}$, we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.*



Domain: $[0, \infty)$
Range: $[0, \infty)$

Figure 1.2.20**:** *For the square root function $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.*



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 1.2.21**:** *For the cube root function $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).*

Example 1.2.8: Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

Solution available at the end of this section.

Example 1.2.9: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution available at the end of this section.

Example 1.2.10: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution available at the end of this section.

Interactive Element: Domain and Range as Shadows of a Graph

The following Interactive Element might help you visualize how the domain and range of $f(x) = 2\sqrt{x+4}$ are just the horizontal and vertical extents of its graph.

Interact: Drag the large orange point along the curve and watch the green bar on the x-axis and the purple bar on the y -axis grow.

Observation: The green bar (the domain) always starts at the closed endpoint $x = -4$ and the purple bar (the range) always starts at $y = 0$; dragging far right shows both go on without bound, so the domain is $[-4, \infty)$ and the range is $[0, \infty)$.

Checkpoint 1.2.10

Find the domain and range of $f(x) = -\sqrt{2-x}$.

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, absolute value can be defined as the **magnitude**, or **modulus**, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0.$$

If we input a negative value, the output is the opposite of the input.

$$f(x) = -x \text{ if } x < 0.$$

Because this requires two different processes, or pieces, the absolute value function is an example of a **piecewise function**. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

Definition: Piecewise Function

A **piecewise-defined function** (also called a **piecewise function** or a **function defined by cases**) is a function specified by two or more sub-functions, where each sub-function applies on its own portion of the domain. The portion of the domain on which a given sub-function applies is called its **subdomain**.

Piecewise functions can be defined using the common functional notation, where the body of the function consists of a list of "subfunctions" and associated subdomains.

Notation: Piecewsie Function

Piecewise functions are almost always written using the following notation:

$$f(x) = \begin{cases} \text{subfunction 1}, & \text{if } x \text{ is in subdomain 1} \\ \text{subfunction 2}, & \text{if } x \text{ is in subdomain 2} \\ \text{subfunction 3}, & \text{if } x \text{ is in subdomain 3} \\ \vdots & \vdots \end{cases}$$

Some authors prefer to use a semicolon instead of a comma following the subfunctions. Moreover, the use of "if" following the comma is a stylistic choice, with some authors preferring to use the word "for" instead (e.g., "for $x < 2$). A piecewise function can also be written more succinctly by leaving out both "if" and "for," as follows:

$$f(x) = \begin{cases} x^2, & x < 2 \\ e^x, & 2 \leq x < 5 \\ \sin(x), & x \geq 5 \end{cases}$$

A value of x at which one subdomain ends and the next begins is called a **breakpoint**.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to \$10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10,000$ and $\$1000 + 0.2(S - \$10,000)$ if $S > \$10,000$.

Example 1.2.11: Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a function relating the number of people, n, to the cost, C.

Solution available at the end of this section.

Example 1.2.12: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25, & \text{if } 0 < g < 2 \\ 25 + 10(g-2), & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Solution available at the end of this section.

While I am not a huge fan of "algorithmic" instructions on how to do each problem (such habits lead to an overwhelming amount of memorization and a subpar level of understanding), such instruction is incredibly helpful when it comes to graphing piecewise functions.

How To

Given a piecewise function, sketch a graph.

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example 1.2.13: Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ 3, & \text{if } 1 < x \leq 2 \\ x, & \text{if } x > 2 \end{cases}$$

Solution available at the end of this section.

Checkpoint 1.2.13

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3, & \text{if } x \leq -1 \\ -2, & \text{if } -1 < x \leq 4 \\ \sqrt{x}, & \text{if } x > 4 \end{cases}$$

Before we leave this section, it's best to preview the *real* definition of the absolute value as a piecewise function.

Definition: Absolute Value Function

The absolute value function is defined to be $f(x) = |x|$, where

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

Its domain is all real numbers, and its range is $[0, \infty)$. Its graph is the **V**-shaped curve formed by the two rays $y = x$ (for $x \geq 0$) and $y = -x$ (for $x < 0$), meeting at a single sharp corner called the **vertex**. For the parent function the vertex is the origin, $(0, 0)$.

Q&A

Can more than one formula from a piecewise function be applied to a value in the domain?

No. Each value corresponds to one equation in a piecewise formula.

Time-Sensitive Advice: HawkConnect

What is HawkConnect?

In the past, navigating through all the student resources (EOPS, TRIO, MI CASA, etc.) required you to go to individual sites and apply individually for each program. HawkConnect now unifies the entire process into a single, guided application process.

To learn more about HawkConnect and to apply, go to the CRC Support Services website.

Success in Mathematics: Show Your Work

Many mathematics exams include the statement, "Show Your Work" (if it's not explicitly written, then it is implied). The exact meaning of these words can change from one professor to another; however, they usually mean that you need to show more than just the answer to a problem. You need to explain or show how you found the answer. Your responsibility is to learn what "Show Your Work" means to your instructor.

Why do you need to show your work?

Your *authentic* work (e.g., without use of A.I., tutors, or any other external help) demonstrates what you know. It shows what you understand and what you do not understand. By looking at your work and the work of others, an instructor can help you improve your understanding.

Everyone makes mistakes. To learn from these mistakes, you need to be able to find them. If you show your work, you can go back and find mistakes. This helps you identify misunderstandings and bad habits. Without your work, all you can do is start over.

Finally, although you may be able to do a problem "in your head," this will not always be possible. The problems get more challenging and involve more detail as we move forward. Showing your work now develops the habits and procedures you will need to do math in the future.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. A student did a homework exercise. She checked her answer with the back of the book. It was wrong. So, she erased her work and started over. Why should she not erase her work?
2. A student did a homework exercise. He checked his answer with the back of the book. It was wrong. He marked the problem on his homework with an asterisk. He went on with the assignment. He never went back to the problem. Why is it essential

for this student to finish the problem?

If you are unsure about the answers to either of these questions, feel free to ask your instructor - that's what they are there for!

Answers

Section Preview

The algebraic functions encountered in this section should *not* involve compositions of functions (e.g., the inverse square function composed with a square root composed with a quadratic). Focus on the domains of basic functions (involving shifts). Piecewise functions and the absolute value represented as a piecewise function are critical topics.

Section Preview

- **Sets and Numbers**
  - Inequalities and Inequality Notation: Domain conditions (radicand ≥ 0, denominator $\neq 0$) and the horizontal/vertical extents of graphs are stated with inequalities, so students must already read order relations fluently.
  - Intervals and Interval Notation: Every domain and range answer is expressed in interval notation; understanding brackets vs. parentheses (inclusion vs. exclusion) is essential to the entire section.
  - Set-Builder Notation: The section specifies domains and ranges in set-builder form and relies on students reading $\{x \mid P(x)\}$.
- **Arithmetic**
  - Mathematical Operations—Higher-Index Roots: The domain rule explicitly distinguishes odd roots (domain all reals) from even roots (restricted), as illustrated by the cube root vs. square root toolkit functions.
  - The Absolute Value of a Number: Absolute value as distance from zero underlies the $\{x \mid |x| \geq 3\}$ set and motivates the piecewise definition of $|x|$ previewed at the end of the section.
- **Graphing Functions**
  - Intercepts and Base Graphs: Finding domains/ranges of the toolkit functions and sketching the pieces in Example 13 relies on already recognizing parent-function shapes; the section states that "each of the component functions is from our library of toolkit functions, so we know their shapes."

Section Preview

- **Sets and Numbers**
  - Sets and Set Notation: Example 1 lists a domain inside braces, and Examples 3 and 5 use the union symbol $\cup$ to combine sets/intervals.
- **Solving Equations**
  - Solving Linear Equations—One or Two-Steps: Example 3 finds the excluded input by setting the denominator $2 - x = 0$ and solving.
- **Solving Inequalities**
  - Solving Linear Inequalities: Examples 4 and 10 obtain the domain by setting the radicand ≥ 0 and solving the resulting linear inequality.
- **Graphs**
  - Closed- and Open-Circle Notations: Graphing the piecewise function in Example 13 requires open circles for excluded endpoints and closed circles for included ones.

Section Preview

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

Example $1.2.1$**: Finding the Domain of a Function as a Set of Ordered Pairs**

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs: $\{2, 3, 4, 5, 6\}$.

Example $1.2.2$**: Finding the Domain of a Function**

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers. In interval form, the domain of f is $(-\infty, \infty)$.

Example $1.2.3$**: Finding the Domain of a Function Involving a Denominator**

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2 - x &= 0 \\ \Longrightarrow \quad 2 &= x \qquad \text{(adding } x \text{ to both sides)} \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$, to combine the two sets. In interval notation, we write the solution: $(-\infty, 2) \cup (2, \infty)$.

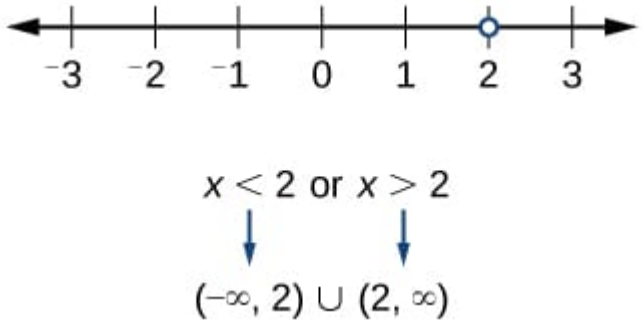


Figure $1.2.4$

Thus, in interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

Example 1.2.4**: Finding the Domain of a Function with an Even Root**

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand. To do this, we set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7-x &\geq 0 && \text{(even-indexed radicals must have nonnegative arguments)} \\ \Rightarrow \quad 7 &\geq x && \text{(adding } x \text{ to both sides)} \end{aligned}$$

Therefore, x must be less than or equal to 7. Hence, we need to exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

Example 1.2.5**: Describing Sets on the Real-Number Line**

To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality	$1 \leq x \leq 3$ or $x > 5$
Set-builder notation	$\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$
Interval notation	$[1, 3] \cup (5, \infty)$

Example 1.2.6**: Finding Domain and Range from a Graph**

We can observe that the horizontal extent of the graph is -3 (exclusive) to 1 (inclusive), so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 (inclusive) to -4 (inclusive), so the range is $[-4, 0]$. See the figure below.

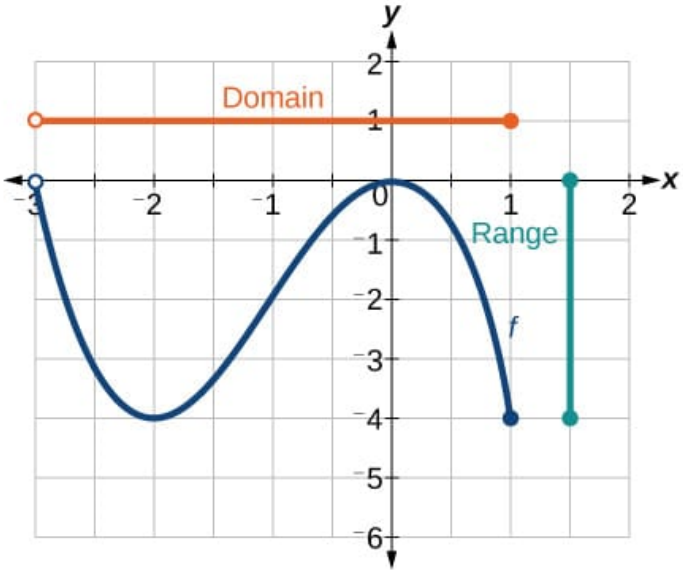


Figure 1.2.10

Example 1.2.7**: Finding Domain and Range from a Graph of Oil Production**

The input quantity along the horizontal axis is "years," which we represent with the variable t for time. The output quantity is "thousands of barrels of oil per day," which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is $[1973, 2008]$, and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Example 1.2.8**: Finding the Domain and Range Using Toolkit Functions**

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result. Therefore, the domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example 1.2.9**: Finding the Domain and Range**

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example 1.2.10**: Finding the Domain and Range**

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$$x + 4 \geq 0 \text{ when } x \geq -4.$$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

The following figure represents the function f.

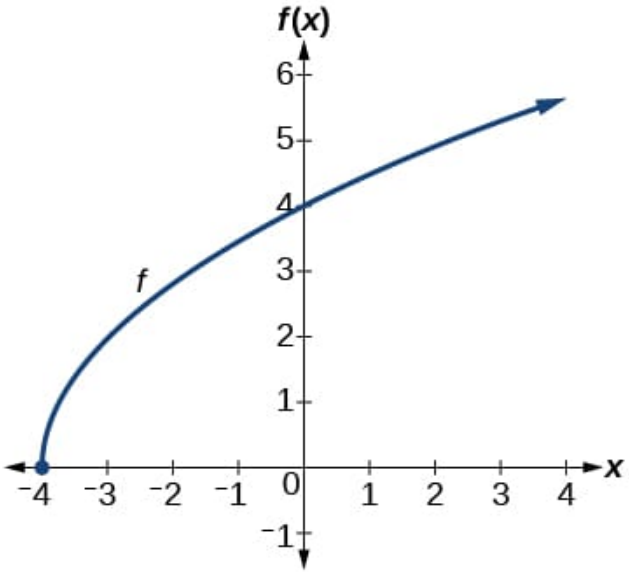


Figure 1.2.22

Example 1.2.11**: Writing a Piecewise Function**

Two different formulas will be needed. For n-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n, & \text{if } 0 < n < 10 \\ 50, & \text{if } n \geq 10 \end{cases}$$

The function is represented in the figure below. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.

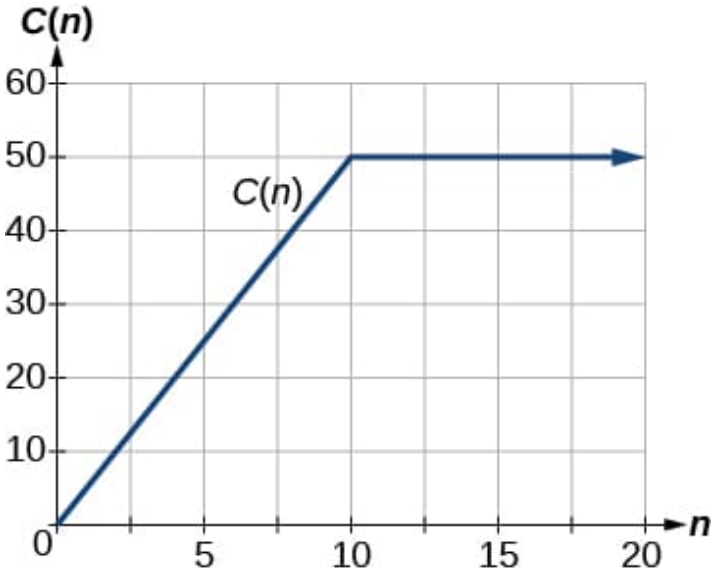


Figure 1.2.23

Example 1.2.12**: Working with a Piecewise Function**

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25.$$

To find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4 - 2) = \$45.$$

The function is represented in the figure below. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

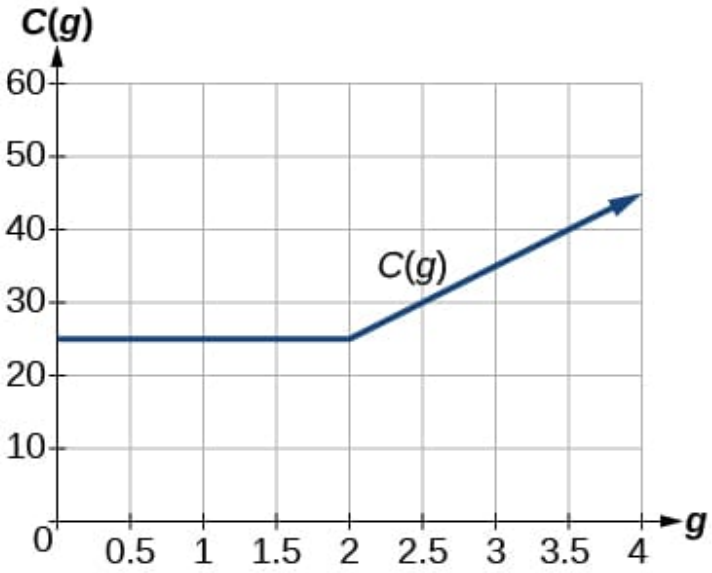


Figure 1.2.24

Example 1.2.13**: Graphing a Piecewise Function**

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

The figure below shows the three components of the piecewise function graphed on separate coordinate systems.

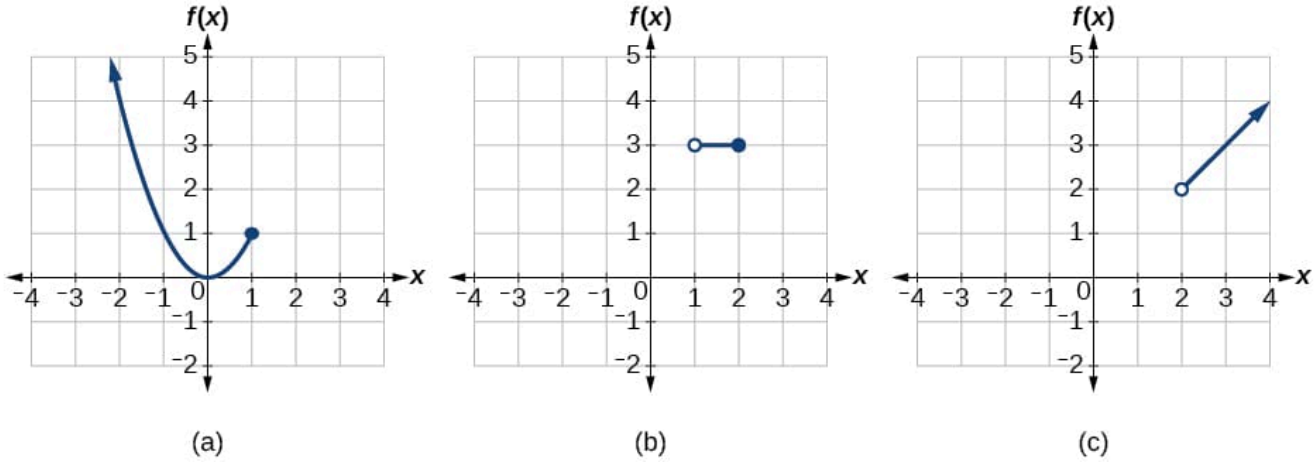


Figure 1.2.25

Now that we have sketched each piece individually, we combine them in the same coordinate plane.

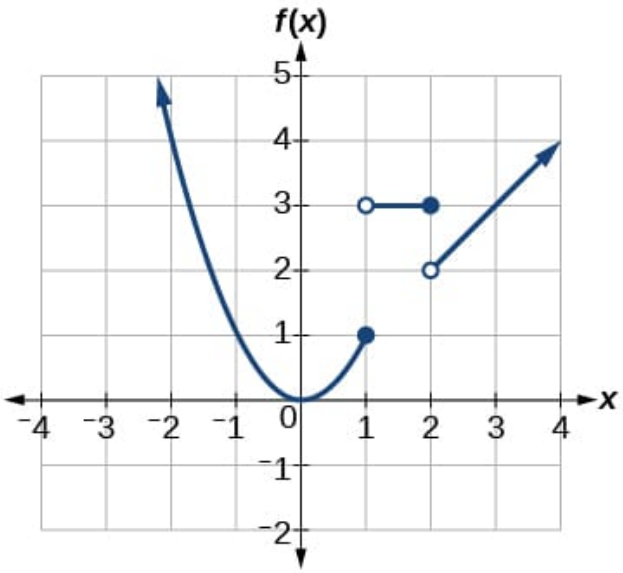


Figure 1.2.26

Note that the graph does pass the Vertical Line Test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

This page titled 1.2: Domain and Range is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

1.2.1: Resources and Key Concepts

Resources

Videos

- Domain and Range of a function
  - Algebra in Review: Functions - Determining Domain
  - Domain and Range - Finding Domain and Range from Graphs of Functions
  - Domain and Range - Finding the Domain of a Function Defined by an Equation
- Interval notation
  - Logic & Set Theory in Review: Sets - Notation (Interval Notation)
- Graphing piecewise functions
  - Domain and Range - Graphing Piecewise-Defined Functions
  - Functions - Overview - A Review for Precalculus: Piecewise-Defined Functions - Graphing

Key Concepts

Definitions

- **Interval Notation:** A method of writing sets of numbers using parentheses $(\ )$ to indicate endpoints that are not included (exclusive) and square brackets $[\ ]$ to indicate endpoints that are included (inclusive). Unbounded intervals use ∞ or $-\infty$.
- **Exclusive Endpoint:** An endpoint of an interval that is not included in the set, denoted by a parenthesis in interval notation.
- **Inclusive Endpoint:** An endpoint of an interval that is included in the set, denoted by a square bracket in interval notation.
- **Union ($\cup$):** A symbol used to combine two or more sets, representing all elements that belong to any of the original sets. For intervals, it connects two or more disjoint segments of the number line.
- **Set-Builder Notation:** A notation used to describe a set by stating the properties that its members must satisfy, typically in the form $\{x \mid \text{condition(s) involving } x\}$, read as "the set of all x such that x satisfies some given condition."
- **Piecewise Function:** A function defined by multiple sub-functions, where each sub-function applies to a different interval in the domain.
- **Magnitude (or Modulus):** For a real number, its absolute value, representing its distance from 0 on the number line, regardless of sign.
- **Absolute Value Function (as a piecewise function):** The function $f(x) = |x|$ defined as $f(x) = x$ if $x \geq 0$ and $f(x) = -x$ if $x < 0$.

Common Mistakes

- **Graphing Multiple Function Pieces Over the Same Interval:** When sketching a piecewise function, graphing more than one of its defining equations over the same segment of the domain violates the definition of a function (as each input would have multiple outputs).
- **Misinterpreting Open and Closed Circles in Piecewise Graphs:** Incorrectly determining whether boundary points are included in a piece of a piecewise function can lead to errors. For example, in Example 13, points like $(1, 3)$ and $(2, 2)$ are excluded (open circles) while $(1, 1)$ and $(2, 3)$ are included (closed circle or part of a continuous segment).
- **Applying Multiple Formulas to a Single Input in a Piecewise Function:** For any given input value in the domain of a piecewise function, only one of its defining formulas/equations applies.

This page titled 1.2.1: Resources and Key Concepts is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

1.2.2: Homework

Reading Questions

1. What are the two main algebraic restrictions you must consider when determining the domain of a function consisting of real numbers given by an equation?
2. Explain the difference between a parenthesis and a square bracket in interval notation.
3. If a graph of a function has an open circle at $x = 3$, is $x = 3$ included in the domain? What if it has a closed circle?
4. What is a piecewise function?
5. How do you find the domain of a function like $f(x) = \frac{5}{x-7}$?
6. How do you find the domain of a function like $g(x) = \sqrt{x+4}$?
7. Can a single input value in the domain of a piecewise function correspond to more than one output value or be defined by more than one of its formulas?
8. What is the domain and range of the identity function, $f(x) = x$?
9. What is the domain and range of the absolute value function, $f(x) = |x|$?
10. When graphing a piecewise function, what do you draw at an endpoint of an interval if the inequality is strict (e.g., $<$ or $>$)? What if it is inclusive (e.g., $\leq$ or $\geq$)?

Homework

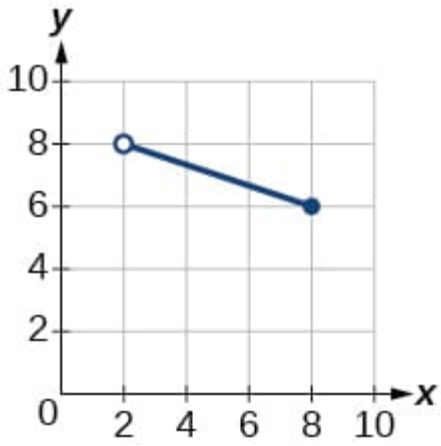
1. Why does the domain differ for different functions?
2. How do we determine the domain of a function defined by an equation?
3. Explain why the domain of $f(x) = \sqrt[3]{x}$ is different from the domain of $f(x) = \sqrt{x}$.
4. When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?
5. How do you graph a piecewise function?

For the following exercises, find the domain of each function using interval notation.

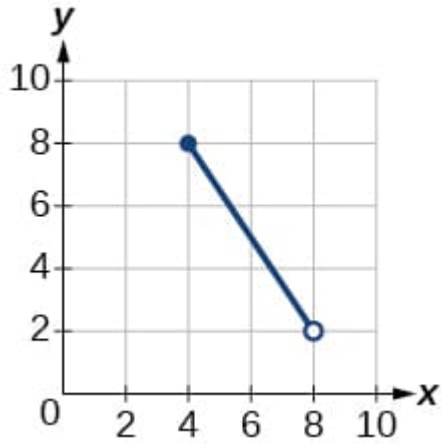
6. $f(x) = -2x(x-1)(x-2)$
7. $f(x) = 5 - 2x^2$
8. $f(x) = 3\sqrt{x-2}$
9. $f(x) = 3 - \sqrt{6-2x}$
10. $f(x) = \sqrt{4-3x}$
11. $f(x) = \sqrt{x^2+4}$
12. $f(x) = \sqrt[3]{1-2x}$
13. $f(x) = \sqrt[3]{x-1}$
14. $f(x) = \frac{9}{x-6}$
15. $f(x) = \frac{3x+1}{4x+2}$
16. $f(x) = \frac{\sqrt{x+4}}{x-4}$
17. $f(x) = \frac{x-3}{x^2+9x-22}$
18. $f(x) = \frac{1}{x^2-x-6}$
19. $f(x) = \frac{2x^3-250}{x^2-2x-15}$
20. $f(x) = \frac{5}{\sqrt{x-3}}$
21. $f(x) = \frac{2x+1}{\sqrt{5-x}}$
22. $f(x) = \frac{\sqrt{x-4}}{\sqrt{x-6}}$
23. $f(x) = \frac{\sqrt{x-6}}{\sqrt{x-4}}$
24. $f(x) = \frac{x}{x}$
25. $f(x) = \frac{x^2-9x}{x^2-81}$

For the following exercises, write the domain and range of each function using interval notation.

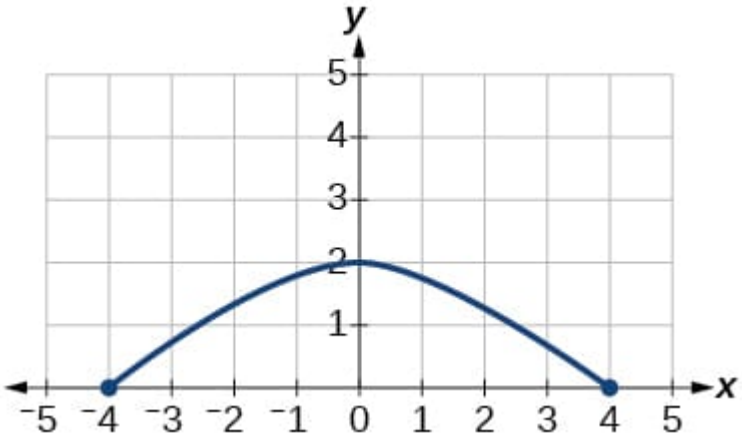
26.



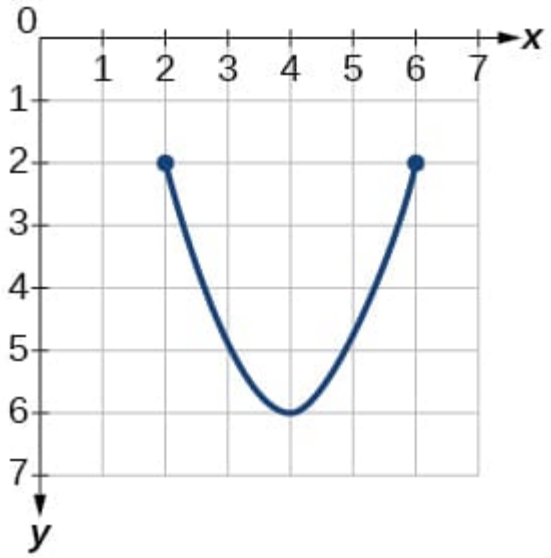
27.



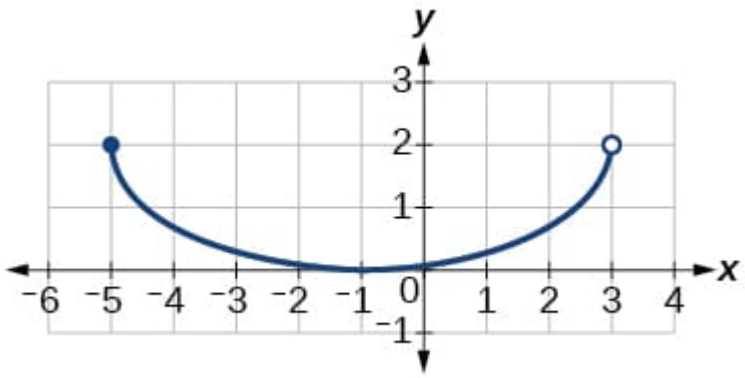
28.



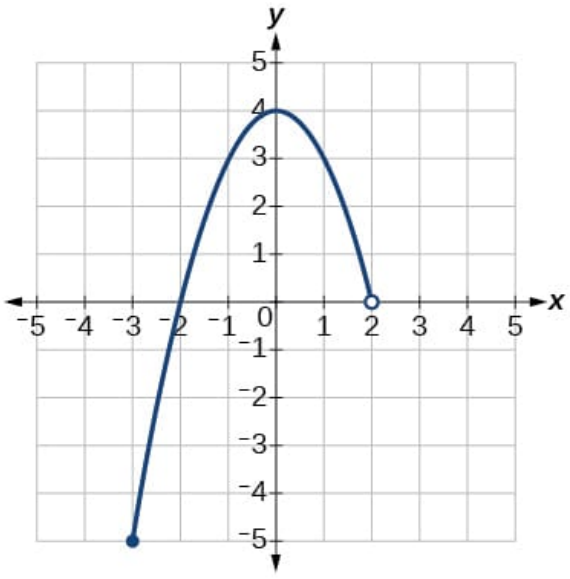
29.


0
1 2 3 4 5 6 7
x
1
2
3
4
5
6
7
y

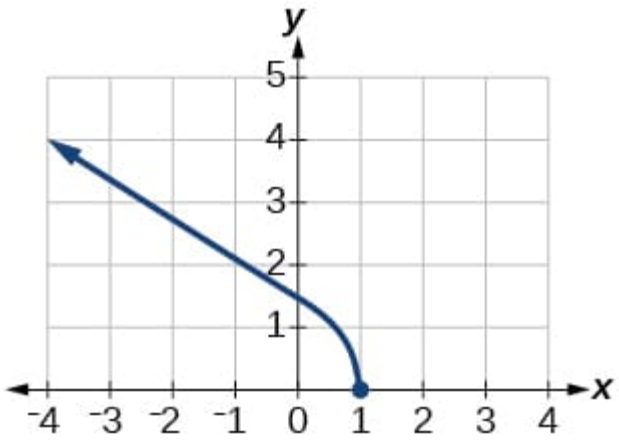
30.


y
3
2
1
-6 -5 -4 -3 -2 -1 0 1 2 3 4
x
-1

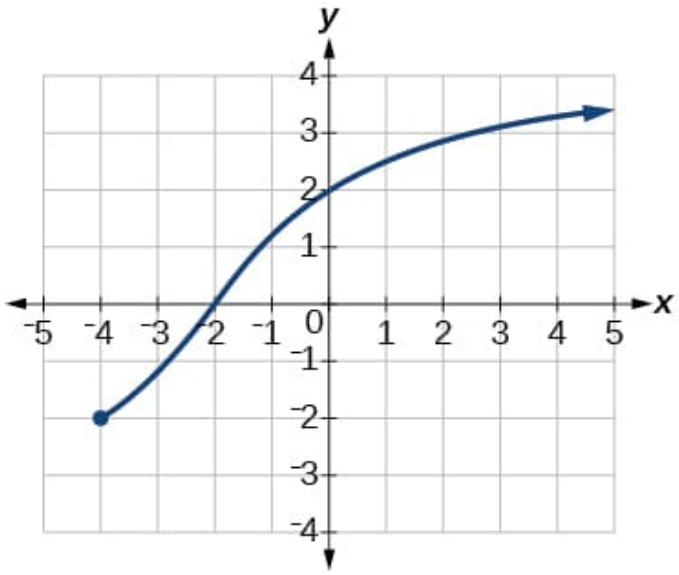
31.


y
5
4
3
2
1
-5 -4 -3 -2 -1 0 1 2 3 4 5
x
-1
-2
-3
-4
-5

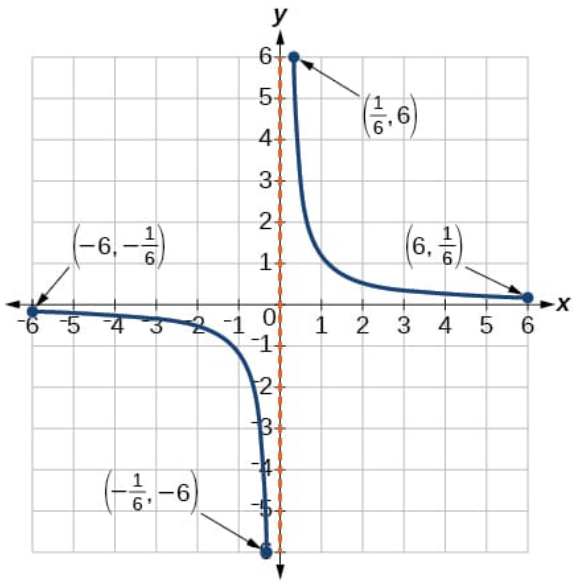
32.


y
x
5
4
3
2
1
-4 -3 -2 -1 0 1 2 3 4

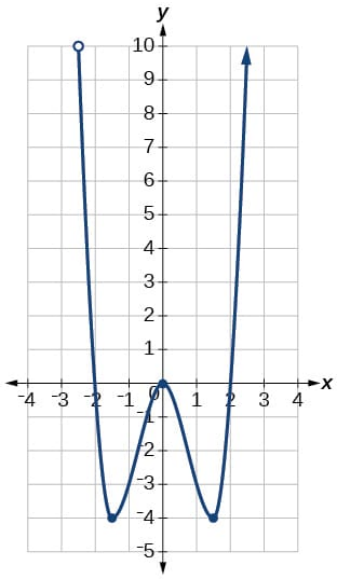
33.


y
x
4
3
2
1
-5 -4 -3 -2 -1 0 1 2 3 4 5
-1
-2
-3
-4

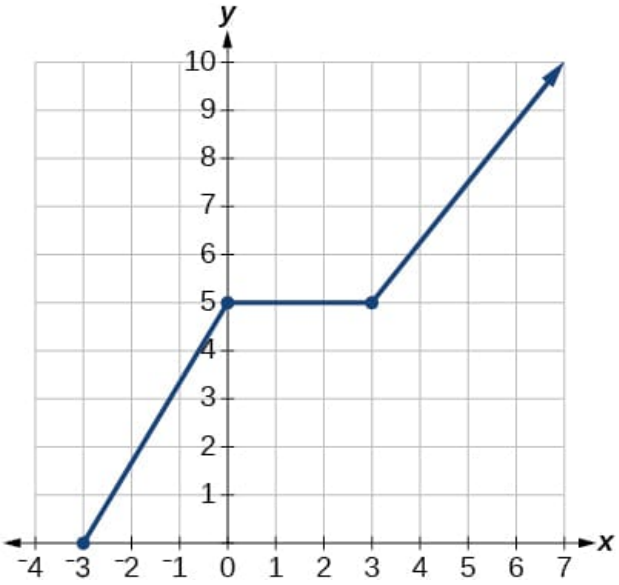
34.


y
x
$\left(\frac{1}{6}, 6\right)$
$\left(-6, -\frac{1}{6}\right)$
$\left(6, \frac{1}{6}\right)$
$\left(-\frac{1}{6}, -6\right)$

35.



36.



For the following exercises, sketch a graph of the piecewise function. Write the domain in interval notation.

37. $f(x) = \begin{cases} x+1, & \text{if } x < -2 \\ -2x-3, & \text{if } x \geq -2 \end{cases}$

38. $f(x) = \begin{cases} 2x-1, & \text{if } x < 1 \\ 1+x, & \text{if } x \geq 1 \end{cases}$

39. $f(x) = \begin{cases} x+1, & \text{if } x < 0 \\ x-1, & \text{if } x > 0 \end{cases}$

40. $f(x) = \begin{cases} 3, & \text{if } x < 0 \\ \sqrt{x}, & \text{if } x \geq 0 \end{cases}$

41. $f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ 1-x, & \text{if } x > 0 \end{cases}$

42. $f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ x+2, & \text{if } x \geq 0 \end{cases}$

43. $f(x) = \begin{cases} x+1, & \text{if } x < 1 \\ x^3, & \text{if } x \geq 1 \end{cases}$

44. $f(x) = \begin{cases} |x|, & \text{if } x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}$

For the following exercises, given each function f, evaluate $f(-3), f(-2), f(-1)$, and $f(0)$.

45. $f(x) = \begin{cases} x+1, & \text{if } x < -2 \\ -2x-3, & \text{if } x \geq -2 \end{cases}$

46. $f(x) = \begin{cases} 1, & \text{if } x \leq -3 \\ 0, & \text{if } x > -3 \end{cases}$

47. $f(x) = \begin{cases} -2x^2+3, & \text{if } x \leq -1 \\ 5x-7, & \text{if } x > -1 \end{cases}$

For the following exercises, given each function f, evaluate $f(-1), f(0), f(2)$, and $f(4)$.

48. $f(x) = \begin{cases} 7x+3, & \text{if } x < 0 \\ 7x+6, & \text{if } x \geq 0 \end{cases}$

49. $f(x) = \begin{cases} x^2-2, & \text{if } x < 2 \\ 4+|x-5|, & \text{if } x \geq 2 \end{cases}$

50. $f(x) = \begin{cases} 5x, & \text{if } x < 0 \\ 3, & \text{if } 0 \leq x \leq 3 \\ x^2, & \text{if } x > 3 \end{cases}$

For the following exercises, write the domain for the piecewise function in interval notation.

51. $f(x) = \begin{cases} x+1, & \text{if } x < -2 \\ -2x-3, & \text{if } x \geq -2 \end{cases}$

52. $f(x) = \begin{cases} x^2-2, & \text{if } x < 1 \\ -x^2+2, & \text{if } x > 1 \end{cases}$

53. $f(x) = \begin{cases} 2x-3, & \text{if } x < 0 \\ -3x^2, & \text{if } x \geq 2 \end{cases}$

54. Graph $y = \frac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

55. Graph $y = \frac{1}{x}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

56. Suppose the range of a function f is $[-5, 8]$. What is the range of $|f(x)|$?

57. Create a function in which the range is all nonnegative real numbers.

58. Create a function in which the domain is $x > 2$.

59. The height h of a projectile is a function of the time t it is in the air. The height in feet for t seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?

60. The cost in dollars of making x items is given by the function $C(x) = 10x + 500$.
    a. The fixed cost is determined when zero items are produced. Find the fixed cost for this item.
    b. What is the cost of making 25 items?
    c. Suppose the maximum cost allowed is $1500. What are the domain and range of the cost function, $C(x)$?

This page titled 1.2.2: Homework is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

1.3: The Arithmetic and Composition of Functions

Note to the Instructor (click to expand)

In addition to introducing compositions (and arithmetic combinations) of functions, you should focus on finding the domains of these new functions.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Simplifying Expressions**
  - Simplifying Ratios of Polynomials: This is directly applied in Example 1 when simplifying $(\frac{g}{f})(x) = \frac{x^2-1}{x-1}$ by canceling common factors.
  - Adding and Subtracting Rational Expressions: Required when combining functions that might result in rational expressions, or when simplifying parts of a composite function. For example, if $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x-1}$, finding $f(x) + g(x)$ would require this skill. (While not explicitly in an example, this is a natural extension of function arithmetic with rational functions like those in the homework).
  - Multiplying and Dividing Rational Expressions: Necessary for performing multiplication or division of functions that are rational expressions. These types of problems occur in the homework.
  - Simplifying Compound Rational Expressions: This skill is necessary when simplifying the result of a composition of rational functions, such as in $f(x)$

Hawk A.I. Section-Specific Tutor

EST. 1970 COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

$= \frac{x}{x-2}$ and $g(x) = \frac{6}{x}$, and one computes $f(g(x))$ or $g(f(x))$.

- **Solving Equations**
  - Solving Rational Equations: Needed to find values of x for which $g(x)$ would be an excluded value from the domain of f when determining the domain of $f(g(x))$, as seen in Example 9 where we solve $g(x) = 1$, which is $\frac{4}{3x-2} = 1$. Also used in the homework.

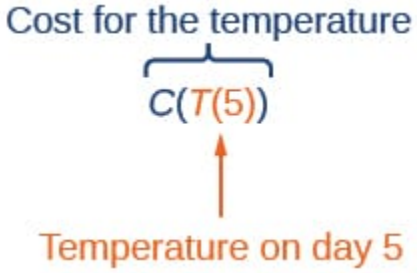
The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Perform arithmetic on functions
- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. This cost will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, \(\text{Cost\} = C (T (d)) \) means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the cost function at that temperature. We would write $C(T(5))$.



By combining these two relationships into one function, we have performed **function composition**, which is the focus of this section.

Combining Functions Using Algebraic Operations

Before we get to function composition, we need to backtrack a little to learn how to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y).$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w.$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of Algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

Definition: Function Arithmetic

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ by the relations

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (f-g)(x) &= f(x)-g(x)\\ (fg)(x) &= f(x)g(x)\\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)},\ \text{if } g(x)\neq 0\end{aligned}$$

Example 1.3.1: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x)=x-1$ and $g(x)=x^2-1$. Are they the same function?

Solution available at the end of this section.

In Example 1.3.1, for $\left(\frac{g}{f}\right)(x)$, the condition $x\neq 1$ is necessary because when $x=1$, the denominator is equal to 0, which makes the function undefined.

Checkpoint 1.3.1

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$ given

$$f(x)=x-1 \text{ and } g(x)=x^2-1.$$

Are they the same function?

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**.

Definition: Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a **composition of functions**. Given functions f and g, the **composite function**, written as $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x)).$$

The function g is the **inner function** (applied first), and f is the **outer function** (applied second).

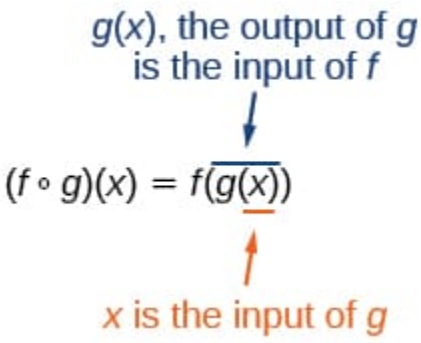
We read the left-hand side as "f composed with g at x," and the right-hand side as "f of g of x." The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the **composition operator**. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value.

Caution

The product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication take two numbers and give a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the Order of Operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.



In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) && \text{(substituting)} \\ &= (x+2)^2 && \text{(substituting)} \\ &= x^2 + 4x + 4 && \text{(distributing)} \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) && \text{(substituting)} \\ &= x^2 + 2 && \text{(substituting)} \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Example 1.3.2: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is commutative.

$$f(x) = 2x + 1, \quad g(x) = 3 - x.$$

Solution available at the end of this section.

Example 1.3.3: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in T minutes. Interpret $c(s(3))$.

Solution available at the end of this section.

Example 1.3.4: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution available at the end of this section.

Q&A

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order.

Checkpoint 1.3.4

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 1.3.5: Using a Table to Evaluate a Composite Function

Using the following table, evaluate $f(g(3))$ and $g(f(3))$.

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

Solution available at the end of this section.

Checkpoint 1.3.5

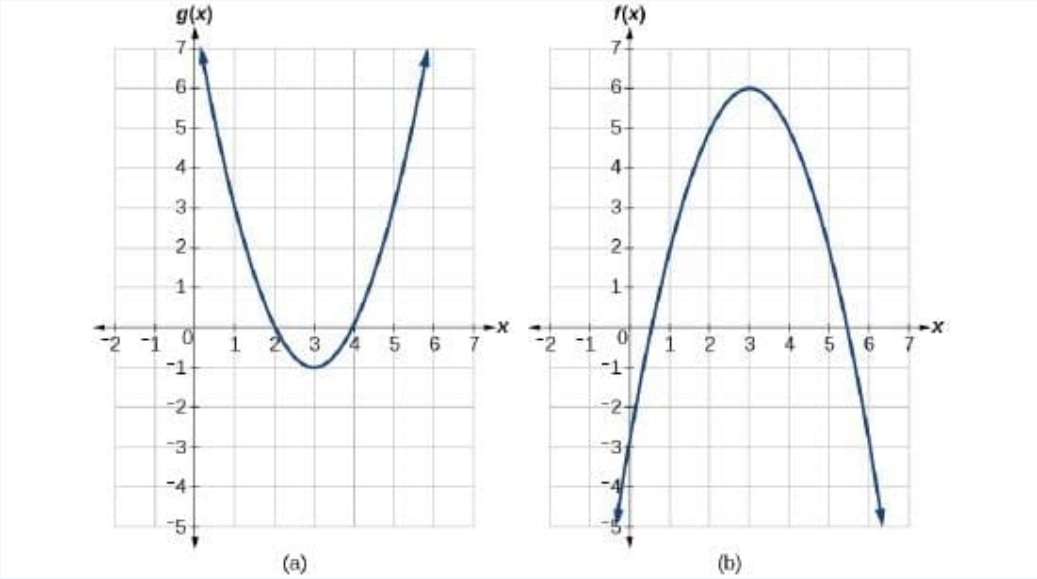
Use the table from Example 1.3.5 to evaluate $f(g(1))$ and $g(f(4))$.

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

Example 1.3.6: Using a Graph to Evaluate a Composite Function

Use the following figure to evaluate $f(g(1))$.



Solution available at the end of this section.

Checkpoint 1.3.6

Use the graph from Example 1.3.6 to evaluate $g(f(2))$.

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

Example 1.3.7: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution available at the end of this section.

Checkpoint 1.3.7

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Finding the Domain of a Composite Function

As we discussed previously, the domain of a composite function such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot. That is, it is important to know the domain of a function such as $f \circ g$.

Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful. Otherwise, we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed and the expression is still undefined. Thus, the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

Theorem: Domain of a Composite Function

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f:

$$\{\, x \mid x \text{ is in the domain of } g \text{ and } g(x) \text{ is in the domain of } f \,\}.$$

To find the domain of the composite function $f(g(x))$, we start by finding the domain of g. Those values of x for which g is defined will be the largest *possible* domain of the composition; however, this set might get restricted once we take the function f into consideration. That said, note the domain of f. Our goal is to find the elements, x, in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$.

Example 1.3.8: Finding the Domain of a Composite Function

Find the domain of $(f \circ g)(x)$ where $f(x) = \frac{5}{x-1}$ and $g(x) = \frac{4}{3x-2}$.

Solution available at the end of this section.

Example 1.3.9: Finding the Domain of a Composite Function Involving Radicals

Find the domain of $(f \circ g)(x)$ where $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{3-x}$.

Solution available at the end of this section.

If we change the function f very slightly in Example 1.3.9 to $f(x) = \sqrt{x-2}$, then the composition, $(f \circ g)(x)$ becomes

$$(f \circ g)(x) = \sqrt{\sqrt{3-x} - 2}.$$

Again, since the radicand is never negative, we know

$$\begin{aligned} \sqrt{3-x} - 2 &\geq 0 \\ \implies \quad \sqrt{3-x} &\geq 2 \qquad (\textbf{adding } 2 \text{ to both sides}) \end{aligned}$$

This makes the problem of finding the domain a more complex. If $\sqrt{3-x} \geq 2$, then

$$\left(\sqrt{3-x}\right)^2 \geq 2^2 \implies 3 - x \geq 4 \implies 3 \geq 4 + x \implies -1 \geq x.$$

Hence, we would require $x \leq 3$ (from the domain of g) and, simultaneously, $-1 \geq x$. The common intersection of these two domain restrictions is $x \leq -1$. Thus, the domain (in this adjusted case) would be $(-\infty, -1]$.

Checkpoint 1.3.9

Find the domain of $(f \circ g)(x)$ where $f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x+4}$.

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to decompose a composite function, so we may choose the decomposition that appears to be most expedient.

Example 1.3.10: Decomposing a Function

Write $f(x) = \sqrt{5 - x^2}$ as the composition of two functions.

Solution available at the end of this section.

Checkpoint 1.3.10

Write $f(x) = \frac{4}{3 - \sqrt{4+x^2}}$ as the composition of two functions.

Answers

Example 1.3.1: Performing Algebraic Operations on Functions

Begin by writing the general form, and then substitute the given functions. For $(g - f)(x)$, we have

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ &= x^2-1-(x-1) && \text{(substituting)}\\ &= x^2-1-x+1 && \text{(distributing)}\\ &= x^2-x && \text{(combining like terms)}\\ &= x(x-1) && \text{(factoring out the GCF)}\end{aligned}$$

For $\left(\frac{g}{f}\right)(x)$, we have

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)}\\ &= \frac{x^2-1}{x-1} && \text{(substituting)}\\ &= \frac{(x-1)(x+1)}{x-1} && \text{(factoring)}\\ &= \frac{\cancel{(x-1)}^{1}(x+1)}{\cancel{(x-1)}^{1}} && \text{(canceling like terms and } \textbf{assuming } x\neq 1)\\ &= x+1 && \text{(where } x\neq 1)\end{aligned}$$

Therefore, the functions are not the same.

Example 1.3.2**: Determining whether Composition of Functions is Commutative**

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned}f(g(x)) &= f(3-x)+1 && \text{(substituting)}\\ &= 2(3-x)+1 && \text{(substituting)}\\ &= 6-2x+1 && \text{(distributing)}\\ &= 7-2x && \text{(combining like terms)}\end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned}g(f(x)) &= g(2x+1) && \text{(substituting)}\\ &= 3-(2x+1) && \text{(substituting)}\\ &= 3-2x-1 && \text{(distributing)}\\ &= 2-2x-1 && \text{(combining like terms)}\end{aligned}$$

We find that $g(f(x))\neq f(g(x))$, so the operation of function composition is not commutative.

Example 1.3.3**: Interpreting Composite Functions**

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t=3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example 1.3.4**: Investigating the Order of Function Composition**

The function $y=f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles} = f(\text{number of hours}).$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons} = g(\text{number of miles}).$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Example 1.3.5**: Using a Table to Evaluate a Composite Function**

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g: $g(3)=2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2)=8$.

$$\begin{aligned}g(3) &= 2\\ \implies f(g(3)) &= f(2)=8\end{aligned}$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3)=3$. Then, using the table for g, we can evaluate

$$g(f(3))=g(3)=2.$$

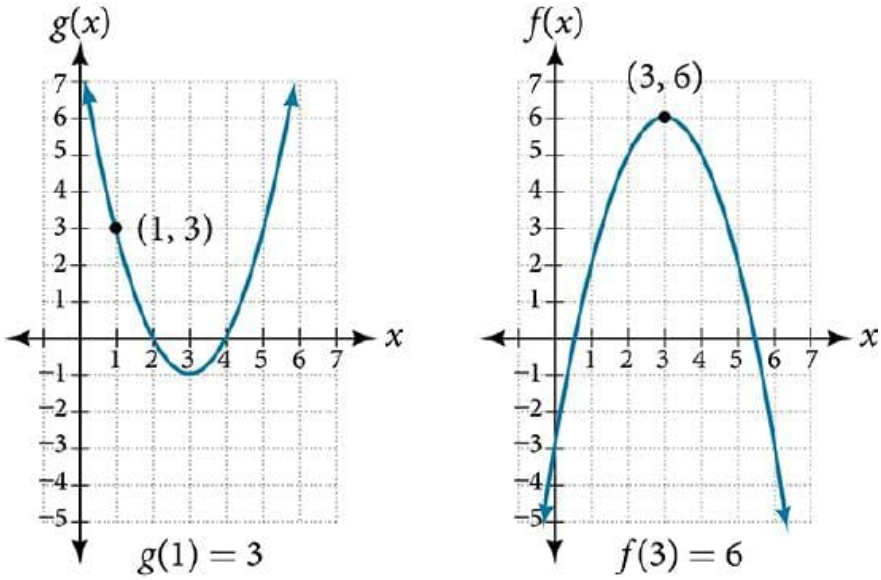
The following table shows the composite functions $f\circ g$ and $g\circ f$ as tables.

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$

3	2	8	3	2

Example 1.3.6: Using a Graph to Evaluate a Composite Function

To evaluate $f(g(1))$, we start with the inside evaluation.

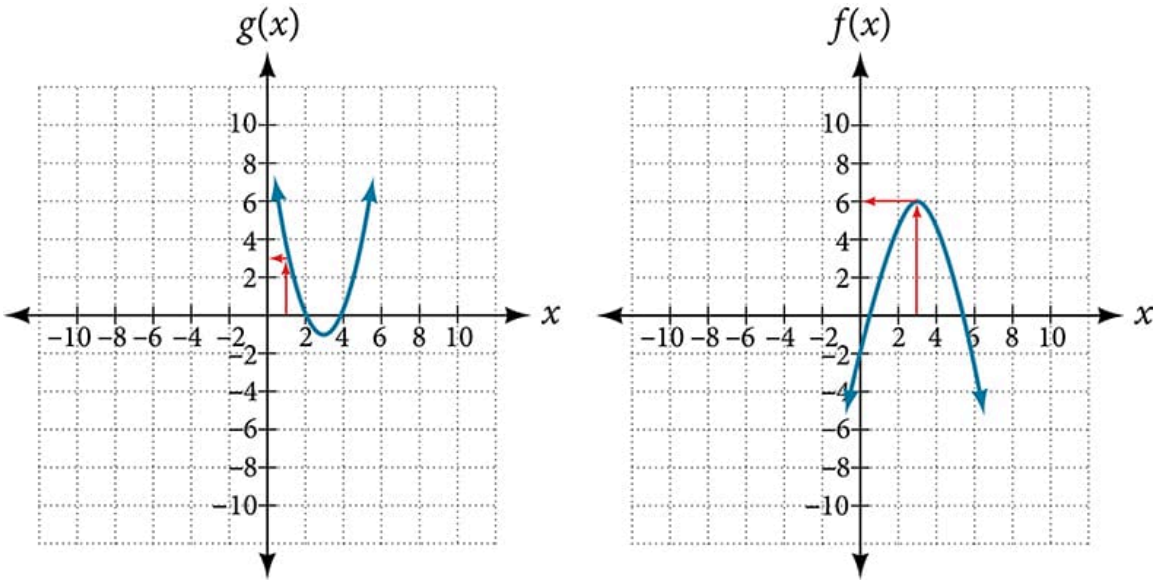


We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

The following figure shows how we can mark the graphs with arrows to trace the path from the input value to the output value.



Example 1.3.7: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$\begin{aligned} h(1) &= 3(1) + 2 && \text{(substituting)} \\ &= 5 && \text{(simplifying)} \end{aligned}$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$$\begin{aligned} f(h(1)) &= f(5) && \text{(substituting)} \\ &= 5^2 - 5 && \text{(substituting)} \\ &= 25 - 5 && \text{(squaring)} \\ &= 20 && \text{(simplifying)} \end{aligned}$$

It makes no difference what the input variables T and x were called in this problem because we evaluated for specific numerical values.

Example 1.3.8**: Finding the Domain of a Composite Function**

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x) = 1$.

$$\begin{aligned} & g(x) = 1 \\ \Longrightarrow \quad & \frac{4}{3x-2} = 1 && \text{(substituting)} \\ \Longrightarrow \quad & 4 = 1(3x-2) && \text{(multiplying both sides by the LCD)} \\ \Longrightarrow \quad & 4 = 3x-2 && \text{(distributing)} \\ \Longrightarrow \quad & 6 = 3x && \text{(adding 2 to both sides)} \\ \Longrightarrow \quad & 2 = x && \text{(dividing both sides by 3)} \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that $x \neq \frac{2}{3}$ o $x \neq 2$.We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty).$$

Example 1.3.9**: Finding the Domain of a Composite Function Involving Radicals**

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}.$$

Since the radicand of a square root must be positive, $\sqrt{3-x}+2 \geq 0$. Hence,

$$\begin{aligned} & \sqrt{3-x}+2 \geq 0 \\ \Longrightarrow \quad & \sqrt{3-x} \geq -2 && \text{(subtracting 2 from both sides)} \end{aligned}$$

Since square roots never return negative numbers, the inequality $\sqrt{3-x} \geq -2$ is always true. Thus, our only restriction on this composition is the original restriction from the domain of g - namely that the values of x must be in the interval $(-\infty, 3]$. This gives a domain of $(-\infty, 3]$.

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

Example 1.3.10**: Decomposing a Function**

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5 - x^2$ is the inside of the square root. We could then decompose the function as

$$h(x) = 5 - x^2 \text{ and } g(x) = \sqrt{x}.$$

We can check our answer by recomposing the functions.

$$g(h(x)) = g(5 - x^2) = \sqrt{5 - x^2}.$$

This page titled 1.3: The Arithmetic and Composition of Functions is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

1.3.1: Resources and Key Concepts

Resources

Videos

- Composition of functions
  - Combining Functions - Domains of Compositions of Functions
  - Combining Functions - Applications of Compositions

Key Concepts

Definitions

- **Function Arithmetic (Sum, Difference, Product, Ratio):**
  - $(f+g)(x) = f(x) + g(x)$ (Sum of functions)
  - $(f-g)(x) = f(x) - g(x)$ (Difference of functions)
  - $(fg)(x) = f(x)g(x)$ (Product of functions)
  - $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$ (Quotient or Ratio of functions
- **Composition of Functions:** The process of combining functions such that the output of one function becomes the input of another. The resulting function is known as a composite function.
- **Composite Function (Notation):** For functions f and g, the composite function, denoted $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$.
- **Composition Operator ($\circ$):** The open circle symbol used to denote function composition, as in $f \circ g$.

Theorems

- **Domain of a Composite Function:** The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

Common Mistakes

- **Confusing Function Composition with Function Multiplication:** The product of functions $(fg)(x) = f(x)g(x)$ is not the same as the function composition $(f \circ g)(x) = f(g(x))$. In general, $f(x)g(x) \neq f(g(x))$.

This page titled 1.3.1: Resources and Key Concepts is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

1.3.2: Homework

Reading Questions

1. If $f(x) = 2x + 3$ and $g(x) = x - 5$, what is $(f + g)(x)$?
2. If $f(x) = x^2$ and $g(x) = x + 1$, what is $(fg)(x)$?
3. What does the notation $(f \circ g)(x)$ mean, and how is it evaluated?
4. Is function composition generally commutative? Explain with an example if $f(x) = x^2$ and $g(x) = x + 2$.
5. If $C(T)$ is the cost of heating a house at temperature T, and $T(d)$ is the temperature on day d, what does $C(T(d))$ represent?
6. What two conditions must be met for an input x to be in the domain of the composite function $(f \circ g)(x)$?
7. How would you decompose the function $H(x) = \sqrt{x^2 + 5}$ into two simpler functions $f(x)$ and $g(x)$ such that $H(x) = f(g(x))$?
8. If you are given functions as tables, how do you evaluate $(f \circ g)(3)$?
9. How does one find the domain of the quotient of two functions, $\frac{f}{g}$?
10. What is the composition of two functions, $f \circ g$?
11. If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.
12. How do you find the domain for the composition of two functions, $f \circ g$?

Homework

Algebraic

For the following exercises, use the pair of functions f and g to find the following values if they exist. Simplify your answers.

- $(f + g)(2)$
- $(f - g)(-1)$
- $(g - f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

15. $f(x) = 3x + 1$ and $g(x) = 4 - x$
16. $f(x) = x^2$ and $g(x) = -2x + 1$
17. $f(x) = x^2 - x$ and $g(x) = 12 - x^2$
18. $f(x) = 2x^3$ and $g(x) = -x^2 - 2x - 3$
19. $f(x) = \sqrt{x + 3}$ and $g(x) = 2x - 1$
20. $f(x) = \sqrt{4 - x}$ and $g(x) = \sqrt{x + 2}$
21. $f(x) = 2x$ and $g(x) = \frac{1}{2x+1}$
22. $f(x) = x^2$ and $g(x) = \frac{3}{2x-3}$
23. $f(x) = x^2$ and $g(x) = \frac{1}{x^2}$
24. $f(x) = x^2 + 1$ and $g(x) = \frac{1}{x^2+1}$

For the following exercises, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f + g)(x)$
- $(f - g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

25. $f(x) = 2x + 1$ and $g(x) = x - 2$
26. $f(x) = 1 - 4x$ and $g(x) = 2x - 1$

27. $f(x) = x^2$ and $g(x) = 3x - 1$
28. $f(x) = x^2 - x$ and $g(x) = 7x$
29. $f(x) = x^2 - 4$ and $g(x) = 3x + 6$
30. $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$
31. $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$
32. $f(x) = x - 1$ and $g(x) = \frac{1}{x-1}$
33. $f(x) = x$ and $g(x) = \sqrt{x+1}$
34. $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$
35. Given $f(x) = x^2 + 2x$ and $g(x) = 6 - x^2$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
36. Given $f(x) = -3x^2 + x$ and $g(x) = 5$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
37. Given $f(x) = 2x^2 + 4x$ and $g(x) = \frac{1}{2x}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
38. Given $f(x) = \frac{1}{x-4}$ and $g(x) = \frac{1}{6-x}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
39. Given $f(x) = 3x^2$ and $g(x) = \sqrt{x-5}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.
40. Given $f(x) = \sqrt{x}$ and $g(x) = |x - 3|$, find $\frac{g}{f}$. Determine the domain of the function in interval notation.
41. Given $f(x) = 2x^2 + 1$ and $g(x) = 3x - 5$, find the following:
    a. $f(g(2))$
    b. $f(g(x))$
    c. $g(f(x))$
    d. $(g \circ g)(x)$
    e. $(f \circ f)(-2)$

For the following exercises, find and simplify $\frac{f(x+h)-f(x)}{h}$ for the given function.

42. $f(x) = 2x - 5$
43. $f(x) = -3x + 5$
44. $f(x) = 6$
45. $f(x) = 3x^2 - x$
46. $f(x) = -x^2 + 2x - 1$
47. $f(x) = 4x^2$
48. $f(x) = x - x^2$
49. $f(x) = x^3 + 1$
50. $f(x) = mx + b$ where $m \neq 0$
51. $f(x) = ax^2 + bx + c$ where $a \neq 0$
52. $f(x) = \frac{2}{x}$
53. $f(x) = \frac{3}{1-x}$
54. $f(x) = \frac{1}{x^2}$
55. $f(x) = \frac{2}{x+5}$
56. $f(x) = \frac{1}{4x-3}$
57. $f(x) = \frac{3x}{x+1}$
58. $f(x) = \frac{x}{x-9}$
59. $f(x) = \frac{x^2}{2x+1}$
60. $f(x) = \sqrt{x-9}$
61. $f(x) = \sqrt{2x+1}$
62. $f(x) = \sqrt{-4x+5}$
63. $f(x) = \sqrt{4-x}$
64. $f(x) = \sqrt{ax+b}$, where $a \neq 0$.
65. $f(x) = x\sqrt{x}$
66. $f(x) = \sqrt[3]{x}$.
    HINT: $(a-b)\left(a^2+ab+b^2\right) = a^3 - b^3$

For the following exercises, use each pair of functions to find $f(g(x))$ and $g(f(x))$. Simplify your answers.

67. $f(x) = x^2 + 1$, $g(x) = \sqrt{x+2}$
68. $f(x) = \sqrt{x} + 2$, $g(x) = x^2 + 3$
69. $f(x) = |x|$, $g(x) = 5x + 1$
70. $f(x) = \sqrt[3]{x}$, $g(x) = \frac{x+1}{x^3}$

71. $f(x)=\frac{1}{x-6}$, $g(x)=\frac{7}{x}+6$
72. $f(x)=\frac{1}{x-4}$, $g(x)=\frac{2}{x}+4$

For the following exercises, use each set of functions to find $f(g(h(x)))$. Simplify your answers.

73. $f(x)=x4+6$, $g(x)=x-6$, and $h(x)=\sqrt{x}$
74. $f(x)=x^2+1$, $g(x)=\frac{1}{x}$, and $h(x)=x+3$
75. Given $f(x)=\frac{1}{x}$ and $g(x)=x-3$, find the following:
    a. $(f\circ g)(x)$
    b. the domain of $(f\circ g)(x)$ in interval notation
    c. $(g\circ f)(x)$
    d. the domain of $(g\circ f)(x)$
    e. $\left(\frac{f}{g}\right)x$
76. Given $f(x)=\sqrt{2-4x}$ and $g(x)=-\frac{3}{x}$, find the following:
    a. $(g\circ f)(x)$
    b. the domain of $(g\circ f)(x)$ in interval notation
77. Given the functions $f(x)=\frac{1-x}{x}$ and \(g(x)= \frac{1}{1+ x^2}, find the following:
    a. $(g\circ f)(x)$
    b. $(g\circ f)(2)$
78. Given functions $p(x)=\frac{1}{\sqrt{x}}$ and $m(x)=x^2-4$, state the domain of each of the following functions using interval notation:
    a. $p(x)m(x)$
    b. $p(m(x))$
    c. $m(p(x))$
79. Given functions $q(x)=\frac{1}{\sqrt{x}}$ and $h(x)=x^2-9$, state the domain of each of the following functions using interval notation.
    a. $\frac{q(x)}{h(x)}$
    b. $q(h(x))$
    c. $h(q(x))$
80. For $f(x)=\frac{1}{x}$ and $g(x)=\sqrt{x-1}$, write the domain of $(f\circ g)(x)$ in interval notation.

For the following exercises, find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x)=f(g(x))$.

81. $h(x)=(x+2)^2$
82. $h(x)=(x-5)^3$
83. $h(x)=\frac{3}{x-5}$
84. $h(x)=\frac{4}{(x+2)^2}$
85. $h(x)=4+\sqrt[3]{x}$
86. $h(x)=\sqrt[3]{\frac{1}{2x-3}}$
87. $h(x)=\frac{1}{(3x^2-4)^{-3}}$
88. $h(x)=\sqrt[4]{\frac{3x-2}{x+5}}$
89. $h(x)=\left(\frac{8+x^3}{8-x^3}\right)^4$
90. $h(x)=\sqrt{2x+6}$
91. $h(x)=(5x-1)^3$
92. $h(x)=\sqrt[3]{x-1}$
93. $h(x)=|x^2+7|$
94. $h(x)=\frac{1}{(x-2)^3}$
95. $h(x)=\left(\frac{1}{2x-3}\right)^2$
96. $h(x)=\sqrt{\frac{2x-1}{3x+4}}$

Graphical

For the following exercises, use the graphs of f, shown in Figure $1.3.2.1$, and g, shown in Figure $1.3.2.2$, to evaluate the expressions.

Figure 1.3.2.1

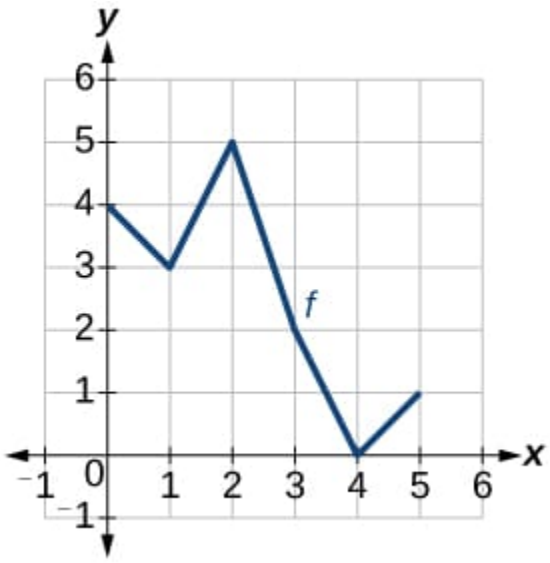
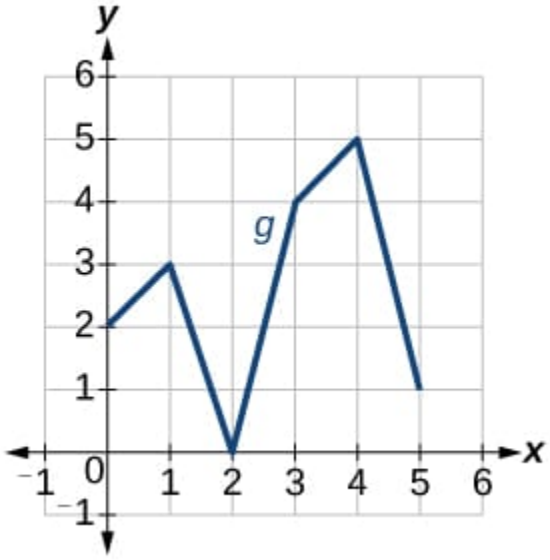


Figure 1.3.2.2



97. $f(g(3))$
98. $f(g(1))$
99. $g(f(1))$
100. $g(f(0))$
101. $f(f(5))$
102. $f(f(4))$
103. $g(g(2))$
104. $g(g(0))$

For the following exercises, use graphs of $f(x)$, shown in Figure 1.3.2.3, $g(x)$, shown in Figure 1.3.2.4, and $h(x)$, shown in Figure 1.3.2.5, to evaluate the expressions.

Figure 1.3.2.3

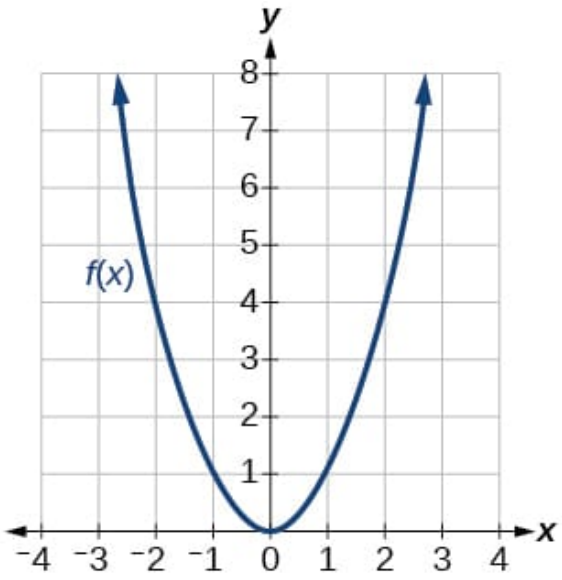


Figure 1.3.2.4

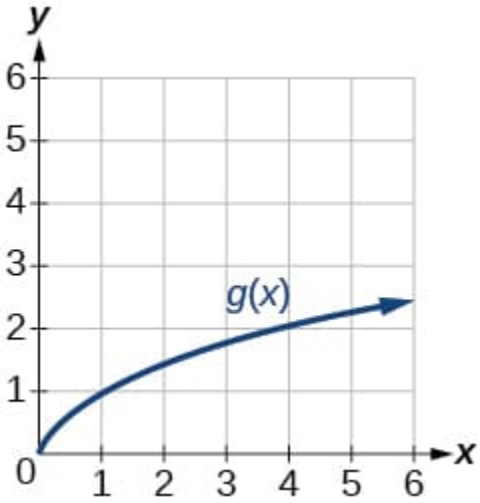
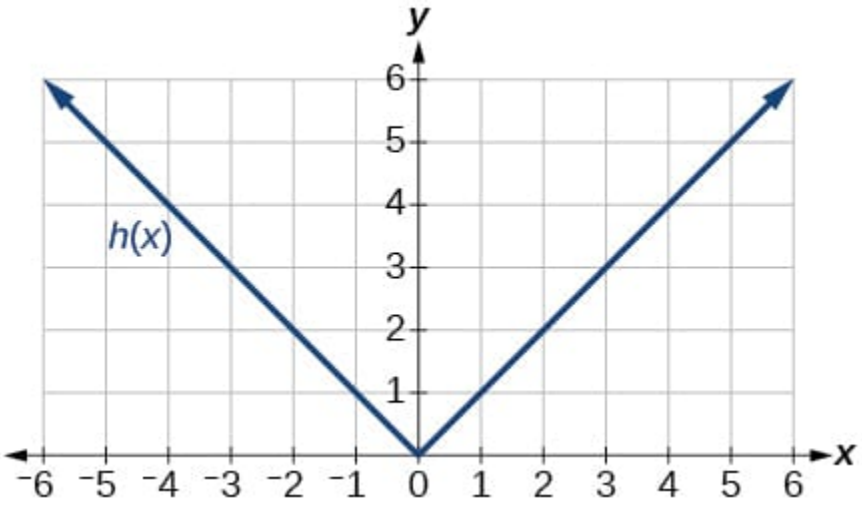


Figure 1.3.2.5



95. $g(f(1))$
96. $g(f(2))$
97. $f(g(4))$
98. $f(g(1))$
99. $f(h(2))$
100. $h(f(2))$
101. $f(g(h(4)))$
102. $f(g(f(-2)))$

Numeric

For the following exercises, use the function values for f and g shown in Table 1.3.2.1 to evaluate each expression.

Table 1.3.2.1

x	$f(x)$	$g(x)$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

103. (f(g(8))\)
104. $f(g(5))$
105. $g(f(5))$
106. $g(f(3))$
107. $f(f(4))$
108. $f(f(1))$
109. $g(g(2))$
110. $g(g(6))$

For the following exercises, use the function values for f and g shown in Table 1.3.2.2 to evaluate the expressions.

Table 1.3.2.2

x	$f(x)$	$g(x)$
-3	11	-8
-2	9	-3
-1	7	0
0	5	1
1	3	0
2	1	-3

x	$f(x)$	$g(x)$
3	-1	-8

21. $(f \circ g)(1)$
22. $(f \circ g)(2)$
23. $(g \circ f)(2)$
24. $(g \circ f)(3)$
25. $(g \circ g)(1)$
26. $(f \circ f)(3)$

For the following exercises, use each pair of functions to find $f(g(0))$ and $g(f(0))$.

27. $f(x) = 4x + 8$, $g(x) = 7 - x^2$
28. $f(x) = 5x + 7$, $g(x) = 4 - 2x^2$
29. $f(x) = \sqrt{x+4}$, $g(x) = 12 - x^3$
30. $f(x) = \frac{1}{x+2}$, $g(x) = 4x + 3$

For the following exercises, use the functions $f(x) = 2x^2 + 1$ and $g(x) = 3x + 5$ to evaluate or find the composite function as indicated.

31. $f(g(2))$
32. $f(g(x))$
33. $g(f(-3))$
34. $(g \circ g)(x)$

Extensions

For the following exercises, let f be the function defined by

$$f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$$

and let g be the function defined

$$g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$$

Compute the indicated value if it exists.

35. $(f+g)(-3)$
36. $(f-g)(2)$
37. $(fg)(-1)$
38. $(g+f)(1)$
39. $(g-f)(3)$
40. $(gf)(-3)$
41. $\left(\frac{f}{g}\right)(-2)$
42. $\left(\frac{f}{g}\right)(-1)$
43. $\left(\frac{f}{g}\right)(2)$
44. $\left(\frac{g}{f}\right)(-1)$
45. $\left(\frac{g}{f}\right)(3)$
46. $\left(\frac{g}{f}\right)(-3)$

For the following exercises, use $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$.

47. Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Compare the two answers.
48. Find $(f \circ g)(2)$ and $(g \circ f)(2)$.
49. What is the domain of $(g \circ f)(x)$?
50. What is the domain of $(f \circ g)(x)$?
51. Let $f(x) = \frac{1}{x}$.
    a. Find $(f \circ f)(x)$.
    b. Is $(f \circ f)(x)$ for any function f the same result as the answer to part (a) for any function? Explain.

For the following exercises, let $F(x) = (x+1)^5$, $f(x) = x^5$, and $g(x) = x+1$.

52. True or False: $(g \circ f)(x) = F(x)$.
53. True or False: $(f \circ g)(x) = F(x)$.

For the following exercises, find the composition when $f(x) = x^2 + 2$ for all $x \geq 0$ and $g(x) = \sqrt{x-2}$.

54. $(f \circ g)(6)$; $(g \circ f)(6)$
55. $(g \circ f)(a)$; $(f \circ g)(a)$
56. $(f \circ g)(11)$; $(g \circ f)(11)$

Real-World Applications

57. The function $D(p)$ gives the number of items that will be demanded when the price is p. The production cost $C(x)$ is the cost of producing x items. To determine the cost of production when the price is $6, you would do which of the following?
    a. Evaluate $D(C(6))$.
    b. Evaluate $C(D(6))$.
    c. Solve $D(C(x)) = 6$.
    d. Solve $C(D(p)) = 6$.
58. The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient's system after t minutes is modeled by $m(t)$. Which of the following would you do in order to determine when the patient will be at a pain level of 4?
    a. Evaluate $A(m(4))$.
    b. Evaluate $m(A(4))$.
    c. Solve $A(m(t)) = 4$.
    d. Solve $m(A(d)) = 4$.
59. A store offers customers a 30% discount on the price x of selected items. Then, the store takes off an additional 15% at the cash register. Write a price function $P(x)$ that computes the final price of the item in terms of the original price x. (Hint: Use function composition to find your answer.)
60. A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25\sqrt{t+2}$, find the area of the ripple as a function of time. Find the area of the ripple at $t = 2$.
61. A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula $r(t) = 2t + 1$, express the area burned as a function of time, t (minutes).
62. Use the function you found in the previous exercise to find the total area burned after 5 minutes.
63. The radius r, in inches, of a spherical balloon is related to the volume, V, by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V(t) = 10 + 20t$.
    a. Find the composite function $r(V(t))$.
    b. Find the *exact* time when the radius reaches 10 inches.
64. The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, \(3
65. Find the composite function $N(T(t))$.
66. Find the time (round to two decimal places) when the bacteria count reaches 6752.

For the following exercises, $C(x)$ denotes the cost to produce x items and $p(x)$ denotes the price-demand function in the given economic scenario. In each Exercise, do the following:

- Find and interpret $C(0)$.
- Find and interpret $\overline{C}(10)$. ($\overline{C}$ is called the **average cost**)
- Find and interpret $p(5)$.
- Find and simplify $R(x)$. ($R(x)$ is the **revenue function**)
- Find and simplify $P(x)$. ($P(x)$ is the **profit function**)
- Solve $P(x) = 0$ and interpret.

65. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, $0 \leq x \leq 15$.
66. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x, 0 \leq x \leq 35$.
67. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x, 0 \leq x \leq 30$.

58. The daily cost, in dollars, to produce x Sasquatch Berry Pies $C(x) = 3x + 36,\ x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x,\ 0 \leq x \leq 24$.
59. The monthly cost, in hundreds of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000,\ x \geq 0$ and the price-demand function, in hundreds of dollars per scooter, is $p(x) = 140 - 2x,\ 0 \leq x \leq 70$.

This page titled 1.3.2: Homework is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

1.4: Transformations of Functions

Note to the Instructor (click to expand)

We start this section with the important conversation about symmetry. The rest of this is the typical section on transformations of functions; however, it is important that students get a strong exposure to understanding horizontal stretches and compressions that are combined with horizontal shifts (everything is assumed to have been seen before).

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Arithmetic**
  - The Order of Operations: Essential for correctly applying sequences of transformations, especially when combining shifts, stretches/compressions, and reflections. For example, understanding whether to apply a vertical stretch before or after a vertical shift.
- **Linear Functions**
  - Slope and Average Rate of Change: The concept of average rate of change introduced in this section is a generalization of the slope of a linear function. Prior understanding of slope as a rate of change is beneficial.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using vertical and horizontal shifts.
- Graph functions using

Hawk A.I. Section-Specific Tutor



COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

reflections about the x-axis and the y-axis. • Graph functions using compressions and stretches. • Combine transformations.	

Figure 1.4.1 **(credit: "Misko"/Flickr)**



We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

Symmetry - Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are **symmetric about the y-axis**. Functions whose graphs are symmetric about the y-axis are called **even functions**.

Definition: Even and Odd Functions

The function f is called an **even function** if for every input x in the domain of f

$$f(-x) = f(x).$$

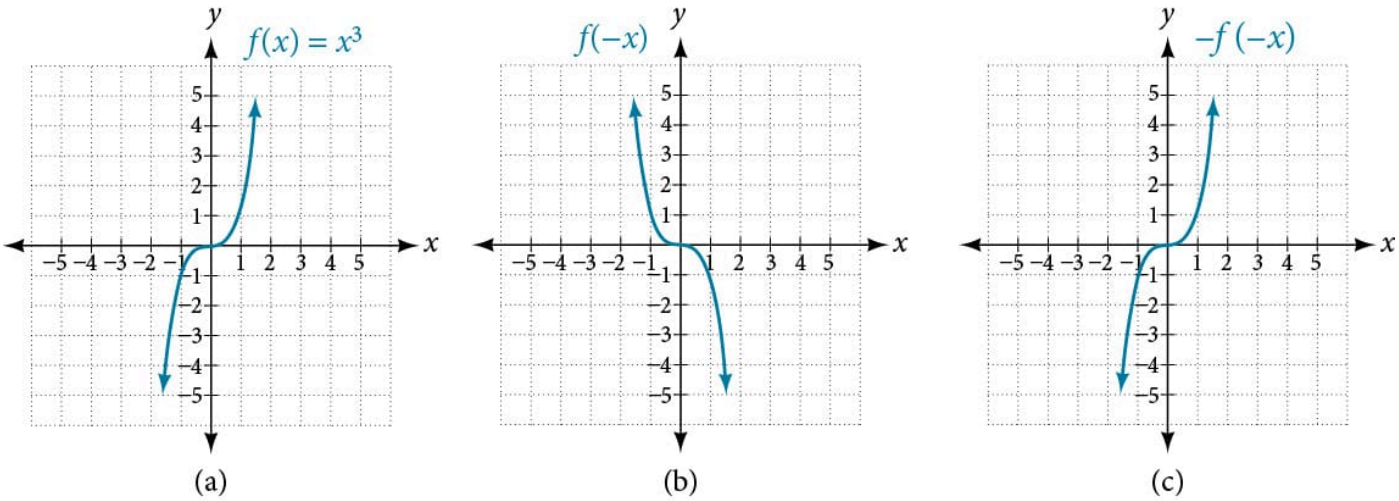
The function f is called an **odd function** if for every input x in the domain of f

$$f(-x) = -f(x).$$

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over *both* axes, the result would be the original graph, as shown in Figre 1.4.2.

Figure 1.4.2

(a) The cubic toolkit function. (b): Horizontal reflection of the cubic toolkit function. (c): Horizontal and vertical reflections reproduce the original cubic function.



We say that these graphs are **symmetric about the origin**. A function with a graph that is symmetric about the origin is called an **odd function**.

Theorem: Symmetry of Even and Odd Functions

Let f be a real function.

- f is even if and only if its graph is symmetric with respect to the y-axis; that is, the graph is unchanged under reflection across the y-axis.
- f is odd if and only if its graph is symmetric with respect to the origin; that is, the graph is unchanged under a 180° rotation about the origin.

Equivalently, f is even when the point $(-x, y)$ lies on the graph whenever (x, y) does, and f is odd when the point $(-x, -y)$ lies on the graph whenever (x, y) does.

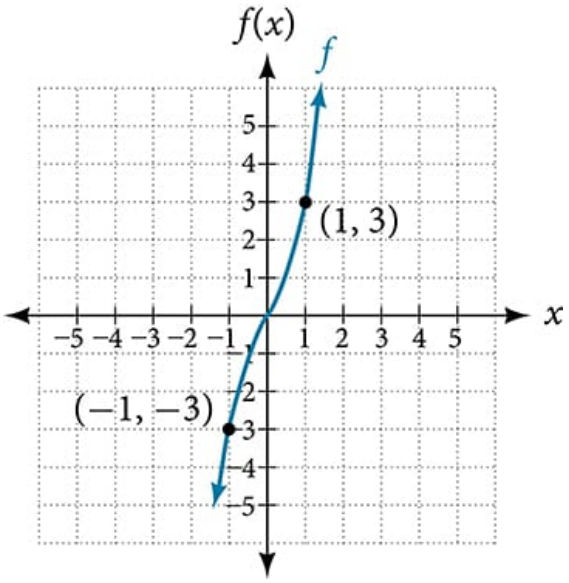
A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Example 1.4.1: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution available at the end of this section.

The graph of the function from Example 1.4.1 is shown below. Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.



Checkpoint 1.4.1

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Graphing Functions Using Vertical and Horizontal Shifts

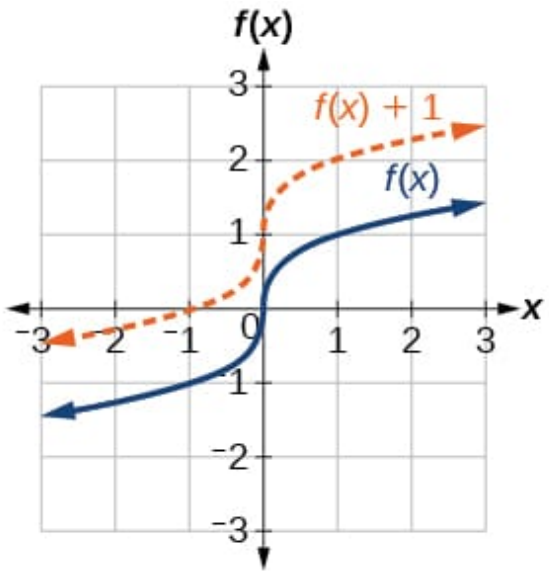
Often when given a problem, we try to model the scenario using Mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest of these is a **vertical shift**, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See Figure $1.4.3$ for an example.

Figure 1.4.3

Vertical shift by $k = 1$ of the cube root function $f(x) = x^3$



To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

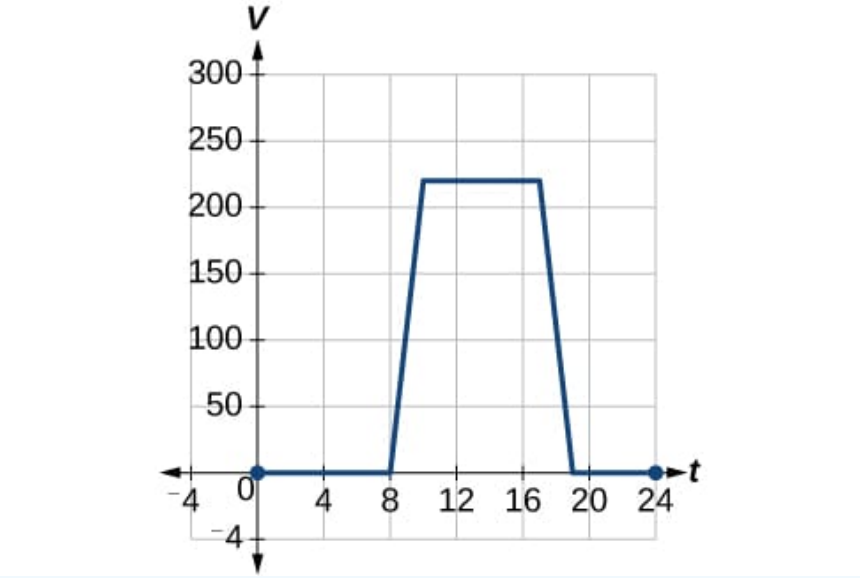
Theorem: Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a vertical shift of $f(x)$. All the output values change by k units. If k is positive, the graph will shift up by k units. If k is negative, the graph will shift down by $|k|$ units.

Example 1.4.2: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 1.4.4 shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

Figure 1.4.4



Solution available at the end of this section.

Example 1.4.3: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 1.4.2. Create a table for the function $g(x) = f(x) - 3$.

Table 1.4.2

x	2	4	6	8
$f(x)$	1	3	7	11

Solution available at the end of this section.

As with Example 1.4.2, the input values in Example 1.4.3 stay the same and only the output values change.

Checkpoint 1.4.3

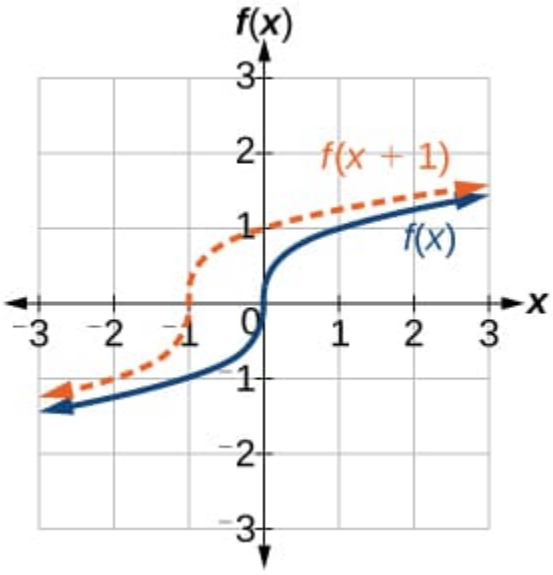
The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 1.4.6.

Figure 1.4.6

Horizontal shift of the function $f(x) = x^3$. Note that $(x + 1)$ means $h = -1$ which shifts the graph to the left, that is, towards negative values of x.



For example, if $f(x) = x^2$, then $g(x) = (x - 2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Theorem: Horizontal Shift

Given a function $f(x)$, a new function $g(x) = f(x - h)$, where h is a constant, is a horizontal shift of $f(x)$. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example 1.4.4: Adding a Constant to an Input

Returning to our building airflow example from Figure 1.4.4, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution available at the end of this section.

It is important to note that, in Example 1.4.4, $V(t + 2)$ has the effect of shifting the graph to the *left*.

Horizontal changes, or "inside changes," affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for $F(t) = V(t + 2)$.

Example 1.4.5: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 1.4.4. Create a table for the function $g(x) = f(x - 3)$.

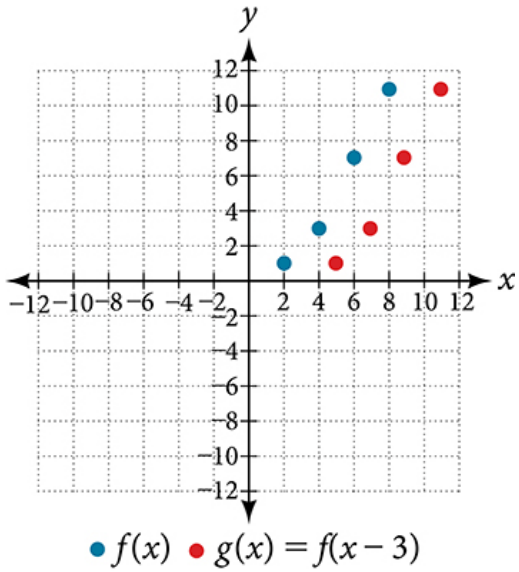
Table 1.4.4

x	2	4	6	8
$f(x)$	1	3	7	11

Solution available at the end of this section.

Figure 1.4.8 represents both of the functions. We can see the horizontal shift in each point.

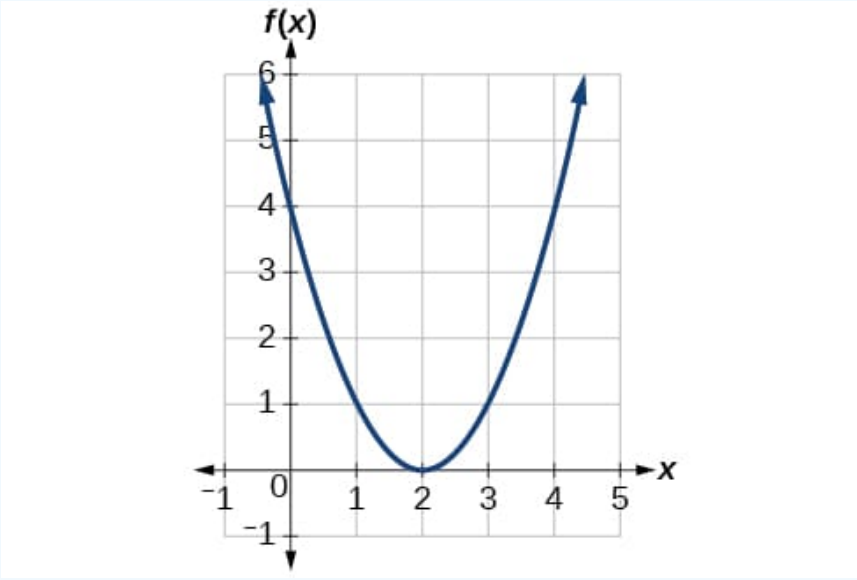
Figure 1.4.8



Example 1.4.6: Identifying a Horizontal Shift of a Toolkit Function

Figure 1.4.9 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

Figure 1.4.9



Solution available at the end of this section.

In Example 1.4.6, to determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the **vertex** point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x - 2) = f(0) = 0$. For this to work, we will need to *subtract* 2 units from our input values.

Example 1.4.7: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m) + 10$ and $G(m + 10)$.

Solution available at the end of this section.

Checkpoint 1.4.7

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output y-axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input x-axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down *and* right or left. Note that these shifts can be performed in either order.

Example 1.4.8: Graphing Combined Vertical and Horizontal Shifts

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x + 1) - 3$.

Solution available at the end of this section.

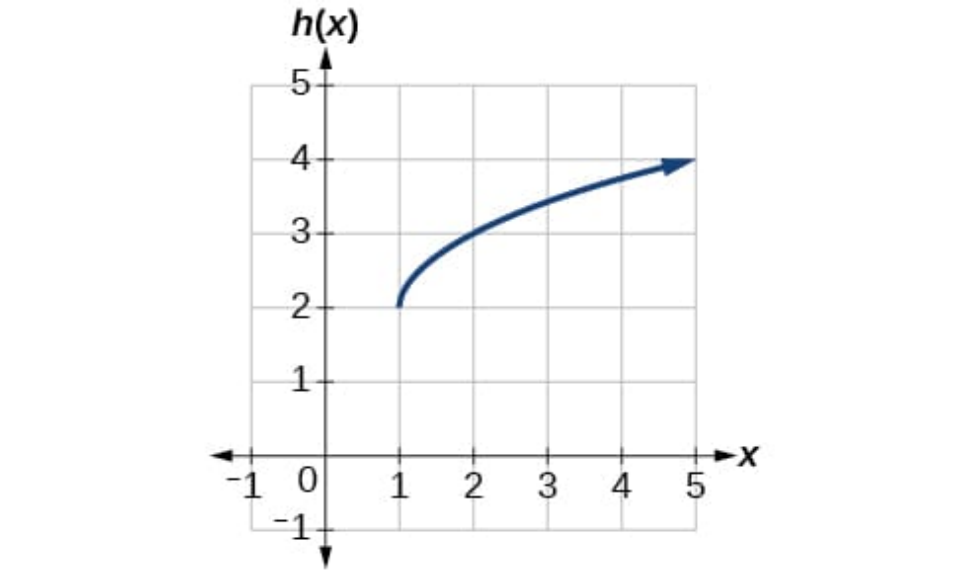
Checkpoint 1.4.8

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x - 2) + 4$.

Example 1.4.9: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure 1.4.12, which is a transformation of the toolkit square root function.

Figure 1.4.12



Solution available at the end of this section.

Note that the transformation in Example 1.4.9 has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

Checkpoint 1.4.9

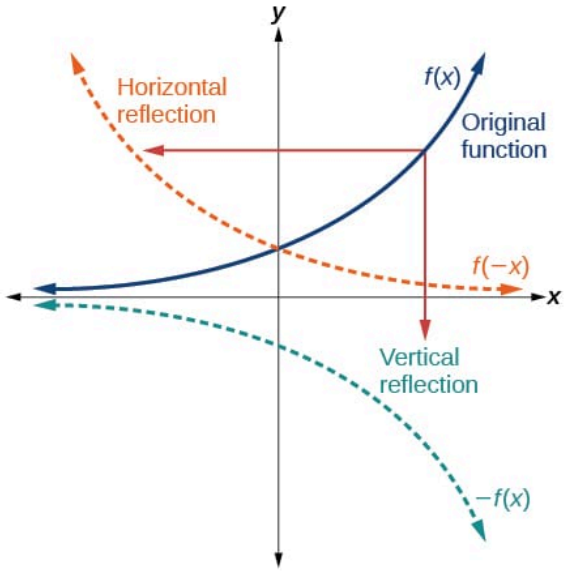
Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 1.4.13.

Figure 1.4.13

Vertical and horizontal reflections of a function



Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

Theorem: Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a vertical reflection of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a horizontal reflection of the function $f(x)$, sometimes called a reflection about the y-axis.

Thus, we multiply all outputs by –1 for a vertical reflection, and we multiply all inputs by –1 for a horizontal reflection.

Example 1.4.10: Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution available at the end of this section.

Note that transformations like those in Example 1.4.10 can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)H(t)$ function the domain $(-\infty, 0]$.

Checkpoint 1.4.10

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

Example 1.4.11: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table 1.4.6. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 1.4.6

x	2	4	6	8
$f(x)$	1	3	7	11

Solution available at the end of this section.

Checkpoint 1.4.11

A function $f(x)$ is given as Table 1.4.9. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

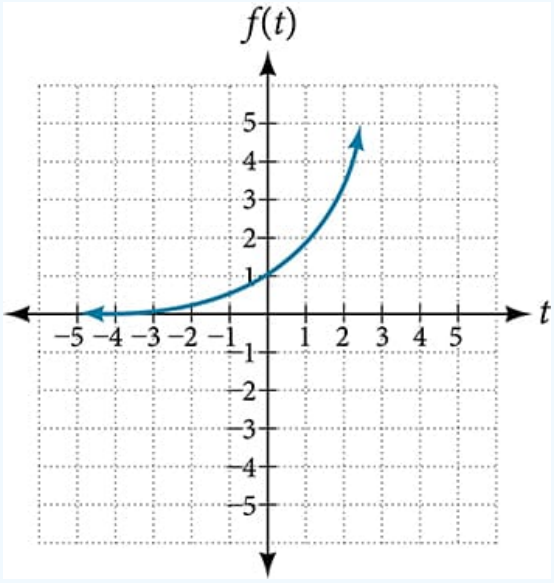
Table 1.4.9

x	−2	0	2	4
$f(x)$	5	10	15	20

Example 1.4.12: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure 1.4.16. Sketch a graph of $k(t)$.

Figure 1.4.16



Solution available at the end of this section.

As a model for learning, the function in Example 1.4.12 would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

Checkpoint 1.4.12

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

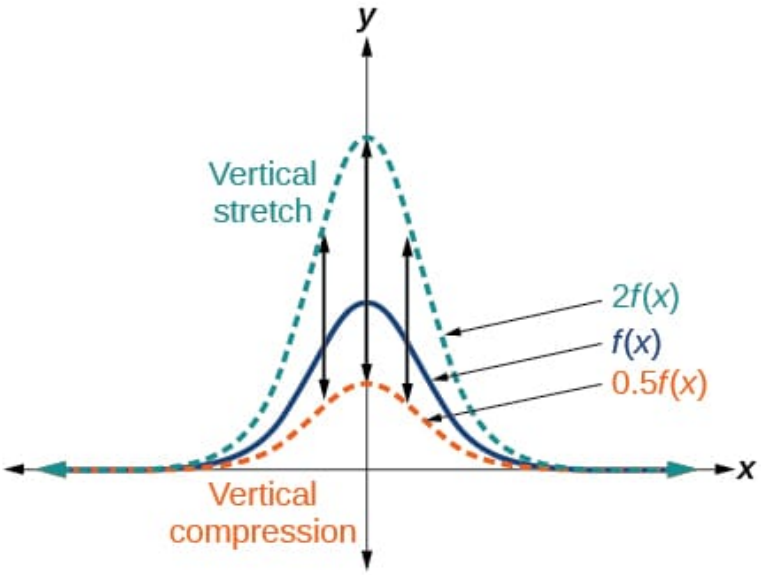
We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 1.4.18 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

Figure 1.4.18

Vertical stretch and compression



Theorem: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = a\ f(x)$, where a is a constant, is a vertical stretch or vertical compression of the function $f(x)$.

- If $a > 1$, then the graph will be stretched vertically by a factor of a.

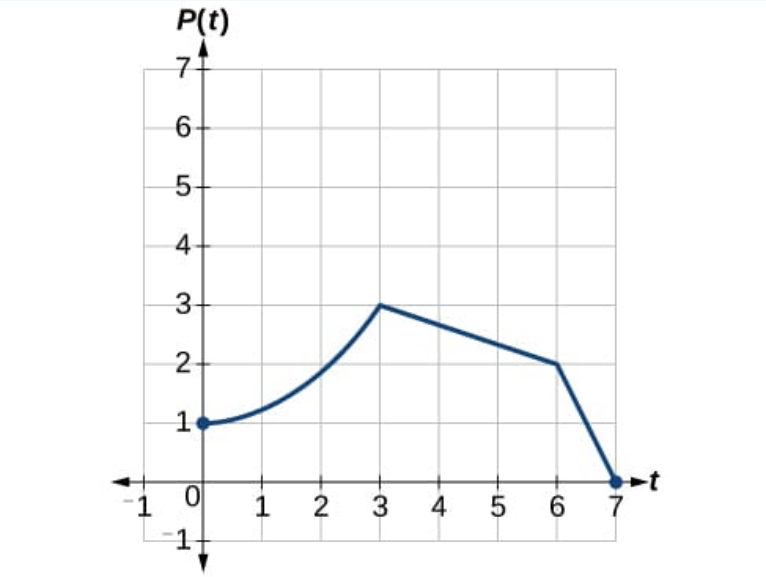
- If \(0
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

A good summary is that, if you have a function multiplied by a, you multiply all range values by a.

Example 1.4.13: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure 1.4.19.

Figure 1.4.19



A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution available at the end of this section.

Example 1.4.14: Finding a Vertical Compression of a Tabular Function

A function f is given as Table 1.4.10. Create a table for the function $g(x) = \frac{1}{2}f(x)$.

Table 1.4.10

x	2	4	6	8
$f(x)$	1	3	7	11

Solution available at the end of this section.

The result in Example 1.4.14 is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Checkpoint 1.4.14

A function f is given as Table 1.4.12. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

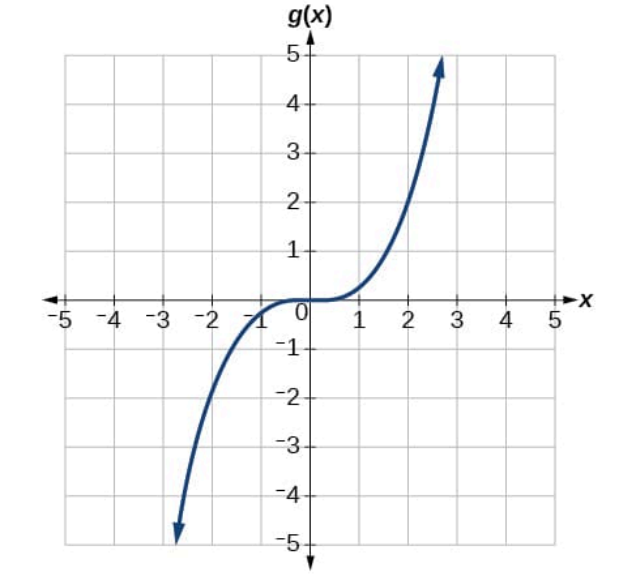
Table 1.4.12

x	2	4	6	8
$f(x)$	12	16	20	0

Example 1.4.15: Recognizing a Vertical Stretch

The graph in Figure 1.4.21 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

Figure 1.4.21



Solution available at the end of this section.

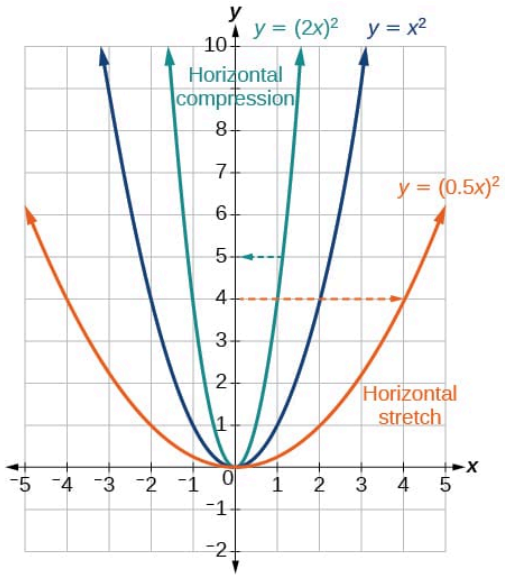
Checkpoint 1.4.15

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

Figure 1.4.22



Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 1.4.22. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of $\frac{1}{2}$.

Theorem: Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a horizontal stretch or horizontal compression of the function $f(x)$.

- If $b > 1$, then the graph will be compressed horizontally by a factor of $\frac{1}{b}$.
- If \(0
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

In effect, we multiply all x-values by $\frac{1}{b}$.

Example 1.4.16: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution available at the end of this section.

Note that, in Example 1.4.16, the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 1.4.17: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 1.4.13. Create a table for the function $g(x) = f\left(\frac{1}{2}x\right)$.

Table 1.4.13

x	2	4	6	8
$f(x)$	1	3	7	11

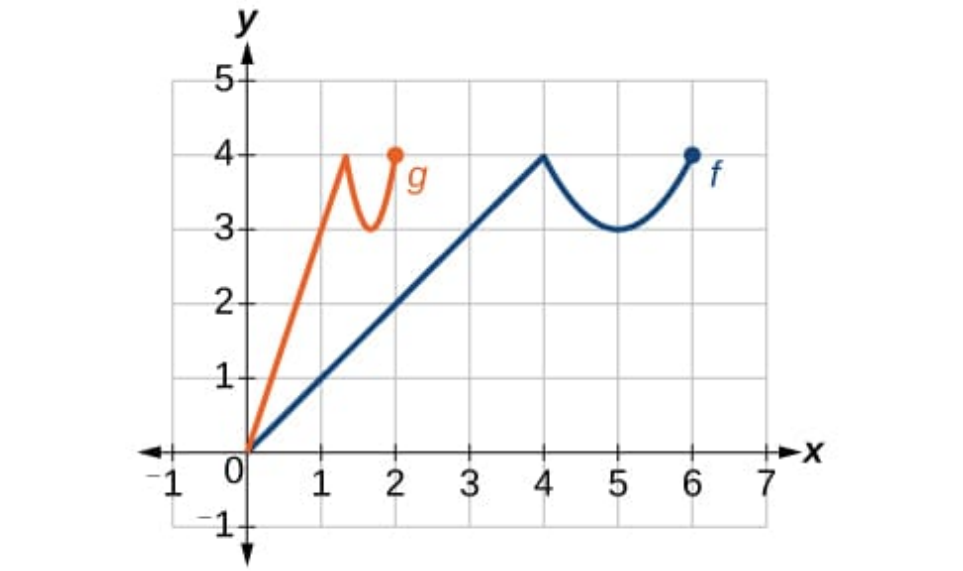
Solution available at the end of this section.

Because each input value has been doubled, the result in Example 1.4.17 is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Example 1.4.18: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 1.4.25.

Figure 1.4.25



Solution available at the end of this section.

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f\left(\frac{1}{4}\right)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Checkpoint 1.4.18

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Performing a Sequence of Transformations

When combining transformations, it is *very* important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the Order of Operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x)=f(2x+3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7)=12$. What input to g would produce that output? In other words, what value of x will allow $g(x)=f(2x+3)=12$? We would need $2x+3=7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p)=f\left(b\left(x+\frac{p}{b}\right)\right).$$

Let's work through an example. If

$$f(x)=(2x+4)^2,$$

then we can factor out a 2 to get

$$f(x)=(2(x+2))^2.$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Combining Transformations

Given a function $g(x)$ of the form

$$g(x)=af(bx+p)+k,$$

where the base graph of $y=f(x)$ is known, we factor b from the argument $bx+p$ to get $b\left(x+\frac{p}{b}\right)$. This gives

$$g(x)=af(bx+p)+k=af\left(b\left(x+\frac{p}{b}\right)\right)+k.$$

With the function rewritten in this specific order, we perform transformations on the graph of f in the order written - from left to right - to arrive at the graph of $g(x)$.

If your original function is not in the form $af(bx+p)+k$, then you will have to "precondition" the function using arithmetic. For example,

$$g(x)=-2-\frac{1}{3}f(7x-8)=-\frac{1}{3}f(7x-8)-2=-\frac{1}{3}f\left(7\left(x-\frac{8}{7}\right)\right)-2.$$

Example 1.4.19: Finding a Triple Transformation of a Tabular Function

Given Table 1.4.15 for the function $f(x)$, create a table of values for the function $g(x)=2f(3x)+1$.

Table 1.4.15

x	6	12	18	24
$f(x)$	10	14	15	17

Solution available at the end of this section.

Example 1.4.20: Graphing a Function with Many Transformations

Sketch a graph of the function

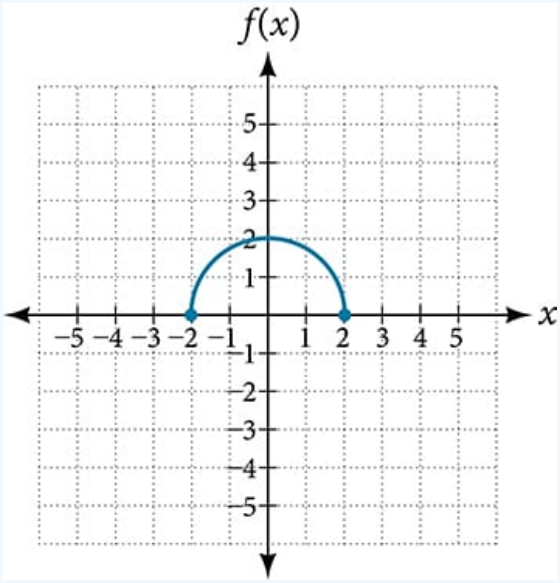
$$f(x) = -4 - \frac{1}{2}\sqrt{12 - 3x}.$$

Solution available at the end of this section.

Example 1.4.21: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 1.4.26 to sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3$.

Figure 1.4.26



Solution available at the end of this section.

Answers

Example 1.4.1: Determining whether a Function Is Even, Odd, or Neither

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

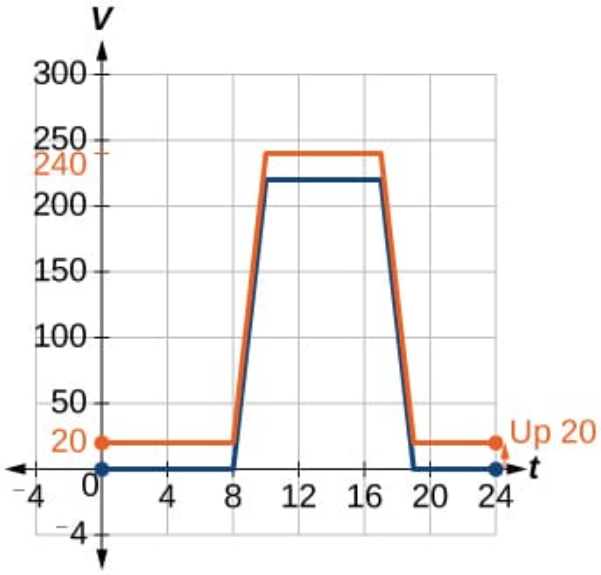
$$\begin{aligned} f(-x) &= (-x)^3 + 2(-x) && \text{(substituting)} \\ &= -x^3 - 2x && \text{(simplifying)} \\ &= -(x^3 + 2x) && \text{(factoring out } -1) \\ &= -f(x) \end{aligned}$$

Because $f(-x) = -f(x)$, this is an odd function.

Example 1.4.2: Adding a Constant to a Function

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 1.4.5.

Figure 1.4.5



Notice that in Figure 1.4.5, for each input value, the output value has increased by 20, so if we call the new function $S(t)$, we could write

$$S(t) = V(t) + 20.$$

This notation tells us that, for any value of t, $S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 1.4.1.

Table 1.4.1

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

Example 1.4.3**: Shifting a Tabular Function Vertically**

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example, we know (from Table 1.4.2) that $f(2) = 1$, and we were given $g(x) = f(x) - 3$. Therefore,

$$\begin{aligned} g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 1.4.3.

Table 1.4.3

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	−2	0	4	8

Example 1.4.4**: Adding a Constant to an Input**

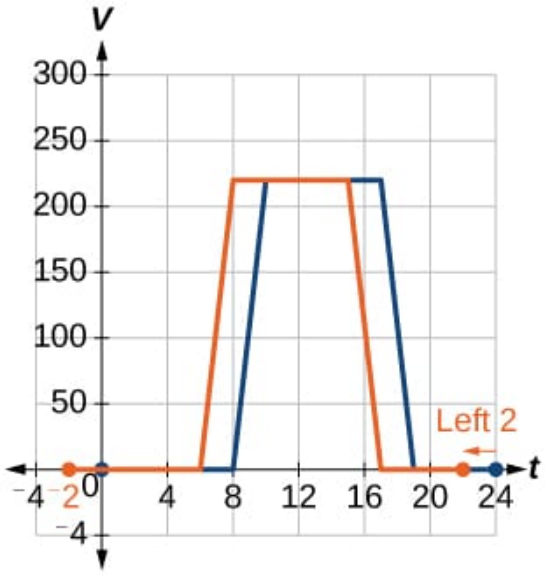
We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$\begin{aligned} V(t) &= \text{the original venting plan} \\ F(t) &= \text{starting 2 hours sooner} \end{aligned}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$ (see Figure 1.4.7). Notice also that the vents first opened to 220 ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220 ft^2 at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t - (-2)) = V(t + 2)$.

Figure 1.4.7



Example 1.4.5**: Shifting a Tabular Function Horizontally**

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 *larger*. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \end{aligned}$$

We continue with the other values to create Table 1.4.5.

Table 1.4.5

x	5	7	9	11
$x-3$	2	4	6	8
$f(x-3)$	1	3	7	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Example 1.4.6**: Identifying a Horizontal Shift of a Toolkit Function**

Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0,0)$, but now the vertex is at $(2,0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x-2) g(x) = f(x-2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$. That is, since $f(x) = x^2$, $g(x) = f(x-2)$. Therefore,

$$g(x) = f(x-2) = (x-2)^2$$

Example 1.4.7**: Interpreting Horizontal versus Vertical Shifts**

$G(m) + 10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m + 10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Example 1.4.8**: Graphing Combined Vertical and Horizontal Shifts**

The function f is our toolkit absolute value function. We know that this graph has a "V" shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 1.4.10.

Let us follow one point of the graph of $f(x) = |x|$.

- The point $(0,0)$ is transformed first by shifting left 1 unit: $(0,0) \to (-1,0)$
- The point $(-1,0)$ is transformed next by shifting down 3 units: $(-1,0) \to (-1,-3)$

Figure 1.4.10

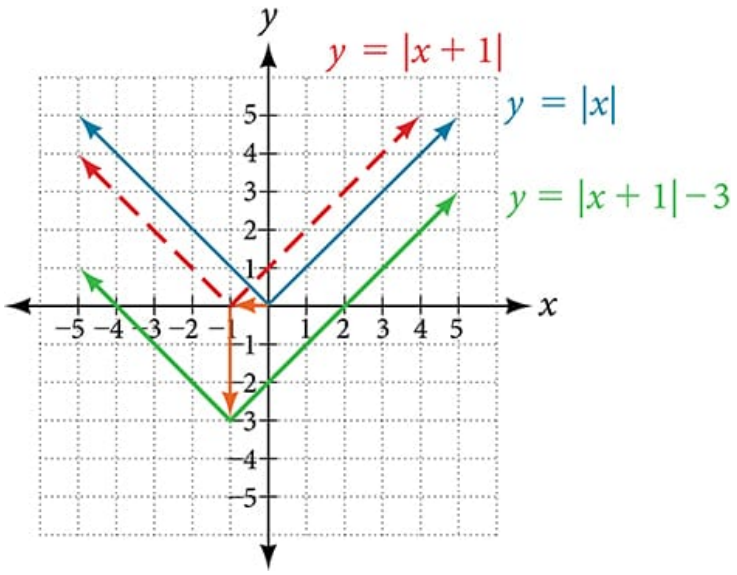
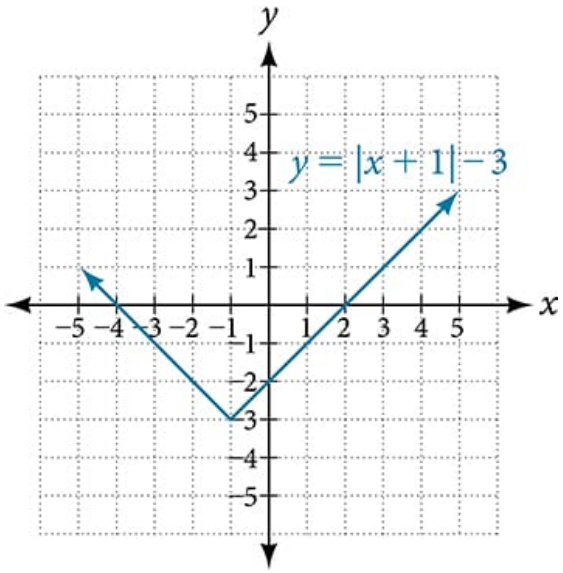


Figure 1.4.11 shows the graph of h.

Figure 1.4.11



Example 1.4.9**: Identifying Combined Vertical and Horizontal Shifts**

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1) + 2.$$

Using the formula for the square root function, we can write

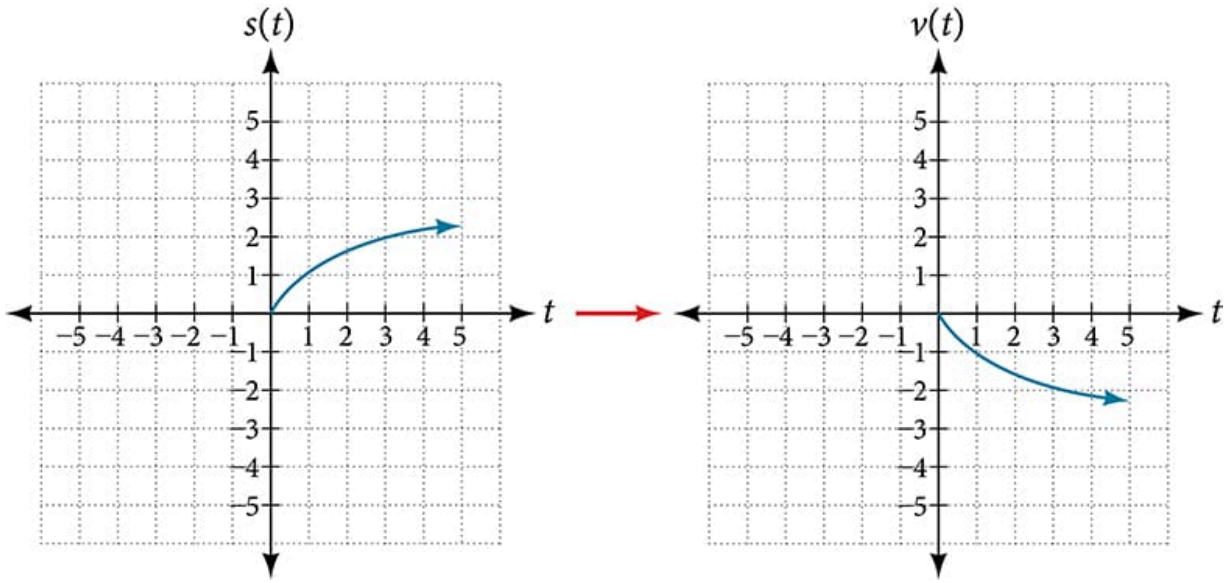
$$h(x) = \sqrt{x-1} + 2.$$

Example 1.4.10**: Reflecting a Graph Horizontally and Vertically**

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure 1.4.14.

Figure 1.4.14

Vertical reflection of the square root function



Because each output value is the opposite of the original output value, we can write

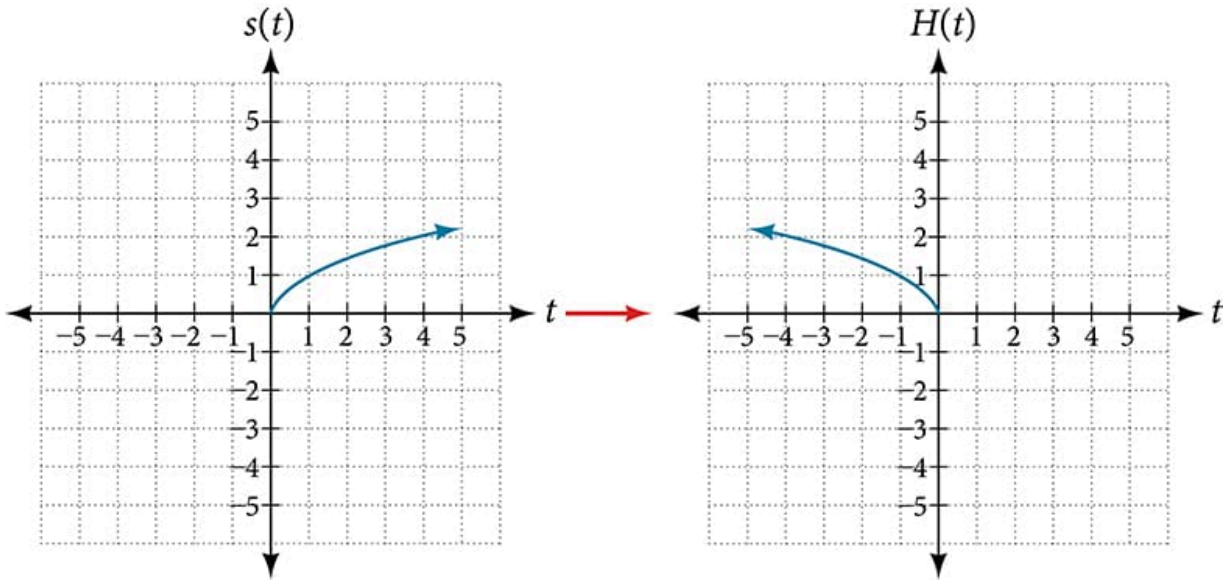
$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}.$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 1.4.15.

Figure 1.4.15

Horizontal reflection of the square root function



Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}.$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Example 1.4.11**: Reflecting a Tabular Function Horizontally and Vertically**

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 1.4.7.

Table 1.4.7

x	2	4	6	8
$g(x)$	–1	–3	–7	–11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 1.4.8.

Table 1.4.8

x	–2	–4	–6	–8
$h(x)$	1	3	7	11

Example 1.4.12**: Applying a Learning Model Equation**

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$
- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t) + 1 = -2^{-t} + 1$

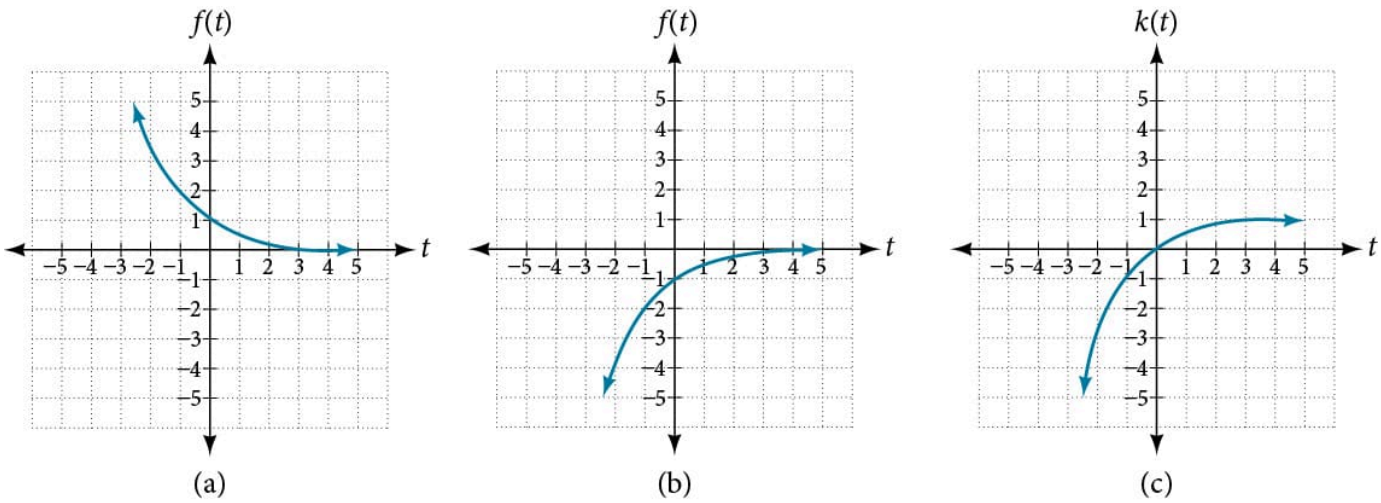
We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points $(0, 1)$ and $(1, 2)$.

1. First, we apply a horizontal reflection: $(0, 1) \to (0, 1)$ and $(1, 2) \to (–1, 2)$.
2. Then, we apply a vertical reflection: $(0, 1) \to (0, -1)$ and $(-1, 2) \to (-1, –2)$.
3. Finally, we apply a vertical shift: $(0, -1) \to (0, 0)$ and $(-1, -2) \to (-1, -1)$.

This means that the original points, $(0, 1)$ and $(1, 2)$ become $(0, 0)$ and $(-1, -1)$ after we apply the transformations.

In Figure 1.4.17, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

Figure 1.4.17



Example 1.4.13**: Graphing a Vertical Stretch**

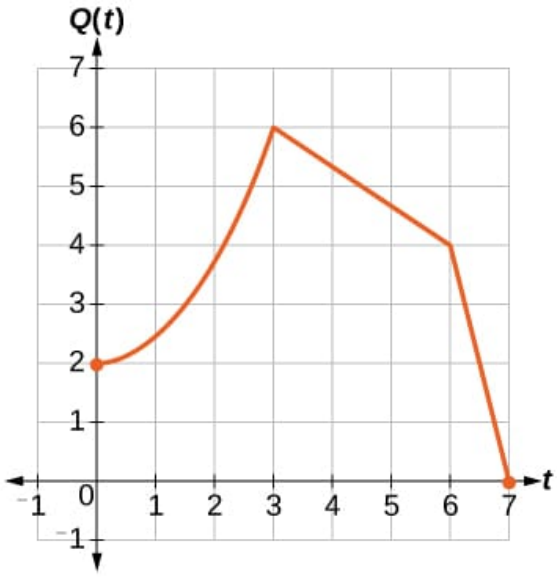
Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure 1.4.20.

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$, and $(7, 0)$, we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \to (0, 2)$$
$$(3, 3) \to (3, 6)$$
$$(6, 2) \to (6, 4)$$
$$(7, 0) \to (7, 0)$$

Figure 1.4.20



Symbolically, the relationship is written as

$$Q(t) = 2P(t).$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

Example 1.4.14**: Finding a Vertical Compression of a Tabular Function**

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}.$$

We do the same for the other values to produce Table 1.4.11.

Table 1.4.11

x	22	44	66	88
$g(x)$	1212	3232	7272	112112

Example 1.4.15**: Recognizing a Vertical Stretch**

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can (somewhat) safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3.$$

Example 1.4.16**: Graphing a Horizontal Compression**

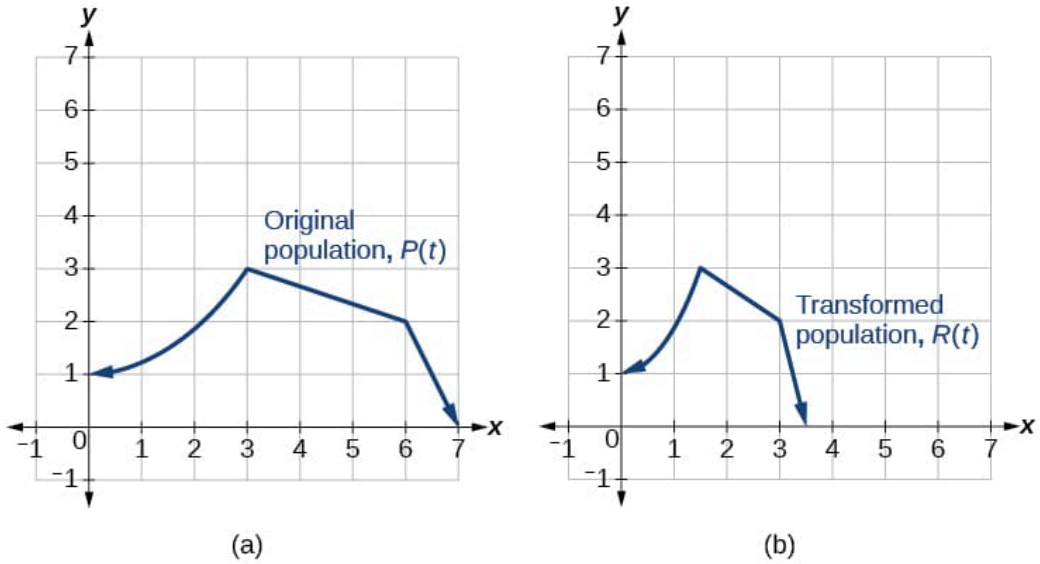
Symbolically, we could write

$$R(1) = P(2),$$
$$R(2) = P(4),$$
$$\text{and in general,}$$
$$R(t) = P(2t).$$

See Figure 1.4.23 for a graphical comparison of the original population and the compressed population.

Figure 1.4.23

(a) Original population graph (b) Compressed population graph



Example 1.4.17**: Finding a Horizontal Stretch for a Tabular Function**

The formula $g(x) = f\left(\frac{1}{2}x\right)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f\left(\frac{1}{2} \cdot 2\right) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f\left(\frac{1}{2} \cdot 4\right) = f(2) = 1.$$

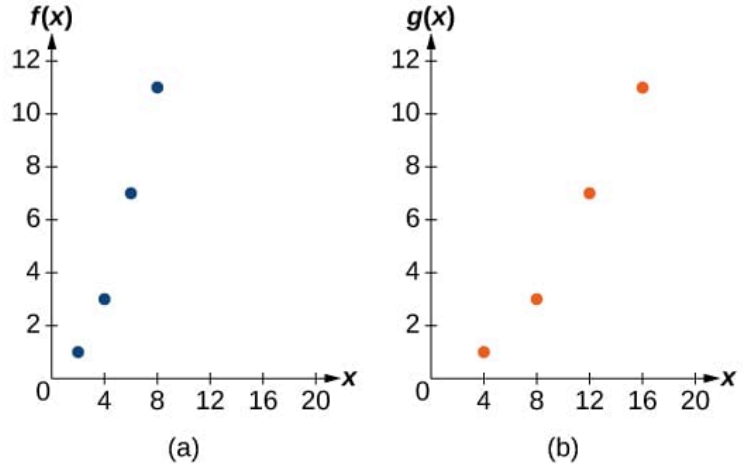
We do the same for the other values to produce Table 1.4.14.

Table 1.4.14

x	4	8	12	16
$g(x)$	1	3	7	11

Figure 1.4.24 shows the graphs of both of these sets of points.

Figure 1.4.24



Example 1.4.18**: Recognizing a Horizontal Compression on a Graph**

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at $(6, 4)$ and $g(x)$ ends at $(2, 4)$, we can see that the x-values have been compressed by $\frac{1}{3}$, because $6\left(\frac{1}{3}\right) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Example 1.4.19**: Finding a Triple Transformation of a Tabular Function**

Using the advice given previously, we make sure the function is written in the form

$$af(bx+p)+k.$$

In this case, we were given $g(x)=2f(3x)+1$, which is already in this form. We then start applying the transformations by reading from left to right.
We start with the vertical stretch, which will multiply the output values by 2. Table 1.4.16.

Table 1.4.16

x	6	12	18	24
$2f(x)$	20	28	30	34

We then move to the right in the function $g(x)=2f(3x)+1$ and deal with the 3. This causes a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$. We apply these transformations to the x-values from Table 1.4.16 to arrive at Table 1.4.17.

Table 1.4.17

x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 1.4.18.

Table 1.4.18

x	2	4	6	8
$g(x)=2f(3x)+1$	21	29	31	35

Example 1.4.20**: Graphing a Function with Many Transformations**

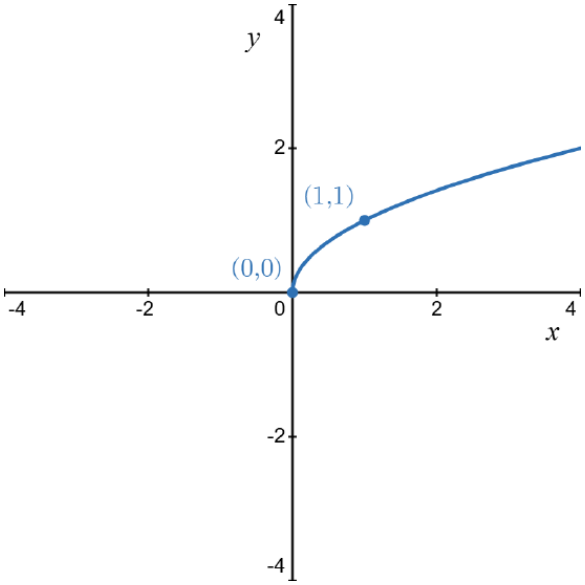
The function given to us needs to be rewritten in the form $af(bx+p)+k=af\left(b\left(x+\frac{p}{b}\right)\right)+k$. Let's start by simply rearranging terms.

$$f(x)=-4-\frac{1}{2}\sqrt{12-3x}=-\frac{1}{2}\sqrt{-3x+12}-4$$

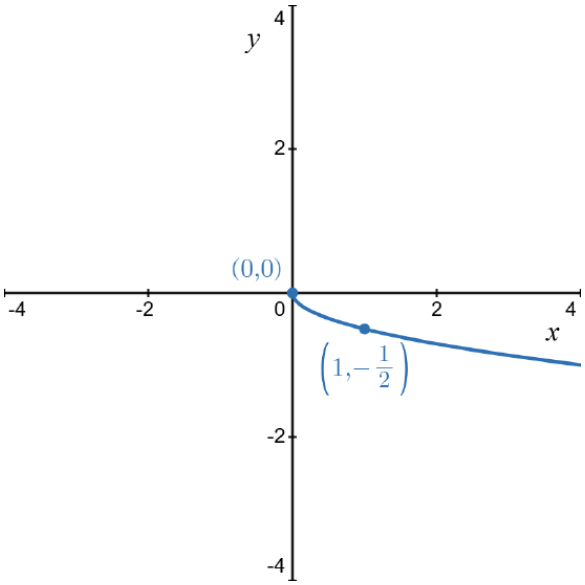
We now take the time to factor the lead coefficient from both terms inside the function.

$$f(x)=-\frac{1}{2}\sqrt{-3\left(x-4\right)}-4.$$

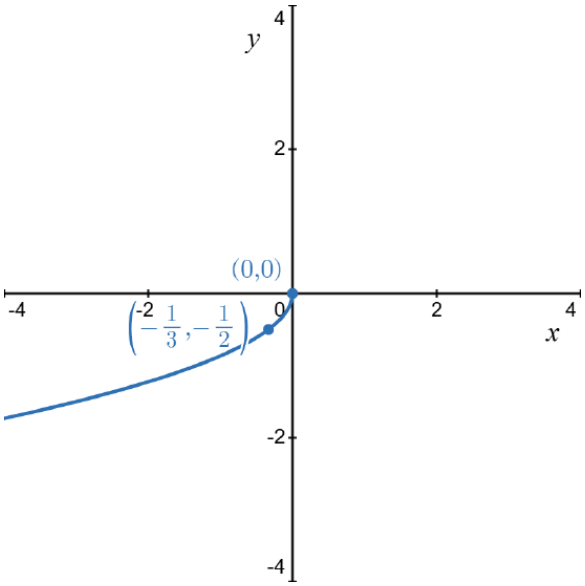
The parent function is $y=\sqrt{x}$. It's best to label two "anchor" points (which will almost always be $(0,0)$ and $(1,1)$ for the Toolkit functions).



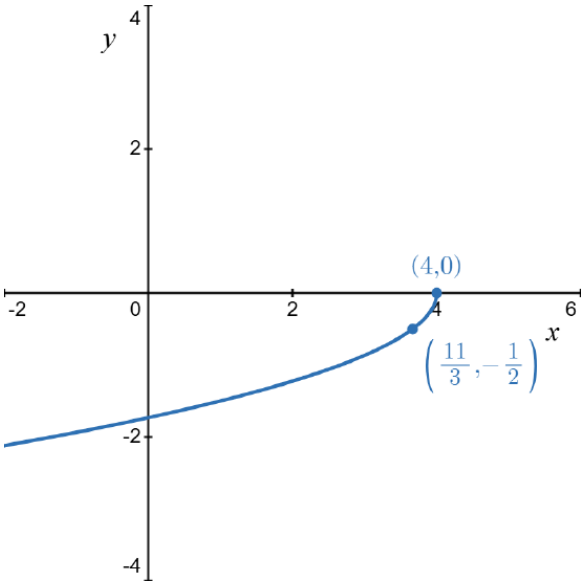
Reading our "cleaned" function from left to right, the coefficient $-\frac{1}{2}$ reflects the graph of $y=\sqrt{x}$ vertically about the x-axis and compresses it vertically (squishing towards the x-axis) by a factor of $\frac{1}{2}$. Another way to think of this is that all of the y-values on the graph of $y=\sqrt{x}$ are multiplied by $-\frac{1}{2}$.



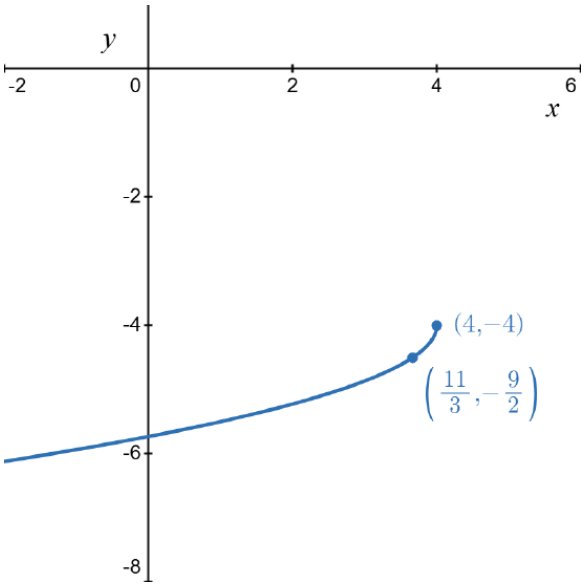
Continuing reading from left to right, we have the factor of -3 on the *inside* of the function. Remember, anything on the inside of the function affects the x-values (and in an opposite way from how they affect the outside of the function). Specifically, the factor of -3 reflects the graph of $y=-\frac{1}{2}\sqrt{x}$ about the y-axis and *compresses* the graph horizontally (squishing it) towards the y-axis by a factor of $\frac{1}{3}$. Another way to think of this is that the factor -3 *divides* all of the x-values on the graph of $y=-\frac{1}{2}\sqrt{x}$ by -3.



Continuing reading our original function from left to right, we have $(x-4)$ on the *inside* of the function. This causes a shift of 4 units horizontally to the right for the graph of $y=-\frac{1}{2}\sqrt{3x}$. Another way to think of this is that we *add* 4 units to each of the x-values.



Finally, we have the subtraction of 4 with which to contend. This subtract occurs *outside* the function (the square root), so it shifts the graph of $y=-\frac{1}{2}\sqrt{3(x-4)}$ vertically *down* 4 units. Our final graph is given below.



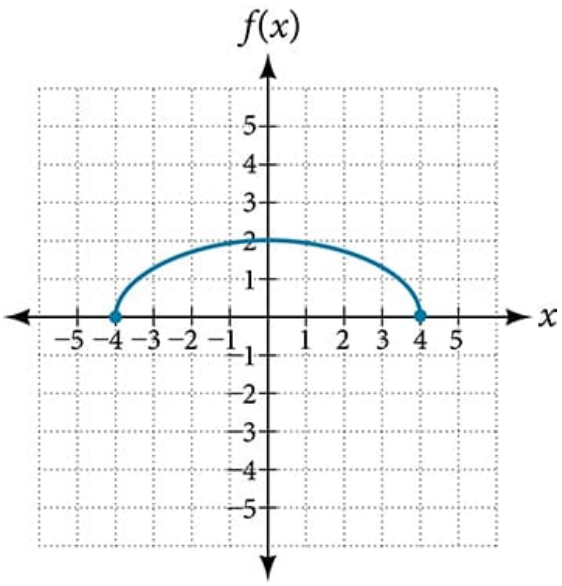
Example $1.4.21$**: Finding a Triple Transformation of a Graph**

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3.$$

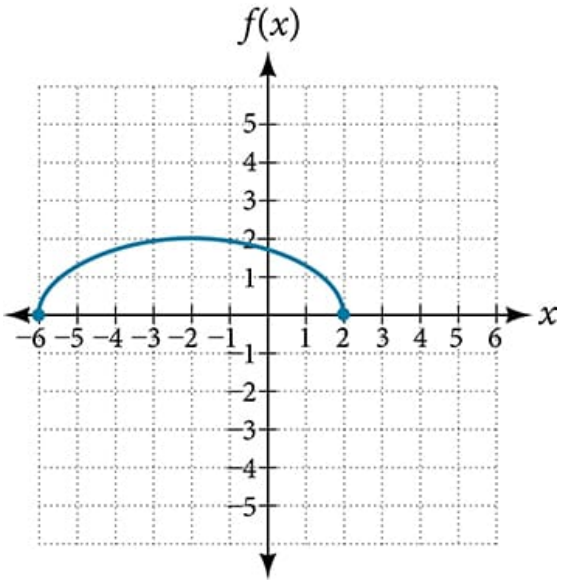
Reading from left to right, we first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0,2)$ remains at $(0,2)$ while the point $(2,0)$ will stretch to $(4,0)$. See Figure $1.4.27$.

Figure 1.4.27



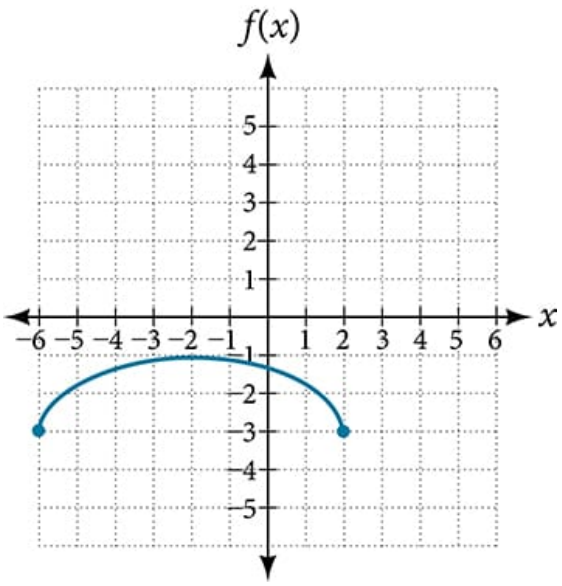
Next, we horizontally shift left by 2 units, as indicated by $x + 2$. See Figure 1.4.28.

Figure 1.4.28



Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See Figure 1.4.29.

Figure 1.4.29



This page titled 1.4: Transformations of Functions is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

1.4.1: Resources and Key Concepts

Resources

Videos

- Transformations of Functions
  - Functions - Graphs - Rigid Transformations - Horizontal and Vertical Shifts
  - Functions - Graphs - Rigid Transformations - Horizontal and Vertical Reflections
  - Functions - Graphs - Non-Rigid Transformations - Stretches and Compressions
  - Functions - Graphs - Performing a Sequence of Rigid and Non-Rigid Transformations (Standard)
  - Functions - Graphs - Sequences of Rigid and Non-Rigid Transformations on Non-Standard Functions
  - Intermediate Algebra - Functions: Transformations (older video)

Key Concepts

Definitions

- **Vertical Shift:** A transformation that moves the graph of a function up or down by adding a constant k to the output: $g(x) = f(x) + k$. If $k > 0$, the shift is upward; if $k < 0$, the shift is downward.
- **Horizontal Shift:** A transformation that moves the graph of a function left or right by adding or subtracting a constant h from the input: $g(x) = f(x - h)$. If $h > 0$, the shift is to the right; if $h < 0$ (e.g., $f(x + c)$ where $c > 0$), the shift is to the left.
- **Vertical Reflection:** A transformation that reflects the graph of a function vertically across the x-axis, given by $g(x) = -f(x)$.
- **Horizontal Reflection:** A transformation that reflects the graph of a function horizontally across the y-axis, given by $g(x) = f(-x)$.
- **Vertical Stretch:** A transformation where the graph of a function is stretched away from the x-axis, caused by multiplying the output by a constant a where $|a| > 1$. If $a > 1$, it's a pure stretch.
- **Vertical Compression:** A transformation where the graph of a function is compressed toward the x-axis, caused by multiplying the output by a constant a where $0 < |a| < 1$. If $0 < a < 1$, it's a pure compression.
- **Horizontal Stretch:** A transformation where the graph of a function is stretched away from the y-axis, caused by multiplying the input by a constant b where $0 < |b| < 1$. If $0 < b < 1$, it's a pure stretch (e.g., $f(\frac{1}{2}x)$).
- **Horizontal Compression:** A transformation where the graph of a function is compressed toward the y-axis, caused by multiplying the input by a constant b where $|b| > 1$. If $b > 1$, it's a pure compression (e.g., $f(2x)$).

Theorems

- **Vertical Shift:** Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a vertical shift of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.
- **Horizontal Shift:** Given a function $f(x)$, a new function $g(x) = f(x - h)$, where h is a constant, is a horizontal shift of $f(x)$. If h is positive, the graph will shift right. If h is negative, the graph will shift left.
- **Reflections:**
  - Given a function $f(x)$, a new function $g(x) = -f(x)$ is a vertical reflection of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.
  - Given a function $f(x)$, a new function $g(x) = f(-x)$ is a horizontal reflection of the function $f(x)$, sometimes called a reflection about the y-axis
- **Vertical Stretches and Compressions:** Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a vertical stretch or vertical compression of the function $f(x)$.
  - If $a > 1$, then the graph will be stretched vertically by a factor of a.
  - If $0 < a < 1$, then the graph will be compressed vertically by a factor of a.
  - If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection
- **Horizontal Stretches and Compressions:** Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a horizontal stretch or horizontal compression of the function $f(x)$.

- If $b > 1$, then the graph will be compressed horizontally by a factor of $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched horizontally by a factor of $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

Common Mistakes

- **Counterintuitive Nature of Horizontal Shifts:** Horizontal changes often seem counterintuitive. For example, $f(x - h)$ shifts right if $h > 0$, and $f(x + h)$ (which is $f(x - (-h))$) shifts left if $h > 0$.
- **Order of Transformations:** When combining multiple transformations, the order in which they are applied is very important, especially when mixing shifts with stretches/compressions. For horizontal transformations, factoring inside the function (e.g., writing $f(b(x + p))$) can make the order easier to manage.
- **Misinterpreting Coefficients for Horizontal Stretch/Compression:** The coefficient b in $f(bx)$ leads to a stretch/compression by a factor of $\frac{1}{b}$, not b. For example, $f(2x)$ is a horizontal compression by a factor of $\frac{1}{2}$, and $f(\frac{1}{2}x)$ is a horizontal stretch by a factor of 2.

This page titled 1.4.1: Resources and Key Concepts is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

1.4.2: Homework

Reading Questions

1. How does the transformation $g(x) = f(x) + 5$ affect the graph of $f(x)$?
2. If the graph of $f(x)$ is shifted 3 units to the left, what is the equation of the new function in terms of $f(x)$?
3. What is the difference in the graphical effect of $g(x) = -f(x)$ versus $h(x) = f(-x)$?
4. If $f(x) = x^2$, describe the transformation that results in $g(x) = 3x^2$. Is this a stretch or a compression? In which direction?
5. If $f(x) = \sqrt{x}$, describe the transformation that results in $g(x) = \sqrt{2x}$. Is this a stretch or a compression? In which direction? By what factor?
6. When transforming a function using the form $y = af(bx + p) + k$, why is it recommended to first factor out b from the term $(bx + p)$?
7. Explain why $f(x - 2)$ shifts the graph of $f(x)$ to the right, even though a subtraction is involved.
8. If you are given the graph of $y = f(x)$ and want to graph $y = 2f(x) - 3$, describe the sequence of transformations you would apply.
9. What is the domain and range of $g(t) = \sqrt{-t}$ if the original function is $s(t) = \sqrt{t}$ with domain $[0, \infty)$ and range $[0, \infty)$?
10. Consider the function $k(t) = -2^{-t} + 1$. Identify the base function and list the transformations applied to it in the correct order to obtain $k(t)$.
11. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?
12. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?
13. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?
14. When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the x-axis from a reflection with respect to the y-axis?

Homework

Algebraic

15. Write a formula for the function obtained when the graph of $f(x) = \sqrt{x}$ is shifted up 1 unit and to the left 2 units.
16. Write a formula for the function obtained when the graph of $f(x) = |x|$ is shifted down 3 units and to the right 1 unit.
17. Write a formula for the function obtained when the graph of $f(x) = \frac{1}{x}$ is shifted down 4 units and to the right 3 units.
18. Write a formula for the function obtained when the graph of $f(x) = \frac{1}{x^2}$ is shifted up 2 units and to the left 4 units.

For the following exercises, describe how the graph of the function is a transformation of the graph of the original function f.

19. $y = f(x - 49)$
20. $y = f(x + 43)$
21. $y = f(x + 3)$
22. $y = f(x - 4)$
23. $y = f(x) + 5$
24. $y = f(x) + 8$
25. $y = f(x) - 2$
26. $y = f(x) - 7$
27. $y = f(x - 2) + 3$
28. $y = f(x + 4) - 1$

For the following exercises, determine the interval(s) on which the function is increasing and decreasing.

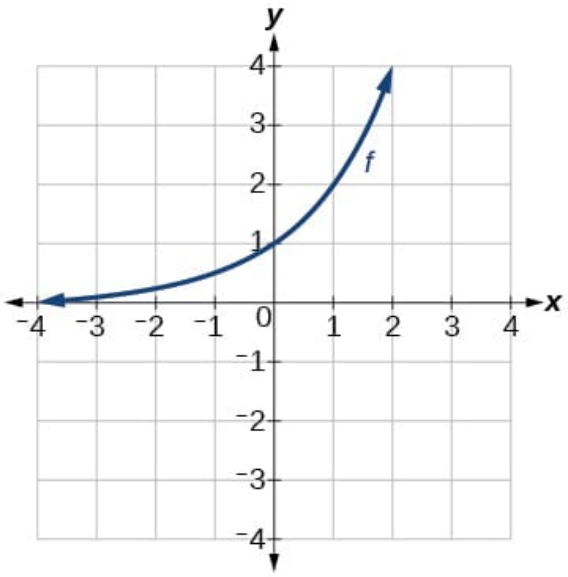
29. $f(x) = 4(x + 1)^2 - 5$

30. $g(x) = 5(x+3)^2 - 2$
31. $a(x) = \sqrt{-x+4}$
32. $k(x) = -3\sqrt{x} - 1$

Graphical

For the following exercises, use the graph of $f(x) = 2^x$ shown in Figure 1.4.2.1 to sketch a graph of each transformation of $f(x)$.

Figure 1.4.2.1



33. $g(x) = 2^x + 1$
34. $h(x) = 2^x - 3$
35. $w(x) = 2^{x-1}$

For the following exercises, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

36. $f(t) = (t+1)^2 - 3$
37. $h(x) = |x-1| + 4$
38. $k(x) = (x-2)^3 - 1$
39. $m(t) = 3 + \sqrt{t+2}$

Numeric

40. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	–2	–1	0	1	2
$f(x)$	–2	–1	–3	1	2

x	–1	0	1	2	3
$g(x)$	–2	–1	–3	1	2

x	–2	–1	0	1	2
$h(x)$	–1	0	–2	2	3

41. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

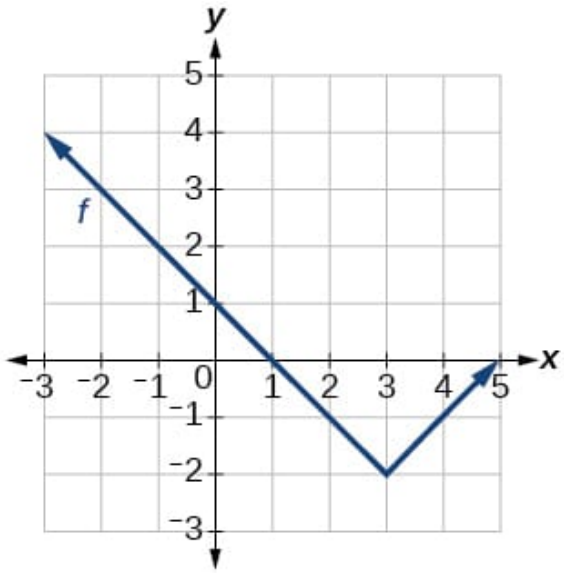
x	–2	–1	0	1	2
$f(x)$	–1	–3	4	2	1

x	–3	–2	–1	0	1
$g(x)$	–1	–3	4	2	1

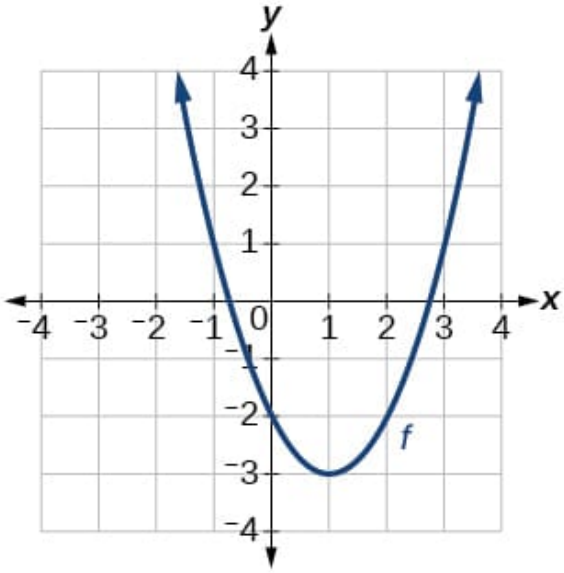
x	–2	–1	0	1	2
$h(x)$	–2	–4	3	1	0

For the following exercises, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.

42.

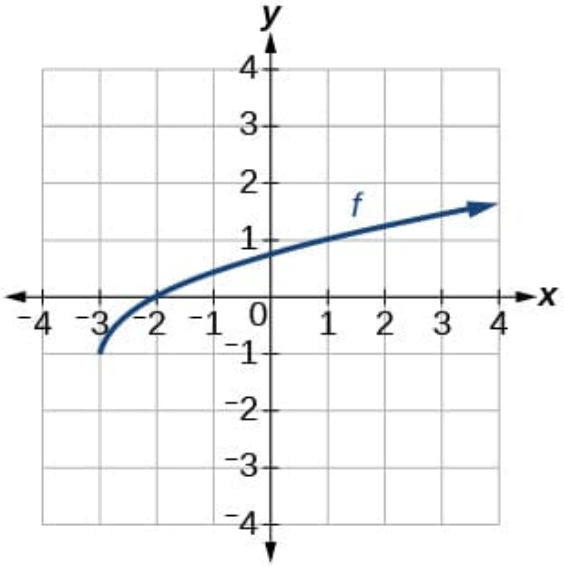


43.

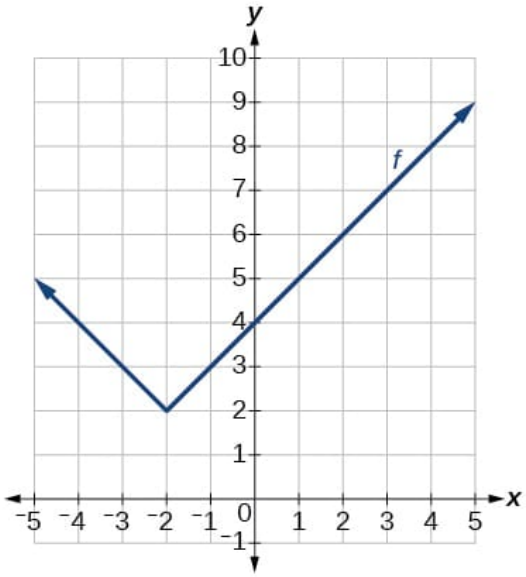


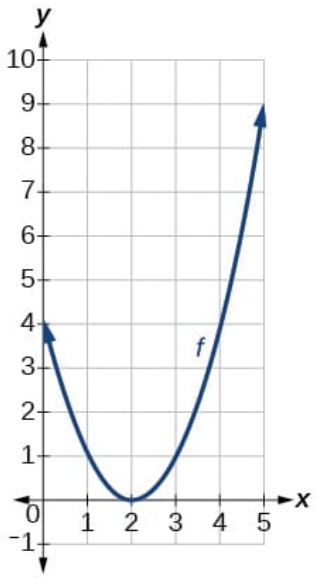
44.

15.

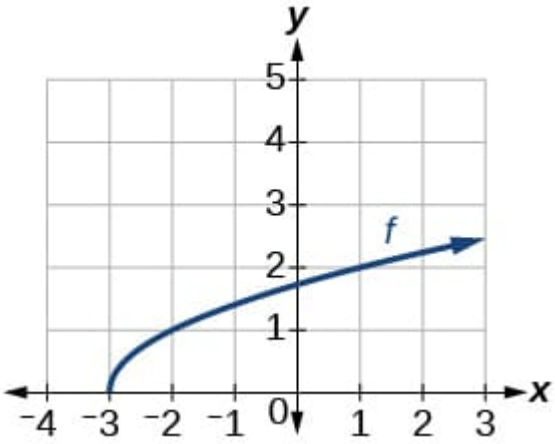


16.

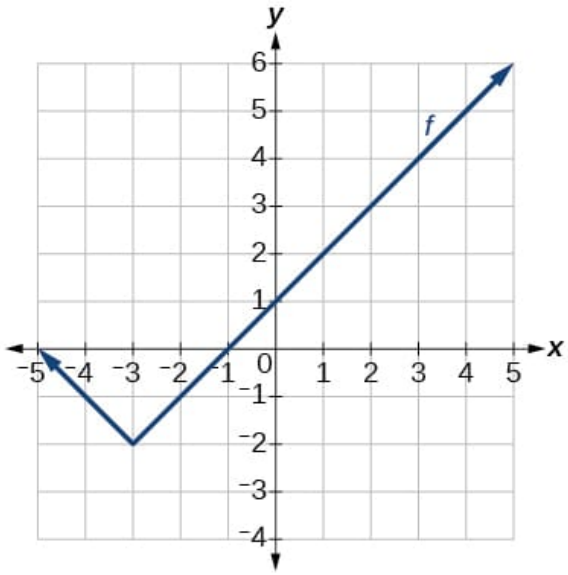



y
10
9
8
7
6
5
4
3
2
1
0
-1
1
2
3
4
5
x
f

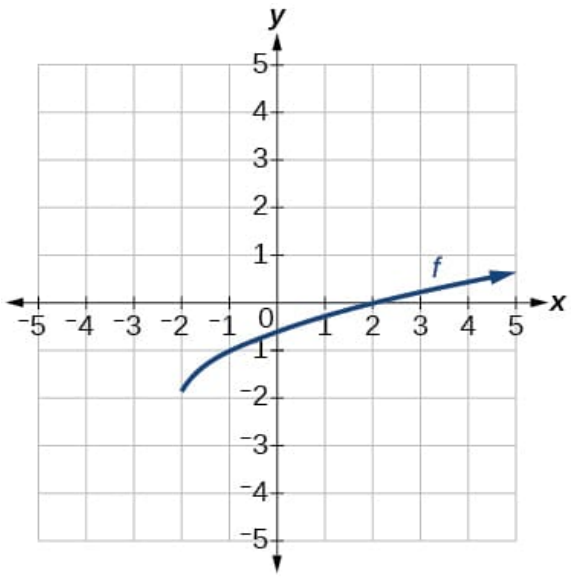
47.


y
5
4
3
2
1
0
-4
-3
-2
-1
1
2
3
x
f

48.

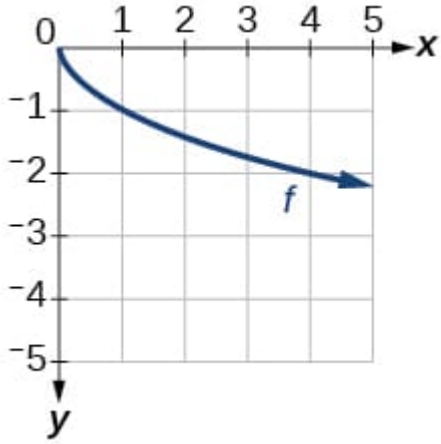

y
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-4
-3
-2
-1
1
2
3
4
5
x
f

49.

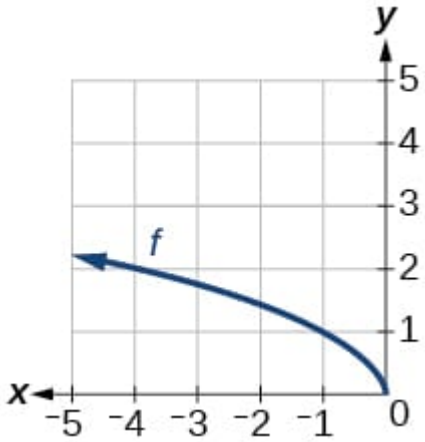


For the following exercises, use the graphs of transformations of the square root function to find a formula for each of the functions.

50.

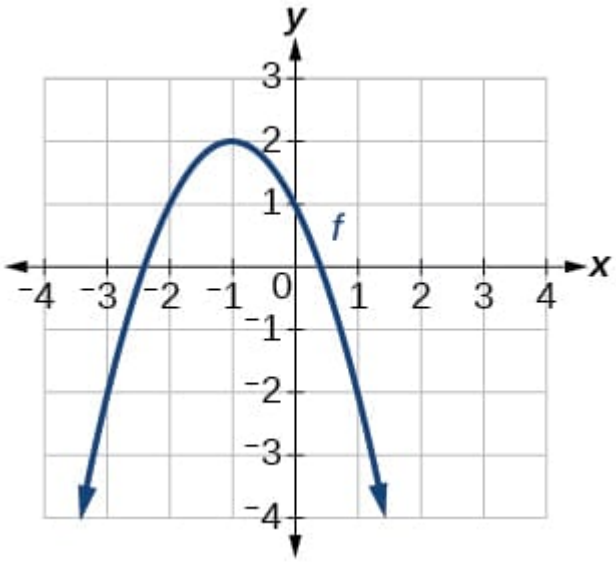


51.

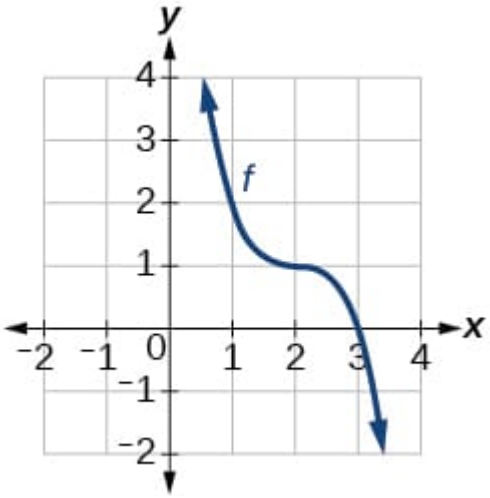


For the following exercises, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

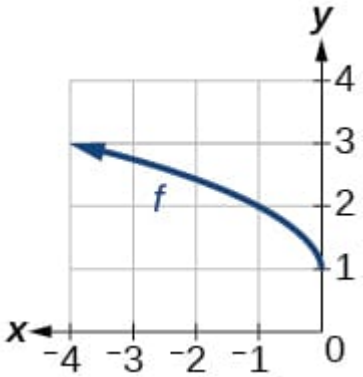
52.



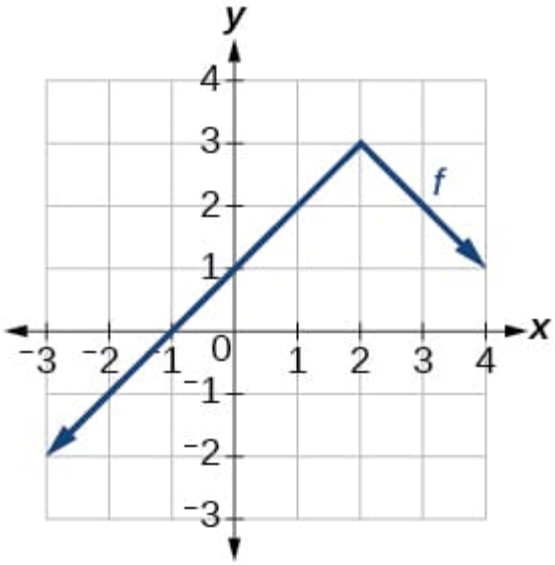
53.



54.



55.



For the following exercises, describe how the graph of each function is a transformation of the graph of the original function f.

56. $g(x) = -f(x)$
57. $g(x) = f(-x)$
58. $g(x) = 4f(x)$
59. $g(x) = 6f(x)$
60. $g(x) = f(5x)$
61. $g(x) = f(2x)$
62. $g(x) = f\left(\frac{1}{3}x\right)$
63. $g(x) = f\left(\frac{1}{5}x\right)$
64. $g(x) = 3f(-x)$
65. $g(x) = -f(3x)$

For the following exercises, write a formula for the function g that results when the graph of a given toolkit function is transformed as described.

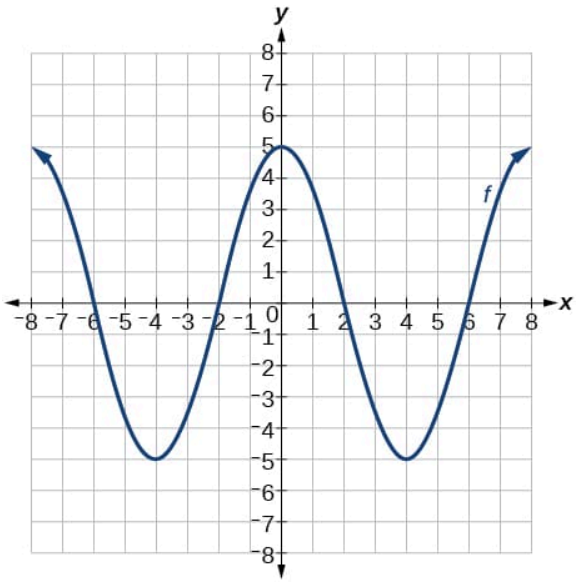
66. The graph of $f(x) = |x|$ is reflected over the y-axis and horizontally compressed by a factor of $\frac{1}{4}$.
67. The graph of $f(x) = \sqrt{x}$ is reflected over the x-axis and horizontally stretched by a factor of 2.
68. The graph of $f(x) = \frac{1}{x^2}$ is vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.
69. The graph of $f(x) = \frac{1}{x}$ is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.
70. The graph of $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$, then shifted to the right 5 units and up 1 unit.
71. The graph of $f(x) = x^2$ is horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

For the following exercises, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

72. $g(x) = 4(x+1)^2 - 5$
73. $g(x) = 5(x+3)^2 - 2$
74. $h(x) = -2|x-4| + 3$
75. $k(x) = -3\sqrt{x} - 1$
76. $m(x) = \frac{1}{2}x^3$
77. $n(x) = \frac{1}{3}|x-2|$
78. $p(x) = \left(\frac{1}{3}x\right)^3 - 3$
79. $q(x) = \left(\frac{1}{4}x\right)^3 + 1$
80. $a(x) = \sqrt{-x+4}$

For the following exercises, use the graph in Figure $1.4.2.2$ to sketch the given transformations.

Figure 1.4.2.2



81. $g(x) = f(x) - 2$
82. $g(x) = -f(x)$
83. $g(x) = f(x + 1)$
84. $g(x) = f(x - 2)$

This page titled 1.4.2: Homework is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

1.5: Absolute Value Functions

Note to the Instructor (click to expand)

Although it was introduced when we introduced piecewise functions, the definition of the absolute value function as a piecewise function is reintroduced (it is that important). The middle part of this section focuses on graphing transformations of the absolute value function, which is a nice bridge from the previous section. The remainder of the section is focused on solving absolute value equations and inequalities (needed for Calculus); however, they only should involve linear expressions (other types are treated in a future section).

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Arithmetic**
  - The Absolute Value of a Number: A fundamental understanding of what absolute value means numerically is essential before discussing absolute value functions.
- **Absolute Value Functions**
  - The Concept of Absolute Value: A deep understanding of absolute value as distance is used in contextual problems like Example 1 and 2.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Graph an absolute value function.
- Solve an absolute value equation.
- Solve an absolute value inequality.

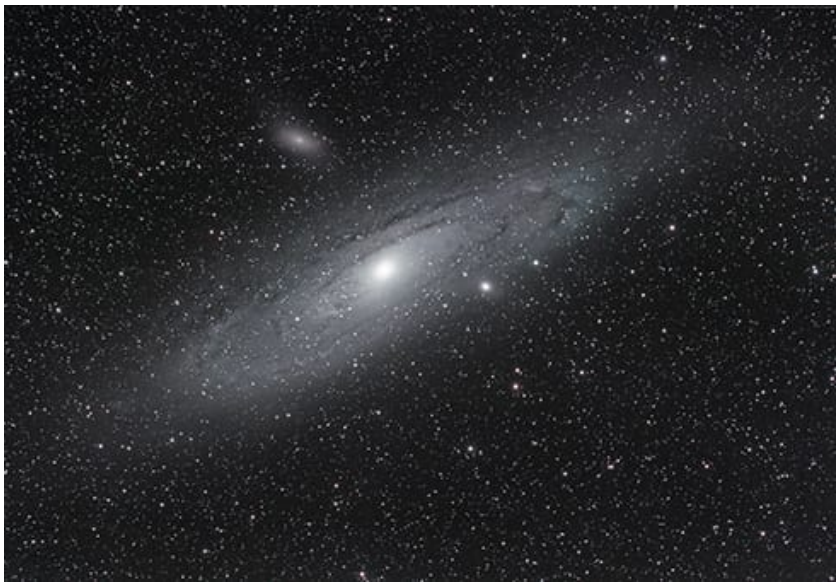
Hawk A.I. Section-Specific Tutor

EST. 1970 COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Figure 1.5.1

Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)



Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate absolute value functions.

Focus on Calculus - Piecewise Definition of the Absolute Value

Recall that in its basic form $f(x) = |x|$, the absolute value function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

Definition: Absolute Value Function

The absolute value function is defined to be $f(x) = |x|$, where

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

Its domain is all real numbers, and its range is $[0, \infty)$. Its graph is the **V**-shaped curve formed by the two rays $y = x$ (for $x \geq 0$) and $y = -x$ (for $x < 0$), meeting at a single sharp corner called the **vertex**. For the parent function the vertex is the origin, $(0, 0)$.

Example 1.5.1: Determine a Number within a Prescribed Distance

Describe all values x within or including a distance of 4 from the number 5.

Solution available at the end of this section.

Note that

$$\begin{aligned} -4 &\leq x-5 \\ 1 &\leq x \end{aligned} \qquad \begin{aligned} x-5 &\leq 4 \\ x &\leq 9. \end{aligned}$$

So $|x-5| \leq 4$ is equivalent to $1 \leq x \leq 9$.

However, mathematicians generally prefer absolute value notation.

Checkpoint 1.5.1

Describe all values x within a distance of 3 from the number 2.

Example 1.5.2: Resistance of a Resistor

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often $\pm 1\%$, $\pm 5\%$, or $\pm 10\%$.

Suppose we have a resistor rated at 680 ohms, $\pm 5\%$. Use the absolute value function to express the range of possible values of the actual resistance.

Solution available at the end of this section.

Checkpoint 1.5.2

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in Figure 1.5.3.

Figure 1.5.3

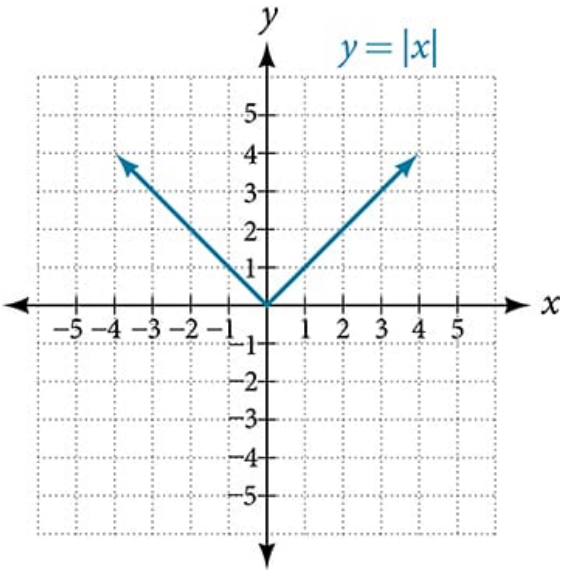
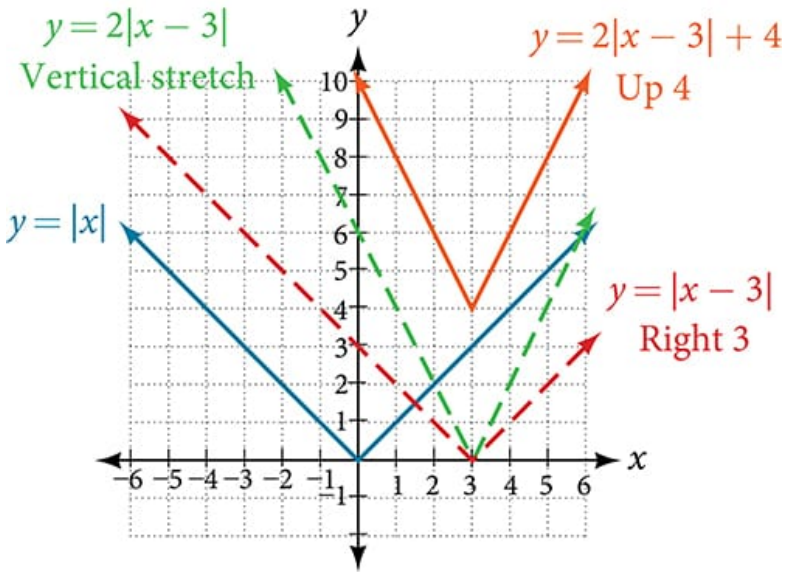


Figure 1.5.4 shows the graph of $y = 2|x–3| + 4$. The graph of $y = |x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3, 4)$ for this transformed function.

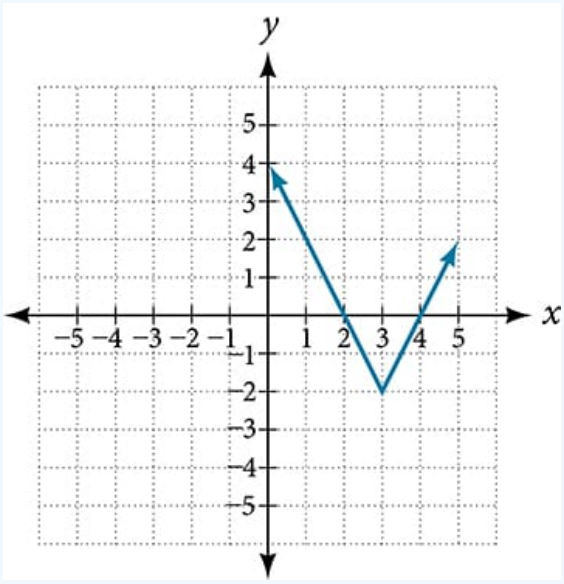
Figure 1.5.4



Example 1.5.3: Writing an Equation for an Absolute Value Function

Write an equation for the function graphed in Figure 1.5.5.

Figure 1.5.5



Solution available at the end of this section.

In Example 1.5.3, we could have opted to treat the stretch as a horizontal compression. In that case, the function would have been $f(x) = |2(x - 3)| - 2$. Note that this equation is algebraically equivalent to what we arrived at in our solution. This result, that the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression, is not a rule for all functions - it just happens to work for absolute value (and linear) functions. Note also that if the vertical stretch factor is negative, there is also a reflection about the x-axis.

Q&A

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for x and $f(x)$.

$$f(x) = a|x-3| - 2.$$

Now substituting in the point $(1, 2)$, we get

$$\begin{aligned} 2 &= a|1-3| - 2 && \text{(substituting)} \\ \implies 4 &= 2a && \text{(simplifying and adding 2 to both sides)} \\ \implies 2 &= a && \text{(dividing both sides by 2)} \end{aligned}$$

Checkpoint 1.5.3A

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

Q&A

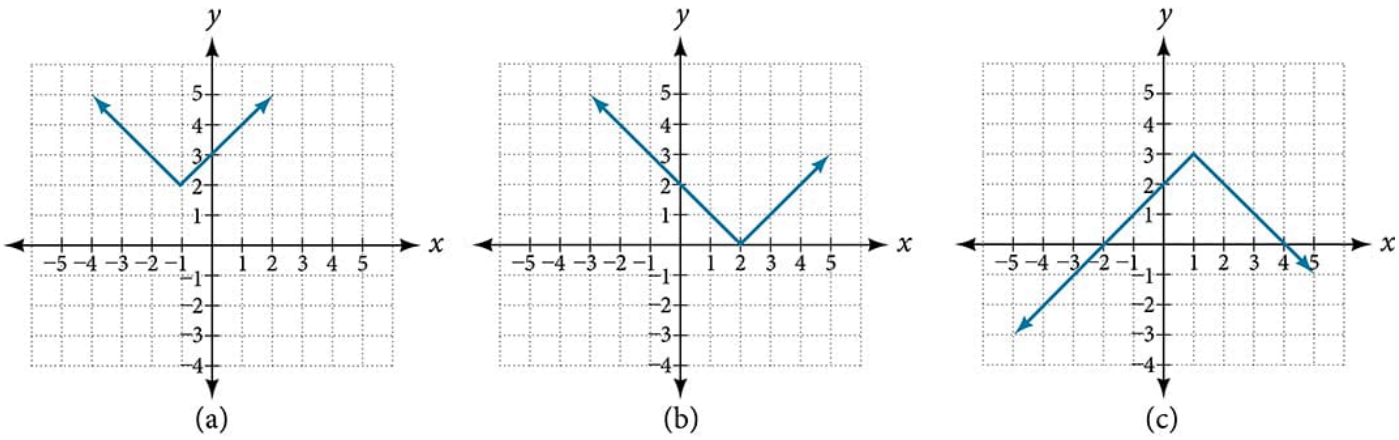
Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (see Figure 1.5.8).

Figure 1.5.8

(a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.



Checkpoint 1.5.3B

Is the absolute value function one-to-one?

Solving an Absolute Value Equation

Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8 = |2x - 6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$\begin{aligned} 2x - 6 &= 8 \quad \text{or} \quad 2x - 6 = -8 \\ \implies 2x &= 14 \qquad\quad 2x = -2 \\ \implies x &= 7 \qquad\quad x = -1 \end{aligned}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

Definition: Absolute Value Equation

An **absolute value equation** is an equation in which the unknown variable appears within the absolute value bars.

For example,

$$\begin{aligned} |x| &= 4, \\ |2x - 1| &= 3, \text{ and} \\ |5x + 2| - 4 &= 9. \end{aligned}$$

How To: Solve Absolute Value Equations

To solve an absolute value equation, $|\blacksquare| = k$, where $\blacksquare$ is an algebraic expression involving a variable, we rely on the piecewise function definition of the absolute value. Namely,

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

This means that to solve $|\blacksquare| = k$, we have two cases to consider:

If $\blacksquare \geq 0$,	If $\blacksquare < 0$,
then we solve	then we solve
$\blacksquare = k$	$-\blacksquare = k$

However, if $k < 0$, we can safely say (without any effort) that the original absolute value equation has no solution. Remember, an absolute value cannot be a negative number.

I specifically chose to use $\blacksquare$ in this version of the definition instead of x because most students new to the concept of the piecewise definition of the absolute value tend to think that the conditions must always be $x \geq 0$ and $x < 0$ - **this is not true**.

Let's illustrate the process through an example.

Example 1.5.4: Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x + 1| - 7$, find the values of x such that $f(x) = 0$.

Solution available at the end of this section.

Checkpoint 1.5.4

For the function $f(x) = |2x - 1| - 3$, find the values of x such that $f(x) = 0$.

Q&A

Should we always expect two answers when solving $|A| = B$?

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x - 5| = 1$.

Example 1.5.5: Solving an Absolute Value Equation

Solve $1 = 4|x - 2| + 2$.

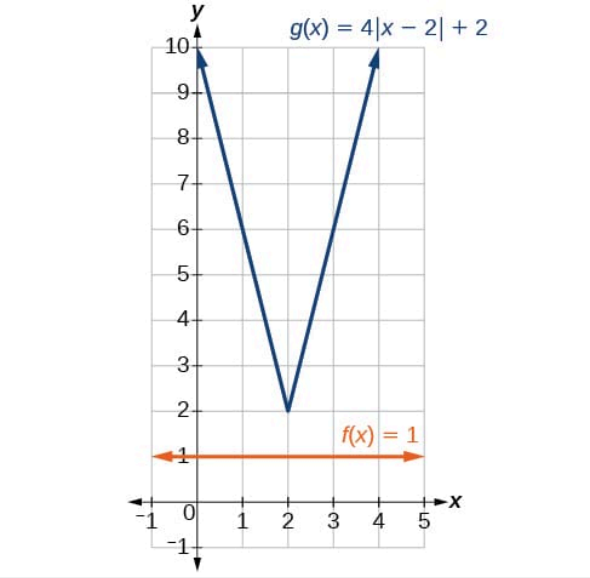
Solution available at the end of this section.

Q&A

In Example 1.5.5, if $f(x) = 1$ and $g(x) = 4|x - 2| + 2$ were graphed on the same set of axes, would the graphs intersect?

No. The graphs of f and g would not intersect, as shown in Figure 1.5.10. This confirms, graphically, that the equation $1 = 4|x - 2| + 2$ has no solution.

Figure 1.5.10



Checkpoint 1.5.5

Find where the graph of the function $f(x) = -|x + 2| + 3$ intersects the horizontal and vertical axes.

Focus on Calculus - Solving Absolute Value Inequalities

Absolute value equations may not always involve equalities. Instead, we may need to solve an equation within a range of values.

We would use an absolute value inequality to solve such an equation. An **absolute value inequality** is an equation of the form\[|A|B, \quad \text{or }|A| \geq B, \nonumber \]where an expression A (and possibly, but not usually, B) depends on a variable x.

Definition: Absolute Value Inequality

An **absolute value inequality** is an inequality in which the unknown variable appears within the absolute value bars.

Solving the inequality means finding the set of all x that satisfy the inequality. Usually this set will be an interval or the union of two intervals.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two functions. The disadvantage of this approach is that, when graphing by hand, the solutions are likely not going to be accurate. The advantage of the algebraic approach is it yields solutions that may be difficult to read from the graph.

For example, we know that all numbers within 200 units of 0 may be expressed as\[| x |<200 \text{ or } −200 & -200 & < & x - 600 & < & 200 & & \[6pt]
\implies & 400 & < & x & < & 800 & \quad & \left(\text{adding }600 \text{ to all three "sides"} \right) \[6pt]
\end{array} \nonumber \]This means our returns would be between $400 and $800.

Sometimes an absolute value inequality problem will be presented to us in terms of a shifted and/or stretched or compressed absolute value function, where we must determine for which values of the input the function's output will be negative or positive.

Example 1.5.6: Solving an Absolute Value Inequality

Solve $|x-5|<4$.

Solution available at the end of this section.

The approach we used in our solution to Example 1.5.6 is what I consider to be a "preferred method." This is because it reinforces the piecewise definition of the absolute value (which is something you must know to be successful in Calculus); however, it is best to show you another approach used by many other instructors and students.

How To: Solve Absolute Value Inequalities

7.03: Solving Absolute Value Inequalities

First, we will need to know where the corresponding equality is true. That is, we need to solve $|x-5|=4$. The process is almost identical to what we did in Example 1.5.6.

$$|x-5| = \begin{cases} x-5, & \text{if } x-5 \geq 0 \implies x \geq 5 \\ -(x-5), & \text{if } x-5 < 0 \implies x < 5 \end{cases}$$

This tells us that our absolute value equation splits into two possibilies:

$$\begin{array}{rrcl} & \text{For } x \geq 5 & & \\ & |x-5| & = & 4 \\ \implies & x-5 & = & 4 \\ \implies & x & = & 9 \end{array} \qquad \begin{array}{rrcl} & \text{For } x < 5 & & \\ & |x-5| & = & 4 \\ \implies & -(x-5) & = & 4 \\ \implies & x-5 & = & -4 \\ \implies & x & = & 1 \end{array}$$

After determining that the absolute value is equal to 4 at $x=1$ and $x=9$, we know the graph can change only from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals:\[x<1, \quad 19. \nonumber \]To determine when the function is less than 4, we could choose a value in each interval and see if the output is less than or greater than 4, as shown in Table 1.5.1.

Table 1.5.1

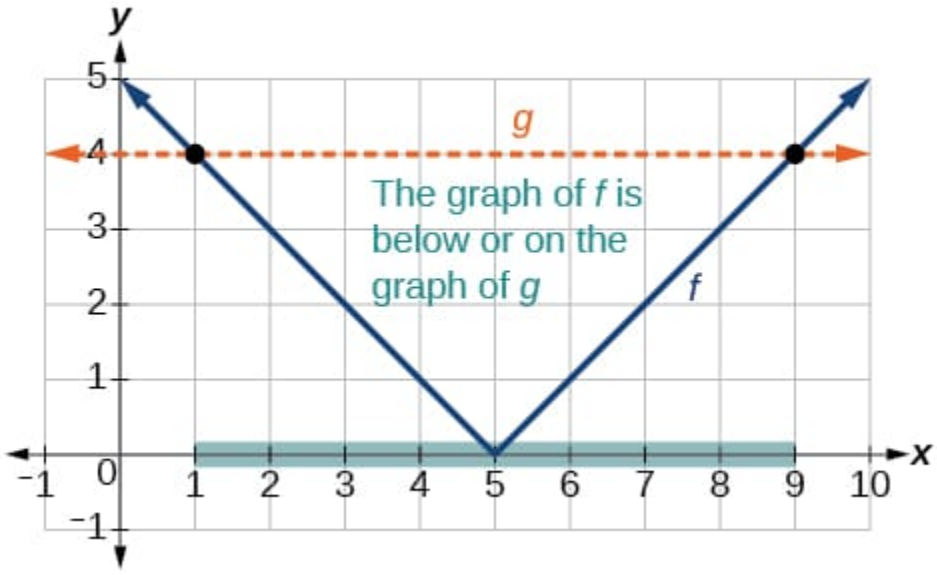
Interval test x	Test value for x	< 4 **or** > 4**?**	
$x < 1$	0	$\|0 - 5\| = 5$	Greater than
\(1	6	$\|6 - 5\| = 1$	Less than
$x > 9$	11	$\|11 - 5\| = 6$	Greater than

Because \(1

To use a graph, we can sketch the function $f(x) = |x - 5|$. To help us see where the outputs are 4, the line $g(x) = 4$ could also be sketched as in Figure 1.5.11.

Figure 1.5.11

Graph to find the points satisfying an absolute value inequality.



We can see the following:

- The output values of the absolute value are equal to 4 at $x = 1$ and $x = 9$.
- The graph of f is below the graph of g on \(1
- The absolute value is less than or equal to 4 between these two points, when \(1

For absolute value inequalities,

$$\begin{array}{rcccl} & |x - A| & < & C & \qquad |x - A| > C \\ \implies & -C < & x - A & < C & \qquad x - A < -C \text{ or } x - A > C \end{array}$$

The $<$ or $>$ symbol may be replaced by $\leq$ or $\geq$.

So, for Example 1.5.6, we could use this alternative approach.

$$\begin{array}{rcccl} & |x - 5| & < & 4 & \\ \implies & -4 < & x - 5 & < 4 & \text{(rewriting by removing the absolute value bars)} \\ \implies & -4 + 5 < & x - 5 + 5 & < 4 + 5 & \text{(adding 5 to all three "sides")} \\ \implies & 1 < & x & < 9 & \end{array}$$

Checkpoint 1.5.6

Solve $|x+2| \leq 6$.

Example 1.5.7: Using a Graphical Approach to Solve Absolute Value Inequalities

Given the function $f(x) = -\frac{1}{2}|4x-5|+3$, determine the x-values for which the function values are negative.

Solution available at the end of this section.

Checkpoint 1.5.7

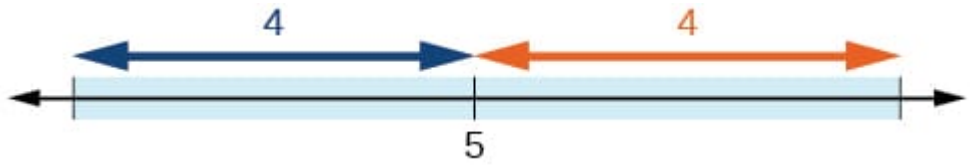
Solve $-2|k-4| \leq -6$.

Answers

Example 1.5.1: Determine a Number within a Prescribed Distance

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as the one in Figure 1.5.2, to represent the condition to be satisfied.

Figure 1.5.2



The distance from x to 5 can be represented using the absolute value as $|x-5|$. We want the values of x that satisfy the condition $|x-5| \leq 4$.

Example 1.5.2: Resistance of a Resistor

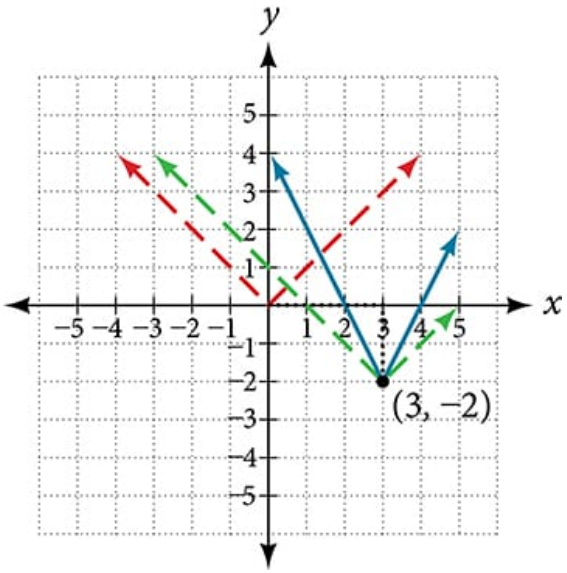
5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,

$$|R-680| \leq 34.$$

Example 1.5.3: Writing an Equation for an Absolute Value Function

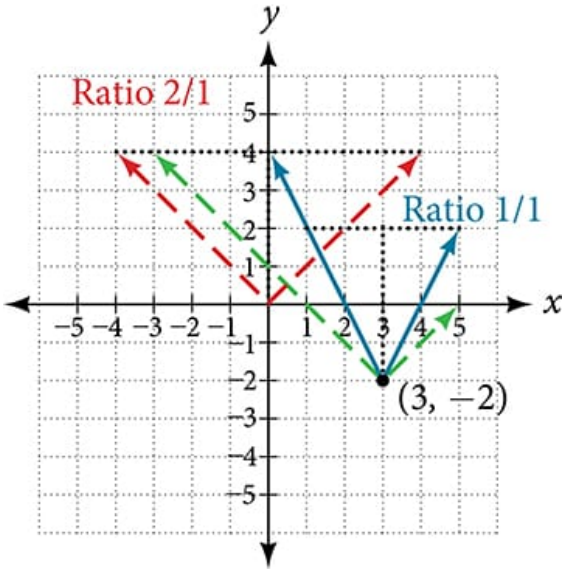
The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See Figure 1.5.6.

Figure 1.5.6



We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in Figure 1.5.7.

Figure 1.5.7



From this information we can write the equation as $f(x) = 2|x - 3| - 2$, treating the stretch as a vertical stretch.

Example 1.5.4**: Finding the Zeros of an Absolute Value Function**

We set $f(x) = 0$ and solve.

$$\begin{aligned} 0 &= |4x+1| - 7 && \text{(setting } f(x) = 0\text{)} \\ \implies 7 &= |4x+1| && \text{(adding 7 to both sides)} \end{aligned}$$

Now that we have the absolute value isolated, let's recall the piecewise function definition of the absolute value.

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

In our case, $\blacksquare = 4x + 1$. Hence,

$$|4x+1| = \begin{cases} 4x+1, & \text{if } 4x+1 \geq 0 \\ -(4x+1), & \text{if } 4x+1 < 0 \end{cases}$$

That is,

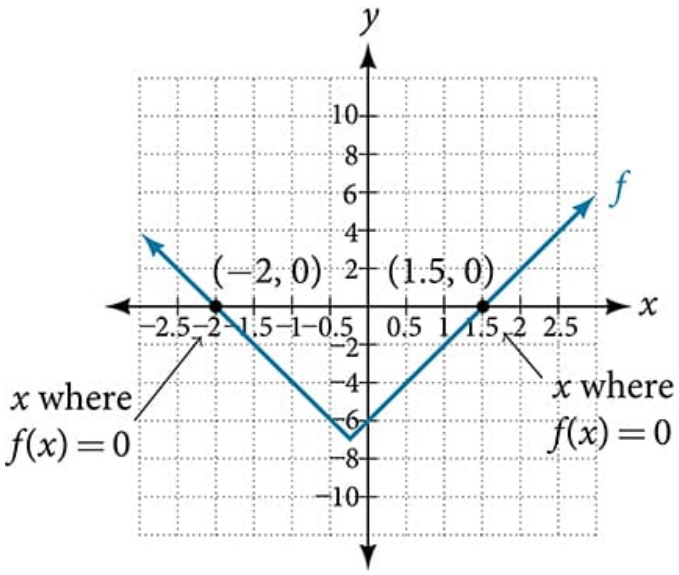
$$|4x+1| = \begin{cases} 4x+1, & \text{if } x \geq -\frac{1}{4} \\ -(4x+1), & \text{if } x < -\frac{1}{4} \end{cases}$$

This tells us that our absolute value equation splits into two possibilies:

For $x \geq -\frac{1}{4}$			For $x < -\frac{1}{4}$		
	7	$= \lvert 4x+1 \rvert$		7	$= \lvert 4x+1 \rvert$
$\implies$	7	$= 4x+1$	$\implies$	7	$= -(4x+1)$
$\implies$	6	$= 4x$	$\implies$	-7	$= 4x+1$
$\implies$	$\frac{3}{2}$	$= x$	$\implies$	-8	$= 4x$
			$\implies$	-2	$= x$

The function outputs 0 when $x = 1.5$ or $x = -2$. See Figure 1.5.9.

Figure 1.5.9



Example 1.5.5**: Solving an Absolute Value Equation**

Isolating the absolute value on one side of the equation gives the following.

$$\begin{aligned} 1 &= 4|x-2|+2 \\ \implies -1 &= 4|x-2| \\ \implies -\frac{1}{4} &= |x-2| \end{aligned}$$

The absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value. At this point, we notice that this equation has no solutions.

Example 1.5.6**: Solving an Absolute Value Inequality**

Our approach will duplicate our work in Example 1.5.5. Since the inequality is already isolated, we immediately dive into the piecewise definition of the absolute value:

$$|\blacksquare| = \begin{cases} \blacksquare, & \text{if } \blacksquare \geq 0 \\ -\blacksquare, & \text{if } \blacksquare < 0 \end{cases}$$

In our case, $\blacksquare = x-5$. Hence,

$$|x-5| = \begin{cases} x-5, & \text{if } x-5 \geq 0 \implies x \geq 5 \\ -(x-5), & \text{if } x-5 < 0 \implies x < 5 \end{cases}$$

This tells us that our absolute value inequality splits into two possibilities:

For $x \geq 5$			For $x < 5$		
	$\lvert x-5\rvert$	< 4		$\lvert x-5\rvert$	< 4
$\implies$	$x-5$	< 4	$\implies$	$-(x-5)$	< 4
$\implies$	x	< 9	$\implies$	$x-5$	> -4
			$\implies$	x	> 1

To summarize, we are okay with $x \geq 5$ as long as $x < 9$. Additionally, we are okay with $x < 5$ as long as $x > 1$. Combining this information, our solution interval is $(1, 9)$.

Example 1.5.7**: Using a Graphical Approach to Solve Absolute Value Inequalities**

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x-5|+3 < 0$. We begin by isolating the absolute value.

$$\begin{aligned} -\frac{1}{2}|4x-5| &< -3 && \text{(subtracting 3 from both sides)} \\ \implies |4x-5| &> 6 && \text{(multiplying both sides by } -2 \text{ and switching the inequality sign)} \end{aligned}$$

Using the definition of the absolute value, we get

$$|4x-5| = \begin{cases} 4x-5, & \text{if } 4x-5 \geq 0 \implies x \geq \frac{5}{4} \\ -(4x-5), & \text{if } 4x-5 < 0 \implies x < \frac{5}{4} \end{cases}$$

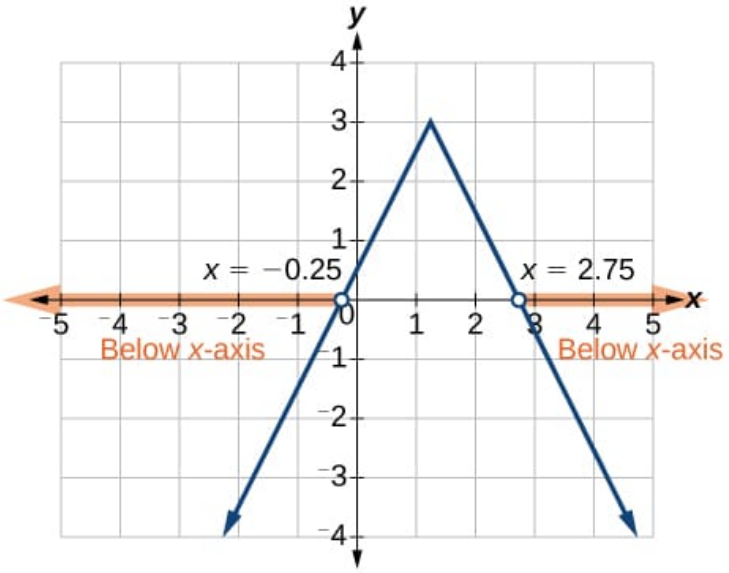
Thus,

$$
\begin{array}{rrcl}
\text{For } x \geq \frac{5}{4} & & & \text{For } x < \frac{5}{4} \\
|4x-5| & > & 6 & |4x-5| > 6 \\
\implies 4x-5 & > & 6 & \implies -(4x-5) > 6 \\
\implies 4x & > & 11 & \implies 4x-5 < -6 \\
\implies x & > & \frac{11}{4} & \implies 4x < -1 \\
 & & & \implies x < -\frac{1}{4}
\end{array}
$$

The leftmost solution tells us that as long as $x \geq \frac{5}{4}$, the original inequality is satisfied for $x > \frac{11}{4}$. The rightmost solution tells us that as long as $x < \frac{5}{4}$, the original inequality is satisfied for $x < -\frac{1}{4}$. Thus, our solution is $\left(-\infty, -\frac{1}{4}\right) \cup \left(\frac{11}{4}, \infty\right)$.

Had we decided to using the graphing method, we can examine the graph of f to observe where the output is negative. We will observe where the branches are below the x-axis. Notice that it is not even important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$ and that the graph has been reflected vertically. See Figure 1.5.12.

Figure 1.5.12



We observe that the graph of the function is below the x-axis left of $x = -\frac{1}{4}$ and right of $x = \frac{11}{4}$. This means the function values are negative to the left of the first horizontal intercept at $x = -\frac{1}{4}$, and negative to the right of the second intercept at $x = \frac{11}{4}$. This gives us the solution to the inequality.

$$x < -\frac{1}{4} \text{ or } x > \frac{11}{4}.$$

In interval notation, this would be $\left(-\infty, -\frac{1}{4}\right) \cup \left(\frac{11}{4}, \infty\right)$.

This page titled 1.5: Absolute Value Functions is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

1.5.1: Resources and Key Concepts

Resources

Videos

- Absolute value as a concept
  - Intermediate Algebra - Absolute Value: The Concepts
- Creating absolute value equations
  - Intermediate Algebra - Absolute Value: Forming Absolute Value Equations
- Solving absolute value equations
  - Equations - Overview - A Review for College Algebra - Solving Absolute Value Equations
- Solving absolute value inequalities
  - Inequalities - Nonlinear - A Review for College Algebra: Solving Absolute Value Inequalities
- Interval notation for solutions
  - Logic & Set Theory in Review: Sets - Notation (Interval Notation)

Key Concepts

Definitions

- **Absolute Value Function:** The function $f(x) = |x|$, which gives the distance of x from zero on a number line. Algebraically, $f(x) = x$ if $x \geq 0$ and $f(x) = -x$ if $x < 0$.
- **Absolute Value Equation:** An equation in which the unknown variable appears within absolute value bars.
- **Absolute Value Inequality:** An inequality in which the unknown variable appears within the absolute value bars, typically of the form $|A| < B$, $|A| \leq B$, $|A| > B$, or $|A| \geq B$.

Common Mistakes

- **Assuming Absolute Value Equations Always Have Two Solutions:** While many absolute value equations yield two solutions (e.g., $|2x - 6| = 8$ in the introduction to solving equations), they can also have one solution (e.g., $|x - 3| = 0$) or no solutions (e.g., $|x - 2| = -\frac{1}{4}$ in Example 5).
- **Incorrectly Handling Negative Numbers in Absolute Value Equations/Inequalities:**
  - An equation of the form $|A| = k$ where $k < 0$ has no solution, because an absolute value cannot be negative.
  - An inequality of the form $|A| < k$ where $k < 0$ has no solution.
  - An inequality of the form $|A| > k$ where $k < 0$ is true for all values of A for which A is defined, because an absolute value is always non-negative and thus always greater than a negative number.
- **Forgetting to Switch Inequality Sign When Multiplying/Dividing by a Negative Number:** When solving absolute value inequalities algebraically, if a step involves multiplying or dividing both sides by a negative number, the direction of the inequality sign must be reversed.
- **Misinterpreting Conditions for Piecewise Definition:** When using the piecewise definition of absolute value, $|■| = ■$ if $■ \geq 0$ and $|■| = -■$ if $■ < 0$, students might incorrectly assume the condition always refers to $x \geq 0$ or $x < 0$ instead of the expression inside the absolute value bars (■).

This page titled 1.5.1: Resources and Key Concepts is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

1.5.2: Homework

Reading Questions

1. How is the absolute value function $f(x) = |x|$ defined as a piecewise function? What does each piece represent?
2. What does the expression $|x - a|$ represent geometrically on a number line?
3. Explain how the parameters a, h, and k in the function $g(x) = a|x - h| + k$ transform the graph of the basic absolute value function $f(x) = |x|$.
4. What is the "corner point" of an absolute value function's graph, and how can its coordinates be determined from the equation $g(x) = a|x - h| + k$?
5. When solving an absolute value equation such as $|ax + b| = c$, where c is a positive number, what are the two distinct linear equations that arise?
6. Under what condition does an absolute value equation of the form $|E| = c$ have no solution? Provide an example.
7. Explain the algebraic steps to solve an absolute value inequality of the form $|E| < c$ where $c > 0$. How does this differ from solving $|E| > c$?
8. What does it mean if the graph of an absolute value function $f(x)$ is below the x-axis for a certain interval of x-values? How does this relate to solving $f(x) < 0$?
9. If you are solving an absolute value inequality and you multiply or divide both sides by -1, what critical adjustment must be made to the inequality symbol?
10. In the piecewise definition of $|E|$, where E is an algebraic expression, when is $|E|$ equal to E, and when is it equal to $-E$?
11. How do you solve an absolute value equation?
12. How can you tell whether an absolute value function has two x-intercepts without graphing the function?
13. When solving an absolute value function, the isolated absolute value term is equal to a negative number. What does that tell you about the graph of the absolute value function?
14. How can you use the graph of an absolute value function to determine the x-values for which the function values are negative?
15. How do you solve an absolute value inequality algebraically?

Homework

Algebraic

16. Describe all numbers x that are at a distance of 4 from the number 8. Express this using absolute value notation.
17. Describe all numbers x that are at a distance of $\frac{1}{2}$ from the number -4. Express this using absolute value notation.
18. Describe the situation in which the distance that point x is from 10 is at least 15 units. Express this using absolute value notation.
19. Find all function values $f(x)$ such that the distance from $f(x)$ to the value 8 is less than 0.03 units. Express this using absolute value notation.

For the following exercises, solve the equations below and express the answer using set notation.

20. $|x + 3| = 9$
21. $|6 - x| = 5$
22. $|5x - 2| = 11$
23. $|4x - 2| = 11$
24. $2|4 - x| = 7$
25. $3|5 - x| = 5$
26. $3|x + 1| - 4 = 5$
27. $5|x - 4| - 7 = 2$
28. $0 = -|x - 3| + 2$
29. $2|x - 3| + 1 = 2$
30. $|3x - 2| = 7$
31. $|3x - 2| = -7$
32. $\left|\frac{1}{2}x - 5\right| = 11$

33. $\left|\frac{1}{3}x+5\right|=14$
34. $-\left|\frac{1}{3}x+5\right|+14=0$

For the following exercises, find the x- and y-intercepts of the graphs of each function.

35. $f(x)=2|x+1|-10$
36. $f(x)=4|x-3|+4$
37. $f(x)=-3|x-2|-1$
38. $f(x)=-2|x+1|+6$

For the following exercises, solve each inequality and write the solution in interval notation.

39. $|x-2|>10$
40. $2|v-7|-4\geq 42$
41. $|3x-4|\leq 8$
42. $|x-4|\geq 8$
43. $|3x-5|\geq 13$
44. $|3x-5|\geq -13$
45. $\left|\frac{3}{4}x-5\right|\geq 7$
46. $\left|\frac{3}{4}x-5\right|+1\leq 16$

Graphical

For the following exercises, graph the absolute value function. Plot at least five points by hand for each graph.

47. $y=|x-1|$
48. $y=|x+1|$
49. $y=|x|+1$

For the following exercises, graph the given functions by hand.

50. $y=|x|-2$
51. $y=-|x|$
52. $y=-|x|-2$
53. $y=-|x-3|-2$
54. $f(x)=-|x-1|-2$
55. $f(x)=-|x+3|+4$
56. $f(x)=2|x+3|+1$
57. $f(x)=3|x-2|+3$
58. $f(x)=|2x-4|-3$
59. $f(x)=|3x+9|+2$
60. $f(x)=-|x-1|-3$
61. $f(x)=-|x+4|-3$
62. $f(x)=\frac{1}{2}|x+4|-3$

Technology

63. Use graphing technology to graph $f(x)=10|x-2|$ on the viewing window $[0,4]$. Identify the corresponding range. Show the graph.
64. Use graphing technology to graph $f(x)=-100|x|+100$ on the viewing window $[-5,5]$. Identify the corresponding range. Show the graph.

For the following exercises, graph each function using graphing technology. Specify the viewing window.

65. $f(x)=-0.1|0.1(0.2-x)|+0.3$
66. $f(x)=4\times 10^9|x-(5\times 10^9)|+2\times 10^9$

Extensions

For the following exercises, solve the inequality.

57. $\left|-2x-\frac{2}{3}(x+1)\right|+3>-1$
58. If possible, find all values of a such that there are no x-intercepts for $f(x)=2|x+1|+a$.
59. If possible, find all values of a such that there are no y-intercepts for $f(x)=2|x+1|+a$.

Real-World Applications

70. Cities A and B are on the same east-west line. Assume that city A is located at the origin. If the distance from city A to city B is at least 100 miles and x represents the distance from city B to city A, express this using absolute value notation.
71. The true proportion p of people who give a favorable rating to Congress is 8% with a margin of error of 1.5%. Describe this statement using an absolute value equation.
72. Students who score within 18 points of the number 82 will pass a particular test. Write this statement using absolute value notation and use the variable x for the score.
73. A machinist must produce a bearing that is within 0.01 inches of the correct diameter of 5.0 inches. Using x as the diameter of the bearing, write this statement using absolute value notation.
74. The tolerance for a ball bearing is 0.01. If the true diameter of the bearing is to be 2.0 inches and the measured value of the diameter is x inches, express the tolerance using absolute value notation.

This page titled 1.5.2: Homework is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

1.6: Inverse Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure $1.6.1$ provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

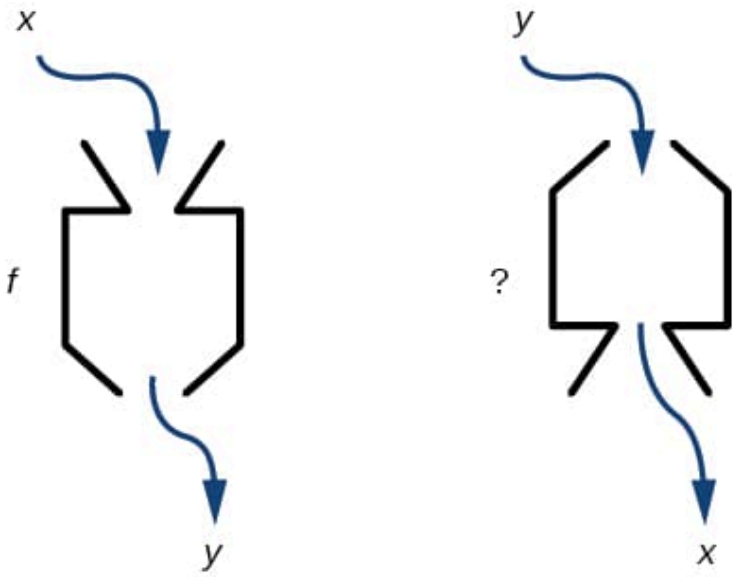


Figure $1.6.1$**:** *Can a function "machine" operate in reverse?*

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the Celsius scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees Fahrenheit to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32)$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24^\circ \text{ C}.$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure $1.6.2$ for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

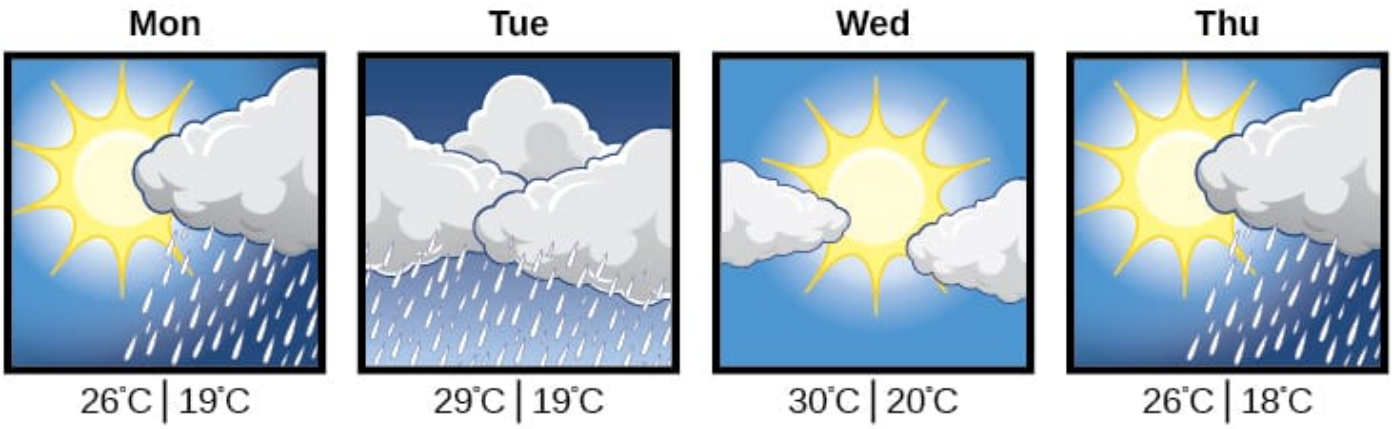


Figure 1.6.2

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her Algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$\begin{aligned} 26 &= \frac{5}{9}(F-32) \\ \implies 26\cdot\frac{9}{5} &= F-32 \\ \implies F &= 26\cdot\frac{9}{5}+32\approx 79 \end{aligned}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Definition: Inverse Function

Let f be a one-to-one function. The **inverse** of f, denoted f^{-1}, is the function that reverses the pairing produced by f: for each y in the range of f,

$$f^{-1}(y)=x \quad \text{precisely when} \quad f(x)=y.$$

f^{-1} is called the **inverse function** of f and the notation, f^{-1}, is read "f-inverse".

Caution

The symbol f^{-1} denotes the inverse function, not a reciprocal. In general, $f^{-1}(x)\neq\frac{1}{f(x)}$. The reciprocal $\frac{1}{f(x)}$ is written $\big(f(x)\big)^{-1}$.

Additionally, not all functions have inverses.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a=1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1}\circ f$ equals the identity function, that is,

$$(f^{-1}\circ f)(x)=f^{-1}\left(f(x)\right)=x.$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a reciprocal, some functions do not have inverses.

Inverse Function Notation

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x."

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

Theorem

If f^{-1} is the inverse of f, then $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}\right) = x.$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Example 1.6.1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution available at the end of this section.

If we show the coordinate pairs from Example 1.6.1 in a table form, the input and output are clearly reversed. See Table 1.6.1.

Table 1.6.1

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Checkpoint 1.6.1

Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

Example 1.6.2: Testing Inverse Relationships Algebraically

If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$, is $g = f^{-1}$?

Solution available at the end of this section.

Checkpoint 1.6.2

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

Example 1.6.3: Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

Solution available at the end of this section.

Checkpoint 1.6.3

If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure 1.6.3.

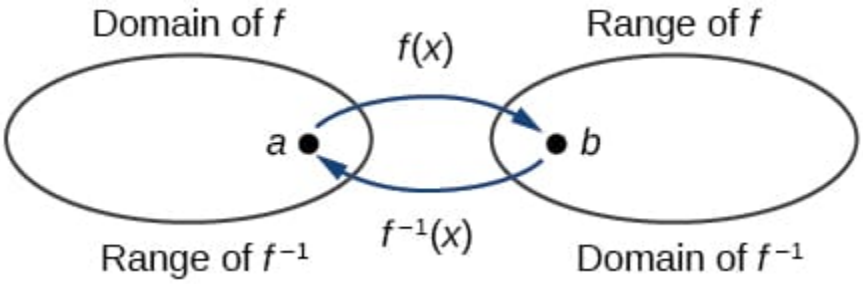


Figure 1.6.3: *Domain and range of a function and its inverse*

Theorem: Domain and Range of Inverse Functions

If f is one-to-one, then the domain of f^{-1} equals the range of f, and the range of f^{-1} equals the domain of f.

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and -3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all!

To put it differently, the quadratic function is not a one-to-one function; it fails the Horizontal Line Test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

Interactive Element: The Horizontal Line Test

The following Interactive Element might help you visualize why the quadratic function is not one-to-one.

Interact: Drag the red point up and down to move the horizontal line.

Observation: Above the vertex the line meets $y = x^2$ at two points, so one output has two inputs — the function fails the Horizontal Line Test.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its domain limited to $[0, \infty)$, which is a one-to-one function (it passes the Horizontal Line Test) and which has an inverse (the square-root function).

If $f(x) = (x-1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of f^{-1} = $[1, \infty)$.
- The domain of f^{-1} = range of f = $[0, \infty)$.

Q&A

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Example 1.6.4: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 1.6.2. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Table 1.6.2

Constant	**Identity**	**Quadratic**	**Cubic**	**Reciprocal**
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	**Cube root**	**Square root**	**Absolute value**	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Solution available at the end of this section.

We can see that the absolute value and reciprocal-squared functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 1.6.4. They both would fail the Horizontal Line Test. However, if a function is restricted to a certain domain so that it passes the Horizontal Line Test, then in that restricted domain, it can have an inverse.

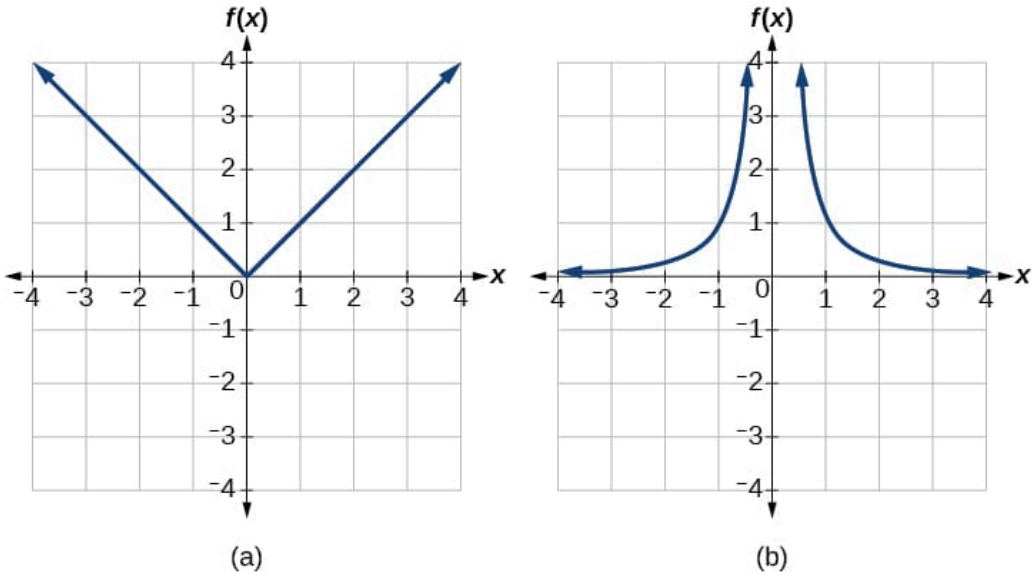


Figure 1.6.4**:** *(a) Absolute value; (b) Reciprocal squared*

Checkpoint 1.6.4

The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example 1.6.5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table 1.6.3, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$.

Table 1.6.3

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Solution available at the end of this section.

Checkpoint 1.6.5

Using Table 1.6.4, find and interpret each of the following:

a. $f(60)$
b. $f^{-1}(60)$

Table $1.6.4$

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We know that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

Example $1.6.6$: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure $1.6.5$. Find $g(3)$ and $g^{-1}(3)$.

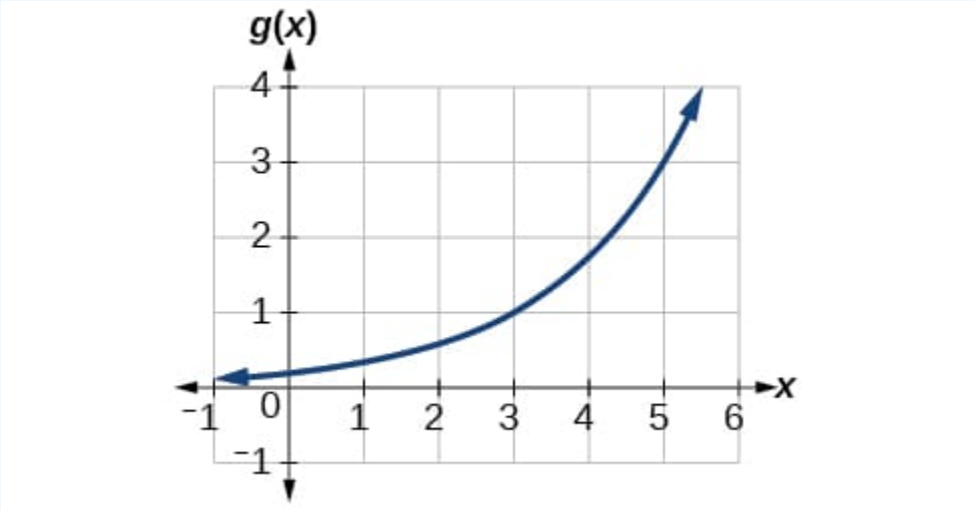


Figure $1.6.5$

Solution available at the end of this section.

Checkpoint $1.6.6$

Using the graph in Figure $1.6.6$ to answer the following questions.

a. Find $g^{-1}(1)$.
b. Estimate $g^{-1}(4)$.

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula—for example, y as a function of x—we can often find the inverse function by solving to obtain x as a function of y.

How To: Find the Equation of an Invertible Function

Given a function, $f(x)$, we find the inverse (if it exists) by doing the following:

1. Let $y = f(x)$.
2. Solve for x.
3. Interchange x and y.

The result of these operations, if it is a function, is $f^{-1}(x)$.

Example 1.6.7: Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature,

$$C = \frac{5}{9}(F - 32).$$

Solution available at the end of this section.

Checkpoint 1.6.7

Solve for x in terms of y given $y = \frac{1}{3}(x - 5)$.

Example 1.6.8: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

Solution available at the end of this section.

Note that the domain and range of f in Example 1.6.8 exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table 1.6.5.

Table 1.6.5

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example 1.6.9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x - 4}$.

Solution available at the end of this section.

The formula we found for $f^{-1}(x)$ in Example 1.6.9 looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

Checkpoint 1.6.9

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in Figure $1.6.7$.

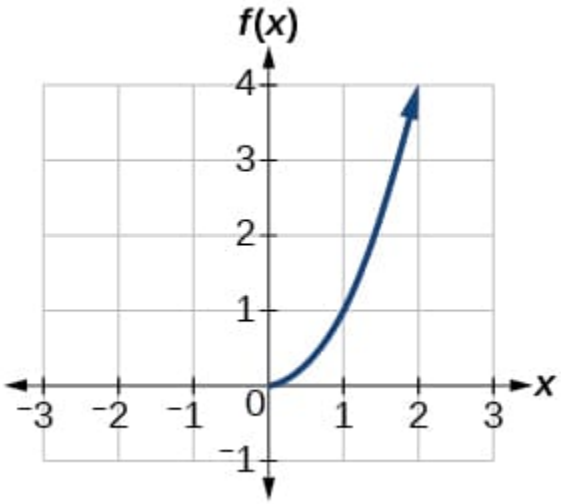


Figure $1.6.7$**:** *Quadratic function with domain restricted to* $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the Horizontal Line Test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure $1.6.8$.

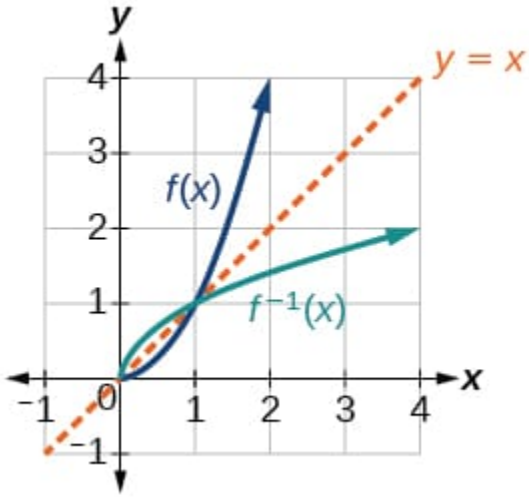


Figure $1.6.8$**:** *Square and square-root functions on the non-negative domain*

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Interactive Element: Reflecting a Function Across the Line y = x

The following Interactive Element might help you visualize how the graph of an inverse function is the mirror image of the original across the line $y = x$.

Interact: Drag the large red point along the blue curve $y = x^2$.

Observation: The orange point always lands on $y = \sqrt{x}$, the mirror image across $y = x$; a pair (a, b) on f becomes (b, a) on f^{-1}.

Example 1.6.10: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure 1.6.9, sketch a graph of $f^{-1}(x)$.

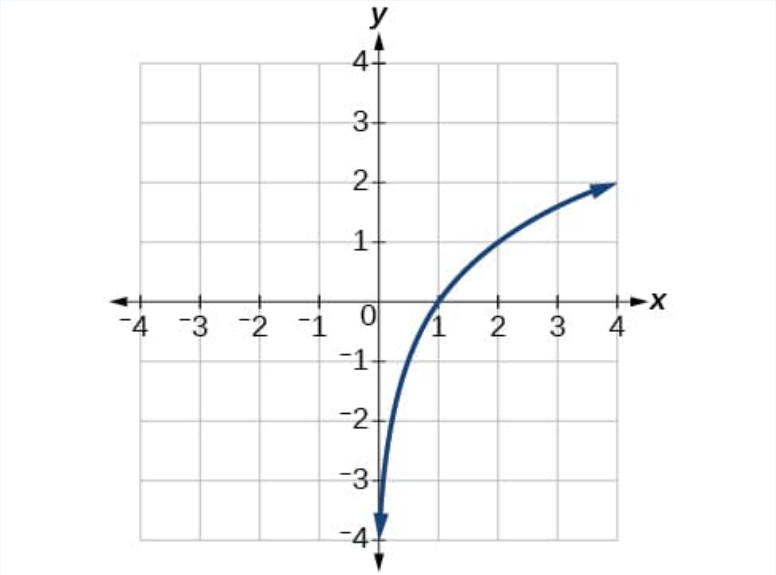


Figure 1.6.9

Solution available at the end of this section.

Checkpoint 1.6.10

Draw graphs of the functions f and f^{-1} from Example 1.6.8.

Q&A

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x.$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Restricting the Domain to Find the Inverse of a Polynomial Function

So far, we have been able to find the inverse functions of cubic functions without having to restrict their domains. However, as we know, not all cubic polynomials are one-to-one. Some functions that are not one-to-one may have their domain restricted so that they are one-to-one, but only over that domain. The function over the restricted domain would then have an inverse function. Since quadratic functions are not one-to-one, we must restrict their domain in order to find their inverses.

Theorem: Forcing Invertibility by Restricting Domain

If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

How To: Restrict Domain to Force Invertibility

Given a function that is not one-to-one,

1. Restrict the domain by determining a domain on which the original function is one-to-one (i.e., a domain where the function passes the Horizontal Line Test).
2. Replace $f(x)$ with y.
3. Solve for x interchange x and y.
4. Rename the function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

Example $1.6.11$: Restricting the Domain to Find the Inverse of a Polynomial Function

Find the inverse function of f :

a. $f(x)=(x-4)^2,\ x \geq 4$
b. $f(x)=(x-4)^2,\ x \leq 4$

Solution available at the end of this section.

On the graphs in Figure $1.6.12$, we see the original function from Example $1.6.11$ graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line $y=x$. The coordinate pair $(4,0)$ is on the graph of f and the coordinate pair $(0,4)$ is on the graph of f^{-1}. For any coordinate pair, if (a,b) is on the graph of f, then (b,a) is on the graph of f^{-1}. Finally, observe that the graph of f intersects the graph of f^{-1} on the line $y=x$. Points of intersection for the graphs of f and f^{-1} will always lie on the line $y=x$.

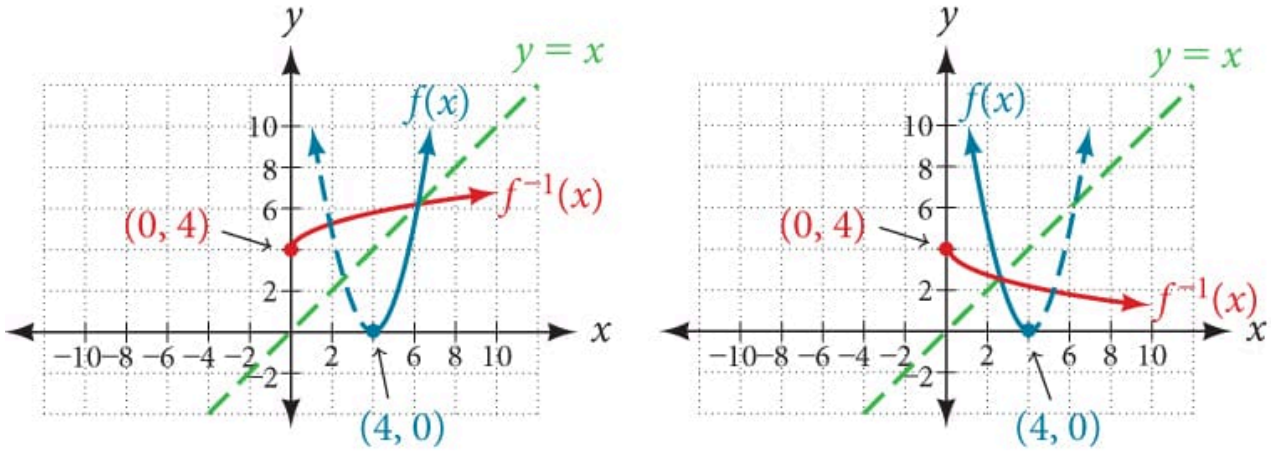


Figure 1.6.12

Example 1.6.12: Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified

Restrict the domain and then find the inverse of

$$f(x) = (x-2)^2 - 3.$$

Solution available at the end of this section.

Notice that we arbitrarily decided to restrict the domain on $x \geq 2$. We could just have easily opted to restrict the domain on $x \leq 2$, in which case $f^{-1}(x) = 2 - \sqrt{x+3}$. Observe the original function graphed on the same set of axes as its inverse function in Figure 1.6.13. Notice that both graphs show symmetry about the line $y = x$. The coordinate pair $(2, -3)$ is on the graph of f and the coordinate pair $(-3, 2)$ is on the graph of f^{-1}. Observe from the graph of both functions on the same set of axes that

$$\text{domain of } f = \text{range of } f^{-1} = [2, \infty)$$

and

$$\text{domain of } f^{-1} = \text{range of } f = [-3, \infty).$$

Finally, observe that the graph of f intersects the graph of f^{-1} along the line $y = x$.

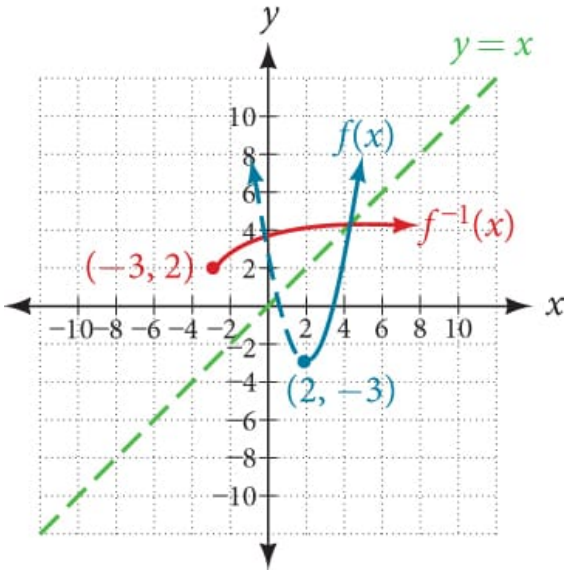


Figure 1.6.13

Checkpoint 1.6.12

Find the inverse of the function $f(x) = x^2 + 1$, on the domain $x \geq 0$.

Solving Applications Involving the Inverse of Polynomial Functions

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

Example 1.6.13: Solving an Application with a Cubic Function

A mound of gravel is in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

$$V = \frac{2}{3}\pi r^3.$$

Find the inverse of this function. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi \approx 3.14$.

Solution available at the end of this section.

Finding Inverses of Rational Functions

As with finding inverses of quadratic functions, it is sometimes desirable to find the inverse of a rational function, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

Example 1.6.14: Finding the Inverse of a Rational Function

The function $C = \frac{20+0.4n}{100+n}$ represents the concentration C of an acid solution after n mL of 40% solution has been added to 100 mL of a 20% solution. First, find the inverse of the function; that is, find an expression for n in terms of C. Then use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

Solution available at the end of this section.

Checkpoint 1.6.14

Find the inverse of the function $f(x) = \dfrac{x+3}{x-2}$.

Success in Mathematics: Mathematics in Other Courses

In other courses, we use mathematics to understand and explore the big ideas of the subject. Professors may use different vocabulary, variables, and notations than you learned in your math classes. The mathematics is the same. The way of talking about it may be different.

Also, some courses will use formulas based on mathematics that you have not yet learned. You may not be ready to understand why these formulas work. Sometimes you have to accept that the formula is correct.

Making connections between different courses takes work. It would be best if you were willing to ask questions. You may need to ask your math instructor about topics in your other classes. You may need to ask your other professors why they use different

words or notations. It is your responsibility to make these connections. Your ability to do critical thinking in other classes depends on it.

1. There are many formulas that we hear about in daily news or history. Of all these famous formulas, name one.
2. Can you describe the use of the formula you mentioned and what each variable stands for?

Hidden Resources: CRC Hub

As a college student, you have access to so many resources that it can be overwhelming. Luckily, the CRC Hub is your "one-stop shop" for all local resources covering various topics (from healthcare to legal fees). Check out the CRC Hub here.

Answers

Section Preview

We begin this chapter with defining and finding the inverse of a function. This is a traditional treatment of inverses, including how to graph them and how to find their equations given the original function. We also include domain restriction (as we are about to encounter radical functions).

Section Preview

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.
- Restrict the domain to find the inverse of a polynomial function.
- Find the inverse of a rational function.

Example $1.6.1$**: Identifying an Inverse Function for a Given Input-Output Pair**

The inverse function reverses the input and output quantities, so if $f(2)=4$, then $f^{-1}(4)=2$; If $f(5)=12$, then $f^{-1}(12)=5$. Alternatively, if we want to name the inverse function g, then $g(4)=2$ and $g(12)=5$.

Example $1.6.2$**: Testing Inverse Relationships Algebraically**

To determine if g is the inverse function of f, we just need to check if g "undoes" f.

$$\begin{aligned} g(f(x)) &= g\left(\frac{1}{x+2}\right) && \text{(substituting)} \\ &= \frac{1}{\frac{1}{x+2}} - 2 && \text{(substituting)} \\ &= x+2-2 && \left(\text{recognizing that } \frac{1}{1/a}=a\right) \\ &= x && \text{(simplifying)} \end{aligned}$$

so

$$g=f^{-1} \text{ and } f=g^{-1}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$\begin{aligned} f(g(x)) &= f\left(\frac{1}{x}-2\right) && \text{(substituting)} \\ &= \frac{1}{\left(\frac{1}{x}+2\right)-2} && \text{(substituting)} \\ &= \frac{1}{\frac{1}{x}} && \text{(simplifying)} \\ &= x && \left(\text{recognizing that } \frac{1}{1/a}=a\right) \end{aligned}$$

Example $1.6.3$**: Determining Inverse Relationships for Power Functions**

$$f(g(x))=f\left(\frac{x}{3}\right)=\left(\frac{x}{3}\right)^3=\frac{x^3}{27}\neq x.$$

No, the functions are not inverses. The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x}=x^{1/3}$, that is, the one-third is an exponent, not a multiplier.

Example $1.6.4$**: Finding the Inverses of Toolkit Functions**

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0,\infty)$, where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0,\infty)$.

Example $1.6.5$**: Interpreting the Inverse of a Tabular Function**

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70

miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Example $1.6.6$**: Evaluating a Function and Its Inverse from a Graph at Specific Points**

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure $1.6.6$.

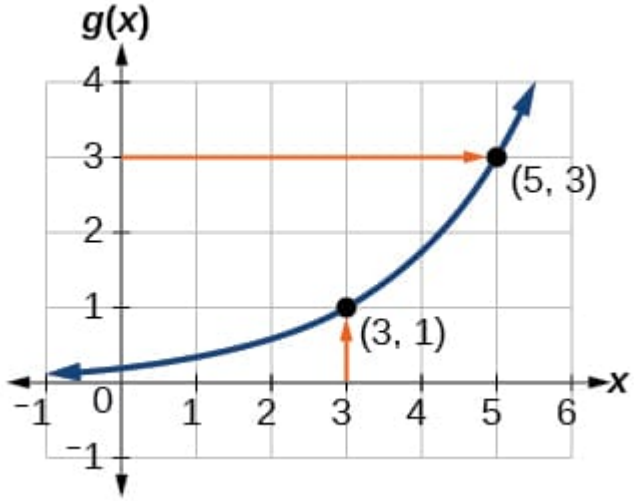


Figure $1.6.6$

Example $1.6.7$**: Inverting the Fahrenheit-to-Celsius Function**

$$\begin{aligned} C &= \frac{5}{9}(F-32) \\ \Longrightarrow \quad \frac{9}{5}C &= F-32 && \left(\text{multiplying both sides by } \frac{9}{5}\right) \\ \Longrightarrow \quad \frac{9}{5}C+32 &= F && (\text{adding } 32 \text{ to both sides}) \end{aligned}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F-32),$$

then

$$F = h^{-1}(C) = \frac{9}{5}C+32.$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Example $1.6.8$**: Solving to Find an Inverse Function**

$$\begin{aligned} y &= \frac{2}{x-3}+4 && (\text{starting with the original equation}) \\ \Longrightarrow \quad y-4 &= \frac{2}{x-3} && (\text{subtracting } 4 \text{ from both sides}) \\ \Longrightarrow \quad (x-3)(y-4) &= 2 && (\text{multiplying both sides by } (x-3)) \\ \Longrightarrow \quad x-3 &= \frac{2}{y-4} && (\text{dividing both sides by } (y-4)) \\ \Longrightarrow \quad x &= \frac{2}{y-4}+3 && (\text{adding } 3 \text{ to both sides}) \end{aligned}$$

So $f^{-1}(y) = \frac{2}{y-4} + 3$ or $f^{-1}(x) = \frac{2}{x-4} + 3$.

Example $1.6.9$**: Solving to Find an Inverse with Radicals**

$$\begin{aligned} y &= 2+\sqrt{x-4} && (\text{starting with the original equation}) \\ \Longrightarrow \quad y-2 &= \sqrt{x-4} && (\text{subtracting } 2 \text{ from both sides}) \\ \Longrightarrow \quad (y-2)^2 &= x-4 && (\text{squaring both sides}) \\ \Longrightarrow \quad (y-2)^2+4 &= x && (\text{adding } 4 \text{ to both sides}) \end{aligned}$$

So $f^{-1}(x) = (x-2)^2 + 4$.

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$.

Example 1.6.10**: Finding the Inverse of a Function Using Reflection about the Identity Line**

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 1.6.10.

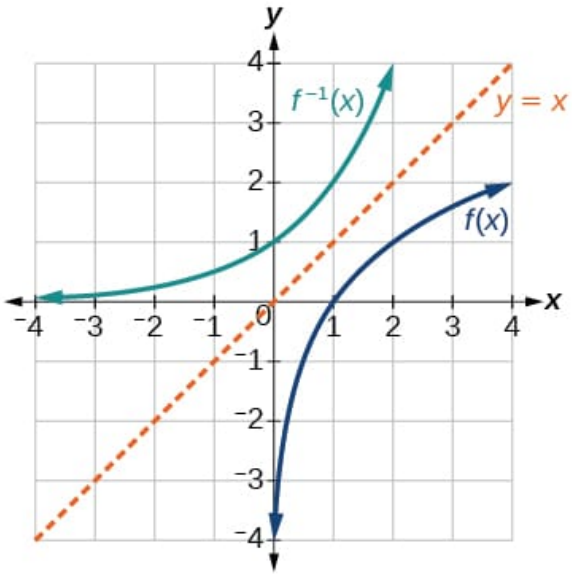


Figure 1.6.10**:** *The function and its inverse, showing reflection about the identity line*

Example 1.6.11**: Restricting the Domain to Find the Inverse of a Polynomial Function**

The original function $f(x) = (x - 4)^2$ is not one-to-one, but the function is restricted to a domain of $x \geq 4$ or $x \leq 4$ on which it is one-to-one. See Figure 1.6.11.

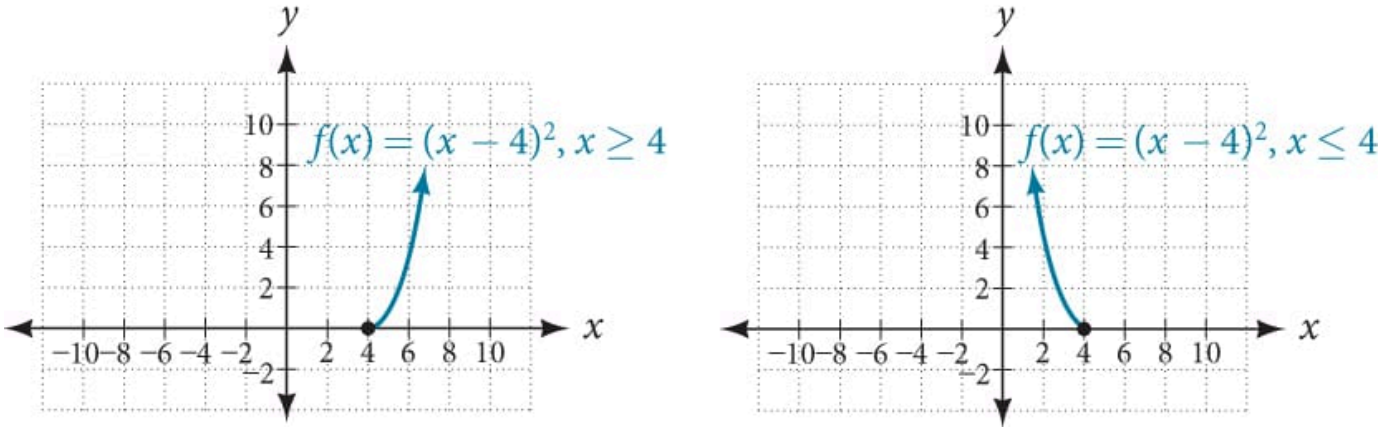


Figure 1.6.11

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$$\begin{aligned} y &= (x-4)^2 && \text{(letting } y = f(x)\text{)} \\ \Rightarrow \quad \pm\sqrt{y} &= x-4 && \text{(Extraction of Roots)} \\ \Rightarrow \quad 4 \pm \sqrt{y} &= x && \text{(adding 4 to both sides)} \\ \Rightarrow \quad 4 \pm \sqrt{x} &= y && \text{(exchanging } y \text{ and } x\text{)} \end{aligned}$$

This is not a function as written. We need to examine the restrictions on the domain of the original function to determine the inverse. Since we reversed the roles of x and y for the original $f(x)$, we need to look at the domain of the original function. When we reversed the roles of x and y, this gave us the values y could assume. For this function, $x \geq 4$, so for the inverse, we should have $y \geq 4$, which is what our inverse function gives.

a. The domain of the original function was restricted to $x \geq 4$, so the outputs of the inverse need to be the same, $f(x) \geq 4$, and we must use the "+" case:

$$f^{-1}(x) = 4 + \sqrt{x}.$$

b. The domain of the original function was restricted to $x \leq 4$, so the outputs of the inverse need to be the same, $f(x) \leq 4$, and we must use the "-" case:

$$f^{-1}(x) = 4 - \sqrt{x}.$$

Example 1.6.12**: Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified**

We can see this is a parabola with vertex at $(2, -3)$ that opens upward. Because the graph will be decreasing on one side of the vertex and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we repeat what we did in Example 1.6.11.

$$\begin{aligned} y &= (x-2)^2-3 && \text{(substituting } y=f(x)\text{)} \\ \Rightarrow \quad y+3 &= (x-2)^2 && \text{(adding 3 to both sides)} \\ \Rightarrow \quad \pm\sqrt{y+3} &= x-2 && \text{(Extraction of Roots)} \\ \Rightarrow \quad 2\pm\sqrt{y+3} &= x && \text{(adding 2 to both sides)} \\ \Rightarrow \quad 2\pm\sqrt{x+3} &= y && \text{(interchanging } x \text{ and } y\text{)} \end{aligned}$$

Now we need to determine which case to use. Because we restricted our original function to a domain of $x \geq 2$, the outputs of the inverse should be the same, telling us to utilize the "+" case. Therefore,

$$f^{-1}(x) = 2+\sqrt{x+3}.$$

If the quadratic had not been given in vertex form, rewriting it into vertex form would be the first step. This way we may easily observe the coordinates of the vertex to help us restrict the domain.

Example $1.6.13$**: Solving an Application with a Cubic Function**

Start with the given function for V. Notice that the meaningful domain for the function is $r \geq 0$ since negative radii would not make sense in this context. Also note the range of the function (hence, the domain of the inverse function) is $V \geq 0$. Solve for r in terms of V, using the method outlined previously.

$$\begin{aligned} V &= \frac{2}{3}\pi r^3 \\ \Rightarrow \quad \frac{3}{2}V &= \pi r^3 \\ \Rightarrow \quad \frac{3}{2\pi}V &= r^3 \\ \Rightarrow \quad \sqrt[3]{\frac{3}{2\pi}V} &= r \end{aligned}$$

This is the result stated in the section opener. Now evaluate this for $V = 100$ and $\pi \approx 3.14$.

$$\begin{aligned} r &= \sqrt[3]{\frac{3V}{2\pi}} \\ &\approx \sqrt[3]{\frac{3(100)}{2(3.14)}} \\ &\approx \sqrt[3]{47.7707} \\ &\approx 3.63 \end{aligned}$$

Therefore, the radius is about 3.63 ft.

Example $1.6.14$**: Finding the Inverse of a Rational Function**

We first want the inverse of the function.

$$\begin{aligned} C &= \frac{20+0.4n}{100+n} \\ \Rightarrow \quad C(100+n) &= 20+0.4n \\ \Rightarrow \quad 100C+Cn &= 20+0.4n \\ \Rightarrow \quad 100C-20 &= 0.4n-Cn \\ \Rightarrow \quad 100C-20 &= (0.4-C)n \\ \Rightarrow \quad \frac{100C-20}{0.4-C} &= n \end{aligned}$$

Now evaluate this function for $C = 0.35$ (35%).

$$n = \frac{100(0.35)-20}{0.4-0.35} = \frac{15}{0.05} = 300.$$

We can conclude that 300 mL of the 40% solution should be added.

This page titled 1.6: Inverse Functions is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

1.6.1: Resources and Key Concepts

Resources

Videos

- Inverse function (concept)
  - Intermediate Algebra - Functions: The Concept of Inverse Functions
- $f^{-1}(x)$ notation
  - Intermediate Algebra - Functions: Inverse Function Notation
- Verifying inverse functions
  - Inverse Functions - Verifying That Two Functions Are Inverse Functions
- One-to-one function
  - One-to-One Functions and Their Inverses - A Brief Review for Precalculus
- Domain and Range of inverse functions
  - Inverse Functions - Finding the Domain and Range of Inverse Functions
- Finding inverse from a formula
  - Inverse Functions - Finding Inverse Functions and Their Graphs
  - Inverse Functions - Finding and Evaluating Inverse Functions
  - Intermediate Algebra - Functions: Finding an Equation for the Inverse Function
- Graph of inverse function
  - Intermediate Algebra - Functions: Graphing the Inverse Function

Key Concepts

Definitions

- **Inverse Function:** For a one-to-one function $y = f(x)$, a function $f^{-1}(x)$ is an inverse function of f if $f^{-1}(f(x)) = x$ for all x in the domain of f, and $f(f^{-1}(y)) = y$ for all y in the domain of f^{-1} (which is the range of f). The input of the original function becomes the output of the inverse function, and the output of the original function becomes the input of the inverse function.
- **Inverse Function Notation (f^{-1}):** The notation used to represent the inverse of a function f. It is read as "f inverse."
- **Identity Line:** The line $y = x$. The graph of a function and its inverse are reflections of each other across this line.

Theorems

- **Inverse Function Property:** If f^{-1} is the inverse of f, then $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}. (And $f^{-1}(f(x)) = x$ for all x in the domain of f).
- **Domain and Range of Inverse Functions:** The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$, and the domain of $f(x)$ is the range of $f^{-1}(x)$.
- **Condition for Existence of an Inverse Function:** In order for a function to have an inverse, it must be a one-to-one function.
- **Forcing Invertibility by Restricting Domain**: If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

Common Mistakes

- **Confusing Inverse Function Notation with Reciprocal:** The notation $f^{-1}(x)$ for an inverse function does not mean $\frac{1}{f(x)}$. These are generally different concepts.
- **Assuming All Functions Have Inverses:** Not all functions have inverses. A function must be one-to-one to have an inverse function.
- **Forgetting to Restrict the Domain for Non-One-to-One Functions:** When finding an inverse for a function that is not naturally one-to-one (like $f(x) = x^2$), failing to restrict the domain of the original function appropriately will result in an inverse

relation that is not a function.

- **Incorrectly Determining the Domain of the Inverse Function:** The domain of f^{-1} is the range of f. Sometimes, after algebraically finding a formula for $f^{-1}(x)$, the apparent domain of this new formula might be larger than the actual range of f. The domain of f^{-1} must be restricted to match the range of f.
- **Forgetting to Restrict the Domain for Non-One-to-One Functions**: Attempting to find an inverse for a function like $f(x) = x^2$ without first restricting its domain to an interval where it is one-to-one. This leads to an inverse relation that is not a function (e.g., $y = \pm\sqrt{x}$).
- **Incorrectly Choosing the Branch of the Inverse**: After restricting the domain of a function (e.g., a quadratic) and finding an inverse involving a square root, selecting the wrong sign (positive or negative) for the square root, which would not correspond to the original restricted domain.
- **Not Verifying if the Original Function is One-to-One**: Proceeding to find an inverse without first confirming that the function (or its restricted version) is one-to-one, for example, by using the Horizontal Line Test.
- **Domain/Range Confusion for Inverse Functions**: Incorrectly stating the domain or range of the inverse function. Remember, the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

This page titled 1.6.1: Resources and Key Concepts is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

1.6.2: Homework

Reading Questions

1. What does it mean for a function f^{-1} to be the inverse of a function f?
2. How can you algebraically verify if two functions, $f(x)$ and $g(x)$, are inverses of each other?
3. Is the notation $f^{-1}(x)$ the same as $\frac{1}{f(x)}$? Explain.
4. What condition must a function satisfy to have an inverse function? How can this be checked graphically?
5. If a function is not one-to-one, what can sometimes be done to allow an inverse function to be found?
6. How are the domain and range of a function f related to the domain and range of its inverse f^{-1}?
7. If the point (a, b) is on the graph of a one-to-one function f, what corresponding point must be on the graph of f^{-1}?
8. Describe the graphical relationship between a function and its inverse. What line is involved?
9. Outline the algebraic steps to find the formula for an inverse function $f^{-1}(x)$ if you are given the formula for $f(x)$.
10. If $f(x) = \sqrt{x}$ has the domain $[0, \infty)$, its inverse is $f^{-1}(x) = x^2$. What must be the domain of this inverse function $f^{-1}(x)$, and why?
11. Can a function have more than one inverse function over the same domain?
12. If you are given a table of values for a one-to-one function $f(t)$ showing distance traveled in t minutes, how would you find $f^{-1}(d)$ for a given distance d? What would $f^{-1}(d)$ represent?
13. Name a toolkit function (other than the identity function) that is its own inverse.
14. Describe why the Horizontal Line Test is an effective way to determine whether a function is one-to-one?
15. Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?
16. Can a function be its own inverse? Explain.
17. Are one-to-one functions either always increasing or always decreasing? Why or why not?
18. How do you find the inverse of a function algebraically?
19. Why is it necessary to restrict the domain of a function like $f(x) = x^2$ before finding its inverse?
20. If you restrict the domain of $f(x) = (x-2)^2 - 3$ to $x \geq 2$, what will be the formula for its inverse $f^{-1}(x)$? What if you restrict the domain to $x \leq 2$?
21. When finding the inverse of a quadratic function, such as $y = ax^2$, you might arrive at $\pm\sqrt{\frac{y}{a}} = x$. How do you decide whether to use the positive or negative root for the inverse function?
22. What is the relationship between the graph of a function and the graph of its inverse with respect to the line $y = x$?
23. Outline the general steps to find the inverse of a polynomial function whose domain has been appropriately restricted.
24. Outline the general steps to find the inverse of a rational function of the form $f(x) = \frac{ax+b}{cx+d}$.
25. If $f(x) = 5x^3 + 1$, and you find its inverse $f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$, how can you verify that these functions are indeed inverses?
26. In the gravel mound problem (Example 5), why is the domain of the volume function $V(r) = \frac{2}{3}\pi r^3$ considered to be $r \geq 0$? What does this imply for the domain of its inverse function?
27. Can all polynomial functions have an inverse if their domain is restricted? Explain why or why not, considering a constant function as an example.
28. When solving for n in terms of C in Example 6 for the function $C = \frac{20+0.4n}{100+n}$, what algebraic steps are involved in isolating n?
29. Explain why we cannot find inverse functions for all polynomial functions.
30. Why must we restrict the domain of a quadratic function when finding its inverse?
31. When finding the inverse of a radical function, what restriction will we need to make?
32. The inverse of a quadratic function will always take what form?

Homework

Algebraic

33. Show that the function $f(x) = a - x$ is its own inverse for all real numbers a.

For the following exercises, find $f^{-1}(x)$ for each function.

34. $f(x) = x + 3$

35. $f(x) = x + 5$
36. $f(x) = 2 - x$
37. $f(x) = 3 - x$
38. $f(x) = \frac{x}{x+2}$
39. $f(x) = \frac{2x+3}{5x+4}$

For the following exercises, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

40. $f(x) = (x + 7)^2$
41. $f(x) = (x - 6)^2$
42. $f(x) = x^2 - 5$
43. Given $f(x) = \frac{x}{2+x}$ and $g(x) = \frac{2x}{1-x}$:
    a. Find $f(g(x))$ and $g(f(x))$.
    b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

For the following exercises, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

44. $f(x) = \sqrt[3]{x - 1}$ and $g(x) = x^3 + 1$
45. $f(x) = -3x + 5$ and $g(x) = \frac{x-5}{-3}$

For the following exercises, find the inverse of the function on the given domain.

46. $f(x) = (x - 4)^2$, $[4, \infty)$
47. $f(x) = (x + 2)^2$, $[-2, \infty)$
48. $f(x) = (x + 1)^2 - 3$, $[-1, \infty)$
49. $f(x) = 2 - \sqrt{3 + x}$, $[-3, \infty)$
50. $f(x) = 3x^2 + 5$, $(-\infty, 0]$
51. $f(x) = 12 - x^2$, $[0, \infty)$
52. $f(x) = 9 - x^2$, $[0, \infty)$
53. $f(x) = 2x^2 + 4$, $[0, \infty)$

For the following exercises, find the inverse of the functions.

54. $f(x) = x^3 + 5$
55. $f(x) = 3x^3 + 1$
56. $f(x) = 4 - x^3$
57. $f(x) = 4 - 2x^3$

For the following exercises, find the inverse of the functions.

58. $f(x) = \sqrt{2x + 1}$
59. $f(x) = \sqrt{3 - 4x}$
60. $f(x) = 9 + \sqrt{4x - 4}$
61. $f(x) = \sqrt{6x - 8} + 5$
62. $f(x) = 9 + 2\sqrt[3]{x}$
63. $f(x) = 3 - \sqrt[3]{x}$
64. $f(x) = \frac{2}{x+8}$
65. $f(x) = \frac{3}{x-4}$
66. $f(x) = \frac{x+3}{x+7}$
67. $f(x) = \frac{x-2}{x+7}$
68. $f(x) = \frac{3x+4}{5-4x}$
69. $f(x) = \frac{5x+1}{2-5x}$
70. $f(x) = x^2 + 2x$, $[-1, \infty)$
71. $f(x) = x^2 + 4x + 1$, $[-2, \infty)$
72. $f(x) = x^2 - 6x + 3$, $[3, \infty)$

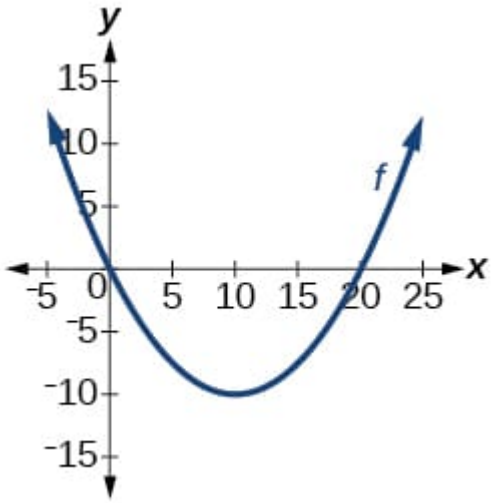
Graphical

For the following exercises, use graphing technology to determine whether each function is one-to-one.

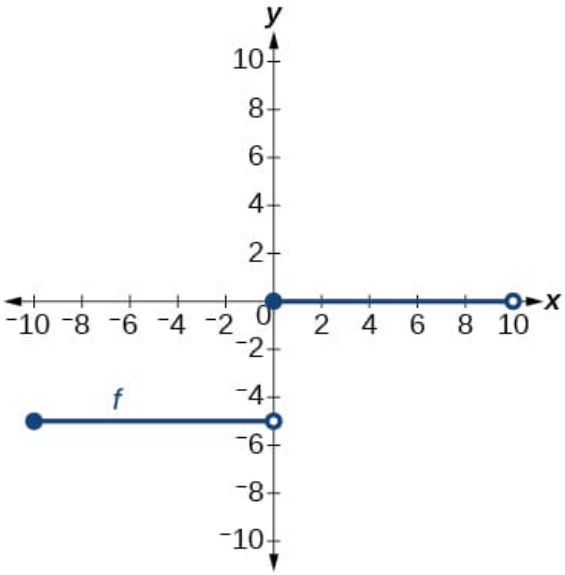
73. $f(x) = \sqrt{x}$
74. $f(x) = \sqrt[3]{3x+1}$
75. $f(x) = -5x + 1$
76. $f(x) = x^3 - 27$

For the following exercises, determine whether the graph represents a one-to-one function.

77.

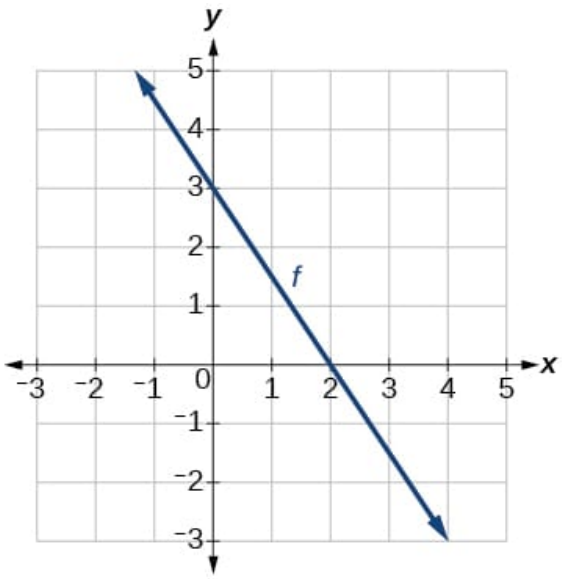


78.



For the following exercises, use the graph of f shown in Figure $1.6.2.1$.

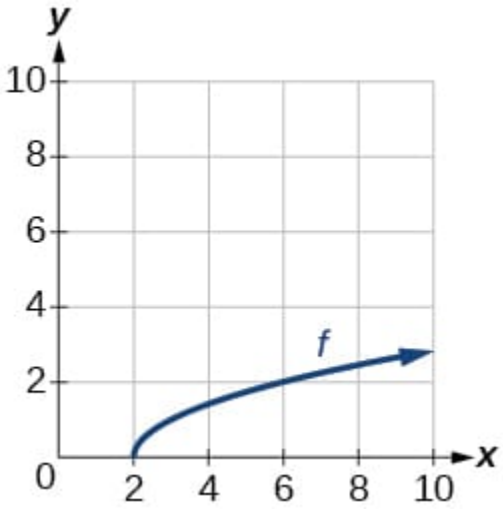
Figure 1.6.2.1



79. Find $f(0)$.
80. Solve $f(x) = 0$.
81. Find $f^{-1}(0)$.
82. Solve $f^{-1}(x) = 0$.

For the following exercises, use the graph of the one-to-one function shown in Figure 1.6.2.2**.**

Figure 1.6.2.2



83. Sketch the graph of f^{-1}.
84. Find $f(6)$ and $f^{-1}(2)$.
85. If the complete graph of f is shown, find the domain of f.
86. If the complete graph of f is shown, find the range of f.

For the following exercises, find the inverse of the function and graph both the function and its inverse.

87. $f(x) = x^2 + 2, x \geq 0$
88. $f(x) = 4 - x^2, x \geq 0$
89. $f(x) = (x + 3)^2, x \geq -3$

90. $f(x) = (x-4)^2$, $x \geq 4$
91. $f(x) = x^3 + 3$
92. $f(x) = 1 - x^3$
93. $f(x) = x^2 + 4x$, $x \geq -2$
94. $f(x) = x^2 - 6x + 1$, $x \geq 3$
95. $f(x) = \frac{2}{x}$
96. $f(x) = \frac{1}{x^2}$, $x \geq 0$

For the following exercises, use a graph to help determine the domain of the functions.

97. $f(x) = \sqrt{\frac{(x+1)(x-1)}{x}}$
98. $f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$
99. $f(x) = \sqrt{\frac{x(x+3)}{x-4}}$
100. $f(x) = \sqrt{\frac{x^2-x-20}{x-2}}$
101. $f(x) = \sqrt{\frac{9-x^2}{x+4}}$

Numeric

For the following exercises, evaluate or solve, assuming that the function f is one-to-one.

102. If $f(6) = 7$, find $f^{-1}(7)$.
103. If $f(3) = 2$, find $f^{-1}(2)$.
104. If $f^{-1}(-4) = -8$, find $f(-8)$.
105. If $f^{-1}(-2) = -1$, find $f(-1)$.

For the following exercises, use the values listed in Table 1.6.2.1 to evaluate or solve.

Table 1.6.2.1

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	8	0	7	4	2	6	5	3	9	1

106. Find $f(1)$.
107. Solve $f(x) = 3$.
108. Find $f^{-1}(0)$.
109. Solve $f^{-1}(x) = 7$.
110. Use the tabular representation of f in Table 1.6.2.2 to create a table for $f^{-1}(x)$.

Table 1.6.2.2

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

Technology

For the following exercises, find the inverse function. Then, graph the function and its inverse.

111. $f(x) = \frac{3}{x-2}$
112. $f(x) = x^3 - 1$
113. Find the inverse function of $f(x) = \frac{1}{x-1}$. Use graphing technology to find its domain and range. Write the domain and range in interval notation.

For the following exercises, use a calculator to graph the function. Then, using the graph, give three points on the graph

of the inverse with y-coordinates given.

14. $f(x) = x^3 - x - 2$, $y = 1, 2, 3$
15. $f(x) = x^3 + x - 2$, $y = 0, 1, 2$
16. $f(x) = x^3 + 3x - 4$, $y = 0, 1, 2$
17. $f(x) = x^3 + 8x - 4$, $y = -1, 0, 1$
18. $f(x) = x^4 + 5x + 1$, $y = -1, 0, 1$

Extensions

For the following exercises, find the inverse of the functions with a, b, c positive real numbers.

19. $f(x) = ax^3 + b$
20. $f(x) = x^2 + bx$
21. $f(x) = \sqrt{ax^2 + b}$
22. $f(x) = \sqrt[3]{ax + b}$
23. $f(x) = \frac{ax+b}{x+c}$

Real-World Applications

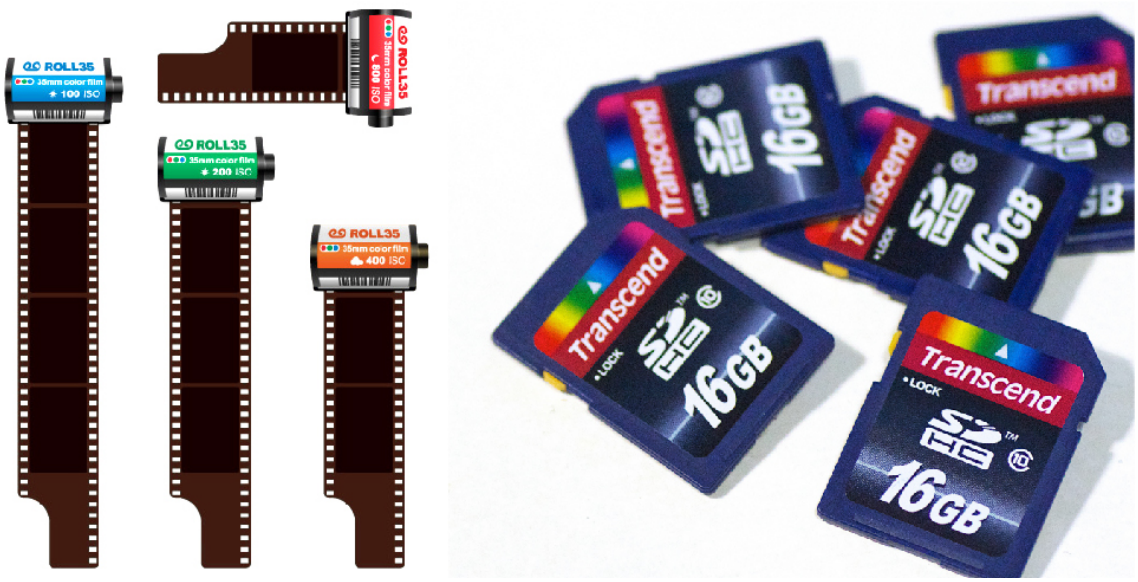
24. To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x) = \frac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.
25. The circumference C of a circle is a function of its radius given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.
26. A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t, in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

For the following exercises, determine the function described and then use it to answer the question.

27. An object dropped from a height of 200 meters has a height, $h(t)$, in meters after t seconds have lapsed, such that $h(t) = 200 - 4.9t^2$. Express t as a function of height, h, and find the time to reach a height of 50 meters.
28. An object dropped from a height of 600 feet has a height, $h(t)$, in feet after t seconds have elapsed, such that $h(t) = 600 - 16t^2$. Express t as a function of height h, and find the time to reach a height of 400 feet.
29. The volume, V, of a sphere in terms of its radius, r, is given by $V(r) = \frac{4}{3}\pi r^3$. Express r as a function of V, and find the radius of a sphere with volume of 200 cubic feet.
30. The surface area, A, of a sphere in terms of its radius, r, is given by $A(r) = 4\pi r^2$. Express r as a function of A, and find the radius of a sphere with a surface area of 1000 square inches.
31. A container holds 100 ml of a solution that is 25 ml acid. If n ml of a solution that is 60% acid is added, the function $C(n) = \frac{25+.6n}{100+n}$ gives the concentration, C, as a function of the number of ml added, n. Express n as a function of C and determine the number of mL that need to be added to have a solution that is 50% acid.
32. The period T, in seconds, of a simple pendulum as a function of its length l, in feet, is given by $T(l) = 2\pi\sqrt{\frac{l}{32.2}}$. Express l as a function of T and determine the length of a pendulum with period of 2 seconds.
33. The volume of a cylinder , V, in terms of radius, r, and height, h, is given by $V = \pi r^2 h$. If a cylinder has a height of 6 meters, express the radius as a function of V and find the radius of a cylinder with volume of 300 cubic meters.
34. The surface area, A, of a cylinder in terms of its radius, r, and height, h, is given by $A = 2\pi r^2 + 2\pi rh$. If the height of the cylinder is 4 feet, express the radius as a function of A and find the radius if the surface area is 200 square feet.
35. The volume of a right circular cone, V, in terms of its radius, r, and its height, h, is given by $V = \frac{1}{3}\pi r^2 h$. Express r in terms of V if the height of the cone is 12 inches and find the radius of a cone with volume of 50 cubic inches.
36. Consider a cone with height of 30 feet. Express the radius, r, in terms of the volume, V, and find the radius of a cone with volume of 1000 cubic feet.

This page titled 1.6.2: Homework is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

2: POLYNOMIAL AND RATIONAL FUNCTIONS



35-mm film, once the standard for capturing photographic images, has been made largely obsolete by digital photography. (credit "film": modification of work by Horia Varlan; credit "memory cards": modification of work by Paul Hudson)

Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by Mathematics. An image becomes a series of numbers, representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects. Inverse functions make it possible to convert from one file format to another. In this chapter, we will learn about these concepts and discover how Mathematics can be used in such applications.

2.1: Quadratic Functions

2.1.1: Resources and Key Concepts

2.1.2: Homework

2.2: Power Functions and Polynomial Functions

2.2.1: Resources and Key Concepts

2.2.2: Homework

2.3: Graphs of Polynomial Functions

2.3.1: Resources and Key Concepts

2.3.2: Homework

2.4: Dividing Polynomials

2.4.1: Resources and Key Concepts

2.4.2: Homework

2.5: Rational Functions

2.5.1: Resources and Key Concepts

2.5.2: Homework

This page titled 2: Polynomial and Rational Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- 3: Polynomial and Rational Functions by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

2.1: Quadratic Functions

Note to the Instructor (click to expand)

This *very dense* sections covers everything the student will need to know about quadratic functions. Plan to spend a bit of time here as the topics go through graphing quadratics, solving equations that are quadratic-in-form, and solving quadratic inequalities. Again, this is a *very* dense section.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Solving Equations**
  - Solving Absolute Value Equations: The section extends this to solving absolute value equations involving quadratic expressions, like $|x^2 - 5x| = 6$ in Example 15, which builds upon basic absolute value equation solving.
- **Quadratic Functions**
  - Vertex Form and Completing the Square: This section uses vertex form $f(x) = a(x - h)^2 + k$ extensively and mentions completing the square as a method to convert from general to vertex form (Example 4). Prior knowledge is assumed.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve equations that are

Hawk A.I. Section-Specific Tutor

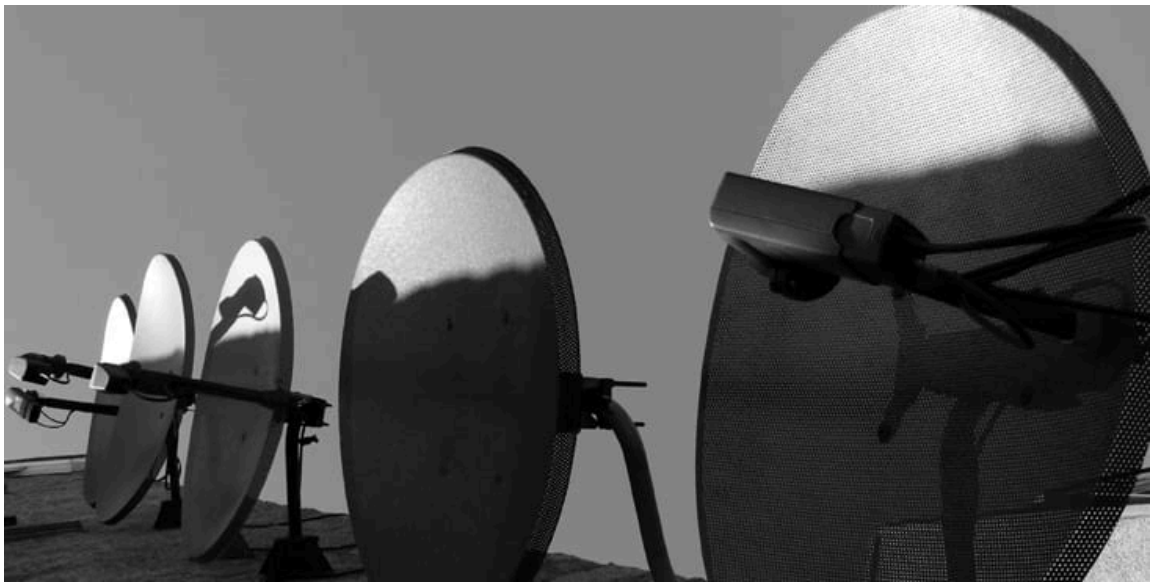


To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

quadratic-in-form. • Solve a quadratic inequality using the graphical and sign chart methods.	

Figure 2.1.1

An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)



Curved antennas, such as the ones shown in Figure 2.1.1, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

It is assumed that you have already worked with quadratic functions in your prerequisite Algebra course. Therefore, a lot of the theory is mentioned in brief.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a **parabola**.

Definition: Parabola

13.02: Graphing Quadratic Functions in General Form

The y-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of x at which $y = 0$.

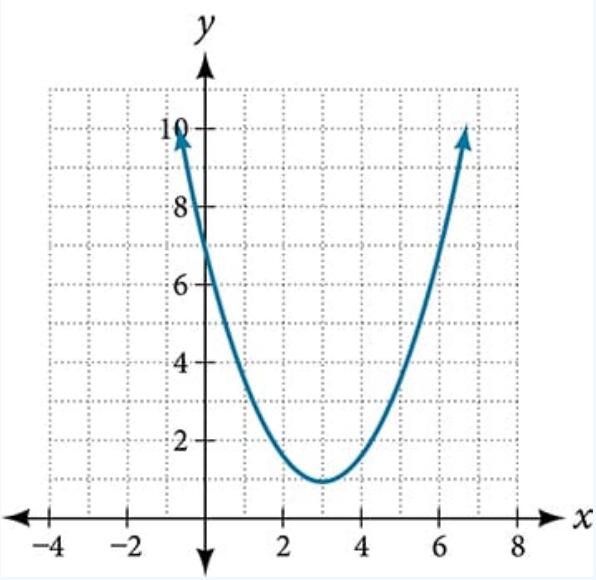
Definition: Zeros of Functions and Roots of Equations

10.01: Graphs of Functions

Example 2.1.1: Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure 2.1.2.

Figure 2.1.2



Solution available at the end of this section.

Quadratic Functions in General Form

Recall from Algebra that the parabola is the graph of a quadratic function. There are two forms for quadratic functions (and polynomials, in general) - **general form** and **standard form**. We begin our review with the general form of a quadratic function.

Definition: General Form of a Quadratic Function

13.02: Graphing Quadratic Functions in General Form

It was shown in your prerequisite Algebra course that the parabola opens upward if $a > 0$. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry. To do so, we first reintroduce a fundamental theorem in all Algebra courses.

Theorem: Quadratic Formula

6.6: Solving Quadratic Equations by Factoring

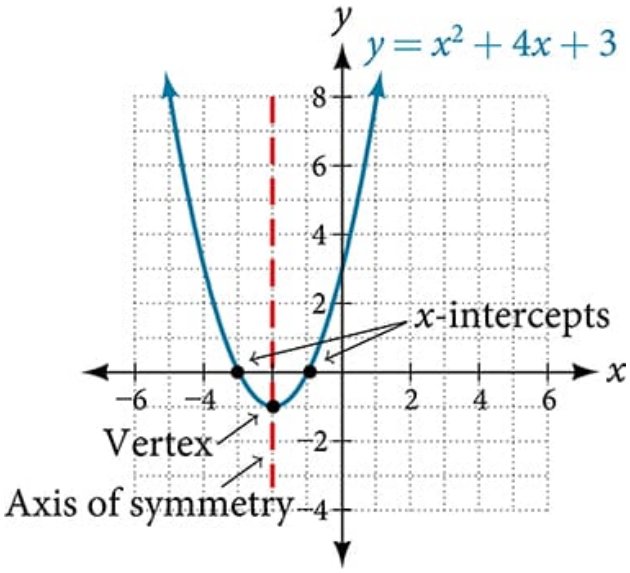
With the Quadratic Formula in hand, we now can find the equation for the axis of symmetry.

Theorem: Axis of Symmetry for a Quadratic Function

13.02: Graphing Quadratic Functions in General Form

Figure 2.1.3 represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense, because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

Figure 2.1.3



Since the vertex always occurs where the axis of symmetry intersects the parabola, we immediately get the following theorem.

Theorem: Coordinates of the Vertex

13.02: Graphing Quadratic Functions in General Form

The general form of a quadratic function is commonly encountered in applications and Physics. While most Algebra textbooks focus on the second form (standard form), you should strive to understand both forms.

Quadratic Functions in Standard Form

The general form of a quadratic can be incredibly useful; however, we cannot escape the need to talk about a quadratic function's alternate identity - the standard form.

Definition: Standard Form of a Quadratic Function

13.01: Vertex Form and Completing the Square

The standard form of a quadratic function has a very appealing feature.

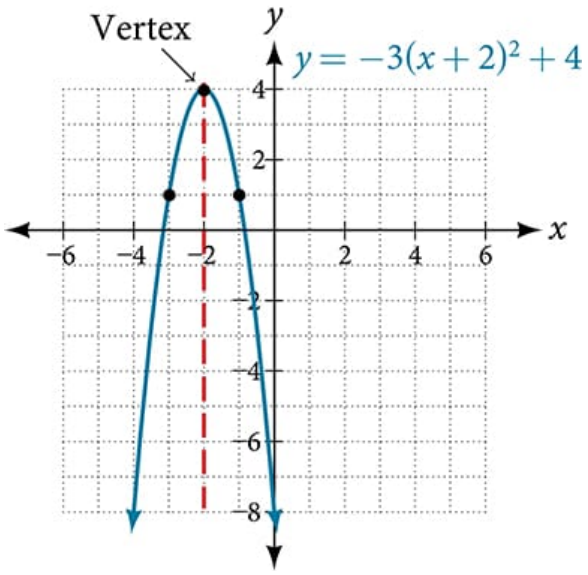
Theorem: Vertex of a Quadratic Function in Standard Form

13.01: Vertex Form and Completing the Square

That is, if given a quadratic function in standard form, you have little-to-no work involved in finding its vertex.

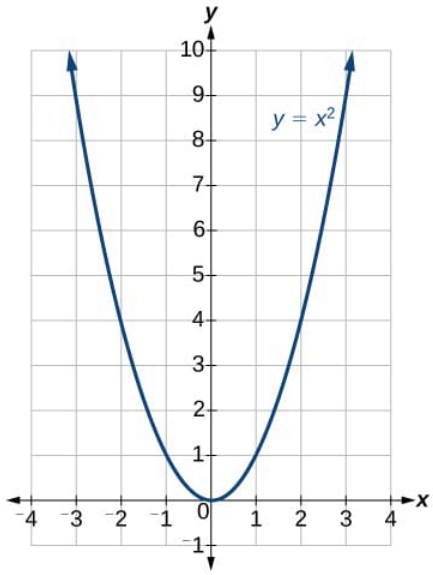
As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure $2.1.4$ represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x–h = x + 2$ in this example, $h = –2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

Figure 2.1.4



The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. Figure 2.1.5 is the graph of this basic function.

Figure 2.1.5



From our previous work with transformations of functions, if $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In Figure 2.1.4, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In Figure 2.1.4, $h < 0$, so the graph is shifted 2 units to the left. Again, from transformations, we know that the magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph *appears* to become narrower (this is an illusion), and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph *appears* to become wider (another illusion), but in fact there is a vertical compression. In Figure 2.1.4, $|a| > 1$, so the graph *appears* narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$\begin{aligned} a(x-h)^2+k &= ax^2+bx+c \\ \implies \quad ax^2-2ahx+(ah^2+k) &= ax^2+bx+c \end{aligned}$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah=b \implies h=-\frac{b}{2a}.$$

This is the axis of symmetry we defined earlier. Setting the constant terms equal, we get

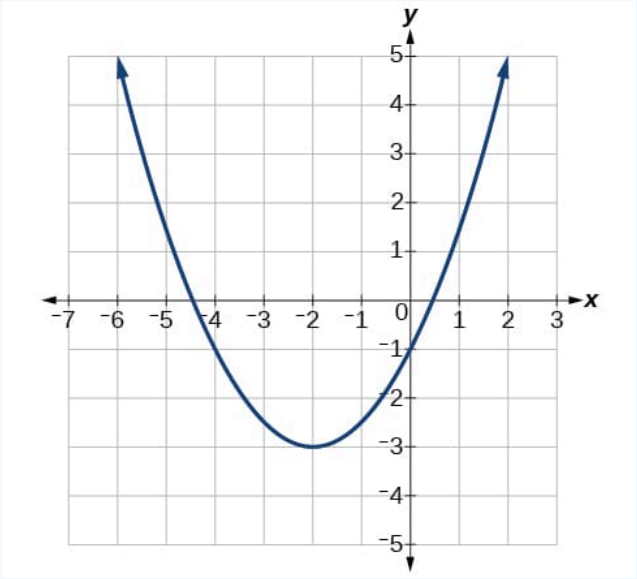
$$\begin{aligned} ah^2+k &= c \\ \implies \quad k &= c-ah^2 \\ &= c-a\left(-\frac{b}{2a}\right)^2 \\ &= c-\frac{b^2}{4a}. \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h)=k$.

Example 2.1.2: Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in Figure 2.1.6 as a transformation of $f(x)=x^2$, and then expand the formula and simplify terms to write the equation in general form.

Figure 2.1.6



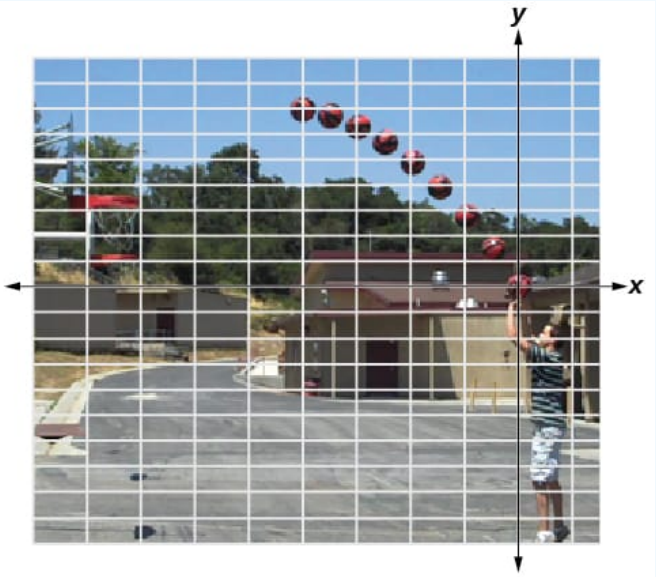
Solution available at the end of this section.

Checkpoint 2.1.2

A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 2.1.7. Find an equation for the path of the ball. Does the shooter make the basket?

Figure 2.1.7

(credit: modification of work by Dan Meyer)



Example 2.1.3: Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2–6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution available at the end of this section.

If given a quadratic function in general form, most students are going to use the formula $-\frac{b}{2a}$ to find the x-value of the vertex (and substitute this into the original quadratic function to find the corresponding y-value for the vertex). This is perfectly fine, and it will always lead you to the vertex of a given quadratic function; however, another approach, which is inevitably going to be necessary in this course, is to **complete the square** like you did in your Elementary Algebra course (and as we did in the proof of the Quadratic Formula).

Example 2.1.4: Rewriting in Standard Form Using Completing the Square

Rewrite the quadratic function $f(x) = 2x^2 - 6x + 7$ in standard (vertex) form by Completing the Square.

Solution available at the end of this section.

Important Skill: Completing the Square

The purpose of being introduced to the process of Completing the Square in your Algebra courses was to prove the Quadratic Formula. We now see that this same process can be used to rewrite a quadratic function in vertex form; however, these are not the only uses of Completing the Square. We will use this skill again in this course when working with conic sections. In Calculus II, you will find many opportunities to use this very same skill when manipulating a seemingly impossible-to-work-with expression into something more simple.

Checkpoint 2.1.4

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers; however, because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

Theorem: Domain and Range of a Quadratic Function

13.02: Graphing Quadratic Functions in General Form

The theorem above is only provided for completeness - it should not be memorized.

Example 2.1.5: Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution available at the end of this section.

Checkpoint 2.1.5

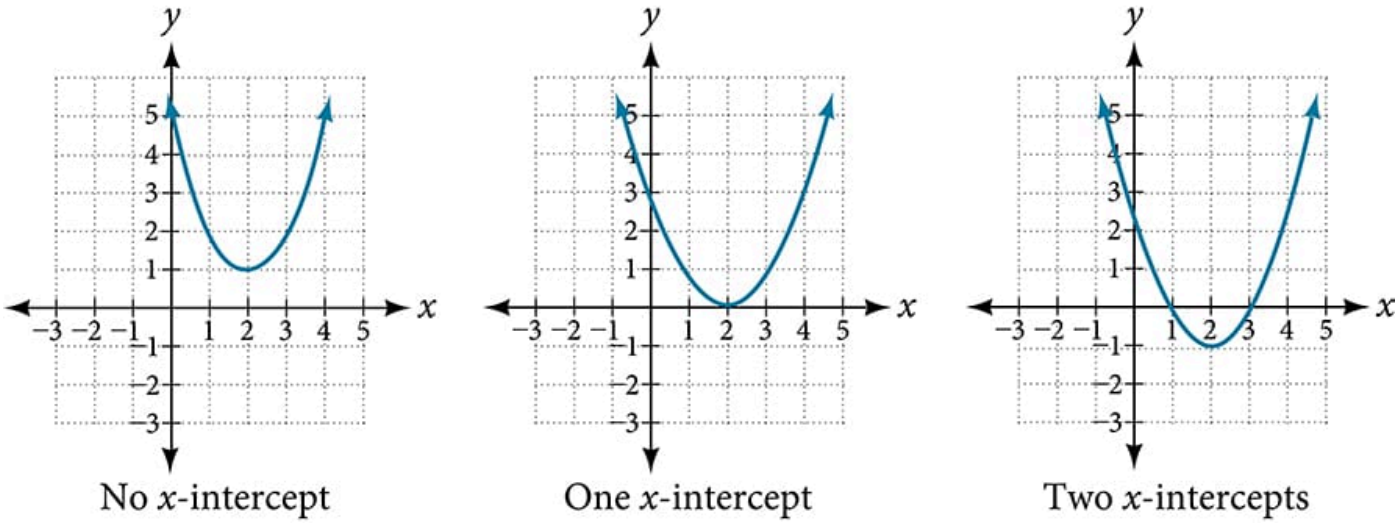
Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure 2.1.8 that the number of x-intercepts can vary depending upon the location of the graph.

Figure 2.1.8

Number of x-intercepts of a parabola



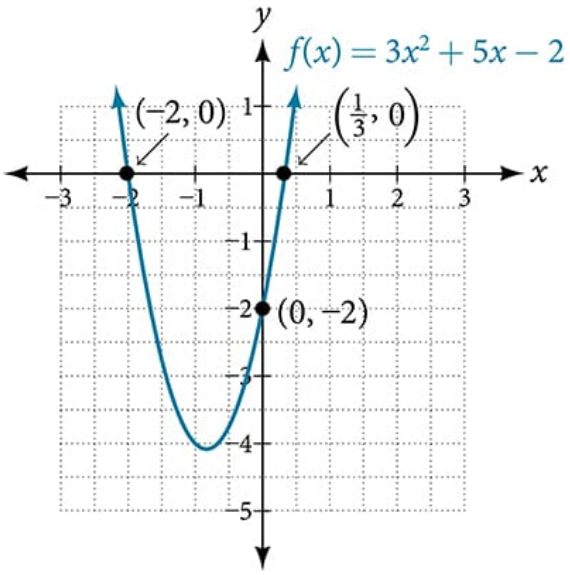
Example 2.1.6: Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution available at the end of this section.

By graphing the function, we can confirm that the graph crosses the y-axis at $(0, -2)$. We can also confirm that the graph crosses the x-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$. See Figure 2.1.9

Figure 2.1.9



Example 9 illustrates a strength of having the general form of a quadratic function - the y-intercept is the constant term. That is, given $f(x) = ax^2 + bx + c$, the y-intercept is $f(0) = c$. This is only true, however, when the quadratic is given in general form.

Another lesson to be learned from Example 2.1.6 is that to solve $ax^2 + bx + c = 0$, we don't have to always resort to the Quadratic Formula. In fact, whenever you need to solve a quadratic equation, you should spend about 30 seconds or so trying to factor the given quadratic before plunging into the algorithm called the Quadratic Formula.

As Figure 2.1.8 illustrates, there are some quadratic functions that do not have x-intercepts. Algebraically, this occurs when the given quadratic function is **prime**. Just like a prime number, a prime quadratic is not factorable (across the real number system). That is, no matter how ingenious we are, we will never be able to write a prime quadratic in the form $(c_1x + d_1)(c_2x + d_2)$ (where c_1, c_2, d_1, and d_2 are real numbers).

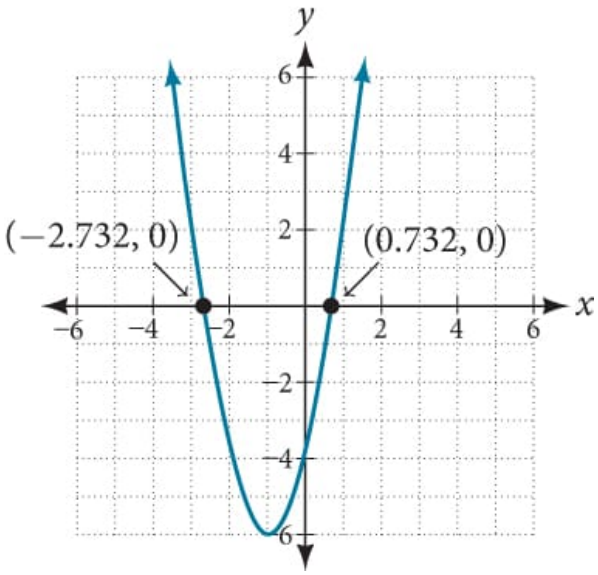
Example 2.1.7: Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution available at the end of this section.

Again, we can check our work for Example 2.1.7 by graphing the given function on a graphing utility and observing the x-intercepts. See Figure 2.1.10.

Figure 2.1.10



Checkpoint 2.1.7

In a separate Checkpoint, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the y- and x -intercepts (if any).

Solving Quadratic Inequalities

We now turn our attention to **quadratic inequalities**.

Definition: Quadratic Inequality

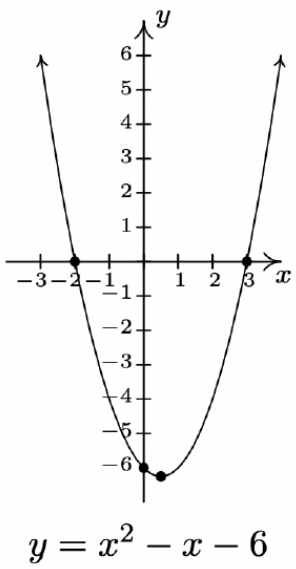
A **quadratic inequality** in x is an inequality that can be written in one of the forms

$$ax^2 + bx + c < 0, \quad ax^2 + bx + c \leq 0, \quad ax^2 + bx + c > 0, \quad \text{or} \quad ax^2 + bx + c \geq 0,$$

where a, b, and c are real numbers with $a \neq 0$.

If we consider $f(x) = x^2 - x - 6$, then solving $x^2 - x - 6 < 0$ corresponds graphically to finding the values of x for which the graph of $y = f(x) = x^2 - x - 6$ (the parabola) is below the graph of $y = 0$ (the x-axis). See Figure 2.1.11.

Figure 2.1.11



We can see that the graph of f dips below the x-axis between its two x-intercepts. The zeros of f are $x = -2$ and $x = 3$ in this case and they divide the domain (the x-axis) into three intervals: $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$. For every number in $(-\infty, -2)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty, -2)$. Similarly, $f(x) < 0$ for all x in $(-2, 3)$, and $f(x) > 0$ for all x in $(3, \infty)$. Thus, the solution to $x^2 - x - 6 < 0$ is all x in the interval $(-2, 3)$.

The method introduced above (graphing the quadratic function and determining when the graph satisfies the inequality) is called the **graphical method** of solving a quadratic inequality. This is the method I prefer to teach in my courses because it simultaneously reinforces graphing and it is *much* faster (in the long run) than the alternative method. Before discussing the alternative method, let's look at a couple examples.

Example 2.1.8: Solving Quadratic Inequalities Using the Graphing Method

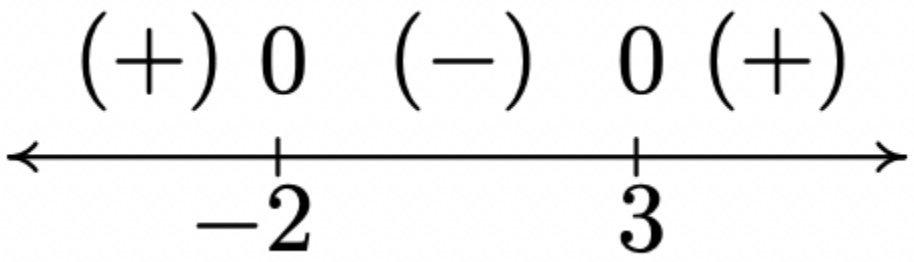
Solve the following inequalities using the graphing method.

a. $2x^2 \leq 3 - x$
b. $x^2 - 2x > 1$

Solutions available at the end of this section.

The alternative method is called the **sign chart method** of solving a quadratic inequality. Like the graphical method, we start by finding the x-intercepts. We then mark those intercepts on a number line. For example, considering our original inequality $x^2 - x - 6 < 0$, we would create the sign chart in Figure 2.1.12.

Figure 2.1.12



Here, the $(+)$ above a portion of the number line indicates $f(x) > 0$ for those values of x; the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the number line are the zeros of f, so we place 0 above them. We see at once that the solution to $f(x) < 0$ is $(-2, 3)$.

An important property of quadratic functions is that if the function is positive at one point and negative at another, the function must have at least one zero in between. Graphically, this means that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis. This allows us to determine the sign of *all* of the function values on a given interval by testing the function at just *one* value in the interval. We use this idea in the following example.

Example 2.1.9: Solving Quadratic Inequalities Using the Sign Chart Method

Solve the following inequalities analytically using sign diagrams. Verify your answer graphically.

a. $2x^2 \leq 3 - x$
b. $x^2 - 2x > 1$

Solutions available at the end of this section.

Applications Involving Quadratic Functions

It should come as no surprise that quadratic functions appear in a lot of applications - from business to physics. You will continue to encounter them in such contexts when you get to Calculus. Therefore, let's take a look at a couple common applications.

Example 2.1.10: Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

a. When does the ball reach the maximum height?
b. What is the maximum height of the ball?
c. When does the ball hit the ground?
d. When will the ball be above 104 feet?

Solutions available at the end of this section.

It's important to note that Figure 2.1.14 in Example 2.1.10 doesn't represent the path of the ball. We arrived at the graph in Figure 2.1.14 by manipulating an inequality based on the function for the path of the ball's travel.

Checkpoint 2.1.10

A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Focus on Calculus - Solving Equations that are Quadratic-in-Form

You have already solved quite a few quadratic equations throughout your academic journey in Mathematics; however, as you approach Calculus, you will need to focus your attention on pattern recognition instead of expecting equations identical in form to $ax^2 = bx + c = 0$. We must begin to recognize that

$$a\blacksquare^2 + b\blacksquare + c = 0$$

is nearly this same equation.

Definition: Quadratic-in-Form

An equation in the variable x is **quadratic-in-form** if it can be rewritten as

$$au^2 + bu + c = 0,$$

where a, b, and c are constants with $a \neq 0$, and u represents an algebraic expression in x.

It turns out that any equation that is quadratic-in-form can be solved using methods similar to those from our Elementary and Intermediate Algebra courses. Let's illustrate the process with a few examples.

Example 2.1.11: Solving an Equation that is Quadratic-in-Form

Solve $3(3x-1)^2 + 13(3x-1) - 10 = 0$.

Solution available at the end of this section.

You will be using the u-substitution process illustrated in Example 2.1.11 often throughout Calculus. As such, it's best to start looking for patterns like this from this point forward.

Example 2.1.12: Solving an Equation that is Quadratic-in-Form

Solve $x^4 - 3x^2 = 45 + x^2$.

Solution available at the end of this section.

Answers

Example 2.1.1: Identifying the Characteristics of a Parabola

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at $(0, 7)$, so this is the y-intercept.

Example 2.1.2: Writing the Equation of a Quadratic Function from the Graph

We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$. Substituting the coordinates of any point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$\begin{aligned} -1 &= a(0+2)^2 - 3 \\ \Longrightarrow \quad 2 &= 4a \\ \Longrightarrow \quad \frac{1}{2} &= a \end{aligned}$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x+2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} g(x) &= \frac{1}{2}(x+2)^2 - 3 \\ &= \frac{1}{2}(x+2)(x+2) - 3 \\ &= \frac{1}{2}(x^2+4x+4) - 3 \\ &= \frac{1}{2}x^2 + 2x + 2 - 3 \\ &= \frac{1}{2}x^2 + 2x - 1 \end{aligned}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Example 2.1.3: Finding the Vertex of a Quadratic Function

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} = -\frac{-6}{2(2)} = \frac{6}{4} = \frac{3}{2}.$$

The vertical coordinate of the vertex will be at

$$k = f(h) = f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 = \frac{5}{2}.$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic. Therefore, using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}.$$

One reason we may want to identify the vertex of the parabola is that this point will inform us where the maximum or minimum value of the output occurs (k) and where it occurs (x).

Example 2.1.4: Rewriting in Standard Form Using Completing the Square

Since Completing the Square is an older skill, we will just use it here rather than explaining the entire process.

$$\begin{aligned} f(x) &= 2x^2 - 6x + 7 \\ &= 2(x^2 - 3x) + 7 && \text{(factoring the lead coefficient from both variable terms)} \\ &= 2\left(x^2 - 3x + \left(\frac{-3}{2}\right)^2\right) + 7 - 2\left(\frac{-3}{2}\right)^2 && \left(\text{Completing the Square and offsetting the } 2\left(\frac{-3}{2}\right)^2 \text{ term with } -2\left(\frac{-3}{2}\right)^2\right) \\ &= 2\left(x - \frac{3}{2}\right)^2 + 7 - 2\left(\frac{9}{4}\right) && \text{(factoring and simplifying)} \\ &= 2\left(x - \frac{3}{2}\right)^2 + \frac{14}{2} - \frac{9}{2} && \text{(common denominator)} \\ &= 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2} && \text{(simplifying)} \end{aligned}$$

Example 2.1.5: Finding the Domain and Range of a Quadratic Function

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$h = -\frac{b}{2a} = -\frac{9}{2(-5)} = \frac{9}{10}.$$

The maximum value is given by $f(h)$.

$$f\left(\frac{9}{10}\right) = -5\left(\frac{9}{10}\right)^2 + 9\left(\frac{9}{10}\right) - 1 = \frac{61}{20}.$$

The range is $\left(-\infty, \frac{61}{20}\right]$.

Example 2.1.6: Finding the y- and x-Intercepts of a Parabola

We find the y-intercept by evaluating $f(0)$.

$$f(0) = 3(0)^2 + 5(0) - 2 = -2.$$

So the y-intercept is at $(0, -2)$.

For the x-intercepts, we find all solutions of $f(x) = 0$. That is, we want to solve

$$0 = 3x^2 + 5x - 2.$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = (3x - 1)(x + 2).$$

Thus, $x = \frac{1}{3}$ or $x = -2$. So the x-intercepts are at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.

Example 2.1.7**: Finding the x-Intercepts of a Parabola**

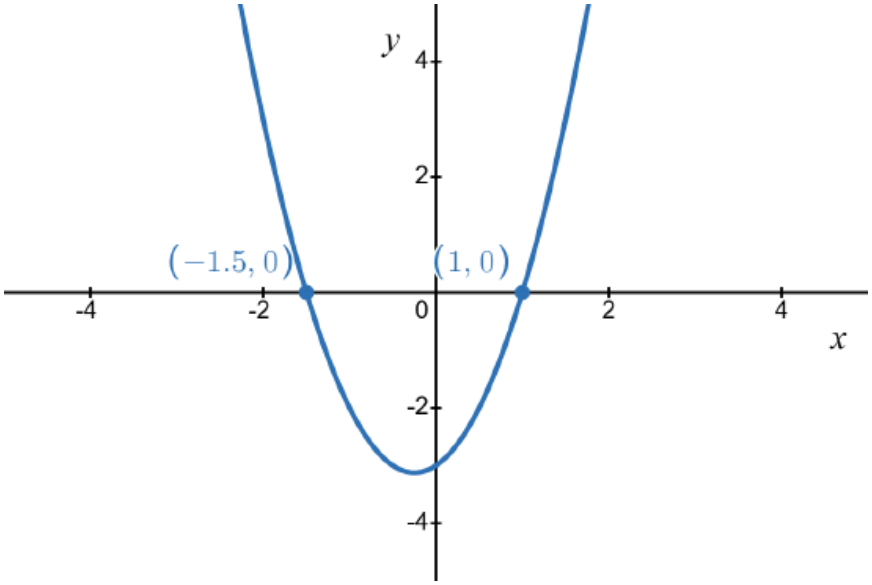
We begin by solving for when the output will be zero; however, because this quadratic is not easily factorable. Using the Quadratic Formula, we get the following:

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{(Quadratic Formula)} \\
&= \frac{-4 \pm \sqrt{4^2 - 4(2)(-4)}}{2(2)} && \text{(substituting)} \\
&= \frac{-4 \pm \sqrt{48}}{4} && \text{(simplifying)} \\
&= \frac{-4 \pm \sqrt{16 \cdot 3}}{4} \\
&= \frac{-4 \pm 4\sqrt{3}}{4} && \text{(simplifying the radical expression)} \\
&= \frac{4(-1 \pm \sqrt{3})}{4} && \text{(factoring out the GCF from the numerator)} \\
&= \frac{\cancel{4}^{1}(-1 \pm \sqrt{3})}{\cancel{4}^{1}} && \text{(canceling like factors)} \\
&= -1 \pm \sqrt{3}
\end{aligned}$$

The graph has x-intercepts at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$.

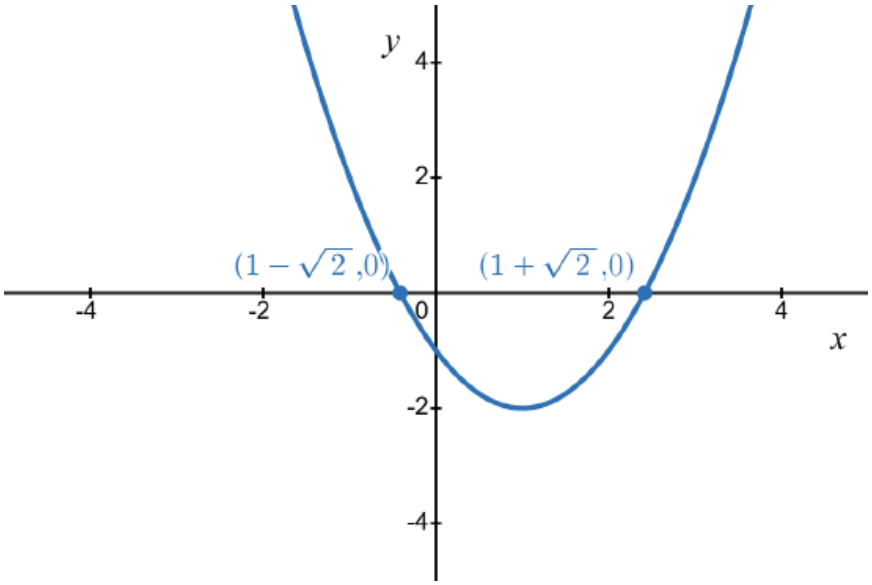
Example 2.1.8**: Solving Quadratic Inequalities Using the Graphing Method**

a. To solve $2x^2 \leq 3 - x$, we first get 0 on one side of the inequality which yields $2x^2 + x - 3 \leq 0$. Therefore, we want to know when the function $f(x) = 2x^2 + x - 3$ is at or below the x-axis. Since the lead coefficient is positive, we know the parabola representing f opens upward. We just need to find the zeros of f by solving $2x^2 + x - 3 = 0$ for x. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$ or $x = 1$. A simple and quick sketch yields the following:



By our sketch, we can see $f(x)$ is at or below the x-axis on the interval $\left[-\frac{3}{2}, 1\right]$. This is the solution to our inequality.

b. Once again, we rewrite $x^2 - 2x > 1$ as $x^2 - 2x - 1 > 0$ and we identify $f(x) = x^2 - 2x - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $x^2 - 2x - 1$ doesn't factor nicely. Hence, we resort to the Quadratic Formula to solve $x^2 - 2x - 1 = 0$, and arrive at $x = 1 \pm \sqrt{2}$. As before, our parabola is opening upward. We spend a moment sketching the graph of f and asking ourselves when the parabola is strictly above the x-axis (because we are solving $f(x) > 0$).



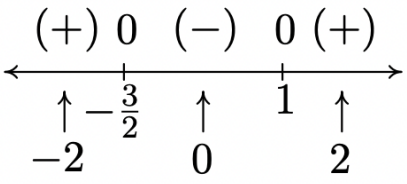
So, we see that the interval notation $\left(-\infty, 1-\sqrt{2}\right) \cup \left(1+\sqrt{2}, \infty\right)$ solves our inequality.

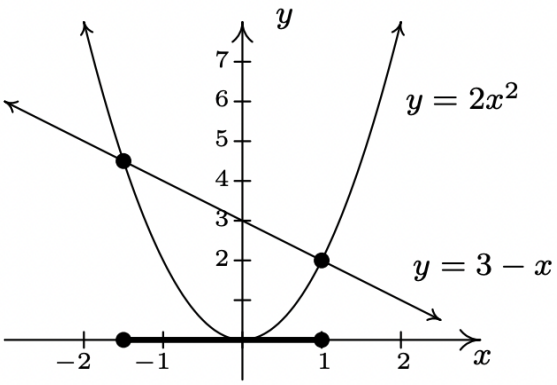
Example 2.1.9**: Solving Quadratic Inequalities Using the Sign Chart Method**

a. To solve $2x^2 \leq 3 - x$, we first get 0 on one side of the inequality which yields $2x^2 + x - 3 \leq 0$. We find the zeros of $f(x) = 2x^2 + x - 3$ by solving $2x^2 + x - 3 = 0$ for x. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$ or $x = 1$. We place these values on the number line with 0 above them and choose test values in the intervals $(-\infty, -\frac{3}{2})$, $(-\frac{3}{2}, 1)$ and $(1, \infty)$. For the interval $(-\infty, -\frac{3}{2})$, we choose $x = -2$; for $(-\frac{3}{2}, 1)$, we pick $x = 0$; and for $(1, \infty)$, $x = 2$. Evaluating the function at the three test values gives us $f(-2) = 3 > 0$, so we place $(+)$ above $(-\infty, -\frac{3}{2})$; $f(0) = -3 < 0$, so $(-)$ goes above the interval $(-\frac{3}{2}, 1)$; and, $f(2) = 7$, which means $(+)$ is placed above $(1, \infty)$.

Since we are solving $2x^2 + x - 3 \leq 0$, we look for solutions to $2x^2 + x - 3 < 0$ as well as solutions for $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which we have a $(-)$. Checking the sign diagram, we see this is $(-\frac{3}{2}, 1)$. We know $2x^2 + x - 3 = 0$ when $x = -\frac{3}{2}$ and $x = 1$, so our final answer is $\left[-\frac{3}{2}, 1\right]$.

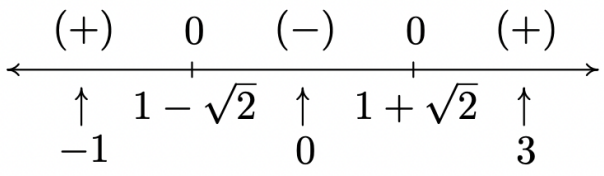
To verify our solution graphically, we refer to the original inequality, $2x^2 \leq 3 - x$. We let $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the x values where the graph of g is below that of h (the solution to $g(x) < h(x)$) as well as the points of intersection (the solutions to $g(x) = h(x)$).

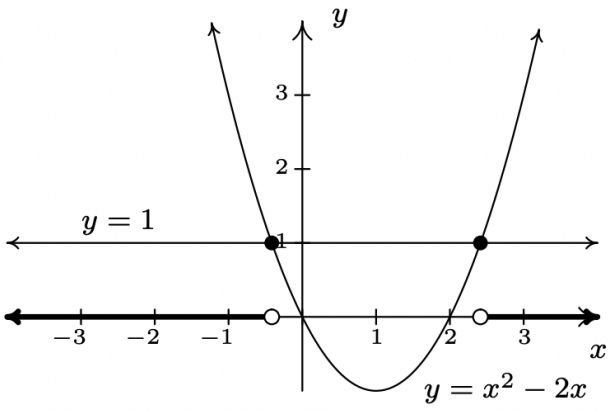




b. Once again, we re-write $x^2 - 2x > 1$ as $x^2 - 2x - 1 > 0$ and we identify $f(x) = x^2 - 2x - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $x^2 - 2x - 1$ doesn't factor nicely. Hence, we resort to the Quadratic Formula to solve $x^2 - 2x - 1 = 0$, and arrive at $x = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $x = -1$, $x = 0$ and $x = 3$ as our test values and find $f(-1) = 2$, which is $(+)$; $f(0) = -1$ which is $(-)$; and $f(3) = 2$ which is $(+)$ again. Our solution to $x^2 - 2x - 1 > 0$ is where we have $(+)$, so, in interval notation $\left(-\infty, 1-\sqrt{2}\right) \cup \left(1+\sqrt{2}, \infty\right)$.

To check the inequality $x^2 - 2x > 1$ graphically, we set $g(x) = x^2 - 2x$ and $h(x) = 1$. We are looking for the x values where the graph of g is above the graph of h.





Example 2.1.10**: Applying the Vertex and x-Intercepts of a Parabola**

a. The ball reaches the maximum height at the vertex of the parabola.

$$h = -\frac{80}{2(-16)} = \frac{80}{32} = \frac{5}{2} = 2.5.$$

The ball reaches a maximum height after 2.5 seconds.

b. To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$k = H\left(-\frac{b}{2a}\right) = H(2.5) = -16(2.5)^2 + 80(2.5) + 40 = 140.$$

The ball reaches a maximum height of 140 feet.

c. We need to solve

$$0 = -16t^2 + 80t + 40$$

which has the same solution as

$$0 = 16t^2 - 80t - 40.$$

Since the coefficients are all divisible by 8, we divide both sides by 8 to get

$$0 = 2t^2 - 10t - 5.$$

Try as you might, the quadratic on the right side doesn't factor. Therefore, we use the Quadratic Formula.

$$t = \frac{10 \pm \sqrt{(-10)^2 - 4(2)(-5)}}{2(2)} \quad \text{(substituting)}$$

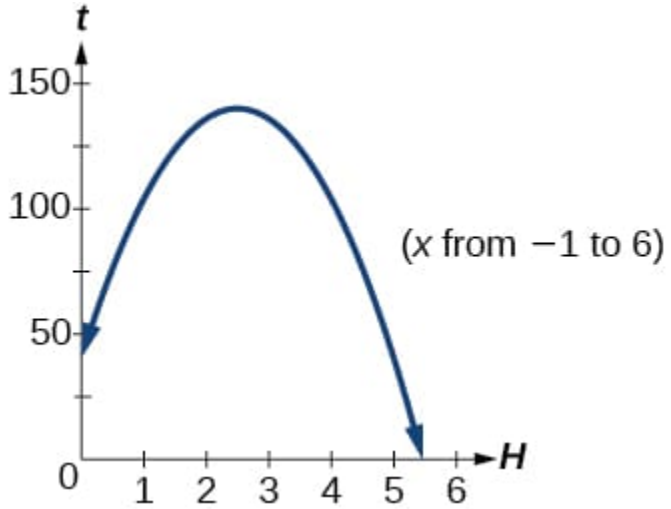
$$= \frac{10 \pm \sqrt{140}}{4} \quad \text{(simplifying)}$$

Because this is an application and the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t \approx 5.458 \quad \text{or} \quad t \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 2.1.13.

Figure 2.1.13



d. This question is akin to asking us to solve $-16t^2 + 80t + 40 > 104$. We starting by getting 0 on one side:

$$-16t^2 + 80t - 64 > 0.$$

Again, it would be better to get that lead coefficient to be positive, so we multiply both sides by -1 (remember that the inequality flips):

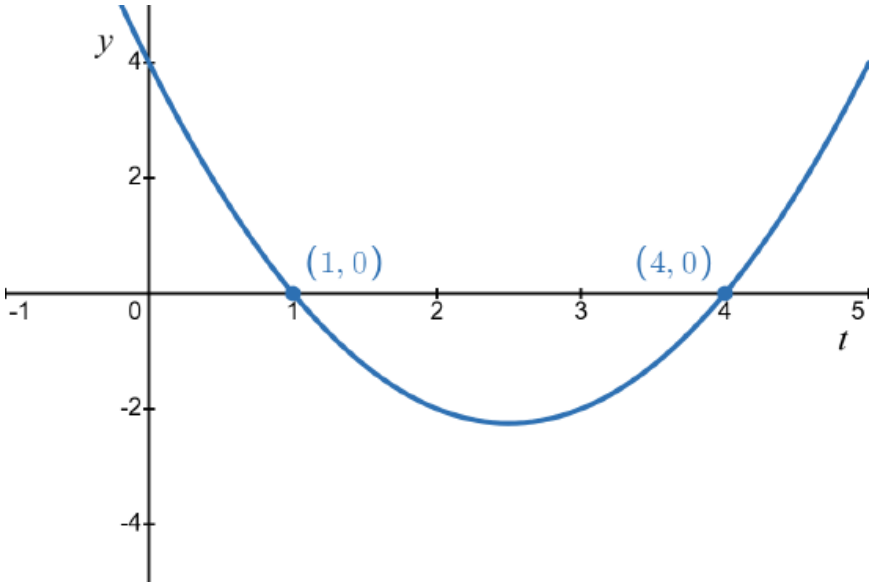
$$16t^2 - 80t + 64 < 0.$$

We then divide both sides by the GCF of each of the coefficients (which is 16):

$$t^2 - 5t + 4 < 0.$$

We want to know when the outputs of this parabola are below the t-axis. We know this parabola is opening upward and the left side factors to $(t-4)(t-1)$. This implies the roots are $t = 1$ and $t = 4$. This means that our solution is the interval $(1, 4)$ (see Figure 2.1.14).

Figure 2.1.14



Example 2.1.11**: Solving an Equation that is Quadratic-in-Form**

Many students would immediately distribute things out to get

$$27x^2 + 21x - 20 = 0$$

and try to factor; however, this is not taking advantage of the form of the original equation.

Notice that there are some common expressions within the original equation - namely $3x - 1$. If we let $u = 3x - 1$, we could rewrite the equation as

$$3u^2 + 13u - 10 = 0.$$

Hopefully, you agree that this equation looks much easier to solve. This type of substitution is formally called a u**-substitution**. The choice of the variable u is arbitrary, and you are encouraged to use whatever variable you want (other than the variable in the original equation).

We can factor the left side of this new equation to get

$$(3u - 2)(u + 5) = 0.$$

Therefore, $u = \frac{2}{3}$ or $u = -5$.

Since the original problem involved the variable x, we need to **resubstitute** $u = 3x - 1$ to arrive at

$$\begin{aligned}
u &= \frac{3}{2} \quad \text{and} \quad u = -5 \\
\implies 3x - 1 &= \frac{3}{2} \quad \text{and} \quad 3x - 1 = -5 && \text{(resubstituting)} \\
\implies 3x &= \frac{3}{2} + 1 \quad \text{and} \quad 3x = -5 + 1 && \text{(adding 1 to both sides)} \\
\implies 3x &= \frac{5}{2} \quad \text{and} \quad 3x = -4 && \text{(simplifying)} \\
\implies x &= \frac{5}{6} \quad \text{and} \quad x = -\frac{4}{3} && \text{(dividing both sides by 3)}
\end{aligned}$$

Thus, the solutions to the original equation are $x = \frac{5}{6}$ and $x = -\frac{4}{3}$.

Example 2.1.12**: Solving an Equation that is Quadratic-in-Form**

Since this equation has several terms involving powers of x, it is best to collect all information on one side (this tends to "reveal" the type of equation we are working with). Subtracting 45 and x^2 from both sides, we get

$$x^4 - 4x^2 - 45 = 0.$$

The key observation here is that the exponent on x in the first term is *twice* that of the exponent on x in the second term. That is, this has the same form as a quadratic equation. Letting $u = x^2$, we get

$$\begin{aligned}
x^4 - 4x^2 - 45 &= \left(x^2\right)^2 - 4x^2 - 45 && \text{(Laws of Exponents: Power Law)} \\
&= u^2 - 4u - 45 && \text{(substituting } u = x^2\text{)}
\end{aligned}$$

Thus, we instead focus on solving

$$\begin{aligned} u^2 - 4u - 45 &= 0 \\ \implies (u-9)(u+5) &= 0 \qquad \text{(factoring)} \end{aligned}$$

Hence, $u = 9$ or $u = -5$. However, we are supposed to be solving for x, so we resubstitute $u = x^2$.

$$\implies x^2 = 9 \quad \text{or} \quad x^2 = -5.$$

While we can easily solve the first equation to get $x = \pm 3$, the second equation doesn't have a solution (over the real number system) because there isn't a real number whose square becomes a negative number. Thus, the solutions to the original equation are $x = -3$ and $x = 3$.

This page titled 2.1: Quadratic Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

2.1.1: Resources and Key Concepts

Resources

Videos

- Completing the square
  - Intermediate Algebra - Completing the Square (Part A)
  - Intermediate Algebra - Completing the Square (Part B)
- Graphing quadratic functions in standard/vertex form
  - Intermediate Algebra - Everything About Graphing Parabolas
  - Intermediate Algebra - Graphing Parabolas in Vertex Form
- Graphing quadratic functions in general form
  - Quadratic Functions - Understanding How Graphs of Parabolas are Related to Their Quadratic Functions
  - Intermediate Algebra - Graphing Quadratic Equations (Part A)
  - Intermediate Algebra - Graphing Quadratic Equations (Part B)
- Solving quadratic equations
  - Intermediate Algebra - Introduction to Quadratic Equations
  - Algebra in Review: Equations - Quadratic (Zero Factor Principle)
  - Intermediate Algebra - Solving Quadratic Equations by Factoring (Part A)
  - Intermediate Algebra - Solving Quadratic Equations by Factoring (Part B)
  - Intermediate Algebra - Quadratic Formula (Part A)
  - Intermediate Algebra - Quadratic Formula (Part B)
  - Intermediate Algebra - Quadratic Formula (Part C)
  - Intermediate Algebra - Quadratic Formula (Part D)
- Solving equations that are quadratic-in-form
  - Equations - Overview - A Review for College Algebra: Solving Equations That Are Quadratic-in-Form
- Solving quadratic inequalities
  - Inequalities - Nonlinear - A Review for College Algebra: Solving Quadratic Inequalities
  - Intermediate Algebra - Quadratic Inequalities (Part A)
  - Intermediate Algebra - Quadratic Inequalities (Part B)
  - Intermediate Algebra - Quadratic Inequalities (Part C)
  - Intermediate Algebra - Quadratic Inequalities (Part D)

Key Concepts

Definitions

- **Quadratic Function:** A function of degree two, which can be written in the general form $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and $a \neq 0$.
- **Parabola:** The U-shaped curve that is the graph of a quadratic function.
- **Vertex (of a parabola):** The point on the parabola where the function reaches its maximum or minimum value. It is also the turning point of the graph.
- **Axis of Symmetry (of a parabola):** A vertical line passing through the vertex of a parabola, dividing it into two mirror images. Its equation is $x = h$, where h is the x-coordinate of the vertex.
- **General Form (of a quadratic function):** $f(x) = ax^2 + bx + c$.
- **Standard Form (or Vertex Form of a quadratic function):** $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex of the parabola.
- **Zeros (or Roots or x-intercepts of a quadratic function):** The values of x for which $f(x) = 0$. These are the points where the parabola intersects the x-axis.
- **y-intercept (of a quadratic function):** The point where the parabola intersects the y-axis, found by evaluating $f(0)$. In general form, this is $(0, c)$.
- **Prime Quadratic:** A quadratic expression that cannot be factored into linear factors with real coefficients. This occurs when

its discriminant is negative.
- u**-substitution:** A technique used to solve equations that are "quadratic-in-form" by substituting a variable (often u) for an expression, transforming the equation into a standard quadratic form.

Theorems

- **Axis of Symmetry and Vertex (from General Form):** For a quadratic function $f(x) = ax^2 + bx + c$, the axis of symmetry is the line $x = -\frac{b}{2a}$. The vertex is at $(-\frac{b}{2a}, f(-\frac{b}{2a}))$.
- **Vertex (from Standard Form):** For a quadratic function $f(x) = a(x-h)^2 + k$, the vertex is (h, k).
- **Relationship between General and Standard Forms:**
  - Given $f(x) = ax^2 + bx + c$, then $h = -\frac{b}{2a}$ and $k = c - ah^2 = c - a(-\frac{b}{2a})^2 = c - \frac{b^2}{4a}$. More simply, $k = f(h)$.
- **Domain and Range of a Quadratic Function:**
  - The domain of any quadratic function is all real numbers, $(-\infty, \infty)$.
  - If $a > 0$, the range is $[k, \infty)$, where k is the y-coordinate of the vertex.
  - If $a < 0$, the range is $(-\infty, k]$, where k is the y-coordinate of the vertex.
- **Quadratic Formula:** The solutions to the quadratic equation $ax^2 + bx + c = 0$ (where $a \neq 0$) are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Common Mistakes to Avoid

- **Confusing h in Standard Form:** In $f(x) = a(x-h)^2 + k$, if the form is, for example, $f(x) = a(x+3)^2 + k$, misidentifying h as 3 instead of -3.
- **Errors in Completing the Square:** Forgetting to factor out the leading coefficient a from the x^2 and x terms before completing the square.
- **Factoring Errors:** Incorrectly factoring a quadratic trinomial when trying to find x-intercepts or solve inequalities.
- **Solving Quadratic Inequalities:** Forgetting to flip the inequality sign if multiplying/dividing the entire inequality by a negative number.
- **Equations Quadratic-in-Form:** After solving for the substituted variable (e.g., u), forgetting to substitute back to solve for the original variable (e.g., x).

This page titled 2.1.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

2.1.2: Homework

Reading Questions

1. What is the shape of the graph of a quadratic function called?
2. What are the key features of a parabola? (List at least three)
3. What is the general form of a quadratic function? What is the standard (or vertex) form?
4. How can you determine from the general form $f(x) = ax^2 + bx + c$ whether the parabola opens upward or downward?
5. What is the formula for the x-coordinate of the vertex (and the axis of symmetry) when a quadratic function is in general form?
6. If a quadratic function is in standard form $f(x) = a(x - h)^2 + k$, what are the coordinates of its vertex?
7. How does the value of a in $f(x) = a(x - h)^2 + k$ affect the width and direction of opening of the parabola compared to $y = x^2$?
8. What is the domain of any quadratic function? How is the range determined?
9. When would you use the Quadratic Formula to find the x-intercepts of a parabola?
10. Describe the process of u-substitution for solving an equation that is quadratic-in-form, like $x^4 - 4x^2 - 45 = 0$.
11. Explain the graphical method for solving a quadratic inequality like $x^2 - x - 6 < 0$.
12. Explain the sign chart method for solving a quadratic inequality.
13. Explain the advantage of writing a quadratic function in standard form.
14. How can the vertex of a parabola be used in solving real world problems?
15. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.
16. What is another name for the standard form of a quadratic function?
17. What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

Homework

Algebraic

For the following exercises, rewrite the quadratic functions in standard form and give the vertex.

18. $f(x) = x^2 - 12x + 32$
19. $g(x) = x^2 + 2x - 3$
20. $f(x) = x^2 - x$
21. $f(x) = x^2 + 5x - 2$
22. $h(x) = 2x^2 + 8x - 10$
23. $k(x) = 3x^2 - 6x - 9$
24. $f(x) = 2x^2 - 6x$
25. $f(x) = 3x^2 - 5x - 1$

For the following exercises, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

26. $y(x) = 2x^2 + 10x + 12$
27. $f(x) = 2x^2 - 10x + 4$
28. $f(x) = -x^2 + 4x + 3$
29. $f(x) = 4x^2 + x - 1$
30. $h(t) = -4t2 + 6t - 1$
31. $f(x) = \frac{1}{2}x^2 + 3x + 1$
32. $f(x) = -\frac{1}{3}x^2 - 2x + 3$

For the following exercises, determine the domain and range of the quadratic function.

33. $f(x) = (x - 3)^2 + 2$
34. $f(x) = -2(x + 3)^2 - 6$

35. $f(x) = x^2 + 6x + 4$
36. $f(x) = 2x^2 - 4x + 2$
37. $k(x) = 3x^2 - 6x - 9$

For the following exercises, use the vertex (h, k) and a point on the graph (x, y) to find the general form of the equation of the quadratic function.

38. $(h, k) = (2, 0)$, $(x, y) = (4, 4)$
39. $(h, k) = (-2, -1)$, $(x, y) = (-4, 3)$
40. $(h, k) = (0, 1)$, $(x, y) = (2, 5)$
41. $(h, k) = (2, 3)$, $(x, y) = (5, 12)$
42. $(h, k) = (-5, 3)$, $(x, y) = (2, 9)$
43. $(h, k) = (3, 2)$, $(x, y) = (10, 1)$
44. $(h, k) = (0, 1)$, $(x, y) = (1, 0)$
45. $(h, k) = (1, 0)$, $(x, y) = (0, 1)$

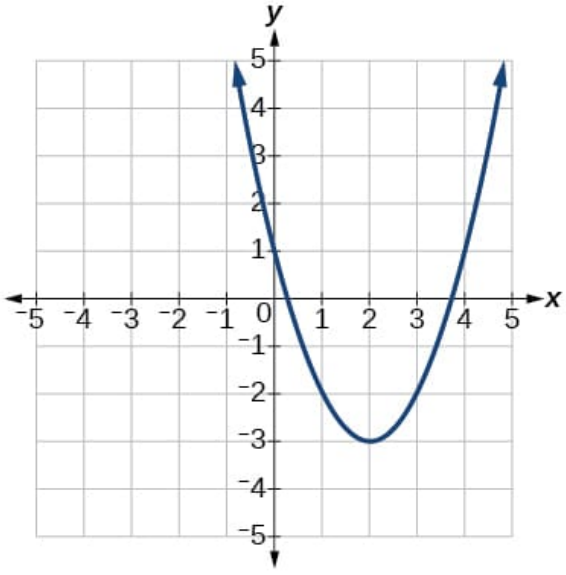
Graphical

For the following exercises, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

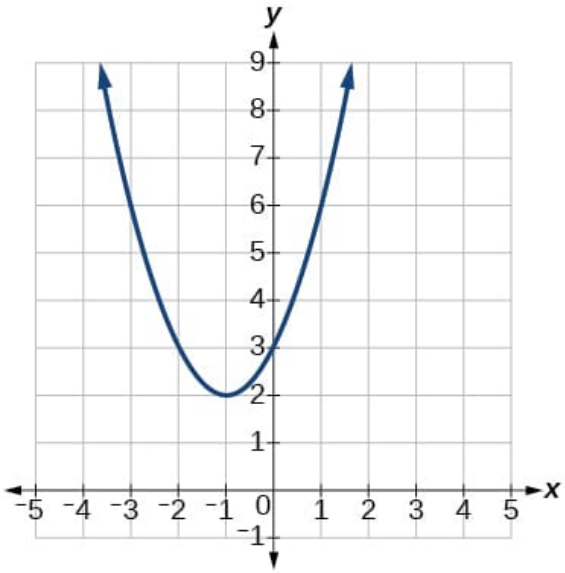
46. $f(x) = x^2 - 2x$
47. $f(x) = x^2 - 6x - 1$
48. $f(x) = x^2 - 5x - 6$
49. $f(x) = x^2 - 7x + 3$
50. $f(x) = -2x^2 + 5x - 8$
51. $f(x) = 4x^2 - 12x - 3$

For the following exercises, write the equation for the graphed function.

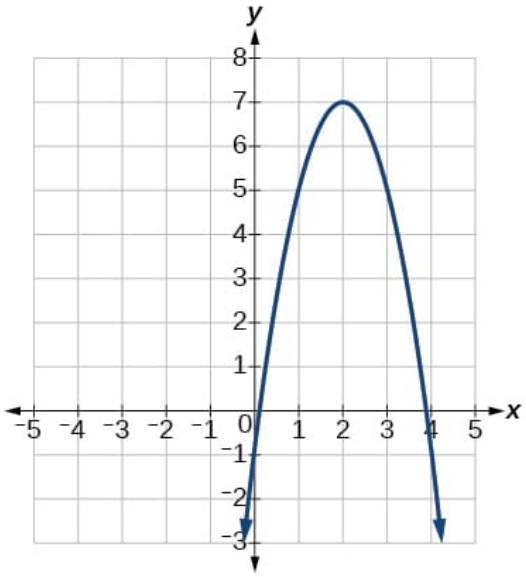
52.



53.

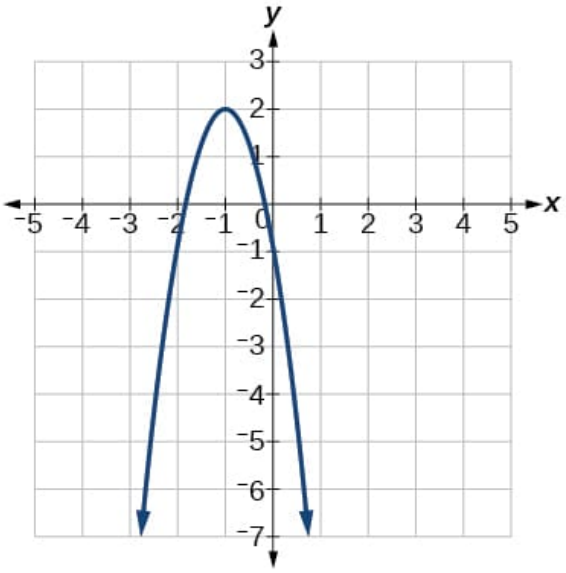


54.

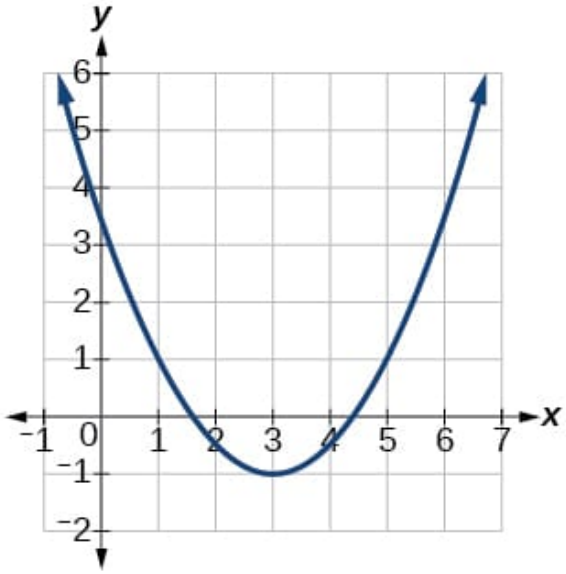


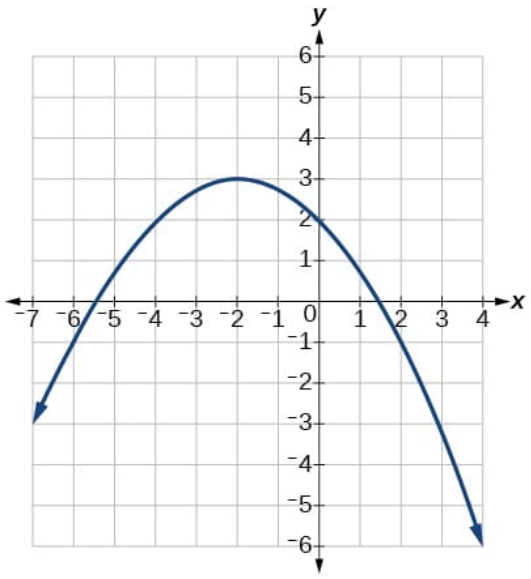
55.

56.



57.





Numeric

For the following exercises, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

58.

x	–2	–1	0	1	2
y	5	2	1	2	5

59.

x	–2	–1	0	1	2
y	1	0	1	4	9

60.

x	–2	–1	0	1	2
y	–2	1	2	1	–2

61.

x	–2	–1	0	1	2
y	–8	–3	0	1	0

62.

x	–2	–1	0	1	2
y	8	2	0	2	8

Technology

For the following exercises, use a calculator to find the answer.

53. Graph on the same set of axes the functions $f(x) = x^2$, $f(x) = 2x^2$, and $f(x) = \frac{1}{3}x^2$. What appears to be the effect of changing the coefficient?
54. Graph on the same set of axes $f(x) = x^2$, $f(x) = x^2 + 2$, $f(x) = x^2 + 5$, and $f(x) = x^2 - 3$. What appears to be the effect of adding a constant?
55. Graph on the same set of axes $f(x) = x^2$, $f(x) = (x-2)^2$, $f(x) = (x-3)^2$, and $f(x) = (x+4)^2$. What appears to be the effect of adding or subtracting those numbers?
56. The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function $h(x) = -\frac{32}{(80)^2}x^2 + x$ where x is the horizontal distance traveled and $h(x)$ is the height in feet. Use graphing technology to determine the height of the object when it has traveled 100 feet away horizontally.
57. A suspension bridge can be modeled by the quadratic function $h(x) = .0001x^2$ with $-2000 \leq x \leq 2000$ where $|x|$ is the number of feet from the center and $h(x)$ is height in feet. Use graphing technology to estimate how far from the center does the bridge have a height of 100 feet.

Extensions

For the following exercises, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

58. Vertex $(1, -2)$, opens up.
59. Vertex $(-1, 2)$ opens down.
70. Vertex $(-5, 11)$, opens down.
71. Vertex $(-100, 100)$, opens up.

For the following exercises, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

72. Contains $(1, 1)$ and has shape of $f(x) = 2x^2$. Vertex is on the y-axis.
73. Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$. Vertex is on the y-axis.
74. Contains $(2, 3)$ and has the shape of $f(x) = 3x^2$. Vertex is on the y-axis.
75. Contains $(1, -3)$ and has the shape of $f(x) = -x^2$. Vertex is on the y-axis.
76. Contains $(4, 3)$ and has the shape of $f(x) = 5x^2$. Vertex is on the y-axis.
77. Contains $(1, -6)$ has the shape of $f(x) = 3x^2$. Vertex has x-coordinate of -1.

Real-World Applications

78. Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?
79. Among all of the pairs of numbers whose difference is 12, find the pair with the smallest product. What is the product?
80. Suppose that the price per unit in dollars of a cell phone production is modeled by $p = 45 - 0.0125x$, where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue.
81. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.
82. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$. How long does it take to reach maximum height?

This page titled 2.1.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

2.2: Power Functions and Polynomial Functions

Note to the Instructor (click to expand)

The purpose of the material in this section is to introduce the vocabulary needed to work with polynomial functions. Students are introduced to power functions, polynomials, and end behavior **but** are not asked to graph at this time.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.

Hawk A.I. Section-Specific Tutor

EST. 1970 COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Figure 2.2.1

credit: Jason Bay, Flickr



Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in Table 2.2.1.

Table 2.2.1

Year	2009	2010	2011	2012	2013

Bird Population	800	897	992	1,083	1,169

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800$, where $P(t)$ represents the bird population on the island t years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.

Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function called a **power function**.

Definition: Power Function

A **power function** is a function of the form

$$f(x) = ax^n,$$

where a and n are real numbers and $a \neq 0$. a is known as the **coefficient** of the power function, and n is called its **exponent**.

As an example, consider functions for area or volume. The function for the area of a circle with radius r is $A(r) = \pi r^2$ and the function for the volume of a sphere with radius r is $V(r) = \frac{4}{3}\pi r^3$. Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a power.

Q&A

Is $f(x) = 2^x$ a power function?

No. A power function contains a variable base raised to a fixed power. This function has a constant base raised to a variable power. This is called an **exponential function**, not a power function. We will deal with these types of functions later.

Example 2.2.1: Identifying Power Functions

Which of the following functions are power functions?

$f(x) = 1$ Constant function

$f(x) = x$ Identify function

$f(x) = x^2$ Quadratic function

$f(x) = x^3$ Cubic function

$f(x) = \frac{1}{x}$ Reciprocal function

$f(x) = \frac{1}{x^2}$ Reciprocal squared function

$f(x) = \sqrt{x}$ Square root function

$f(x) = \sqrt[3]{x}$ Cube root function

Solution available at the end of this section.

Checkpoint 2.2.1

Which functions are power functions?

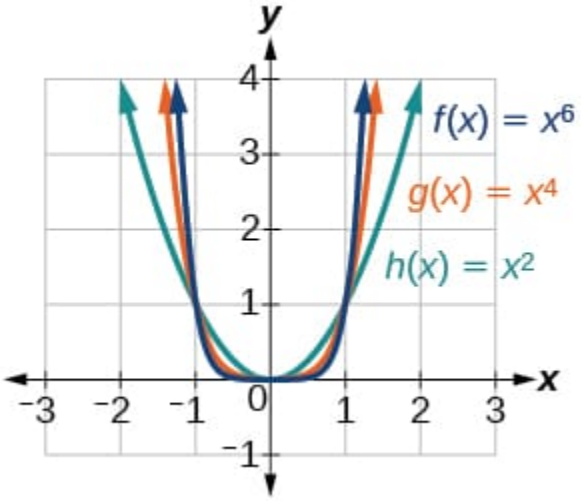
$$f(x) = 2x^2 \cdot 4x^3$$
$$g(x) = -x^5 + 5x^3 - 4x$$
$$h(x) = \frac{2x^5-1}{3x^2+4}$$

Identifying End Behavior of Power Functions

Figure 2.2.2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$, and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

Figure 2.2.2

Even-power functions



To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

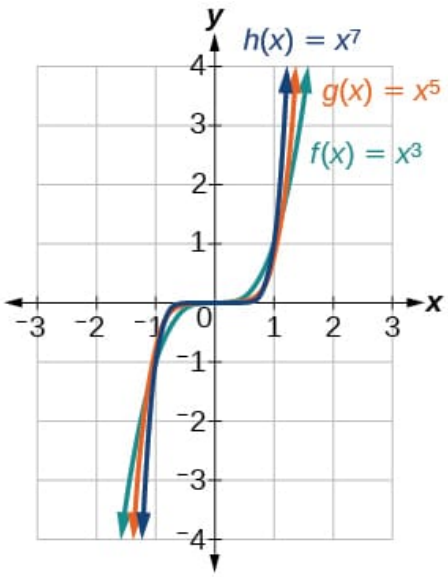
With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty, \quad f(x) \to \infty.$$

Figure 2.2.3 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

Figure 2.2.3

Odd-power functions



These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure 2.2.2 we see that functions of the form $f(x) = x^n$, where n is even, are symmetric about the y-axis. In Figure 2.2.3, we see that functions of the form $f(x) = x^n$, where n is odd, are symmetric about the origin.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form, we write

$$\text{as } x \to -\infty, \quad f(x) \to -\infty \quad \text{and as } x \to \infty, \quad f(x) \to \infty.$$

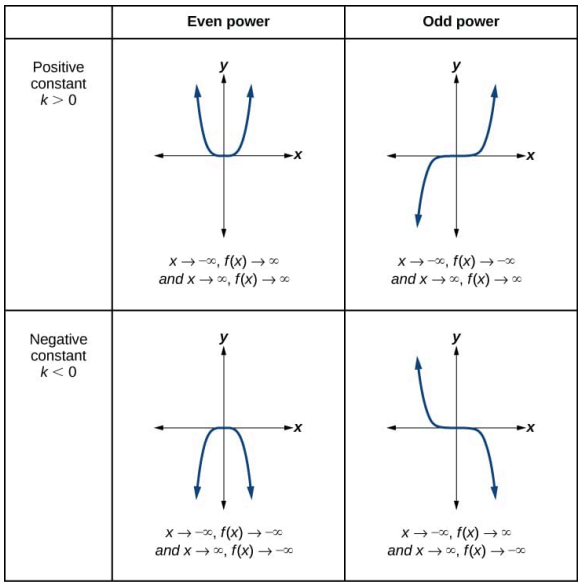
The behavior of the graph of a function as the input values get very negative ($x \to -\infty$) and get very large ($x \to \infty$) is referred to as the **end behavior** of the function. We can use words or symbols to describe end behavior.

Definition: End Behavior of a Function

The **end behavior** of a function refers to the way the graph of the function behaves as the input (typically the x-value) approaches positive or negative infinity (denoted $x \to \infty$ or $x \to -\infty$).

Figure 2.2.4 shows the end behavior of power functions in the form $f(x) = kx^n$, where n is a non-negative integer depending on the power and the constant.

Figure 2.2.4



Example 2.2.2: Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution available at the end of this section.

Example 2.2.3: Identifying the End Behavior of a Power Function.

Describe the end behavior of the graph of $f(x) = -x^9$.

Solution available at the end of this section.

Checkpoint 2.2.3

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius, r, of the spill depends on the number of weeks, w, that have passed. This relationship is linear.

$$r(w) = 24 + 8w.$$

We can combine this with the formula for the area A of a circle, $A(r) = \pi r^2$. Composing these functions gives a formula for the area in terms of weeks.

$$A(w) = A(r(w)) = A(24 + 8w) = \pi(24 + 8w)^2.$$

Distributing gives the formula

$$A(w) = 576\pi + 384\pi w + 64\pi w^2.$$

This formula is an example of a **polynomial function**.

Definition: Polynomial Function

A **polynomial function** is a function that can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0,$$

where $n \in \mathbb{N}$. This is called the **general form of a polynomial function**. Each a_i is a **coefficient** and can be any real number. Each product $a_i x_i$ is called a **term** of the function. The largest n for which $a_n \neq 0$ is called the **degree** of the polynomial function. The **leading term** is the term containing the highest power of the variable, or the term with the highest degree. The **leading coefficient** is the coefficient of the leading term.

Example 2.2.4: Identifying Polynomial Functions

Which of the following are polynomial functions?

$$\begin{aligned} f(x) &= 2x^3 \cdot 3x + 4 \\ g(x) &= -x(x^2 - 4) \\ h(x) &= 5\sqrt{x} + 2 \end{aligned}$$

Solution available at the end of this section.

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power (this is what we call, in the definition of the polynomial function, **general form**).

Example 2.2.5: Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$\begin{aligned} f(x) &= 3 + 2x^2 - 4x^3 \\ g(t) &= 5t^5 - 2t^3 + 7t \\ h(p) &= 6p - p^3 - 2 \end{aligned}$$

Solution available at the end of this section.

Checkpoint 2.2.5

Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 6$.

Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its **end behavior**. Recall the following definition.

Definition: End Behavior of a Function

The **end behavior** of a function refers to the way the graph of the function behaves as the input (typically the x-value) approaches positive or negative infinity (denoted $x \to \infty$ or $x \to -\infty$).

To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate

the graph. We state this in the following theorem along with a formal proof of the result.

Theorem: End Behavior of a Polynomial Function

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,$$

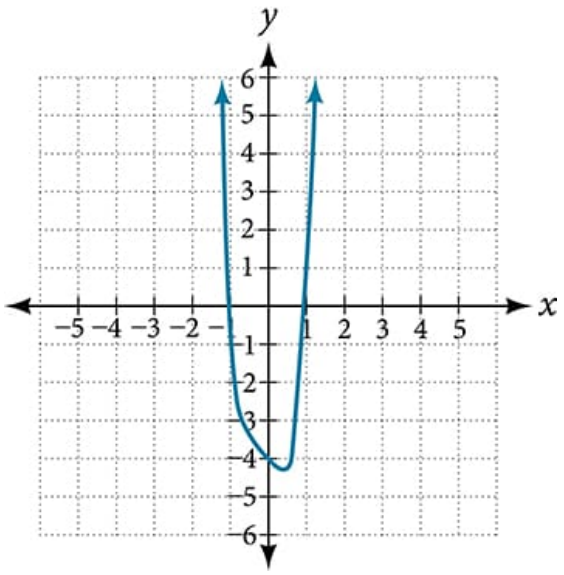
where $a_n \neq 0$ and $n \geq 1$. The end behavior of the graph of $P(x)$ matches the end behavior of the graph of $y = a_n x^n$. Hence,

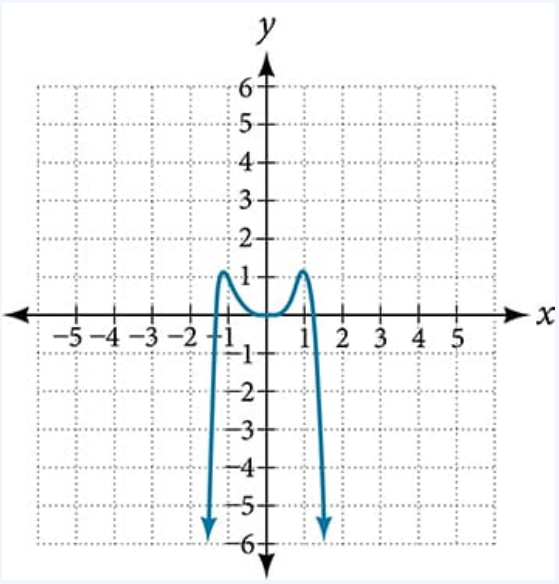
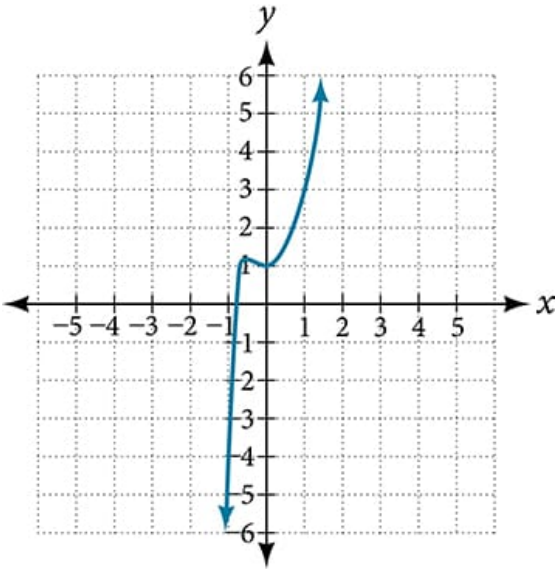
a. If n is even and $a_n > 0$, then $P(x) \to \infty$ as $x \to -\infty$ and as $x \to \infty$.
b. If n is even and $a_n < 0$, then $P(x) \to -\infty$ as $x \to -\infty$ and as $x \to \infty$.
c. If n is odd and $a_n > 0$, then $P(x) \to -\infty$ as $x \to -\infty$, and $P(x) \to \infty$ as $x \to \infty$.
d. If n is odd and $a_n < 0$, then $P(x) \to \infty$ as $x \to -\infty$, and $P(x) \to -\infty$ as $x \to \infty$.

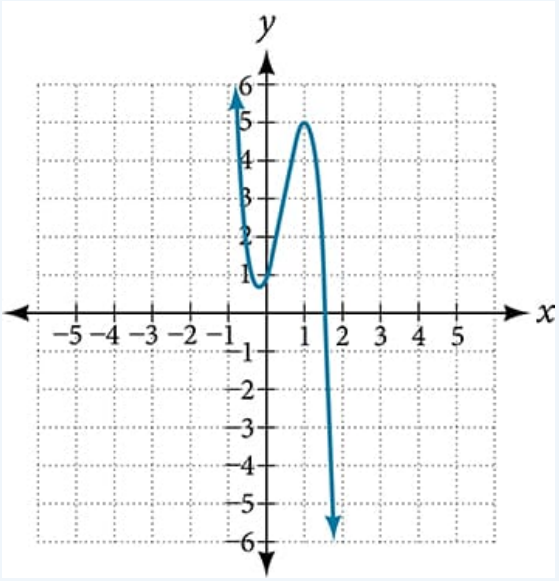
Proof available at the end of this section.

Table 2.2.3 illustrates some examples.

Table 2.2.3

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$	

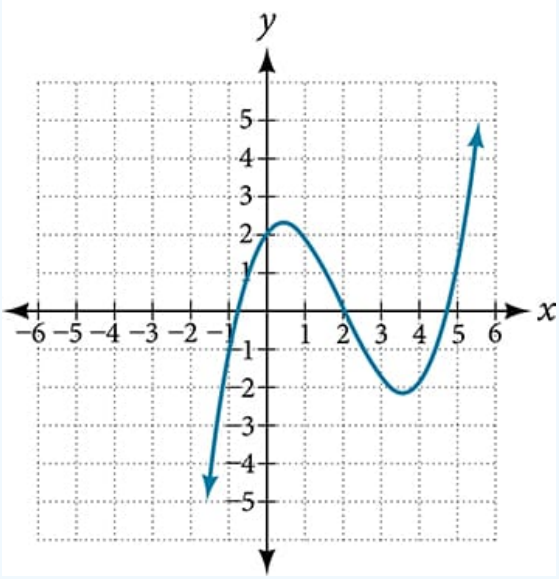
Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$	
$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$	$3x^5$	

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = -6x^3 + 7x^2 + 3x + 1$	$-6x^3$	

Example 2.2.6: Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure 2.2.7.

Figure 2.2.7

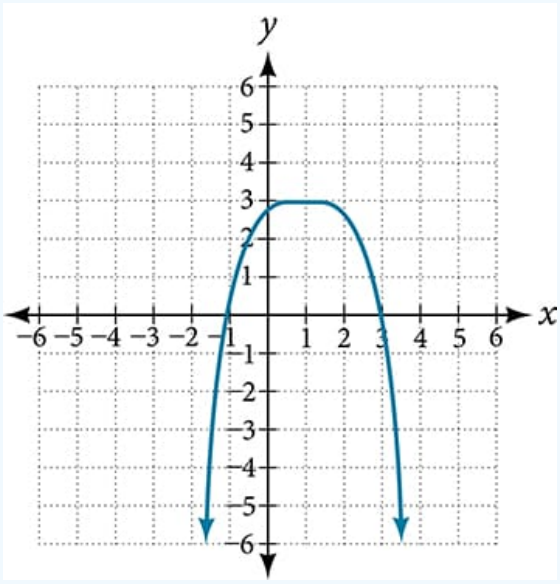


Solution available at the end of this section.

Checkpoint 2.2.6

Describe the end behavior, and determine a possible degree of the polynomial function in Figure 2.2.8.

Figure 2.2.8



Example 2.2.7: Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Solution available at the end of this section.

Checkpoint 2.2.7

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. At first glance, most people think we are immediately going to find the **turning points** of the function.

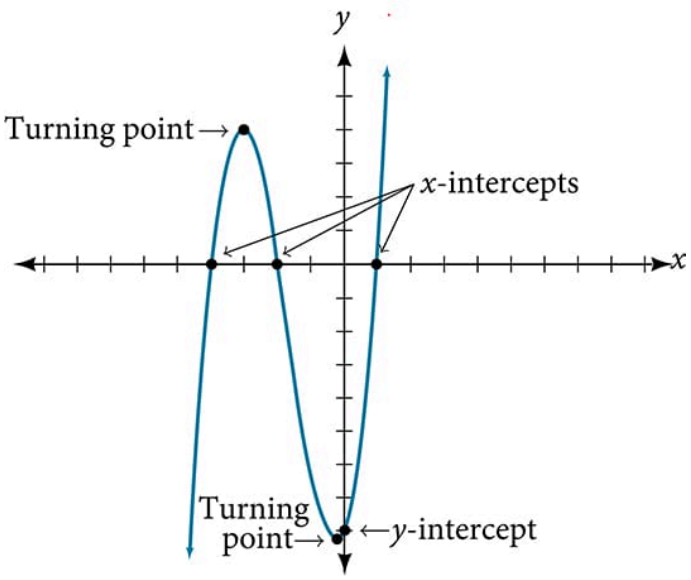
Definition: Turning Point

A **turning point** of a graph is a point at which the graph changes from increasing to decreasing, or from decreasing to increasing.

Unfortunately, determining the exact locations of the turning points requires Calculus (however, we will soon be able to determine the maximum number of turning points for any given polynomial!).

While any further discussion of turning points must be placed on hold, we can (at the very least) concern ourselves with the intercepts. As with all functions, the y-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one y-intercept, $(0, a_0)$. The x-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one x-intercept. See Figure 2.2.9.

Figure 2.2.9



Example 2.2.8: Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the y- and x-intercepts.

Solution available at the end of this section.

Example 2.2.9: Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the y- and x-intercepts.

Solution available at the end of this section.

Checkpoint 2.2.9

Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the y- and x-intercepts.

Comparing Smooth and Continuous Graphs

We already know that the degree of a polynomial function helps us to determine its end behavior; however, the degree also helps us to determine the number of x-intercepts and, as such, a rough approximation to the number of turning points. A polynomial function of nth degree is the product of n factors, so it will have at most n zeros, or x-intercepts. Therefore, the graph of the polynomial function of degree n must have at most $n–1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

Theorem: Number of Turning Points for a Polynomial Function

The graph of a polynomial function of degree $n \geq 1$ will have, at most, $n-1$ turning points and n horizontal intercepts.

Proof available at the end of this section.

The proof of this theorem requires more mathematical "technology" than we currently have access to, so we hold off until Calculus.

While not technically true, we will, for the sake of simplicity, define a **continuous function** to be one that has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A **smooth curve** is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Example 2.2.10: Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of x-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

Solution available at the end of this section.

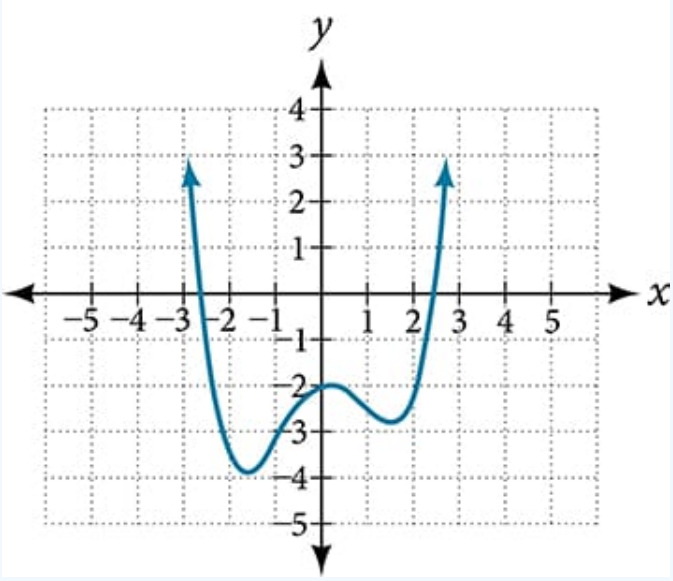
Checkpoint 2.2.10

Without graphing the function, determine the maximum number of x-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$.

Example 2.2.11: Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure 2.2.12 based on its intercepts and turning points?

Figure 2.2.12

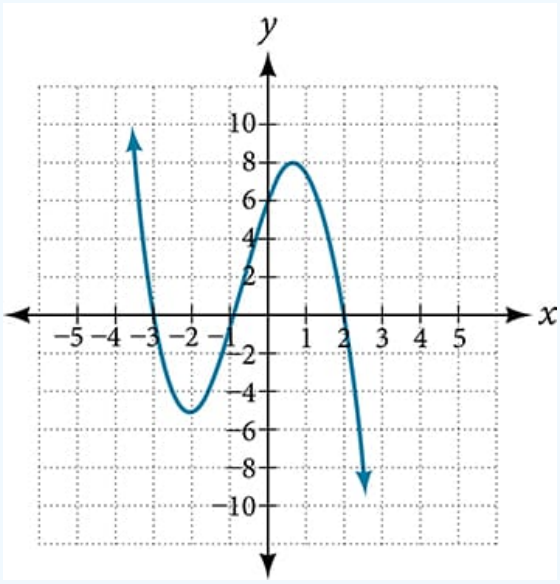


Solution available at the end of this section.

Checkpoint 2.2.11

What can we conclude about the polynomial represented by the graph shown in Figure 2.2.14 based on its intercepts and turning points?

Figure 2.2.14



Example 2.2.12: Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

Solution available at the end of this section.

Checkpoint 2.2.12

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

Answers

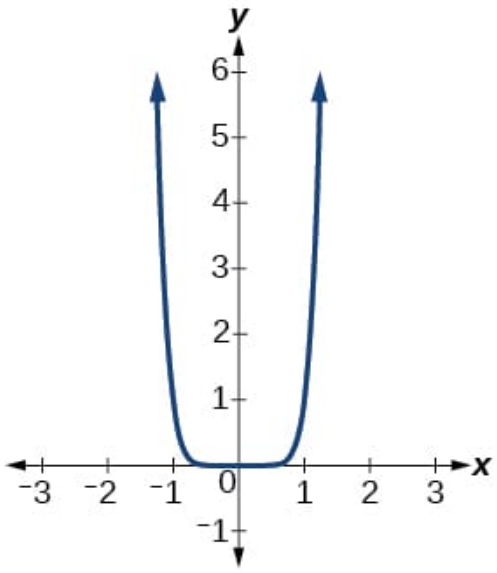
Example 2.2.1: Identifying Power Functions

All of the listed functions are power functions. The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$, respectively. The quadratic and cubic functions are power functions with whole number powers. The reciprocal and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$. The square and cube root functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Example 2.2.2: Identifying the End Behavior of a Power Function

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. Therefore, as $x \to \infty$, $f(x) \to \infty$. As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty$, $f(x) \to \infty$. We can graphically represent the function as shown in Figure 2.2.5.

Figure 2.2.5

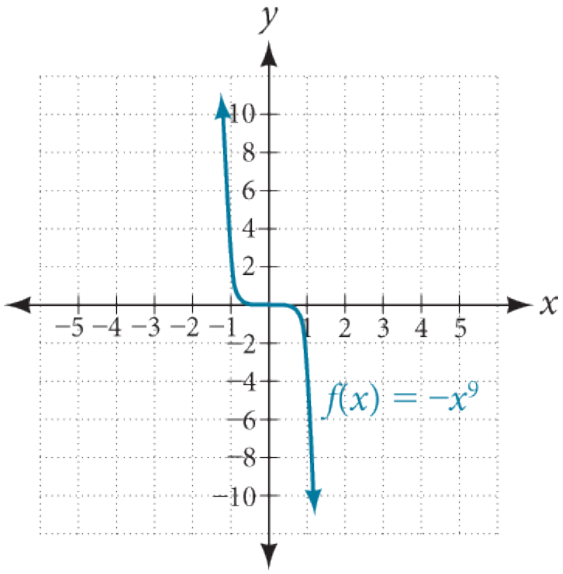


Example 2.2.3**: Identifying the End Behavior of a Power Function.**

The exponent of the power function is 9 (an odd number). Because the coefficient is -1 (negative), the graph is the reflection about the x-axis of the graph of $f(x) = x^9$. Figure 2.2.6 shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty,\ f(x) \to \infty;\ \text{as } x \to \infty,\ f(x) \to -\infty.$$

Figure 2.2.6



We can check our work by using the table feature on a graphing utility.

Table 2.2.2

x	$f(x)$
–10	1,000,000,000
–5	1,953,125
0	0
5	–1,953,125

x	$f(x)$
10	-1,000,000,000

We can see from Table 2.2.2 that, when we substitute very small values for x, the output is very large, and when we substitute very large values for x, the output is very small (meaning that it is a very large negative value).

Example 2.2.4: Identifying Polynomial Functions

The first two functions are examples of polynomial functions because they can be written in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Example 2.2.5: Identifying the Degree and Leading Coefficient of a Polynomial Function

For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, -1.

Theorem: End Behavior of a Polynomial Function

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_n \neq 0$. Then

$$\begin{aligned} P(x) &= a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0 \\ &= a_n x^n \left(1 + \frac{a_{n-1}x^{n-1}}{a_n x^n} + \frac{a_{n-2}x^{n-2}}{a_n x^n} + \cdots + \frac{a_2 x^2}{a_n x^n} + \frac{a_1 x}{a_n x^n} + \frac{a_0}{a_n x^n}\right) && \text{(factoring out } a_n x^n) \\ &= a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \frac{a_{n-2}}{a_n x^2} + \cdots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right) && \text{(Laws of Exponents)} \end{aligned}$$

Since $a_n \neq 0$, the fractions $\frac{a_k}{a_n}$ are finite constants. For notational efficiency, let's denote $\frac{a_k}{a_n}$ as b_k. Therefore,

$$P(x) = a_n x^n \left(1 + b_{n-1}\cdot\frac{1}{x} + b_{n-2}\cdot\frac{1}{x^2} + \cdots + b_2\cdot\frac{1}{x^{n-2}} + b_1\cdot\frac{1}{x^{n-1}} + b_0\frac{1}{x^n}\right).$$

(The next step is proved in Calculus, but can be understood here as a strong possibility.) As long as the power on x is not negative, as $x \to \pm\infty$, $\frac{1}{x^p} \to 0$. Since all the powers on x in

$$P(x) = a_n x^n \left(1 + b_{n-1}\cdot\frac{1}{x} + b_{n-2}\cdot\frac{1}{x^2} + \cdots + b_2\cdot\frac{1}{x^{n-2}} + b_1\cdot\frac{1}{x^{n-1}} + b_0\frac{1}{x^n}\right)$$

are not negative, all terms involving x in the denominator must tend to 0 as $x \to \pm\infty$. That is, as $x \to \pm\infty$, $P(x) \to a_n x^n(1 + 0 + 0 + \cdots + 0 + 0 + 0) = a_n x^n$. Hence, the end behavior of $P(x)$ is $y = a_n x^n$.

Example 2.2.6: Identifying End Behavior and Degree of a Polynomial Function

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very negative, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to -\infty,\ f(x) \to -\infty;\ \text{as } x \to \infty,\ f(x) \to \infty.$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

Example 2.2.7: Identifying End Behavior and Degree of a Polynomial Function

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2 + 3x - 4) \\ &= -3x^4 - 9x^3 + 12x^2 \end{aligned}$$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x \to -\infty,\ f(x) \to -\infty;\ \text{as } x \to \infty,\ f(x) \to -\infty.$$

Example 2.2.8: Determining the Intercepts of a Polynomial Function

The y-intercept occurs when the input is zero so substitute 0 for x.

Illegal pream-token (f)

The y-intercept is $(0, 8)$.

The x-intercepts occur when the output is zero.

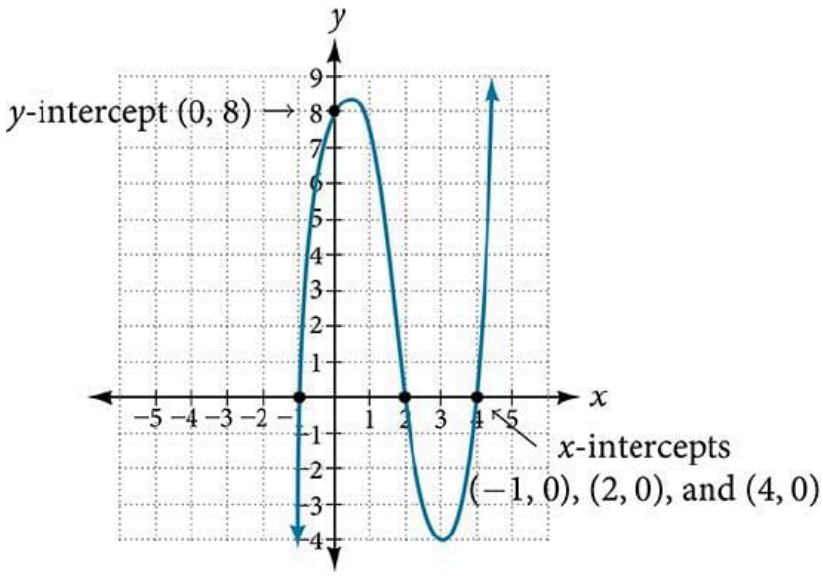
$$0 = (x-2)(x+1)(x-4)$$

$$\implies \quad x-2=0 \quad \text{or} \quad x+1=0 \quad \text{or} \quad x-4=0$$
$$\implies \quad x=2 \quad \text{or} \quad x=-1 \quad \text{or} \quad x=4$$

The x-intercepts are $(2,0)$, $(-1,0)$, and $(4,0)$.

We can see these intercepts on the graph of the function shown in Figure 2.2.10.

Figure 2.2.10



Example 2.2.9**: Determining the Intercepts of a Polynomial Function with Factoring**

The y-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4-4(0)^2-45 \\ &= -45 \end{aligned}$$

The y-intercept is $(0,-45)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$f(x)=x^4-4x^2-45=(x^2-9)(x^2+5)=(x-3)(x+3)(x^2+5).$$

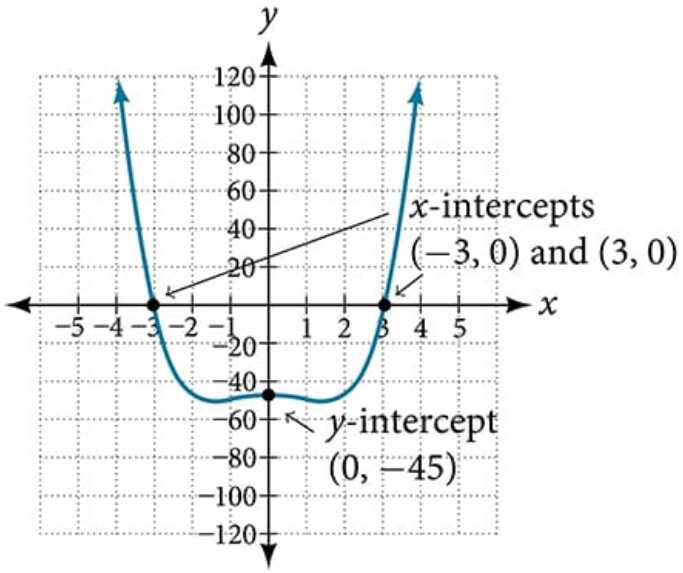
Therefore, we solve

$$0=(x-3)(x+3)(x^2+5).$$

This mean $x-3=0$ or $x+3=0$ ($x^2+5=0$ would require $x^2=-5$, which has no real solution). Hence, $x=3$ or $x=-3$. The x-intercepts are $(3,0)$ and $(-3,0)$.

We can see these intercepts on the graph of the function shown in Figure 2.2.11. We can see that the function is even because $f(x)=f(-x)$.

Figure 2.2.11



Theorem: Number of Turning Points for a Polynomial Function

This proof must wait until Calculus.

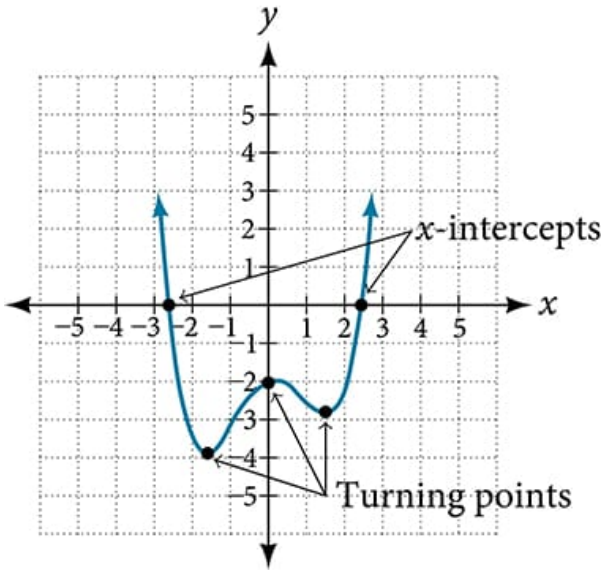
Example $2.2.10$**: Determining the Number of Intercepts and Turning Points of a Polynomial**

The polynomial has a degree of 10, so there are at most 10 x-intercepts and at most $10 - 1 = 9$ turning points.

Example $2.2.11$**: Drawing Conclusions about a Polynomial Function from the Graph**

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See Figure $2.2.13$.

Figure $2.2.13$



The graph has 2 x-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

Example $2.2.12$**: Drawing Conclusions about a Polynomial Function from the Factors**

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The y-intercept is $(0, 0)$.

The x-intercepts are found by determining the zeros of the function.

$$0 = -4x(x+3)(x-4) \implies x = 0 \text{ or } x+3 = 0 \text{ or } x-4 = 0 \implies x = 0 \text{ or } x = -3 \text{ or } x = 4.$$

The x-intercepts are $(0, 0), (–3, 0)$, and $(4, 0)$.

The degree is 3 so the graph has at most 2 turning points.

This page titled 2.2: Power Functions and Polynomial Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

2.2.1: Resources and Key Concepts

Resources

Videos

- Power function
  - Intermediate Algebra - Integer Exponents: Power Functions
- End behavior
  - Power Functions and Polynomial Functions - Identifying End Behavior of Polynomial Functions
  - Polynomial Functions and Their Graphs - End Behavior of Polynomials
- Polynomial function
  - Polynomial Functions - The Concept
- Zeros (of a polynomial)
  - Graphs of Polynomial Functions - Using Factoring to Find Zeros of Polynomial Functions

Key Concepts

Definitions

- **Power Function**: A function that can be represented in the form $f(x) = kx^p$, where k and p are real numbers, and k is known as the coefficient.
- **End Behavior of a Function**: The way the graph of the function behaves as the input (typically the x-value) approaches positive or negative infinity (denoted $x \to \infty$ or $x \to -\infty$).
- **Polynomial Function**: A function that can be written in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where $n^{\text{int}}\mathbb{N}$ (a non-negative integer) and the coefficients a_i are real numbers.
- **General Form of a Polynomial Function**: The form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, typically with terms arranged in descending order of power.
- **Coefficient (of a term in a polynomial)**: The real number factor multiplying the variable part of a term (e.g., in $a_i x^i$, a_i is the coefficient).
- **Term (of a polynomial function)**: Each product $a_i x^i$ in the polynomial function.
- **Degree (of a polynomial function)**: The largest exponent n for which the coefficient a_n is not zero.
- **Leading Term (of a polynomial function)**: The term containing the highest power of the variable (the term with the highest degree).
- **Leading Coefficient (of a polynomial function)**: The coefficient of the leading term.
- **Turning Point (of a polynomial graph)**: A point at which the function values change from increasing to decreasing or decreasing to increasing.
- **Continuous Function (informal definition)**: A function whose graph has no breaks; it can be drawn without lifting the pen from the paper.
- **Smooth Curve (informal definition)**: A graph that has no sharp corners.

Theorems

- **End Behavior of a Polynomial Function**: Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_n \neq 0$. Then the end behavior of the graph of $P(x)$ will match the end behavior of the graph of its leading term $y = a_n x^n$.
- **Number of Turning Points for a Polynomial Function**: A polynomial of degree n will have, at most, $n - 1$ turning points and n horizontal intercepts (zeros).

Common Mistakes

- **Confusing Power Functions with Exponential Functions**: A power function has a variable base and a fixed power (e.g., x^2),

while an exponential function has a fixed base and a variable power (e.g., 2^x). The function $f(x) = 2^x$ is not a power function.

- **Incorrectly Identifying the Degree or Leading Coefficient**: When a polynomial is not written in standard general form (descending powers), care must be taken to identify the term with the highest power to determine the degree and leading coefficient.
- **Errors in Determining End Behavior**: Misinterpreting the combined effect of the sign of the leading coefficient and the parity (even/odd) of the degree of the polynomial on its end behavior.
- **Assuming Maximum Number of Turning Points/Intercepts is Always Achieved**: A polynomial of degree n has *at most* $n - 1$ turning points and *at most* n x-intercepts. It does not necessarily have exactly that many.

This page titled 2.2.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

2.2.2: Homework

Reading Questions

1. What is the general form of a power function? Give an example of a toolkit function that is a power function.
2. Is $f(x) = 2^x$ a power function? Why or why not?
3. How does the end behavior of an even-degree power function $f(x) = kx^n$ (where n is even) differ based on the sign of k?
4. How does the end behavior of an odd-degree power function $f(x) = kx^n$ (where n is odd) differ based on the sign of k?
5. What is the general form of a polynomial function? What restriction is placed on the exponents of the variable?
6. Define the "degree" of a polynomial.
7. Define the "leading term" and "leading coefficient" of a polynomial.
8. How does the leading term of a polynomial function relate to its end behavior?
9. What is a "turning point" of a polynomial graph?
10. What is the maximum number of x-intercepts a polynomial of degree n can have?
11. What is the maximum number of turning points a polynomial of degree n can have?
12. How do you find the y-intercept of a polynomial function given in general form?
13. If a polynomial function is given in factored form, like $f(x) = (x-2)(x+1)(x-4)$, how can you find its x-intercepts?
14. Are the graphs of polynomial functions smooth and continuous? What do these terms mean informally?
15. Explain the difference between the coefficient of a power function and its degree.
16. If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?
17. In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.
18. What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph?
19. What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$.

Homework

Algebraic

For the following exercises, identify the function as a power function, a polynomial function, or neither.

21. $f(x) = x^5$
22. $f(x) = (x^2)^3$
23. $f(x) = x - x^4$
24. $f(x) = \frac{x^2}{x^2-1}$
25. $f(x) = 2x(x+2)(x-1)^2$
26. $f(x) = 3^{x+1}$

For the following exercises, find the degree and leading coefficient for the given polynomial.

27. $-3x^4$
28. $7 - 2x^2$
29. $-2x^2 - 3x^5 + x - 6$
30. $x(4 - x^2)(2x + 1)$
31. $x^2(2x - 3)^2$

For the following exercises, determine the end behavior of the functions.

32. $f(x) = x^4$
33. $f(x) = x^3$
34. $f(x) = -x^4$
35. $f(x) = -x^9$

36. $f(x) = -2x^4 - 3x^2 + x - 1$
37. $f(x) = 3x^2 + x - 2$
38. $f(x) = x^2(2x^3 - x + 1)$
39. $f(x) = (2 - x)^7$

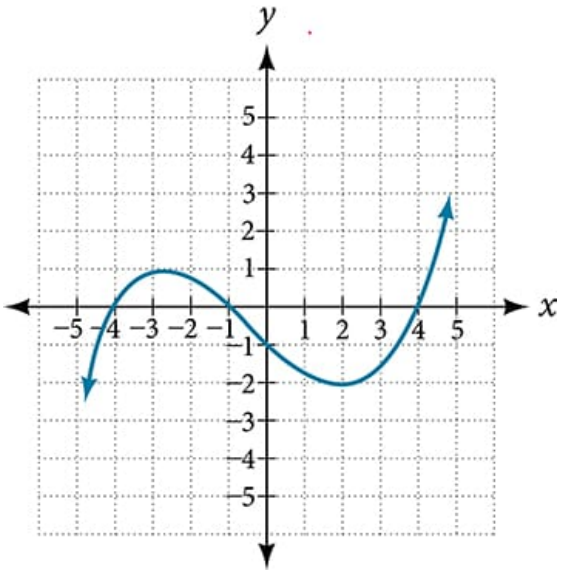
For the following exercises, find the intercepts of the functions.

40. $f(t) = 2(t - 1)(t + 2)(t - 3)$
41. $g(n) = -2(3n - 1)(2n + 1)$
42. $f(x) = x^4 - 16$
43. $f(x) = x^3 + 27$
44. $f(x) = x(x^2 - 2x - 8)$
45. $f(x) = (x + 3)(4x^2 - 1)$

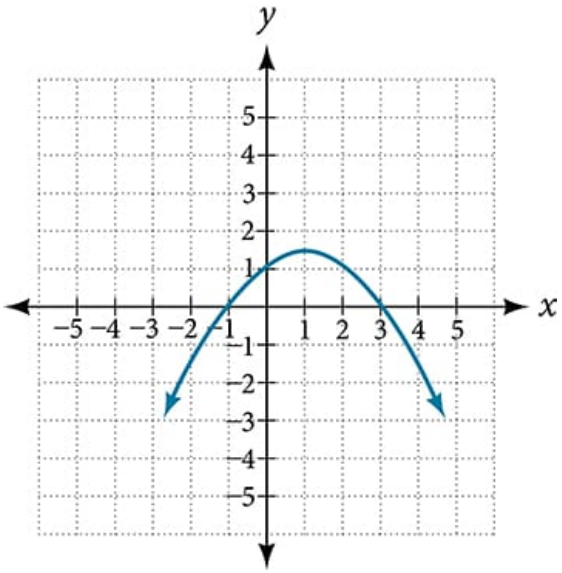
Graphical

For the following exercises, determine the least possible degree of the polynomial function shown.

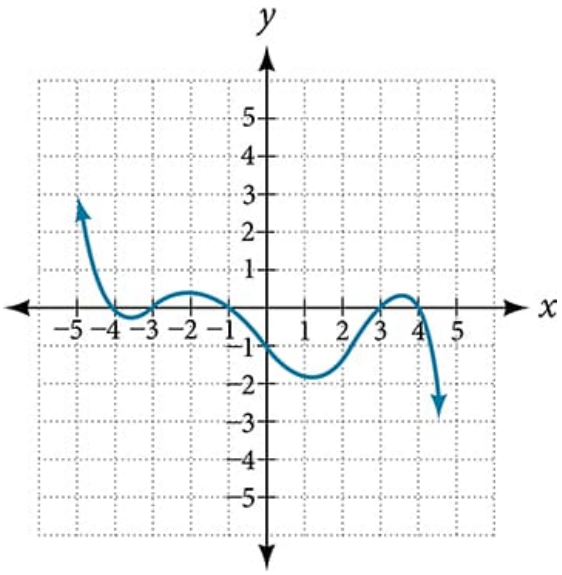
46.



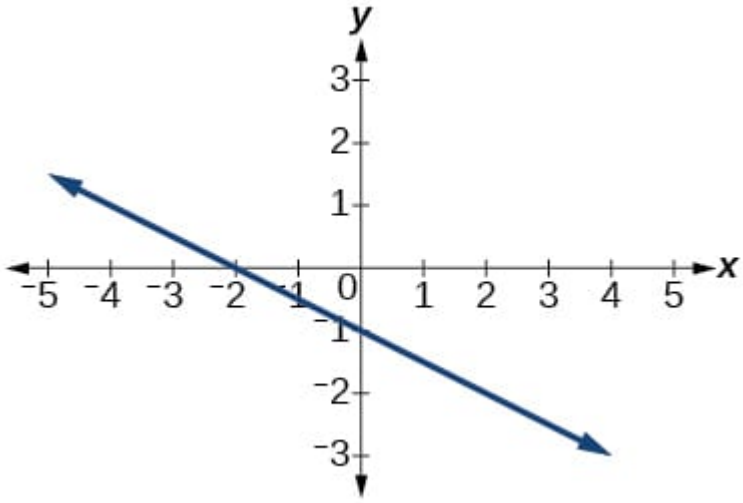
47.



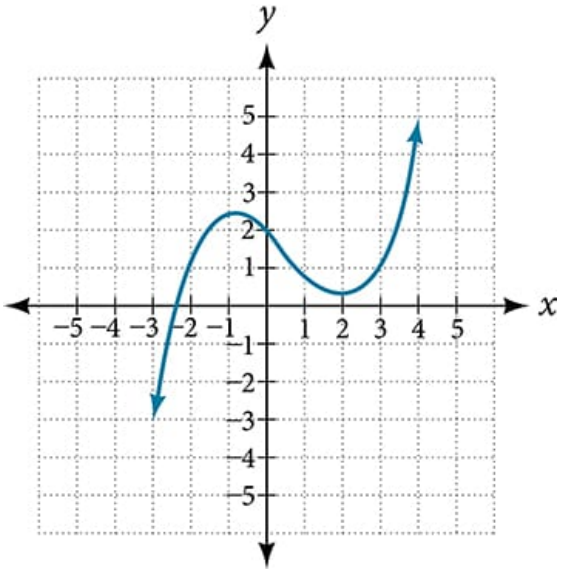
18.


y
x
5
4
3
2
1
−5 −4 −3 −2 −1
1 2 3 4 5
−1
−2
−3
−4
−5

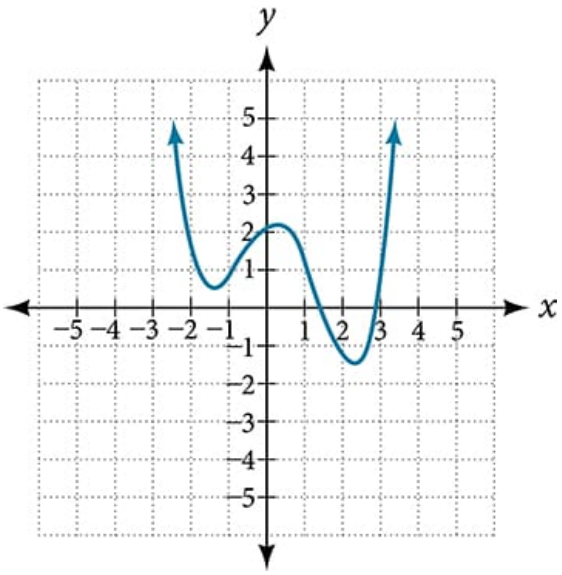
19.


y
x
3
2
1
−5 −4 −3 −2 −1 0 1 2 3 4 5
−1
−2
−3

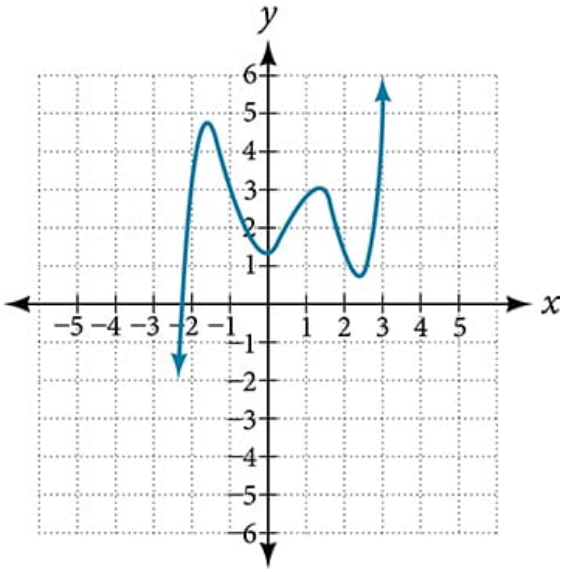
50.


y
x
5
4
3
2
1
−5 −4 −3 −2 −1
1 2 3 4 5
−1
−2
−3
−4
−5

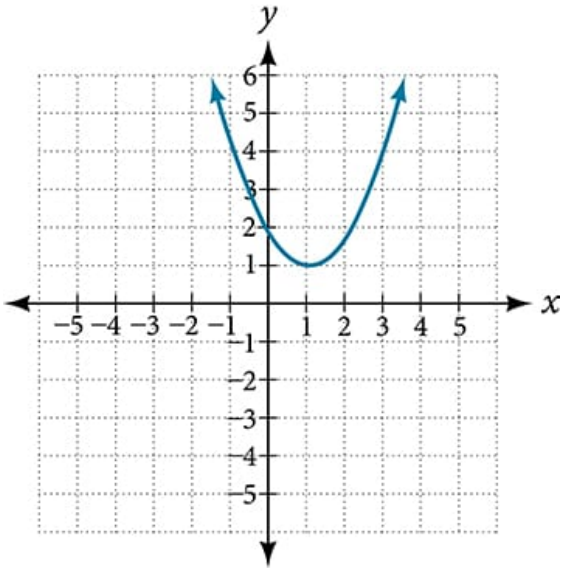
51.



52.

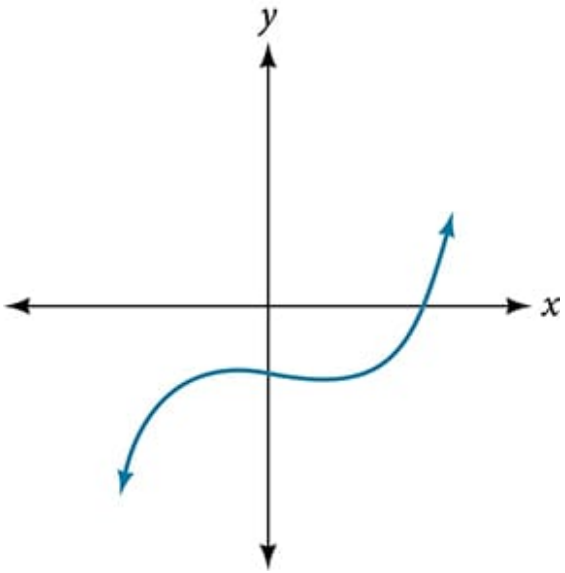


53.



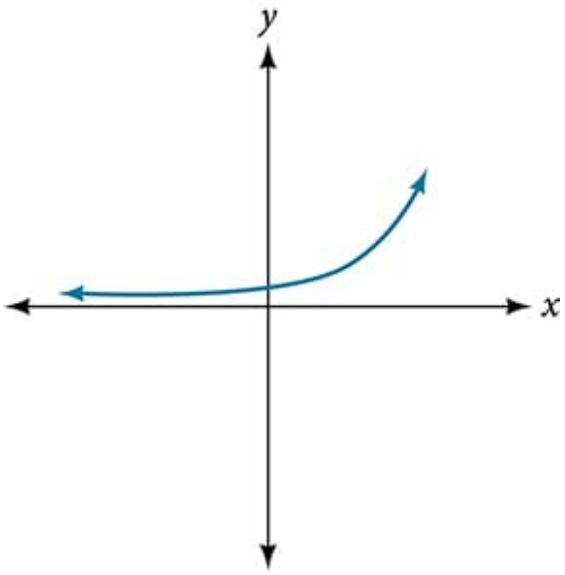
For the following exercises, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.

54.

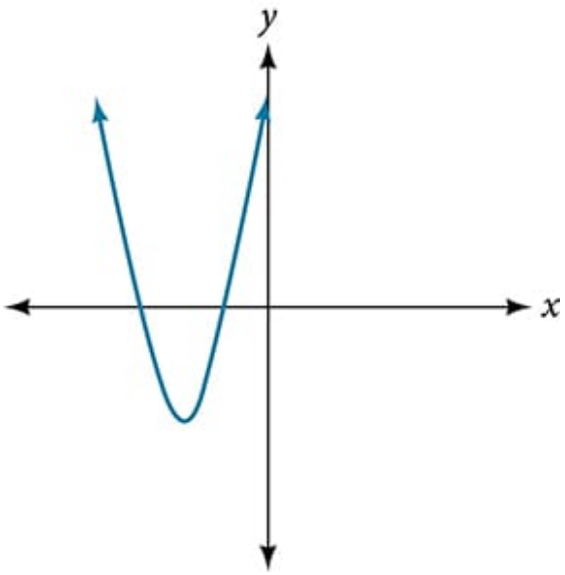


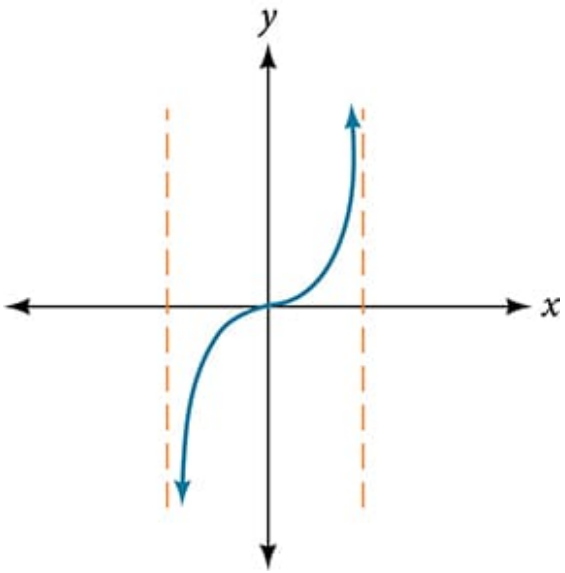
55.

56.

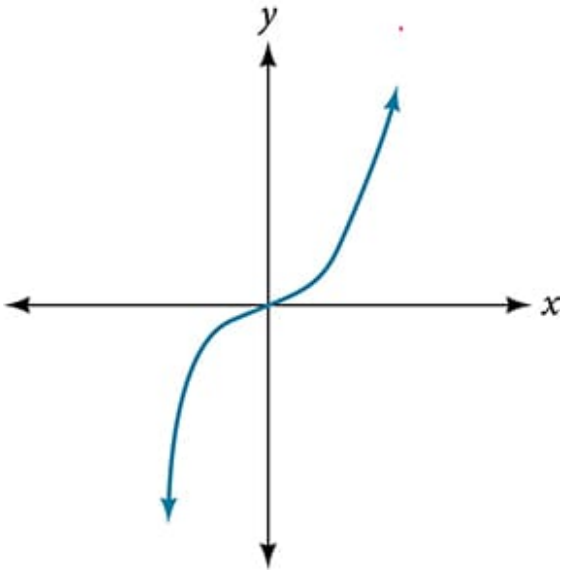


57.

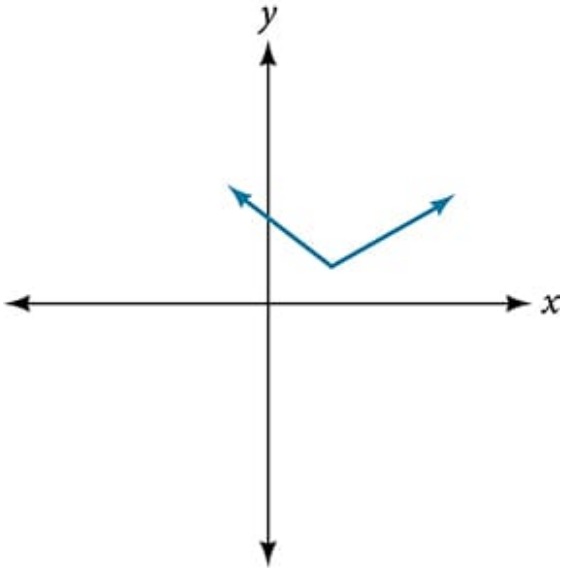




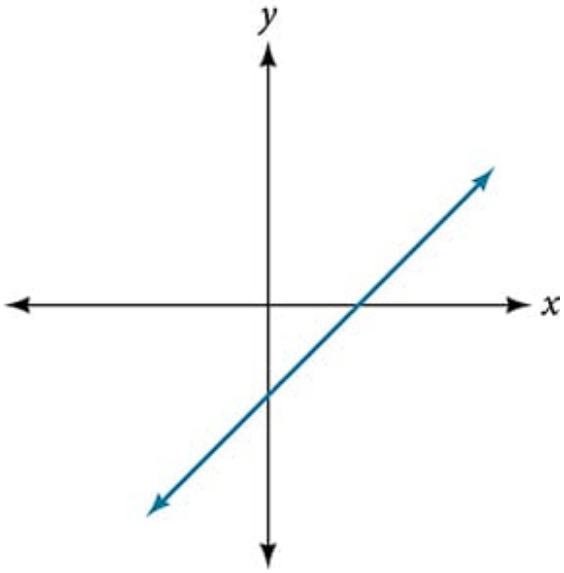
58.



59.



60.



Numeric

For the following exercises, make a table to confirm the end behavior of the function.

61. $f(x) = -x^3$
62. $f(x) = x^4 - 5x^2$
63. $f(x) = x^2(1-x)^2$
64. $f(x) = (x-1)(x-2)(3-x)$
65. $f(x) = \frac{x^5}{10} - x^4$

Technology

For the following exercises, graph the polynomial functions using graphing technology. Based on the graph, determine the intercepts and the end behavior.

66. $f(x) = x^3(x-2)$
67. $f(x) = x(x-3)(x+3)$

58. $f(x) = x(14 - 2x)(10 - 2x)$
59. $f(x) = x(14 - 2x)(10 - 2x)^2$
70. $f(x) = x^3 - 16x$
71. $f(x) = x^3 - 27$
72. $f(x) = x^4 - 81$
73. $f(x) = -x^3 + x^2 + 2x$
74. $f(x) = x^3 - 2x^2 - 15x$
75. $f(x) = x^3 - 0.01x$

Extensions

For the following exercises, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is 1 or –1. There may be more than one correct answer.

76. The y-intercept is $(0, -4)$. The x-intercepts are $(-2, 0)$, $(2, 0)$. Degree is 2. End behavior: as $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to \infty$.
77. The y-intercept is $(0, 9)$. The x-intercepts are $(-3, 0)$, $(3, 0)$. Degree is 2. End behavior: as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to -\infty$.
78. The y-intercept is $(0, 0)$. The x-intercepts are $(0, 0)$, $(2, 0)$. Degree is 3. End behavior: as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.
79. The y-intercept is $(0, 1)$. The x-intercept is $(1, 0)$. Degree is 3. End behavior: as $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to -\infty$.
80. The y-intercept is $(0, 1)$. There is no x-intercept. Degree is 4. End behavior: as $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to \infty$.

Real-World Applications

For the following exercises, use the written statements to construct a polynomial function that represents the required information.

81. An oil slick is expanding as a circle. The radius of the circle is increasing at the rate of 20 meters per day. Express the area of the circle as a function of d, the number of days elapsed.
82. A cube has an edge of 3 feet. The edge is increasing at the rate of 2 feet per minute. Express the volume of the cube as a function of m, the number of minutes elapsed.
83. A rectangle has a length of 10 inches and a width of 6 inches. If the length is increased by x inches and the width increased by twice that amount, express the area of the rectangle as a function of x.
84. An open box is to be constructed by cutting out square corners of x-inch sides from a piece of cardboard 8 inches by 8 inches and then folding up the sides. Express the volume of the box as a function of x.
85. A rectangle is twice as long as it is wide. Squares of side 2 feet are cut out from each corner. Then the sides are folded up to make an open box. Express the volume of the box as a function of the width (x).

This page titled 2.2.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

2.3: Graphs of Polynomial Functions

Note to the Instructor (click to expand)

This section introduces multiplicities of zeros for polynomials and goes over the process of *accurately* graphing a polynomial. Make sure you focus on the local behavior of the function near a zero with multiplicities greater than 2 (discussing the "flattening" as the multiplicity increases). Solving polynomial inequalities using graphs is also a required topic.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Solving Equations**
  - Solving Polynomial Equations: While this section focuses on finding zeros (which is equivalent to solving $P(x) = 0$), prior experience with solving polynomial equations by factoring is beneficial.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Recognize characteristics of graphs of polynomial functions.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Use the Factor Theorem.
- Determine end behavior.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.
- Solve a polynomial inequality (using a graph).

Hawk A.I. Section-Specific Tutor

EST. 1970 COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in Table 2.3.1.

Table 2.3.1

Year	2006	2007	2008	2009	2010	2011	2012	2013
Revenues	52.4	52.8	51.2	49.5	48.6	48.6	48.7	47.1

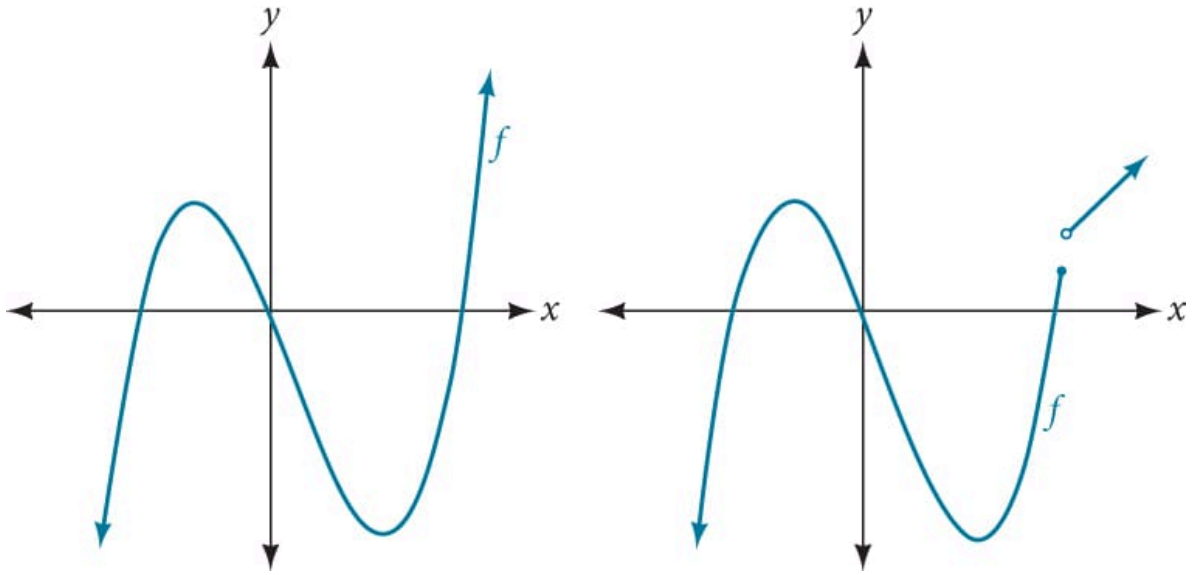
The revenue can be modeled by the polynomial function $R(t) = -0.037t^4 + 1.414t^3 - 19.777t^2 + 118.696t - 205.332$, where R represents the revenue in millions of dollars and t represents the year, with $t = 6$ corresponding to 2006. Over which intervals is the revenue for the company increasing? Over which intervals is the revenue for the company decreasing?

These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous (again, as mention in the previous section, this is not technically true; however, we will use this flawed definition of continuity until we reach Calculus). Figure 2.3.1 shows a graph that represents a polynomial function and a graph that represents a function that is not a polynomial.

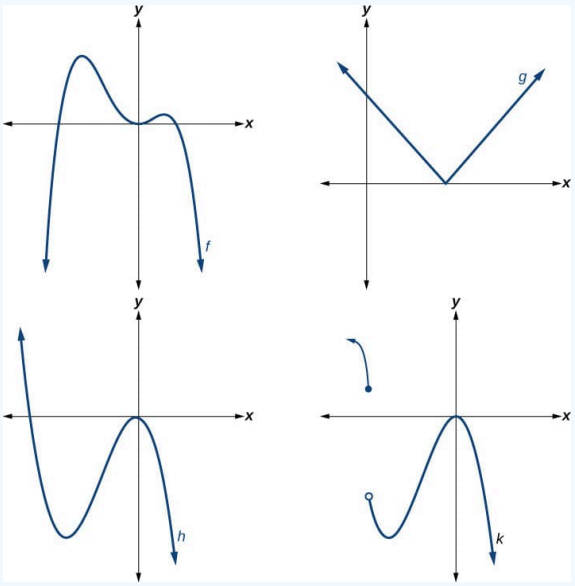
Figure 2.3.1



Example 2.3.1: Recognizing Polynomial Functions

Which of the graphs in Figure 2.3.2 represents a polynomial function?

Figure 2.3.2



Solution available at the end of this section.

Q&A

Do all polynomial functions have as their domain all real numbers?

Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f(x) = 0$ are called zeros of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple Quadratic Formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

1. The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.
3. Technology is used to determine the intercepts.

Example 2.3.2: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Solution available at the end of this section.

Example 2.3.3: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

Solution available at the end of this section.

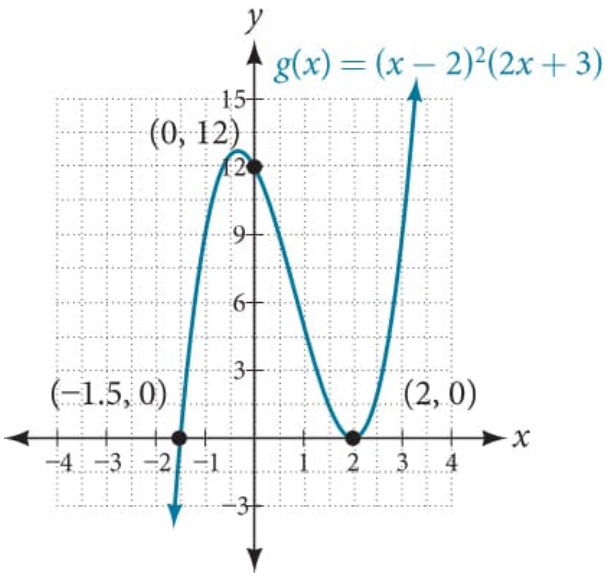
Example 2.3.4: Finding the y- and x-Intercepts of a Polynomial in Factored Form

Find the y- and x-intercepts of $g(x) = (x-2)^2(2x+3)$.

Solution available at the end of this section.

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial. For example, Figure 2.3.5 shows that our answers for Example 2.3.4 are correct.

Figure 2.3.5



Example 2.3.5: Finding the x-Intercepts of a Polynomial Function Using a Graph

Find the x-intercepts of $h(x) = x^3 + 4x^2 + x - 6$.

Solution available at the end of this section.

Checkpoint 2.3.5

Find the y- and x-intercepts of the function $f(x) = x^4 - 19x^2 + 30x$.

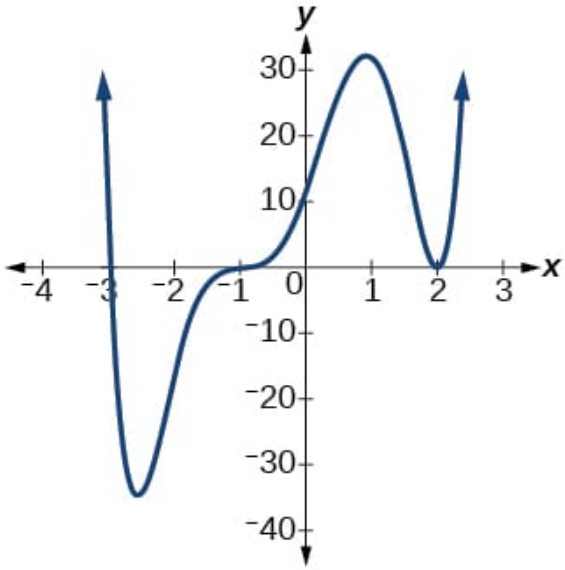
Identifying Zeros and Their Multiplicities

Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and bounce off.

Suppose, for example, we graph the function $f(x) = (x+3)(x-2)^2(x+1)^3$. Notice in Figure 2.3.7 that the behavior of the function at each of the x-intercepts is different.

Figure 2.3.7

Identifying the behavior of the graph at an x-intercept by examining the multiplicity of the zero.



The x-intercept at $x = -3$ is the solution of equation $(x + 3) = 0$. The graph passes directly through the x-intercept at $x = -3$. We say the factor is **linear** (has a degree of 1), so the behavior near the intercept is like that of a line - it passes directly through the intercept. We call this a **single zero** because the zero corresponds to a single factor of the function.

The x-intercept at $x = 2$ is the repeated solution of equation $(x - 2)^2 = 0$. The graph touches the axis at the intercept and changes direction. We say the factor is **quadratic** (degree 2), so the behavior near the intercept is like that of a quadratic - it bounces off of the horizontal axis at the intercept. Since $(x - 2)^2 = (x - 2)(x - 2)$, the factor is repeated. This leads us to an important definition that will be useful when it comes to polynomial functions.

Definition: Multiplicity of a Zero

Let h be a zero of a nonzero polynomial function f. The **multiplicity** of h is the largest positive integer p for which $(x - h)^p$ is a factor of $f(x)$. We say that $x = h$ is a **zero of multiplicity** p. A zero of multiplicity 1 is called a **simple zero**; a zero of multiplicity greater than 1 is called a **repeated zero** (or **multiple zero**).

Hence, the zero associated with $x = 2$ has multiplicity 2 because the factor $(x - 2)$ occurs twice.

The x-intercept at $x = -1$ is the repeated solution of factor $(x + 1)^3 = 0$. The graph passes through the axis at the intercept, but flattens out a bit first. We say this factor is **cubic** (degree 3), so the behavior near the intercept is like that of a cubic - with the same "S"-shape near the intercept as the toolkit function $f(x) = x^3$. By definition, this is a zero of multiplicity 3.

Now that we know what it means for a zero to have multiplicity p, it is natural to ask if there is any ramification of the multiplicity of a zero.

Theorem: Local Behavior Near a Zero

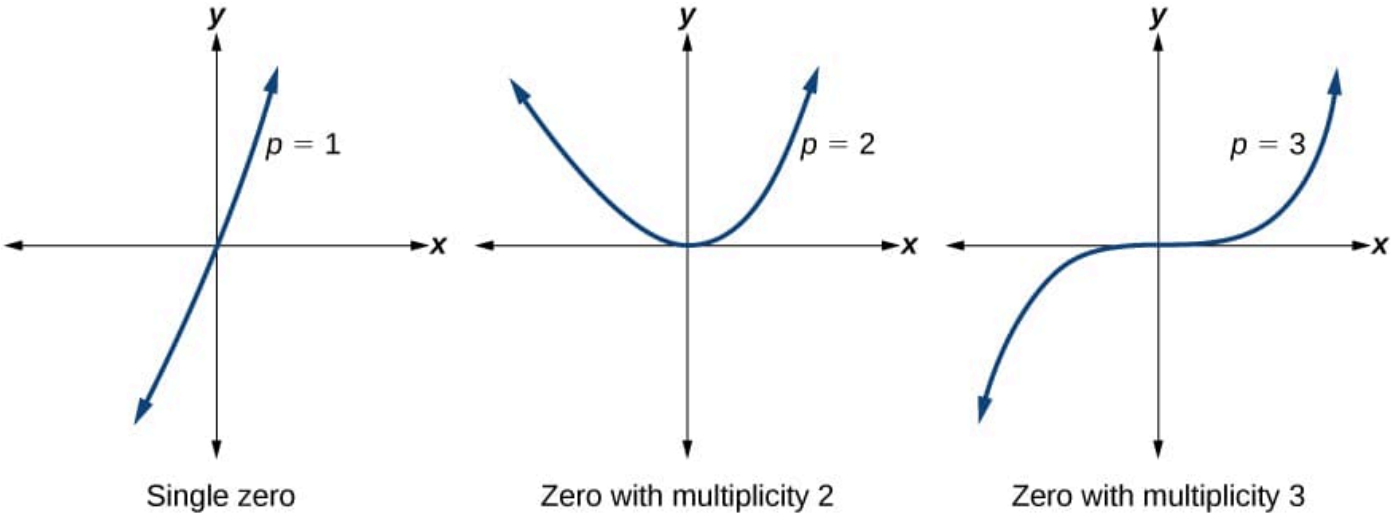
If $x = h$ is a zero of multiplicity p for a given polynomial, then the local behavior of the graph of the polynomial near $x = h$ matches the behavior of $y = \pm(x - h)^p$, where the sign of the coefficient is determined by other characteristics of the given polynomial.

In truth, most people (myself included) would say that the behavior of the polynomial near to a zero of multiplicity p matches the graph of $y = x^p$, but possibly reflected about the x-axis. What we *mean* is that the behavior of the graph near to the zero matches

the look of $y = x^p$ near the origin.

For zeros with even multiplicities, the graphs *touch* or are **tangent** to the x-axis. For zeros with odd multiplicities, the graphs *cross* or intersect the x-axis. See Figure $2.3.8$ for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.

Figure $2.3.8$



For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis. For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

Caution: Graph Accurately Near Zeros!

You will eventually be asked to make accurate sketches of polynomial functions. Part of this skill is to show that you know the impact of high multiplicity on a zero. The higher the multiplicity, the flatter the polynomial graph is near to the zero - be prepared to show this.

Finally, you will inevitably notice the following fact.

Theorem: Sum of Multiplicities and the Degree of a Polynomial

The degree of a polynomial factored into the product of linear and irreducible quadratic factors is sum of the multiplicities of the linear factors and the twice the sum of the multiplicities of the irreducible quadratic factors. That is, if

$$P(x) = (a_1x + b_1)^{m_1}(a_2x + b_2)^{m_2} \cdots (a_kx + b_k)^{m_k}(c_1x^2 + d_1)^{p_1}(c_2x^2 + d_2)^{p_2} \cdots (c_nx^2 + d_n)^{p_n},$$

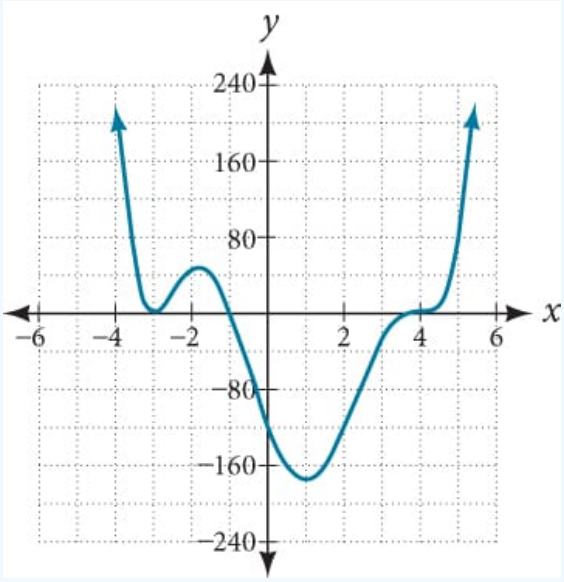
then the degree of $P(x)$ is

$$m_1 + m_2 + \cdots + m_k + 2(p_1 + p_2 + \cdots p_n).$$

Example $2.3.6$: Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in Figure $2.3.9$ to identify the zeros of the function and their possible multiplicities.

Figure 2.3.9

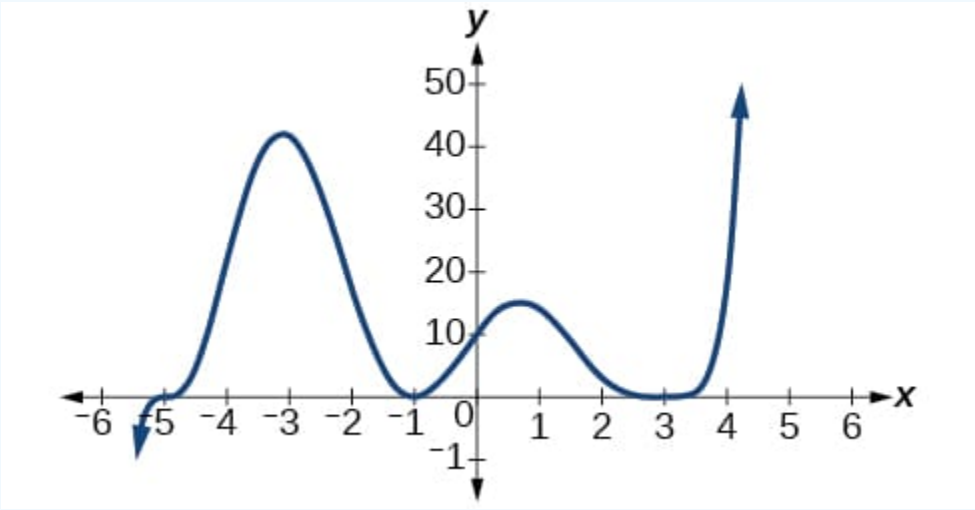


Solution available at the end of this section.

Checkpoint 2.3.6

Use the graph of the function of degree 7 in Figure 2.3.10 to identify the zeros of the function and their multiplicities.

Figure 2.3.10



Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors.

Theorem: Factor Theorem

Let $f(x)$ be a nonzero polynomial function having degree greater than zero. c is a zero of $f(x)$ if and only if $(x-c)$ is a factor of $f(x)$.

Proof available at the end of this section.

We have been "dancing around" the Factor Theorem for a a little while now; however, it wasn't until now that we formally stated the result. We can use the Factor Theorem to completely factor a polynomial into the product of n factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

Example 2.3.7: Using the Factor Theorem

Show that $(x+2)$ is a factor of x^3-6x^2-x+30.

Solution available at the end of this section.

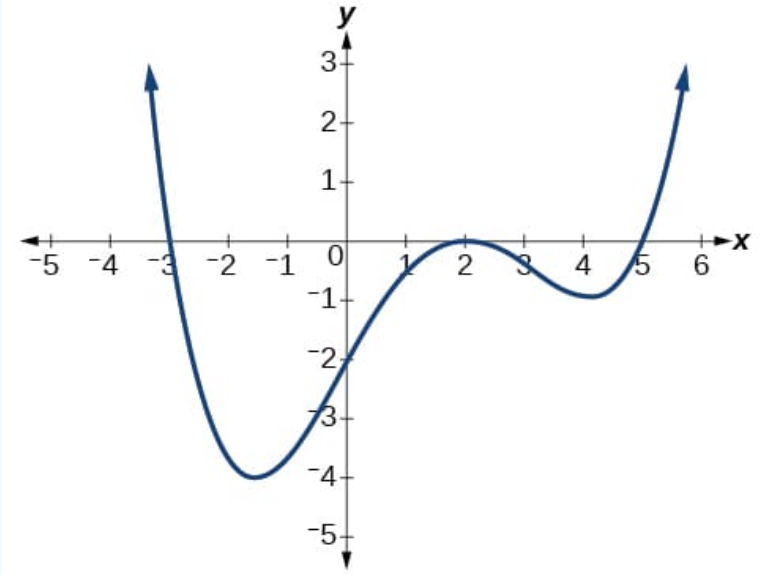
Checkpoint 2.3.7

Use the Factor Theorem to show that $(x-2)$ is a factor of $f(x)=x^3+4x^2-4x-16$.

Example 2.3.8: Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in Figure 2.3.11.

Figure 2.3.11

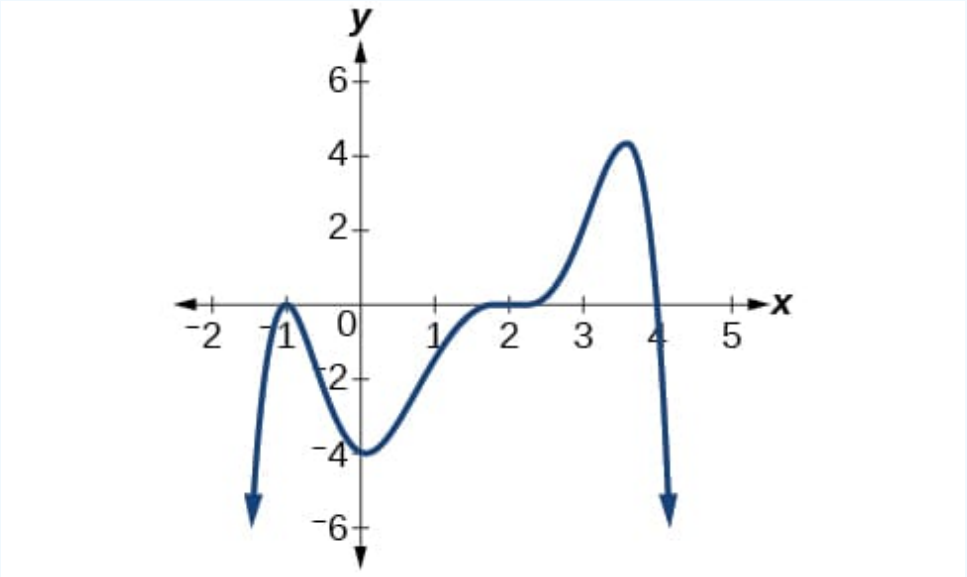


Solution available at the end of this section.

Checkpoint 2.3.8

Given the graph shown in Figure 2.3.12, write a formula for the function shown.

Figure 2.3.12



Determining End Behavior

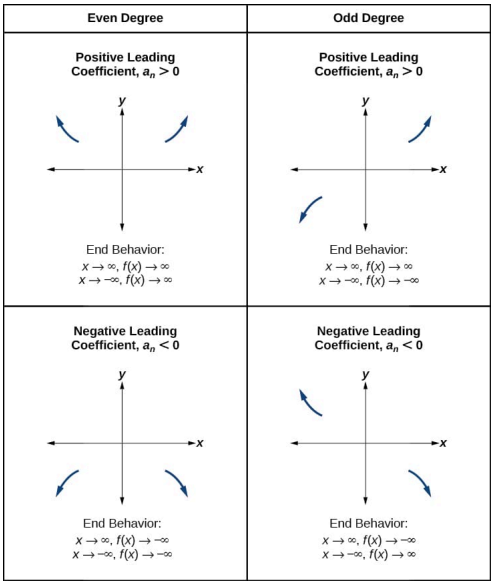
As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

will either ultimately rise or fall as x increases or decreases without bound. This is because, for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very negative inputs, say –100 or -1,000.

Recall that we call this behavior the end behavior of a function. As we pointed out when discussing power functions, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure 2.3.13 summarizes all four cases.

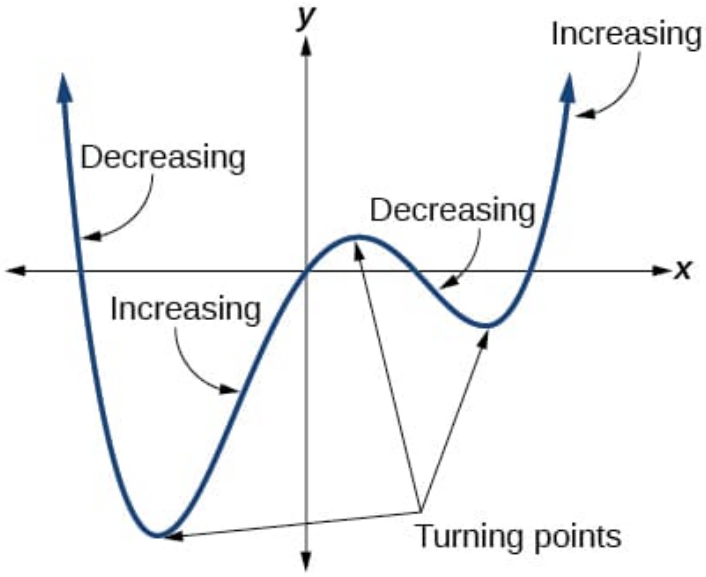
Figure 2.3.13



Understanding the Relationship Between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's *local behavior*. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure $2.3.14$. The graph has three turning points.

Figure $2.3.14$



This function is a 4th degree polynomial function and has 3 turning points. This should be familiar because, as mentioned in the previous section, the maximum number of turning points of a polynomial function is always one less than the degree of the function.

Example $2.3.9$: Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

1. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$
2. $f(x) = -(x-1)^2(1+2x^2)$

Solutions available at the end of this section.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

How To: Graph a polynomial function

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetric about the y-axis. That is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetric about the origin. That is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
4. Determine the end behavior by examining the leading term.

5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

Example 2.3.10: Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution available at the end of this section.

Checkpoint 2.3.10

Sketch a graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$.

While you will not be graphing the absolute value of polynomial functions in Calculus, you *will* be graphing the absolute value of other functions. Therefore, it's a good idea to keep that skill fresh (it's a simple skill once you know how to graph the non-absolute value function).

Example 2.3.11: Revisiting the Graphs of Absolute Value Functions

Graph

$$f(x) = \left|-0.9(x+2)^3(x-1)^6(x-5)^5\right|.$$

Solution available at the end of this section.

Solving Polynomial Inequalities

We finish this section with a topic that might not seem related to graphing polynomial functions - solving polynomial inequalities.

There are those who insist on students solving inequalities using sign charts, and then there are those who want to reinforce a student's graphing skills. If you cannot tell, I am in the latter camp. To solve *any* inequality, you could move all terms to one side of the inequality, graph the resulting expression on the non-zero side, and use that graph to solve the inequality. The key to being able to do this is know how to graph that non-zero expression.

Let's see this in an example.

Example 2.3.12

Solve

$$3x^3 - 4x \geq 6x^2 - 8.$$

Solution available at the end of this section.

Answers

Example 2.3.1: Recognizing Polynomial Functions

The graphs of f and h are graphs of polynomial functions. They are smooth and continuous.

The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

Example 2.3.2: Finding the x-Intercepts of a Polynomial Function by Factoring

We can attempt to factor this polynomial to find solutions for $f(x) = 0$.

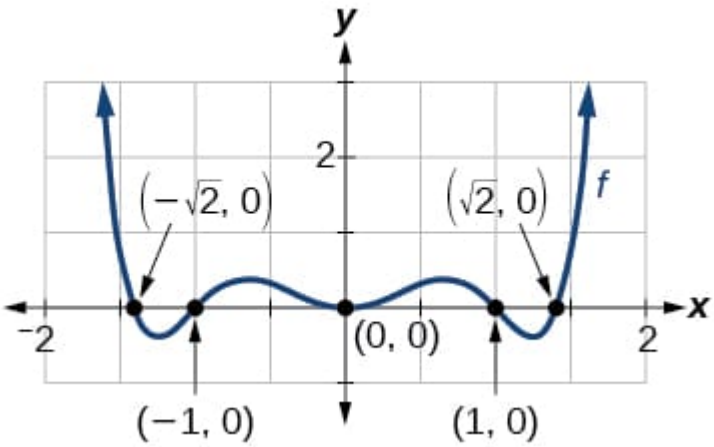
$$
\begin{aligned}
x^6 - 3x^4 + 2x^2 &= 0 \\
\Longrightarrow \quad x^2(x^4 - 3x^2 + 2) &= 0 && \text{(factoring out the GCF)} \\
\Longrightarrow \quad x^2(u^2 - 3u + 2) &= 0 && \text{(substituting } u = x^2 \text{ in the trinomial)} \\
\Longrightarrow \quad x^2(u-2)(u-1) &= 0 && \text{(factoring the trinomial)} \\
\Longrightarrow \quad x^2(x^2-2)(x^2-1) &= 0 && \text{(resubstituting } u = x^2) \\
\Longrightarrow \quad x^2(x^2-2)(x-1)(x+1) &= 0 && \text{(factoring a Difference of Squares)}
\end{aligned}
$$

Thus,

$$
\begin{array}{rcccccc}
 & x^2 = 0 & \text{or} & x^2 - 2 = 0 & \text{or} & x - 1 = 0 & \text{or} \quad x + 1 = 0 \\
\Longrightarrow & x = 0 & & x^2 = 2 & & x = 1 & x = -1 \\
\Longrightarrow & x = 0 & & x = \pm\sqrt{2} & & x = 1 & x = -1
\end{array}
$$

This gives us five x-intercepts: $(0,0)$, $(1,0)$, $(-1,0)$, $(\sqrt{2},0)$, and $(-\sqrt{2},0)$ (see Figure 2.3.3). We can see that this is an even function.

Figure 2.3.3



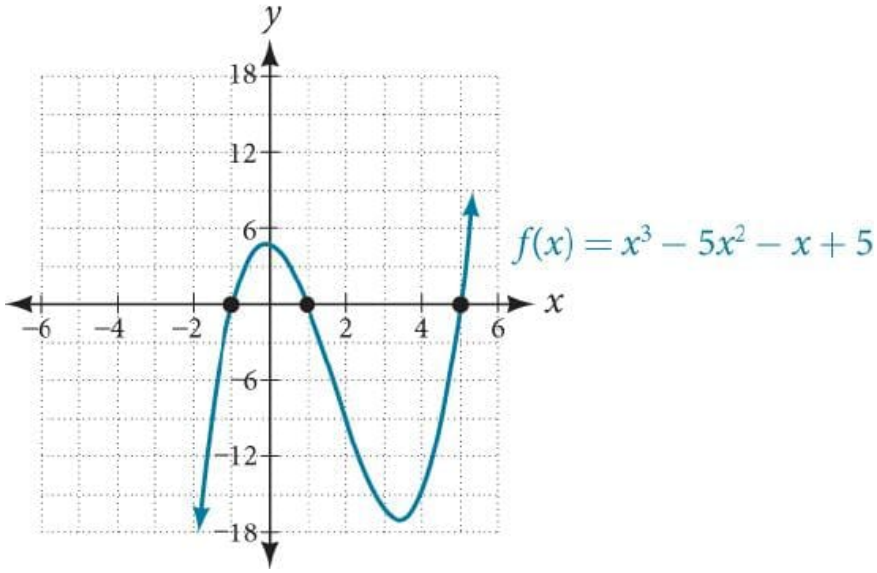
Example 2.3.3**: Finding the x-Intercepts of a Polynomial Function by Factoring**

Find solutions for $f(x) = 0$ by factoring.

$$
\begin{aligned}
x^3 - 5x^2 - x + 5 &= 0 \\
\Longrightarrow \quad x^2(x-5) - 1(x-5) &= 0 && \text{(factoring by grouping)} \\
\Longrightarrow \quad (x-5)(x^2-1) &= 0 && \text{(factor out the GCF)} \\
\Longrightarrow \quad (x-5)(x-1)(x+1) &= 0 && \text{(factoring the Difference of Squares)}
\end{aligned}
$$

Hence, $x = 5$, or $x = 1$, or $x = -1$. There are three x-intercepts: $(-1,0)$, $(1,0)$, and $(5,0)$ (see Figure 2.3.4).

Figure 2.3.4



Example 2.3.4: Finding the y- and x-Intercepts of a Polynomial in Factored Form

The y-intercept can be found by evaluating $g(0)$.

$$g(0) = (0-2)^2(2(0)+3) = 12.$$

So the y-intercept is $(0, 12)$.

The x-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3) = 0 \implies x = 2 \text{ or } x = -\frac{3}{2}.$$

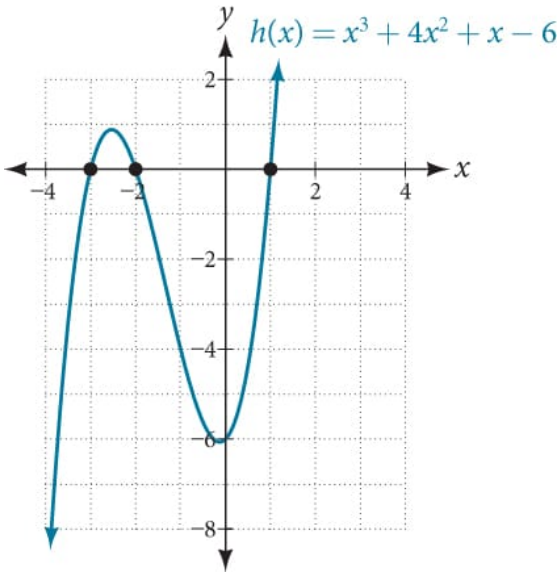
So the x-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

Example 2.3.5: Finding the x-Intercepts of a Polynomial Function Using a Graph

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in Figure 2.3.6, it appears that there are x-intercepts at $x = -3$, -2, and 1.

Figure 2.3.6



We can check whether these are correct by substituting these values for x and verifying that

$$h(-3) = h(-2) = h(1) = 0.$$

Since $h(x) = x^3 + 4x^2 + x - 6$, we have:

$$\begin{aligned} h(-3) &= (-3)^3 + 4(-3)^2 + (-3) - 6 = -27 + 36 - 3 - 6 = 0 \\ h(-2) &= (-2)^3 + 4(-2)^2 + (-2) - 6 = -8 + 16 - 2 - 6 = 0 \\ h(1) &= (1)^3 + 4(1)^2 + (1) - 6 = 1 + 4 + 1 - 6 = 0 \end{aligned}$$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$h(x) = x^3 + 4x^2 + x - 6 = (x+3)(x+2)(x-1).$$

Example 2.3.6: Identifying Zeros and Their Multiplicities

The polynomial function is of degree n. The sum of the multiplicities must be n.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of -3 could have multiplicity 2, 4, or 6 (remember, we were told this polynomial has degree 6, so the sum of the multiplicities of the zeros must be 6).

The next zero occurs at $x = -1$. The graph looks almost linear at this point, so this is likely a zero of multiplicity 1.

The last zero occurs at $x = 4$. The graph crosses the x-axis, so the multiplicity of the zero must be odd. From the shape of the curve near to $x = 4$, we know the multiplicity cannot be 1 (the graph is not *locally linear* near $x = 4$). Hence, $x = 4$ must be a zero of multiplicity either 3 or 5.

Now that we have an idea of the *possible* multiplicities, let's build a table and use a little deductive logic to identify the exact multiplicities.

Zero	Possible Multiplicities
-3	$2, 4, 6$
-1	1
4	$3, 5$

Since the sum of the multiplicities must be 6, and we have three zeros, the maximum multiplicity for any given zero is 3. This cleans up our table quite a bit:

Zero	Possible Multiplicities
-3	2
-1	1
4	3

Thus, we have identified the zeros are their exact multiplicities.

Theorem: Factor Theorem

Let $f(x)$ be a polynomial function of degree greater than zero.

( ): Suppose c is a zero of $f(x)$. Then $f(c) = 0$. By the Remainder Theorem, if $f(x)$ is divided by $x - c$, then

$$\frac{f(x)}{x-c} = Q(x) + \frac{R(x)}{x-c} = Q(x) + \frac{f(c)}{x-c} = Q(x) + \frac{0}{x-c} = Q(x).$$

Multiplying both sides by $x - c$, we get

$$f(x) = q(x)(x-c).$$

Hence, $x - c$ is a factor of $f(x)$.

( ): Suppose $x - c$ is a factor of $f(x)$. Then

$$f(x) = (x-c)g(x),$$

for some polynomial $g(x)$. Therefore,

$$f(c) = (c-c)g(c) = 0 \cdot g(c) = 0.$$

Thus, $f(c) = 0$, which means c is a zero of $f(x)$.

Example 2.3.7: Using the Factor Theorem

We let $f(x) = x^3 - 6x^2 - x - 30$ and evaluate $f(-2)$ to find that, indeed, $f(-2) = 0$. Therefore, $(x + 2)$ is a factor of the polynomial.

Example 2.3.8: Writing a Formula for a Polynomial Function from the Graph

This graph has three x-intercepts: $x = -3$, 2, and 5. The y-intercept is located at $(0, -2)$. At $x = -3$ and $x = 5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

$$f(x) = a(x+3)(x-2)^2(x-5).$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0,-2)$, to solve for a.

$$\begin{aligned} f(0) &= a(0+3)(0-2)^2(0-5) && \text{(substituting)} \\ \implies -2 &= a(0+3)(0-2)^2(0-5) && \text{(substituting)} \\ \implies -2 &= -60a && \text{(simplifying)} \\ \implies \frac{1}{30} &= a && \text{(dividing both sides by } -60) \end{aligned}$$

The graphed polynomial appears to represent the function $f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$.

Example 2.3.9: Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

1. First, rewrite the polynomial function in descending order: $f(x) = 4x^5 - x^3 - 3x^2 + 1$. Identify the degree of the polynomial function. This polynomial function is of degree 5. The maximum number of turning points is $5 - 1 = 4$.
2. First, identify the leading term of the polynomial function if the function were expanded. Looking at the given function, $f(x) = -(x-1)^2(1+2x^2)$, we can see that the factor $(x-1)^2$ will contribute x^2 (if we multiplied it out) and the factor $1 + 2x^2$ contributes $2x^2$. Therefore, the leading term is

   $$a_n x^n = -(x^2)(2x^2) = -2x^4.$$

   Therefore, this polynomial function is of degree 4. As such, the maximum number of turning points is $4 - 1 = 3$.

Example 2.3.10: Sketching the Graph of a Polynomial Function

This graph has two x-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x = 5$, the function has a multiplicity of one, indicating the graph will cross through the axis like a line at this intercept.

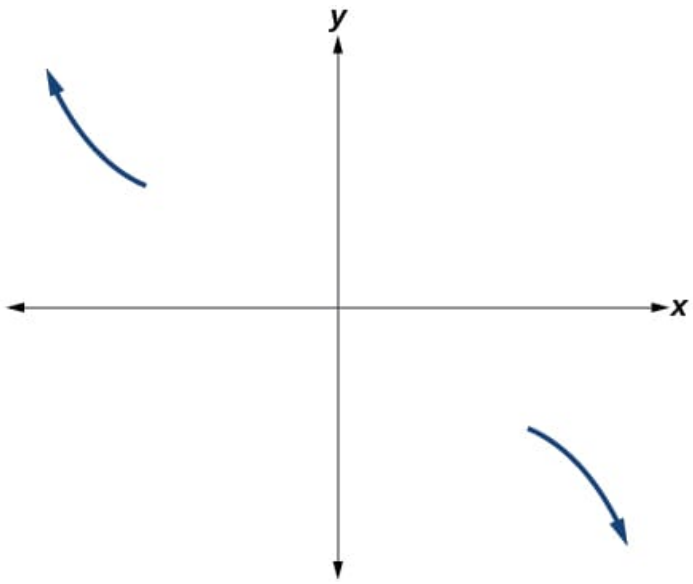
The y-intercept is found by evaluating $f(0)$.

$$f(0) = -2(0+3)^2(0-5) = -2 \cdot 9 \cdot (-5) = 90.$$

The y- intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See Figure 2.3.15.

Figure 2.3.15

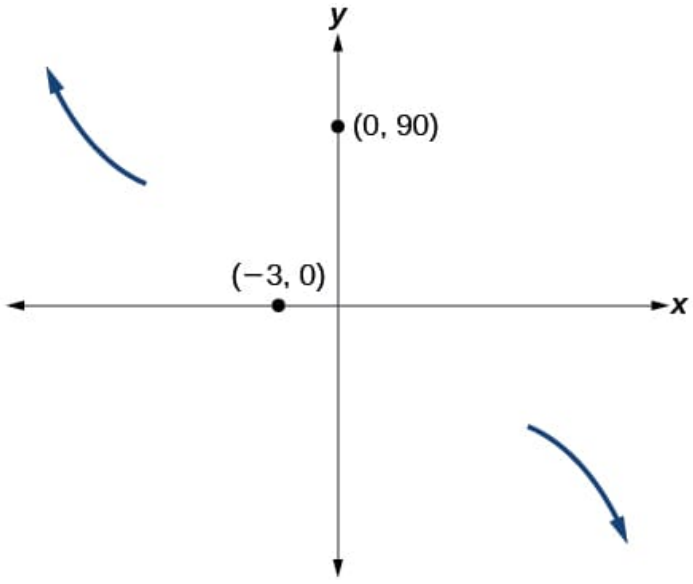


To sketch this, we consider that:

- As $x \to -\infty$, $f(x) \to \infty$. So we know the graph starts in the second quadrant and is decreasing toward the x-axis.
- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $\pm f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of the x-axis, so the function must start increasing.

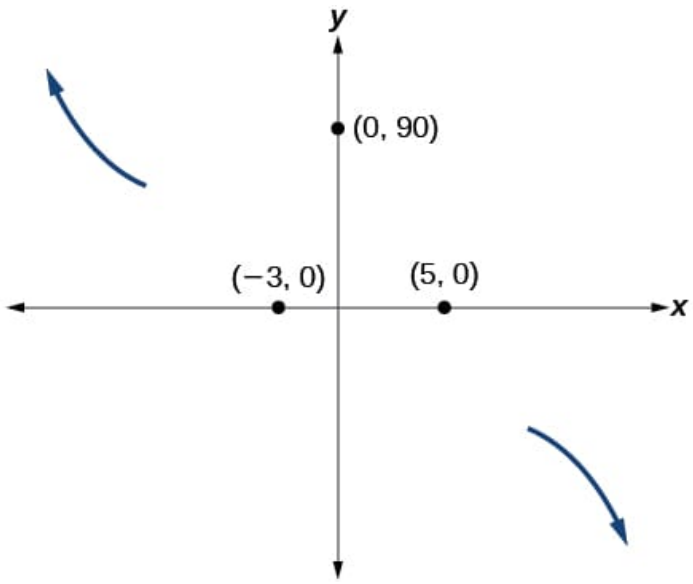
  At $(0, 90)$, the graph crosses the y-axis at the y-intercept. See Figure 2.3.16.

Figure 2.3.16



Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5, 0)$. See Figure 2.3.17.

Figure 2.3.17

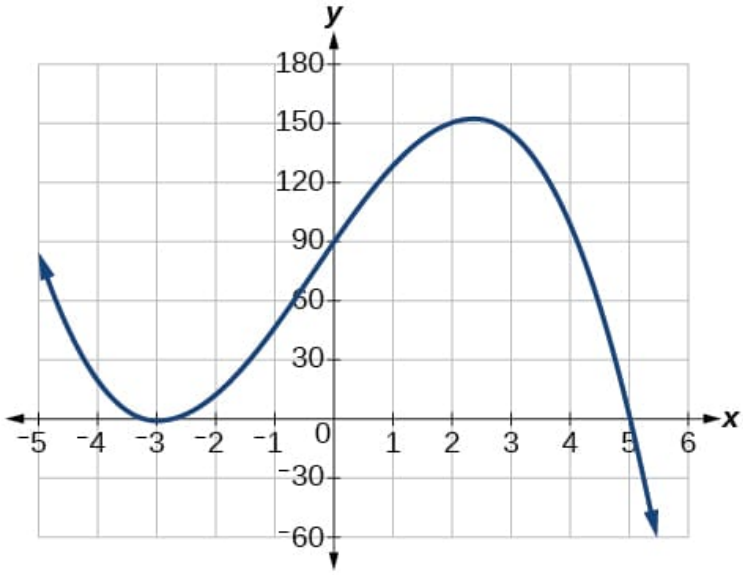


As $x \to \infty$, the function $f(x) \to -\infty$, so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure 2.3.18, and verify that the resulting graph looks like our sketch in Figure 2.3.17.

Figure 2.3.18

The complete graph of the polynomial function $f(x) = -2(x+3)^2(x-5)$



Example 2.3.11**: Revisiting the Graphs of Absolute Value Functions**

As we have done in previous sections, we initially concern ourselves with the *non-absolute value* function

$$f(x) = -0.9(x+2)^3(x-1)^6(x-5)^5.$$

We list the roots and their multiplicities, as well as the behavior of the graph near that root:

Root	Multiplicity	Behavior
-2	3	like x^3
1	6	like x^6 (a very flat parabola)
5	5	like x^5 (a flat cubic)

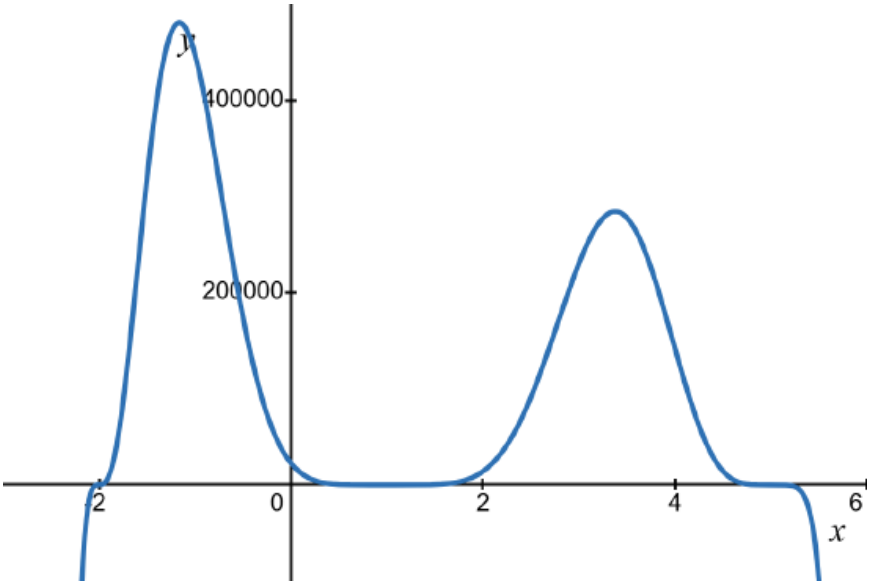
Next, we note that the degree of the polynomial is $3+6+5=14$, so the end behavior is like that of x^2 *except* the leading coefficient is negative. Therefore, the end behavior is more like $-x^2$. That is, as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to -\infty$. Reading from left to right, the graph of f should

- start low
- increase until it crosses the x-axis at $x = -2$ like a cubic
- climb for a bit, but then turn to fall back to the x-axis to "bounce" like a very flat parabola at $x = 1$

- climb for a bit, but then turn to fall back to the x-axis to cross at $x = 5$ like a flat cubic
- it then continues to fall forever.

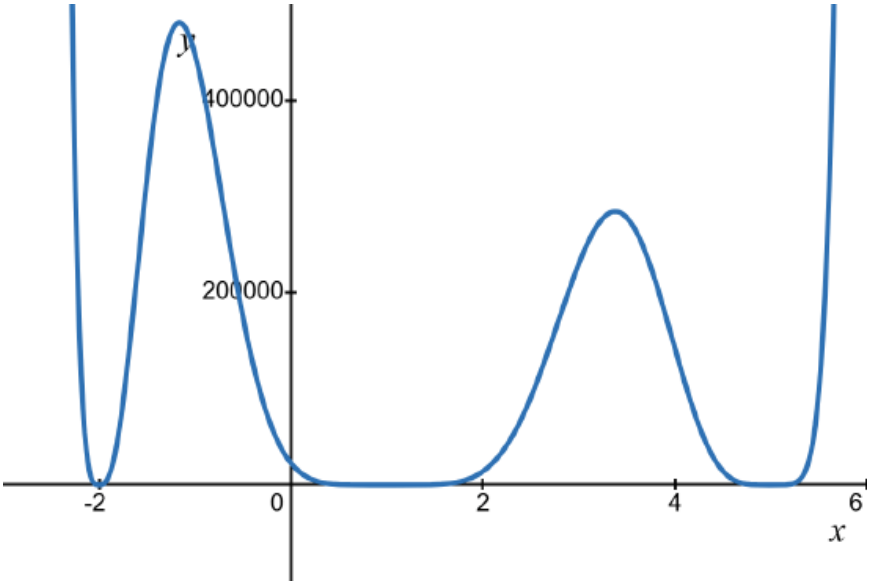
Figure $2.3.19$ shows the graph of the non-absolute value function.

Figure $2.3.19$



To graph the absolute value of this function, we take all the function outputs (y-values) that are negative and we force them to be positive. In effect, this keeps the part of the graph from Figure $2.3.19$ that is already above the x-axis in place; however, all the parts that are below the x-axis get reflected about the y-axis to become positive (above the x-axis). See Figure $2.3.20$.

Figure $2.3.20$



Example $2.3.12$

Let's start by getting all terms to one side.

$$\begin{aligned} & 3x^3 - 4x \geq 6x^2 - 8 \\ \implies \quad & 3x^3 - 6x^2 - 4x + 8 \geq 0 \end{aligned}$$

Now we want to graph the expression on the left side. That is, let's graph

$$g(x) = 3x^3 - 6x^2 - 4x + 8.$$

Since this is a polynomial, we would like to find the roots and their multiplicities. Let's try factoring by grouping:

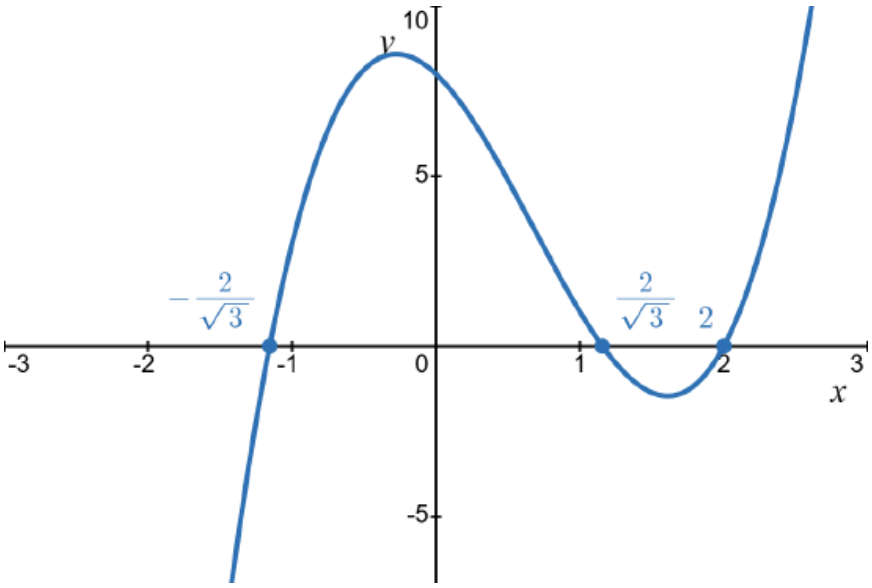
$$\begin{aligned} g(x) &= 3x^3 - 6x^2 - 4x + 8 \\ &= 3x^2(x-2) - 4(x-2) \\ &= (x-2)(3x^2 - 4) \end{aligned}$$

Hence, this polynomial has the following roots (and multiplicities):

Root	Multiplicity
2	1
$\frac{2}{\sqrt{3}}$	1
$-\frac{2}{\sqrt{3}}$	1

Since $g(x)$ is a cubic, we know it "starts low and ends high." A quick sketch yields the graph in Figure 2.3.21.

Figure 2.3.21



Remember, we are being asked to solve $3x^3 - 6x^2 - 4x + 8 \geq 0$. This is equivalent to asking, "When is the graph of $g(x) = 3x^3 - 6x^2 - 4x + 8$ at or above the x -axis?" The answer to this question is simple once you have the graph. We can see from Figure 2.3.21, that $g(x) \geq 0$ when

$$x^{\text{int}} \left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right] \cup [2, \infty) .$$

This page titled 2.3: Graphs of Polynomial Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

2.3.1: Resources and Key Concepts

Resources

Videos

- Graphs of polynomial functions
  - Graphs of Polynomial Functions - Graphing Polynomial Functions
  - Polynomial Functions and Their Graphs - Graphing Polynomials Using End Behavior and Zeros
- Zeros (and their multiplicities) of polynomial functions
  - Graphs of Polynomial Functions - Identifying Zeros and Their Multiplicities
  - Polynomial Functions and Their Graphs - Zeros of Functions and Their Multiplicities
- Factoring polynomials
  - Graphs of Polynomial Functions - Using Factoring to Find Zeros of Polynomial Functions
- Polynomial modeling
  - Polynomial Functions - Applications

Key Concepts

Definitions

- **Zeros of a Polynomial Function**: The values of x for which $f(x) = 0$. These correspond to the x-intercepts of the graph.
- **Multiplicity of a Zero**: If a polynomial contains a factor of the form $(x - h)^p$, and if this is the greatest such factor of $(x - p)$, we say that $x = h$ is a zero of multiplicity p.
- **Single Zero**: A zero of multiplicity 1. The graph passes directly through the x-axis at a single zero.

Theorems

- **Local Behavior Near a Zero**: If $x = h$ is a zero of multiplicity p for a given polynomial, then the local behavior of the graph of the polynomial near $x = h$ matches the behavior of $y = \pm(x - h)^p$, where the sign of the coefficient is determined by other characteristics of the given polynomial.
- **Sum of Multiplicities and the Degree of a Polynomial**: The sum of the multiplicities of the factors for a polynomial is the degree of the polynomial function.
- **Factor Theorem**: Let $f(x)$ be a polynomial function of degree greater than or equal to one. k is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$.

Common Mistakes

- **Incorrectly Identifying Multiplicity**: Miscounting the multiplicity of a zero, especially when a factor is repeated or when a zero comes from a factor like $(x^2 - k)$ which might yield two distinct zeros each of multiplicity 1.
- **Misinterpreting Behavior at Zeros based on Multiplicity**: Confusing whether the graph crosses or touches/bounces at an x-intercept based on the multiplicity of the zero (odd multiplicity means cross, even multiplicity means touch/bounce).
- **Errors in Determining End Behavior**: Incorrectly identifying the leading term (especially if the polynomial is not in standard form) or misapplying the rules for end behavior based on the degree and sign of the leading coefficient.
- **Exceeding Maximum Number of Turning Points**: Sketching a graph with more turning points than $n - 1$, where n is the degree of the polynomial.

This page titled 2.3.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

2.3.2: Homework

Reading Questions

1. What are two key visual characteristics of the graphs of polynomial functions of degree 2 or more?
2. What are the "zeros" of a polynomial function $f(x)$, and how do they relate to the graph's x-intercepts?
3. Describe three methods mentioned for finding the zeros of a polynomial function.
4. What does the "multiplicity" of a zero tell us about the behavior of the graph of a polynomial function at that x-intercept?
   - What happens if the multiplicity is odd?
   - What happens if the multiplicity is even?
5. How does the graph's behavior near an x-intercept change for higher even or odd multiplicities?
6. What determines the end behavior of a polynomial function's graph?
7. What is the maximum number of turning points a polynomial function of degree n can have?
8. List the general steps suggested for sketching the graph of a polynomial function.
9. If you know the x-intercepts and their multiplicities for a polynomial, how can you write a general formula for that polynomial function? What additional piece of information is needed to find the specific stretch factor a?
10. Describe a graphical approach to solving a polynomial inequality like $P(x) > 0$.
11. What is the difference between an x-intercept and a zero of a polynomial function f?
12. If a polynomial function of degree n has n distinct zeros, what do you know about the graph of the function?
13. Explain how the factored form of the polynomial helps us in graphing it.
14. If the graph of a polynomial just touches the x-axis and then changes direction, what can we conclude about the factored form of the polynomial?

Homework

Algebraic

For the following exercises, find the x- or t-intercepts of the polynomial functions.

15. $C(t) = 2(t-4)(t+1)(t-6)$
16. $C(t) = 3(t+2)(t-3)(t+5)$
17. $C(t) = 4t(t-2)^2(t+1)$
18. $C(t) = 2t(t-3)(t+1)^2$
19. $C(t) = 2t^4 - 8t^3 + 6t^2$
20. $C(t) = 4t^4 + 12t^3 - 40t^2$
21. $f(x) = x^4 - x^2$
22. $f(x) = x^3 + x^2 - 20x$
23. $f(x) = x^3 + 6x^2 - 7x$
24. $f(x) = x^3 + x^2 - 4x - 4$
25. $f(x) = x^3 + 2x^2 - 9x - 18$
26. $f(x) = 2x^3 - x^2 - 8x + 4$
27. $f(x) = x^6 - 7x^3 - 8$
28. $f(x) = 2x^4 + 6x^2 - 8$
29. $f(x) = x^3 - 3x^2 - x + 3$
30. $f(x) = x^6 - 2x^4 - 3x^2$
31. $f(x) = x^6 - 3x^4 - 4x^2$
32. $f(x) = x^5 - 5x^3 + 4x$

For the following exercises, find the zeros and give the multiplicity of each.

33. $f(x) = (x+2)^3(x-3)^2$
34. $f(x) = x^2(2x+3)^5(x-4)^2$
35. $f(x) = x^3(x-1)^3(x+2)$

36. $f(x) = x^2(x^2 + 4x + 4)$
37. $f(x) = (2x + 1)^3(9x^2 - 6x + 1)$
38. $f(x) = (3x + 2)^5(x^2 - 10x + 25)$
39. $f(x) = x(4x^2 - 12x + 9)(x^2 + 8x + 16)$
40. $f(x) = x^6 - x^5 - 2x^4$
41. $f(x) = 3x^4 + 6x^3 + 3x^2$
42. $f(x) = 4x^5 - 12x^4 + 9x^3$
43. $f(x) = 2x^4(x^3 - 4x^2 + 4x)$
44. $f(x) = 4x^4(9x^4 - 12x^3 + 4x^2)$

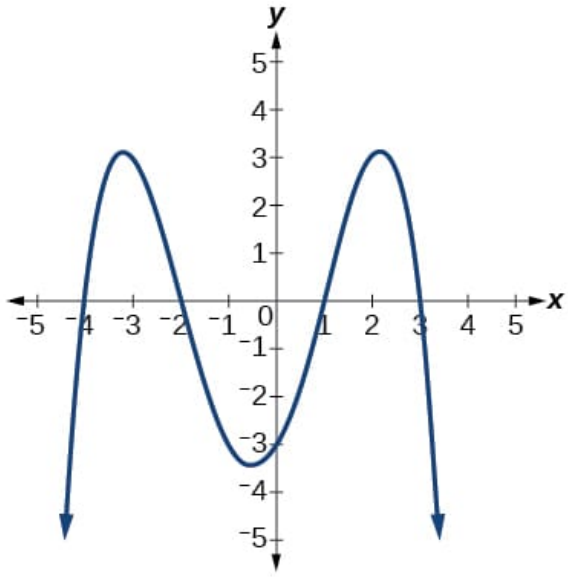
Graphical

For the following exercises, graph the polynomial functions. Note x- and y-intercepts, multiplicity, and end behavior.

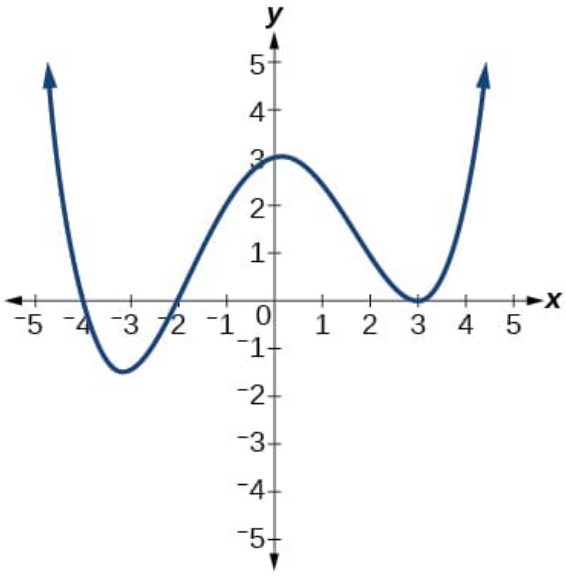
45. $f(x) = (x + 3)^2(x - 2)$
46. $g(x) = (x + 4)(x - 1)^2$
47. $h(x) = (x - 1)^3(x + 3)^2$
48. $k(x) = (x - 3)^3(x - 2)^2$
49. $m(x) = -2x(x - 1)(x + 3)$
50. $n(x) = -3x(x + 2)(x - 4)$

For the following exercises, use the graph to identify zeros and multiplicity.

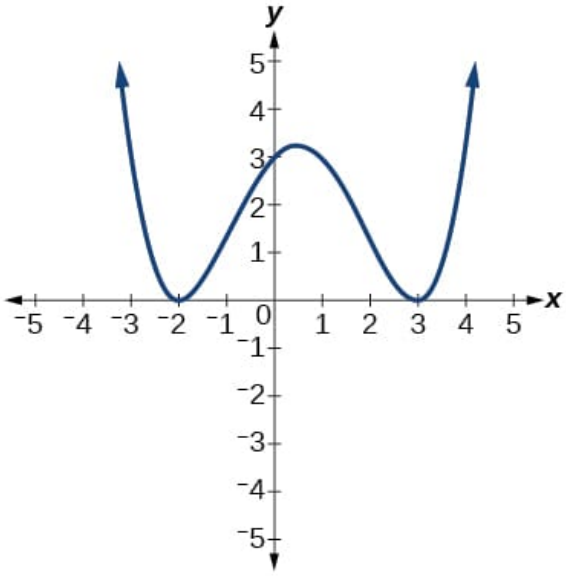
51.



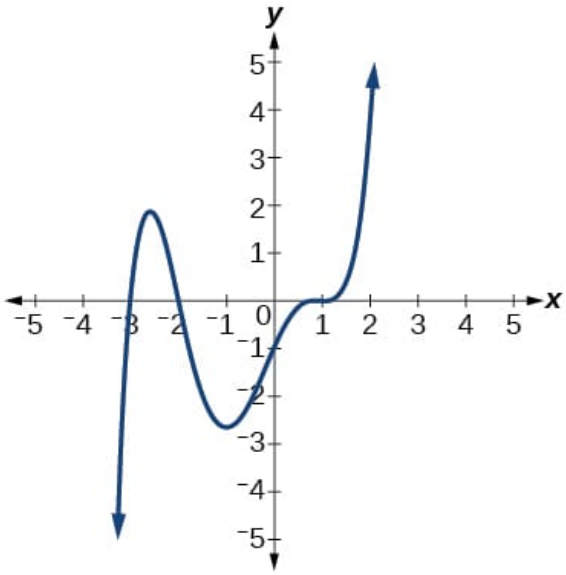
52.



53.



54.



Real-World Applications

For the following exercises, write the polynomial function that models the given situation.

55. A rectangle has a length of 10 units and a width of 8 units. Squares of x by x units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of x.
56. Consider the same rectangle of the preceding problem. Squares of $2x$ by $2x$ units are cut out of each corner. Express the volume of the box as a polynomial in terms of x.
57. A square has sides of 12 units. Squares $x+1$ by $x+1$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of x.
58. A cylinder has a radius of $x+2$ units and a height of 3 units greater. Express the volume of the cylinder as a polynomial function.
59. A right circular cone has a radius of $3x+6$ and a height 3 units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V=\frac{1}{3}\pi r^2 h$ for radius r and height h.

This page titled 2.3.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

2.4: Dividing Polynomials

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure 2.4.1**:** *Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)*

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m, and height 30 m. We can easily find the volume using elementary Geometry.

$$V = l \cdot w \cdot h = 61.5 \cdot 40 \cdot 30 = 73,800.$$

So the volume is 73,800 cubic meters (m^3). Suppose we knew the volume, length, and width. We could divide to find the height.

$$h = \frac{V}{l \cdot w} = \frac{73,800}{61.5 \cdot 40} = 30.$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any or all of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$ and the width is given by $x - 2$. To find the height of the solid, we can use polynomial division, which is the focus of this section.

Using Long Division to Divide Polynomials

Suppose we wish to find the zeros of $f(x) = x^3 + 4x^2 - 5x - 14$. Setting $f(x) = 0$ results in the polynomial equation $x^3 + 4x^2 - 5x - 14 = 0$. Despite all the factoring techniques we learned in Intermediate Algebra, this equation foils us at every turn. If we graph f using Desmos, we get the graph in Figure 2.4.2.

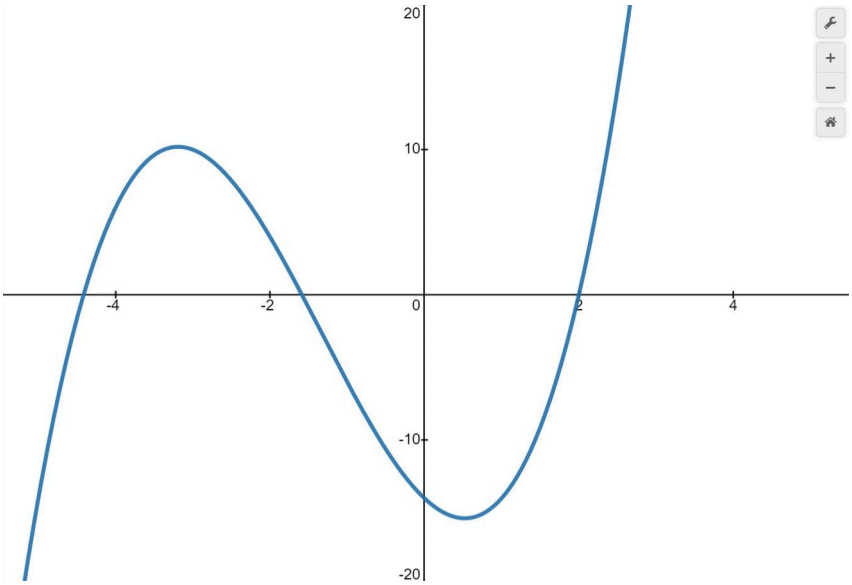


Figure 2.4.2**:** $f(x) = x^3 + 4x^2 - 5x - 14$

The graph suggests that the function has three zeros, one of which is $x = 2$. It's easy to show that $f(2) = 0$, but the other two zeros seem less friendly. Even though we could click on the curve in Desmos to find decimal approximations for these, we seek an analytical method (a method by hand) to find the remaining zeros exactly.

Since f is a polynomial function and $f(2) = 0$, it should seem realistic that $(x - 2)$ is a factor of f. That is, if $x = 2$ is a zero, it seems that there should be a factor of $(x - 2)$ lurking around in the factorization of $f(x)$. Therefore, we should expect that $x^3 + 4x^2 - 5x - 14 = (x - 2)q(x)$, where $q(x)$ is some other polynomial. If it even exists, how could we find such a $q(x)$? The answer comes from our old friend, polynomial division. Dividing $x^3 + 4x^2 - 5x - 14$ by $x - 2$ gives

$$
\begin{array}{rr|l}
 & & x^2 + 6x + 7 \\
\cline{2-3}
x - 2 & \big) & x^3 + 4x^2 - 5x - 14 \\
 & & -(x^3 - 2x^2) \\
\cline{3-3}
 & & 6x^2 - 5x \\
 & & -(6x^2 - 12x) \\
\cline{3-3}
 & & 7x - 14 \\
 & & -(7x - 14) \\
\cline{3-3}
 & & 0
\end{array}
$$

Thus, $\frac{x^3+4x^2-5x-14}{x-2} = x^2 + 6x + 7$. This means $x^3 + 4x^2 - 5x - 14 = (x - 2)(x^2 + 6x + 7)$, so to find the zeros of f, we now solve $(x - 2)(x^2 + 6x + 7) = 0$. We get $x - 2 = 0$ (which gives us our known zero, $x = 2$) as well as $x^2 + 6x + 7 = 0$. The latter doesn't factor nicely, so we apply the Quadratic Formula to get $x = -3 \pm \sqrt{2}$.

Interactive Element: When Does $x - k$ Divide Evenly?

The following Interactive Element might help you visualize why dividing $f(x) = x^3 + 4x^2 - 5x - 14$ by $x - k$ leaves no remainder exactly when k is a zero of f.

Interact: Drag the red point along the x-axis to choose the divisor $x - k$; the green bar's height is the remainder $f(k)$.

Observation: The remainder bar collapses to 0 exactly when the red point sits on an x-intercept — for example at $k = 2$, where $(x - 2)$ is a factor and the division comes out even.

The point of this section is to prepare us to be able to find as many zeros of a polynomial as possible by using division. We begin

with a friendly reminder of what we can expect when we divide polynomials.

Theorem: Division Algorithm

Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. Then there exist two unique polynomials, $q(x)$ and $r(x)$, such that

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)},$$

where either $r(x) = 0$ or the degree of r is strictly less than the degree of d. By multiplying both sides of this equation by $d(x)$, we get the equivalent form

$$p(x) = q(x)d(x) + r(x).$$

Each of these polynomials is given a special name:

- $d(x)$ is called the **divisor**
- $p(x)$ is called the **dividend**
- $q(x)$ is called the **quotient**
- $r(x)$ is called the **remainder**

For example, we just performed long division and discovered that $\frac{x^3+4x^2-5x-14}{x-2} = x^2 + 6x + 7$, where the remainder is 0. In this equation, the dividend is $x^3 + 4x^2 - 5x - 14$, the divisor is $x - 2$, the quotient is $x^2 + 6x + 7$, and (as was just mentioned) the remainder is 0.

Example 2.4.1: Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$.

Solution available at the end of this section.

The division problem in Example 2.4.1 had a remainder of 0 and the result was that our original polynomial (the dividend) had a factor of $(x + 1)$ (our divisor). This is a preview of an important theorem introduced in the next section.

Example 2.4.2: Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$.

Solution available at the end of this section.

Checkpoint 2.4.2

Divide $16x^3 - 12x^2 + 20x - 3$ by $4x + 5$.

Focus on Calculus - Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and can be quite cumbersome. As such, we hunt for a better method. Fortunately, people like Ruffini and Horner have already blazed this trail. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1 (although, it can be adjusted to divide by linear factors like $3x - 2$).

Let's look at the long division we performed at the beginning of the section and try to streamline it. First off, let's change all of the subtractions into additions by distributing through the negative signs.

$$
\begin{array}{r|l}
x-2 & x^2+6x+7 \\
\hline
 & x^3+4x^2-5x-14 \\
 & -x^3+2x^2 \\
\hline
 & 6x^2-5x \\
 & -6x^2+12x \\
\hline
 & 7x-14 \\
 & -7x+14 \\
\hline
 & 0
\end{array}
$$

Next, observe that the terms $-x^3$, $-6x^2$ and $-7x$ are the exact opposite of the terms above them.

$$
\begin{array}{r|l}
x-2 & x^2+6x+7 \\
\hline
 & x^3+4x^2-5x-14 \\
 & \boxed{-x^3}+2x^2 \\
\hline
 & 6x^2-5x \\
 & \boxed{-6x^2}+12x \\
\hline
 & 7x-14 \\
 & \boxed{-7x}+14 \\
\hline
 & 0
\end{array}
$$

Our algorithm ensures this is always the case, so we can omit them without losing any information. Also, note that the terms we "bring down" (namely the $-5x$ and -14) aren't essential to recopy, so we omit them, too.

$$
\begin{array}{r|l}
x-2 & x^2+6x+7 \\
\hline
 & x^3+4x^2-5x-14 \\
 & 2x^2 \\
\hline
 & 6x^2 \\
 & 12x \\
\hline
 & 7x \\
 & 14 \\
\hline
 & 0
\end{array}
$$

Now, let's move things up a bit and, for reasons that will become clear in a moment, copy the x^3 into the last row.

$$
\begin{array}{r|l}
x-2 & x^2+6x+7 \\
\hline
 & x^3+4x^2-5x-14 \\
\hline
 & 2x^2 \quad 12x \quad 14 \\
\hline
x^3 & 6x^2 \quad 7x \quad 0
\end{array}
$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. Take the time to work back through the original division problem. You will find that this is exactly the way we determined the quotient polynomial. This means we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$
\begin{array}{r|l}
\underline{-2} & x^3+4x^2-5x-14 \\
\hline
 & 2x^2 \quad 12x \quad 14 \\
\hline
x^3 & 6x^2 \quad 7x \quad 0
\end{array}
$$

We've streamlined things quite a bit, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, $6x$ and 7 , respectively, by the -2 in $x-2$, then by -1 when we changed the subtraction to addition. Multiplying by -2 then by -1 is the same as multiplying by 2, so we replace the -2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$\begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & 0 \end{array}$$

We have constructed a synthetic division tableau for this polynomial division problem. Let's rework our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3+4x^2-5x-14$ by $x-2$, we write 2 in the place of the divisor and the coefficients of $x^3+4x^2-5x-14$ in for the dividend. Then, "bring down" the first coefficient of the dividend.

$$\begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \end{array} \qquad \begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was "brought down" to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} \underline{2} & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ & 1 & 6 & 7 & \boxed{0} \end{array}$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third-degree polynomial and divided by a first-degree polynomial, so the quotient is a second-degree polynomial. Hence the quotient is x^2+6x+7. The number in the box is the remainder.

Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x-c$. It is important to note that it works only for these kinds of divisors (you'll need to use good old-fashioned polynomial long division for divisors of degree larger than 1). Also, take note that when a polynomial (of degree at least 1) is divided by $x-c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division. While the authors have done their best to indicate where the algorithm comes from, there is no substitute for working through it yourself.

How To: Use synthetic division to divide a polynomial by a binomial of the form $x-k$.

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder, the next number from the right has degree 0, the next number from the right has degree 1, and so on.

Example 2.4.3: Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

Solution available at the end of this section.

Example 2.4.4: Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

Solution available at the end of this section.

Example 2.4.5: Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

Solution available at the end of this section.

Checkpoint 2.4.5

Use synthetic division to divide $3x^4 + 18x^3 - 3x + 40$ by $x + 7$.

Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

Example 2.4.6: Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$ and the width is given by $x - 2$. Find the height of the solid.

Solution available at the end of this section.

Checkpoint 2.4.6

The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6$. The width of the rectangle is given by $x + 6$. Find an expression for the length of the rectangle.

Miscellaneous Topics: Thinking About Mathematics (Part 1)

1. What is it about math that you feel is difficult?
2. Would you think it an odd statement if someone told you that they suck at reading and could barely read a comic book? Why do you think people are more apt to proclaim that they suck at math?

Hidden Resources: Microsoft Office 365

Los Rios students have free access to Microsoft Office 365 for as long as they remain enrolled! This is an incredible deal! A warning is necessary - do not use the included OneDrive storage as it gets deleted each semester. Other than that minor issue, this is a fantastic resource. Check out the details on this CRC webpage.

Answers

Section Preview

The entire purpose of this section is to introduce the student to synthetic division, which is preferable to know prior to moving forward with polynomial theory.

Section Preview

- **Arithmetic**
  - Mathematical Operations — Addition, Subtraction, Multiplication, and Division: Every synthetic-division step multiplies by k and adds signed coefficients; fluency with signed-number arithmetic is exercised throughout Examples 2.4.3–2.4.6.
- **Geometry**
  - Volumes and Surface Areas of Common Shapes: The volume formula $V = l \cdot w \cdot h$ for a rectangular solid drives both the section opener (Lincoln Memorial) and the application in Example 2.4.6.

Section Preview

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

Example 2.4.1: Using Long Division to Divide a Second-Degree Polynomial

$$\begin{array}{rl} & \quad 5x \;-\; 2 \\ x+1 \;\big)\; & 5x^2 + 3x - 2 \\ -\; & (5x^2 + 5x) \\ \hline & \quad -2x - 2 \\ -\; & (-2x - 2) \\ \hline & \quad 0 \end{array}$$

The quotient is $5x - 2$. The remainder is 0. We write the result as

$$\frac{5x^2 + 3x - 2}{x + 1} = 5x - 2$$

or

$$5x^2 + 3x - 2 = (x + 1)(5x - 2).$$

Example 2.4.2: Using Long Division to Divide a Third-Degree Polynomial

$$\begin{array}{rl} & \quad 2x^2 + 5x - 7 \\ 3x-2 \;\big)\; & 6x^3 + 11x^2 - 31x + 15 \\ -\; & (6x^3 - 4x^2) \\ \hline & \quad 15x^2 - 31x \\ -\; & (15x^2 - 10x) \\ \hline & \quad -21x + 15 \\ -\; & (-21x + 14) \\ \hline & \quad 1 \end{array}$$

There is a remainder of 1. We can express the result as:

$$\frac{6x^3 + 11x^2 - 31x + 15}{3x - 2} = 2x^2 + 5x - 7 + \frac{1}{3x - 2}$$

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x - 2)(2x^2 + 5x - 7) + 1 = 6x^3 + 11x^2 - 31x + 15.$$

Notice, as we write our result,

- the dividend is $6x^3 + 11x^2 - 31x + 15$
- the divisor is $3x - 2$
- the quotient is $2x^2 + 5x - 7$
- the remainder is 1

Example 2.4.3: Using Synthetic Division to Divide a Second-Degree Polynomial

Begin by setting up the synthetic division. Write k and the coefficients.

$$\begin{array}{r|rrr} \underline{3} & 5 & -3 & -36 \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by k.

$$\begin{array}{r|rrr} \underline{3} & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by k. Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} \underline{3} & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x+12$. The remainder is 0. So $x-3$ is a factor of the original polynomial.

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x-3)(5x+12)+0=5x^2-3x-36.$$

Example $2.4.4$**: Using Synthetic Division to Divide a Third-Degree Polynomial**

The binomial divisor is $x+2$ so $k=-2$. Add each column, multiply the result by -2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} \underline{-2} & 4 & 10 & -6 & -20 \\ & & -8 & -4 & 20 \\ \hline & 4 & 2 & -10 & 0 \end{array}$$

The result is $4x^2+2x-10$. The remainder is 0. Thus, $x+2$ is a factor of $4x^3+10x^2-6x-20$.

The graph of the polynomial function $f(x)=4x^3+10x^2-6x-20$ in Figure $2.4.3$ shows a zero at $x=k=-2$. This confirms that $x+2$ is a factor of $4x^3+10x^2-6x-20$.

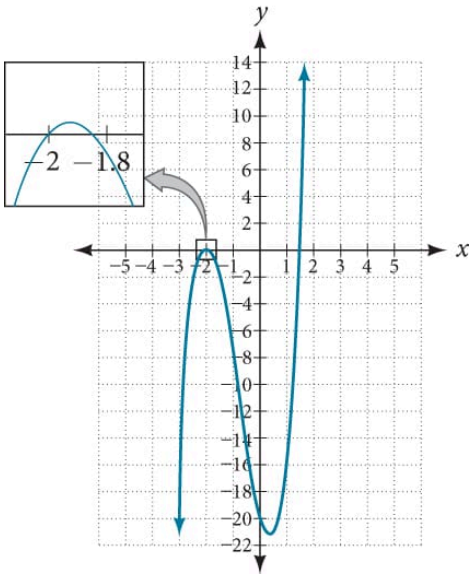


Figure $2.4.3$

Example $2.4.5$**: Using Synthetic Division to Divide a Fourth-Degree Polynomial**

Notice there is no x-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} \underline{1} & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3+x^2+8x+8+\frac{2}{x-1}$.

Example $2.4.6$**: Using Polynomial Division in an Application Problem**

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in Figure $2.4.4$.

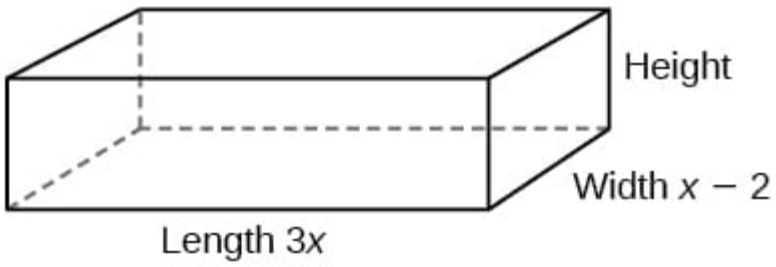


Figure 2.4.4

We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$\begin{aligned} V &= l \cdot w \cdot h \\ \implies 3x^4 - 3x^3 - 33x^2 + 54x &= 3x \cdot (x-2) \cdot h \end{aligned}$$

To solve for h, first divide both sides by $3x$.

$$\begin{aligned} \frac{3x \cdot (x-2) \cdot h}{3x} &= \frac{3x^4 - 3x^3 - 33x^2 + 54x}{3x} \\ \implies (x-2)h &= x^3 - x^2 - 11x + 18 \end{aligned}$$

Now solve for h using synthetic division.

$$h = \frac{x^3 - x^2 - 11x + 18}{x - 2}$$

$$\begin{array}{r|rrrr} \underline{2} & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The quotient is $x^2 + x - 9$ and the remainder is 0. The height of the solid is $x^2 + x - 9$.

This page titled 2.4: Dividing Polynomials is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

2.4.1: Resources and Key Concepts

Resources

Videos

- Synthetic division
  - Dividing Polynomials - Synthetic Division

Key Concepts

Definitions

- **Dividend**: In polynomial division $p(x) = d(x)q(x) + r(x)$, $p(x)$ is the dividend (the polynomial being divided).
- **Divisor**: In polynomial division $p(x) = d(x)q(x) + r(x)$, $d(x)$ is the divisor (the polynomial by which another polynomial is divided).
- **Quotient**: In polynomial division $p(x) = d(x)q(x) + r(x)$, $q(x)$ is the quotient (the result of the division, excluding the remainder).
- **Remainder**: In polynomial division $p(x) = d(x)q(x) + r(x)$, $r(x)$ is the remainder (the part left over after division if the divisor does not divide the dividend exactly).
- **Synthetic Division**: A shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1 (e.g., $x - k$).

Theorems

- **Division Algorithm**: Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. Then there exist two unique polynomials, $q(x)$ and $r(x)$, such that $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$ where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

Common Mistakes

- **Sign Errors in Long Division**: When subtracting partial products in polynomial long division, errors often occur if the signs of all terms in the polynomial being subtracted are not correctly changed.
- **Misalignment of Terms in Long Division**: Not aligning like terms vertically during the long division process, which can lead to errors in subtraction or in bringing down the next term.
- **Incorrectly Handling Missing Terms**: When setting up long division or synthetic division, failing to include a zero coefficient as a placeholder for any missing terms (e.g., if a cubic polynomial has no x^2 term, a $0x^2$ should be used).
- **Errors in Synthetic Division Setup**:
  - Using the wrong value for k when dividing by $x - k$. For example, if dividing by $x + 2$, then $k = -2$, not 2.
  - Incorrectly listing the coefficients of the dividend.
- **Arithmetic Errors in Synthetic Division**: Making mistakes in the multiply-add steps of synthetic division.
- **Misinterpreting the Result of Synthetic Division**: Incorrectly determining the degree of the quotient (it is one less than the degree of the dividend when dividing by a linear factor).
- **Using Synthetic Division for Non-Linear Divisors**: Attempting to use synthetic division when the divisor is not a linear factor of the form $x - k$ (or adaptable to it). Synthetic division is a specialized method.

This page titled 2.4.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

2.4.2: Homework

Reading Questions

1. What are the four main components in the Division Algorithm for polynomials: $f(x) = d(x)q(x) + r(x)$? Define each term.
2. When performing long division of polynomials, how do you know when the division process is complete? What condition must the remainder satisfy?
3. If the remainder of a polynomial division is 0, what does this imply about the relationship between the divisor, the dividend, and the quotient?
4. Describe the first step in setting up polynomial long division (e.g., dividing $2x^3 - 3x^2 + 4x + 5$ by $x + 2$).
5. In polynomial long division, after multiplying the current term of the quotient by the entire divisor, what is the next arithmetic operation performed?
6. What is synthetic division, and for what special case of divisor is it typically used?
7. In synthetic division, if you are dividing by $x - k$, what value representing the divisor do you write outside the division symbol? What if you are dividing by $x + k$?
8. Describe the process of bringing down the leading coefficient and the subsequent multiply-add steps in synthetic division.
9. How do you interpret the bottom row of numbers in a completed synthetic division problem to find the quotient and remainder?
10. If a polynomial dividend is missing a term (e.g., no x^2 term in a cubic polynomial), how should this be handled when setting up either long division or synthetic division?
11. How can you check your work after performing polynomial division?
12. Can polynomial division be used to find unknown dimensions of a geometric solid if its volume and some other dimensions are given as polynomials?
13. If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?
14. If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

Homework

Algebraic

For the following exercises, use long division to divide. Specify the quotient and the remainder.

15. $(x^2 + 5x - 1) \div (x - 1)$
16. $(2x^2 - 9x - 5) \div (x - 5)$
17. $(3x^2 + 23x + 14) \div (x + 7)$
18. $(4x^2 - 10x + 6) \div (4x + 2)$
19. $(6x^2 - 25x - 25) \div (6x + 5)$
20. $(-x^2 - 1) \div (x + 1)$
21. $(2x^2 - 3x + 2) \div (x + 2)$
22. $(x^3 - 126) \div (x - 5)$
23. $(3x^2 - 5x + 4) \div (3x + 1)$
24. $(x^3 - 3x^2 + 5x - 6) \div (x - 2)$
25. $(2x^3 + 3x^2 - 4x + 15) \div (x + 3)$

For the following exercises, use synthetic division to find the quotient.

26. $(3x^3 - 2x^2 + x - 4) \div (x + 3)$
27. $(2x^3 - 6x^2 - 7x + 6) \div (x - 4)$
28. $(6x^3 - 10x^2 - 7x - 15) \div (x + 1)$
29. $(4x^3 - 12x^2 - 5x - 1) \div (2x + 1)$
30. $(9x^3 - 9x^2 + 18x + 5) \div (3x - 1)$
31. $(3x^3 - 2x^2 + x - 4) \div (x + 3)$
32. $(-6x^3 + x^2 - 4) \div (2x - 3)$

33. $(2x^3 + 7x^2 - 13x - 3) \div (2x - 3)$
34. $(3x^3 - 5x^2 + 2x + 3) \div (x + 2)$
35. $(4x^3 - 5x^2 + 13) \div (x + 4)$
36. $(x^3 - 3x + 2) \div (x + 2)$
37. $(x^3 - 21x^2 + 147x - 343) \div (x - 7)$
38. $(x^3 - 15x^2 + 75x - 125) \div (x - 5)$
39. $(9x^3 - x + 2) \div (3x - 1)$
40. $(6x^3 - x^2 + 5x + 2) \div (3x + 1)$
41. $(x^4 + x^3 - 3x^2 - 2x + 1) \div (x + 1)$
42. $(x^4 - 3x^2 + 1) \div (x - 1)$
43. $(x^4 + 2x^3 - 3x^2 + 2x + 6) \div (x + 3)$
44. $(x^4 - 10x^3 + 37x^2 - 60x + 36) \div (x - 2)$
45. $(x^4 - 8x^3 + 24x^2 - 32x + 16) \div (x - 2)$
46. $(x^4 + 5x^3 - 3x^2 - 13x + 10) \div (x + 5)$
47. $(x^4 - 12x^3 + 54x^2 - 108x + 81) \div (x - 3)$
48. $(4x^4 - 2x^3 - 4x + 2) \div (2x - 1)$
49. $(4x^4 + 2x^3 - 4x^2 + 2x + 2) \div (2x + 1)$

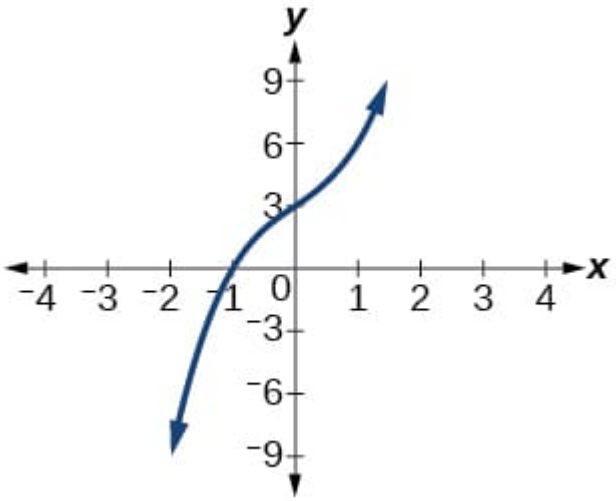
For the following exercises, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.

50. $x - 2$, $4x^3 - 3x^2 - 8x + 4$
51. $x - 2$, $3x^4 - 6x^3 - 5x + 10$
52. $x + 3$, $-4x^3 + 5x^2 + 8$
53. $x - 2$, $4x^4 - 15x^2 - 4$
54. $x - \frac{1}{2}$, $2x^4 - x^3 + 2x - 1$
55. $x + \frac{1}{3}$, $3x^4 + x^3 - 3x + 1$

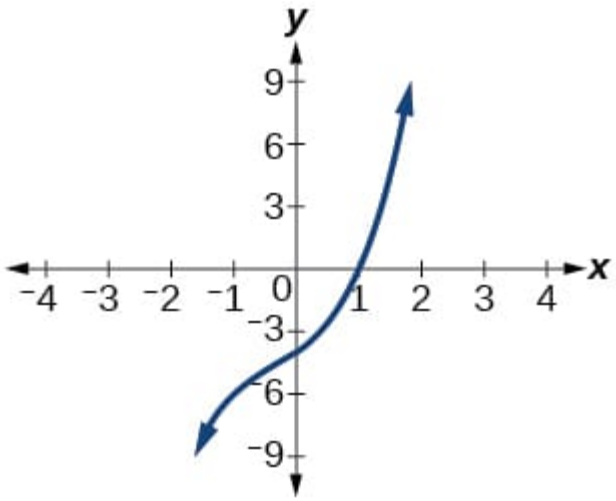
Graphical

For the following exercises, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

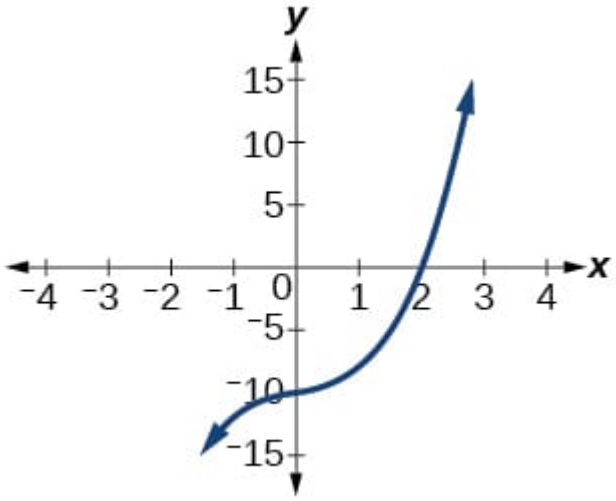
56. Factor is $x^2 - x + 3$



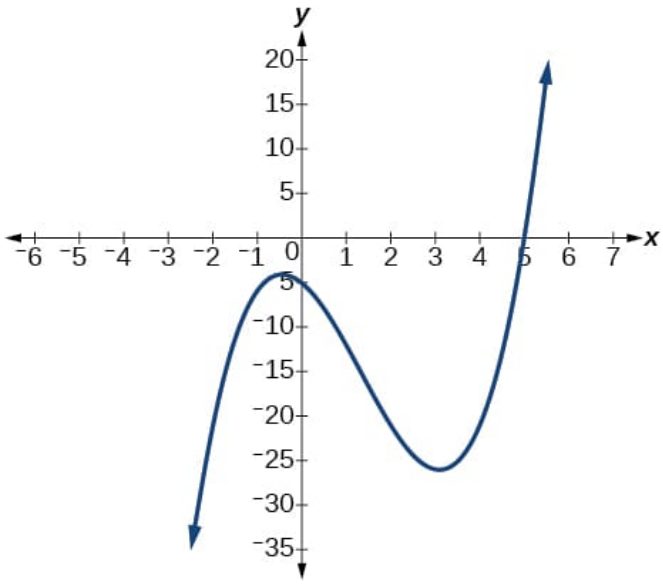
57. Factor is $x^2 + 2x + 4$



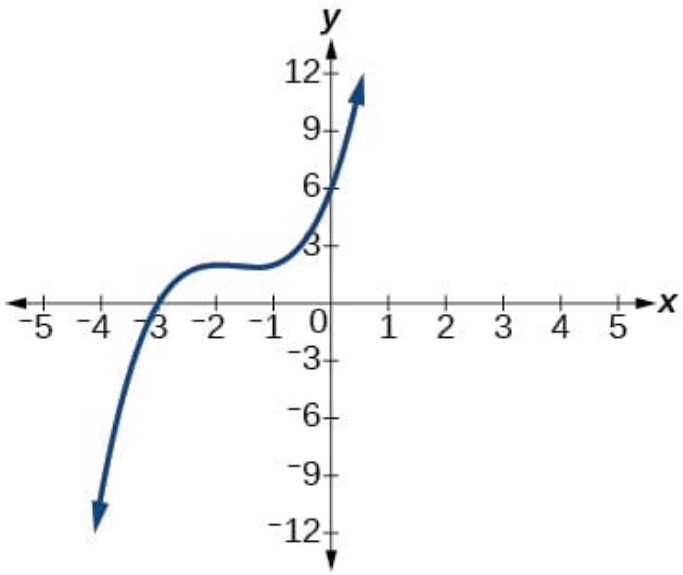
58. Factor is $x^2 + 2x + 5$



59. Factor is $x^2 + x + 1$



60. Factor is $x^2 + 2x + 2$



For the following exercises, use synthetic division to find the quotient and remainder.

61. $\frac{4x^3-33}{x-2}$
62. $\frac{2x^3+25}{x+3}$
63. $\frac{3x^3+2x-5}{x-1}$
64. $\frac{-4x^3-x^2-12}{x+4}$
65. $\frac{x^4-22}{x+2}$

Technology

For the following exercises, use technology to answer the questions.

66. Consider $\frac{x^k-1}{x-1}$ with $k = 1$, 2, and 3. What do you expect the result to be if $k = 4$?
67. Consider $\frac{x^k+1}{x+1}$ for $k = 1$, 3, and 5. What do you expect the result to be if $k = 7$?
68. Consider $\frac{x^4-k^4}{x-k}$ for $k = 1$, 2, and 3. What do you expect the result to be if $k = 4$?
69. Consider $\frac{x^k}{x+1}$ with $k = 1$, 2, and 3. What do you expect the result to be if $k = 4$?
70. Consider $\frac{x^k}{x-1}$ with $k = 1$, 2, and 3. What do you expect the result to be if $k = 4$?

Extensions

For the following exercises, use synthetic division to determine the quotient involving a complex number.

71. $\frac{x+1}{x-i}$
72. $\frac{x^2+1}{x-i}$
73. $\frac{x+1}{x+i}$
74. $\frac{x^2+1}{x+i}$
75. $\frac{x^3+1}{x-i}$

Real-World Applications

For the following exercises, use the given length and area of a rectangle to express the width algebraically.

76. Length is $x + 5$, area is $2x^2 + 9x - 5$.
77. Length is $2x + 5$, area is $4x^3 + 10x^2 + 6x + 15$

78. Length is $3x–4$, area is $6x^4 - 8x^3 + 9x^2 - 9x - 4$

For the following exercises, use the given volume of a box and its length and width to express the height of the box algebraically.

79. Volume is $12x^3 + 20x^2 - 21x - 36$, length is $2x + 3$, width is $3x - 4$.
80. Volume is $18x^3 - 21x^2 - 40x + 48$, length is $3x–4$, width is $3x–4$.
81. Volume is $10x^3 + 27x^2 + 2x - 24$, length is $5x–4$, width is $2x + 3$.
82. Volume is $10x^3 + 30x^2 - 8x - 24$, length is 2, width is $x + 3$.

For the following exercises, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.

83. Volume is $\pi(25x^3 - 65x^2 - 29x - 3)$, radius is $5x + 1$.
84. Volume is $\pi(4x^3 + 12x^2 - 15x - 50)$, radius is $2x + 5$.
85. Volume is $\pi(3x^4 + 24x^3 + 46x^2 - 16x - 32)$, radius is $x + 4$.

This page titled 2.4.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

2.5: Rational Functions

Note to the Instructor (click to expand)

This section covers rational functions and their graphs. You should use the Calculus "arrow" notation as often as possible so that it is not foreign to students once they get into Calculus. Solving rational inequalities by using graphs is also covered (and is a great reinforcement of the need to graph).

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Solving Equations**
  - Solving Rational Equations: While not directly solving equations in this section, the process of finding x -intercepts (setting numerator to zero) and vertical asymptotes/holes (setting denominator to zero) involves solving polynomial equations derived from the rational function.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Use arrow notation.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.
- Solve a rational inequality by graphing.

Hawk A.I. Section-Specific Tutor



To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Suppose we know that the cost of making a product is dependent on the number of items, x, produced. This is given by the equation $C(x) = 15,000x - 0.1x^2 + 1000$. If we want to know the average cost for producing x items, we would divide the cost

function by the number of items, x.

The average cost function, which yields the average cost per item for x items produced, is $f(x) = \frac{15{,}000x - 0.1x^2 + 1000}{x}$. Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

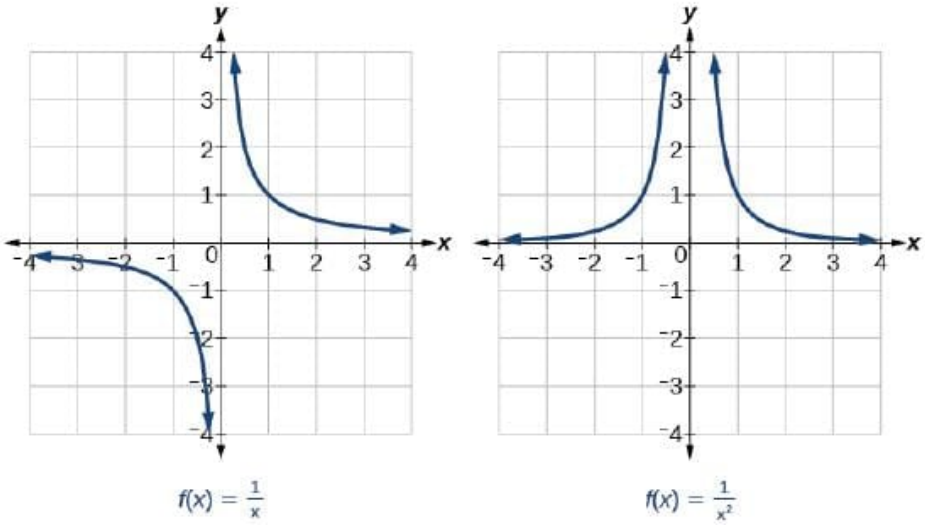
In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore **rational functions**, which have variables in the denominator.

Focus on Calculus - Using Arrow Notation

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our study of toolkit functions. Let's examine these graphs and notice some of their features.

Figure 2.5.1

Graphs of Toolkit Functions



Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

1. On the left branch of the graph, the curve approaches the x-axis ($y = 0$) as $x \to -\infty$.
2. As the graph approaches $x = 0$ from the left ($x \to 0^-$), the curve drops ($f(x) \to -\infty$), but as we approach zero from the right ($x \to 0^+$), the curve rises ($f(x) \to \infty$).
3. Finally, on the right branch of the graph, the curves approaches the x-axis ($y = 0$) as $x \to \infty$.

To summarize, we use arrow notation to show that x or $f(x)$ is approaching a particular value. See Table 2.5.1.

Table 2.5.1

Defining Arrow Notation

Symbol	Meaning
$x \to a^-$	x approaches a from the left (\(x
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)

Symbol	Meaning
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table 2.5.2.

Table 2.5.2

Numerical Investigation of $f(x) = \frac{1}{x}$ *as* $x \to 0^-$.

x	-0.1	-0.01	-0.001	-0.0001
$f(x) = \frac{1}{x}$	-10	-100	-1000	-10,000

We would indicate this behavior using the arrow notation: "as $x \to 0^-$, $f(x) \to -\infty$."

On the other hand, as the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in Table 2.5.3.

Table 2.5.3

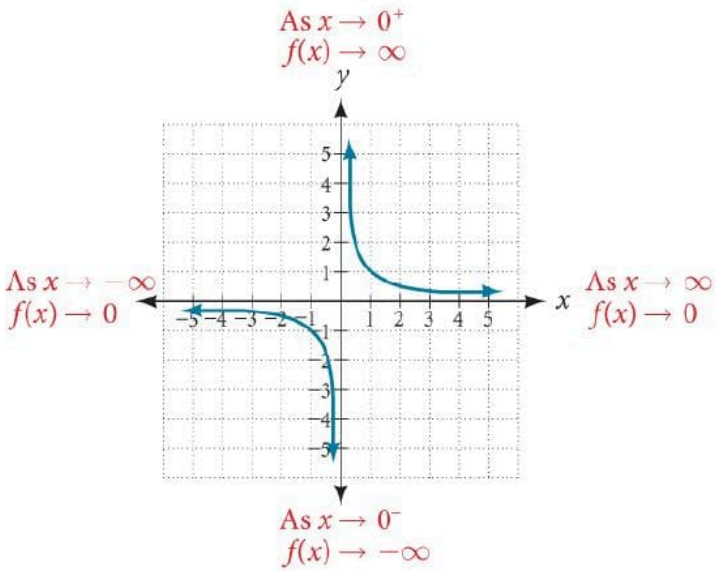
Numerical Investigation of $f(x) = \frac{1}{x}$ *as* $x \to 0^+$.

x	0.1	0.01	0.001	0.0001
$f(x) = \frac{1}{x}$	10	100	1000	10,000

This is interpreted using the following arrow notation: "as $x \to 0^+$, $f(x) \to \infty$." See Figure 2.5.2.

Figure 2.5.2

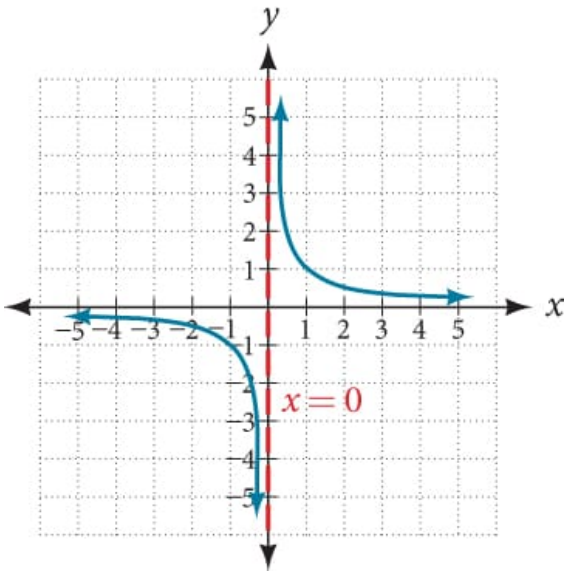
Graphical Investigation of $f(x) = \frac{1}{x}$.



This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero. See Figure $2.5.3$.

Figure $2.5.3$

Showing the vertical asymptote, $x = 0$, for the function $f(x) = \frac{1}{x}$.



Definition: Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of f if the values of $f(x)$ increase or decrease without bound as x approaches a from the left, from the right, or from both sides. We write

$f(x) \to \infty$ as $x \to a^-$ if the graph increases without bound as x approaches a from the left,

$f(x) \to -\infty$ as $x \to a^-$ if the graph decreases without bound as x approaches a from the left,

$f(x) \to \infty$ as $x \to a^+$ if the graph increases without bound as x approaches a from the right,

and

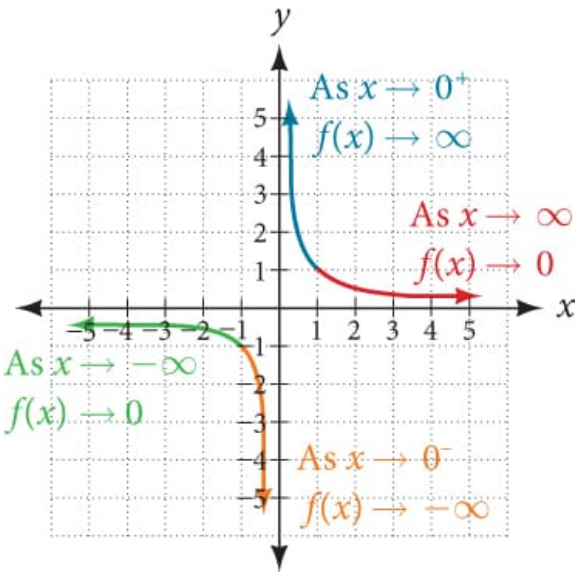
$f(x) \to -\infty$ as $x \to a^+$ if the graph decreases without bound as x approaches a from the right.

End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. Likewise, as the values of x approach negative infinity, the function values approach 0. See Figure $2.5.4$. Symbolically, using arrow notation, we write: "as $x \to \infty$, $f(x) \to 0$," and "as $x \to -\infty$, $f(x) \to 0$."

Figure 2.5.4

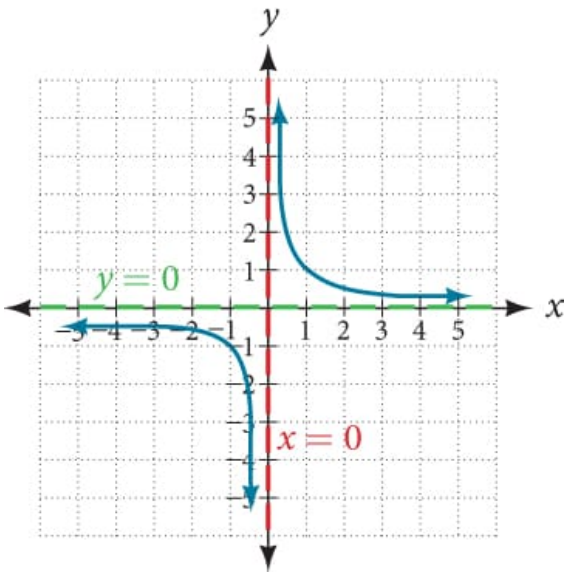
Graphical Investigation of $f(x) = \frac{1}{x}$.



Based on this overall behavior (and the graph), we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure 2.5.5.

Figure 2.5.5

Showing the vertical and horizontal asymptotes for the function $f(x) = \frac{1}{x}$.



Definition: Horizontal Asymptote

The line $y = c$ is a **horizontal asymptote** of the graph of f if the values of $f(x)$ approach b as x increases without bound, as x decreases without bound, or both. We write

$f(x) \to b$ as $x \to \infty$ if the graph approaches the line $y = b$ as x increases without bound

and

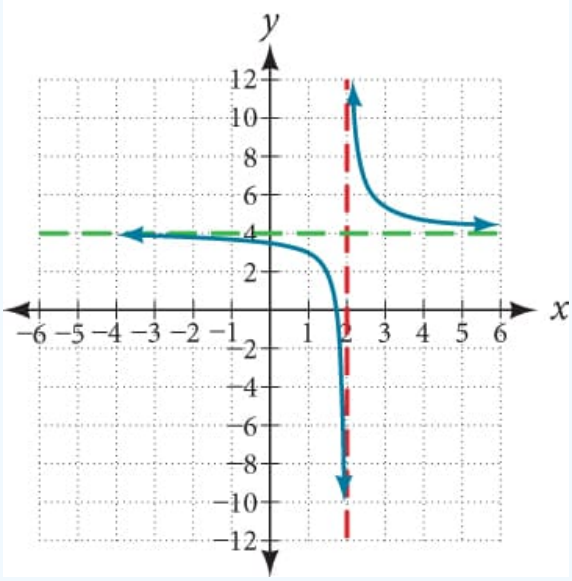
$f(x) \to b$ as $x \to -\infty$ if the graph approaches the line $y = b$ as x decreases without bound.

Example 2.5.1: Using Arrow Notation

Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure 2.5.6.

Figure 2.5.6

A transformation of $y = \frac{1}{x}$.



Solution available at the end of this section.

Checkpoint 2.5.1

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

Example 2.5.2: Using Transformations to Graph a Rational Function

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution available at the end of this section.

Checkpoint 2.5.2

Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Rational Functions

In Example 2.5.2, we shifted a toolkit function in a way that resulted in the function $f(x) = \frac{3x+7}{x+2}$. This is an example of a rational function. A **rational function** is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

Definition: Rational Function

A **rational function** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1}x^{p-1} + ... + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + ... + b_1 x + b_0}, \text{ where } Q(x) \neq 0.$$

A vertical asymptote represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Theorem: Domain of a Rational Function

The domain of a rational function includes all real numbers except those that cause the denominator of the unsimplified rational function to equal zero.

Caution: Domain is Always Considered Prior to Simplifications

A common mistake is for students to simplify or otherwise perform Algebra on an expression and *then* state the domain. In this simplification process, you might "remove" a domain issue that originally existed. Therefore, you need to state the domain *prior* to simplifying the function.

For example, if $f(x) = \frac{x+2}{(x+2)(x+3)}$, you would need to state the domain of the function *prior* to simplifying. In this case, $x = -2$ and $x = -3$ cause division by zero. Therefore, the domain is $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$.

Had you simplified the function first, you would have arrived at

$$f(x) = \frac{x+2}{(x+2)(x+3)} = \frac{1}{x+3},$$

and you would have (incorrectly) stated the domain as $(-\infty, -3) \cup (-3, \infty)$. In truth, $f(x)$ is *not* equal to $\frac{1}{x+3}$ because, if it were, we would be able to evaluate $f(-2)$. However, looking at the original function, evaluating f at $x = -2$ causes division by zero!

Example 2.5.3: Finding the Domain of a Rational Function

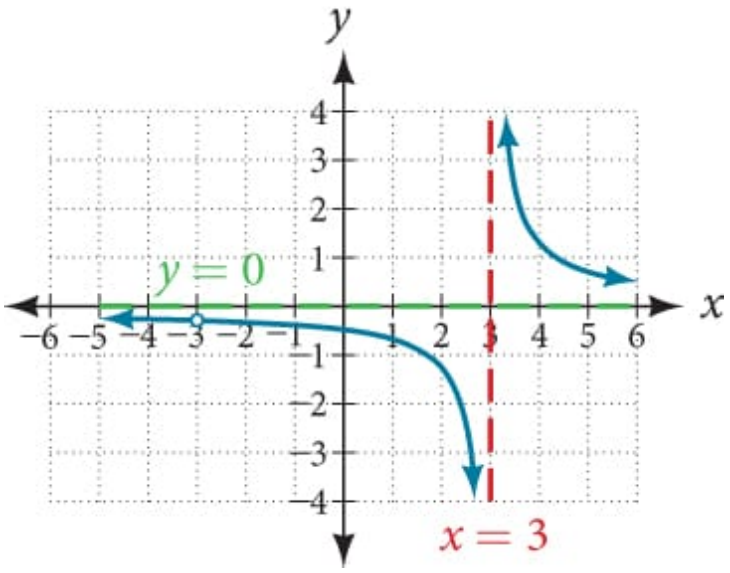
Find the domain of $f(x) = \frac{x+3}{x^2-9}$.

Solution available at the end of this section.

A graph of the function from Example 2.5.3 is shown in Figure 2.5.8. It confirms that the function is not defined when $x = \pm 3$.

Figure 2.5.8

Graph of $f(x) = \frac{x+3}{x^2+9}$.



There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

Checkpoint 2.5.3

Find the domain of $f(x) = \frac{4x}{5(x-1)(x-5)}$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

Example 2.5.4: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \frac{5+2x^2}{2-x-x^2}$.

Solution available at the end of this section.

Removable Discontinuities

Occasionally, a graph will contain a hole.

Definition: Removable Discontinuity

A single point where the graph of a function is not defined, indicated by an open circle on the graph, is called a **removable discontinuity**.

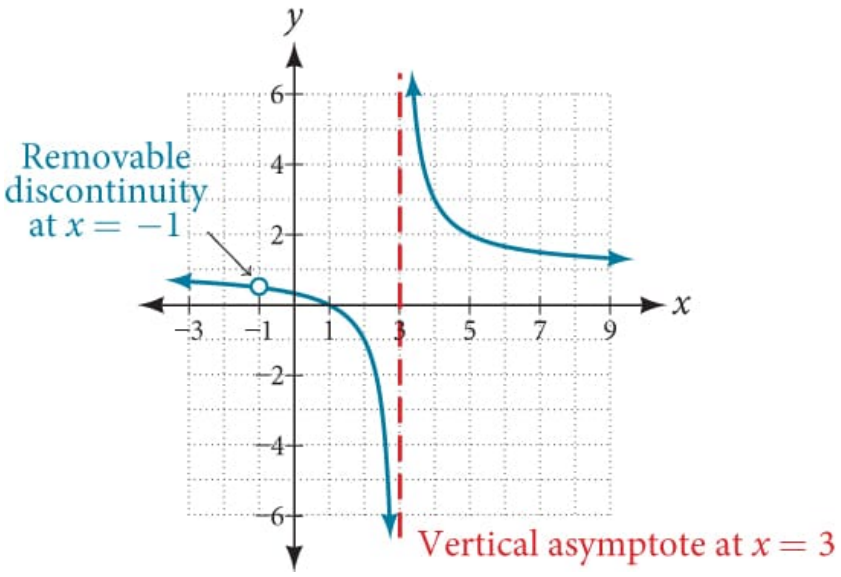
For example, the function $f(x) = \frac{x^2-1}{x^2-2x-3}$ may be rewritten by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}.$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x=-1$, is the location of the removable discontinuity. Notice also that x–3 is *not* a factor in both the numerator and denominator. The zero of this factor, $x=3$, is the vertical asymptote. See Figure $2.5.10$.

Figure $2.5.10$

Showcasing the removable discontinuity for the graph of the function $f(x) = \frac{x^2-1}{x^2-2x-3}$.



Definition: Removable Discontinuity of a Rational Function

A removable discontinuity for the graph of a rational function occurs at $x=a$ if

1. a is a zero for a factor in the denominator that is common with a factor in the numerator, and
2. after simplification, a is no longer a zero of the denominator.

Proof available at the end of this section.

An clean way to summarize this theorem is to first factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This x-value leads to a removable discontinuity if the multiplicity of the factor in the numerator is greater than or equal to that in the denominator. If the multiplicity of the factor is greater in the denominator, then there is still an asymptote at that value.

Example $2.5.5$: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \frac{x-2}{x^2-4}$.

Solution available at the end of this section.

Checkpoint 2.5.5

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \frac{x^2-25}{x^3-6x^2+5x}$.

Identifying End Behavior of Rational Functions

As we have discussed, horizontal asymptotes help describe the behavior of a graph as the input gets infinitely large (either positive or negative); however, it is possible that the graph of a function doesn't tend to a horizontal line as $x \to \pm\infty$. In fact, there are an infinite number of cases in which graphs tend to non-horizontal lines or, in more advanced cases, other higher-degreed functions. Therefore, rather than speaking of identifying "horizontal asymptotes" of graphs, it's best to keep things general and speak of the **end behavior** of a graph.

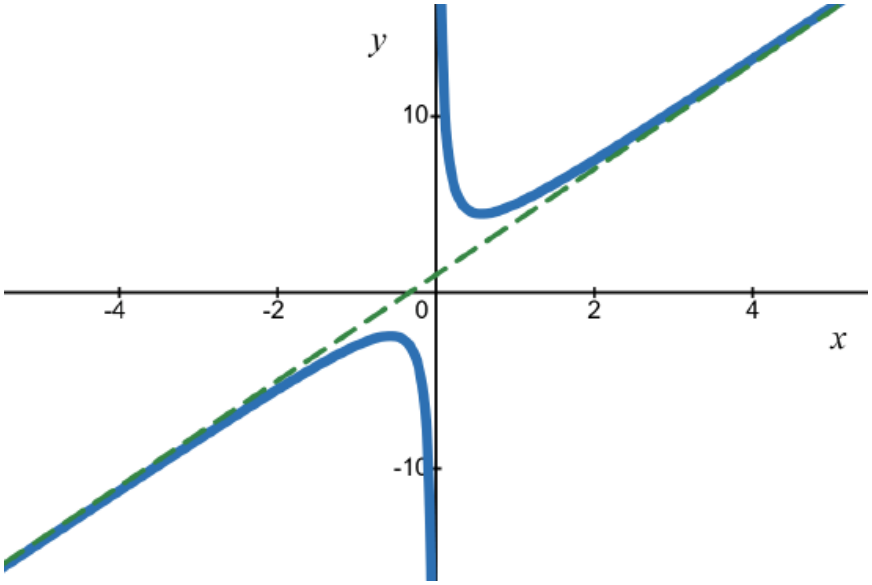
Definition: End Behavior

The **end behavior** of a graph is a description, usually as a function, of what the function values of the graph tend to approach as the inputs increase or decrease without bound.

For example, the function $f(x) = 3x + 1 + \frac{1}{x}$ is the solid curve graphed in Figure 2.5.12.

Figure 2.5.12

Graph of $f(x) = 3x + 1 + \frac{1}{x}$ showcasing a slant asymptote.



As $x \to \pm\infty$, the graph quickly begins to "mimic" the dashed line, which happens to have equation $y = 3x + 1$. Why does this happen? The third term of the $f(x)$, which is $\frac{1}{x}$, tends to 0 (it tends to disappear) as $x \to \pm\infty$. This forces $f(x)$ to eventually "mimic" what *doesn't* disappear (specifically, $y = 3x + 1$). Hence, we would say that the end behavior of $f(x) = 3x + 1 + \frac{1}{x}$ is $y = 3x + 1$. This type of end behavior is called a **slant asymptote** because it is a line.

With this stated, we can see that a horizontal asymptote is a *type* of end behavior, but it is not the *only* type. Our goal at this point is to discover the types of end behaviors a rational function might have.

Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror

that of the ratio of the leading terms of the numerator and denominator functions. There are three distinct outcomes when checking for the end behavior of a rational function; however, let's define a new notation that will simplify some of our discussion.

Definition: Polynomial Degree Notation

The degree of a polynomial $f(x)$ is denoted $\deg(f(x))$.

For example, $\deg(1-x^{17}+x)=17$.

With that definition, we are ready to explore the end behavior possibilities of rational functions. We begin by letting $f(x)=\frac{N(x)}{D(x)}$, where N and D are polynomials and $D(x)\neq 0$.

Case 1 ($\deg(D(x))>\deg(N(x))$)

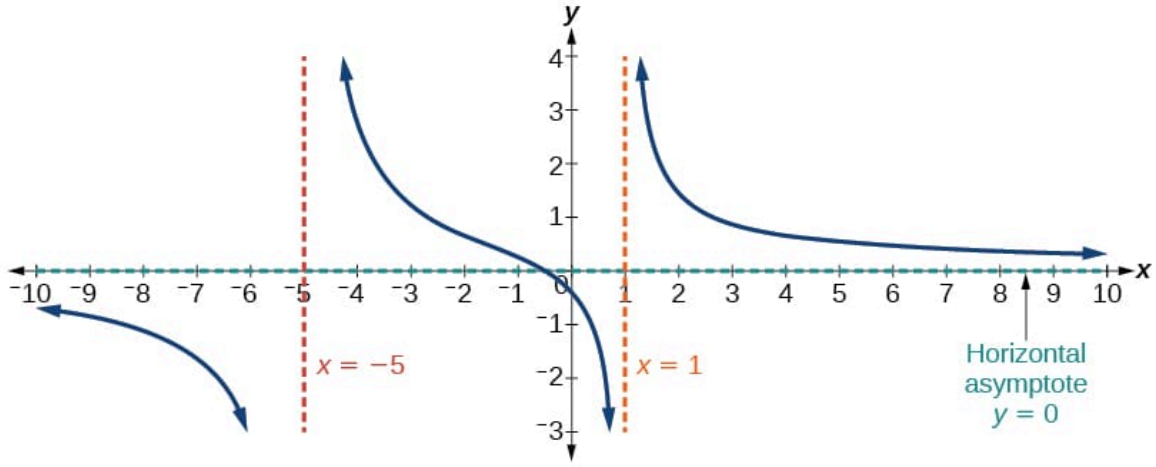
Consider the function $f(x)=\frac{4x+2}{x^2+4x-5}$. A little Algebra goes a long way:

$$\begin{aligned}\frac{4x+2}{x^2+4x-5} &= \frac{\frac{1}{x^2}}{\frac{1}{x^2}}\cdot\frac{4x+2}{x^2+4x-5} && \left(\text{multiplying numerator and denominator by } \tfrac{1}{x^2}\right)\\ &= \frac{\frac{4}{x}+\frac{2}{x^2}}{1+\frac{4}{x}-\frac{5}{x^2}} && (\text{distributing})\end{aligned}$$

As $x\to\pm\infty$, all expressions of the form $\frac{\text{constant}}{x}$ and $\frac{\text{constant}}{x^2}$ will tend to 0. Thus, as $x\to\pm\infty$, $f(x)\to\frac{0+0}{1+0+0}=\frac{0}{1}=0$. That is, $y=0$ is a horizontal asymptote for $f(x)$. See Figure 2.5.13. Note that this graph crosses the horizontal asymptote.

Figure 2.5.13

Horizontal Asymptote $y=0$ when $f(x)=\frac{N(x)}{D(x)}$, where $D(x)\neq 0$ and $\deg(N(x))<\deg(D(x))$.



Case 2 $\deg(D(x))<\deg(N(x))$ (by 1)

In this textbook, we will only consider the possibility of the numerator being one degree more than the denominator. Your instructor might spend a few minutes of class time to discuss what happens when the difference of degrees between numerator and denominator is greater than 1, but that is not a major topic in Calculus.

Consider the function $f(x)=\frac{3x^2-2x+1}{x-1}$. Since we can think of this as an improper fraction (where the numerator might be somewhat divisible by the denominator), we perform synthetic division to arrive at

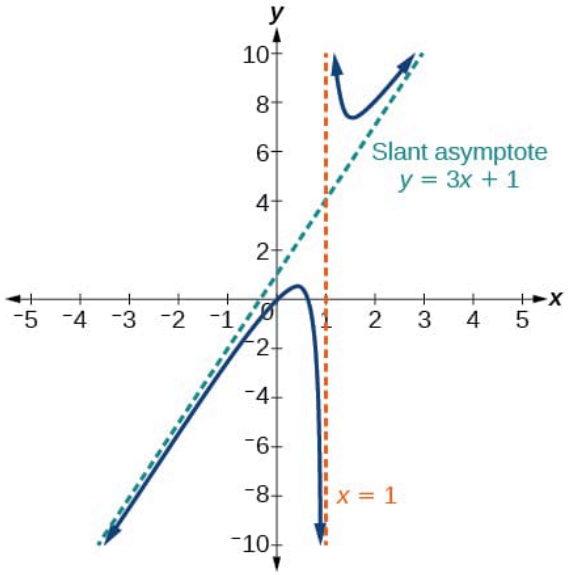
$$\frac{3x^2-2x+1}{x-1}=3x+1+\frac{2}{x-1}.$$

As $x\to\pm\infty$, the expression $\frac{2}{x-1}\to 0$. Therefore, our function will "mimic" $y=3x+1$. That is, $f(x)\to 3x+1$ as $x\to\pm\infty$. Thus, the

line $y = 3x + 1$ is a slant asymptote of $f(x)$ (see Figure 2.5.14).

Figure 2.5.14

Slant Asymptote when $f(x) = \frac{N(x)}{D(x)}$, *where* $D(x) \neq 0$ *and* $\deg(N(x)) > \deg(D(x))$ *by 1.*



Case 3 $\deg(D(x)) = \deg(N(x))$

Consider the function $f(x) = \frac{3x^2+2}{x^2+4x-5}$. Performing some Algebra, we get

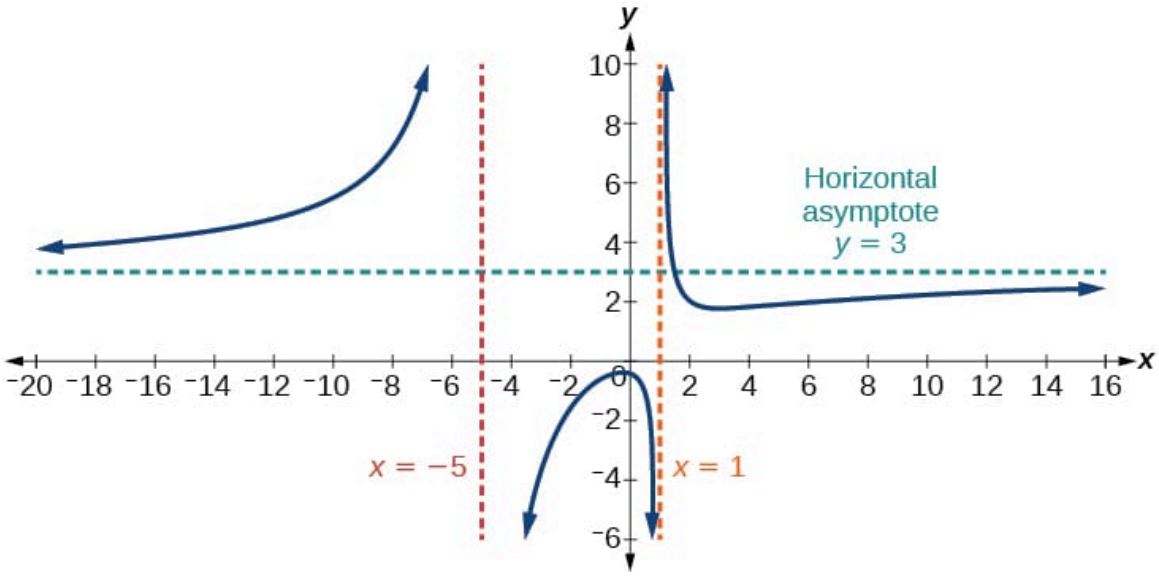
$$\frac{3x^2+2}{x^2+4x-5} = \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \cdot \frac{3x^2+2}{x^2+4x-5} \qquad \left(\text{multiplying numerator and denominator by } \tfrac{1}{x^2}\right)$$

$$= \frac{3+\frac{2}{x^2}}{1+\frac{4}{x}-\frac{5}{x^2}} \qquad (\text{distributing})$$

As $x \to \pm\infty$, all expressions of the form $\frac{\text{constant}}{x}$ and $\frac{\text{constant}}{x^2}$ will tend to 0. Thus, as $x \to \pm\infty$, $f(x) \to \frac{3+0}{1+0-0} = \frac{3}{1} = 3$. That is, $y = 3$ is a horizontal asymptote for $f(x)$. See Figure 2.5.15. Again, note that this graph crosses the horizontal asymptote.

Figure 2.5.15

Horizontal Asymptote when $f(x) = \frac{N(x)}{D(x)}$, *where* $D(x) \neq 0$ *and* $\deg(N(x)) = \deg(D(x))$.



Notice that, while the graph of a rational function will never cross a *vertical* asymptote, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph of a *rational function* will have at most one horizontal (or slant) asymptote. As an aside, there exist *non-rational* functions that have two horizontal or slant asymptotes (and there are some that have both a horizontal *and* a slant asymptote); however, we are only focusing our attention on rational functions for this discussion.

Theorem: End Behavior of a Rational Function

Let $f(x) = \frac{N(x)}{D(x)}$ be a rational function. Then

- $f(x)$ has a horizontal asymptote of $y = 0$ if $\deg(D(x)) > \deg(N(x))$
- $f(x)$ has a horizontal asymptote of $y = \frac{a}{b}$ if $\deg(D(x)) = \deg(N(x))$, where a is the lead coefficient of $N(x)$ and b is the lead coefficient of $D(x)$
- $f(x)$ has a slant or oblique asymptote if $\deg(D(x)) < \deg(N(x))$. The asymptote is a slant asymptote if $\deg(N(x)) = \deg(D(x)) + 1$; otherwise, it is an oblique asymptote. In either case, the equation of the asymptote is the quotient after dividing $N(x)$ by $D(x)$.

Example 2.5.6: Identifying Horizontal and Slant Asymptotes

For the functions below, identify the horizontal or slant asymptote.

a. $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$
b. $h(x) = \frac{x^2 - 4x + 1}{x + 2}$
c. $k(x) = \frac{x^2 + 4x}{x^3 - 8}$

Solution available at the end of this section.

Example 2.5.7: Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \frac{5 + t}{100 + 10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

Solution available at the end of this section.

Example 2.5.8: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

Solution available at the end of this section.

Checkpoint 2.5.8

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Like all functions, a rational function will have a y-intercept when the input is zero, if the function is defined at zero. A rational function will not have a y-intercept if the function is not defined at zero. Likewise, a rational function will have x-intercepts at the inputs that cause the output to be zero; however, since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

Example 2.5.9: Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution available at the end of this section.

Checkpoint 2.5.9

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

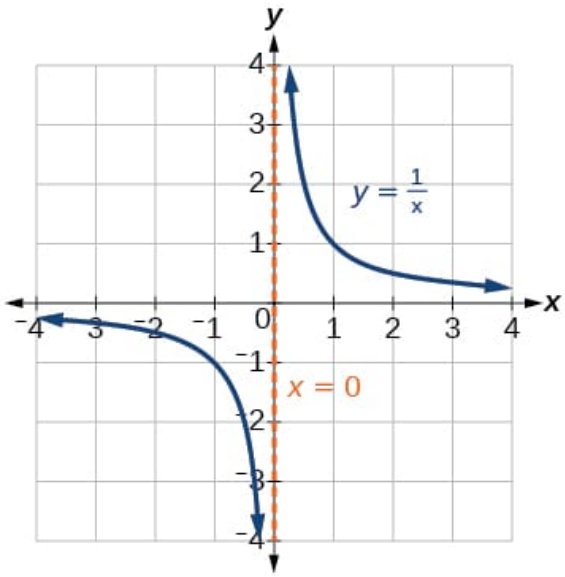
Graphing Rational Functions

In Example 2.5.8, we see that the numerator of a rational function reveals the x-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure 2.5.18.

Figure 2.5.18

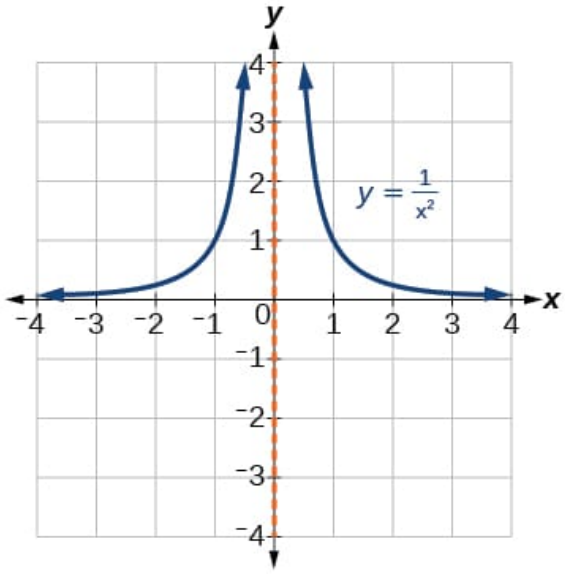
Graph of the toolkit function $y = \frac{1}{x}$.



When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure 2.5.19.

Figure 2.5.19

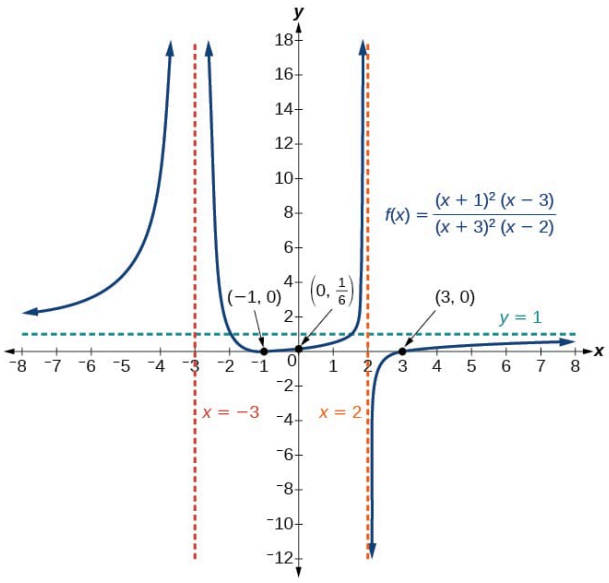
Graph of the toolkit function $y = \frac{1}{x^2}$.



For example, the graph of $f(x) = \frac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure 2.5.20.

Figure 2.5.20

Graph of $f(x) = \frac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$.



- At the x-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph bounces (like a parabola), consistent with the quadratic nature of the factor.
- At the x-intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis (like a line) as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \frac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of $f(x) = \frac{1}{x}$.

How To: Graph a rational function

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Example 2.5.10: Graphing a Rational Function

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution available at the end of this section.

Checkpoint 2.5.10

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an x-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of x-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

Writing Rational Functions from Intercepts and Asymptotes

If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and $xi \neq v_j$ for all i and j, then the function can be written in the form:

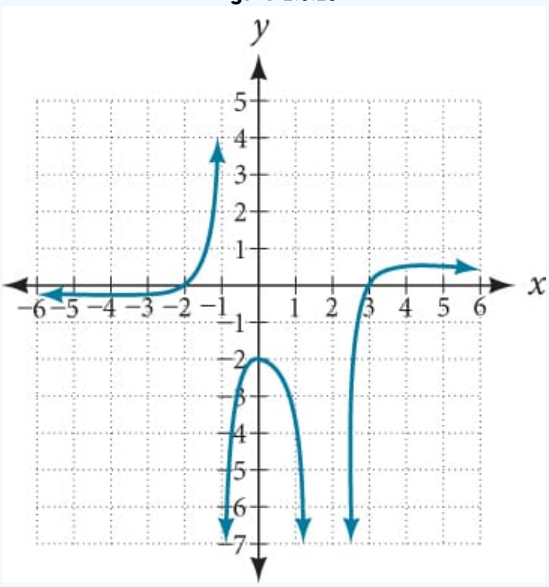
$$f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_m}},$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the x-intercept or by the horizontal asymptote if it is nonzero.

Example 2.5.11: Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in Figure 2.5.23.

Figure 2.5.23



Solution available at the end of this section.

Revisiting Solving Inequalities

As we did with polynomials, we can solve inequalities involving rational expressions using our graphing skills. Again, many instructors would have you solve rational inequalities using sign charts; however, I find solving via graphing (1) is faster, (2) leads to a more natural understanding of what the solution to a rational inequality means, and (3) reinforces graphing skills. Let's see how this is done via an example.

Example 2.5.12: Solving a Rational Inequality

Find all solutions to the inequality.

$$\frac{6x^2+14x-34}{x+4} > 5x-9$$

Solution available at the end of this section.

Answers

Example 2.5.1: Using Arrow Notation

Notice that the graph is showing a vertical asymptote at $x=2$, which tells us that the function is undefined at $x=2$. We can see that as $x \to 2^-$, $f(x) \to -\infty$, and as $x \to 2^+$, $f(x) \to \infty$. Moreover, as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y=4$. Likewise, as the inputs increase without bound, the graph levels off at 4. That is, $f(x) \to 4$ as $x \to \pm\infty$.

Example 2.5.2: Using Transformations to Graph a Rational Function

Shifting the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3$$

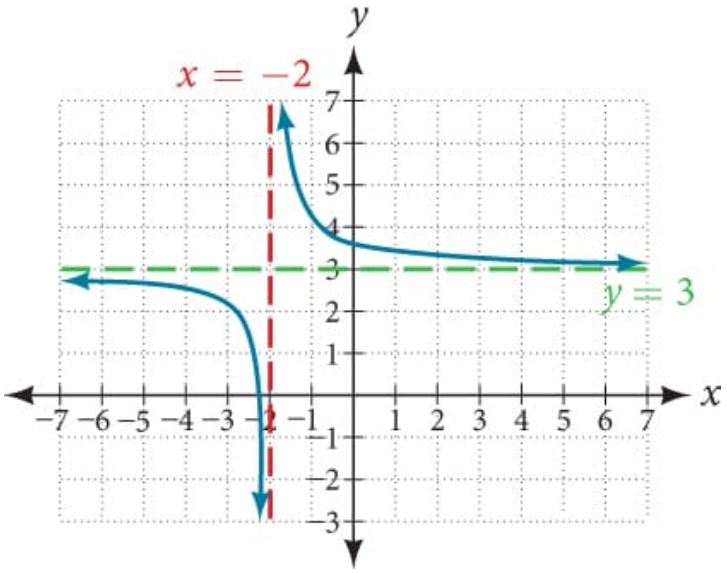
or equivalently, by giving the terms a common denominator,

$$f(x) = \frac{3x+7}{x+2}.$$

The graph of the shifted function is displayed in Figure 2.5.7.

Figure 2.5.7

The graph of $f(x) = \frac{1}{x+2} + 3$.



As $x \to -2^-$, $f(x) \to -\infty$. Therefore, the graph has a vertical asymptote at $x=-2$. We could have arrived at this same conclusion by considering that $f(x) \to \infty$ as $x \to -2^+$.

As $x \to \pm\infty$, $f(x) \to 3$. Thus, the graph has a horizontal asymptote at $y=3$.

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

Example 2.5.3: Finding the Domain of a Rational Function

Begin by setting the denominator equal to zero and solving.

$$\begin{aligned} x^2 - 9 &= 0 \\ \Longrightarrow \quad x^2 &= 9 \\ \Longrightarrow \quad x &= \pm 3 \end{aligned}$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$. Writing this in interval notation, we say the domain of f is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

Example 2.5.4: Identifying Vertical Asymptotes

First, factor the numerator and denominator.

$$\begin{aligned} k(x) &= \frac{5 + 2x^2}{2 - x - x^2} \\ &= \frac{5 + 2x^2}{(2 + x)(1 - x)} \end{aligned}$$

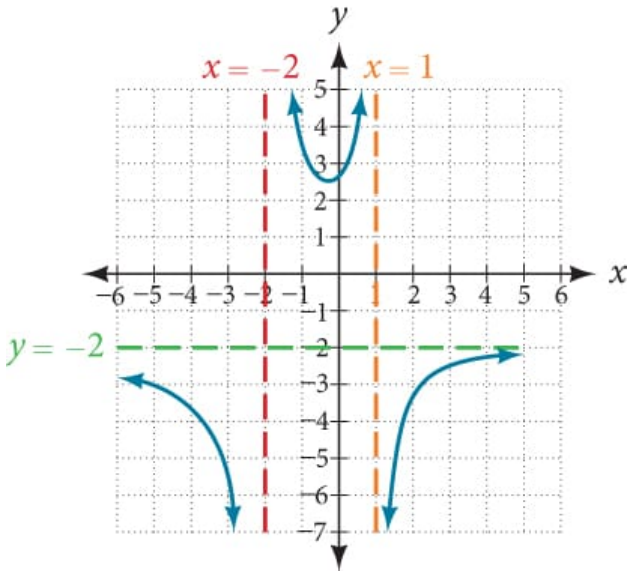
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$\begin{aligned} (2 + x)(1 - x) &= 0 \\ \Longrightarrow \quad x &= -2, 1 \end{aligned}$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 2.5.9 confirms the location of the two vertical asymptotes.

Figure 2.5.9

Graph of $k(x) = \frac{5+2x^2}{2-x-x^2}$.



Definition: Removable Discontinuity of a Rational Function

The most elegant proof of this theorem occurs in Calculus.

Example 2.5.5: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

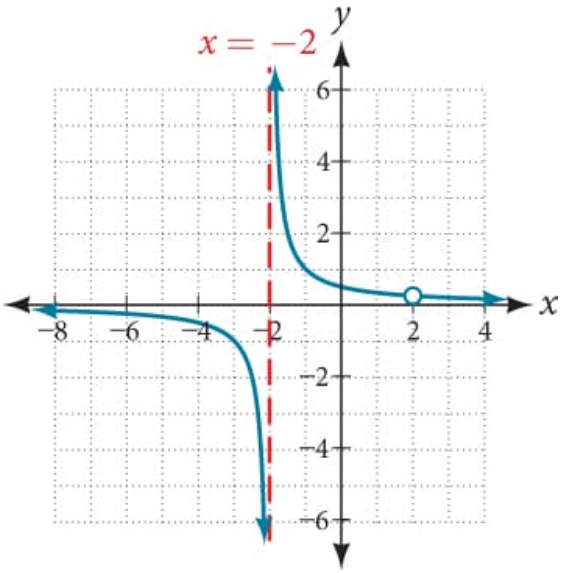
Factor the numerator and the denominator.

$$k(x) = \frac{x - 2}{(x - 2)(x + 2)}.$$

Notice that there is a common factor in the numerator and the denominator, x–2. The zero for this factor is $x = 2$. This is the location of the removable discontinuity. Notice that there is a factor in the denominator that is not in the numerator, $x + 2$. The zero for this factor is $x = -2$. Therefore, the vertical asymptote is $x = -2$. See Figure 2.5.11.

Figure 2.5.11

Graph of $k(x) = \frac{x-2}{x^2-4}$ showing the vertical asymptote and removable discontinuity.



The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole.

Example 2.5.6**: Identifying Horizontal and Slant Asymptotes**

For these solutions, we will let $p(x)$ represent the numerator and $q(x)$ the denominator.

a. The degree of p is equal to the degree of q (3) so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.

b. The degree of p is greater than the degree of q by 1. Therefore, this rational function has a slant asymptote. To find the equation of the slant asymptote, we divide $\frac{x^2-4x+1}{x+2}$. This is easily done using synthetic division.

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

The quotient is $x - 6$ and the remainder is 13. Thus, the equation of the slant asymptote is $y = x - 6$.

c. The degree of p is less than the degree of q, so there is a horizontal asymptote at $y = 0$.

Example 2.5.7**: Identifying Horizontal Asymptotes**

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty \implies C(t) \to \frac{1}{10}.$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Example 2.5.8**: Identifying Horizontal and Vertical Asymptotes**

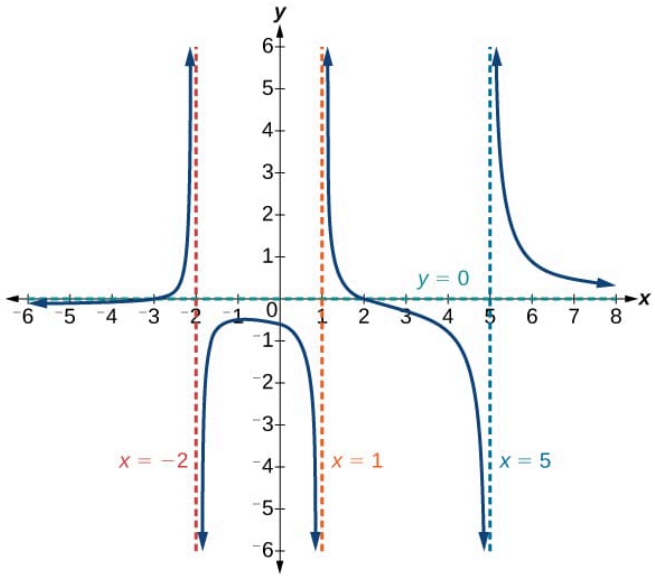
First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1$,–2, and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure 2.5.16.

Figure 2.5.16

Graph of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.



Example 2.5.9**: Finding the Intercepts of a Rational Function**

We can find the y-intercept by evaluating the function at zero

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} = \frac{-6}{10} = -\frac{3}{5} = -0.6.$$

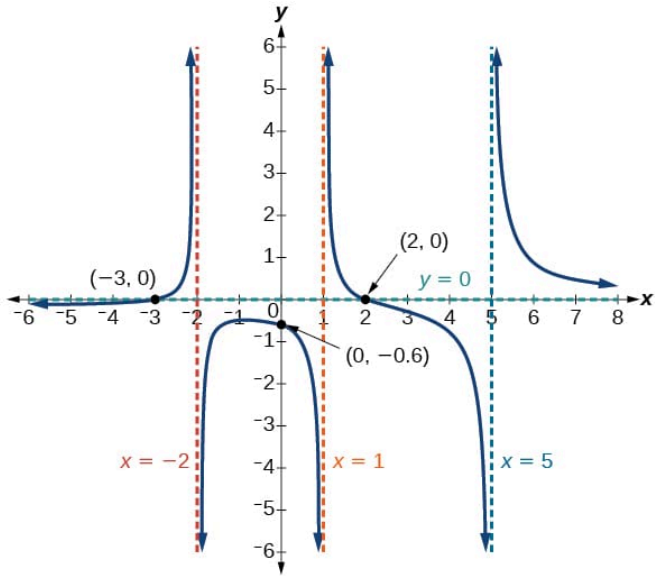
The x-intercepts will occur when the function is equal to zero:

$$\begin{aligned} 0 &= \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \\ \implies 0 &= (x-2)(x+3) \qquad \text{(a fraction is zero only when the numerator is zero)} \end{aligned}$$

This happens when $x = 2$ or when $x = -3$. Therefore, the y-intercept is $(0, -0.6)$, and the x-intercepts are $(2, 0)$ and $(-3, 0)$. See Figure 2.5.17.

Figure 2.5.17

Graph of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$



Example 2.5.10**: Graphing a Rational Function**

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the y-intercept:

$$f(0) = \frac{(0+2)(0-3)}{(0+1)^2(0-2)} = 3.$$

To find the x-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find x-intercepts at $x = -2$ and $x = 3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a y-intercept at $(0, 3)$ and x-intercepts at $(-2, 0)$ and $(3, 0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x + 1 = 0$ and when $x - 2 = 0$, giving us vertical asymptotes at $x = -1$ and $x = 2$.

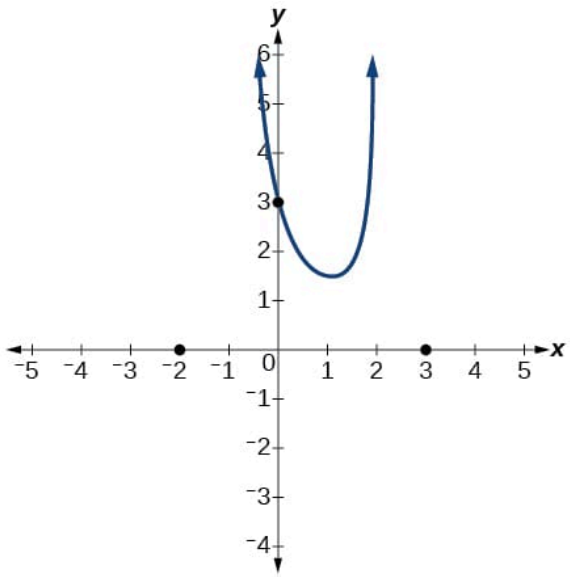
There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no x-intercepts between the vertical asymptotes, and the y-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure $2.5.21$.

Figure $2.5.21$

An unfinished graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$ *(the middle portion).*

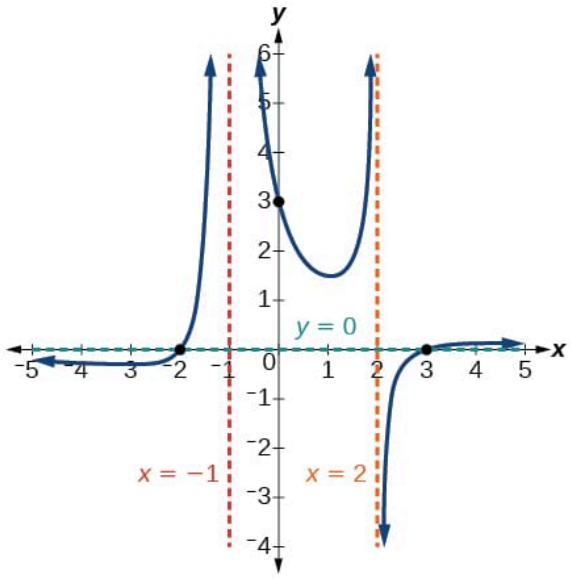


The factor associated with the vertical asymptote at $x = -1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x = 2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure $2.5.22$. After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

Figure $2.5.22$

Graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.



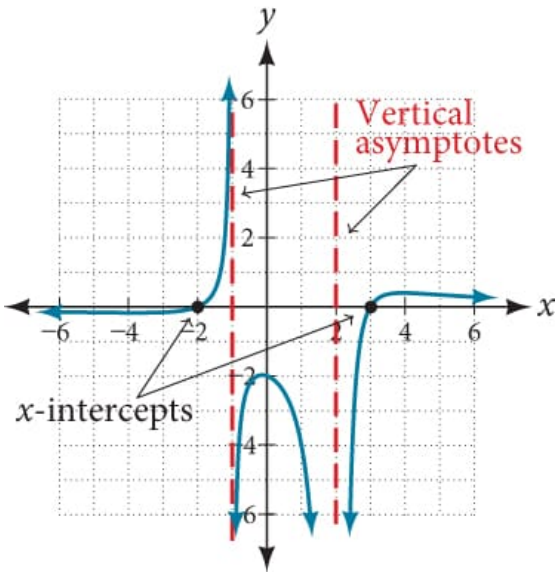
Example $2.5.11$**: Writing a Rational Function from Intercepts and Asymptotes**

The graph appears to have x-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward

negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See Figure 2.5.24.

Figure 2.5.24

The original graph with asymptotes drawn in for guidance.



We can use this information to write a function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}.$$

To find the stretch factor, we can use another clear point on the graph, such as the y-intercept $(0, -2)$.

$$\begin{aligned} -2 &= a\frac{(0+2)(0-3)}{(0+1)(0-2)^2} \\ \Longrightarrow \quad -2 &= a\frac{-6}{4} \\ \Longrightarrow \quad a &= \frac{-8}{-6} \\ \Longrightarrow \quad a &= \frac{4}{3} \end{aligned}$$

This gives us a final function of

$$f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}.$$

Example 2.5.12**: Solving a Rational Inequality**

We begin as we did when solving polynomial inequalities - we move all terms to one side and simplify as much as possible.

$$\begin{aligned} & \frac{6x^2+14x-34}{x+4} > 5x-9 & & \\ \Longrightarrow \quad & \frac{6x^2+14x-34}{x+4} - (5x-9) > 0 & & \text{(subtracting } 5x-9 \text{ from both sides)} \\ \Longrightarrow \quad & \frac{6x^2+14x-34}{x+4} - \frac{(5x-9)(x+4)}{x+4} > 0 & & \text{(rewriting with a common denominator)} \\ \Longrightarrow \quad & \frac{6x^2+14x-34}{x+4} - \frac{5x^2+11x-36}{x+4} > 0 & & \text{(distributing)} \\ \Longrightarrow \quad & \frac{6x^2+14x-34-(5x^2+11x-36)}{x+4} > 0 & & \text{(combining fractions)} \\ \Longrightarrow \quad & \frac{6x^2+14x-34-5x^2-11x+36}{x+4} > 0 & & \text{(distributing)} \\ \Longrightarrow \quad & \frac{x^2+3x+2}{x+4} > 0 & & \text{(combining like terms)} \\ \Longrightarrow \quad & \frac{(x+2)(x+1)}{x+4} > 0 & & \text{(factoring the trinomial)} \end{aligned}$$

It's important to note that you would have done all of this work if you were going to solve the inequality using sign charts or graphing - there is no difference in "grunt work" up to this point.

As we did with polynomial inequalities, we graph the expression on the left side. That is, we want to sketch $f(x) = \frac{(x+2)(x+1)}{x+4}$. Our goal is to identify when the graph of f is *above* the x-axis (because we are solving $f(x) > 0$).

- **Horizontal Intercepts:** The numerator of f is zero at $x = -2$ and $x = -1$. Therefore, the graph has x-intercepts at $(-2, 0)$ and $(-1, 0)$. Both of these zeros have multiplicity 1, so the graph will pass through each like a line.
- **Vertical Intercept:** $f(0) = \frac{2}{4} = \frac{1}{2}$, so the y-intercept is $\left(0, \frac{1}{2}\right)$.
- **Vertical Asymptote:** The denominator is zero when $x = -4$, which is not a zero shared with the numerator. Therefore, f has a vertical asymptote at $x = -4$. Additionally, the degree of the factor, $x + 4$, in the denominator is 1, so the vertical asymptote shares the behavior of $\frac{1}{x}$.
- **Horizontal/Slant Asymptote:** Since the numerator of f has degree 2 and the denominator has degree 1, there will be a slant asymptote. To find the equation of the slant asymptote, it's best to use the unfactored form, $f(x) = \frac{x^2+3x+2}{x+4}$, and divide.

$$\begin{array}{r|rrr} \underline{-4} & 1 & 3 & 2 \\ & & -4 & 4 \\ & 1 & -1 & 6 \end{array}$$

  Hence, the quotient is $x - 1$ and the remainder is 6. Therefore, the equation of the slant asymptote is $y = x - 1$.

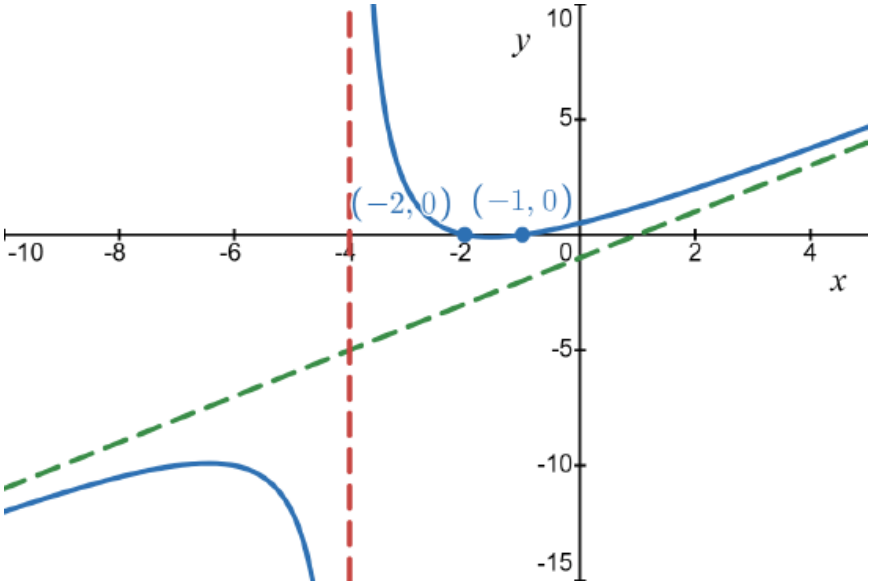
We now have enough information to sketch the graph. Starting at the y-intercept (which is slightly above the x-axis), we have a choice, start sketching to the left or right. For no better reason than random choice, I will choose to sketch to the right. Does the graph rise or fall? If we started drawing to the right and had the graph fall, it would eventually intersect the x-axis; however, we do not have any x-intercepts greater than 0. Thus, the graph climbs and eventually tends to the line $y = x - 1$ (see Figure $2.5.25$).

Going back to the y-intercept, we now start sketching the curve to the left. Since there is an x-intercept at $x = -1$, our curve falls (as we draw to the left), goes through $(-1, 0)$, and (since there is another x-intercept) climbs to strike the x-axis at $x = -2$. It goes through this intercept and we are now above the x-axis. If we continue to draw to the left (again, see Figure $2.5.25$), our curve would approach the vertical asymptote, $x = -4$. We have a choice - have the graph rise to infinity as $x \to -4^+$, or have the graph fall to negative infinity as $x \to -4^+$. Since we do not have any more x-intercepts, the choice is out of our hands. The function grows without bound (to infinity) as $x \to -4^+$.

As stated in the list above, the behavior of the function near to the vertical asymptote mimics that of $\frac{1}{x}$, so we know $f(x) \to -\infty$ as $x \to -4^-$. That is, since the graph of f is rising on one side of $x = -4$, it must be falling on the other side. Finally, as we continue drawing to the left, the function tends to the slant asymptote.

Figure $2.5.25$

Graph of $y = \frac{(x+2)(x+1)}{x+4}$



Trust me when I say that you will be able to sketch a function like this in a couple of minutes once you practice these types of problems. Do not freak out.

Our ultimate goal was to find out when $f(x) > 0$. Now that we have the graph, we can see this occurs when $x^{\text{int}}\,(-4, -2) \cup (-1, \infty)$.

This page titled 2.5: Rational Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

2.5.1: Resources and Key Concepts

Resources

Videos

- Rational function
  - Rational Functions - A Brief Review of Rational Functions
- Vertical asymptote
  - Rational Functions - Identifying Vertical Asymptotes of Rational Functions
  - Rational Functions - Asymptotic Behavior of Rational Functions
- Horizontal asymptote
  - Rational Functions - Identifying Horizontal Asymptotes of Rational Functions
- Slant asymptote (oblique asymptote)
  - Rational Functions - Identifying Slant Asymptotes of Rational Functions
- Domain of a rational function
  - Rational Functions - Finding the Domain of Rational Functions
- Graphing rational functions
  - Rational Functions - Graphing Rational Functions
  - Rational Functions - Graphing Rational Functions
  - Algebra Lecture: Graphing Rational Functions
- Solving rational inequalities
  - Rational Functions - Solving Rational Inequalities
  - Inequalities - Nonlinear - A Review for College Algebra: Solving Rational Inequalities

Key Concepts

Definitions

- **Arrow Notation**: A symbolic way to describe the behavior of a function as its input x approaches a certain value or infinity. Examples: $x \to a^-$ (approaches a from the left), $x \to \infty$ (approaches infinity), $f(x) \to L$ (the output approaches L).
- **Vertical Asymptote**: A vertical line $x = a$ where the graph of a function tends toward positive or negative infinity as the inputs approach a from either the left or the right.
- **Horizontal Asymptote**: A horizontal line $y = b$ where the graph of a function approaches the line as the inputs increase or decrease without bound ($x \to \infty$ or $x \to -\infty$).
- **Rational Function**: A function that can be written as the quotient of two polynomial functions, $f(x) = \frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$.
- **Removable Discontinuity (Hole)**: A single point where the graph of a function is not defined, indicated by an open circle on the graph. It occurs in a rational function at $x = a$ if $(x - a)$ is a factor of both the numerator and the denominator, and after simplification, a is no longer a zero of the denominator.
- **End Behavior (of a rational function's graph)**: A description, usually as a function (e.g., horizontal or slant asymptote), of what the function values of the graph tend to approach as the inputs increase or decrease without bound.
- **Polynomial Degree Notation ($deg(f(x))$)**: Denotes the degree of a polynomial $f(x)$.
- **Slant Asymptote (Oblique Asymptote)**: A non-horizontal, non-vertical line that the graph of a rational function approaches as $x \to \infty$ or $x \to -\infty$. It occurs when the degree of the numerator is exactly one more than the degree of the denominator.

Theorems

- **Domain of a Rational Function**: The domain of a rational function includes all real numbers except those that cause the denominator of the unsimplified rational function to equal zero.
- **Vertical Asymptotes of Rational Functions**: Vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

- **Removable Discontinuity of a Rational Function**: A removable discontinuity for the graph of a rational function occurs at $x = a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator, and after simplification, a is no longer a zero of the denominator.
- **End Behavior of a Rational Function** $f(x) = \frac{N(x)}{D(x)}$:
  - If $deg(D(x)) > deg(N(x))$, then $f(x)$ has a horizontal asymptote at $y = 0$.
  - If $deg(D(x)) = deg(N(x))$, then $f(x)$ has a horizontal asymptote at $y = \frac{a}{b}$, where a is the lead coefficient of $N(x)$ and b is the lead coefficient of $D(x)$.
  - If $deg(N(x)) = deg(D(x)) + 1$, then $f(x)$ has a slant (oblique) asymptote whose equation is the quotient after dividing $N(x)$ by $D(x)$ (ignoring the remainder).

Common Mistakes

- **Incorrectly Identifying Domain Before Simplification**: Finding the domain of a rational function *after* simplifying it by canceling common factors. The domain must be determined from the original, unsimplified function.
- **Confusing Holes with Vertical Asymptotes**: If a factor $(x - a)$ cancels from the numerator and denominator, there is a hole at $x = a$, not a vertical asymptote, provided a is not still a zero of the simplified denominator.
- **Errors in Determining Horizontal/Slant Asymptotes**:
  - Making errors in calculating the ratio of leading coefficients for horizontal asymptotes when degrees are equal.
  - Errors in polynomial division when finding slant asymptotes.
- **Assuming Graphs Cannot Cross Horizontal/Slant Asymptotes**: While graphs of rational functions never cross their vertical asymptotes, they can cross their horizontal or slant asymptotes.
- **Errors in Simplifying Difference Quotients for Rational Functions**: Mistakes in finding a common denominator or in distributing terms when simplifying the complex fraction that arises.
- **Solving Rational Inequalities Incorrectly**: Making errors when finding critical points (zeros of numerator and denominator) or when testing intervals, or misinterpreting the graph if using the graphical method.

This page titled 2.5.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

2.5.2: Homework

Reading Questions

1. What is a rational function? Give its general form.
2. Explain what arrow notation like $x \to a^+$ and $f(x) \to -\infty$ means.
3. What is a vertical asymptote of a rational function? How do you typically find the location(s) of vertical asymptotes from the function's equation?
4. What is a horizontal asymptote? Describe the three cases for determining the horizontal asymptote of a rational function based on the degrees of the numerator and denominator.
5. Under what condition does a rational function have a slant asymptote instead of a horizontal asymptote? How is its equation found?
6. What is a removable discontinuity (or hole) in the graph of a rational function? How is it identified from the function's equation?
7. How do you find the domain of a rational function?
8. How are the x-intercepts of a rational function typically found? What about the y-intercept?
9. Describe how the multiplicity of a factor in the denominator affects the behavior of the graph near a vertical asymptote (odd vs. even multiplicity).
10. When writing the equation of a rational function from its graph, what information do the x-intercepts provide? What about the vertical asymptotes? How can the stretch factor a be determined?
11. When solving a rational inequality graphically, like $\frac{P(x)}{Q(x)} > 0$, what are you looking for on the graph of $f(x) = \frac{P(x)}{Q(x)}$?
12. What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?
13. What is the fundamental difference in the graphs of polynomial functions and rational functions?
14. If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?
15. Can a graph of a rational function have no vertical asymptote? If so, how?
16. Can a graph of a rational function have no x-intercepts? If so, how?

Homework

Algebraic

For the following exercises, find the domain of the rational functions.

17. $f(x) = \frac{x-1}{x+2}$
18. $f(x) = \frac{x+1}{x^2-1}$
19. $f(x) = \frac{x^2+4}{x^2-2x-8}$
20. $f(x) = \frac{x^2+4x-3}{x^4-5x^2+4}$

For the following exercises, find the domain, vertical asymptotes, and horizontal asymptotes of the functions.

21. $f(x) = \frac{4}{x-1}$
22. $f(x) = \frac{2}{5x+2}$
23. $f(x) = \frac{x}{x^2-9}$
24. $f(x) = \frac{x}{x^2+5x-36}$
25. $f(x) = \frac{3+x}{x^3-27}$
26. $f(x) = \frac{3x-4}{x^3-16x}$
27. $f(x) = \frac{x^2-1}{x^3+9x^2+14x}$
28. $f(x) = \frac{x+5}{x^2-25}$
29. $f(x) = \frac{x-4}{x-6}$
30. $f(x) = \frac{4-2x}{3x-1}$

For the following exercises, find the x-and y-intercepts for the functions.

31. $f(x) = \frac{x+5}{x^2+4}$
32. $f(x) = \frac{x}{x^2-x}$
33. $f(x) = \frac{x^2+8x+7}{x^2+11x+30}$
34. $f(x) = \frac{x^2+x+6}{x^2-10x+24}$
35. $f(x) = \frac{94-2x^2}{3x^2-12}$

For the following exercises, describe the local and end behavior of the functions.

36. $f(x) = \frac{x}{2x+1}$
37. $f(x) = \frac{2x}{x-6}$
38. $f(x) = \frac{-2x}{x-6}$
39. $f(x) = \frac{x^2-4x+3}{x^2-4x-5}$
40. $f(x) = \frac{2x^2-32}{6x^2+13x-5}$

For the following exercises, find the slant asymptote of the functions.

41. $f(x) = \frac{24x^2+6x}{2x+1}$
42. $f(x) = \frac{4x^2-10}{2x-4}$
43. $f(x) = \frac{81x^2-18}{3x-2}$
44. $f(x) = \frac{6x^3-5x}{3x^2+4}$
45. $f(x) = \frac{x^2+5x+4}{x-1}$

Graphical

For the following exercises, use the given transformation to graph the function. Note the vertical and horizontal asymptotes.

46. The reciprocal function shifted up two units.
47. The reciprocal function shifted down one unit and left three units.
48. The reciprocal squared function shifted to the right 2 units.
49. The reciprocal squared function shifted down 2 units and right 1 unit.

For the following exercises, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal or slant asymptote of the functions. Use that information to sketch a graph.

50. $p(x) = \frac{2x-3}{x+4}$
51. $q(x) = \frac{x-5}{3x-1}$
52. $s(x) = \frac{4}{(x-2)^2}$
53. $r(x) = \frac{5}{(x+1)^2}$
54. $f(x) = \frac{3x^2-14x-5}{3x^2+8x-16}$
55. $g(x) = \frac{2x^2+7x-15}{3x^2-14x+15}$
56. $a(x) = \frac{x^2+2x-3}{x^2-1}$
57. $b(x) = \frac{x^2-x-6}{x^2-4}$
58. $h(x) = \frac{2x^2+x-1}{x-4}$
59. $k(x) = \frac{2x^2-3x-20}{x-5}$
60. $w(x) = \frac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$
61. $z(x) = \frac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

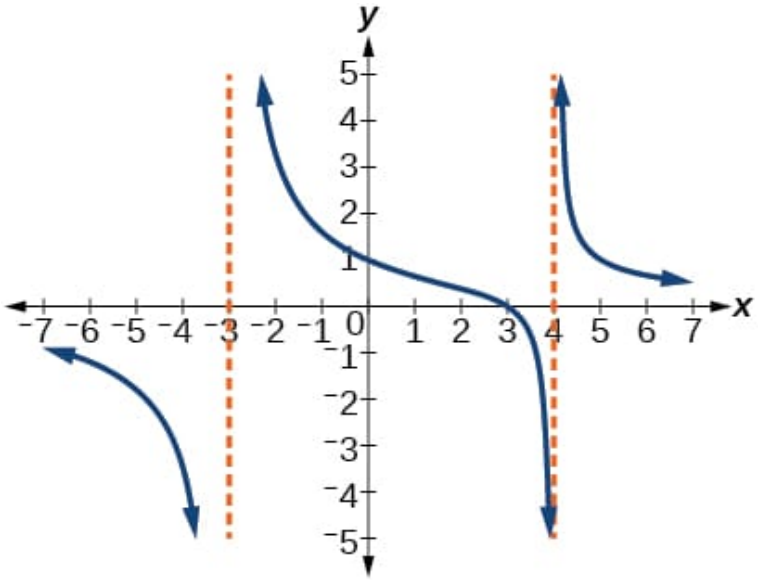
For the following exercises, write an equation for a rational function with the given characteristics.

62. Vertical asymptotes at $x = 5$ and $x = -5$, x-intercepts at $(2, 0)$ and $(-1, 0)$, y-intercept at $(0, 4)$
63. Vertical asymptotes at $x = -4$ and $x = -1$, x-intercepts at $(1, 0)$ and $(5, 0)$, y-intercept at $(0, 7)$
64. Vertical asymptotes at $x = -4$ and $x = -5$, x-intercepts at $(4, 0)$ and $(-6, 0)$, Horizontal asymptote at $y = 7$
65. Vertical asymptotes at $x = -3$ and $x = 6$, x-intercepts at $(-2, 0)$ and $(1, 0)$, Horizontal asymptote at $y = -2$
66. Vertical asymptote at $x = -1$, Double zero at $x = 2$, y-intercept at $(0, 2)$

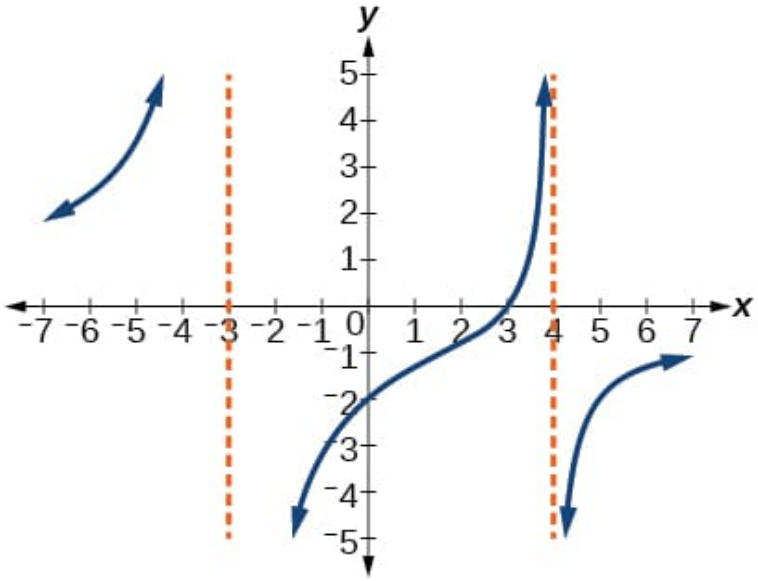
57. Vertical asymptote at $x = 3$, Double zero at $x = 1$, y-intercept at $(0, 4)$

For the following exercises, use the graphs to write an equation for the function.

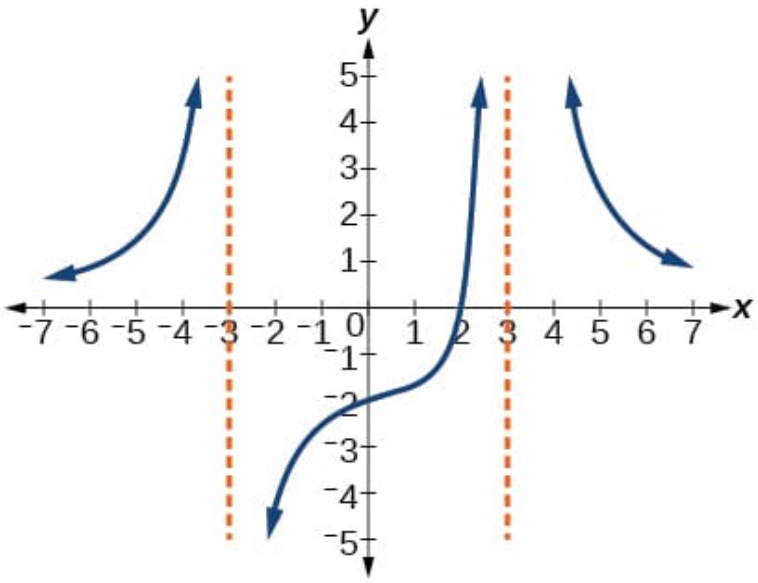
58.



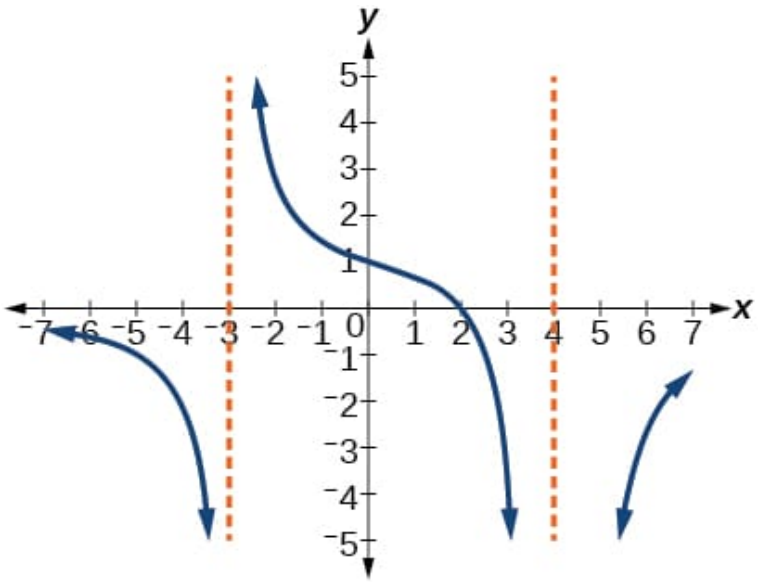
59.



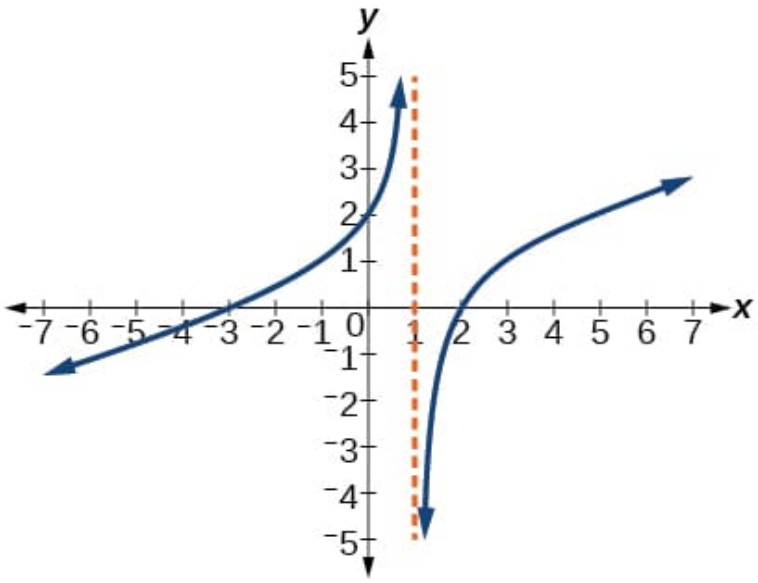
70.


y
x
5
4
3
2
1
−1
−2
−3
−4
−5
−7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7

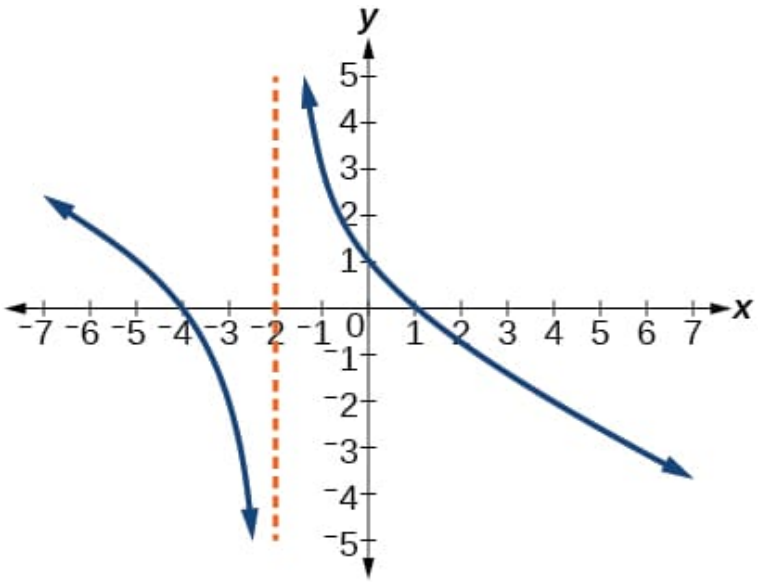
71.


y
x
5
4
3
2
1
−1
−2
−3
−4
−5
−7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7

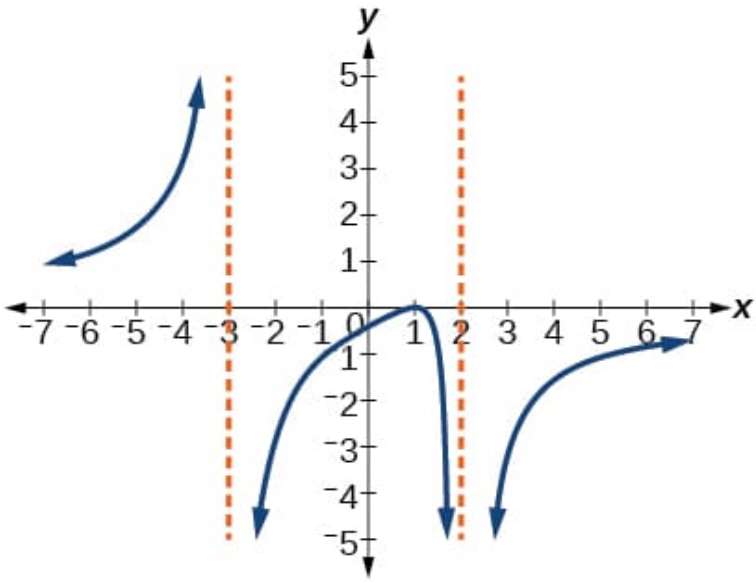
72.


y
x
5
4
3
2
1
−1
−2
−3
−4
−5
−7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7

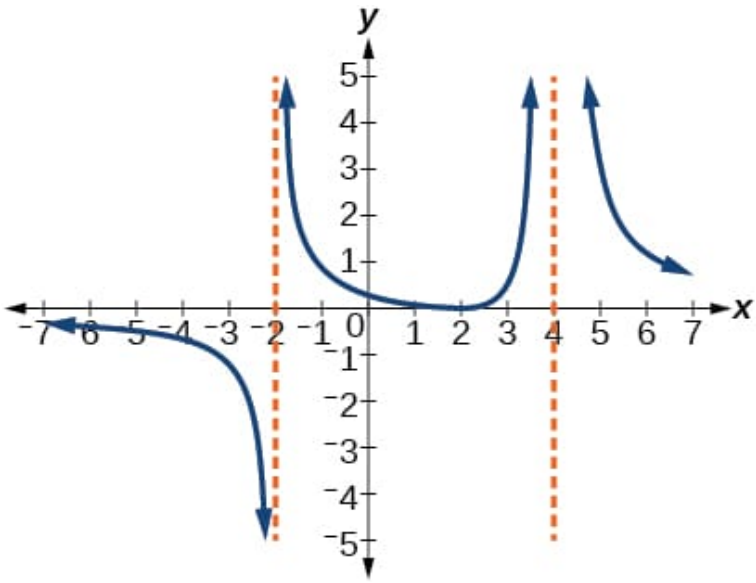
73.


y
x
5
4
3
2
1
−1
−2
−3
−4
−5
−7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7

74.



75.



Numeric

For the following exercises, make tables to show the behavior of the function near the vertical asymptote and reflecting the horizontal asymptote

76. $f(x) = \frac{1}{x-2}$
77. $f(x) = \frac{x}{x-3}$
78. $f(x) = \frac{2x}{x+4}$
79. $f(x) = \frac{2x}{(x-3)^2}$
80. $f(x) = \frac{x^2}{x^2+2x+1}$

Technology

For the following exercises, use a calculator to graph $f(x)$. Use the graph to solve $f(x) > 0$.

81. $f(x) = \frac{2}{x+1}$

32. $f(x) = \frac{4}{2x-3}$
33. $f(x) = \frac{2}{(x-1)(x+2)}$
34. $f(x) = \frac{x+2}{(x-1)(x-4)}$
35. $f(x) = \frac{(x+3)^2}{(x-1)^2(x+1)}$

Extensions

For the following exercises, identify the removable discontinuity.

36. $f(x) = \frac{x^2-4}{x-2}$
37. $f(x) = \frac{x^3+1}{x+1}$
38. $f(x) = \frac{x^2+x-6}{x-2}$
39. $f(x) = \frac{2x^2+5x-3}{x+3}$
40. $f(x) = \frac{x^3+x^2}{x+1}$

Real-World Applications

For the following exercises, express a rational function that describes the situation.

91. A large mixing tank currently contains 200 gallons of water, into which 10 pounds of sugar have been mixed. A tap will open, pouring 10 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 3 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.
92. A large mixing tank currently contains 300 gallons of water, into which 8 pounds of sugar have been mixed. A tap will open, pouring 20 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 2 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

For the following exercises, use the given rational function to answer the question.

93. The concentration C of a drug in a patient's bloodstream t hours after injection in given by $C(t) = \frac{2t}{3+t^2}$. What happens to the concentration of the drug as t increases?
94. The concentration C of a drug in a patient's bloodstream t hours after injection is given by $C(t) = \frac{100t}{2t^2+75}$. Use a calculator to approximate the time when the concentration is highest.

For the following exercises, construct a rational function that will help solve the problem. Then, use a calculator to answer the question.

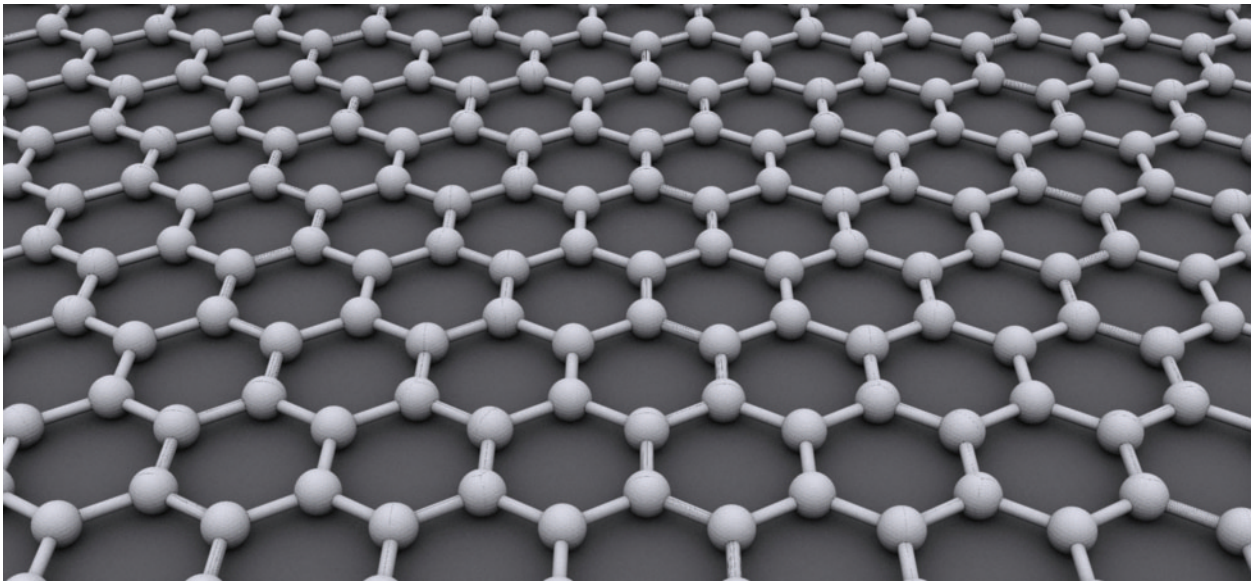
95. An open box with a square base is to have a volume of 108 cubic inches. Find the dimensions of the box that will have minimum surface area. Let x = length of the side of the base.
96. A rectangular box with a square base is to have a volume of 20 cubic feet. The material for the base costs 30 cents/ square foot. The material for the sides costs 10 cents/square foot. The material for the top costs 20 cents/square foot. Determine the dimensions that will yield minimum cost. Let x = length of the side of the base.
97. A right circular cylinder has volume of 100 cubic inches. Find the radius and height that will yield minimum surface area. Let x = radius.
98. A right circular cylinder with no top has a volume of 50 cubic meters. Find the radius that will yield minimum surface area. Let x = radius.
99. A right circular cylinder is to have a volume of 40 cubic inches. It costs 4 cents/square inch to construct the top and bottom and 1 cent/square inch to construct the rest of the cylinder. Find the radius to yield minimum cost. Let x = radius.

This page titled 2.5.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

3: INVERSE AND RADICAL FUNCTIONS

Imagine charging your cell phone is less than five seconds. Consider cleaning radioactive waste from contaminated water. Think about filtering salt from ocean water to make an endless supply of drinking water. Ponder the idea of bionic devices that can repair spinal injuries. These are just of few of the many possible uses of a material called graphene. Materials scientists are developing a material made up of a single layer of carbon atoms that is stronger than any other material, completely flexible, and conducts electricity better than most metals. Research into this type of material requires a solid background in mathematics, including understanding roots and radicals. In this chapter, you will learn to simplify expressions containing roots and radicals, perform operations on radical expressions and equations, and evaluate radical functions.

Graphene is an incredible strong and flexible material made from carbon. It can also conduct electricity. Notice the hexagonal grid pattern. (credit: "AlexanderAIUS" / Wikimedia Commons)



3.1: Inverse Functions

3.1.1: Resources and Key Concepts

3.1.2: Homework

3.2: Radical Functions

3.2.1: Resources and Key Concepts

3.2.2: Homework

3.3: Radical Equations

3.3.1: Resources and Key Concepts

3.3.2: Homework

3.4: Rational Exponent Expressions Encountered in Calculus

3.4.1: Resources and Key Concepts

3.4.2: Homework

This page titled 3: Inverse and Radical Functions is shared under a license and was authored, remixed, and/or curated by LibreTexts.

3.1: Inverse Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure $3.1.1$ provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

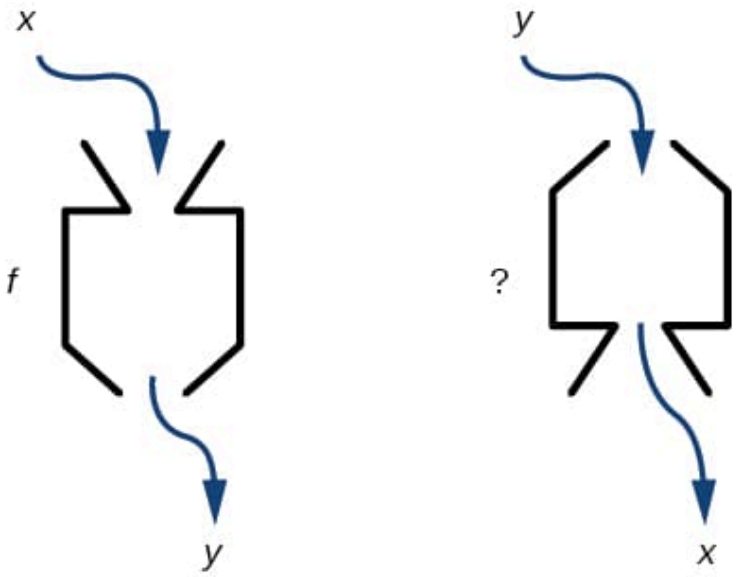


Figure $3.1.1$**:** *Can a function "machine" operate in reverse?*

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the Celsius scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees Fahrenheit to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32)$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24^\circ \text{ C}.$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure $3.1.2$ for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

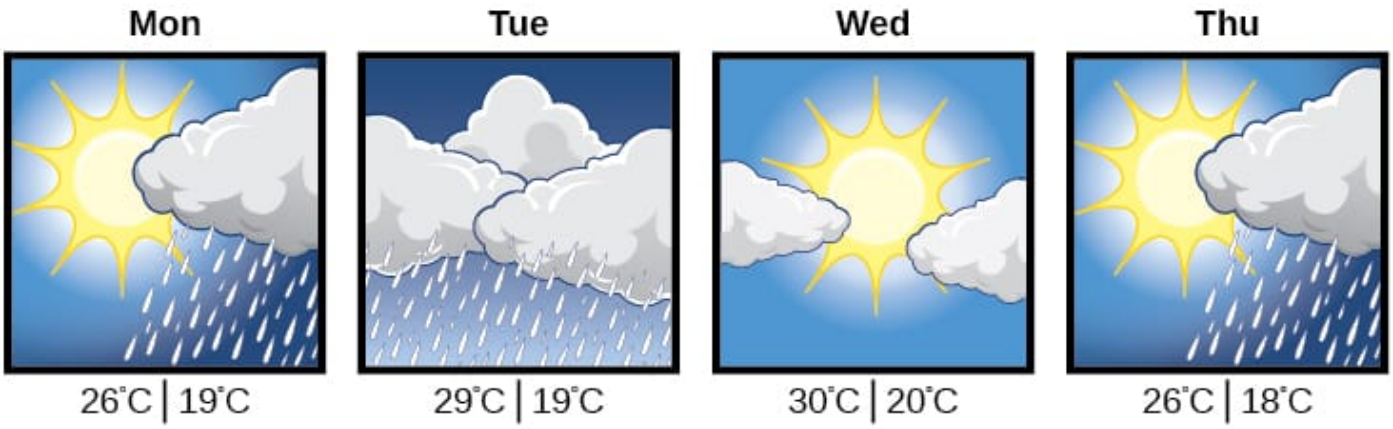


Figure 3.1.2

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her Algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$\begin{aligned} 26 &= \frac{5}{9}(F-32) \\ \implies 26\cdot\frac{9}{5} &= F-32 \\ \implies F &= 26\cdot\frac{9}{5}+32\approx 79 \end{aligned}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Definition: Inverse Function

Let f be a one-to-one function. The **inverse** of f, denoted f^{-1}, is the function that reverses the pairing produced by f: for each y in the range of f,

$$f^{-1}(y)=x \quad \text{precisely when} \quad f(x)=y.$$

f^{-1} is called the **inverse function** of f and the notation, f^{-1}, is read "f-inverse".

Caution

The symbol f^{-1} denotes the inverse function, not a reciprocal. In general, $f^{-1}(x)\neq\frac{1}{f(x)}$. The reciprocal $\frac{1}{f(x)}$ is written $\big(f(x)\big)^{-1}$.

Additionally, not all functions have inverses.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a=1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1}\circ f$ equals the identity function, that is,

$$(f^{-1}\circ f)(x)=f^{-1}\left(f(x)\right)=x.$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a reciprocal, some functions do not have inverses.

Inverse Function Notation

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x."

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

Theorem

If f^{-1} is the inverse of f, then $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}\right) = x.$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Example 3.1.1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution available at the end of this section.

If we show the coordinate pairs from Example 3.1.1 in a table form, the input and output are clearly reversed. See Table 3.1.1.

Table 3.1.1

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Checkpoint 3.1.1

Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

Example 3.1.2: Testing Inverse Relationships Algebraically

If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$, is $g = f^{-1}$?

Solution available at the end of this section.

Checkpoint 3.1.2

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

Example 3.1.3: Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

Solution available at the end of this section.

Checkpoint 3.1.3

If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure 3.1.3.

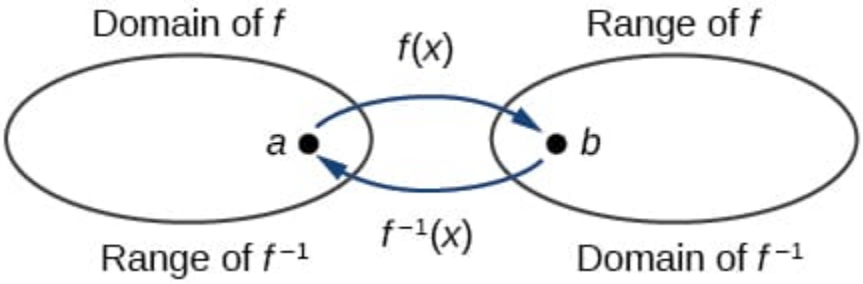


Figure 3.1.3: *Domain and range of a function and its inverse*

Theorem: Domain and Range of Inverse Functions

If f is one-to-one, then the domain of f^{-1} equals the range of f, and the range of f^{-1} equals the domain of f.

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and -3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all!

To put it differently, the quadratic function is not a one-to-one function; it fails the Horizontal Line Test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

Interactive Element: The Horizontal Line Test

The following Interactive Element might help you visualize why the quadratic function is not one-to-one.

Interact: Drag the red point up and down to move the horizontal line.

Observation: Above the vertex the line meets $y = x^2$ at two points, so one output has two inputs — the function fails the Horizontal Line Test.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its domain limited to $[0, \infty)$, which is a one-to-one function (it passes the Horizontal Line Test) and which has an inverse (the square-root function).

If $f(x) = (x - 1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of f^{-1} = $[1, \infty)$.
- The domain of f^{-1} = range of f = $[0, \infty)$.

Q&A

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Example 3.1.4: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 3.1.2. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Table 3.1.2

Constant	**Identity**	**Quadratic**	**Cubic**	**Reciprocal**
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	**Cube root**	**Square root**	**Absolute value**	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Solution available at the end of this section.

We can see that the absolute value and reciprocal-squared functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 3.1.4. They both would fail the Horizontal Line Test. However, if a function is restricted to a certain domain so that it passes the Horizontal Line Test, then in that restricted domain, it can have an inverse.

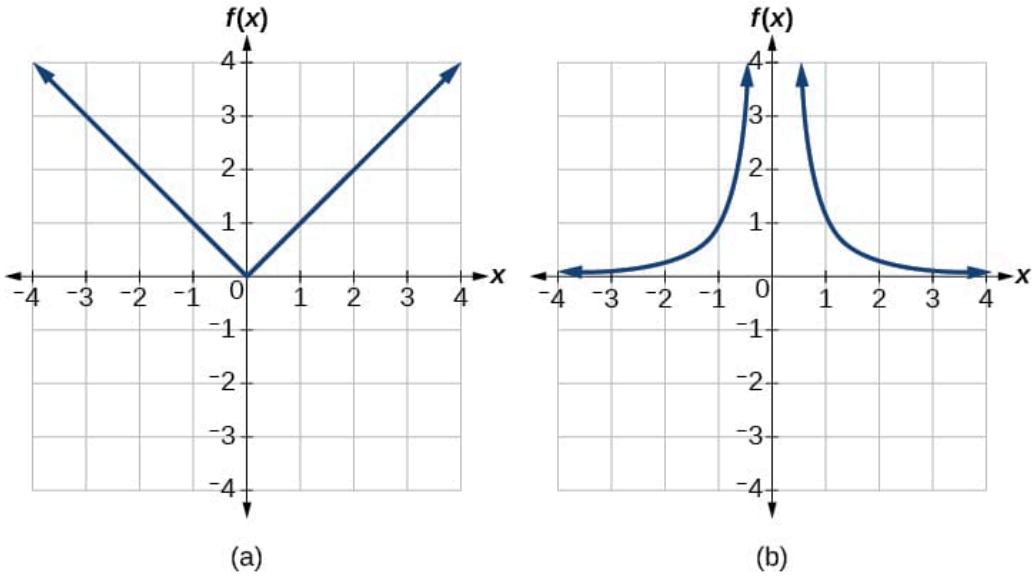


Figure 3.1.4**:** *(a) Absolute value; (b) Reciprocal squared*

Checkpoint 3.1.4

The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example 3.1.5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table 3.1.3, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$.

Table 3.1.3

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Solution available at the end of this section.

Checkpoint 3.1.5

Using Table 3.1.4, find and interpret each of the following:

a. $f(60)$
b. $f^{-1}(60)$

Table 3.1.4

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We know that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

Example 3.1.6: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 3.1.5. Find $g(3)$ and $g^{-1}(3)$.

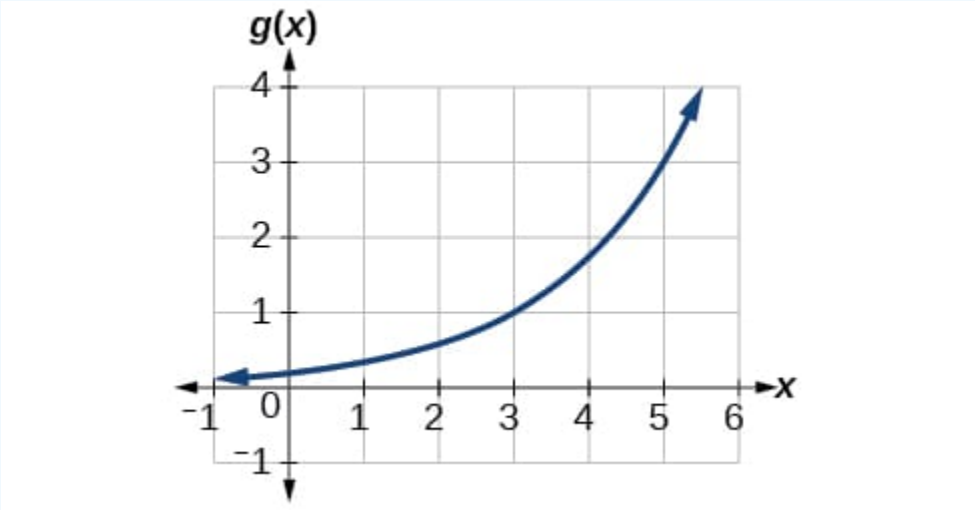


Figure 3.1.5

Solution available at the end of this section.

Checkpoint 3.1.6

Using the graph in Figure 3.1.6 to answer the following questions.

a. Find $g^{-1}(1)$.
b. Estimate $g^{-1}(4)$.

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula—for example, y as a function of x—we can often find the inverse function by solving to obtain x as a function of y.

How To: Find the Equation of an Invertible Function

Given a function, $f(x)$, we find the inverse (if it exists) by doing the following:

1. Let $y = f(x)$.
2. Solve for x.
3. Interchange x and y.

The result of these operations, if it is a function, is $f^{-1}(x)$.

Example 3.1.7: Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature,

$$C = \frac{5}{9}(F - 32).$$

Solution available at the end of this section.

Checkpoint 3.1.7

Solve for x in terms of y given $y = \frac{1}{3}(x - 5)$.

Example 3.1.8: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

Solution available at the end of this section.

Note that the domain and range of f in Example 3.1.8 exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table 3.1.5.

Table 3.1.5

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example 3.1.9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x - 4}$.

Solution available at the end of this section.

The formula we found for $f^{-1}(x)$ in Example 3.1.9 looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

Checkpoint 3.1.9

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in Figure 3.1.7.

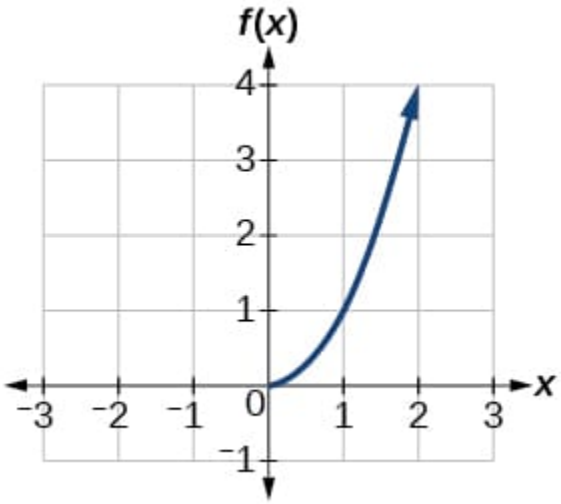


Figure 3.1.7**:** *Quadratic function with domain restricted to* $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the Horizontal Line Test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure 3.1.8.

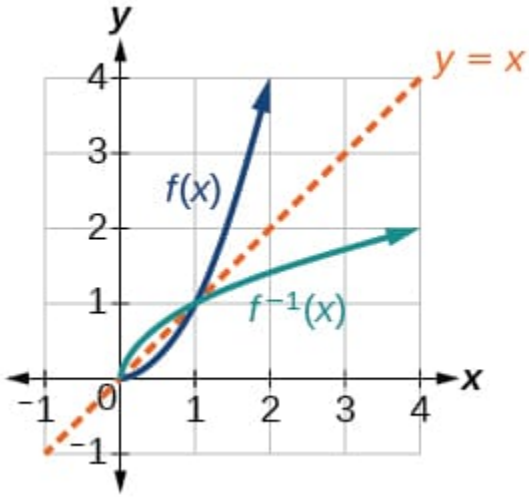


Figure 3.1.8**:** *Square and square-root functions on the non-negative domain*

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Interactive Element: Reflecting a Function Across the Line y = x

The following Interactive Element might help you visualize how the graph of an inverse function is the mirror image of the original across the line $y = x$.

Interact: Drag the large red point along the blue curve $y = x^2$.

Observation: The orange point always lands on $y = \sqrt{x}$, the mirror image across $y = x$; a pair (a, b) on f becomes (b, a) on f^{-1}.

Example 3.1.10: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure 3.1.9, sketch a graph of $f^{-1}(x)$.

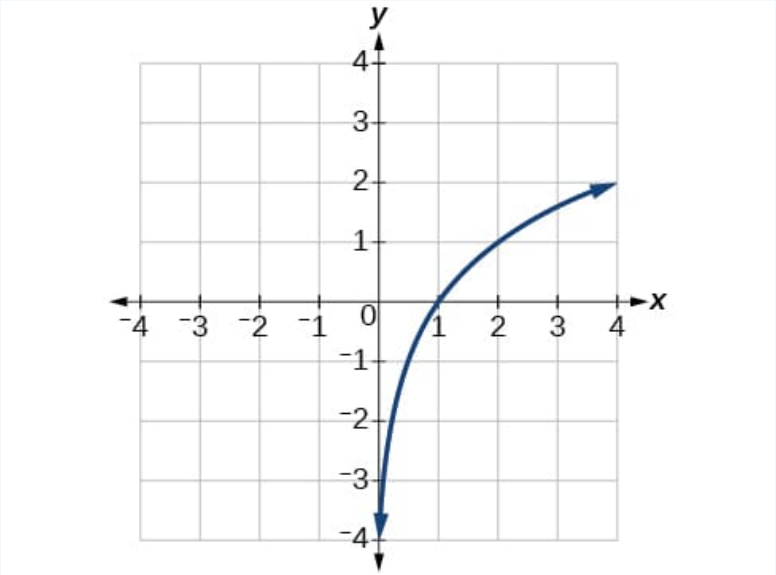


Figure 3.1.9

Solution available at the end of this section.

Checkpoint 3.1.10

Draw graphs of the functions f and f^{-1} from Example 3.1.8.

Q&A

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x.$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Restricting the Domain to Find the Inverse of a Polynomial Function

So far, we have been able to find the inverse functions of cubic functions without having to restrict their domains. However, as we know, not all cubic polynomials are one-to-one. Some functions that are not one-to-one may have their domain restricted so that they are one-to-one, but only over that domain. The function over the restricted domain would then have an inverse function. Since quadratic functions are not one-to-one, we must restrict their domain in order to find their inverses.

Theorem: Forcing Invertibility by Restricting Domain

If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

How To: Restrict Domain to Force Invertibility

Given a function that is not one-to-one,

1. Restrict the domain by determining a domain on which the original function is one-to-one (i.e., a domain where the function passes the Horizontal Line Test).
2. Replace $f(x)$ with y.
3. Solve for x interchange x and y.
4. Rename the function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

Example 3.1.11: Restricting the Domain to Find the Inverse of a Polynomial Function

Find the inverse function of f :

a. $f(x)=(x-4)^2,\ x\geq 4$
b. $f(x)=(x-4)^2,\ x\leq 4$

Solution available at the end of this section.

On the graphs in Figure 3.1.12, we see the original function from Example 3.1.11 graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line $y=x$. The coordinate pair $(4,0)$ is on the graph of f and the coordinate pair $(0,4)$ is on the graph of f^{-1}. For any coordinate pair, if (a,b) is on the graph of f, then (b,a) is on the graph of f^{-1}. Finally, observe that the graph of f intersects the graph of f^{-1} on the line $y=x$. Points of intersection for the graphs of f and f^{-1} will always lie on the line $y=x$.

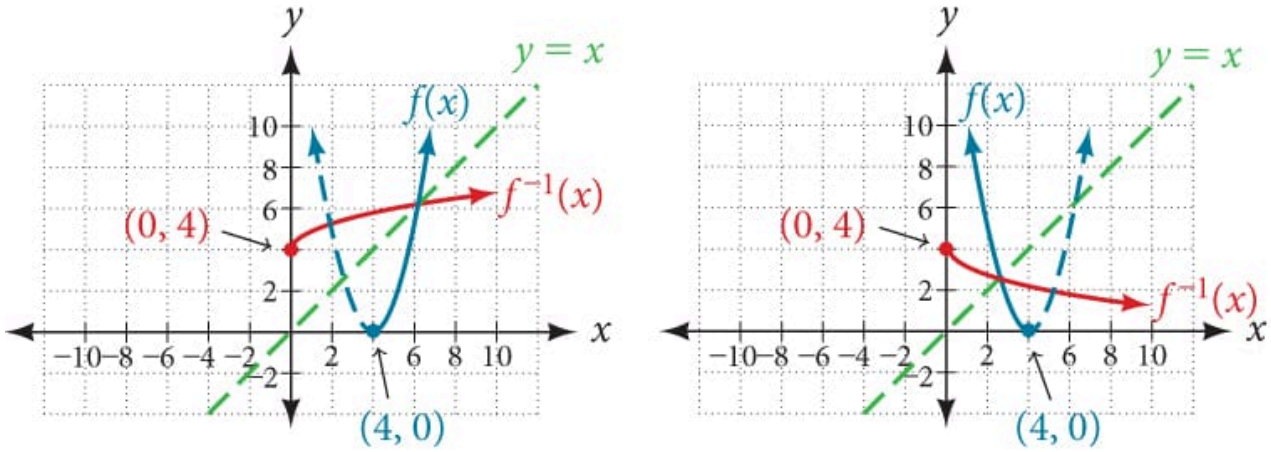


Figure 3.1.12

Example 3.1.12: Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified

Restrict the domain and then find the inverse of

$$f(x) = (x-2)^2 - 3.$$

Solution available at the end of this section.

Notice that we arbitrarily decided to restrict the domain on $x \geq 2$. We could just have easily opted to restrict the domain on $x \leq 2$, in which case $f^{-1}(x) = 2 - \sqrt{x+3}$. Observe the original function graphed on the same set of axes as its inverse function in Figure 3.1.13. Notice that both graphs show symmetry about the line $y = x$. The coordinate pair $(2, -3)$ is on the graph of f and the coordinate pair $(-3, 2)$ is on the graph of f^{-1}. Observe from the graph of both functions on the same set of axes that

$$\text{domain of } f = \text{range of } f^{-1} = [2, \infty)$$

and

$$\text{domain of } f^{-1} = \text{range of } f = [-3, \infty).$$

Finally, observe that the graph of f intersects the graph of f^{-1} along the line $y = x$.

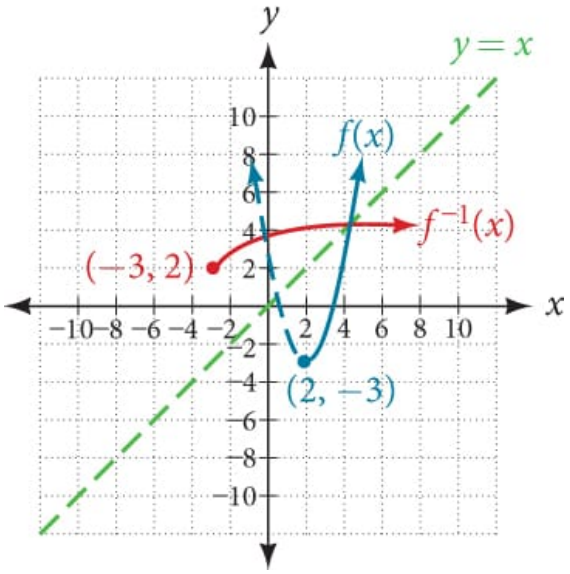


Figure 3.1.13

Checkpoint 3.1.12

Find the inverse of the function $f(x) = x^2 + 1$, on the domain $x \geq 0$.

Solving Applications Involving the Inverse of Polynomial Functions

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

Example 3.1.13: Solving an Application with a Cubic Function

A mound of gravel is in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

$$V = \frac{2}{3}\pi r^3.$$

Find the inverse of this function. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi \approx 3.14$.

Solution available at the end of this section.

Finding Inverses of Rational Functions

As with finding inverses of quadratic functions, it is sometimes desirable to find the inverse of a rational function, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

Example 3.1.14: Finding the Inverse of a Rational Function

The function $C = \frac{20+0.4n}{100+n}$ represents the concentration C of an acid solution after n mL of 40% solution has been added to 100 mL of a 20% solution. First, find the inverse of the function; that is, find an expression for n in terms of C. Then use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

Solution available at the end of this section.

Checkpoint 3.1.14

Find the inverse of the function $f(x) = \dfrac{x+3}{x-2}$.

Success in Mathematics: Mathematics in Other Courses

In other courses, we use mathematics to understand and explore the big ideas of the subject. Professors may use different vocabulary, variables, and notations than you learned in your math classes. The mathematics is the same. The way of talking about it may be different.

Also, some courses will use formulas based on mathematics that you have not yet learned. You may not be ready to understand why these formulas work. Sometimes you have to accept that the formula is correct.

Making connections between different courses takes work. It would be best if you were willing to ask questions. You may need to ask your math instructor about topics in your other classes. You may need to ask your other professors why they use different

words or notations. It is your responsibility to make these connections. Your ability to do critical thinking in other classes depends on it.

1. There are many formulas that we hear about in daily news or history. Of all these famous formulas, name one.
2. Can you describe the use of the formula you mentioned and what each variable stands for?

Hidden Resources: CRC Hub

As a college student, you have access to so many resources that it can be overwhelming. Luckily, the CRC Hub is your "one-stop shop" for all local resources covering various topics (from healthcare to legal fees). Check out the CRC Hub here.

Answers

Section Preview

We begin this chapter with defining and finding the inverse of a function. This is a traditional treatment of inverses, including how to graph them and how to find their equations given the original function. We also include domain restriction (as we are about to encounter radical functions).

Section Preview

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.
- Restrict the domain to find the inverse of a polynomial function.
- Find the inverse of a rational function.

Example 3.1.1: Identifying an Inverse Function for a Given Input-Output Pair

The inverse function reverses the input and output quantities, so if $f(2) = 4$, then $f^{-1}(4) = 2$; If $f(5) = 12$, then $f^{-1}(12) = 5$. Alternatively, if we want to name the inverse function g, then $g(4) = 2$ and $g(12) = 5$.

Example 3.1.2: Testing Inverse Relationships Algebraically

To determine if g is the inverse function of f, we just need to check if g "undoes" f.

$$\begin{aligned} g(f(x)) &= g\left(\frac{1}{x+2}\right) && \text{(substituting)} \\ &= \frac{1}{\frac{1}{x+2}} - 2 && \text{(substituting)} \\ &= x + 2 - 2 && \left(\text{recognizing that } \frac{1}{1/a} = a\right) \\ &= x && \text{(simplifying)} \end{aligned}$$

so

$$g = f^{-1} \text{ and } f = g^{-1}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$\begin{aligned} f(g(x)) &= f\left(\frac{1}{x} - 2\right) && \text{(substituting)} \\ &= \frac{1}{\left(\frac{1}{x} + 2\right) - 2} && \text{(substituting)} \\ &= \frac{1}{\frac{1}{x}} && \text{(simplifying)} \\ &= x && \left(\text{recognizing that } \frac{1}{1/a} = a\right) \end{aligned}$$

Example 3.1.3: Determining Inverse Relationships for Power Functions

$$f(g(x)) = f\left(\frac{x}{3}\right) = \left(\frac{x}{3}\right)^3 = \frac{x^3}{27} \neq x.$$

No, the functions are not inverses. The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x} = x^{1/3}$, that is, the one-third is an exponent, not a multiplier.

Example 3.1.4: Finding the Inverses of Toolkit Functions

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0, \infty)$, where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

Example 3.1.5: Interpreting the Inverse of a Tabular Function

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70

miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Example 3.1.6**: Evaluating a Function and Its Inverse from a Graph at Specific Points**

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 3.1.6.

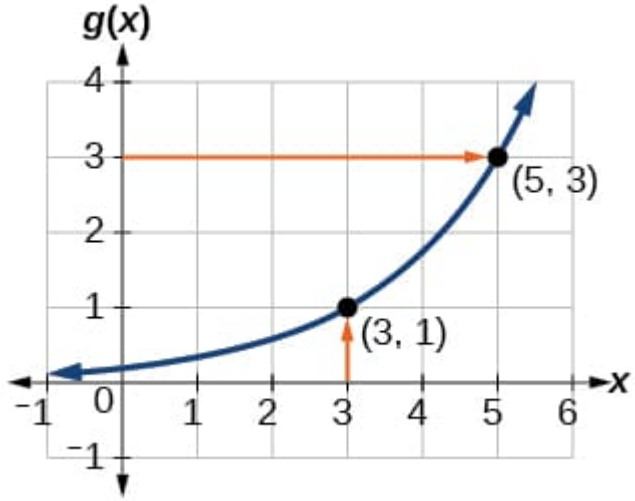


Figure 3.1.6

Example 3.1.7**: Inverting the Fahrenheit-to-Celsius Function**

$$\begin{aligned} C &= \frac{5}{9}(F-32) \\ \Longrightarrow \quad \frac{9}{5}C &= F-32 && \left(\text{multiplying both sides by } \frac{9}{5}\right) \\ \Longrightarrow \quad \frac{9}{5}C+32 &= F && (\text{adding 32 to both sides}) \end{aligned}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F-32),$$

then

$$F = h^{-1}(C) = \frac{9}{5}C+32.$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Example 3.1.8**: Solving to Find an Inverse Function**

$$\begin{aligned} y &= \frac{2}{x-3}+4 && (\text{starting with the original equation}) \\ \Longrightarrow \quad y-4 &= \frac{2}{x-3} && (\text{subtracting 4 from both sides}) \\ \Longrightarrow \quad (x-3)(y-4) &= 2 && (\text{multiplying both sides by } (x-3)) \\ \Longrightarrow \quad x-3 &= \frac{2}{y-4} && (\text{dividing both sides by } (y-4)) \\ \Longrightarrow \quad x &= \frac{2}{y-4}+3 && (\text{adding 3 to both sides}) \end{aligned}$$

So $f^{-1}(y) = \frac{2}{y-4} + 3$ or $f^{-1}(x) = \frac{2}{x-4} + 3$.

Example 3.1.9**: Solving to Find an Inverse with Radicals**

$$\begin{aligned} y &= 2+\sqrt{x-4} && (\text{starting with the original equation}) \\ \Longrightarrow \quad y-2 &= \sqrt{x-4} && (\text{subtracting 2 from both sides}) \\ \Longrightarrow \quad (y-2)^2 &= x-4 && (\text{squaring both sides}) \\ \Longrightarrow \quad (y-2)^2+4 &= x && (\text{adding 4 to both sides}) \end{aligned}$$

So $f^{-1}(x) = (x-2)^2 + 4$.

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$.

Example 3.1.10**: Finding the Inverse of a Function Using Reflection about the Identity Line**

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 3.1.10.

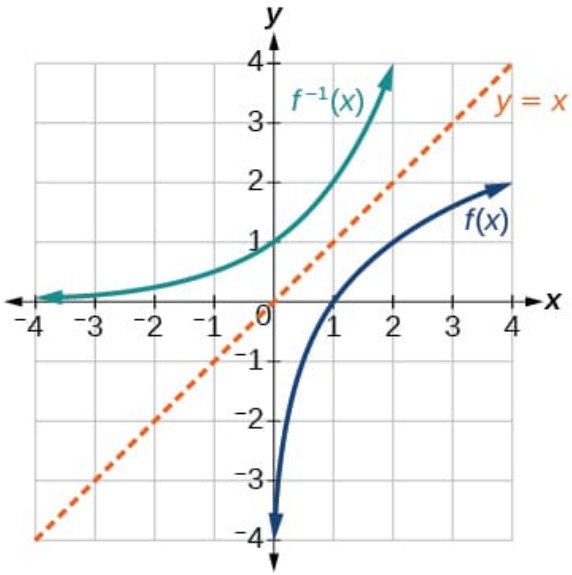


Figure 3.1.10**:** *The function and its inverse, showing reflection about the identity line*

Example 3.1.11**: Restricting the Domain to Find the Inverse of a Polynomial Function**

The original function $f(x) = (x - 4)^2$ is not one-to-one, but the function is restricted to a domain of $x \geq 4$ or $x \leq 4$ on which it is one-to-one. See Figure 3.1.11.

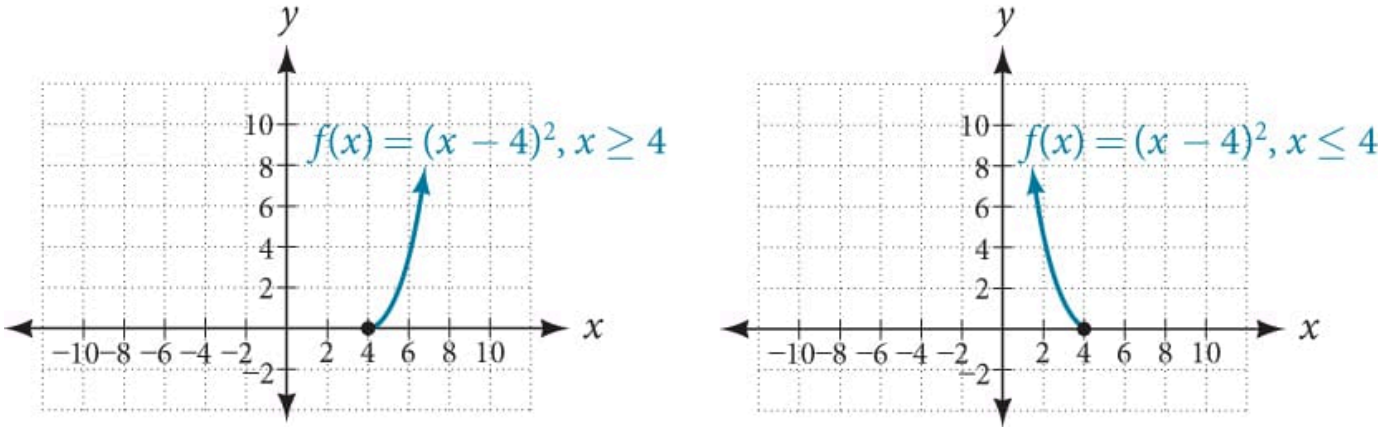


Figure 3.1.11

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$$\begin{aligned} y &= (x-4)^2 && \text{(letting } y = f(x)\text{)} \\ \Rightarrow \quad \pm\sqrt{y} &= x-4 && \text{(Extraction of Roots)} \\ \Rightarrow \quad 4 \pm \sqrt{y} &= x && \text{(adding 4 to both sides)} \\ \Rightarrow \quad 4 \pm \sqrt{x} &= y && \text{(exchanging } y \text{ and } x\text{)} \end{aligned}$$

This is not a function as written. We need to examine the restrictions on the domain of the original function to determine the inverse. Since we reversed the roles of x and y for the original $f(x)$, we need to look at the domain of the original function. When we reversed the roles of x and y, this gave us the values y could assume. For this function, $x \geq 4$, so for the inverse, we should have $y \geq 4$, which is what our inverse function gives.

a. The domain of the original function was restricted to $x \geq 4$, so the outputs of the inverse need to be the same, $f(x) \geq 4$, and we must use the "+" case:

$$f^{-1}(x) = 4 + \sqrt{x}.$$

b. The domain of the original function was restricted to $x \leq 4$, so the outputs of the inverse need to be the same, $f(x) \leq 4$, and we must use the "-" case:

$$f^{-1}(x) = 4 - \sqrt{x}.$$

Example 3.1.12**: Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified**

We can see this is a parabola with vertex at $(2, -3)$ that opens upward. Because the graph will be decreasing on one side of the vertex and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we repeat what we did in Example 3.1.11.

$$\begin{aligned} y &= (x-2)^2-3 && \text{(substituting } y=f(x)\text{)} \\ \Longrightarrow \quad y+3 &= (x-2)^2 && \text{(adding 3 to both sides)} \\ \Longrightarrow \quad \pm\sqrt{y+3} &= x-2 && \text{(Extraction of Roots)} \\ \Longrightarrow \quad 2\pm\sqrt{y+3} &= x && \text{(adding 2 to both sides)} \\ \Longrightarrow \quad 2\pm\sqrt{x+3} &= y && \text{(interchanging } x \text{ and } y\text{)} \end{aligned}$$

Now we need to determine which case to use. Because we restricted our original function to a domain of $x \geq 2$, the outputs of the inverse should be the same, telling us to utilize the "+" case. Therefore,

$$f^{-1}(x) = 2+\sqrt{x+3}.$$

If the quadratic had not been given in vertex form, rewriting it into vertex form would be the first step. This way we may easily observe the coordinates of the vertex to help us restrict the domain.

Example $3.1.13$**: Solving an Application with a Cubic Function**

Start with the given function for V. Notice that the meaningful domain for the function is $r \geq 0$ since negative radii would not make sense in this context. Also note the range of the function (hence, the domain of the inverse function) is $V \geq 0$. Solve for r in terms of V, using the method outlined previously.

$$\begin{aligned} V &= \frac{2}{3}\pi r^3 \\ \Longrightarrow \quad \frac{3}{2}V &= \pi r^3 \\ \Longrightarrow \quad \frac{3}{2\pi}V &= r^3 \\ \Longrightarrow \quad \sqrt[3]{\frac{3}{2\pi}V} &= r \end{aligned}$$

This is the result stated in the section opener. Now evaluate this for $V = 100$ and $\pi \approx 3.14$.

$$\begin{aligned} r &= \sqrt[3]{\frac{3V}{2\pi}} \\ &\approx \sqrt[3]{\frac{3(100)}{2(3.14)}} \\ &\approx \sqrt[3]{47.7707} \\ &\approx 3.63 \end{aligned}$$

Therefore, the radius is about 3.63 ft.

Example $3.1.14$**: Finding the Inverse of a Rational Function**

We first want the inverse of the function.

$$\begin{aligned} C &= \frac{20+0.4n}{100+n} \\ \Longrightarrow \quad C(100+n) &= 20+0.4n \\ \Longrightarrow \quad 100C+Cn &= 20+0.4n \\ \Longrightarrow \quad 100C-20 &= 0.4n-Cn \\ \Longrightarrow \quad 100C-20 &= (0.4-C)n \\ \Longrightarrow \quad \frac{100C-20}{0.4-C} &= n \end{aligned}$$

Now evaluate this function for $C = 0.35$ (35%).

$$n = \frac{100(0.35)-20}{0.4-0.35} = \frac{15}{0.05} = 300.$$

We can conclude that 300 mL of the 40% solution should be added.

This page titled 3.1: Inverse Functions is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

3.1.1: Resources and Key Concepts

Resources

Videos

- Inverse function (concept)
  - Intermediate Algebra - Functions: The Concept of Inverse Functions
- $f^{-1}(x)$ notation
  - Intermediate Algebra - Functions: Inverse Function Notation
- Verifying inverse functions
  - Inverse Functions - Verifying That Two Functions Are Inverse Functions
- One-to-one function
  - One-to-One Functions and Their Inverses - A Brief Review for Precalculus
- Domain and Range of inverse functions
  - Inverse Functions - Finding the Domain and Range of Inverse Functions
- Finding inverse from a formula
  - Inverse Functions - Finding Inverse Functions and Their Graphs
  - Inverse Functions - Finding and Evaluating Inverse Functions
  - Intermediate Algebra - Functions: Finding an Equation for the Inverse Function
- Graph of inverse function
  - Intermediate Algebra - Functions: Graphing the Inverse Function

Key Concepts

Definitions

- **Inverse Function:** For a one-to-one function $y = f(x)$, a function $f^{-1}(x)$ is an inverse function of f if $f^{-1}(f(x)) = x$ for all x in the domain of f, and $f(f^{-1}(y)) = y$ for all y in the domain of f^{-1} (which is the range of f). The input of the original function becomes the output of the inverse function, and the output of the original function becomes the input of the inverse function.
- **Inverse Function Notation (f^{-1}):** The notation used to represent the inverse of a function f. It is read as "f inverse."
- **Identity Line:** The line $y = x$. The graph of a function and its inverse are reflections of each other across this line.

Theorems

- **Inverse Function Property:** If f^{-1} is the inverse of f, then $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}. (And $f^{-1}(f(x)) = x$ for all x in the domain of f).
- **Domain and Range of Inverse Functions:** The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$, and the domain of $f(x)$ is the range of $f^{-1}(x)$.
- **Condition for Existence of an Inverse Function:** In order for a function to have an inverse, it must be a one-to-one function.
- **Forcing Invertibility by Restricting Domain**: If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

Common Mistakes

- **Confusing Inverse Function Notation with Reciprocal:** The notation $f^{-1}(x)$ for an inverse function does not mean $\frac{1}{f(x)}$. These are generally different concepts.
- **Assuming All Functions Have Inverses:** Not all functions have inverses. A function must be one-to-one to have an inverse function.
- **Forgetting to Restrict the Domain for Non-One-to-One Functions:** When finding an inverse for a function that is not naturally one-to-one (like $f(x) = x^2$), failing to restrict the domain of the original function appropriately will result in an inverse

relation that is not a function.

- **Incorrectly Determining the Domain of the Inverse Function:** The domain of f^{-1} is the range of f. Sometimes, after algebraically finding a formula for $f^{-1}(x)$, the apparent domain of this new formula might be larger than the actual range of f. The domain of f^{-1} must be restricted to match the range of f.
- **Forgetting to Restrict the Domain for Non-One-to-One Functions**: Attempting to find an inverse for a function like $f(x) = x^2$ without first restricting its domain to an interval where it is one-to-one. This leads to an inverse relation that is not a function (e.g., $y = \pm\sqrt{x}$).
- **Incorrectly Choosing the Branch of the Inverse**: After restricting the domain of a function (e.g., a quadratic) and finding an inverse involving a square root, selecting the wrong sign (positive or negative) for the square root, which would not correspond to the original restricted domain.
- **Not Verifying if the Original Function is One-to-One**: Proceeding to find an inverse without first confirming that the function (or its restricted version) is one-to-one, for example, by using the Horizontal Line Test.
- **Domain/Range Confusion for Inverse Functions**: Incorrectly stating the domain or range of the inverse function. Remember, the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

This page titled 3.1.1: Resources and Key Concepts is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

3.1.2: Homework

Reading Questions

1. What does it mean for a function f^{-1} to be the inverse of a function f?
2. How can you algebraically verify if two functions, $f(x)$ and $g(x)$, are inverses of each other?
3. Is the notation $f^{-1}(x)$ the same as $\frac{1}{f(x)}$? Explain.
4. What condition must a function satisfy to have an inverse function? How can this be checked graphically?
5. If a function is not one-to-one, what can sometimes be done to allow an inverse function to be found?
6. How are the domain and range of a function f related to the domain and range of its inverse f^{-1}?
7. If the point (a, b) is on the graph of a one-to-one function f, what corresponding point must be on the graph of f^{-1}?
8. Describe the graphical relationship between a function and its inverse. What line is involved?
9. Outline the algebraic steps to find the formula for an inverse function $f^{-1}(x)$ if you are given the formula for $f(x)$.
10. If $f(x) = \sqrt{x}$ has the domain $[0, \infty)$, its inverse is $f^{-1}(x) = x^2$. What must be the domain of this inverse function $f^{-1}(x)$, and why?
11. Can a function have more than one inverse function over the same domain?
12. If you are given a table of values for a one-to-one function $f(t)$ showing distance traveled in t minutes, how would you find $f^{-1}(d)$ for a given distance d? What would $f^{-1}(d)$ represent?
13. Name a toolkit function (other than the identity function) that is its own inverse.
14. Describe why the Horizontal Line Test is an effective way to determine whether a function is one-to-one?
15. Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?
16. Can a function be its own inverse? Explain.
17. Are one-to-one functions either always increasing or always decreasing? Why or why not?
18. How do you find the inverse of a function algebraically?
19. Why is it necessary to restrict the domain of a function like $f(x) = x^2$ before finding its inverse?
20. If you restrict the domain of $f(x) = (x-2)^2 - 3$ to $x \geq 2$, what will be the formula for its inverse $f^{-1}(x)$? What if you restrict the domain to $x \leq 2$?
21. When finding the inverse of a quadratic function, such as $y = ax^2$, you might arrive at $\pm\sqrt{\frac{y}{a}} = x$. How do you decide whether to use the positive or negative root for the inverse function?
22. What is the relationship between the graph of a function and the graph of its inverse with respect to the line $y = x$?
23. Outline the general steps to find the inverse of a polynomial function whose domain has been appropriately restricted.
24. Outline the general steps to find the inverse of a rational function of the form $f(x) = \frac{ax+b}{cx+d}$.
25. If $f(x) = 5x^3 + 1$, and you find its inverse $f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$, how can you verify that these functions are indeed inverses?
26. In the gravel mound problem (Example 5), why is the domain of the volume function $V(r) = \frac{2}{3}\pi r^3$ considered to be $r \geq 0$? What does this imply for the domain of its inverse function?
27. Can all polynomial functions have an inverse if their domain is restricted? Explain why or why not, considering a constant function as an example.
28. When solving for n in terms of C in Example 6 for the function $C = \frac{20+0.4n}{100+n}$, what algebraic steps are involved in isolating n?
29. Explain why we cannot find inverse functions for all polynomial functions.
30. Why must we restrict the domain of a quadratic function when finding its inverse?
31. When finding the inverse of a radical function, what restriction will we need to make?
32. The inverse of a quadratic function will always take what form?

Homework

Algebraic

33. Show that the function $f(x) = a - x$ is its own inverse for all real numbers a.

For the following exercises, find $f^{-1}(x)$ for each function.

34. $f(x) = x + 3$

35. $f(x) = x + 5$
36. $f(x) = 2 - x$
37. $f(x) = 3 - x$
38. $f(x) = \frac{x}{x+2}$
39. $f(x) = \frac{2x+3}{5x+4}$

For the following exercises, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

40. $f(x) = (x + 7)^2$
41. $f(x) = (x - 6)^2$
42. $f(x) = x^2 - 5$
43. Given $f(x) = \frac{x}{2+x}$ and $g(x) = \frac{2x}{1-x}$:
    a. Find $f(g(x))$ and $g(f(x))$.
    b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

For the following exercises, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

44. $f(x) = \sqrt[3]{x - 1}$ and $g(x) = x^3 + 1$
45. $f(x) = -3x + 5$ and $g(x) = \frac{x-5}{-3}$

For the following exercises, find the inverse of the function on the given domain.

46. $f(x) = (x - 4)^2$, $[4, \infty)$
47. $f(x) = (x + 2)^2$, $[-2, \infty)$
48. $f(x) = (x + 1)^2 - 3$, $[-1, \infty)$
49. $f(x) = 2 - \sqrt{3 + x}$, $[-3, \infty)$
50. $f(x) = 3x^2 + 5$, $(-\infty, 0]$
51. $f(x) = 12 - x^2$, $[0, \infty)$
52. $f(x) = 9 - x^2$, $[0, \infty)$
53. $f(x) = 2x^2 + 4$, $[0, \infty)$

For the following exercises, find the inverse of the functions.

54. $f(x) = x^3 + 5$
55. $f(x) = 3x^3 + 1$
56. $f(x) = 4 - x^3$
57. $f(x) = 4 - 2x^3$

For the following exercises, find the inverse of the functions.

58. $f(x) = \sqrt{2x + 1}$
59. $f(x) = \sqrt{3 - 4x}$
60. $f(x) = 9 + \sqrt{4x - 4}$
61. $f(x) = \sqrt{6x - 8} + 5$
62. $f(x) = 9 + 2\sqrt[3]{x}$
63. $f(x) = 3 - \sqrt[3]{x}$
64. $f(x) = \frac{2}{x+8}$
65. $f(x) = \frac{3}{x-4}$
66. $f(x) = \frac{x+3}{x+7}$
67. $f(x) = \frac{x-2}{x+7}$
68. $f(x) = \frac{3x+4}{5-4x}$
69. $f(x) = \frac{5x+1}{2-5x}$
70. $f(x) = x^2 + 2x$, $[-1, \infty)$
71. $f(x) = x^2 + 4x + 1$, $[-2, \infty)$
72. $f(x) = x^2 - 6x + 3$, $[3, \infty)$

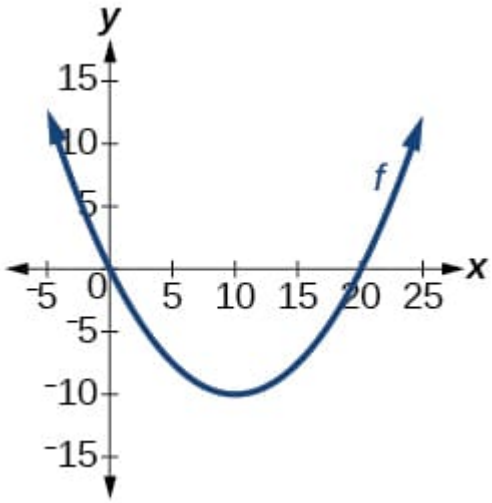
Graphical

For the following exercises, use graphing technology to determine whether each function is one-to-one.

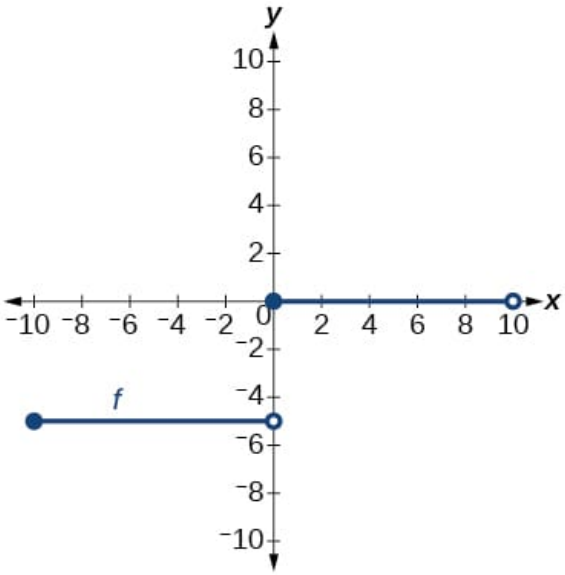
73. $f(x) = \sqrt{x}$
74. $f(x) = \sqrt[3]{3x+1}$
75. $f(x) = -5x + 1$
76. $f(x) = x^3 - 27$

For the following exercises, determine whether the graph represents a one-to-one function.

77.

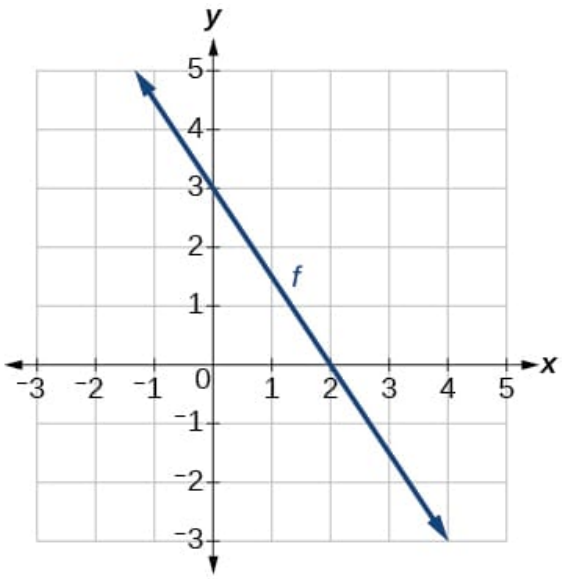


78.



For the following exercises, use the graph of f shown in Figure 3.1.2.1.

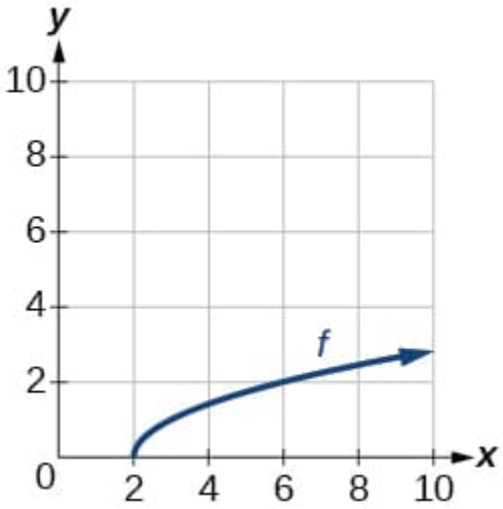
Figure 3.1.2.1



79. Find $f(0)$.
80. Solve $f(x) = 0$.
81. Find $f^{-1}(0)$.
82. Solve $f^{-1}(x) = 0$.

For the following exercises, use the graph of the one-to-one function shown in Figure 3.1.2.2**.**

Figure 3.1.2.2



83. Sketch the graph of f^{-1}.
84. Find $f(6)$ and $f^{-1}(2)$.
85. If the complete graph of f is shown, find the domain of f.
86. If the complete graph of f is shown, find the range of f.

For the following exercises, find the inverse of the function and graph both the function and its inverse.

87. $f(x) = x^2 + 2, x \geq 0$
88. $f(x) = 4 - x^2, x \geq 0$
89. $f(x) = (x + 3)^2, x \geq -3$

90. $f(x) = (x-4)^2$, $x \geq 4$
91. $f(x) = x^3 + 3$
92. $f(x) = 1 - x^3$
93. $f(x) = x^2 + 4x$, $x \geq -2$
94. $f(x) = x^2 - 6x + 1$, $x \geq 3$
95. $f(x) = \frac{2}{x}$
96. $f(x) = \frac{1}{x^2}$, $x \geq 0$

For the following exercises, use a graph to help determine the domain of the functions.

97. $f(x) = \sqrt{\frac{(x+1)(x-1)}{x}}$
98. $f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$
99. $f(x) = \sqrt{\frac{x(x+3)}{x-4}}$
100. $f(x) = \sqrt{\frac{x^2-x-20}{x-2}}$
101. $f(x) = \sqrt{\frac{9-x^2}{x+4}}$

Numeric

For the following exercises, evaluate or solve, assuming that the function f is one-to-one.

102. If $f(6) = 7$, find $f^{-1}(7)$.
103. If $f(3) = 2$, find $f^{-1}(2)$.
104. If $f^{-1}(-4) = -8$, find $f(-8)$.
105. If $f^{-1}(-2) = -1$, find $f(-1)$.

For the following exercises, use the values listed in Table 3.1.2.1 to evaluate or solve.

Table 3.1.2.1

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	8	0	7	4	2	6	5	3	9	1

106. Find $f(1)$.
107. Solve $f(x) = 3$.
108. Find $f^{-1}(0)$.
109. Solve $f^{-1}(x) = 7$.
110. Use the tabular representation of f in Table 3.1.2.2 to create a table for $f^{-1}(x)$.

Table 3.1.2.2

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

Technology

For the following exercises, find the inverse function. Then, graph the function and its inverse.

111. $f(x) = \frac{3}{x-2}$
112. $f(x) = x^3 - 1$
113. Find the inverse function of $f(x) = \frac{1}{x-1}$. Use graphing technology to find its domain and range. Write the domain and range in interval notation.

For the following exercises, use a calculator to graph the function. Then, using the graph, give three points on the graph

of the inverse with y-coordinates given.

14. $f(x) = x^3 - x - 2$, $y = 1, 2, 3$
15. $f(x) = x^3 + x - 2$, $y = 0, 1, 2$
16. $f(x) = x^3 + 3x - 4$, $y = 0, 1, 2$
17. $f(x) = x^3 + 8x - 4$, $y = -1, 0, 1$
18. $f(x) = x^4 + 5x + 1$, $y = -1, 0, 1$

Extensions

For the following exercises, find the inverse of the functions with a, b, c positive real numbers.

19. $f(x) = ax^3 + b$
20. $f(x) = x^2 + bx$
21. $f(x) = \sqrt{ax^2 + b}$
22. $f(x) = \sqrt[3]{ax + b}$
23. $f(x) = \frac{ax+b}{x+c}$

Real-World Applications

24. To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x) = \frac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.
25. The circumference C of a circle is a function of its radius given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.
26. A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t, in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

For the following exercises, determine the function described and then use it to answer the question.

27. An object dropped from a height of 200 meters has a height, $h(t)$, in meters after t seconds have lapsed, such that $h(t) = 200 - 4.9t^2$. Express t as a function of height, h, and find the time to reach a height of 50 meters.
28. An object dropped from a height of 600 feet has a height, $h(t)$, in feet after t seconds have elapsed, such that $h(t) = 600 - 16t^2$. Express t as a function of height h, and find the time to reach a height of 400 feet.
29. The volume, V, of a sphere in terms of its radius, r, is given by $V(r) = \frac{4}{3}\pi r^3$. Express r as a function of V, and find the radius of a sphere with volume of 200 cubic feet.
30. The surface area, A, of a sphere in terms of its radius, r, is given by $A(r) = 4\pi r^2$. Express r as a function of A, and find the radius of a sphere with a surface area of 1000 square inches.
31. A container holds 100 ml of a solution that is 25 ml acid. If n ml of a solution that is 60% acid is added, the function $C(n) = \frac{25+.6n}{100+n}$ gives the concentration, C, as a function of the number of ml added, n. Express n as a function of C and determine the number of mL that need to be added to have a solution that is 50% acid.
32. The period T, in seconds, of a simple pendulum as a function of its length l, in feet, is given by $T(l) = 2\pi\sqrt{\frac{l}{32.2}}$. Express l as a function of T and determine the length of a pendulum with period of 2 seconds.
33. The volume of a cylinder , V, in terms of radius, r, and height, h, is given by $V = \pi r^2 h$. If a cylinder has a height of 6 meters, express the radius as a function of V and find the radius of a cylinder with volume of 300 cubic meters.
34. The surface area, A, of a cylinder in terms of its radius, r, and height, h, is given by $A = 2\pi r^2 + 2\pi rh$. If the height of the cylinder is 4 feet, express the radius as a function of A and find the radius if the surface area is 200 square feet.
35. The volume of a right circular cone, V, in terms of its radius, r, and its height, h, is given by $V = \frac{1}{3}\pi r^2 h$. Express r in terms of V if the height of the cone is 12 inches and find the radius of a cone with volume of 50 cubic inches.
36. Consider a cone with height of 30 feet. Express the radius, r, in terms of the volume, V, and find the radius of a cone with volume of 1000 cubic feet.

This page titled 3.1.2: Homework is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

3.2: Radical Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Evaluate a Radical Function

In this section we will extend our previous work with functions to *officially* include radicals. A lot of the material in this section will feel like a review of material from early in the course; however, after a few brief definitions, theorems, and examples at this "review" level, we will increase the level to be a bit more challenging.

Recall from Algebra that if a function is defined by a radical expression, we call it a **radical function**. In particular,

- the square root function is $f(x) = \sqrt{x}$.
- the cube root function is $f(x) = \sqrt[3]{x}$.

For completeness, we include the official definition of a radical function here.

Definition: Radical Function

A **radical function** is a function that is defined by a radical expression.

Let's take a moment to review how to evaluate radical functions.

Example 3.2.1

For the function $f(x) = \sqrt{2x-1}$, find

a. $f(5)$
b. $f(-2)$

Solutions available at the end of this section.

Example $3.2.1b$ illustrates an important point—when working with functions (radical or not), we are assuming that the number system we're interested in is the real number system. Therefore, we do not want imaginary numbers as function values.

Checkpoint 3.2.1

For the function $f(x) = \sqrt{3x-2}$, find

a. $f(6)$
b. $f(0)$

Answers available at the end of this section.

Given a cube root function, we follow the same procedure.

Example 3.2.2

For the function $g(x) = \sqrt[3]{x-6}$, find

a. $g(14)$
b. $g(-2)$

Solutions available at the end of this section.

As Example $3.2.2b$ shows, one of the major differences between a square root function and a cube root function is that we can evaluate the cube root of a negative number. In fact, as you (hopefully) remember from Algebra, the domain of a radical function with an odd index is all real numbers, while the domain of an even-indexed radical function is restricted so that the radicand (the argument of the radical function) is non-negative. We will write this as a theorem in a little bit.

Checkpoint 3.2.2

For the function $g(x) = \sqrt[3]{3x-4}$, find

a. $g(4)$
b. $g(1)$

Answers available at the end of this section.

Again, as you should recall from Algebra, radical functions need not be restricted to square roots and cube roots.

Example 3.2.3

For the function $f(x) = \sqrt[4]{5x-4}$, find

a. $f(4)$
b. $f(-12)$

Solutions available at the end of this section.

Checkpoint 3.2.3

For the function $f(x) = \sqrt[4]{3x+4}$, find

a. $f(4)$
b. $f(-1)$

Answers available at the end of this section.

The Domain of a Radical Function

To find the domain of a radical function, we use our properties of radicals from Algebra. For a radical with an even index, we said the radicand had to be greater than or equal to zero. This is because even roots of negative numbers are not real numbers. For an odd index, the radicand can be any real number. We restate the properties here for reference.

Theorem: Properties of $\sqrt[n]{a}$

When n is an even number and:

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.

Therefore, to find the domain of a radical function with an even index, we set the radicand to be greater than or equal to zero. For an odd index radical, the radicand can be any real number. Again, for completeness, we include this statement as a theorem.

Theorem: Domain of a Radical Function

When the index of the radical is even, the domain of the radical function consists of all real numbers for which the radicand is greater than or equal to zero. When the index of the radical is odd, the domain of a radical function is all real numbers.

Example 3.2.4

Find the domain of the function, $f(x) = \sqrt{3x - 4}$. Write the domain in interval notation.

Solution available at the end of this section.

Checkpoint 3.2.4

Find the domain of the function, $f(x) = \sqrt{6x - 5}$. Write the domain in interval notation.

Answer available at the end of this section.

Up to this point in this section, we really have not stated anything new. Everything we have covered should feel like a repeat from your previous Algebra classes. Now we begin to improve upon these older skills to introduce examples more relevant to Calculus.

Example 3.2.5

Find the domain of the function, $g(x) = \sqrt{\frac{6}{x-1}}$. Write the domain in interval notation.

Solution available at the end of this section.

Checkpoint 3.2.5

Find the domain of the function, $f(x) = \sqrt{\frac{4}{x+3}}$. Write the domain in interval notation.

Answer available at the end of this section.

Example 3.2.6

Find the domain of the function, $h(x) = \sqrt[6]{-21x^2 + 2x + 8}$. Write the domain in interval notation.

Solution available at the end of this section.

Checkpoint 3.2.6

Find the domain of the function, $f(x) = \sqrt{x^3 + x^2 - 4x - 4}$. Write the domain in interval notation.

Answer available at the end of this section.

Example 3.2.6 and Checkpoint 3.2.6 illustrate the need to keep our skills with the material we previously covered in this course sharp. This is the way of Calculus—once you have learned a skill, it is likely going to come into play at some point in the future.

Example 3.2.7: Finding the Domain of a Radical Function Composed with a Rational Function

Find the domain of the function $f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$.

Solution available at the end of this section.

Domains of Non-Radical Functions Containing Radicals

In Calculus, you often work with functions that *contain* radical expressions *in addition* to containing non-radical expressions. Technically speaking, these are not called radical functions—they are simply referred to as compositions of functions (a catchall phrase encompassing most functions). Examples of these styles of functions are:

- $f(x) = 3x\sqrt[3]{2 - x}$
- $g(x) = \sqrt{2 - \sqrt[4]{x + 3}}$
- $k(x) = \frac{2x}{\sqrt{x^2 - 1}}$

Being able to identify the domains of functions like these is going to be a necessary skill for success in Calculus.

To find the domains of functions like these, we use a method akin to the Order of Operations—working from the innermost function to the outermost. It is often easier to use the method of sign charts when finding the domains of composite functions.

When creating a sign chart for a given function, we are looking for when a denominator becomes zero, when an even-indexed radical has a negative as its radicand, or when the argument of a logarithm (which we haven't encountered yet) is zero or negative. There are other, more rare, cases where domain issues occur, but we will approach those only as needed.

Let's spend some time investigating how such domains can be determined.

Example 3.2.8

State the domains of the following functions.

a. $f(x) = 3x\sqrt[3]{2 - x}$
b. $g(x) = \sqrt{2 - \sqrt[4]{x + 3}}$
c. $k(x) = \frac{2x}{\sqrt{x^2 - 1}}$

Solutions available at the end of this section.

Graphing Higher-Indexed Radical Functions

We already learned the graphs of the square root and cube root functions, however, we have yet to graph higher-indexed radical functions.

It is worth remarking that we could define $f(x) = \sqrt[n]{x}$ functionally as the inverse of $g(x) = x^n$ with the stipulation that when n is even, the domain of g is restricted to $[0, \infty)$. From what we know about $g(x) = x^n$ from our investigations of polynomials and their inverses, we can produce the graphs of $f(x) = \sqrt[n]{x}$ by reflecting the graphs of $g(x) = x^n$ across the line $y = x$. Below are the graphs of $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$. The point $(0, 0)$ is indicated as a reference. The axes are hidden so we can see the vertical steepening near $x = 0$ and the horizontal flattening as $x \to \infty$.

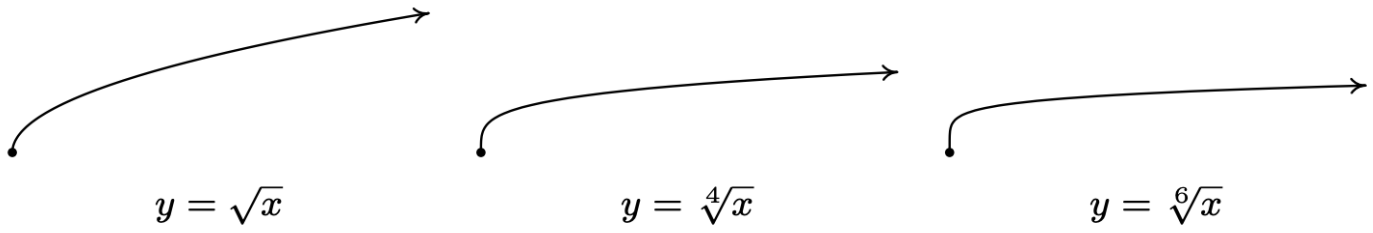


Figure 3.2.5

The odd-indexed radical functions also follow a predictable trend—steepening near $x = 0$ and flattening as $x \to \pm\infty$. In the exercises, you'll have a chance to graph some basic radical functions using transformations.

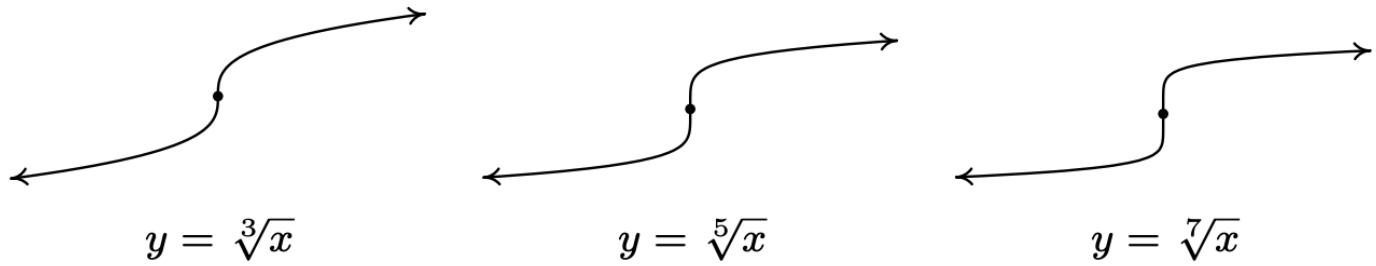


Figure 3.2.6

Interactive Element: Shapes of nth-Root Functions

The following Interactive Element might help you visualize how the shape of $y = \sqrt[n]{x}$ changes as the index n increases.

Interact: Drag the red handle left and right to change the index n in $y = \sqrt[n]{x}$ (shown for $x \geq 0$).

Observation: Every n^{th}-root curve passes through $(0, 0)$ and $(1, 1)$; as n grows the curve steepens near $x = 0$ and flattens as x increases.

Example 3.2.9

Sketch a graph of

$$f(x) = -\frac{1}{2}\sqrt[6]{2x + 8} + 1.$$

Solution available at the end of this section.

Applications Involving Radicals

Radical functions often arise in Calculus as a result of a process called **optimization**. The following example illustrates one such optimization problem (but we are allowing ourselves technology because we don't have Calculus yet).

Example 3.2.10

Carl wishes to get high speed internet service installed in his remote Sasquatch observation post located 30 miles from Route 117. The nearest junction box is located 50 miles down the road from the post, as indicated in the diagram below. Suppose it costs $\$15$ per mile to run cable along the road and $\$20$ per mile to run cable off of the road.

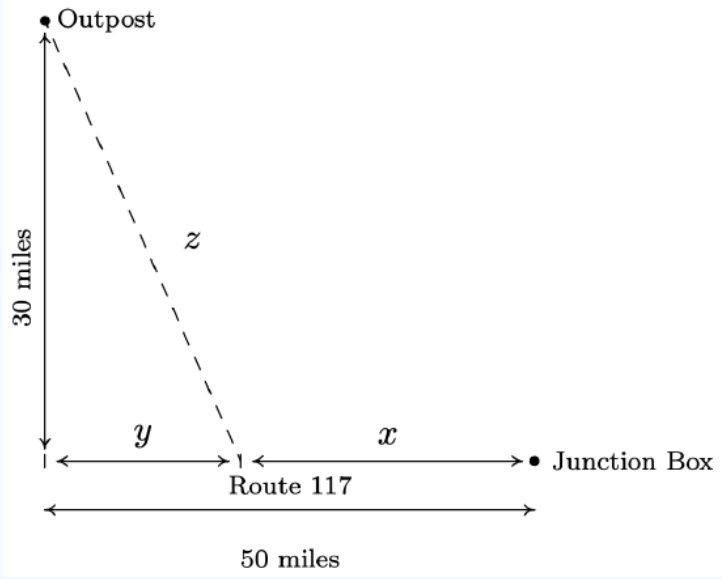


Figure 3.2.8

a. Express the total cost C of connecting the junction box to the post as a function of x, the number of miles the cable is run along Route 117 before heading off road directly towards the post. Determine a reasonable applied domain for the problem.
b. Use graphing technology to graph $y = C(x)$ on its domain. What is the minimum cost? How far along Route 117 should the cable be run before turning off the road?

Solutions available at the end of this section.

Success in College: Dealing With Failure (Part 1)

People sometimes do not achieve their goals. Athletes lose games. Students fail tests or classes. A job interview does not go well. It is your responsibility to learn how to cope with failure.

Former Chicago Bears football coach, Mike Ditka, said, "Success isn't permanent, and failure isn't fatal." This idea applies beyond the game of football. Failure on one test or even failing a class does not mean you cannot earn your degree. However, a continuing pattern of failures can prevent success.

When failure happens, you need to take action. Change the way you study. Redo the problems you missed on a test. Make an appointment with your professor to learn what you don't understand. If you are failing a course, meet with your advisor. Perhaps you should consider withdrawing. Make sure you know the effect of withdrawing or failing on any financial aid you receive.

Failure is not fun. It is tempting to skip class the day after a nasty test. It is easy to avoid talking to your professor about it. However, it is your responsibility to be honest about failure. Ignoring failure does not make it go away.

1. A class has four tests. A student taking this class fails one test with a score of 45%. The student's other test scores were 72%, 72%, and 79%. Find the student's test average for this class.
2. The effect of a bad test score depends on the number of tests in the class. It also depends on how much test scores contribute to the final course grade. If you had the same test average in this class as the student in the last question, could you still pass? Explain.

Success in College: Memorization Learning Preferences

> Students may be inclined to approach their courses in one of three ways. Those with a reproducing orientation tend to take a surface approach to learning, relying on rote memorization and mechanical formula substitution and making little or no effort to understand the material being taught. Those with a meaning orientation tend to adopt a deep approach, probing and questioning and exploring the limits of applicability of new material. Those with an achieving orientation tend to use a strategic approach, doing whatever is necessary to get the highest grade they can, taking a surface approach if that suffices and a deep approach when necessary.
>
> Source: R. Felder, R. Brent , Journal of Engineering Education, 94 (1), 57-72 (2005).

1. Students have different motivations for going to math class. What do you think the main motivation is for students who have a reproducing orientation and who take a surface approach?
2. What do you think is the main motivation for students who have a meaning orientation and who take a deep approach?

Answers

Section Preview

Typical textbooks do not have a full chapter dedicated to radical functions and equations involving radicals; however, this topic just seemed disjoint enough to warrant its own chapter. In this section, we officially introduce the concept of the radical function. While they have seen most of this before, it's best to spend some extra time here on domains of radical functions as well as difference quotients. Finally, getting a basic idea of the shapes of the nth root functions is useful for Calculus.

Section Preview

- **Geometry**
  - The Pythagorean Theorem, Distance Formula, and Midpoint Formula: Setting up the cost model in the optimization application (Example 10) requires $y^2 + 30^2 = z^2$; used only in that example.

Section Preview

- Evaluate a radical function
- Find the domain of a radical function
- Graph radical functions

Example 3.2.1

a. To evaluate $f(5)$, substitute 5 for x.

$$\begin{aligned} f(5) &= \sqrt{2 \cdot 5 - 1} && \text{(substituting)} \\ &= \sqrt{10 - 1} && \text{(simplifying)} \\ &= \sqrt{9} && \text{(simplifying)} \\ &= 3 && \text{(simplifying)} \end{aligned}$$

b. We follow the same process as in part a.

$$\begin{aligned} f(-2) &= \sqrt{2 \cdot (-2) - 1} && \text{(substituting)} \\ &= \sqrt{-4 - 1} && \text{(simplifying)} \\ &= \sqrt{-5} && \text{(simplifying)} \end{aligned}$$

Since the square root of a negative number is not a real number, the function does not have a (real) value at $x = -2$.

Checkpoint 3.2.1

a. $f(6) = 4$
b. no value at $x = 0$

Example 3.2.2

a. We substitute 14 into $g(x)$.

$$\begin{aligned} g(14) &= \sqrt[3]{14 - 6} && \text{(substituting)} \\ &= \sqrt[3]{8} && \text{(simplifying)} \\ &= 2 && \text{(simplifying)} \end{aligned}$$

b. As we did with part a, we substitute.

$$\begin{aligned} g(-2) &= \sqrt[3]{-2-6} && \text{(substituting)} \\ &= \sqrt[3]{-8} && \text{(simplifying)} \\ &= -2 && \text{(simplifying)} \end{aligned}$$

Checkpoint 3.2.2

a. $g(4) = 2$
b. $g(1) = -1$

Example 3.2.3

a.

$$\begin{aligned} f(4) &= \sqrt[4]{5 \cdot 4 - 4} && \text{(substituting)} \\ &= \sqrt[4]{20 - 4} && \text{(simplifying)} \\ &= \sqrt[4]{16} && \text{(simplifying)} \\ &= 2 && \text{(simplifying)} \end{aligned}$$

b.

$$\begin{aligned} f(-12) &= \sqrt[4]{5 \cdot (-12) - 4} && \text{(substituting)} \\ &= \sqrt[4]{-60 - 4} && \text{(simplifying)} \\ &= \sqrt[4]{-64} && \text{(simplifying)} \end{aligned}$$

Since the fourth root of a negative number is not a real number, the function does not have a (real) value at $x = -12$.

Checkpoint 3.2.3

a. $f(4) = 2$
b. $f(-1) = 1$

Example 3.2.4

Since the function, $f(x) = \sqrt{3x - 4}$ has a radical with an index of 2, which is even, we know the radicand must be greater than or equal to 0.

$$\begin{aligned} 3x - 4 &\geq 0 && \text{(setting the radicand greater than or equal to 0)} \\ \implies 3x &\geq 4 && \text{(adding 4 to both sides)} \\ \implies x &\geq \frac{4}{3} && \text{(dividing both sides by 3)} \end{aligned}$$

The domain of $f(x) = \sqrt{3x - 4}$ is all values $x \geq \frac{4}{3}$, and we write it in interval notation as $\left[\frac{4}{3}, \infty\right)$.

Checkpoint 3.2.4

$\left[\frac{5}{6}, \infty\right)$

Example 3.2.5

Since the function, $g(x) = \sqrt{\frac{6}{x-1}}$, has a radical with an even index, we know the radicand must be greater than or equal to 0. Moreover, since the numerator is a nonzero constant, the radicand cannot be zero.

For $\frac{6}{x-1}$ to be greater than zero, the denominator must be positive (since the numerator is positive). Therefore, we set $x - 1 > 0$ and solve. This yields $x > 1$. So far, our *candidate* domain is $(1, \infty)$.

An additional consideration when dealing with functions involving fractions is that the denominator cannot be zero. Therefore, we know $x \neq 1$; however, since 1 is not in the interval $(1, \infty)$, we don't have to worry about this extra restriction.

Putting this together we get the domain is $(1, \infty)$.

Checkpoint 3.2.5

$(-3, \infty)$

Example 3.2.6

The even index on this radical function tells us that the radicand must be greater than or equal to zero. Therefore, we need to solve

$$-21x^2 + 2x + 8 \geq 0.$$

Proceeding as we did when solving quadratic inequalities, we factor this quadratic to get

$$(-3x + 2)(7x + 4) \geq 0.$$

We then sketch a graph of the quadratic expression on the left side of the inequality to find when the curve is at or above the x-axis.

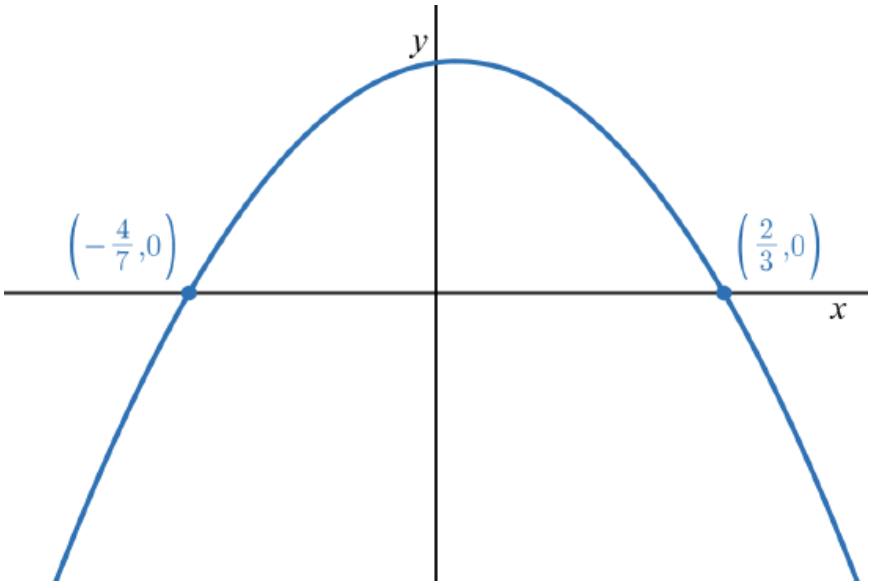


Figure 3.2.1

We can see from the graph that the inequality is satisfied when $-\frac{4}{7} \leq x \leq \frac{2}{3}$. Hence, the domain of the radical function is $\left[-\frac{4}{7}, \frac{2}{3}\right]$.

Checkpoint 3.2.6

$[-2,-1] \cup [2,\infty)$

Example 3.2.7**: Finding the Domain of a Radical Function Composed with a Rational Function**

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where $\frac{(x+2)(x-3)}{x-1} \geq 0$. To determine the intervals on which the rational expression is positive, we could test some values in the expression or sketch a graph. While both approaches work equally well, we will continue to reinforce the graphing skills we learned previously and choose to sketch the rational function (see Figure 3.2.2).

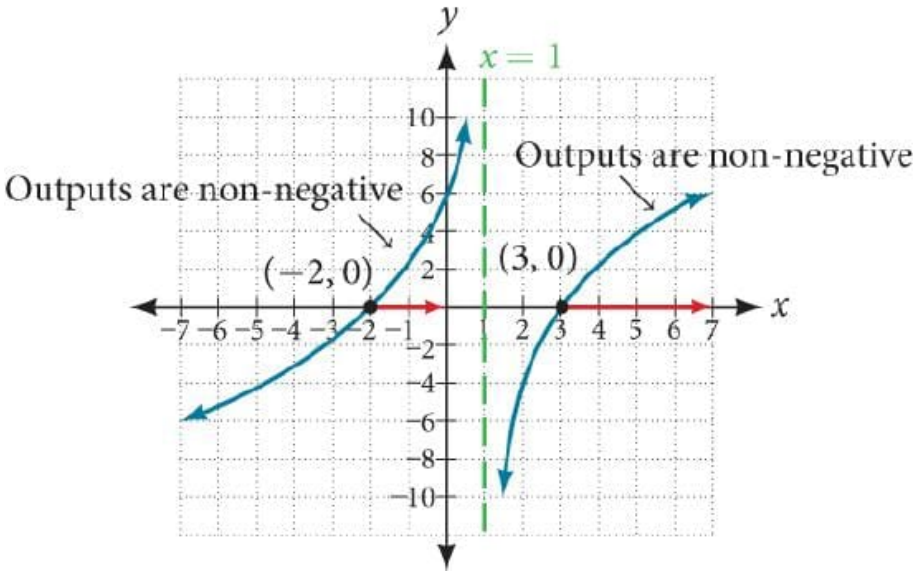


Figure 3.2.2

From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f(x)$ will be defined. $f(x)$ has domain $-2 \leq x < 1$ or $x \geq 3$, or in interval notation, $[-2,1) \cup [3,\infty)$.

Example 3.2.8

a. As far as domain is concerned, $f(x)$ has no denominators and no even roots, which means there are no domain issues. Hence, the domain of this function is $(-\infty,\infty)$.

b. In $g(x) = \sqrt{2-\sqrt[4]{x+3}}$, we have two radicals both of which are even-indexed. To satisfy the innermost radical, $\sqrt[4]{x+3}$, we require $x+3 \geq 0$ or $x \geq -3$. Thus, our *candidate* domain of $g(x)$ is $[-3,\infty)$. This is the largest possible domain for the function g; however, we have more to consider, and this domain might shrink. "Zooming out" in our function, we see we need $2-\sqrt[4]{x+3} \geq 0$. While it may be tempting to write this as $2 \geq \sqrt[4]{x+3}$ and take both sides to the fourth power, there are times when this technique will produce erroneous results. Instead, we solve $2-\sqrt[4]{x+3} \geq 0$ using a sign chart. We take a moment to find out when $2-\sqrt[4]{x+3} = 0$.

$$\begin{aligned} 2-\sqrt[4]{x+3} &= 0 \\ \Longrightarrow \quad 2 &= \sqrt[4]{x+3} && \left(\text{adding } \sqrt[4]{x+3} \text{ to both sides}\right) \\ \Longrightarrow \quad 2^4 &= \left(\sqrt[4]{x+3}\right)^4 && (\text{raising both sides to the fourth power}) \\ \Longrightarrow \quad 16 &= x+3 && (\text{simplifying}) \\ \Longrightarrow \quad 13 &= x && (\text{subtracting 3 from both sides}) \end{aligned}$$

One thing to be careful of throughout all of Mathematics is raising both sides of an equation to an even power—this *sometimes* introduces extraneous solutions. Therefore, we need to check to see if $x = 13$ is an extraneous solution. Substituting $x = 13$ into the expression $\sqrt{2-\sqrt[4]{x+3}}$ does check since

$$2-\sqrt[4]{x+3} = 2-\sqrt[4]{13+3} = 2-\sqrt[4]{16} = 2-2 = 0.$$

Remember, we are currently trying to figure out when $2-\sqrt[4]{x+3} \geq 0$. We make a sign chart with marked with $x = 13$ (where the expression on the left side of the inequality is zero) and, since the *candidate* domain is $[-3, \infty)$, we start our number line at $x = -3$.

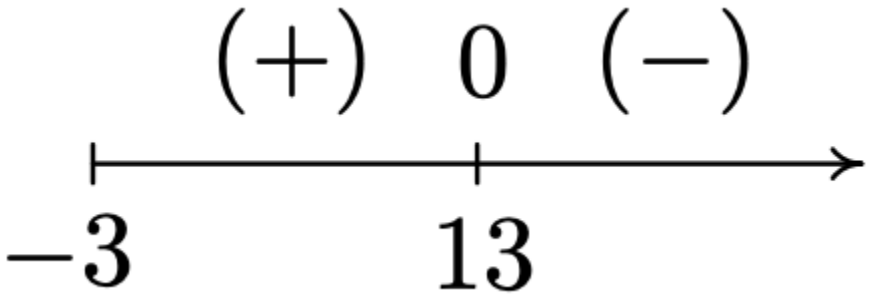


Figure 3.2.3

We find $2-\sqrt[4]{x+3} \geq 0$ on $[-3, 13]$ so this is the domain of g.

c. To find the domain of k, we have both an even root and a denominator to concern ourselves with. To satisfy the square root, $x^2 - 1 \geq 0$. Setting $r(x) = x^2 - 1$, we find the zeros of r to be $x = \pm 1$, and we find the sign chart of r to be

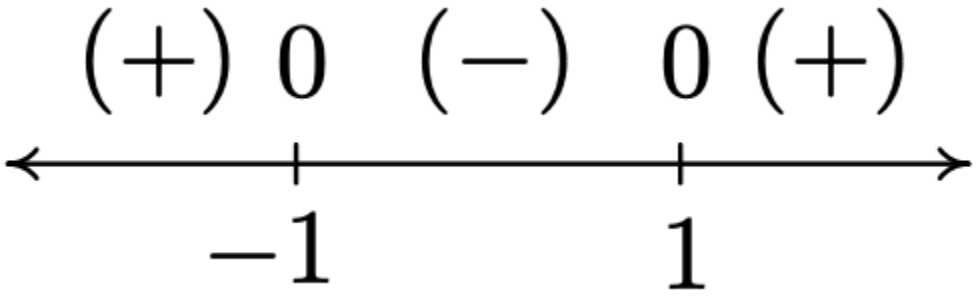


Figure 3.2.4

We find $x^2 - 1 \geq 0$ for $(-\infty, -1] \cup [1, \infty)$. This is our *candidate* domain, which might shrink as we look further into the problem.

To keep the denominator of $k(x)$ away from zero, we set $\sqrt{x^2-1} = 0$. We leave it to the reader to verify the solutions are $x = \pm 1$, both of which must be excluded from the domain. Hence, the domain of k is $(-\infty, -1) \cup (1, \infty)$.

Example 3.2.9

We know the base graph of this function (it is in the images above), so we just need to apply the transformations. We begin by modifying the function to get it into an appropriate form.

$$f(x) = -\frac{1}{2}\sqrt[6]{2x+8}+1 = -\frac{1}{2}\sqrt[6]{2(x+4)}+1.$$

Reading from left to right, the base graph of $y = \sqrt[6]{x}$:

- gets reflected about the x-axis ($y = -\sqrt[6]{x}$)
- gets compressed vertically by a factor of $\frac{1}{2}$ ($y = -\frac{1}{2}\sqrt[6]{x}$)
- gets compressed horizontally by a factor of $\frac{1}{2}$ ($y = -\frac{1}{2}\sqrt[6]{2x}$)
- gets shifted 4 units to the left ($y = -\frac{1}{2}\sqrt[6]{2(x+4)}$)
- gets shifted 1 unit up ($y = -\frac{1}{2}\sqrt[6]{2(x+4)}+1$)

The graph of $f(x)$ is shown below.

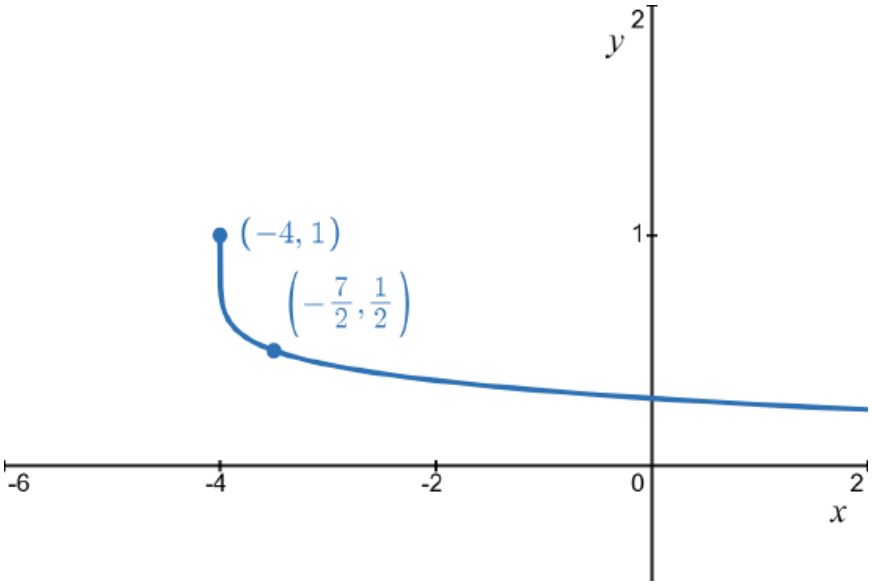


Figure 3.2.7

Example 3.2.10

a. The cost is broken into two parts: the cost to run cable along Route 117 at $\$15$ per mile, and the cost to run it off road at $\$20$ per mile. Since x represents the miles of cable run along Route 117, the cost for that portion is $15x$. From the diagram, we see that the number of miles the cable is run off road is z, so the cost of that portion is $20z$. Hence, the total cost is $C = 15x + 20z$.

Our next goal is to determine z as a function of x. The diagram suggests we can use the Pythagorean Theorem to get

$$y^2 + 30^2 = z^2.$$

But we also see $x + y = 50$, so that $y = 50 - x$. Hence,

$$z^2 = (50 - x)^2 + 900.$$

Solving for z, we obtain

$$z = \pm\sqrt{(50 - x)^2 + 900}.$$

Since z represents a distance, we choose $z = \sqrt{(50 - x)^2 + 900}$ so that our cost as a function of x only is given by

$$C(x) = 15x + 20\sqrt{(50 - x)^2 + 900}.$$

From the context of the problem, we have the applied domain of $x^{\text{int}}\,[0, 50]$.

b. Graphing $y = C(x)$ using Desmos, we find the relative minimum (which is also the absolute minimum in this case) to two decimal places to be $(15.98, 1146.86)$.

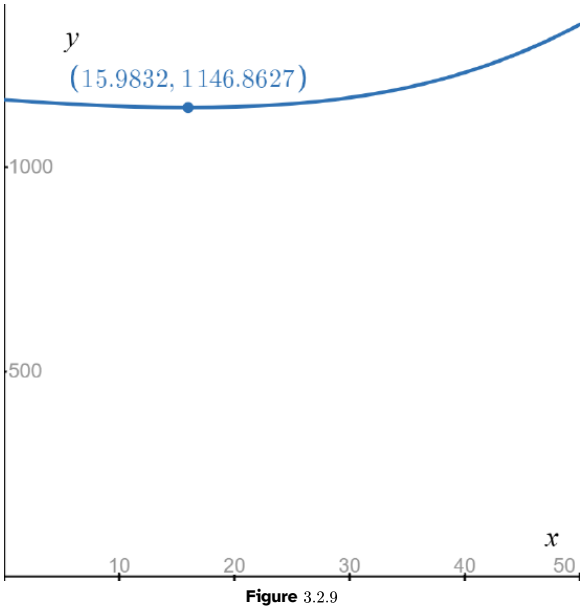


Figure 3.2.9

Here the x-coordinate tells us that in order to minimize cost, we should run 15.98 miles of cable along Route 117 and then turn off of the road and head towards the outpost. The y-coordinate tells us that the minimum cost, in dollars, to do so is $\$1146.86$. The ability to stream live SasquatchCasts? Priceless. The following Interactive Element shows the actual physical situation.

Interactive Element: Minimizing Cable Cost

The following Interactive Element might help you visualize how the total cost changes with where the cable leaves the road and heads for the post.

Interact: Drag the blue point along Route 117 to change x, the number of miles of cable run along the road before turning off toward the post.

Observation: The total cost falls to a minimum near $x \approx 15.98$ miles (about $\$1146.86$); turning off too early forces a long expensive diagonal, while running too far along the road overshoots.

This page titled 3.2: Radical Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

3.2.1: Resources and Key Concepts

Resources

Key Concepts

Definitions

- **Radical Function**: A function that is defined by a radical expression.

Theorems

- **Properties of $\sqrt[n]{a}$ (Regarding Real Numbers)**:
  - When n is an even number and $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
  - When n is an even number and $a < 0$, then $\sqrt[n]{a}$ is not a real number.
  - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.
- **Domain of a Radical Function**:
  - When the index of the radical is even, the domain of the radical function consists of all real numbers for which the radicand is greater than or equal to zero.
  - When the index of the radical is odd, the domain of a radical function consists of all real numbers (assuming the radicand itself is defined for all real numbers).

Common Mistakes

- **Ignoring Domain Restrictions for Even-Indexed Radicals**: Attempting to evaluate an even-indexed radical function for an input that makes the radicand negative, or failing to correctly identify the domain by setting the radicand ≥ 0.
- **Forgetting Denominator Restrictions within Radicands**: When the radicand of an even-indexed root is a rational expression, not only must the entire rational expression be ≥ 0, but the denominator of that rational expression also cannot be zero.

This page titled 3.2.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

3.2.2: Homework

Reading Questions

1. What is a radical function? Give an example of a square root function and a cube root function.
2. When evaluating radical functions, why is it important to consider the index of the radical (even vs. odd) if the radicand might be negative?
3. What is the primary condition that the radicand of an even-indexed radical function must satisfy for the function to yield a real number?
4. What is the domain of an odd-indexed radical function like $f(x) = \sqrt[3]{x-2}$?
5. How do you find the domain of an even-indexed radical function like $f(x) = \sqrt{3x-4}$?
6. If the radicand of an even-indexed radical function is a rational expression, such as $f(x) = \sqrt{\frac{x+2}{x-1}}$, what conditions must be met to determine the domain?
7. Describe the general shape of the graph of $y = \sqrt{x}$ and $y = \sqrt[3]{x}$.
8. How do transformations (shifts, stretches, reflections) affect the graph of a basic radical function like $y = \sqrt{x}$?
9. When finding the domain of a function like $g(x) = \sqrt[4]{2 - \sqrt[4]{x+3}}$, which radical's domain requirements should you address first?
10. What does it mean if, when evaluating a radical function for a specific input, you find the radicand of an even root is negative (e.g., $f(-2)$ for $f(x) = \sqrt{2x-1}$)?

Homework

In the following exercises, evaluate each function.

1. $f(x) = \sqrt{4x-4}$, find
   a. $f(5)$
   b. $f(0)$
2. $f(x) = \sqrt{6x-5}$, find
   a. $f(5)$
   b. $f(-1)$
3. $g(x) = \sqrt{6x+1}$, find
   a. $g(4)$
   b. $g(8)$
4. $g(x) = \sqrt{3x+1}$, find
   a. $g(8)$
   b. $g(5)$
5. $F(x) = \sqrt{3-2x}$, find
   a. $F(1)$
   b. $F(-11)$
6. $F(x) = \sqrt{8-4x}$, find
   a. $F(1)$
   b. $F(-2)$
7. $G(x) = \sqrt{5x-1}$, find
   a. $G(5)$
   b. $G(2)$
8. $G(x) = \sqrt{4x+1}$, find
   a. $G(11)$
   b. $G(2)$
9. $g(x) = \sqrt[3]{2x-4}$, find
   a. $g(6)$
   b. $g(-2)$
10. $g(x) = \sqrt[3]{7x-1}$, find
    a. $g(4)$

   b. $g(-1)$
11. $h(x)=\sqrt[3]{x^2-4}$, find
   a. $h(-2)$
   b. $h(6)$
12. $h(x)=\sqrt[3]{x^2+4}$, find
   a. $h(-2)$
   b. $h(6)$
13. For the function $f(x)=\sqrt[4]{2x^3}$, find
   a. $f(0)$
   b. $f(2)$
14. For the function $f(x)=\sqrt[4]{3x^3}$, find
   a. $f(0)$
   b. $f(3)$
15. For the function $g(x)=\sqrt[4]{4-4x}$, find
   a. $g(1)$
   b. $g(-3)$
16. For the function $g(x)=\sqrt[4]{8-4x}$, find
   a. $g(-6)$
   b. $g(2)$

In the following exercises, find the domain of the function and write the domain in interval notation.

17. $f(x)=\sqrt{3x-1}$
18. $f(x)=\sqrt{4x-2}$
19. $g(x)=\sqrt{2-3x}$
20. $g(x)=\sqrt{8-x}$
21. $h(x)=\sqrt{\frac{5}{x-2}}$
22. $h(x)=\sqrt{\frac{6}{x+3}}$
23. $f(x)=\sqrt{\frac{x+3}{x-2}}$
24. $f(x)=\sqrt{\frac{x-1}{x+4}}$
25. $g(x)=\sqrt[3]{8x-1}$
26. $g(x)=\sqrt[3]{6x+5}$
27. $f(x)=\sqrt[3]{4x^2-16}$
28. $f(x)=\sqrt[3]{6x^2-25}$
29. $F(x)=\sqrt[4]{8x+3}$
30. $F(x)=\sqrt[4]{10-7x}$
31. $G(x)=\sqrt[5]{2x-1}$
32. $G(x)=\sqrt[5]{6x-3}$

In the following exercises, find the domain of the function, graph the function, and use the graph to determine the range.

33. $f(x)=\sqrt{x+1}$
34. $f(x)=\sqrt{x-1}$
35. $g(x)=\sqrt{x+4}$
36. $g(x)=\sqrt{x-4}$
37. $f(x)=\sqrt{x}+2$
38. $f(x)=\sqrt{x}-2$
39. $g(x)=2\sqrt{x}$
40. $g(x)=3\sqrt{x}$
41. $f(x)=\sqrt{3-x}$
42. $f(x)=\sqrt{4-x}$
43. $g(x)=-\sqrt{x}$
44. $g(x)=-\sqrt{x}+1$
45. $f(x)=\sqrt[3]{x+1}$
46. $f(x)=\sqrt[3]{x-1}$
47. $g(x)=\sqrt[3]{x+2}$

48. $g(x) = \sqrt[3]{x-2}$
49. $f(x) = \sqrt[3]{x}+3$
50. $f(x) = \sqrt[3]{x}-3$
51. $g(x) = \sqrt[3]{x}$
52. $g(x) = -\sqrt[3]{x}$
53. $f(x) = 2\sqrt[3]{x}$
54. $f(x) = -2\sqrt[3]{x}$

This page titled 3.2.2: Homework is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

3.3: Radical Equations

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

As is the theme in College Algebra, we begin this section with the assumption that you already have exposure to solving simple radical equations. We still spend a few moments recalling old techniques, but quickly jump into solving more complex radical equations.

Solving Radical Equations Containing One Radical

We start our trip down memory lane by recalling an old definition.

Definition: Radical Equation

An equation in which variables are part of a radicand is called a **radical equation**.

When solving these equations, our goal is to isolate the radical. At that point, our strategy will be to raise both sides of the equation to the power of the index. This will, in turn, eliminate the radical; however, we have something to watch out for when solving radical equations!

Caution: Solving Equations Containing Even-Indexed Radicals

Care must be taken when working with expressions and equations involving even-indexed radicals. This is due to the nature of such beasts. Remember, the argument of an even-indexed radical cannot be negative, nor can an even-indexed radical return a negative value.

If, while solving a radical equation, we obtain a "solution" that would otherwise cause an even-indexed radical to be negative or the argument of the radical to be negative, then we know this "solution" will not work. In such cases we call this an **extraneous solution** as we did when we solved rational equations.

Definition: Extraneous Solution

An **extraneous solution**, also known as a **spurious solution**, is the solution of a transformed version of the original equation that is not an actual solution of the original equation because it was excluded from the domain (of the original equation).

As the definition suggests, extraneous solutions are always a concern when functions within an equation have implicit domain restrictions (e.g., rational and radical functions).

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index n to the nth power. This will eliminate the radical.

Example 3.3.1: Solving a radical equation involving a single radical expression

Solve: $\sqrt{5n-4}-9=0$.

Solution available at the end of this section.

Checkpoint 3.3.1

Solve: $\sqrt{3m+2}-5=0$.

Answer available at the end of this section.

It is important to remember the following advice:

> Whenever you raise both sides of an equation to an even power, you must check your final solutions.

In truth, the reason why we (as mathematicians) ask you to do this is to catch any inconsistencies that might have slipped past you. For example, suppose we were asked to solve $\sqrt{3x-1}-7=-9$. We would start by isolating the radical to get $\sqrt{3x-1}=-2$. If you are paying attention (and not asleep at the "mathematical wheel"), you would notice that you have an even-indexed radical equaling a negative number—**this cannot happen!** At this point, you would stop the solution process and state that the solution doesn't exist (some people use the notation "DNE" for "does not exist").

However, if you didn't notice the negative on the right side, you would continue to solve by squaring both sides to get

$$3x-1=4 \implies 3x=5 \implies x=\frac{5}{3}.$$

As long as we develop the habit to check our solutions whenever we square both sides of an equation, we have a second safety net.

$$\sqrt{3\left(\frac{5}{3}\right)-1}-7=\sqrt{5-7}-7=\sqrt{-2}-7$$

which is not a real number. As such, it cannot become -9.

I often try to get my students to stop at each step of the solution process to think about whether or not the statement before them makes sense, but checking your work at the end is your second opportunity to catch the logical flaw.

Interactive Element: Why Squaring Can Create Extra Solutions

The following Interactive Element might help you visualize why squaring both sides of a radical equation can introduce "solutions" that don't actually work. The purpose is to "visually" solve $\sqrt{x}=x-b$. That is, we want to know when the blue curve meets the green line. This obviously happens at the green dot (unless you drag the purple point too far up), so we expect only one solution (or none, again, if the purple point is dragged too far up).

Interact: Drag the purple point along the y-axis to slide the line $y=x-b$ by changing the value of b.

Observation: The equation $\sqrt{x}=x-b$ is solved where the blue curve meets the green line (green point). Squaring both sides also allows $\sqrt{x}=-(x-b)$, the gray dashed mirror line. This introduces the false solution at the red point on that dashed line. That is, an *extraneous* solution. Drag the purple point far enough up and the curve and line no longer meet: no solution.

Example 3.3.2

Solve: $\sqrt{9k-2}+1=0$.

Solution available at the end of this section.

Having recognized the square root equaling a negative number in the last step of Example 3.3.2 saved us time. Had we not seen that issue, we would have continued on to get

$$\begin{aligned} & \sqrt{9k-2}+1 &=& \ 0 & \\ \implies & \sqrt{9k-2} &=& -1 & \text{(isolating the radical by subtracting 1 from both sides)} \\ \implies & 9k-2 &=& \ 1 & \text{(squaring both sides)} \\ \implies & 9k &=& \ 3 & \text{(adding 2 to both sides)} \\ \implies & k &=& \ \frac{1}{3} & \text{(dividing both sides by 9)} \end{aligned}$$

Because we squared both sides, we go back to the original equation and check our answer.

$$\sqrt{9\left(\frac{1}{3}\right)-2}+1=\sqrt{3-2}+1=\sqrt{1}+1=1+1=2\neq 0.$$

This is a **contradiction**. Hence, $k=\frac{1}{3}$ is an extraneous solution.

Checkpoint 3.3.2

Solve: $\sqrt{2r-3}+5=0$.

Answer available at the end of this section.

Example 3.3.3

Solve: $\sqrt{p-1}+1=p$.

Solution available at the end of this section.

Checkpoint 3.3.3

Solve: $\sqrt{x-2}+2=x$.

Answer available at the end of this section.

It should be obvious that when the index of the radical is n, we eventually raise both sides of the radical equation to the nth power.

Example 3.3.4

Solve: $\sqrt[3]{5x+1}+8=4$.

Solution available at the end of this section.

It's always a good idea to check your answers when solving equations in Mathematics; however, it's only truly required (at least for now) when we have raised both sides of an equation to an even power (e.g., when solving radical equations) or when we have

multiplied both sides of an equation by an expression containing a variable (e.g., when solving a rational equation). This is why we did not bother checking our solution in Example 3.3.4. We leave it to the reader to check that $x = -13$ works.

Checkpoint 3.3.4

Solve: $\sqrt[3]{4x-3}+8=5$

Answer available at the end of this section.

Solving Equations Containing Rational Exponents

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical equation. Since $a^{m/n} = \sqrt[n]{a^m}$, we first raise each side of the equation to the power of the denominator of the rational exponent. Once that is done, we take the mth root of both sides of the equation (or raise both sides to the $\frac{1}{m}{}^{\text{th}}$ power).

Example 3.3.5

Solve: $(3x-2)^{1/4}+3=5$.

Solution available at the end of this section.

Checkpoint 3.3.5

Solve: $(9x+9)^{1/4}-2=1$

Answer available at the end of this section.

As with radical equations, care must be taken when solving equations containing rational exponents. We must remember all of the techniques we previously learned. For example, in the equation $a^{4/7}=1$, we start by raising both sides to the 7th power to get

$$a^4 = 1^7 = 1.$$

However, we now have the variable a raised to an even power. Extraction of Roots tells us that $a = \pm 1$. This is important!

Example 3.3.6

Solve: $3\left(\frac{1}{2}x+1\right)^{2/3}-5=7$

Solution available at the end of this section.

Checkpoint 3.3.6

Solve: $6\left(3x-5\right)^{3/2}-4=2$

Answer available at the end of this section.

Solving Radical Equations with Two Radicals

If we have a radical equation containing two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first. In the next example, when one radical is isolated, the second radical is also isolated.

Example 3.3.7

Solve: $\sqrt[3]{4x-3}-\sqrt[3]{3x+2}=0$.

Solution available at the end of this section.

Checkpoint 3.3.7

Solve: $\sqrt[3]{7x+1}=\sqrt[3]{2x-5}$.

Answer available at the end of this section.

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we "rinse and repeat" by isolating the remaining radical and raising both sides of the equation to the power of the index again.

Example 3.3.8

Solve: $\sqrt{m}+1=\sqrt{m+9}$.

Solution available at the end of this section.

Checkpoint 3.3.8

Solve: $3-\sqrt{x}=\sqrt{x-3}$.

Answer available at the end of this section.

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation. This procedure will now work for any radical equation.

How To: Solve a Radical Equation

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
   If yes, repeat Step 1 and Step 2 again.
   If no, solve the new equation.
4. Check the answer in the original equation.

Be careful as you square binomials (as in the next example). Remember the pattern in $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Example 3.3.9

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution available at the end of this section.

Checkpoint 3.3.9

Solve: $\sqrt{x-1}+2=\sqrt{2x+6}$

Answer available at the end of this section.

Equations involving rational exponents are fertile territory to run into equations that are quadratic-in-form. Not only must you recognize their form, but you also must watch out for all the pitfalls mentioned throughout this section concerning checking your solutions.

Example 3.3.10: Revisiting Solving Equations Quadratic-in-Form

Solve: $x-2\sqrt{x}-8=0$.

Solution available at the end of this section.

Checkpoint 3.3.10

Solve: $\sqrt{x}-\sqrt[4]{x}-2=0$

Answer available at the end of this section.

While Example 3.3.10 is interesting, the following example showcases a more frequent occurrence.

Example 3.3.11: Revisiting Solving Equations Quadratic-in-Form

Solve: $x^{-2/3}-3x^{-1/3}-10=0$.

Solution available at the end of this section.

Applications Involving Radicals

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. The following Problem Solving Strategy is incredibly helpful as you progress through College Algebra and into Calculus.

Problem Solving Strategy

1. **Read** the problem carefully and get clarification on words or concepts that are foreign to you.
2. **Understand** what is being stated and asked by rephrasing statements into language you connect with. Draw a figure to represent the situation, if possible.
3. **Label** the given and wanted information. The wanted information should have a variable representing it.
4. **Translate** into an equation relating the given and wanted information by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good Algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **State** the answer to the question with a complete sentence.

One application of radicals has to do with the effect of gravity on falling objects. The following (non-memorizable) formula allows us to determine how long it will take a fallen object to hit the ground.

Theorem: Time for a Dropped Object to Strike the Ground

On Earth, if an object is dropped from a height of h feet (where h is not too large) and if we ignore air resistance, the time (in seconds) it will take to reach the ground is approximated by using the formula $t = \frac{\sqrt{h}}{4}$.

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h = 64$ into the formula.

$$\begin{aligned} & t = \frac{\sqrt{h}}{4} & \\ \Longrightarrow \quad & t = \frac{\sqrt{64}}{4} & (\text{substituting}) \\ \Longrightarrow \quad & t = \frac{8}{4} & (\text{simplifying}) \\ \Longrightarrow \quad & t = 2 & (\text{simplifying}) \end{aligned}$$

Therefore, it would take 2 seconds for an object dropped from a height of 64 feet to reach the ground (ignoring deceleration from air resistance).

Don't Memorize!

The previous theorem is (for most instructors) a non-memorizable theorem. It's just a shortcut to approximate the time it takes for an object to fall a certain distance (and, in truth, it is only partially accurate for short falls).

Example 3.3.12

Kaitlyn dropped her sunglasses from a bridge 400 feet above a river. How many seconds does it take for the sunglasses to reach the river?

Solution available at the end of this section.

Checkpoint 3.3.12

A window washer dropped a squeegee from a platform 196 feet above the sidewalk. How many seconds does it take for the squeegee to reach the sidewalk?

Answer available at the end of this section.

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Theorem: Skid Marks and Speed of a Car

If the length of the skid marks is d feet, then the speed, s, of the car (in miles per hour) before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

Again, this formula is not memorizable *and* it is only accurate under special circumstances (e.g., it is not accurate at all if the roadway was covered in ice).

Example 3.3.13

After a car accident, the skid marks for one car measured 190 feet. Find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution available at the end of this section.

Checkpoint 3.3.13

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer available at the end of this section.

Success in College: Learning From Tests (Part 1)

1. A student scored 75% on their last math test. They tried to redo the questions and problems that they missed. Now they need to know if their new answers are correct. For your current math class, how could they find this out?
2. Look over your last math test. Did you miss any points because you did not show enough work? If so, explain why you didn't show the work on the test.

Success in Mathematics: Math Anxiety (Part 2)

I'm going to keep this short (for a change). Please take this math anxiety test to help you better understand your math anxiety. While at that website, check out the hints on overcoming math anxiety.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. A suggestion for reducing math anxiety is to come to class prepared. If you are prepared, then you are ready to be successful. When you are successful, you start learning that you do not have to be anxious. What do you think it means to be prepared for your math class?
2. A suggestion for reducing math anxiety is to practice math every day. How many days a week do you do math? How long, on average, do you spend doing math on those days?

Answers

Section Preview

This entire section is focused on solving equations involving radicals and rational exponents. While not terribly common in Calculus, these occur enough that exposure is critical.

Section Preview

- **Arithmetic**
  - Measurement, Approximation, Rounding, and Significant Figures: Rounding to the nearest tenth is required in the skid-mark application (Example 13 and Checkpoint 13).
- **Solving Equations**
  - Solving Linear Equations — Multistep with Variables on Both Sides: Once radicals are removed, the resulting linear equation must be solved, including the variables-on-both-sides case in Example 7.

Section Preview

- Solve radical equations involving one or two radicals.
- Solve equations involving rational exponents.
- Solve equations that are quadratic-in-form involving radicals or rational exponents.
- Use radicals in applications.

Example 3.3.1: Solving a radical equation involving a single radical expression

$$\begin{aligned}
& \sqrt{5n-4}-9 &&= 0 && \\
\Longrightarrow\quad & \sqrt{5n-4} &&= 9 && \text{(isolating the radical by adding 9 to both sides)} \\
\Longrightarrow\quad & \left(\sqrt{5n-4}\right)^2 &&= 9^2 && \text{(squaring both sides. } \textbf{Warning:} \text{ This forces us to check our final solution)} \\
\Longrightarrow\quad & 5n-4 &&= 81 && \text{(simplifying)} \\
\Longrightarrow\quad & 5n &&= 85 && \text{(adding 4 to both sides)} \\
\Longrightarrow\quad & n &&= 17 && \text{(dividing both sides by 5)}
\end{aligned}$$

Now we **must** check the solution. When doing so, you need to go back to the original equation to check your answer.

$$\sqrt{5(17)-4}-9=\sqrt{85-4}-9=\sqrt{81}-9=9-9=0.\checkmark$$

Thus, $n=17$ is the solution to the equation.

Checkpoint 3.3.1

$m=\frac{23}{3}$

Example 3.3.2

$$\begin{aligned}
& \sqrt{9k-2}+1 &&= 0 && \\
\Longrightarrow\quad & \sqrt{9k-2} &&= -1 && \text{(isolating the radical by subtracting 1 from both sides)}
\end{aligned}$$

At this point, if we were paying attention, we could stop because a square root is never equal to -1 (at least, not over the real numbers).

Checkpoint 3.3.2

no solution

Example 3.3.3

$$\begin{aligned}
& \sqrt{p-1}+1 &&= p && \\
\Longrightarrow\quad & \sqrt{p-1} &&= p-1 && \text{(isolating the radical by subtracting 1 from both sides)} \\
\Longrightarrow\quad & \left(\sqrt{p-1}\right)^2 &&= (p-1)^2 && \text{(squaring both sides)} \\
\Longrightarrow\quad & p-1 &&= p^2-2p+1 && \text{(simplifying and distributing)} \\
\Longrightarrow\quad & 0 &&= p^2-3p+2 && \text{(subtracting } p \text{ and adding 1 to both sides)} \\
\Longrightarrow\quad & 0 &&= (p-2)(p-1) && \text{(factoring)}
\end{aligned}$$

Therefore, our *candidate*solutions are $p=2$ or $p=1$. However, because we squared both sides, we need to check these results (using the original equation).

$$\begin{array}{lll}
p=2 & & p=1 \\
\Longrightarrow\quad \sqrt{2-1}+1 \stackrel{?}{=} 2 & & \Longrightarrow\quad \sqrt{1-1}+1 \stackrel{?}{=} 1 \\
\Longrightarrow\quad \sqrt{1}+1 \stackrel{?}{=} 2 & & \Longrightarrow\quad \sqrt{0}+1 \stackrel{?}{=} 1 \\
\Longrightarrow\quad 1+1 \stackrel{?}{=} 2 & & \Longrightarrow\quad 0+1 \stackrel{?}{=} 1 \\
\Longrightarrow\quad 2=2\checkmark & & \Longrightarrow\quad 1=1\checkmark
\end{array}$$

Therefore, the solutions are $p=1$ and $p=2$.

Checkpoint 3.3.3

$x=2, x=3$

Example 3.3.4

$$\begin{aligned}
& \sqrt[3]{5x+1}+8 &&= 4 && \\
\Longrightarrow\quad & \sqrt[3]{5x+1} &&= -4 && \text{(isolating the radical by subtracting 4 from both sides)} \\
\Longrightarrow\quad & \left(\sqrt[3]{5x+1}\right)^3 &&= (-4)^3 && \text{(raising both sides to the third power)} \\
\Longrightarrow\quad & 5x+1 &&= -64 && \text{(simplifying)} \\
\Longrightarrow\quad & 5x &&= -65 && \text{(subtracting 1 from both sides)} \\
\Longrightarrow\quad & x &&= -13 && \text{(dividing both sides by 5)}
\end{aligned}$$

Checkpoint 3.3.4

$x=-6$

Example 3.3.5

$$
\begin{aligned}
(3x-2)^{1/4}+3 &= 5 \\
\Longrightarrow \quad (3x-2)^{1/4} &= -2 && \text{(subtracting 3 from both sides)} \\
\Longrightarrow \quad \left((3x-2)^{1/4}\right)^4 &= (-2)^4 && \text{(raising both sides to the fourth power)} \\
\Longrightarrow \quad (3x-2)^{4/4} &= 16 && \text{(Laws of Exponents)} \\
\Longrightarrow \quad (3x-2)^{1} &= 16 && \text{(simplifying)} \\
\Longrightarrow \quad 3x-2 &= 16 && \text{(simplifying)} \\
\Longrightarrow \quad 3x &= 18 && \text{(adding 2 to both sides)} \\
\Longrightarrow \quad x &= 6 && \text{(dividing both sides by 3)}
\end{aligned}
$$

Since we raised both sides to an even power, we need to check for extraneous solutions. Going back to the original equation and substituting $x=6$, we get

$$(3(6)-2)^{1/4}+3=(18-2)^{1/4}+3=(16)^{1/4}+3=\sqrt[4]{16}+3=2+3=5\checkmark.$$

Checkpoint 3.3.5

$x=8$

Example 3.3.6

$$
\begin{aligned}
3\left(\frac{1}{2}x+1\right)^{2/3}-5 &= 7 \\
\Longrightarrow \quad 3\left(\frac{1}{2}x+1\right)^{2/3} &= 12 && \text{(begin isolating the rational exponent expression by adding 5 to both sides)} \\
\Longrightarrow \quad \left(\frac{1}{2}x+1\right)^{2/3} &= 4 && \text{(continue isolating the rational exponent expression by dividing both sides by 3)} \\
\Longrightarrow \quad \left(\left(\frac{1}{2}x+1\right)^{2/3}\right)^3 &= (4)^3 && \text{(raise both sides to the third power)} \\
\Longrightarrow \quad \left(\frac{1}{2}x+1\right)^{6/3} &= 64 && \text{(Laws of Exponents)} \\
\Longrightarrow \quad \left(\frac{1}{2}x+1\right)^{2} &= 64 && \text{(simplifying)} \\
\Longrightarrow \quad \frac{1}{2}x+1 &= \pm 8 && \text{(Extraction of Roots)} \\
\Longrightarrow \quad \frac{1}{2}x &= -1\pm 8 && \text{(subtracting 1 from both sides)} \\
\Longrightarrow \quad x &= -2\pm 16 && \text{(multiplying both sides by 2)}
\end{aligned}
$$

Thus, the solutions are $x=-2-16=-18$ and $x=-2+16=14$.

Checkpoint 3.3.6

$x=2$

Example 3.3.7

$$
\begin{aligned}
\sqrt[3]{4x-3}-\sqrt[3]{3x+2} &= 0 \\
\Longrightarrow \quad \sqrt[3]{4x-3} &= \sqrt[3]{3x+2} && \text{(isolating one of the radicals (which actually isolates the other in this example))} \\
\Longrightarrow \quad \left(\sqrt[3]{4x-3}\right)^3 &= \left(\sqrt[3]{3x+2}\right)^3 && \text{(raising both sides to the third power)} \\
\Longrightarrow \quad 4x-3 &= 3x+2 && \text{(simplifying)} \\
\Longrightarrow \quad x &= 5 && \text{(subtracting } 3x \text{ and adding 3 to both sides)}
\end{aligned}
$$

The solution is $x=5$. We leave it to you to show that 5 checks.

Checkpoint 3.3.7

$x=-\frac{6}{5}$

Example 3.3.8

$$\begin{aligned}
\sqrt{m}+1 &= \sqrt{m+9} \\
\implies (\sqrt{m}+1)^2 &= \left(\sqrt{m+9}\right)^2 && \text{(since a radical was already isolated, square both sides)} \\
\implies (\sqrt{m}+1)(\sqrt{m}+1) &= m+9 && \text{(simplifying)} \\
\implies m+2\sqrt{m}+1 &= m+9 && \text{(distributing)} \\
\implies 2\sqrt{m} &= 8 && \text{(begin isolating the remaining radical by subtracting both } m \text{ and 1 from both sides)} \\
\implies \sqrt{m} &= 4 && \text{(dividing both sides by 2)} \\
\implies (\sqrt{m})^2 &= 4^2 && \text{(squaring both sides)} \\
\implies m &= 16 && \text{(simplifying)}
\end{aligned}$$

Since we squared both sides during our solution process, we need to check that our solution works in the original equation.

$$\begin{aligned}
\sqrt{16}+1 &\stackrel{?}{=} \sqrt{16+9} \\
\implies 4+1 &\stackrel{?}{=} \sqrt{25} \\
\implies 5 &= 5 \checkmark
\end{aligned}$$

Therefore, the solution is $m = 16$.

Checkpoint 3.3.8

$x = 4$

Example 3.3.9

$$\begin{aligned}
\sqrt{q-2}+3 &= \sqrt{4q+1} \\
\implies \left(\sqrt{q-2}+3\right)^2 &= \left(\sqrt{4q+1}\right)^2 && \text{(squaring both sides since one radical is already isolated)} \\
\implies \left(\sqrt{q-2}+3\right)\left(\sqrt{q-2}+3\right) &= 4q+1 && \text{(simplifying)} \\
\implies q-2+6\sqrt{q-2}+9 &= 4q+1 && \text{(distributing)} \\
\implies q+7+6\sqrt{q-2} &= 4q+1 && \text{(combining like terms)} \\
\implies 6\sqrt{q-2} &= 3q-6 && \text{(begin isolating the remaining radical by subtracting } q \text{ and 7 from both sides)} \\
\implies 2\sqrt{q-2} &= q-2 && \text{(noticing that all coefficients share a common divisor, we divide both sides by 3)} \\
\implies \left(2\sqrt{q-2}\right)^2 &= (q-2)^2 && \text{(instead of dividing by 2 (which would introduce fractions), we square both sides)} \\
\implies 2^2\left(\sqrt{q-2}\right)^2 &= (q-2)(q-2) && \text{(Laws of Exponents)} \\
\implies 4(q-2) &= q^2-4q+4 && \text{(simplifying and distributing)} \\
\implies 4q-8 &= q^2-4q+4 && \text{(distributing)} \\
\implies 0 &= q^2-8q+12 && \text{(subtracting } 4q \text{ and adding 8 to both sides)} \\
\implies 0 &= (q-6)(q-2) && \text{(factoring the trinomial)}
\end{aligned}$$

Therefore, our *candidate* solutions are $q = 6$ or $q = 2$. We leave it to the reader to check that neither is extraneous.

Checkpoint 3.3.9

$x = 5$

Example 3.3.10**: Revisiting Solving Equations Quadratic-in-Form**

This is a radical equation that can be written in quadratic form. If we let $u = \sqrt{x}$, then $u^2 = \left(\sqrt{x}\right)^2 = x$ and we can write

$$\begin{aligned}
x-2\sqrt{x}-8 &= 0 \\
\implies u^2-2u-8 &= 0 && \text{(substituting)} \\
\implies (u-4)(u+2) &= 0 && \text{(factoring the trinomial)}
\end{aligned}$$

Therefore, $u = 4$ or $u = -2$. Since $u = \sqrt{x}$, we resubstitute to get $\sqrt{x} = 4$ or $\sqrt{x} = -2$; however, square roots are never negative. Hence, we throw out that possibility and focus on $\sqrt{x} = 4 \implies x = 16$. Checking this solution in the *original* equation, we get

$$16-2\sqrt{16}-8 = 16-2(4)-8 = 16-8-8 = 0. \checkmark$$

Thus, the solution to the equation is $x = 16$.

Checkpoint 3.3.10

$x = 16$

Example 3.3.11**: Revisiting Solving Equations Quadratic-in-Form**

If we let $u = x^{-1/3}$, then $u^2 = \left(x^{-1/3}\right)^2 = x^{-2/3}$ and we can write

$$\begin{aligned} x^{-2/3} - 3x^{-1/3} - 10 &= 0 \\ \Longrightarrow \quad u^2 - 3u - 10 &= 0 \qquad \text{(substituting)} \\ \Longrightarrow \quad (u-5)(u+2) &= 0 \qquad \text{(factoring the trinomial)} \\ \Longrightarrow \quad \left(x^{-1/3}-5\right)\left(x^{-1/3}+2\right) &= 0 \qquad \text{(resubstituting)} \end{aligned}$$

Therefore, $x^{-1/3} = 5$ or $x^{-1/3} = -2$.

$$\begin{aligned} x^{-1/3} &= 5 \qquad\qquad x^{-1/3} = -2 \\ \Longrightarrow \quad \frac{1}{x^{1/3}} &= 5 \qquad\qquad \frac{1}{x^{1/3}} = -2 \qquad \text{(Laws of Exponents)} \\ \Longrightarrow \quad x^{1/3} &= \frac{1}{5} \qquad\qquad x^{1/3} = -\frac{1}{2} \qquad \left(\text{if } a=b, \text{ where neither are zero, then } \tfrac{1}{a} = \tfrac{1}{b}\right) \\ \Longrightarrow \quad \left(x^{1/3}\right)^3 &= \left(\frac{1}{5}\right)^3 \qquad \left(x^{1/3}\right)^3 = \left(-\frac{1}{2}\right)^3 \qquad \text{(raising both sides to the third power)} \\ \Longrightarrow \quad x^{3/3} &= \frac{1^3}{5^3} \qquad\qquad x^{3/3} = -\frac{1^3}{2^3} \qquad \text{(Laws of Exponents)} \\ \Longrightarrow \quad x &= \frac{1}{125} \qquad\qquad x = -\frac{1}{8} \qquad \text{(simplifying)} \end{aligned}$$

The domain of the original equation only has the restriction that $x \neq 0$ (can you see why?). Furthermore, the solution process involved cubing, not squaring. Squaring can introduce extraneous solutions, but cubing does not, so a check is not necessary. We leave it to the reader to check that these solutions work in the original equation.

Example 3.3.12

1. **Read** the problem.
2. **Understand** the situation. We are looking for the time it takes for Kaitlyn's sunglasses to reach the river.
3. **Label** the given and wanted information. Let t represent the unknown time it will take the sunglasses to fall to the river.

Given	Want
$h = 400$	t

4. **Translate** into an equation. The recent theorem allows us to relate h and t by the formula

$$t = \frac{\sqrt{h}}{4}.$$

Substituting in the given information, we get

$$t = \frac{\sqrt{400}}{4}.$$

5. **Solve** the equation.

$$\begin{aligned} t &= \frac{\sqrt{400}}{4} \\ &= \frac{20}{4} \qquad \text{(simplifying)} \\ &= 5 \qquad \text{(simplifying)} \end{aligned}$$

6. **Check** the solution.

$$\begin{aligned} 5 &\stackrel{?}{=} \frac{\sqrt{400}}{4} \\ \Longrightarrow \quad 5 &\stackrel{?}{=} \frac{20}{4} \\ \Longrightarrow \quad 5 &= 5 \checkmark \end{aligned}$$

Thus, 5 is correct *mathematically*. It also makes sense logically (for example, if the answer was negative, that would not make sense).

7. **State** the answer. It takes 5 seconds for Kaitlyn's sunglasses to reach the river.

Checkpoint 3.3.12

3.5 seconds

Example 3.3.13

1. **Read!**
2. **Understand:** We are looking for the initial speed of the car before the brakes were applied.
3. **Label:** Let s represent the unknown speed of the car and d represent the length of the skid.

Given	Want
$d = 190$	s

4. **Translate:**

$$s = \sqrt{24d} = \sqrt{24(190)}.$$

5. **Solve:**

$$\begin{aligned} s &= \sqrt{24(190)} \\ &= \sqrt{4560} && \text{(simplifying)} \\ &\approx 67.5277... && \text{(approximating)} \\ &\approx 67.5 && \text{(rounding to the nearest tenth)} \end{aligned}$$

6. **Check:**

$$\begin{aligned} 67.5 &\overset{?}{\approx} \sqrt{24(190)} \\ \implies 67.5 &\overset{?}{\approx} \sqrt{4560} \\ \implies 67.5 &\approx 67.5277... \checkmark \end{aligned}$$

   Thus, 67.5 is correct *mathematically*. It also makes sense logically that the car could have been going 67.5 miles per hour.
7. **State:** The speed of the car before the brakes were applied was roughly 67.5 miles per hour.

Checkpoint 3.3.13

42.7 mph

This page titled 3.3: Radical Equations is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

3.3.1: Resources and Key Concepts

Resources

Videos

- Solving radical equations
  - Equations - Overview - A Review for College Algebra: Solving Radical Equations
  - Intermediate Algebra - Radical Equations: Solving

Key Concepts

Definitions

- **Radical Equation**: An equation in which unknowns (variables) are part of a radicand.
- **Extraneous Solution (Spurious Solution)**: A root of a transformed (manipulated) equation that is not a root of the original equation because it was excluded from the domain of the original equation or introduced by operations like squaring both sides.
- **Contradiction**: A statement that is both true and false simultaneously (e.g., arriving at $2 \neq 0$ after assuming a solution exists and checking it).

Common Mistakes

- **Forgetting to Check for Extraneous Solutions**: When raising both sides of an equation to an even power (e.g., squaring), it is crucial to check all candidate solutions in the original equation, as this process can introduce extraneous solutions.
- **Incorrectly Isolating the Radical**: Failing to isolate the radical term on one side of the equation before raising both sides to a power. This leads to more complicated expressions that still contain radicals.
- **Errors in Raising Both Sides to a Power**: Not raising the entire side to the power, especially if it's a binomial (e.g., if $\sqrt{A} = B + C$, then $A = (B + C)^2$, not $B^2 + C^2$).
- **Errors with Rational Exponents**: Incorrectly applying powers to "undo" rational exponents (e.g., for $x^{m/n} = k$, raise both sides to n/m).
- **Not Recognizing Equations Quadratic-in-Form**: Failing to see that a radical or rational exponent equation can be solved using u-substitution if it's quadratic-in-form.

This page titled 3.3.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

3.3.2: Homework

Reading Questions

1. What is a radical equation?
2. What is the primary strategy for solving a radical equation containing one radical?
3. Why is it crucial to check your solutions when solving radical equations, especially if you've raised both sides to an even power? What kind of solutions might arise?
4. If, after isolating an even-indexed radical, you find it is equal to a negative number (e.g., $\sqrt{x} = -2$), what can you conclude about the real solutions to the equation?
5. When solving an equation with a rational exponent, like $(3x-2)^{1/4}+3=5$, what is the first step after isolating the term with the rational exponent?
6. If an equation contains two radicals, what is the general approach to solving it?
7. When solving an equation like $x - 2\sqrt{x} - 8 = 0$, what substitution could you make to transform it into a quadratic equation?
8. What does the term "extraneous solution" mean in the context of solving radical equations?
9. If you are solving an equation and raise both sides to an odd power (e.g., cubing both sides), is it still mandatory to check for extraneous solutions for the same reasons as with even powers?
10. Describe the general steps for solving a radical equation as outlined in the "How To: Solve a Radical Equation" box.

Homework

In the following exercises, solve.

11. $\sqrt{5x-6}=8$
12. $\sqrt{4x-3}=7$
13. $\sqrt{5x+1}=-3$
14. $\sqrt{3y-4}=-2$
15. $\sqrt[3]{2x}=-2$
16. $\sqrt[3]{4x-1}=3$
17. $\sqrt{2m-3}-5=0$
18. $\sqrt{2n-1}-3=0$
19. $\sqrt{6v-2}-10=0$
20. $\sqrt{12u+1}-11=0$
21. $\sqrt{4m+2}+2=6$
22. $\sqrt{6n+1}+4=8$
23. $\sqrt{2u-3}+2=0$
24. $\sqrt{5v-2}+5=0$
25. $\sqrt{u-3}+3=u$
26. $\sqrt{v-10}+10=v$
27. $\sqrt{r-1}=r-1$
28. $\sqrt{s-8}=s-8$
29. $\sqrt[3]{6x+4}=4$
30. $\sqrt[3]{11x+4}=5$
31. $\sqrt[3]{4x+5}-2=-5$
32. $\sqrt[3]{9x-1}-1=-5$
33. $(6x+1)^{1/2}-3=4$
34. $(3x-2)^{1/2}+1=6$
35. $(8x+5)^{1/3}+2=-1$
36. $(12x-5)^{1/3}+8=3$
37. $(12x-3)^{1/4}-5=-2$
38. $(5x-4)^{1/4}+7=9$
39. $\sqrt{x+1}-x+1=0$
40. $\sqrt{y+4}-y+2=0$
41. $\sqrt{z+100}-z=-10$

42. $\sqrt{w+25}-w=-5$
43. $3\sqrt{2x-3}-20=7$
44. $2\sqrt{5x+1}-8=0$
45. $2\sqrt{8r+1}-8=2$
46. $3\sqrt{7y+1}-10=8$

In the following exercises, solve.

47. $\sqrt{3u+7}=\sqrt{5u+1}$
48. $\sqrt{4v+1}=\sqrt{3v+3}$
49. $\sqrt{8+2r}=\sqrt{3r+10}$
50. $\sqrt{10+2c}=\sqrt{4c+16}$
51. $\sqrt[3]{5x-1}=\sqrt[3]{x+3}$
52. $\sqrt[3]{8x-5}=\sqrt[3]{3x+5}$
53. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$
54. $\sqrt[3]{x^2-x+18}=\sqrt[3]{2x^2-3x-6}$
55. $\sqrt{a}+2=\sqrt{a+4}$
56. $\sqrt{r}+6=\sqrt{r+8}$
57. $\sqrt{u}+1=\sqrt{u+4}$
58. $\sqrt{x}+1=\sqrt{x+2}$
59. $\sqrt{a+5}-\sqrt{a}=1$
60. $-2=\sqrt{d-20}-\sqrt{d}$
61. $\sqrt{2x+1}=1+\sqrt{x}$
62. $\sqrt{3x+1}=1+\sqrt{2x-1}$
63. $\sqrt{2x-1}-\sqrt{x-1}=1$
64. $\sqrt{x+1}-\sqrt{x-2}=1$
65. $\sqrt{x+7}-\sqrt{x-5}=2$
66. $\sqrt{x+5}-\sqrt{x-3}=2$

In the following exercises, solve. Round approximations to one decimal place.

67. **Landscaping.** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.
68. **Landscaping.** Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.
69. **Gravity.** A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.
70. **Gravity.** A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.
71. **Accident investigation.** The skid marks for a car involved in an accident measured 216 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.
72. **Accident investigation.** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

This page titled 3.3.2: Homework is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

3.4: Rational Exponent Expressions Encountered in Calculus

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

While you will be presented with radical expressions in Calculus, more often than not these will be written with rational exponents. In fact, it is advantageous in Calculus to use rational exponents instead of radical notation.

When working with expressions in Calculus involving rational exponents, there are three skills we need to review (or introduce). In order of frequency and importance, these are:

1. understanding the meaning and impact of a negative exponent,
2. factoring out the GCF from an expression, and
3. knowing how to multiply the numerator and denominator of an expression by a unit fraction to simplify.

The first of these is addressed with a quick reminder:

$$a^{-n} = \frac{1}{a^n} \quad \text{and} \quad \frac{1}{a^{-n}} = a^n$$

We will be using this Law of Exponents frequently in this course and in Calculus.

The second might *seem* silly to mention (because you likely have been factoring out the GCF from algebraic expressions since elementary school); however, a simple change in the *look* of an expression throws many students into a downward spiral. The critical concept to remember when factoring out the GCF from an algebraic expression is that factoring is a division process. That is, when we factor $3x - 6$, we *divide out* the 3 from both $3x$ and -6. Hence,

$$3x - 6 = 3\left(\frac{3x}{3} - \frac{6}{3}\right) = 3(x - 2).$$

Of course, most people skip that middle step. With this being said, we can focus on factoring more complex algebraic expressions.

When factoring the GCF out of an algebraic expression, we were taught to factor out the smallest-powered versions of the shared factors. For example, given $a^M b^n + a^m b^N$, where $m < M$ and $n < N$, we identify the common factors as a (to a power) and b (to a power). Since the smallest power on a is m and the smallest power on b is n, we factor out $a^m b^n$.

$$a^M b^n + a^m b^N = a^m b^n \left(\frac{a^M b^n}{a^m b^n} + \frac{a^m b^N}{a^m b^n}\right) = a^m b^n \left(a^{M-m} + b^{N-n}\right).$$

To memorize this formula is preposterous, so we will focus on the concept rather than the rote memorization.

Example 3.4.1

Simplify the expression by factoring:

$$3(1+x)^{1/3} - x(1+x)^{-2/3}$$

Solution available at the end of this section.

If you pay close attention, Example $3.4.1$ reveals a shortcut to factoring out the smallest-powered version of each shared factor: factoring out $(1+x)^{-2/3}$ ends up subtracting $-\frac{2}{3}$ from the remaining exponents. That is, it *adds* the $\frac{2}{3}$ to those exponents. If you pay attention and do a few examples of your own, this trick speeds up factoring problems like this significantly.

Another thing to note from Example 3.4.1 is that we left our answer with a negative exponent—this is perfectly fine (and actually preferred in Calculus).

The next example demonstrates that we always try to factor completely, if possible.

Example 3.4.2

Factor:

$$x\left(2x+7\right)^{1/3}+5\left(2x+7\right)^{-2/3}$$

Solution available at the end of this section.

Checkpoint 3.4.2

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online



Expressions like those in Examples 3.4.1 and 3.4.2 commonly occur as the result of the Product Rule in Differential Calculus (Calculus I). You will have need to simplify those without any prompting from your instructor.

Another style of algebraic expression that frequently occurs in Calculus is in the following example.

Example 3.4.3

Simplify the expression.

$$\frac{2\left(18+x\right)^{1/2}-x\left(18+x\right)^{-1/2}}{x+18}$$

Solution available at the end of this section.

While both approaches in Example 3.4.3 look like they take the same amount of work, the second (Method #2) is often a little faster. Expressions of the form in Example 3.4.3 are the result of the Quotient Rule in Differential Calculus. As before, you will have need to simplify these without any prompting from your instructor.

Checkpoint 3.4.3

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online



Success in College: Learning From Tests (Part 2)

1. A student earned 75% of the points on her last math test. On the first class day after the test, the class started a new chapter

with new material. She does not understand several vital procedures from the last test. What do you think she should do about this situation?

2. A student usually finishes tests right as class ends. He doesn't have time to check for careless mistakes. He looks carefully at his corrected tests. He realizes that many of his errors are arithmetic or algebra mistakes. He knew how to do the problems; he didn't do them well enough. What do you think he should do?

Miscellaneous Topics: Thinking About Mathematics (Part 2)

A significant concern in mathematics education is the trend that students rely on calculators more. An argument for calculator usage is that, in "real life," a person will always have access to a computer or a calculator. Therefore, forcing students to do problems "by hand" is useless.

1. Give an argument against high calculator usage in math classes.
2. Do you think that your professor relies too much on calculator usage in class? What about their use of technology? Would you like more or less?

Answers

Section Preview

The purpose of this *very short* section is to reintroduce students to rational exponents and to acquaint them with factoring expressions having rational exponents. These occur somewhat frequently in Calculus.

Section Preview

- Factor and simplify common expressions found in Calculus involving rational powers

Example 3.4.1

Both terms in the expression share the factor $(1+x)$; however, the second term has the smallest-powered version of this factor. Therefore, we factor out $(1+x)^{-2/3}$.

$$\begin{aligned}
3(1+x)^{1/3} - x(1+x)^{-2/3} &= (1+x)^{-2/3}\left(\frac{3(1+x)^{1/3}}{(1+x)^{-2/3}} - \frac{x(1+x)^{-2/3}}{(1+x)^{-2/3}}\right) && \text{(factoring out the GCF (which is just division))}\\
&= (1+x)^{-2/3}\left(3(1+x)^{1/3+2/3} - x(1+x)^{-2/3+2/3}\right) && \text{(Laws of Exponents)}\\
&= (1+x)^{-2/3}\left(3(1+x) - x(1+x)^{0}\right) && \text{(simplifying)}\\
&= (1+x)^{-2/3}\left(3(1+x) - x\right) && \text{(Laws of Exponents)}\\
&= (1+x)^{-2/3}\left(3+2x\right) && \text{(distributing and combining like terms)}
\end{aligned}$$

Example 3.4.2

$$\begin{aligned}
x(2x+7)^{1/3} + 5(2x+7)^{-2/3} &= (2x+7)^{-2/3}\left(x(2x+7)^{3/3} + 5(2x+7)^{0}\right) && \text{(factoring out the GCF)}\\
&= (2x+7)^{-2/3}\left(x(2x+7) + 5\right) && \text{(simplifying)}\\
&= (2x+7)^{-2/3}\left(2x^2 + 7x + 5\right) && \text{(distributing)}\\
&= (2x+7)^{-2/3}(2x+5)(x+1) && \text{(factoring the trinomial)}
\end{aligned}$$

Example 3.4.3

There are two approaches to simplifying expressions like these. The first is to mimic what we did in Examples 3.4.1 and 3.4.2.

Method #1

$$\begin{aligned}
\frac{2(18+x)^{1/2} - x(18+x)^{-1/2}}{x+18} &= \frac{(18+x)^{-1/2}\left(2(18+x)^{2/2} - x(18+x)^{0}\right)}{x+18} && \text{(factoring the GCF from the numerator)}\\
&= \frac{(18+x)^{-1/2}\left(2(18+x) - x\right)}{x+18} && \text{(simplifying)}\\
&= \frac{(18+x)^{-1/2}(36+2x-x)}{x+18} && \text{(distributing)}\\
&= \frac{(18+x)^{-1/2}(36+x)}{x+18} && \text{(combining like terms)}\\
&= \frac{(36+x)}{(18+x)^{1/2}(x+18)} && \text{(Laws of Exponents)}\\
&= \frac{36+x}{(18+x)^{3/2}} && \text{(Laws of Exponents)}
\end{aligned}$$

While this method works, and there is nothing wrong with it, it is helpful to show another approach.

Method #2

This approach will use the fact that a^{-n} is the fraction $\frac{1}{a^n}$.

$$\begin{aligned}
\frac{2(18+x)^{1/2}-x(18+x)^{-1/2}}{x+18} &= \frac{2(18+x)^{1/2}-x\frac{1}{(18+x)^{1/2}}}{x+18} && \text{(Laws of Exponents)}\\
&= \frac{\left(2(18+x)^{1/2}-x\frac{1}{(18+x)^{1/2}}\right)}{(x+18)}\cdot\frac{(18+x)^{1/2}}{(18+x)^{1/2}} && \text{(simplifying the compound fraction)}\\
&= \frac{2(18+x)^{2/2}-x}{(x+18)^{3/2}} && \text{(distributing)}\\
&= \frac{2(18+x)-x}{(x+18)^{3/2}} && \text{(simplifying)}\\
&= \frac{36+2x-x}{(x+18)^{3/2}} && \text{(distributing)}\\
&= \frac{36+x}{(x+18)^{3/2}} && \text{(combining like terms)}
\end{aligned}$$

This page titled 3.4: Rational Exponent Expressions Encountered in Calculus is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College.

3.4.1: Resources and Key Concepts

Resources

Videos

- Factoring with rational exponents
  - Factoring Polynomials - Factoring Expressions with Rational or Negative Exponents
  - Algebraic Expressions - Factoring Expressions with Negative Rational Exponents
  - Intermediate Algebra - Rational Exponents: Factoring/Distributing Expressions

Key Concepts

Common Mistakes

- **Errors with Negative Exponents**: Incorrectly interpreting or manipulating negative exponents, such as $a^{-n} \neq -a^n$ or misapplying rules when factoring.
- **Incorrectly Identifying the GCF with Rational/Negative Exponents**: When factoring expressions like $3(1+x)^{1/3} - x(1+x)^{-2/3}$, choosing the wrong power for the GCF $(1+x)$. The rule is to factor out the term with the smallest (most negative or least positive) exponent.
- **Errors in Subtracting Exponents During Factoring**: When factoring out a GCF like A^m from A^M, the remaining exponent is $M-m$. Mistakes often occur with signs, especially if m is negative (e.g., $1/3-(-2/3)=1/3+2/3$).
- **Distributing Unnecessarily in Denominators**: When simplifying complex fractions or expressions resulting from the quotient rule, distributing terms in the overall denominator can make further simplification or cancellation harder. It's often better to keep denominators factored.
- **Algebraic Errors in Simplifying Compound Fractions**: When multiplying by LCD to clear minor denominators, making errors in distribution or combining terms.
- **Not Factoring Completely**: After an initial factoring step, failing to see if any resulting polynomial factors (like a quadratic) can be factored further.

This page titled 3.4.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

3.4.2: Homework

Reading Questions

1. What does a negative exponent, such as a^{-n}, signify? How can it be rewritten with a positive exponent?
2. When factoring out the Greatest Common Factor (GCF) from an algebraic expression where terms have different powers of a common base (e.g., $x^{1/3}$ and $x^{-2/3}$), which power of the common base should you factor out?
3. If you factor $(1+x)^{-2/3}$ from $3(1+x)^{1/3}$, what is the exponent of $(1+x)$ in the remaining factor inside the parentheses? Show the subtraction of exponents.
4. When simplifying an expression like $\frac{2(18+x)^{1/2}-x(18+x)^{-1/2}}{x+18}$ by first factoring out the GCF from the numerator, what is a common GCF to choose?
5. Describe an alternative method for simplifying an expression like $\frac{2(18+x)^{1/2}-x(18+x)^{-1/2}}{x+18}$ that involves treating negative exponents as fractions.
6. Why is it often preferred in Calculus to leave answers with negative exponents rather than converting them to fractions?
7. When simplifying expressions that arise from the Quotient Rule in Calculus, such as the one in Example 3, why is it generally advisable not to distribute terms in the final overall denominator?
8. What is the critical concept to remember when factoring out a GCF from an algebraic expression? (Hint: "factoring is a _____ process")

Homework

The textbook homework has not been written at this time.

This page titled 3.4.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

4: TRIANGLES AND CIRCLES

Circles and triangles are the most straightforward geometric figures. A circle is the simplest curve, and a triangle is the simplest polygon - the one with the fewest sides. Each has properties that make it useful in many fields of endeavor. A triangle is the most stable polygon because once its sides are fixed in length, its angles cannot change. A triangular truss bridge is a stable structure. A three-legged stool will not wobble.

Figure 1





A circle encloses more area than any figure of the same length or perimeter, and a sphere encloses more space. A geodesic dome is a portion of a sphere constructed with triangles. It has been called "the strongest, lightest, and most efficient means of enclosing space known to man."

Geodesic domes may also help us learn how to live on another planet. In 2017, the United Arab Emirates began construction on Mars Science City, a series of interconnecting geodesic domes designed to be a realistic model for living on Mars. The city will cover 1.9 million (1.9×10^6) square feet, and its walls will be 3D-printed using sand from the desert.

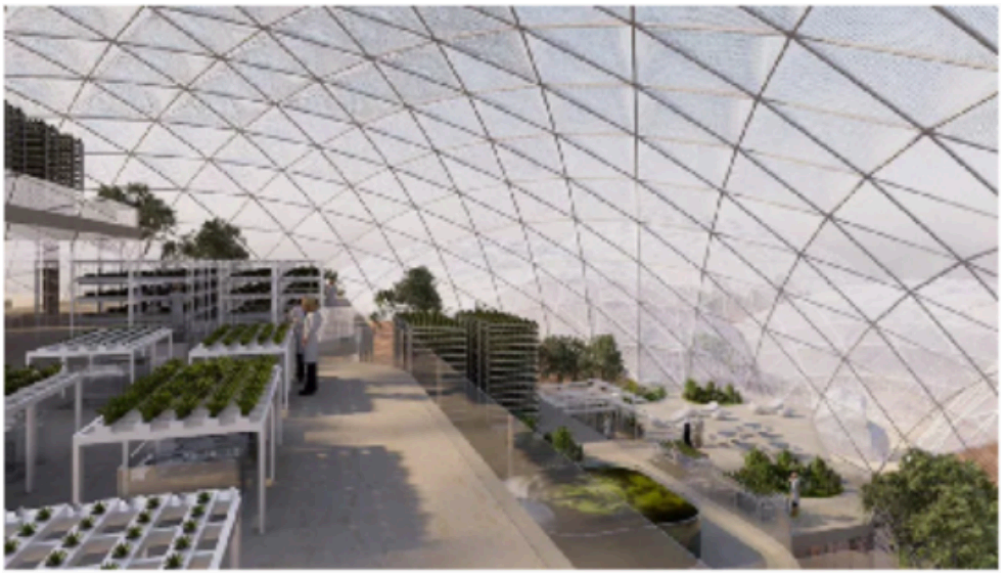
Figure 2



The city will contain laboratories to address food, sustainability, and energy issues worldwide. Finally, the project will implement

an experiment where a team will spend a year living on the simulated planet.

Figure 3



Activity: Properties of Triangles

Materials available at the end of this section.

A. What do we know about the sides of a triangle?
   1. a. Can you make a triangle with sides of length 2 inches, 3 inches, and 6 inches?
      b. Use the pieces of length 2 inches and 3 inches to form two sides of a triangle. What happens to the length of the third side as you increase the angle between the first two sides?
      c. What is the longest that the third side could be? What is the smallest?
   2. Two sides of a triangle are 6 centimeters and 8 centimeters long. What are the possible lengths of the third side?
   3. Two sides of a triangle are p units and q units long. What are the possible lengths of the third side?

B. What do we know about the angles of a triangle?
   1. Use a protractor to measure the three angles of the paper triangle in degrees. Now, add up the three angles. What is their sum?
   2. Tear off the three corners of the triangle. Place them side-by-side with their vertices (tips) at the same point. What do you find?
   3. How are your answers to parts (1) and (2) related?

C. How are the sides and angles of a triangle related?
   1. A standard way to label a triangle is to call the angles A, B, and C. The side opposite angle A is called a, the side opposite angle B is called b, and the side opposite angle C is called c. Sketch a triangle and label it with standard notation.
   2. Using a ruler, carefully draw a triangle and label it with standard notation so that $a > b > c$. Now, use a protractor to measure the angles and list them from largest to smallest. What do you observe?
   3. Using a ruler, carefully draw another triangle and label it with standard notation so that $A > B > C$. Now, use a ruler to measure the sides and list them in order from largest to smallest. What do you observe?

D. What do we know about right triangles?
   1. The **hypotenuse** is the side opposite the 90° angle in a right triangle. Why is the hypotenuse always the longest side of a right triangle?
   2. The **Pythagorean Theorem** states the following:
      IF: a, b, and c are the sides of a right triangle, and c is the hypotenuse $a^2 + b^2 = c^2$
      THEN: $a^2 + b^2 = c^2$
      The "if" part of the theorem is called the **hypothesis**, and the "then" part is called the **conclusion**. The **converse** of a theorem is the new statement you obtain when you interchange the hypothesis and the conclusion. Write the converse of the Pythagorean Theorem.

3. The converse of the Pythagorean Theorem is also true.[1] Use the converse to decide whether each of the following triangles is a right triangle. Support your conclusions with calculations.
   a. $a = 9, b = 16, c = 25$
   b. $a = 12, b = 16, c = 20$
   c. $a = \sqrt{8}, b = \sqrt{5}, c = \sqrt{13}$
   d. $a = \frac{\sqrt{3}}{2}, b = \frac{1}{2}, c = 1$

4.1: Angles and Basic Geometry

4.1.1: Resources and Key Concepts

4.1.2: Homework

4.2: Triangles

4.2.1: Resources and Key Concepts

4.2.2: Homework

4.3: Similar Triangles

4.3.1: Resources and Key Concepts

4.3.2: Homework

4.4: Circles and Angles in the Rectangular Coordinate System

4.4.1: Resources and Key Concepts

4.4.2: Homework

Footnotes

[1] In general, the converse of a theorem is not necessarily true. For example, someone might say, "If it is raining, it must be cloudy." While this is true, the converse, "If it is cloudy, then it must be raining," is not necessarily true. A theorem in which the converse is also true is called a **bijection** (but most students know this as an "if and only if").

Answers

Activity: Properties of Triangles

You will need paper and pencil, scissors, a ruler and compass, and two plastic straws.

Cut the straw into pieces of length 2 inches, 3 inches, and 6 inches. Use the ruler to draw a large triangle and cut it out.

This page titled 4: Triangles and Circles is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Katherine Yoshiwara & Roy Simpson.

4.1: Angles and Basic Geometry

Section Preview

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Angles

Before we can jump into Trigonometry, we must define angles; however, before defining angles, we need to define what makes up the components of an angle.

Definition: Ray

A **ray** is a line segment that has one endpoint and extends infinitely in one direction. The point from which the ray originates is called the **initial point** of the ray. A ray can also be described as a **half-line**.



Figure 4.1.1**:** *A ray with initial point* P

Note that we label the initial point P. The choice of letter is arbitrary, so we could just have easily chosen Q, R, or S. The choice of using capital letters for points on a line or ray, however, is tradition and we stick with that convention in this text. We now move on to defining an angle.

Definition: Angle

When two rays share a common initial point, they form an **angle**, and the shared initial point is called the angle's **vertex**.

Figure 4.1.2 shows two examples of what are commonly considered as angles.



Figure 4.1.2: *An angle with vertex P (left) and an angle with vertex Q (right)*

Figure 4.1.3 also depicts angles—albeit these are, in some sense, extreme cases. In the first case, the two rays are directly opposite each other, forming what is known as a **straight angle**; in the second, the rays are identical, so the "angle" is indistinguishable from the ray itself.



Figure 4.1.3: *A straight angle (left) and a "closed" angle (right)*

Definition: Straight Angle

A **straight angle** is defined as an angle that is formed when the two rays forming the angle extend in opposite directions.

Now that we have an idea of what makes an angle, and a "beginner's notation," it's time we define the **measure of an angle**.

Definition: Angle Measure

The **measure of an angle** is a number that indicates the amount of rotation that separates the rays of the angle.

There is one immediate problem with this definition, as should be evident from Figure 4.1.4 below.



Figure 4.1.4: *The ambiguity of angles*

Which amount of rotation are we attempting to quantify?

We have just discovered that this diagram describes at least two angles. Clearly, these two angles have different measures because one appears to represent a larger rotation than the other, so we must label them differently.

In Trigonometry, we often use lowercase Greek letters to represent the measure of the angle. For example, we use the Greek letters α (alpha) and β (beta) as representations of the unknown angle measures in Figure 4.1.5.

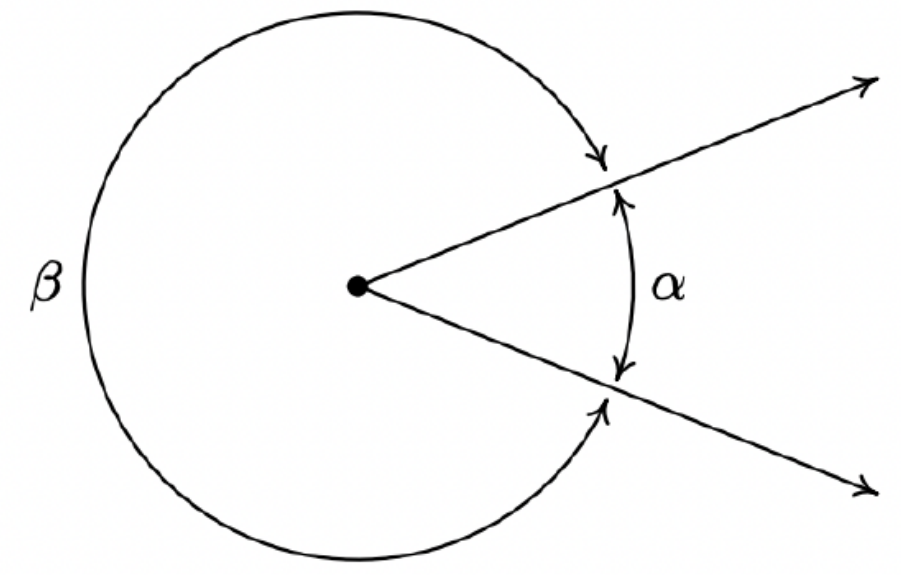


Figure 4.1.5

Identifying the lowercase Greek letters will be beneficial in this course. Table 4.1.1 summarizes the (lowercase) Greek alphabet. The letters in **bold** are the ones most commonly used within Trigonometry.

Table 4.1.1**:** *Lowercase Letters in the Greek Alphabet*

Greek Alphabet
[redacted]

Caution

The lowercase Greek letter π is *never* used as a symbolic representation of an *unknown* angle. π is reserved for the universal constant $\pi \approx 3.141592654$ (which is an **irrational number**).

Another ambiguity that sneaks into angular measure is which direction the angle is "opening." We need to extend our notion of "angle" from merely measuring an extent of rotation to quantities associated with real numbers. To that end, we introduce the concept of an **oriented angle**.

Definition: Oriented Angle

An **oriented angle** refers to the measure of rotation between two rays with a common vertex, taking into account the direction of rotation (clockwise or counterclockwise). The angle is swept out starting from an **initial side** and ending at a **terminal side**. When the rotation is counterclockwise from the initial side to the terminal side, we say that the angle is **positive**; when the rotation is clockwise, we say that the angle is **negative**.

As its name suggests, in an oriented angle, the direction of the rotation is essential.



Figure 4.1.6**:** *A positive angle (left) and a negative angle (right)*

Alternative Angle Notations

We have already mentioned that we will frequently use lowercase Greek letters when referencing angles; however, this is not the only convention that exists when referencing an angle. Angles may be labeled with a single letter at the vertex as long as it is clear that there is only one such angle. For example, in Figure 4.1.7 (below), it is clear that angle A is the vertex. There is no ambiguity about this fact. Therefore, an acceptable notation to reference this angle is $\angle A$ (some texts might use the notation $m\angle A$, meaning "the measure of angle A").

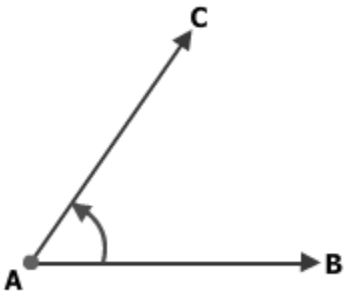


Figure 4.1.7

However, if there is the possibility for confusion, angles are labeled by specifying 3 points, with the center point being the angle's vertex. Therefore, we could reference the angle in Figure 4.1.7 as $\angle BAC$. The first letter indicates the initial side of the angle, the middle is the vertex, and the last letter indicates the terminal side. It is crucial that the center letter in this notation is the angle's vertex.

Common Angle Notations

The common notations used for angles are as follows:

- Lowercase Greek letters (most commonly α, β, and θ)
- $\angle A$ or $m\angle A$, where A is the vertex of the rays where the angle is formed
- $\angle BAC$, where A is the vertex where the rays emanating from A and going through B and C meet.

Degree Measure

One commonly used system to measure angles is **degree measure**. Quantities measured in degrees are denoted by the familiar "°" symbol.

Definition: Degree Measure

We define **degree measure** by dividing the full rotation around the circle into 360 segments. The angular measure of one of these segments is defined to be one degree, denoted 1°. That is, 1° represents the measure of an angle which constitutes $\frac{1}{360}$ of a revolution.

Figure $4.1.8$ shows the degree measures for a full, half, and quarter revolution. Since we define a full revolution as being 360°, half of a revolution measures $\frac{1}{2}(360^\circ) = 180^\circ$, a quarter of a revolution measures $\frac{1}{4}(360^\circ) = 90^\circ$, and so on.

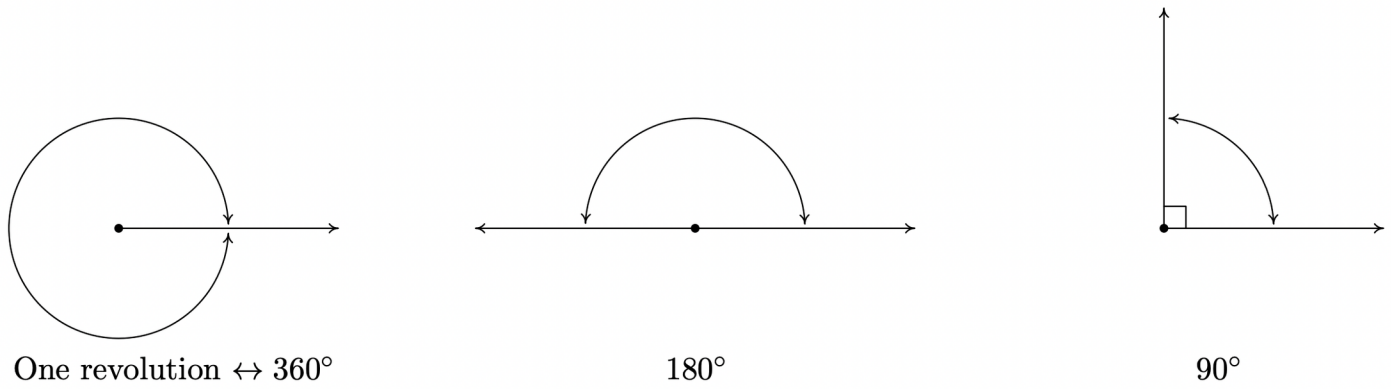


Figure $4.1.8$**:** *A full revolution (left), one-half a revolution (middle), and one-quarter of a revolution (right)*

The following theorem is a direct result of Figure $4.1.8$.

Theorem: Degree Measure of a Straight Angle

The degree measure of a straight angle is 180°.

In Figure $4.1.8$, we have used the small square "□" to denote a very special angle—the **right angle**.

Definition: Right Angle

A **right angle** is defined to be the angle measure of a quarter-circle, which is 90°.

The right angle is a very special angle in Trigonometry. As such, we use it as a sort of "measuring stick" against which we judge other angles. One such judgment is whether the measure of an angle is smaller or larger than the measure of a right angle.

Definition: Acute Angle

An angle that measures strictly between 0° and 90° is called an **acute angle**.

Definition: Obtuse Angle

An angle that measures strictly between 90° and 180° is called an **obtuse angle**.

It is important to note that, theoretically, we can know the measure of any angle as long as we know the proportion it represents of the entire revolution. For instance, the measure of an angle which represents a rotation of $\frac{2}{3}$ of a revolution would measure $\frac{2}{3}(360^\circ) = 240^\circ$, the measure of an angle which constitutes only $\frac{1}{12}$ of a revolution measures $\frac{1}{12}(360^\circ) = 30^\circ$, and an angle which indicates no rotation at all is measured as 0° (see Figure $4.1.9$).

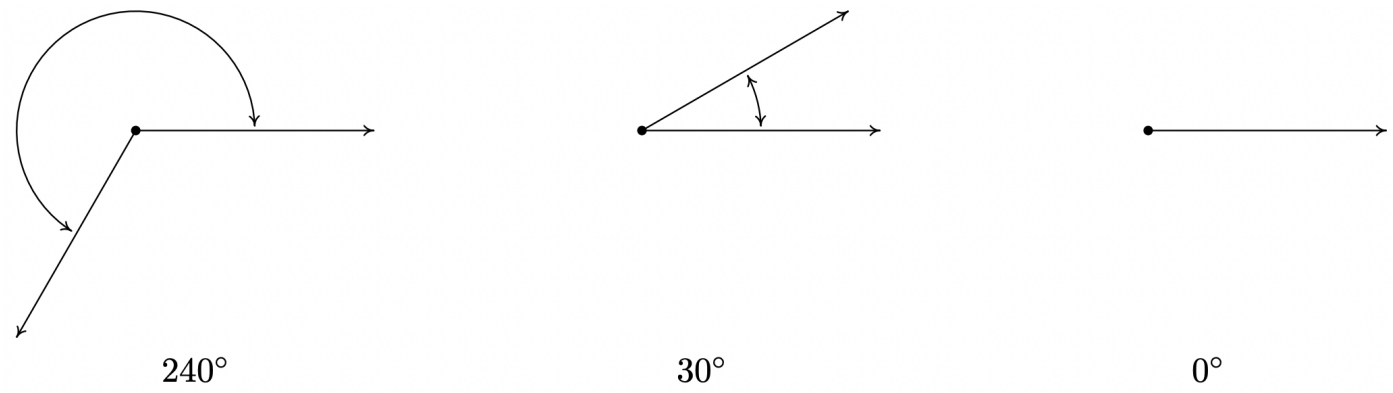


Figure 4.1.9**:** *An obtuse angle (left), an acute angle (middle), and a "closed" angle (right)*

Interactive Element: Angle as an Amount of Rotation

The following Interactive Element might help you visualize how an angle's measure is just the amount its terminal ray has rotated from the initial ray.

Interact: Drag the orange point to swing the terminal ray around the vertex.

Observation: The measure grows smoothly as you rotate. Watch it pass 90° (right), climb through the obtuse range, and reach 180° (straight)—and notice that sweeping the "long way" gives a very different measure for the same terminal ray.

We now investigate the relationship between two angles whose measures add to 90° or 180°.

Definition: Complementary Angles

Two acute angles are called **complementary angles** if their measures add to 90°.

Definition: Supplementary Angles

Two angles, either a pair of right angles or one acute angle and one obtuse angle, are called **supplementary angles** if their measures add to 180°.

In the diagram below, the angles α and β are supplementary while the pair γ and θ are complementary angles.

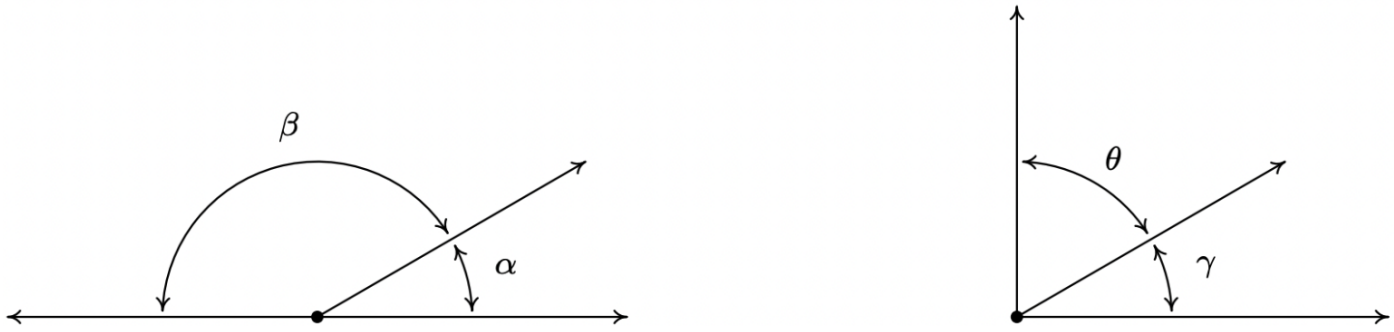


Figure 4.1.10**:** *Supplementary angles, α and β (left) and complementary angles, γ and θ (right)*

Caution

In this text, we will restrict the definition of complementary and supplementary angles to positive angles only. That is, despite their sum being 90°, we will *not* say that $\alpha = -20^\circ$ and $\beta = 110^\circ$ are complementary. Other textbooks allow for this silliness, but in truth there is no need to expand these definitions to include negative angles.

Interactive Element: Supplementary Angles

The following Interactive Element might help you visualize why two angles that sit on a straight line are supplementary.

Interact: Drag the red point to tilt the ray standing on the line.

Observation: Both α and β change as you drag, but $\alpha + \beta$ is always 180°. The same idea over a right angle gives complementary angles, which sum to 90°.

Finally, in practice, the distinction between the angle itself and its measure is blurred so that the sentence "α is an angle measuring 42°" is often abbreviated as "$\alpha = 42^\circ$."

Example 4.1.1

Consider Figure 4.1.11 below.

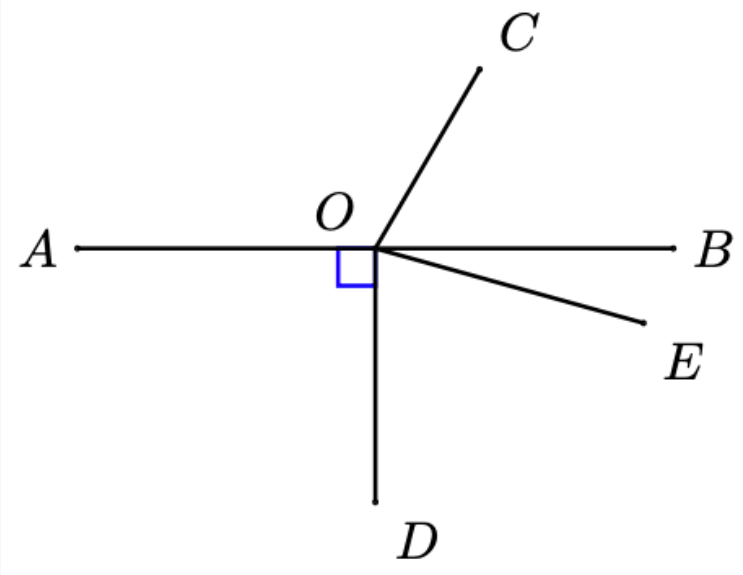


Figure 4.1.11

a. Which angle pairs are supplementary?
b. Which angle pairs are complementary?
c. Which angles are obtuse?
d. Which angles are acute?

Solutions available at the end of this section.

It could be argued in Example $4.1.1d$ that $\angle COE$ is acute; however, there is no valid reference for the measure of $\angle COB$ and $\angle BOE$—only a visual cue. Therefore, we avoid assuming that $\angle COE$ is acute.

Checkpoint 4.1.1

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

The measuring "system" we use to measure angles is formally called the **Decimal Degree** (**DD**) system.

Definition: Decimal Degree

An angle written as a decimal is said to belong to the **decimal degree (DD)** system of measuring angles.

Therefore, angles like 38°, 23.1°, and 123.456° belong to the decimal degree system. There are other angle measure systems:

- **Degree-Minute-Second** (**DMS**)**:** While we will eventually use this angular measurement system (specifically, in some applications), it is not necessary to introduce it at this time.
- **Radians:** We will introduce this "better" angular measurement system later. Without this system, we could not perform Calculus.

Angles in Rotations

Now that we have some basic information about angles and have reintroduced ourselves to a measurement system for angles, it's time to introduce an application of angular measure. Specifically, we will learn to use angles to describe rotation. For example, think of the minute hand on a clock. The minute hand moves through one complete rotation every hour, or 360°. In two hours, the minute hand rotates through 720°.

Example 4.1.2

Through how many degrees does the minute hand rotate in an hour and a half? In forty minutes?

Solution available at the end of this section.

Checkpoint 4.1.2

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Geometry Necessary for Trigonometry

Trigonometry requires a decent grasp of basic Geometry. As such, let's review some pertinent topics.

Definition: Vertical Angles

When two lines intersect, they form four angles. The two pairs of nonadjacent angles—those lying opposite each other at the point of intersection—are called **vertical angles** (or **vertically opposite angles**).

For example, $\angle 1$ and $\angle 3$ in Figure $4.1.13$ are vertical angles, as are the angles labeled $\angle 2$ and $\angle 4$.

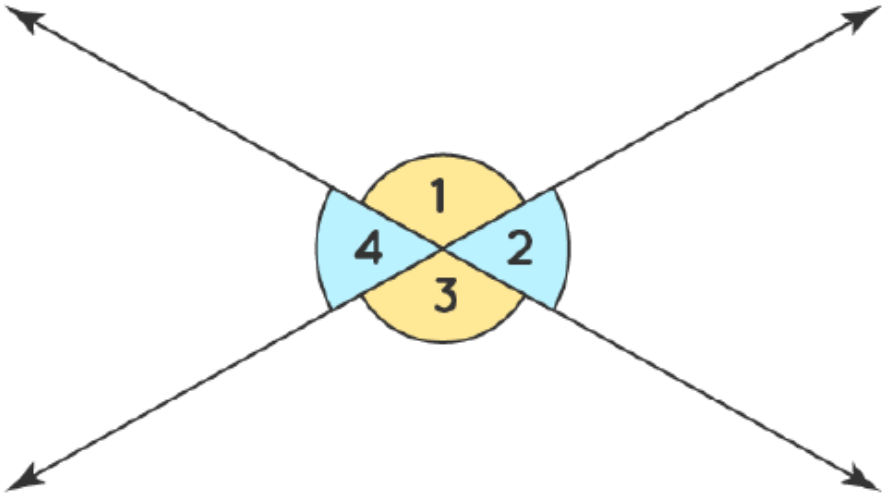


Figure $4.1.13$

Before introducing a theorem relating vertical angles, we need to define what it means for angles to be **equal**.

Definition: Equality of Angles

Two angles are defined to be **equal** if their measures are equal.

Having clarified this, we introduce a theorem that is very useful for deriving other concepts from Geometry.

Theorem: Congruency of Vertical Angles

Vertical angles are equal (also known in Geometry as **congruent**).

Proof available at the end of this section.

A line intersecting two parallel lines forms eight angles, as shown in Figure $4.1.14$.

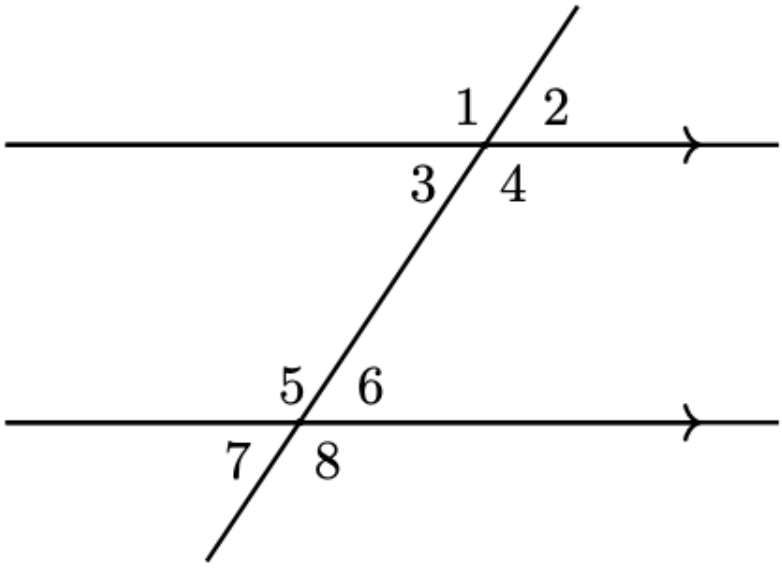


Figure 4.1.14**:** *Two parallel lines "cut" by a line*

The line cutting through the parallel lines in Figure 4.1.14 gets a special name.

Definition: Transversal

A **transversal** is a line that intersects two or more lines at distinct points.

It's important to note that transversals need not cut through parallel lines. That is, a line cutting through any other two lines (whether those lines are parallel or not) is also called a transversal.

Figure 4.1.14 shows that there are four pairs of vertical angles formed by a transversal—$\angle 1$ and $\angle 4$, $\angle 2$ and $\angle 3$, $\angle 5$ and $\angle 8$, and finally $\angle 6$ and $\angle 7$; however, there are other relationships that can be seen in Figure 4.1.14.

Definition: Corresponding Angles

Corresponding angles are a pair of angles, one at each intersection, that lie on the same side of the transversal with one angle interior and the other exterior. Equivalently, they occupy the same relative position at the two intersections.

For example, $\angle 1$ and $\angle 5$ are corresponding angles, as are $\angle 4$ and $\angle 8$. When the lines the transversal is cutting through are parallel, we get the following theorem.

Theorem: Corresponding Angles Theorem

Two lines cut by a transversal are parallel if and only if each pair of corresponding angles is congruent.

The "if and only if" part of this theorem is very important. It states that, "If two lines cut by a transversal are parallel, then each pair of corresponding angles is congruent." Moreover, it states that, "If each pair of corresponding angles is congruent, then the two lines cut by a transversal are parallel."

Our final definition concerning the transversals is one of the most popular from Geometry.

Definition: Alternate Interior Angles

Alternate interior angles are a pair of interior angles that lie on opposite sides of the transversal, with one angle at each intersection.

In Figure 4.1.14, $\angle 3$ and $\angle 6$ are alternate interior angles, as are $\angle 4$ and $\angle 5$.

Theorem: Alternate Angles Theorem

If two parallel lines are cut by a transversal, then each pair of alternate interior angles is congruent, and each pair of alternate exterior angles is congruent.

Proof available at the end of this section.

Interactive Element: Angles Formed by a Transversal

The following Interactive Element might help you visualize the angle relationships formed when a transversal cuts two parallel lines.

Interact: Drag the orange point to tilt the transversal.

Observation: However you tilt it, the two corresponding angles stay equal, and only two values ever appear among all eight angles: θ and its supplement $180^\circ - \theta$. Alternate interior angles are equal for the same reason.

Example 4.1.3

The parallelogram $ABCD$ shown below is formed by the intersection of two sets of parallel lines. Show that the opposite angles of the parallelogram are equal.

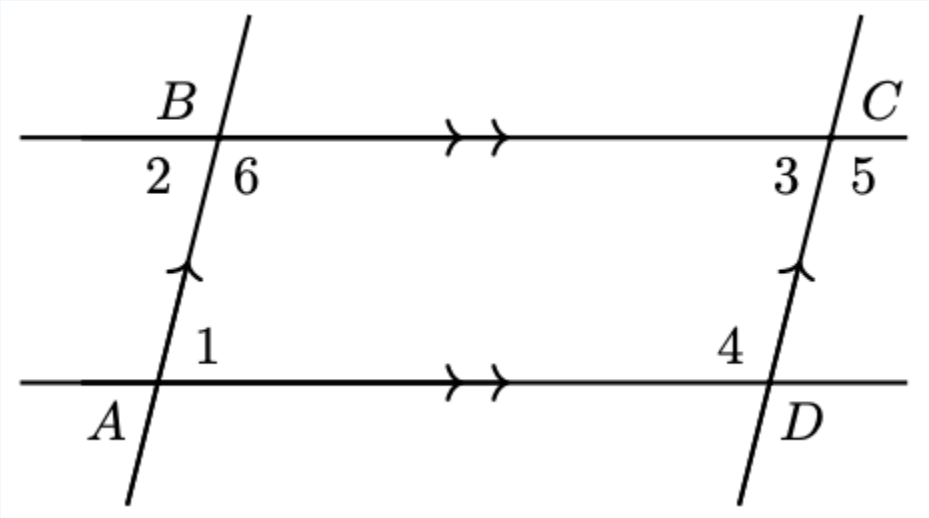


Figure 4.1.15

Solution available at the end of this section.

Checkpoint $4.1.3$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Circles

We end this section with one of the defining geometric structures in all of Mathematics—the **circle**.

Definition: Circle

The shape created from all points in a plane that are equidistant from a given point is called a **circle**. The given point is called the **center** of the circle, and the distance from the center to any point along the circle is called the **radius** of the circle.

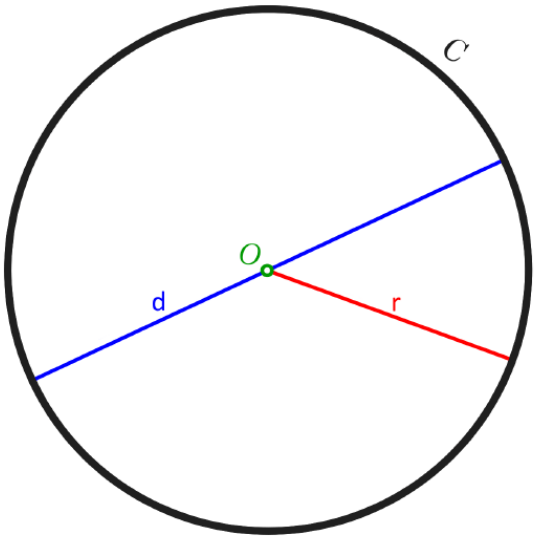


Figure $4.1.16$**:** *A circle with center* O *and radius* r

Figure $4.1.16$ shows a circle of radius r centered at a point O. In fact, the word "radius" plays two roles in Geometry. By definition, the radius is the distance between the center of a circle and its edge; however, *any* line segment joining the center of a circle with any single point on the circle itself is called a radius. Therefore, the word radius simultaneously refers to a distance and a line segment.

Definition: Radius (as a line segment)

A line segment with one endpoint at the center of a circle and the other on the circle is called a **radius** of the circle.

Just like the radius, the **diameter** of a circle, denoted as d in Figure $4.1.16$, has two interpretations. Our first is in terms of distance.

Definition: Diameter (as a distance)

The **diameter** of a circle is the length of any line segment going through the center of a circle and whose endpoints are on the circle.

This definition naturally leads to the following theorem.

Theorem: Diameter in Terms of Radius

Given a circle of radius r, the diameter, d, of the circle satisfies

$$d = 2r.$$

As with the radius of a circle, the second interpretation of the diameter is in terms of a line segment.

Definition: Diameter (as a line segment)

A line segment going through the center of a circle and whose endpoints are on the circle is called a **diameter** of the circle.

The terminology from Geometry related to circles extends beyond center, radius, and diameter (e.g., arc, chord, sector, and tangent); however, we will introduce these only as needed.

Circumference

Now that we have the basic definition of a circle, let's dive into some measurements.

Definition: Circumference

The **circumference** is the distance around a circle.

In Figure $4.1.16$, the circumference is denoted using the letter C.

Theorem: Circumference

A circle's circumference, C, is proportional to its radius, r. That is,

$$C = kr.$$

The constant of proportionality is 2π. Hence,

$$C = 2\pi r.$$

A proof of this fundamental theorem in Mathematics is beyond the scope of this course.

In the theorem presented for the circumference, it would be more impactful to state the circumference is proportional to the diameter. That is,

$$C = \text{ some number } \times d.$$

It turns out that the value of "some number" (which is commonly called the **constant of proportionality**) is the universal constant π. That is,

$$C = \pi d \implies \pi = \frac{C}{d}.$$

This fascinating fact tells us that no matter the size of the circle, the ratio of its circumference to its diameter is always this magical number that we call π.

Language Note: "Circumference of a Circle" is Redundant

It is common to hear someone say, "the circumference of a circle." However, the word circumference *only applies to circles*. Therefore, saying "the circumference of a circle" is redundant. For non-circular two-dimensional shapes, we use the word **perimeter** instead of circumference.

Example 4.1.4

Loi buys a ring light for her YouTube videos. The light is circular and has a radius of 7 inches. What is the circumference of her ring light?

Solution available at the end of this section.

Checkpoint 4.1.4

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Area of a Circle

The area of a circle is proportional to the square of its radius.

Theorem: Area of a Circle

The area of a circle of radius r is given by

$$A = \pi r^2.$$

Again, the proof of this theorem is beyond the scope of this course.

Interactive Element: Circumference vs. Area

The following Interactive Element might help you visualize how a circle's circumference and area both depend on its radius.

Interact: Drag the red point along the axis to change the radius.

Observation: Both C and A grow as the radius increases, but the area grows faster—$A = \pi r^2$ depends on the square of r, while $C = 2\pi r$ depends on r directly.

Example 4.1.5

Phuong is making a circular tablecloth for her grandfather's antique oak table. If she wants the tablecloth to have a diameter of at least 7 feet (the bare minimum to cover the table), but no more than 9 feet (allowing a little bit of tablecloth to hang over the edge of the table), between what two values must the area of the tablecloth be? Round your answers to the nearest tenth of a square foot.

Solution available at the end of this section.

Checkpoint 4.1.5

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Hidden Resources: Borrowing Technology

Having a computer is a requirement for any college class these days; however, what do you do if you don't have a computer? Cosumnes River College (and the Los Rios Community College District) has pulled together a great list of resources for students needing a computer. Whether you want to borrow a Chromebook from the CRC library for the semester or you prefer to purchase a discounted computer, the CRC Library has all the information you need!

MyOpenMath Advice: "Message Instructor" Within Questions

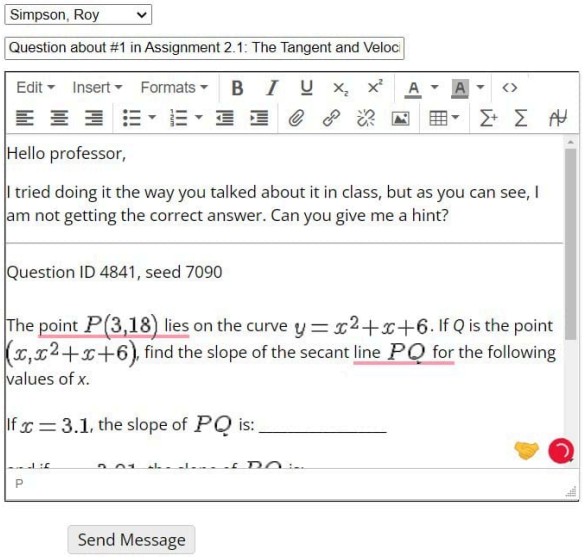
What is it?

If you are stuck on a homework question, and if your professor has set up a discussion area on Canvas, post your question to that area first; however, not all professors have enabled a discussions area. In this case, you can ask your professor for help directly by clicking on the "Message instructor" link at the bottom of the question.



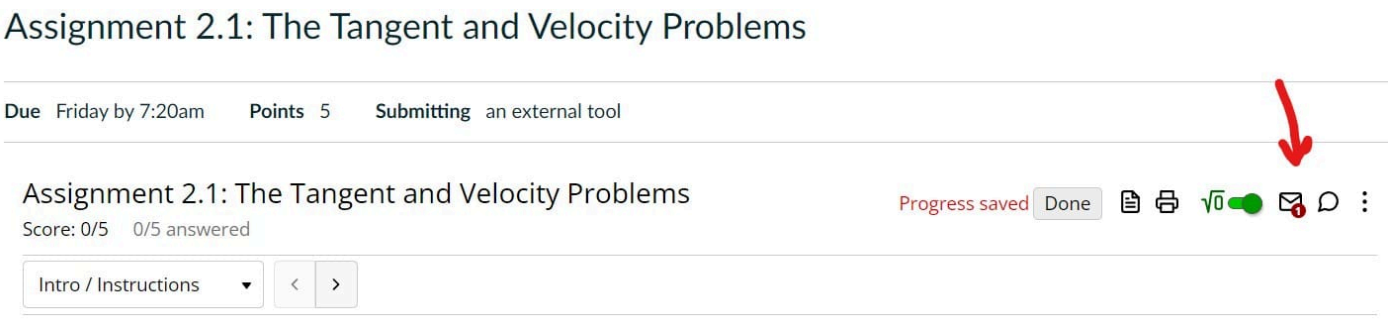
Why use it?

The nice thing about using the "Message instructor" link is that it sends *your version* of the question you are asking about directly to your professor. This helps them better understand your question. Be sure to include some information as to what might be causing your confusion.



How do you know when your professor has responded?

This is the only tricky part. When your professor responds, you will *not* get an email notification. This is a weakness of the MyOpenMath embed into Canvas (the messaging systems do not work well together); however, when your professor responds, you will see a message flag within *any* homework assignment you open.



Just click on the envelope and you will see your professor's response.

Answers

Section Preview

- **Sets and Numbers**
  - The Language of Algebra (focus: definitions and theorems): This section is definition- and theorem-heavy from the very first paragraph; students need to understand what a definition and theorem are in order to engage with the material as presented.

Section Preview

- **Arithmetic**
  - Mathematical Operations - Addition and Subtraction: Computing complements/supplements requires addition/subtraction with signed numbers.
  - Mathematical Operations - Powers: Finding circumference and area requires the multiplication of powers since $A=\pi r^2$ involves squaring.
  - Multiplying and Dividing Fractions: Computing partial rotations (e.g., $\frac{2}{3}\times 360^\circ$) requires multiplying and dividing fractions.

Section Preview

- Identify acute, obtuse, right, and straight angles.
- Compute the complement and supplement of an angle.
- Determine the number of degrees in a partial rotation.
- Use vertical, alternate interior, or corresponding angles to solve for an angle.
- Find the circumference and area of a circle.

Example 4.1.1

a. $\angle AOC$ and $\angle BOC$ are supplementary, $\angle AOD$ and $\angle BOD$ are supplementary, and $\angle AOE$ and $\angle BOE$ are supplementary.
b. $\angle DOE$ and $\angle BOE$ are complementary.

c. $\angle AOC$, $\angle AOE$, and $\angle DOC$ are each obtuse angles.
d. $\angle BOC$, $\angle BOE$, and $\angle DOE$ are each acute angles.

Example 4.1.2

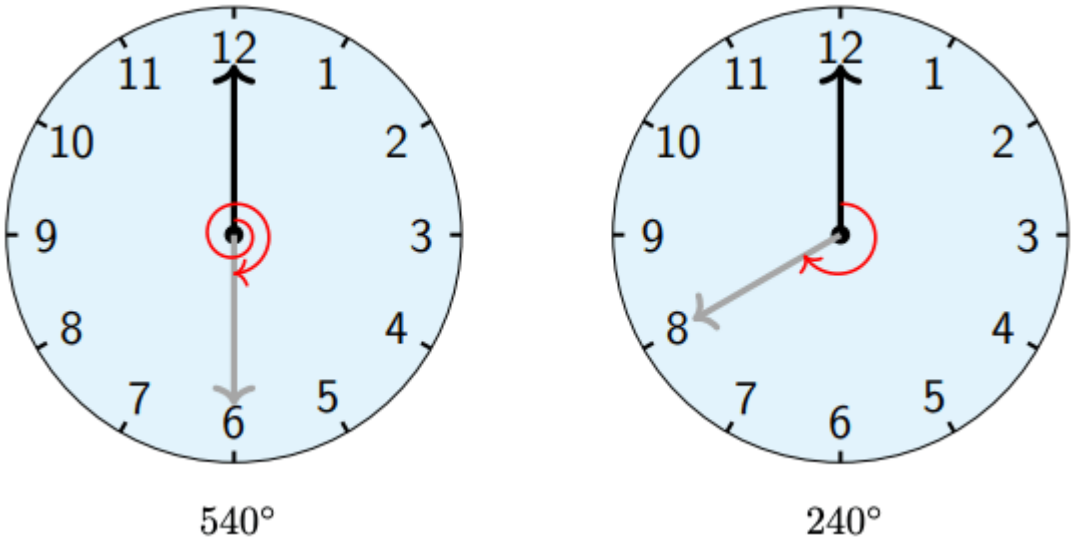


Figure 4.1.12

Look at Figure 4.1.12. An hour and a half represents 1.5 complete rotations, or

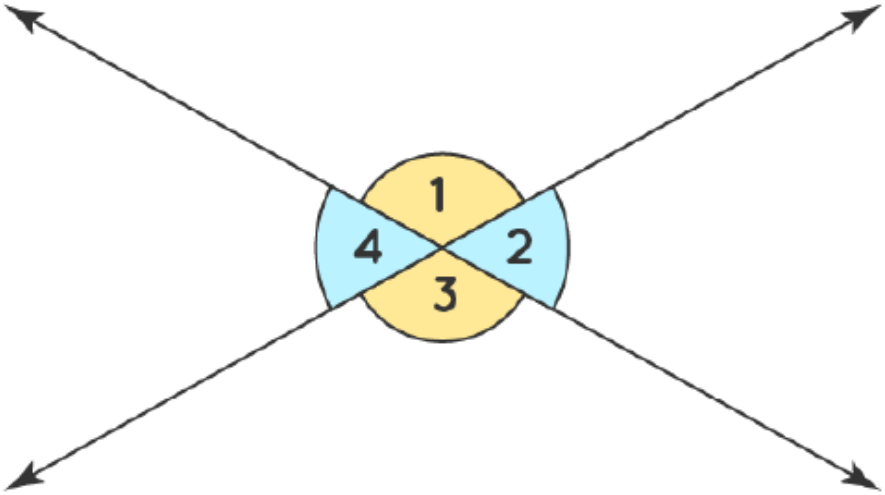
$$1.5(360^\circ) = 540^\circ.$$

Forty minutes is two-thirds of an hour, so the minute hand rotates through

$$\frac{2}{3}(360^\circ) = 240^\circ.$$

Theorem: Congruency of Vertical Angles

Suppose two lines intersect, as shown in the figure below.



Since $\angle 1 + \angle 2$ forms a straight angle,

$$\angle 1 + \angle 2 = 180^\circ.$$

By a similar argument,

$$\angle 1 + \angle 4 = 180^\circ.$$

Therefore,

$$\angle 1 + \angle 2 = 180^\circ = \angle 1 + \angle 4.$$

That is,

$$\angle 2 = \angle 4.$$

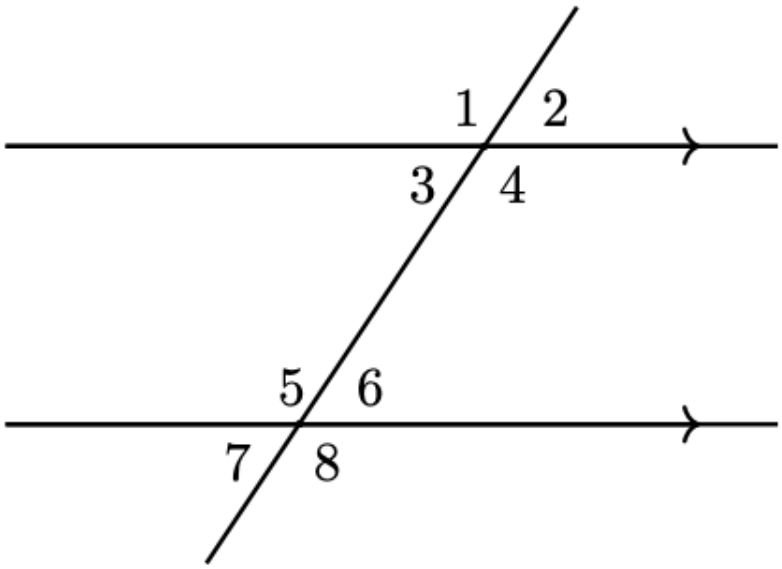
By a similar set of arguments, we can show that

$$\angle 1 = \angle 3.$$

Hence, it must always be the case that vertical angles are congruent.

Theorem: Alternate Angles Theorem

Suppose a transversal cuts two parallel lines as shown below.



We know that the corresponding and vertical angles are congruent. Therefore,

$$\angle 1 = \angle 5$$

and

$$\angle 1 = \angle 4.$$

Hence,

$$\angle 5 = \angle 4.$$

Since our choice of angles was arbitrary, this shows that alternate interior angles are always congruent. A similar argument shows the alternate exterior angles are congruent.

Example 4.1.3

Angles 1 and 2 are equal because they are alternate interior angles, and angles 2 and 3 are equal because they are corresponding angles. Therefore, angles 1 and 3, the opposite angles of the parallelogram, are equal. A similar argument shows that angles 4, 5, and 6 are equal.

Example 4.1.4

We are given the fact that the radius is 7 inches. Therefore, $r = 7$ (inches). Using the formula for circumference, we find that Loi's ring light has a circumference of

$$C = 2\pi r = 2\pi(7 \text{ inches}) = 14\pi \text{ inches} \approx 44 \text{ inches}.$$

Example 4.1.5

To barely cover the table, Phuong needs a tablecloth with radius 3.5 feet. Therefore, the minimum area of the tablecloth should be

$$A_{\text{minimum}} = \pi r^2_{\text{minimum}} = \pi(3.5)^2 \approx 38.5 \text{ square feet}.$$

At most, if she wants a little bit of the tablecloth to hang over the edge, Phuong will want the radius to be 4.5 feet. This leads to an area of

$$A_{\text{maximum}} = \pi r^2_{\text{maximum}} = \pi(4.5)^2 \approx 63.6 \text{ square feet}.$$

Any amount of circular tablecloth between 38.5 and 63.6 square feet should work nicely for Phuong's project.

This page titled 4.1: Angles and Basic Geometry is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

4.1.1: Resources and Key Concepts

Resources

Videos

- Lines, Rays, Vertices, Angles, Oriented Angles, Initial Side, Terminal Side
  - Angles - Creating Angles
- Degree Measure
  - Angles - Angle Measure: Degrees
- Circles and Their Properties
  - Intermediate Algebra - Circles: The Concept

Key Concepts

Definitions

- **Ray:** A line segment with one starting point (the initial point) that extends infinitely in one direction. Also called a half-line.
- **Angle:** A figure formed when two rays share a common initial point.
- **Vertex:** The shared initial point of the two rays that form an angle.
- **Straight Angle:** An angle formed when its two rays extend in opposite directions, forming a straight line.
- **Angle Measure:** A numerical value indicating the amount of rotation that separates the two rays of an angle.
- **Oriented Angle:** An angle whose measure considers the direction of rotation from an initial side to a terminal side.
- **Initial Side:** The starting ray of an oriented angle.
- **Terminal Side:** The ending ray of an oriented angle after rotation.
- **Positive Angle:** An angle formed by a counterclockwise rotation from the initial side to the terminal side.
- **Negative Angle:** An angle formed by a clockwise rotation from the initial side to the terminal side.
- **Degree Measure:** A system for measuring angles where a full rotation is divided into 360 equal parts, with each part being one degree (1°).
- **Right Angle:** An angle that measures exactly 90°, representing a quarter-circle rotation.
- **Acute Angle:** An angle whose measure is strictly between 0° and 90°.
- **Obtuse Angle:** An angle whose measure is strictly between 90° and 180°.
- **Complementary Angles:** Two positive acute angles whose measures sum to 90°.
- **Supplementary Angles:** Two positive angles whose measures sum to 180°.
- **Decimal Degree (DD) system:** A system of measuring angles where the angle is written as a decimal number.
- **Vertical Angles:** The non-adjacent angles that are formed by the intersection of two straight lines.
- **Equality of Angles:** A condition where two angles are considered equal if their measures are equal.
- **Transversal:** A line that intersects two parallel lines.
- **Corresponding Angles:** Angles that are in the same relative position at each intersection where a transversal crosses two lines.
- **Alternate Interior Angles:** A pair of angles on opposite sides of the transversal and inside the two parallel lines.
- **Circle:** The set of all points in a plane that are the same distance (equidistant) from a central point.
- **Center:** The given point from which all points on a circle are equidistant.
- **Radius:** The distance from the center to any point on the circle. The term also refers to any line segment connecting the center to a point on the circle.
- **Diameter:** The length of a line segment that passes through the center of a circle and has both endpoints on the circle. The term also refers to the line segment itself.
- **Circumference:** The distance around a circle.

Theorems

- **Degree Measure of a Straight Angle:** The degree measure of a straight angle is 180°.

- **Congruency of Vertical Angles:** Vertical angles are equal (congruent).
- **Congruency of Corresponding Angles:** Corresponding angles are congruent.
- **Alternate Interior Angles:** If a transversal intersects parallel lines, the alternate interior angles are congruent.
- **Diameter in Terms of Radius:** For a circle with radius r and diameter d, the relationship is $d = 2r$.
- **Circumference:** The circumference C of a circle with radius r is given by the formula $C = 2\pi r$.
- **Area of a Circle:** The area A of a circle with radius r is given by the formula $A = \pi r^2$.

Common Mistakes

- **Using π for an angle:** The lowercase Greek letter π is reserved for the constant approximately equal to 3.14159 and should never be used to represent an unknown angle.
- **Using negative angles for complements/supplements:** In this textbook, complementary and supplementary angles are restricted to be positive angles only.
- **Relying on visual appearance:** Do not assume angles are acute, obtuse, or have a specific measure based on a diagram alone. Conclusions must be supported by definitions or theorems. For example, in Example 1.1.1, one cannot conclude $\angle COE$ is acute without more information.
- **Ambiguous angle notation:** Using a single letter to name an angle can be ambiguous if multiple angles share the same vertex. Use the three-point notation (e.g., $\angle BAC$) to be clear, ensuring the middle letter is the vertex.
- **Redundant language:** The word "circumference" applies only to circles, so saying "the circumference of a circle" is redundant. For other shapes, the term "perimeter" is used.

This page titled 4.1.1: Resources and Key Concepts is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.1.2: Homework

Skills Refresher

The following is a set of review exercises you will need for this section.

Skills Refresher

For the following exercises, solve the equation.

1. $x - 8 = 19 - 2x$
2. $2x - 9 = 12 - x$
3. $13x + 5 = 2x - 28$
4. $4 + 9x = -7 + x$

For the following exercises, solve the system.

5. $\begin{cases} 5x - 2y &= -13 \\ 2x + 3y &= -9 \end{cases}$
6. $\begin{cases} 4x + 3y &= 9 \\ 3x + 2y &= 8 \end{cases}$

Answers available at the end of this section.

Homework

Reading Questions

1. What is a "ray"?
2. How is the "vertex" of an angle defined?
3. What distinguishes a "positive angle" from a "negative angle"?
4. How many degrees constitute a full revolution?
5. What is an "obtuse angle"?
6. What are "supplementary angles"?
7. According to the theorem presented, what is true about "vertical angles"?
8. What does a "transversal" line do?
9. What is the formula for the circumference of a circle in terms of its radius, r?
10. If a circle has a diameter d, how is its radius r related to d?

Vocabulary Check

11. (MyOpenMath) When two rays share a common initial point they form a(n) ___.
12. (MyOpenMath) Trigonometry focuses on oriented angles. An oriented angle starts from an ___ side and ends at a ___ side.
13. (MyOpenMath) ___ angles start at an initial side and rotate clockwise to the terminal side.
14. (MyOpenMath) A positive angle opens with a ___ rotation.
15. We define ___ measure by dividing the full rotation around the circle into 360 segments.
16. One-quarter of a complete revolution around a circle forms a ___ angle, which measures ___ degrees.
17. ___ angles measure between 0° and 90°, while ___ angles measure between 90° and 180°.
18. The degree system for measuring the size of an angle is also known as the ___ system.
19. When two lines intersect, the non-adjacent angles formed from this intersection are called ___ angles.
20. A line intersecting two parallel lines is called a ___.
21. The set of all points equidistant to a single point on a plane is called a ___.
22. The ___ of a circle is simultaneously the largest distance between any two points on a circle and is the name of the line segment that passes through the center of the circle whose endpoints lie on the circle.
23. π is a(n) ___ number (unlike 3, which is a natural number).

Concept Check

24. Greek letters like α, β, and θ are often used to represent angles in Trigonometry. Name one Greek letter that is *never* used as a label for an angle.
25. If a positive angle rotates from its initial to its terminal side, what direction is the rotation?
26. If a negative angle rotates from its initial to its terminal side, what direction is the rotation?
27. Suppose you have a straight angle and you subtract a right angle. What type of angle is the result?
28. Can a right angle be acute? How about obtuse?
29. Suppose the vertex of an angle is at the point P, and the points R and S lie on the initial and terminal sides of the angle, respectively. Which of the following are correct notations for the angle?
    i. $\angle P$
    ii. P
    iii. $m\angle P$
    iv. $\triangle P$
    v. $\angle PRS$
    vi. $\angle RPS$
    vii. $\angle SPR$
30. Quantities measured in degrees use the ___ symbol.

31. According to this text, can 107° have a complement? If so, what is the value of this complement?

32. According to this text, can 107° have a supplement? If so, what is the value of this supplement?

33. Can two acute angles be supplementary?

34. Choose two of the eight angles formed by a pair of parallel lines cut by a transversal. Those two angles are either equal or ___.

35. Write two formulas for the circumference of a circle - one involving the radius of the circle and the other involving the diameter.

36. The formula for the circumference of a circle is ___.

37. The formula for the area of a circle is ___.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

38. If α and β are vertical angles, then $\alpha + \beta = 90^\circ$.

39. (MyOpenMath) The complement to 47° is 43°.

40. The supplement to 190° is -10°.

41. Alternate interior angles sum to 180°.

42. The distance from the center to the edge of a circle is called the radius of the circle.

43. The line connecting the center of a circle to its edge is called a radius.

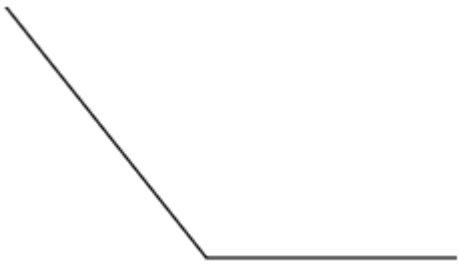
44. The circumference of a square is the sum of the lengths of its sides.

45. $\pi = 3.14$

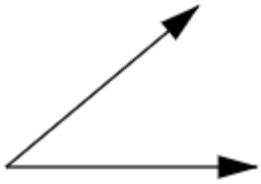
46. $\pi = \frac{22}{7}$

Basic Skills

47. (MyOpenMath) State (if possible) if the angle is acute or obtuse.



48. (MyOpenMath) Which is the closest measure to the angle shown below: 40°, 65°, 95°, or 120^{ \circ }?



For the following exercises, state (if possible) which angles are acute and which are obtuse. Give the complement and the supplement of each angle, if applicable.

49. (MyOpenMath) 30°

50. 45°

51. 60°

52. 43°

53. 90°

54. 120°

55. 135°

56. 143°

57. 150°

For the following exercises, give the complement of each angle.

58. 60°

59. 80°

60. 25°

61. 18°

62. 64°

63. 47°

For the following exercises, give the supplement of each angle.

64. 30°

65. 45°

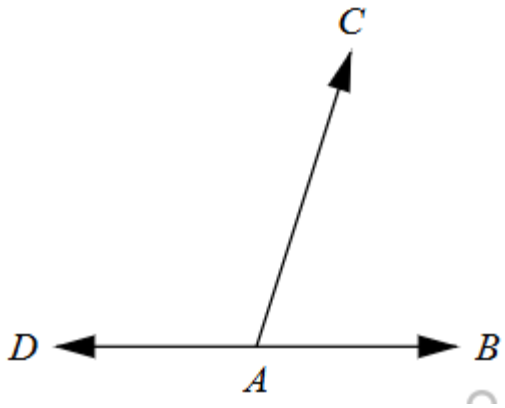
66. 120°

67. 25°

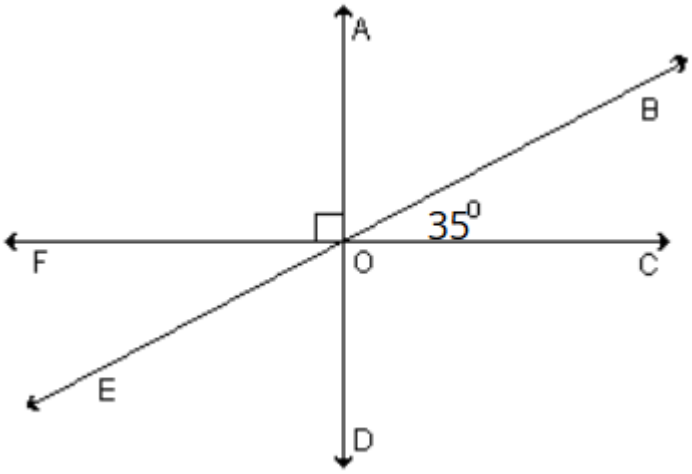
68. 165°

69. 110°

70. (MyOpenMath) If the measure of $\angle CAB = 62^\circ$, then find the measure of $\angle CAD$.



71. (MyOpenMath)



a. Which angle is supplementary to $\angle BOC$?

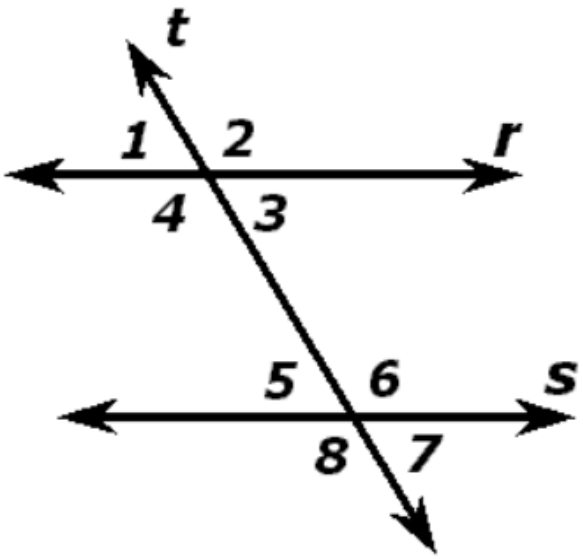
b. Which angle is complementary to $\angle BOC$?

c. What is the measure of $\angle EOF$?

d. What is the measure of $\angle AOE$?

e. What is the measure of $\angle BOF$?

For the following exercises, use the following graph to answer the question. Assume lines r and s are parallel.



72. (MyOpenMath) $\angle 3$ is alternate interior to which other angle(s)?

73. (MyOpenMath) $\angle 5$ is a corresponding angle to which other angle(s)?

74. (MyOpenMath) $\angle 7$ and what other angle(s) are vertical angles?

75. (MyOpenMath) $\angle 8$ is alternate exterior to which other angle(s)?

76. (MyOpenMath) Find $m\angle 5$ given that $m\angle 1 = 46^\circ$.

77. (MyOpenMath) Find $m\angle 8$ given that $m\angle 6 = 154^\circ$.

78. (MyOpenMath) Find $m\angle 5$ given that $m\angle 3 = 82^\circ$.

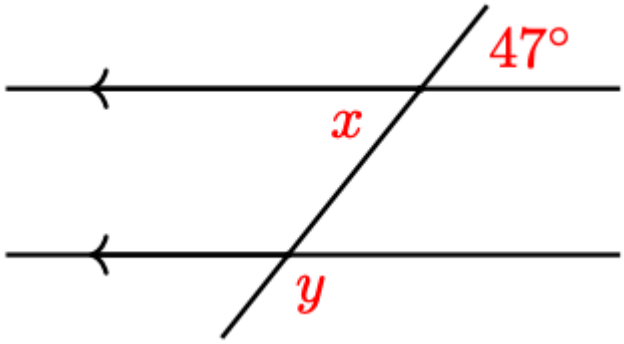
79. (MyOpenMath) Find $m\angle 3$ given that $m\angle 6 = 105^\circ$.

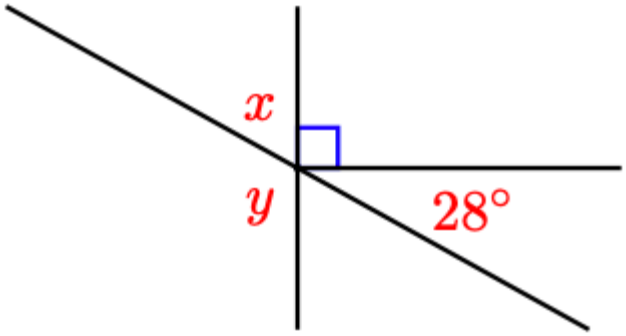
80. (MyOpenMath) Find $m\angle 8$ given that $m\angle 3 = 76^\circ$.

81. (MyOpenMath) Find the remaining angles given that $m\angle 1 = 15^\circ$.

For the following exercises, arrows on a pair of lines indicate that they are parallel. Find x and y.

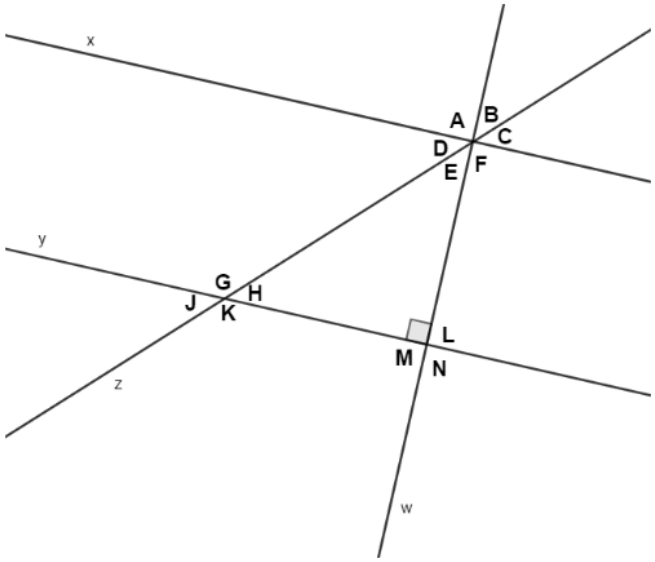
82.



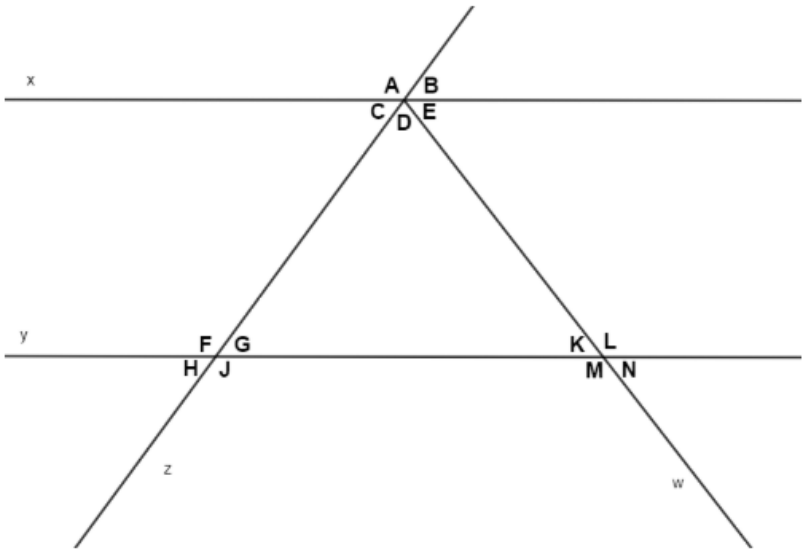


33.

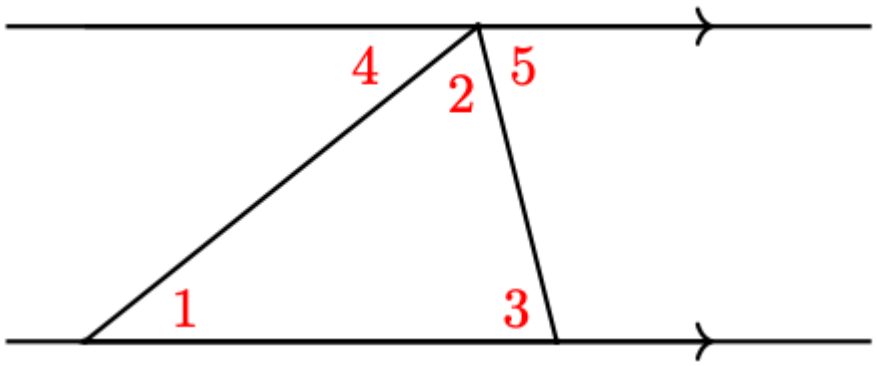
34. (MyOpenMath) The lines x and y are parallel, and are cut by transversals z and w. Given $\angle E = 57°$, find the measure of all remaining angles.



35. (MyOpenMath) The lines x and y are parallel, and are cut by transversals z and w. Given $\angle H = 36°$ and $\angle E = 40°$, find the measure of all remaining angles.



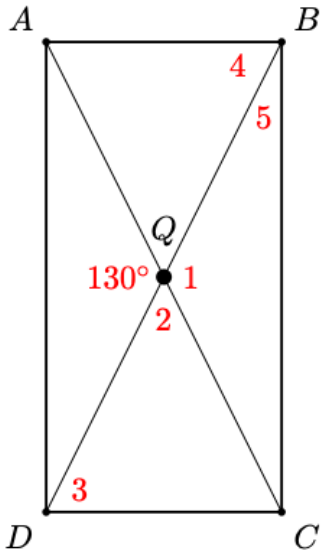
36. a. Among the angles labeled 1 through 5 in the figure below, find two pairs of equal angles.



b. $\angle 4 + \angle 2 + \angle 5 =$_____.

c. Use parts (a) and (b) to explain why the sum of the angles of a triangle is 180°.

37. $ABCD$ is a rectangle. The diagonals of a rectangle bisect each other. In the figure, $\angle AQD = 130^\circ$. Find the angles labeled 1 through 5 in order, and give a reason for each answer.

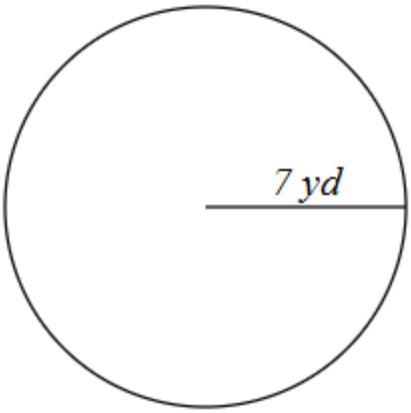


For the following exercises, give an exact answer, and then round your answer to hundredths.

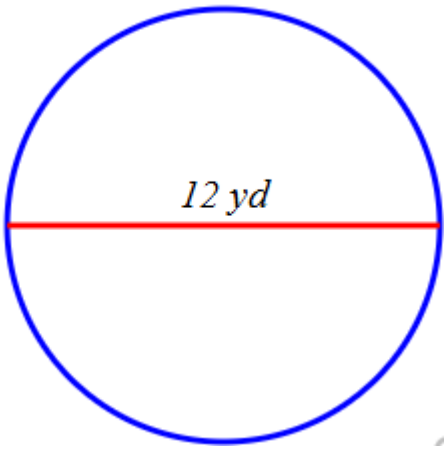
38. What is the area of a circle whose radius is 5 inches?

39. What is the circumference of a circle whose radius is 5 meters?

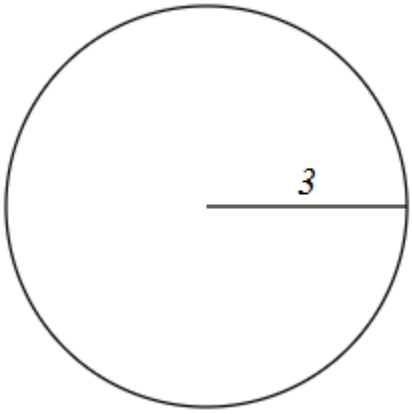
40. (MyOpenMath) What is the circumference of the following circle?



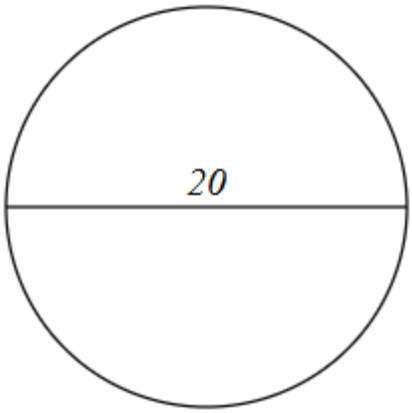
91. (MyOpenMath) What is the circumference of the following circle?



92. (MyOpenMath) What is the area of the following circle?



93. (MyOpenMath) What is the area of the following circle?



For the following exercises, find the circumference and area for each circle having the given radius r or diameter d.

94. (MyOpenMath) $r = 7$ feet

95. $r = 8.4$ centimeters

96. $d = 16$ yards

97. $d = 12$ meters

98. Find the diameter of a circle having a circumference of 4π kilometers.

99. (MyOpenMath) Find the radius of a circle having a circumference of 10 inches.

100. (MyOpenMath) Find the radius of a circle having an area of 24π square meters.

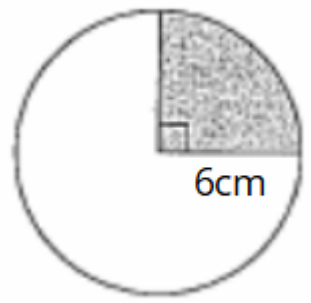
101. (MyOpenMath) Find the radius of a circle having an area of 272 square meters.

102. (MyOpenMath) Find the diameter of a circle having an area of 64 square feet.

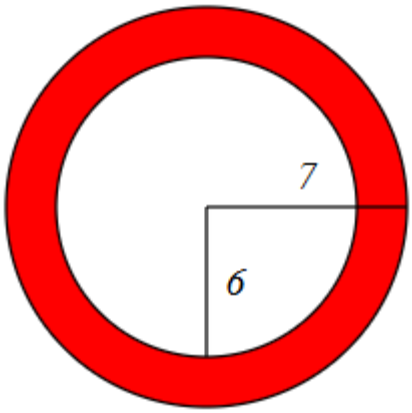
103. Find the circumference of a circle having an area of 81π square miles.

Synthesis Questions

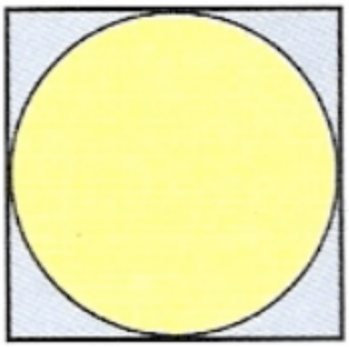
104. (MyOpenMath) Find the area of the shaded region. Round your answer to the nearest tenth.



105. (MyOpenMath) Find the area of the shaded region. Round your answer to the nearest tenth.



96. (MyOpenMath) Suppose the sides of the square are 8 miles in length.



a. What is the area of the square?

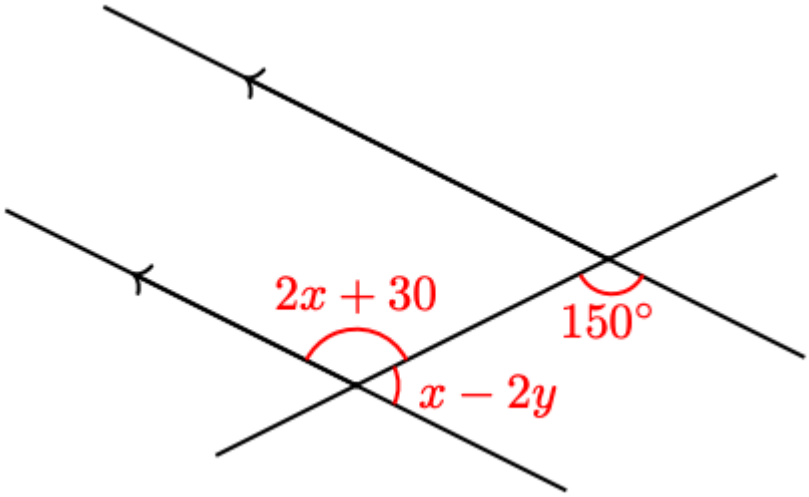
b. What is the area of the circle? Round your answer to the nearest tenth.

c. What is the area of the blue shaded region? Round your answer to the nearest tenth.

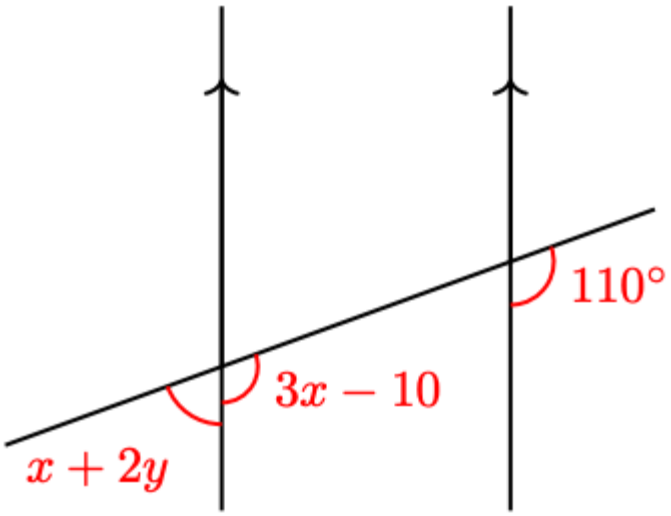
97. State the condition(s) for which α is an acute angle, and for which α is an obtuse angle. In each case, give the complement and the supplement of α.

For the following exercises, arrows on a pair of lines indicate that they are parallel. Find x and y.

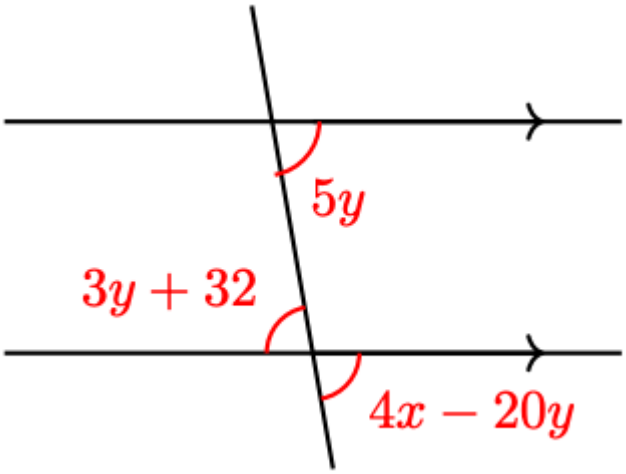
98.



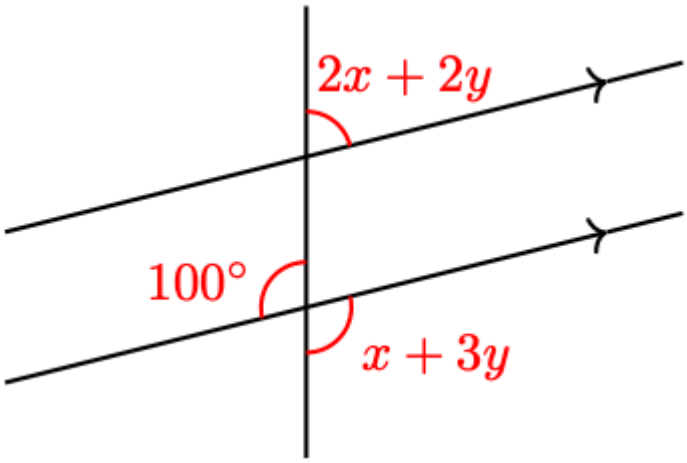
9.



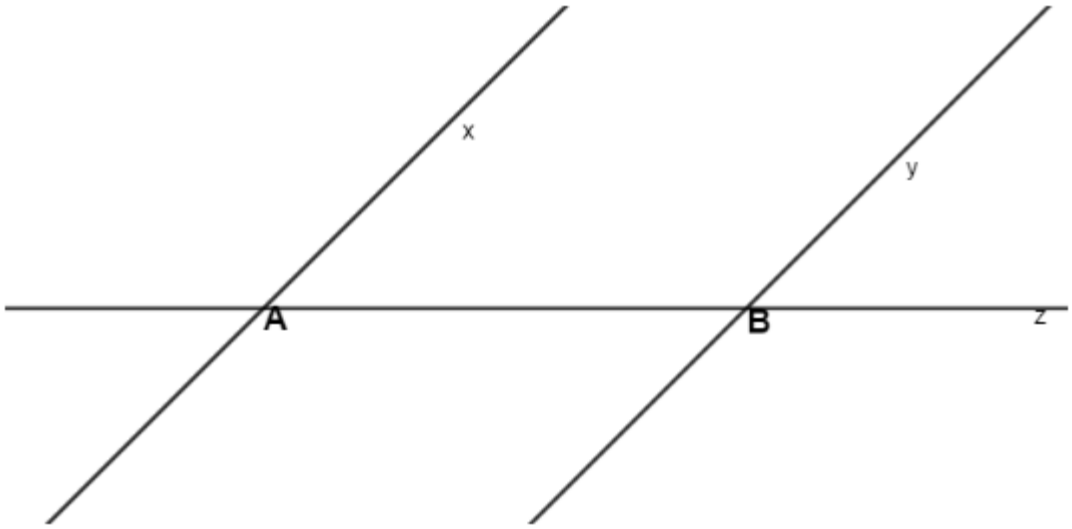
10.



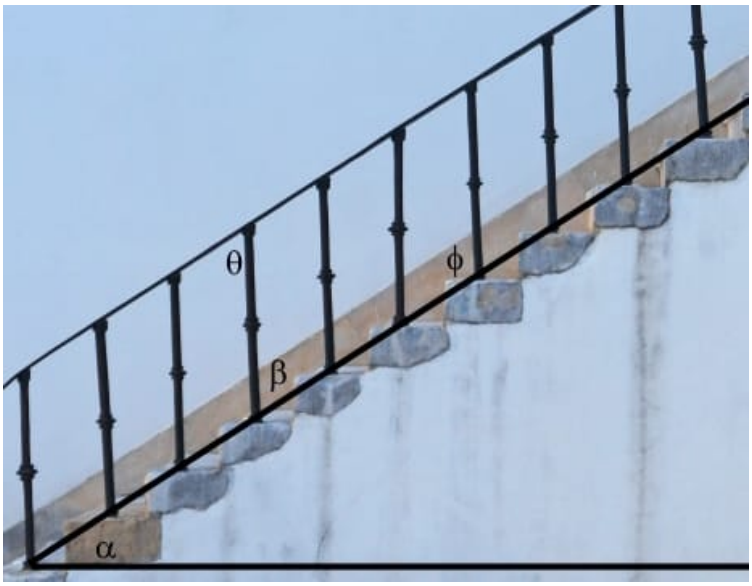
11.



12. (MyOpenMath) The lines x and y are parallel and cut by a transversal z. Given that $\angle A = k^2 + 22k$ and $\angle B = 14k + 48$, find k.



13. **Climbing Stairs.** In each of the following problems, refer to the following image of a staircase and rails. Assume the rails are perpendicular to the ground, they are parallel to each other, and that the top rail (handrail) is parallel to the bottom rail (the black line along the steps).



a. What is the relationship between α and β?

b. What is the relationship between β and θ?

c. What is the relationship between β and ϕ?

d. What is the relationship between θ and ϕ?

e. Find α if $\beta = 48^\circ$.

f. Find θ if $\alpha = 30^\circ$.

Applications

14. **Emergency Light.** An ambulance has a rotating light on its roof. The light rotates through one complete revolution every 2 seconds. How long does it take the light to rotate through 90°?

15. **Rotation of the Earth (MyOpenMath).** The Earth goes through a complete circuit around the sun in approximately 365.25 days. Through how many degrees does the Earth move in one week?

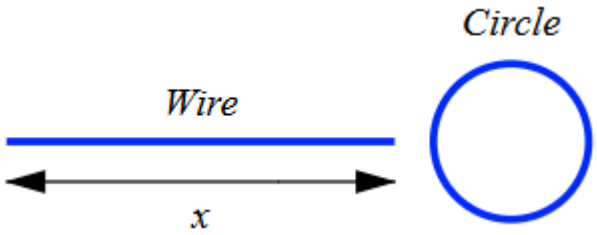
16. **Earth's Rotation About the Sun.** The Earth takes 24 hours to make a complete rotation on its axis. If your math class is 2 hours and 20 minutes, through how many degrees does the Earth turn during your math class?

17. **Circumference of the Earth.** The radius of the Earth is approximately 3,960 miles. Using this approximation, find the circumference of the Earth.

18. **Earth's Rotation About the Sun.** The Earth has a nearly circular orbit about the sun. The radius of this near-circular orbit is approximately 93 million miles. Find the distance the Earth travels when it completes one full orbit about the sun.

Challenge Problems

19. (MyOpenMath) A straight piece of wire, x centimeters long, is bent into a perfect circle. Express the area of the circle, A, in terms of x.



Answers

Skills Refresher

1. 9
2. 2
3. -3
4. -2
5. $x=-3, y=-1$
6. $x=6, y=-5$

This page titled 4.1.2: Homework is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.2: Triangles

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Triangle Basics

The study of Trigonometry inevitably involves understanding everything we possibly can about **triangles**.

Definition: Triangle

When three line segments bound a portion of the plane, the resulting shape is called a **triangle** (also known as a **three-sided polygon**). The line segments are called the **sides** of the triangle. A point where two sides meet is called a **vertex** of the triangle. Finally, the angle formed in the triangle's interior where two sides meet is called an **angle** of the triangle.

In Figure $4.2.1$, the sides are denoted $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. The vertices (plural of vertex) are A, B, and C. The angles are denoted $\angle A$, $\angle B$, and $\angle C$. Using the notation $\triangle ABC$, we refer to the triangle by its vertices.

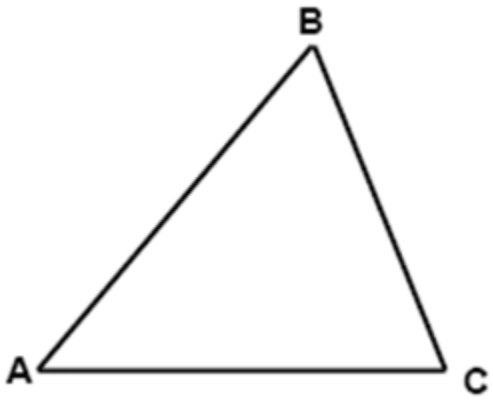


Figure $4.2.1$: $\triangle ABC$

If one tears off the corners of any triangle and lines them up, as shown in Figure $4.2.2$, they will always form a straight angle.

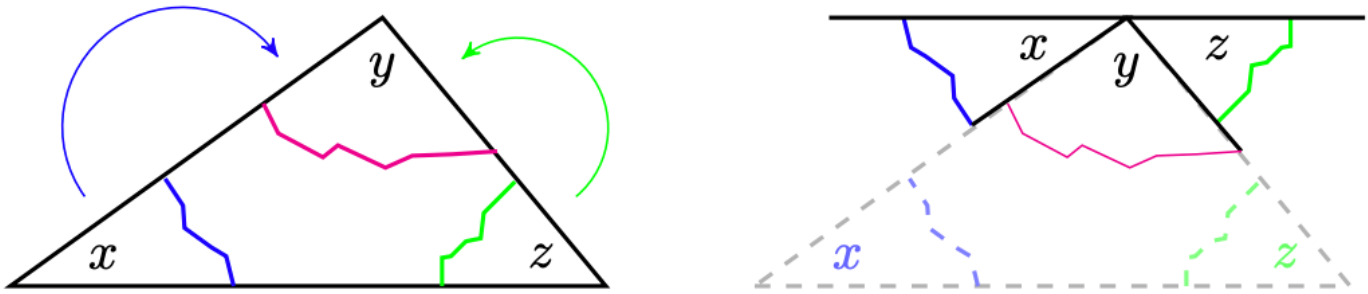


Figure 4.2.2**:** *Tearing corners from a triangle to form a straight angle*

Theorem: Triangle Sum

The sum of the interior angles in a triangle is 180°.

Proof available at the end of this section.

Example 4.2.1

Two of the angles in the triangle shown below are 25° and 115°. Find the third angle.

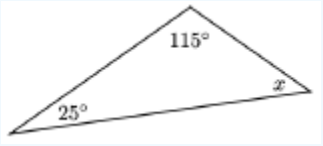


Figure 4.2.3

Solution available at the end of this section.

What do Arithmetic and ~~Arithmetic~~ mean?

To get you used to the Mathematical Mantra, every once in a while I will include the "thought processes" during solutions. Just to review, we perform Mathematics in the order we learned it—Arithmetic, Algebra, Trigonometry, ... At each step during a "mechanical" solution process, we should pause and ask ourselves if there is some simple Arithmetic to be done. If not, we move on to any Algebra that can be done. If there is no Algebra to be done, we then move on to Trigonometry.

Thus,

- "$\text{Arithmetic} \implies$ " in a solution means that there is some Arithmetic we could do to clean up the previous expression.
- "$\cancel{\text{Arithmetic}} \to \text{Algebra} \implies$ " means there isn't any Arithmetic but there *is* some Algebra we could use to clean up the previous expression or equation.
- "$\cancel{\text{Arithmetic}} \to \cancel{\text{Algebra}} \to \text{Trigonometry} \implies$ " means there isn't any Arithmetic nor is there any Algebra that can be done to clean up the previous step, so we need to start looking at doing something from Trigonometry.

Checkpoint 4.2.1

Find all the unknown angles in each figure.

a.

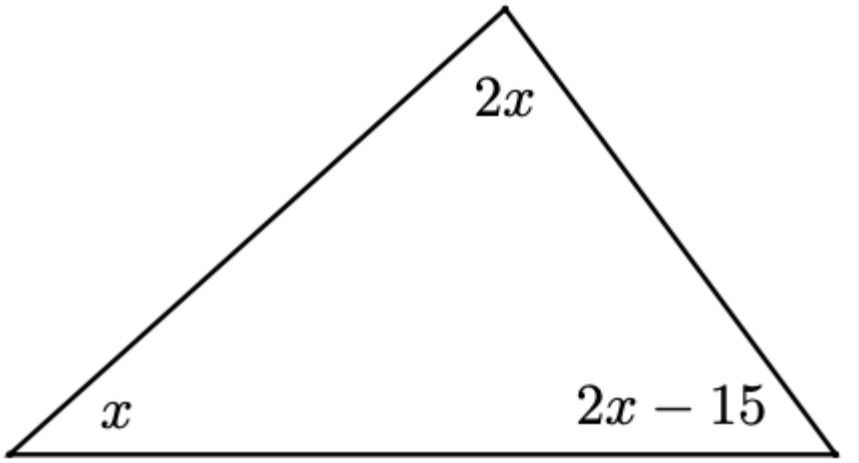


Figure 4.2.4

b.

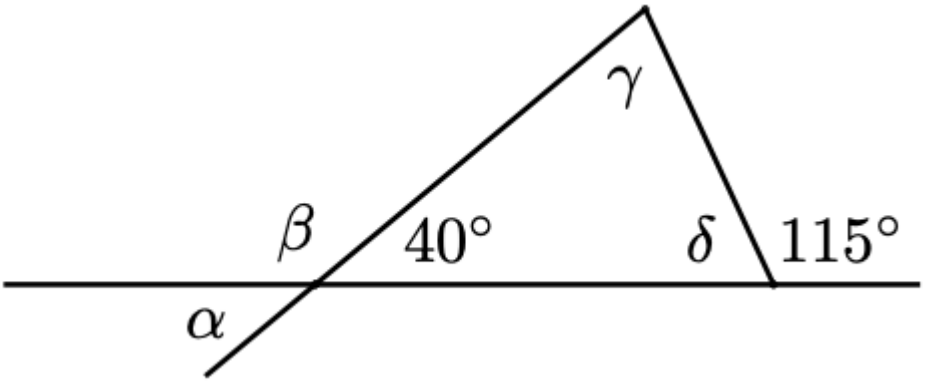
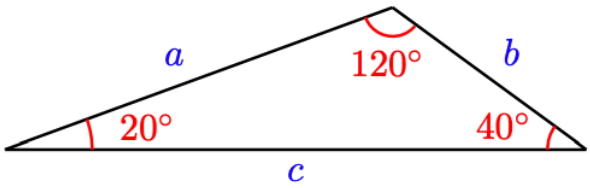


Figure 4.2.5

Answer available at the end of this section.

In any triangle, the longest side is always opposite the largest angle, and the shortest side is opposite the smallest, as illustrated in Figure 4.2.6. Although obvious to most people, this fact will become invaluable as we move through Trigonometry.



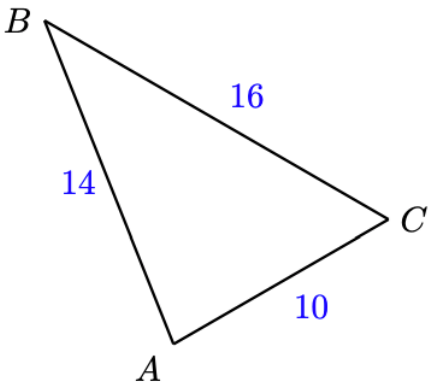


Figure 4.2.6

Notation: Angles and Sides of a Triangle

It is common (though not necessary) to label the angles of a triangle with capital letters and the side opposite each angle with the corresponding lowercase letter, as shown in Figure 4.2.7. We will follow this practice unless indicated otherwise.

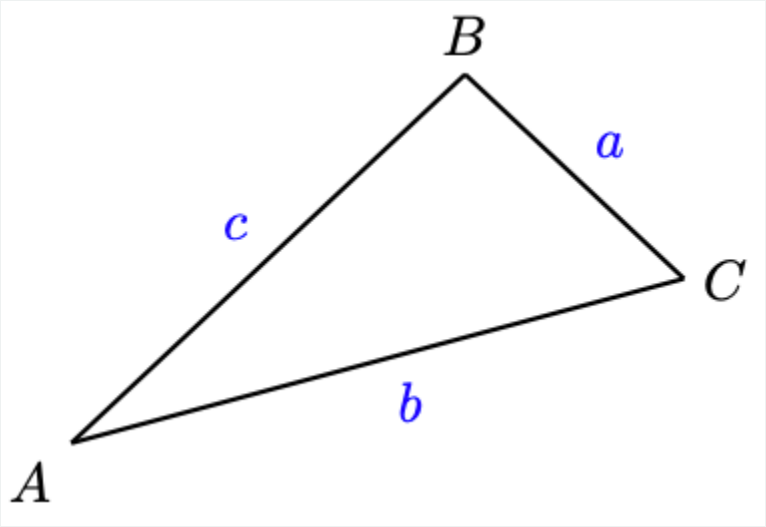


Figure 4.2.7

Example 4.2.2

In $\triangle FGH$, $\angle F = 48^\circ$, and $\angle G$ is obtuse. Side f is 6 feet long. What can be concluded about the other sides?

Solution available at the end of this section.

Checkpoint 4.2.2

In $\triangle RST$, $\angle S$ is 72°. The remaining two angles have the same size. Which side is longer, s or t?

Answer available at the end of this section.

The Triangle Inequality

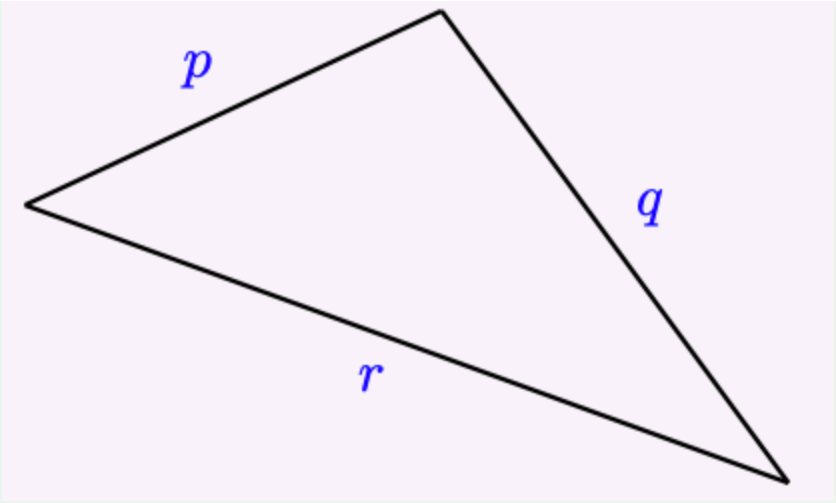
The sum of the lengths of any two sides of a triangle must be greater than the third side; otherwise, the two sides will not meet to form a triangle. This fact is known as the **triangle inequality**. The theorem is presented without proof (however, you can always find a suitable proof online).

Theorem: Triangle Inequality

In any triangle, we must have that

$$p + q > r,$$

where p, q, and r are the lengths of the sides of the triangle (see the figure below).



That is, the sum of the lengths of any two sides of a triangle is greater than the length of the other side.

We cannot use the triangle inequality to find the *exact* lengths of a triangle's sides, but we can find the largest and smallest possible values for the length.

Example 4.2.3

Two sides of a triangle have lengths 7 inches and 10 inches, as shown below. What can be said about the length of the third side?

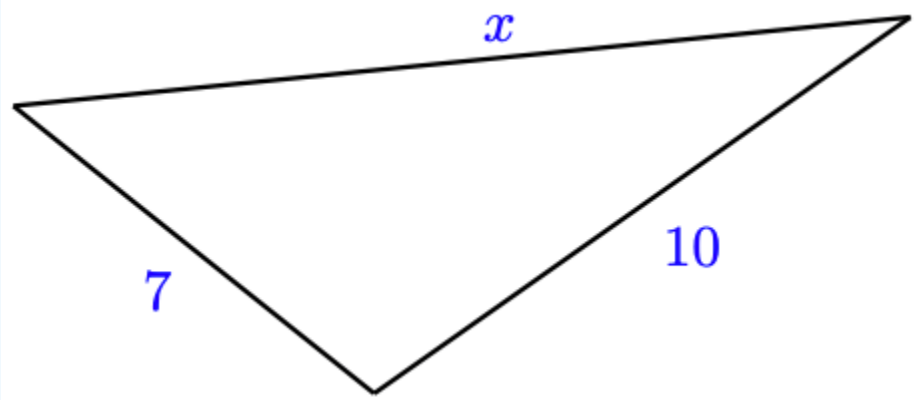


Figure 4.2.8

Solution available at the end of this section.

Interactive Element: The Triangle Inequality

The following Interactive Element might help you visualize why the third side of a triangle cannot be too long or too short.

Interact: Drag the large red vertex left and right to change the length of the third side.

Observation: The two fixed sides form a triangle only while the third side stays between about 3 and 17; beyond either limit, the two sides can no longer reach each other.

Checkpoint 4.2.3

Can a triangle be made with three wooden sticks of lengths 14 feet, 26 feet, and 10 feet? Sketch a picture, and explain why or why not.

Answer available at the end of this section.

Types of Triangles

There are three types of triangles classified by whether they have three equal side lengths, two equal side lengths, or no side lengths that are equal. These are called the equilateral, isosceles, and scalene triangles, respectively.

Equilateral Triangles

Definition: Equilateral Triangle

A triangle in which all three sides have the same length is called an **equilateral triangle**.

Since all three sides of an equilateral triangle have the same length, there is no smallest side for which the smallest angle will be opposite, and there is no largest side for which the largest angle will be opposite. This gives us the following theorem.

Theorem: Angles of an Equilateral Triangle

A triangle is equilateral if and only if it is equiangular, in which case each interior angle measures 60°.

Example 4.2.4

All three sides of a triangle are 4 feet long. Find the angles.

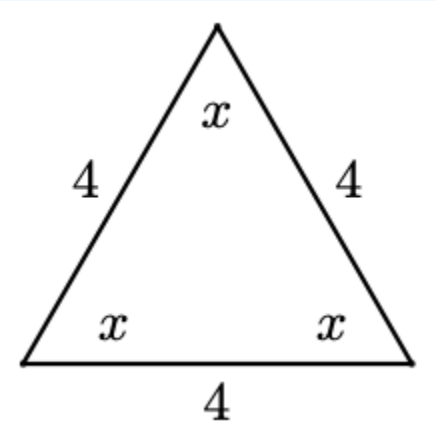


Figure 4.2.9

Solution available at the end of this section.

Checkpoint 4.2.4

Find x, y, and z in the triangle.

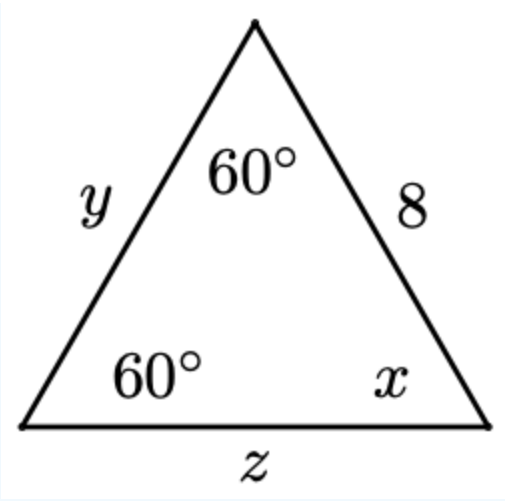


Figure 4.2.10

Answer available at the end of this section.

Isosceles Triangles

Definition: Isosceles Triangle

A triangle in which at least two sides are of equal length is called an **isosceles triangle**. The angle between the equal sides is called the **vertex angle**, and the other two angles are called the **base angles**.

Theorem: Angles of an Isosceles Triangle

The base angles of an isosceles triangle are equal.

In Example 4.2.5, we introduce another common notation used throughout Geometry—the **tick mark**. We use tick marks on line segments to denote they have equal length. Similarly, tick marks on angles denote that the angles have the same measure.

Example 4.2.5

Explain why $\alpha = \beta$ in the triangle below.

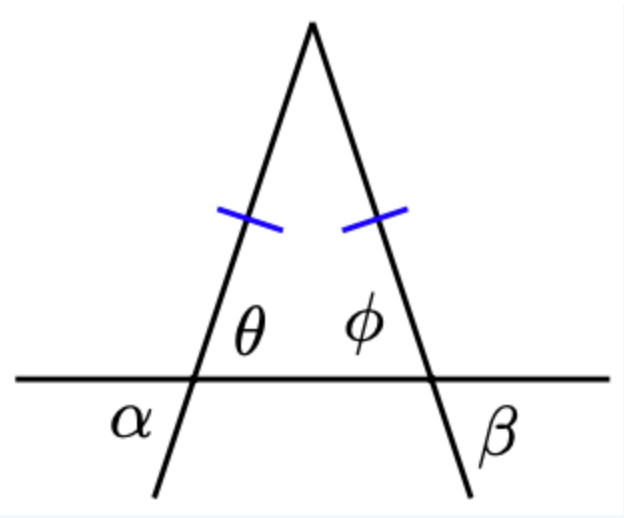


Figure 4.2.11

Solution available at the end of this section.

Example 4.2.6

Find x and y in the triangle.

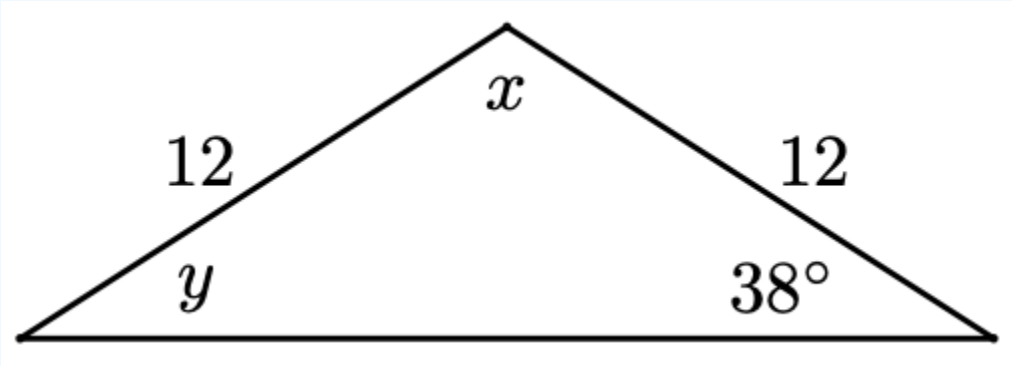


Figure 4.2.12

Solution available at the end of this section.

Checkpoint 4.2.6

Find x and y.

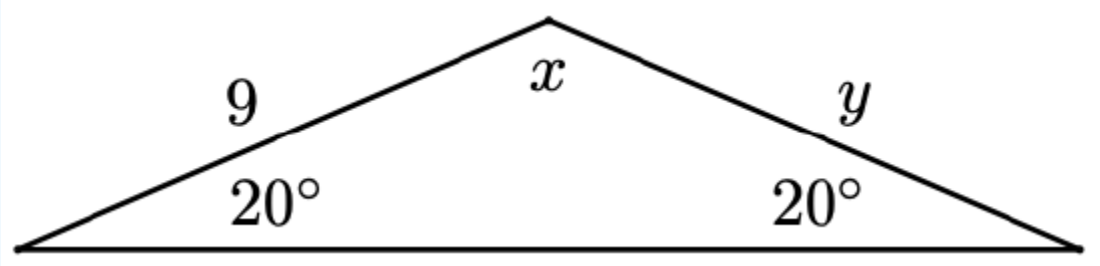


Figure 4.2.13

Answer available at the end of this section.

Scalene Triangles

Finally, we have the **scalene** triangles.

Definition: Scalene Triangle

A triangle in which none of the sides have equal length (and, therefore, none of the angles are equal) is called a **scalene triangle**.

Figure $4.2.14$ shows a general scalene triangle. The different "hash marks" on the sides indicate the side lengths are different. The same is true for the angles.

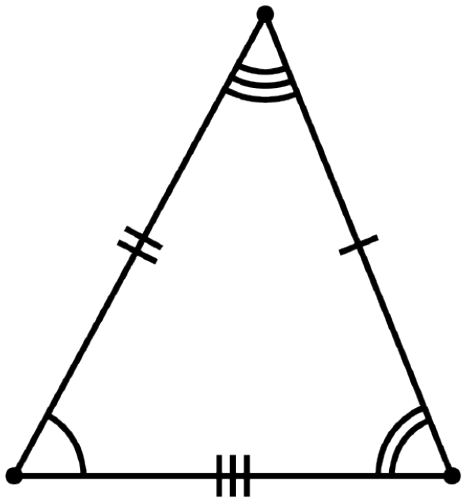


Figure $4.2.14$: *A general scalene triangle*

Scalene triangles become important in this course when we learn the Law of Sines and the Law of Cosines.

Classification of Triangles by Largest Angle

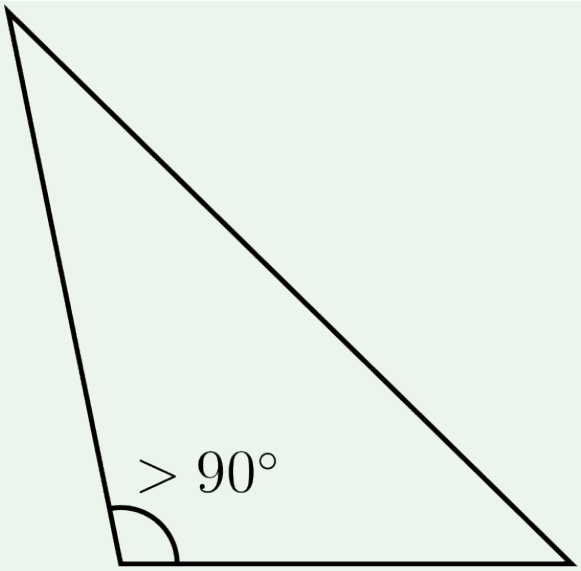
Most people are not likely to refer to a triangle by its type (equilateral, isosceles, or scalene); however, knowing the features of each type of triangle is important in Trigonometry. Another classification system for triangles, and one that will be referenced constantly in Trigonometry, is based on the size of the triangle's largest angle. Three classifications exist: right, obtuse, and acute triangles.

Obtuse Triangles

We start with the class of triangles having their largest angle greater than 90°.

Definition: Obtuse Triangle

An **obtuse triangle** is a triangle that has one interior angle measuring more than 90°.



Recall that an obtuse angle has measure strictly between $90°$ and $180°$ (meaning that we exclude $90°$ and $180°$). Since the sum of the interior angles of a triangle must be $180°$, and one of the angles in an obtuse triangle is greater than $90°$, the two smaller angles in an obtuse triangle must sum to strictly less than $90°$.

Theorem: Angles of an Obtuse Triangle

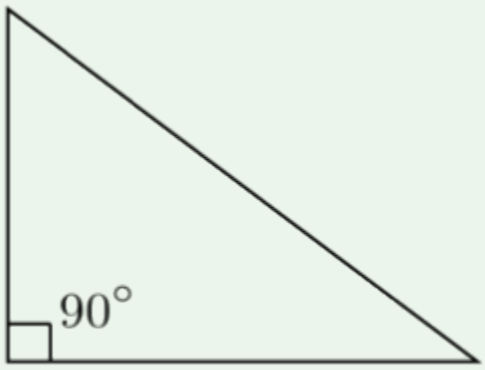
In an obtuse triangle, one angle must be greater than $90°$, and the remaining two angles must each be less than $90°$.

Right Triangles

Reducing the largest possible angle to *exactly* $90°$, we arrive at arguably the most essential triangle classification in Trigonometry—the **right triangles**.

Definition: Right Triangle

A **right triangle** is a triangle that has one interior angle measuring exactly $90°$. The side opposite the right angle is the **hypotenuse**. The two remaining sides are the **legs**.



Example 4.2.7

One of the smaller angles of a right triangle is $34°$. What is the third angle?

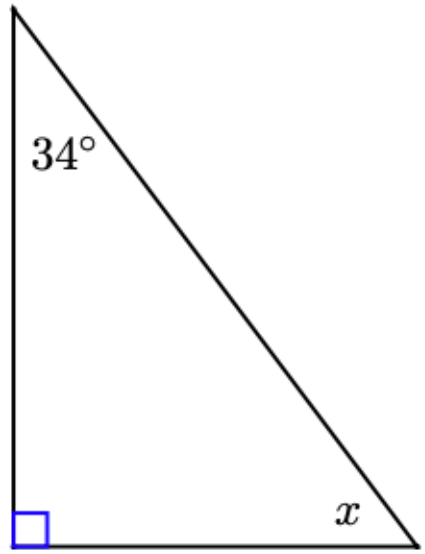


Figure 4.2.15

Solution available at the end of this section.

Checkpoint 4.2.7

Two angles of a triangle are 35° and 45°. Can it be a right triangle?

Answer available at the end of this section.

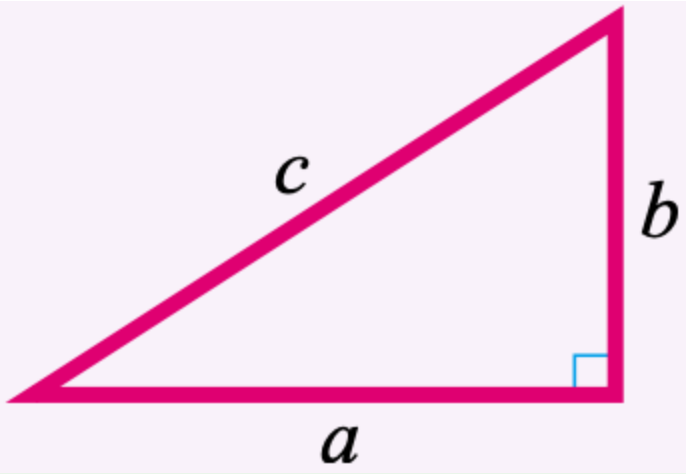
Right Triangles: The Pythagorean Theorem

Arguably, the most important theorem in Trigonometry is the **Pythagorean Theorem**, mainly because without it, Trigonometry would not exist.

Theorem: Pythagorean Theorem

In a right triangle, if c is the length of the hypotenuse, and the lengths of the two legs are denoted by a and b (as shown in the figure below), then

$$a^2 + b^2 = c^2.$$



Interactive Element: The Pythagorean Theorem as Areas

The following Interactive Element might help you visualize what $a^2 + b^2 = c^2$ means in terms of area.

Interact: Drag either red endpoint to change the lengths of the two legs.

Observation: The squares on the two legs always add up in area to the square on the hypotenuse, no matter how the right triangle is reshaped — that is $a^2 + b^2 = c^2$.

The proof of the Pythagorean Theorem is left as a guided exercise in the homework.

Example 4.2.8

A 25-foot ladder is placed against a wall so that its foot is 7 feet from the base of the wall. How far up the wall does the ladder reach?

Solution available at the end of this section.

Checkpoint 4.2.8

A baseball diamond is a square with sides that are 90 feet long. When the catcher sees a runner on first trying to steal second, he throws the ball to the second baseman. Find the straight-line distance from home plate to second base.

Answer available at the end of this section.

Caution

The Pythagorean Theorem applies only to right triangles, and c must always be the hypotenuse—the side opposite the right angle, which is also the longest side.

Most people do not realize that the Pythagorean Theorem is a biconditional. That is, the converse of the theorem is also true. If the sides of a triangle satisfy the relationship $a^2 + b^2 = c^2$, then the triangle must be a right triangle with hypotenuse c. We can use this fact to test whether a given triangle has a right angle.

Example 4.2.9

Mike is paving a patio in his backyard and would like to know if the corner at C is a right angle. He measures 20 cm along one side from the corner and 48 cm along the other, placing pegs P and Q at each position, as shown below. The line joining those two pegs is 52 cm long. Is the corner a right angle?

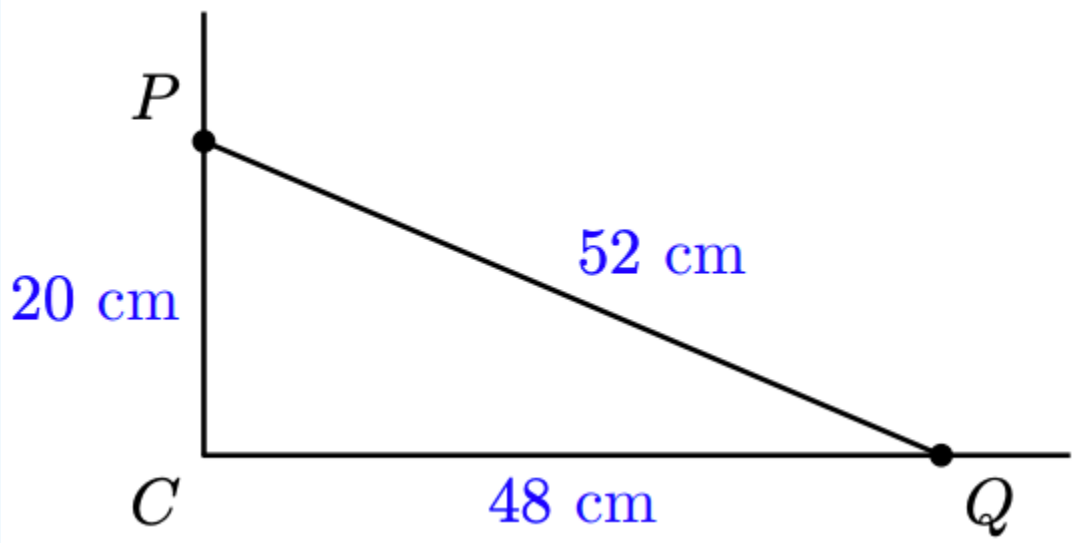


Figure 4.2.17

Solution available at the end of this section.

Checkpoint 4.2.9

The sides of a triangle measure 15, 25, and 30 inches long. Is the triangle a right triangle?

Answer available at the end of this section.

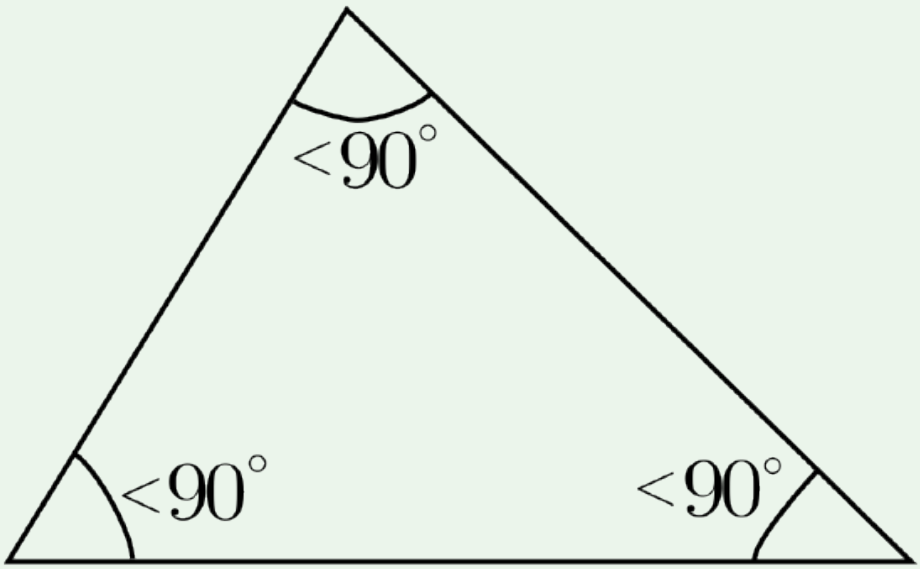
The Pythagorean Theorem relates the sides of *right* triangles. However, for information about the sides of other triangles, the best we can do (without Trigonometry) is the triangle inequality. Moreover, the Pythagorean Theorem doesn't help us find the *angles* in a triangle—that comes later.

Acute Triangles

Finally, we reduce the largest possible angle within a triangle to strictly less than 90° and obtain the **acute triangles**.

Definition: Acute Triangle

An **acute triangle** is a triangle whose three interior angles each measure less than 90°.



Special Triangles

We finally arrive at what could be considered the most essential triangles in elementary Trigonometry—the **special triangles**. These triangles are revered in Trigonometry because they allow the learner (read as "you") to perform computations without needing a calculator. This, in turn, allows you to focus on concepts rather than computations.

In order to derive one of the theorems we are about to introduce, we need a quick definition.

Definition: Altitude of a Triangle

The **altitude** of a triangle is a line segment drawn from a vertex of the triangle perpendicular to the opposite side (or the line containing the opposite side).

The altitude of a triangle introduces a right angle, as the following theorem states.

Theorem: Altitudes and Right Angles

The altitude of a triangle represents the shortest distance from the vertex to the opposite side and forms a 90° angle with that side.

The 30°-60°-90° Triangle

The first of our two special triangles is the 30°-60°-90° **triangle**.

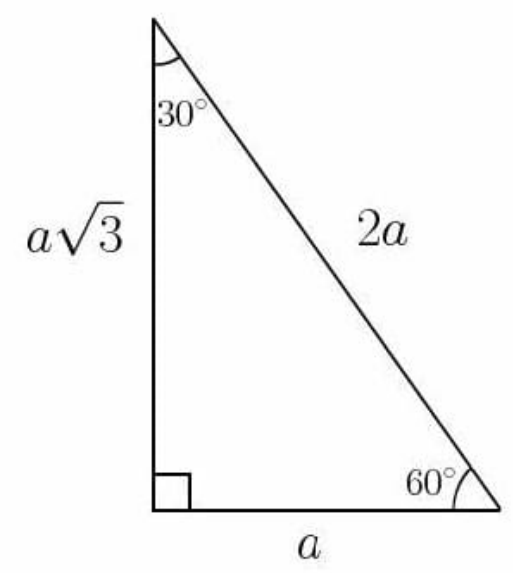
Definition: 30°-60°-90° Triangle

A right triangle in which one angle is 30° and another angle is 60° is called a 30°-60°-90° **triangle**.

Working with this triangle allows us to know the length of all three sides if only given the length of one side.

Theorem: Side Relationships for a $30°$-$60°$-$90°$ Triangle

In any right triangle in which the two acute angles are $30°$ and $60°$, the hypotenuse is always twice the length of the shortest side (the side opposite the $30°$ angle), and the remaining side (opposite the $60°$ angle) is always $\sqrt{3}$ times the shortest side.



Proof available at the end of this section.

Before we go any further, notice in the proof of this theorem the convention of writing $x = a\sqrt{3}$ instead of $x = \sqrt{3}a$. This is purposeful. The latter often leads the reader to think that the a is "under" the square root. It's good form always to write the radical at the end of a term to avoid this visual ambiguity.

Example $4.2.10$

Matterhorn Chocolate bars are sold in boxes shaped like triangular prisms. The two triangular ends of a box are equilateral triangles with 8-centimeter sides. When stacking the boxes on a shelf, the second level will be a triangle's altitude above the bottom level. Find the exact length of the triangle's altitude, h.

Solution available at the end of this section.

Checkpoint $4.2.10$

A ladder is leaning against a house so that the top is 12 feet above the ground, and the bottom makes a $60°$ angle with the ground. Assuming the wall of the house is perpendicular to the ground (and the ground is level), how long is the ladder? State your answer in exact form and round to two decimal places.

Answer available at the end of this section.

Note that the exact form of the answer in Checkpoint $4.2.10$ was left as $\frac{24}{\sqrt{3}}$. While it is true that you could rationalize the denominator to get

$$\frac{24}{\sqrt{3}} = \frac{24}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{24\sqrt{3}}{3} = 8\sqrt{3},$$

we will commonly leave radicals in the denominators of purely numeric expressions.

The $45°$-$45°$-$90°$ Triangle

The second of our two special triangles is the $45°$-$45°$-$90°$ **triangle**. This is also known as an **isosceles right triangle**.

Definition: Isosceles Right Triangle

An isosceles triangle in which the vertex angle is $90°$ is called an **isosceles right triangle**.

Since the right angle takes $90°$, the remaining two angles must share the remaining $90°$ evenly. That is, the remaining two angles must be $45°$ each.

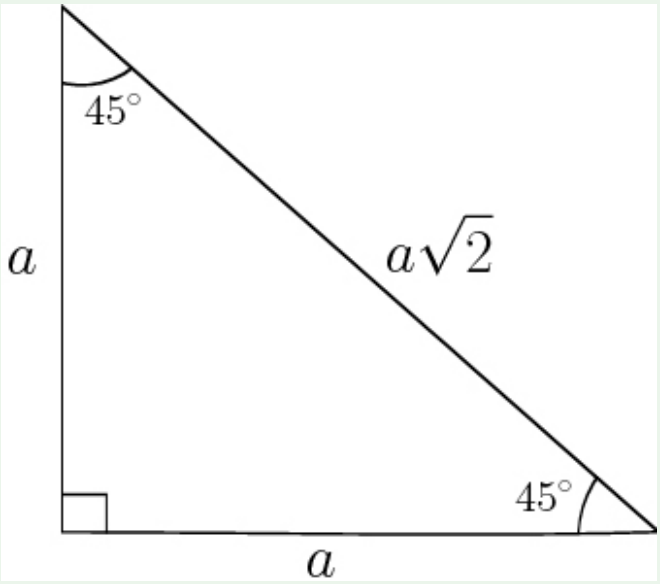
Definition: $45°$-$45°$-$90°$ Triangle

Since an isosceles right triangle has two angles that are $45°$, it is commonly called a $45°$-$45°$-$90°$ **triangle**.

Just as with our first special triangle, knowing one side of a $45°$-$45°$-$90°$ triangle allows us to easily compute the lengths of the remaining sides.

Theorem: Side Relationships for a $45°$-$45°$-$90°$ Triangle

In any right triangle in which the two acute angles are $45°$, the hypotenuse is always $\sqrt{2}$ times the side length.



Proof available at the end of this section.

Example 4.2.11

You are hired to build a truss to support a balcony. Figure $4.2.19$ shows the design plans you have submitted for the project. What is the length of the beam needed from point A to point D?

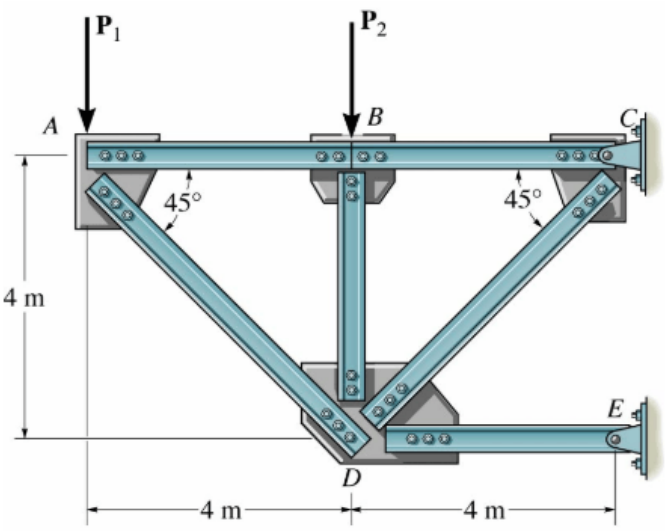


Figure 4.2.19

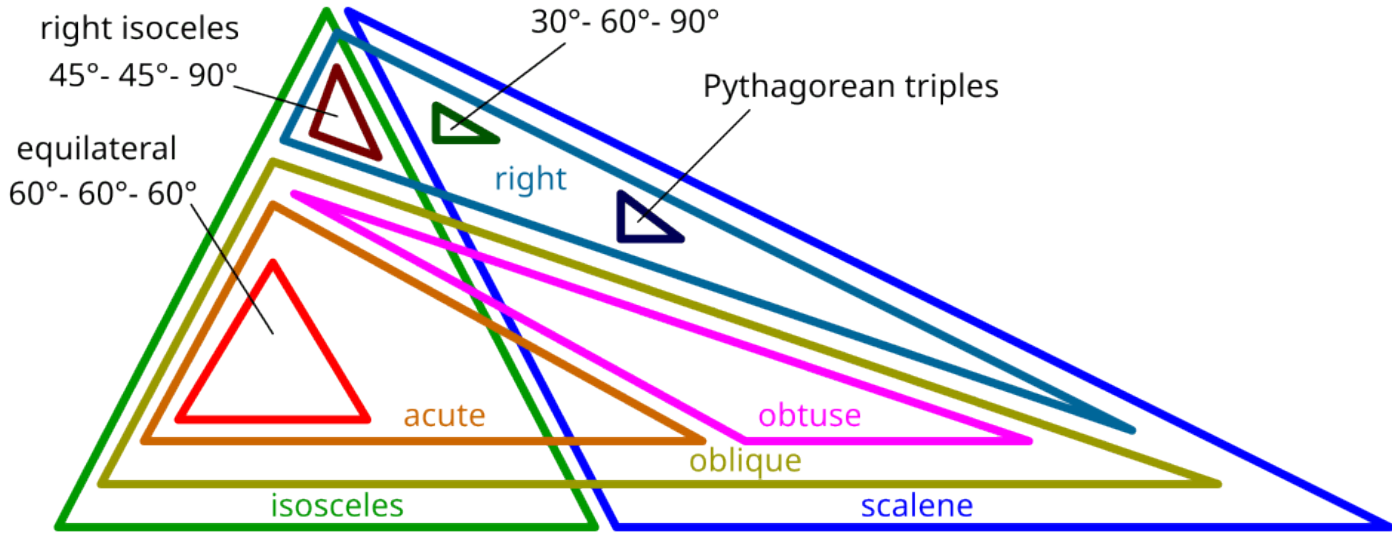
Solution available at the end of this section.

Checkpoint 4.2.11

Suppose, instead of knowing the 4-meter lengths in Example 4.2.11, we knew that the length of the beam from A to D was 12 meters. How long is the beam from B to D?

Answer available at the end of this section.

Now that we have successfully covered all of the types of triangles, we have the ability to appreciate the Euler diagram of the types of triangles (don't worry—you are not responsible for memorizing this).



Area of a Triangle

Our final review topic for triangles in this section is something necessary at multiple points throughout Trigonometry and

Calculus—the area of a triangle.

Theorem: Area of a Triangle

Let b be the length of one side of a triangle and h be the altitude height from that side to its opposite vertex. Then the area of the triangle is given by

$$A = \frac{1}{2}bh.$$

Proof available at the end of this section.

Interactive Element: Base, Height, and Area

The following Interactive Element might help you visualize why a triangle's area depends only on its base and height.

Interact: Drag the red apex left and right along the dashed rail.

Observation: The triangle changes shape, but with the base and height unchanged, the area $\frac{1}{2}bh = \frac{1}{2}(8)(4) = 16$ stays the same.

Example 4.2.12

Compute the area of the triangle in Figure 4.2.20.

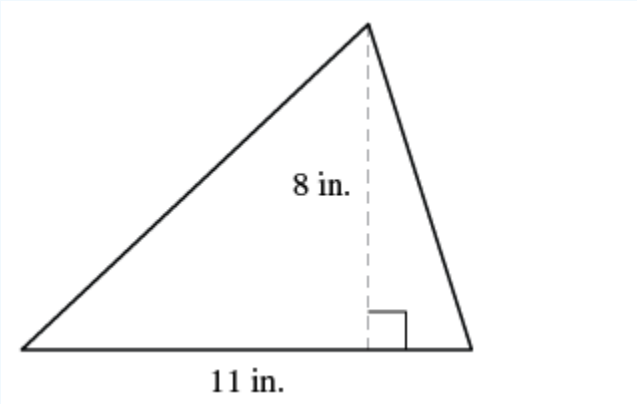


Figure 4.2.20

Solution available at the end of this section.

Checkpoint 4.2.12

Compute the area of the triangle in Figure 4.2.21 if we know that $c = 5.24$ and $h = 1.25$.

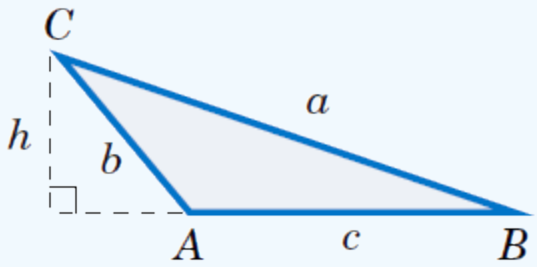


Figure 4.2.21

Answer available at the end of this section.

Time-Sensitive Advice: HawkConnect

What is HawkConnect?

In the past, navigating through all the student resources (EOPS, TRIO, MI CASA, etc.) required you to go to individual sites and apply individually for each program. HawkConnect now unifies the entire process into a single, guided application process.

To learn more about HawkConnect and to apply, go to the CRC Support Services website.

Success in Mathematics: Show Your Work

Many mathematics exams include the statement, "Show Your Work" (if it's not explicitly written, then it is implied). The exact meaning of these words can change from one professor to another; however, they usually mean that you need to show more than just the answer to a problem. You need to explain or show how you found the answer. Your responsibility is to learn what "Show Your Work" means to your instructor.

Why do you need to show your work?

Your *authentic* work (e.g., without use of A.I., tutors, or any other external help) demonstrates what you know. It shows what you understand and what you do not understand. By looking at your work and the work of others, an instructor can help you improve your understanding.

Everyone makes mistakes. To learn from these mistakes, you need to be able to find them. If you show your work, you can go back and find mistakes. This helps you identify misunderstandings and bad habits. Without your work, all you can do is start over.

Finally, although you may be able to do a problem "in your head," this will not always be possible. The problems get more challenging and involve more detail as we move forward. Showing your work now develops the habits and procedures you will need to do math in the future.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. A student did a homework exercise. She checked her answer with the back of the book. It was wrong. So, she erased her work and started over. Why should she not erase her work?
2. A student did a homework exercise. He checked his answer with the back of the book. It was wrong. He marked the problem on his homework with an asterisk. He went on with the assignment. He never went back to the problem. Why is it essential for this student to finish the problem?

If you are unsure about the answers to either of these questions, feel free to ask your instructor - that's what they are there for!

Answers

Section Preview

Triangles, being the focus of Trigonometry, deserve their own full section. Focus here is on types of triangles, the Pythagorean Theorem, the special triangles, and the area of a triangle (not requiring trigonometric functions). Many exercises involving the 30°-60°-90° and 45°-45°-90° triangles that would normally be presented in Right Triangle Trigonometry are found in this section and its homework (but without reference to the trigonometric functions). Similar triangles are held off until next section.

Section Preview

- **Sets and Numbers**
  - Inequalities and Inequality Notation: For understanding the Triangle Inequality.
- **Simplifying Expressions**
  - Simplifying Products of Radical Expressions: For following the special triangle theorems and their proofs.

Section Preview

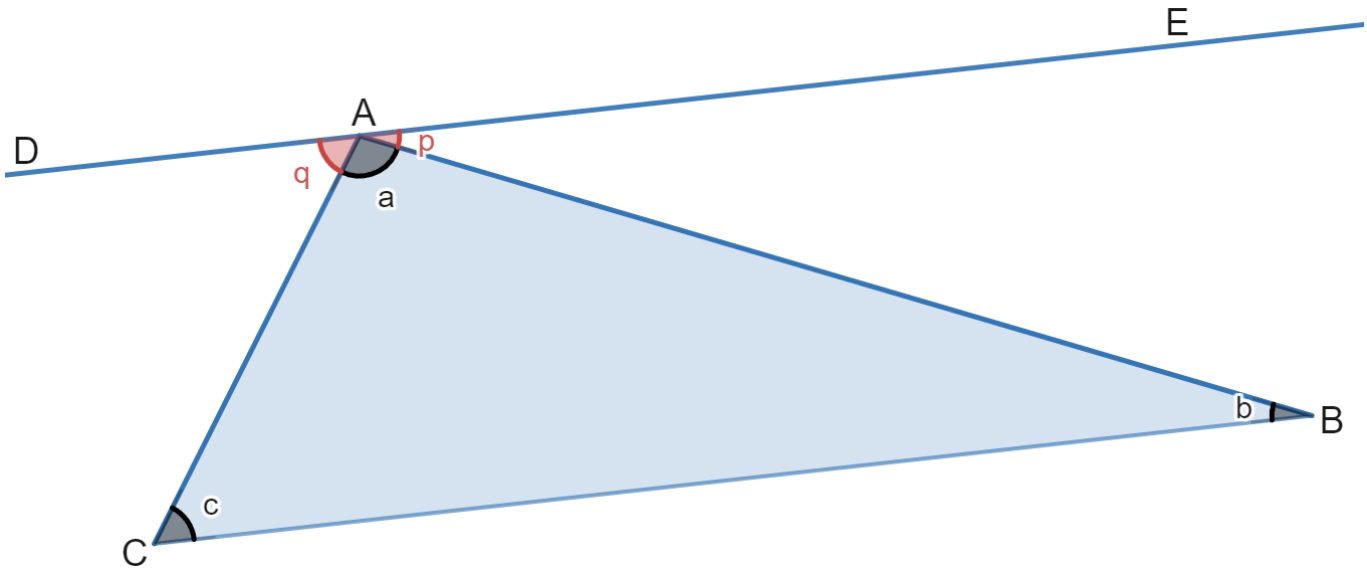
- **Arithmetic**
  - Mathematical Operations - Square Roots: For applying the Pythagorean Theorem numerically.
- **Geometry**
  - Volumes and Surface Areas of Common Shapes: For specific homework applications.
- **Solving Equations**
  - Solving Quadratic Equations Using Extraction of Roots: For solving $x^2 = k$.
- **Solving Inequalities**
  - Solving Linear Inequalities: For the Triangle Inequality problems.
  - Solving Compound Inequalities: For the Triangle Inequality problems.

Section Preview

- Find the missing angle within a triangle using the Triangle Sum.
- Use the Triangle Inequality to determine constraints on an unknown side length of a triangle.
- Use the Pythagorean Theorem to find the third side of a right triangle.
- Find the missing sides of a 30°-60°-90° or 45°-45°-90° triangle given one side.
- Solve real-world applications requiring the Pythagorean Theorem or the special triangles.

Theorem: Triangle Sum

Given the triangle, $\triangle ABC$, draw a line parallel to side $\overline{BC}$ going through vertex A (see the figure below).



Since $\angle q + \angle a + \angle p$ forms a straight angle,

$$\angle q + \angle a + \angle p = 180^\circ.$$

Moreover, the line segment $\overline{AC}$ is a transversal for the parallel lines $\overline{BC}$ and $\overline{DE}$. Therefore, $\angle c = \angle p$. By a similar argument, $\angle b = \angle q$. Hence,

$$\angle b + \angle a + \angle c = 180^\circ.$$

Thus, the sum of the angles of a triangle is 180°.

Example 4.2.1

To find the third angle, we write an equation.

$$\begin{array}{rcrcrl} & & x + 25 + 115 & = & 180 & \text{(Triangle Sum)} \\ \text{Arithmetic} & \Longrightarrow & x + 140 & = & 180 & \text{(add)} \\ \text{Arithmetic}\to\text{Algebra} & \Longrightarrow & x & = & 40 & \text{(subtract 140 from both sides)} \end{array}$$

The third angle is 40°.

Checkpoint 4.2.1

a. $x = 39^\circ, 2x = 78^\circ, 2x - 15 = 63^\circ$
b. $\alpha = 40^\circ, \beta = 140^\circ, \gamma = 75^\circ, \delta = 65^\circ$

Example 4.2.2

Because $\angle G$ is greater than 90°, we know that $\angle F + \angle G$ is greater than $90^\circ + 48^\circ = 138^\circ$, so $\angle H$ is less than $180^\circ - 138^\circ = 42^\circ$. Thus, $\angle H < \angle F < \angle G$, and consequently \(h6\) feet long.

Checkpoint 4.2.2

s is longer

Example 4.2.3

We let x represent the length of the third side of the triangle. By looking at each side in turn, we can apply the triangle inequality in three different ways to get\[7-3\) because x must be positive, but the other two inequalities do give us new information. The third side must be greater than 3 inches but less than 17 inches long.

Checkpoint 4.2.3

No, $10 + 14$ is not greater than 26.

Example 4.2.4

The triangle is equilateral, so all of its angles are equal. Thus

$$\begin{aligned} & & 3x &= 180 && \text{(Triangle Sum)} \\ \xRightarrow{\text{Arithmetic} \to \text{Algebra}} & & x &= 60 && \text{(divide both sides by 3)} \end{aligned}$$

Each of the angles is 60°.

Checkpoint 4.2.4

$x = 60^\circ, y = 8, z = 8$

Example 4.2.5

Because they are the base angles of an isosceles triangle, θ (theta) and ϕ (phi) are equal. Also, $\alpha = \theta$ because they are vertical angles, and similarly $\beta = \phi$. Therefore, $\alpha = \beta$ because they are equal to equal quantities.

Example 4.2.6

The triangle is isosceles, so the base angles are equal. Therefore, $y = 38^\circ$. To find the vertex angle, we solve

$$\begin{aligned} & x + 38 + 38 &= 180 && \text{(Triangle Sum)} \\ \xRightarrow{\text{Arithmetic}} & x + 76 &= 180 && \text{(add)} \\ \xRightarrow{\text{Arithmetic} \to \text{Algebra}} & x &= 104 && \text{(subtract 76 from both sides)} \end{aligned}$$

The vertex angle is 104°.

Checkpoint 4.2.6

$x = 140^\circ, y = 9$

Example 4.2.7

The sum of the two smaller angles in a right triangle is 90°. So

$$\begin{aligned} & x + 34 &= 90 \\ \xRightarrow{\text{Arithmetic} \to \text{Algebra}} & x &= 56 && \text{(subtract 34 from both sides)} \end{aligned}$$

The unknown angle must be 56°.

Checkpoint 4.2.7

No.

Example 4.2.8

We sketch the situation, as shown below, and label any known dimensions. We will call the unknown height h.

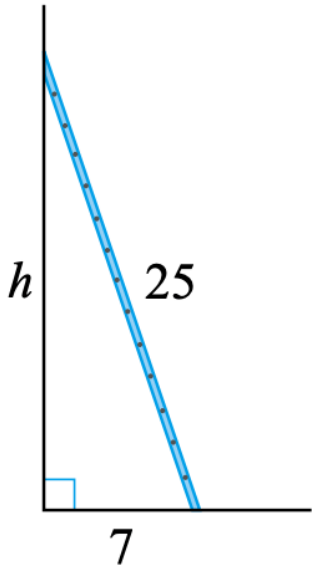


Figure 4.2.16

The ladder forms the hypotenuse of a right triangle, so we can apply the Pythagorean Theorem. Substituting 25 for c, 7 for b, and h for a, we get the following:

$$\begin{array}{rcrcll} & & a^2+b^2 & = & c^2 & \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \Longrightarrow & h^2+7^2 & = & 25^2 & \text{(substitution)} \\ \xrightarrow{\text{Arithmetic}} & \Longrightarrow & h^2+49 & = & 625 & \text{(apply powers)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \Longrightarrow & h^2 & = & 576 & \text{(subtract 49 from both sides)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \Longrightarrow & h & = & \pm\sqrt{576} & \text{(Extraction of Roots)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \Longrightarrow & h & = & \pm 24 & \text{(simplify the radical)} \end{array}$$

The height must be a positive number, so the solution -24 does not make sense for this problem. The ladder reaches 24 feet up the wall.

Checkpoint $4.2.8$

$90\sqrt{2} \approx 127.3$ feet

Example $4.2.9$

If it is a right triangle, its sides must satisfy $p^2+q^2=c^2$. We find

$$\begin{aligned} p^2+q^2 &= 20^2+48^2 \\ &= 400+2304 \\ &= 2704 \end{aligned}$$

and

$$\begin{aligned} c^2 &= 52^2 \\ &= 2704 \end{aligned}$$

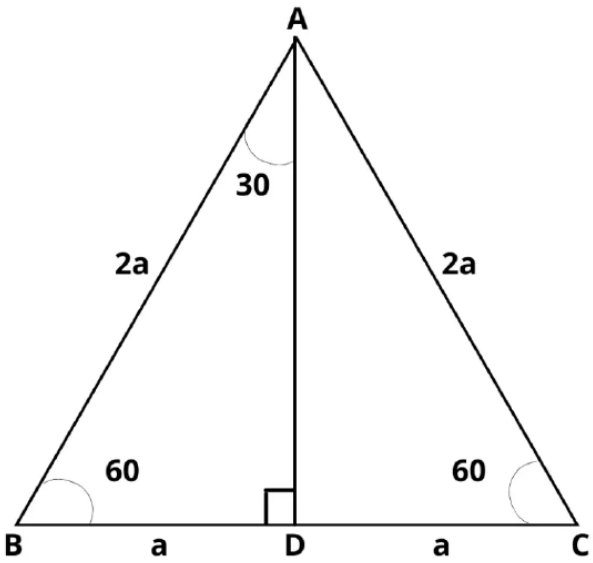
Therefore, yes, because $p^2+q^2=c^2$, the corner at C is a right angle.

Checkpoint $4.2.9$

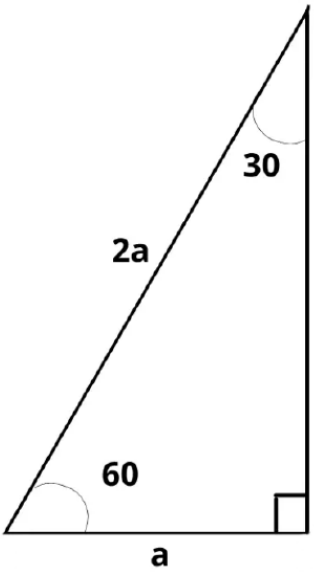
No

Theorem: Side Relationships for a 30°-60°-90° Triangle

Consider an equilateral triangle with side lengths $2a$, as shown in the figure below.



Since this triangle is equilateral, each angle is 60°. Drawing an altitude divides the triangle into two smaller triangles. Since the altitude is perpendicular to the base, each of these smaller triangles is a right triangle. Moreover, since one angle in each of these smaller right triangles is 60°, the remaining angle must be 30°. Furthermore, the side opposite the 30° angle must have length a (in the original equilateral triangle, the side opposite 60° had length $2a$, so it should make sense that the side opposite 30° is half of this). We now consider one of these smaller right triangles (see the figure below).



The missing side length, which we will call x for now, can be found using the Pythagorean Theorem.

$$\begin{aligned} & x^2 + a^2 &=& (2a)^2 \\ \Longrightarrow\quad & x^2 + a^2 &=& 4a^2 \\ \Longrightarrow\quad & x^2 &=& 3a^2 \\ \Longrightarrow\quad & x &=& a\sqrt{3} \end{aligned}$$

Example 4.2.10

The altitude divides the triangle into two $30^\circ - 60^\circ - 90^\circ$ right triangles, as shown in Figure 4.2.18 below.

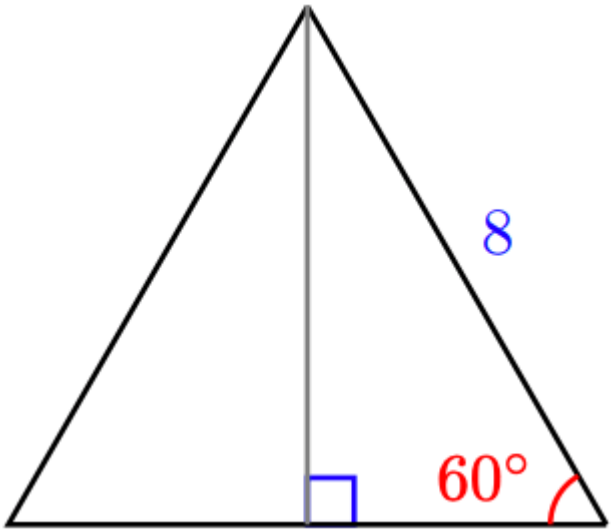


Figure 4.2.18

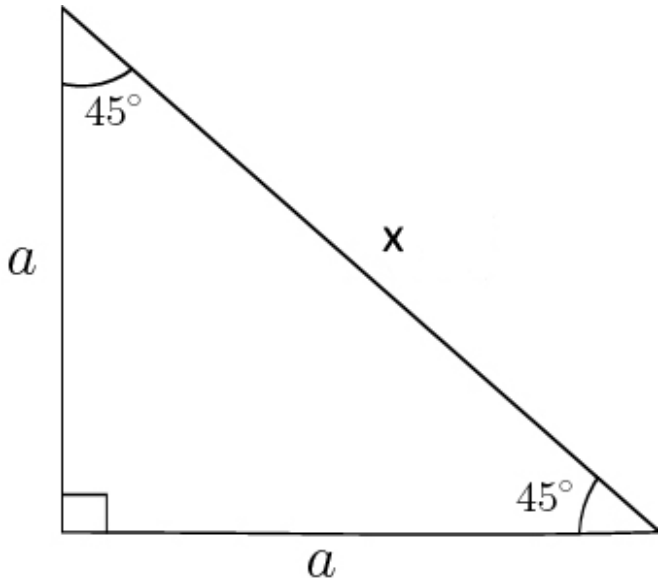
We see that the hypotenuse is $2a = 8$, which implies the shortest side (opposite the 30° angle) is $a = 4$. Hence, the side opposite the 60° angle is $a\sqrt{3} = 4\sqrt{3}$ ≈ 6.9282 centimeters long.

Checkpoint 4.2.10

$\frac{24}{\sqrt{3}} \approx 13.86$ feet

Theorem: Side Relationships for a 45°-45°-90° Triangle

The sides opposite the 45° angles in a 45°-45°-90° triangle must have the same length. Let a be the lengths of these sides and x be the length of the hypotenuse, as shown in the figure below.



Since this is a right triangle, we can use the Pythagorean Theorem to find the length of the hypotenuse in terms of a.

$$\begin{aligned} a^2 + a^2 &= x^2 \\ \Longrightarrow \quad 2a^2 &= x^2 \\ \Longrightarrow \quad a\sqrt{2} &= x \end{aligned}$$

Example 4.2.11

$\triangle ABD$ is a 45°-45°-90° triangle. Therefore, if the leg length (opposite the 45° angle) is $a = 4$ meters, then the beam between points A and D must measure $a\sqrt{2} = 4\sqrt{2} \approx 5.66$ meters.

Checkpoint 4.2.11

$\frac{12}{\sqrt{2}} \approx 8.49$ meters

Theorem: Area of a Triangle

To be done in homework.

Example 4.2.12

We are given the length of one leg of this triangle as 11 inches. Let's call this leg the base. Thus, $b = 11$. The altitude from the base to its opposite vertex has length 8 inches. Hence, $h = 8$. Therefore, the area of this triangle is

$$A = \frac{1}{2}bh = \frac{1}{2}(11)(8) = 44 \,\text{square inches}.$$

Checkpoint 4.2.12

3.275

This page titled 4.2: Triangles is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

4.2.1: Resources and Key Concepts

Resources

Videos

- Triangles, their types, and the Pythagorean Theorem
  - Right Triangle Trigonometry - Triangles
- Special Triangles
  - Right Triangle Trigonometry - Special Triangles in Trigonometry

Key Concepts

Definitions

- **Triangle:** A three-sided polygon formed by three line segments bounding a portion of a plane. The line segments are its sides, the points where sides meet are its vertices, and the interior angles are its angles.
- **Equilateral Triangle:** A triangle where all three sides have the same length.
- **Isosceles Triangle:** A triangle where at least two sides have equal length. The angle between the two equal sides is the vertex angle, and the other two angles are the base angles.
- **Scalene Triangle:** A triangle where no sides have equal length.
- **Obtuse Triangle:** A triangle that contains one obtuse angle (an angle greater than 90°).
- **Right Triangle:** A triangle that contains one right angle (an angle of exactly 90°).
- **Acute Triangle:** A triangle in which all three angles are acute (less than 90°).
- **30°-60°-90° Triangle:** A right triangle whose acute angles measure 30° and 60°.
- **Isosceles Right Triangle:** An isosceles triangle where the vertex angle is 90°.
- **45°-45°-90° Triangle:** A common name for an isosceles right triangle, since its two other angles must be 45°.

Theorems

- **Triangle Sum:** The sum of the interior angles in a triangle is 180°.
- **Triangle Inequality:** In any triangle with side lengths p, q, and r, the sum of the lengths of any two sides is greater than the length of the third side ($p + q > r$).
- **Angles of an Equilateral Triangle:** All the angles of an equilateral triangle are equal.
- **Angles of an Isosceles Triangle:** The base angles of an isosceles triangle are equal.
- **Angles of an Obtuse Triangle:** In an obtuse triangle, one angle must be greater than 90°, and the remaining two angles must each be less than 90°.
- **Pythagorean Theorem:** In a right triangle, if c is the length of the hypotenuse and the lengths of the two legs are a and b, then $a^2 + b^2 = c^2$.
- **Converse of the Pythagorean Theorem:** If the sides of a triangle satisfy the relationship $a^2 + b^2 = c^2$, then the triangle must be a right triangle with hypotenuse c.
- **Side Relationships for a 30°-60°-90° Triangle:** In any 30°-60°-90° triangle, the hypotenuse is twice the length of the shortest side (the side opposite the 30° angle), and the side opposite the 60° angle is $\sqrt{3}$ times the length of the shortest side.
- **Side Relationship for a 45°-45°-90° Triangle:** In any 45°-45°-90° triangle, the hypotenuse is $\sqrt{2}$ times the length of a leg.

Common Mistakes

- **Misapplying the Pythagorean Theorem:** The Pythagorean Theorem is only valid for right triangles. It cannot be used to find side lengths in acute or obtuse triangles.
- **Confusing the Triangle Inequality with the Pythagorean Theorem:** The Triangle Inequality theorem determines if a

triangle can be formed from three side lengths. The Converse of the Pythagorean Theorem determines if a triangle is a right triangle. A triangle can exist without being a right triangle.

- **Radical Notation Ambiguity:** When a term includes a variable and a radical, it is best to write the radical last to avoid confusion. For example, write $a\sqrt{3}$ instead of $\sqrt{3}a$, as the latter might be misinterpreted as $\sqrt{3a}$.
- **Confusing Isosceles Triangle Angles:** The vertex angle of an isosceles triangle is the angle between the two equal sides. The base angles are opposite the equal sides. The base angles are equal to each other, but not necessarily equal to the vertex angle.

This page titled 4.2.1: Resources and Key Concepts is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.2.2: Homework

Skills Refresher

The following is a set of review exercises you will need for this section.

Skills Refresher

For the following exercises, solve the inequality.

1. $6 - x > 3$
2. $\frac{-3x}{4} \geq -6$
3. $3x - 7 \leq -10$
4. $4 - 3x < 2x + 9$

For the following exercises, assume that $x < 0$. If this is the case, is the expression positive or negative?

5. $-x$
6. $-(-x)$
7. $|x|$
8. $-|x|$
9. $-|-x|$
10. x^{-1}

Answers available at the end of this section.

Homework

Reading Questions

1. What is the sum of the interior angles in any triangle?
2. What does the Triangle Inequality Theorem state about the side lengths of a triangle?
3. What is the definition of an equilateral triangle?
4. In an isosceles triangle, which angles are referred to as the "base angles," and what is true about their measures?
5. What is the name for the side opposite the right angle in a right triangle?
6. State the Pythagorean Theorem using the terms hypotenuse and legs.
7. If you are given the three side lengths of a triangle, how can you test if it is a right triangle?
8. Is it possible for a triangle to contain two right angles? Why or why not?

9. In a 30°-60°-90° triangle, which side is the longest?
10. What is the relationship between the hypotenuse and the shortest side in a 30°-60°-90° triangle?
11. What is another name for a 45°-45°-90° triangle?
12. If the two legs of a 45°-45°-90° triangle each have length a, what is the length of the hypotenuse?

Vocabulary Check

13. A three-sided polygon is another way of saying ___.
14. (MyOpenMath) If given the triangle $\triangle ABC$, it is common notation to label the side opposite $\angle A$ with the letter ___.
15. (MyOpenMath) A triangle where all three sides have the same length is called a ___ triangle.
16. An isosceles triangle has two ___ angles.
17. The ___ angle of an isosceles triangle lies between the two sides of equal length.
18. (MyOpenMath) A(n) ___ triangle must have one angle ___ 90°.
19. A(n) ___ triangle must have all angles ___ 90°.
20. The two special triangles are the ___-90° and ___-90° triangles.
21. Another name for the 45°-45°-90° triangle is a(n) ___ triangle.

Concept Check

22. Is it always true that the hypotenuse is the longest side in a right triangle? Why or why not?
23. In $\triangle DEF$, is it possible that $d + e > f$ and $e + f > d$ are both true? Explain your answer.
24. In a right triangle with hypotenuse c, we know that $a^2 + b^2 = c^2$. Is it also true that $a + b = c$? Why or why not?
25. Is having more than one obtuse angle in a triangle possible? Why or why not?
26. Draw any quadrilateral (a four-sided polygon) and divide it into two triangles by connecting two opposite vertices by a diagonal. What is the sum of the angles in your quadrilateral?
27. What is the difference between a vertex and vertical angles?
28. In any triangle, the sum of the three angles is ___.
29. The longest side in a triangle is opposite the ___ angle, and the shortest side is opposite the ___ angle.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

30. A right triangle always has one angle of 90°.
31. Given a triangle with side lengths a, b, and c, where c is the longest side,
$$a^2 + b^2 = c^2.$$
32. All of the angles of an equilateral triangle are equal.
33. The vertex angles of an isosceles triangle are equal.

34. If the sum of the lengths of any two sides is greater than the length of the third side, then the triangle must be a right triangle.

35. In $\triangle ABC$, $a^2 + b^2 = c^2$.

36. If the sides of a triangle satisfy the relationship $a^2 + b^2 = c^2$, then the triangle is a right triangle.

Basic Skills

For the following exercises, sketch and label a triangle with the given properties.

37. An isosceles triangle with vertex angle 30°

38. A scalene triangle with one obtuse angle

39. (MyOpenMath) A right triangle with legs 4 and 7

40. (MyOpenMath) A right triangle with hypotenuse 12, where one leg is of length 8.

41. An isosceles right triangle

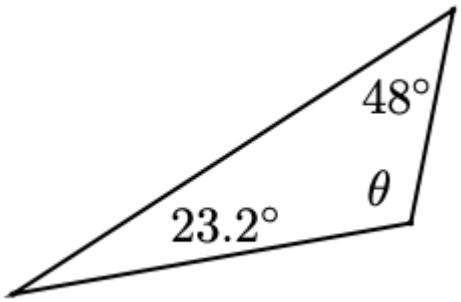
42. An isosceles triangle with one obtuse angle

43. A right triangle with one angle 20°

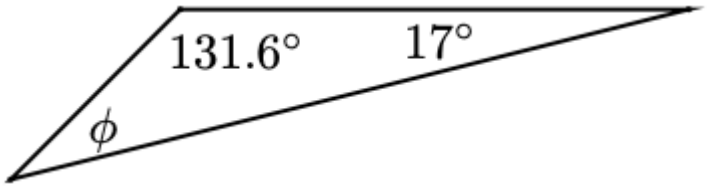
44. (MyOpenMath) The triangle $\triangle ABC$ has $m\angle A = 88^\circ$ and $m\angle B = 87^\circ$, find $m\angle C$.

For the following exercises, find the unknown angle.

45. (MyOpenMath)

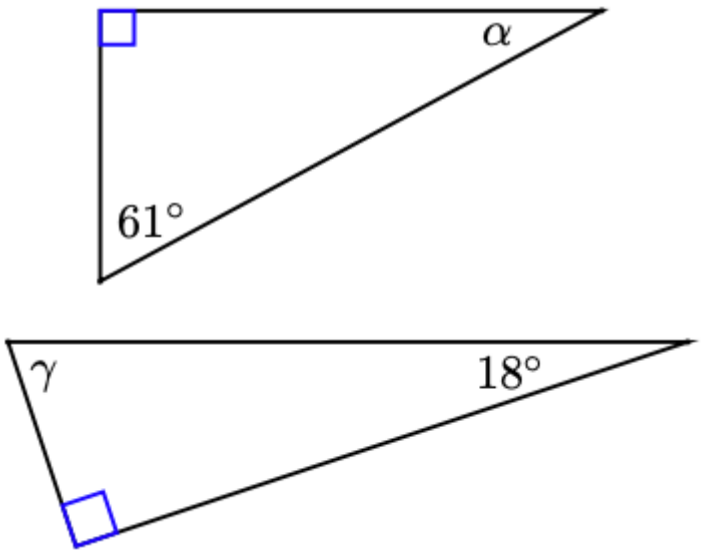


46.

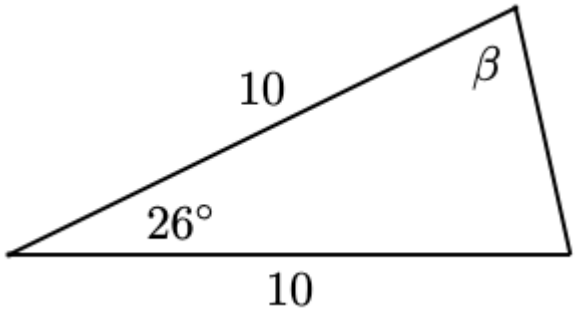


47. (MyOpenMath)

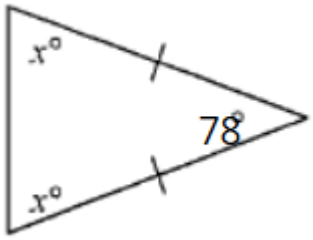
18.



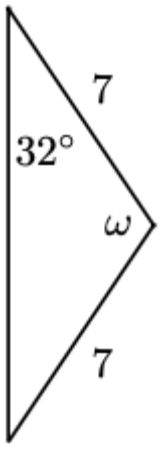
19.



50. (MyOpenMath)



51. (MyOpenMath)



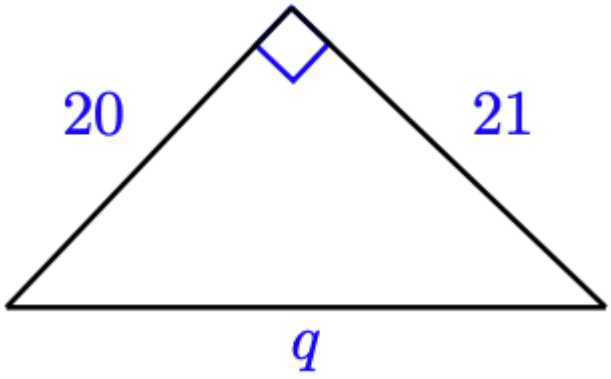
52. If two sides of a triangle are 6 feet and 10 feet long, what are the largest and smallest possible values for the length of the third side?

53. Two adjacent sides of a parallelogram are 3 cm and 4 cm long. What are the largest and smallest possible values for the length of the diagonal?

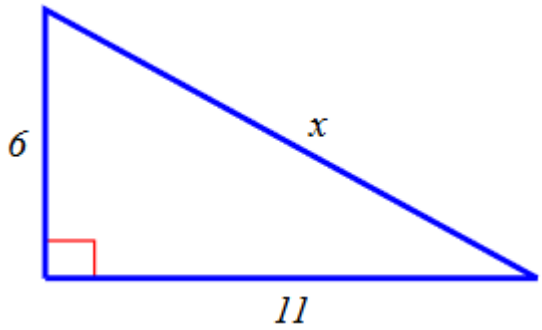
54. If one of the equal sides of an isosceles triangle is 8 millimeters long, what are the largest and smallest possible values for the length of the base?

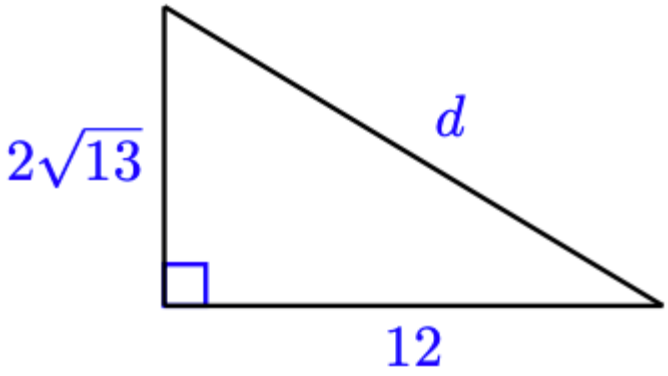
For the following exercises, find the unknown side of the triangle.

55.



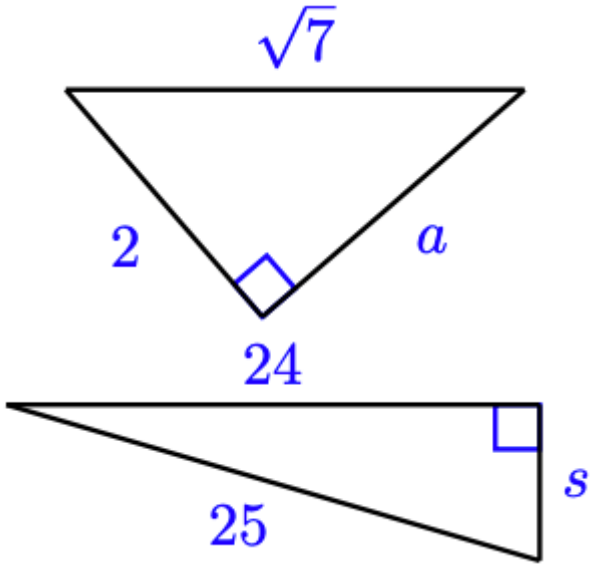
56. (MyOpenMath)





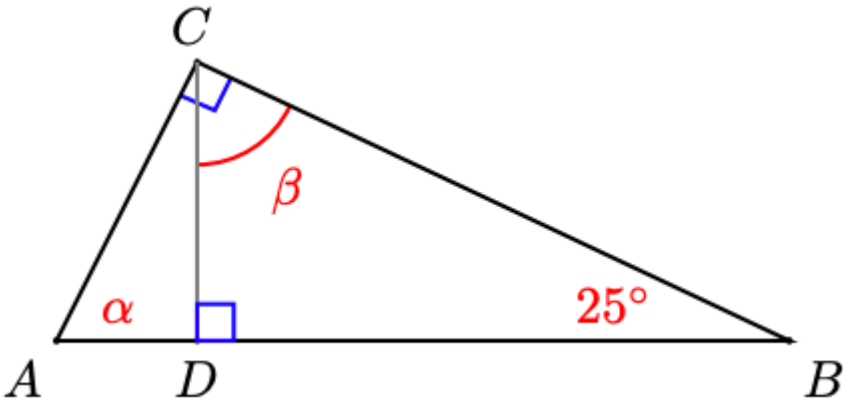
57.

58. (MyOpenMath)

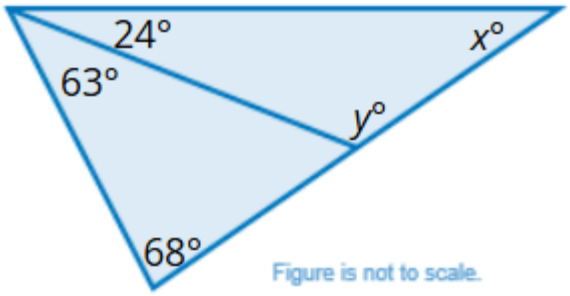


59.

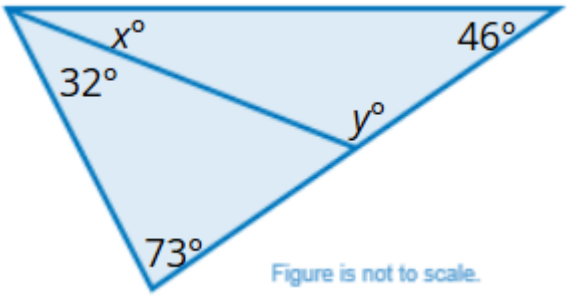
60. Find α and β.



61. (MyOpenMath) Find x and y.



62. (MyOpenMath) Find x and y.



For the following exercises, decide whether a triangle with the given sides is a right triangle.

63. (MyOpenMath) 9 in, 16 in, 25 in

64. 12 m, 16 m, 20 m

65. 5 m, 12 m, 13 m

66. 5 ft, 8 ft, 13 ft

67. 5^2 ft, 8^2 ft, 13^2 ft

68. $\sqrt{5}$ ft, $\sqrt{8}$ ft, $\sqrt{13}$ ft

69. (MyOpenMath) Compute the exact value of the altitude h in an equilateral triangle, where all sides are 14 inches long.

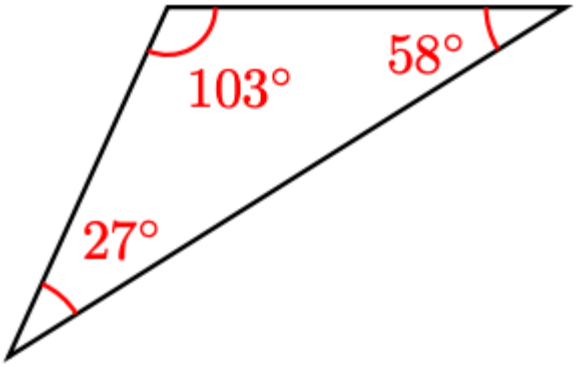
Synthesis Questions

70. (MyOpenMath) In a certain triangle, angle A is three times as big as angle C and angle B is 55° more than angle C. What is the degree measure of angle B?

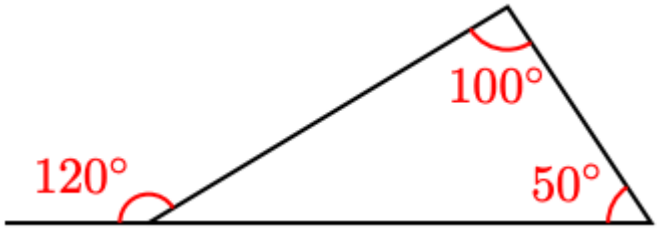
71. The two shorter sides of an obtuse triangle are 3 in and 4 in. What are the possible lengths for the third side?

For the following exercises, explain why the measurements shown cannot be accurate.

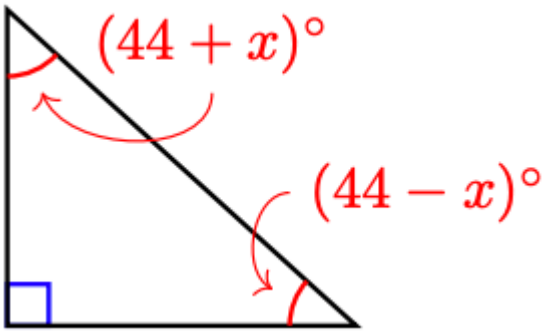
72.



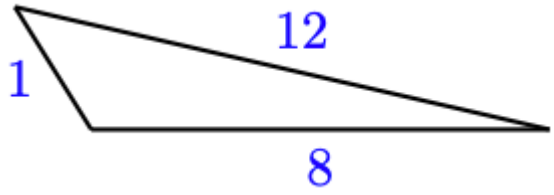
73.



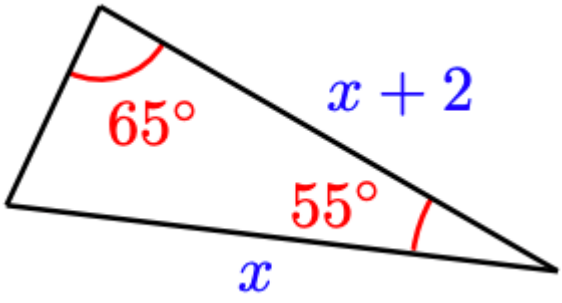
74.



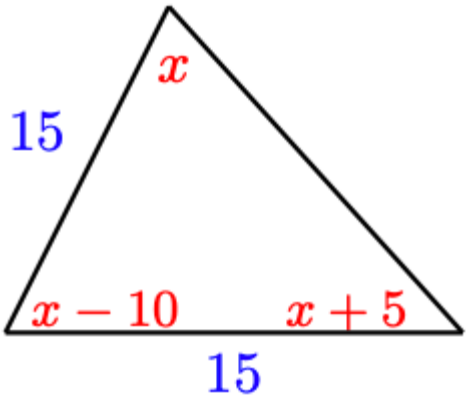
75.



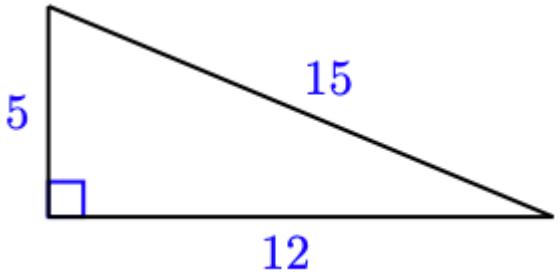
76.



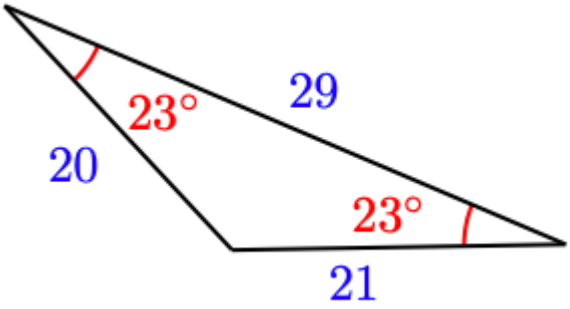
77.



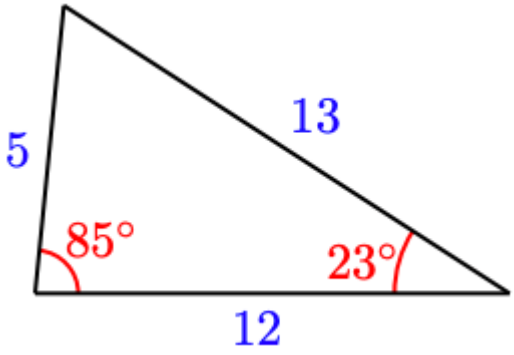
78.



79.

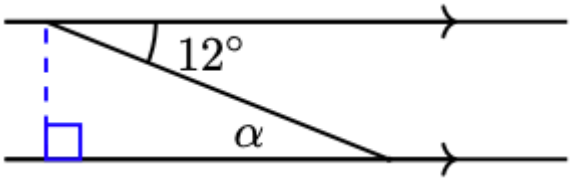


80.

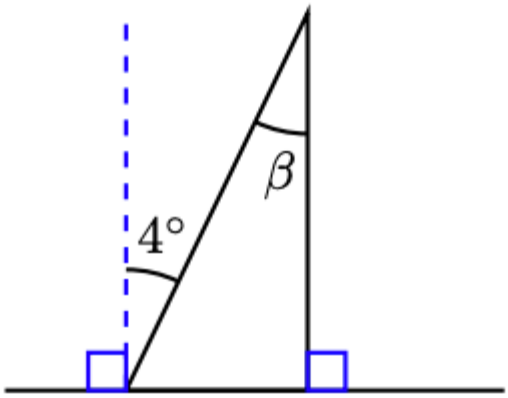


For the following exercises, find the unknown angle. Arrows on a pair of lines indicate that they are parallel.

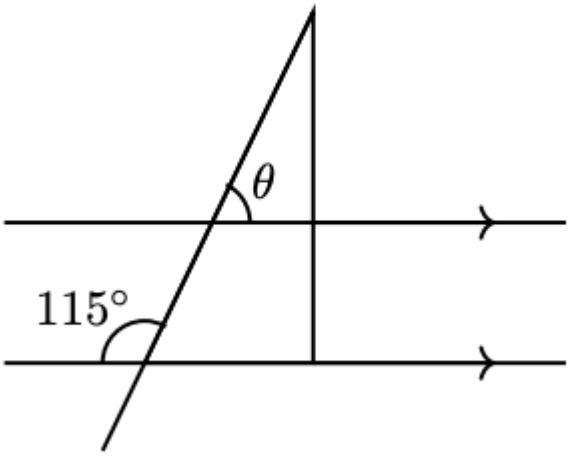
81.



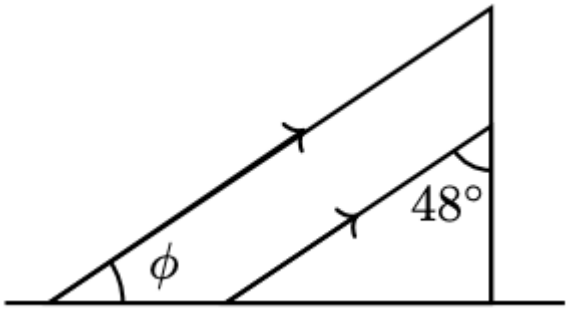
32.



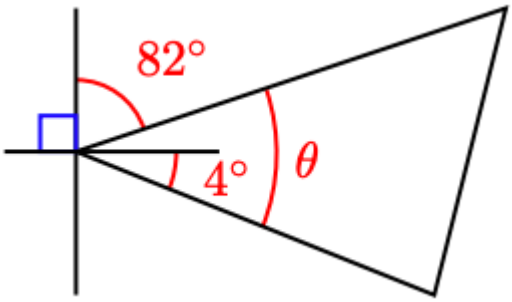
33.

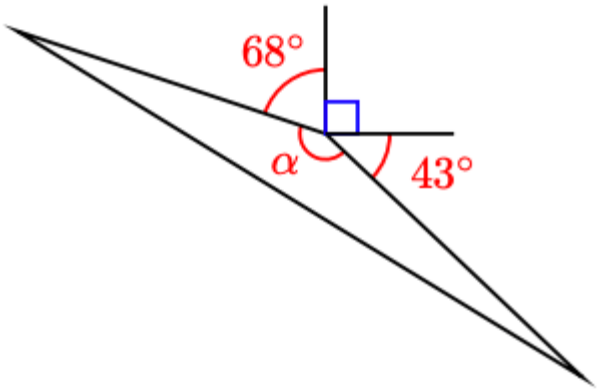


34.

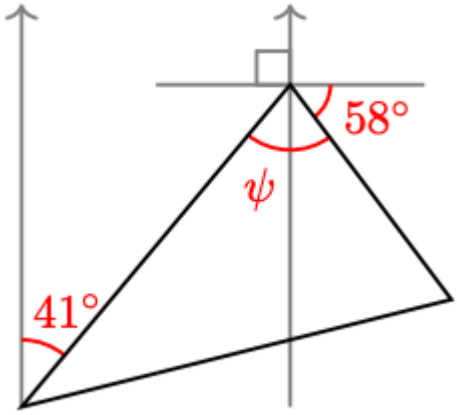


35.

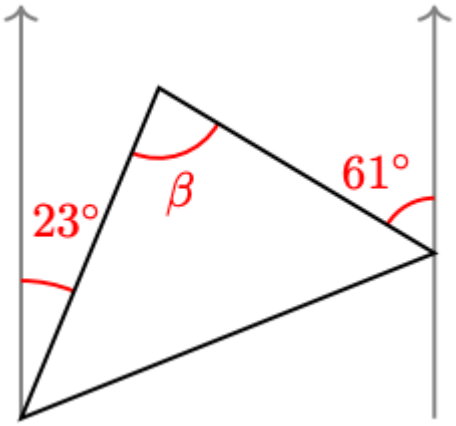




36.

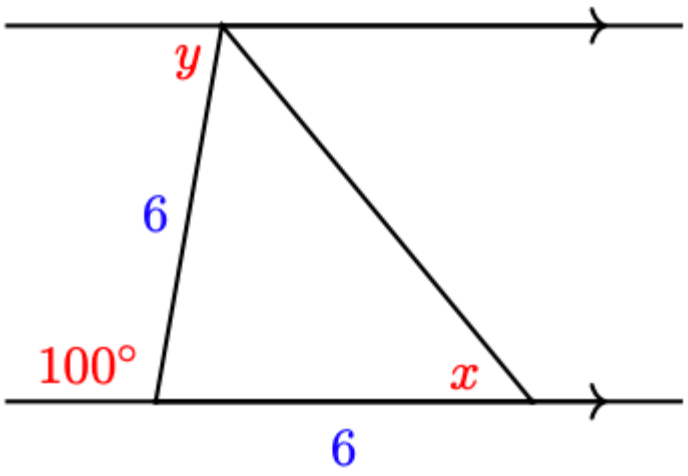


37.

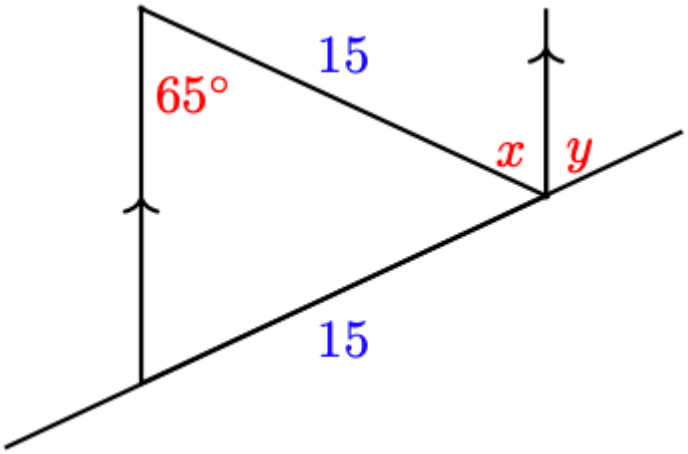


38.

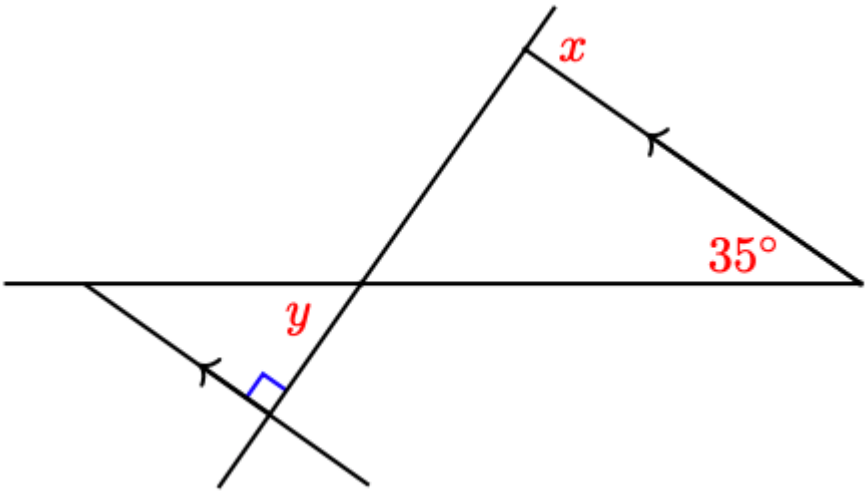
For the following exercises, arrows on a pair of lines indicate that they are parallel. Find x and y.



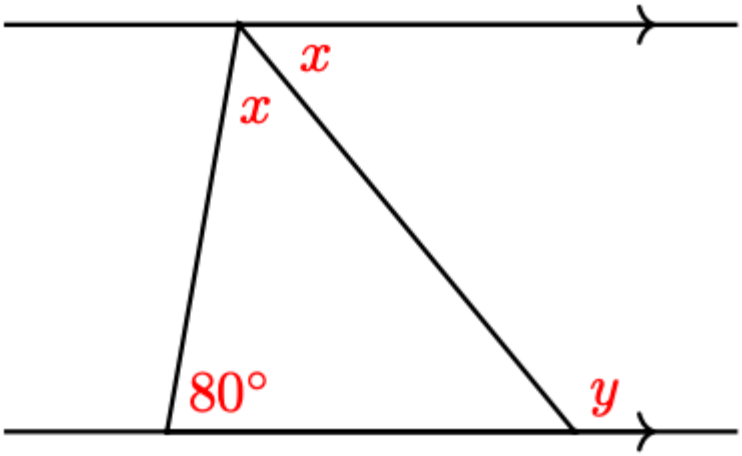
39.



90.



91.



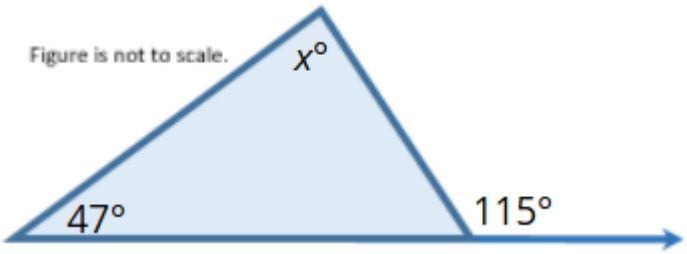
92.

93. How long is the diagonal of a square whose side is 6 centimeters?

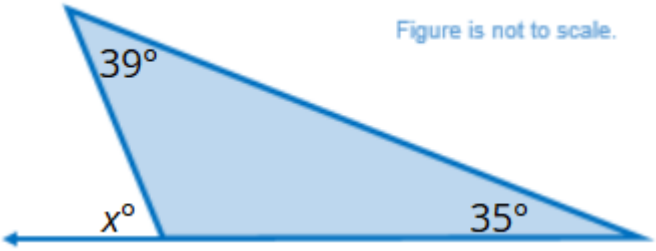
94. How long is the side of a cube whose volume is 80 cubic feet?

For the following exercises, find the value of x.

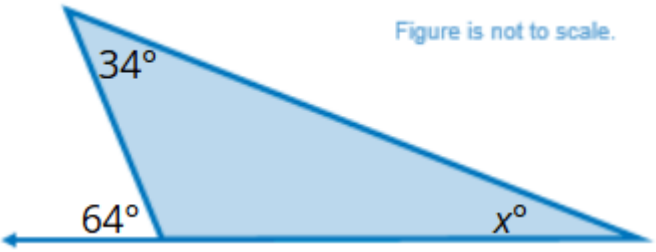
95. (MyOpenMath)



96. (MyOpenMath)

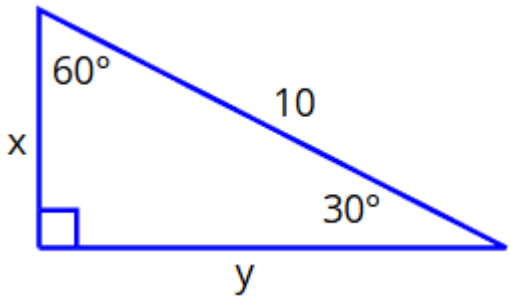


97. (MyOpenMath)

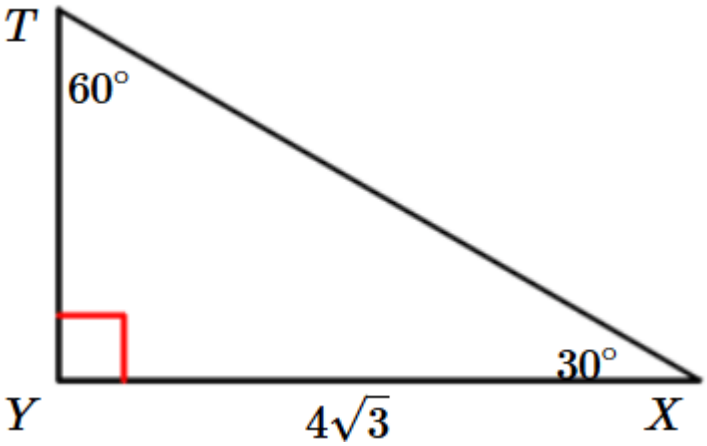


For the following exercises, find the length of the indicated side(s).

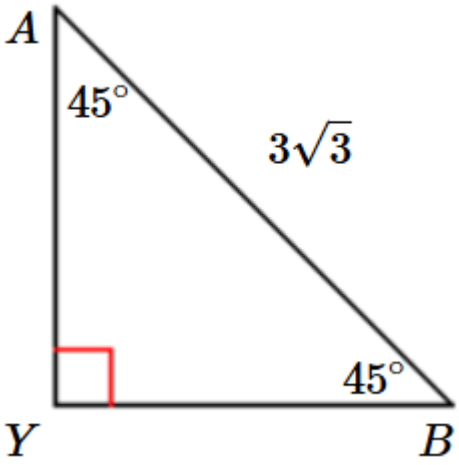
28. (MyOpenMath) x and y



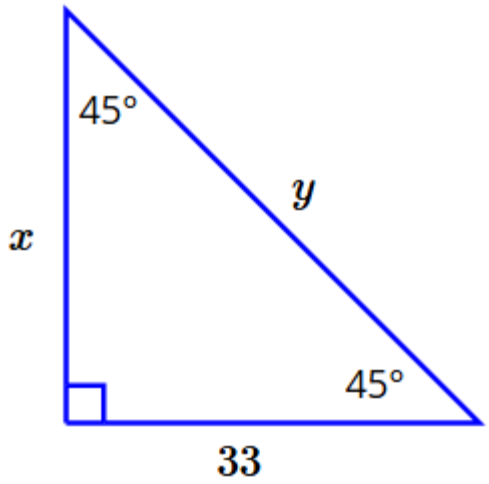
29. (MyOpenMath) $\overline{TY}$ and $\overline{TX}$



30. (MyOpenMath) $\overline{AY}$ and $\overline{YB}$

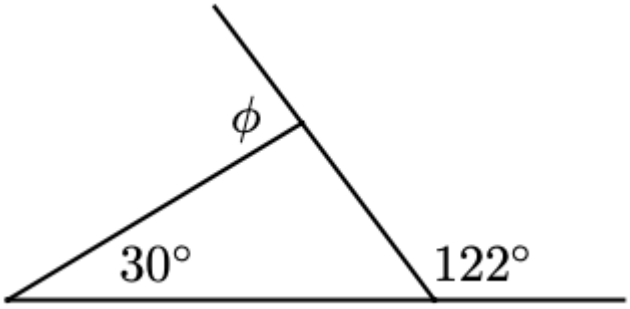


31. (MyOpenMath) x and y

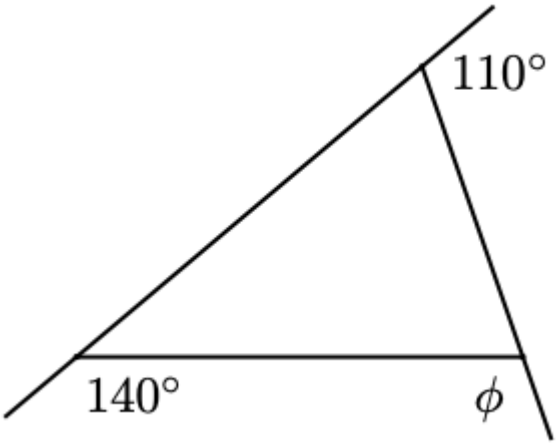


For the following exercises, the angle labeled ϕ is called an exterior angle of the triangle, formed by one side and the extension of an adjacent side. Find ϕ.

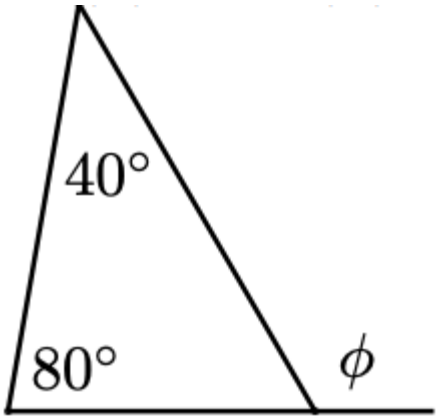
92.



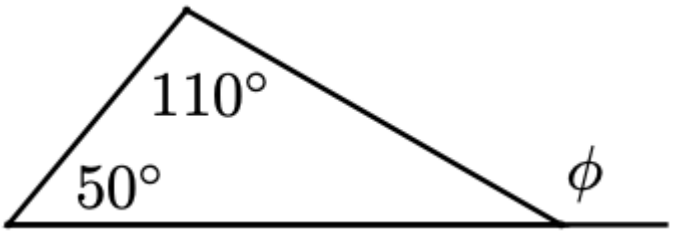
93.



94.



95.



For the following exercises, make a sketch of the situation described and label a right triangle. Then use the Pythagorean Theorem to solve the problem.

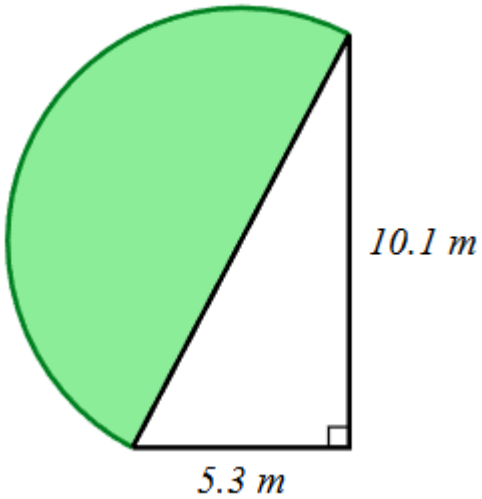
96. The diagonal of a square is 12 inches long. How long is the side of the square?

97. The length of a rectangle is twice its width, and its diagonal is meters long. Find the dimensions of the rectangle.

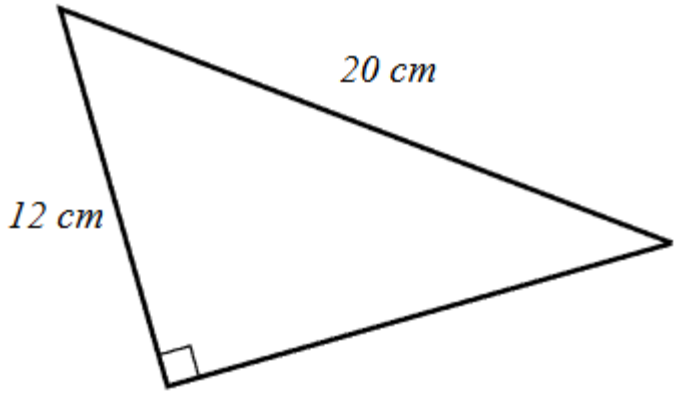
98. (MyOpenMath) The hypotenuse of right triangle is 146 centimeters long. The difference between the other two sides is 14 centimeters. Find the exact values of the missing sides.

99. (MyOpenMath) The shortest side of a right triangle is 9 centimeters long. The difference between the lengths of the other two sides is 1 centimeter. Find the exact values of the missing side and hypotenuse.

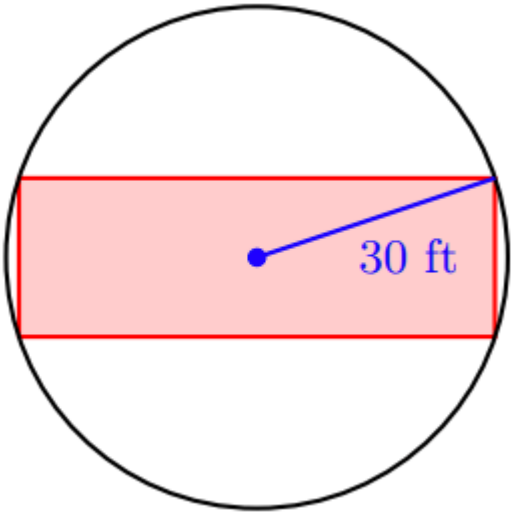
100. (MyOpenMath) The shaded, half circular disk is drawn on the side of a right triangle, whose legs are labeled 5.3 meters and 10.1 meters. Find the area of this half disk. Round your answer to one decimal place.



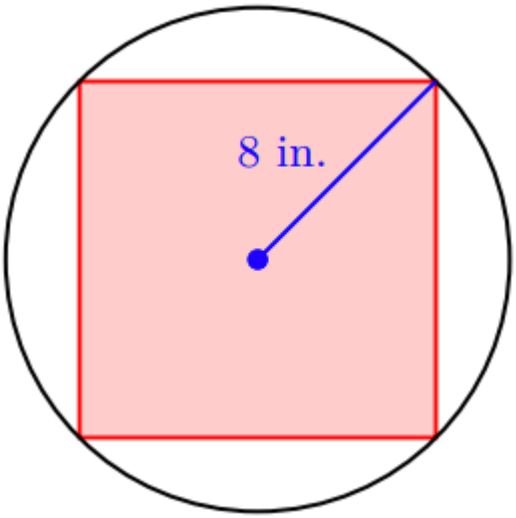
101. (MyOpenMath) Imagine that for the right triangle shown below, each of its three sides is bent into a perfect circle. What would be the *combined total area* of all three circles? Round your answer to one decimal place.



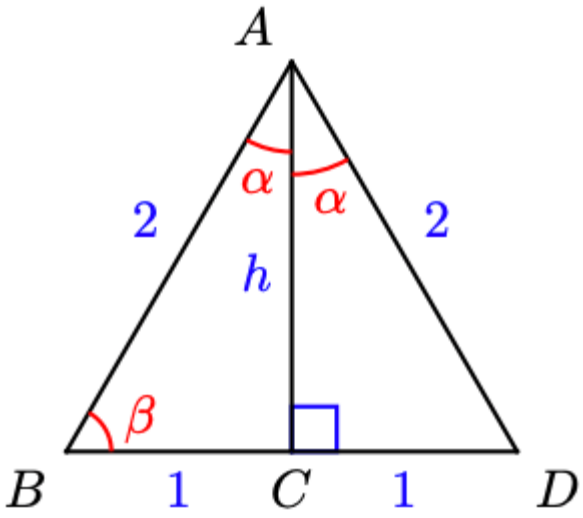
12. What size rectangle can be inscribed in a circle of radius 30 feet if the length of the rectangle must be three times its width?



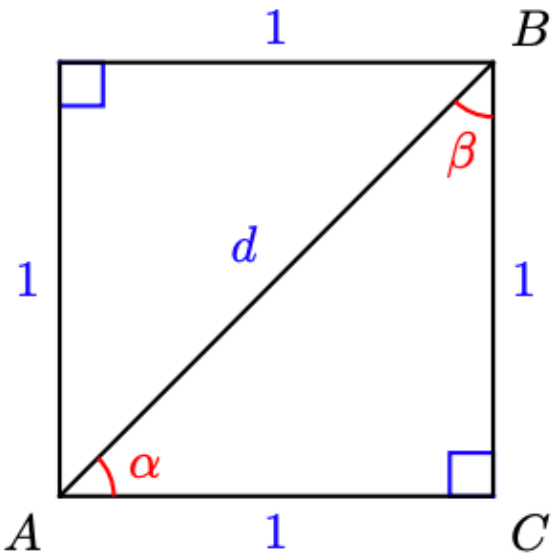
13. What size square can be inscribed inside a circle of radius 8 inches, so that its vertices just touch the circle?



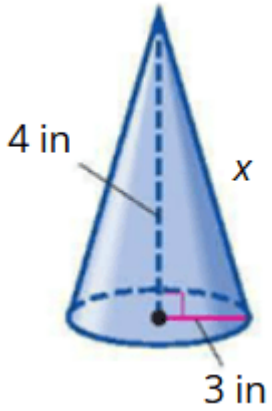
14. Find α, β and h.



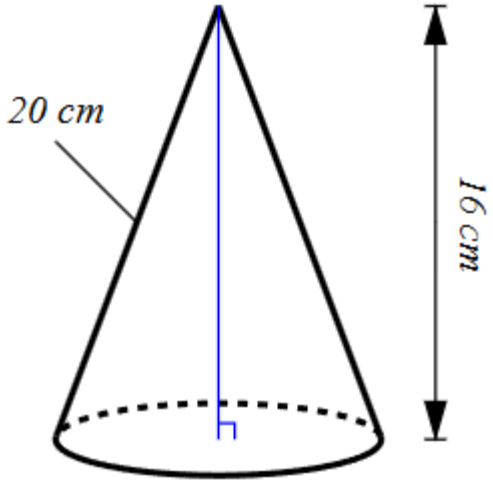
15. Find α, β and d.



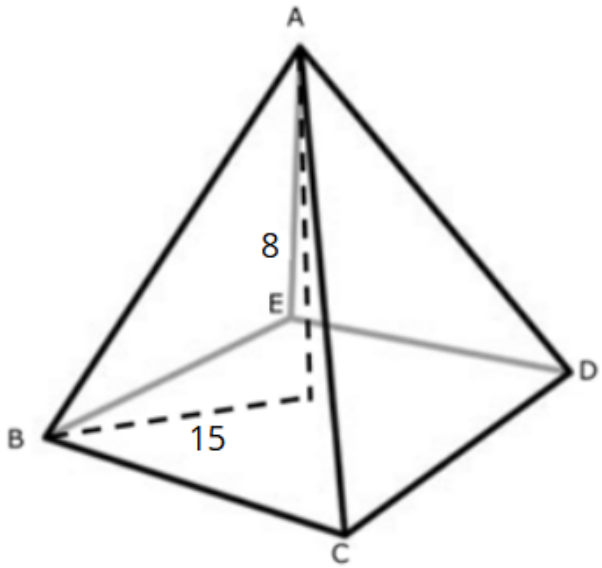
16. (MyOpenMath) Find the slant height (labeled x) of the right circular cone.



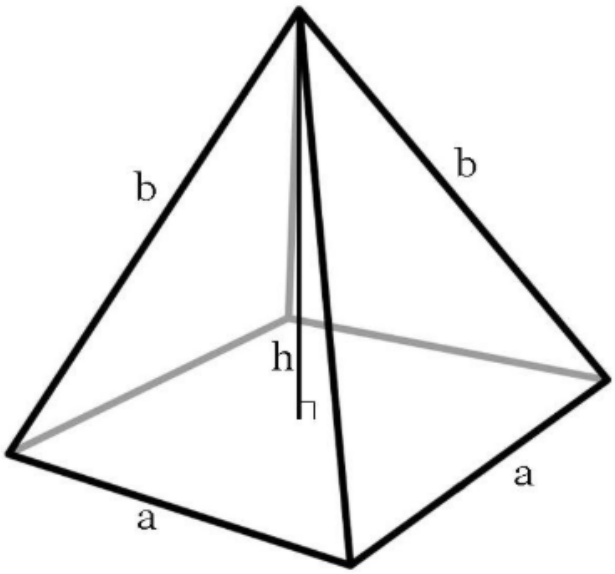
17. (MyOpenMath) Compute the radius of the base of the right circular cone if given the slant height of 20 cm and the cone height of 16 cm.



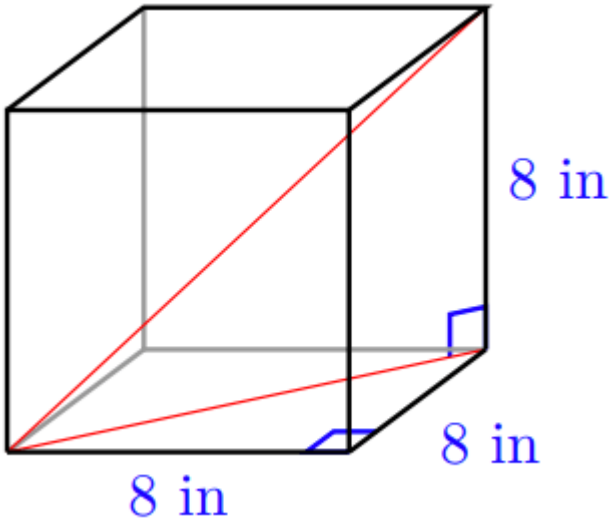
18. (MyOpenMath) A right square pyramid is shown. The height of the pyramid is 8 units. The distance from the center of the base of the pyramid to vertex B is 15 units, as shown. Find the length of segment $\overline{AB}$.



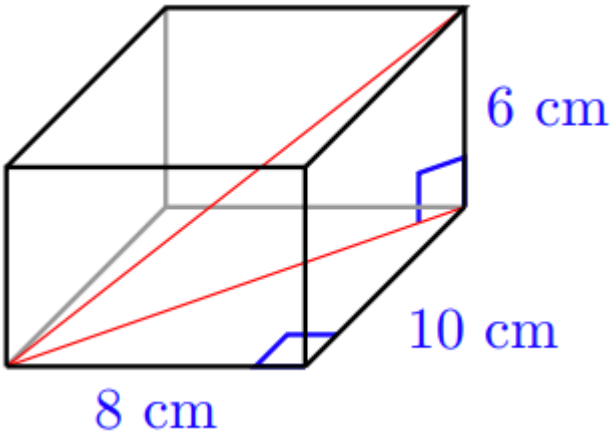
19. (MyOpenMath) Compute the exact value of the height h of the square-based straight pyramid, given that the base is a square with sides $a = 36$ inches long, and all other edges are $b = 54$ inches long.



20. Find the diagonal of a cube of side 8 inches. Hint: Find the diagonal of the base first.

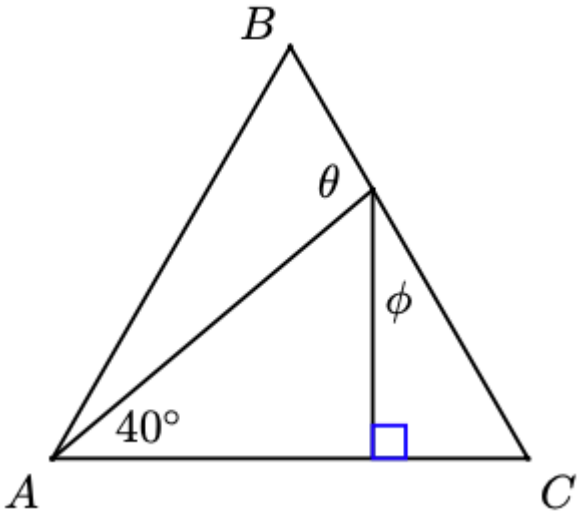


21. Find the diagonal of a rectangular box whose sides are 6 cm by 8 cm by 10 cm. Hint: Find the diagonal of the base first.

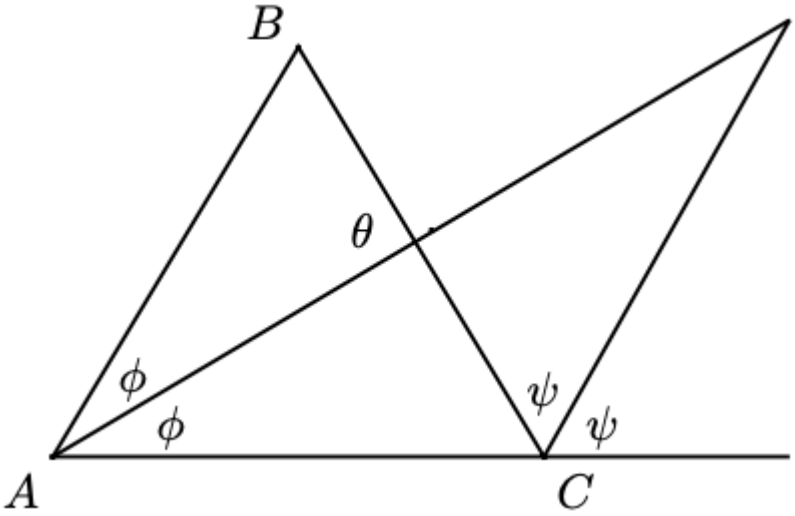


For the following exercises, ΔABC is equilateral. Find the unknown angles.

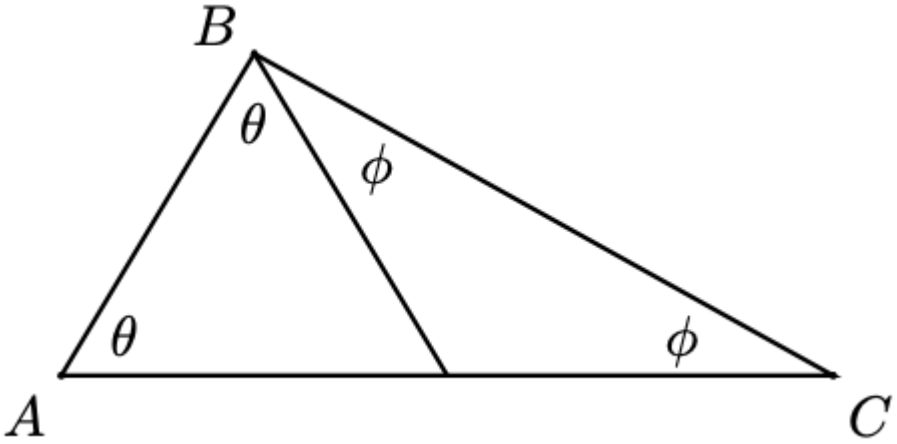
22.



23.



24. Consider the following figure.

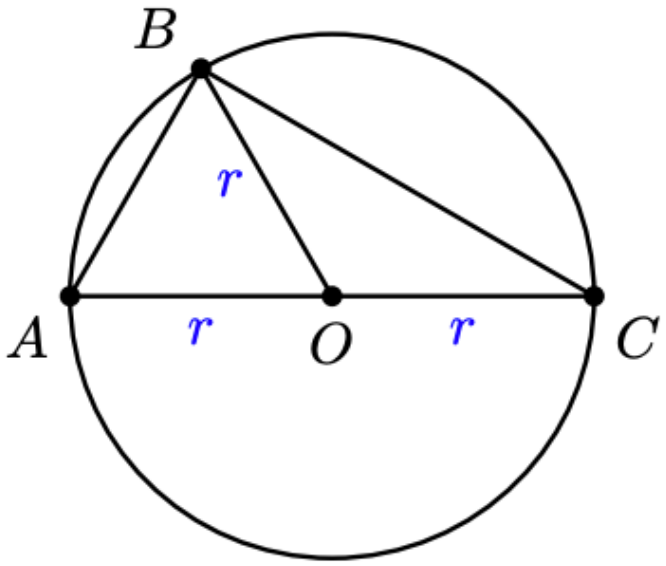


a. Compute $2\theta + 2\phi$.

b. Compute $\theta + \phi$.

c. What must be true about $\triangle ABC$?

25. Consider the following figure.

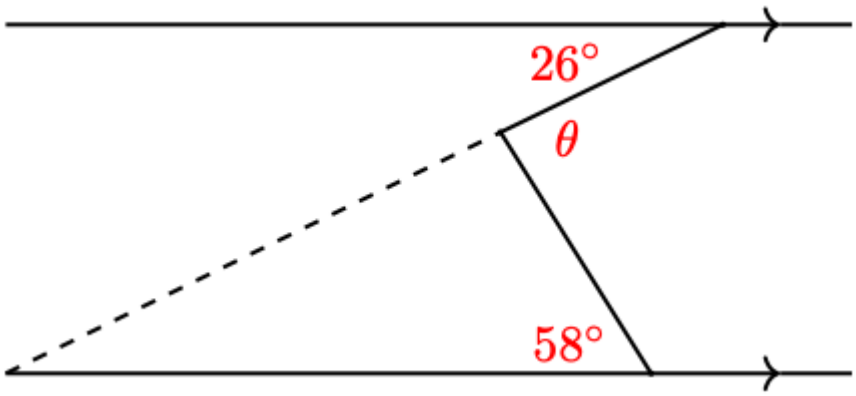


a. Explain why $\angle OAB$ and $\angle ABO$ are equal in measure.

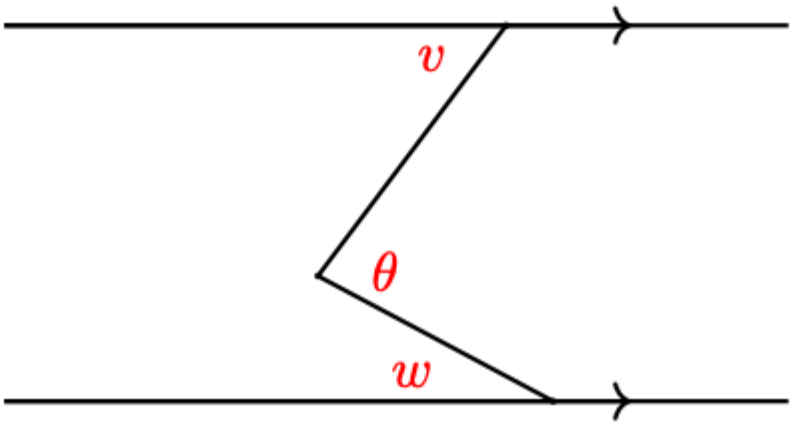
b. Explain why $\angle OBC$ and $\angle BCO$ are equal in measure.

c. Explain why $\angle ABC$ is a right angle.

26. a. In the figure below, find θ, and justify your answer.



b. Write an algebraic expression for θ in the figure below.



Applications

27. **Bounding Distances.** The town of Madison is 15 miles from Newton, and 20 miles from Lewis. What are the possible values for the distance from Lewis to Newton?

For the following exercises, make a sketch of the situation described and label a right triangle. Then use the Pythagorean Theorem to solve the problem.

28. **TV Dimensions.** The size of a TV screen is the length of its diagonal. If the width of a 35-inch TV screen is 28 inches, what is its height?

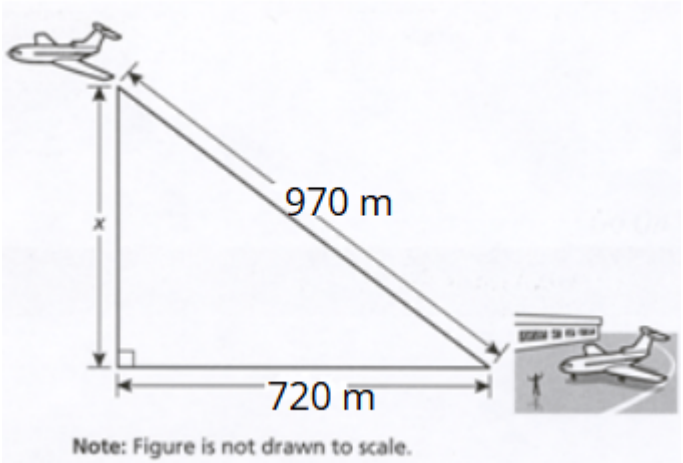
29. **TV Dimensions (MyOpenMath).** Piyali goes to the store to buy a new TV. She tells the salesperson that she needs a TV that has a maximum width of 37 inches and a maximum height of 28 inches. Assuming that she can get a TV with these maximum dimensions, and knowing that TVs are measured by the diagonal of the screen, what size TV can Piyali buy?

30. **Tree Shadows.** If a 30-meter pine tree casts a shadow of 30 meters, how far is the tip of the shadow from the top of the tree?

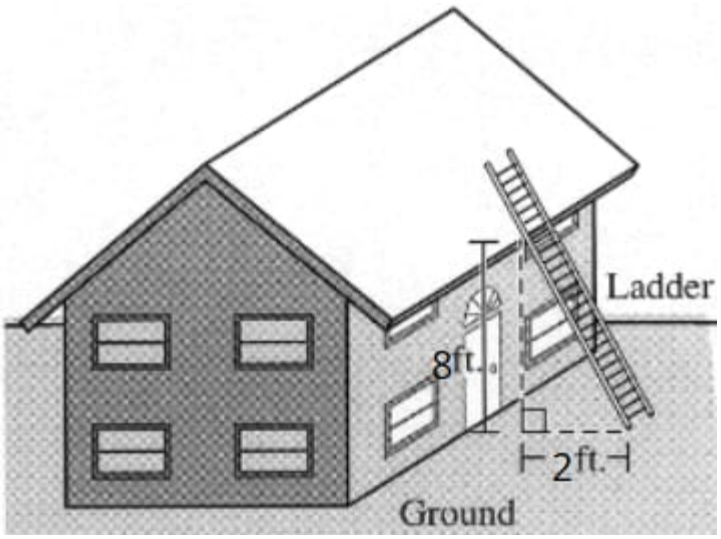
31. **Utility Pole (MyOpenMath).** A utility pole is 9 m high. A cable is stretched from the top of the pole to a point in the ground that is 8 m from the bottom of the pole. How long is the cable?

32. **Distance Between Towns (MyOpenMath).** If Nhat is 22 miles due west of Sacramento and Loi is 30 miles due north of Sacramento, what is the shortest distance from Loi to Nhat?

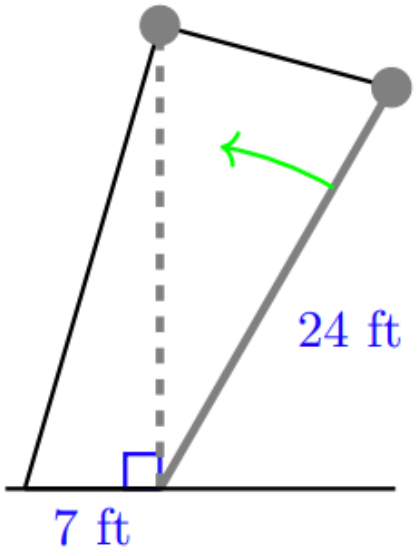
33. **Altitude (MyOpenMath).** What is the altitude of the plane?



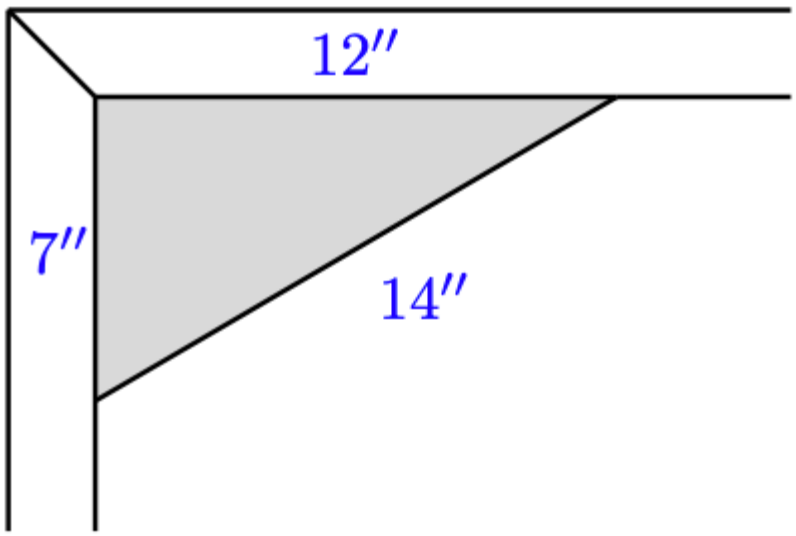
34. **Ladder Length (MyOpenMath).** What is the minimum length the ladder must be to reach the roof?



35. A 24-foot flagpole is being raised by a rope and pulley, as shown in the figure. The loose end of the rope can be secured to a ring on the ground 7 feet from the base of the pole. From the ring to the top of the pulley, how long should the rope be when the flagpole is vertical?



36. To check whether the corners of a frame are square, carpenters sometimes measure the sides of a triangle, with two sides meeting at the join of the boards. Is the corner shown in the figure square?



For the following exercises, make a sketch and solve.

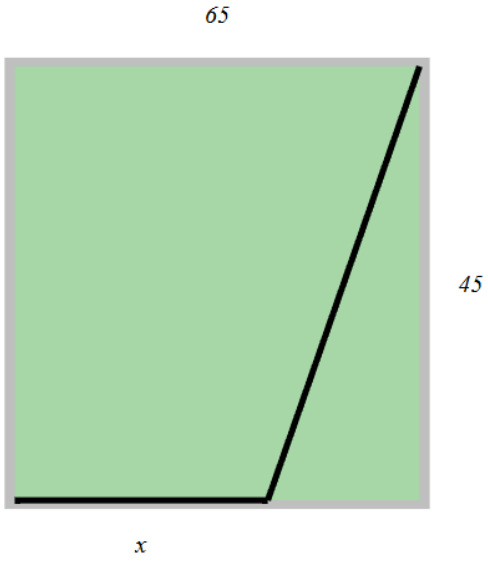
37. **Pesky Pipe.** The back of Ron's pickup truck is five feet wide and seven feet long. He wants to bring home a 9-foot length of copper pipe. Will it lie flat on the floor of the truck?

38. **Pesky Pipe.** Mary has a pickup truck with a camper shell. The camper shell floor is five feet wide and seven feet long, but the shell is only three feet tall. Will a 9-foot copper pipe fit diagonally across the back of the truck?

39. What is the longest curtain rod that will fit inside a box 60 inches long by 10 inches wide by 4 inches tall?

Challenge Problems

40. (MyOpenMath) A sidewalk goes all the way around a rectangular field, as shown below (not drawn to scale). The field is 65 feet long and 45 feet wide. You need to get from the bottom-left corner to the top-right corner. Your walking speed is 7 feet per second along the sidewalk and 4 feet per second through the field.



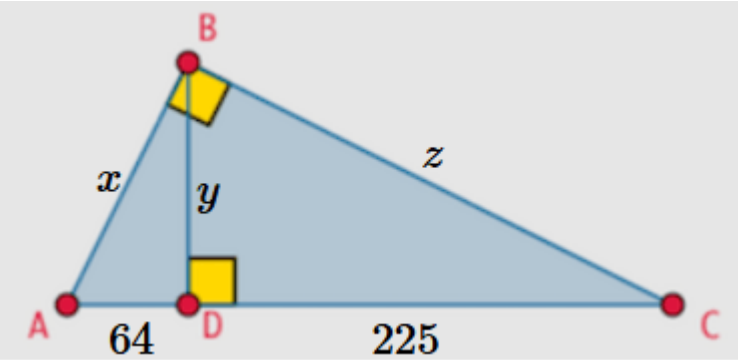
a. How many seconds will it take you if you walk from the bottom-left corner to the top-right corner directly across the field?

b. How many seconds will it take you if you walk all 65 feet along the sidewalk (from bottom-left to bottom-right) and then

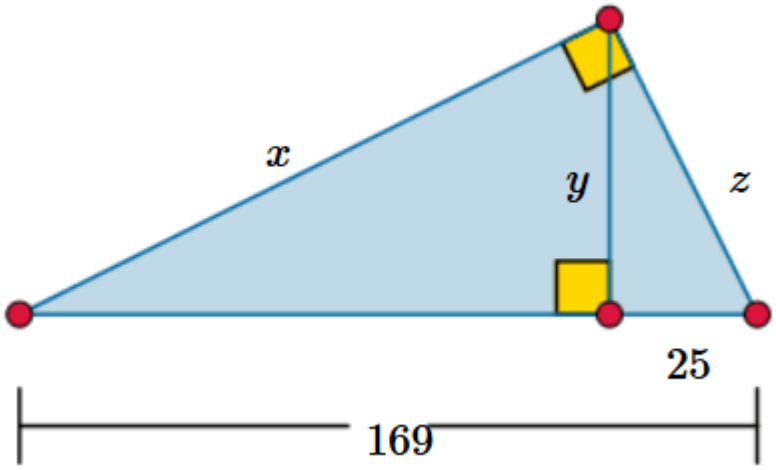
45 feet along the sidewalk (from bottom-right to top-right)?

c. Suppose you walk x feet along the sidewalk and then turn to walk the remaining distance through the field directly to the top-right corner. Write an expression in x representing the number of seconds it would take you to walk this path.

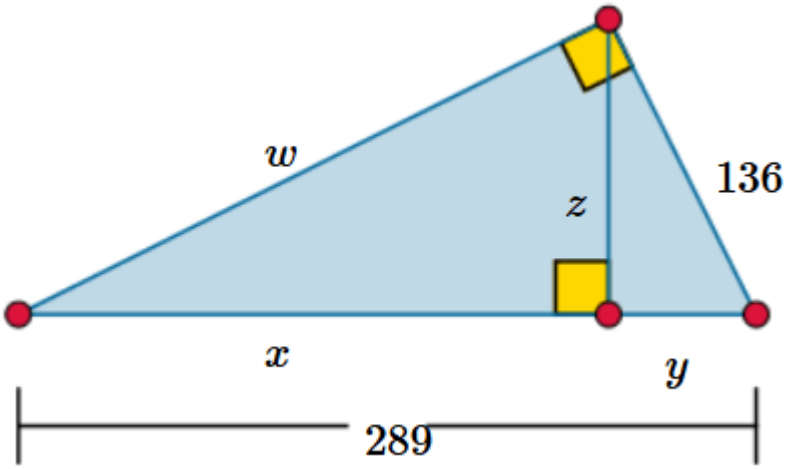
41. (MyOpenMath) Find the values of x, y, and z to two decimal places.



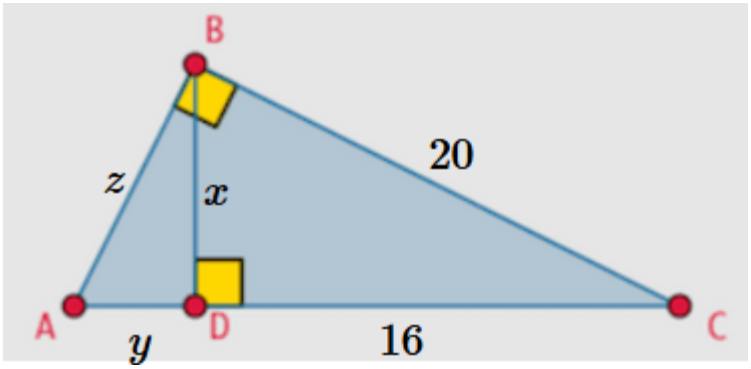
42. (MyOpenMath) Find the values of x, y, and z to two decimal places.



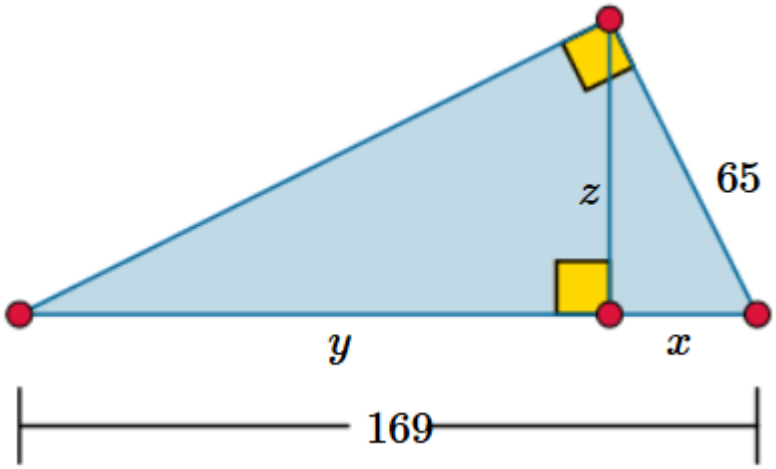
43. (MyOpenMath) Find the values of x, y, and z to two decimal places.



44. (MyOpenMath) Find the values of x, y, and z to two decimal places.

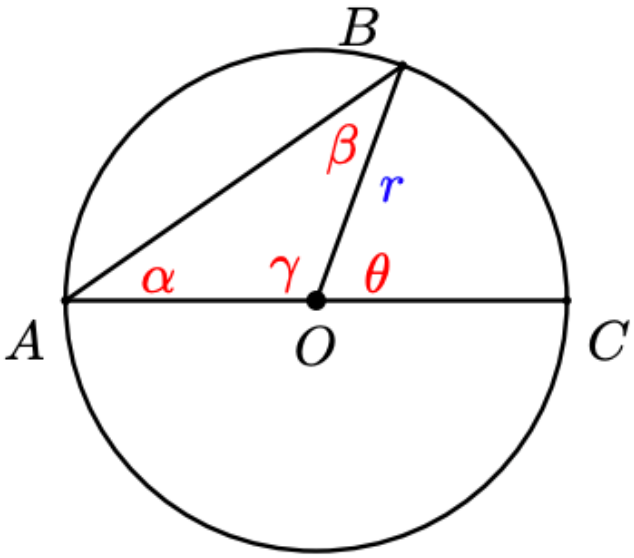


15. (MyOpenMath) Find the values of x, y, and z to two decimal places.



16. (MyOpenMath) An arch is in the shape of a semicircle. At a point along the base 4 feet from an end of the arch, the height of the arch is 8 feet. Find the exact value of the maximum height of the arch.

17. Consider the following figure.

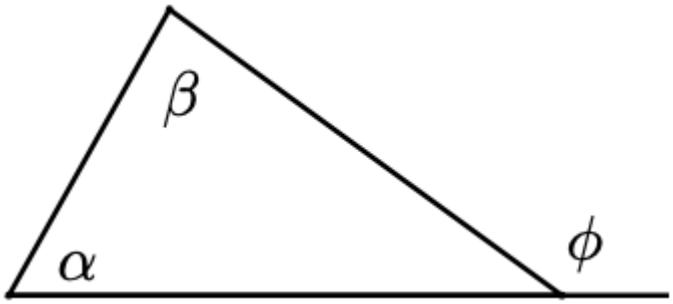


a. Compare θ with $\alpha + \beta$. (Hint: What do you know about supplementary angles and the sum of angles in a triangle?

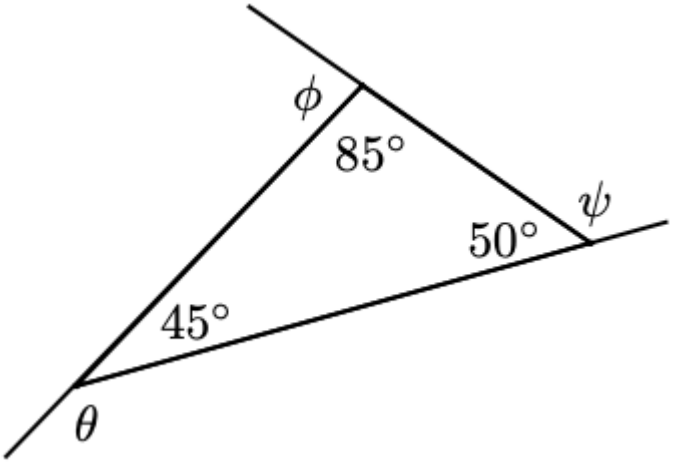
b. Compare α and β.

c. Explain why the inscribed angle $\angle BAO$ is half the size of the central angle $\angle BOC$.

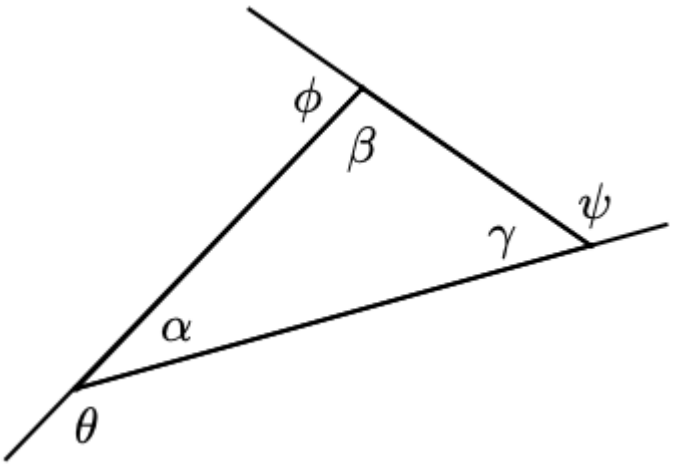
18. Find an algebraic expression for ϕ and use your answer to write a rule for finding an exterior angle of a triangle.



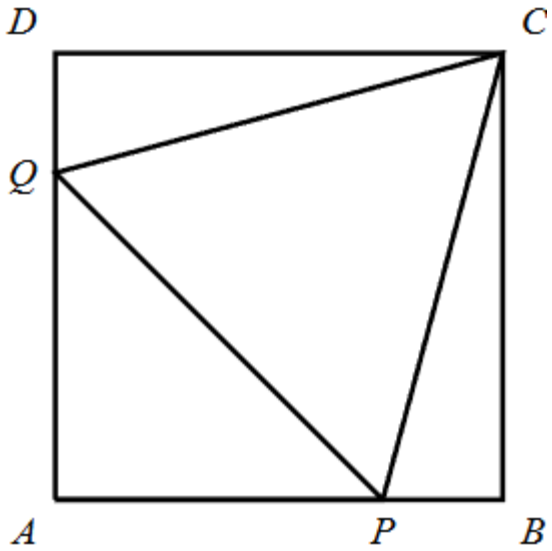
19. a. Find the three exterior angles of the triangle. What is the sum of the exterior angles?



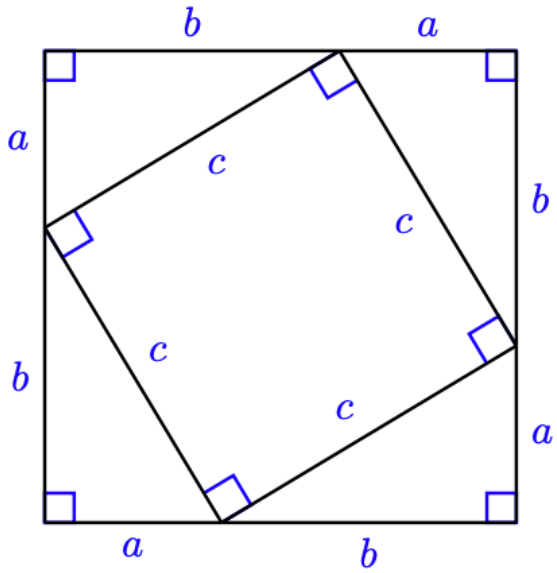
b. Write an algebraic expression for each exterior angle in terms of one of the angles of the triangle. What is the sum of the exterior angles?



20. (MyOpenMath) The area of square $ABCD$ is one square unit. Triangle $\triangle PCQ$ is equilateral. What are the exact lengths of the sides of the triangle?



51. **Proving the Pythagorean Theorem.** There are many proofs of the Pythagorean Theorem. Here is a simple visual argument.



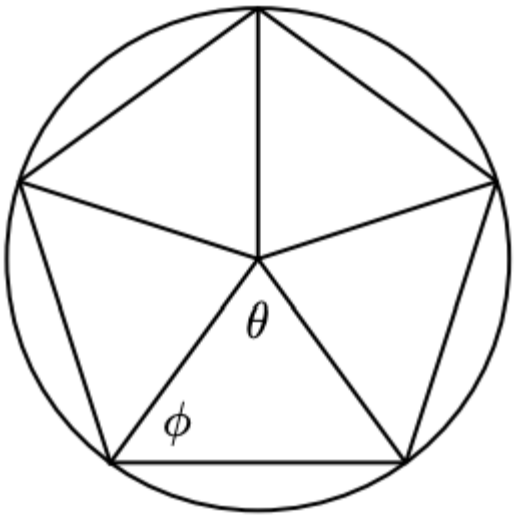
a. What is the length of the side of the large square in the figure? Write an expression for its area.

b. Write another expression for the area of the large square by adding the areas of the four right triangles and the smaller central square.

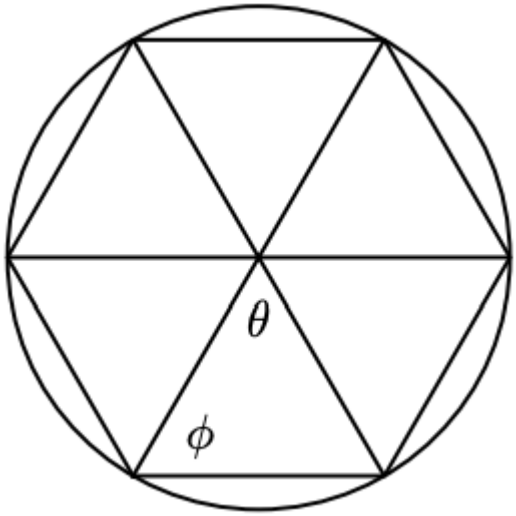
c. Equate your two expressions for the area of the large square, and deduce the Pythagorean Theorem.

For the following exercises, the figures inscribed in each problem are regular polygons, which means that all their sides are the same length, and all the angles have the same measure. Find the angles θ and ϕ.

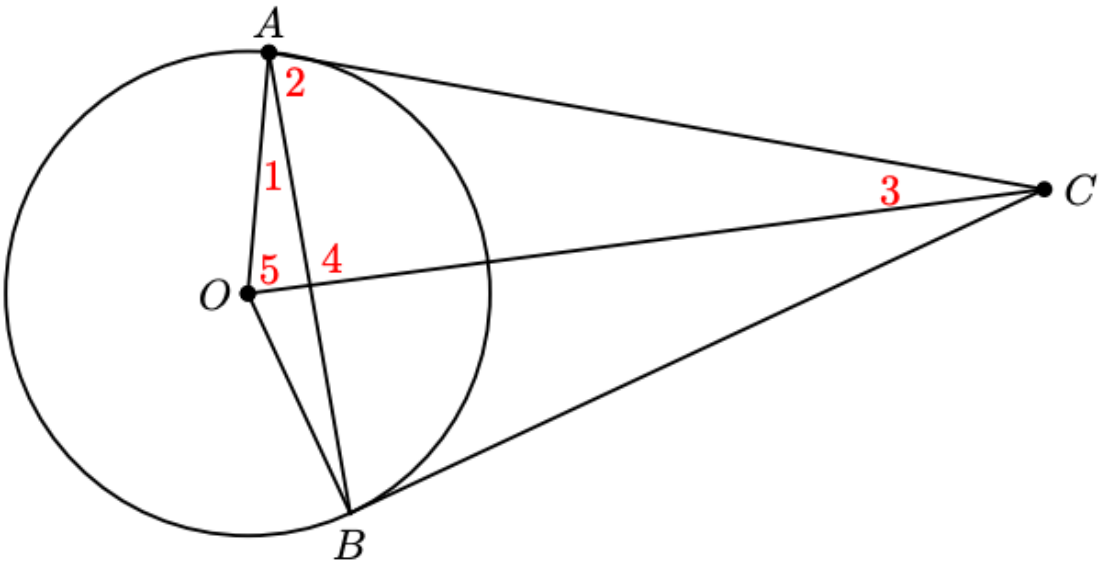
52.



53.



54. A tangent meets the radius of a circle at a right angle. In the figure, $\angle AOB = 140^\circ$. Find the angles labeled 1 through 5 in order, and give a reason for each answer.



Answers

Skills Refresher

1. $x < 3$
2. $x \leq 8$
3. $x \leq -1$
4. $x > -1$
5. Positive
6. Negative
7. Positive
8. Negative
9. Negative
10. Negative

This page titled 4.2.2: Homework is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.3: Similar Triangles

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Congruent Triangles

We begin this section by identifying what it means for two triangles to be **congruent**.

Definition: Congruent Triangles

Two triangles are said to be **congruent** if their corresponding angles are equal, and their corresponding sides have the same lengths. Symbolically, we use the congruency symbol $\cong$ to state that two triangles are congruent.

In simple terms, two triangles are considered congruent if they have the same size and shape. For example, it looks as though $\triangle ABC$ and $\triangle DEF$ in Figure $4.3.1$ are identical in both shape and size ($\triangle DEF$ has been rotated from $\triangle ABC$), so they are congruent. We write $\triangle ABC \cong \triangle DEF$.

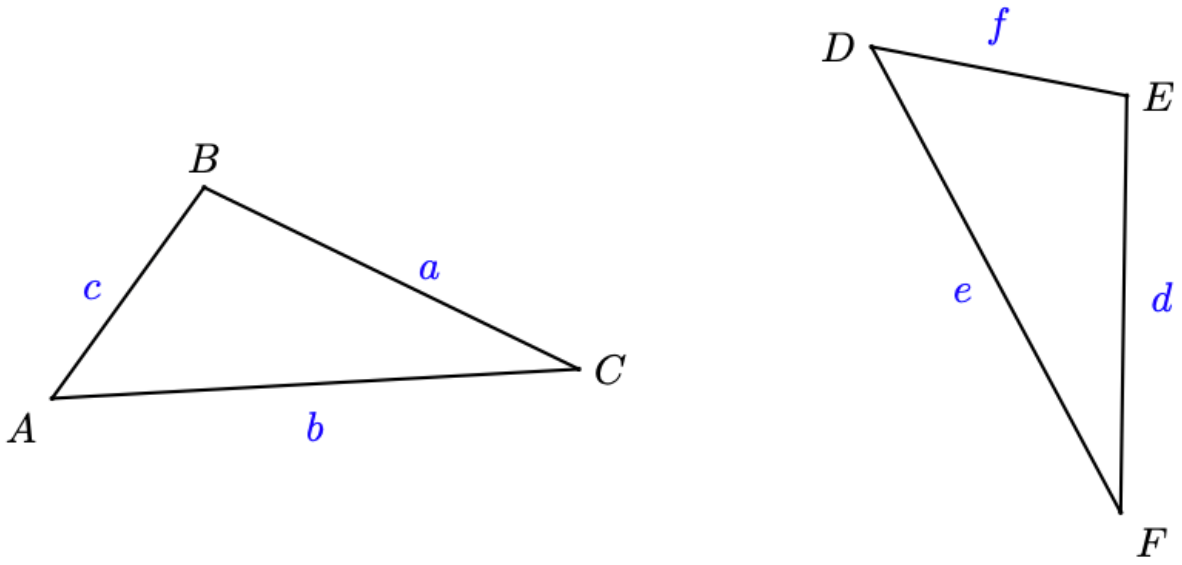


Figure $4.3.1$**:** *Congruent triangles* $\triangle ABC$ *and* $\triangle DEF$

Example 4.3.1

The two triangles in Figure 4.3.2 below are congruent. List the corresponding parts, and find the angles θ, ϕ, and χ, and side z.

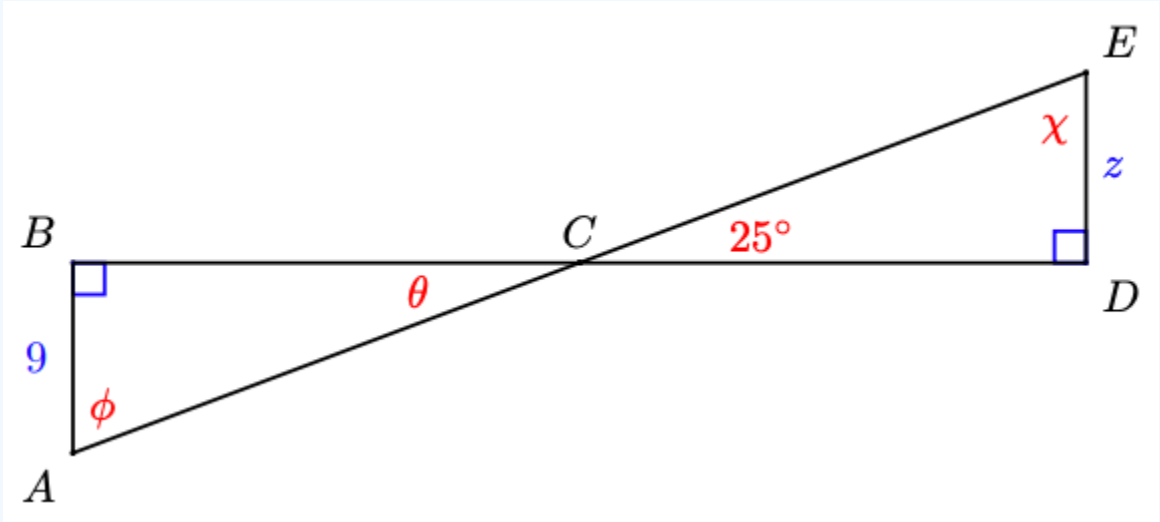


Figure 4.3.2

Solution available at the end of this section.

Checkpoint 4.3.1

The two triangles in Figure 4.3.3 below are congruent. Find the values of α, β, and γ.

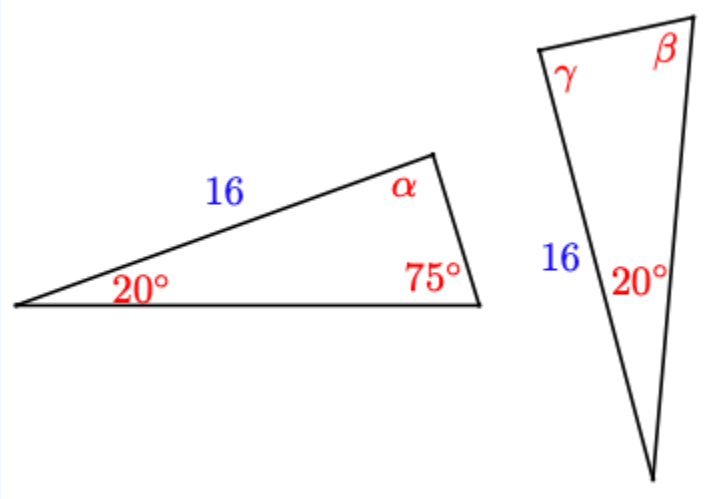


Figure 4.3.3

Answer available at the end of this section.

Recall that the altitude of a triangle is the segment from one vertex of the triangle perpendicular to the opposite side.

Example 4.3.2

Show that the altitude of an equilateral triangle divides it into two congruent right triangles.

Solution available at the end of this section.

For the following Checkpoint, you should recall the following definition from your Geometry class.

Definition: Parallelogram

A **parallelogram** is a quadrilateral with two pairs of parallel sides. The opposite or facing sides of a parallelogram are of equal length and the opposite angles of a parallelogram are of equal measure.

Checkpoint 4.3.2

The diagonal of a parallelogram divides it into two congruent triangles, as shown in Figure 4.3.5 below. List the corresponding parts of the two triangles and explain why each pair is equal.

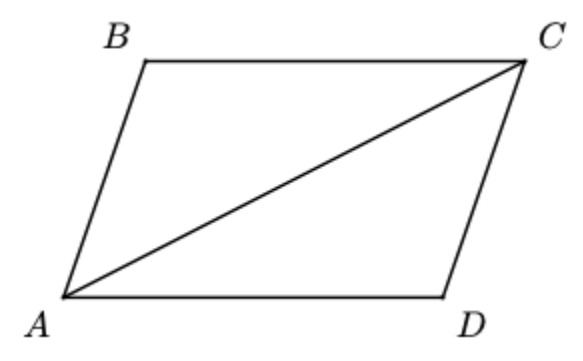


Figure 4.3.5

Answer available at the end of this section.

Similar Triangles

The most common comparison of triangles in both Trigonometry and Calculus will be with **similar triangles**.

Definition: Similar Triangles

Two triangles are **similar** if their corresponding angles are congruent and the lengths of their corresponding sides are proportional. If $\triangle ABC$ is similar to $\triangle DEF$, you write $\triangle ABC \sim \triangle DEF$, where the order of the vertices specifies the correspondence: A with D, B with E, and C with F.

The definition of similar triangles looks almost like the definition of congruent triangles; however, similarity between triangles is a weaker form of congruence—the triangles have the same shape but might be sized differently. This flexibility grants us much power in mathematics.

Fun Fact!

Congruent triangles are similar, but similar triangles need not be congruent.

The corresponding angles between similar triangles are equal, and the corresponding sides are proportional. We can think of one similar triangle as an enlargement or a reduction of the other (see Figure $4.3.6$ below).

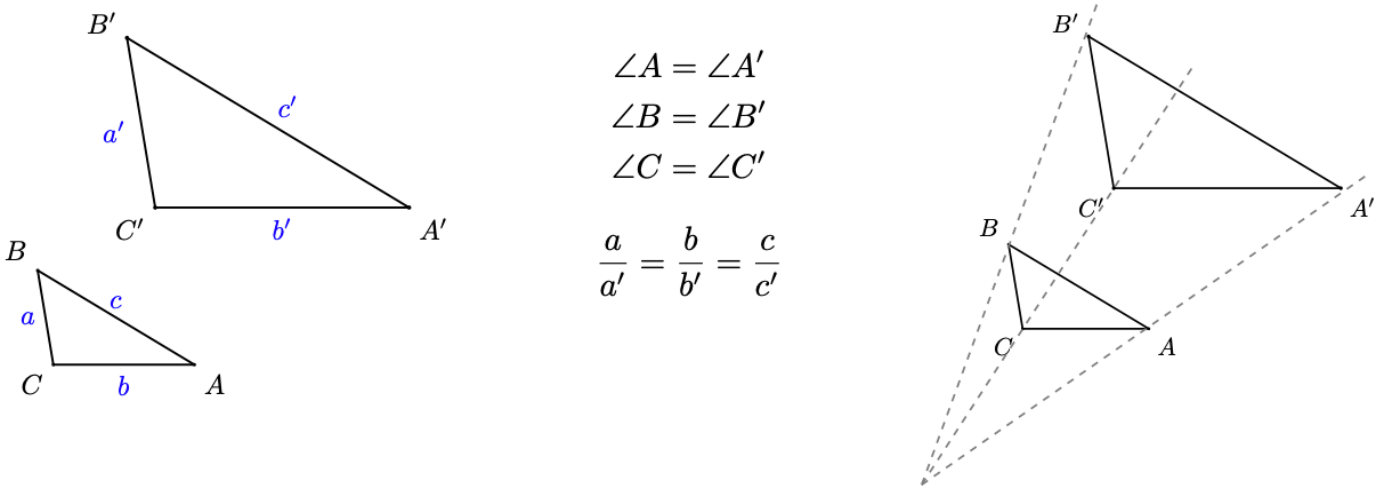


Figure $4.3.6$**:** *Proportional relationships between similar triangles*

Interactive Element: Similar Triangles as Scaled Copies

The following Interactive Element might help you visualize how one similar triangle is just an enlargement or reduction of another.

Interact: Drag the green apex point of the right-hand triangle up and down to change the scale factor k.

Observation: The triangle changes size but keeps its shape — corresponding angles stay equal and every corresponding side ratio equals k. When $k > 1$ it is an enlargement; when $k < 1$, a reduction.

To decide whether two triangles are similar, we must verify *only one* of the two similarity conditions (the other condition will be true automatically).

Theorem: Similarity Conditions

Two triangles are similar whenever any one of the following conditions holds:

- **(AA)** Two angles of one triangle are congruent to two angles of the other. The third pair of angles is then automatically congruent.
- **(SSS)** All three pairs of corresponding sides are proportional.
- **(SAS)** Two pairs of corresponding sides are proportional, and the angles included between those sides are congruent.

Example $4.3.3$

Which of the pairs of triangles shown below are similar?

a.

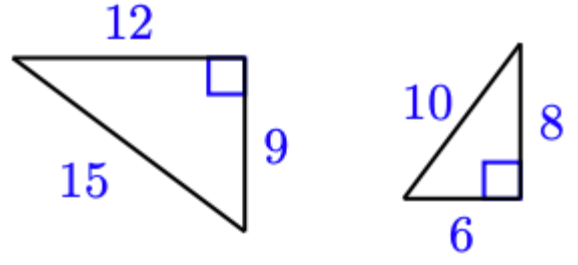


Figure 4.3.7

b.

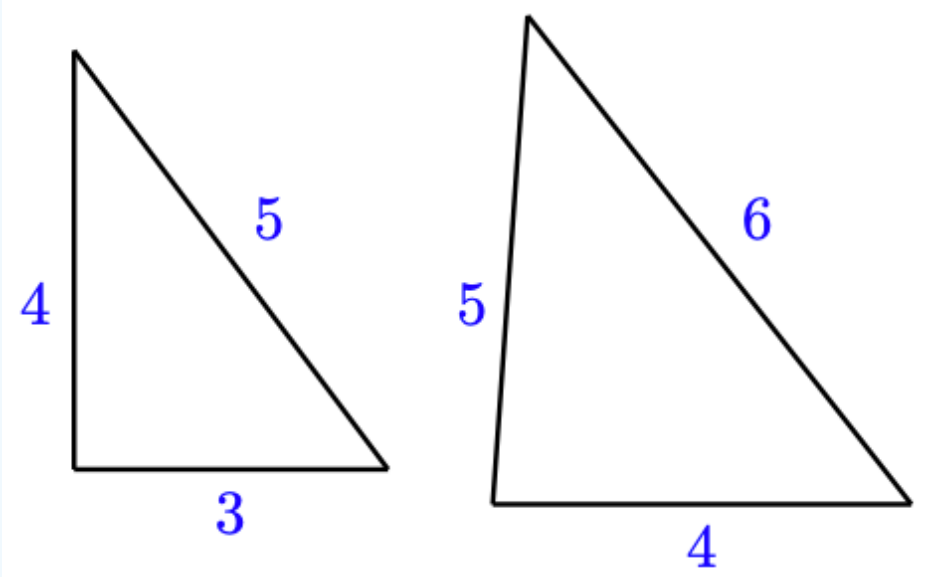


Figure 4.3.8

c.

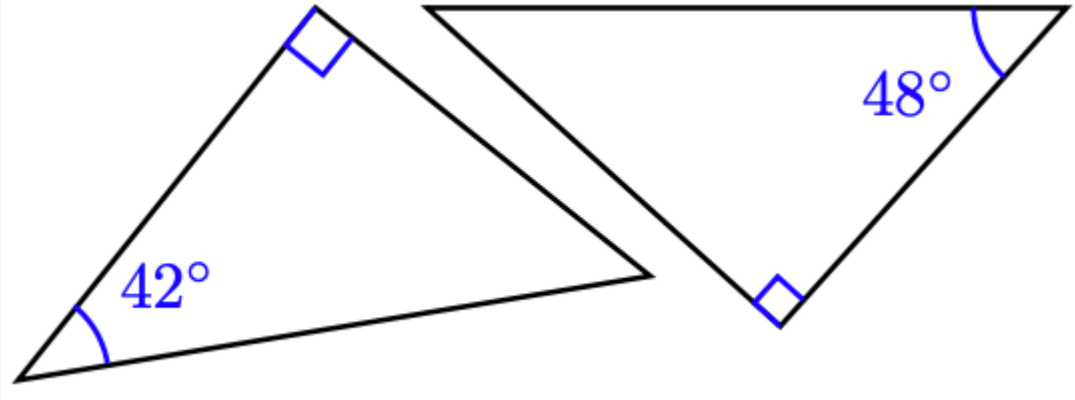


Figure 4.3.9

Solutions available at the end of this section.

Checkpoint 4.3.3

Are the triangles below similar? Explain why or why not in each case.

a.

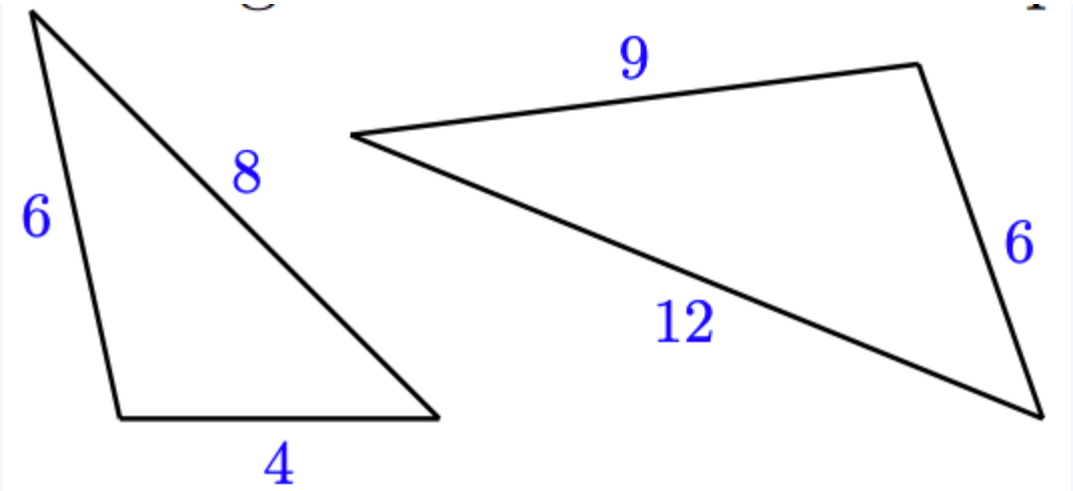


Figure 4.3.10

b.

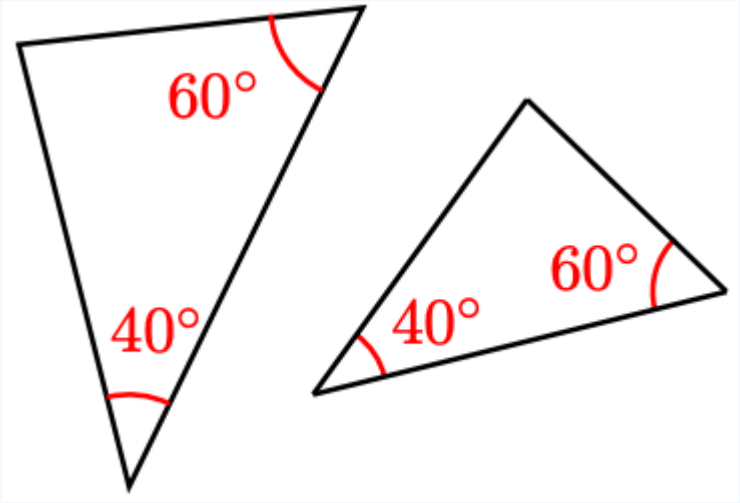


Figure 4.3.11

Answers available at the end of this section.

In part (b) of Checkpoint 4.3.3, the third angle in each triangle must be 80° because the sum of the angles is 180°. Thus, we only need to show that *two* pairs of angles are equal to show that two triangles are similar.

Using Proportions with Similar Triangles

Figure 4.3.12 shows a parallelogram $ABCD$ and two triangles, $\triangle ABE$ and $\triangle FCE$. Can we find the unknown lengths x and y in the larger triangle?

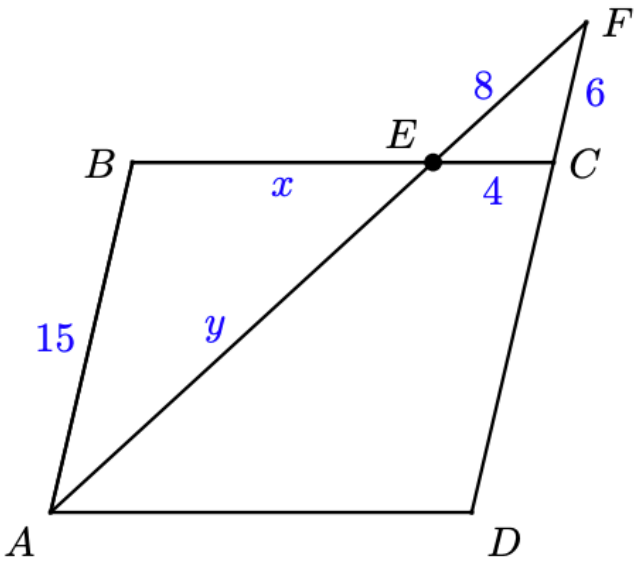


Figure 4.3.12

First, note that two pairs of corresponding angles in the triangles are equal: $\angle BEA$ and $\angle FEC$ are vertical angles, and $\angle EFC$ and $\angle BAE$ are alternate interior angles. However, if two pairs of corresponding angles are equal, the third pair must also be equal. This means that the two triangles are similar, and we can use the fact that their corresponding sides are proportional to find x and y.

Example 4.3.4

Find the value of x in Figure 4.3.12.

Solution available at the end of this section.

What do Arithmetic and $\cancel{\text{Arithmetic}}$ mean?

To get you used to the Mathematical Mantra, every once in a while I will include the "thought processes" during solutions. Just to review, we perform Mathematics in the order we learned it—Arithmetic, Algebra, Trigonometry, ... At each step during a "mechanical" solution process, we should pause and ask ourselves if there is some simple Arithmetic to be done. If not, we move on to any Algebra that can be done. If there is no Algebra to be done, we then move on to Trigonometry.

Thus,

- "$\text{Arithmetic} \implies$ " in a solution means that there is some Arithmetic we could do to clean up the previous expression.
- "$\cancel{\text{Arithmetic}} \to \text{Algebra} \implies$ " means there isn't any Arithmetic but there *is* some Algebra we could use to clean up the previous expression or equation.
- "$\cancel{\text{Arithmetic}} \to \cancel{\text{Algebra}} \to \text{Trigonometry} \implies$ " means there isn't any Arithmetic nor is there any Algebra that can be done to clean up the previous step, so we need to start looking at doing something from Trigonometry.

Checkpoint 4.3.4

Find the value of y in the previous example.

Answer available at the end of this section.

Similar Right Triangles

If two *right* triangles have one pair of corresponding acute angles with the same measure, they are similar. We can use this fact about right triangles to make indirect measurements.

Example 4.3.5

Ivan wants to know the height of a particular building. He asks Kim to hold up a 5-foot pole near the building and measure the length of its shadow. The shadow of the pole is 3 feet long, and the shadow of the building is 12 feet long.

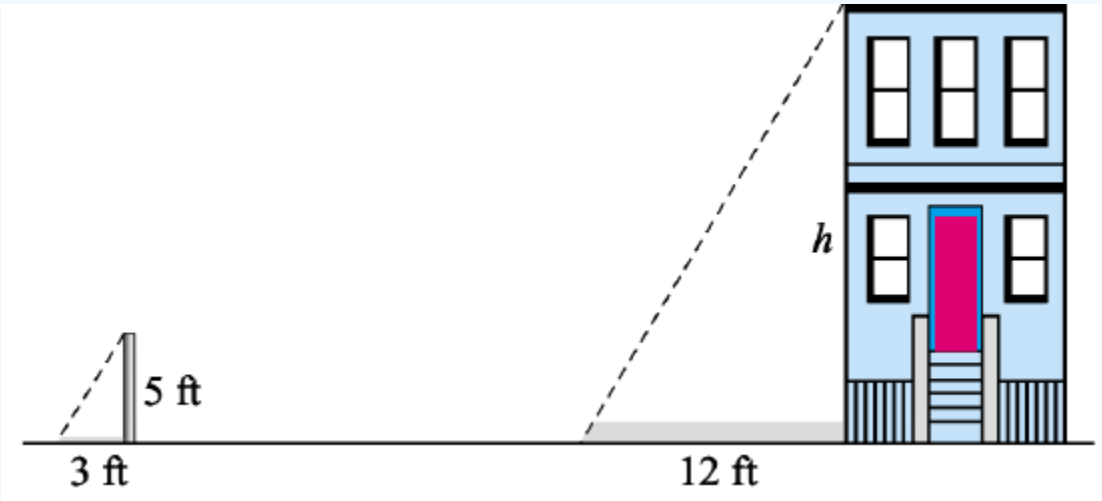


Figure 4.3.13

a. Use similar triangles to write a proportion involving the height of the building.
b. Solve the proportion to find the height of the building.

Solutions available at the end of this section.

Interactive Element: Shadows and Similar Right Triangles

The following Interactive Element might help you visualize why shadows let you measure the height of an object you can't reach directly.

Interact: Drag the green point — the tip of the pole's shadow — left and right to change the sun's angle.

Observation: Both shadows grow and shrink together with the sun rays parallel, so the two right triangles stay similar. The building's shadow stays 4 times the pole's, equal to the ratio of their heights $\left(\frac{20}{5}\right)$ — which is exactly what lets you solve for the unknown height.

Checkpoint 4.3.5

In Example 4.3.2, we created a $30°$-$60°$-$90°$ triangle in which the shorter leg was 4 inches and the hypotenuse was 8 inches. The hypotenuse of another $30°$-$60°$-$90°$ triangle is 5 feet. What is the length of the side opposite the $30°$ angle?

Answer available at the end of this section.

Overlapping Triangles

In some applications, similar triangles may share a side or an angle.

Example 4.3.6

Identify two similar triangles in Figure 4.3.14 below, and write a proportion to find H.

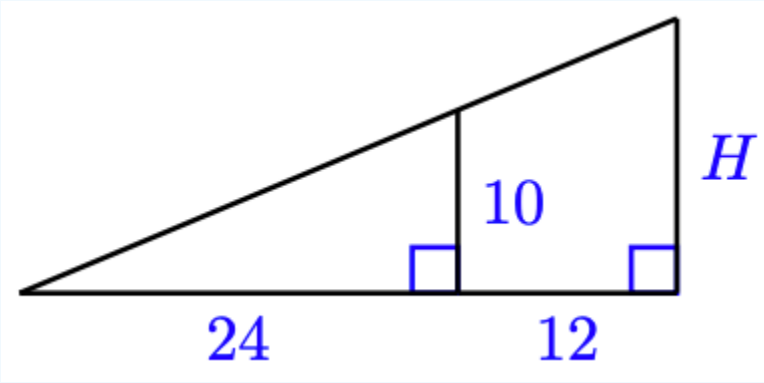


Figure 4.3.14

Solution available at the end of this section.

Checkpoint 4.3.6

Min wants to know the height of a street lamp. She discovers that when she is 12 feet from the lamp, her shadow is 6 feet long. Find the height of the street lamp.

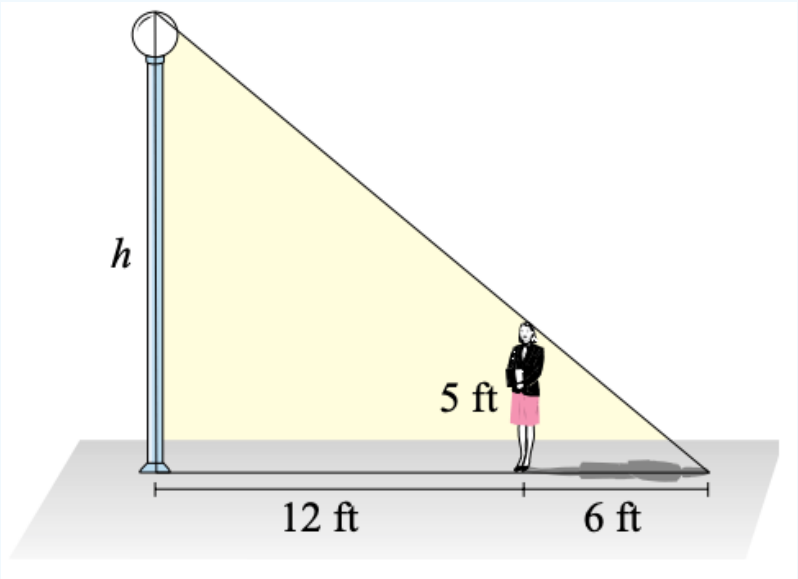


Figure 4.3.16

Answer available at the end of this section.

Time-Sensitive Advice: "Full Refund" and "No Notation" Drop Deadlines

"Full Refund" Drop Deadline

Within the Los Rios Community College District, the last day you can drop a class for a full refund is the Friday of the second week of class. Any drops processed after that point will only receive a *partial* refund.

"No Notation" Drop Deadline

The Sunday following the "full refund" drop deadline is the last day to drop without a "W" (withdrawal) notation on your transcript. You can only take a course, and receive a notation, a maximum of three times within a given community college district in California - this includes notations of D, F, *and* W.

Once you withdraw (W) or fail (D or F) a course three times within the Los Rios Community College District (ARC, CRC, FLC, or SCC), you are *never* allowed to take that course again at any of the colleges within the District. If you find yourself in that unfortunate position, you would need to take this course outside the Los Rios Community College District.

Take a moment of self-reflection to determine if you feel ready to continue this journey for the rest of this long semester. If not, consider dropping by this course by the "full refund" deadline or the "no notation" deadline, at the latest.

Success in College: What is Tutoring?

Tutoring is an age-old practice. The dictionary definition describes a tutor as a person who gives individual or, in some cases, small group instruction. The purpose of tutoring is to help students help themselves or assist or guide them to the point at which they become independent learners and thus no longer need a tutor.

Content knowledge is an essential ingredient for a tutor; however, to be truly effective, a tutor must combine content knowledge with empathy, honesty, and humor. Empathy requires tutors to "read" their students' emotional states, attitudes, and perceptions. Empathy is the ability to see others from their frame of reference and communicate this understanding to the person involved. To establish a supportive relationship with their students, tutors must be open and honest. Students are often reluctant to talk with a stranger about their academic problems. If a tutor is perceived as genuine and strongly desires to listen, students will be more willing to open up and discuss their problems. Humor can also play an essential part in a tutoring session. Humor can reduce tension. Shared laughter is a powerful way to reinforce learning, and humor can set students at ease and increase rapport. Tutors can also use humor to compliment, guide, or positively provide feedback.

In addition, a successful tutor demonstrates a caring attitude. Caring consists of being punctual and organized for the tutoring session, establishing a learning relationship with the student, developing unique teaching strategies, and becoming familiar with the learning process. Ultimately, tutoring is sharing yourself with another student in a way that makes a difference in both your lives.

There are many benefits to tutoring.

- Heightens a sense of competency/adequacy in conforming to a new role.
- Encourages higher levels of thinking.
- Permits more advanced students to study below-level material without embarrassment.
- Increases motivation to learn to maintain the new role.
- Increases ability to manage own learning and study strategies.
- Increases subject-specific knowledge.
- Increases related general knowledge.
- Increases understanding of the subject area.
- Improves attitude toward the subject area.
- Provides more empathy with students.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Does your college have a place such as a tutoring center where you can get help?
2. If so, what are the hours that this tutoring center is open?
3. If you are unsure about the answers to either of these questions, feel free to ask your professor - that's what they are there for!

Answers

Section Preview

This entire section focuses solely on similar triangles and their applications.

Section Preview

- **Arithmetic**
  - Ratios, Proportions, and Proportional Reasoning: Understanding what a proportion *is* conceptually (ratios and proportional reasoning) is the prerequisite for grasping why similar triangles have proportional sides. Proportions are also used within the examples.

Section Preview

- **Arithmetic**
  - Simplifying Fractions and Mixed Numbers: For verifying equal ratios.
- **Solving Equations**
  - Solving Rational Equations: For solving proportions algebraically.

Section Preview

- Determine when two triangles are congruent.
- Use proportion to find a side given two similar triangles.
- Solve application problems involving similar right triangles.

Example 4.3.1

In these triangles, $\angle B = \angle D$ because they are both right angles and $\angle BCA = \angle DCE$ because they are vertical angles, so $\theta = 25^\circ$. The third angles, $\angle A$ and $\angle E$, must also be equal, so $\phi = \chi = 65^\circ$. (Do you see why?) The sides opposite each pair of corresponding angles are equal, so $AB = DE, BC = CD$ and $AC = CE$. In particular, we find that $z = 9$.

Checkpoint 4.3.1

$\alpha = 85^\circ, \beta = 75^\circ, \gamma = 85^\circ$

Example 4.3.2

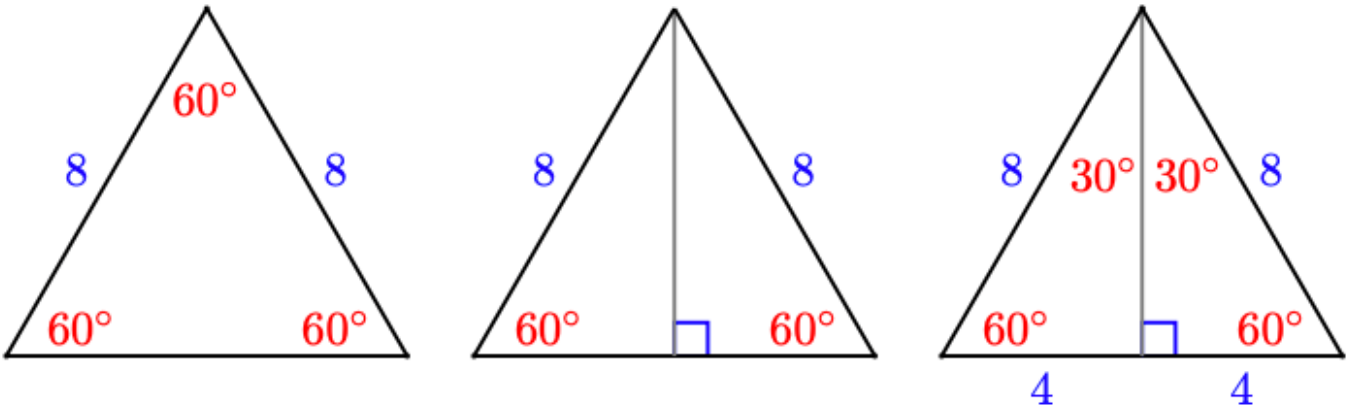


Figure 4.3.4

Consider, for example, an equilateral triangle of side 8 inches, as shown above. The altitude is perpendicular to the base, so each half of the original triangle is a right triangle. Because each right triangle contains a 60° angle, the remaining angle in each triangle must be $90^\circ - 60^\circ = 30^\circ$. Both triangles have a side of length 8 between the angles of 30° and 60°, so they are congruent. (Consequently, the short sides of the congruent triangles are equal, so each is half the original base.)

Checkpoint 4.3.2

$\angle BCA = \angle CAD$ and $\angle BAC = \angle ACD$ because they are alternate interior angles. If two angles in a triangle are equal, so is the third pair, so $\angle B = \angle D$. $BC = AD$ and $AB = CD$ because they are opposite sides of a parallelogram, and $AC = AC$.

Example 4.3.3

a. We will check whether the corresponding sides are proportional. We compute the ratios of the corresponding sides, making sure to write each ratio in the same order, $\frac{\text{larger triangle}}{\text{smaller triangle}}$. (The other order, $\frac{\text{smaller triangle}}{\text{larger triangle}}$, would also work, as long as we use the same order for all the ratios.)

$$\text{shorter legs: } \frac{9}{6} \quad \text{longer legs: } \frac{12}{8} \quad \text{hypotenuses: } \frac{15}{10}$$

Because all of these ratios are equal to 1.5, the triangles are similar.

b. The ratios of corresponding sides are not equal: the ratio of the longest sides is $\frac{6}{5}$, but the ratio of the smallest sides is $\frac{4}{3}$. The triangles are not similar.

c. The missing angle of the first right triangle is 48°, and the missing angle in the second right triangle is 42°, so three pairs of angles match. The triangles are similar.

Checkpoint 4.3.3

a. The triangles are similar because $\frac{4}{6} = \frac{6}{9} = \frac{8}{12}$, so the sides are proportional.

b. The third angle in both triangles is $80°$, so the triangles are similar because their corresponding angles are equal.

Example 4.3.4

We see that x is the length of the shortest side in $\triangle ABE$. We know the short side in $\triangle FCE$ and the lengths of the medium sides in each triangle. If we form the ratios of the short and medium sides, we obtain the following proportion:

$$\frac{\text{larger triangle}}{\text{smaller triangle}} : \frac{x}{4} = \frac{15}{6}$$

To solve the proportion, we multiply both sides by the LCD to get

$$\begin{aligned} & & 6x &= 4(15) & \\ \text{Arithmetic} \implies & & 6x &= 60 & \text{(multiply)} \\ \cancel{\text{Arithmetic}}\text{→Algebra} \implies & & x &= 10 & \text{(divide both sides by 6)} \end{aligned}$$

Checkpoint 4.3.4

$y = 20$

Example 4.3.5

a. In Figure 4.3.13, we see two right triangles: One triangle is formed by the building and its shadow, and the pole and its shadow form the other. Because the light rays from the sun are parallel, the two angles at the tips of the shadows are equal. Thus, the two right triangles are similar, and their corresponding sides are proportional. The ratios of heights and bases in the two triangles yield the proportion:

$$\frac{\text{larger triangle}}{\text{smaller triangle}} : \frac{h}{5} = \frac{12}{3}$$

b. To solve the proportion, we multiply both sides by the LCD to get

$$\begin{aligned} \cancel{\text{Arithmetic}}\text{→Algebra} \implies & 3h = 5(12) & \text{(multiply both sides by the LCD of all terms in the equation)} \\ \text{Arithmetic} \implies & 3h = 60 & \text{(multiply)} \\ \cancel{\text{Arithmetic}}\text{→Algebra} \implies & h = 20 & \text{(divide both sides by 3)} \end{aligned}$$

The building is 20 feet tall.

Checkpoint 4.3.5

2.5 feet

Example 4.3.6

The two triangles overlap, sharing the marked angle, as shown below. Because each triangle also has a right angle, they are similar.

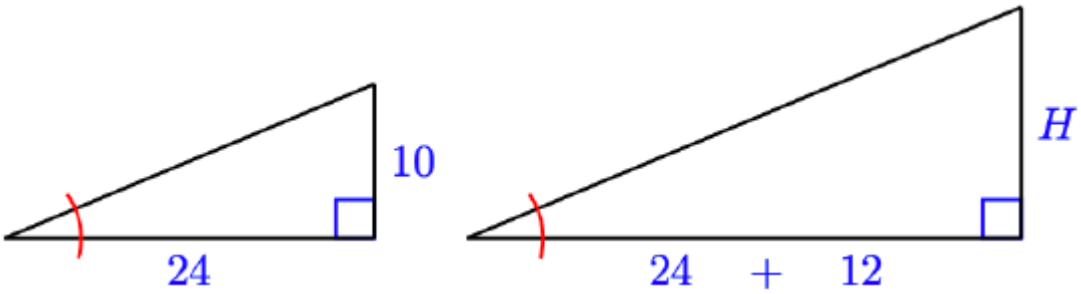


Figure 4.3.15

Note that the base of the larger triangle is $24 + 12 = 36$. The ratio of the heights and the ratio of the bases must be equal, so we write the following proportion:

$$\begin{aligned} & \frac{H}{10} = \frac{36}{24} & \text{(start with the proportion)} \\ \cancel{\text{Arithmetic}}\text{→Algebra} \implies & 24H = 360 & \text{(multiply both sides by the LCD)} \\ \cancel{\text{Arithmetic}}\text{→Algebra} \implies & H = \frac{360}{24} & \text{(divide both sides by 24)} \\ \text{Arithmetic} \implies & H = 15 & \text{(division)} \end{aligned}$$

Checkpoint 4.3.6

15 feet

This page titled 4.3: Similar Triangles is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

4.3.1: Resources and Key Concepts

Key Concepts

Definitions

- **Congruent Triangles:** Two triangles that have equal corresponding angles and identical corresponding side lengths. They are the same shape and size.
- **Parallelogram:** A four-sided figure (quadrilateral) in which both pairs of opposite sides are parallel and have equal length.
- **Similar Triangles:** Two triangles whose corresponding angles are equal. This implies they have the same shape but may be different sizes.

Theorems

- **Similarity Conditions:** Two triangles are determined to be similar if one of two conditions is met: either their corresponding angles are equal, or their corresponding sides are proportional. If one condition is true, the other is automatically true.

Common Mistakes

- **Confusing Congruent and Similar:** While congruent triangles are always similar, similar triangles are not necessarily congruent. Similarity requires only the same shape (equal angles), not necessarily the same size.
- **Incorrectly Setting Up Proportions:** When creating ratios of corresponding sides, the order must be consistent. For example, if the first ratio is $\frac{\text{larger triangle side}}{\text{smaller triangle side}}$, all subsequent ratios in the proportion must follow the same order.
- **Using Incorrect Side Lengths in Overlapping Triangles:** When triangles overlap, be careful to identify the full side length of the larger triangle. For instance, in Example 6, the base of the larger triangle is the sum of two segments $(24 + 12)$, not just one of the individual segments.

This page titled 4.3.1: Resources and Key Concepts is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.3.2: Homework

Reading Questions

1. What two conditions must be met for two triangles to be congruent?
2. What is the symbol used to indicate that two triangles are congruent?
3. What is the key difference between congruent triangles and similar triangles?
4. Are all congruent triangles similar? Are all similar triangles congruent?
5. What does it mean for the corresponding sides of two triangles to be "proportional"?
6. According to the Similarity Conditions theorem, what are the two ways to verify that two triangles are similar?
7. If two right triangles share one pair of equal acute angles, are the triangles similar?
8. In Example 2, how is the altitude of an equilateral triangle used to create two congruent right triangles?
9. In Example 5, what assumption is made about the sun's light rays that allows us to conclude the triangles are similar?
10. When triangles overlap, as in Example 6, what must you be careful about when determining the side lengths for your proportion?

Skills Refresher

The following is a set of review exercises you will need for this section.

Skills Refresher

For each of the following exercises, solve the equation.

1. $\frac{x}{12} = \frac{3}{x}$
2. $1 + \frac{x}{2} = \frac{2x}{5}$
3. $\frac{x}{6} = \frac{1}{6} + \frac{x}{6}$
4. $x + \frac{6}{x} = -5$
5. $\frac{5}{2x} = \frac{17}{18} - \frac{1}{3x}$

Answers available at the end of this section.

Homework

Vocabulary Check

1. If given two triangles that are the same size and shape, we say they are ___; however, if their size is different but their shape remains the same, we call them ___ triangles.
2. The ___ of a triangle is the segment from one vertex of the triangle perpendicular to the ___ side.

Concept Check

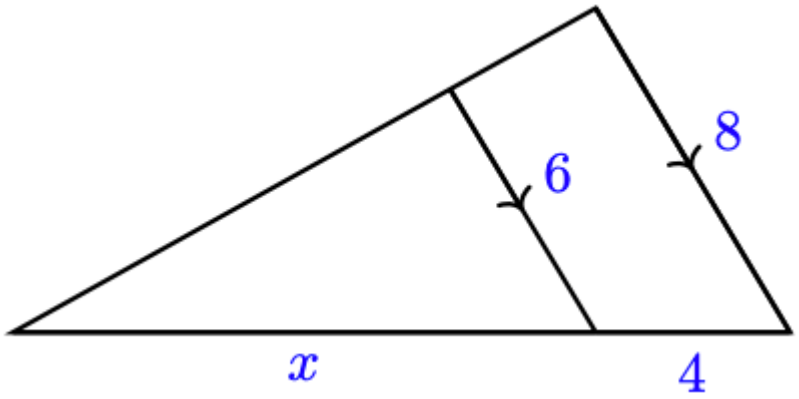
3. What is the difference between congruent triangles and similar triangles?
4. What is the name of the short-cut method for solving proportions? Why does the method work?
5. In two triangles, are the triangles similar if two corresponding pairs of angles are equal? How do you know?

True or False? For each of the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

6. The diagonal of a parallelogram splits the shape into two congruent triangles.
7. In a $30°$-$60°$-$90°$ triangle, the leg opposite the $60°$ angle is half the length of the hypotenuse.
8. If two right triangles have one pair of angles with the same measure, then the triangles are similar.

Basic Skills

9. For the triangles shown, which of the following equations is true? Explain why.



a. $\frac{4}{x} = \frac{6}{8}$

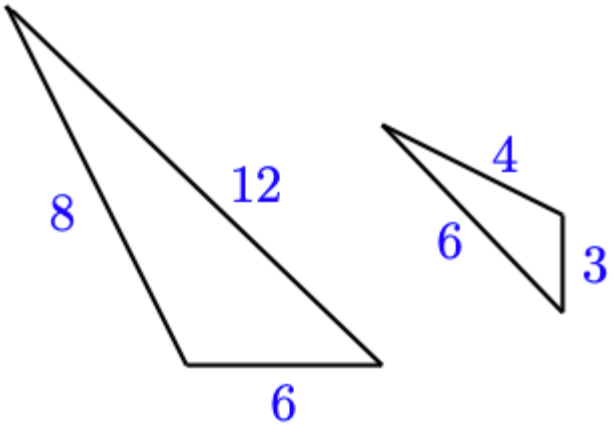
b. $\frac{x}{4} = \frac{6}{8}$

c. $\frac{x}{x+4} = \frac{6}{8}$

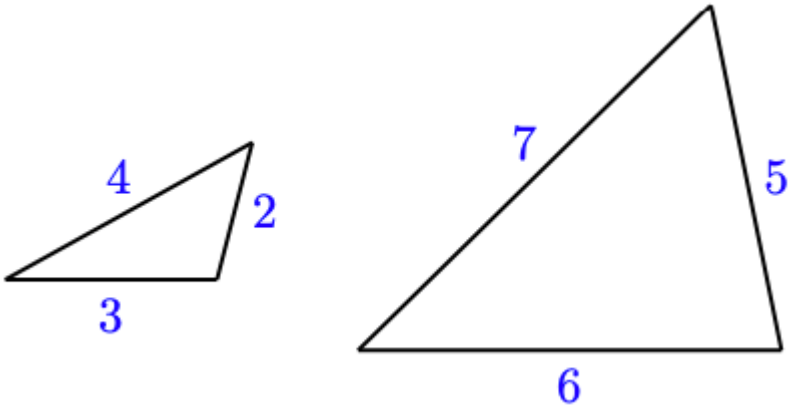
d. $\frac{x}{x+4} = \frac{6}{14}$

For each of the following exercises, decide whether the triangles are similar and explain why or why not.

10.

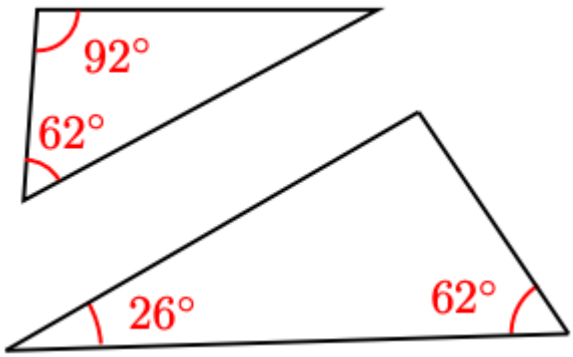


11.

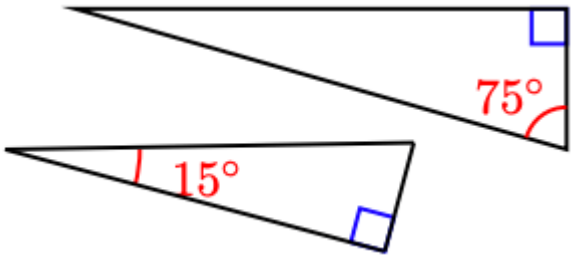


12.

MyOpenMath

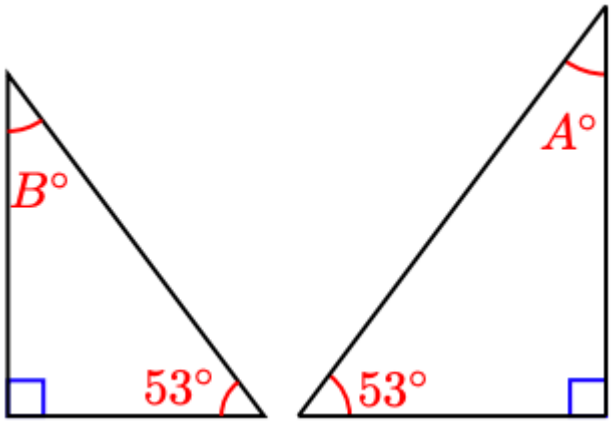


13.

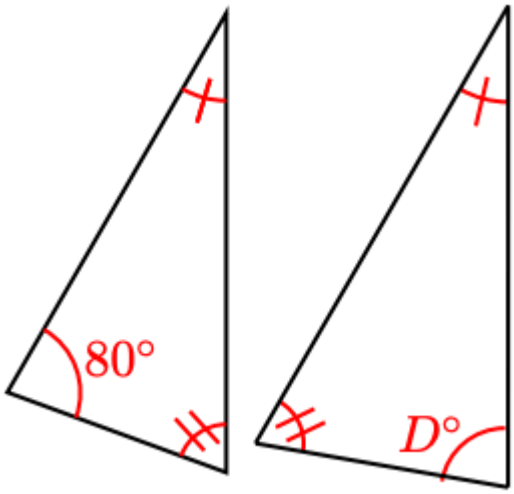


For each of the following exercises, assume the triangles are similar. Solve for the variables. (Figures are not drawn to scale.)

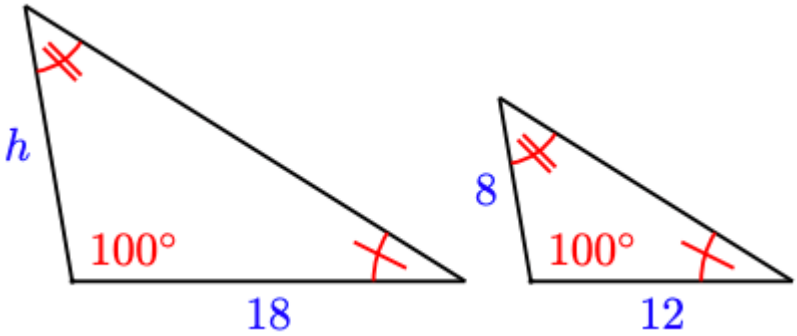
14.



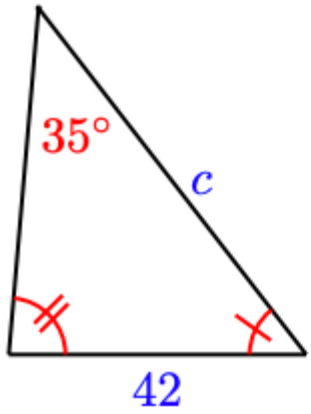
15.

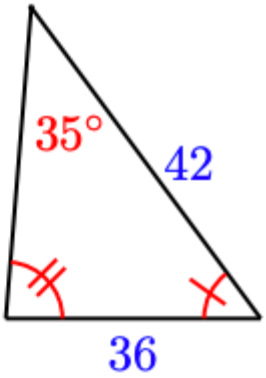


16.



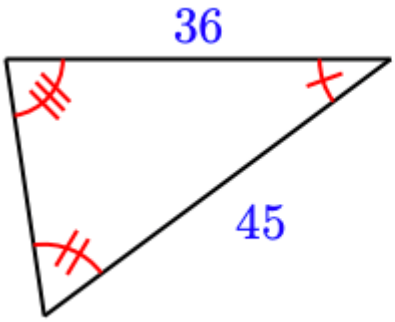
17.

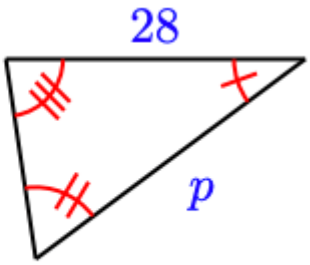




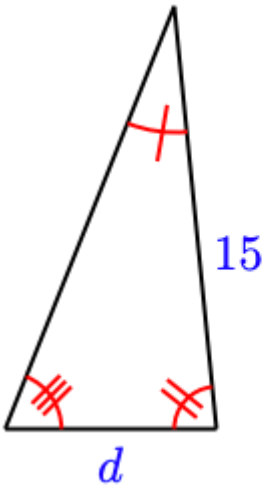
For each of the following exercises, use properties of similar triangles to solve for the variable.

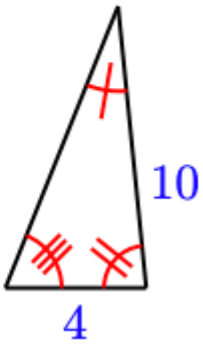
18.





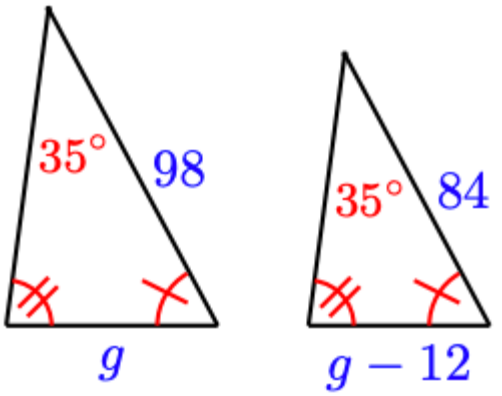
19.



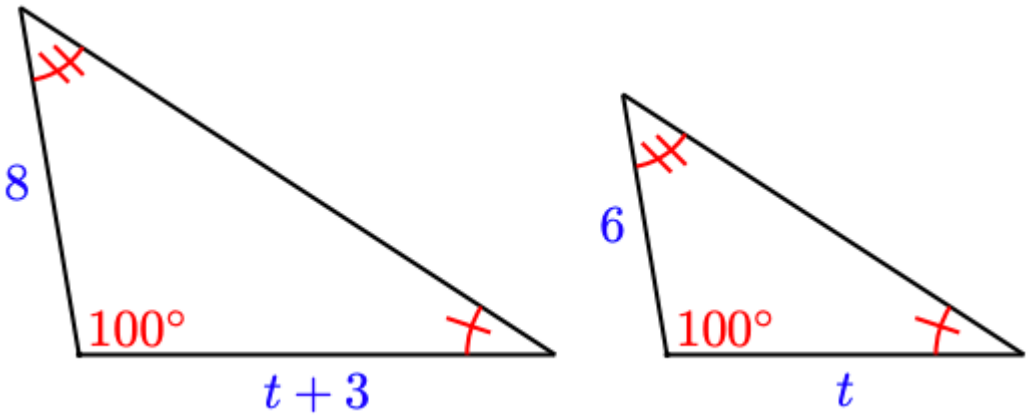


20.

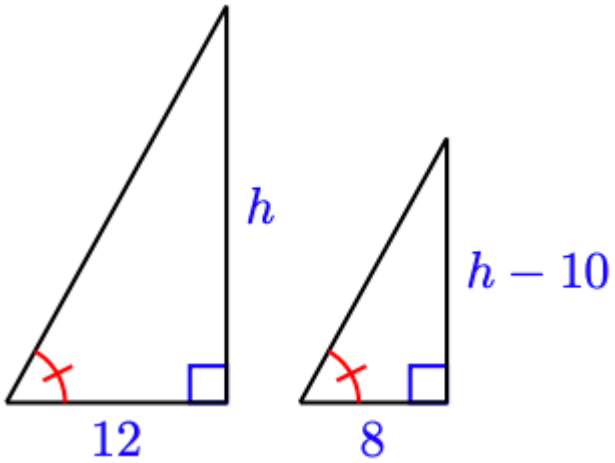
21.

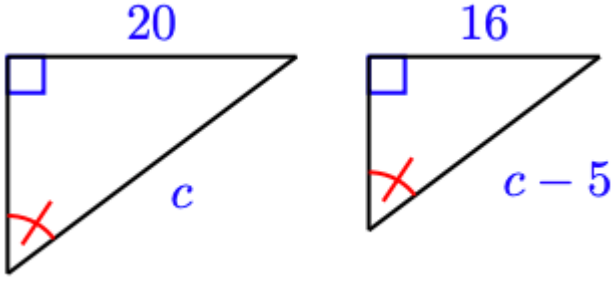


22.



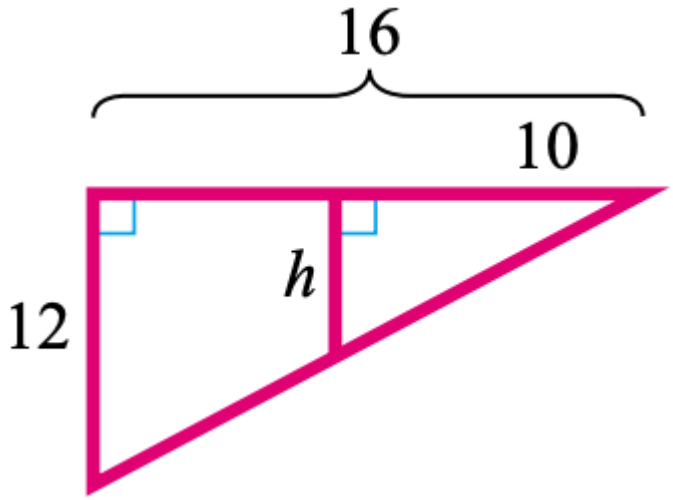
23.



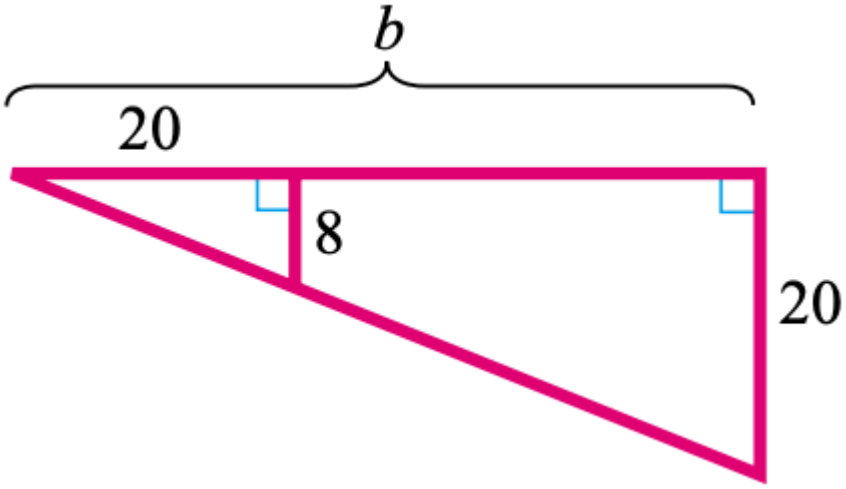


For each of the following exercises, use properties of similar triangles to solve for the variable.

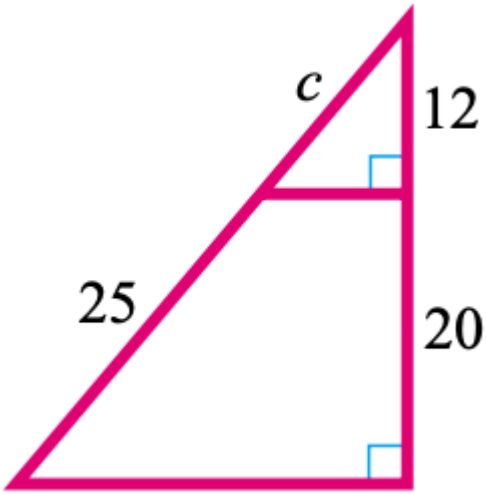
24.



25.

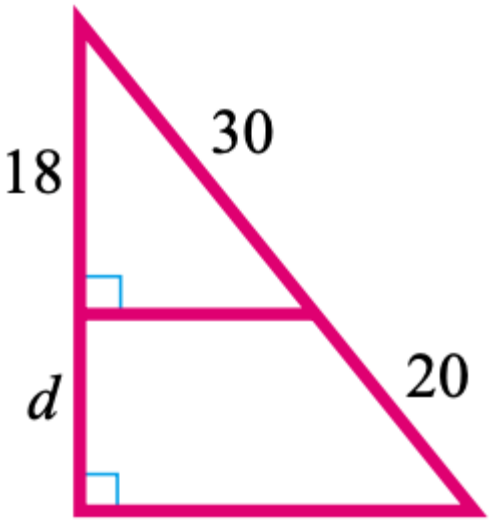


26.

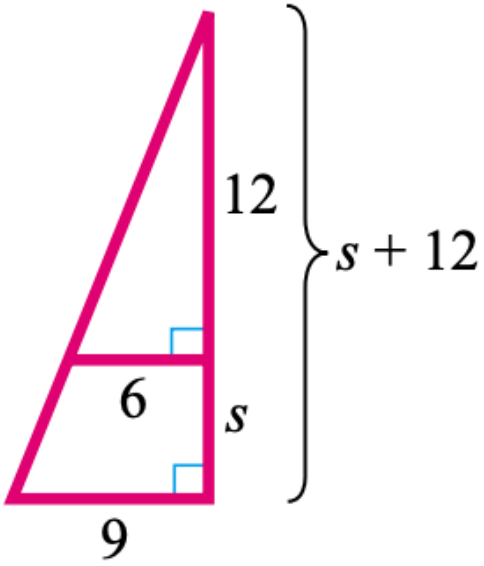


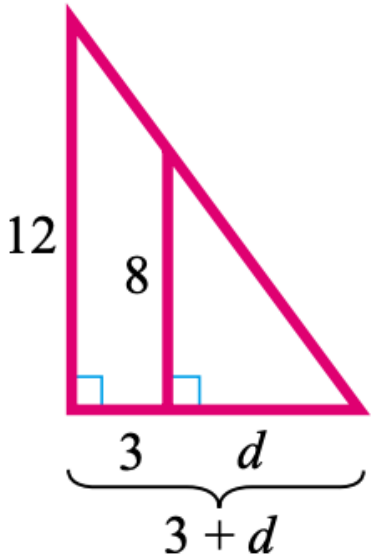
27.

28.



29.

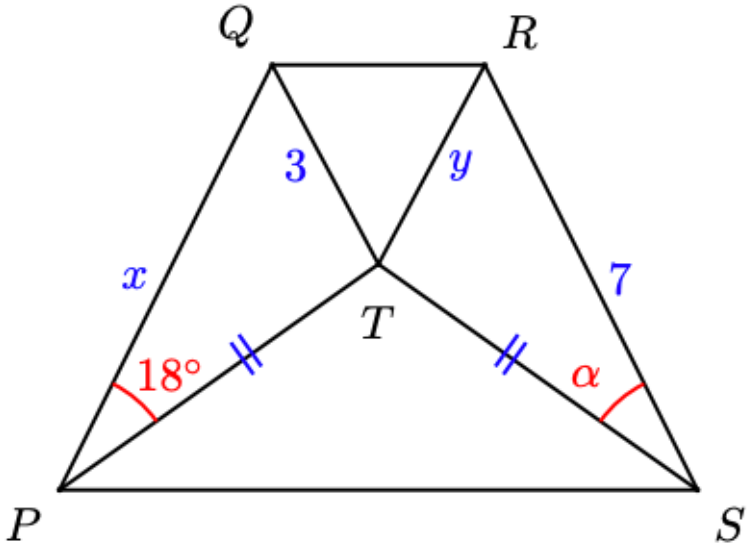




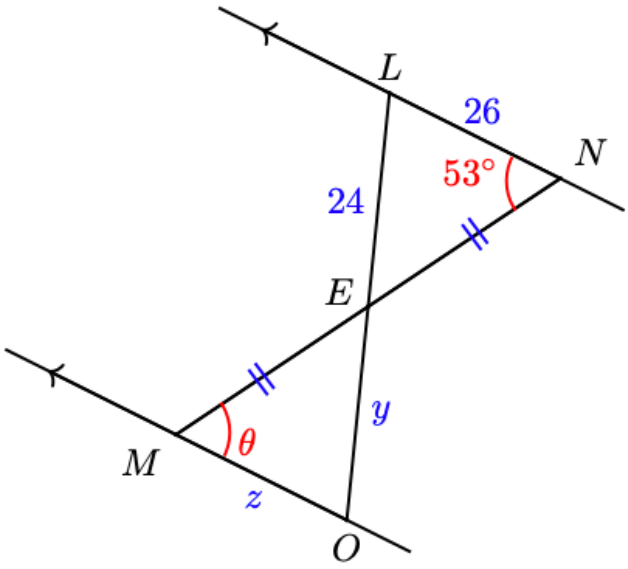
Synthesis Questions

For each of the following exercises, name two congruent triangles and find the unknown quantities.

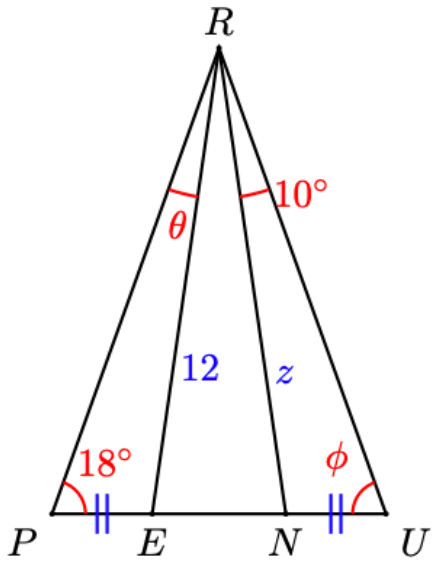
30. $PQRS$ is an isosceles trapezoid.



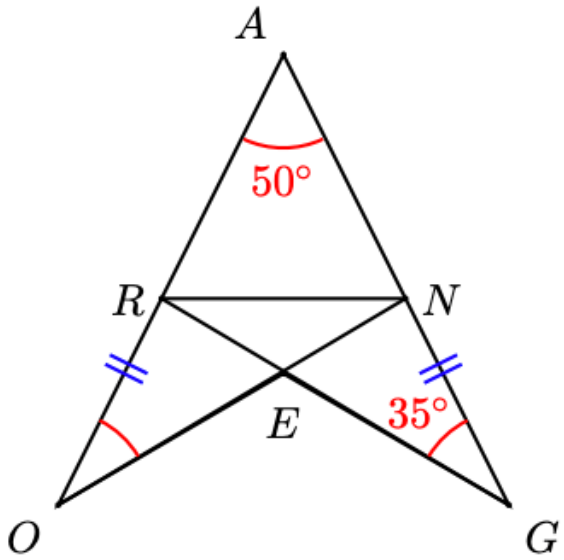
31.



32. ΔPRU is isosceles.

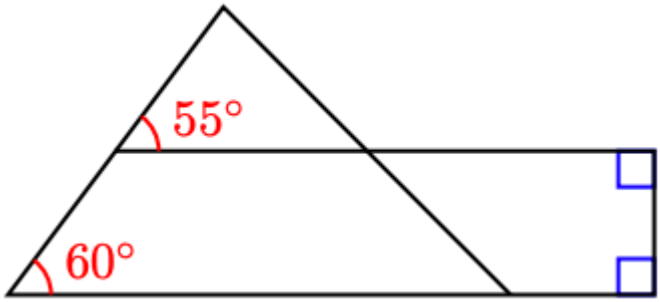


33. ΔARN is isosceles and $OR = NG$. Find $\angle RNG$ and $\angle RNO$.

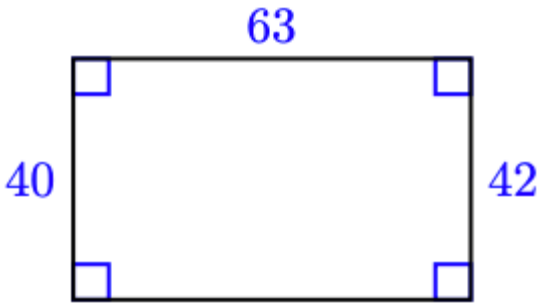


For each of the following exercises, explain why the measurements shown are inaccurate.

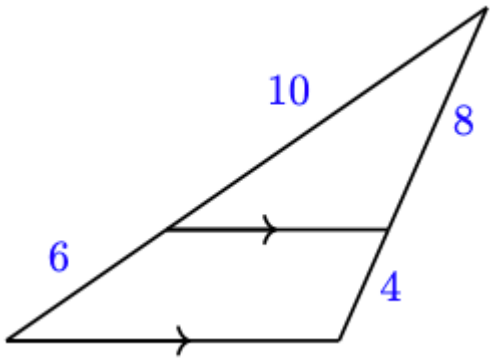
34.



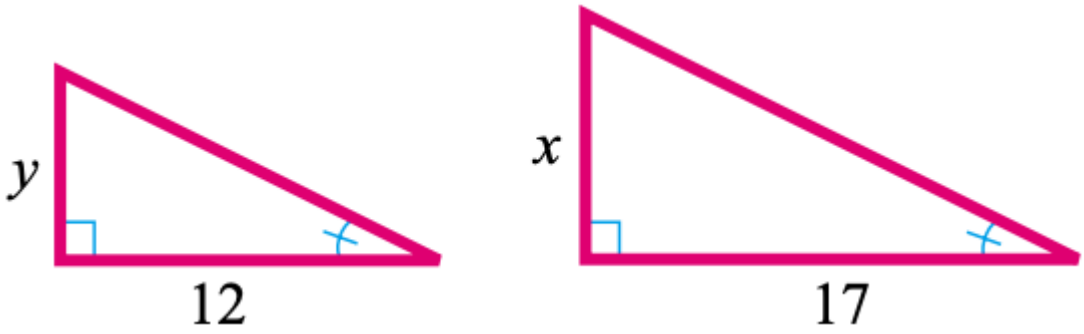
35.



36.

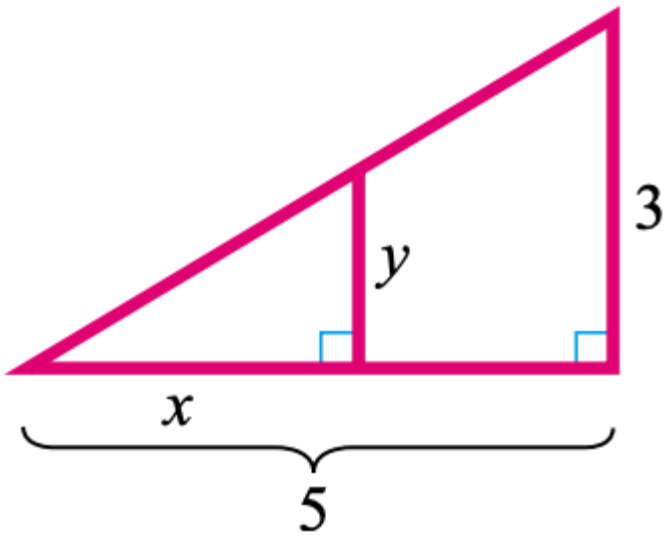


37. The follow pairs of triangles are similar. Solve for y in terms of x.

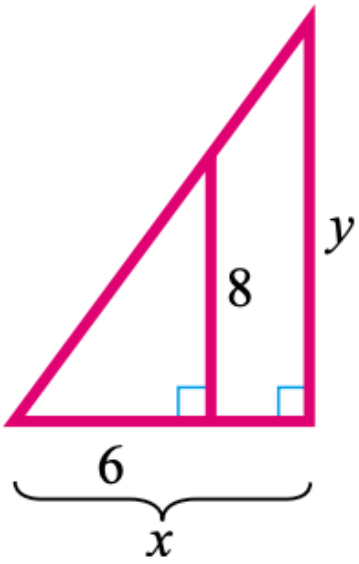


For each of the following exercises, solve for y in terms of x.

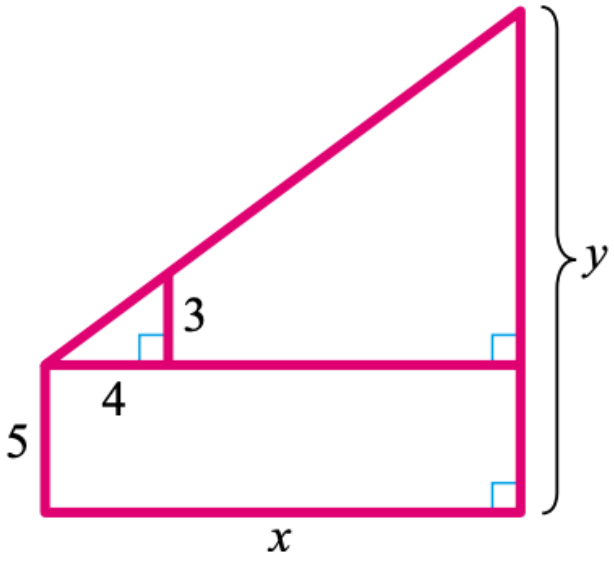
38.



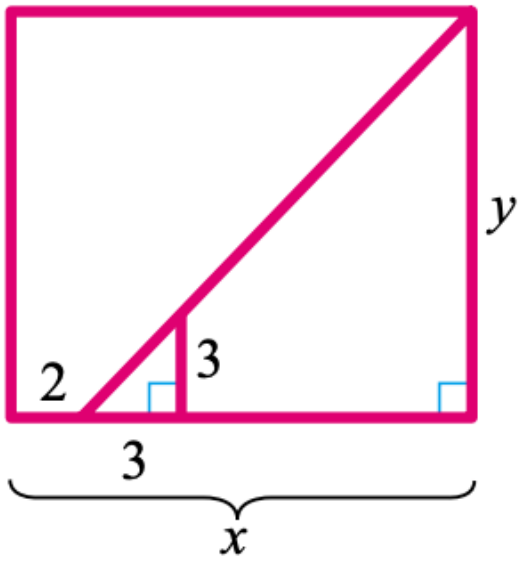
39.



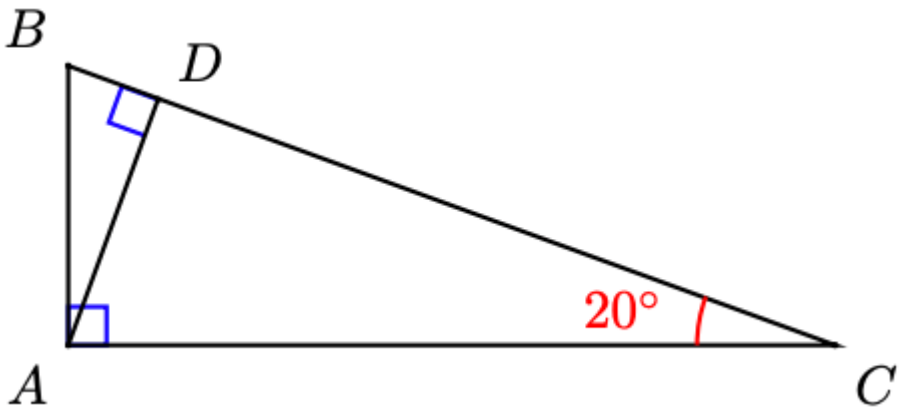
40.



41.



42. Triangle ABC is a right triangle, and AD meets the hypotenuse BC at a right angle.



a. If $\angle ACB = 20^\circ$, find $\angle B$, $\angle CAD$, and $\angle DAB$.

b. Find two triangles similar to ΔABC. List the corresponding sides in each of the triangles.

Applications

43. **Measuring Distances.** Ivan and Kim want to measure the distance across a stream. They mark point A directly across the stream from a tree at point T on the opposite bank. Ivan walks from point A down the bank a short distance to point B and sights the tree. He measures the angle between his line of sight and the stream bank.

a. Draw a figure showing the stream, the tree, and the right triangle $\triangle ABT$.

b. Meanwhile, Kim, still standing at point A, walks away from the stream at right angles to Ivan's path. Ivan watches her progress and tells her to stop at point C when the angle between the stream bank and his line of sight to Kim is the same as the angle from the stream bank to the tree. Add triangle $\triangle ABC$ to your figure.

c. Ivan now measures the distance from point A to Kim at point C. Explain why this distance is the same as the distance across the stream.

44. **Measuring Distances.** If you have a baseball cap, here is another way to measure the distance across a river. Stand at point

A directly across the river from a convenient landmark, say a large rock, on the other side. Tilt your head down so that the brim of the cap points directly at the base of the rock, R.

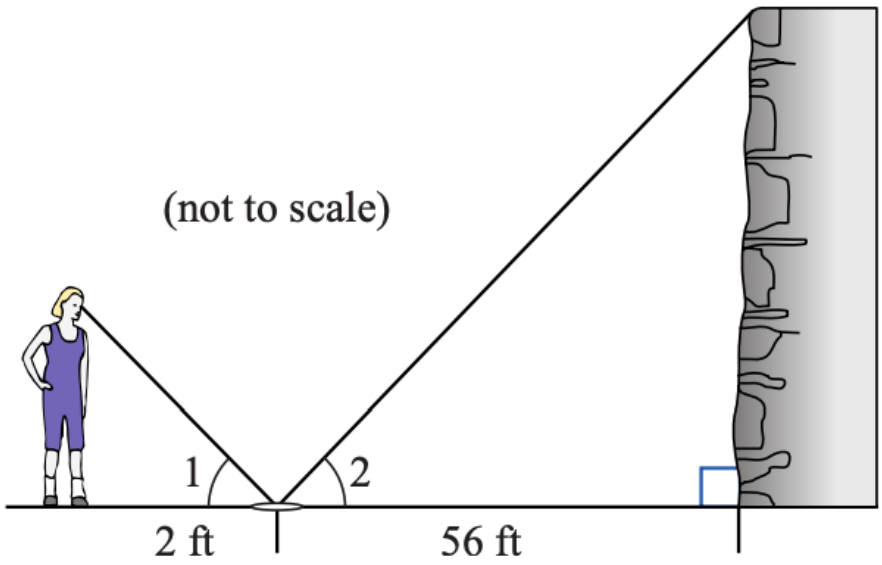
a. Draw a figure showing the river, the rock, and the right triangle $\triangle ABR$, where B is the location of your baseball cap on your head.

b. Now, without changing the angle of your head, rotate 90° and sight along the bank on your side of the river. Have a friend mark the spot C on the ground where the brim of your cap points. Add triangle $\triangle ABC$ to your figure.

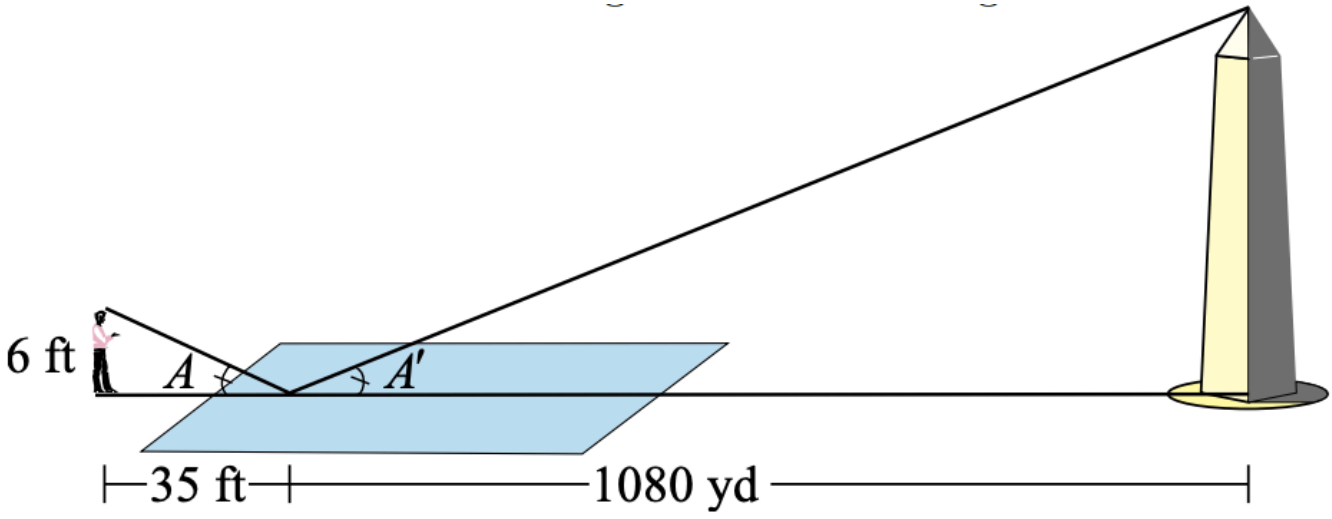
c. Finally, you can measure the distance from point A to point C. Explain why this distance is the same as the distance across the river.

For each of the following exercises, use properties of similar triangles to solve.

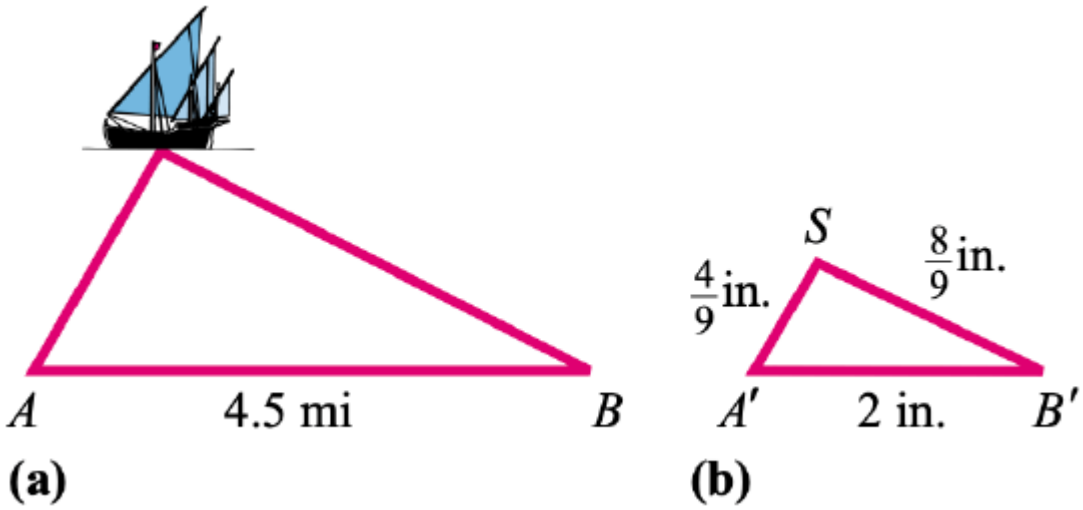
15. **Heights.** A rock climber estimates the height of a cliff she plans to scale. She places a mirror on the ground to see the top of the cliff in the mirror while she stands straight. The angles 1 and 2 formed by the light rays are equal, as shown in the figure. She then measures the distance to the mirror (2 feet) and the distance from the mirror to the base of the cliff (56 feet). How high is the cliff if she is 5 feet 6 inches tall?



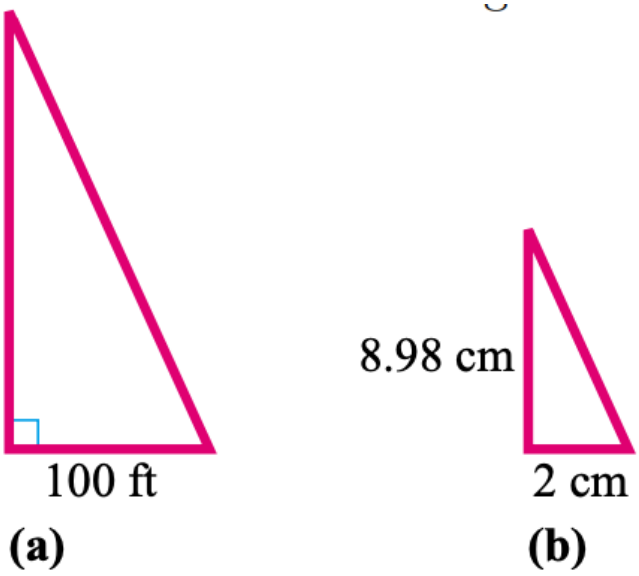
16. **Heights.** Lap wants to estimate the height of the Washington Monument. He notices that he can see the reflection of the top of the monument in the reflecting pool. He is 35 feet from the tip of the reflection, and that point is 1080 yards from the base of the monument, as shown below. From his physics class, Lap knows that the angles marked (in the image below) are equal. If Lap is 6 feet tall, what is his estimate for the height of the Washington Monument?



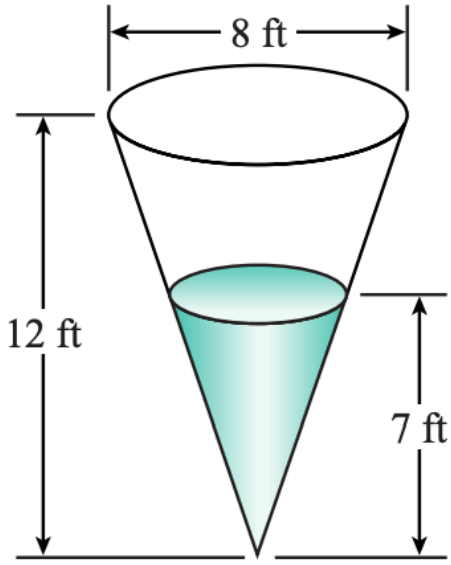
17. **Distances.** In the sixth century BC, the Greek philosopher and mathematician Thales used similar triangles to measure the distance to a ship at sea. Two observers on the shore at points A and B would sight the ship and measure the angles formed, as shown in Figure (a). They would then construct a similar triangle, as shown in Figure (b), with the same angles at A and B , and measure its sides. (This method is called *triangulation*.) Use the lengths given in the figures to find the distance from the observer at location B to the ship.



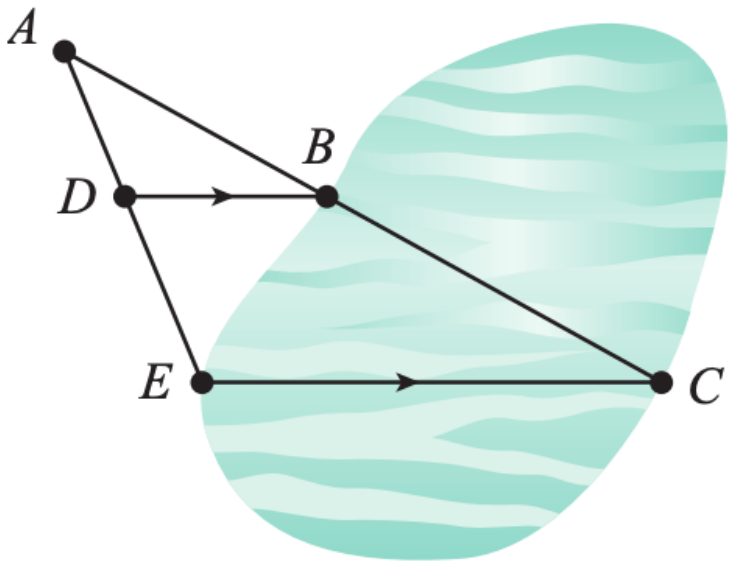
18. **Distances.** The Capilano Suspension Bridge is a footbridge that spans a 230-foot gorge north of Vancouver, British Columbia. Before crossing the bridge, you decide to estimate its length. You walk 100 feet downstream from the bridge and sight its far end, noting the angle formed by your line of sight, as shown in Figure (a). You then construct a similar right triangle with a two-centimeter base, as shown in Figure (b). You find that the height of your triangle is 8.98 centimeters. How long is the Capilano Suspension Bridge?



49. **Surface Area.** A conical tank is 12 feet deep, and the top's diameter is 8 feet. If the tank is filled with water to a depth of 7 feet, as shown in the figure at right, what is the area of the exposed surface of the water?

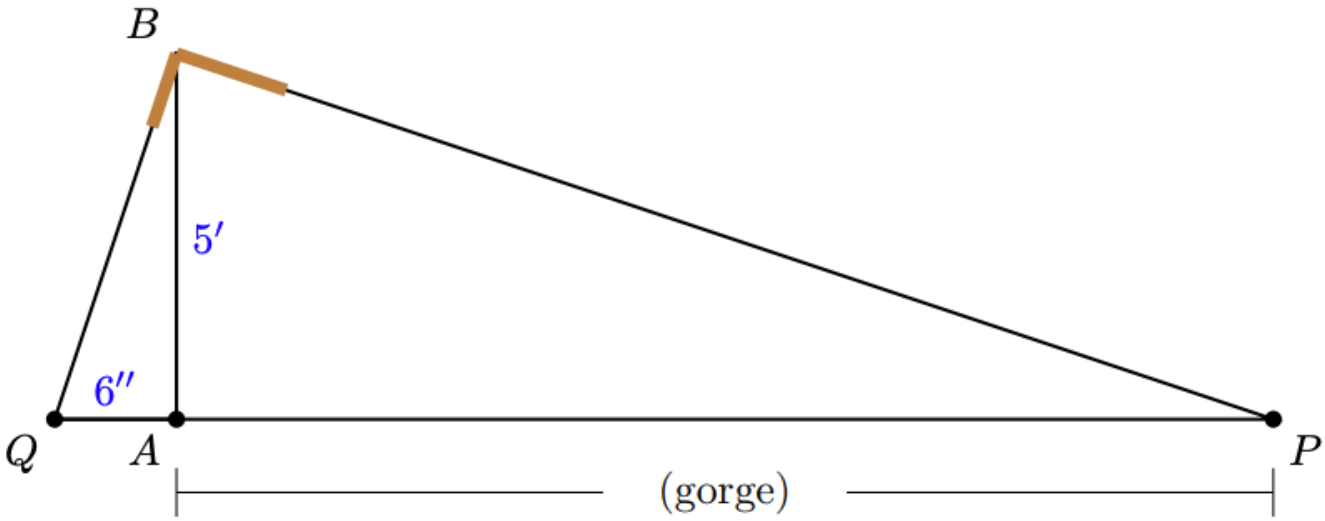


50. **Distances.** To measure the distance EC across the lake shown in the figure at right, stand at A and sight point C across the lake, then mark point B. Then sight to point E and mark point D so that DB is parallel to EC. If $AD = 25$ yards, $AE = 60$ yards, and $BD = 30$ yards, how wide is the lake?



Challenge Problems

51. **Distances.** Here is a way to find the distance across a gorge using a carpenter's square and a five-foot pole. Plant the pole vertically on one side of the gorge at point A and place the angle of the carpenter's square on top of the pole at point B, as shown in the figure. Sight along one side of the square so that it points to the opposite side of the gorge at point P. Without moving the square, sight along the other side and mark point Q. If the distance from Q to A is six inches, calculate the width of the gorge. Explain your method.



Answers

Skills Refresher

1. $x = \pm 6$
2. $x = -10$
3. $x = 4$
4. $x = -3$ and $x = -2$
5. $x = 3$

This page titled 4.3.2: Homework is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.4: Circles and Angles in the Rectangular Coordinate System

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The Distance Formula

Matt is hiking in the Santa Monica mountains. He would like to know the distance from the Sycamore Canyon trailhead, located at $12-\mathrm{C}$ on his map, to the Coyote Trail junction, located at $8-\mathrm{F}$, as shown below.

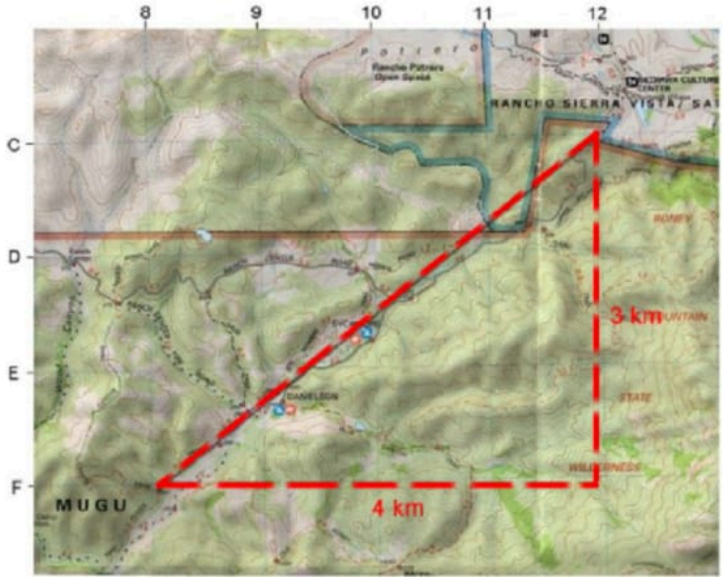


Figure 4.4.1

Each interval on the map represents one kilometer. Matt remembers the Pythagorean Theorem and uses the map coordinates to label the sides of a right triangle. The distance he wants is the hypotenuse of the triangle, so

$$\begin{aligned} & d^2 &=& 4^2+3^2 \\ \Longrightarrow\quad & d^2 &=& 16+9 \\ \Longrightarrow\quad & d^2 &=& 25 \\ \Longrightarrow\quad & d &=& \sqrt{25} \\ \Longrightarrow\quad & d &=& 5 \end{aligned}$$

The straight-line distance to Coyote junction is about 5 kilometers. The formula for the distance between two points is obtained similarly.

We first label a right triangle with points P_1 and P_2 on opposite ends of the hypotenuse (see Figure 4.4.2). The sides of the triangle have lengths $|x_2 - x_1|$ and $|y_2 - y_1|$. We can use the Pythagorean Theorem to calculate the distance between P_1 and P_2: $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

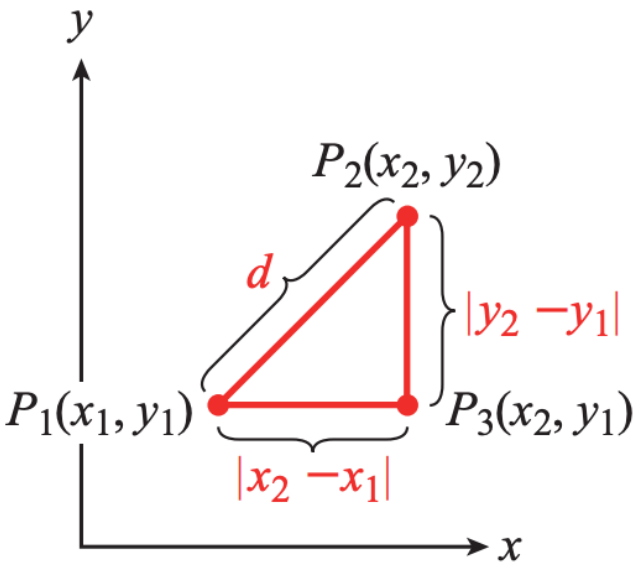


Figure 4.4.2

Taking the (positive) square root of each side of this equation gives us the **Distance Formula**.

Theorem: Distance Formula

The distance d between two points $P_1\,(x_1, y_1)$ and $P_2\,(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 4.4.1

Find the distance between $(2, -1)$ and $(4, 3)$.

Solution available at the end of this section.

Critical Mistake

In Example 4.4.1, the radical $\sqrt{4+16}$ cannot be simplified to $\sqrt{4}+\sqrt{16}$. (Do you remember why not?)

Checkpoint 4.4.1

a. Find the distance between the points $(-5, 3)$ and $(3, -9)$.
b. Plot the points on a Cartesian coordinate system and show how the Pythagorean Theorem is used to calculate the distance.

Answers available at the end of this section.

Equation for a Circle

The Distance Formula is required for us to find the equation of a circle.

Theorem: Equation of a Circle Centered at the Origin

The equation for a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2.$$

Proof available at the end of this section.

Example 4.4.2

Find two points on the circle $x^2 + y^2 = 4$ with an x-coordinate of -1.

Solution available at the end of this section.

Checkpoint 4.4.2

Find the coordinates of two points on the circle $x^2 + y^2 = 1$ with y-coordinate $\frac{1}{2}$.

Answer available at the end of this section.

The circle in Checkpoint 4.4.2 holds a special place in Trigonometry.

Definition: Unit Circle

The circle centered at the origin and having radius 1, is called the **unit circle**. The unit circle has equation

$$x^2 + y^2 = 1.$$

While the definition of the unit circle is only mentioned in passing here, it will become one of the foundational pieces of Trigonometry. We will revisit this later.

Later in this course, we will need to work with circles that do not have centers at the origin.

Theorem: Equation of a Circle Centered Off-Origin

The equation for a circle of radius r centered at the point (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2.$$

This is also know as the **standard form** of the equation of a circle.

Proof available at the end of this section.

Example 4.4.3

Write the standard equation of the circle with center $(-2, 3)$ and radius 5.

Solution available at the end of this section.

Checkpoint 4.4.3

Graph $(x+2)^2 + (y-1)^2 = 4$. Find the center and radius.

Answer available at the end of this section.

Interactive Element: Circle in Standard Form

The following Interactive Element might help you visualize how the numbers h, k, and r in $(x-h)^2 + (y-k)^2 = r^2$ control a circle's center and size.

Interact: Drag the red center point to move the circle; drag the green handle to change the radius r.

Observation: The center sits exactly at (h, k) — so $(x+2)^2 + (y-3)^2 = 25$ is centered at $(-2, 3)$, not $(2, 3)$. The radius is r, and $r = 1$ at the origin gives the unit circle.

Angles in the Cartesian Coordinate System

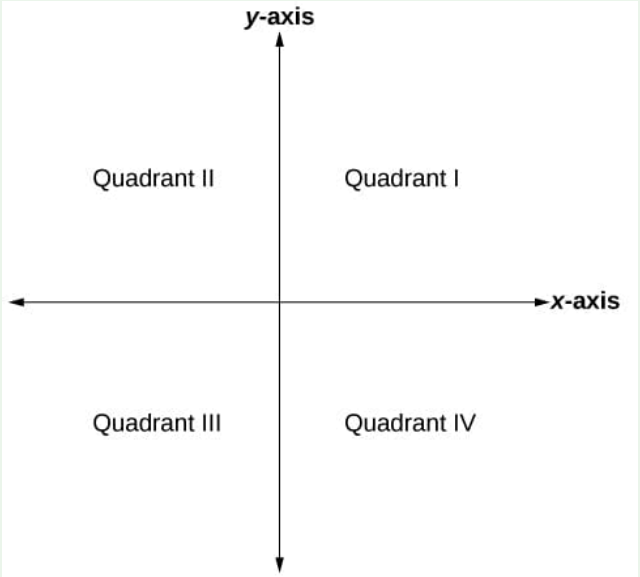
Now that we have reacquainted ourselves with angles, circles, and the Cartesian coordinate system, the next natural evolution toward Trigonometry is introducing angles to the Cartesian coordinate plane. Before we begin, however, it will be instrumental in this course to divide the Cartesian coordinate system into four sections, called **quadrants**.

Definition: Quadrants

In the Cartesian coordinate system, the section of the plane where

- x and y are both positive is called **Quadrant I** (**QI**),
- $x < 0$ and $y > 0$ is called **Quadrant II** (**QII**),
- x and y are both negative is called **Quadrant III** (**QIII**), and
- $x > 0$ and $y < 0$ is called **Quadrant IV** (**QIV**).

The four quadrants of the Cartesian coordinate system



Knowing this definition and understanding how the signs of x and y relate to these quadrants will be critical as we move forward.

Angles in Standard Position

In general, and without worrying about a coordinate system, the degree measure of an angle depends only on the fraction of a whole rotation between its sides, not on the angle's location or position. To compare and analyze angles, we place them in **standard position**.

Definition: Standard Position of an Angle

An angle is said to be in **standard position** if the vertex of the angle is at the origin and the initial side of the angle is on the positive x-axis.

Figure 4.4.6 below shows several angles moved into standard position.

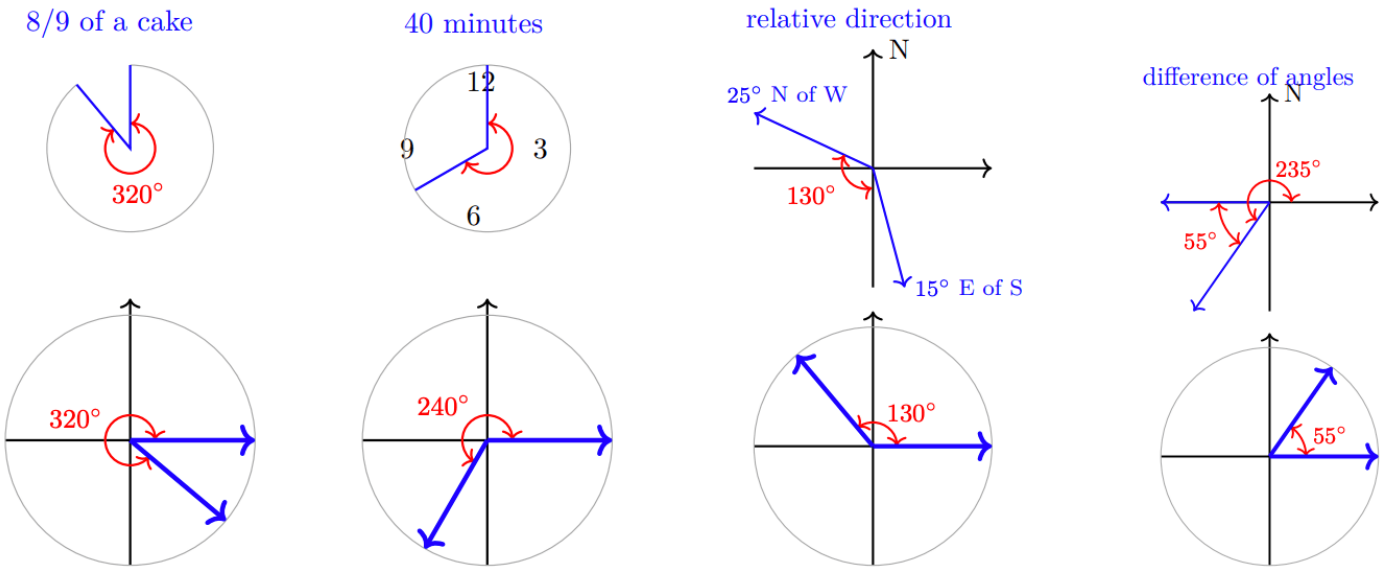


Figure 4.4.6**:** *Angles in nonstandard position (top row) and their equivalent angles in standard position (bottom row)*

The 320° in Figure 4.4.6 has its terminal side in quadrant IV. We denote this as Double exponent: use braces to clarify . Likewise, the remaining angles in Figure 4.4.6 are listed as Double exponent: use braces to clarify , Double exponent: use braces to clarify , and Double exponent: use braces to clarify .

Example 4.4.4

Find the degree measure of each angle shown below, state the quadrant where the angle terminates, and sketch the angle in standard position.

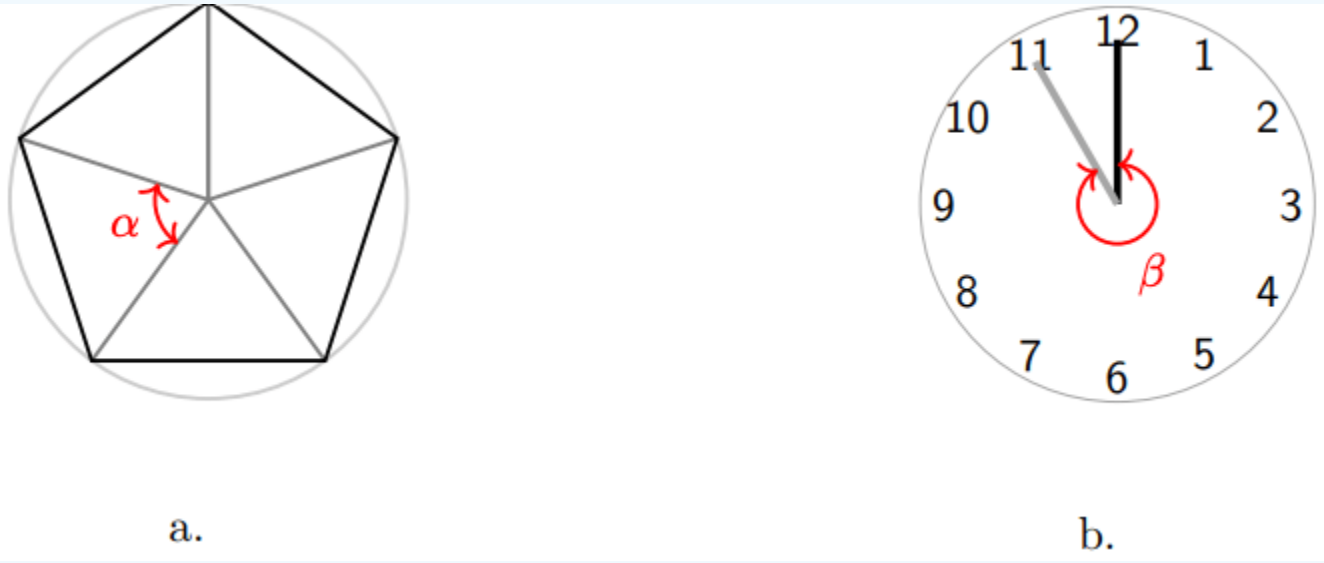


Figure 4.4.7

Solutions available at the end of this section.

Checkpoint 4.4.4

Find the degree measure of each angle below, sketch the angle in standard position, and state the quadrant in which the angle terminates.

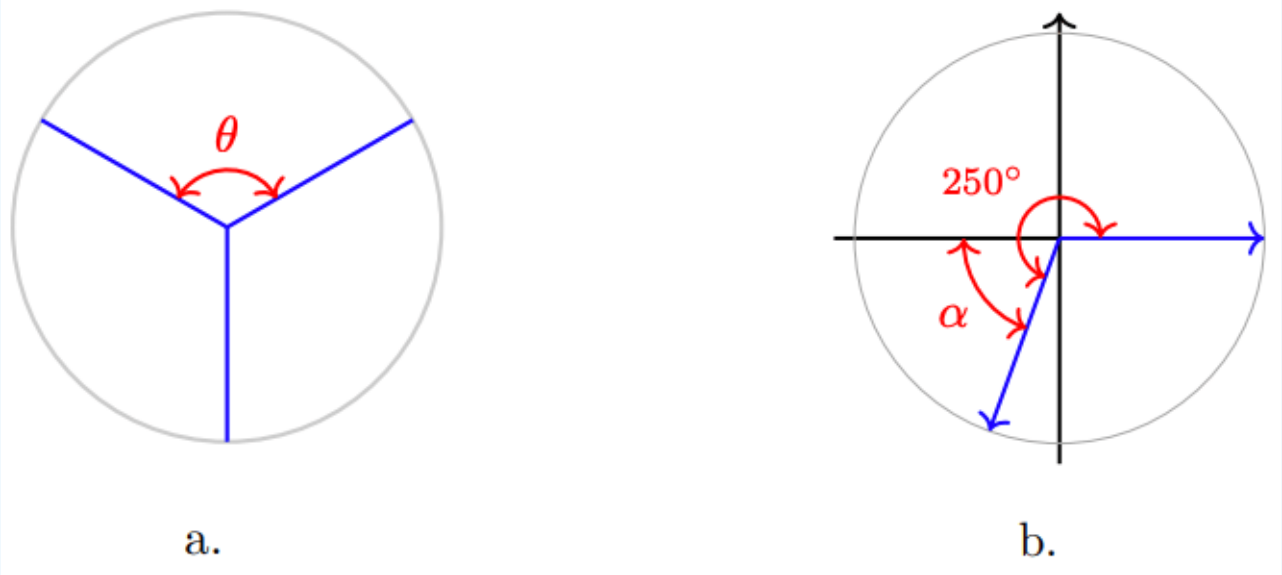


Figure 4.4.9

Answers available at the end of this section.

Quadrantal Angles

Now that we are acquainted with angles in standard position, it's time we start classifying these angles by type. First up are the **quadrantal angles**.

Definition: Quadrantal Angle

An angle in standard position having terminal side on an axis is called a **quadrantal angle**.

For example, 270° has a terminal side on the negative y-axis. Therefore, we would call 270° a quadrantal angle. Figure $4.4.12$ illustrates a few quadrantal angles; however, as we will learn shortly, there are infinite quadrantal angles.

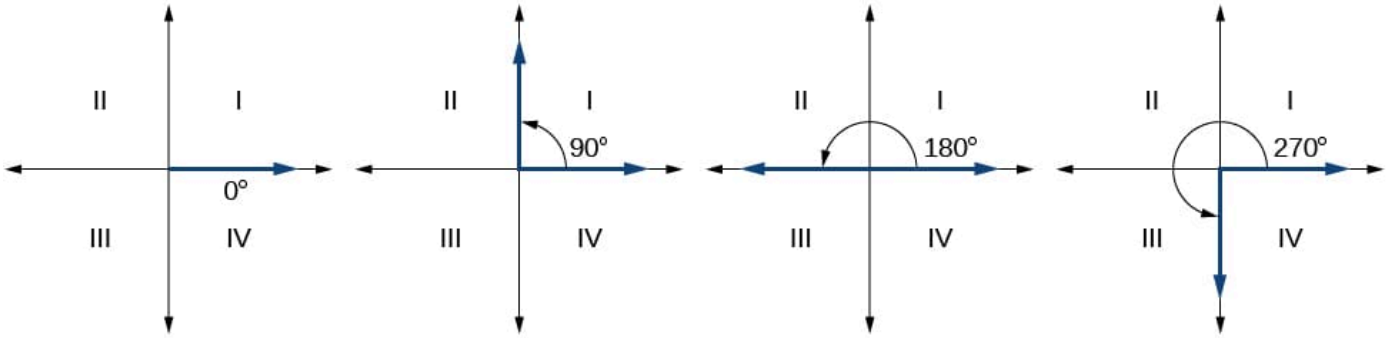


Figure $4.4.12$**:** *Some quadrantal angles*

Negative Angles

Aligning with what we said earlier in this textbook, an angle is positive if it opens counterclockwise. If an angle opens clockwise, the angle is considered negative. For example, Figure $4.4.13$ shows the angle $\alpha = -135^\circ$, which terminates in QIII.

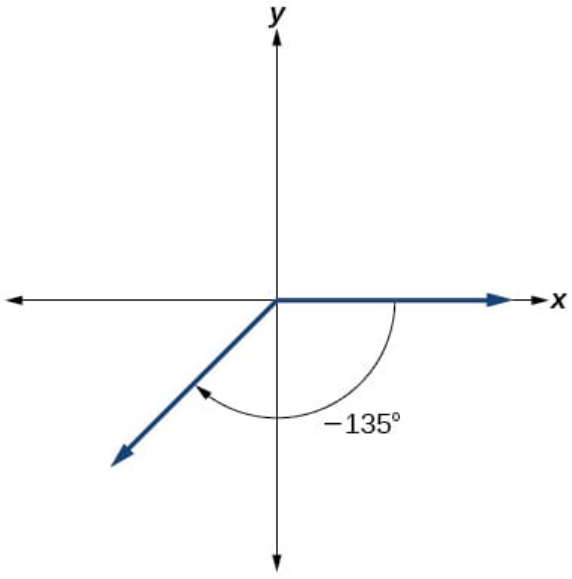


Figure $4.4.13$**:** *A negative angle*

Graphing positive and negative angles in standard position will be a critical skill in Trigonometry.

Coterminal Angles

Because 360° represents one complete revolution, we can add or subtract a multiple of 360° to any angle and the terminal side of the resulting angle will arrive at the same position. For example, the angles 70° and 430° have the same terminal side because $430^\circ = 70^\circ + 360^\circ$. Such angles are called **coterminal**.

Definition: Coterminal Angles

Angles in standard position that share a terminal side are called **coterminal angles**.

The angle 790° is also coterminal with 70°, because $790^\circ = 70^\circ + 2\,(360^\circ)$ (see Figure $4.4.14$ below).



Figure $4.4.14$**:** *Graphical representations of coterminal angles*

Coterminal angles need not be positive. For example, we could have been given $\alpha = 140^\circ$ and chosen to *subtract* 360° to get the coterminal angle -220° as shown in Figure $4.4.15$.

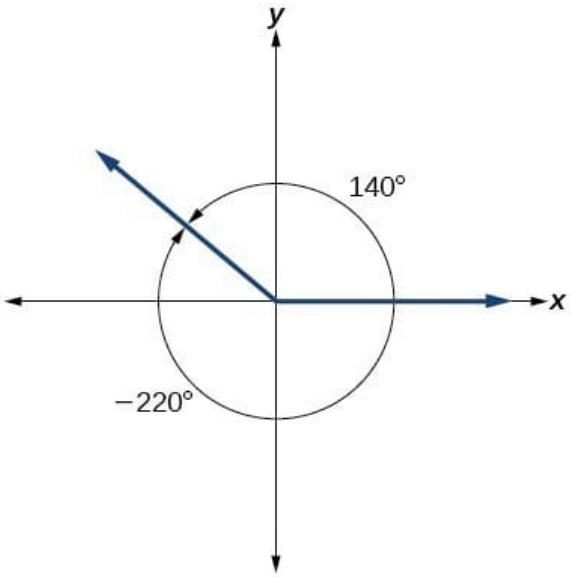


Figure $4.4.15$

In fact, we could denote all angles coterminal to $\alpha = 140^\circ$ using the notation

$$140^\circ + 360^\circ k, \text{ where } k \in \mathbb{Z}.$$

The symbol, $\mathbb{Z}$, denotes the set of all integers (the set of numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$). This notation gets used frequently in Mathematics from this point forward, so it is best to get used to it quickly.

Interactive Element: Standard-Position and Coterminal Angles

The following Interactive Element might help you visualize how a single terminal side can be named by infinitely many angle measures.

Interact: Drag the red point around the circle — spin it past a full turn, then back the other way.

Observation: The measure changes as you rotate, but every time you add or subtract 360° you land on the exact same terminal side — those are coterminal angles. Notice which quadrant the terminal side falls in.

Theorem: Coterminal Angle Formula

All angles coterminal to the angle α (in standard position) have the form

$$\alpha + 360^\circ k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

Previously, we mentioned that the angles illustrated in Figure $4.4.12$ were only a "few quadrantal angles," implying there were more than what we listed. This was because, for example, given the quadrantal angle 90°, we could generate an infinite number of other quadrantal angles with the same terminal side by considering

$$90^\circ + 360^\circ k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

Moreover, since the quadrantal angles only differ from each other by 90°, we could generate *all* quadrantal angles with the single statement

$$0^\circ + 90^\circ k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

There is no need for the 0° angle in that formula, so we could shorten it up to just

$$90^\circ k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

Example $4.4.5$

Graph each of the angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

a. $\alpha = 60^\circ$
b. $\beta = -225^\circ$
c. $\gamma = 540^\circ$
d. $\phi = -750^\circ$

Solutions available at the end of this section.

Checkpoint $4.4.5$

Find two angles coterminal with 102°, one positive and one negative.

Answer available at the end of this section.

MyOpenMath Advice: Check For Feedback

When grading your uploaded (handwritten) work, your professor might have left feedback. Sometimes this feedback can be as

simple as, "Please read the written solution," but it can be as robust as a breakdown of your mistake and how you can avoid such a mistake in the future. To check a past due assignment for any feedback, go to the Grades area in Canvas, click on the assignment in question, and you should be able to see any feedback there. If you cannot see feedback, please let your professor know.

Success in College: Meeting With Your Professor

Meeting with a faculty member might be intimidating. Students are encouraged to use the following guidelines to make the faculty meeting productive. These strategies were developed by the Retention Office at U. of Toledo (OH) and first appeared in National On-Campus Report's sister publication, Recruitment and Retention in Higher Education.

Step 1: Make an appointment

- Review the syllabus for the professor's office hours and make an appointment.
- If you're uncertain of your professor's office hours, ask.
- If your schedule conflicts with the office hours, tell your professor why you can't make the scheduled office hours. Then tell them about your specific concern.
- Set up a specific time to meet with your professor. Be sure to indicate the amount of time you will need.

Step 2: Establish rapport

- Be on time for your appointment.
- When you arrive, be pleasant, smile, introduce yourself again (include your name and class), and shake hands if appropriate.

Step 3: Present your concern

- Focus on the specific questions you've identified as problematic.
- Have your problem(s) written out so that the professor can see where your difficulties are.

Step 4: Provide background information

- Briefly tell your professor about your high school background and preparation for the course – if relevant to solving the problem.
- Explain the study strategies you've used to understand the material.

Step 5: Redirect for clarification

- If the professor's explanation isn't clear, redirect their attention to the specific point where you became confused.
- "Talk through" your problem.

Step 6: Summarize Resolution of the Problem

- "I was missing this step in... I need to apply this formula..."

Step 7: Thank your Professor

- Ask to come back if necessary.

Armed with this information, think about the following questions:

1. How would you make an appointment to talk to your professor if your schedule prevents you from visiting during office hours? If it is not possible to make such an appointment or your professor does not have office hours, think about another way you can get extra help.
2. Some professors prefer you talk to them using their title and last name. For example, you might say Dr. Rodriguez or Ms.

Smith. Others prefer that you use just their first name. When asking your professor a question, what name or title do they prefer that you use?

If you are unsure about the answers to either of these questions, feel free to ask your professor - that's what they are there for!

Answers

Section Preview

Up to this point, all your instruction should have omitted the Cartesian coordinate system and equations depicting solutions on the coordinate plane. We now glue our beloved coordinate system to the Geometry reviewed up to this point. Much of the content here cannot be done without mastering the previous sections.

Section Preview

- **Sets and Numbers**
  - Sets of Numbers: For understanding and properly using the notations for sets of numbers (e.g., $\mathbb{Z}$ for the integers).
- **Graphs**
  - The Cartesian Coordinate System and the Quadrants: Students must arrive already knowing how to work in the coordinate plane to engage with any of this content.
  - Ordered Pairs and Graphing Relations by Point-Plotting: Students must arrive already knowing how to work in the coordinate plane to engage with any of this content.

Section Preview

- **Arithmetic**
  - Mathematical Operations - Multiplication: Computing coterminal angles requires multiplying $360^\circ \times k$.

Section Preview

- Compute the distance between two points on the Cartesian plane.
- Verify a point lies on the unit circle.
- Graph a circle given its equation.
- Determine in which quadrant an angle in standard position terminates.
- Sketch an angle in standard position.
- Find an angle coterminal with a given angle.

Example 4.4.1

We substitute $(2, -1)$ for (x_1, y_1) and $(4, 3)$ for (x_2, y_2) in the Distance Formula to obtain

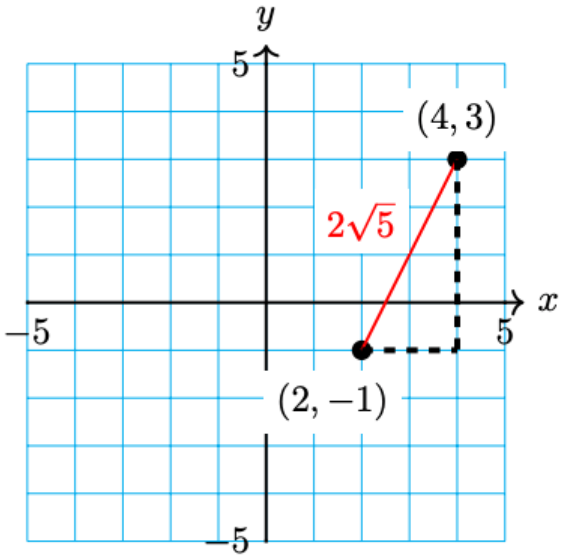


Figure 4.4.3

$$
\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{(Distance Formula)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad &= \sqrt{(4-2)^2 + [3-(-1)]^2} && \text{(substitution)} \\
\xRightarrow{\text{Arithmetic}} \quad &= \sqrt{2^2 + 4^2} && \text{(addition and subtraction)} \\
\xRightarrow{\text{Arithmetic}} \quad &= \sqrt{4 + 16} && \text{(applying powers)} \\
\xRightarrow{\text{Arithmetic}} \quad &= \sqrt{20} && \text{(addition)} \\
\xRightarrow{\text{Arithmetic}} \quad &= 2\sqrt{5} && \text{(simplify the radical)}
\end{aligned}
$$

The exact value of the distance is $2\sqrt{5}$ units. We obtain the same answer if we use $(4, 3)$ for P_1 and $(2, -1)$ for P_2:

$$\begin{aligned} d &= \sqrt{(2-4)^2+[(-1)-3]^2} \\ &= \sqrt{4+16} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

We can use a calculator to obtain an approximation for this value and find $2\sqrt{5} \approx 2(2.236) = 4.472$.

Checkpoint 4.4.1

a. $4\sqrt{13}$

b. The points $(-5, 3)$ and $(3, -9)$ are two vertices of a right triangle with one horizontal leg of length 8 and one vertical leg of length 12. The distance between $(-5, 3)$ and $(3, -9)$ is the hypotenuse c.

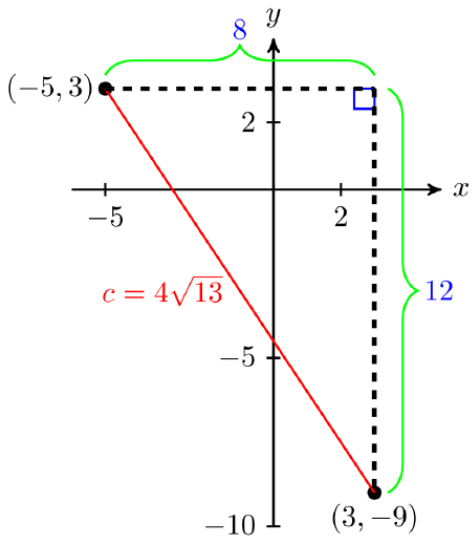
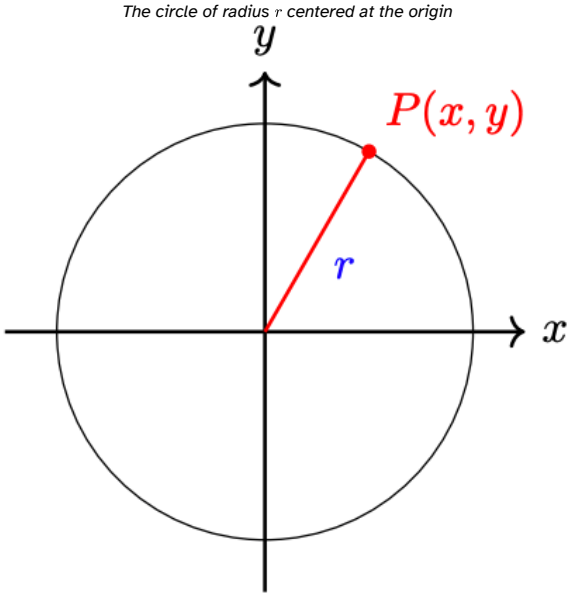


Figure 4.4.4

Theorem: Equation of a Circle Centered at the Origin

Consider a circle of radius r centered at the origin, as shown in the following figure.

The circle of radius r centered at the origin



The distance from the origin to any point $P(x, y)$ on the circle is r. Therefore, by the Distance Formula,

$$\sqrt{(x-0)^2+(y-0)^2} = r$$

or, squaring both sides,

$$(x-0)^2+(y-0)^2 = r^2.$$

Because every point on the circle must satisfy this equation, we have found an equation for the circle.

Example 4.4.2

We substitute $x = -1$ into the equation for the circle and solve for y.

$$\begin{aligned} \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad (-1)^2 + y^2 &= 4 && \text{(substitution)} \\ \xRightarrow{\text{Arithmetic}} \quad 1 + y^2 &= 4 && \text{(applying the power)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad y^2 &= 3 && \text{(subtract 1 from both sides)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad y &= \pm\sqrt{3} && \text{(Extraction of Roots)} \end{aligned}$$

The points are $(-1, \sqrt{3})$ and $(-1, -\sqrt{3})$. Note that $\sqrt{3} \approx 1.732$.

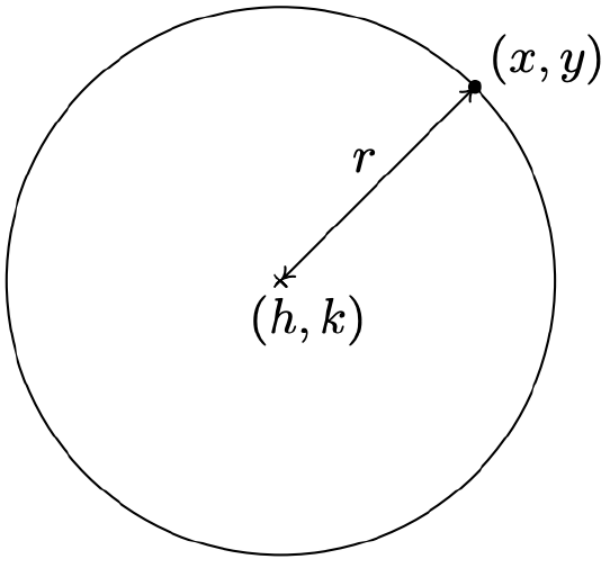
Checkpoint 4.4.2

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Theorem: Equation of a Circle Centered Off-Origin

Consider a circle of radius r centered at the point (h, k), as shown in the following figure.

A circle of radius r centered at the point (h, k)



Since this is a circle of radius r, the point (x, y) is on the circle if and only if its distance to (h, k) is r. By the Distance Formula, this means

$$r = \sqrt{(x-h)^2 + (y-k)^2}.$$

By squaring both sides of this equation, we get

$$r^2 = (x-h)^2 + (y-k)^2.$$

Example 4.4.3

Here, $(h, k) = (-2, 3)$ and $r = 5$, so we get

$$\begin{aligned} (x-(-2))^2 + (y-3)^2 &= (5)^2 \\ \Longrightarrow \quad (x+2)^2 + (y-3)^2 &= 25 \end{aligned}$$

Checkpoint 4.4.3

From the standard form of a circle we have that $x + 2$ is $x - h$, so $h = -2$ and $y - 1$ is $y - k$ so $k = 1$. This tells us that our center is $(-2, 1)$. Furthermore, $r^2 = 4$, so $r = 2$. Thus we have a circle centered at $(-2, 1)$ with a radius of 2. Graphing gives us

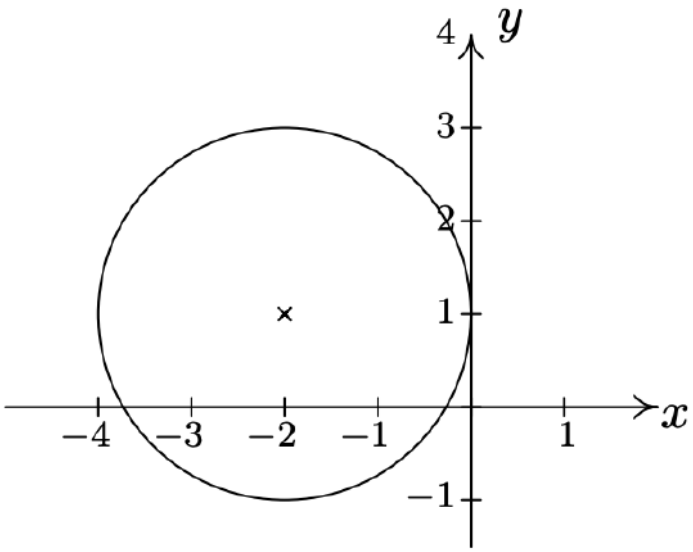


Figure 4.4.5

Example 4.4.4

a. The angle α is one-fifth of a complete revolution, or

$$\frac{1}{5}(360^\circ) = 72^\circ.$$

In standard position, Double exponent: use braces to clarify, as shown in Figure $4.4.8(a)$ below.

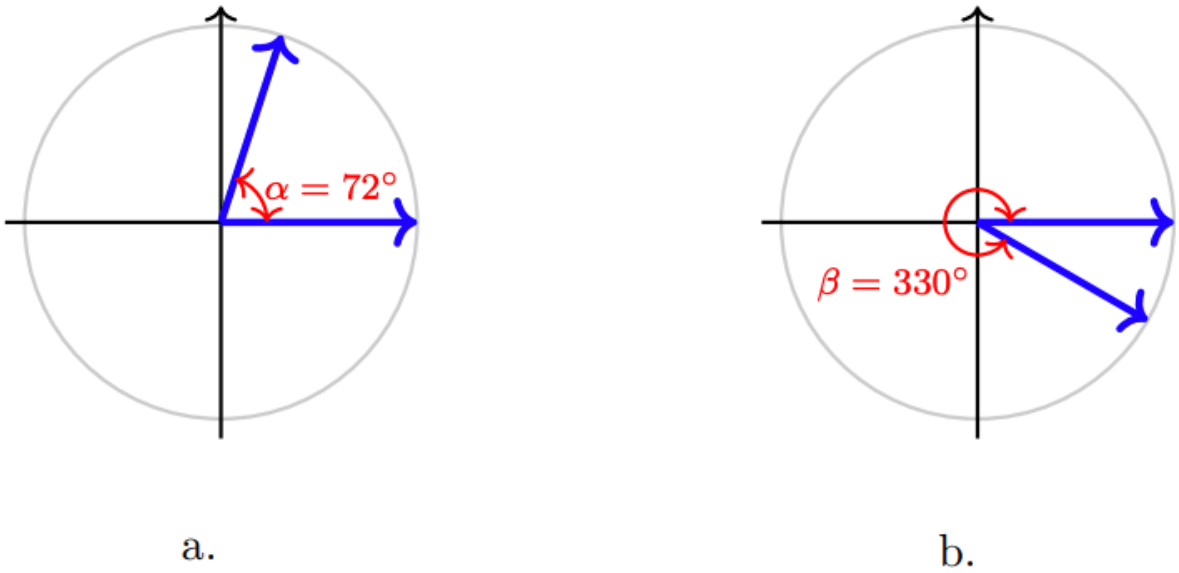


Figure 4.4.8

b. The angle β is $\frac{11}{12}$ of a complete revolution, or

$$\frac{11}{12}(360^\circ) = 330^\circ.$$

In standard position, Double exponent: use braces to clarify. (See Figure $4.4.8(b)$.)

Checkpoint 4.4.4

a. Double exponent: use braces to clarify

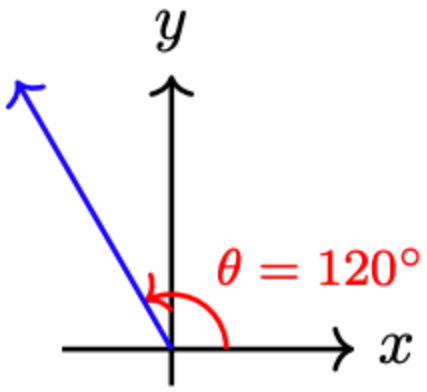


Figure 4.4.10

b. Double exponent: use braces to clarify

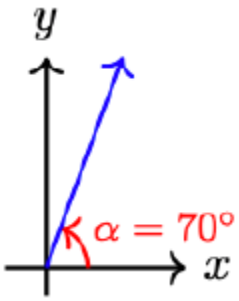
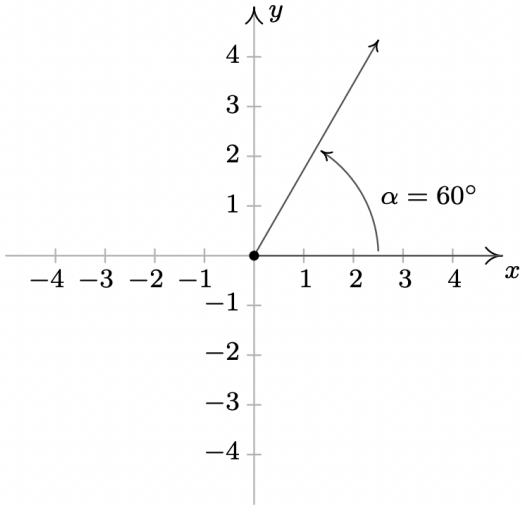


Figure 4.4.11

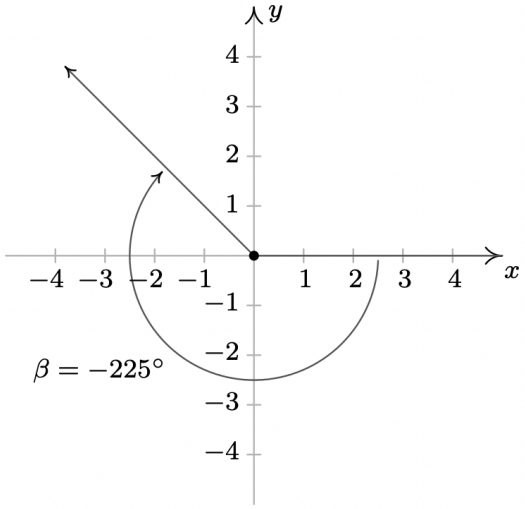
Example 4.4.5

a. To graph $\alpha = 60^\circ$, we draw an angle with its initial side on the positive x-axis and rotate counterclockwise $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of a revolution. We see that α is a Quadrant I angle. To find coterminal angles, we look for angles θ of the form $\theta = \alpha + 360^\circ \cdot k$, for some integer k. When $k = 1$, we get $\theta = 60^\circ + 360^\circ = 420^\circ$. Substituting $k = -1$ gives $\theta = 60^\circ - 360^\circ = -300^\circ$. Finally, if we let $k = 2$, we get $\theta = 60^\circ + 720^\circ = 780^\circ$.

b. Since $\beta = -225^\circ$ is negative, we start at the positive x-axis and rotate *clockwise* $\frac{225^\circ}{360^\circ} = \frac{5}{8}$ of a revolution. We see that β is a Quadrant II angle. To find coterminal angles, we proceed as before and compute $\theta = -225^\circ + 360^\circ \cdot k$ for integer values of k. We find 135°, -585° and 495° are all coterminal with -225°.



$\alpha = 60^\circ$ in standard position.

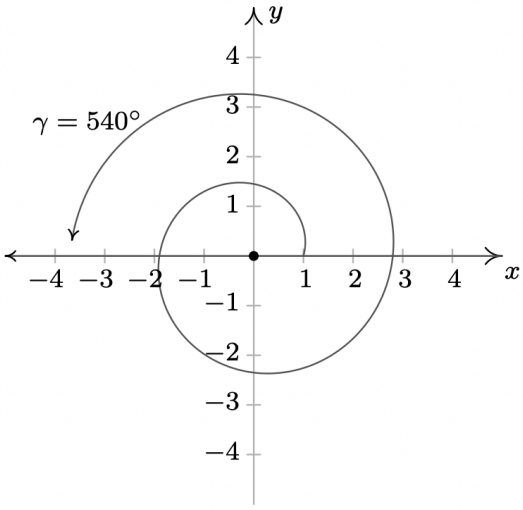


$\beta = -225^\circ$ in standard position.

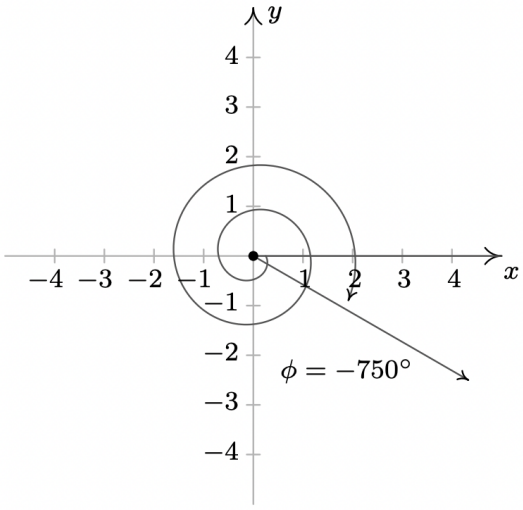
Figure 4.4.16

c. Since $\gamma = 540^\circ$ is positive, we rotate counterclockwise from the positive x-axis. One full revolution accounts for 360°, with 180°, or $\frac{1}{2}$ of a revolution remaining. Since the terminal side of γ lies on the negative x-axis, γ is a quadrantal angle. All angles coterminal with γ are of the form $\theta = 540^\circ + 360^\circ \cdot k$, where k is an integer. Working through the arithmetic, we find three such angles: 180°, -180° and 900°.

d. The Greek letter ϕ is pronounced "fee" or "fie," and since ϕ is negative, we begin our rotation clockwise from the positive x-axis. Two full revolutions account for 720°, with just 30° or $\frac{1}{12}$ of a revolution to go. We find that ϕ is a Quadrant IV angle. To find coterminal angles, we compute $\theta = -750^\circ + 360^\circ \cdot k$ for a few integers k and obtain -390°, -30° and 330°.



$\gamma = 540°$ in standard position.



$\phi = -750°$ in standard position.

Figure 4.4.17

Checkpoint 4.4.5

$462°, -258°$

This page titled 4.4: Circles and Angles in the Rectangular Coordinate System is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

4.4.1: Resources and Key Concepts

Resources

Videos

- Distance Formula
  - Intermediate Algebra - Distance Formula: The Concept
  - Intermediate Algebra - Distance Formula: General and Advanced Uses
- Equation of a Circle
  - Intermediate Algebra - Circles: The Concept
  - Intermediate Algebra - Circles: Equations and Graphs

Key Concepts

Definitions

- **Unit Circle:** The circle with a radius of 1 that is centered at the origin of the Cartesian plane. Its equation is $x^2 + y^2 = 1$.
- **Quadrants:** The four regions into which the Cartesian coordinate system is divided by the x- and y-axes. They are numbered I, II, III, and IV, starting from the top right and moving counterclockwise.
- **Standard Position of an Angle:** An angle is in standard position if its vertex is located at the origin and its initial side lies along the positive x-axis.
- **Quadrantal Angle:** An angle in standard position whose terminal side lies on either the x-axis or the y-axis.
- **Coterminal Angles:** Angles that are in standard position and share the same terminal side.

Theorems

- **Distance Formula:** The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- **Equation of a Circle Centered at the Origin:** The equation for a circle with radius r centered at the origin is $x^2 + y^2 = r^2$.
- **Equation of a Circle Centered Off-Origin:** The standard form for the equation of a circle with radius r centered at the point (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.
- **Coterminal Angle Formula:** All angles that are coterminal with an angle α can be expressed in the form $\alpha + 360°k$, where k is any integer.

Common Mistakes

- **Distributing a Radical over a Sum:** A square root cannot be distributed across terms that are added or subtracted. For example, $\sqrt{a^2 + b^2}$ is not equal to $a + b$, and $\sqrt{4 + 16}$ is not equal to $\sqrt{4} + \sqrt{16}$. This is a critical mistake when using the Distance Formula.
- **Direction of Rotation for Angles:** Positive angles are generated by a counterclockwise rotation from the initial side. Negative angles are generated by a clockwise rotation.

This page titled 4.4.1: Resources and Key Concepts is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.4.2: Homework

Reading Questions

1. What fundamental theorem is used as the basis for deriving the Distance Formula?
2. Write the equation for a circle that has its center at the origin and a radius of r.
3. What is a "unit circle"?
4. What is the "standard form" of the equation for a circle with center (h, k) and radius r?
5. In which quadrant are the x-coordinates negative and the y-coordinates positive?
6. What two conditions define an angle in "standard position"?
7. Where is the terminal side of a 90° angle in standard position?
8. What is a "quadrantal angle"?
9. What does it mean for two angles to be "coterminal"?
10. Describe the direction of rotation (clockwise or counterclockwise) for a negative angle.
11. If you are given an angle of 50°, how would you find a negative angle that is coterminal with it?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, find the slope and y-intercept of the graph for the given equation (do not graph).

1. $y = 5.2x - 1.9$
2. $y = x$
3. $y = -8$
4. $x = 7$
5. $12y - 10x = 5$

For the following exercises, write the equation of the line that satisfies the given conditions.

6. The line goes through the points $(-2, 6)$ and $(3, -4)$.
7. The line goes through the points $(5, -7)$ and $(5, -5)$.

For the following exercises, determine if the statement is true or false.

8. $\sqrt{a^2 + b^2} = a + b$
9. $\sqrt{36 + 64} = 6 + 8$
10. $\sqrt{16x^4} = 4x^2$
11. $\sqrt{2x}\sqrt{3y} = \sqrt{6xy}$
12. $\sqrt{5x} + \sqrt{3x} = \sqrt{8x}$
13. $\sqrt{4 + N} = 2 + \sqrt{N}$

14. $\sqrt{\frac{x}{4}} = \frac{\sqrt{x}}{2}$

15. $\sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$

For the following exercises, solve the equation. If no solution exists, state why.

16. $x^2 - 81 = 0$

17. $2y^2 - 98 = 0$

18. $x^2 + 12 = 8$

19. $-3x^2 + 19 = 10$

Answers available at the end of this section.

Homework

Concept Check

1. Explain why the Distance Formula,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

cannot be simplified to

$$(x_2 - x_1) + (y_2 - y_1).$$

2. What is a unit circle, and what is its equation?

3. In which quadrant are the x-values for points negative but the y-values are positive?

4. Explain what it means for angles to be coterminal. Demonstrate by creating three coterminal angles, one of which is a negative angle.

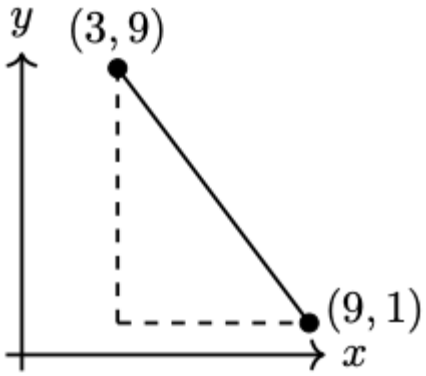
True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

5. Positive angles open clockwise.

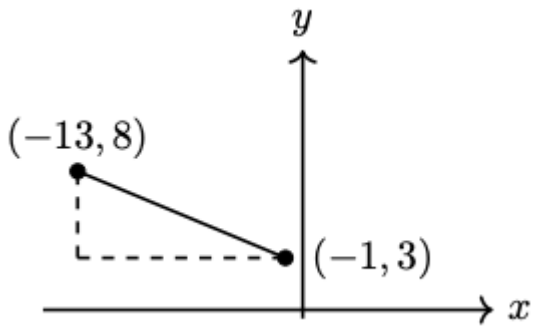
Basic Skills

For the following exercises, find the distance between the points. Give your answer as an exact value, then as a decimal rounded to hundredths.

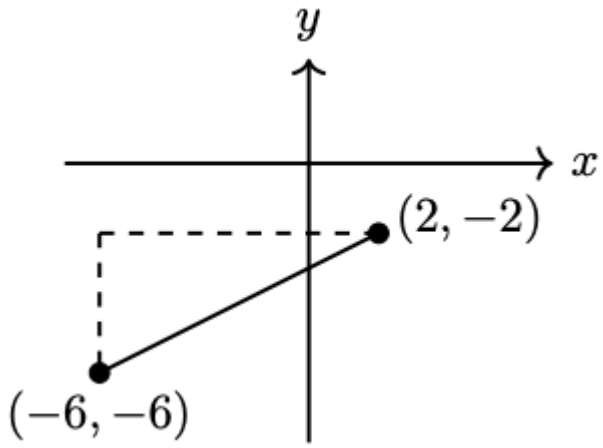
6.



7.



8. (MyOpenMath)



For the following exercises, find the distance between the points.

9. (1, 1), (4, 5)
10. (−1, 1), (5, 9)
11. (MyOpenMath) (2, −3), (−2, −1)
12. (5, −4), (−1, 1)
13. (3, 5), (−2, 5)
14. (−2, −5), (−2, 3)

15. a. Write an expression for the distance between the points (x, y) and $(-3, 4)$.

    b. Write an equation that says "The distance between the points (x, y) and $(-3, 4)$ is 5 units."

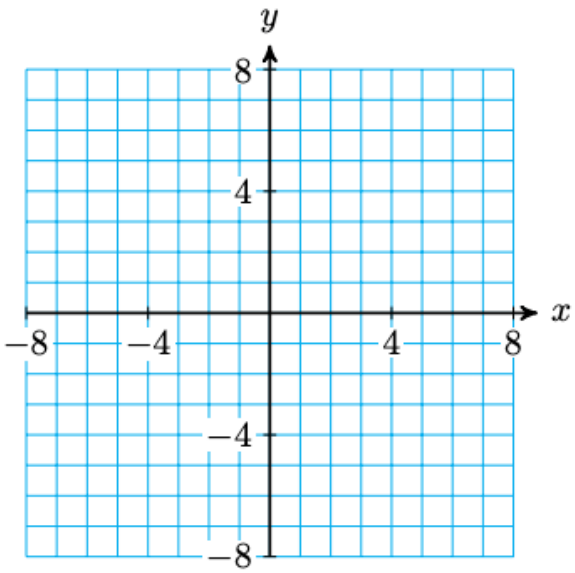
16. a. Write an expression for the distance between the points $(-6, -1)$ and (h, k).

    b. Write an equation that says, "The point $(-6, -1)$ is 3 units from the point (h, k)."

17. a. Complete the table of values for the equation $x^2 + y^2 = 25$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											

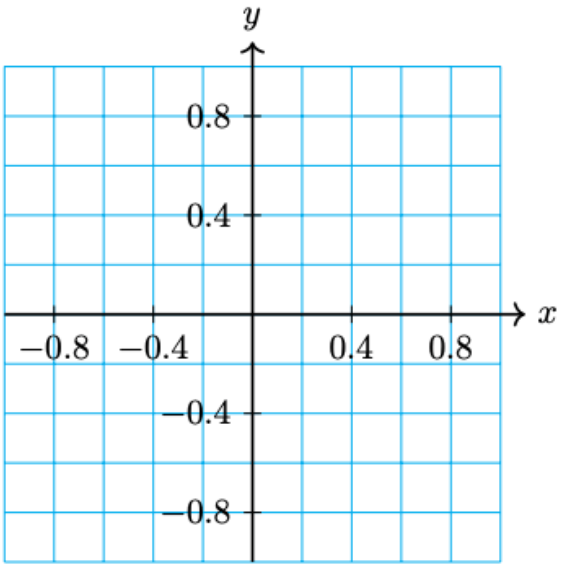
    b. Plot the points and graph the equation.



18. a. Complete the table of values for the equation $x^2 + y^2 = 1$.

x	-1	-0.8	-0.6	0	0.6	0.8	2
y							

    b. Plot the points and graph the equation.



19. a. Sketch a graph of all points in the plane that lie 6 units from the origin.

    b. Write an equation for your graph.

20. a. Sketch a graph of all points in the plane that lie $\sqrt{12}$ units from the origin.

    b. Write an equation for your graph.

21. a. Sketch a graph of all points in the plane that lie less than 3 units from the origin.

    b. Write an inequality for your graph.

22. a. Sketch a graph of all points in the plane that lie more than 5 units from the origin.

    b. Write an inequality for your graph.

23. a. Explain why the solutions of the equation $x^2 + y^2 = 16$ must have $-4 \leq x \leq 4$.

    b. What does part (a) tell you about the graph of the equation?

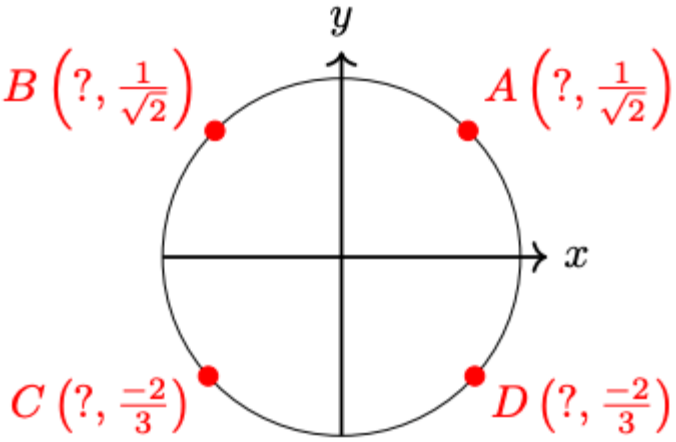
24. a. Explain why the solutions of the equation $x^2 + y^2 = 100$ must have $-10 \leq x \leq 10$

    b. What does part (a) tell you about the graph of the equation?

25. The point $(1, -3)$ lies on a circle centered at the origin. What is the radius of the circle?

26. The point $(-2, \sqrt{6})$ lies on a circle centered at the origin. What is the radius of the circle?

27. Find the coordinates of the points on the unit circle.

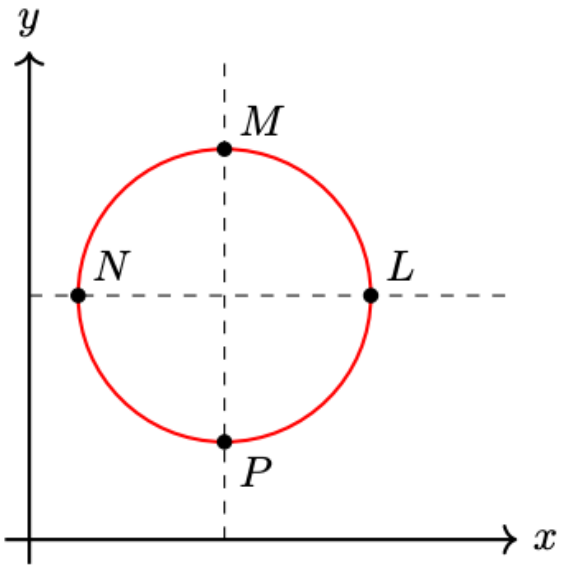


28. a. Use the Distance Formula to write an equation for the circle of radius 6 centered at the point $(3, -2)$.

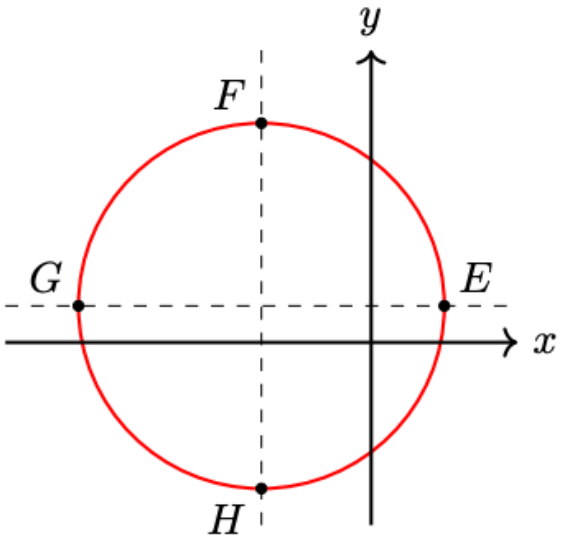
b. Use the Distance Formula to derive an equation for the circle of radius r centered at the point (h, k).

For the following exercises, find the coordinates of the indicated points on each circle.

29. $(x-4)^2 + (y-5)^2 = 9$



30. $(x+3)^2 + (y-1)^2 = 25$



31. How many degrees are in each angle?

    a. $\frac{3}{5}$ of one rotation

    b. $\frac{3}{10}$ of one rotation

    c. $\frac{4}{3}$ of one rotation

    d. $\frac{8}{3}$ of one rotation

32. How many degrees are in each angle?

    a. $\frac{5}{6}$ of one rotation

    b. $\frac{3}{8}$ of one rotation

    c. $\frac{7}{4}$ of one rotation

    d. $\frac{7}{12}$ of one rotation

33. What fraction of a complete rotation is represented by each angle?

    a. 45°

    b. 300°

    c. 540°

    d. 420°

34. What fraction of a complete rotation is represented by each angle?

    a. 60°

    b. 240°

    c. 450°

    d. 150°

For the following exercises, find two angles, one positive and one negative, that are coterminal with the given angle.

35. 40°

36. 160°

37. 215°

38. 250°

39. 305°

40. 340°

For the following exercises, find a positive angle between 0° and 360° that is coterminal with the given angle.

41. -65°

42. -140°

43. -290°

44. -325°

45. (MyOpenMath) 655°

46. (MyOpenMath) -750°

47. (MyOpenMath) Write an expression describing all the angles that are coterminal with 252°.

Synthesis Questions

47. Show that the triangle with vertices $(0,0)$, $(6,0)$, and $(3,3)$ is an isosceles right triangle, that is, a right triangle with two sides of the same length.

48. Two opposite vertices of a square are $A(-9,-5)$ and $C(3,3)$.

    a. Find the length of a diagonal of the square.

    b. Find the length of the side of the square.

49. Find all of the points on the line $y = 2x + 1$ which are 4 units from the point $(-1,3)$.

50. Sketch a triangle with vertices $(10,1)$, $(3,1)$, and $(5,9)$, and find its perimeter. Round your answer to tenths.

51. Sketch a triangle with vertices $(-1,5)$, $(8,-7)$, and $(4,1)$, and find its perimeter. Round your answer to tenths.

For the following exercises, interpret the equations as statements about distance.

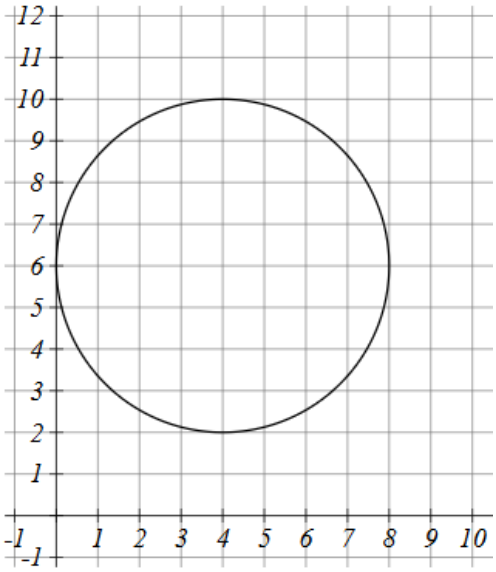
52. $\sqrt{(x-4)^2+(y+1)^2} = 3$

53. $\sqrt{(-2-h)^2+(5-k)^2} = l$

For the following exercises, use the graph to answer the question.

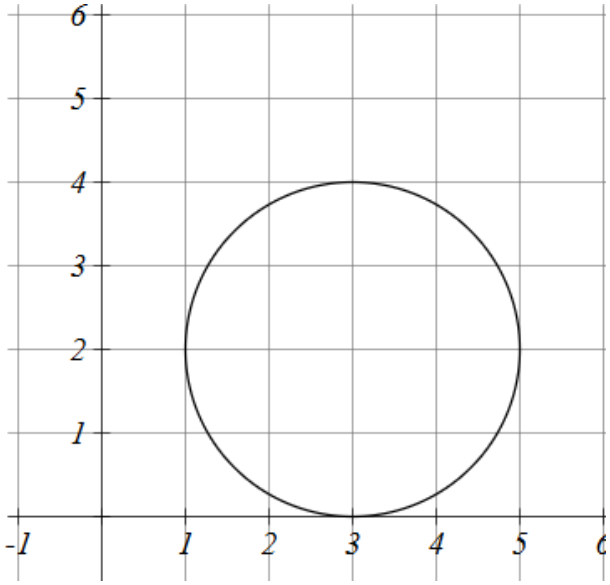
54. Find the circumference of the circle.

MyOpenMath



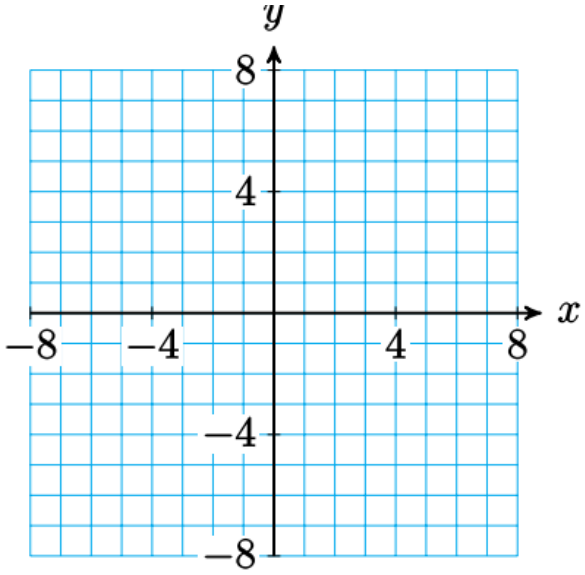
55. Find the area of the circle.

MyOpenMath

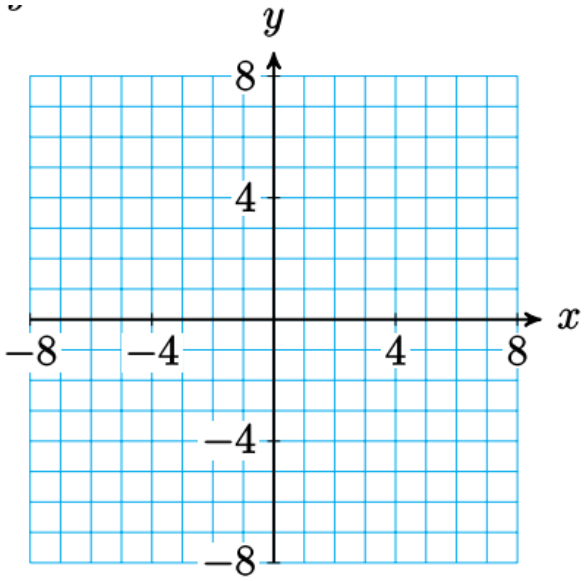


For the following exercises, graph the equation and find the circumference of the circle.

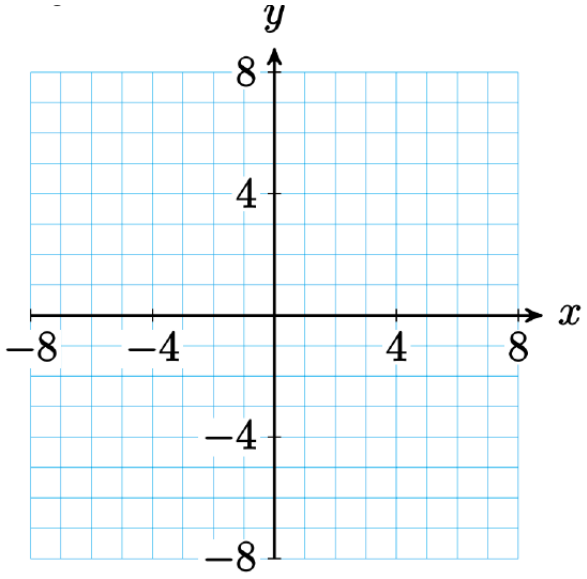
56. $x^2 + y^2 = 36$



57. $x^2 + y^2 = 16$

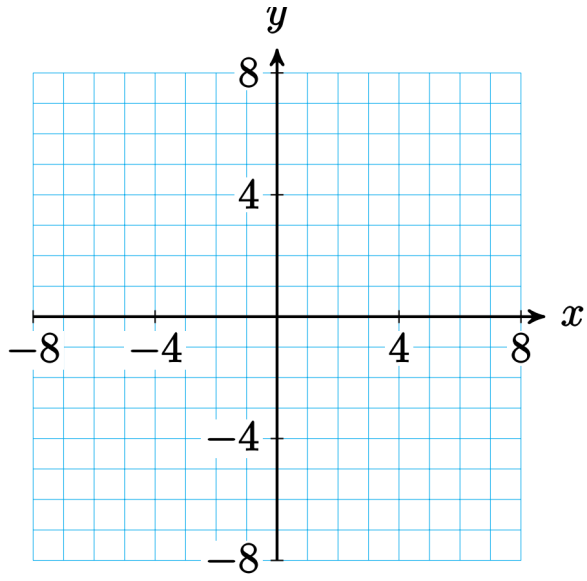


58. $4x^2 + 4y^2 = 16$



59. Give the coordinates of two points on the circle in the previous exercise that have $y = -4$. Plot those points on your graph.

60. $2x^2 + 2y^2 = 18$



61. Give the coordinates of two points on the circle in the previous exercise that have $x = -2$. Plot those points on your graph.

Applications

62. **Distance.** Paige is sailing and is currently 3 miles west and 5 miles south of the harbor. She heads directly towards an island 8 miles west and 7 miles north of the harbor. How far is Paige from the island?

63. **Distance.** Nam is 100 meters east and 250 meters north of Kristen. He walks directly towards a tree 220 meters east and 90 meters north of Kristen. How far is Nam from the tree?

64. **Computing Angles (MyOpenMath).**

    a. Through what angle does the hour hand of a clock rotate between 2 pm and 10 pm?

    b. Through what angle does the hour hand of a clock rotate between 2 am and 10 pm?

35. **Computing Angles.**

    a. Through what angle does the hour hand of a clock rotate between 3:25 am and 3:50 am?

    b. Through what angle does the hour hand of a clock rotate between 4:10 pm and 6:25 pm?

36. **Astronomy.** The Distance Formula can be modified to compute distances between objects in three-dimensional space. To do so, we must expand the Cartesian coordinate system to include one more spatial dimension. In three-space, coordinates are listed in **ordered triples**, such as (x, y, z). If we have two points, $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, in three-dimensional space, the distance between these points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

    Suppose our Sun is given the very special coordinate of $(0, 0, 0)$ (thereby affirming that we are the center of the universe). Compute the distance between each star (these computed distances are in light-years). Round answers to the nearest hundredth of a light-year.

    a. Our Sun, $(0, 0, 0)$, and Alpha Centauri, $(-1.8, 0, 3.9)$.

    b. Our Sun, $(0, 0, 0)$, and Sirius, $(-3.4, -3.1, 7.3)$.

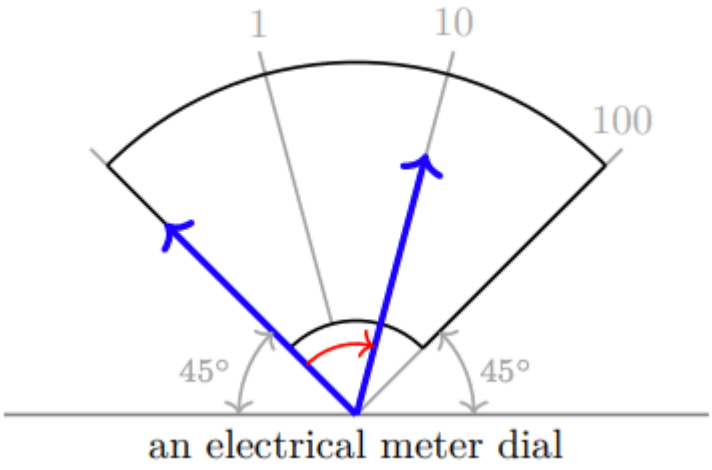
    c. Our Sun, $(0, 0, 0)$, and Polaris, $(99.6, 28.2, 376.0)$.

    d. Alpha Centauri, $(-1.8, 0, 3.9)$, and Sirius, $(-3.4, -3.1, 7.3)$.

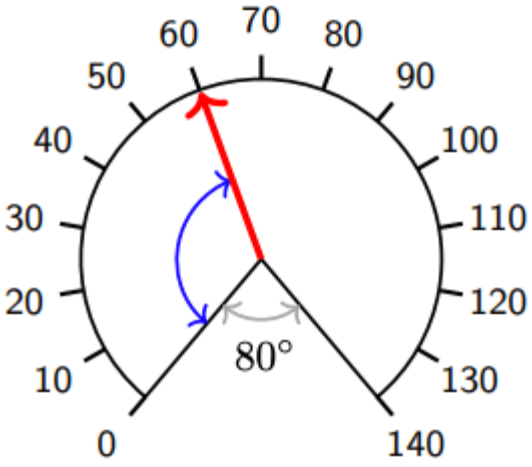
    e. Zubenelgenubi, $(64.6, -22.0, 23.0)$, and Polaris, $(99.6, 28.2, 376.0)$.

Computing Angles. For the following exercises, calculate the degree measure of the unknown angle and sketch the angle in standard position.

37.

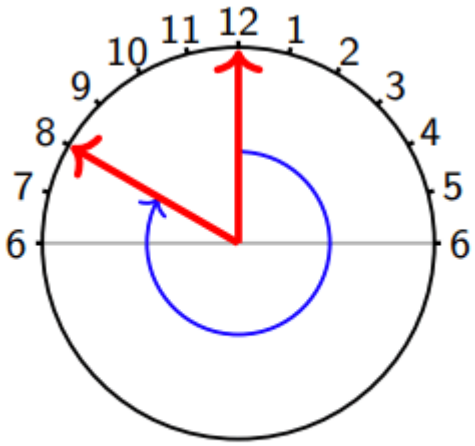


38.



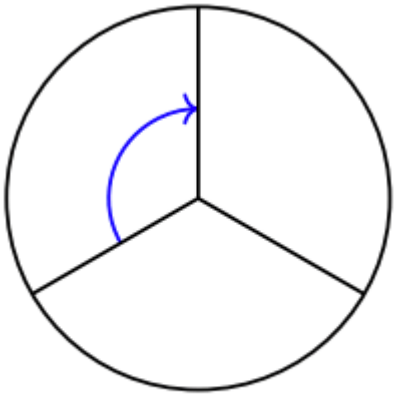
a speedometer

69.



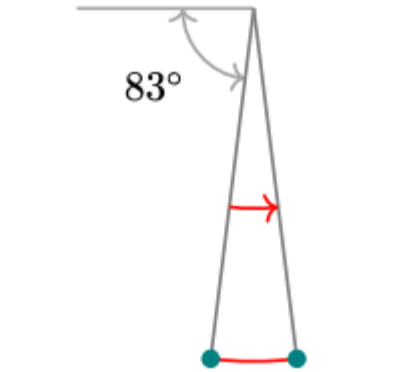
a sundial

70.



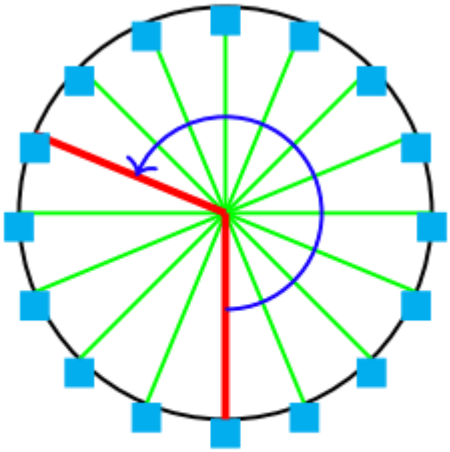
a revolving door

71.



a swinging pendulum

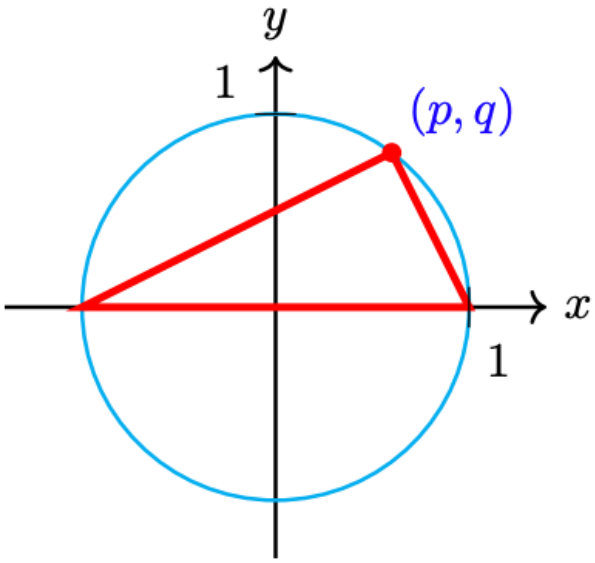
72.



a Ferris wheel

Challenge Problems

73. In this problem, we'll show that any angle inscribed in a semi-circle must be a right angle. The figure shows a triangle inscribed in a unit circle, one side lying on the diameter of the circle and the opposite vertex at point (p, q) on the circle.

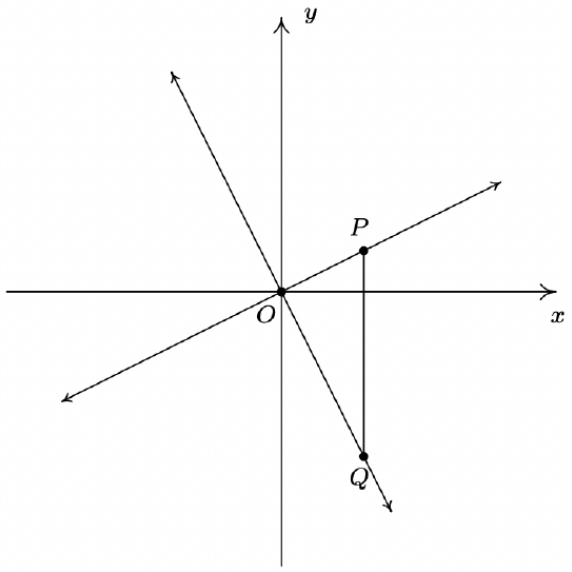


a. What are the coordinates of the other two vertices of the triangle? What is the length of the side joining those vertices?

b. Use the Distance Formula to compute the lengths of the other two sides of the triangle.

c. Show that the sides of the triangle satisfy the Pythagorean Theorem, $a^2 + b^2 = c^2$.

74. We shall now prove that $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ if and only if $m_1 \cdot m_2 = -1$. To make our lives easier we shall assume that $m_1 > 0$ and $m_2 < 0$. We can also "move" the lines so that their point of intersection is the origin without messing things up, so we'll assume $b_1 = b_2 = 0$. (Take a moment with your classmates to discuss why this is okay.) Graphing the lines and plotting the points $O(0, 0)$, $P(1, m_1)$ and $Q(1, m_2)$ gives us the following set up.



The line $y = m_1x$ will be perpendicular to the line $y = m_2x$ if and only if $\triangle OPQ$ is a right triangle. Let d_1 be the distance from O to P, let d_2 be the distance from O to Q and let d_3 be the distance from P to Q. Use the Pythagorean Theorem to show that $\triangle OPQ$ is a right triangle if and only if $m_1 \cdot m_2 = -1$ by showing $d_1^2 + d_2^2 = d_3^2$ if and only if $m_1 \cdot m_2 = -1$.

Answers

Skills Refresher

1. $m = 5.2 \quad (0, -1.9)$
2. $m = 1 \quad (0, 0)$

3. $m = 0 \quad (0, -8)$
4. The slope is undefined and there is not a y-intercept.
5. $m = \frac{5}{6} \quad \left(0, \frac{5}{12}\right)$
6. $y = -2x + 2$
7. $x = 5$
8. False
9. False
10. True
11. True
12. False
13. False
14. True
15. True
16. $x = \pm 9$
17. $y = \pm 7$
18. No solution because x^2 should never be negative, but when solving, we get $x^2 = -4$.
19. $x = \pm\sqrt{3}$

This page titled 4.4.2: Homework is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

5: COORDINATE TRIGONOMETRY

5.1: The Trigonometric Functions - Coordinate Definition

5.1.1: Resources and Key Concepts

5.1.2: Homework

5.2: Algebra with Trigonometric Functions

5.2.1: Resources and Key Concepts

5.2.2: Homework

5.3: Introduction to Trigonometric Identities

5.3.1: Resources and Key Concepts

5.3.2: Homework

5.4: Introduction to Proofs in Trigonometry

5.4.1: Resources and Key Concepts

5.4.2: Homework

This page titled 5: Coordinate Trigonometry is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

- Chapter 4: Trigonometric Functions by Katherine Yoshiwara is licensed GNU FDL. Original source: https://yoshiwarabooks.org/trig/trig.html.

5.1: The Trigonometric Functions - Coordinate Definition

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

Defining Trigonometric Functions in Terms of Coordinates

Armed with our knowledge from the previous chapter, we now define the **trigonometric functions**. We will be viewing the trigonometric functions from three different perspectives in this course—**coordinate trigonometry**, right triangle trigonometry, and unit circle trigonometry (since we are not covering right triangle trigonometry and unit circle trigonometry in this chapter, I chose to only bold coordinate trigonometry.). For the most part, each perspective is mathematically equivalent to the others; however, each has an inherent weakness as a standalone perspective of the trigonometric functions. While you can find a favorite perspective and stick with that one for most of the course, understanding all three is the only way to succeed in Trigonometry.

Definition: Trigonometric Functions (Coordinate Definition)

Let θ be an angle in standard position with the point $P(x, y)$ on the terminal side of θ, where P is not the origin. Define $r = \sqrt{x^2 + y^2}$. The trigonometric functions of the angle θ are defined as follows:



A few notes are in order. First, the trigonometric functions are simply the ratios of x, y, and r, where $P(x, y)$ is a point on the terminal side of θ and r is the distance from the origin to the point P (see Figure $5.1.1$ below).

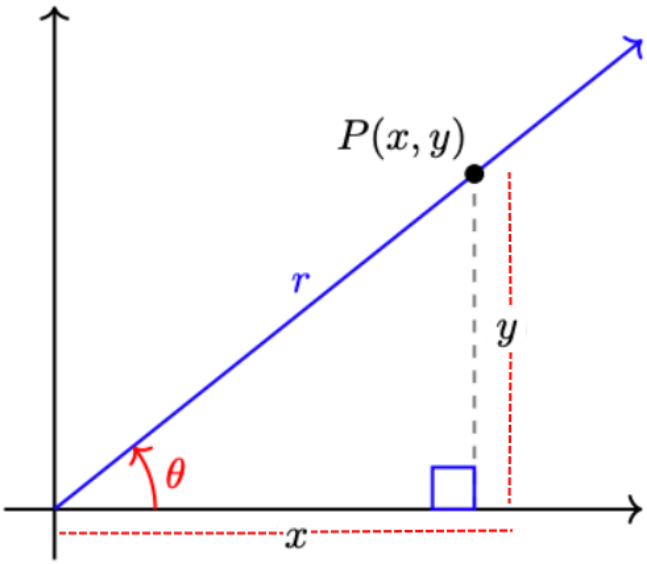


Figure 5.1.1

Second, because it's the distance from the origin to P, r is always positive. However, x and y can be positive, negative, or zero (but not both zero), depending on the angle θ. For example, in the second quadrant, x is negative, but y is positive, so the cosine and the tangent of angles between 90° and 180° are negative, but their sines are positive.

Third, referencing Figure 5.1.1, since the tangent function is defined to be $\frac{y}{x}$ and the terminal side of θ is always a ray emanating from the origin, we get that

$$\frac{y}{x} = \frac{\text{rise}}{\text{run}} = \text{slope of the terminal side of } \theta.$$

The interpretation of the value of the tangent function at θ being the slope of the ray that is the terminal side of θ grants us a potent conceptual understanding of the tangent function. Specifically, since a vertical line has an undefined slope, it makes sense that the tangent function is undefined when $x = 0$. Any ray emanating from the origin and going through a point $P\,(0, y)$, where $y \neq 0$, will be vertical. While the secant function is also undefined when $x = 0$ (and the cosecant and cotangent functions are undefined when $y = 0$), we will hold off on gaining a conceptual understanding of why until we familiarize ourselves with more of the basics of the trigonometric functions.

Interactive Element: The Tangent Is the Slope of the Terminal Side

The following Interactive Element might help you visualize how the tangent of an angle is just the slope of its terminal side—and why it is undefined precisely when that side is vertical.

Interact: Drag the red point P; the blue ray is the terminal side of the angle, and the green and orange legs show the run x and the rise y.

Observation: The slope of the terminal side is $\text{rise}/\text{run} = y/x = \tan\theta$. As you drag P toward the y-axis, the run shrinks to 0, the ray stands vertical, and $\tan\theta$ grows without bound and is undefined; along the x-axis the slope is 0.

Note: Angles are in Standard Position

Unless otherwise stated, all angles from this point forward will be considered angles in standard position.

Example 5.1.1

Find the values of the six trigonometric functions if $P(-3,-8)$ is on the terminal side of θ.

Solution available at the end of this section.

Note: Rationalizing Denominators

In Algebra, we learned to rationalize denominators so we could "simplify" expressions like $-\frac{8}{\sqrt{73}}$; however, in Trigonometry, such manipulations often are pointless. The rule of thumb I will adhere to in this textbook is to rationalize denominators *if* the denominator involves a variable (e.g., $\frac{3}{\sqrt{x}-2}$). In all other cases (specifically, when the numerator and denominator do *not* contain variables), I will only rationalize denominators if there is an "obvious" benefit.

In the case of $-\frac{8}{\sqrt{73}}$, if we tried to rationalize the denominator, we would get

$$-\frac{8}{\sqrt{73}}\cdot\frac{\sqrt{73}}{\sqrt{73}}=-\frac{8\sqrt{73}}{73},$$

which is no better than the original fraction.

Checkpoint 5.1.1

a. Find the equation of the terminal side of the angle in the previous example. (Hint: The terminal side lies on a line that goes through the origin and the point $(-3,-8)$).
b. Show that the point $P'(-6,-16)$ also lies on the terminal side of the angle.
c. Compute the trigonometric ratios for θ using the point P' instead of P.

Answers available at the end of this section.

Example 5.1.2

Find the values of $\cos(\theta)$ and $\tan(\theta)$ if θ is an obtuse angle with $\sin(\theta)=\frac{1}{3}$.

Solution available at the end of this section.

Example 5.1.2 illustrates a common theme in Trigonometry—we will often use the quadrant of the terminal side of an angle to inform our decision on the sign of x- and y-values.

If given a negative trigonometric ratio, as in Checkpoint 5.1.2, it is important to remember that the value of r is always non-negative (at least, prior to introducing something called the polar coordinate system).

Checkpoint 5.1.2

a. Sketch an obtuse angle θ whose cosine is $\frac{-8}{17}$.
b. Find the cosecant and the tangent of θ.

Answers available at the end of this section.

Our use of function notation (e.g., the parentheses in $\sin(\theta)=\frac{y}{r}$) is purposeful. This is because, a given angle θ (which is considered the independent variable) returns a single value for the associated trigonometric ratio (which we consider to be the dependent variable). Thus, by definition, the trigonometric functions are... functions (of angles).

Caution: Notational Abuse

We will sometimes write $\cos\theta$ when we really mean $\cos(\theta)$. This is a common, but accepted, abuse of notation reserved specifically for the trigonometric functions (and, at times, the logarithmic functions).

A common mistake is to think of $\cos\theta$ or $\cos(\theta)$ as a product, cos *times* θ, but this makes no sense, because "cos" by itself has no meaning. Remember that $\cos\theta$ represents a single number, namely the output of the cosine function.

Trigonometric Values at Special Angles

As was mentioned earlier in this textbook, special triangles play a crucial role in Trigonometry. In fact, the angles 30°, 45°, and 60° are just a few of the so-called **special angles** because we can express the exact values of their trigonometric functions in terms of radicals. In the following example, we explore the values of the trigonometric functions at one of the special angles.

Example 5.1.3

Find the values of all six trigonometric functions of 30°.

Solution available at the end of this section.

At this point, you should be asking yourself why we arbitrarily chose $a = 1$ in Example 5.1.3 to "lock in" the value of the point $P\left(\sqrt{3}, 1\right)$. Had we chosen, for example, $a = 5$, we would have gotten

$$x = 5\sqrt{3}, \quad y = 5, \quad \text{and} \quad r = 10.$$

Therefore,

$$\sin(30^\circ) = \frac{y}{r} = \frac{5}{10} = \frac{1}{2} \qquad \csc(30^\circ) = \frac{r}{y} = \frac{10}{5} = 2$$

$$\cos(30^\circ) = \frac{x}{r} = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2} \qquad \sec(30^\circ) = \frac{r}{x} = \frac{10}{5\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$\tan(30^\circ) = \frac{y}{x} = \frac{5}{5\sqrt{3}} = \frac{1}{\sqrt{3}} \qquad \cot(30^\circ) = \frac{x}{y} = \frac{5\sqrt{3}}{5} = \sqrt{3}$$

Notice that the values of the trigonometric functions didn't change? In fact, if we had chosen to leave a "as is," we would have arrived at

$$x = a\sqrt{3}, \quad y = a, \quad \text{and} \quad r = 2a.$$

Therefore,

$$\sin(30^\circ) = \frac{y}{r} = \frac{a}{2a} = \frac{1}{2} \qquad \csc(30^\circ) = \frac{r}{y} = \frac{2a}{a} = 2$$

$$\cos(30^\circ) = \frac{x}{r} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2} \qquad \sec(30^\circ) = \frac{r}{x} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$\tan(30^\circ) = \frac{y}{x} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} \qquad \cot(30^\circ) = \frac{x}{y} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

The choice of a doesn't change the values of the trigonometric functions at all! This demonstration *implies* (but does not prove) that the values of the trigonometric functions do not depend on which point you choose on the terminal side of 30°—just as long as that point is *not* the origin.

Note

The result of this last statement is beneficial. If we have angles from our special triangles, we can just set $a = 1$ in those special triangles to get the x- and y-values of the related point.

Checkpoint 5.1.3

Find the values of the six trigonometric functions at $\theta = 45^\circ$.

Answer available at the end of this section.

We do not need to limit our use of the special angles and special triangles to angles given in the first quadrant. The following example illustrates our tactics if given an angle terminating in a quadrant other than the first, where we can use our knowledge of the special triangles.

Example 5.1.4

Find exact values for all six trigonometric functions of 240°.

Solution available at the end of this section.

Checkpoint 5.1.4

Find exact values for the sine, cosine, and tangent of 300°.

Answer available at the end of this section.

All special angles between 0° and 360° are shown in Figure 5.1.12 below.

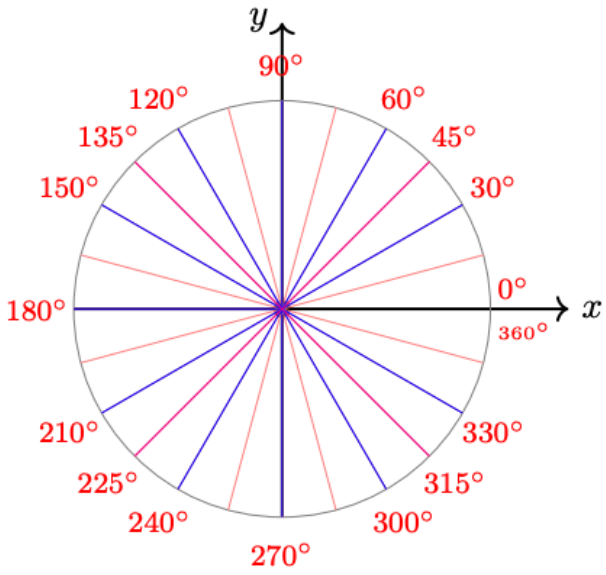


Figure 5.1.12

The astute student will notice that these can be succinctly written using the following definition.

Definition: Special Angles

The **special angles** are defined to be any angle that can be written as either $\pm 30^\circ k$ or $\pm 45^\circ k$, where $k \overset{\text{int}}{\in} \mathbb{Z}$.

It would be best if you memorized these values or be able to calculate them quickly.

Trigonometric Values at Quadrantal Angles

Evaluating trigonometric functions at quadrantal angles is much easier than at non-quadrantal angles. However, we have to watch out for division by zero.

Example 5.1.5

Find the values of the six trigonometric functions at -270°.

Solution available at the end of this section.

Example 5.1.5 illustrates the fact that there are angles for which some of the trigonometric functions are not defined. In terms of functions, we would say that these functions have domain restrictions. We will discuss this in more detail later in this textbook. It suffices to know that the tangent, cotangent, secant, and cosecant functions each have angles for which the functions are undefined. You can see from the definitions of these trigonometric functions where these domain issues occur (either when $x = 0$ or when $y = 0$, depending on the trigonometric function).

Checkpoint 5.1.5

Find the values of the six trigonometric functions at 540°.

Answer available at the end of this section.

Caution: Picky Language

Very technically, "$\cot(540^\circ) = \text{ UNDEFINED}$" is not correct. It not only sounds odd (read the following sentence out loud, and you will understand what we mean, "This value equals undefined"), but it also is not mathematically accurate. Once you use the equals sign ($=$), you are assigning a value; however, "UNDEFINED" is not a value.

A mathematically accurate response would be, "$\cot(540^\circ)$ is undefined."

Comparing Values of the Trigonometric Functions

Understanding how to compare the values of a trigonometric function for different angles can be beneficial in future math courses. For example, in Calculus, you will need to have an innate understanding that $\sin(50^\circ) < \sin(80^\circ)$. The following example illustrates the thought process.

Example 5.1.6

Without reaching for technology, can you determine which is greater, $\sin(50^\circ)$ or $\sin(80^\circ)$?

Solution available at the end of this section.

Checkpoint 5.1.6

Without reaching for technology, can you determine which is greater, $\cos(100^\circ)$ or $\cos(170^\circ)$?

Answer available at the end of this section.

Quadrants and the Signs of the Trigonometric Functions

Example 5.1.7

Give the sign of the sine, cosine, and tangent of the angle.

a. 200°
b. 300°

Solutions available at the end of this section.

Example 5.1.7 demonstrates a common need in Trigonometry—to determine the sign of a trigonometric function. Luckily, our task is fairly easy.

Theorem: Signs of the Trigonometric Functions

The signs of the trigonometric functions depend on the trigonometric function in question and the quadrant where the terminal side of θ lies.

All trigonometric functions are positive when the terminal side of θ is in the first quadrant.

Sine and the cosecant functions are positive when the terminal side of θ is in the second quadrant.

Tangent and the cotangent functions are positive when the terminal side of θ is in the third quadrant.

Cosine and the secant functions are positive when the terminal side of θ is in the fourth quadrant.

Proof available at the end of this section.

Figure 5.1.17 summarizes these results.

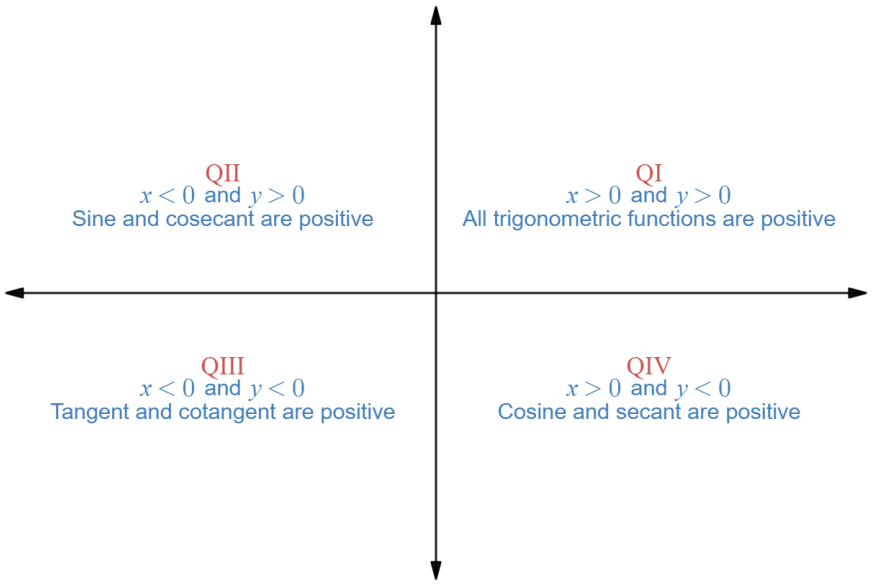


Figure 5.1.17

The bold lettering in the previous theorem is purposeful. Figure 5.1.18 gives the lovely mnemonic, "All Student Take Calculus," to remember that **a**ll trigonometric functions are positive in the first quadrant, the **s**ine and cosecant functions are positive in the second quadrant, the **t**angent and cotangent functions are positive in the third quadrant, and the **c**osine and secant functions are positive in the fourth quadrant.

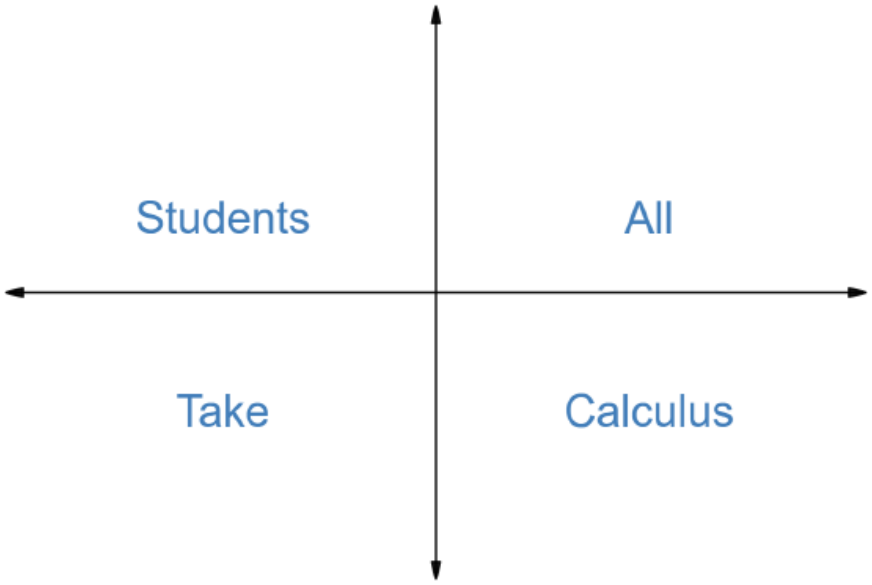


Figure 5.1.18

Example 5.1.8

Given that $\cos(\theta) = -\frac{3}{5}$ and θ terminates in QIII, find the values of the remaining trigonometric functions of θ.

Solution available at the end of this section.

Checkpoint 5.1.8

In which quadrant(s) is the sine negative but the cosine positive?

Answer available at the end of this section.

Proving that Values of the Trigonometric Functions are Independent of P

Up to this point, we have avoided *proving* that the choice of P on the terminal side of θ makes no difference (other than choosing the origin—which is not allowed). We now officially prove the result so we can use it from this point forward.This proves the following theorem.

Theorem: Trigonometric Ratios are Independent of P

Let $P(x,y)$ and $\overline{P}(\overline{x},\overline{y})$ be two distinct points on the terminal side of θ, neither of which are the origin, and let $r=\sqrt{x^2+y^2}$ and $\overline{r}=\sqrt{\overline{x}^2+\overline{y}^2}$. Then

$$\begin{aligned}
\sin(\theta) &= \frac{y}{r} = \frac{\overline{y}}{\overline{r}} & \csc(\theta) &= \frac{r}{y} = \frac{\overline{r}}{\overline{y}}\\
\cos(\theta) &= \frac{x}{r} = \frac{\overline{x}}{\overline{r}} & \sec(\theta) &= \frac{r}{x} = \frac{\overline{r}}{\overline{x}}\\
\tan(\theta) &= \frac{y}{x} = \frac{\overline{y}}{\overline{x}} & \cot(\theta) &= \frac{x}{y} = \frac{\overline{x}}{\overline{y}}
\end{aligned}$$

Proof available at the end of this section.

Hidden Resources: Tutoring Services

Technically, this is not a "hidden resource," per se, but the tutoring services at CRC are a bit spread out between programs. We have the Tutoring Center, Math Center at CRC, Math Center at Elk Grove, Reading and Writing Center, Accounting Lab, Embedded Tutoring (Supplemental Instruction), and NetTutor. If you're an athlete, you might have additional access to tutors through the athletics program. Needless to say, being overwhelmed by the number of overlapping tutoring services is completely understandable. Luckily, CRC has a single webpage with all of the links to these services. Check out the details on this CRC webpage.

Hidden Resources: Student Health and Wellness Services

Cosumnes River College has amazing support services for students in need of emotional and/or mental health services. Our Student Health and Wellness Services include the following:

Physical Health Services (no cost)

- Health and illness assessments
- First aid for minor injuries & illness
- Over-the-counter medications
- Party Safe Kits - Narcan, fentanyl & drink spike test strips
- Sexual health education
- Condoms (internal & external) & lubricant
- Opill (birth control)
- Emergency contraception (Plan B)
- STI testing
- Pregnancy testing
- Flu A/B & Covid testing
- Flu vaccine
- Strep throat testing

- TB risk assessments
- Health requirements for academic programs

Physical Health Services (low cost)

- Immunizations/titers
- TB skin test

Mental Health Services (no cost)

- **Togetherall:** This is a safe online space to connect anonymously with other students to get and give support.
- **Therapy:** Meet with a Los Rios Student Health and Wellness Center (SHWC) therapist or advocate. Support is available in person and virtually.
- **BetterMynd:** Online Therapy: Talk to an online licensed therapist during the day, evenings or weekends. Or call 24/7 support line: (844) BTR-MYND.

For more information, please see this page on the CRC website.

Answers

Section Preview

The definition of the trigonometric functions introduced in this section focuses on trigonometric functions of *any* angle (in degrees)—not just acute, obtuse, or positive angles; however, if given an angle (rather than a coordinate) and asked to evaluate a trigonometric function, the angle will strictly be a *special angle*. Therefore, using technology (e.g., calculators) to evaluate any trigonometric function in this section is *strictly avoided* so as to get the student familiar with these functions at a fundamental level. Simply stated, **students should not require a calculator to evaluate any of the trigonometric functions within this entire chapter!**

Section Preview

- **An Overview of Functions**
  - The Definition of a Function: Trig functions are defined as ratios y/r, x/r, etc., where $r = \sqrt{x^2 + y^2}$—so students need to already understand what a function is before the section can define trig functions *as* functions.
  - The Domain and Range of a Function: Domain and range are mentioned when discussing undefined values at quadrantal angles.

Section Preview

- **Simplifying Expressions**
  - Rationalizing Denominators and Numerators; Simplifying Quotients of Radical Expressions: Ratios involving radicals appears in this material, which may involve radical denominators requiring rationalization.
- **An Overview of Functions**
  - Evaluating Functions: Required to understand the notation, and resulting values, of computations in this section.

Section Preview

- Find the value of a trigonometric function of an angle given a point on the terminal side of the angle.
- Use the coordinate definition of the trigonometric functions to answer a conceptual question about a trigonometric function.
- Determine the quadrants in which an angle could terminate.
- Evaluate the trigonometric functions at the special angles and the quadrantal angles using the coordinate definition of the trigonometric functions.
- Find the value of a trigonometric function given one of the other trigonometric values.
- Compare the values of trigonometric functions at "non-special" angles without using technology.
- Use the quadrant in which an angle terminates to restrict the sign of a trigonometric function.

Example 5.1.1

It's a good habit in Trigonometry to start by sketching a graph of the situation.

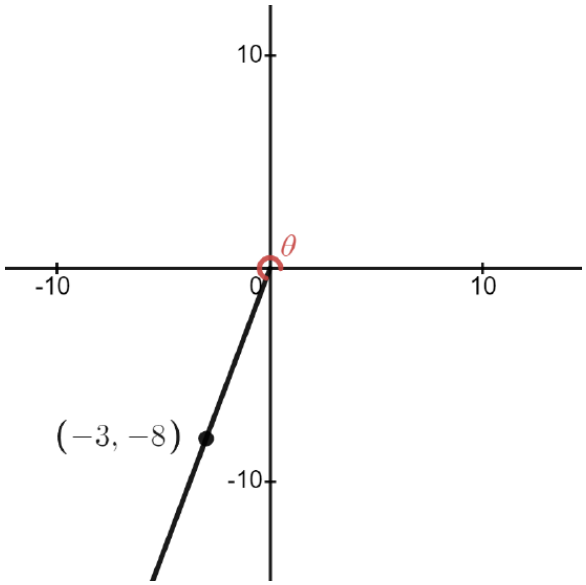


Figure 5.1.2

We have $x=-3$, $y=-8$, and $r=\sqrt{(-3)^2+(-8)^2}=\sqrt{9+64}=\sqrt{73}$. Therefore,

$$\sin(\theta) = \frac{y}{r} = \frac{-8}{\sqrt{73}} = -\frac{8}{\sqrt{73}} \qquad \csc(\theta) = \frac{r}{y} = \frac{\sqrt{73}}{-8} = -\frac{\sqrt{73}}{8}$$

$$\cos(\theta) = \frac{x}{r} = \frac{-3}{\sqrt{73}} = -\frac{3}{\sqrt{73}} \qquad \sec(\theta) = \frac{r}{x} = \frac{\sqrt{73}}{-3} = -\frac{\sqrt{73}}{3}$$

$$\tan(\theta) = \frac{y}{x} = \frac{-8}{-3} = \frac{8}{3} \qquad \cot(\theta) = \frac{x}{y} = \frac{-3}{-8} = \frac{3}{8}$$

Checkpoint 5.1.1

a. $y=\frac{8}{3}x$

b. $(-6,-16)$ satisfies $y=\frac{8}{3}x$. That is, the equation $-16=\frac{8}{3}(-6)$ is true.

c. $r^2=(-6)^2+(-16)^2=292$, so $r=\sqrt{292}=2\sqrt{73}$. Then

$$\sin(\theta) = \frac{y}{r} = \frac{-16}{2\sqrt{73}} = -\frac{8}{\sqrt{73}} \qquad \csc(\theta) = \frac{r}{y} = \frac{2\sqrt{73}}{-16} = -\frac{\sqrt{73}}{8}$$

$$\cos(\theta) = \frac{x}{r} = \frac{-6}{2\sqrt{73}} = -\frac{3}{\sqrt{73}} \qquad \sec(\theta) = \frac{r}{x} = \frac{2\sqrt{73}}{-6} = -\frac{\sqrt{73}}{3}$$

$$\tan(\theta) = \frac{y}{x} = \frac{-16}{-6} = \frac{8}{3} \qquad \cot(\theta) = \frac{x}{y} = \frac{-6}{-16} = \frac{3}{8}$$

Example 5.1.2

Because θ is obtuse, the terminal side of the angle lies in the second quadrant, as shown in Figure 5.1.3.

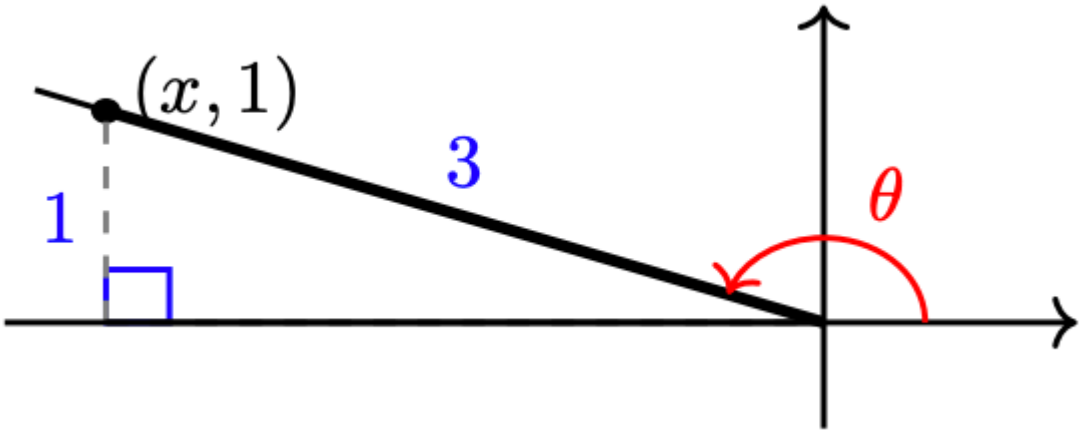


Figure 5.1.3

Because $\sin(\theta)=\frac{1}{3}$, we know that $\frac{y}{r}=\frac{1}{3}$, so we can choose a point P with $y=1$ and $r=3$. To find $\cos(\theta)$ and $\tan(\theta)$, we need to know the value of x. From the Pythagorean Theorem,

$$\begin{array}{rcrcll} & & x^2+1^2 & = & 3^2 & \text{(substitute known values)} \\ \xrightarrow{\text{Arithmetic}} & \Longrightarrow & x^2+1 & = & 9 & \text{(Laws of Exponents)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \Longrightarrow & x^2 & = & 8 & \text{(subtract 1 from both sides)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \Longrightarrow & x & = & \pm\sqrt{8} & \text{(Extraction of Roots)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \Longrightarrow & x & = & \pm 2\sqrt{2} & \text{(simplifying radicals)} \end{array}$$

Remembering that x is negative in the second quadrant, we get that $x = -2\sqrt{2}$. Thus,

$$\cos(\theta) = \frac{x}{r} = \frac{-2\sqrt{2}}{3} \quad \text{and} \quad \tan(\theta) = \frac{y}{x} = \frac{-1}{2\sqrt{2}}.$$

Checkpoint 5.1.2

a.

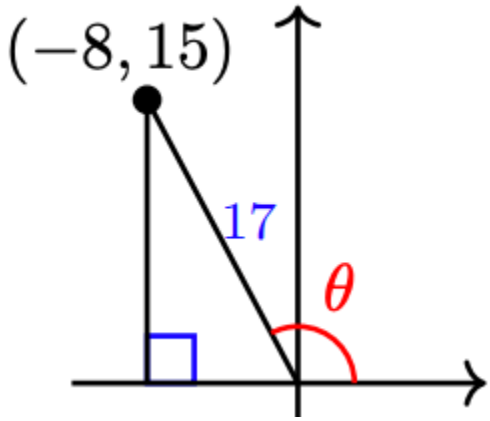


Figure 5.1.4

b. $\csc(\theta) = \frac{17}{15}, \tan(\theta) = \frac{-15}{8}$

Example 5.1.3

Like any good Trigonometry student, we start with a sketch.

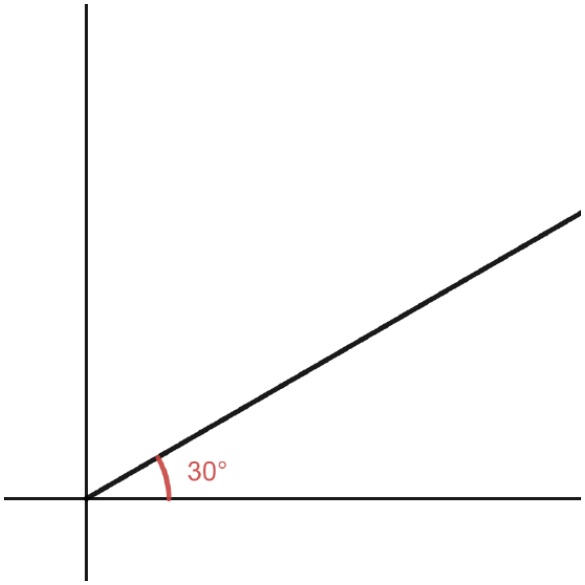


Figure 5.1.5

Unfortunately, we were not given a value for a point on the terminal side of $\theta = 30°$; however, recall the following figure from our exploration of the special triangles last chapter.

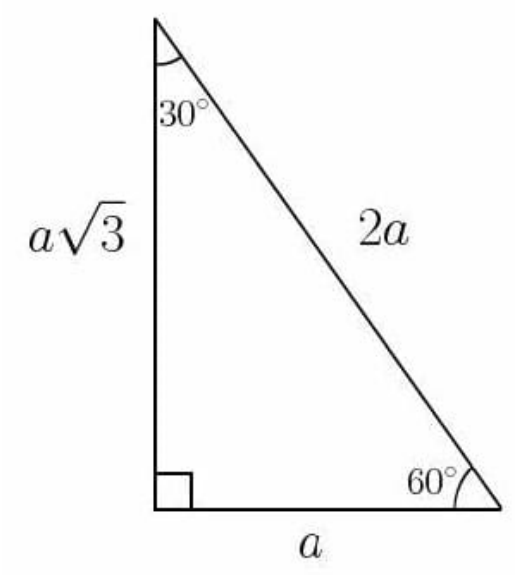


Figure 5.1.6

Rotating this special triangle counterclockwise by 90°, we can visualize fitting it to our first sketch, almost like a jigsaw puzzle.

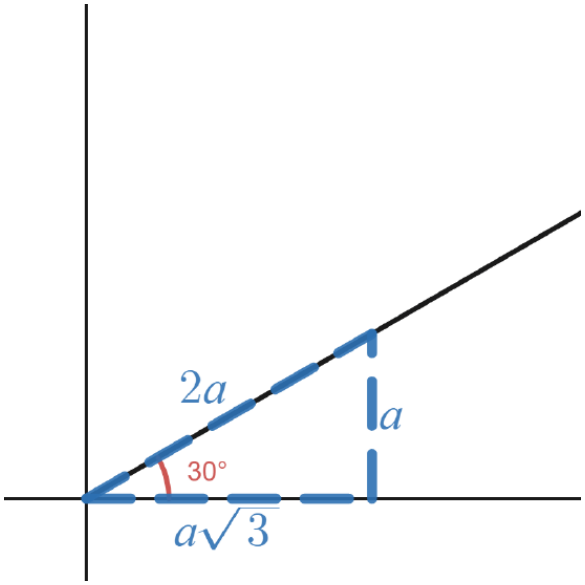


Figure 5.1.7

If we let $a = 1$ (a completely arbitrary choice), we get

$$x = \sqrt{3}, \quad y = 1, \quad \text{and} \quad r = 2.$$

Therefore,

$$\sin(30^\circ) = \frac{y}{r} = \frac{1}{2} \qquad \csc(30^\circ) = \frac{r}{y} = 2$$

$$\cos(30^\circ) = \frac{x}{r} = \frac{\sqrt{3}}{2} \qquad \sec(30^\circ) = \frac{r}{x} = \frac{2}{\sqrt{3}}$$

$$\tan(30^\circ) = \frac{y}{x} = \frac{1}{\sqrt{3}} \qquad \cot(30^\circ) = \frac{x}{y} = \sqrt{3}$$

Checkpoint 5.1.3

$$\sin(45^\circ) = \frac{1}{\sqrt{2}} \qquad \csc(45^\circ) = \sqrt{2}$$

$$\cos(45^\circ) = \frac{1}{\sqrt{2}} \qquad \sec(45^\circ) = \sqrt{2}$$

$$\tan(45^\circ) = 1 \qquad \cot(45^\circ) = 1$$

Example 5.1.4

An angle of 240° lies in the third quadrant, as seen in Figure 5.1.8.

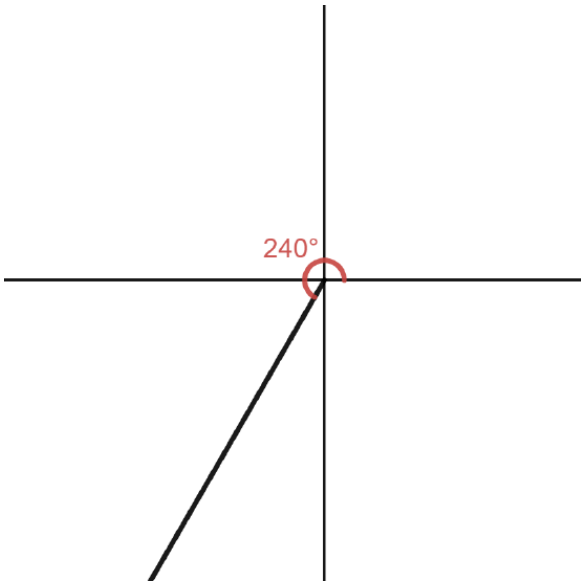


Figure 5.1.8

If we want to work with a smaller angle, we consider the angle between the terminal side of 240° and the negative x-axis. This can be found by taking the larger angle (240°) and subtracting off the smaller angle (the 180° angle that gets to the negative x-axis).

$$240^\circ - 180^\circ = 60^\circ.$$

This action is illustrated in Figure 5.1.9.

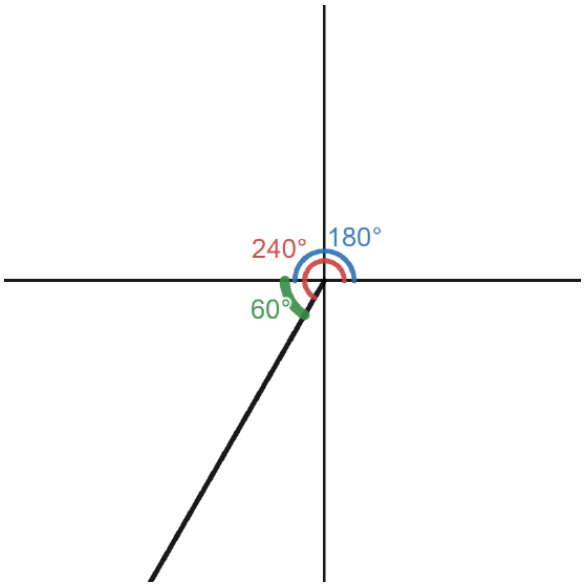


Figure 5.1.9

That 60° angle between 240° and 180° should be familiar.

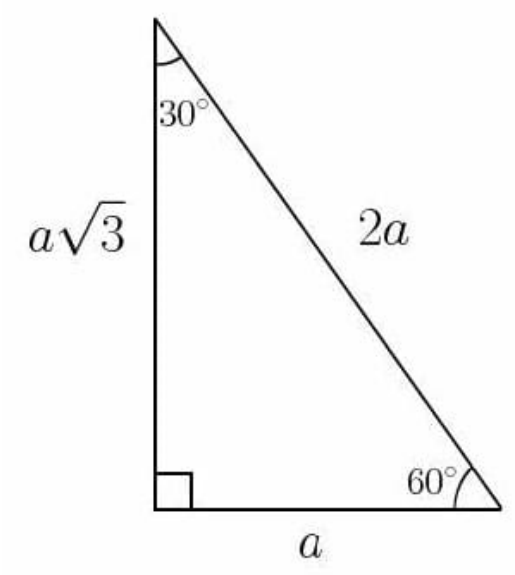


Figure 5.1.10

Superimposing this $30°$-$60°$-$90°$ special triangle into our graph, we get Figure 5.1.11.

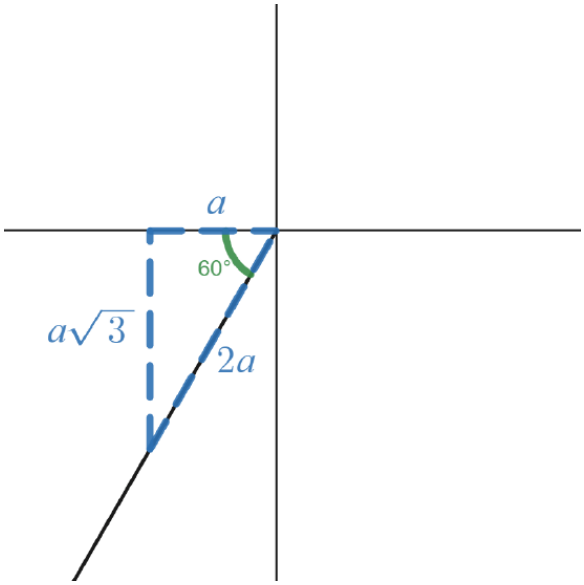


Figure 5.1.11

Using our new knowledge that we can let a be any convenient number, we choose $a = 1$. **However**, here is where we need to be careful! If $a = 1$, it is not true that $x = a = 1$. This is because the point $P(x, y)$ on the terminal side of $240°$ must be in QIII. Therefore, $x = -1$. Likewise, $y = -\sqrt{3}$. No matter what, however, $r > 0$. Thus, $r = 2a = 2$. We now compute the values of the trigonometric functions.

$$\sin(240°) = \frac{y}{r} = -\frac{\sqrt{3}}{2} \qquad \csc(240°) = \frac{r}{y} = -\frac{2}{\sqrt{3}}$$

$$\cos(240°) = \frac{x}{r} = -\frac{1}{2} \qquad \sec(240°) = \frac{r}{x} = -2$$

$$\tan(240°) = \frac{y}{x} = \sqrt{3} \qquad \cot(240°) = \frac{x}{y} = \frac{1}{\sqrt{3}}$$

Checkpoint 5.1.4

$\sin(300°) = \dfrac{-\sqrt{3}}{2}$, $\cos(300°) = \dfrac{1}{2}$, $\tan(300°) = -\sqrt{3}$

Example 5.1.5

We get Figure 5.1.13 by creating a quick sketch for this situation and remembering that negative angles open clockwise.

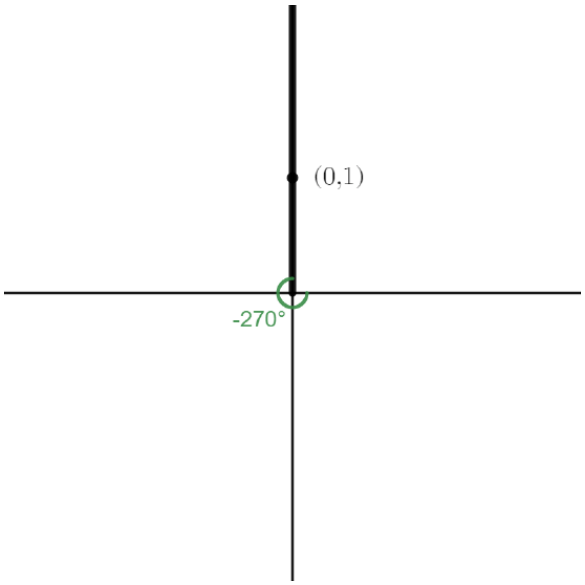


Figure 5.1.13

We took the liberty to label a point on the terminal side of -270°. In this case, $x = 0$, $y = 1$, and $r = 1$. Therefore,

$$\sin(-270^\circ) = \frac{y}{r} = \frac{1}{1} = 1 \qquad \csc(-270^\circ) = \frac{r}{y} = \frac{1}{1} = 1$$

$$\cos(-270^\circ) = \frac{x}{r} = \frac{0}{1} = 0 \qquad \sec(-270^\circ) = \frac{r}{x} = \frac{1}{0} = \text{UNDEFINED}$$

$$\tan(-270^\circ) = \frac{y}{x} = \frac{1}{0} = \text{UNDEFINED} \qquad \cot(-270^\circ) = \frac{x}{y} = \frac{0}{1} = 0$$

Checkpoint 5.1.5

$$\sin(540^\circ) = 0 \qquad \csc(540^\circ) = \text{UNDEFINED}$$
$$\cos(540^\circ) = -1 \qquad \sec(540^\circ) = -1$$
$$\tan(540^\circ) = 0 \qquad \cot(540^\circ) = \text{UNDEFINED}$$

Example 5.1.6

Quickly sketching two angles, $\alpha = 50^\circ$ and $\beta = 80^\circ$, we get the following graph.

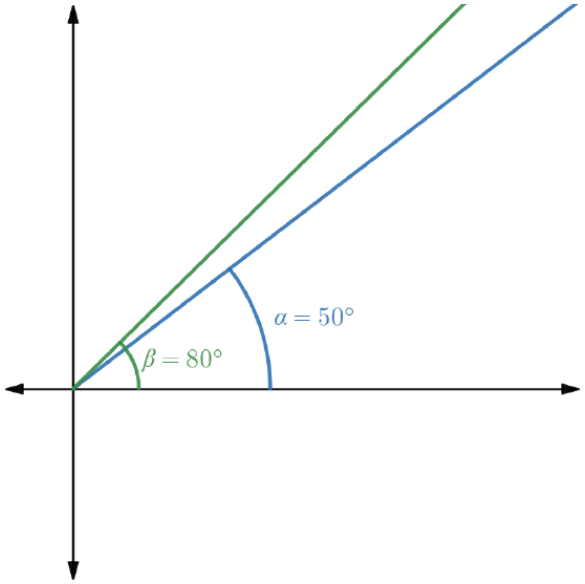


Figure 5.1.14

By itself, this graph is not helpful; however, if we recall that the sine of an angle is defined to be $\frac{y}{r}$, where (x, y) is *any* point on the terminal side of the angle and r is the distance from the origin to the point, then we could choose points on the terminal sides of $\alpha = 50^\circ$ and $\beta = 80^\circ$ such that $r = 1$. This gives us $\sin(50^\circ) = \frac{y_1}{r} = \frac{y_1}{1} = y_1$ and $\sin(80^\circ) = \frac{y_2}{r} = \frac{y_2}{1} = y_2$ (see Figure 5.1.15).

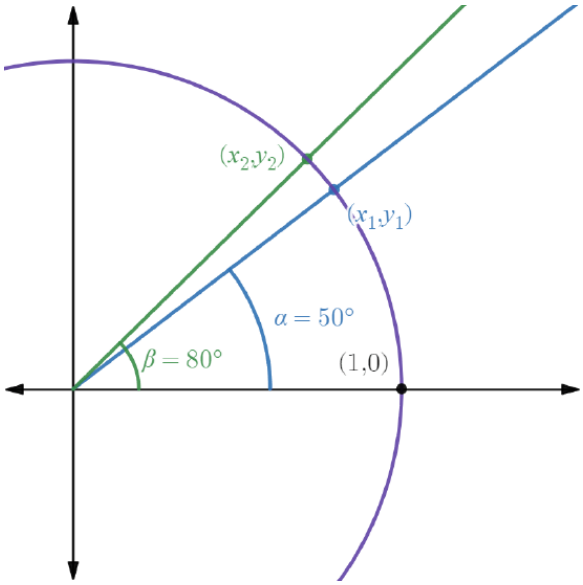


Figure 5.1.15

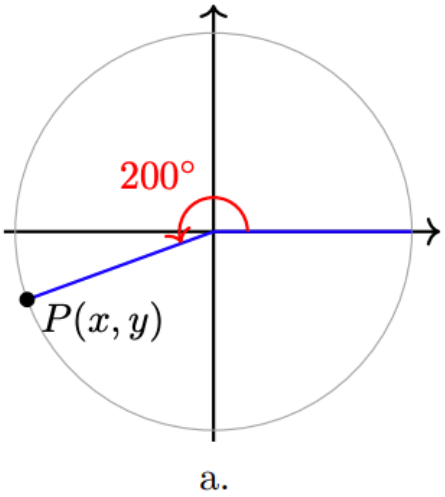
From Figure 5.1.15, we can see that $y_1 < y_2$. Therefore, $\sin(50°) < \sin(80°)$.

Checkpoint 5.1.6

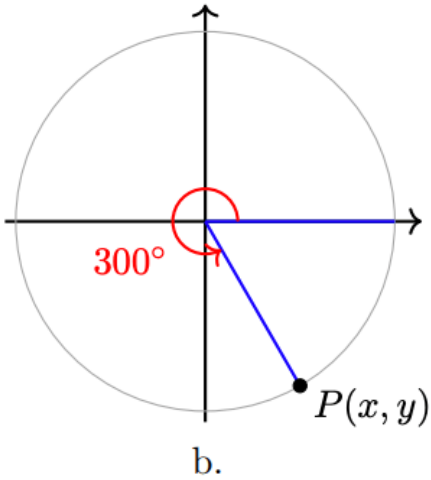
$\cos(100°)$ is greater than $\cos(170°)$

Example 5.1.7

a. In standard position, the terminal side of an angle of $200°$ lies in QIII (see Figure $5.1.16(a)$ below). In the third quadrant, $x < 0$ and $y < 0$, but $r > 0$. Thus, $\sin(200°)$ is negative, $\cos(200°)$ is negative, and $\tan(200°)$ is positive.



a.



b.

Figure 5.1.16

b. Double exponent: use braces to clarify, so $x > 0$ and $y < 0$, and $r > 0$. Thus, $\sin(300°)$ is negative, $\cos(300°)$ is positive, and $\tan(300°)$ is negative.

Theorem: Signs of the Trigonometric Functions

Since the value of r is always positive (unless the point is the origin), the signs of the trigonometric functions are completely dependent on the signs of the x- and y-coordinates of the point on the terminal side of θ.



Therefore, the signs of the sine and cosecant functions completely depend on the signs of the y-value of the point on the terminal side of θ. Hence, the sine and

cosecant are positive in QI and QII. On the other hand, these functions are negative when the angle terminates in either QIII or QIV.

Likewise, the signs of the cosine and secant functions completely depend on the signs of the x-value of the point on the terminal side of θ. Therefore, the cosine and secant functions are positive in QI and QIV - they are negative in QII and QIII.

Finally, since the tangent and cotangent functions rely on the values of both x and y, we must consider what would make these ratios positive and negative. If the values of x and y are both positive or both negative, their ratios will be positive. This occurs in the first and third quadrants. Thus, the tangent and cotangent functions are positive in QI and QIII - they are negative in QII and QIV.

Example 5.1.8

Since we are given $\cos(\theta) = -\frac{3}{5}$, we know the ratio of x to r is $-\frac{3}{5}$. Letting $x = -3$ and $r = 5$, we find y by solving

$$\begin{array}{rrcll} & r & = & \sqrt{x^2+y^2} & \text{(definition of } r\text{)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & 5 & = & \sqrt{(-3)^2+y^2} & \text{(substitute values)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & 25 & = & 9+y^2 & \text{(Laws of Exponents and squaring both sides)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & 16 & = & y^2 & \text{(subtracting 9 from both sides)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \pm 4 & = & y & \text{(Extraction of Roots)} \end{array}$$

Using the fact that $\theta \overset{\text{int}}{\text{QIII}}$, we throw out the positive possibility and set $y = -4$. We now have everything we need!

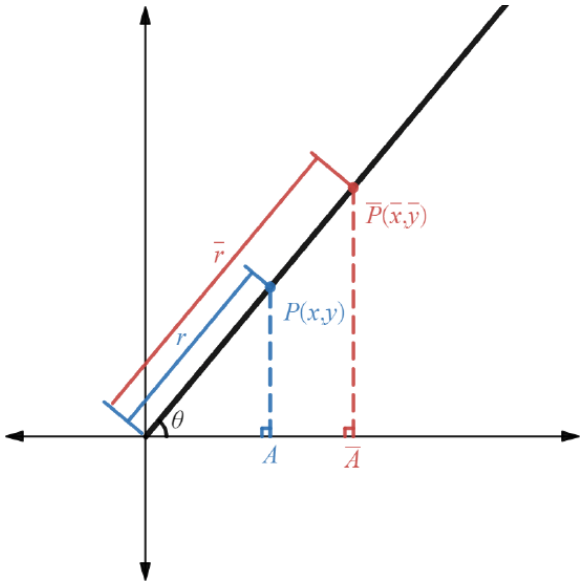
$$\begin{array}{rclrcl} \sin(\theta) & = & -\frac{4}{5} & \csc(\theta) & = & -\frac{5}{4} \\ \cos(\theta) & = & -\frac{3}{5} & \sec(\theta) & = & -\frac{5}{3} \\ \tan(\theta) & = & \frac{4}{3} & \cot(\theta) & = & \frac{3}{4} \end{array}$$

Checkpoint 5.1.8

QIV

Theorem: Trigonometric Ratios are Independent of P

Let $P(x,y)$ and $\overline{P}(\overline{x},\overline{y})$ be two distinct points on the terminal side of θ, neither of which are the origin, and let $r = \sqrt{x^2+y^2}$ and $\overline{r} = \sqrt{\overline{x}^2+\overline{y}^2}$ (see the following figure).



Since $\triangle AOP$ and $\triangle \overline{A}O\overline{P}$ are similar,

$$\begin{array}{rclrcl} \sin(\theta) & = & \frac{y}{r} = \frac{\overline{y}}{\overline{r}} & \csc(\theta) & = & \frac{r}{y} = \frac{\overline{r}}{\overline{y}} \\ \cos(\theta) & = & \frac{x}{r} = \frac{\overline{x}}{\overline{r}} & \sec(\theta) & = & \frac{r}{x} = \frac{\overline{r}}{\overline{x}} \\ \tan(\theta) & = & \frac{y}{x} = \frac{\overline{y}}{\overline{x}} & \cot(\theta) & = & \frac{x}{y} = \frac{\overline{x}}{\overline{y}} \end{array}$$

This page titled 5.1: The Trigonometric Functions - Coordinate Definition is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

5.1.1: Resources and Key Concepts

Resources

Videos

- Trigonometric Functions (Coordinate Definition)
  - Trigonometric Functions of Angles - Review of Trigonometric Functions - The Point Definition

Key Concepts

Definitions

- **Trigonometric Functions (Coordinate Definition):** Let θ be an angle in standard position and let $P(x, y)$ be a point on its terminal side. If $r = \sqrt{x^2 + y^2}$ is the distance from the origin to P, the six trigonometric functions are defined as the ratios of x, y, and r. For example, $\sin(\theta) = \frac{y}{r}$ and $\cos(\theta) = \frac{x}{r}$.
- **Special Angles:** Angles that can be expressed in the form $\pm 30^\circ k$ or $\pm 45^\circ k$, where k is an integer. The exact values of their trigonometric functions can be found using special triangles.

Theorems

- **Signs of the Trigonometric Functions:** The signs of the trigonometric function values depend on the quadrant in which the angle's terminal side lies.
  - **QI:** All functions are positive.
  - **QII:** Sine and cosecant are positive.
  - **QIII:** Tangent and cotangent are positive.
  - **QIV:** Cosine and secant are positive.
- **Trigonometric Ratios are Independent of P:** The values of the trigonometric functions of an angle θ depend only on the angle itself, not on the specific point $P(x, y)$ chosen on the terminal side of θ (provided the point is not the origin).

Common Mistakes

- **Confusing Function Notation with Multiplication:** The notation $\cos(\theta)$ represents a single value, which is the output of the cosine function for the input angle θ. It is not a product of "cos" and "θ". The term "cos" by itself has no mathematical meaning.
- **Incorrectly Stating Undefined Values:** Mathematically, it is more accurate to state that a function "is undefined" at a certain value rather than saying it "equals UNDEFINED". For example, the statement "$\cot(540^\circ)$ is undefined" is preferred over "$\cot(540^\circ) = \text{UNDEFINED}$".
- **Assuming r Can Be Negative:** In the coordinate definition of trigonometric functions, r represents a distance and is always non-negative.
- **Ignoring the Quadrant for Signs:** When finding trigonometric values, the quadrant of the angle is crucial for determining the correct signs of the x and y coordinates. For example, for an obtuse angle (in QII), the x-coordinate must be negative.

This page titled 5.1.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

5.1.2: Homework

Reading Questions

1. Using the variables x, y, and r, what are the definitions for $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$?
2. What is the formula for r in the coordinate definition of the trigonometric functions?
3. The value of $\tan(\theta)$ is equivalent to what geometric property of the angle's terminal side?
4. Why are the values of the trigonometric functions independent of the specific point chosen on the terminal side?
5. Why are some trigonometric functions, like $\tan(90^\circ)$, undefined?
6. What makes an angle a "special angle"?
7. In which quadrant are the sine and cosine functions both negative?
8. What does the mnemonic "All Students Take Calculus" help you remember?
9. If you are given that $\sin(\theta) > 0$ and $\cos(\theta) < 0$, in which quadrant must the terminal side of θ lie?
10. Explain the "rule of thumb" this textbook will use for deciding when to rationalize denominators.

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, evaluate the function.

1. $f(x) = x^2 - 2x$
   a. $f(-3)$
   b. $f(a - 3)$
   c. $f(a) - 5$
   d. $f(a) - f(5)$
2. $g(x) = \sqrt{x + 4}$
   a. $g(9)$
   b. $g(4h)$
   c. $g(0) + g(1)$
   d. $g(c^2)$
3. $F(x) = \dfrac{2}{x}$
   a. $F\left(\dfrac{-1}{2}\right)$
   b. $F\left(\dfrac{w}{2}\right)$
   c. $F(w + 2)$
   d. $F(w) + F(2)$

4. $G(x) = 2^x$
   a. $G(-3)$
   b. $G(a+3)$
   c. $G(a) + G(3)$
   d. $G\left(\frac{3}{2}\right)$

For the following exercises, rationalize the denominator.

5. $\frac{7}{\sqrt{5}}$
6. $\sqrt{\frac{2}{7}}$
7. $\frac{\sqrt{3}}{\sqrt{32}}$
8. $\frac{3}{2\sqrt{5x}}$
9. $\frac{11}{6-\sqrt{3}}$
10. $\frac{\sqrt{3}}{3-\sqrt{5}}$
11. $\frac{\sqrt{5}-2}{\sqrt{3}-\sqrt{5}}$
12. $\frac{-4}{\sqrt[4]{3x^3}}$

For the following exercises, rationalize the numerator.

13. $\frac{11-\sqrt{2}}{6}$
14. $\frac{\sqrt{x}-\sqrt{h}}{\sqrt{x}}$

Answers available at the end of this section.

Homework

Vocabulary Check

1. ___ defines new functions in terms of the coordinates of points on the Cartesian Plane and their distance to the origin.
2. In the function notation $f(x)$, x is called the ___ variable.
3. 30°, 45°, and 60° are examples of ___ angles.
4. A helpful way to remember in which quadrants the trigonometric functions are positive is ASTC. This stands for **A**___ **S**___

T__ C__.

Concept Check

5. We define the trigonometric ratios of any angle by placing the angle in standard position and choosing a point on the terminal side, with $r = \sqrt{x^2 + y^2}$.

6. We use the notation $y = f(x)$ to indicate that y is a function of x, that is, x is the input variable and y is the output variable.

7. The trigonometric ratios $\sin\theta$ and $\cos\theta$ are functions of the angle θ.

8. If θ is an angle in standard position, and (x, y) is a point on its terminal side, with $r = \sqrt{x^2 + y^2}$, then $\sin(\theta) = \frac{y}{r}$ $\cos(\theta) = \frac{x}{r}$ $\tan(\theta) = \frac{y}{x}$ $\csc(\theta) = \frac{r}{y}$ $\sec(\theta) = \frac{r}{x}$ $\cot(\theta) = \frac{x}{y}$

9. There are always two angles between 0° and 360° (except for the quadrantal angles) with a given trigonometric ratio.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

10. The angle used in the coordinate definition of the trigonometric functions is measured from the positive y-axis.

11. The value, $\frac{1}{\sqrt{2}}$, is "more correct" than its rationalized form $\frac{\sqrt{2}}{2}$.

12. $\cos\theta$ is another notation for $\cos(\theta)$.

13. $3\sin(30^\circ) = \sin(3 \cdot 30^\circ) = \sin(90^\circ) = 1$.

14. $\sin(\theta)$ can be undefined.

15. $\tan(\theta)$ can be undefined.

16. All students take Calculus.

17. If P and Q are points on the terminal side of θ, where neither are the origin, and Q is three times the distance from the origin as P, the values of the trigonometric functions defined by Q are three times the values of the same trigonometric functions defined by P.

Basic Skills

For the following exercises, find the exact values of all six trigonometric functions of θ if the given point is on the terminal side of θ.

18. $(3, 4)$

19. $\left(-\sqrt{3}, 1\right)$

20. $(-5, -5)$

21. $(15, -8)$

22. $(2, 3)$

23. $\left(-\sqrt{5}, 1\right)$

24. $(-6, 0)$

25. $(0, -2)$

26. (A, B)

27. $(-6t, 9t)$ (assume $t > 0$)

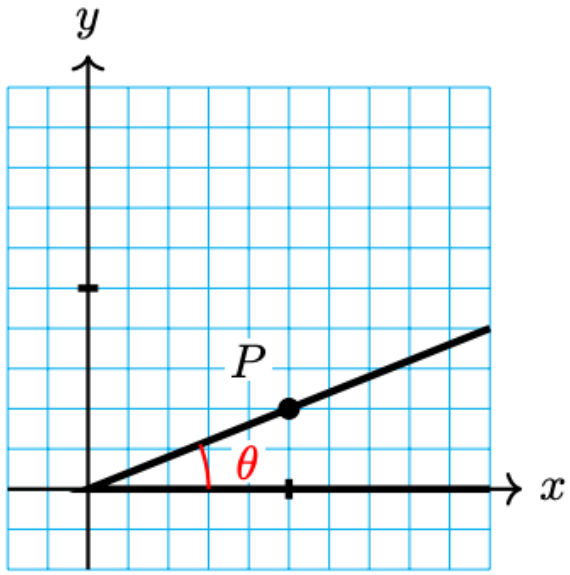
For the following exercises, use technology to find the approximate values of all six trigonometric functions of θ if the given point is on the terminal side of θ.

28. $(1.27, 5.03)$

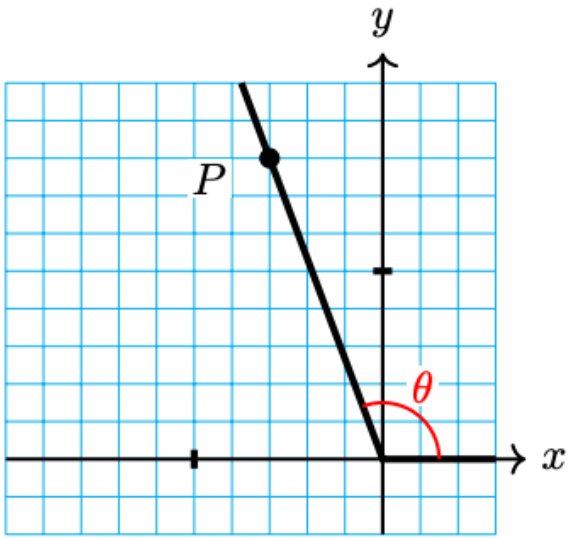
29. $(3.39, -7.46)$

For the following exercises,
(a) Give the coordinates of point P on the terminal side of the angle.
(b) Find the distance from the origin to point P.
(c) Find $\cos\theta, \sin\theta$, and $\tan\theta$.

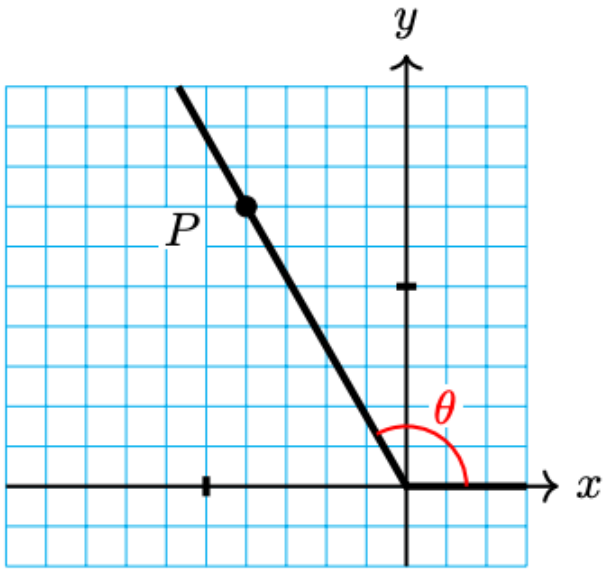
30.



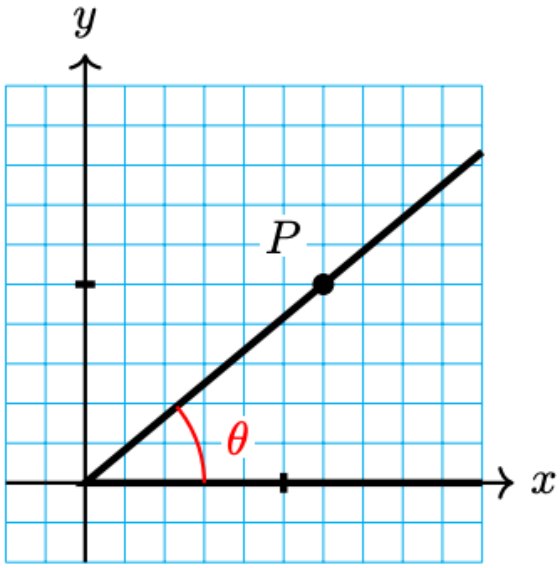
31.



32.



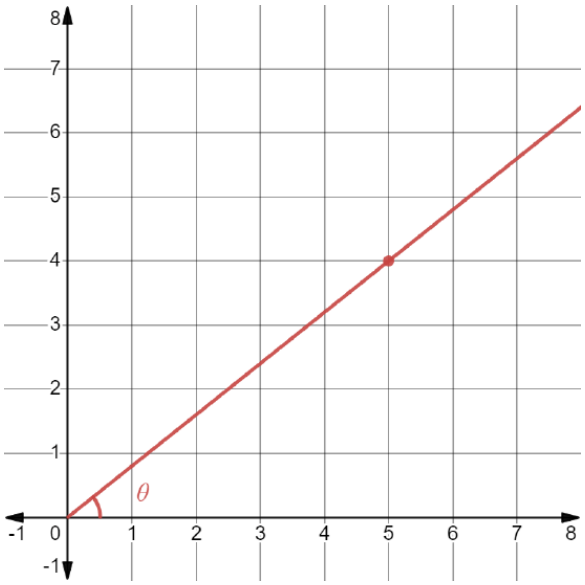
33.



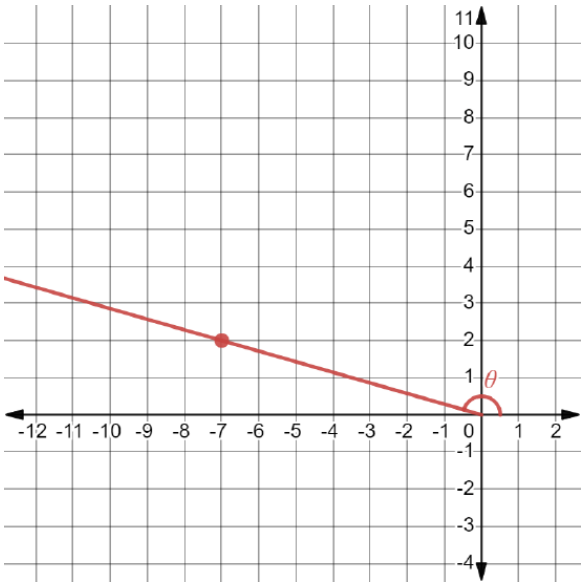
For the following exercises, determine the $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for the θ in the given graph. Assume θ is in standard position.

34.

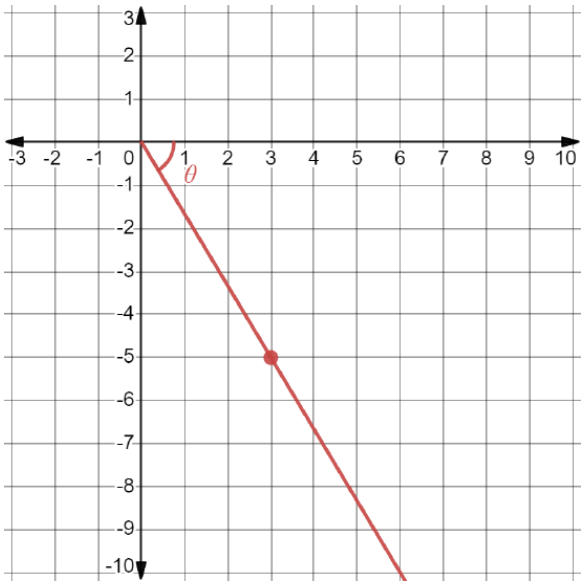
35.

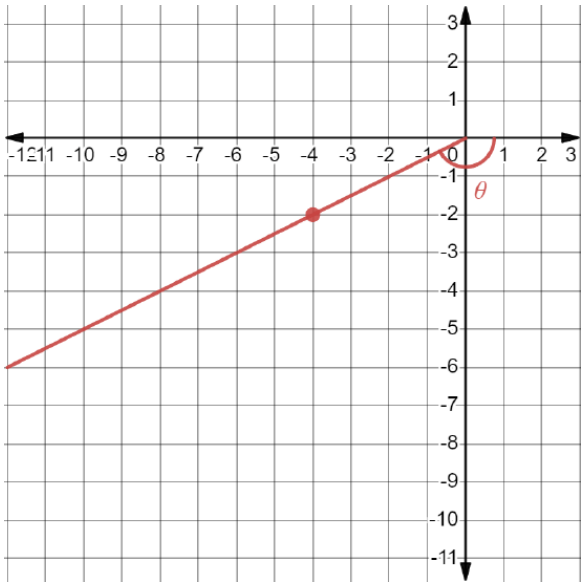


36.



37.





38. Complete the table with exact values.

θ	30°	45°	60°	120°	135°	150°	210°	225°	240°	300°	315°	330°
$\cos(\theta)$												
$\sin(\theta)$												
$\tan(\theta)$												

39. Fill in the table from memory with exact values. Do you notice any patterns that might help you memorize the values?

θ	0°	30°	45°	60°	90°
$\sin(\theta)$					
$\cos(\theta)$					
$\tan(\theta)$					

40. Fill in the table from memory with decimal approximations to four places.

θ	0°	30°	45°	60°	90°
$\sin(\theta)$					
$\cos(\theta)$					
$\tan(\theta)$					

41. Complete the table with exact values.

θ	0	30°	45°	60°
$\sec(\theta)$				
$\csc(\theta)$				
$\cot(\theta)$				

For the following exercises, choose all values from the list below that are exactly equal to, or decimal approximations for, the given trigonometric ratio. (Try not to use a calculator!)



42. $\cos 30^\circ$

43. $\sin 45^\circ$

44. $\tan 30^\circ$

45. $\cos 60^\circ$

46. $\sin 90^\circ$

47. $\cos 0^\circ$

For the following exercises, evaluate. Give exact values.

48. $\sin(30^\circ)$

49. $\cos(0^\circ)$

50. $\tan(45^\circ)$

51. $\sin(60^\circ)$

52. $\csc(30^\circ)$

53. $\sec(0^\circ)$

54. $\cot(45^\circ)$

55. $\csc(60^\circ)$

For the following exercises, draw the angle in standard position, find a point on the terminal side of the angle, and find the value of all six trigonometric functions of the angle.

56. 150°

57. -90°

58. 225°

59. 180°

60. -300°

61. 90°

62. 315°

63. Explain why it makes sense that $\sin 0^\circ = 0$ and $\sin 90^\circ = 1$. Use a figure to illustrate your explanation.

64. Explain why it makes sense that $\cos 0^\circ = 1$ and $\cos 90^\circ = 0$. Use a figure to illustrate your explanation.

For the following exercises, state two possible quadrants where θ could terminate.

65. $\sin(\theta) = \frac{7}{12}$

66. $\cos(\theta) = -\frac{\sqrt{2}}{9}$

67. $\tan(\theta) = -1.2$

68. $\csc(\theta) = -3.87$

69. $\sec(\theta) = \sqrt{2}$

70. $\cot(\theta) = \frac{3}{2}$

For the following exercises, determine in which two quadrants is the statement true.

71. The sine is negative.

72. The cosine is negative.

73. The tangent is positive.

For the following exercises, state the quadrant(s) where θ must terminate.

74. $\sin(\theta) < 0$ and $\cos(\theta) > 0$

75. $\sec(\theta) > 0$ and $\tan(\theta) < 0$

76. $\csc(\theta) < 0$ and $\cot(\theta) < 0$

77. $\sin(\theta)$ and $\cot(\theta)$ have the same sign

For the following exercises, find the values of the remaining trigonometric functions of θ based on the given information.

78. $\sin(\theta) = \frac{21}{29}$, where $\theta^{\text{int}}\text{QI}$

79. $\cos(\theta) = -\frac{3}{5}$, where $\theta^{\text{int}}\text{QIII}$

80. $\tan(\theta) = \frac{15}{8}$, where $\theta^{\text{int}}\text{QIII}$

81. $\csc(\theta) = -\frac{2}{\sqrt{3}}$, where $\tan(\theta) > 0$

82. $\sec(\theta) = \sqrt{2}$, where $\theta^{\text{int}}\text{QIV}$

83. $\cot(\theta) = -\frac{1}{\sqrt{3}}$, where $\theta^{\text{int}}\text{QII}$

84. $\sin(\theta) = -\frac{1}{4}$, where $\cos(\theta) < 0$

85. $\tan(\theta) = \frac{a}{b}$, where both a and b are positive

For the following exercises, find all angles between 0° and 360° for which the statement is true.

86. $\cos(\theta) = -1$

87. $\sin(\theta) = -1$

88. $\tan(\theta) = -1$

89. Find two angles, $0 \le \theta \le 360^\circ$, with $\sin(\theta) = 0$.

90. Find two angles, $0 \le \theta \le 360^\circ$, with $\cos(\theta) = 0$.

91. Find all angles, $0 \le \theta \le 360^\circ$, with $\sin(\theta) = -1$.

92. Find all angles, $0 \le \theta \le 360^\circ$, with $\cos(\theta) = -1$.

Synthesis Questions

For the following exercises, determine if the statement is true or false. Justify your response.

93. If $\alpha < \beta$, then $\sin(\alpha) < \sin(\beta)$.

94. If $0 \le \alpha < \beta \le 90^\circ$, then $\sin(\alpha) < \sin(\beta)$.

95. If $0 \le \alpha < \beta \le 90^\circ$, then $\cos(\alpha) < \cos(\beta)$.

96. If $0 \le \alpha < \beta \le 90^\circ$, then $\csc(\alpha) < \csc(\beta)$.

97. Is there an angle θ such that $\cos(\theta) = 3$? Why or why not?

98. Is there an angle θ such that $\csc(\theta) = \frac{1}{5}$? Why or why not?

99. Explain why $|\cos(\theta)| \le 1$ for any angle θ.

100. Is it true that $\tan(\theta) \le 1$ for any angle θ? Explain.

101. Use a sketch to explain why $\cos 90^\circ = 0$.

102. Use a sketch to explain why $\cos 180^\circ = 1$.

For the following exercises, compare the given value with the trigonometric ratios of the special angles to answer the questions.

103. Is the acute angle larger or smaller than 45°?

a. $\sin\alpha = 0.7$

b. $\tan\beta = 1.2$

c. $\cos\gamma = 0.65$

94. Is the acute angle larger or smaller than 60°?

a. $\cos\theta = 0.75$

b. $\tan\phi = 1.5$

c. $\sin\psi = 0.72$

For the following exercises, describe what happens to the function values as θ increases from 0° to 90°.

95. $\sin(\theta)$

96. $\cos(\theta)$

97. $\tan(\theta)$

98. $\csc(\theta)$

99. Find two angles, $0 \le \theta \le 360^\circ$, with $\sin(\theta) = \cos(\theta)$.

100. Find two angles, $0 \le \theta \le 360^\circ$, with $\sin(\theta) = -\cos(\theta)$.

101. Explain why the definitions of the trigonometric ratios for a third-quadrant angle (between 180° and 270°) are independent of the point P chosen on the terminal side. Illustrate with a figure.

102. Explain why the definitions of the trigonometric ratios for a fourth-quadrant angle (between 270° and 360°) are independent of the point P chosen on the terminal side. Illustrate with a figure.

103. Find the values of all the trigonometric functions of θ if the terminal side of θ lies on the line $y = 3x$ in QI.

104. Find the values of all the trigonometric functions of θ if the terminal side of θ lies on the line $y = -\frac{1}{3}x$ in QII.

105. Find the values of all the trigonometric functions of θ if the terminal side of θ lies on the line $y = \frac{4}{5}x$ in QIII.

106. Find the values of all the trigonometric functions of θ if the terminal side of θ lies on the line $y = -6x$ in QIV.

For the following exercises, evaluate the expression for $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$.

107. $3 + f(30^\circ)$

108. $3f(30^\circ)$

109. $4g(225^\circ) - 1$

110. $-4 + 2g(225^\circ) - 1$

111. $-2f(3\theta)$, for $\theta = 90^\circ$

112. $6f\left(\frac{\theta}{2}\right)$, for $\theta = 90^\circ$

113. $8 - 5g\left(\frac{\theta}{3}\right)$, for $\theta = 360^\circ$

114. $1 - 4g(4\theta)$, for $\theta = 135^\circ$

For the following exercises,
(a) Sketch an angle θ in standard position, $0^\circ \le \theta \le 180^\circ$, with the given properties.
(b) Find expressions for $\cos\theta, \sin\theta$, and $\tan\theta$ in terms of the given variable.

25. $\cos\theta = \dfrac{x}{3}, x < 0$

26. $\tan\theta = \dfrac{4}{\alpha}, \alpha < 0$

27. θ is obtuse and $\sin\theta = \dfrac{y}{2}$

28. θ is obtuse and $\tan\theta = \dfrac{q}{-7}$

29. θ is obtuse and $\tan\theta = m$

30. $\cos\theta = h$

Answers

Skills Refresher

1. a. 15
   b. $a^2 - 8a + 15$
   c. $a^2 - 2a - 5$
   d. $a^2 - 2a - 15$
2. a. $\sqrt{13}$
   b. $2\sqrt{h+1}$
   c. $2 + \sqrt{5}$
   d. $\sqrt{c^2+4}$
3. a. -4
   b. $\dfrac{4}{w}$
   c. $\dfrac{2}{w+2}$
   d. $\dfrac{2}{w} + 1$
4. a. $\dfrac{1}{8}$
   b. $8(2^a)$
   c. $2^a + 8$
   d. $2\sqrt{2}$
5. $\dfrac{7\sqrt{5}}{5}$
6. $\dfrac{\sqrt{14}}{7}$
7. $\dfrac{\sqrt{6}}{8}$
8. $\dfrac{3\sqrt{5x}}{10x}$
9. $\dfrac{6+\sqrt{3}}{3}$
10. $\dfrac{3\sqrt{3}+\sqrt{15}}{4}$
11. $\dfrac{-\sqrt{15}+2\sqrt{3}-5+2\sqrt{5}}{2}$
12. $\dfrac{-4\sqrt[4]{27x^9}}{3x^3}$
13. $\dfrac{3}{22+2\sqrt{2}}$
14. $\dfrac{x-h}{x+\sqrt{hx}}$

This page titled 5.1.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

5.2: Algebra with Trigonometric Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

As we move forward, we will need to apply some techniques from Algebra to analyze more complicated trigonometric expressions. Therefore, let's review some algebraic terminology.

Definition: Algebraic Expression

An **algebraic expression** is a well-formed mathematical phrase that combines numbers, variables, and mathematical operations. It does not include an equals sign.

For example, the height of a golf ball is given in feet by the expression $-16t^2 + 64t$, where t is the number of seconds after the ball is hit.

Definition: Evaluate (an algebraic expression)

To **evaluate** an algebraic expression means to substitute a specific value for the variable or variables involved.

Thus, after 1 second, the height of the golf ball is $-16(1)^2 + 64(1) = -16 + 64 = 48$ feet, and after 2 seconds, the height is $-16(2)^2 + 64(2) = -64 + 128 = 64$ feet, and so on.

Evaluating Trigonometric Expressions

The value of a trigonometric function represents a number, and it may appear as part of an algebraic expression. Expressions containing trigonometric functions can be simplified or evaluated like other algebraic expressions.

Example 5.2.1

Evaluate each expression for $X = 30^\circ$ and $Y = 240^\circ$.

a. $\frac{1}{\sqrt{3}}\tan(Y) + 3\sin(X)$
b. $6\tan(X)\cos(Y)$

Solutions available at the end of this section.

Checkpoint 5.2.1

Evaluate each expression for $X = 30^\circ, Y = 60^\circ$.

a. $4\sin(3X + 45^\circ)$
b. $1 - \cos(4Y)$

Answers available at the end of this section.

Caution: Do NOT "Split" Arguments of Functions

In Checkpoint 5.2.1, $\sin(3X + 45^\circ)$ is *not* equal to $\sin(3X) + \sin(45^\circ)$. That is,

$$\sin(90^\circ + 45^\circ) \neq \sin(90^\circ) + \sin(45^\circ)$$

(You can check this for yourself.) Before applying the sine function, we must follow the Order of Operations and evaluate the expression $3X + 45^\circ$ inside parentheses.

Powers of Trigonometric Functions

Compare the two expressions

$$(\cos(\theta))^2 \quad \text{and} \quad \cos(\theta^2).$$

They are not the same. The first expression, $(\cos(\theta))^2$, says to compute $\cos(\theta)$ and then square the result. $\cos(\theta^2)$, on the other hand, says to square the angle first and then compute the cosine.

For example, if $\theta = 30^\circ$, then

$$(\cos(30^\circ))^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$$

but

$$\cos\left((30^\circ)^2\right) = \cos(900^\circ)$$

We have not delved into the details of how to compute the cosine of such a large angle (yet). Still, it should make sense that if the point $P(x, y)$ is on the terminal side of $\theta = 900^\circ$, then P must be on the terminal side of any angle coterminal with $\theta = 900^\circ$. Specifically, P must be on the terminal side of $900^\circ - 360^\circ = 540^\circ$. Still, this angle is too unwieldy to deal with, so we find another, smaller angle coterminal to 540°. Luckily, $540^\circ - 360^\circ = 180^\circ$ fits that bill! Hence, if $P(x, y)$ is on the terminal side of 900°, it must also be on the terminal side of 180°. Since the point $(-1, 0)$ is on the terminal side of 180°, we can now compute the cosine of 900°.

$$\cos\left((30^\circ)^2\right) = \cos(900^\circ) = \cos(180^\circ) = \frac{x}{r} = \frac{-1}{1} = -1.$$

Notice this is not the same value as $\cos(30^\circ)$.

This brings us to a new notation reserved specifically for trigonometric functions.

Notation: Powers of Trigonometric Functions

$$\sin^2(\theta) \quad \text{means} \quad (\sin(\theta))^2 \qquad \csc^2(\theta) \quad \text{means} \quad (\csc(\theta))^2$$
$$\cos^2(\theta) \quad \text{means} \quad (\cos(\theta))^2 \qquad \sec^2(\theta) \quad \text{means} \quad (\sec(\theta))^2$$
$$\tan^2(\theta) \quad \text{means} \quad (\tan(\theta))^2 \qquad \cot^2(\theta) \quad \text{means} \quad (\cot(\theta))^2$$

Example 5.2.2

Evaluate $\sin^2(45^\circ)$.

Solution available at the end of this section.

Other powers are written in the same fashion. Thus, for example, $\sin^3(\theta) = (\sin(\theta))^3$.

Checkpoint 5.2.2

Evaluate $\tan^4(60^\circ)$

Answer available at the end of this section.

Caution: Negative Exponents

We have already stated that $[\cos(\theta)]^2 = \cos^2(\theta)$ and $[\sin(\theta)]^5 = \sin^5(\theta)$. **However**, we will **not** adopt such notation for negative exponents. That is,

$$[\cos(\theta)]^{-1} \neq \cos^{-1}(\theta).$$

The reason for excluding the negative exponents from this "shorthand" notation will become apparent when we discuss inverse trigonometric functions.

Products of Trigonometric Functions

We can multiply trigonometric expressions just as we multiply algebraic expressions. Recall that we use the Distributive Law in computing products such as $x(3x-2) = 3x^2 - 2x$ and $(x-3)(x+5) = x^2 + 2x - 15$.

Example 5.2.3

Using the Distributive Law, multiply

$$\cos(t)(3\cos(t) - 2).$$

Solution available at the end of this section.

Checkpoint 5.2.3

Multiply $2\tan(\beta)(4\tan^2(\beta) + \tan(\alpha))$

Answer available at the end of this section.

We can also use the Distributive Law to multiply binomials that include trigonometric functions.

Example 5.2.4

Multiply $(4\sin(C)-1)(3\sin(C)+2)$.

Solution available at the end of this section.

Checkpoint 5.2.4

Expand $(4\cos(\alpha)+3)^2$

Answer available at the end of this section.

Factoring Expressions Involving Trigonometric Functions

We can factor trigonometric expressions with the same techniques we use for algebraic expressions. The next few subsections of this topic quickly review all our previous factorization techniques. It is important to note that each of these factoring techniques will be needed at some point during Trigonometry.

Factoring Out a Common Expression

Likely, the first technique you encountered when learning to factor was factoring out the greatest common factor (GCF). I call this factoring out a common expression.

In the next two examples, compare the familiar algebraic factoring with a similar trigonometric expression.

Example 5.2.5

Factor.

a. $6w^2-9w$
b. $6\sin^2(\theta)-9\sin(\theta)$

Solutions available at the end of this section.

Checkpoint 5.2.5

Factor.

a. $2a^2-ab$
b. $2\cos^2(\phi)-\cos(\phi)\sin(\phi)$

Answers available at the end of this section.

Factoring by Grouping

Once you mastered factoring out common expressions from a collection of terms, you moved to factoring by grouping. We only review via example here as we did in the previous topic.

Example 5.2.6

Factor.

a. $10x^2 + 2x - 15xy - 3y$
b. $10\sin^2(\theta) + 2\sin(\theta) - 15\sin(\theta)\cos(\theta) - 3\cos(\theta)$

Solutions available at the end of this section.

Example $5.2.6b$ illustrates a common tactic we will use as we move through mathematics. Namely, when an expression (or equation) looks overly complicated, we make it easier to work with by performing substitutions. In Calculus, this will be tied to a technique called u**-Substitutions** (where we use the variable u instead of x).

Checkpoint 5.2.6

Factor.

a. $12rt + 9r - 20t - 15$
b. $12\sec(\alpha)\tan(\beta) + 9\sec(\alpha) - 20\tan(\beta) - 15$

Answers available at the end of this section.

Factoring Trinomials

Despite being more complex than factoring a difference of squares (which we will talk about next), the next factorization technique you learned was factoring trinomials.

Example 5.2.7

Factor.

a. $t^2 - 3t - 10$
b. $\tan^2(\alpha) - 3\tan(\alpha) - 10$

Solutions available at the end of this section.

Checkpoint 5.2.7

Factor.

a. $3z^2 - 2z - 1$
b. $3\sin^2(\beta) - 2\sin(\beta) - 1$

Answers available at the end of this section.

Factoring Differences of Squares

In addition to factoring out common expressions and factoring trinomials, special factoring formulas for the Difference of Squares and the Sums and Differences of Cubes will be needed. Recall the following theorem from Algebra.

Theorem: Difference of Squares

For any algebraic expressions F and L,

$$F^2 - L^2 = (F - L)(F + L).$$

Proof available at the end of this section.

Example 5.2.8

Factor completely.

$$16\cos^4(\theta) - 1$$

Solution available at the end of this section.

A couple of notes are in order for Example 5.2.8. First and foremost, while we took our time with substitutions and being meticulous with our work, you need to reach a level of proficiency in mathematics where you do not need to perform those substitutions to do the factoring. That is, you need to reach the point (and very soon) where you see $16\cos^4(\theta) - 1$ and immediately rewrite it (without substitutions) as

$$\left(4\cos^2(\theta) - 1\right)\left(4\cos^2(\theta) + 1\right) = (2\cos(\theta) - 1)(2\cos(\theta) + 1)\left(4\cos^2(\theta) + 1\right).$$

I cannot express how important it is for you, as the student, to reach that level of mathematical fluency as quickly as possible (truthfully, in a college-level mathematics course, you should already have this level of fluency).

The second note is better mentioned as a cautionary statement.

Caution: $F^2 + L^2$ is NOT Factorable

A common mistake of students in lower levels of Algebra is to think that $F^2 + L^2$ is factorable—this is **not** true (unless we are factoring over the complex number system). That is,

$$4x^2 + 1$$

cannot be factored (over the real number system).

Checkpoint 5.2.8

Factor completely.

$$\sin^4(\alpha) - 81\cos^4(\beta)$$

Answer available at the end of this section.

Factoring Sums and Differences of Cubes

We have now reached the final factorization method, which we must recall from Algebra.

Theorem: Sums and Differences of Cubes

For any algebraic expressions F and L,

$$\begin{aligned} F^3 + L^3 &= (F + L)(F^2 - FL + L^2) \\ F^3 - L^3 &= (F - L)(F^2 + FL + L^2) \end{aligned}$$

Proof available at the end of this section.

Example 5.2.9

Factor.

$$27\sin^3(A) - 8$$

Solution available at the end of this section.

Checkpoint 5.2.9

Factor.

$$125\csc^3(\theta) + 64\cot^3(\phi)$$

Answer available at the end of this section.

Believe it or not, other factorization methods exist beyond the basic techniques reviewed in this section. For example, in College Algebra (and in Precalculus), you learn about factoring high-degree polynomials with integer coefficients using the Rational Roots Theorem. Moreover, there are critically important methods for factoring expressions involving negative rational exponents—these are encountered frequently in Calculus.

Hidden Resources: The Hawk Spot Food Pantry

The Hawk Spot Food Pantry is a program at CRC that gives food once per week to CRC students.

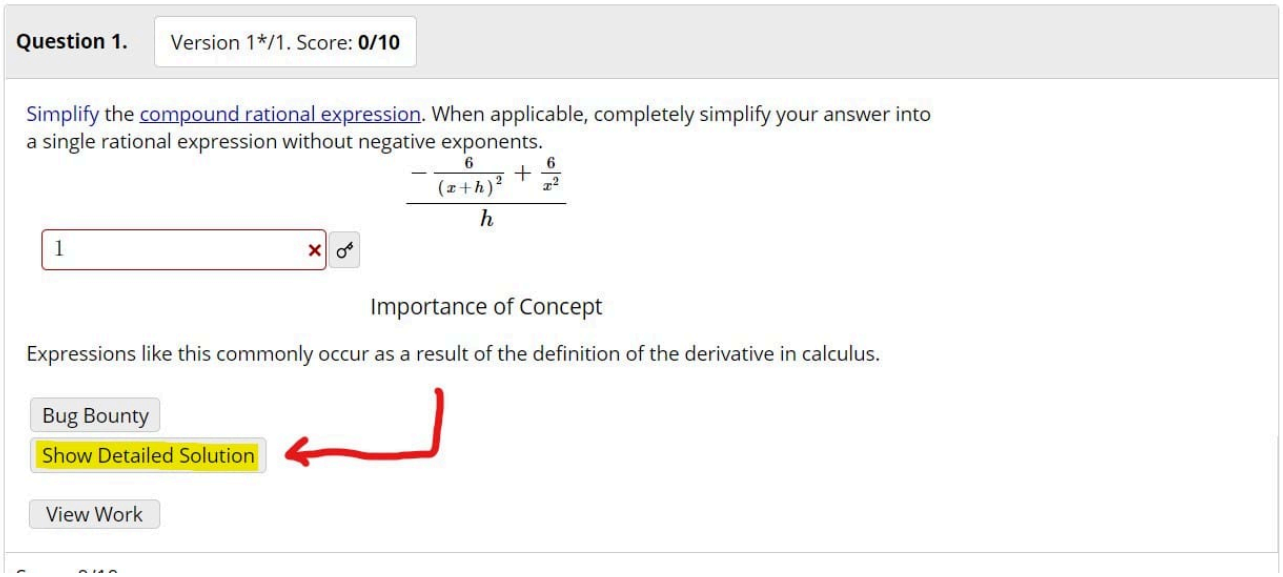
How Does CalFresh Work?

Details may change, so use the link below to verify what is stated here; however, you show your Student ID, fill out the intake form (on first visit), and pick-up your premade bag. Each student is allowed to be served once per week.

For more information, please see this page on the CRC website.

MyOpenMath Advice: Reading Solutions to Exercises

Some problems have complete worked-out solutions. If your feedback has a comment like, "Please see the solutions," then you should probably look through these solutions. You can see solutions by going into the Grades area, finding the assignment, and clicking "Show Detailed Solution" on an exercise with a solution. Solutions are generally not shown until after the due date has passed.



I usually leave a comment of the form, "Please see the written solutions," if there were too many mistakes for me to cover by typing or if I could not follow your logic.

- Professor Simpson, Cosumnes River College

Answers

Section Preview

This section focuses on performing our old Algebra (factoring, distribution, powers, etc.) on expressions involving trigonometric functions. The results will not require any trigonometric identities. The point here is for the student to realize that Algebra still works, even though we are not using x.

Section Preview

- **Simplifying Expressions**
  - Laws of Exponents — Product and Quotient Rules: Understanding that trig functions behave as algebraic objects requires knowing the Product and Quotient Rules from the Laws of Exponents—without them, a student cannot follow the conceptual framework of the section.
  - Laws of Exponents — Power Rules: Understanding that trig functions behave as algebraic objects requires knowing the Power Rules from the Laws of Exponents—without them, a student cannot follow the conceptual framework of the section.
  - Multiplying Polynomials and the Distributive Property: Understanding that trig functions behave as algebraic objects requires knowing the Distributive Property—without this, a student cannot follow the conceptual framework of the section.

Section Preview

- **Arithmetic**
  - The Order of Operations: This skill is explicitly called out in the cautionary statement regarding not splitting functions across arguments.
- **Factoring Techniques**
  - Greatest Common Factor: This mechanical skill is needed to complete the homework problems.
  - Factoring by Grouping: This mechanical skill is needed to complete the homework problems.
  - Factoring Trinomials: This mechanical skill is needed to complete the homework problems.
  - Factoring Differences of Squares: This mechanical skill is needed to complete the homework problems.
  - Factoring Sums and Differences of Cubes: This mechanical skill is needed to complete the homework problems.

Section Preview

- Evaluate expressions involving trigonometric functions.
- Compute powers of trigonometric functions.
- Evaluate products of trigonometric functions.
- Perform factorization on expressions involving trigonometric functions.

Example 5.2.1

a. Substituting the values for X and Y, we get the following:

$$\xrightarrow{\text{Arithmetic}\to\text{Algebra}} \implies \frac{1}{\sqrt{3}}\tan(Y)+3\sin(X) = \frac{1}{\sqrt{3}}\tan(240^\circ)+3(\sin(30^\circ)) \quad \text{(substitute)}$$

$$\xrightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} \implies = \frac{1}{\sqrt{3}}(\sqrt{3})+3\left(\frac{1}{2}\right) \quad \text{(evaluating the trigonometric functions)}$$

$$\xrightarrow{\text{Arithmetic}} \implies = 1+\frac{3}{2} \quad \text{(canceling like factors)}$$

$$\xrightarrow{\text{Arithmetic}} \implies = \frac{5}{2} \quad \text{(addition)}$$

b. This expression includes the product of two trigonometric functions.

$$\begin{aligned}
\cancel{\text{Arithmetic}}\to\text{Algebra} \implies 6\tan(X)\cos(Y) &= 6(\tan(30^\circ))(\cos(240^\circ)) && \text{(substitute)}\\
\cancel{\text{Arithmetic}}\to\cancel{\text{Algebra}}\to\text{Trigonometry} \implies &= 6\left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{2}\right) && \text{(evaluating the trigonometric functions)}\\
\text{Arithmetic} \implies &= -\frac{3}{\sqrt{3}} && \text{(simplify)}\\
\cancel{\text{Arithmetic}}\to\text{Algebra} \implies &= -\sqrt{3} && \text{(rationalizing the denominator)}
\end{aligned}$$

Checkpoint 5.2.1

a. $2\sqrt{2}$

b. $\frac{3}{2}$

Example 5.2.2

$$\begin{aligned}
\sin^2(45^\circ) &= (\sin(45^\circ))^2 && \text{(notational definition)}\\
\cancel{\text{Arithmetic}}\to\cancel{\text{Algebra}}\to\text{Trigonometry} \implies &= \left(\frac{1}{\sqrt{2}}\right)^2 && \text{(evaluating the trigonometric function)}\\
\cancel{\text{Arithmetic}}\to\text{Algebra} \implies &= \frac{1}{2} && \text{(Laws of Exponents)}
\end{aligned}$$

Checkpoint 5.2.2

9

Example 5.2.3

Think of $\cos(t)$ as a single variable, and multiply by each term inside parentheses. (The algebraic form of the calculation is shown on the right in bold).

$$\begin{aligned}
\cancel{\text{Arithmetic}}\to\text{Algebra} \implies \cos(t)(3\cos(t)-2) &= (\cos(t))(3\cos(t)) - (\cos(t)\cdot 2) && \text{(distribution: } \boldsymbol{x(3x-2) = x\cdot 3x - x\cdot 2}\text{)}\\
\cancel{\text{Arithmetic}}\to\text{Algebra} \implies &= 3\cos^2(t) - 2\cos(t) && \text{(Commutative Property of Multiplication)}
\end{aligned}$$

Notice that we write $(\cos(t))^2$ as $\cos^2(t)$.

Checkpoint 5.2.3

$8\tan^3(\beta) + 2\tan(\beta)\tan(\alpha)$

Example 5.2.4

This calculation is similar to the product $(4x-1)(3x+2)$, except that the variable x has been replaced by $\sin(C)$. Compare the calculations for the two products; first the familiar algebraic product:

$$\begin{aligned}
(4x-1)(3x+2) &= 4x\cdot 3x + 4x\cdot 2 - 1\cdot 3x - 1\cdot 2\\
&= 12x^2 + 8x - 3x - 2 = 12x^2 + 5x - 2
\end{aligned}$$

We compute the product in this example in the same way, but replacing x by $\sin(C)$.

$$\begin{aligned}
(4\sin(C)-1)(\sin(C)+2) &= (4\sin(C))(3\sin(C)) + (4\sin(C)\cdot 2) - 1(3\sin(C)) - 1\cdot 2\\
&= 12\sin^2(C) + 5\sin(C) - 2
\end{aligned}$$

Checkpoint 5.2.4

$16\cos^2(\alpha) + 24\cos(\alpha) + 9$

Example 5.2.5

a. We factor out the common factor, $3w$.

$$6w^2 - 9w = 3w(2w-3).$$

b. We factor out the common factor, $3\sin(\theta)$.

$$6\sin^2(\theta) - 9\sin(\theta) = 3\sin(\theta)(2\sin(\theta) - 3).$$

Checkpoint 5.2.5

a. $a(2a-b)$

b. $\cos(\phi)(2\cos(\phi) - \sin(\phi))$

Example 5.2.6

a. Grouping terms, we get

$$\begin{aligned}
10x^2 + 2x - 15xy - 3y &= (10x^2 + 2x) + (-15xy - 3y) && \text{(Grouping)}\\
&= 2x(5x+1) - 3y(5x+1) && \text{(Factoring out the GCF from each group)}\\
&= 2x(\mathbf{5x+1}) - 3y(\mathbf{5x+1}) && \text{(Noticing the result has two common factors)}\\
&= (5x+1)(2x-3y) && \text{(Factoring out the common factor of } (5x+1)\text{)}
\end{aligned}$$

b. The structure of this expression is almost identical to that of the previous problem. Let $x = \sin(\theta)$ and $y = \cos(\theta)$ to get

$$\begin{aligned}
10\sin^2(\theta) + 2\sin(\theta) - 15\sin(\theta)\cos(\theta) - 3\cos(\theta) &= 10x^2 + 2x - 15xy - 3y && \text{(Substitution: } x = \sin(\theta) \text{ and } y = \cos(\theta)\text{)}\\
&= (5x+1)(2x-3y) && \text{(From part (a))}\\
&= (5\sin(\theta)+1)(2\sin(\theta) - 3\cos(\theta)) && \text{("Back-substitute:" } \sin(\theta) = x \text{ and } \cos(\theta) = y\text{)}
\end{aligned}$$

Checkpoint 5.2.6

a. $(3r-5)(4t+3)$
b. $(3\sec(\alpha)-5)(4\tan(\beta)+3)$

Example 5.2.7

a. We look for numbers p and q so that $(t+p)(t+q)=t^2-3t-10$. Their product is $pq=-10$, and their sum is $p+q=-3$. By checking the factors of -10 for the correct sum, we find $p=-5$ and $q=2$. Thus,

$$t^2-3t-10=(t-5)(t+2).$$

b. Now replace t by $\tan(\alpha)$ to find

$$\tan^2(\alpha)-3\tan(\alpha)-10=(\tan(\alpha)-5)(\tan(\alpha)+2).$$

Checkpoint 5.2.7

a. $(3z+1)(z-1)$
b. $(3\sin(\beta)+1)(\sin(\beta)-1)$

Theorem: Difference of Squares

$$\begin{aligned}(F-L)(F+L) &= F^2+FL-FL-L^2 && \text{(distributing)}\\ &= F^2-L^2 && \text{(simplifying)}\end{aligned}$$

Example 5.2.8

By this point, you (hopefully) understand what we will do.

$$\begin{aligned}16\cos^4(\theta)-1 &= 16x^4-1 && \text{(Substitute: } x=\cos(\theta)\text{)}\\ &= (4x^2)^2-1^2 && \text{(Laws of Exponents)}\\ &= F^2-L^2 && \text{(Substitute: } F=4x^2 \text{ and } L=1\text{)}\\ &= (F-L)(F+L) && \text{(Difference of Squares)}\\ &= (4x^2-1)(4x^2+1) && \text{(Resubstitute: } 4x^2=F \text{ and } 1=L\text{)}\\ &= ((2x)^2-1^2)(4x^2+1) && \text{(Laws of Exponents)}\\ &= \left(\overline{F}^2-\overline{L}^2\right)(4x^2+1) && \left(\text{Substitute: } \overline{F}=2x \text{ and } \overline{L}=1\right)\\ &= \left(\overline{F}-\overline{L}\right)\left(\overline{F}+\overline{L}\right)(4x^2+1) && \text{(Difference of Squares)}\\ &= (2x-1)(2x+1)(4x^2+1) && \left(\text{Resubstitute: } 2x=\overline{F} \text{ and } 1=\overline{L}\right)\\ &= (2\cos(\theta)-1)(2\cos(\theta)+1)(4\cos^2(\theta)+1) && \text{(Resubstitute: } \cos(\theta)=x\text{)}\end{aligned}$$

Checkpoint 5.2.8

$(\sin(\alpha)-3\cos(\beta))(\sin(\alpha)+3\cos(\beta))(\sin^2(\alpha)+9\cos^2(\beta))$

Theorem: Sums and Differences of Cubes

$$\begin{aligned}(F+L)(F^2-FL+L^2) &= F^3-F^2L+FL^2+F^2L-FL^2+L^3 && \text{(distributing)}\\ &= F^3+L^3 && \text{(simplifying)}\end{aligned}$$

and

$$\begin{aligned}(F-L)(F^2+FL+L^2) &= F^3+F^2L+FL^2-F^2L-FL^2-L^3 && \text{(distributing)}\\ &= F^3-L^3 && \text{(simplifying)}\end{aligned}$$

Example 5.2.9

This time, we will speed things up a bit.

$$\begin{aligned}27\sin^3(A)-8 &= 27x^3-8 && \text{(Substitute: } x=\sin(A)\text{)}\\ &= (3x)^3-2^3 && \text{(Laws of Exponents)}\\ &= (3x-2)((3x)^2+(3x)(2)+2^2) && \text{(Difference of Cubes)}\\ &= (3x-2)(9x^2+6x+4)\\ &= (3\sin(A)-2)(9\sin^2(A)+6\sin(A)+4) && \text{(Resubstitute: } \sin(A)=x\text{)}\end{aligned}$$

Checkpoint 5.2.9

$(5\csc(\theta)+4\cot(\phi))(25\csc^2(\theta)-20\csc(\theta)\cot(\phi)+16\cot^2(\phi))$

This page titled 5.2: Algebra with Trigonometric Functions is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

5.2.1: Resources and Key Concepts

Resources

Key Concepts

Definitions

- **Algebraic Expression:** A mathematical phrase that is a combination of numbers, variables, and mathematical operations, but does not contain an equals sign.
- **Evaluate (an algebraic expression):** The process of substituting a given value for the variable(s) in an algebraic expression and simplifying the result.

Theorems

- **Difference of Squares:** For any algebraic expressions F and L, the formula for factoring a difference of squares is $F^2 - L^2 = (F - L)(F + L)$.
- **Sums and Differences of Cubes:** For any algebraic expressions F and L, the formulas for factoring are:
  - Sum of Cubes: $F^3 + L^3 = (F + L)(F^2 - FL + L^2)$
  - Difference of Cubes: $F^3 - L^3 = (F - L)(F^2 + FL + L^2)$

Common Mistakes

- **Do NOT "Split" Arguments of Functions:** A function cannot be distributed across terms inside its argument. For example, $\sin(A + B)$ is not equal to $\sin(A) + \sin(B)$. The expression inside the parentheses must be evaluated first.
- **Incorrect Notation for Negative Exponents:** The shorthand notation $\cos^2(\theta)$ for $(\cos(\theta))^2$ does not apply to negative exponents. Specifically, $[\cos(\theta)]^{-1}$ is not the same as $\cos^{-1}(\theta)$, which represents the inverse cosine function.
- - **Factoring a Sum of Squares:** A sum of two squares, such as $F^2 + L^2$, is not factorable over the real numbers. Attempting to factor it as $(F + L)(F + L)$ or in any other way is incorrect.

This page titled 5.2.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

5.2.2: Homework

Reading Questions

1. What is the difference between an algebraic expression and an equation?
2. Explain the difference between evaluating $(\cos(30^\circ))^2$ and $\cos((30^\circ)^2)$.
3. What does the notation $\sin^2(\theta)$ mean?
4. Why is the notation $\cos^{-1}(\theta)$ not used to represent $\frac{1}{\cos(\theta)}$?
5. When multiplying expressions like $\cos(t)(3\cos(t)-2)$, what algebraic property is used?
6. What is the purpose of using a substitution, like letting $x=\sin(\theta)$, when factoring a complex trigonometric expression?
7. What is the formula for factoring a difference of squares, F^2-L^2?
8. Is an expression like $\sin^2(\theta)+9$ factorable over the real numbers? Why or why not?
9. What is the formula for factoring a sum of cubes, F^3+L^3?
10. What is the formula for factoring a difference of cubes, F^3-L^3?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, factor completely.

1. $2x^2+5x-12$
2. $3x^2-2x-8$
3. $12x^2+x-1$
4. $6x^2-13x-15$
5. $6x^2-x-12$
6. $24x^2+10x-21$
7. $9y^2-16$
8. $12m^2-75n^2$
9. $81x^4-1$
10. x^8-100y^4
11. x^3+27
12. x^6-64y^3
13. y^9-1
14. $16m^3-54y^3$

Answers available at the end of this section.

Homework

Concept Check

1. To evaluate $\cos^2(30^\circ)$, Wyatt used the keystrokes

$$\text{COS} \quad 30 \quad \wedge \quad 2 \quad \text{ENTER}$$

and got the answer -1. Were his keystrokes correct? Why or why not?

2. Make up an example to show that $\tan(\theta+\phi) \neq \tan(\theta)+\tan(\phi)$.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

3. Expressions containing trigonometric functions can be simplified or evaluated like other algebraic expressions.

4. We can *solve* expressions like $3\sin(x)+2\sin(x)$.

5. $\sin(X+Y)=\sin(X)+\sin(Y)$

6. $\cos^2(\theta)=(\cos(\theta))^2$

7. $\sin^{-1}(\theta)=\dfrac{1}{\sin(\theta)}$

8. $64x^2-9=(8x-3)(8x+3)$

9. $64x^2+9=(8x+3)(8x+3)=(8x+3)^2$

Basic Skills

For the following exercises, evaluate the expressions, using exact values for the trigonometric ratios.

10. $5\tan 135^\circ+6\cos 60^\circ$

11. $3\tan 240^\circ+8\sin 300^\circ$

12. $\sin(15^\circ+30^\circ)$

13. $\cos(2\cdot 75^\circ)$

14. $8\cos^2(30^\circ)$

15. $12\sin^2(315^\circ)$

16. $3\tan^2(150^\circ)-\sin^2(45^\circ)$

17. $1+\tan^2(120^\circ)$

For the following exercises, evaluate the expressions for $x=30^\circ, y=45^\circ$, and $z=60^\circ$. Give exact values for your answers.

18. $3\sin x+5\cos y$

19. $4\tan y+6\cos y$

20. $-2\tan 3y$

21. $\sin(3x-2x)$

22. $\cos^2 x + \sin^2 x$

23. $7\sin^2 y + 7\cos^2 y$

24. $\cos x \cos x - \sin x \sin z$

25. $\tan(180^\circ - x)\tan x$

For the following exercises, combine like terms.

26. a. $3x^2 - x - 5x^2$

    b. $3\cos^2\theta - \cos\theta - 5\cos^2\theta$

27. a. $4x + 5y - y$

    b. $4\cos\theta + 5\sin\theta - \sin\theta$

28. a. $-3SC + 7SC$

    b. $-3\sin\theta\cos\theta + 7\sin\theta\cos\theta$

29. a. $4SC + 11SC - 17SC$

    b. $4\sin\theta\cos\theta + 11\sin\theta\cos\theta - 17\sin\theta\cos\theta$

30. a. $-C^2S^3 + 6C^2S^3$

    b. $-\cos^2\theta\sin^3\theta + 6\cos^2\theta\sin^3\theta$

31. a. $7C^2S^2 - (2CS)^2$

    b. $7\cos^2\theta\sin^2\theta - (2\cos\theta\sin\theta)^2$

For the following exercises, simplify the expression, then evaluate.

32. $\cos t + 2\cos t\sin t - 3\cos t$, for $t = 30^\circ$

33. $11\tan w - 4\tan w + 6\tan w\cos w$, for $w = 60^\circ$

34. $7\tan\theta - 4\tan\phi + 3\tan\phi - 6\tan\theta$, for $\theta = 45^\circ, \phi = 120^\circ$

35. $5\sin A + \sin B - 6\sin B + 6\sin A$, for $A = 210^\circ, B = 135^\circ$

36. $-\sin x\cos x - \sin(2x) + 3\sin x\cos x$, for $x = 45^\circ$

37. $4\cos u\sin u + 3\sin 2u - 10\sin u\cos u$, for $u = 270^\circ$

For the following exercises, multiply or expand.

38. a. $x(2x - 1)$

    b. $\sin A(2\sin A - 1)$

39. a. $q(5r - q)$

    b. $\cos\theta(5\sin\theta - \cos\theta)$

40. a. $a(b - 3a)$

    b. $\tan A(\tan B - 3\tan A)$

41. a. $3w(2w - z)$

    b. $3\sin\alpha(2\sin\alpha - \sin\beta)$

42. a. $(C + 1)(2C - 1)$

    b. $(\cos\phi + 1)(2\cos\phi - 1)$

43. a. $(3S - 2)(S + 1)$

    b. $(3\sin B - 2)(\sin B + 1)$

44. a. $(a + b)(a - b)$

    b. $(\cos\theta + \cos\phi)(\cos\theta - \cos\phi)$

45. a. $(t - w)(t + 4w)$

    b. $(\tan\alpha - \tan\beta)(\tan\alpha + 4\tan\beta)$

46. a. $(1 - T)^2$

    b. $(1 - \tan\theta)^2$

47. a. $(2 + 3S)^2$

    b. $(2 + 3\sin\theta)^2$

48. a. $\left(T^2 + 2\right)\left(T^2 - 2\right)$

    b. $\left(\tan^2\theta + 2\right)\left(\tan^2\theta - 2\right)$

49. a. $\left(2c^2 - 3\right)\left(2c^2 + 3\right)$

    b. $\left(2\cos^2\phi - 3\right)\left(2\cos^2\phi + 3\right)$

For the following exercises, decide whether or not the expressions are equivalent. Justify your response by citing a theorem or providing a counterexample.

50. $\cos(x + y), \quad \cos x + \cos y$

51. $\sin(\theta - \phi), \quad \sin\theta - \sin\phi$

52. $\tan(2A), \quad 2\tan A$

53. $\cos\left(\frac{1}{2}\beta\right), \quad \frac{1}{2}\cos\beta$

54. $(\sin\alpha)^2, \quad \sin^2\alpha$

55. $(\tan B)^2, \quad \tan\left(B^2\right)$

56. $\sin 3t + \sin 5t, \quad \sin 8t$

57. $3\cos 2x + 5\cos 2x, \quad 8\cos 2x$

58. $\tan\left(\theta + 45^\circ\right) - \tan\theta, \quad \tan 45^\circ$

59. $\sin\left(30^\circ + z\right) + \sin\left(30^\circ - z\right), \quad \sin 60^\circ$

60. Factor each expression, if possible.

a. $x^2 - 4x$

b. $x^2 - 4$

c. $x^2 + 4$

d. $x^2 + 4x + 4$

e. $x^2 - 4x + 4$

f. $x^2 + 4x$

g. $x^2 - 4x - 4$

h. $-x^2 + 4$

For the following exercises, factor.

61. a. $9m + 15n$

b. $9\cos\alpha + 15\cos\beta$

62. a. $12p - 20q$

b. $12\sin\theta - 20\sin\phi$

63. a. $5r^2 - 10qr$

b. $5\tan^2 C - 10\tan B\tan C$

64. a. $2x^2 - 6xy$

b. $2\sin^2 A - 6\sin A\cos A$

65. a. $9C^2 - 1$

b. $9\cos^2\beta - 1$

66. a. $25S^2 - 16S$

b. $25\sin^2\beta - 16\sin\beta$

67. a. $6T^3 - 8T^2$

b. $6\tan^3 A - 8\tan^2 A$

68. a. $9T^2 - 15T^3$

b. $9\tan^2 B - 15\tan^3 B$

69. a. $t^2 - t - 20$

b. $\tan^2\theta - \tan\theta - 20$

70. a. $s^2 + 5s - 6$

b. $\sin^2 A + 5\sin A - 6$

71. a. $3c^2 + 2c - 1$

b. $3\cos^2 B + 2\cos B - 1$

72. a. $8S^2 - 6S + 1$

    b. $8\sin^2\phi - 6\sin\phi + 1$

73. a. $6S^2 - 5S - 1$

    b. $6\sin^2\alpha - 5\sin\alpha - 1$

74. a. $T^2 - 4T - 12$

    b. $\tan^2\alpha - 4\tan\alpha - 12$

75. a. $9C^2 - 1$

    b. $9\cos^2(\theta) - 1$

76. a. $16 - 9S^2$

    b. $16 - 9\sin^2(\alpha)$

77. a. $81T^2 - 16C^2$

    b. $81\tan^2(A) - 16\cos^2(B)$

78. a. $81T^4 - 16C^4$

    b. $81\tan^4(A) - 16\cos^4(B)$

79. a. $\frac{1}{9}S^2 - \frac{4}{25}C^2$

    b. $\frac{1}{9}\sin^2(\phi) - \frac{4}{25}\cos^2(\lambda)$

80. a. $27C^3 - 1$

    b. $27\cos^3(\theta) - 1$

81. a. $64 + 27S^3$

    b. $64 + 27\sin^3(\alpha)$

82. a. $729T^3 + 64C^3$

    b. $729\tan^3(A) + 64\cos^3(B)$

83. a. $729T^6 - 64C^6$

    b. $729\tan^6(A) - 64\cos^6(B)$

84. a. $\frac{1}{27}S^3 - \frac{8}{125}C^3$

    b. $\frac{1}{27}\sin^3(\phi) - \frac{8}{125}\cos^3(\lambda)$

Answers

Skills Refresher

1. $(2x - 3)(x + 4)$
2. $(3x + 4)(x - 2)$
3. $(4x - 1)(3x + 1)$

4. $(6x+5)(x-3)$
5. $(2x-3)(3x+4)$
6. $(6x+7)(4x-3)$
7. $(3y-4)\,(3y+4)$
8. $3\,(2m-5n)\,(2m+5n)$
9. $(3x-1)\,(3x+1)\,(9x^2+1)$
10. $(x^4-10y^2)\,(x^4+10y^2)$
11. $(x+3)\,(x^2-3x+9)$
12. $(x^2-4y)\,(x^4+4x^2y+16y^2)$
13. $(y-1)\,(y^2+y+1)\,(y^6+y^3+1)$
14. $2\,(2m-3y)\,(4m^2+6my+9y^2)$

This page titled 5.2.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

5.3: Introduction to Trigonometric Identities

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

What is an Identity?

Recall that an equation may be true or false, depending on the values of any variables involved. For example, the equation $x^2 + 3x = 10$ is true only if $x = 2$ or $x = -5$. You might have forgotten a bit of language from the past, so let's quickly redefine **conditional equations** and **identities**.

Definition: Conditional Equation

An equation that is true only for certain variable values (and, therefore, is false for others) is called a **conditional equation**.

When you solve a conditional equation, you find the variable values that make the equation true.

Definition: Identity

An equation that is true for all allowable variable values is called an **identity**.

Here are some examples of identities:

$$\begin{aligned} 3(x+y) &= 3x+3y \\ (x+1)^2 &= x^2+2x+1 \end{aligned}$$

In an identity, the statements on either side of the equal sign are called **equivalent expressions**.

Definition: Equivalent Expressions

Two algebraic expressions are called **equivalent** if they have the same value for all allowable variable values.

Example 5.3.1

Which of the following equations are identities?

a. $3s + 7s = 10s$
b. $5c(c - 2s) = 5c^2 - 10cs$

c. $2t - 1 = 3$

Solutions available at the end of this section.

Checkpoint 5.3.1

Which of the following equations are identities?

a. $(c - s)(c + s) = c^2 - s^2$
b. $3t^2 = 1$
c. $(2c + 1) + (s - 3) = 2c + s - 2$

Answers available at the end of this section.

Of course, you wouldn't be asked to *solve* an identity because all values of the variable are solutions. Instead, we use identities to replace one form of an expression with a more useful form. You do this when you solve a quadratic equation by factoring. For example,

$$\begin{aligned} 2x^2 - x - 1 &= 0 \\ \implies (2x + 1)(x - 1) &= 0 \qquad \text{(factoring)} \end{aligned}$$

Because $(2x + 1)(x - 1)$ is equivalent to $2x^2 - x - 1$, we have not changed the equation or its solutions; however, we now can apply the Zero Factor Principle and solve the equation.

Using identities is especially useful when we are working with trigonometric ratios.

Using Trigonometric Ratios in Identities

Because the identity $2x^2 - x - 1 = (2x + 1)(x - 1)$ is true for any value of x, it is true when x is replaced, for instance, by $\cos(\theta)$. This gives us a new identity:

$$2\cos^2(\theta) - \cos(\theta) - 1 = (2\cos(\theta) + 1)(\cos(\theta) - 1).$$

Expressions involving the trigonometric functions can be manipulated by the same rules that we use with simple variables (such as the Distributive Law or the Laws of Exponents).

Example 5.3.2

Which of the following equations are identities?

a. $3\sin(\theta) + 7\sin(\theta) = 10\sin(\theta)$
b. $5\cos(\theta)(\cos(\theta) - 2\sin(\theta)) = 5\cos^2(\theta) - 10\cos(\theta)\sin(\theta)$
c. $2\tan(\theta) - 1 = 3$

Solutions available at the end of this section.

Checkpoint 5.3.2

Which of the following equations are identities?

a. $(\cos(\theta) - \sin(\theta))(\cos(\theta) + \sin(\theta)) = \cos^2(\theta) - \sin^2(\theta)$
b. $3\tan^2(\theta) = 1$
c. $(2\cos(\theta) + 1) + (\sin(\theta) - 3) = 2\cos(\theta) + \sin(\theta) - 2$

Answers available at the end of this section.

Caution: Examples are Not Proofs

To show that an equation is *not* an identity, we need only find one value of the variable for which the equation is false (this would create what we call a **contradiction**). However, to show that a given equation *is* an identity, it is not enough to show that the equation is valid for one, or even a dozen, values of the variable. The equation must be true for *all* legitimate values of the variable. Because we cannot check all values of the variable numerically, we must use algebraic methods to *prove* that an equation is an identity.

Ratio Identities

All of the trigonometric functions are related. We begin by considering the relationship between the tangent and the sine and cosine functions (we will find a similar relationship for the cotangent). Complete the following table with exact values.

θ	$\cos(\theta)$	$\sin(\theta)$	$\frac{\sin(\theta)}{\cos(\theta)}$	$\tan(\theta)$
0°				
30°				
45°				
60°				

For the angles in the table, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. The question you should have is, "Is this true for all angles?" The short answer is no, but before we describe why, let's understand why this relationship exists between the tangent, sine, and cosine.

The sine function for an angle θ is defined as $\sin(\theta) = \frac{y}{r}$, where (x, y) is any point on the terminal side of θ (other than the origin) and $r = \sqrt{x^2 + y^2}$. Likewise, the cosine is defined to be $\cos(\theta) = \frac{x}{r}$. If we look at the ratio of the sine to the cosine, we get

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{y/r}{x/r} = \frac{(y/r)}{(x/r)} \cdot \frac{r}{r} = \frac{(y/\cancel{r})}{(x/\cancel{r})} \cdot \frac{\cancel{r}}{\cancel{r}} = \frac{y}{x} = \tan(\theta).$$

A similar argument shows that $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$.

One essential item to note, however, is that because $\cos(90^\circ) = 0$, the quotient $\frac{\sin(\theta)}{\cos(\theta)}$ is undefined for $\theta = 90^\circ$, and, as such, $\tan(90^\circ)$ is undefined. The same applies for $\theta = 270^\circ$. For all other angles between 0° and $360^\circ, \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. A similar argument shows that $\cot(0^\circ)$ and $\cot(180^\circ)$ are undefined.

We are now ready to formalize these statements as a theorem.

Theorem: Ratio Identities

For any angle not coterminal with 90° or 270°,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

For any angle not coterminal with 0° or 180°,

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}.$$

Advice: Memorize These Identities

The Ratio Identities are needed throughout the rest of Mathematics, Science, and Engineering. The need to commit them to memory and their importance cannot be overstated.

A cute little problem:

$$\frac{\sin(\text{gerine})}{\cos(\text{gerine})} = ???$$

Example 5.3.3

If $\sin(\theta) = \frac{12}{13}$ and $\cos(\theta) = -\frac{5}{13}$, find the values of $\tan(\theta)$ and $\cot(\theta)$.

Solution available at the end of this section.

Checkpoint 5.3.3

If $\sin(\theta) = -\frac{1}{5}$ and $\cos(\theta) = -\frac{2\sqrt{6}}{5}$, find the values of $\tan(\theta)$ and $\cot(\theta)$.

Answer available at the end of this section.

Reciprocal Identities

We now state one of the more obvious identities. We find the following relationships by comparing the definitions of cosecant, secant, and cotangent to the three fundamental trigonometric functions (sine, cosine, and tangent).

Theorem: Reciprocal Identities

$$\sin(\theta) = \frac{1}{\csc(\theta)} \quad (\text{provided } \csc(\theta) \neq 0) \qquad \csc(\theta) = \frac{1}{\sin(\theta)} \quad (\text{provided } \sin(\theta) \neq 0)$$

$$\cos(\theta) = \frac{1}{\sec(\theta)} \quad (\text{provided } \sec(\theta) \neq 0) \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \quad (\text{provided } \cos(\theta) \neq 0)$$

$$\tan(\theta) = \frac{1}{\cot(\theta)} \quad (\text{provided } \cot(\theta) \neq 0) \qquad \cot(\theta) = \frac{1}{\tan(\theta)} \quad (\text{provided } \tan(\theta) \neq 0)$$

You likely already figured out the Reciprocal Identities because, for example, $\sin(\theta) = \frac{y}{r}$ and $\csc(\theta) = \frac{r}{y}$ are obvious reciprocals.

Advice: Memorize These Identities

The Reciprocal Identities are needed throughout the rest of Mathematics, Science, and Engineering. The need to commit them to memory and their importance cannot be overstated.

Example 5.3.4

Compute $\cos(\theta)$ given $\sec(\theta) = -\frac{10}{3}$.

Solution available at the end of this section.

Checkpoint 5.3.4

If $\cot(\theta) = \frac{8}{5}$, find $\tan(\theta)$.

Answer available at the end of this section.

Caution: Cosecant, Secant, and Cotangent are NOT Inverses

Refrain from calling the cosecant, secant, and cotangent inverse trigonometric ratios. The word "inverse" is very special in Mathematics. While it is true that $\sin(\theta)$ and $\csc(\theta)$ are **multiplicative inverses**, they are not *functional* inverses.

Pythagorean Identities

All of the trigonometric functions are related. We begin by considering the relationship between $\sin(\theta)$ and $\cos(\theta)$. Complete the following table with exact values.

θ	$\cos(\theta)$	$\sin(\theta)$	$\cos^2(\theta)$	$\sin^2(\theta)$	$\cos^2(\theta)+\sin^2(\theta)$
0°					
30°					
45°					
60°					
90°					

For all the angles in the table, $\cos^2(\theta)+\sin^2(\theta)=1$. *This observation leads to the most important result in all of Trigonometry.*

Theorem: The Pythagorean Identity

For any angle θ,

$$\cos^2(\theta)+\sin^2(\theta)=1.$$

Proof available at the end of this section.

Interactive Element: The Pythagorean Identity on the Unit Circle

The following Interactive Element might help you visualize why $\cos^2(\theta)+\sin^2(\theta)=1$ holds for *every* angle, not just the ones in the table above.

Interact: Drag the orange point around the circle to change the angle θ.

Observation: The green leg is $\cos(\theta)$ and the red leg is $\sin(\theta)$; the hypotenuse is the radius, which is always 1. Since (leg)^2 +(leg)^2 =(hypotenuse)^2, we get $\cos^2(\theta)+\sin^2(\theta)=1$ at every position — even when $\cos(\theta)$ or $\sin(\theta)$ is negative.

From its name, the Pythagorean Identity is intimately related to the Pythagorean Theorem.

Advice: Memorize This Identity

The Pythagorean Identity (also known as the Fundamental Pythagorean Identity) is needed throughout the rest of Mathematics. The need to commit this to memory and its importance cannot be overstated.

Example 5.3.5

Evaluate $\cos^2(27^\circ) + \sin^2(27^\circ)$ without using a calculator.

Solution available at the end of this section.

Checkpoint 5.3.5

Evaluate $5\left[\cos^2(12^\circ) + \sin^2(12^\circ)\right]$ without using a calculator.

Answer available at the end of this section.

When we solve more complicated trigonometric equations in later chapters, we will need to simplify trigonometric expressions so that they involve only one of the trigonometric functions. The Pythagorean Identity is useful when we wish to write an equivalent expression for either $\cos^2(\theta)$ or for $\sin^2(\theta)$. The following equivalent forms to the Pythagorean Identity will be helpful throughout Trigonometry.

Corollary: Alternate Forms of the Pythagorean Identity

$$\begin{aligned} \sin^2(\theta) &= 1 - \cos^2(\theta) \implies \sin(\theta) = \pm\sqrt{1-\cos^2(\theta)} \\ \cos^2(\theta) &= 1 - \sin^2(\theta) \implies \cos(\theta) = \pm\sqrt{1-\sin^2(\theta)} \end{aligned}$$

The Pythagorean Identity, along with the Ratio and Reciprocal Identities, show us that all trigonometric functions of an angle are related. If we know the value of one trigonometric function at θ and where θ terminates, we can calculate the remaining trigonometric functions. We do *not* need to find the angle itself to do this. We only need to know which quadrant the angle lies in to determine the correct sign for the trigonometric ratios.

Advice: Memorize These Identities

The alternate forms of the Pythagorean Identity are needed throughout the rest of Mathematics, Science, and Engineering. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Example 5.3.6 shows that we do not need to memorize the alternate forms of the Pythagorean Identity as long as we are confident manipulating mathematical expressions.

Example 5.3.6

If $\sin(\theta) = \frac{7}{25}$ and $90^\circ < \theta < 180^\circ$, find $\cos(\theta)$ and $\tan(\theta)$.

Solution available at the end of this section.

We could have solved Example 5.3.6 by creating a quick sketch of θ terminating in QII. Because $\sin(\theta) = \frac{7}{25}$, we can choose a point on the terminal side to have y-coordinate 7 and $r = 25$ (see Figure 5.3.1 below).

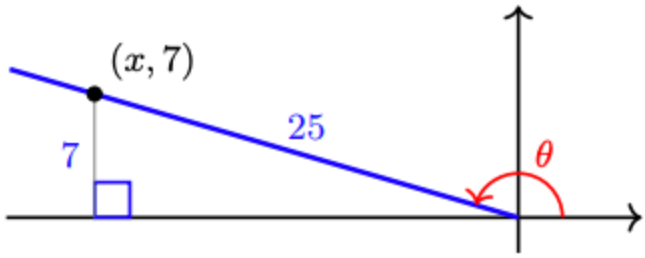


Figure 5.3.1

In order to calculate $\cos(\theta)$ and $\tan(\theta)$, we must find the x-coordinate of the point. By the Pythagorean Theorem,

$$\begin{aligned} x^2 + 7^2 &= 25^2 \\ \Longrightarrow \quad x^2 &= 576 && \text{(isolating } x^2\text{)} \\ \Longrightarrow \quad x &= -24 && \text{(taking the square root of both sides)} \end{aligned}$$

Note that x is negative because the point $(-24, 7)$ is in the second quadrant. Then

$$\cos(\theta) = \frac{x}{r} = -\frac{24}{25} \quad \text{and} \quad \tan(\theta) = \frac{y}{x} = \frac{7}{-24} = -\frac{7}{24}.$$

Checkpoint 5.3.6

a. If $\cos(\theta) = \frac{5}{13}$ and $-90^\circ < \theta < 0^\circ$, find $\sin(\theta)$ and $\tan(\theta)$.
b. Find $\sin(\theta)$ and $\tan(\theta)$ by using an appropriate sketch.

Answers available at the end of this section.

You might have noticed that this subsection is "Pythagorean Identities" (plural). Yet, we only named a single Pythagorean Identity (purposefully titled **The** Pythagorean Identity). We now introduce two additional Pythagorean Identities that will be used frequently throughout Trigonometry.

$$\begin{aligned} \cos^2(\theta) + \sin^2(\theta) &= 1 && \text{(the Pythagorean Identity)} \\ \Longrightarrow \quad \frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} &= \frac{1}{\cos^2(\theta)} && \text{(dividing both sides by } \cos^2(\theta)\text{)} \\ \Longrightarrow \quad 1 + \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2 &= \left(\frac{1}{\cos(\theta)}\right)^2 && \text{(cancelling like factors and Laws of Exponents)} \\ \Longrightarrow \quad 1 + (\tan(\theta))^2 &= (\sec(\theta))^2 && \text{(Ratio and Reciprocal Identities)} \\ \Longrightarrow \quad 1 + \tan^2(\theta) &= \sec^2(\theta) && \text{(Laws of Exponents)} \end{aligned}$$

A similar process shows

$$1 + \cot^2(\theta) = \csc^2(\theta).$$

We formalize these statements in a summary theorem.

Theorem: Pythagorean Identities

For any angle θ where the functions are defined,

$$\begin{aligned} \cos^2(\theta) + \sin^2(\theta) &= 1 \\ 1 + \tan^2(\theta) &= \sec^2(\theta) \\ \cot^2(\theta) + 1 &= \csc^2(\theta) \end{aligned}$$

Advice: Memorize These Identities

Each of the Pythagorean Identities are needed throughout the rest of Mathematics. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Success in College: Study Time

Believe it or not, there is a direct link between the number of units a class is worth and how many hours you should be studying (outside of class). For each hour of class time, the California state education code suggests at least two hours of studying outside of class. If your math background is weak, you may need to spend more time studying.

At Cosumnes River College, each unit of a math course corresponds to one hour of in-class time per week. Thus, a five-unit math course requires five hours of in-class interaction per week and *ten* hours of study time outside of class per week.

One way to study is to do what you're doing now - read! Class time is an excellent opportunity to introduce new concepts and see how they fit with those previously learned; however, there is usually not enough time to address details with the depth that a textbook can. In this way, your professor and the textbook should be considered as a combined primary source of information in your course.

Here are some questions to consider regarding how you study.

1. Do you study at least two hours for every one hour of class time? If not, why?
2. How many hours do you dedicate to TV, YouTube, and social media every week (on average)?
3. How many hours do you dedicate to leisure activities per week (e.g., hanging out with friends, going to the movies, gaming)?
4. Would it be too difficult to trade a few hours for study time?

Success in College: I Understood It... Until I Left Class

You understand a concept when the professor does it in class, but you're lost when trying it on your own - this is very common. There are two general remedies to this problem.

The first revolves around how the information is delivered and how you engage with it. We tend to retain the least when only *listening* (maybe this is why those who passively sit in the back of a classroom and zone out end up having the highest failure rate). We gain a little more retention when we *see* someone do something (e.g., watching your professor, but not taking notes). A substantial leap in retention and understanding occurs when we *write* concepts down (e.g., taking notes during lectures). Finally, the biggest gains in understanding and retention occur when we *do* work on our own (e.g., practicing, doing homework, independently investigating a concept).

This kind of makes sense, right? This is why you see the most successful students in class attempting the examples before the professor starts them. It forces the information into their memory.

It gets even better if we involve "real-world" examples with manipulatives. Imagine learning about mathematical models that deal with finance (exponential and simple interest formulas) by *actually* investing. How cool would it be to build a fence that borders a river by using minimization techniques learned in Algebra? The problem with math and how it is taught is that we hear about the applications but nobody wants to go out and do them. If you had more physical or engaging examples, your chance of success would skyrocket. Unfortunately, there is only a limited amount of time in class, and this kind of teaching would require students to allot several more hours a day to their math class (however, the benefit would be enormous).

The second remedy is squarely on your shoulders. You will never master a skill by putting only half of your effort into it. To excel and enjoy mathematics, you must immerse yourself in it. Imagine yourself as a mathematician and look at the world differently. As you drive your car to work, make a mental graph of the distance you have traveled versus the time you have been driving (if you're in Calculus, then interpret the slope of the tangent line to this graph at any given point as the derivative). When you purchase items, do the math in your head... yes, even the tax! When you buy a car, make a function that describes the cost of owning the car! You'd be surprised at how helpful mathematics is.

In college, class time is limited. Often, class is over before students feel confident about new concepts. They leave class worried they won't be able to do the homework. These are just some suggestions, but others in your class might have some wonderful additions to this list. Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. How often does this happen to you (understanding in class but completely lost out of class)? When it does, what do you do about it?
2. How often do you think this happens to the other class members?

Answers

Section Preview

The purpose of this section is to *introduce* the student to identities and contradictions. We officially introduce the Reciprocal, Ratio, and Pythagorean Identities to allow the student a different way to *compute* the value of a trigonometric function given the value of a different trigonometric function at the same angle—***proofs are not done by the student here***!

Section Preview

- **Simplifying Expressions**
  - Multiplying and Dividing Rational Expressions: Used when simplifying the compound fraction in the Ratio Identity derivation, as well as in the homework.
  - Adding and Subtracting Rational Expressions: This skill is needed in the homework.
  - Simplifying Compound Rational Expressions: The derivation of the Ratio Identities involves simplifying the compound fraction $\frac{y/r}{x/r}$; a student must understand compound fractions to follow this.
- **Solving Equations**
  - Solving Literal Equations: The concept of an identity versus a conditional equation requires understanding what it means for an equation to be "true for all values"—while not an obvious placement, this appears in the Solving Literal Equations material.

Section Preview

- Determine if an algebraic equation represents an identity.
- Find the value of a trigonometric function using the Reciprocal, Ratio, and Pythagorean Identities.

Example $5.3.1$

Many of the algebraic operations you have already learned, such as combining like terms or applying the Distributive Law, produce equivalent expressions.

a. Equation (a) is an identity obtained by combining like terms on the left side.
b. Equation (b) is an identity obtained by applying the Distributive Law on the left side.
c. Equation (c) is not an identity because the equation is valid only for $t = 2$.

Checkpoint $5.3.1$

(a) and (c)

Example $5.3.2$

These equations are obtained from the equations in Example $5.3.1$ by replacing the variable s with $\sin(\theta)$, the variable c with $\cos(\theta)$, and the variable t with $\tan(\theta)$. Equations (a) and (b) are identities, and equation (c) is not, for the same reasons as before.

Checkpoint $5.3.2$

(a) and (c)

Example $5.3.3$

We apply the Ratio Identities to obtain

$$\begin{aligned} \tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \frac{12/13}{-5/13} \\ &= \frac{12/13}{-5/13} \cdot \frac{13}{13} \\ &= \frac{12}{-5} \\ &= -\frac{12}{5} \end{aligned}$$

and

$$\begin{aligned}\cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)}\\ &= \frac{-5/13}{12/13}\\ &= \frac{-5/13}{12/13}\cdot\frac{13}{13}\\ &= \frac{-5}{12}\\ &= -\frac{5}{12}\end{aligned}$$

Checkpoint 5.3.3

$\tan(\theta) = \frac{1}{2\sqrt{6}}$ and $\cot(\theta) = 2\sqrt{6}$

Example 5.3.4

$$\cos(\theta) = \frac{1}{\sec(\theta)} = \frac{1}{-10/3} = -\frac{3}{10}$$

Checkpoint 5.3.4

$\tan(\theta) = \frac{5}{8}$

Theorem: The Pythagorean Identity

From the coordinate definition of the trigonometric functions, we know that $r = \sqrt{x^2+y^2}$. Thus,

$$\begin{aligned} & r = \sqrt{x^2+y^2} & \\ \Longrightarrow\quad & r^2 = x^2+y^2 & \text{(squaring both sides)}\\ \Longrightarrow\quad & 1 = \frac{x^2}{r^2}+\frac{y^2}{r^2} & \text{(dividing both sides by } r^2)\\ \Longrightarrow\quad & 1 = \left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2 & \text{(Laws of Exponents)}\\ \Longrightarrow\quad & 1 = (\cos(\theta))^2+(\sin(\theta))^2 & \text{(definitions of the cosine and sine functions)}\\ \Longrightarrow\quad & 1 = \cos^2(\theta)+\sin^2(\theta) & \text{(using a better notation)}\end{aligned}$$

Example 5.3.5

Because the Pythagorean Identity, $\cos^2(\theta)+\sin^2(\theta) = 1$, is true for any value of θ, it is true in particular for $\theta = 27^\circ$. So $\cos^2(27^\circ)+\sin^2(27^\circ) = 1$.

Checkpoint 5.3.5

5

Example 5.3.6

We substitute $\sin(\theta) = \frac{7}{25}$ into the Pythagorean Identity $\cos^2(\theta)+\sin^2(\theta) = 1$ to get

$$\begin{aligned} \cos^2(\theta)+\left(\frac{7}{25}\right)^2 &= 1 && \text{(substituting)}\\ \Longrightarrow\quad \cos^2(\theta) &= 1-\left(\frac{7}{25}\right)^2 && \text{(isolating } \cos^2(\theta))\\ \Longrightarrow\quad \cos^2(\theta) &= 1-\frac{49}{625} &&\\ \Longrightarrow\quad \cos^2(\theta) &= \frac{576}{625} &&\\ \Longrightarrow\quad \cos(\theta) &= \pm\sqrt{\frac{576}{625}} && \text{(taking the square root of both sides)}\\ \Longrightarrow\quad \cos(\theta) &= \pm\frac{24}{25} &&\end{aligned}$$

Because θ is between 90° and 180°, $\cos(\theta)$ must be negative. Therefore, $\cos(\theta) = -\frac{24}{25}$. To find $\tan(\theta)$, we use the Ratio Identities to get

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{7/25}{-24/25} = -\frac{7}{24}.$$

Checkpoint 5.3.6

$\sin(\theta) = -\frac{12}{13}$, $\tan(\theta) = -\frac{12}{5}$

This page titled 5.3: Introduction to Trigonometric Identities is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

5.3.1: Resources and Key Concepts

Resources

Videos

- Pythagorean Identities
  - Fundamental Identities - The Pythagorean Identities: Theorem, Proof, and Examples

Key Concepts

Definitions

- **Conditional Equation:** An equation that is true for only a specific set of variable values and is false for others.
- **Identity:** An equation that holds true for all permissible values of the variables involved.
- **Equivalent Expressions:** Two algebraic expressions that produce the same value for all allowable inputs. They form the two sides of an identity.

Theorems

- **Ratio Identities:**
  - For any angle θ not coterminal with 90° or 270°, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.
  - For any angle θ not coterminal with 0° or 180°, $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$.
- **Reciprocal Identities:**
  - $\sin(\theta) = \frac{1}{\csc(\theta)}$ and $\csc(\theta) = \frac{1}{\sin(\theta)}$
  - $\cos(\theta) = \frac{1}{\sec(\theta)}$ and $\sec(\theta) = \frac{1}{\cos(\theta)}$
  - $\tan(\theta) = \frac{1}{\cot(\theta)}$ and $\cot(\theta) = \frac{1}{\tan(\theta)}$
- **The Pythagorean Identities:** For any angle θ where the functions are defined:
  - $\cos^2(\theta) + \sin^2(\theta) = 1$
  - $1 + \tan^2(\theta) = \sec^2(\theta)$
  - $\cot^2(\theta) + 1 = \csc^2(\theta)$
- **Alternate Forms of the Pythagorean Identity:**
  - $\sin^2(\theta) = 1 - \cos^2(\theta) \implies \sin(\theta) = \pm\sqrt{1 - \cos^2(\theta)}$
  - $\cos^2(\theta) = 1 - \sin^2(\theta) \implies \cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$

Common Mistakes

- **Using Examples as Proofs:** To prove an equation is an identity, you must use algebraic methods. Showing that an equation works for a few numerical examples is not sufficient, as it does not prove it is true for all possible values.
- **Incorrectly Taking Square Roots:** Taking the square root of both sides of an equation like $\cos^2(\theta) + \sin^2(\theta) = 1$ does not result in $\cos(\theta) + \sin(\theta) = 1$. The square root operation does not distribute over addition.
- **Confusing Reciprocals with Inverses:** The cosecant, secant, and cotangent functions are the multiplicative reciprocals of the sine, cosine, and tangent functions, respectively. They are not functional inverses. The term "inverse" has a specific meaning for functions that will be discussed later.

This page titled 5.3.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

5.3.2: Homework

Reading Questions

1. What is the difference between a conditional equation and an identity?
2. To prove that an equation is an identity, is it enough to show that it is true for several different values of the variable? Why or why not?
3. State the Ratio Identity for $\tan(\theta)$.
4. For which angles is the identity for $\tan(\theta)$ not defined?
5. State the three Reciprocal Identities that relate the six trigonometric functions.
6. What is the Fundamental Pythagorean Identity?
7. If you know that $\sin(\theta) = \frac{7}{25}$, why is it also necessary to know the quadrant of θ to find the exact value of $\cos(\theta)$?
8. What are the two Pythagorean Identities that involve the tangent, cotangent, secant, and cosecant functions?
9. Are $\csc(\theta)$ and $\sin(\theta)$ multiplicative inverses or functional inverses?
10. How can you use the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ to derive the identity involving $\tan^2(\theta)$ and $\sec^2(\theta)$?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, write the expression as a single fraction in simplest form.

1. $\frac{1}{x} + \frac{1}{y}$
2. $1 - \frac{1}{x}$
3. $\frac{1}{x-1} + \frac{1}{x+1}$
4. $x - \frac{x}{x+1}$
5. $\frac{\frac{y}{x} - \frac{x}{y}}{\frac{y}{x} + 1}$
6. $\frac{1 - \frac{x^2}{y^2}}{1 + \frac{x^2}{y^2}}$
7. $\frac{\frac{2a}{x}}{1 - \frac{a^2}{x^2}}$
8. $\frac{\frac{a}{x} + \frac{b}{y}}{1 - \frac{ab}{xy}}$

Answers available at the end of this section.

Homework

Vocabulary Check

1. A ___ equation is true only for certain variable values and false for others.
2. An equation that is true for all legitimate values of the variables is called a(n) ___.
3. The expressions on either side of the equal sign in an identity are called ___ because they have the same value for all values of the variable.

Concept Check

4. What is the difference between a conditional equation and an identity? Give an example of each.
5. What happens when you try to "solve" an identity?
6. Nhat claims that since $\cos^2(\theta) + \sin^2(\theta) = 1$, we can take the square root of both sides to get the simpler form $\cos(\theta) + \sin(\theta) = 1$. Is she correct? Why or why not?
7. If $\frac{a}{b} = \frac{2}{5}$, is it necessarily true that $a = 2$ and $b = 5$? Explain.
8. If $\tan(\theta) = \frac{3}{11}$, is it true that $\sin(\theta) = 3$ and $\cos(\theta) = 11$? Explain.
9. Explain how you would arrive at the identity
$$1 + \cot^2(\theta) = \csc^2(\theta)$$
given that you know $\cos^2(\theta) + \sin^2(\theta) = 1$.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

10. The reciprocal of the secant is the cosecant.
11. For any angle θ,
$$\cos^2(\theta) + \sin^2(\theta) = 1.$$
12. For any angle θ,
$$\cos(\theta) + \sin(\theta) = 1.$$
13. For any angle θ,
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$
14. To show that an equation is an identity, we just need to find one value of the unknown that makes the statement true.

Basic Skills

For the following exercises, decide which of the following equations are identities. Explain your reasoning.

15. $(\sqrt{a} + \sqrt{b})^2 = a + b$
16. $\sqrt{a^2 - b^2} = a - b$
17. $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$

18. $\dfrac{a+b}{a} = b$

19. $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

20. $\dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

21. $(1+\tan\theta)^2 = 1+\tan^2\theta$

22. $\sqrt{1-\sin^2\phi} = 1-\sin\phi$

For the following exercises, evaluate (without using a calculator).

23. $3\cos^2 1.7^\circ + 3\sin^2 1.7^\circ$

24. $4-\cos^2 338^\circ - \sin^2 338^\circ$

25. $\left(\cos^2 20^\circ + \sin^2 20^\circ\right)^4$

26. $\dfrac{18}{\cos^2 17^\circ + \sin^2 17^\circ}$

27. $\dfrac{6}{\cos^2 53^\circ} - 6\tan^2 53^\circ$

28. $\dfrac{1}{\sin^2 102^\circ} - \dfrac{\cos^2 102^\circ}{\sin^2 102^\circ}$

For the following exercises, find exact values for the other trigonometric functions of the given angle.

29. $\cos A = \dfrac{12}{13}$ and \(270^{\circ}

30. $\sin B = \dfrac{-3}{5}$ and \(180^{\circ}

31. $\sin\phi = \dfrac{1}{7}$ and $90^\circ < \phi < 180^\circ$

32. $\cos t = \dfrac{-2}{3}$ and \(180^{\circ}

33. $\tan\theta = -\dfrac{1}{2}$ and $270^\circ < \theta < 360^\circ$

34. $\tan\theta = 2$ and $180^\circ < \theta < 270^\circ$

35. $\tan\theta = \dfrac{3}{4}$ and $180^\circ < \theta < 270^\circ$

36. $\tan\theta = -3$ and $90^\circ < \theta < 180^\circ$

37. $2\cos A + 9 = 8$ and \(90^{\circ}

38. $25\sin B + 8 = -12$ and \(180^{\circ}

39. $8\tan\beta + 5 = -11$ and $90^\circ < \beta < 180^\circ$

40. $6(\tan\beta - 4) = -24$ and $90^\circ < \beta < 270^\circ$

41. $\tan^2 C - 3 = 0$ and \(90^{\circ}

42. $4\cos^2 A - \cos A = 0$ and \(0^{\circ}

Synthesis Questions

13. **Proving the Pythagorean Identity.** Prove the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$ by carrying out the following steps. Sketch an angle θ in standard position, and label a point (x, y) on the terminal side, at a distance r from the vertex.

    a. Begin with the equation $\sqrt{x^2 + y^2} = r$, and square both sides.

    b. Divide both sides of your equation from part (a) by r^2.

    c. Write the left side of the equation as the sum of the squares of two fractions.

    d. Substitute the appropriate trigonometric ratio for each fraction.

14. **Proving the Ratio Identity.** Prove the Ratio Identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ by carrying out the following steps. Sketch an angle θ in standard position, and label a point (x, y) on the terminal side, at a distance r from the vertex.

    a. Write $\sin\theta$ in terms of y and r, and solve for y.

    b. Write $\cos\theta$ in terms of x and r, and solve for x.

    c. Write $\tan\theta$ in terms of x and y, then substitute your results from parts (a) and (b).

    d. Simplify your fraction in part (c).

Answers

Skills Refresher

1. $\frac{x+y}{xy}$
2. $\frac{x-1}{x}$
3. $\frac{2x}{x^2-1}$
4. $\frac{x^2}{x+1}$
5. $\frac{y-x}{y}$
6. $\frac{y^2-x^2}{y^2+x^2}$
7. $\frac{2ax}{x^2-a^2}$
8. $\frac{ay+bx}{xy-ab}$

This page titled 5.3.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

5.4: Introduction to Proofs in Trigonometry

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We have already been introduced to several fundamental identities in Trigonometry. Namely, the Ratio, Reciprocal, and Pythagorean Identities. We then spent most of that section using those identities to help us evaluate trigonometric functions. In this section, we learn how to use those identities to simplify expressions involving trigonometric functions. We then start dipping our toes into proving identities.

Simplifying Expressions Involving Trigonometric Functions

Recall that an algebraic expression (or mathematical expression) is a combination of symbols that are mathematically "well-formed." The mathematical symbols can include numbers (constants), variables, operations (e.g., addition, subtraction, multiplication, etc.), functions, brackets, and other grouping symbols to help determine the order of operations. The one symbol that is always missing from an expression is the equals sign ($=$).

Reminder: Equations versus Expressions

Equations have equals signs—expressions do not.

You **solve** equations, but you **simplify** expressions.

When we simplify an algebraic expression, we obtain a new expression that has the same values as the old one, but is easier to work with. For example, we can apply the Distributive Law and combine like terms to simplify

$$\begin{aligned} 2x(x-6)+3(x+2) &= 2x^2-12x+3x+6 \\ &= 2x^2-9x+6 \end{aligned}$$

The new expression is equivalent to the old one, that is, the expressions have the same value when we evaluate them at any value of x. For instance, you can check that, at $x=3$, the expressions $2x(x-6)+3(x+2)$ and $2x^2-9x+6$ become

$$\begin{aligned} 2(3)(3-6)+3(3+2) &= 6(-3)+3(5) &= -3 \\ 2(3)^2-9(3)+6 &= 18-27+6 &= -3 \end{aligned}$$

To simplify an expression containing trigonometric functions, we treat each function as a single variable. Compare the two calculations below:

$$\begin{array}{ccccc} 8xy & - & 6xy & = & 2xy \\ 8\cos(\theta)\sin(\theta) & - & 6\cos(\theta)\sin(\theta) & = & 2\cos(\theta)\sin(\theta) \end{array}$$

Both calculations are examples of combining like terms. In the second calculation, we treat $\cos(\theta)$ and $\sin(\theta)$ as variables, just as we treat x and y in the first calculation.

In Trigonometry, we are often tasked with simplifying expressions involving trigonometric functions. In doing so, we will use many of our skills from Algebra (e.g., simplifying compound rational expressions, factoring, distributing, etc.) in combination with the identities we have recently discovered (along with those we will soon discover). One strategy for simplifying a trigonometric expression is to reduce the number of different trigonometric functions involved. The following example showcases this process using the Ratio Identities.

Example 5.4.1

a. Simplify: $\cos(\theta)\tan(\theta)+\sin(\theta)$
b. Multiply: $(\cos(\theta)+\sin(\theta))^2$
c. Add: $\frac{\cos(\theta)}{\sin(\theta)}+\frac{\sin(\theta)}{\cos(\theta)}$
d. Simplify: $3\tan(A)+4\tan(A)-2\cos(A)$
e. Simplify: $2-\sin(B)+2\sin(B)$

Solutions available at the end of this section.

It is often easier to know how a **trigonometric expression** (an expression involving trigonometric functions) will simplify once you try simplification techniques. In Example $5.4.1a$, most students new to Trigonometry would likely never have looked at $\cos(\theta)\tan(\theta)+\sin(\theta)$ and thought, "Hey, I bet that simplifies down to something nice... like $2\sin(\theta)$." Luckily, as you move forward in Trigonometry (and mathematics), you develop an intuition for when an expression can be simplified; however, cultivating this intuition takes time and experimentation.

Checkpoint 5.4.1

Simplify each expression.

a. $(1-\cos(\theta))(1+\cos(\theta))$
b. $2\cos(t)-4\cos(w)\sin(w)+3\cos(t)-2\cos(w)$

Answers available at the end of this section.

Caution

In Checkpoint $5.4.1b$, note that $\cos(t)$ and $\cos(w)$ are not like terms. (We can choose values for t and w so that $\cos(t)$ and $\cos(w)$ have different values.)

Rewriting Trigonometric Expressions

It is often necessary, especially in Calculus, to rewrite a trigonometric expression in terms of a single trigonometric function. To do so, we must use identities.

Example 5.4.2

a. Rewrite $\sin(\theta)\cos^2(\theta)$ as an expression involving only sums or differences of powers of $\sin(\theta)$.
b. Rewrite $\cot(\theta)$ in terms of only $\cos(\theta)$.

Solutions available at the end of this section.

If we graph the original expression from Example $5.4.2a$ as $y_1 = \sin(x)\cos^2(x)$ and our resulting equivalent expression as $y_2 = \sin(x) - \sin^3(x)$, we see that they have the same graph, as shown in Figure $5.4.1$ below.

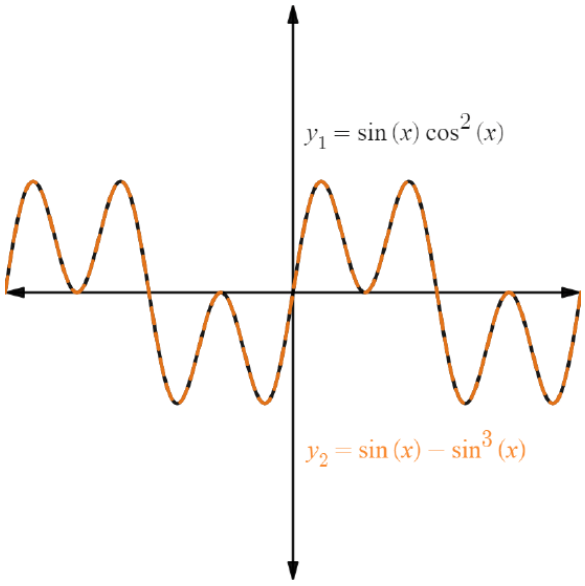


Figure $5.4.1$

This should convince us that $\sin(\theta)\cos^2(\theta)$ truly is equivalent to $\sin(x) - \sin^3(x)$.

Caution

If you tried to graph the functions above using your graphing calculator or another graphing technology and didn't get graphs similar to those in Figure $5.4.1$, it is likely because your graphing device is in degree mode. The graphs we create in Trigonometry require a mode called radian mode. The meaning of these modes will be explained later.

The results of Example $5.4.2$ can be thought of as two new identities:

$$\sin(\theta)\cos^2(\theta) = \sin(\theta) - \sin^3(\theta) \quad \text{and} \quad \cot(\theta) = \pm\frac{\cos(\theta)}{\sqrt{1-\cos^2(\theta)}},$$

however, before you get too concerned with having to memorize these as two more identities, let's be clear:

> Unless formally stated as a theorem, there is no need to memorize the hundreds of identities we will create, prove, or encounter in Trigonometry.

This means that the only identities you are responsible for memorizing (so far) are the Reciprocal, Ratio, and Pythagorean Identities.

Checkpoint $5.4.2$

a. Rewrite $\sin^2(\alpha)\cos^2(\alpha)$ as an expression involving only sums or differences of powers of $\cos(\alpha)$.
b. Verify your identity by graphing.

Answers available at the end of this section.

The following example is extremely useful in Calculus II (Integral Calculus).

Example 5.4.3

Simplify the expression

$$\frac{3}{\sqrt{x^2-4}}$$

as much as possible by substituting $2\sec(\theta)$ for x.

Solution available at the end of this section.

Our solution to Example 5.4.3 needs some clarification to ensure you understand what happened. Most of the work should be understandable; however, two steps might throw you off.

First, from the Pythagorean Identities, we used the fact that

$$1+\tan^2(\theta)=\sec^2(\theta);$$

however, we modified this identity slightly by subtracting 1 from both sides to get

$$\tan^2(\theta)=\sec^2(\theta)-1.$$

The implication of that subtle modification cannot be overstated.

> Being comfortable with the available identities and willing to manipulate them as needed will play a critical role in your success in Trigonometry.

The second item that needs our attention is the mathematical equivalence

$$\sqrt{\tan^2(\theta)}=|\tan(\theta)|.$$

A lot of students forget about the absolute values. Let's focus on what is happening here.

Suppose a friend of yours is claiming that the equation $\sqrt{x^2}=x$ is an identity (this is the same as someone saying $\sqrt{\tan^2(\theta)}=\tan(\theta)$). As an astute mathematics student, you know that even though the equation is true for all *positive* values of x, it is false for negative values of x. For example, if $x=-3$, then

$$\sqrt{x^2}=\sqrt{(-3)^2}=\sqrt{9}=3$$

so $\sqrt{x^2}\neq x$. The radical symbol $\sqrt{\ }$ stands for the **principal square root**—that is, the nonnegative square root. Therefore, the left side of the equation, $\sqrt{x^2}$, is never negative. Thus, $\sqrt{x^2}$ cannot equal x when x is a negative number. The equation is false for $x<0$.

One way to see that $\sqrt{x^2}$ and x are not equivalent is to compare the graphs of $Y_1=\sqrt{x^2}$ and $Y_2=x$, shown in Figure 5.4.2 below. You can see that $\sqrt{x^2}$ and x do not have the same value for $x<0$.

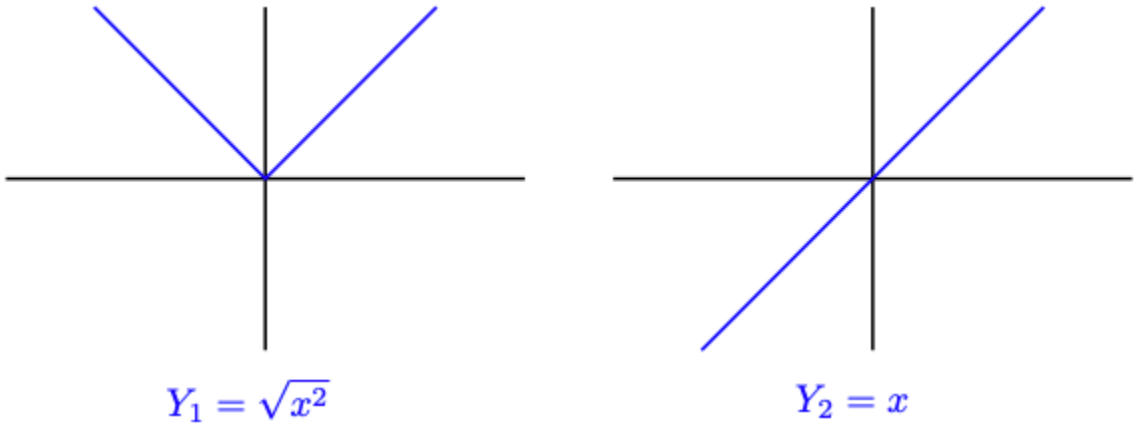


Figure 5.4.2

Interactive Element: $\sqrt{x^2}$ Is $|x|$, Not x

The following Interactive Element might help you visualize why $\sqrt{x^2} = |x|$ rather than x.

Interact: Drag the red point along the x-axis. Two markers ride the curves $y = \sqrt{x^2}$ and $y = x$; watch the segment between them.

Observation: For $x \geq 0$ the two markers coincide, but for $x < 0$ they split apart—$\sqrt{x^2}$ stays positive while x is negative. So $\sqrt{x^2} = |x|$, and it equals x only when $x \geq 0$.

Coming back to the last step in the solution of Example $5.4.3$, we should now feel comfortable saying that $\sqrt{\tan^2(\theta)} \neq \tan(\theta)$ and we should be okay with saying

$$\sqrt{\tan^2(\theta)} = |\tan(\theta)|.$$

Checking the Plausibility of Identities Graphically

Before jumping into how to rigorously prove a trigonometric identity, let's focus on ways to show that a *claimed* identity is *not* an identity.

From the discussion after Example $5.4.3$, we can see that, to check whether an equation *might be* an identity, we can compare graphs of $Y_1 =$(left side of the equation) and $Y_2 =$ (right side of the equation). If the two graphs are identical, it is plausible that the equation is an identity. If the two graphs differ, the equation is *not* an identity.

Read that last paragraph again.

> You cannot use a graph to prove an equation is an identity; however, you can use a graph to demonstrate it is *not* an identity.

This is crucial to understand. The following example provides some clarity.

Example $5.4.4$

Which of the following equations *might* be identities?

a. $\sin(2\alpha) = 2\sin(\alpha)$
b. $\cos\left(x + \frac{\pi}{180}\right) = \cos(x)$
c. $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos(\theta)}{2}$

Solutions available at the end of this section.

Example $5.4.4$ has a few cautionary tales.

Caution: The Trouble with Technology

1. Example $5.4.4b$ illustrates that graphs can be deceiving: even if two graphs look identical, it is always a good idea to zoom in or check some numerical values.
2. The related equation is *not* an identity if the two graphs are different.
3. Graphs cannot be used to prove that an equation is an identity.

Checkpoint 5.4.4

Use graphs to decide which of the following equations *might* be identities.

a. $\cos(2\theta) = 2\cos(\theta)$
b. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
c. $\cos(\theta^2) = \cos^2(\theta)$

Answers available at the end of this section.

Proving Identities

We have proved a trigonometric identity when we show that one trigonometric expression is equivalent to another. In Example 5.4.1a we proved that the equation

$$\cos(\theta)\tan(\theta) + \sin(\theta) = 2\sin(\theta)$$

is an identity; it is valid for all values of θ (as long as the tangent function is defined).

A common strategy for proving an identity is to transform one side of the equation using equivalent expressions until it is identical to the other side. To help us choose the transformations at each step of the proof, we try to match the algebraic form of the final expression.

Example 5.4.5

Prove the identity:

$$1 + \tan^2(t) = \frac{1}{\cos^2(t)}.$$

Solution available at the end of this section.

When you write out the proof of an identity, your goal is to transform the expression on one side of the identity into the expression on the other, showing one step of the calculation on each line of your proof. You can justify each step to the right of the calculation. The proof of the identity in the previous example would look like this:

$$\begin{aligned}
\text{LHS} &= 1 + \tan^2(t) \\
&= 1 + \left(\frac{\sin(t)}{\cos(t)}\right)^2 && \text{(Ratio Identities)} \\
&= 1 + \frac{\sin^2(t)}{\cos^2(t)} && \text{(Laws of Exponents)} \\
&= \frac{\cos^2(t)}{\cos^2(t)} + \frac{\sin^2(t)}{\cos^2(t)} && \text{(Get a common denominator to combine fractions)} \\
&= \frac{\cos^2(t) + \sin^2(t)}{\cos^2(t)} && \text{(adding fractions)} \\
&= \frac{1}{\cos^2(t)} && \text{(Pythagorean Identity)} \\
&= \text{RHS}
\end{aligned}$$

For now, we will focus on transforming the left side of an equation into the right side; however, as we move forward in Trigonometry, we will adopt the rule of thumb of transforming the more "complicated" side of the equation into the other side.

Example 5.4.6

Prove the identity:

$$(\sin(\theta) - \cos(\theta))^2 - 1 = -2\sin(\theta)\cos(\theta)$$

Solution available at the end of this section.

Checkpoint 5.4.6

Prove the identity: $2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

Answer available at the end of this section.

Before we leave this section, it's important to note that you will be proving many identities in Trigonometry. It takes PRACTICE! We have introduced the idea of a proof early so that we spend a good deal of time sharpening those skills and giving advice on tactics as we move forward. For now, the best advice is to do all the proofs in the homework section—even if your instructor does not assign them.

MyOpenMath Advice: Latepasses and File Uploads

Sometimes, life can get overwhelmingly busy, and missing a due date happens. In those instances, you can use a Latepass to extend the original due date of the homework (see the syllabus for how long a single Latepass extends the original due date by).

Warning: Hand-Written Solutions

Read the syllabus to find out what happens to the questions that must be graded by hand (file uploads) - most professors are not likely to grade the late submitted hand-written homework solutions due to time constraints and fairness to other students.

Success in Mathematics: Holding Onto Old, Broken Techniques

It's common for students to learn a problem-solving technique years ago and not want to learn a different method because they are comfortable with their old habits. Unfortunately, we often fail to recognize that if our old, comfortable technique worked well, we should be much further along in our mathematical journey.

This adherence to comfort is widespread for all human beings in various situations. Whether it's a dead-end job, an abusive relationship, or a bad habit, we tend to ignore signals—remaining in a known situation rather than taking the chance to improve and evolve.

In general, when your professor gives you advice or a different approach to doing something, they are doing so from the lens of years of experience seeing students repeat old, broken techniques or habits that tend to lead students to make mistakes. Paying attention to these words of wisdom is critical.

EXAMPLE

Simplify the compound rational expression

$$\frac{-\frac{2}{(x+h)^2} + \frac{2}{x^2}}{h}$$

Students stuck in old habits tend to find the common denominator of the fractions in the numerator, make equivalent fractions, add fractions, divide the result by h, then simplify. This entire process looks like the following:

$$\begin{aligned}
\frac{-\frac{2}{(x+h)^2}+\frac{2}{x^2}}{h} &= \frac{-\frac{2x^2}{x^2(x+h)^2}+\frac{2(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \frac{\frac{-2x^2+2(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \frac{-2x^2+2(x+h)^2}{x^2(x+h)^2} \div h \\
&= \frac{-2x^2+2(x+h)^2}{x^2(x+h)^2} \cdot \frac{1}{h} \\
&= \frac{-2x^2+2(x+h)^2}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2(x^2+2hx+h^2)}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2x^2+4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{h(4x+2h)}{hx^2(x+h)^2} \\
&= \frac{\cancel{h}(4x+2h)}{\cancel{h}x^2(x+h)^2} \\
&= \frac{(4x+2h)}{x^2(x+h)^2}
\end{aligned}$$

Compare this to the advice given by your math professor to begin by multiplying both numerator and denominator of the original expression by the LCD of all minor fractions within the compound rational expression.

$$\begin{aligned}
\frac{-\frac{2}{(x+h)^2}+\frac{2}{x^2}}{h} &= \frac{-2x^2+2(x+h)^2}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2(x^2+2hx+h^2)}{hx^2(x+h)^2} \\
&= \frac{-2x^2+2x^2+4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{4hx+2h^2}{hx^2(x+h)^2} \\
&= \frac{h(4x+2h)}{hx^2(x+h)^2} \\
&= \frac{\cancel{h}(4x+2h)}{\cancel{h}x^2(x+h)^2} \\
&= \frac{(4x+2h)}{x^2(x+h)^2}
\end{aligned}$$

As you can see, the second method is cleaner and less complex.

The moral of the story is, when learning, try to be open to... learn.

Answers

Section Preview

The proofs done by the student in this section only require the Ratio, Reciprocal, and Pythagorean Identities. The material here is also meant to give the student experience in simplifying complex-looking expressions involving trigonometric functions into much cleaner expressions. Algebra involving trigonometric functions (e.g., multiplying expressions or adding rational expressions involving trigonometric functions) is treated. Finally, the student should be able to write a trigonometric function in terms of another (using the Pythagorean Identities).

Section Preview

- **Sets and Numbers**
  - The Language of Algebra (focus: proofs): The proof-writing framework requires understanding what a mathematical proof is and the logical structure of an argument.
- **Graphs**
  - Reading Values from Graphs: Examples within the text and lecture use graphs to check plausibility of identities.

Section Preview

- **Arithmetic**
  - The Absolute Value of a Number: The section makes a notable point that $\sqrt{\tan^2(\theta)} = |\tan(\theta)|$, explicitly discussing the principle square root and why $\sqrt{x^2} \neq x$ for negative x. Students must already understand numerical absolute value to follow this argument.
- **Simplifying Expressions**
  - Simplifying Ratios of Polynomials: The proofs in several examples (and homework questions) require simplifying ratios of polynomial-like expressions.

Section Preview

- Simplify a trigonometric expression.
- Rewrite a trigonometric expression in terms of sines and cosines.
- Simplify an algebraic expression by performing a trigonometric substitution.
- Determine the plausibility of an identity graphically and understand this does not prove an identity (but *can* disprove an identity).
- Prove basic trigonometric identities.

Example 5.4.1

a. We apply the Ratio Identities to replace $\tan(\theta)$ by $\frac{\sin(\theta)}{\cos(\theta)}$ and obtain

$$\begin{aligned}\cos(\theta)\tan(\theta)+\sin(\theta) &= \cos(\theta)\left(\tfrac{\sin(\theta)}{\cos(\theta)}\right)+\sin(\theta)\\ &= \sin(\theta)+\sin(\theta)\\ &= 2\sin(\theta)\end{aligned}$$

b. The instruction to multiply implies we need to square the binomial and simplify.

$$\begin{aligned}(\cos(\theta)+\sin(\theta))^2 &= \cos^2(\theta)+2\cos(\theta)\sin(\theta)+\sin^2(\theta) && \text{(distributing)}\\ &= \cos^2(\theta)+\sin^2(\theta)+2\cos(\theta)\sin(\theta) && \text{(Commutative Property of Addition)}\\ &= 1+2\cos(\theta)\sin(\theta) && \text{(Pythagorean Identity)}\end{aligned}$$

c. While we *could* use the Ratio Identities to make an equivalent expression, the result would be $\cot(\theta)+\tan(\theta)$. At that point, we would be stuck. Instead, let's use the Mathematical Mantra and perform some arithmetic before trying our new Trigonometry skills. Specifically, let's get a common denominator and perform the subtraction.

$$\begin{aligned}\frac{\cos(\theta)}{\sin(\theta)}+\frac{\sin(\theta)}{\cos(\theta)} &= \frac{\cos(\theta)}{\sin(\theta)}\cdot\frac{\cos(\theta)}{\cos(\theta)}+\frac{\sin(\theta)}{\cos(\theta)}\cdot\frac{\sin(\theta)}{\sin(\theta)} && \text{(multiplying each fraction by an expression}\\ & && \text{equivalent to 1 to get common denominators)}\\ &= \frac{\cos^2(\theta)}{\cos(\theta)\sin(\theta)}+\frac{\sin^2(\theta)}{\cos(\theta)\sin(\theta)}\\ &= \frac{\cos^2(\theta)+\sin^2(\theta)}{\cos(\theta)\sin(\theta)} && \text{(adding fractions with like denominators)}\\ &= \frac{1}{\cos(\theta)\sin(\theta)} && \text{(Pythagorean Identity)}\end{aligned}$$

d. Combine like terms:

$$3\tan(A)+4\tan(A)-2\cos(A)=7\tan(A)-2\cos(A)$$

Note that $\tan(A)$ and $\cos(A)$ are not like terms.

e. Combine like terms:

$$2-\sin(B)+2\sin(B)=2+\sin(B)$$

Note that $-\sin(B)$ means $-1\cdot\sin(B)$.

Checkpoint 5.4.1

a. $\sin^2(\theta)$

b. $5\cos(t)-4\cos(w)\sin(w)-2\cos(w)$

Example 5.4.2

a. Using one of the alternate forms of the Pythagorean Identity, we replace $\cos^2(\theta)$ with $1-\sin^2(\theta)$ to get

$$\begin{aligned}\sin(\theta)\cos^2(\theta) &= \sin(\theta)\left(1-\sin^2(\theta)\right) && \text{(Pythagorean Identity)}\\ &= \sin(\theta)-\sin^3(\theta) && \text{(Distributive Law)}\end{aligned}$$

b.

$$\begin{aligned}\cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)} && \text{(Ratio Identities)}\\ &= \frac{\cos(\theta)}{\pm\sqrt{1-\cos^2(\theta)}} && \text{(Pythagorean Identities (alternate form))}\\ &= \pm\frac{\cos(\theta)}{\sqrt{1-\cos^2(\theta)}}\end{aligned}$$

Checkpoint 5.4.2

$\cos^2(\alpha)-\cos^4(\alpha)$

Example 5.4.3

$$\begin{aligned}\frac{3}{\sqrt{x^2-4}} &= \frac{3}{\sqrt{(2\sec(\theta))^2-4}} && \text{(substituting } 2\sec(\theta) \text{ for } x)\\ &= \frac{3}{\sqrt{4\sec^2(\theta)-4}} && \text{(Laws of Exponents)}\\ &= \frac{3}{\sqrt{4\left(\sec^2(\theta)-1\right)}} && \text{(factoring)}\\ &= \frac{3}{2\sqrt{\sec^2(\theta)-1}} && \text{(simplifying)}\\ &= \frac{3}{2\sqrt{\tan^2(\theta)}} && \text{(Pythagorean Identity: } \tan^2(\theta)=\sec^2(\theta)-1)\\ &= \frac{3}{2\left|\tan(\theta)\right|} && \text{(see comments below)}\end{aligned}$$

Example 5.4.4

a. Compare the graphs of $y_1=\sin(2x)$ and $y_2=2\sin(x)$. The Desmos graphs for both equations are shown in the figure below.

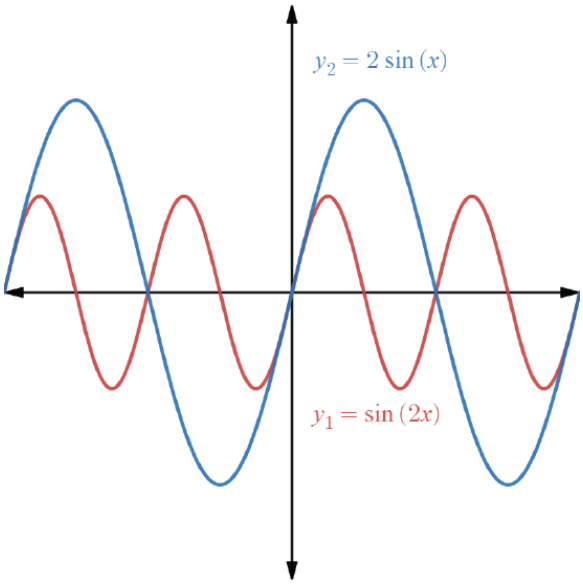


Figure 5.4.3

Because there are two distinct graphs, the expressions $\sin(2x)$ and $2\sin(x)$ are not equivalent, and consequently, $\sin(2\alpha)=2\sin(\alpha)$ is not an identity.

b. This time we graph $y_1=\cos\left(x+\frac{\pi}{180}\right)$ and $y_2=\cos(x)$.

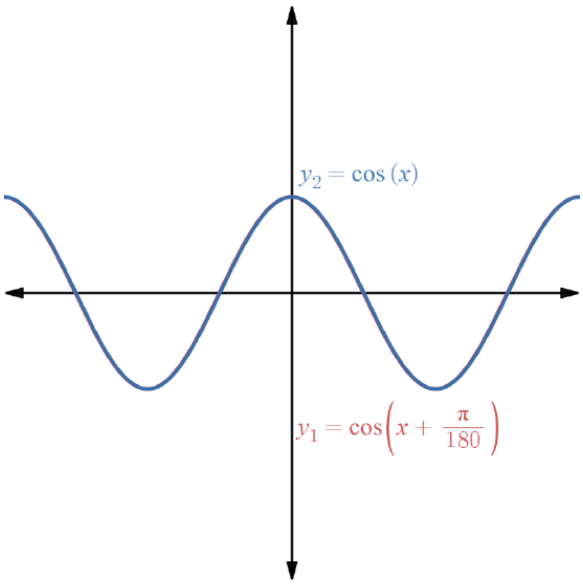


Figure 5.4.4

Although the graphs appear identical, when we zoom in, we see that the graphs are, indeed, not the same.

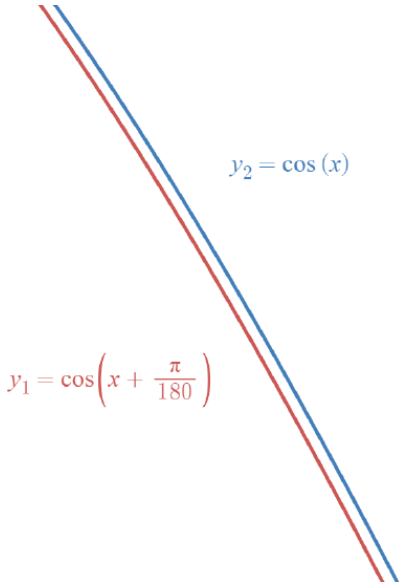


Figure 5.4.5

The graphs are so close together that Desmos' resolution does not distinguish them, but zooming in reveals that they are not identical. Because the two graphs differ, the equation $\cos\left(x + \frac{\pi}{180}\right) = \cos(x)$ is not an identity.

c. Letting $y_1 = \cos^2\left(\frac{x}{2}\right)$ and $y_2 = \frac{1+\cos(x)}{2}$, we get the following graph from Desmos.

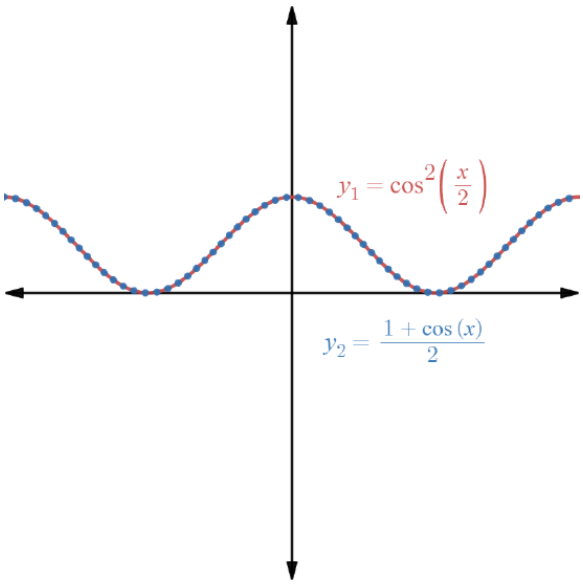


Figure 5.4.6

These two graphs *look* identical; however, it is best to zoom in to double-check.

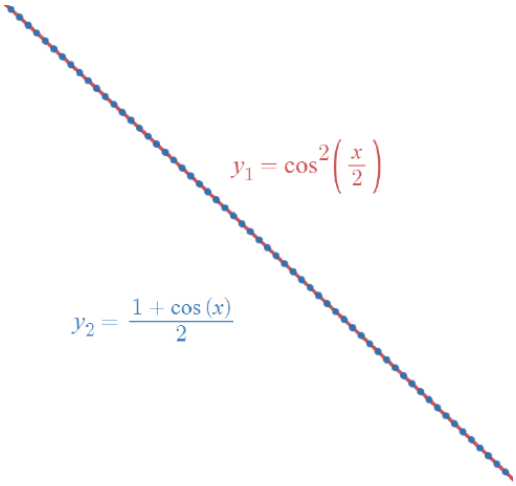


Figure 5.4.7

No matter how much we zoom in, the two graphs *appear* identical. Therefore, we can say that it is *plausible* that $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos(\theta)}{2}$ is an identity.

Checkpoint 5.4.4

(b)

Example 5.4.5

By manipulating the *left side* of the equation, we will show that the expression $1+\tan^2(t)$ is equivalent to $\frac{1}{\cos^2(t)}$. First, we use the Ratio Identities to write the expression in terms of sines and cosines:

$$1+\tan^2(t) = 1+\left(\frac{\sin(t)}{\cos(t)}\right)^2 = 1+\frac{\sin^2(t)}{\cos^2(t)}$$

Next, we notice that the right side of the proposed identity has only one term, so we combine the terms on the *left side*. So that the fractions have the same denominator, we write 1 as $\frac{\cos^2(t)}{\cos^2(t)}$.

$$1+\frac{\sin^2(t)}{\cos^2(t)} = \frac{\cos^2(t)}{\cos^2(t)}+\frac{\sin^2(t)}{\cos^2(t)} = \frac{\cos^2(t)+\sin^2(t)}{\cos^2(t)}$$

Finally, we apply the Pythagorean Identity to the numerator.

$$\frac{\cos^2(t)+\sin^2(t)}{\cos^2(t)} = \frac{1}{\cos^2(t)}$$

Thus, $1+\tan^2(t) = \frac{1}{\cos^2(t)}$, and the identity is proved.

Example 5.4.6

$$\begin{aligned}
\text{LHS} &= (\sin(\theta) - \cos(\theta))^2 - 1 \\
&= \sin^2(\theta) - 2\cos(\theta)\sin(\theta) + \cos^2(\theta) - 1 && \text{(distributing)} \\
&= \sin^2(\theta) + \cos^2(\theta) - 1 - 2\cos(\theta)\sin(\theta) && \text{(Commutative Property of Addition)} \\
&= 1 - 1 - 2\cos(\theta)\sin(\theta) && \text{(Pythagorean Identity)} \\
&= -2\cos(\theta)\sin(\theta) && \text{(simplifying)} \\
&= \text{RHS}
\end{aligned}$$

Checkpoint 5.4.6

$2\cos^2(x) - 1 = 2\left(1 - \sin^2(x)\right) - 1 = 2 - 2\sin^2(x) - 1 = 1 - 2\sin^2(x)$

This page titled 5.4: Introduction to Proofs in Trigonometry is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

5.4.1: Resources and Key Concepts

Resources

Videos

- Proving Identities
  - Fundamental Identities - Tactics for Proving Trigonometric Identities
- Trigonometric Substitution and Simplifying Trigonometric Expressions
  - Fundamental Identities - The Pythagorean Identities: Theorem, Proof, and Examples

Key Concepts

Common Mistakes

- **Confusing Expressions with Equations:** Expressions do not have equals signs and are simplified, whereas equations have equals signs and are solved.
- **Treating Different Angle Variables as Like Terms:** Trigonometric functions with different angle variables, such as $\cos(t)$ and $\cos(w)$, are not like terms and cannot be combined.
- **Forgetting Absolute Value with Square Roots:** The square root of a squared expression is its absolute value, not just the expression itself. For example, $\sqrt{\tan^2(\theta)} = |\tan(\theta)|$, not just $\tan(\theta)$.
- **The Trouble with Technology:**
  - Graphs can be deceiving; two different graphs may appear identical unless you zoom in.
  - If the graphs of the two sides of an equation are different, the equation is not an identity.
  - You cannot use a graph to prove that an equation is an identity, only to show it is plausible or to disprove it.

This page titled 5.4.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

5.4.2: Homework

Reading Questions

1. What is the difference between simplifying an expression and solving an equation?
2. What is a common first strategy for simplifying a trigonometric expression that involves multiple trigonometric functions?
3. In the expression $5\cos(t) - 2\cos(w)$, can the two terms be combined? Why or why not?
4. Explain the difference between $\sqrt{x^2}$ and x. Why is this important when simplifying expressions like $\sqrt{\tan^2(\theta)}$?
5. Can you use a graph to prove that an equation is an identity? Explain your answer.
6. How can you use a graph to prove that an equation is *not* an identity?
7. What does it mean if the graphs of the two sides of an equation appear to be identical?
8. What is the general strategy described in this section for proving a trigonometric identity?
9. In Example 1b, what algebraic process is used to multiply $(\cos(\theta) + \sin(\theta))^2$?
10. What fundamental identity is used to simplify $\cos^2(\theta) + \sin^2(\theta)$ to 1 in Example 1c?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, simplify the expression.

1. $\dfrac{\frac{1}{x}}{\frac{1}{y}}$
2. $\dfrac{1}{x} - \dfrac{1}{y}$
3. $\dfrac{1}{x} - x$
4. $\dfrac{y}{x} + \dfrac{1}{y}$
5. $\dfrac{x}{y} - \dfrac{y}{x}$

For the following exercises, multiply.

6. $(x - y)^2$
7. $(x - 1)(x + 1)$
8. $(x - 3)(x + 5)$

Answer available at the end of this section.

Homework

Vocabulary Check

1. A(n) ___ is a combination of symbols that are mathematically "well-formed."
2. You ___ equations and ___ expressions.

Concept Check

3. Explain how to use graphs to disprove an identity.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

4. To check whether an equation is an identity, we can compare graphs of the left side of the equation and the right side of the equation. If the two graphs agree, the equation is an identity.
5. To prove an identity, we write one side of the equation in equivalent forms until it is identical to the other side of the equation.

Basic Skills

For the following exercises, use identities to rewrite each expression.

6. $\cos(\theta)$ as an expression in $\sin(\theta)$ only
7. $\sin(\theta)$ as an expression in $\cos(\theta)$ only
8. $\cot(\theta)$ as an expression in $\sin(\theta)$ only
9. $\csc(\theta)$ as an expression in $\cos(\theta)$ only

For the following exercises, use graphs to decide which of the following equations *might be* identities.

10. $\sin 2t = 2\sin t$
11. $\cos\theta + \sin\theta = 1$
12. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
13. $\dfrac{\tan^2 x}{1+\tan^2 x} = \sin^2 x$
14. $\tan x + \dfrac{1}{\tan x} = \sin x \cos x$

For the following exercises, write each expression in terms of $\sin(\theta)$ and $\cos(\theta)$ and simplify, if possible.

15. $\csc(\theta)\tan(\theta)$
16. $\dfrac{\sec(\theta)}{\csc(\theta)}$
17. $\dfrac{\sec(\theta)}{\tan(\theta)}$
18. $\dfrac{\cot(\theta)}{\tan(\theta)}$

19. $\sec(\theta) + \tan(\theta)$

20. $\sin(\theta)\tan(\theta) + \sec(\theta)$

21. $\csc(\theta) - \cot(\theta)\cos(\theta)$

For the following exercises, add or subtract the expressions and simplify completely. Leave your answers in terms of sine or cosine.

22. $\cos(\theta) - \dfrac{1}{\sin(\theta)}$

23. $\dfrac{1}{\sin(\theta)} - \dfrac{1}{\cos(\theta)}$

24. $\tan(\theta) + \csc(\theta)$

25. $\dfrac{1}{\cos^2\beta} - \dfrac{\sin^2\beta}{\cos^2\beta}$

26. $\dfrac{1}{\sin^2\phi} - \dfrac{1}{\tan^2\phi}$

For the following exercises, multiply and simplify completely.

27. $(\sec(\theta) - 1)(\sec(\theta) + 1)$

28. $(\sin(\theta) + 3)(\sin(\theta) + 5)$

29. $(\cos(\theta) + 1)(\cos(\theta) - 1)$

30. $(\sin(\theta) - \cos(\theta))^2$

31. $(\sin(\theta) - 9)^2$

For the following exercises, simplify the expression completely. Write your answers in terms of a single trigonometric function or a constant.

32. $\cos^2\alpha(1 + \tan^2\alpha)$

33. $\cos^3\phi + \sin^2\phi\cos\phi$

34. $\tan^2 A - \tan^2 A\sin^2 A$

35. $\cos^2 B\tan^2 B + \cos^2 B$

36. $\dfrac{1 - \cos^2 z}{\cos^2 z}$

37. $\dfrac{\sin t}{\cos t\tan t}$

For the following exercises, use identities to rewrite each expression.

38. $2 - \cos^2\theta + 2\sin\theta$ as an expression in $\sin\theta$ only

39. $3\sin^2 B + 2\cos B - 4$ as an expression in $\cos B$ only

40. $\cos^2\phi - 2\sin^2\phi$ as an expression in $\cos\phi$ only

41. $\cos^2\phi\sin^2\phi$ as an expression in $\sin\phi$ only

For the following exercises, use the given substitution and completely simplify the result.

42. $\sqrt{25-x^2}$; substituting $5\sin(\theta)$ for x

43. $\sqrt{x^2+1}$; substituting $\tan(\theta)$ for x

44. $\sqrt{x^2-9}$; substituting $3\sec(\theta)$ for x

45. $\sqrt{81-9x^2}$; substituting $3\sin(\theta)$ for x

46. $\sqrt{36x^2+1296}$; substituting $6\tan(\theta)$ for x

For the following exercises, show that the equation is an identity by transforming the left side into the right side.

47. $\cos(\theta)\csc(\theta)\tan(\theta)=1$

48. $\dfrac{\cos(\theta)}{\sec(\theta)}=\cos^2(\theta)$

49. $\dfrac{\csc(\theta)\tan(\theta)}{\sec(\theta)}=1$

50. $\cos(\theta)\cot(\theta)+\sin(\theta)=\csc(\theta)$

51. $\tan^2(\theta)+1=\sec^2(\theta)$

52. $\csc(\theta)\tan(\theta)-\cos(\theta)=\dfrac{\sin^2(\theta)}{\cos(\theta)}$

53. $(1+\sin w)(1-\sin w)=\cos^2 w$

54. $(\cos\theta-1)(\cos\theta+1)=-\sin^2\theta$

55. $(\cos\theta-\sin\theta)^2=1-2\sin\theta\cos\theta$

56. $\sin^2 x-\cos^2 x=1-2\cos^2 x$

57. $\sec(\theta)(\sin(\theta)+\cos(\theta))=\tan(\theta)+1$

58. $\tan\theta\cos\theta=\sin\theta$

59. $\dfrac{\sin\mu}{\tan\mu}=\cos\mu$

60. $\cos^4 x-\sin^4 x=\cos^2 x-\sin^2 x$

61. $1-2\cos^2 v+\cos^4 v=\sin^4 v$

62. $\dfrac{\sin u}{1+\cos u}=\dfrac{1-\cos u}{\sin u}$
Hint. Multiply the numerator and denominator of the left side by $1-\cos u$.

63. $\dfrac{\sin v}{1-\sin v}=\dfrac{\tan v(1+\sin v)}{\cos v}$
Hint. Multiply the numerator and denominator of the left side by $1+\sin v$.

For the following exercises, rewrite tangents in terms of sines and cosines, then simplify each side separately (without adding, subtracting, multiplying, or dividing between sides). Finally, show the equation is an identity by transforming the left side into the right side. (These problems involve simplifying complex fractions. See the Algebra Refresher to review this skill.)

54. $\dfrac{\tan\alpha}{1+\tan\alpha} = \dfrac{\sin\alpha}{\sin\alpha+\cos\alpha}$

55. $\dfrac{1-\tan u}{1+\tan u} = \dfrac{\cos u-\sin u}{\cos u+\sin u}$

56. $\dfrac{1+\tan^2\beta}{1-\tan^2\beta} = \dfrac{1}{\cos^2\beta-\sin^2\beta}$

57. $\tan^2 v-\sin^2 v=\tan^2 v\sin^2 v$

Answers

Skills Refresher

1. $\dfrac{y}{x}$
2. $\dfrac{y-x}{xy}$
3. $\dfrac{(1-x)(1+x)}{x}$
4. $\dfrac{y^2+x}{xy}$
5. $\dfrac{(x-y)(x+y)}{xy}$
6. $x^2-2xy+y^2$
7. x^2-1
8. $x^2+2x-15$

This page titled 5.4.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

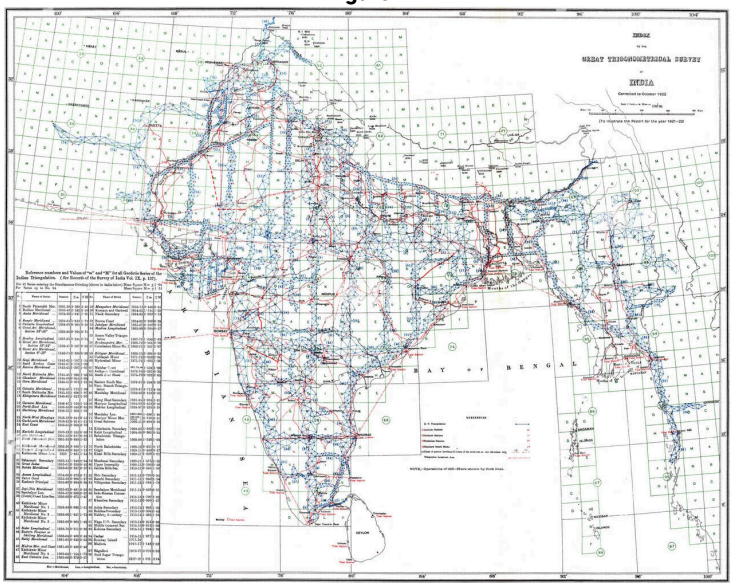
6: RIGHT TRIANGLE TRIGONOMETRY

How would one measure the distance to an inaccessible object, such as a ship at sea? In the 6th century BC, the Greek philosopher Thales estimated the distances to ships at sea using **triangulation**, a method for calculating distances by forming triangles. Using Trigonometry and the measured length of just one side, the lengths of the other sides can be calculated.

Triangulation has been used to compute distances ever since. In the 16th century, mapmakers used triangulation to position distant places accurately. As new methods in navigation and astronomy required greater precision, the idea of a survey using chains of triangles was developed.

In 1802, the East India Company embarked on the Great Trigonometrical Survey of India, which aimed to measure the entire Indian subcontinent with scientific precision.

Figure 1



The surveyors began by measuring a baseline near Madras. The baseline was the only distance they measured; all other distances were calculated from it using measured angles. Each calculated distance became the base side of another triangle used to calculate the distance to another point, which in turn started another triangle. Eventually this process formed a chain of triangles connecting the origin point to other locations.

Because of the size of the area to be surveyed, the surveyors did not triangulate the whole of India. Instead, they created a "gridiron" of triangulation chains running from north to south and east to west, as seen on the survey map above.

The survey, completed in 1871, calculated the heights of the Himalayan giants: Everest, K2, and Kanchenjunga. It also provided one of the first accurate measurements of a section of an arc of longitude.

Today, triangulation is used for many purposes, including surveying, navigation, metrology, astrometry, binocular vision, and location of earthquakes.

6.1: The Trigonometric Functions - Right Triangle Definition

6.1.1: Resources and Key Concepts

6.1.2: Homework

6.2: Technology and Right Triangle Trigonometry

6.2.1: Resources and Key Concepts
6.2.2: Homework

6.3: Solving Right Triangles

6.3.1: Resources and Key Concepts
6.3.2: Homework

6.4: Applications Involving Right Triangles

6.4.1: Resources and Key Concepts
6.4.2: Homework

This page titled 6: Right Triangle Trigonometry is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Katherine Yoshiwara.

6.1: The Trigonometric Functions - Right Triangle Definition

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

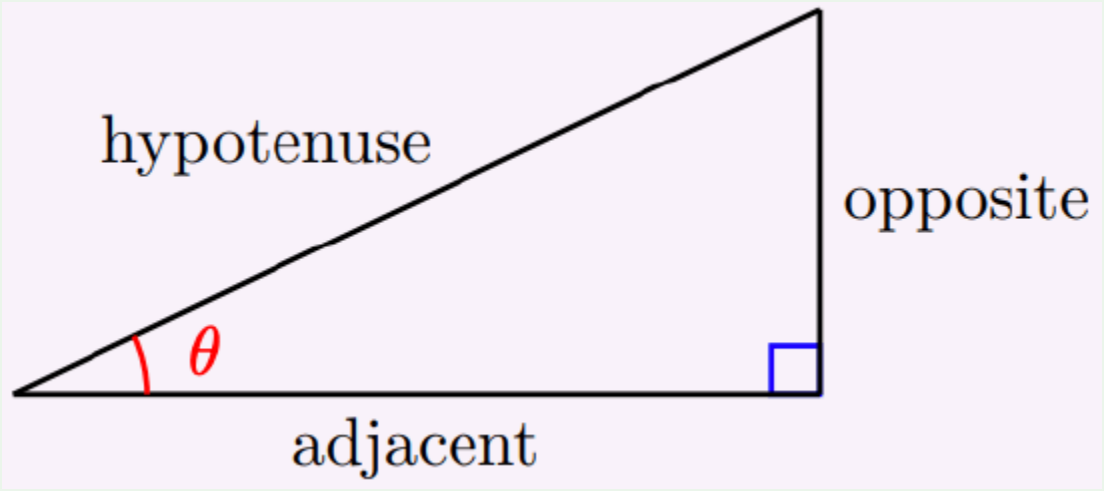
Learning Objectives (click to expand) available at the end of this section.

Introduction to Right Triangle Trigonometry

Our initial foray into Trigonometry involved defining the trigonometric functions in terms of points on the Cartesian plane and the distances from those points to the origin. In this chapter, we focus on Trigonometry outside of a coordinate system. Specifically, we redefine the trigonometric functions in terms of ratios of sides of a right triangle.

Definition: Trigonometric Functions (Right Triangle Definition)

If θ is one of the angles in a right triangle, where the side opposite θ is called "opposite," the side adjacent to θ is called "adjacent," and the hypotenuse is named as such, we define the trigonometric functions of θ as follows:



While this definition looks different than our coordinate definition of the trigonometric functions, it is, in fact, the same definition

if we restrict the coordinates in our original definition to the first quadrant. While it might seem limiting for us to create a new definition of the trigonometric functions by restricting our original definition, this new definition serves a greater purpose—it simplifies Trigonometry immensely. It allows us to separate the triangle from the Cartesian coordinate system.

Example 6.1.1

Find all trigonometric ratios of the labeled angle in each triangle.

a.

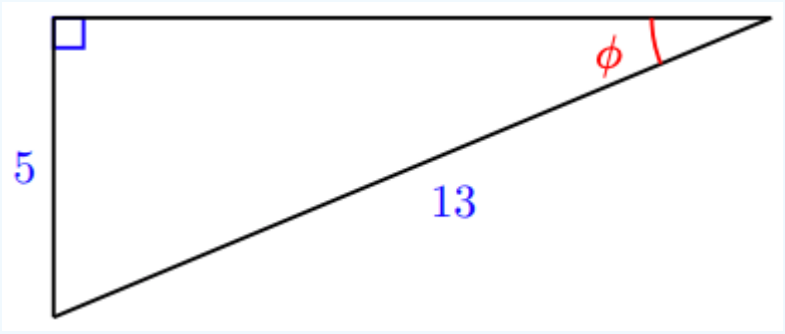


Figure 6.1.1

b.

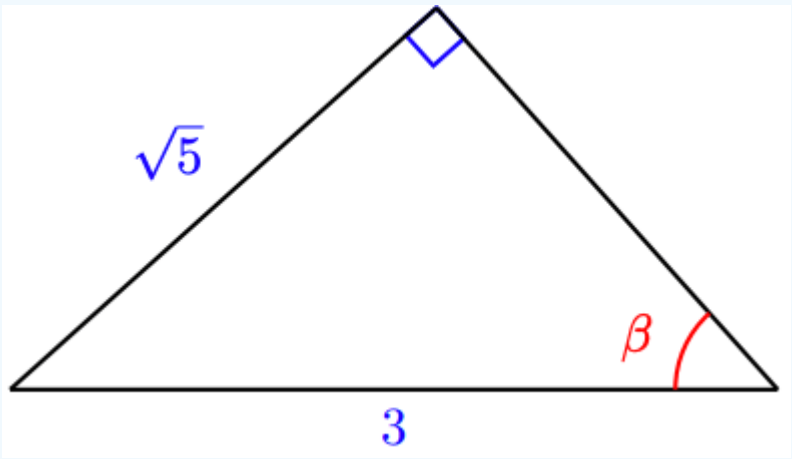


Figure 6.1.2

Solutions available at the end of this section.

The result of the computation to find the missing side in Example 6.1.1 part (a) reveals one of the **Pythagorean triples**, $(5, 12, 13)$.

Definition: Pythagorean Triple

A **Pythagorean triple** consists of three positive integers a, b, and c, such that $a^2 + b^2 = c^2$.

There are an infinite number of Pythagorean triples, but the common ones found in Trigonometry textbooks are $(3, 4, 5)$, $(5, 12, 13)$, and $(7, 24, 25)$. Triangles having sides lengths matching Pythagorean triples get a special name.

Definition: Pythagorean Triangle

A right triangle whose sides form a Pythagorean triple is called a **Pythagorean triangle**.

It can be proved that every Pythagorean triangle must be a right triangle. For more information, see this Wikipedia page.

Checkpoint 6.1.1

Find the sine of the labeled angle in the triangle below. Round the answer to 4 decimal places.

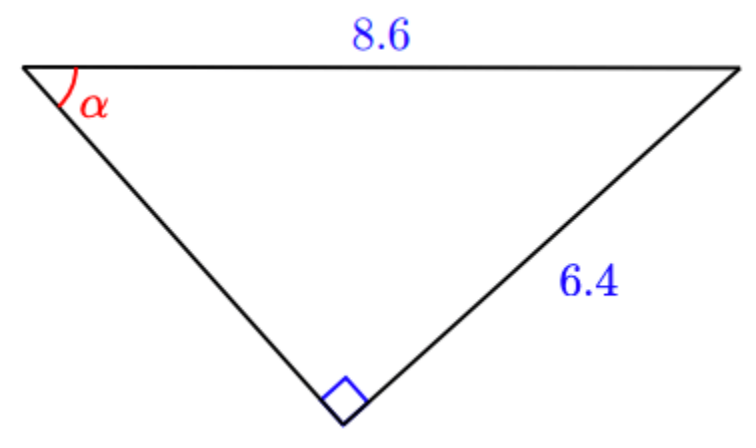


Figure 6.1.3

Answer available at the end of this section.

The following example illustrates a necessary skill for us as we move forward in Trigonometry.

Example 6.1.2

Write algebraic expressions for the six trigonometric ratios of the angle θ if the hypotenuse has length 2 and the side adjacent to θ has length D.

Solution available at the end of this section.

Checkpoint 6.1.2

If the hypotenuse of a right triangle is h and the side opposite θ is L, write expressions in h and L for all six trigonometric functions of θ.

Answer available at the end of this section.

Caution: Right Triangle Trigonometry Works For... Right Triangles

To calculate the trigonometric function of an angle within a triangle, it is crucial that the triangle be a *right* triangle. For example, in the triangle below, $\sin(\alpha) = \frac{4}{7}$, because $\triangle ABC$ is a right triangle. It is *not* true that $\sin(\alpha) = \frac{5}{7}$, or $\sin(\alpha) = \frac{6}{7}$.

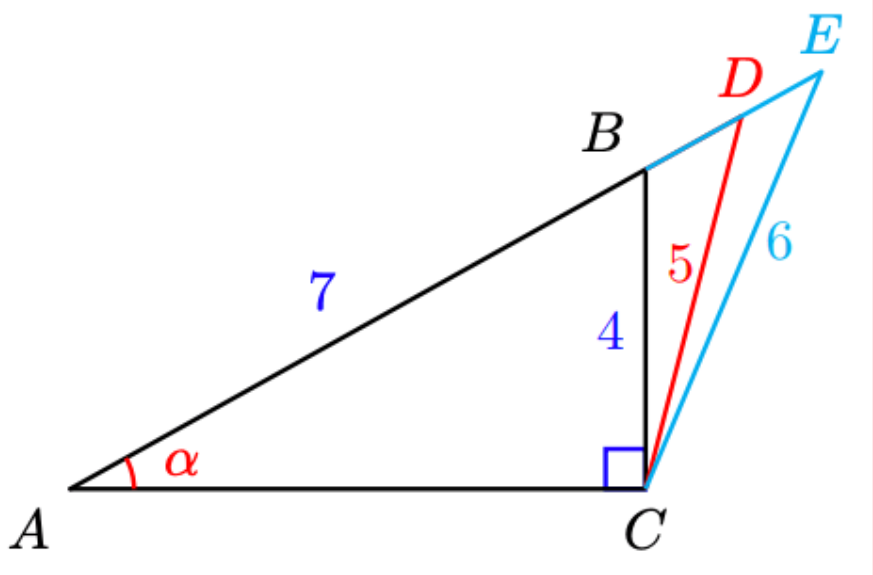


Figure 6.1.4

Revisiting the Fundamental Identities

Our previous identities (the Ratio, Reciprocal, and Pythagorean Identities) still hold when using the right triangle definition of the trigonometric functions. For example, notice that:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\frac{\text{hypotenuse}}{\text{opposite}}} = \frac{1}{\csc(\theta)}.$$

We leave it to the reader to show that all other Reciprocal Identities also hold.

Likewise:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\sin(\theta)}{\cos(\theta)}.$$

Again, we leave it to the reader to finish showing that both Ratio Identities still hold.

Finally, it is left as a homework exercise to show that the Pythagorean Identity still holds with this new definition.

Example 6.1.3

If θ is an acute angle in a right triangle and $\tan(\theta) = \frac{15}{8}$, compute the remaining trigonometric ratios of θ.

Solution available at the end of this section.

Checkpoint 6.1.3

If β is an acute angle in a right triangle and $\csc(\beta) = \frac{2}{\sqrt{3}}$, compute the remaining trigonometric ratios of β.

Answer available at the end of this section.

Example 6.1.4

If $\sin(\theta) = \frac{3}{7}$ and $\cos(\theta) = \frac{2\sqrt{10}}{7}$, where θ is an acute angle within a right triangle, find the value of $\cot(\theta)$ without drawing a triangle.

Solution available at the end of this section.

Checkpoint 6.1.4

If λ is an acute angle in a right triangle, $\tan(\lambda) = \frac{2}{3}$, and $\cos(\lambda) = \frac{3}{\sqrt{13}}$, find $\sin(\lambda)$ using only the Ratio Identities.

Answer available at the end of this section.

Right Triangle Trigonometry and the Special Angles

Now that we have a right triangle definition for the trigonometric functions, we can use our special triangles to help us quickly evaluate the trigonometric functions at 30°, 45°, and 60° angles. These angles are foundations upon which we build the rest of Trigonometry.

Table 6.1.1: *Fundamental Trigonometric Ratios for the Special Angles*



The exact values for these trigonometric ratios should be committed to memory. A great way to remember them is to know the two special triangles in Figure 6.1.6 below. Writing down the three trigonometric ratios for the special angles from these triangles is simple.



Figure 6.1.6: *Left: The 45°-45°-90° special triangle, Right: The 30°-60°-90° special triangle*

For example, we can quickly compute $\tan(60^\circ)$ and $\sec(30^\circ)$ by sketching the 30°-60°-90° triangle and finding the ratios of the relevant sides:

$$\sec(30^\circ) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{\sqrt{3}} \qquad \tan(60^\circ) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3}. \tag{6.1.1}$$

These are *exact* values for the trigonometric ratios, but we can also find decimal approximations. Use a calculator to verify the following approximate values.

$$\begin{array}{rcccc} & & \text{exact value} & & \text{approximation} \\ \sec(30^\circ) & = & \dfrac{2}{\sqrt{3}} & \approx & 1.1547 \\ \tan(60^\circ) & = & \sqrt{3} & \approx & 1.7321 \end{array}$$

Caution: Approximations are NOT Exact

It is essential to understand the difference between *exact* and *approximate* values. The decimal approximations above are rounded off. Even if a calculator shows ten or twelve digits, the values are not precisely correct—although they are adequate for most practical calculations.

Computational Note: To Rationalize or Not To Rationalize... That is the Question

Did you notice that we left $\sec(30^\circ) = \frac{2}{\sqrt{3}}$ with a radical in its denominator? This is quite common in Trigonometry. While one can rationalize the denominator to get

$$\sec(30^\circ) = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3},$$

the need to do so with numerical expressions (expressions *not* involving variables) is an antiquated "throwback" to the days when calculators did not exist. As such, unless it truly simplifies an expression, we will be okay with leaving radicals in the denominators of numerical expressions for the rest of mathematics.

Example 6.1.5

Matterhorn Chocolate bars are sold in boxes shaped like triangular prisms. The two triangular ends of a box are equilateral triangles with 8-centimeter sides. When stacking the boxes on a shelf, the second level will be a triangle's altitude above the bottom level. Find the exact length of the triangle's altitude, h.

Solution available at the end of this section.

Checkpoint 6.1.5

Show that $\cos^2(60^\circ) + \sin^2(60^\circ) = 1$.

Answer available at the end of this section.

Example 6.1.6

Let $x = 60^\circ$ and $y = 45^\circ$. Evaluate:

$$\sqrt{3}\csc(x) - \frac{5}{\sqrt{2}}\sec(y) + \sqrt{6}\tan(90^\circ - x)\sin(y)$$

Solution available at the end of this section.

Checkpoint 6.1.6

Evaluate and write as a single expression:

$$3\cos(45^\circ) - 2\cot(30^\circ)$$

Answer available at the end of this section.

Cofunction Identities

The following definition may answer a burning question you have had since being introduced to trigonometric functions.

Definition: Cofunction

The sine and cosine are called **cofunctions**. Likewise, the tangent and cotangent, as well as the secant and cosecant, are called cofunctions.

We say the sine is the cofunction of the cosine, and the cosine is the cofunction of the sine. The same holds for tangent and cotangent, and secant and cosecant. Now that we have a definition for cofunctions, what's the big deal?

The right triangle definition of the trigonometric functions reveals a new identity that was not so obvious when working with the coordinate definition of trigonometric functions. Consider the right triangle in Figure 6.1.8 below.

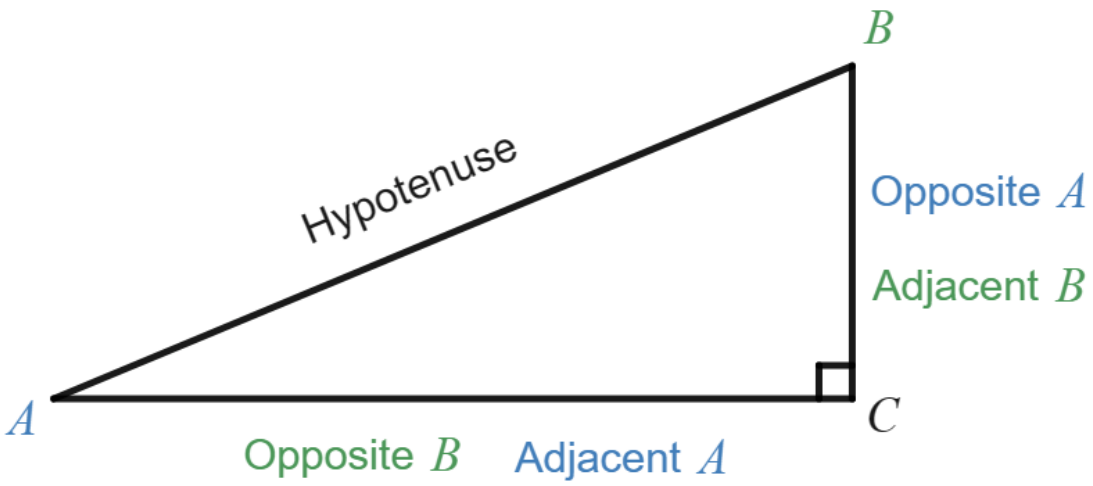


Figure 6.1.8

By the Triangle Sum, $\angle A + \angle B + \angle C = 180°$; however, $\angle C = 90°$. Therefore, $\angle A + \angle B = 90°$. That is, $\angle A = 90° - \angle B$. By the right triangle definition,

$$\sin(\angle A) = \frac{\text{Opposite } A}{\text{Hypotenuse}} = \frac{\text{Adjacent } B}{\text{Hypotenuse}} = \cos(\angle B) = \cos(90° - \angle A)$$

$$\sec(\angle A) = \frac{\text{Hypotenuse}}{\text{Adjacent } A} = \frac{\text{Hypotenuse}}{\text{Opposite } B} = \csc(\angle B) = \csc(90° - \angle A)$$

$$\tan(\angle A) = \frac{\text{Opposite } A}{\text{Adjacent } A} = \frac{\text{Adjacent } B}{\text{Opposite } B} = \cot(\angle B) = \cot(90° - \angle A)$$

At this point, I hope you can see a budding relationship between a function and its cofunction. The following theorem summarizes the results (the proof of which will be held off until we have more mathematics to work with).

Theorem: Cofunction Identities

The trigonometric function of an angle is equal to the cofunction evaluated at the complement of the angle. That is,

$$\begin{aligned} \sin(\theta) &= \cos(90^\circ - \theta) \quad \text{and} \quad \cos(\theta) = \sin(90^\circ - \theta) \\ \sec(\theta) &= \csc(90^\circ - \theta) \quad \text{and} \quad \csc(\theta) = \sec(90^\circ - \theta) \\ \tan(\theta) &= \cot(90^\circ - \theta) \quad \text{and} \quad \cot(\theta) = \tan(90^\circ - \theta) \end{aligned}$$

Example 6.1.7

Evaluate (by hand):

$$\sin^2(13^\circ) + \sin^2(77^\circ)$$

Solution available at the end of this section.

Checkpoint 6.1.7

Write the function in terms of its cofunction.

a. $\sin(13^\circ)$
b. $\cot(60^\circ)$
c. $\sec(y)$

Answers available at the end of this section.

Observations of Values for Trigonometric Functions Using Right Triangle Trigonometry

We have now spent a decent amount of time with the trigonometric functions—both defined in terms of coordinates on the Cartesian plane and in terms of ratios of sides of right triangles. It's now time to formalize some observations.

Trigonometric Functions are Invariant When Scaling Right Triangles

The trigonometric function of an angle is the same for any right triangle, no matter the size or orientation of the triangle. Another way to say this is the values of the trigonometric functions are **invariant** under scaling of the right triangle. For example, Figure 6.1.9 (below) shows three right triangles, each with a 50° angle. Although the sides of the triangle may be bigger or smaller, the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ is always the same for that angle because the triangles are similar. Therefore, $\sin(50^\circ)$ is the same for all three right triangles.

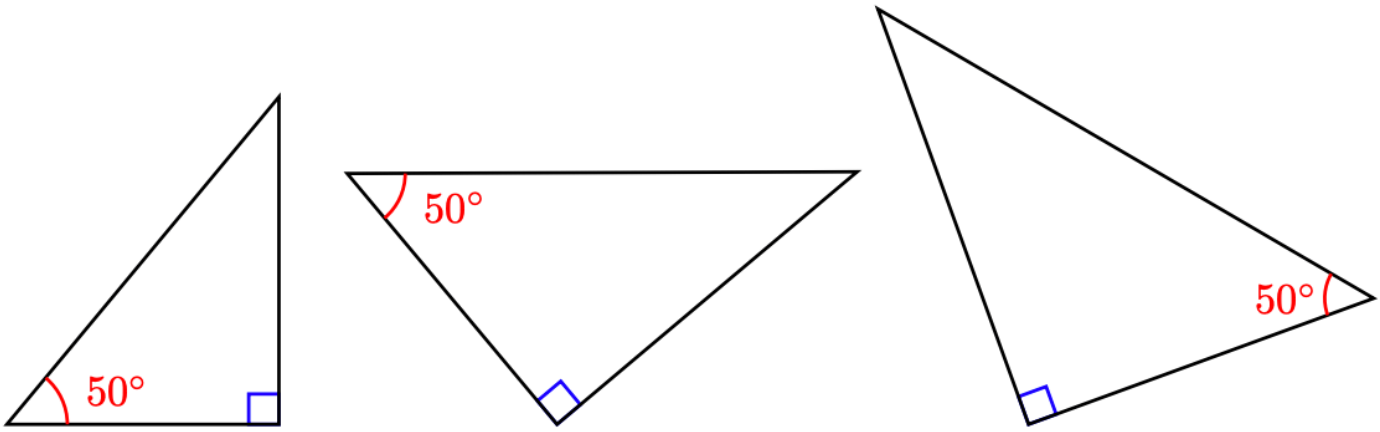


Figure 6.1.9

Similarly, the values of all the other trigonometric functions at 50° remain constant when we scale the triangles to be larger or smaller.

Interactive Element: Trig Ratios Are Invariant Under Scaling

The following Interactive Element might help you visualize why the trigonometric ratios of a fixed angle do not depend on the size of the right triangle.

Interact: Drag the red point along the horizontal base to grow or shrink the triangle; the 50° angle at the corner stays fixed.

Observation: Every side length changes as you resize the triangle, yet the ratio $\frac{\text{opposite}}{\text{hypotenuse}} = \sin(50^\circ)$ stays the same—similar triangles share the same trigonometric ratios.

Some Trigonometric Functions Have Limited Ranges

Consider the right triangle definition of the trigonometric functions. Since the hypotenuse is the longest side of a right triangle, the ratios

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{and} \quad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

will always be less than 1.

Moreover, if we think about the coordinate definition of the trigonometric functions,

$$\sin(\theta) = \frac{y}{r} \quad \text{and} \quad \cos(\theta) = \frac{x}{r}.$$

Since $r = \sqrt{x^2+y^2} \geq \sqrt{y^2} = |y|$ and $r = \sqrt{x^2+y^2} \geq \sqrt{x^2} = |x|$,

$$|y| \leq r \implies \frac{|y|}{r} \leq 1 \implies \left|\frac{y}{r}\right| \leq 1 \implies |\sin(\theta)| \leq 1 \implies -1 \leq \sin(\theta) \leq 1$$

and

$$|x| \leq r \implies \frac{|x|}{r} \leq 1 \implies \left|\frac{x}{r}\right| \leq 1 \implies |\cos(\theta)| \leq 1 \implies -1 \leq \cos(\theta) \leq 1$$

The only times the sine or cosine become 1 or -1 is when the given angle is a quadrantal angle.

Example 6.1.8

If θ is an angle within a right triangle, is it possible for $\sec(\theta) = \frac{1}{2}$? Explain.

Solution available at the end of this section.

Checkpoint 6.1.8

Can $\csc(\theta)$ ever be 2?

Answer available at the end of this section.

Using Special Angles to Approximate Trigonometric Functions at Other Angles

In a right triangle, as the angle θ increases, $\sin(\theta)$ increases but $\cos(\theta)$ decreases. Figure 6.1.10 (below) demonstrates why this is true. In each right triangle, the hypotenuse has the same length. However, as the angle increases, the opposite side gets longer, and the adjacent side gets shorter.

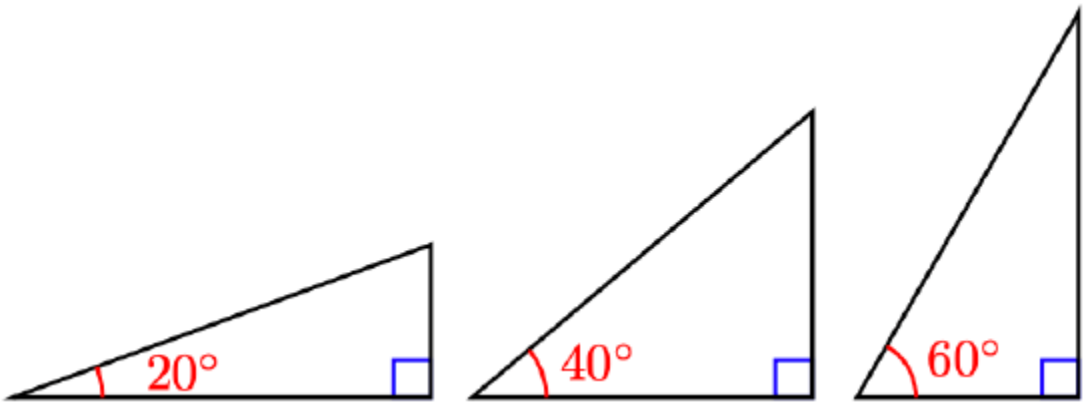


Figure 6.1.10

Interactive Element: As the Angle Opens, Sine Rises and Cosine Falls

The following Interactive Element might help you visualize how sine and cosine change as the acute angle grows while the hypotenuse is held fixed.

Interact: Drag the red point along the base; the top corner glides along the arc of the fixed-length hypotenuse.

Observation: As the angle increases, the opposite side lengthens and the adjacent side shortens, so $\sin$ increases toward 1 while $\cos$ decreases toward 0.

We can use the special angles as benchmarks for estimating and mental calculation. For example, we know that $\sin(60°) \approx 0.8660$, so if $\sin(\theta) = 0.95$ for some unknown angle θ within a right triangle, we know that $\theta > 60°$, because as θ increases, the sine of θ increases as well.

Example 6.1.9

If $\cos(\alpha) > \frac{\sqrt{3}}{2}$, where α is an angle within a right triangle, what can we say about α?

Solution available at the end of this section.

Checkpoint 6.1.9

If $1 < \tan(\beta) < \sqrt{3}$, where β is one of the angles in a right triangle, what can we say about β?

Answer available at the end of this section.

Hidden Resources: CalFresh

CalFresh (formerly known as Food Stamps) is a nutrition program that helps students and their families buy healthy foods.

How Does CalFresh Work?

CalFresh benefits are issued on an electronic benefit transfer (EBT) card that works like a debit card and can be used at most grocery stores and many farmers' markets to buy food. You can even use CalFresh to purchase food online through Amazon and Walmart.com.

If you are on a limited income and are struggling to buy the healthy foods that you need, then CalFresh may be able to help! Plus, CalFresh benefits do not impact your financial aid, and you do not need your parents' tax return information to apply.

For more information, please see this page on the CRC website.

Success in Mathematics: Why Study Mathematics? (Part 1)

Until modern times, only a tiny percent of people had the opportunity to be "liberally educated." These people were not studying to learn a job or trade - they wanted to learn what other humans had thought and created. They were also trying to develop new knowledge. Menaechmus (350 B.C.E) was this kind of student. As he studied geometry, he discovered the properties of conic sections. Others built on his knowledge. Apollonius (350 B.C.E) wrote an eight-chapter book about conic sections.

Today we use our knowledge of conic sections to predict orbits, build telescopes, locate ships at sea, make "eyes" for robots, and collect solar energy. But we also study conic sections as an accomplishment of our human ancestors. In developing their equations, we also develop logic and creative problem-solving skills. Think about the football coach who has his team doing agility drills and running through tires. Why? When has a football team ever had to run through tires during a game? Similarly, with the right attitude, you can develop your ability to reason and think in your life outside of college by doing mathematics. You are responsible for recognizing that learning and studying may be valuable even when it is not immediately practical.

> One problem with teaching mathematics in the K-12 system - and I see it as a major difficulty - is that there is virtually nothing the pupils learn that has a non-trivial application in today's world. The most a teacher can tell a student who inquires, entirely reasonably, "How is this useful?" is that almost all mathematics finds uses, in many cases important ones, and that what they learn in school leads on to mathematics that definitely is used.
>
> - Keith Devlin, Standford University

By "non-trivial," Dr. Devlin means that many real-life applications require mathematics beyond what is studied in high school.

1. Younger students often ask, " When will I use this math?" Their teachers often reply, "You need to learn this math to go on to the next course. In college, you can take math that is very powerful for doing all kinds of practical things." Do you think this is the best reply to give to the student? Explain.
2. I've often pondered the best mixture of application and theory in a mathematics course. My love of mathematics lies in theory; however, my research in mathematics is in applications. As a student, which do you prefer? Explain.

Answers

Section Preview

This section introduces the student to the Right Triangle definition of the trigonometric functions. As such, all angles will be acute (and in degrees); however, if given an angle (rather than the sides of a triangle) and asked to evaluate a trigonometric function, the angle will strictly be an acute *special angle*. Therefore, using technology (e.g., calculators) to evaluate any trigonometric function in this section is *strictly avoided* so as to get the student familiar with right triangle trigonometry at a fundamental level.

Section Preview

- **Geometry**
  - The Pythagorean Theorem, Distance Formula, and Midpoint Formula: Students must understand the Pythagorean Theorem before the right triangle definition can be applied.

Section Preview

- Find the value of a trigonometric function of an unknown angle in a right triangle given at least two sides of the triangle.
- Compute the values of the remaining trigonometric functions of an unknown angle within a right triangle given one trigonometric ratio.
- Use the Ratio, Reciprocal, and Pythagorean Identities, along with Right Triangle Trigonometry, to compute trigonometric values.
- Find the exact value of a trigonometric function for a special angle.
- Use the Cofunction Identities to find the value of a trigonometric function.
- Use exact values to simplify an expression involving trigonometric functions.
- Recognize and understand the constraints on the values of certain trigonometric functions.

Example 6.1.1

a. The side opposite angle ϕ is 5 and the hypotenuse is 13. Let x be the length of the side adjacent to ϕ. To find x, we invoke the Pythagorean Theorem.

$$\begin{aligned} & x^2+5^2 = 13^2 && \text{(Pythagorean Theorem)} \\ \Longrightarrow\ & x^2+25 = 169 && \text{(simplifying)} \\ \Longrightarrow\ & x^2 = 144 && \text{(subtracting 25 from both sides)} \\ \Longrightarrow\ & x = \pm 12 && \text{(Extraction of Roots)} \\ \Longrightarrow\ & x = 12 && (x \text{ must be positive---it's a side length}) \end{aligned}$$

Therefore,

$$\sin(\phi) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{13} \qquad \csc(\phi) = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{13}{5}$$

$$\cos(\phi) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{12}{13} \qquad \sec(\phi) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{13}{12}$$

$$\tan(\phi) = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12} \qquad \cot(\phi) = \frac{\text{adjacent}}{\text{opposite}} = \frac{12}{5}$$

b. The side opposite angle β is $\sqrt{5}$ and the hypotenuse is 3. Labeling the side adjacent to β as x, we get

$$\begin{aligned} & x^2+\left(\sqrt{5}\right)^2 = 3^2 && \text{(Pythagorean Theorem)} \\ \Longrightarrow\ & x^2+5 = 9 && \text{(simplifying)} \\ \Longrightarrow\ & x^2 = 4 && \text{(subtracting 5 from both sides)} \\ \Longrightarrow\ & x = 2 && \text{(Extraction of Roots)} \end{aligned}$$

Therefore,

$$\sin(\beta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{5}}{3} \qquad \csc(\beta) = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{3}{\sqrt{5}}$$

$$\cos(\beta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2}{3} \qquad \sec(\beta) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{3}{2}$$

$$\tan(\beta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{5}}{2} \qquad \cot(\phi) = \frac{\text{adjacent}}{\text{opposite}} = \frac{2}{\sqrt{5}}$$

Checkpoint 6.1.1

$\sin(\alpha) \approx 0.7442$

Example 6.1.2

We begin by finding the missing side, which we shall call x.

$$\begin{aligned} & D^2+x^2 = 2^2 && \text{(Pythagorean Theorem)} \\ \Longrightarrow\ & D^2+x^2 = 4 \\ \Longrightarrow\ & x^2 = 4-D^2 \\ \Longrightarrow\ & x = \sqrt{4-D^2} \end{aligned}$$

We can now fill in the values of the trigonometric ratios.

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{4-D^2}}{2} \qquad \csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{2}{\sqrt{4-D^2}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{D}{2} \qquad \sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{D}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{4-D^2}}{D} \qquad \cot(\theta) = \frac{\text{adjacent}}{\text{opposite}} = \frac{D}{\sqrt{4-D^2}}$$

Checkpoint 6.1.2

$$\sin(\theta) = \frac{L}{h} \qquad \csc(\theta) = \frac{h}{L}$$

$$\cos(\theta) = \frac{\sqrt{h^2-L^2}}{h} \qquad \sec(\theta) = \frac{h}{\sqrt{h^2-L^2}}$$

$$\tan(\theta) = \frac{L}{\sqrt{h^2-L^2}} \qquad \cot(\theta) = \frac{\sqrt{h^2-L^2}}{L}$$

Example 6.1.3

It is best to sketch the right triangle with θ and where the side adjacent to θ has length 8 and the side opposite θ has length 15.

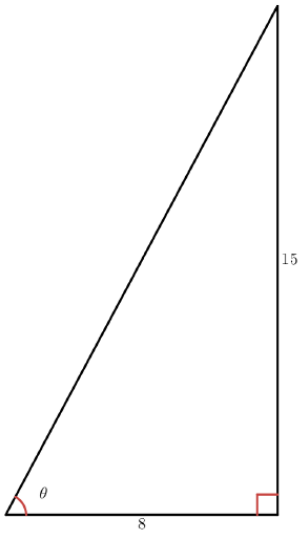


Figure 6.1.5

Using the Pythagorean Theorem, we get

$$\begin{aligned} 8^2 + 15^2 &= h^2 \\ \Longrightarrow \quad 289 &= h^2 \\ \Longrightarrow \quad 17 &= h \end{aligned}$$

Hence,

$$\sin(\theta) = \frac{15}{17} \Longrightarrow \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{15/17} = \frac{17}{15}$$

$$\cos(\theta) = \frac{8}{17} \Longrightarrow \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{8/17} = \frac{17}{8}$$

$$\tan(\theta) = \frac{15}{8} \Longrightarrow \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{15/8} = \frac{8}{15}$$

Checkpoint 6.1.3

$$\sin(\beta) = \frac{\sqrt{3}}{2} \qquad \csc(\beta) = \frac{2}{\sqrt{3}}$$

$$\cos(\beta) = \frac{1}{2} \qquad \sec(\beta) = 2$$

$$\tan(\beta) = \sqrt{3} \qquad \cot(\beta) = \frac{1}{\sqrt{3}}$$

Example 6.1.4

$$\begin{aligned}\cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)}\\ &= \frac{2\sqrt{10}/7}{3/7}\\ &= \frac{2\sqrt{10}}{3}\end{aligned}$$

Checkpoint 6.1.4

$\sin(\lambda) = \frac{2}{\sqrt{13}}$

Example 6.1.5

We have seen this question before (near the beginning of the textbook). The altitude divides the triangle into two $30°$-$60°$-$90°$ right triangles, as shown in Figure 6.1.7 below.

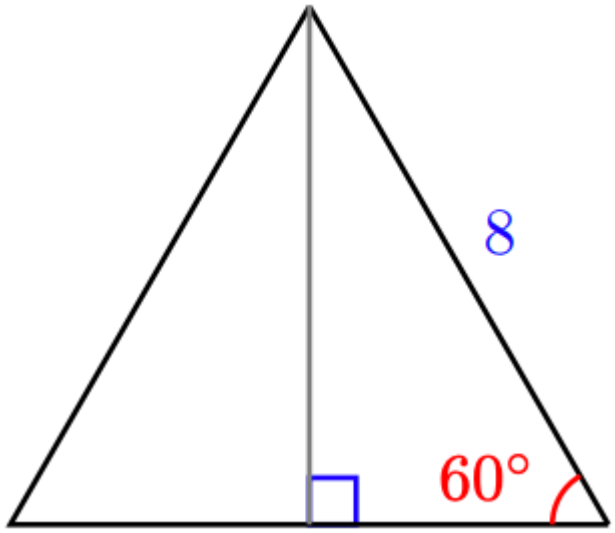


Figure 6.1.7

The altitude is adjacent to the $30°$ angle and the hypotenuse of the right triangle is 8 centimeters. Thus, using the fact that we already know $\cos(30°) = \frac{\sqrt{3}}{2}$, we get

$$\begin{aligned}\cos(30°) &= \frac{\text{adjacent}}{\text{hypotenuse}} && \text{(Fill in the values.)}\\ \Rightarrow \quad \frac{\sqrt{3}}{2} &= \frac{h}{8}\\ \Rightarrow \quad h &= 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3} && \text{(multiplying both sides by 8)}\end{aligned}$$

The altitude is *exactly* $4\sqrt{3}$ centimeters long. From this exact answer, we can find approximations to any degree of accuracy we like. You can check that $4\sqrt{3} \approx 6.9282$, so the altitude is approximately 6.9 centimeters long.

Checkpoint 6.1.5

$\cos^2(60°) + \sin^2(60°) = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1.$

Example 6.1.6

Using our two special triangles in Figure 6.1.6, we get

$$\begin{aligned}\sqrt{3}\csc(x) - \frac{5}{\sqrt{2}}\sec(y) + \sqrt{6}\tan(90° - x)\sin(y) &= \sqrt{3}\csc(60°) - \frac{5}{\sqrt{2}}\sec(45°) + \sqrt{6}\tan(90° - 60°)\sin(45°)\\ &= \sqrt{3}\csc(60°) - \frac{5}{\sqrt{2}}\sec(45°) + \sqrt{6}\tan(30°)\sin(45°)\\ &= \cancel{\sqrt{3}}\left(\frac{2}{\cancel{\sqrt{3}}}\right) - \left(\frac{5}{\cancel{\sqrt{2}}}\right)\left(\frac{\cancel{\sqrt{2}}}{1}\right) + \left(\cancel{\sqrt{6}}\right)\left(\frac{1}{\cancel{\sqrt{3}}}\right)\left(\frac{1}{\cancel{\sqrt{2}}}\right)\\ &= 2 - 5 + 1\\ &= -2\end{aligned}$$

Checkpoint 6.1.6

$\frac{3 - 2\sqrt{6}}{\sqrt{2}}$

Example 6.1.7

$$\begin{aligned}\sin^2(13^\circ)+\sin^2(77^\circ) &= \sin^2(13^\circ)+(\sin(77^\circ))^2\\ &= \sin^2(13^\circ)+(\cos(90^\circ-77^\circ))^2\\ &= \sin^2(13^\circ)+(\cos(13^\circ))^2\\ &= \sin^2(13^\circ)+\cos^2(13^\circ)\\ &= 1.\end{aligned}$$

Checkpoint 6.1.7

a. $\sin(13^\circ)=\cos(90^\circ-13^\circ)=\cos(77^\circ)$
b. $\cot(60^\circ)=\tan(90^\circ-60^\circ)=\tan(30^\circ)$
c. $\sec(y)=\csc(90^\circ-y)$

Example 6.1.8

$\sec(\theta)$ is the ratio of the hypotenuse to the adjacent side of θ; however, the hypotenuse of a right triangle is always longer than the other sides. Therefore, the ratio $\frac{\text{hypotenuse}}{\text{adjacent}}$ must be greater than 1 (by the discussion before this example, the secant can attain a value of 1 only if the given angle is a quadrantal angle).

Checkpoint 6.1.8

Yes!

Example 6.1.9

As an angle increases from 0° to 90°, its cosine decreases (why?). Now, $\cos(30^\circ)=\dfrac{\sqrt{3}}{2}$, so if $\cos(\alpha)>\frac{\sqrt{3}}{2}$, then α must be less than 30°.

Checkpoint 6.1.9

$45^\circ<\beta<60^\circ$

This page titled 6.1: The Trigonometric Functions - Right Triangle Definition is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

6.1.1: Resources and Key Concepts

Resources

Videos

- Right Triangle Trigonometry
  - Right Triangle Trigonometry - Trigonometric Functions – A Right Triangle Approach
  - Trigonometry of Right Triangles - Review of Trigonometric Functions - A Right Triangle Approach
- Special Angles
  - Right Triangle Trigonometry - Evaluating Trigonometric Functions for the Special Acute Angles
  - Trigonometry of Right Triangles - Review of Special Triangles in Trigonometry
- Cofunction Identities
  - Right Triangle Trigonometry - The Cofunction Identities

Key Concepts

Definitions

- **Trigonometric Functions (Right Triangle Definition):** For an angle θ in a right triangle, the trigonometric functions are defined as ratios of the side opposite θ, the side adjacent to θ, and the hypotenuse. For example, $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$.
- **Pythagorean Triple:** A set of three positive integers, a, b, and c, such that $a^2 + b^2 = c^2$.
- **Pythagorean Triangle:** A right triangle whose side lengths form a Pythagorean triple.
- **Cofunction:** Pairs of trigonometric functions consisting of sine and cosine, tangent and cotangent, and secant and cosecant. For example, the cofunction of sine is cosine.

Theorems

- **Cofunction Identities:** The value of a trigonometric function of an angle is equal to the value of its cofunction at the complement of the angle. For example, $\sin(\theta) = \cos(90^\circ - \theta)$ and $\tan(\theta) = \cot(90^\circ - \theta)$.

Common Mistakes

- **Applying Right Triangle Trigonometry to Non-Right Triangles:** The right triangle definitions of trigonometric functions are only valid for angles within a right triangle. They cannot be applied to obtuse or acute triangles in general.
- **Confusing Exact and Approximate Values:** An exact value, like $\frac{\sqrt{3}}{2}$, is precise. A decimal approximation, like 0.8660, is a rounded value. For theoretical work, exact values are required.
- **Rationalizing Numerical Denominators Unnecessarily:** While rationalizing is sometimes useful, expressions like $\frac{2}{\sqrt{3}}$ are perfectly acceptable in trigonometry, and rationalizing them to $\frac{2\sqrt{3}}{3}$ is often an unnecessary step.

This page titled 6.1.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

6.1.2: Homework

Reading Questions

1. In a right triangle, what are the definitions of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ in terms of the opposite side, adjacent side, and hypotenuse?
2. What is a Pythagorean triple?
3. Can the right triangle definitions of the trigonometric functions be applied to an obtuse triangle?
4. If you are given the lengths of the opposite and adjacent sides in a right triangle, what theorem must you use to find the length of the hypotenuse?
5. What is the cofunction of the tangent function?
6. What does the Cofunction Identity theorem state about $\sin(20^\circ)$?
7. Why is the value of $\cos(\theta)$ always less than or equal to 1 for any acute angle θ?
8. Is it possible for $\sec(\theta)$ to equal $\frac{1}{2}$ for an angle θ in a right triangle? Explain why or why not.
9. What happens to the value of $\sin(\theta)$ as θ increases from 0° to 90°?
10. What is the difference between an exact value, like $\sqrt{2}$, and an approximate value, like 1.414?

Skills Refresher

Review the following skills you will need for this section.

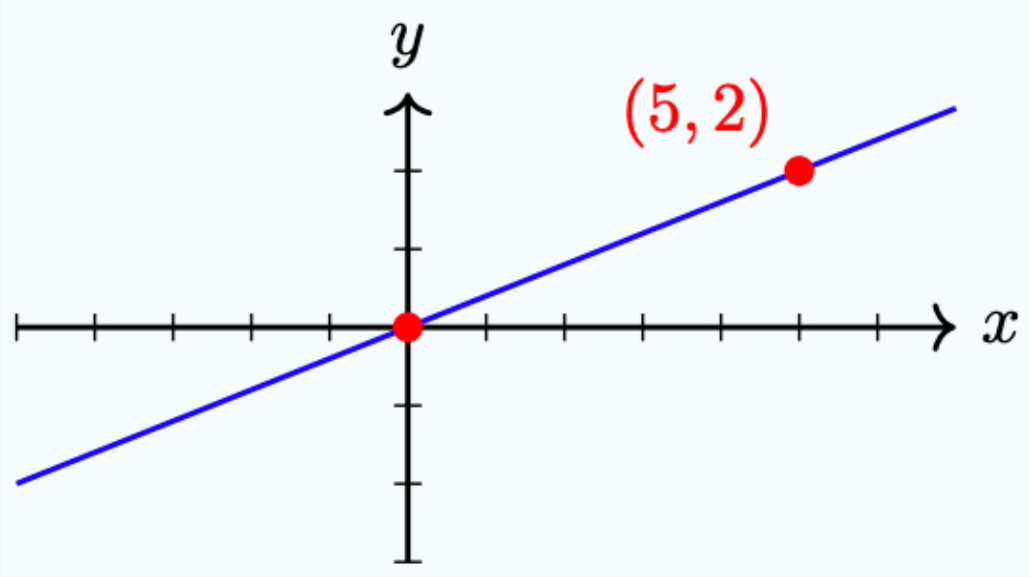
Skills Refresher

For the following exercises, write two more ratios equivalent to the given fraction.

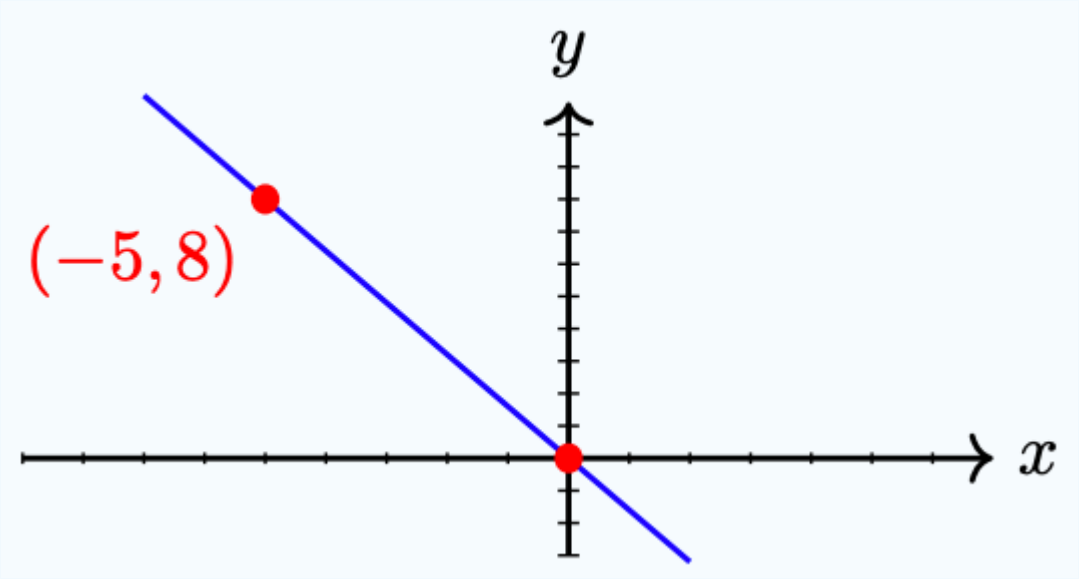
1. $\frac{10}{4}$
2. $\frac{6}{8}$
3. 0.6
4. 1.5

For the following exercises, compute the slope of the line.

5.

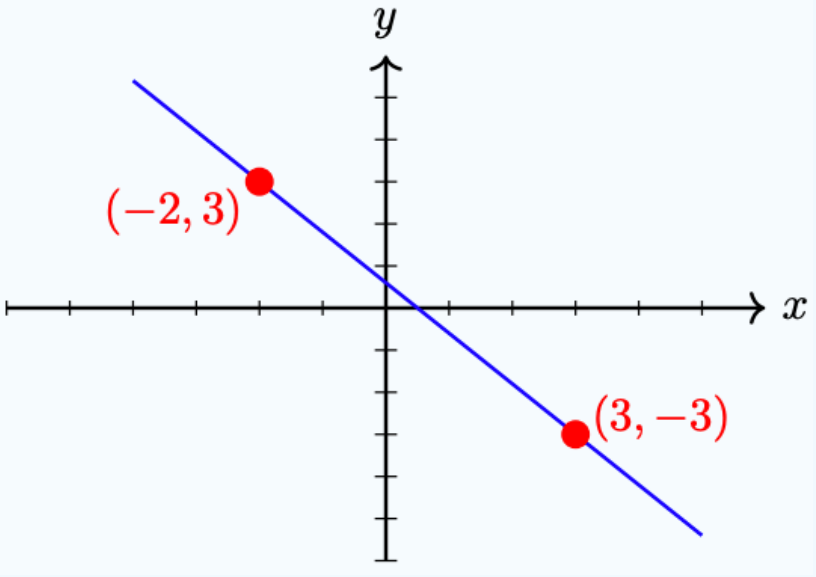


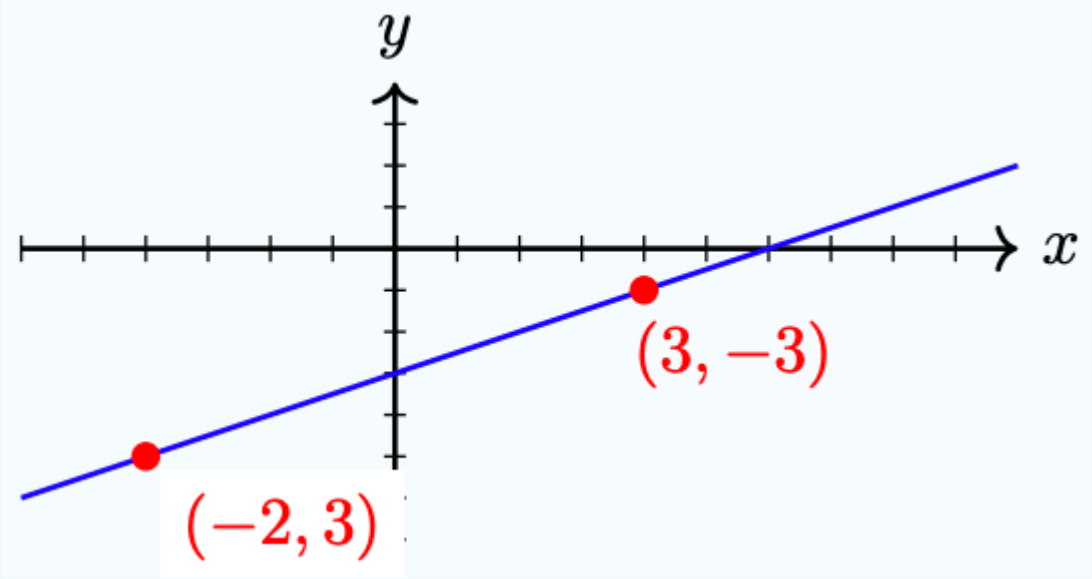
6.



7.

8.





For the following exercises, solve.

9. $\frac{12}{x} = 48$

10. $\frac{60}{x} = 80$

For the following exercises, solve the formula for the specified variable.

11. $C = 2\pi r$ for r (Circumference of a Circle)

12. $r = \frac{d}{t}$ for t

13. $P = 2a + b - c$ for a

14. $x = \frac{2}{3}y + 10$ for y

15. $S = 2\pi rh + 2\pi r^2$ for h (Surface Area of a Right Circular Cylinder)

Answers available at the end of this section.

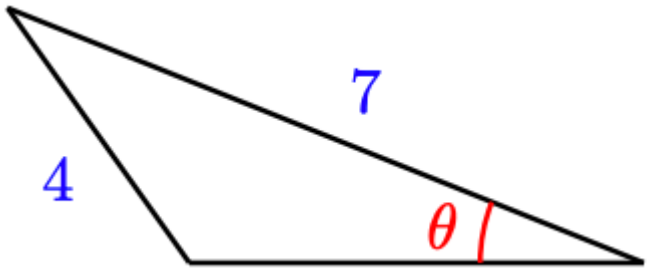
Homework

Vocabulary Check

1. The sequences of natural numbers, (3, 4, 5) and (7, 24, 25), are both examples of ___.
2. Just like the tangent and cotangent, the secant and cosecant are called ___.

Concept Check

3. Sketch a figure that illustrates why $\cos 25^\circ$ is the same for every right triangle with a 25° angle.
4. Sketch a figure that illustrates why $\cos\theta$ decreases as θ increases from 0° to 90°.
5. Linda says that $\sin\theta = \frac{4}{7}$ in the figure. Is she correct? Why or why not?



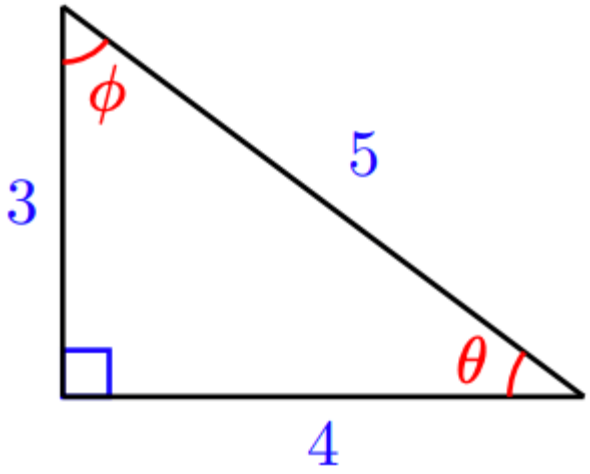
True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

6. Given a triangle having θ as one of its interior angles, the tangent of the angle θ is the ratio of the side opposite θ to the side adjacent to θ.
7. $\sin(60^\circ) = 0.8660$
8. The value of a trigonometric function of an angle is equal to the value of its cofunction at the complement of the angle.

Basic Skills

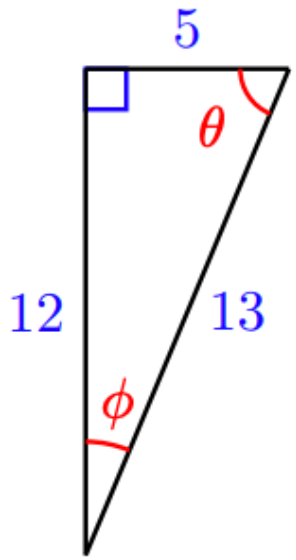
For the following exercises, fill in the table.

9.



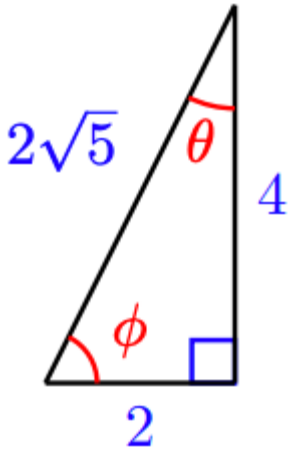
	sin	cos	tan
θ			
ϕ			

10.



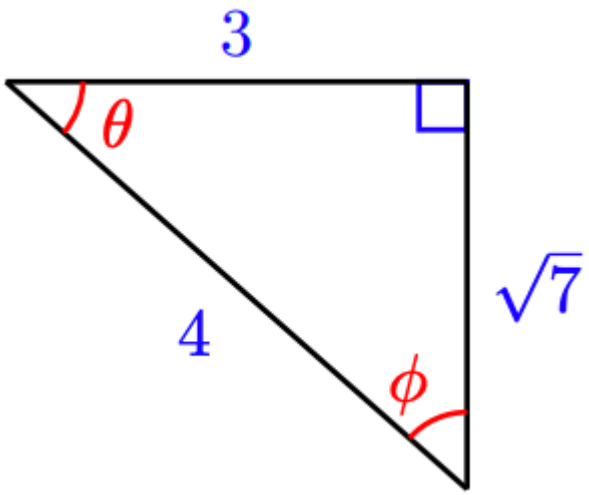
	sin	cos	tan
θ			
ϕ			

11.



	sin	cos	tan
θ			
ϕ			

12.



	sin	cos	tan
θ			
ϕ			

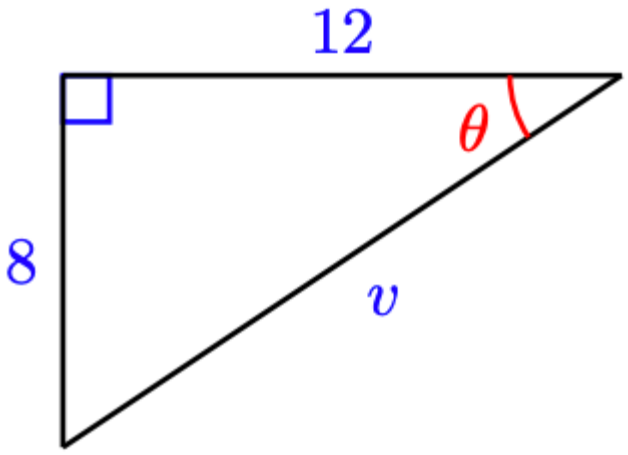
13. In each of the figures for the previous exercises, what is the relationship between the angles θ and ϕ?

14. Study the tables for the previous problems. What do you notice about the values of the sine and cosine for the angles θ and ϕ? Explain why this is true.

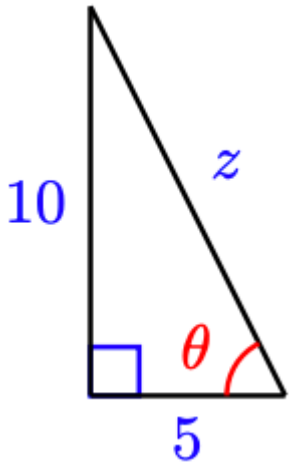
15. There is a relationship between the tangent, the sine, and the cosine of any angle. Study the tables for the previous problems to discover this relationship. Write your answer as an equation.

For the following exercises,
(a) Find the length of the unknown side.
(b) Find the sine, cosine, and tangent of θ. Round your answers to four decimal places.

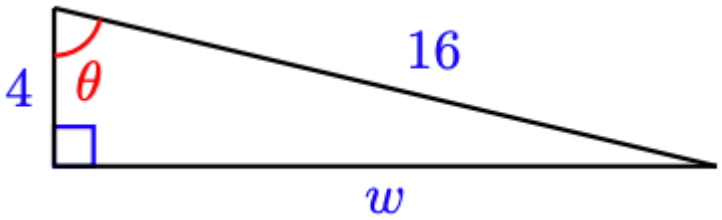
16.



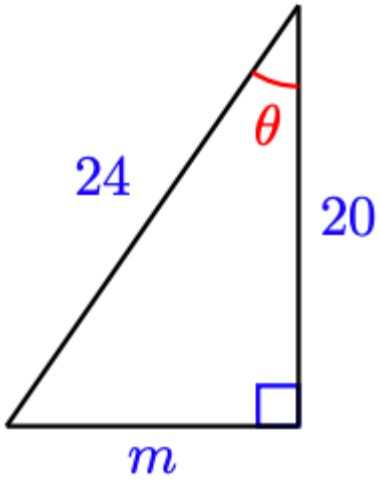
17.



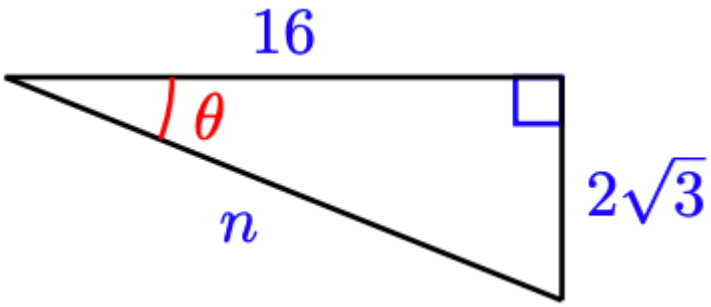
18.



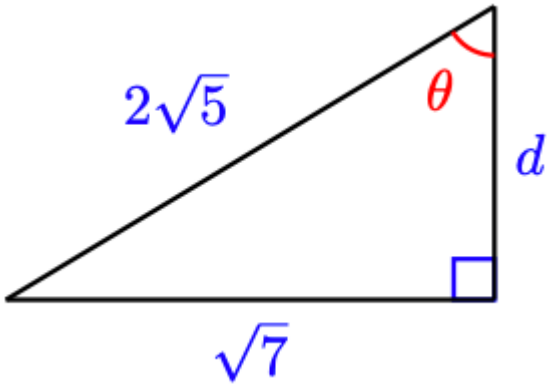
19.



20.

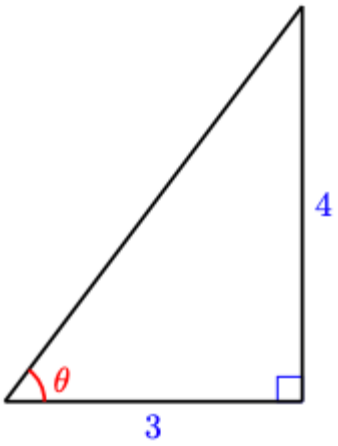


21.

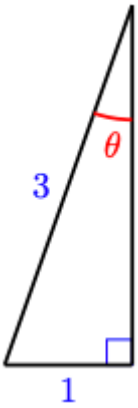


For the following exercises, find exact values for the six trigonometric ratios of the angle θ.

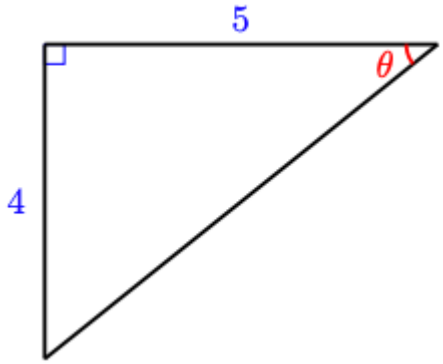
22.



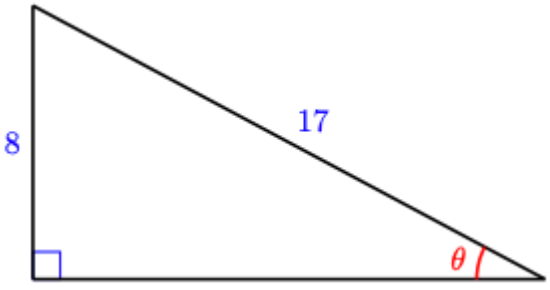
23.



24.



25.



For the following exercises,
(a) Sketch and label the sides of a right triangle with angle θ.

(b) Sketch and label another right triangle with angle θ and longer sides.

26. $\cos\theta = \frac{3}{5}$

27. $\tan\theta = \frac{7}{2}$

28. $\tan\theta = \frac{11}{4}$

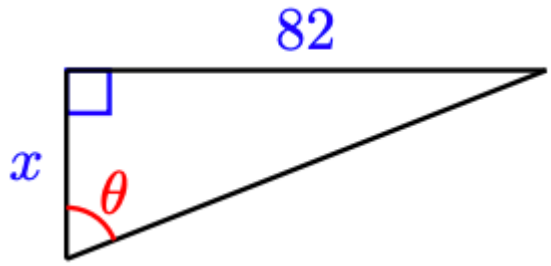
29. $\sin\theta = \frac{4}{9}$

30. $\sin\theta = \frac{1}{9}$

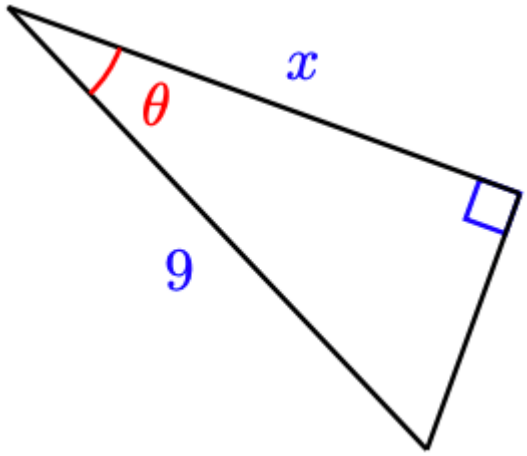
31. $\cos\theta = \frac{7}{8}$

For the following exercises, use a trigonometric ratio to write an equation for x in terms of θ.

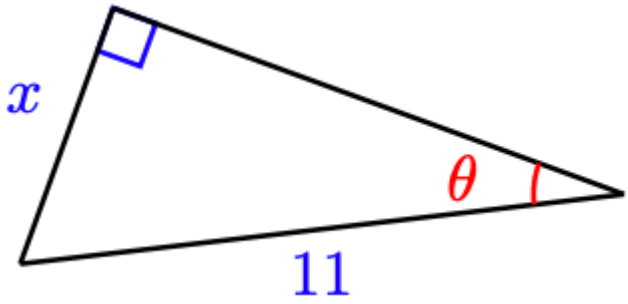
32.



33.

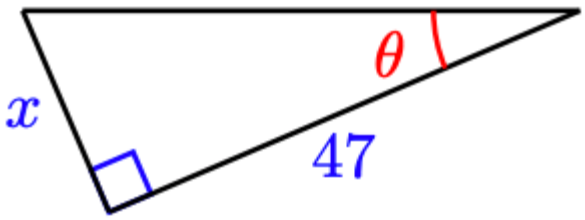


34.

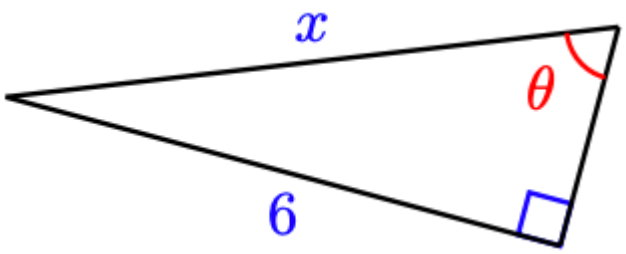


35.

36.



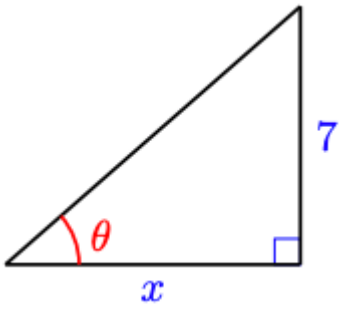
37.



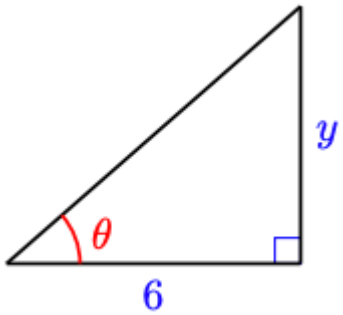
For the following exercises, write algebraic expressions for the six trigonometric ratios of the angle θ.

38.

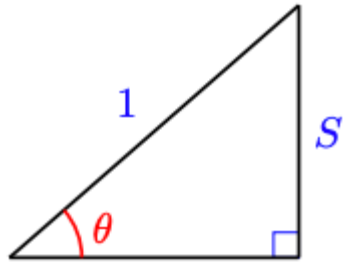
39.

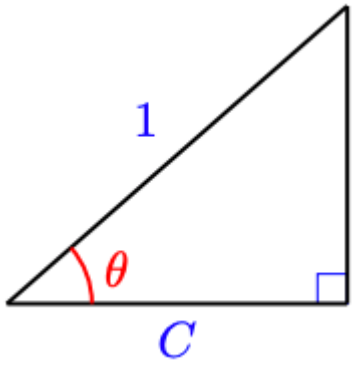


40.



41.





For the following exercises, write the expression in terms of sine and cosine, and simplify.

42. $\dfrac{\csc t}{\cot t}$

43. $\dfrac{\tan v}{\sec v}$

44. $\sec \beta - \tan \beta$

45. $\cot \alpha + \csc \alpha$

For the following exercises, choose all values from the list below that are exactly equal to, or decimal approximations for, the given trigonometric ratio. (Try not to use a calculator!)



46. $\cos 30^\circ$

47. $\sin 45^\circ$

48. $\tan 30^\circ$

49. $\cos 60^\circ$

For the following exercises, evaluate. Give exact values.

50. $\sin(30^\circ)$

51. $\tan(45^\circ)$

52. $\sin(60^\circ)$

53. $\csc(30^\circ)$

54. $\cot(45^\circ)$

55. $\csc(60^\circ)$

Synthesis Questions

For the following exercises, evaluate.

56. $4\cot(60^\circ) + 2\sec(45^\circ)$

57. $\frac{1}{2}\csc(30^\circ) - \frac{1}{4}\cot(30^\circ)$

For the following exercises, solve the equation assuming that θ is an acute angle in a right triangle.

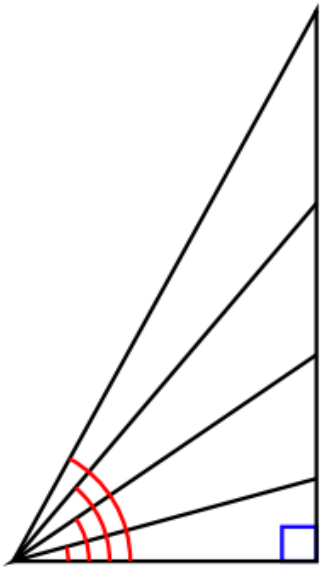
58. $3\csc(\theta) + 2 = 8$

59. $-2\sec(\theta) + 8 = 6$

60. $\sqrt{2}\sec(\theta) = 2$

61. $\sqrt{3}\cot(\theta) = 1$

62.

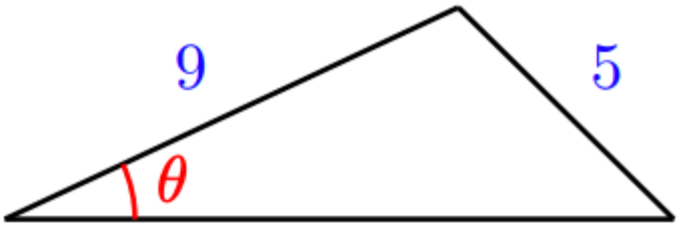


a. Use the figure to explain what happens to $\tan\theta$ as θ increases, and why.

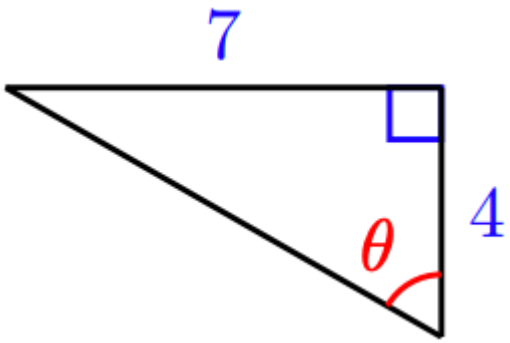
b. Use the figure to explain what happens to $\cos\theta$ as θ increases, and why.

For the following exercises, explain why the trigonometric ratio is *not* correct.

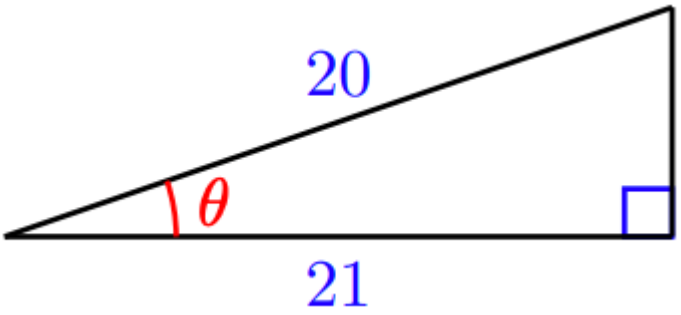
63. $\sin\theta = \frac{5}{9}$



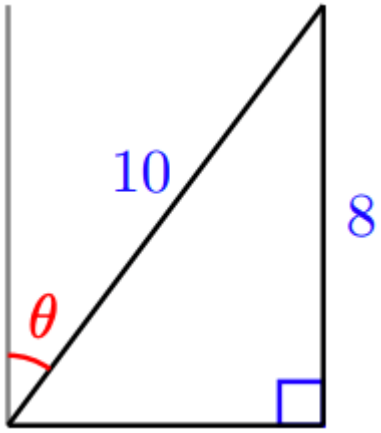
54. $\tan\theta = \frac{4}{7}$



55. $\cos\theta = \frac{21}{20}$



56. $\sin\theta = \frac{8}{10}$



For the following exercises, sketch and label a right triangle, then fill in the blank.

57. a. If $\sin\theta = 0.2358$, then $\cos(90^\circ - \theta) =$_____.

b. If $\cos\alpha = \frac{3}{11}$, then _____ $(90^\circ - \alpha) = \frac{3}{11}$.

c. If $\sin 42^\circ = n$, then $\cos$_____ $= n$.

d. If $\cos 13^\circ = z$, then $\sin$_____ $= z$.

58. a. If $\cos\beta = \frac{2}{\sqrt{7}}$, then $\sin(90^\circ - \beta) =$_____.

b. If $\sin\phi = 0.693$, then _____$-(90^\circ - \phi) = 0.693$.

c. If $\cos 87^\circ = p$, then $\sin$_____ $= p$.

d. If $\sin 59^\circ = w$, then $\cos$_____ $= w$.

59. a. If $\sin\phi = \frac{5}{13}$ and $\cos\phi = \frac{12}{13}$, then $\tan\phi =$_____.

b. If $\cos\beta = \frac{1}{\sqrt{10}}$, and $\sin\beta = \frac{3}{\sqrt{10}}$, then $\tan\beta =$_____.

c. If $\tan B = \frac{2}{\sqrt{5}}$ and $\cos B = \frac{\sqrt{5}}{3}$, then $\sin B =$_____.

d. If $\sin W = \sqrt{\frac{3}{7}}$ and $\tan W = \frac{\sqrt{3}}{2}$, then $\cos W =$_____.

60. a. If $\cos\theta = \frac{2}{\sqrt{10}}$ and $\sin\theta = \sqrt{\frac{3}{5}}$, then $\tan\theta =$_____.

b. If $\sin\alpha = \frac{\sqrt{2}}{4}$, and $\cos\alpha = \frac{\sqrt{14}}{4}$, then $\tan\alpha =$_____.

c. If $\tan A = \frac{\sqrt{7}}{3}$ and $\cos A = \frac{3}{4}$, then $\sin A =$_____.

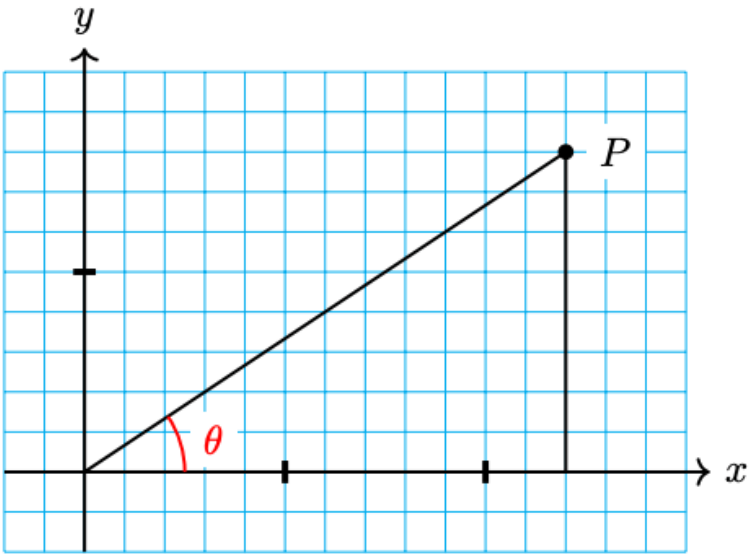
d. If $\sin V = \sqrt{\frac{10}{5}}$ and $\tan V = \frac{2}{5}$, then $\cos V =$_____.

71. Explain why the cosine of a 73° angle is always the same, no matter what size triangle the angle is in. Illustrate your explanation with a sketch.

72. a. What is the slope of the line through the origin and point P?

b. What is the tangent of the angle θ?

c. On the same grid, sketch an angle whose tangent is $\frac{8}{5}$



73. Which of the following numbers are equal to $\cos 45^\circ$?

a. $\frac{\sqrt{2}}{2}$

b. $\frac{1}{\sqrt{2}}$

c. $\frac{2}{\sqrt{2}}$

d. $\sqrt{2}$

74. Which of the following numbers are equal to $\tan 30^\circ$?

a. $\sqrt{3}$

b. $\frac{1}{\sqrt{3}}$

c. $\frac{\sqrt{3}}{3}$

d. $\frac{3}{\sqrt{3}}$

75. Which of the following numbers are equal to $\tan 60^\circ$?

a. $\sqrt{3}$

b. $\frac{1}{\sqrt{3}}$

c. $\frac{\sqrt{3}}{3}$

d. $\frac{3}{\sqrt{3}}$

76. Which of the following numbers are equal to $\sin 60^\circ$?

a. $\frac{3}{\sqrt{2}}$

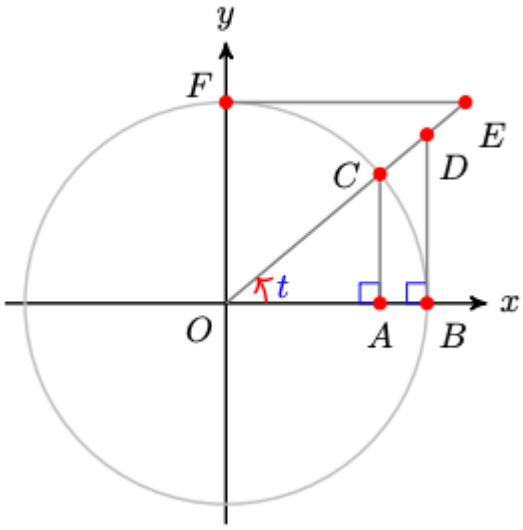
b. $\frac{\sqrt{3}}{2}$

c. $\frac{\sqrt{2}}{3}$

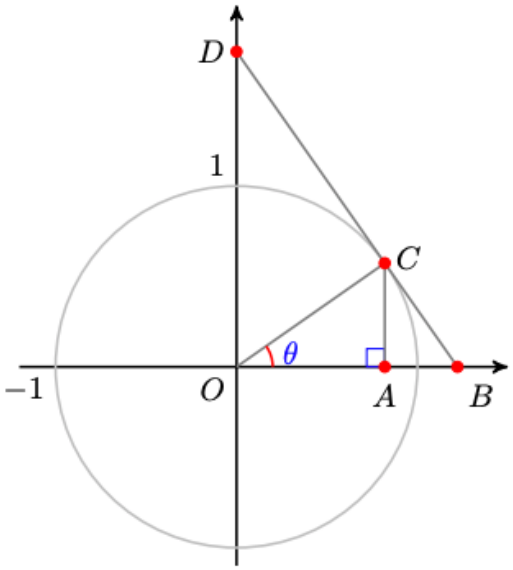
d. $\frac{2}{\sqrt{3}}$

Challenge Problems

77. The diagram shows a unit circle. Find six line segments whose lengths are, respectively, $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.



78. The figure shows a unit circle and an angle θ in standard position. Each of the six trigonometric ratios for θ is represented by the length of a line segment in the figure. Find the line segment for each ratio, and explain your choice.



Answers

Skills Refresher

(Many answers are possible for 1-4.)

1. $\frac{5}{2}, \frac{20}{8}$
2. $\frac{3}{4}, \frac{12}{16}$
3. $\frac{3}{5}, \frac{12}{20}$
4. $\frac{6}{4}, \frac{12}{8}$
5. $\frac{2}{5}$
6. $\frac{-8}{5}$
7. $\frac{-6}{5}$
8. $\frac{1}{2}$
9. $\frac{1}{4}$
10. $\frac{3}{4}$
11. $r = \frac{C}{2\pi}$
12. $t = \frac{d}{r}$
13. $a = \frac{P-b+c}{2}$
14. $y = \frac{3}{2}(x - 10)$
15. $h = \frac{S-2\pi r^2}{2\pi r}$

This page titled 6.1.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

6.2: Technology and Right Triangle Trigonometry

Note to the Instructor (click to expand)

We finally reach the moment where we are evaluating trigonometric functions at non-special *acute* angles. As such, technology (e.g., Desmos or a calculator) will be necessary. This will also be needed to find the *acute* angle within a right triangle given the sides of the triangle.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Use a calculator to approximate the value of a trigonometric function.
- Use a calculator to approximate an acute angle given the value of a trigonometric function.

Hawk A.I. Section-Specific Tutor



COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Using a Calculator to Compute the Values of Trigonometric Functions

The trigonometric values for non-special angles were originally collected into tables and are available on scientific calculators (and other, more recent, technology). For example, let's find the sine of 50°.

Note: Change to Degree Mode

Before evaluating any trigonometric function, ensure your technology is set to the correct "mode." In Trigonometry, the two standard modes are **degree mode** and **radian mode**. At this point, we will only be working with angles measured in degrees, so you must only concern yourself with degree mode. If using a scientific or graphing calculator, refer to your instruction manual on setting the calculator to degree mode. The scientific calculator in Desmos is set to degree mode by default, so you do not have to make any adjustments if using that technology; however, the *graphing* calculator in Desmos defaults to radian mode. To change Desmos' graphing calculator to degree mode, open the "Graph Settings" and select "Degrees."

Example 6.2.1

Approximate $\sin(50^\circ)$.

Scientific and Graphing Calculators available at the end of this section.

It is important to note that the values returned by the technology used in Example $6.2.1$ are not exact values; it turns out that the sine of 50° is an irrational number, and your calculator shows as many digits as its display will allow (not all sine values are as "nice" as the sine of 30°). Unless otherwise instructed, we round to four decimal places, so we write

$$\sin(50^\circ) \approx 0.7660.$$

Take note of using the approximation sign, $\approx$, instead of the equal sign - this is important!

Checkpoint 6.2.1

a. Use a calculator to complete the table, rounding answers to four decimal places.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	89°
$\tan(\theta)$										

b. What do you notice about the values of $\tan(\theta)$ as θ increases from 0° to 89°? If the values of $\tan(\theta)$ are plotted against the values of θ, will the graph be a straight line? Why or why not?

Answers available at the end of this section.

Example 6.2.2

Use technology to compute the value of each trigonometric function.

a. $\cos(23.7^\circ)$
b. $\tan(72.13^\circ)$
c. $\sin^2(10^\circ)$

Solutions available at the end of this section.

Caution: Save Rounding for the Final Answer

Example $6.2.2c$ demonstrates a significant point that bears repeating - **rounding is for display purposes only!** Never round a result and then use that rounded result for other computations.

For example, had we rounded our approximation of $\sin(10^\circ)$ to be 0.1736 and then squared, we would have incorrectly stated $\sin^2(10^\circ) \approx 0.0301$. While the difference between this result and the more accurate result of 0.0304 may seem insignificant, these rounding errors can propagate in such a way as to make massive differences between answers.

Checkpoint 6.2.2

a. Use a calculator to complete the table. Round the values of sine and cosine to four decimal places.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$\sin(\theta)$										
$\cos(\theta)$										

b. Is there a pattern to the values of sine and cosine? Why this is true? (Hint: If one (non-right) angle of a right triangle measures x degrees, how big is the other angle? Sketch the triangle and label each angle's opposite and adjacent sides.)

Answers available at the end of this section.

Now that we have access to technology to approximate the value of trigonometric functions at non-special angles, we can revisit simplifying and evaluating trigonometric expressions.

Example 6.2.3

Simplify, and evaluate for $z = 40^\circ$.

$$3\sin(z) - \sin(z)\tan(z) + 3\sin(z)$$

Solution available at the end of this section.

Checkpoint 6.2.3

Simplify, and evaluate for $x = 25^\circ, y = 70^\circ$.

$$3\cos(x) + \cos(y) - 2\cos(y) + \cos(x)$$

Answer available at the end of this section.

Unlike the fundamental trigonometric functions, calculators do not have keys for the cosecant, secant, and cotangent functions. Instead, we calculate their values as reciprocals.

Example 6.2.4

Use technology to approximate $\sec(47^\circ)$ to three decimal places.

Solution available at the end of this section.

Checkpoint 6.2.4

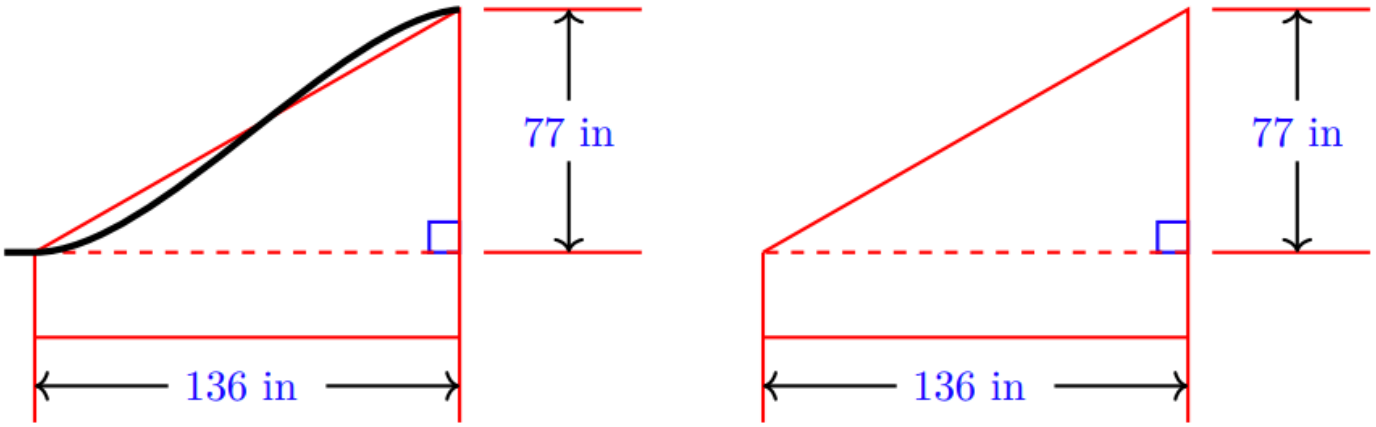
Use technology to approximate $\csc(82^\circ)$ to three decimal places.

Answer available at the end of this section.

Finding an Acute Angle Using a Calculator

While watching her niece at the playground, Nhat wonders how steep the slide is. She takes out her measuring tape and finds the slide to be 77 inches high over a horizontal distance of 136 inches, as shown below.

Figure 6.2.1



Nhat knows that one way to describe the steepness of an incline is to calculate its slope, which, in this case, is

$$\frac{\Delta y}{\Delta x} = \frac{77}{136} \approx 0.5662.$$

However, she wants to know what angle the slide makes with the horizontal. Thinking about it for a moment, she realizes that the slope she had just calculated is also the tangent of the angle she wants (can you see this from the figures above?). This begs the question: If we know the tangent of an angle, can we find the angle? The answer is yes!

Locate the key labeled TAN^{-1} on your calculator; it is probably the second function above the TAN key. Enter

$$\text{2nd} \quad \text{TAN} \quad 0.5662$$

to find that

$$\tan^{-1}(0.5662) \approx 29.5185^\circ.$$

This means that 29.5185° is the approximate angle whose tangent is 0.5662. We read the notation, $\tan^{-1}(0.5662) \approx 29.5185^\circ$ as "**inverse tangent** of 0.5662 is approximately 29.5185 degrees."

Note: Use Degree Mode

Until later in this course, you should make sure your calculator is set to "degree" mode. If you do not do this, your answers, while correct in a different angular measure system, will be incorrect.

Caution: Do Not Use Rounded Values for Calculations

Before describing the inverse tangent, we must note the flaw in our work. We *approximated* the value of the fraction, $\frac{77}{136}$, and then used that approximation to approximate the angle - this is very bad form. In truth, we should not have approximated $\frac{77}{136}$, but instead just computed

$$\tan^{-1}\left(\frac{77}{136}\right) \approx 29.5175^\circ.$$

Do you see that the returned angle from the inverse tangent is slightly different from our calculations using the rounded approximation of the fraction?

The moral of the story is that, whenever possible, **do not use rounded or approximate values to get yet another approximation** - this leads to a propagation of rounding errors.

When we find $\tan^{-1}$ of a number, we find an angle whose tangent is that number. Similarly, $\sin^{-1}$ and $\cos^{-1}$ are read as "**inverse sine**" and "**inverse cosine**." They find an angle with the given sine or cosine.

Example 6.2.5

Find the angle whose sine is 0.6834.

Solution available at the end of this section.

Note: Equivalent Statements

In Example $6.2.5$, the two equations

$$\sin(43.11^\circ) \approx 0.6834 \quad \text{and} \quad \sin^{-1}(0.6834) \approx 43.11^\circ$$

say the same thing in different ways.

Caution: Inverse Trigonometric Functions are NOT Reciprocals

The notation $\sin^{-1}(x)$ does *not* mean $\frac{1}{\sin(x)}$!

We indeed use negative exponents to indicate reciprocals of numbers, for example, $a^{-1} = \frac{1}{a}$ and $3^{-1} = \frac{1}{3}$. However, "sin" by itself is not a variable.

- $\sin^{-1}(x)$ means "the angle whose sine is x"
- $\frac{1}{\sin(x)}$ means "the reciprocal of the sine of angle x"

(You may recall that $f^{-1}(x)$ denotes the inverse function for $f(x)$. We will study inverse trigonometric functions in a later chapter.)

Checkpoint 6.2.5

Write the following fact in two different ways: 68° is the angle whose cosine is 0.3746.

Answer available at the end of this section.

Note: Inputs and Outputs of Inverse Trigonometric Functions

Without going into too much detail (yet), it is essential to note that an inverse trigonometric function's input (a.k.a., the argument) is a ratio. The output is an angle.

Example 6.2.6

Find the acute angle for which the equation is true. Round answers to the nearest tenth.

a. $\cos(\beta) = 0.28$
b. $\tan(\kappa) = 1.93$

Solutions available at the end of this section.

Checkpoint 6.2.6

Find the acute angle for which $\tan(\theta) = 1.23$. Round your answer to the nearest tenth.

Answer available at the end of this section.

Finding acute angles is slightly more complex when the equation involves the cosecant, secant, or cotangent. A good rule of thumb is to rewrite the equation in terms of the fundamental trigonometric functions using the Reciprocal Identities.

Example 6.2.7

Suppose x is an acute angle within a right triangle. Solve each equation. Round answers to two decimal places.

a. $\sec(x) = 8.104$
b. $3\cot(x) - 2 = 7$

Solutions available at the end of this section.

Checkpoint 6.2.7

Let ω be an acute angle within a right triangle. Solve $2\csc(\omega) + 3 = 11$. Round the answer to two decimal places.

Answer available at the end of this section.

Activities

Activity: Trigonometric Ratios

A. Using Ratios and Proportions

Recall that two related quantities or variables are proportional if their ratio is always the same.

1. a. On any given day, the cost of filling up your car's gas tank is proportional to the number of gallons of gas you buy. For each purchase below, compute the ratio

$$\frac{\text{total cost of gasoline}}{\text{number of gallons}}$$

Gallons of Gas Purchased	Total Cost	$\frac{\text{Dollars}}{\text{Gallon}}$
5	\$14.45	
12	\$34.68	
18	\$52.02	

   b. Write an equation that you can solve to answer the question: How much does 21 gallons of gas cost? Use the ratio $\frac{\text{Dollars}}{\text{Gallon}}$ in your equation.

   c. Write an equation that you can solve to answer the question: How many gallons of gas can you buy for $\$46.24$? Use the ratio $\frac{\text{Dollars}}{\text{Gallon}}$ in your equation.

2. A recipe for coffee cake calls for $\frac{3}{4}$ cup of sugar and $1\frac{3}{4}$ cup of flour.

   a. What is the ratio of sugar to flour? Write your answer as a common fraction, then give a decimal approximation rounded to four places.

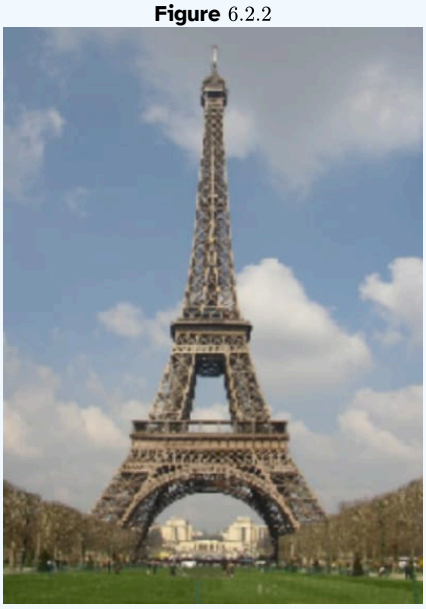
   For parts (b) and (c) below, write an equation that you can solve to answer the question. Use the ratio $\frac{\text{Amount of sugar}}{\text{Amount of flour}}$ your equation.

   b. How much sugar should you use if you use 4 cups of flour? Compute your answer in two ways: write the ratio as a common fraction and then write the ratio as a decimal approximation. Are your answers the same?

   c. How much flour should you use if you use 4 cups of sugar? Compute your answer in two ways: write the ratio as a common fraction and then write the ratio as a decimal approximation. Are your answers the same?

3. You are making a scale model of the Eiffel Tower, which is 324 meters tall and 125 meters wide at its base.

Figure 6.2.2



a. Compute the ratio of the base width to the tower's height. Round your answer to four decimal places.

Use your ratio to write equations and answer the questions below:

b. If the base of your model is 8 inches wide, how tall should the model be?

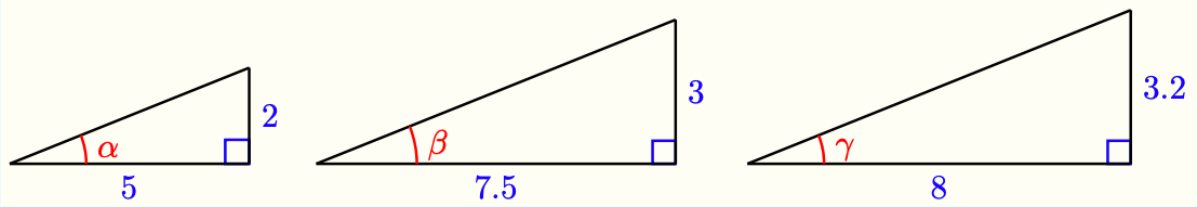
c. If you make a larger model that is 5 feet tall, how wide will the base be?

B. Similar Triangles

1. Recall that two triangles are similar if their corresponding sides are proportional. The corresponding angles of similar triangles are equal.

a. What is the ratio of the two given sides in each triangle? Are the corresponding sides of the three triangles proportional? How do we know that $\alpha = \beta = \gamma$?

Figure 6.2.3

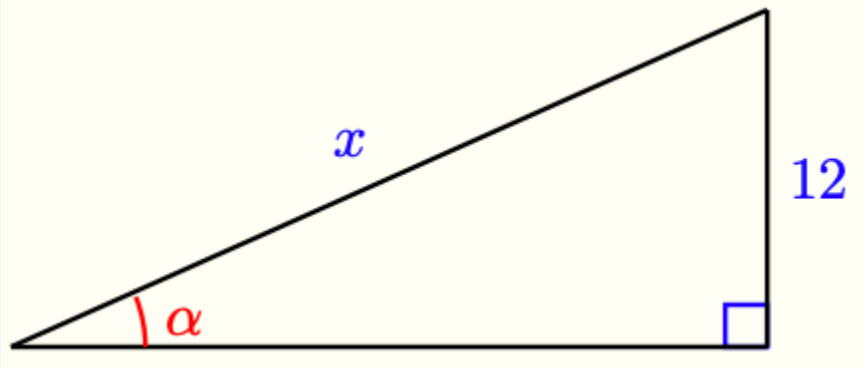


b. Find the hypotenuse of each right triangle.

c. Use the sides of the appropriate triangle to compute $\sin\alpha, \sin\beta$, and $\sin\gamma$. Round your answers to four decimal places. Does the sine of an angle depend on the lengths of its sides?

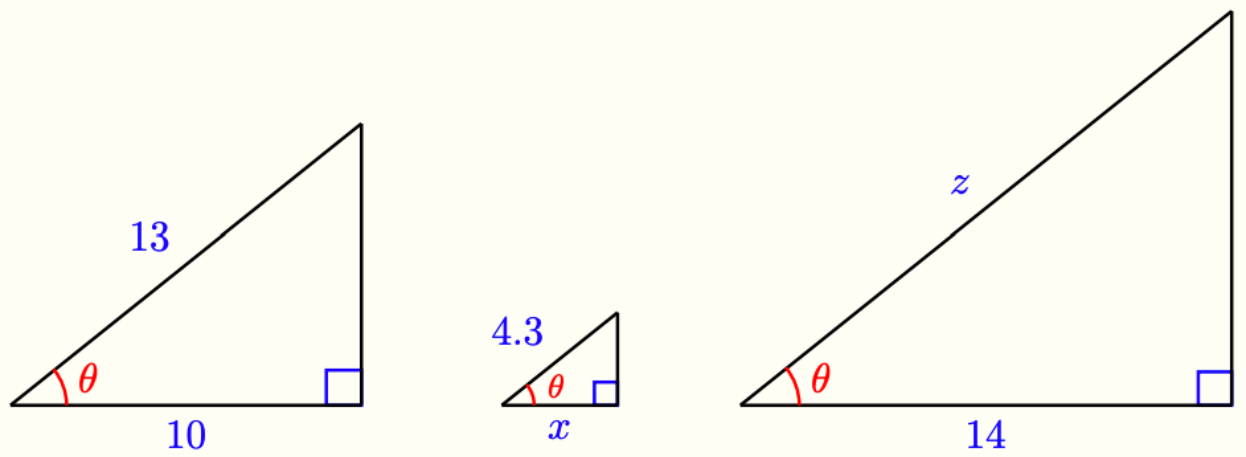
d. How do you know that the triangle below is similar to the three triangles in part (a)? Write an equation using the ratio from part (c) to find x.

Figure 6.2.4



2. In the three right triangles below, the angle θ is the same size.

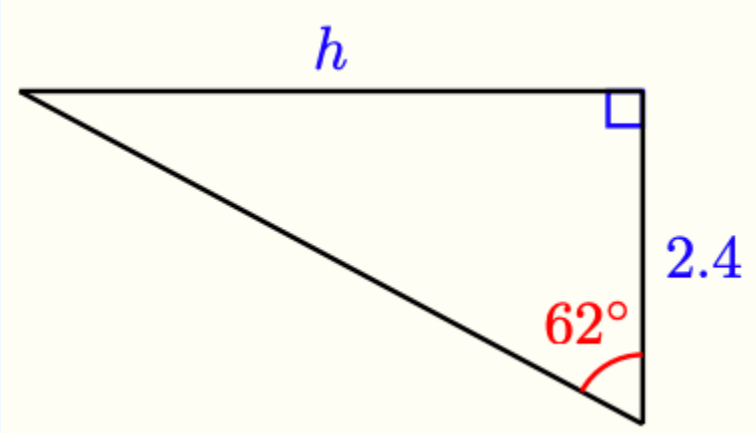
Figure 6.2.5



   a. Use the first triangle to calculate $\cos\theta$. Round your answer to four decimal places.
   b. In the second triangle, explain why $\frac{x}{4.3} = \frac{10}{13}$. Write an equation using your answer to part (a) and solve it to find x.
   c. Write and solve an equation to find z in the third triangle.

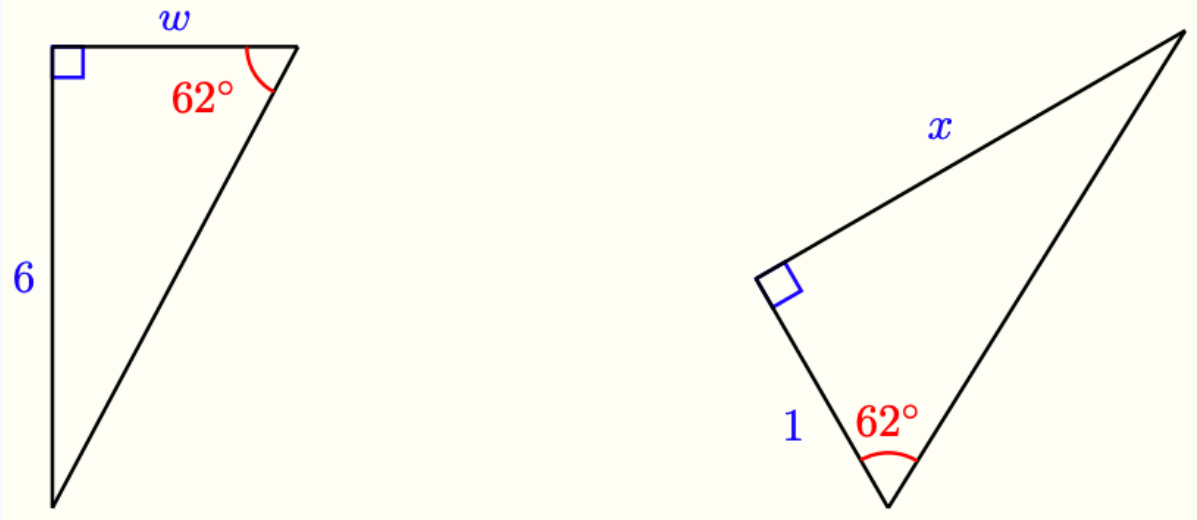
3. a. Use your calculator to find the value of $\frac{h}{2.4}$. (Hint: Which trigonometric ratio should you use?) What is the length of side h?

Figure 6.2.6



   b. What is the value of $\frac{6}{w}$ for the triangle below? Write an equation and solve for w.

Figure 6.2.7

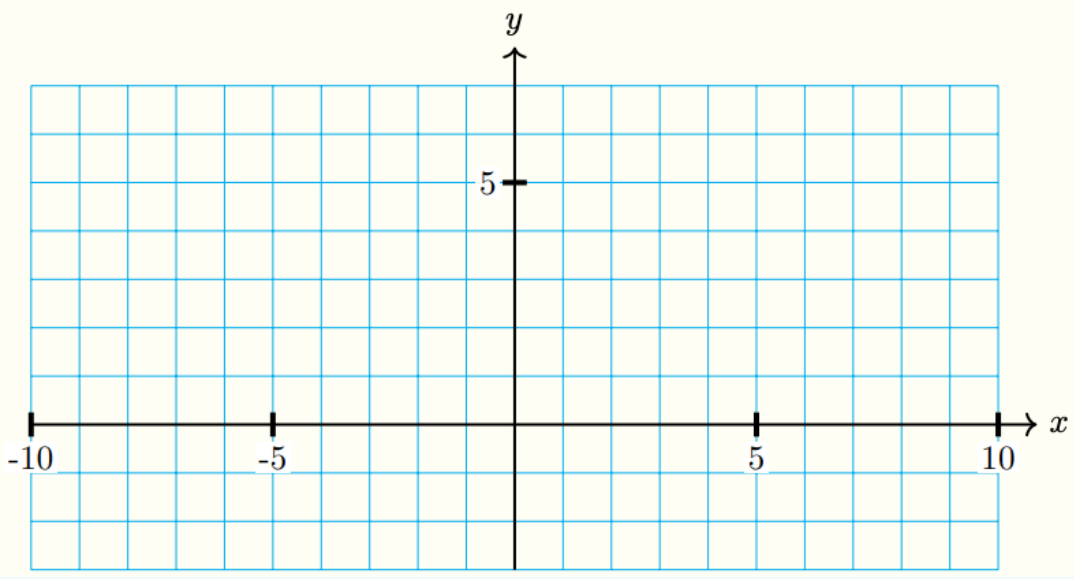


c. Write an equation and solve it to find x in the triangle above.

Activity: Obtuse Angles

An obtuse angle has measure between 90° and 180°, exclusively. In this activity, we will investigate the trigonometric functions of an obtuse angle.

Figure 6.2.8



a. Draw an angle θ in standard position with the point $P(6, 4)$ on its terminal side.
b. Find r, the distance from the origin to P.
c. Calculate all six trigonometric functions of θ. Give both exact answers and decimal approximations rounded to four places.
d. Use the inverse cosine key on your calculator to find θ. Use your calculator to verify the values of $\sin(\theta), \cos(\theta)$, and $\tan(\theta)$ that you found in part (c).
e. Draw another angle ϕ in standard position with the point $Q(-6, 4)$ on its terminal side. Explain why ϕ is the supplement of θ. (Hint: Consider the right triangles formed by drawing vertical lines from P and Q).
f. Can you use the right triangle definition of the trigonometric functions to compute the sine and cosine of ϕ? Why or why not?

g. Calculate all six trigonometric functions of ϕ. How are these trigonometric values of ϕ related to the trigonometric values of θ?
h. Explain why θ and ϕ have the same sine but different cosines.
i. Use the inverse cosine key on your calculator to find ϕ. Use your calculator to verify the values of $\sin(\phi), \cos(\phi)$, and $\tan(\phi)$ that you found in part (g).
j. Compute $180^\circ - \phi$. What answer should you expect to get?

Answers

Example 6.2.1

Press the SIN button. Your calculator will display

$$\sin($$

with an open parenthesis as the prompt to enter an angle. Type 50 and press ENTER (since your calculator should already be in degree mode, it will interpret the 50 as 50°). The calculator returns 0.7660444431 (please verify).

Using the scientific calculator in Desmos, type $\sin(50)$ and hit ENTER (since the scientific calculator in Desmos defaults to degree mode, it will interpret the 50 as 50°). Desmos returns 0.7660444431 (please verify).

Checkpoint 6.2.1

a.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$\tan(\theta)$	0	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321	2.7475	5.6713	57.2900

b. The values of $\tan(\theta)$ increase seemingly without bound as θ increases from 0° to 89°. The graph will not be a line because the slopes between successive points are not constant.

Example 6.2.2

a. $\cos(23.7^\circ) \approx 0.9156625933 \approx 0.9157$
b. $\tan(72.13^\circ) \approx 3.101611189 \approx 3.1016$
c. To compute $\sin^2(10^\circ)$, we need to recall that

$$\sin^2(10^\circ) = [\sin(10^\circ)]^2.$$

Therefore, we start by getting the approximation $\sin(10^\circ) \approx 0.1736481777$. Using as many of these digits as possible, we then square the result to get

$$\sin^2(10^\circ) \approx (0.1736481777)^2 \approx 0.03015368961 \approx 0.0304.$$

Checkpoint 6.2.2

a.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$\sin(\theta)$	0	0.1737	0.3420	0.5	0.6428	0.7660	0.8660	0.9397	0.9848	1
$\cos(\theta)$	1	0.9849	0.9397	0.8660	0.7660	0.6428	0.5	0.3420	0.1737	0

b. The cosine of θ is equal to the sine of the complement of θ, or $\cos(\theta) = \sin(90 - \theta)$. This result confirms the Cofunction Identity from the previous section.

Example 6.2.3

We can combine like terms to get

$$6\sin(z) - \sin(z)\tan(z)$$

Because 40° is not one of the angles for which we know exact trig values, we use a calculator to evaluate the expression. It is *very* important to make sure we save any rounding until the very end of the computation. Therefore, we enter the following keystrokes into our calculator (or other technology):

$$6 \quad \times \quad \text{SIN} \quad 40 \quad) \quad - \quad \text{SIN} \quad 40 \quad) \quad \times \quad \text{TAN} \quad 40 \quad \text{ENTER}$$

Doing so, we get approximately 3.3174.

Checkpoint 6.2.3

3.2832

Example 6.2.4

Enter

$$1 \div \cos 47) \text{ ENTER}$$

to obtain $\sec(47^\circ) \approx 1.466$. Or we can calculate $\cos(47^\circ)$ first, and then use the reciprocal key:

$$\cos 47) \text{ ENTER } x^{-1} \text{ ENTER}$$

Checkpoint 6.2.4

1.010

Example 6.2.5

Enter

$$2\text{nd} \quad \text{SIN} \quad 0.6834$$

into your calculator to find

$$\sin^{-1}(0.6834) \approx 43.11^\circ.$$

So 43.11° is the angle whose sine is *approximately* 0.6834. Alternatively, we can say that

$$\sin(43.11^\circ) \approx 0.6834.$$

You can check the last equation on your calculator.

Checkpoint 6.2.5

$\cos(68^\circ) = 0.3746$ or $\cos^{-1}(0.3746) = 68^\circ$

Example 6.2.6

a. We use the following button sequence:

$$2\text{nd} \quad \text{COS} \quad 0.28 \quad \text{ENTER}.$$

This returns 73.73979529. Hence, $\beta \approx 73.7^\circ$.

b. Using the sequence

$$2\text{nd} \quad \text{TAN} \quad 1.93 \quad \text{ENTER},$$

we get 62.60975803. Hence, $\kappa \approx 62.6^\circ$.

Checkpoint 6.2.6

$\theta \approx 50.9^\circ$

Example 6.2.7

a. We first rewrite the equation as

$$\frac{1}{\cos(x)} = 8.104 \implies \cos(x) = \frac{1}{8.104}$$

We then enter

$$2\text{nd} \quad \text{COS} \quad 1 \quad \div \quad 8.104$$

into the calculator to find

$$\cos^{-1}\left(\frac{1}{8.104}\right) \approx 82.91^\circ.$$

Therefore, the solution to $\sec(x) = 8.104$ is $x \approx 82.91^\circ$.

b. The equation handed to us needs to be simplified into the form $\cot(\text{angle}) = \text{number}$ before we can find the angle.

$$\begin{aligned} & 3\cot(x) - 2 = 7 \\ \implies & 3\cot(x) = 9 \\ \implies & \cot(x) = 3 \end{aligned}$$

We now rewrite the cotangent as the tangent using the Reciprocal Identities.

$$\cot(x) = 3 \implies \frac{1}{\tan(x)} = 3 \implies \tan(x) = \frac{1}{3}$$

Finally, we grab our technology and enter

$$2\text{nd} \quad \text{TAN} \quad 1 \quad \div \quad 3$$

into the calculator to find

$$\tan^{-1}\left(\frac{1}{3}\right) \approx 18.43^\circ$$

Therefore, the solution to $3\cot(x) - 2 = 7$ is $x \approx 18.43^\circ$.

Checkpoint 6.2.7

$\omega \approx 14.48^\circ$

This page titled 6.2: Technology and Right Triangle Trigonometry is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

6.2.1: Resources and Key Concepts

Key Concepts

Common Mistakes

- **Using the Wrong Calculator Mode:** Before evaluating any trigonometric function, you must ensure your calculator is set to the correct mode (degree mode for this section). Using the wrong mode will result in incorrect answers.
- **Rounding Intermediate Steps:** Rounding should only be done for the final answer. If you round a value and then use that rounded value in another calculation, the rounding error can propagate and lead to a significantly different and less accurate final result.
- **Confusing Inverse Functions with Reciprocals:** The notation $\sin^{-1}(x)$ represents the inverse sine, which finds the angle whose sine is x. It does not mean the reciprocal, $\frac{1}{\sin(x)}$, which is $\csc(x)$.
- **Using Rounded Values in Inverse Functions:** When finding an angle, you should use the exact ratio if possible. For example, to find the angle whose tangent is $\frac{77}{136}$, you should compute $\tan^{-1}\left(\frac{77}{136}\right)$ directly rather than first approximating $\frac{77}{136}$ as a decimal and then taking the inverse tangent of the rounded decimal.

This page titled 6.2.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

6.2.2: Homework

Reading Questions

1. What calculator mode must you be in to evaluate trigonometric functions of angles given in degrees?
2. When a calculator displays a value for $\sin(50^\circ)$, is this an exact value or an approximation?
3. What does the text mean by "rounding is for display purposes only"?
4. How do you calculate the value of $\sec(47^\circ)$ on a calculator that does not have a SEC button?
5. What does the notation $\tan^{-1}(0.5662)$ represent?
6. What is the input for an inverse trigonometric function (an angle or a ratio)? What is the output?
7. Explain the difference in meaning between $\sin^{-1}(x)$ and $(\sin(x))^{-1}$.

Skills Refresher

Review the following skills you will need for this section.

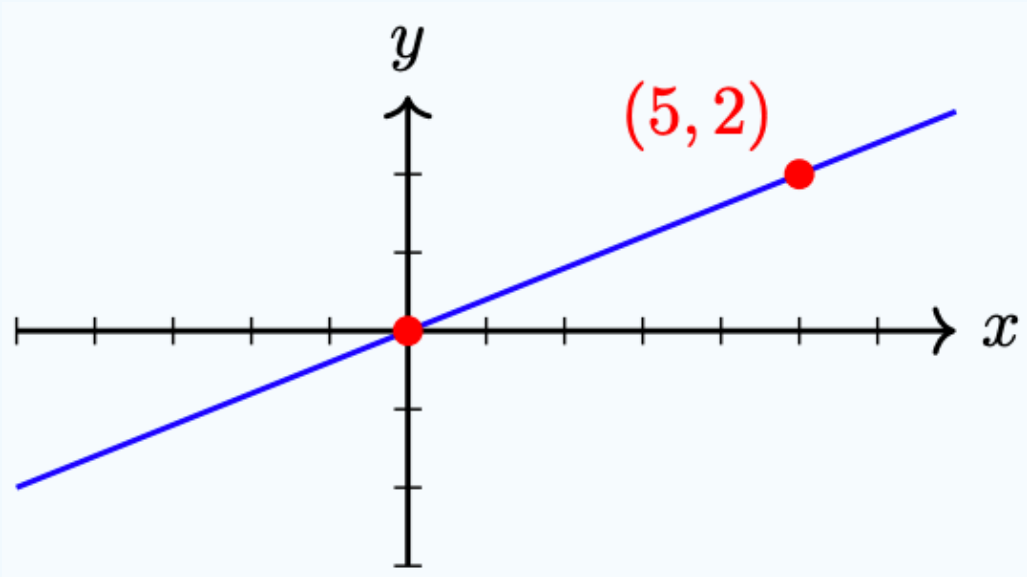
Skills Refresher

For the following exercises, write two more ratios equivalent to the given fraction.

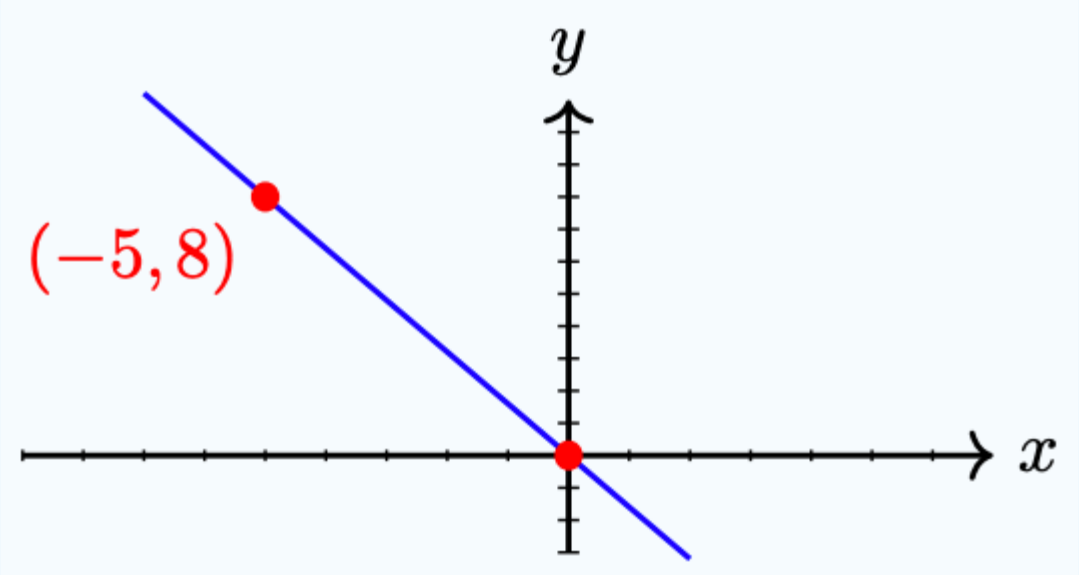
1. $\frac{10}{4}$
2. $\frac{6}{8}$
3. 0.6
4. 1.5

For the following exercises, compute the slope of the line.

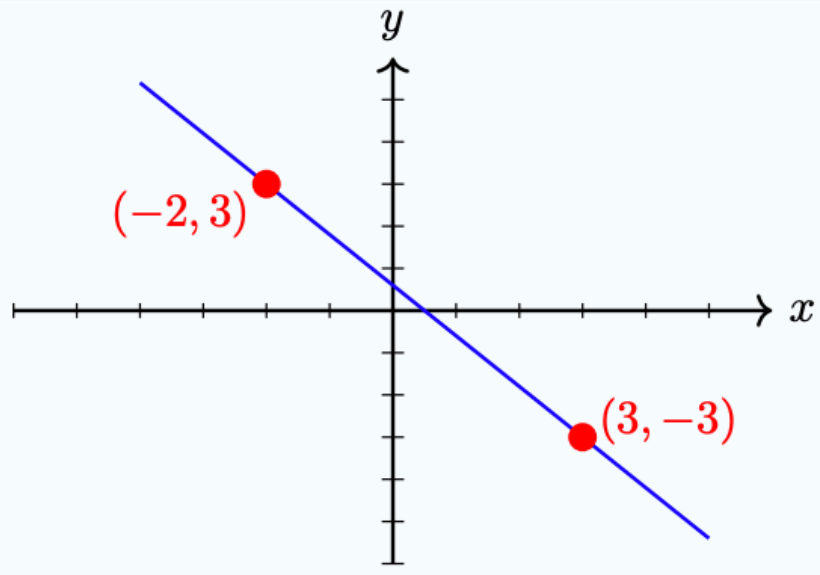
5.



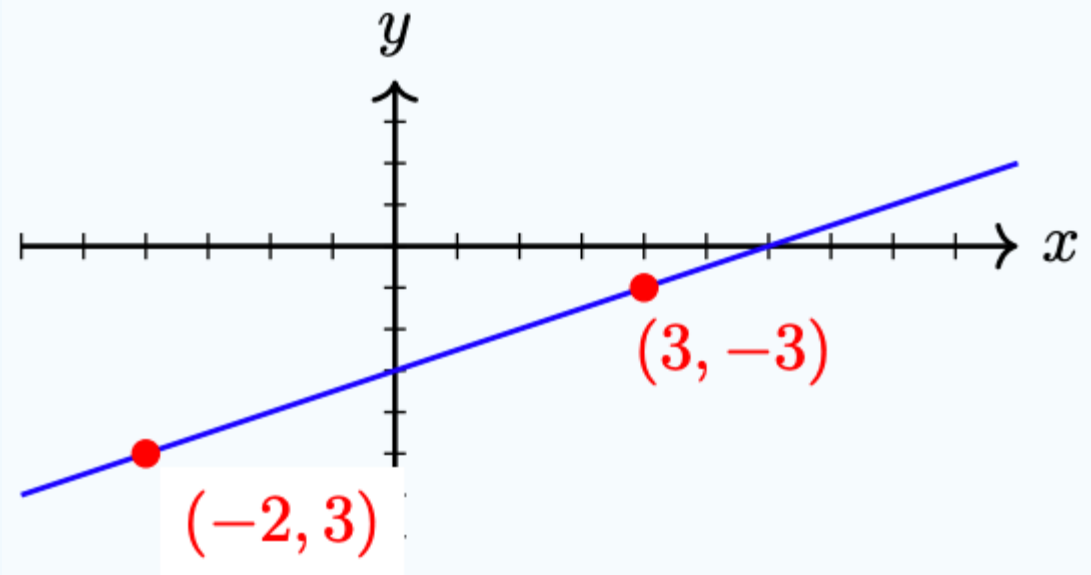
6.



7.



8.



For the following exercises, solve.

9. $\frac{12}{x} = 48$

10. $\frac{60}{x} = 80$

Answers available at the end of this section.

Homework

Vocabulary Check

1. Calculators and other technology have two modes that deal with the values of trigonometric functions - ___ mode and ___ mode. At this point, we only use ___ mode.

Concept Check

2. Does your calculator give you the exact decimal values for the trigonometric ratios of acute angles?

3. The SIN, COS, and TAN keys on a calculator (or other technology) returns

   the value of the trigonometric function an angle

   (circle one) when you input

   the value of the trigonometric function an angle

   (circle one).

4. The SIN^{-1}, COS^{-1}, and TAN^{-1} keys on a calculator (or other technology) returns

   the value of the trigonometric function an angle

   (circle one) when you input

   the value of the trigonometric function an angle

   (circle one).

5. Min and Paige try to approximate $2.1\tan(37^\circ)$. Min types the following into her calculator

$$2.1 \quad \times \quad \text{TAN} \quad 37 \quad \text{ENTER},$$

and the calculator returns approximately 1.5825; however, Paige types

$$\text{TAN} \quad 37 \quad \times \quad 2.1 \quad \text{ENTER}$$

into her calculator and gets an approximation of 4.5864. They each have their calculators set to Degree Mode, so that's not the issue. Who has the wrong approximation? If she cannot change the order of what she types into her calculator, but *can* add or remove a button press, what can she do to fix her error?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

6. $\cot^{-1}(\theta) = \dfrac{1}{\cot(\theta)} = \tan(\theta)$.

7. $\cos^{-1}(\theta) = \sin(\theta)$

Basic Skills

For the following exercises, evaluate. Round answers to 3 decimal places.

8. $\sin(53^\circ)$

9. $\cos(18^\circ)$

10. $\tan(5^\circ)$

11. $\sin(85^\circ)$

12. $\csc(27^\circ)$

13. $\sec(8^\circ)$

14. $\cot(65^\circ)$

15. $\csc(11^\circ)$

16. a. Fill in the table for values of $\tan\theta$. Round your answers to four decimal places.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°
$\tan\theta$									

b. Fill in the table for values of $\tan\theta$. Round your answers to three decimal places.

θ	81°	82°	83°	84°	85°	86°	87°	88°	89°
$\tan\theta$									

c. What happens to $\tan\theta$ as θ increases?

d. What value does your calculator give for $\tan 90^\circ$? Why?

17. a. Use your calculator to fill in a table of values for $\cos\theta$, rounded to hundredths.

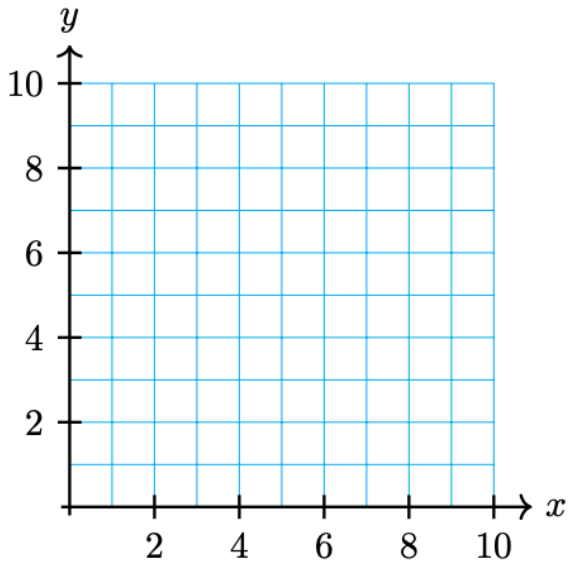
θ	0°	15°	30°	45°	60°	75°	90°
$\cos\theta$							

b. If you plotted the points in your table, would they lie on a straight line? Why or why not?

18. a. Use your calculator to complete the table. Rounded your answers to hundredths

θ	14°	22°	35°	42°	58°	78°
$\tan\theta$						

b. Use the values of $\tan\theta$ to sketch all the angles listed in the table. Locate the vertex of each angle at the origin, and the initial side along the positive x-axis.



For the following exercises, evaluate the expressions using a calculator.

19. a. $\sin(10^\circ + 40^\circ)$

b. $\sin 10^\circ + \sin 40^\circ$

c. $\sin 10^\circ \cos 40^\circ + \cos 10^\circ \sin 40^\circ$

20. a. $\cos(20^\circ + 50^\circ)$

b. $\cos 20^\circ + \cos 50^\circ$

c. $\cos 20^\circ \cos 50^\circ - \sin 20^\circ \sin 50^\circ$

21. a. $\cos(2 \cdot 24^\circ)$

b. $2\cos 24^\circ$

c. $2\cos^2(24^\circ) - 1$

22. a. $\cos(3 \cdot 19^\circ)$

b. $3\cos 19^\circ$

c. $4\cos^3(19^\circ) - 3\cos 19^\circ$

23. a. $\cos^2(17^\circ) + \sin^2(17^\circ)$

b. $\sec^2(86^\circ) - \tan^2(86^\circ)$

c. $\csc^2(32.1^\circ) - \cot^2(32.1^\circ)$

24. a. $\sin^2(25^\circ) + \cos^2(25^\circ)$

b. $\dfrac{1}{\cos^2(80^\circ)} - \tan^2(80^\circ)$

c. $\dfrac{1}{\sin^2(8^\circ)} - \cot^2(8^\circ)$

For the following exercises, evaluate the expression.

25. $\sin^{-1} 0.2$

26. $\cos^{-1} 0.8$

27. $\tan^{-1} 1.5$

28. $\tan^{-1} 2.5$

29. $\cos^{-1} 0.2839$

30. $\sin^{-1} 0.4127$

31. Evaluate the expressions, and explain what each means.

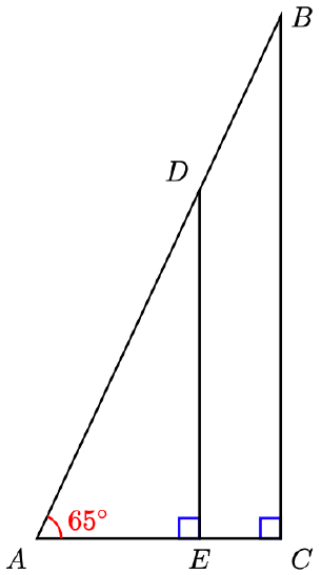
$$\sin^{-1}(0.6), \quad (\sin 6^\circ)^{-1}$$

32. Evaluate the expressions, and explain what each means.

$$\cos^{-1}(0.36), \quad (\cos 36^\circ)^{-1}$$

Synthesis Questions

33. Here are two right triangles with a 65° angle.



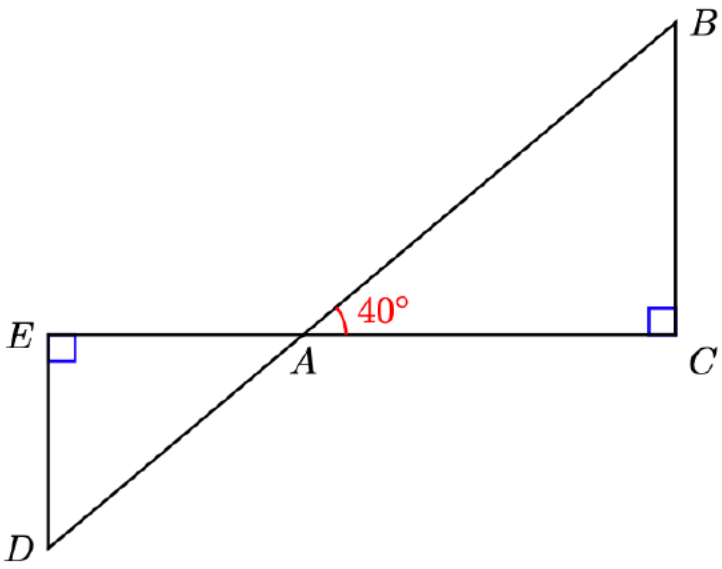
a. Measure the sides AB and BC with a ruler. Use the lengths to estimate $\sin 65^\circ$.

b. Measure the sides AD and DE with a ruler. Use the lengths to estimate $\sin 65^\circ$.

c. Use your calculator to look up $\sin 65^\circ$. Compare your answers. How close were your estimates?

34. Use the figure in the previous problem to calculate two estimates each for the cosine and tangent of 65°. Compare your estimates to your calculator's values for $\cos 65^\circ$ and $\tan 65^\circ$.

35. Here are two right triangles with a 40° angle.



a. Measure the sides AB and AC with a ruler. Use the lengths to estimate $\cos 40^\circ$.

b. Measure the sides AD and AE with a ruler. Use the lengths to estimate $\cos 40^\circ$.

c. Use your calculator to look up $\cos 40^\circ$. Compare your answers. How close were your estimates?

36. Use the figure in the previous problem to calculate two estimates each for the cosine and tangent of 40°. Compare your estimates to your calculator's values for $\sin 40^\circ$ and $\tan 40^\circ$.

For the following exercises, write two different equations for the statement.

37. The cosine of 15° is 0.9659.

38. The sine of 70° is 0.9397.

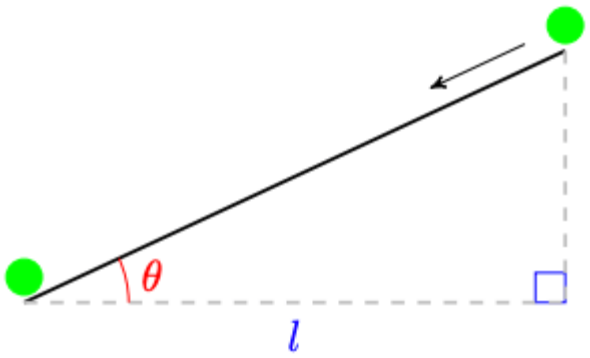
39. The angle whose tangent is 3.1445 is 65°.

40. The angle whose cosine is 0.0872 is 85°.

Applications

41. When a plane is tilted by an angle θ from the horizontal, the time required for a ball starting from rest to roll a horizontal distance of l feet on the plane is

$$t = \sqrt{\frac{l}{8}\csc(2\theta)} \text{ seconds}$$



a. How long, to the nearest 0.01 second, will it take the ball to roll 2 feet horizontally on a plane tilted by 12°?

b. Solve the formula for l in terms of t and θ.

12. After a heavy rainfall, the depth, D, of the runoff flow at a distance x feet from the watershed down a slope at angle α is given by

$$D = (kx)^{0.6}(\cot(\alpha))^{0.3} \text{ inches}$$

where k is a constant determined by the surface roughness and the intensity of the runoff.

a. How deep, to the nearest 0.01 inch, is the runoff 100 feet down a slope of 10° if $k = 0.0006$?

b. Solve the formula for x in terms of D and α.

Answers

Skills Refresher

(Many answers are possible for 1-4.)

1. $\frac{5}{2}, \frac{20}{8}$
2. $\frac{3}{4}, \frac{12}{16}$
3. $\frac{3}{5}, \frac{12}{20}$
4. $\frac{6}{4}, \frac{12}{8}$
5. $\frac{2}{5}$
6. $\frac{-8}{5}$
7. $\frac{-6}{5}$
8. $\frac{1}{2}$
9. $\frac{1}{4}$
10. $\frac{3}{4}$

This page titled 6.2.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

6.3: Solving Right Triangles

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

With the Pythagorean Theorem, we can find one side of a right triangle if we know the other two sides. For example, the missing side in Figure $6.3.1$ can be found as follows:

$$\begin{aligned} & b^2 + 5^2 &&= 12^2 \\ \implies & b^2 + 25 &&= 144 \\ \implies & b^2 &&= 119 \\ \implies & b &&= \sqrt{119} \end{aligned}$$

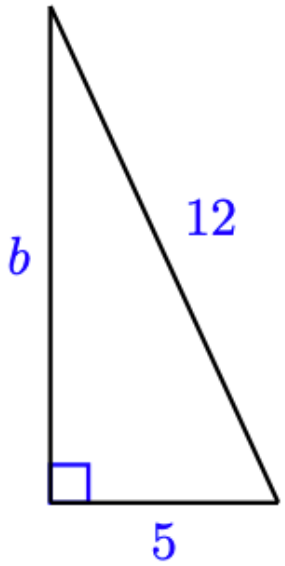


Figure $6.3.1$

However, is there a way to find the length of side b in Figure $6.3.2$, given that we only know the length of one side of the right triangle and one of the non-right angles?

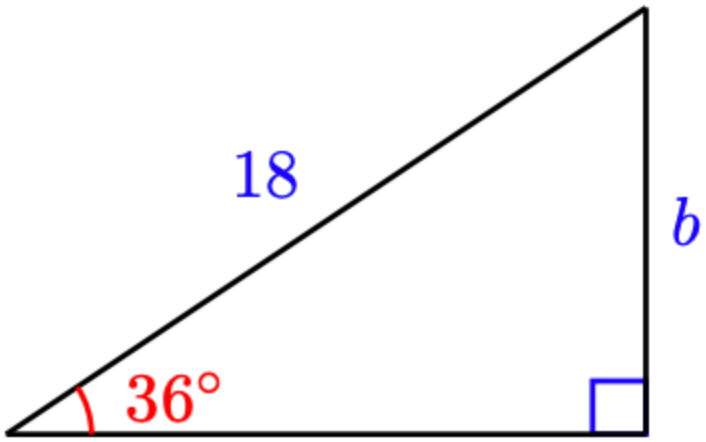


Figure 6.3.2

Using a Trigonometric Function to Find an Unknown Side

When finding the length of a side of a right triangle, we must be given at least two pieces of information. If we are given two side lengths (as in Figure 6.3.1), we use the Pythagorean Theorem. If, on the other hand, we are given an angle and a side length, we determine the trigonometric function relating the missing side and the provided information.

Example 6.3.1

Find the side length opposite the 50° angle in Figure 6.3.3.

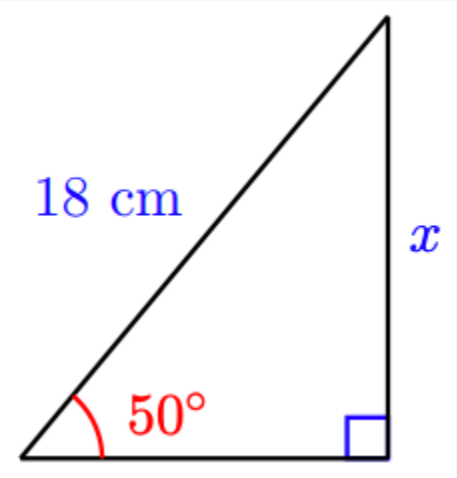


Figure 6.3.3

Solution available at the end of this section.

Caution: Save Rounding for Display Purposes Only

In the previous example, even though we showed only four decimal places in $\sin(50^\circ)$, you should not round off intermediate steps in a calculation—the answer loses accuracy with each rounding. The following keystrokes can be used to avoid entering an extended approximation for $\sin(50^\circ)$:

$$18 \quad \times \quad \text{SIN} \quad 50 \quad \text{ENTER}$$

The calculator returns 13.78879998.

As an additional warning, recall that *most* scientific and graphing calculators return $\text{SIN}($ when you press the SIN key. If you intend to perform the multiplication in the opposite order (i.e., you intend to compute $\sin(50^\circ) \times 18$), you must close the parentheses on the argument of the sine. That is, you would use the following keystrokes in your calculator:

$$\text{SIN} \quad 50 \quad) \quad \times \quad 18.$$

Without the closing parenthesis, the calculator computes $\sin(50 \times 18^\circ)$, which is not equivalent to $\sin(50^\circ) \times 18$.

Checkpoint 6.3.1

a. Use the tangent ratio to find x in Figure 6.3.4.

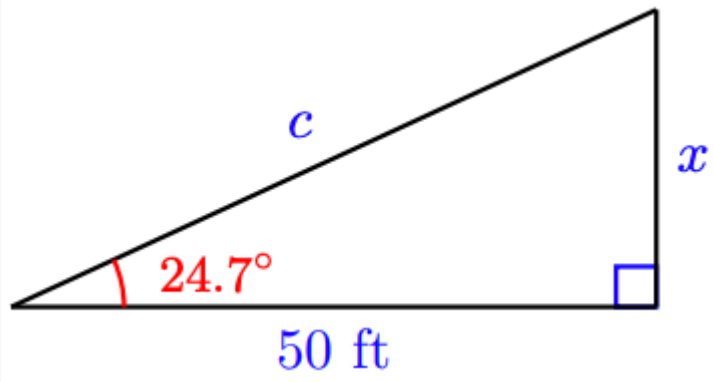


Figure 6.3.4

b. Use the sine ratio to find the hypotenuse, c, of the triangle.
c. Use the Pythagorean Theorem to find the hypotenuse of the triangle. Is the answer the same with both methods? Explain why the calculations might give (slightly) different answers.

Answers available at the end of this section.

Solving Triangles

A triangle has six parts: three sides and three angles. In a right triangle, we know that one of the angles is 90°. If given enough information, we can find the measures of the remaining angles within the right triangle and the lengths of the sides. This is called **solving the triangle**.

Definition: Solving a Triangle

Solving a triangle means finding all the missing side lengths and angle measures, given some initial information about the triangle.

Example 6.3.2

In the right triangle $\triangle ABC$, $\angle A = 37^\circ$ and (non-hypotenuse) side $b = 11$. Solve the triangle.

Solution available at the end of this section.

Traditional Notation: The Right Angle of $\triangle ABC$

Given the right triangle $\triangle ABC$, we adopt the convention used throughout most math textbooks that $\angle C$ is the 90° angle and side c is the hypotenuse.

Checkpoint 6.3.2

Sketch a right triangle with one angle of 37° and the side adjacent to that angle of length 5 centimeters. Without doing the calculations, list the steps used to solve the triangle.

Answer available at the end of this section.

In Example 6.3.2 and Checkpoint 6.3.2, we were given an angle and a side. Using those two pieces of information (along with the fact that one of the other angles in the right triangle is 90°), we were able to solve the triangles completely. What if we were not given any angles (other than the right angle) but two side lengths instead?

Example 6.3.3

Solve $\triangle ABC$ given that $a = 3$ and $c = 5$.

Solution available at the end of this section.

Checkpoint 6.3.3

Solve the right triangle $\triangle ABC$ given $b = 1.12$ and $c = 3.25$. Round your *displayed* answers to the nearest hundredth.

Answer available at the end of this section.

The power of Trigonometry comes from its usefulness in helping us solve physical applications.

Example 6.3.4

A 10-foot ladder is leaning against a wall. If the ladder's base is too far from the wall, the base can slide out. If the ladder base is too close to the wall, there's a risk that the ladder could tip over backward. The leaning ladder should make a 75° angle with the ground for safety reasons.

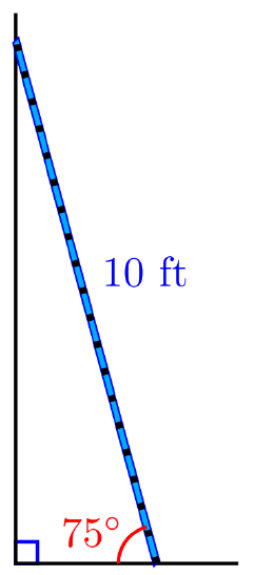


Figure 6.3.7

a. How far should the base of the ladder be from the wall?

b. How far up the wall will the top of the ladder reach?

Solutions available at the end of this section.

Checkpoint 6.3.4

The length, L, of the shadow cast by a flagpole on a sunny day depends on the height, h, of the flagpole and the angle, θ, that the sun's rays make with the ground (see Figure 6.3.8).

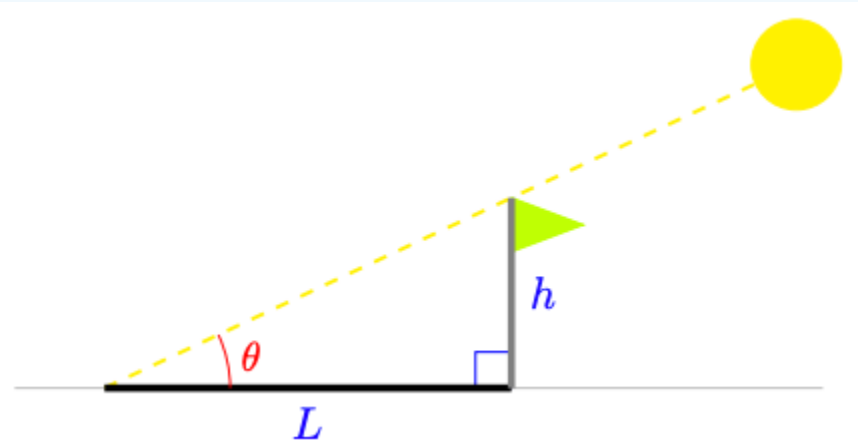


Figure 6.3.8

a. Write an expression for the length, L, of the shadow cast by a flagpole of height h when the sun makes an angle of θ from the ground.
b. Find the length (to the nearest 0.01 meter) of the shadow cast by a 3-meter flagpole when the sun makes an angle of 20° from the ground.

Answers available at the end of this section.

Interactive Element: Sun Angle and Shadow Length

The following Interactive Element might help you visualize how a flagpole's shadow length depends on the sun's angle, θ, above the ground.

Interact: Drag the bright point at the tip of the shadow left and right along the ground.

Observation: The lower the sun sits (smaller θ), the longer the shadow — and near a low sun a tiny drop in angle stretches the shadow dramatically, matching $L = h\cot(\theta)$.

Introducing Right Triangles to Non-Right Triangles

While the trigonometric functions are defined in terms of ratios of sides of a right triangle, they are useful on many occasions when a non-right triangle is involved.

Example 6.3.5

The vertex angle of an isosceles triangle is 34°, and the equal sides are 16 meters long. Find the altitude of the triangle.

Solution available at the end of this section.

Checkpoint 6.3.5

Another isosceles triangle has base angles of 72° and equal sides of length 6.8 centimeters. Find the size of the base.

Answer available at the end of this section.

Thought-Provoking Right Triangle Geometry

Part of your excursion into Trigonometry (and higher-level mathematics) is to think critically and problem-solve. One way to engage your imagination and thinking skills is through abstract geometry problems that are often fun!

Example 6.3.6

Figure $6.3.10$ shows a circle of radius r and a right triangle with side a along the radius of the circle and side b tangent to the circle.

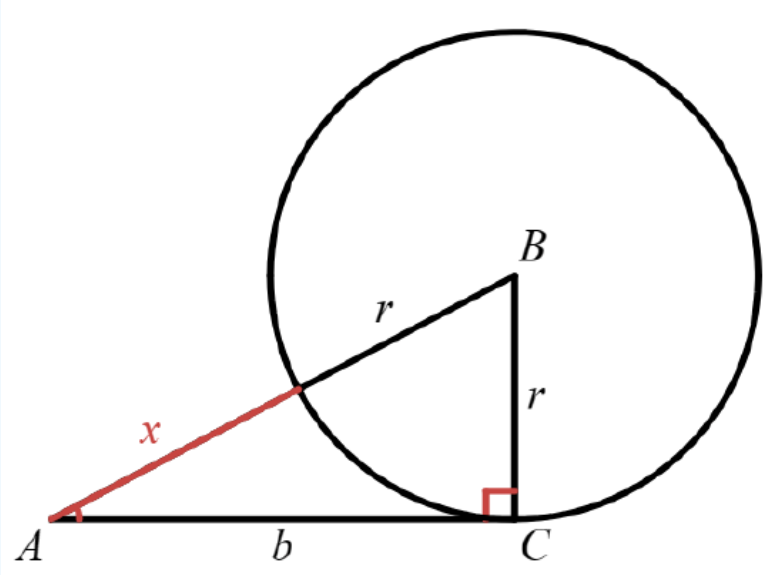


Figure 6.3.10

If $\angle B = 50^\circ$ and $b = 10$ inches, find the radius of the circle and the partial length of the hypotenuse, x, that sits outside the circle. Round your answers to the nearest tenth.

Solution available at the end of this section.

We continually harp on this but notice that we used the *exact* value of r when doing our final computation in Example $6.3.6$. This is because we always want to avoid using rounded values for subsequent computations.

Checkpoint 6.3.6

Let $r = 5$ and $\angle A = 24^\circ$ in Figure $6.3.10$ (Example $6.3.6$), find x. Round your answer to the nearest tenth.

Answer available at the end of this section.

Example 6.3.7

In Figure 6.3.11, $x = 5$, $\angle A = 25°$, and $\angle BDC = 48°$. Find y.

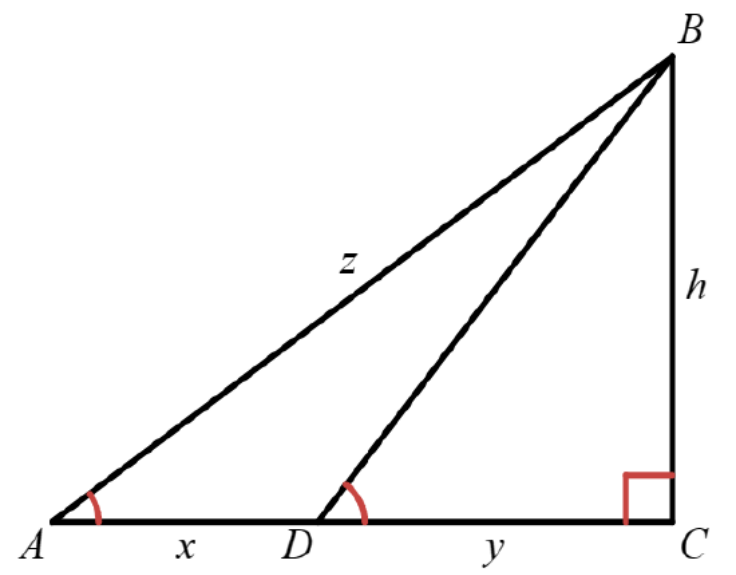


Figure 6.3.11

Solution available at the end of this section.

Checkpoint 6.3.7

Let $\angle A = 24°$, $y = 12$, and $\angle BDC = 62°$ in Figure 6.3.11 (Example 6.3.7). Find x. Round your answer to the nearest tenth.

Answer available at the end of this section.

Success in College: Learning From Tests (Part 1)

1. A student scored 75% on their last math test. They tried to redo the questions and problems that they missed. Now they need to know if their new answers are correct. For your current math class, how could they find this out?
2. Look over your last math test. Did you miss any points because you did not show enough work? If so, explain why you didn't show the work on the test.

Success in Mathematics: Math Anxiety (Part 1)

The following is an excerpt of an article written at MathAcademy.com.

> A famous stage actress was once asked if she had ever suffered from stage fright and, if so, how she had gotten over it. She laughed at the interviewer's naive assumption that she must not feel that kind of anxiety since she was an accomplished actress. She assured him that she had always had stage fright and had never gotten over it. Instead, despite it, she had learned to walk on stage and perform.
>
> Like stage fright, math anxiety can be a disabling condition, causing humiliation, resentment, and even panic.
>
> Even though we all suffer from math anxiety to some degree – just as anyone feels at least a little nervous when speaking

to an audience – for some of us, it is a severe problem. This burden interferes with our lives, preventing us from achieving our goals. The first step, without which no further progress is possible, is to recognize that math anxiety is an emotional response. (In fact, severe math anxiety is a learned emotional response.) As with any strong emotional reaction, there are constructive and unconstructive ways to manage math anxiety. Unconstructive (and damaging) ways include rationalization, suppression, and denial.

By "rationalization," we mean finding reasons why it is okay and perhaps even inevitable – and therefore justified – for you to have this reaction. The myths of "men are better at math than women" or "math is for logical people and not creative people" are examples of rationalizations. While they may make you feel better (or at least less bad) about having math anxiety, they will do nothing to lessen it or to help you get it under control. Therefore, rationalization is unconstructive.

"Suppression" means being aware of the anxiety – but trying very hard not to feel it. I have found that students very commonly attempt this, and some severe self-criticism accompanies it. Students feel that they shouldn't feel this anxiety, that it's a weakness they should overcome by brute force if necessary. When this effort doesn't succeed (as invariably it doesn't), the self-criticism becomes ever harsher, leading to a deep sense of frustration and often a severe loss of self-esteem – mainly if the stakes for a student are high, as when their career or personal goals are riding on a successful outcome in a math class, or when parental disapproval is a factor. Consequently, suppression of math anxiety is not only unconstructive but can be damaging.

Finally, there is denial. People using this approach probably aren't likely to see this essay, much less read it, for they carefully construct their lives to avoid all mathematics as much as possible. They choose college majors and careers that don't require math and let the bank or their spouse balance the checkbook. This approach has the advantage that feelings of frustration and anxiety about math are mostly avoided. However, their lives are drastically constrained, for, in our society, fewer than 25% of all careers are, so to speak, "math-free." Thus, their choices of personal and professional goals are severely limited. (Incidentally, most math-free jobs are low-status and low-pay.)

People in denial about mathematics miss out on something else, too, for the student of mathematics learns to see aspects of the structure and beauty of our world that can be seen in no other way and to which the "innumerate" necessarily remain forever blind. It would be a lot like never hearing music or never seeing colors. (I understand that some people have these disabilities but didn't choose to have them.)

Okay, so what is the constructive way to manage math anxiety? I call it "taking possession." It involves identifying the sources of math anxiety in one's life, accepting those feelings without self-criticism, and then learning strategies for disarming math anxiety's influence on one's future study of mathematics.

Begin by understanding that your feelings of math anxiety are not uncommon and do not indicate that there is anything wrong with you or inferior about your ability to learn math. For some, this can be hard to accept, but it is worth trying to accept – since, after all, it happens to be true. This can be made more accessible by exploring your own "math history." Think back across your career as a math student, and identify those experiences which have contributed most to your feelings of frustration about math. For some, this will be a memory of a humiliating experience in school, such as being made to stand at the blackboard and embarrassed in front of one's peers. For others, it may involve interaction with a parent. Whatever the principle episodes are, recall them as vividly as you can. Then, write them down. This is important. After you have written the episode on a sheet(s) of paper, write down your reaction to the episode, both at the time and how it makes you feel to recall it now.

After completing this exercise, take a fresh sheet of paper and try to sum up in a few words what your feelings about math are at this point in your life, together with the reason or reasons you wish to succeed at math. This, too, is important. Not until after we lay out for ourselves consciously and deliberately what our feelings and desires are towards mathematics will it become possible to take possession of our feelings of math anxiety and become free to implement strategies for coping with those feelings.

At this point, sharing your memories, feelings, and goals with others can be enormously helpful. In a math class I teach for art majors, I hand out a questionnaire early in the semester asking students to do precisely what is described above. After spending about twenty minutes writing down their recollections and goals, I lead them in a classroom discussion on math anxiety. This process of dialogue and sharing – though it may seem just a bit on the goopy side – invariably brings out of each student their barriers to math, often helping them become completely conscious of these barriers for the first time. Just as important, it helps all my students understand that the negative experiences they have had, and their reactions to them, are shared one way or another by almost everyone else in the room.

If you do not have the opportunity to engage in a group discussion in a classroom setting, find friends or relatives you trust to respect your feelings and induce them to talk about their own math anxiety experiences and listen to yours.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Are there counselors at your school that are available to help students who are struggling with math anxiety? If so, where are they located, and how to make an appointment to see one?
2. Estimate the percent of students in your class you think are negatively affected by math anxiety.

If you are unsure about the answers to either of these questions, feel free to ask your instructor - that's what they are there for!

Answers

Section Preview

This section focuses on solving right triangles. As such, all angles will be acute. In addition, applications and concept exercises require the student to find missing angles and/or sides within a right triangle; however, more robust applications involving angles of elevation or depression, navigation, and problems involving more than a single right triangle in a three-dimensional environment are reserved for the next section.

Section Preview

- **Simplifying Expressions**
  - Simplifying Sums and Differences of Radical Expressions: For the Skills Refresher homework.
- **Solving Equations**
  - Solving Linear Equations — Multistep with Variables on Both Sides: For Example 7's algebraic chain.

Section Preview

- Solve a right triangle for a missing side.
- Solve a right triangle for a missing angle.
- Solve an application using Right Triangle Trigonometry.
- Use Right Triangle Trigonometry in a robust geometry problem.

Example $6.3.1$

In this triangle, the trigonometric function that relates the missing side to the given information is the sine:

$$\sin(50^\circ) = \frac{\text{opposite}}{\text{hypotenuse}}$$

We use a calculator to find an approximate value for the sine of 50°, filling in the lengths of the hypotenuse and the opposite side to get

$$0.7660 \approx \frac{x}{18}.$$

We solve for x to find

$$x \approx 18(0.7660) = 13.788.$$

To two decimal places, the length of the opposite side is approximately 13.79 centimeters.

Checkpoint $6.3.1$

a. $x \approx 23$ ft
b. $c \approx 55$ ft
c. The answers agree when rounded to units. Rounding during calculation can cause the results to differ.

Example $6.3.2$

Since $\angle A = 37^\circ$, side a is not the hypotenuse of the right triangle. Moreover, we were told that side b is not the hypotenuse. Thus, c must be the hypotenuse and $\angle C = 90^\circ$. Creating a rough sketch of $\triangle ABC$, we get

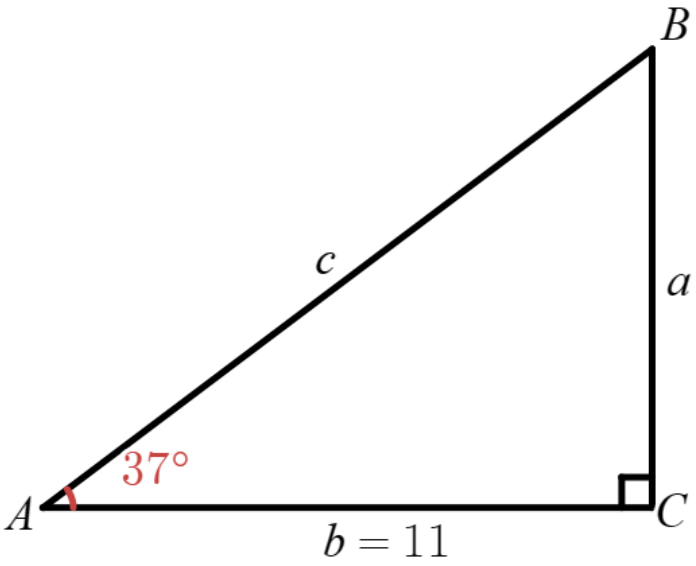


Figure 6.3.5

Using the Triangle Sum, we quickly compute $\angle B = 90^\circ - 37^\circ = 53^\circ$. We now concentrate on finding the missing sides by using the trigonometric functions and the given information. The tangent relates 37° to sides a and b. Therefore,

$$\tan(37^\circ) = \frac{a}{11} \implies a = 11\tan(37^\circ) \approx 8.2891.$$

Finally, we use the cosine to relate 37° to the given side b and the hypotenuse c.

$$\cos(37^\circ) = \frac{11}{c} \implies c = \frac{11}{\cos(37^\circ)} \approx 13.7735.$$

Checkpoint 6.3.2

Use $\tan(37^\circ)$ to find the opposite side. Use $\cos(37^\circ)$ to find the hypotenuse. Subtract 37° from 90° to find the third angle.

Example 6.3.3

It's always best to start by drawing a triangle.

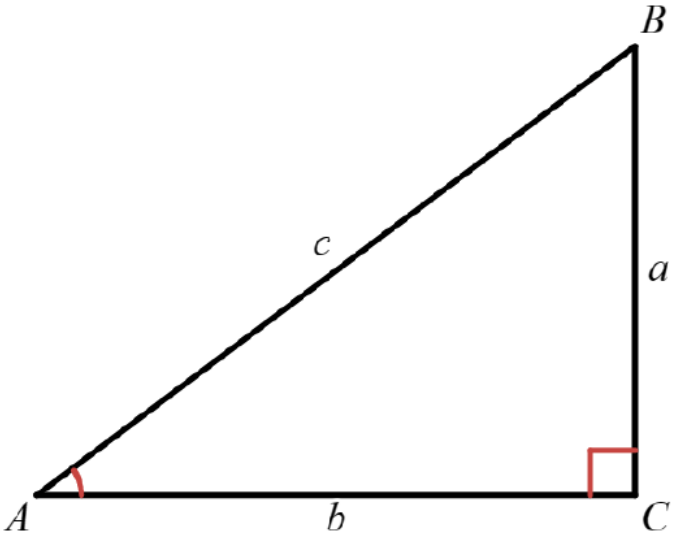


Figure 6.3.6

We can use the Pythagorean Theorem to compute the length of side b, or we could recognize that this is a Pythagorean Triple. No matter how we compute it, $b = 4$. Now, we need to find the missing angles. Luckily, we were given that this is a right triangle, so once we find either $\angle A$ or $\angle B$, we can use the Triangle Sum to find the other angle. Let's go ahead and find $\angle A$. Since $\sin(\angle A) = \frac{a}{c} = \frac{3}{5}$, we use a calculator to compute $\angle A = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.8699^\circ$. Thus,

$$\angle B = 180^\circ - \angle C - \angle A \approx 180^\circ - 90^\circ - 36.8699^\circ \approx 53.1301^\circ.$$

It's often best to organize all of this information in a table like so:

$$\begin{array}{lcllcl} \angle A & \approx & 36.8699^\circ & a & = & 3 \\ \angle B & \approx & 53.1301^\circ & b & = & 4 \\ \angle C & = & 90^\circ & c & = & 5 \end{array}$$

Checkpoint 6.3.3

$$\begin{aligned} \angle A &\approx 69.84^\circ & a &\approx 3.05 \\ \angle B &\approx 20.16^\circ & b &= 1.12 \\ \angle C &= 90^\circ & c &= 3.25 \end{aligned}$$

Example 6.3.4

a. The distance between the ladder's base and the wall is the side adjacent to the 75° angle. We can find the side adjacent to the 75° angle using the cosine ratio.

$$\begin{aligned} \cos(75^\circ) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ \implies 0.2588 &\approx \frac{b}{10} \\ \implies 10(0.2588) &\approx b \\ \implies b &\approx 2.588 \end{aligned}$$

The ladder's base should be about 2.6 feet from the wall.

b. We could use the Pythagorean Theorem to find how far up the wall the ladder will reach. However, using the given information rather than the calculated values to find the other unknown parts is better. We will use the sine ratio.

$$\begin{aligned} \sin(75^\circ) &= \frac{\text{opposite}}{\text{hypotenuse}} \\ \implies 0.9659 &\approx \frac{a}{10} \\ \implies 10(0.9659) &\approx a \\ \implies a &\approx 9.659 \end{aligned}$$

The ladder will reach about 9.7 feet.

Checkpoint 6.3.4

a. $L = h\cot(\theta)$

b. The shadow is about 8.24 meters long.

Example 6.3.5

The triangle described is *not* a right triangle. However, the altitude of an isosceles triangle bisects the vertex angle. It divides the triangle into two congruent right triangles, as shown in Figure 6.3.9. The 16-meter side becomes the hypotenuse of the right triangle. The altitude, h, of the original triangle is the side adjacent to the 17° angle.

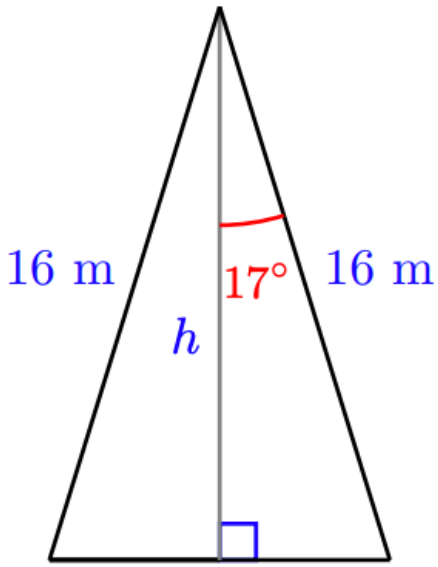


Figure 6.3.9

Which of the three trigonometric ratios is helpful in this problem? The cosine is the ratio that relates the hypotenuse and the adjacent side. Therefore, we will begin with the equation:

$$\cos(17^\circ) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

We use a calculator to find $\cos(17^\circ)$ and fill in the lengths of the sides to get

$$0.9563 \approx \frac{h}{16}.$$

Solving for h gives

$$h \approx 16(0.9563) = 15.3008.$$

The altitude of the triangle is about 15.3 meters long.

Checkpoint 6.3.5

4 cm

Example 6.3.6

Throughout Trigonometry, you will be faced with thought-provoking questions. The beautiful thing about right triangle trigonometry is that you typically need to find a trigonometric function that relates the given information to the missing information. In this case, we are given $\angle B$ and side b. We are initially asked for the radius of the circle, which is the same as the side length a of the right triangle. The trigonometric function relating $\angle B$ to sides a and b is the tangent:

$$\tan(\angle B) = \frac{b}{a} = \frac{b}{r}.$$

Therefore,

$$\tan(50^\circ) = \frac{10}{r} \implies r = \frac{10}{\tan(50^\circ)}.$$

Hence, the radius of the circle is $\frac{10}{\tan(50^\circ)} \approx 8.4$ inches.

Now we need to find the length x. Again, we want to find a trigonometric function that relates the information to what we seek. Unfortunately, we are looking for only *part* of the hypotenuse of the right triangle; however, the *entire* hypotenuse is $x + r$, and we just found r. The sine is the trigonometric function that relates $\angle B$ to side b and the hypotenuse:

$$\sin(\angle B) = \frac{b}{x+r} \implies x + r = \frac{b}{\sin(\angle B)} \implies x = \frac{b}{\sin(\angle B)} - r.$$

Substituting the given values for $\angle B$ and side b, along with the **exact** value for r, we get

$$x = \frac{10}{\sin(50^\circ)} - \frac{10}{\tan(50^\circ)} \approx 4.66 \text{ inches}.$$

Checkpoint 6.3.6

$x \approx 7.3$

Example 6.3.7

It is helpful to redraw only $\triangle BCD$ and then $\triangle ABC$, noticing that they share the common side h. For $\triangle BCD$, we only have one piece of given information, $\angle BDC$; however, we need to find y, and it will be useful to work with h since both triangles share that side (I call this the "link" between the triangles). The trigonometric function relating these is the tangent:

$$\tan(\angle BDC) = \frac{h}{y} \implies y\tan(\angle BDC) = h.$$

Hence,

$$h = y\tan(48^\circ).$$

Moving to $\triangle ABC$, since we were given x and $\angle A$, it seems best to use the tangent again to get h involved. Therefore,

$$\tan(\angle A) = \frac{h}{x+y} \implies (x+y)\tan(\angle A) = h.$$

That is,

$$h = (5+y)\tan(25^\circ).$$

We now equate these two versions of h to each other (since they both equal h, they are both equal to each other).

$$\begin{aligned} y\tan(48^\circ) = (5+y)\tan(25^\circ) &\implies y\tan(48^\circ) = 5\tan(25^\circ) + y\tan(25^\circ) \\ &\implies y\tan(48^\circ) - y\tan(25^\circ) = 5\tan(25^\circ) \\ &\implies y(\tan(48^\circ) - \tan(25^\circ)) = 5\tan(25^\circ) \\ &\implies y = \frac{5\tan(25^\circ)}{\tan(48^\circ) - \tan(25^\circ)} \\ &\implies y \approx 3.62 \end{aligned}$$

Checkpoint 6.3.7

$x \approx 38.7$

This page titled 6.3: Solving Right Triangles is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

6.3.1: Resources and Key Concepts

Resources

Videos

- Trigonometric Applications
  - Trigonometry of Right Triangles - Review of Trigonometric Functions - A Right Triangle Approach
  - Right Triangle Trigonometry - Applications of Right Triangle Trigonometry (Part 1)
  - Trigonometry of Right Triangles - Review of Applications of Right Triangle Trigonometry (Part 1)
  - Inverse Trigonometric Functions - Applications of Right Triangle Trigonometry (Part 2)

Key Concepts

Definitions

- **Solving a Triangle:** The process of finding the measures of all unknown side lengths and angles of a triangle, based on some given initial information.

Common Mistakes

- **Save Rounding for Display Purposes Only:** You should not round off intermediate steps in a calculation. Using a rounded value in a subsequent calculation introduces and propagates errors, leading to a less accurate final answer. Use your calculator's stored values or the exact expressions whenever possible.
- **Using Calculated Values Instead of Given Values:** When solving a triangle, it is better to use the original given information for each calculation rather than a value you calculated in a previous step. This helps avoid carrying forward any rounding errors.
- **Incorrect Calculator Entry:** Be careful with the order of operations on your calculator. For example, to compute $\sin(50^\circ) \times 18$, you may need to close the parenthesis after the 50, as in `SIN(50) * 18`. Without the parenthesis, the calculator might compute $\sin(50^\circ \times 18)$, which is incorrect.

This page titled 6.3.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

6.3.2: Homework

Reading Questions

1. What does it mean to "solve a triangle"?
2. If you are given one acute angle and the length of the hypotenuse in a right triangle, which trigonometric function would you use to find the length of the side opposite the given angle?
3. If you are given the lengths of the two legs of a right triangle, how do you find the measure of one of the acute angles?
4. Why is it generally better to use the given information for each calculation, rather than using values you have already calculated?
5. What is the first step you should always take when starting to solve a triangle, especially in an application problem?
6. How can right triangle trigonometry be used to find the altitude of an isosceles triangle that is not a right triangle?
7. In a right triangle $\triangle ABC$ with right angle at C, if you are given sides a and c, what inverse trigonometric function would you use to find $\angle A$?
8. The text warns that when calculating $\sin(50^\circ) \times 18$, you might need to enter `SIN(50) * 18` into your calculator. What might the calculator compute if you enter `SIN(50 * 18)` instead?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, simplify.

1. $\sqrt{2}\sqrt{2}$
2. $\frac{3}{\sqrt{3}}$
3. $\sqrt{8}$
4. $\sqrt{\frac{3}{4}}$

For the following exercises, rationalize the denominator.

5. $\dfrac{1}{\sqrt{2}}$
6. $\dfrac{2}{\sqrt{3}}$
7. $\dfrac{6}{\sqrt{3}}$
8. $\dfrac{4}{\sqrt{8}}$

Answers available at the end of this section.

Homework

Vocabulary Check

1. Given enough initial information about a triangle, we can find the lengths of all remaining sides and angles. This process is known as ___.

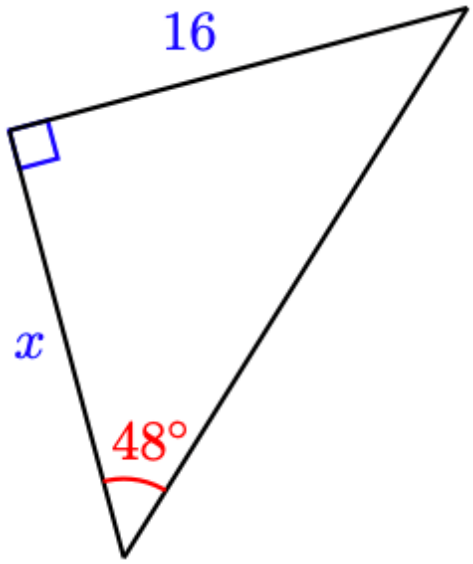
Concept Check

2. How many parts of a right triangle (including the right angle) do you need to know to solve it?
3. Why is it better to use the given values when solving a triangle than the ones you calculated?
4. What is the $\sin^{-1}$ (or $\cos^{-1}$ or $\tan^{-1}$) button on the calculator used for?
5. If you knew one acute angle and the side opposite that angle, which trigonometric ratio would you use to find the hypotenuse of a right triangle?

Basic Skills

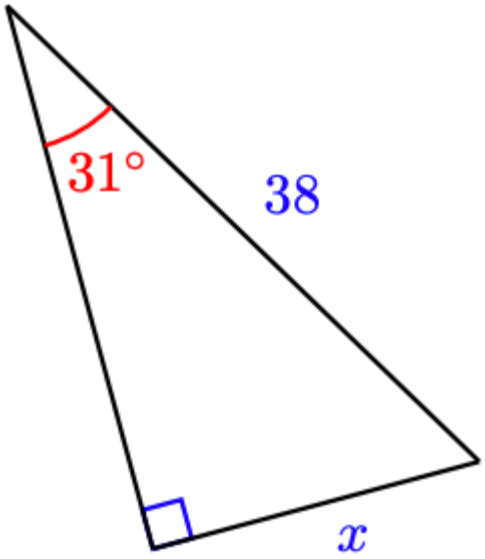
For the following exercises, use one of the three trigonometric ratios to find the unknown side of the triangle. Round your answer to hundredths.

6.

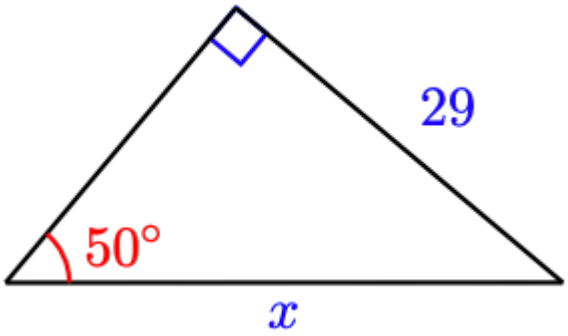


7.

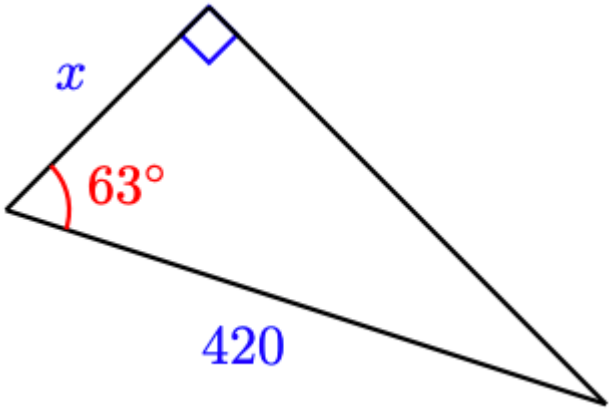
8.



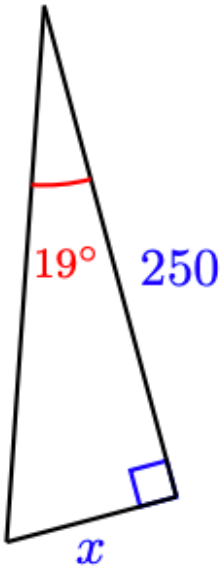
9.

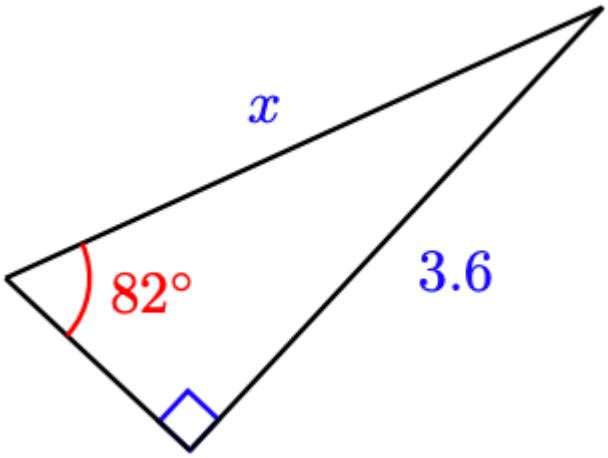


10.



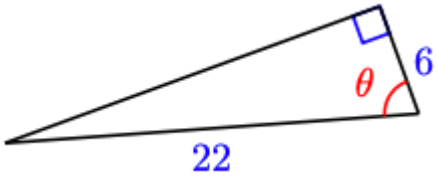
11.



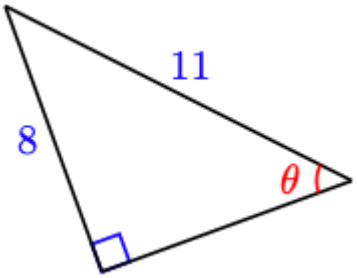


For the following exercises, find the labeled angle. Round your answer to tenths of a degree.

12.

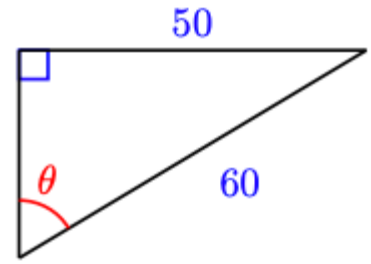


13.

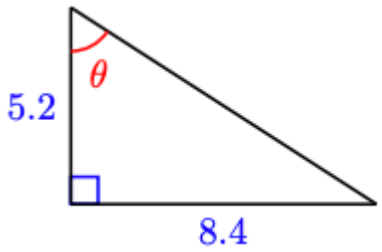


14.

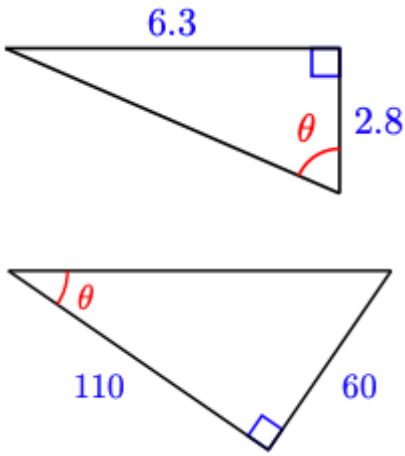
15.



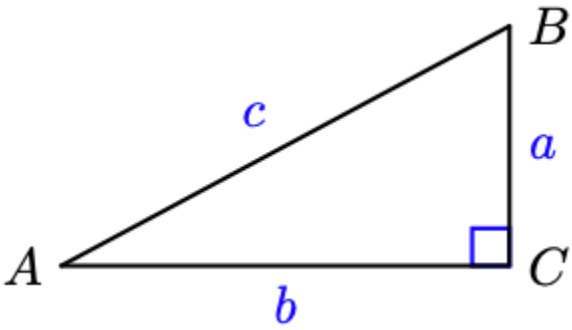
16.



17.



In these Homework Problems, we use the following standard notation for a right triangle: in $\triangle ABC, \angle C$ is a right angle. The side opposite $\angle C$ has length c, and so on. (See the figure at below.)



For the following exercises,
(a) Sketch the right triangle described.
(b) List the steps you would use to solve the triangle without doing the calculations.

18. $B = 53.7^\circ, b = 8.2$

19. $B = 80^\circ, a = 250$

20. $A = 25^\circ, b = 40$

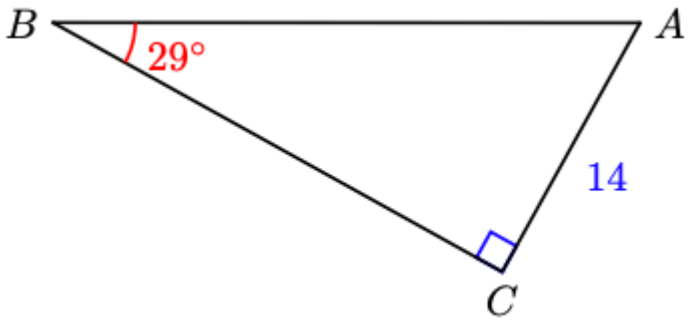
21. $A = 15^\circ, c = 62$

22. $A = 64.5^\circ, c = 24$

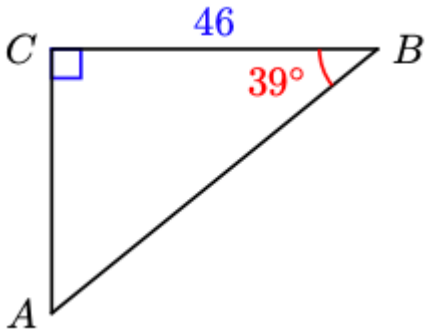
23. $B = 44^\circ, b = 0.6$

For the following exercises, solve the triangle. Round answers to hundredths.

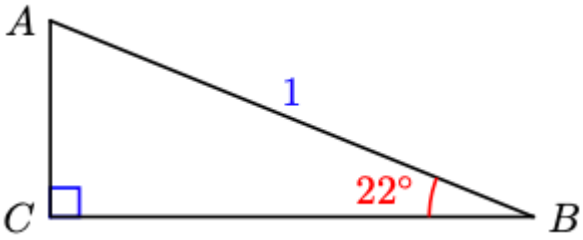
24.



25.



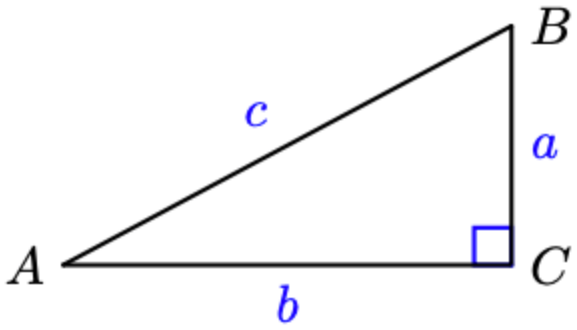
26.



27.



In these Homework Problems, we use the following standard notation for a right triangle: in $\triangle ABC, \angle C$ is a right angle. The side opposite $\angle C$ has length c, and so on. (See the figure at below.)



For the following exercises,
(a) Sketch the right triangle described.
(b) Solve the triangle.

28. $A = 42^\circ, c = 26$

29. $B = 28^\circ, c = 6.8$

30. $B = 33^\circ, a = 300$

31. $B = 79^\circ, a = 116$

32. $A = 12^\circ, a = 4$

33. $A = 50^\circ, a = 10$

For the following exercises, sketch and label a right triangle with the given properties. Solve the triangle.

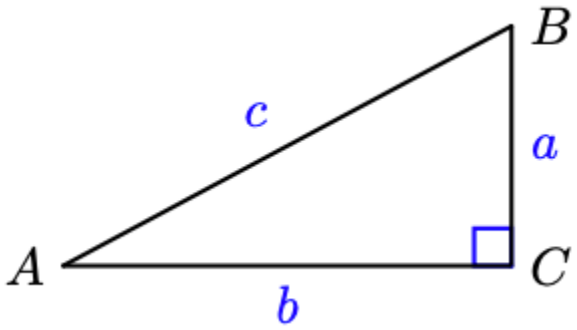
34. One angle is 40°, and the side opposite that angle is 8 inches

35. One angle is 65°, and the side adjacent to that angle is 30 yards

36. One angle is 28°, and the hypotenuse is 56 feet

37. One leg is 15 meters, and the hypotenuse is 18 meters

In these Homework Problems, we use the following standard notation for a right triangle: in $\triangle ABC, \angle C$ is a right angle. The side opposite $\angle C$ has length c, and so on. (See the figure at below.)



For the following exercises,
(a) Sketch the right triangle described.
(b) Solve the triangle.

38. $a = 18, b = 26$

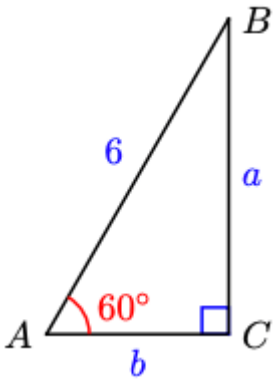
39. $a = 35, b = 27$

10. $b = 10.6, c = 19.2$

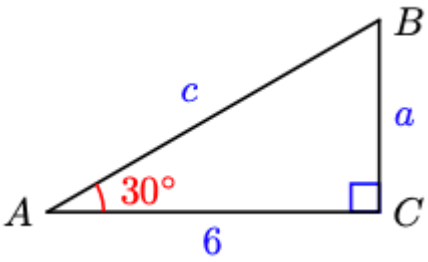
11. $a = 88, c = 132$

For the following exercises, solve the triangle. Give your answers as exact values.

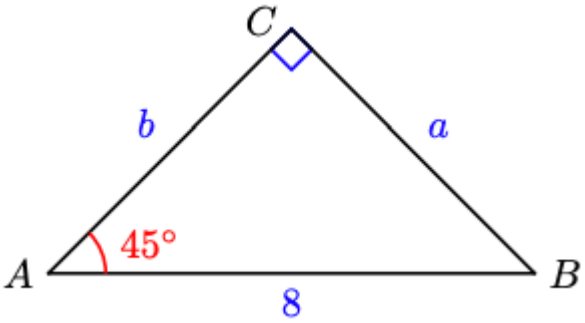
12.



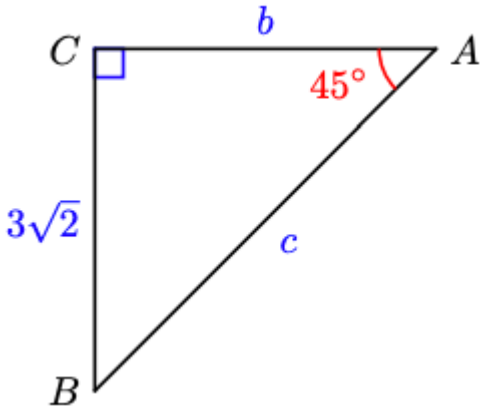
13.



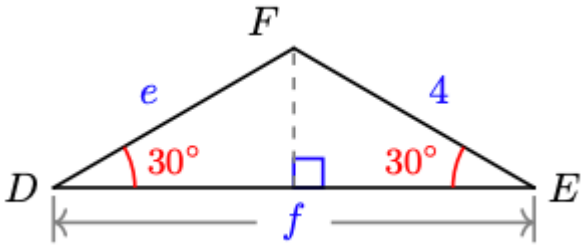
14.



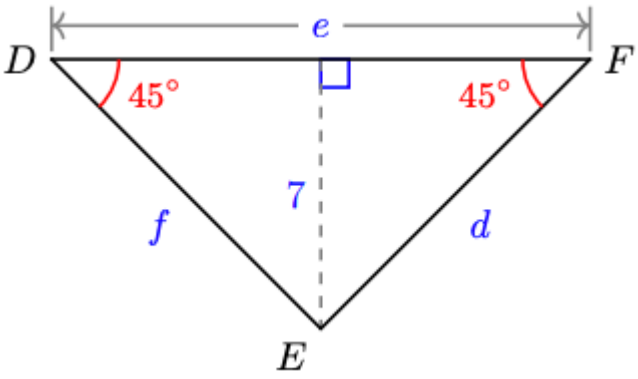
15.



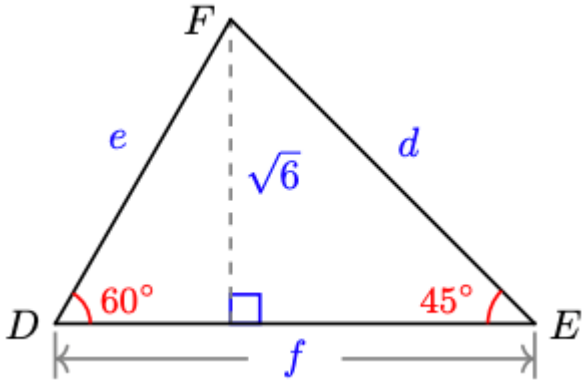
16.



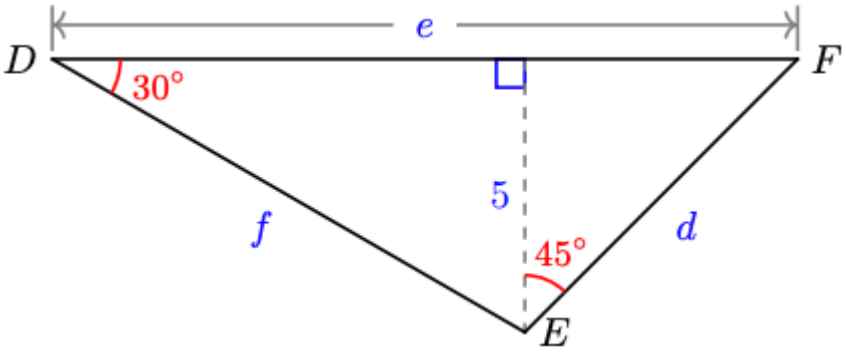
17.



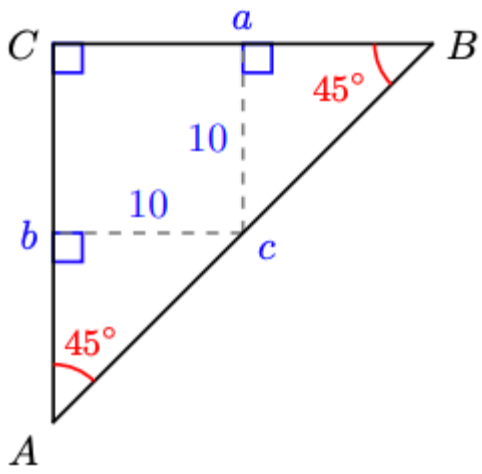
18.



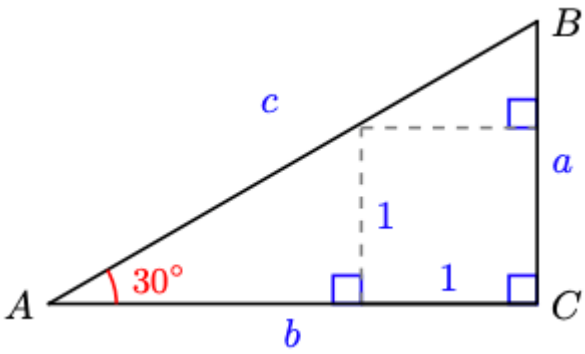
19.



20.



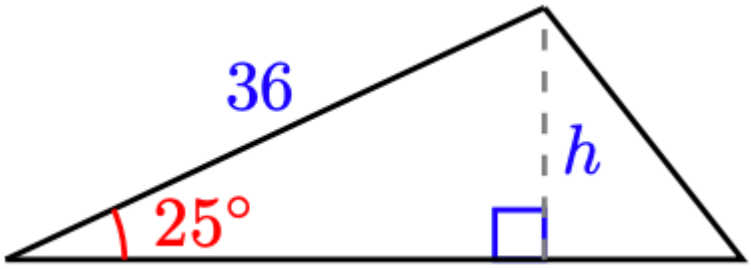
51.



Synthesis Questions

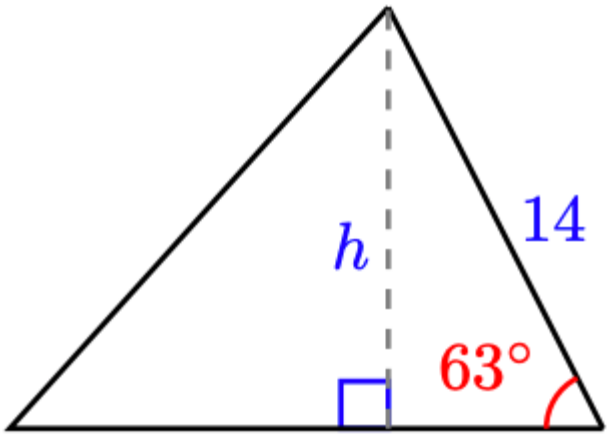
For the following exercises, find the triangle's altitude. Round your answer to two decimal places.

52.

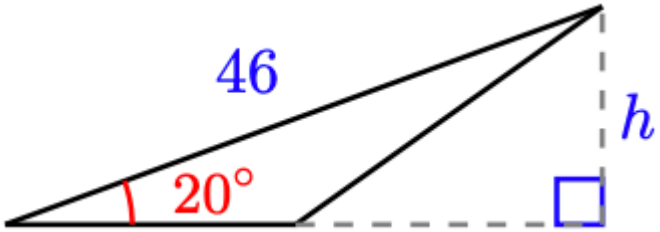


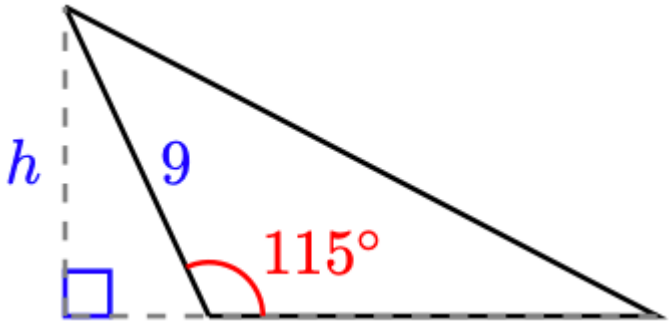
53.

54.



55.

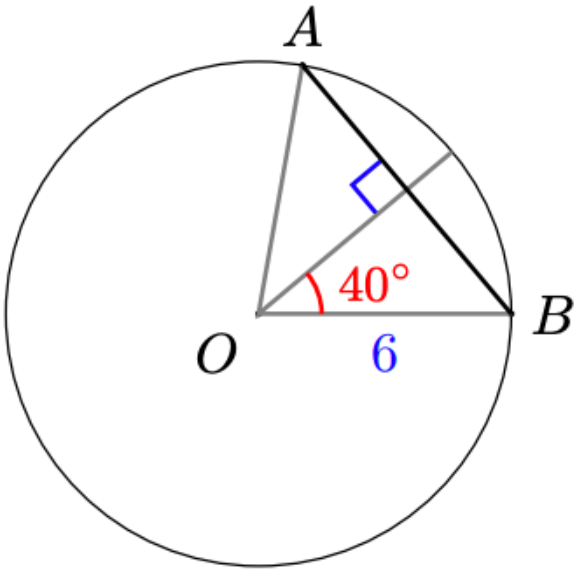


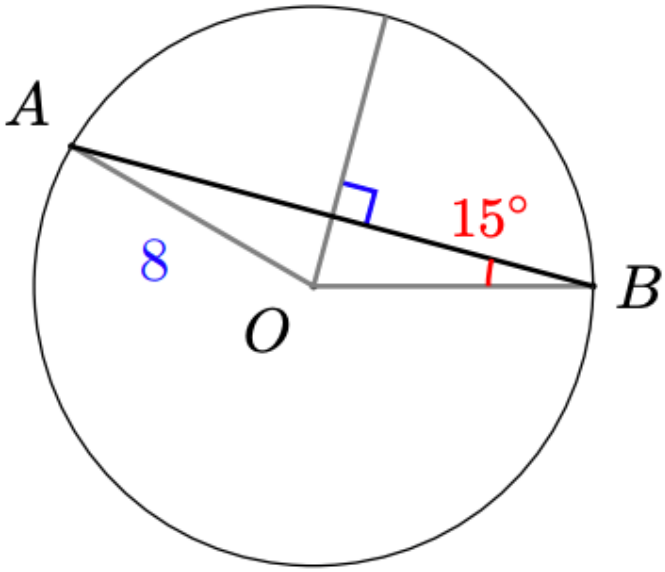


For the following exercises, find the chord length AB. Round your answer to two decimal places.

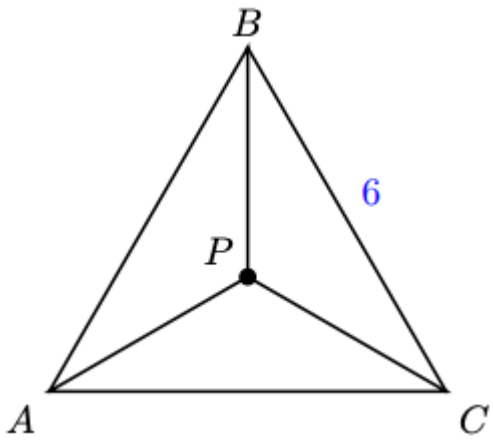
56.

57.



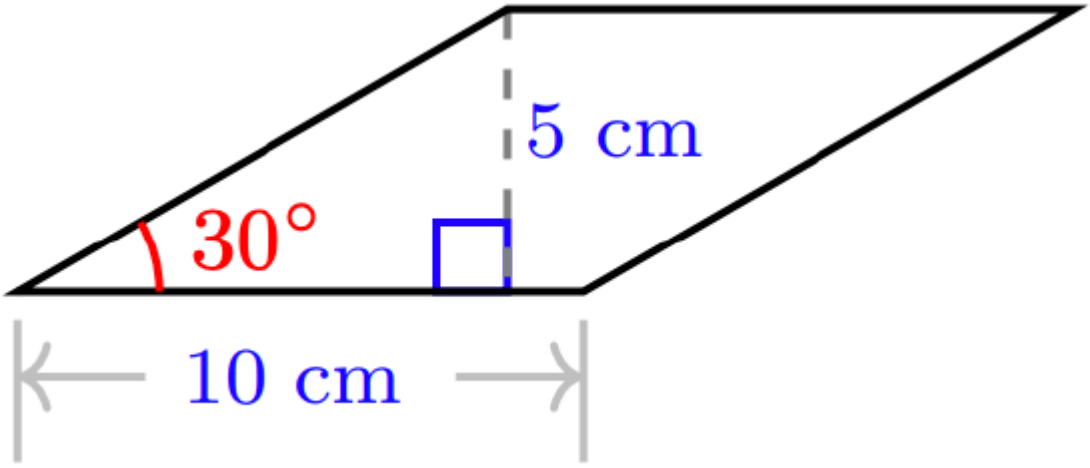


58. Triangle *ABC* is equilateral, and its angle bisectors meet at point *P*. The sides of $\triangle ABC$ are 6 inches long. Find the length of AP.

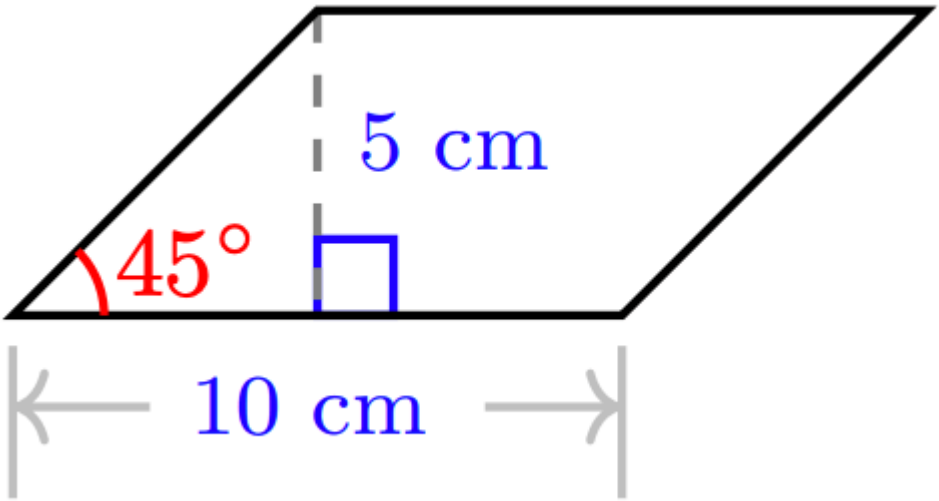


59. Find an exact value for the perimeter of each parallelogram.

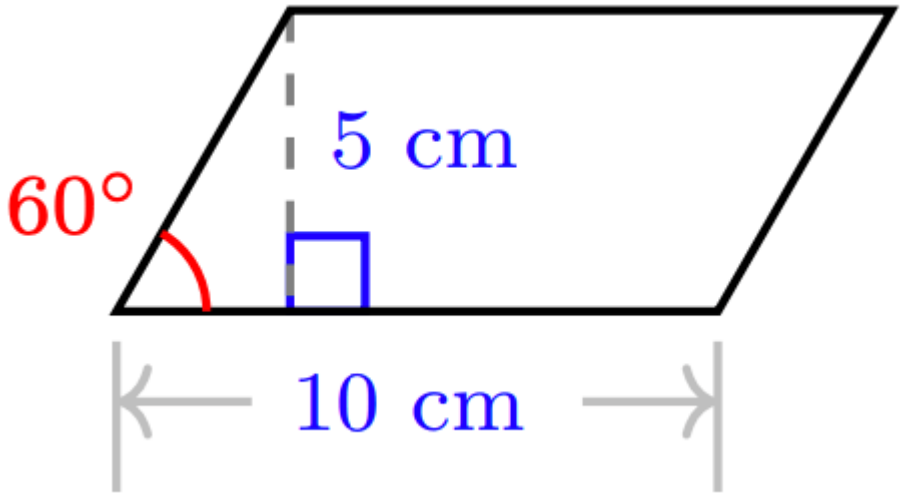
a.



b.



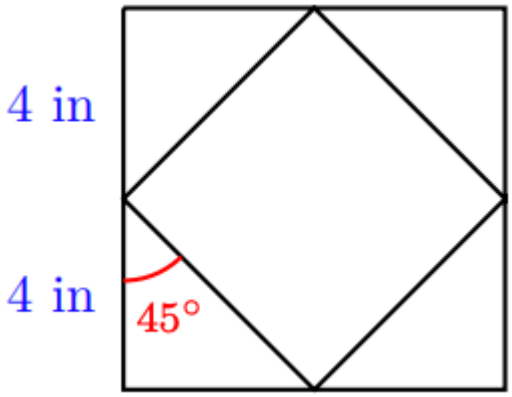
c.



60. a. Find the area of the outer square.

b. Find the dimensions and the area of the inner square.

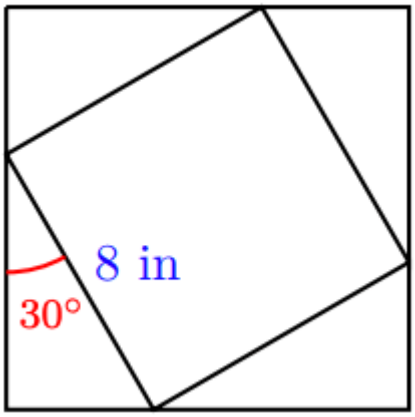
c. What is the ratio of the area of the outer square to the area of the inner square?



61. a. Find the area of the inner square.

b. Find the dimensions and the area of the outer square.

c. What is the ratio of the area of the outer square to the area of the inner square?



Applications

For the following exercises,
(a) Sketch a right triangle that illustrates the situation. Label your sketch with the given information.
(b) Choose the appropriate trigonometric ratio, write an equation, and solve the problem.

62. **Distance Across a Canyon.** To measure the distance across a canyon, Roy first sights an interesting rock directly opposite on the other side. He then walks 200 yards down the canyon's rim and sights the rock again, this time at an angle of 18.5° from the canyon rim. What is the width of the canyon?

63. **Engineering Accessibility.** Ramps for wheelchairs should be no steeper than an angle of 6°. How much horizontal distance should a ramp that rises 5 feet in height be allowed?

64. **Cable Length.** A cable car on a ski lift traverses a horizontal distance of 1800 meters at an angle of 38°. How long is the cable?

65. **Cable Angle.** The gondola cable for the ski lift at Snowy Peak is 2458 yards long and climbs 1860 feet. What angle with the horizontal does the cable make?

66. **Construction.** Loi is building the loft on her summer cottage. At its central point, the height of the loft is 8 feet, and the roof's

pitch should be 24°. How long should the rafters be?

67. **Leaning Tower of Pisa.** The Leaning Tower of Pisa, if standing perfectly vertical, would have a height of 55 meters - unfortunately, the tower is leaning (thus, its famous name). An object dropped from the top of the tower lands 4.8 meters from the base of the tower. At what angle from the horizontal does the tower lean?

68. **Mining.** A mining company locates a vein of minerals at a depth of 32 meters. However, a layer of granite is directly above the minerals, so they decide to drill at an angle, starting 10 meters from their original location. At what angle from the horizontal should they drill?

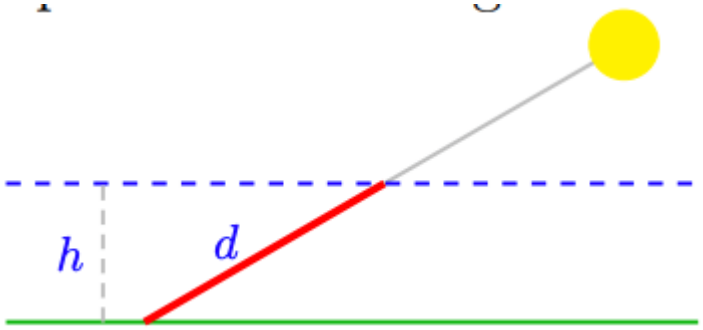
69. **Chichen Itza.** The Mayan pyramid of El Castillo in Chichen Itza, Mexico, has 91 steps, each 26 cm high and 30 cm deep.

    a. What angle does the side of the pyramid make with the horizontal?

    b. What is the distance up the face of the pyramid, from the base to the top platform?

70. **Ferris Wheel.** The first Ferris Wheel was built for the 1893 Chicago World's Fair. It had a diameter of 250 feet, and the boarding platform at the wheel's base was 14 feet above the ground. What would be your height above the ground if you boarded the wheel and rotated through an angle of 50°?

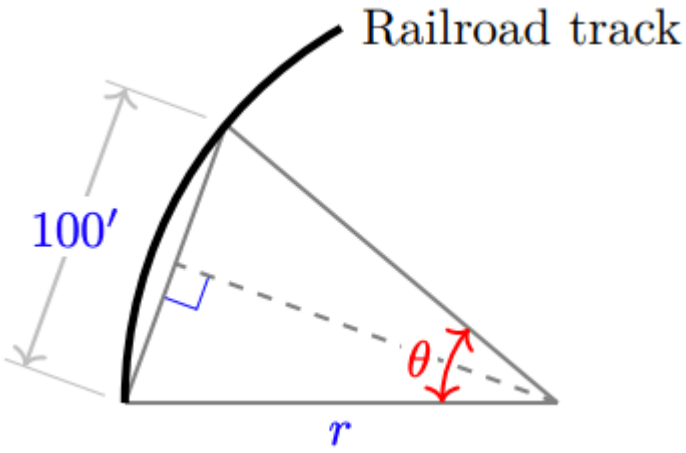
71. **The Physics of Light.** The distance that sunlight must travel to pass through a layer of Earth's atmosphere depends on the thickness of the atmosphere and the sun's angle.



    a. Write an expression for the distance, d, that sunlight travels through a layer of atmosphere of thickness h.

    b. Find the distance (to the nearest mile) that sunlight travels through a 100-mile layer of atmosphere when the sun is 40° above the horizon.

72. **Railroad Engineering.** In railroad design, the degree of curvature of a section of track is the angle subtended by a chord 100 feet long.



a. Use the figure to write an expression for the radius, r, of a curve whose degree of curvature is θ. (Hint: The bisector of the angle θ is perpendicular to the chord.)

b. Find the radius of a curve whose degree of curvature is 43°.

Answers

Skills Refresher

1. 2
2. $\sqrt{3}$
3. $2\sqrt{2}$
4. $\frac{\sqrt{3}}{2}$
5. $\frac{\sqrt{2}}{2}$
6. $\frac{2\sqrt{3}}{3}$
7. $2\sqrt{3}$
8. $\sqrt{2}$

This page titled 6.3.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

6.4: Applications Involving Right Triangles

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We now venture into practical applications of right triangle trigonometry. The concepts introduced in this section are used throughout the rest of this text and are the most common applications (although not the only applications) within Trigonometry.

Angles of Elevation or Depression

Suppose you want to measure the height of a tall tree without climbing it (see the figure below).

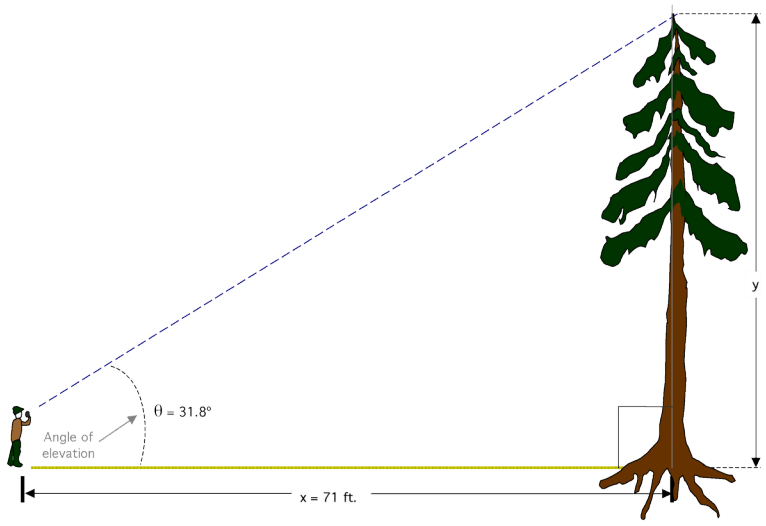


Figure 6.4.1

We can measure the distance to the tree's base and use a surveying tool called a theodolite to measure the angle between the ground and our line of sight to the top of the tree. This angle is called the **angle of elevation**.

Definition: Angle of Elevation

The **angle of elevation** (or **angle of inclination**) is the angle formed between a horizontal line and the line of sight when an observer looks upwards at an object. It represents the amount an observer would need to elevate their gaze (or line of sight) from a horizontal plane to see an object above them.

Since the height we want is the side opposite the angle of elevation, and we already measured the distance to the base of the tree, we could relate the quantities using the tangent function.

Example 6.4.1

Find the height of a cliff if the angle of elevation to the top of the cliff is 58° at a distance of 300 feet from the base of the cliff.

Solution available at the end of this section.

Checkpoint 6.4.1

The tallest living tree is a coastal redwood named Hyperion, measuring 378.1 feet tall. From a distance of 100 feet from the base of the tree, what is the angle of elevation to the top of the tree? Round the answer to the nearest degree.

Answer available at the end of this section.

The angle of elevation has a "sister" concept called the **angle of depression**.

Definition: Angle of Depression

The **angle of depression** (or **angle of declination**) is the angle formed between a horizontal line and the line of sight when an observer looks downwards at an object. It represents the amount an observer would need to lower their gaze (or line of sight) from a horizontal plane to see an object below them.

The two concepts are illustrated in Figure 6.4.3.

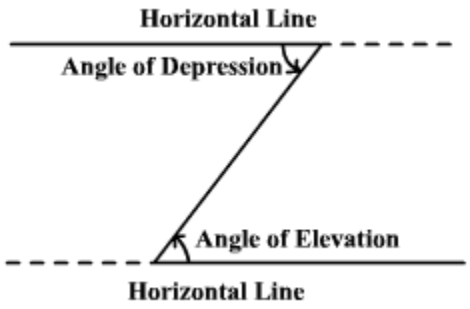


Figure 6.4.3

Example 6.4.2

Phuong takes a helicopter tour through the Grand Canyon. At one point, the pilot hovers 500 feet from the canyon wall. Phuong, being a prepared mathematician, takes out her protractor and measures the angle of elevation to Pima Point (the top of the canyon wall) as 67° and the angle of depression to the canyon floor as 82°. To the nearest foot, what is the elevation gain from the canyon floor to Pima Point?

Solution available at the end of this section.

Checkpoint 6.4.2

On August 7th, 1974, French street performer Philippe Petit began the most famous high-wire walk in history, calmly traversing the space between the former Twin Towers (in New York City) at an elevation of 1,350 feet without any safety gear. Not long after stepping out onto the thin wire, the angle of depression to the base of one building was 88°, and the angle of depression to the base of the other building was 86°. How many feet was his tightrope walk?

Answer available at the end of this section.

Angles of Lines

As a rare moment in Mathematics, we have an alternate definition of **angle of inclination** when it comes to lines.

Definition: Angle of Inclination of a Line

The **angle of inclination of a line** is the measure of the angle in the positive direction from the positive x-axis to the line.

Figure 6.4.5 demonstrates the angle of inclination of a line.

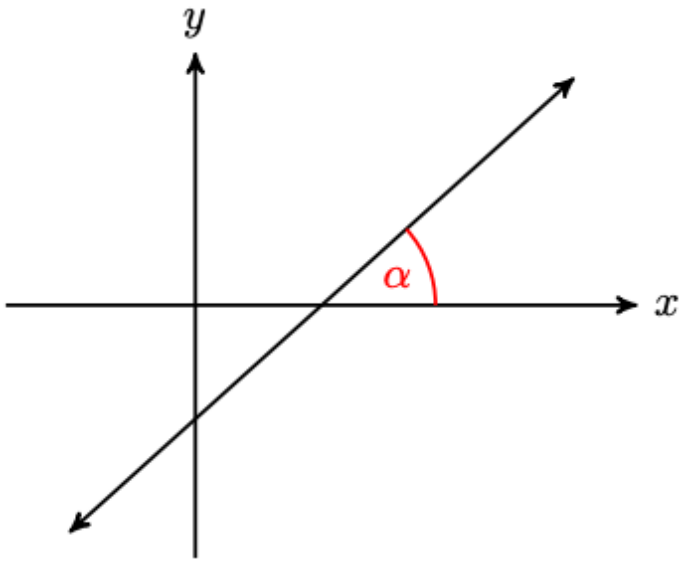


Figure 6.4.5

It is important to note that the angle of inclination to the line is measured from where the line crosses the x-axis. For example, the angle of inclination in Figure 6.4.6 is the *positive* angle, β, between the x-axis (in the positive direction) and the line, and this happens to be an obtuse angle.

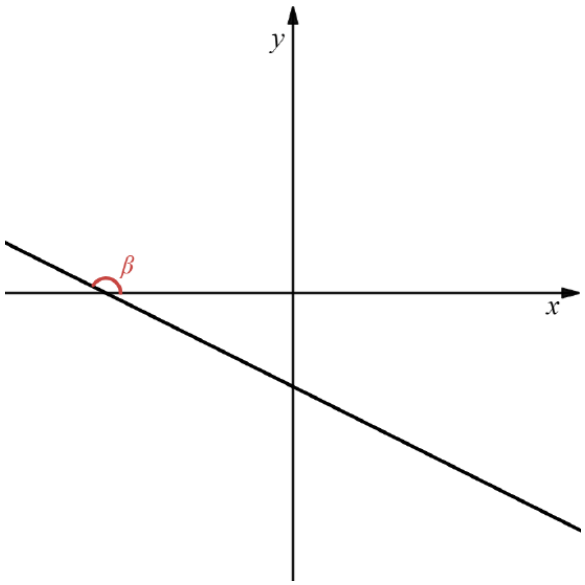


Figure 6.4.6

Recall that the slope of a line is given by the ratio

$$\frac{\text{change in } y}{\text{change in } x}$$

as we move from one point to another on the line. So, if we create a right triangle by dropping a perpendicular segment from the line to the x-axis, the ratio of sides $\frac{\text{opposite}}{\text{adjacent}}$ gives the slope of the line.

Theorem: The Tangent and the Angle of Inclination of a Line

If the slope of the line is m, then the angle of inclination of the line, α, is related to the slope by the formula

$$\tan(\alpha) = m$$

where $0^\circ \leq \alpha \leq 180^\circ$.

Interactive Element: Angle of Inclination and Slope

The following Interactive Element might help you visualize how a line's angle of inclination depends on its slope.

Interact: Drag the large red point to tilt the line, and watch the angle of inclination at the origin, the reference right triangle, and the slope readout update together.

Observation: A positive slope gives an *acute* angle of inclination equal to $\tan^{-1}(\text{slope})$; dragging the point below the x-axis makes the slope negative and the angle of inclination *obtuse*. A vertical line has an inclination of 90° and no defined slope.

Example 6.4.3

Find the angle of inclination of the line $y = \frac{3}{4}x - 3$.

Solution available at the end of this section.

Checkpoint 6.4.3

Find the angle of inclination of the line in Figure 6.4.8.

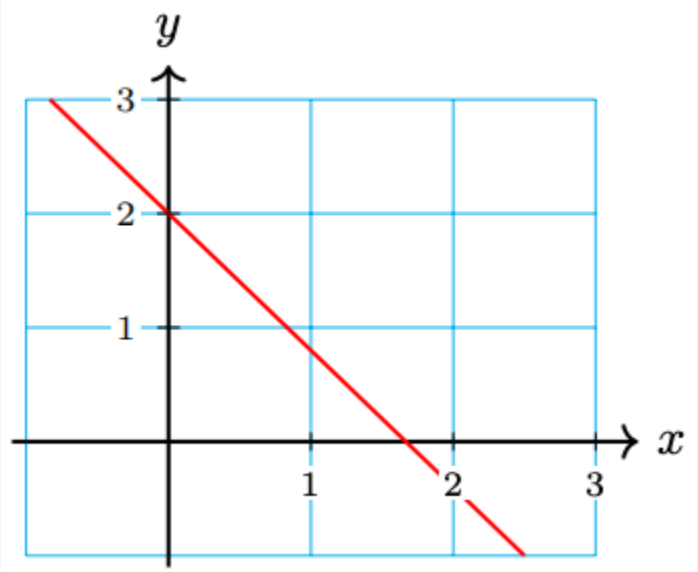


Figure 6.4.8: $y = -\frac{6}{5}x + 2$

Answer available at the end of this section.

Navigation Problems

Up to this point, we have presented two definitions for the trigonometric functions. Coordinate trigonometry allows us to work with any angle (positive, negative, acute, or obtuse) as long as it is in standard position. Right triangle trigonometry will enable us to work with acute angles within a right triangle without a coordinate system. With navigation problems, we officially begin to marry the two definitions into a unified approach to Trigonometry.

Heading

Navigational directions for vessels or aircraft are often given as a **heading**.

Definition: Heading (navigational)

In aviation and nautical navigation, the **heading** of a vessel or aircraft is the compass direction measured clockwise from due north in which the craft's bow or nose is pointed.

For example, a heading of 110° is shown in Figure 6.4.9.

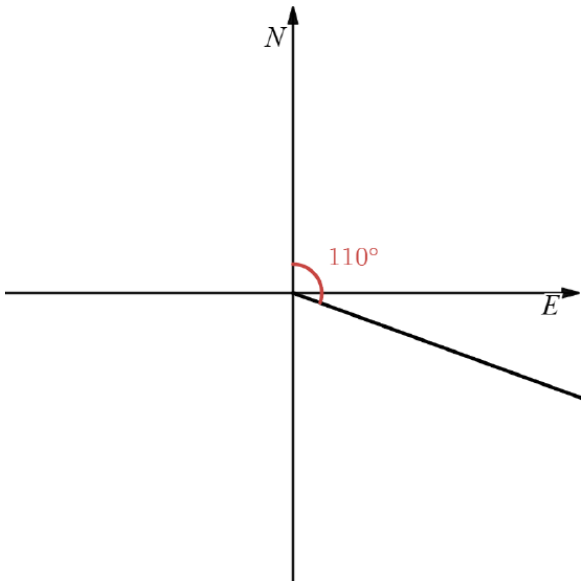


Figure 6.4.9

However, as shown in Figure 6.4.10, this heading generates a terminal side shared with the angle -20° in standard position.

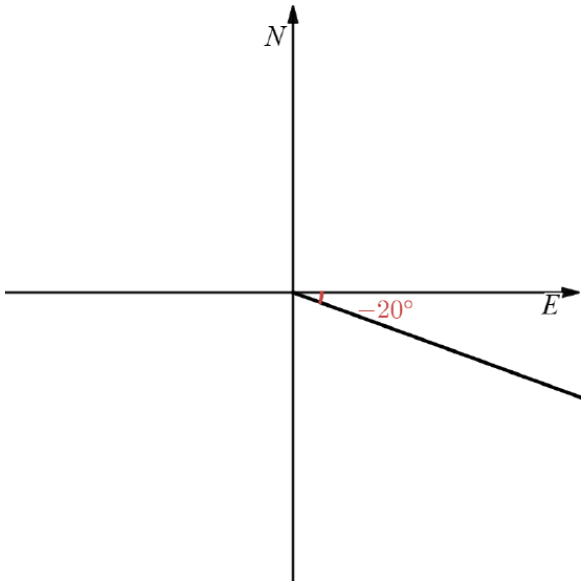


Figure 6.4.10

While there is a method for converting a heading to an angle in standard position, it is much better to get used to making a sketch of the given situation and superimposing a right triangle with hypotenuse along the terminal side of the given heading angle and adjacent side along the x-axis, as shown in Figure 6.4.11.

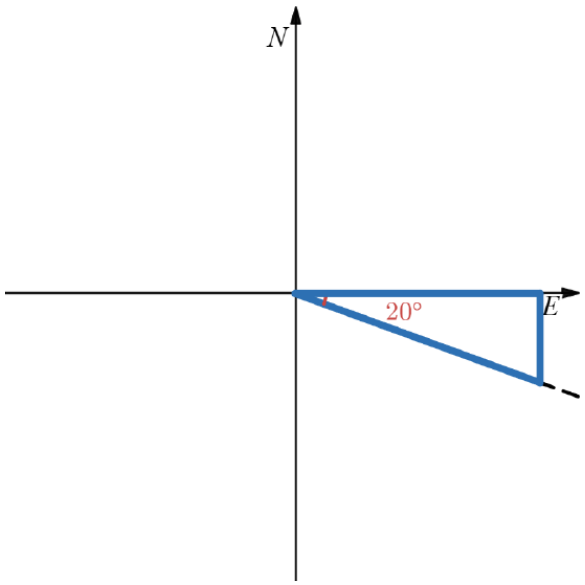


Figure 6.4.11

Once we have the superimposed right triangle, we can use right triangle trigonometry to answer all questions about this situation.

Example 6.4.4

Piyali leaves the airport at a heading of 245° and flies 60 miles. How far south of the airport is she at that time?

Solution available at the end of this section.

Checkpoint 6.4.4

Lap leaves the airport and flies 150 miles at a heading of 132°. How far east of the airport is he at that time?

Answer available at the end of this section.

Bearing

Land-based navigation often uses **bearing** instead of heading.

Definition: Bearing (navigational)

In land-based navigation, the **bearing** (also called the **quadrant bearing**) of an object (person, vehicle, etc.) represents the direction that object is facing or moving. The notation for bearing is a letter followed by an acute angle followed by a letter. These are known as **Meridian - Angle - Direction**.

- **Meridian:** The first letter indicates whether the angle is measured from North (N) or South (S), whichever is closer to the direction.
- **Angle:** This is the acute angle measured from the chosen meridian to the direction of interest.
- **Direction:** The second letter indicates whether the angle is measured towards the East (E) or West (W).

An example of a bearing is $\mathrm{S}\ 20^\circ\ \mathrm{W}$ (read as, "south by 20 degrees west"). Conceptually, this means the following: initially face south, then turn by 20° in the westerly direction. Thus, a bearing describes a line as heading north or south and deflected some degrees toward the east or west. Since a bearing always starts with a north or south facing and then a turn toward east or west, bearings will always have an acute angle.

Figure 6.4.15 illustrates some directions given in bearings.

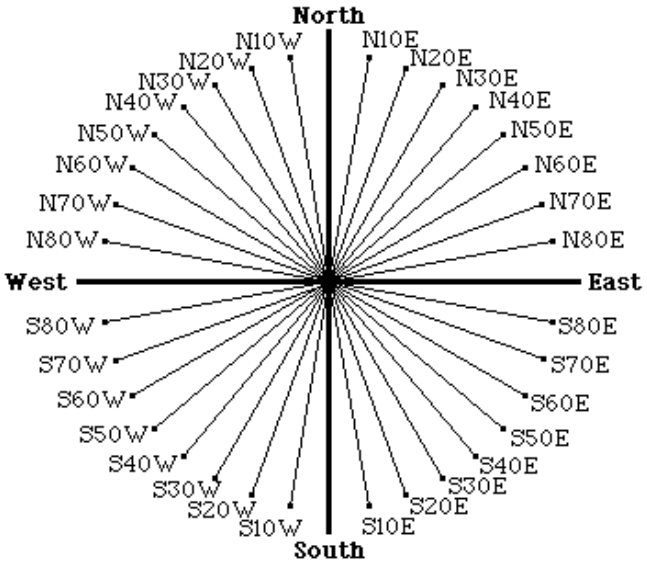


Figure 6.4.15

When working with directions given in bearings, it is crucial that you understand the angle given will always be between the y-axis (the north-south line) and the terminal side of the angle. This is the only time in Trigonometry you will see such a convention. As long as you keep your thinking cap on and do not try to memorize your way through Trigonometry, you should be fine (especially if you get in the habit of superimposing right triangles on your sketches as we did with headings).

Example 6.4.5

Roy is hiking on the Pacific Crest Trail. He leaves his campsite at 6:00 A.M., walking at a bearing of $\mathrm{N}\ 37^\circ\ \mathrm{W}$. His average hiking speed is 3.4 miles per hour. At 9:00 A.M., he decides to eat breakfast at Scissor's Crossing. At this time, how far north is he from the campsite?

Solution available at the end of this section.

Caution: Avoid Creating Reference Triangles Along the y-axis

You would not be incorrect if you superimposed a right triangle with a side along the y-axis. If you do this, you will use the cosine of 37° to determine how far north Roy is from his campsite; however, superimposing right triangles with sides along the y-axis will eventually lead to some confusion and mistakes. It's best to nip that bad habit in the bud and always superimpose our right triangles with sides along the x-axis.

Checkpoint 6.4.5

Ivan leaves the airport and flies 150 miles at a bearing of $\mathrm{S}\ 48^\circ\ \mathrm{E}$. How far east of the airport is he at that time?

Answer available at the end of this section.

Example 6.4.6

Mount Whitney is 5.86 miles due east of the Pacific Crest Trail, labeled as point P. The John Muir Trail, labeled as point J, is 1.32 miles due south of Mount Whitney. You hike from Mount Whitney to point J on the John Muir Trail. Once at point J, you decide to hike to point P on the Pacific Crest Trail. What is the bearing in which you must hike *from* point J on the John Muir Trail *to* point P on the Pacific Crest Trail, and how many miles will your hike be? (For simplicity, we will ignore the fact that there are significant elevation gains and losses between each of these points.)

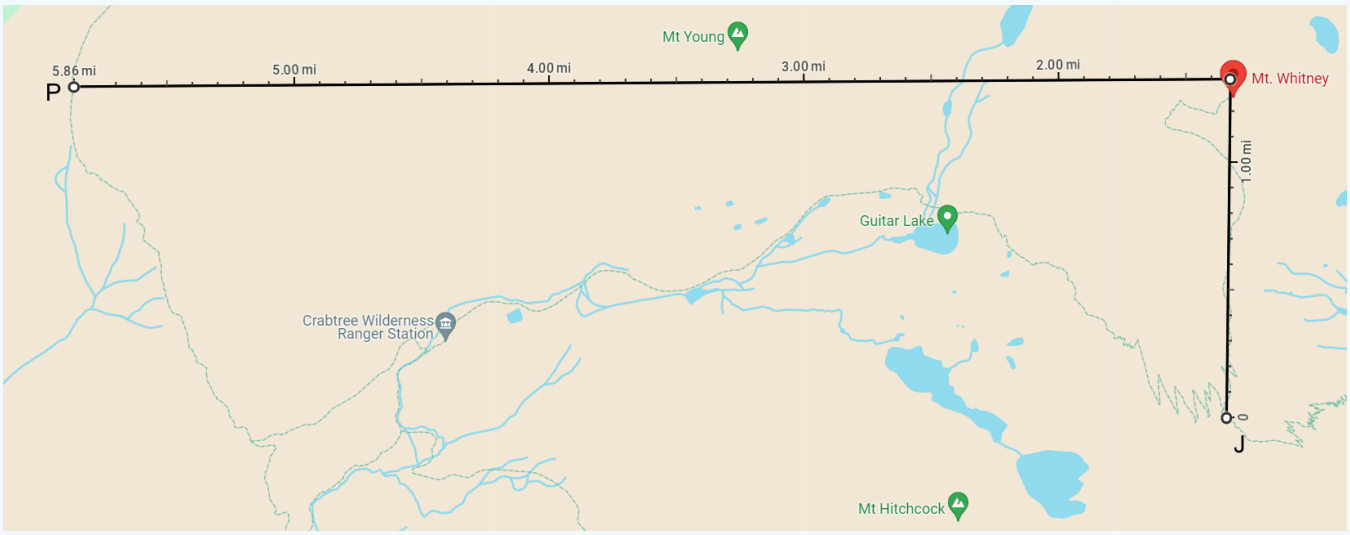


Figure 6.4.18

Solution available at the end of this section.

Checkpoint 6.4.6

Sutter County Airport is 28.4 miles due north of Sacramento International Airport. If Placerville Airport is 45.3 miles due east of Sacramento International Airport, what is the bearing from Sutter County Airport to Placerville Airport, and what is the distance between these two airports?

Answer available at the end of this section.

In Examples 6.4.5 and 6.4.6, we were given a single destination for a bearing. We were essentially told, "Go this direction for so many miles." However, an important application within navigation occurs when you have been given "stop-over points" where the bearing is changed. The following example illustrates, in great detail, how to manage these problems.

Example 6.4.7

Roy has made significant progress along the Pacific Crest Trail and has reached the Sierra Nevada range. Last night, he made camp at Junction A. In the morning, he breaks camp at 6:00 A.M. and hikes at a bearing of $\mathrm{N}\ 42^\circ\ \mathrm{E}$. After two hours, he reaches Junction B, where the trail turns 90° in the bearing of $\mathrm{N}\ 48^\circ\ \mathrm{W}$. After another hour, he runs into Junction C, which is a "shortcut" trail that directly connects his current location back to his original campsite at Junction A.

Normally, Roy would never consider "back-tracking" as it's a complete waste of time and energy; however, seeing the "shortcut" trail sparks his memory that he left his trash bag back at his campsite (Junction A)! Being a conscientious hiker, he decides to burn the time and return to ensure he "leaves no trace" on the trail.

If Roy hikes at an average pace of 3.4 miles per hour, what is the distance between Junction C and Junction A, and what is his bearing from Junction C to Junction A?

Solution available at the end of this section.

Note: Place Small Sets of Axes at Each Stop-Over Point

As Example 6.4.7 illustrates, bearing problems where direction changes occur are more manageable if you get into the good habit of drawing small "north-south, east-west" axes at each stop-over point.

Checkpoint 6.4.7

A ship leaves port bearing $\mathrm{S}\ 32^{\circ}\ \mathrm{E}$. It travels in this direction for 1 mile and changes its bearing to $\mathrm{S}\ 58^{\circ}\mathrm{W}$. It then travels an additional 10 miles. At this point, the captain remembers he left his phone back at port (and he wanted to post a ton of videos of this trip on his social media platforms). Find the direct distance and the bearing from this point back to port so that the "social-influencer" captain can get his phone. (Round answers to the nearest tenth.)

Answer available at the end of this section.

Compound Right Triangle Applications

An application that requires you to use more than a single right triangle is called a **compound right triangle application** (don't bother looking for that phrase elsewhere, we just made it up to label applications that fit into this category). These entertaining applications often require you to think in three dimensions, and they help build your ability to break larger problems into small pieces.

Example 6.4.8

Camille is worried about her neighbor's tree eventually falling and hitting her house. Unfortunately, her neighbor has refused to allow her to pay someone to measure the exact height of the tree, so Camille resorts to her top-notch math skills. Since she cannot step onto her neighbor's property, Camille stands on her property, facing the tree, and finds the angle of elevation to the top of the tree to be 38°. She then turns 90° and walks for 32 feet (staying on her property the entire time). At this point, she turns to face her neighbor's tree and measures the angle between the path she just walked and her neighbor's tree to be 50°. Use this information and Figure 6.4.25 to find the tree's height.

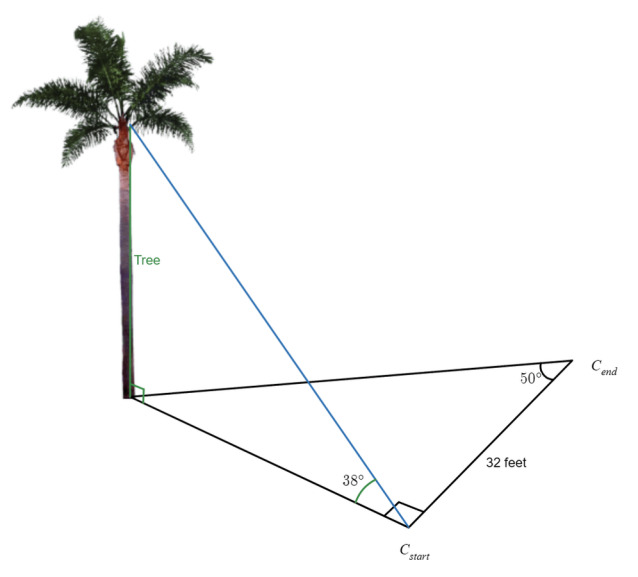


Figure 6.4.25

Solution available at the end of this section.

Checkpoint 6.4.8

Sang's boat loses power while on a boating trip out on the ocean. She radios her position to a pair of radar stations 43 miles apart along the coastline. The radio stations are directly north and south of each other. The bearing of her boat from the southern radio station is $\mathrm{N}\ 54^\circ\ \mathrm{W}$, while the bearing from the northern radio station is $\mathrm{S}\ 72^\circ\ \mathrm{W}$. To the nearest mile, how far from the coastline is Sang's boat?

Answer available at the end of this section.

Success in College: Learning From Tests (Part 2)

1. A student earned 75% of the points on her last math test. On the first class day after the test, the class started a new chapter with new material. She does not understand several vital procedures from the last test. What do you think she should do about this situation?
2. A student usually finishes tests right as class ends. He doesn't have time to check for careless mistakes. He looks carefully at his corrected tests. He realizes that many of his errors are arithmetic or algebra mistakes. He knew how to do the problems; he didn't do them well enough. What do you think he should do?

Success in College: Cheating and Consequences

The following is an excerpt from an article at WorldNetDaily.com by Dennis Prager.

> A report by Donald McCabe, professor of organization management at Rutgers University and founding president of the Center for Academic Integrity at Duke University, describes a pretty bad ethical situation in America's high schools. Seventy-five percent of students engage in "serious cheating" (such as repeated cheating and significant plagiarism).
>
> It gets worse.
>
> First, according to the professor, this epidemic is the culmination of three decades of increased cheating in high schools.
>
> Second, about half of all students (not just half of all cheaters) see nothing wrong with cheating.
>
> As one who has written and lectured on ethics for 30 years, during which time I, too, have spoken at high schools across America, I am certain that these statistics are accurate. I also believe that I know the major reasons.
>
> One reason is that parents emphasize many things over character.
>
> A second reason is the absurd emphasis on getting into the best college. This preoccupation on the part of parents (which often begins in preschool) has utterly distorted our children's (and high schools') priorities. Because students are taught that getting into the right college is the most important thing in their lives, they readily compromise their character to achieve this goal. And they could not care less about learning. Since the purpose of school is to get good grades, learning is incidental, and loving learning is a non-issue.
>
> A third and final reason is the death of the concept of sin, a product of the thoroughly secular education American high schools offer. As noted, far worse than the number of students who cheat is the number of students who think that there is nothing wrong with cheating. I can testify from my talks at high schools that the vast majority of students believe what they have been taught - that right and wrong - are entirely subjective concepts. There is no morality higher than one's own; therefore, right and wrong are defined by the individual, who can rationalize any behavior.

When students do not understand a concept, they need help. Instead, some students in this situation choose to cheat. In college, any form of cheating is "academic dishonesty."

1. Has your professor discussed or posted their cheating policy?
2. What happens if you cheat in this math class?

   To be clear, cheating - in any form - results in more than just a "zero" on that assignment. This includes, but is not limited to, using notes on exams, using a device that is not allowed during an exam, or using unapproved technology to help during an exam. If I suspect a student of cheating, I will have them meet me outside of class time in my office to discuss what happened and why it is considered cheating. They will receive a zero on the assignment/exam and will be reported to the Student Discipline Officer for further discussions. The Student Discipline Officer is involved for two reasons: to give the student tools to help them break the "cheating habit," and to create an official record within the Los Rios Community College District to track that student's cheating offenses - if multiple offenses have occurred, the student could like be placed on academic probation or, in severe cases, completely expelled from the District.

   - Professor Simpson, Cosumnes River College

Answers

Section Preview

This section expands on the applications and methods introduced in the previous section. The applications treated here cover angles of elevation and depression, navigation, and right triangle applications in three dimensions. As is the case with everything in this chapter, all angles are acute.

Section Preview

- **Linear Functions**
  - The Slope of a Line: To understand the angles of inclination and declination, students must have a strong understanding of slope.

Section Preview

- **Miscellaneous Applications**
  - Rates, Unit Rates, and Uniform Motion (Distance, Rate, Time): Students who have not seen $d = rt$ will be stumped by navigation setups in this section.
- **Linear Functions**
  - Equations of Lines — Slope-Intercept Form: Students will be asked to find the equation of a line given its angle of inclination.

Section Preview

- Solve an application problem involving an angle of elevation or depression.
- Determine the angle of inclination (or declination) of a line.
- Solve a navigation problem involving bearing or heading.
- Solve a compound right triangle application.

Example 6.4.1

As always, we begin by sketching the situation.

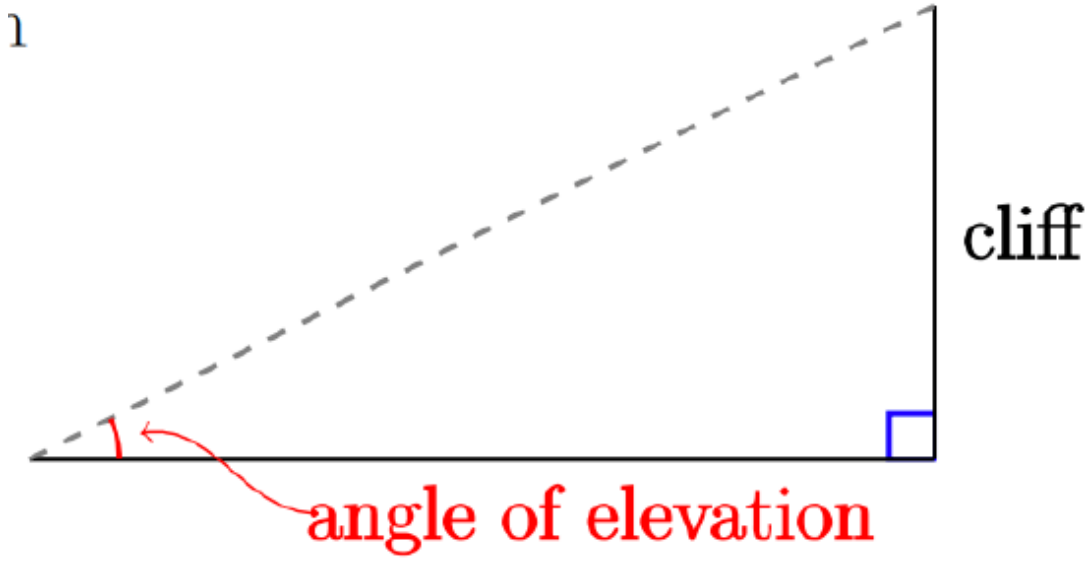


Figure 6.4.2

We know the angle of elevation, 58°, and the side length adjacent to that angle, 300 feet. We use the tangent function to find the cliff's height (which is opposite the given angle). Letting h be the height of the cliff, we get

$$\tan(58^\circ) = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{300} \implies h = 300\tan(58^\circ) \approx 480 \text{ feet.}$$

Checkpoint 6.4.1

75°

Example 6.4.2

We start by sketching the situation.

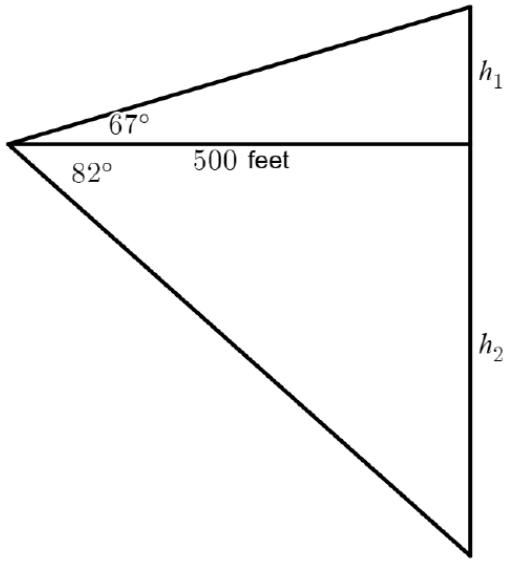


Figure 6.4.4

We must find the value of $h_1 + h_2$. The trigonometric function relating 67° to h_1 and 500 is the tangent. Thus,

$$\tan(67^\circ) = \frac{h_1}{500} \implies 500\tan(67^\circ) = h_1.$$

Similarly,

$$\tan(82^\circ) = \frac{h_2}{500} \implies 500\tan(82^\circ) = h_2.$$

Therefore, the elevation gain from the canyon floor to Pima Point is

$$h_1 + h_2 = 500\tan(67^\circ) + 500\tan(82^\circ) \approx 4,736 \text{ feet}.$$

Checkpoint 6.4.2

Roughly 141.5 feet.

Example 6.4.3

The slope of the line is $\frac{3}{4}$. Therefore,

$$\begin{aligned} \tan(\alpha) &= \frac{3}{4} \\ \implies \alpha &= \tan^{-1}\left(\frac{3}{4}\right) \approx 36.9^\circ \end{aligned}$$

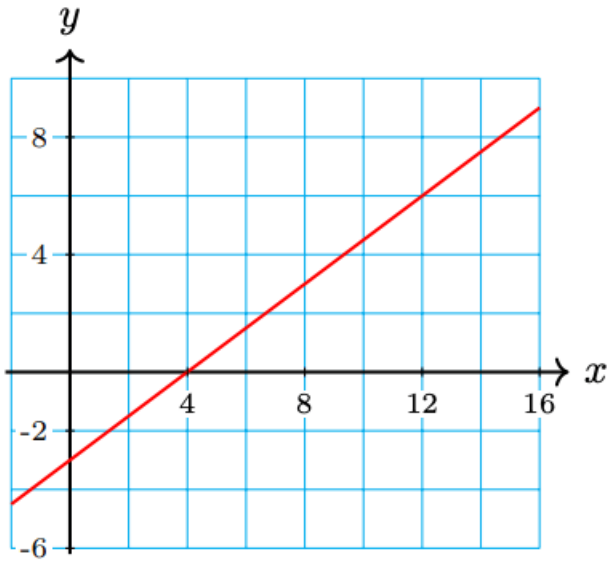


Figure 6.4.7

The angle of inclination is approximately 36.9°.

Checkpoint $6.4.3$

$\approx 129.8^\circ$

Example $6.4.4$

Sketching a heading angle of 245°, we get

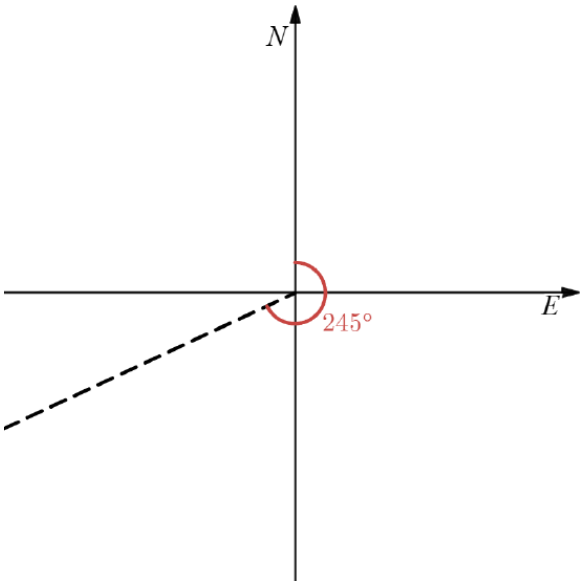


Figure $6.4.12$

We now superimpose a right triangle on this graph, where the hypotenuse is on the terminal side of the heading angle and the adjacent side is on the x-axis.

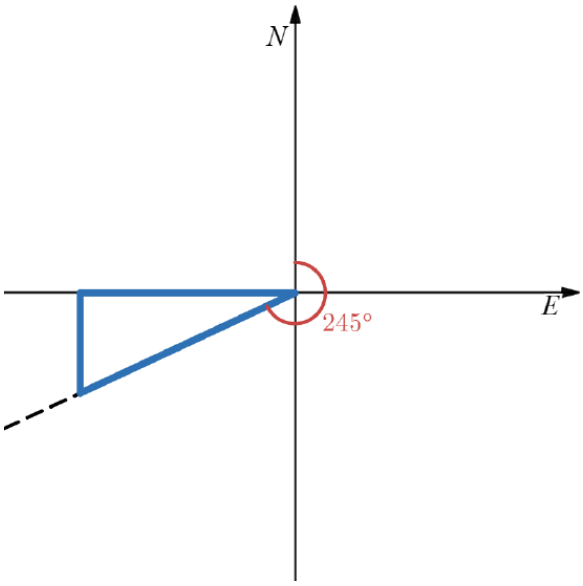


Figure $6.4.13$

The key to using this superimposed right triangle is to use our *deductive* skills to determine the angle between the hypotenuse (terminal side of the heading angle) and the adjacent side (x-axis). Had the heading angle been 270°, Piyali's plane would be heading directly west along the negative x-axis (the adjacent side of the right triangle). This means there must be a small angle of 25° between the adjacent side of the right triangle and the hypotenuse. Said differently, to get to the 245° heading, we could have turned clockwise from due north by 270° and then backed off by 25°. Figure $6.4.14$ shows the resulting angle within the superimposed right triangle.

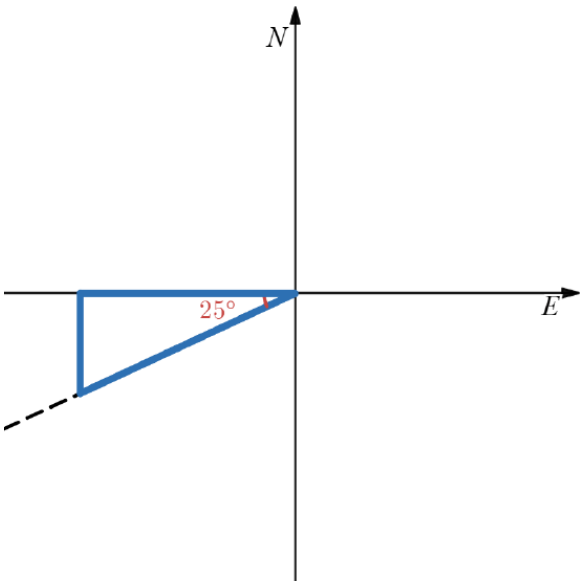


Figure 6.4.14

Piyali's plane flew along the hypotenuse of this right triangle for 60 miles. To determine how far south she is from the airport (located at the origin), we need to compute the length of the side opposite the 25° angle. The trigonometric function relating this angle, the opposite side, and the hypotenuse is the sine.

$$\sin(25^\circ) = \frac{\text{opposite}}{60} \implies \text{opposite} = 60\sin(25^\circ) \approx 25.36.$$

Hence, Piyali is about 25.4 miles south of the airport.

Checkpoint 6.4.4

111.5 miles

Example 6.4.5

Quickly sketching the situation, we get Figure 6.4.16.

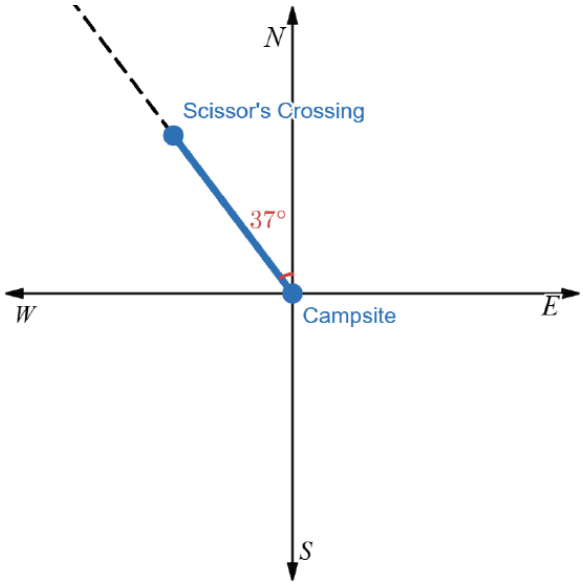


Figure 6.4.16

We superimpose a right triangle on this figure, where the hypotenuse is along the terminal side of the bearing angle, and one side is on the x-axis. The result is Figure 6.4.17.

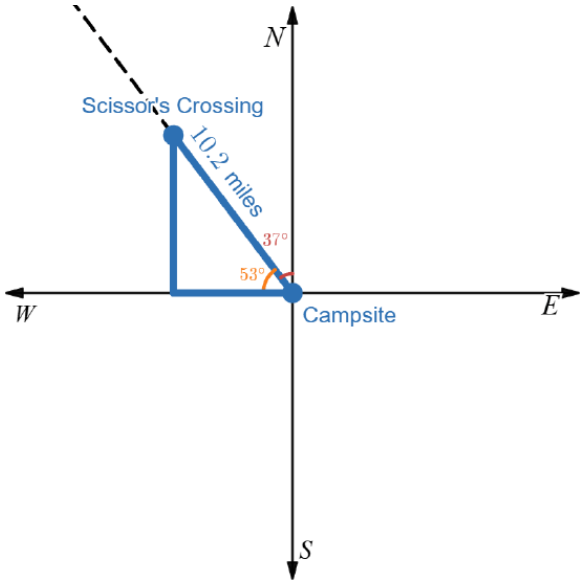


Figure 6.4.17

Notice that we already included that Roy has hiked a total of $3 \times 3.4 = 10.2$ miles by 9:00 A.M. Also, from the superimposed right triangle, we can see that the trigonometric function relating the hypotenuse to the vertical distance and the 53° angle (which is the complement of the given 37° angle) is the sine function. Hence,

$$\sin(53^\circ) = \frac{y}{10.2} \implies y = 10.2\sin(53^\circ) \approx 8.1461.$$

Therefore, Roy is approximately 8 miles north of his campsite at 9:00 A.M.

Checkpoint 6.4.5

111.5 miles

Example 6.4.6

With bearing problems, it is beneficial to draw a small set of axes at the point we are moving *from*, as seen in Figure 6.4.19.

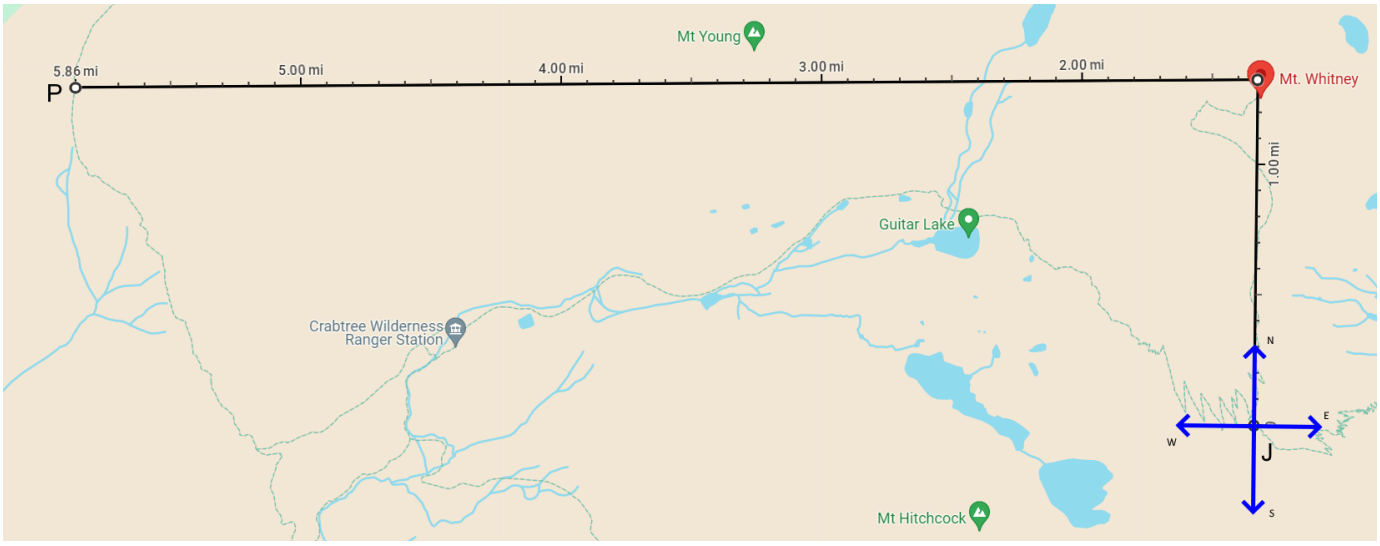


Figure 6.4.19

We then draw a line connecting points J and P, as follows.

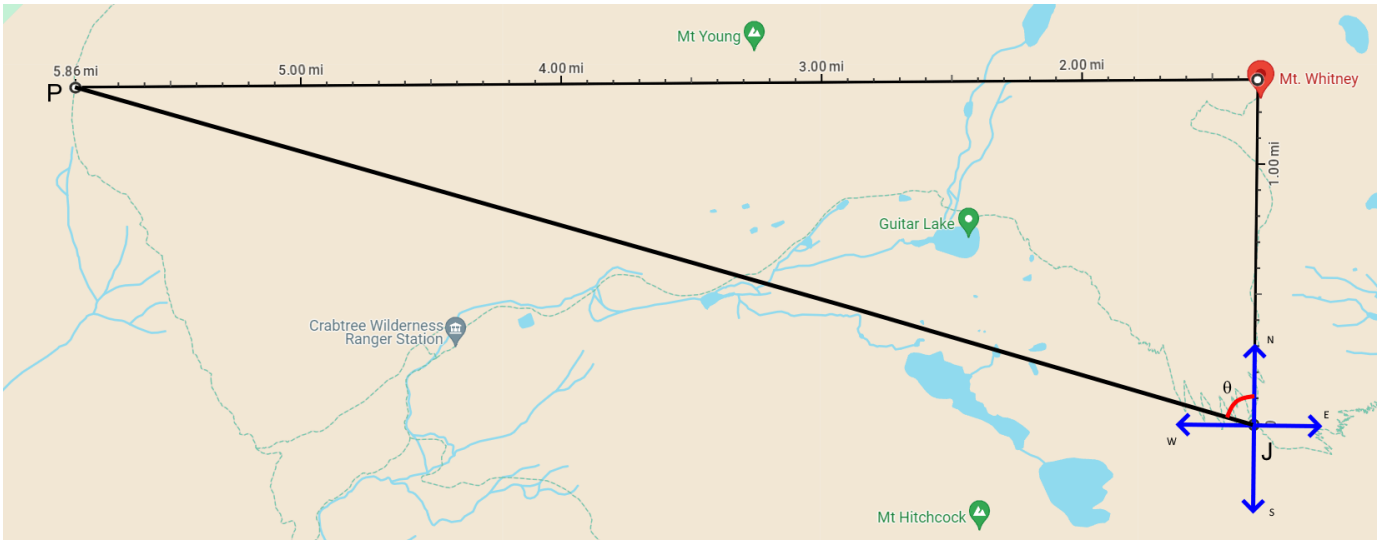


Figure 6.4.20

From the given image, the bearing *from* point J *to* point P has the form $\mathrm{N}\ \theta^\circ\ \mathrm{W}$. The tangent is the trigonometric function that relates the given sides of the right triangle and the angle θ. Hence,

$$\tan(\theta) = \frac{5.86}{1.32} \implies \theta = \tan^{-1}\left(\frac{5.86}{1.32}\right) \approx 77.3^\circ.$$

Hence, to hike from point J on the John Muir Trail to point P on the Pacific Crest Trail, we must maintain an approximate bearing of $\mathrm{N}\ 77.3^\circ\ \mathrm{W}$. The distance covered during that hike (if there was a direct trail between points J and P) is the hypotenuse of the right triangle.

$$d = \sqrt{5.86^2 + 1.32^2} \approx 6 \text{ miles}.$$

Checkpoint 6.4.6

$\mathrm{S}\ 57.9^\circ\ \mathrm{E}$, and 53.5 miles.

Example 6.4.7

This is an excellent example for several reasons. First, in the "real world," the application of mathematics often occurs when you have a lot of extraneous information and "word noise" in a problem. It would be best to get used to simplifying the language in a situation and removing extraneous bits. Here's a great rewording:

"The bearing from A to B is $\mathrm{N}\ 42^\circ\ \mathrm{E}$ and the distance is 6.8. The bearing from B to C is $\mathrm{N}\ 48^\circ\mathrm{W}$ and the distance is 3.4. Find the distance between A and C, and the bearing from C to A."

Now that we have removed as much extraneous wording as possible, let's draw a picture of the situation. We begin with a simple sketch of the bearing from A, including that Roy hiked for 2 hours at 3.4 miles per hour.

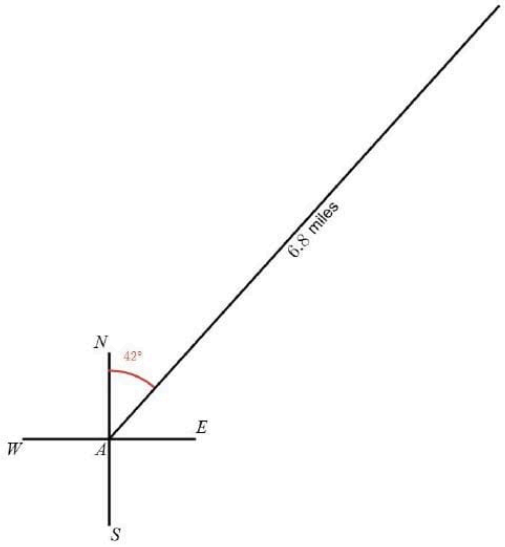


Figure 6.4.21

It's important to note that we drew the north-south and east-west axes at the initial point in Figure 6.4.21. Once we have drawn the angle and its terminal side, we place B at the tip of that terminal side (see Figure 6.4.22. We also draw in another set of north-south and east-west axes.

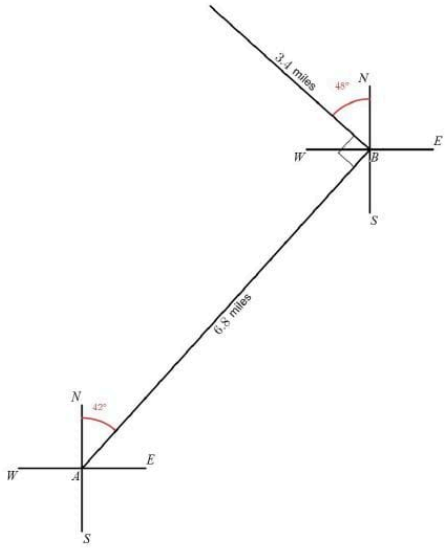


Figure 6.4.22

The next "stop-over" point, at the end of that last 3.4-mile hike, is C. Plotting that point and drawing in a final set of north-south and east-west axes, we arrive at Figure 6.4.23 below.

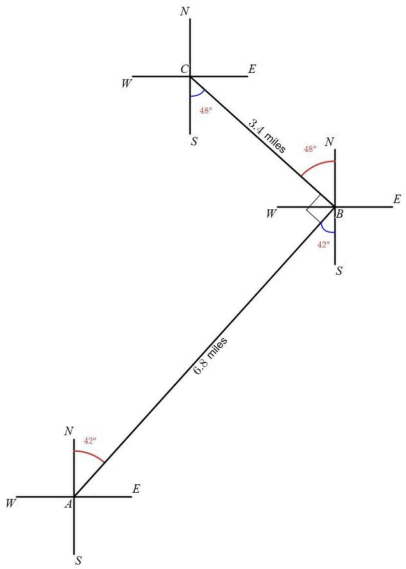


Figure 6.4.23

Take a moment to compare Figures 6.4.22 and 6.4.23. The latter figure has more information than just an extra set of axes—we have included alternate interior angles! As you will soon see, this information will be critical in helping us finish the problem. Speaking of which, let's draw a line connecting C to A.

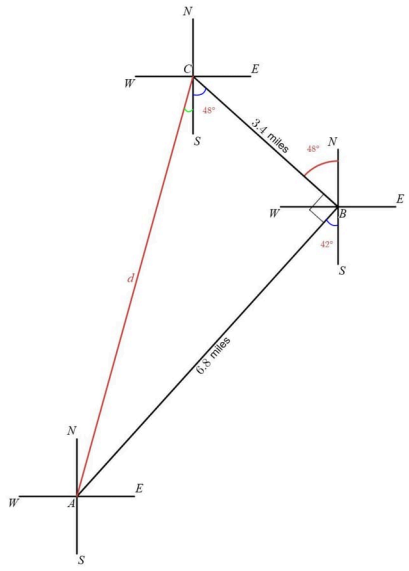


Figure 6.4.24

Figure 6.4.24 is the most important for you to be able to draw. It would be best if you had it to complete this problem successfully. The triangle we have drawn is a right triangle. We should immediately know this from the clue that Roy turns 90° once he arrives at Junction B ($\angle ABC + 42^\circ + 48^\circ = 180^\circ \implies \angle ABC = 90^\circ$). The distance back to A from C is the length, d, of the hypotenuse of the right triangle. Hence,

$$d = \sqrt{6.8^2 + 3.4^2} \approx 7.6 \text{ miles}.$$

To find the bearing, we first must recognize that, from C, Roy must face south and turn through a small angle in the westerly direction to face A. We need that little sliver of a green angle. Focusing only on the right triangle, we can see that

$$\tan(\angle ACB) = \frac{6.8}{3.4} = 2 \implies \angle ACB = \tan^{-1}(2) \approx 63.4^\circ.$$

However, if we call that little sliver of an angle α, then $\angle ACB = \alpha + 48^\circ$. Therefore,

$$63.4^\circ \approx \alpha + 48^\circ \implies \alpha \approx 15.4^\circ.$$

Thus, Roy's bearing from C to A is $\text{S } 15.4^\circ \text{ W}$.

Checkpoint 6.4.7

10.0 miles at a bearing of $\text{N } 52.3^\circ \text{ E}$.

Example 6.4.8

This is a compound right triangle problem because the situation requires *at least* two right triangles. The key to solving such problems is to recognize that these triangles share a common side. If we call this shared side s, we can "separate" the triangles, as seen in Figure 6.4.26.

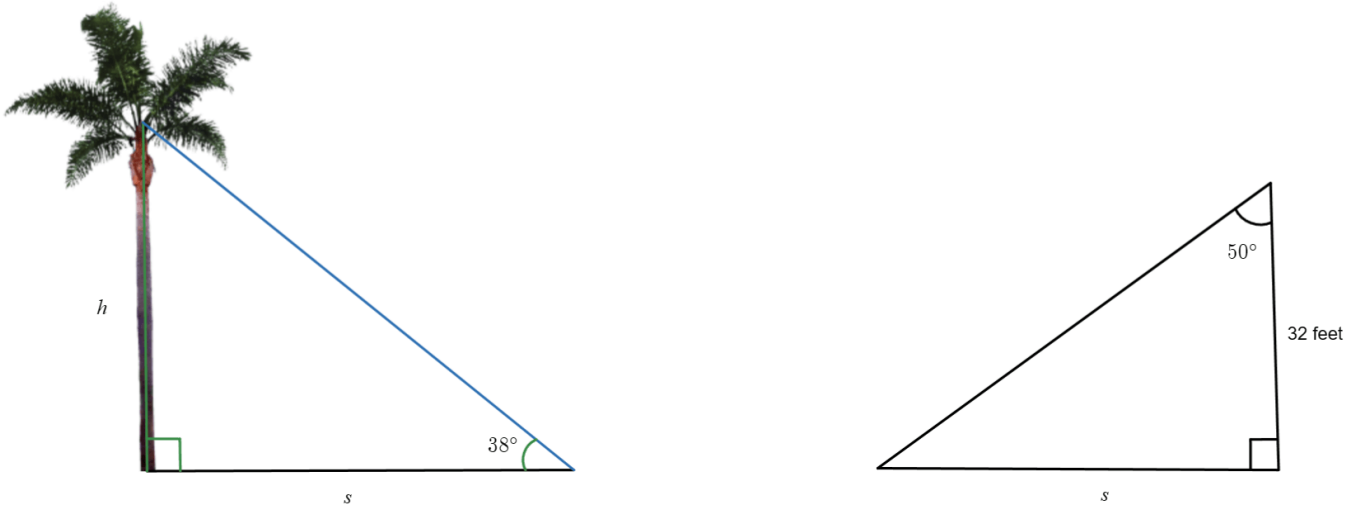


Figure 6.4.26

In both cases, we can use the tangent function to relate the given angle and the two sides so that

$$\tan(38^\circ) = \frac{h}{s} \quad \text{and} \quad \tan(50^\circ) = \frac{s}{32}.$$

We now use the shared side, s, as the common link between the triangles. That is, solve both equations for s.

$$s = \frac{h}{\tan(38^\circ)} \quad \text{and} \quad s = 32\tan(50^\circ).$$

Therefore,

$$\frac{h}{\tan(38^\circ)} = 32\tan(50^\circ).$$

We solve this final equation for h to find that the height of the tree is

$$h = 32\tan(50^\circ)\tan(38^\circ) \approx 29.8 \text{ feet}.$$

Checkpoint 6.4.8

41 miles

This page titled 6.4: Applications Involving Right Triangles is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

6.4.1: Resources and Key Concepts

Resources

Videos

- Angle of Elevation and Depression
  - Trigonometry of Right Triangles - Review of Applications of Right Triangle Trigonometry (Part 1)
- Compound Right Triangle
  - Right Triangle Trigonometry - Applications of Right Triangle Trigonometry (Part 1)

Key Concepts

Definitions

- **Angle of Elevation (or Angle of Inclination):** The angle formed between a horizontal line and the line of sight when an observer looks upwards at an object.
- **Angle of Depression (or Angle of Declination):** The angle formed between a horizontal line and the line of sight when an observer looks downwards at an object.
- **Angle of Inclination of a Line:** The measure of the angle in the positive direction from the positive x-axis to the line.
- **Heading (navigation):** The compass direction in which a vessel or aircraft is pointed, measured clockwise from due north.
- **Bearing (navigational):** The direction of an object, given as an acute angle measured from a north-south line toward either east or west. It is also called a quadrant bearing.

Theorems

- **The Tangent and the Angle of Inclination of a Line:** If a line has a slope of m, then its angle of inclination, α, is related to the slope by the formula $\tan(\alpha) = m$, where $0^\circ \leq \alpha < 180^\circ$.

Common Mistakes

- **Confusing Heading and Bearing:** A heading is a single angle measured clockwise from due north. A bearing is an acute angle measured from the north-south line toward the east or west.
- **Incorrectly Orienting Triangles in Navigation Problems:** When superimposing a right triangle onto a navigation sketch, it is best to always place one of the legs on a horizontal (east-west) axis to avoid confusion, rather than on a vertical (north-south) axis.
- **Forgetting to Place Axes at Stop-Over Points:** In navigation problems with multiple parts or changes in direction, it is much easier to manage the geometry if you draw a new set of small north-south, east-west axes at each point where a turn is made.

This page titled 6.4.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

6.4.2: Homework

Reading Questions

1. What is the difference between an angle of elevation and an angle of depression?
2. How is the angle of inclination of a line defined?
3. How is the slope of a line, m, related to its angle of inclination, α?
4. In navigation, what is a "heading"? From what direction and in what manner is it measured?
5. What are the three components of a navigational "bearing"?
6. What is the primary difference between how a heading is measured and how a bearing is measured?
7. In a multi-part navigation problem, what is a helpful drawing strategy to use at each "stop-over" point?
8. What defines a "compound right triangle application"?
9. In a compound right triangle problem, what is the key to solving the problem?

Homework

Vocabulary Check

1. The angle between the horizon and the line of sight above the horizon is called the angle of ___, also known as the angle of ___.
2. A(n) ___ is an angle measured clockwise from due north.
3. A(n) ___ is a direction given in the form north or south by a certain number of degrees east or west (denoted, for example, by $\mathrm{S}\ 13^\circ\ \mathrm{W}$).

Concept Check

4. If the angle of inclination of a line is greater than 45°, what can you say about its slope?
5. What is the bearing that shares a terminal side with the standard position angle $\theta = 45^\circ$?
6. What is the bearing that shares a terminal side with the standard position angle $\theta = 120^\circ$?
7. What is the bearing that shares a terminal side with the standard position angle $\theta = -150^\circ$?

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

8. An angle of depression is measured from due south to an individual's line of sight below the horizon.
9. The angle given in a bearing must be acute.
10. The angle given in a heading must be acute.
11. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle 120°.
12. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle -120°.
13. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle 60°.
14. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle -60°.
15. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle 240°.

16. The angle $\theta = 120^\circ$ in standard position has a terminal side shared with the heading angle -240°.

Basic Skills

For the following exercises, sketch the situation using a right triangle and label each angle.

17. The angle of elevation from the ground to the top of a power pole is 50°.

18. The angle of depression from the eye of a hiker on a cliff to the bottom of the canyon is 85°.

19. Two people are looking up at a hot air balloon. They are positioned on opposite sides of the ballon. The angle of elevation to the ballon for one person is 57°, and the angle of elevation to the ballon for the other person is 30°.

For the following exercises, sketch a diagram illustrating the given bearing.

20. The bearing from Sacramento to South Lake Tahoe is $\text{N } 72^\circ \text{ E}$.

21. The bearing from New York City to Philadelphia is $\text{S } 48^\circ \text{ W}$.

22. The bearing from the Southern Terminus of the Pacific Crest Trail to the Northern Terminus is $\text{N } 10^\circ \text{ W}$.

23. The bearing from Lincoln Airport to Placerville Airport is $\text{S } 69^\circ \text{ E}$.

For the following exercises, find the angle of inclination of the line.

24. $y = \frac{5}{4}x - 3$

25. $y = 6 + \frac{2}{9}x$

26. $y = -2 - \frac{3}{8}x$

27. $y = \frac{-7}{2}x + 1$

For the following exercises, find an equation for the line passing through the given point with an angle of inclination α.

28. $(3, -5), \alpha = 28^\circ$

29. $(-2, 6), \alpha = 67^\circ$

30. $(-8, 12), \alpha = 112^\circ$

31. $(-4, -1), \alpha = 154^\circ$

Synthesis Questions

32. The slope of a line is a function of its angle of inclination, $m = f(\alpha)$. Complete the table and sketch a graph of the function.

α	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
m													

   a. What happens to the slope of the line as α increases toward 90°?

   b. What happens to the slope of the line as α decreases toward 90°?

33. The angle of inclination of a line is a function of its slope, $\alpha = g(m)$. Complete the table and sketch a graph of the function.

m	-20	-10	-5	-2	-1	-0.75	-0.5	-0.25	0
α									

m	0.25	0.5	0.75	1	2	5	10	20
α								

a. What happens to the angle of inclination as the slope increases toward infinity?

b. What happens to the angle of inclination as the slope decreases toward negative infinity?

Applications

Angles of Elevation and Angles of Depression

For the following exercises,
(a) Sketch a right triangle that illustrates the situation. Label your sketch with the given information.
(b) Choose the appropriate trigonometric ratio, write an equation, and solve the problem.

34. **Estimating Height.** The birdhouse in Phuong's front yard is 12 feet tall, and its shadow at 4 P.M. is 15 feet 4 inches long. What is the angle of elevation of the sun at 4 P.M.?

35. **Estimating the Angle of Elevation.** If a 65.2-foot sculpture casts a shadow 42.1 feet long, find the sun's angle of elevation to the nearest tenth of a degree.

36. **Air Traffic Control.** To measure the height of cloud cover, airport controllers fix a searchlight to shine a vertical beam on the clouds. The searchlight is 120 yards from the office. A technician in the office measures the angle of elevation to the light on the cloud cover at 54.8°. What is the height of the cloud cover?

37. **Salvage Crew.** A salvage ship is searching for the wreck of a pirate vessel on the ocean floor. Using sonar, they locate the wreck at an angle of depression of 36.2°. The depth of the ocean at their location is 260 feet. How far should they move to be directly above the wrecked vessel?

38. **Volcanology.** Cinder cones are the most common type of volcano in the world. They may look like an idealized depiction of a volcano as they are steep, conical hills that usually have a prominent crater at the top. Maha decides to hike to the top of Capulin Volcano, a cinder cone in New Mexico. Once at the rim, she measures the angle of depression to the surrounding plain to be 26.3°. Her pedometer shows that her hike up to the top of Capulin Volcano covered 2,939 feet. Assuming the hike up the volcano was along a straight path, what was her elevation gain?

39. **Meteorology.** The radio signal from a weather balloon indicates that it is 1500 meters from a meteorologist on the ground. The angle of elevation to the balloon is 48°. What is the balloon's altitude?

40. **Ancient Warfare.** According to Chinese legend, around 200 BC, the general Han Xin used a kite to determine the distance from his location to an enemy palace. He then dug a secret tunnel which emerged inside the palace. When the kite was directly above the palace, its angle of elevation was 27° and the string to the kite was 1850 feet long. How far did Han Xin's troops have to dig?

41. **Ocean Rescue.** A hot air balloon was forced far off course, and the occupants had to make a forced landing at sea. They sent out a distress signal before taking the balloon down, and, very luckily, a United States Lifesaving Association boat was nearby. The distress signal the boat receives gives the balloon's altitude before descending as 1,128 feet. Immediately, an observer on the boat could see the balloon and measure the angle of elevation to the balloon as 22.1°. How far would the boat travel to reach any survivors if the balloon went directly (vertically) down and crashed into the water?

42. **Airplane Navigation.** An airplane begins its descent when its altitude is 10 kilometers. The angle of descent should be 3° from horizontal.

a. How far from the airport (measured along the ground) should the airplane begin its descent?

b. How far will the airplane travel on its descent to the airport?

13. **Estimating Height.** Linda is on the upper floor of a tall building. She measures the angle of elevation to the building across the street to be 38° and the angle of depression to the base of that building as 65°. If she knows the two buildings are 50 feet apart, how tall is the building across the street?

14. **Height of a Flagpole.** To estimate the height of a flagpole, Kim positions herself due south of the pole, and Matt stands due east of the pole. If they are the same distance from the pole and 50 feet from each other, find the height of the pole if the angle of elevation from the ground to the top of the pole at each person's position is 49°.

15. **Estimating Height - D&D Edition.** Deitrich Wyvernfang bravely stands in defense against a massive stone giant. He is waiting for the precise moment when the creature steps within reach of his trusty glaive; however, as with most stone giants, the beast is being measured in its approach and purposefully keeps its distance. In the intervening moments, Hoot de'La Temp, Deitrich's trusty but enigmatic companion, is overwhelmed with a momentary curiosity. In his studies of all creatures magical and curious, Hoot had never read of a stone giant this tall.

    Always taking advantage of the moment, Hoot places his hand on Deitrich's shoulder, casually saying, "Hold the giant off for me. This is a rare opportunity to measure the stature of such a creature in its non-deceased form." Deitrich's surprised, wide-eyed glance at Hoot is all the wizard needs to know that his friend must share his excitement about such an opportunity. Kneeling, Hoot quickly utters the arcane words for his *calculate* spell and measures the angle of elevation to the stone giant's head to be $52^\circ\ 34'$. Standing, he nods at Deitrich (who, as Hoot thinks about it, now looks disgusted for some reason) and says, "Keep him still for just a few moments longer."

    Hoot walks directly away from the stone giant, keeping Deitrich between himself and the dangerous creature. Each step he takes is a measured 3-foot distance. He smiles warmly as his friend decides to help him by barking "But... But... But..." in time with his steps - what a great companion. After ten such steps, Hoot turns, kneels, and casts his *calculate* spell again. This time, the angle of elevation to the now confused giant's head measures $23^\circ\ 31'$. Standing, the wizard shouts victoriously, "Got it! That beast is precisely ___ feet tall! Well, to the nearest tenth of a foot, that is. You may slay him now."

    What is the number of feet that Hoot estimated the height of this stone giant to be?

16. **Neighborly Dispute.** Nam's neighbors have built a massive pile of junk in their yard topped with a standard door frame. He would like to report them to the county for the eyesore, but the county will only give the neighbor a citation if the structure is more than 30 feet tall. Unfortunately, Nam's neighbors are not allowing him to measure the height of the junk pile on their property, so he resorts to his mathematics training. From a point on the ground (level with the base of the junk pile), Nam measures the angle of elevation to the top of the entire pile to be 37°. From that same point, he measures the angle of elevation to the base of the door frame (on the top of the junk pile) to be 32°. Assuming the door frame is made to industry standards, it should be 80 inches tall. How tall is the entire pile of junk (in feet)? Can the county cite the neighbor?

17. **Estimating Distance.** To find the distance across a ravine, Roy takes some measurements from a small airplane. When he is a short distance from the ravine at an altitude of 500 feet, he finds that the angle of depression to the near side of the ravine is 56°, and the angle of depression to the far side is 32°. What is the width of the ravine? (Hint: First find the horizontal distance from Roy to the near side of the ravine.)

18. **Radius of the Earth.** A communications satellite is in a low earth orbit (LOE) at an altitude of 400 km. From the satellite, the angle of depression to the earth's horizon is 19.728°. Use this information to calculate the radius of the planet.

19. **Sunlit Area.** The window in Nhat's office is 4 feet wide and 5 feet tall. The bottom of the window is 3 feet from the floor. When the sun is at an angle of elevation of 64°, what is the area of the sunny spot on the floor?

Heading Problems

For the following exercises, a ship sails from the seaport on the given heading for the given distance.
(a) Make a sketch showing the ship's current location relative to the seaport.
(b) How far east or west of the seaport is the ship's present location? How far north or south?

20. $36^\circ, 26$ miles

51. $124^\circ, 80$ km

52. $230^\circ, 120$ km

53. $318^\circ, 75$ miles

54. $285^\circ, 32$ km

55. $192^\circ, 260$ miles

56. A private plane flies 2.5 hours at 110 mph on a heading of 45°. It then turns and continues another 1.5 hours at the same speed but on a bearing of 135°.

    a. How far is the plane from its starting point at the end of this time?

    b. What is the heading from the pilot's starting location to her ending location?

Bearing Problems

57. **Ship Navigation.** A ship travels on a course of bearing $\mathrm{N}\ 62^\circ\ \mathrm{E}$ for 83.1 miles. How many miles north and how many miles east has the boat traveled?

58. **Ship Navigation.** A ship leaves port and travels 20 miles in the direction $\mathrm{S}\ 29^\circ\ \mathrm{W}$. It then turns 90° and travels another 14.4 miles in the direction $\mathrm{N}\ 61^\circ\ \mathrm{W}$. At that time, how far is the ship from port, and what is its bearing back to the port?

59. **Hiking Navigation.** Mike parks his car on a road that runs east-west. From this road, he sets out on a straight trail at a bearing of $\mathrm{N}\ 65^\circ\ \mathrm{E}$ and hikes for 25 miles to a water cache. After a short break, Mike turns onto another straight trail with a bearing of $\mathrm{S}\ 25^\circ\ \mathrm{E}$ for an additional 5 miles to a lunch spot.

    a. To the nearest tenth of a mile, how far was Mike from the east-west road when he reached the water cache?

    b. By the time Mike reaches the lunch spot, what is the bearing from the lunch spot back to his car?

    c. How far away (in terms of direct distance) is Mike's car from the lunch spot?

60. ***Locate Object* - D&D Edition.** The Bandits, as the heroes are known, are searching for a *wand of water control* that has been lost to the overgrowth of the Amedio Jungle. Hoot de'La Temp, the group's enigmatic wizard, casts a *locate object* spell multiple times to find the missing wand. Unfortunately, this spell only reveals the *direction* of an object - not its distance (another issue is that the spell can only find an object once the object is within 1,000 feet of the wizard). Frustrated and tired, the Bandits begin trekking back to town; however, after a short time, Hoot decides to give the spell one more try.

    "Wait! I sense the wand - it's that way," Hoot exclaims, pointing deeper into the jungle.

    Anlie, the Bandit's gruff ranger quickly snaps, "How far? Not all the way back to that group of gnolls, I hope."

    "I don't know. This magic only gives me direction - not distance," Hoot responds, frustratedly shaking his head. "This particular issue concerns the bending of the Gygaxian weaves with those of the natural weaves present in the surrounding region. I have often thought there might be a better..."

    Cutting the wizard off, Anlie tersely states, "I don't need a magic lesson, Hoot. A simple, 'I don't know' works."

    The ranger looks up through the jungle canopy, searching for signs only someone with her skill could observe. "Okay, you're pointing roughly north by 45° east." She looks around, spotting a path with less vegetation, "How much time do you have left on that spell?"

    Hoot, unruffled by the interruption, shrugs, "Seven minutes." Pausing, he adds, "And 36 seconds?"

    Rolling her eyes, Anlie quickly moves down the trail. "Follow me."

After a few minutes of hiking through the jungle, Anlie turns to Hoot. "Now point toward the wand."

Furrowing his brow and giving the ranger an appraising look, Hoot absentmindedly raises his arm to the side, pointing his index finger in the perceived direction of the magic item.

Anlie looks at the wizard accusingly, "Are you sure?"

Feigning a look of innocence, the wizard responds, "Nobody can be absolutely certain about these things, but I have the utmost confidence that it's roughly out there," wiggling his finger.

Sighing, Anlie looks around, spending a minute looking for those hidden signs only she seems to see. "Okay, you're pointing north by 45° west. It's 1,050 feet back to where we started in the direction north by 78° west."

"Are *you* sure," the wizard asks with a grin.

Giving him a flat stare, Anlie sighs again, "Yes, and I know how far away the wand is."

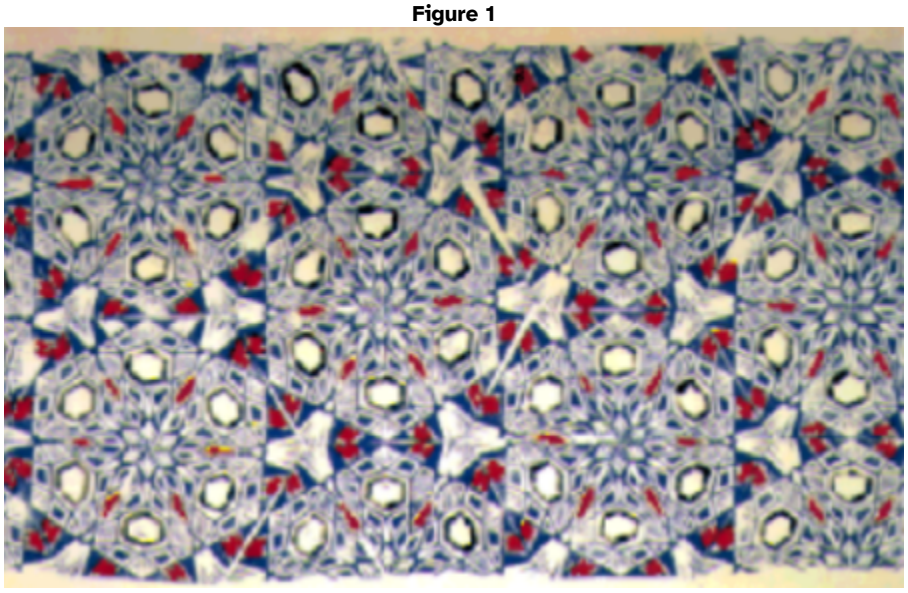
How far away is the *wand of water control* from their current position?

61. **Distance.** From two observation points 25 miles apart on a straight shoreline running north and south, the bearings to a boat at sea are $\mathrm{N}\ 26^\circ\ \mathrm{W}$ and $\mathrm{S}\ 56^\circ\ \mathrm{W}$. What is the shortest distance from the boat to the shore?

This page titled 6.4.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

7: RADIAN MEASURE AND THE CIRCULAR FUNCTIONS

Figure 1



Have you ever wondered why we divide the circle into 360 degrees? Nobody knows the answer, but it may have started around 600 BCE with the Babylonians.

The Babylonians lived between the Tigris and Euphrates rivers in present-day Turkey and Syria. They kept written records using a stylus to press cuneiform, or wedge-shaped, symbols into wet clay tablets, which were then baked in the sun. Thousands of these tablets have survived and give us detailed information about the mathematical practices of the time.

Figure 2

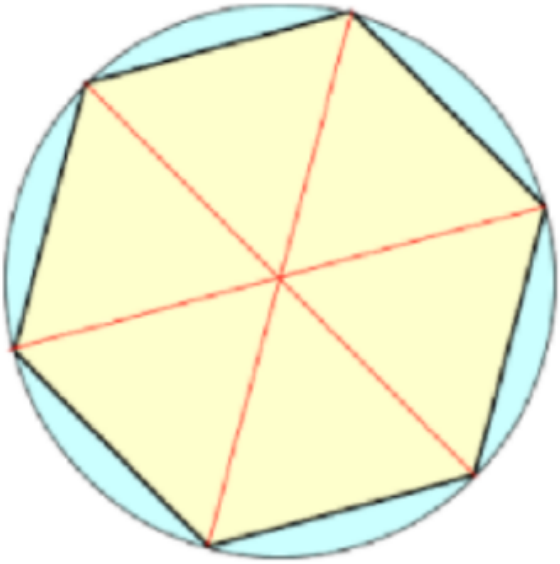


The Babylonians used a base 60 number system because the number 60 has many factors. They did not invent decimal fractions, so they found it challenging to deal with remainders when dividing. But 60 can be divided evenly by 2, 3, 4, 5, and 6, which made calculations with common fractions much more manageable.

We still have traces of their base 60 system: there are 60 seconds in a minute and 60 minutes in an hour.

In Geometry, Babylonian mathematicians used the corner of an equilateral triangle as their basic unit of angular measure. Naturally, they divided that angle into 60 smaller angles. Now, suppose the corners of six equilateral triangles are placed together. In that case, they form a complete circle, so there are six times 60, or 360, degrees of arc in a circle.

Figure 3

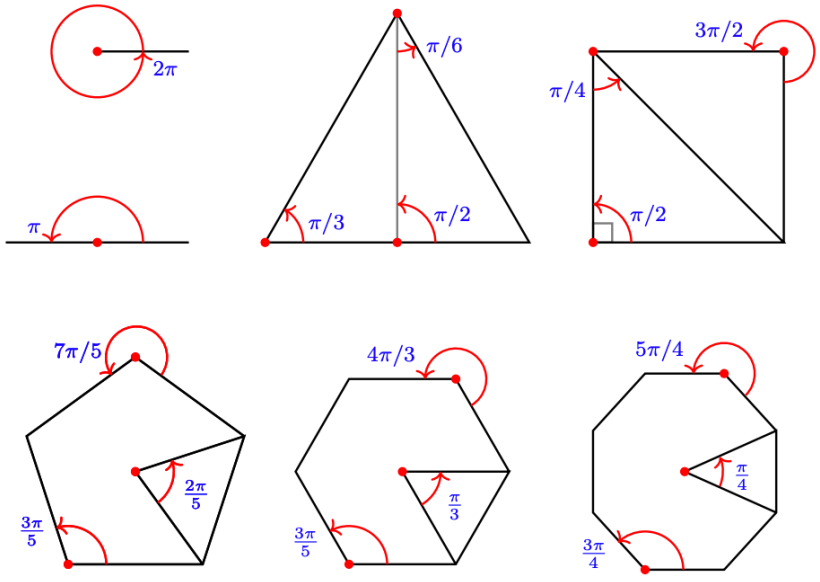


During the reign of Nebuchadnezzar, using the tools and technology available to them, Babylonian astronomers calculated that a complete year numbered 360 days. This made dividing the circle into 360 degrees even more helpful.

The number 360 is not fundamental to the nature of a circle. If ancient civilizations had defined the whole circle as some other number of degrees, we'd probably be using that number today.

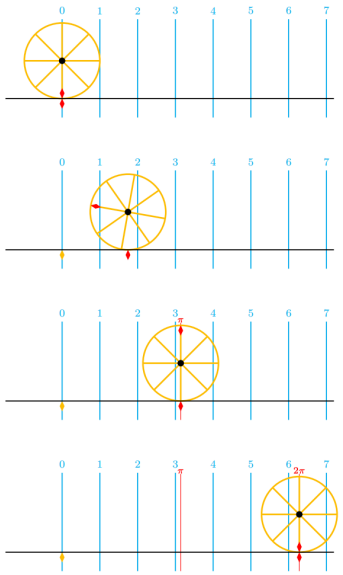
But why do we need another, different way to measure angles? In this chapter, we'll study **radian measure**, which initially may seem awkward and unnatural. As a hint, consider that although 360 is not fundamental to circles, the number π is.

Figure 4



Radians connect the measure of an angle with the arc length it cuts out on a circle.

Figure 5



Imagine a circle with a radius of 1 unit rolling along a straight line. The circumference of a circle is 2π times the length of its radius, so when the circle has turned through one complete revolution, it has traveled a distance of 2π units. When the circle has turned through half a revolution, it has traveled a distance of π units.

In radian measure, the angle through which the circle has turned is equal to the distance the circle has traveled or, put another way, the length of the arc that has unrolled on the line.

Roger Cotes, an English mathematician who worked closely with Isaac Newton, developed the idea of radian measure. He described the radian in everything but name and recognized its naturalness as a unit of angular measure.

The term **radian** first appeared in print on June 5, 1873, on an exam written by James Thomson, the brother of Lord Kelvin, at Queen's College in Belfast.

In Calculus and most other branches of mathematics beyond practical Geometry, angles are nearly always measured in radians. Because radians arise naturally when dealing with circles, essential relationships are expressed more concisely in radians. In particular, results involving trigonometric functions are more straightforward when the variables are expressed in radians.

7.1: Reference Angles

7.1.1: Resources and Key Concepts

7.1.2: Homework

7.2: Radian Measure, Arc Length, and Sector Area

7.2.1: Resources and Key Concepts

7.2.2: Homework

7.3: The Trigonometric Functions - Unit Circle Definition

7.3.1: Resources and Key Concepts

7.3.2: Homework

This page titled 7: Radian Measure and the Circular Functions is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Katherine Yoshiwara & Roy Simpson.

7.1: Reference Angles

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The town of Avery lies 48 miles due east of Baker, and Clio is 34 miles in the bearing $\mathrm{N}\ 35^\circ\ \mathrm{W}$ from Baker. How far is it from Avery to Clio?

We know how to solve right triangles using the trigonometric ratios; however, the triangle formed by the three towns is *not* a right triangle because it includes an obtuse angle of 125° at B, as shown in Figure 7.1.1.

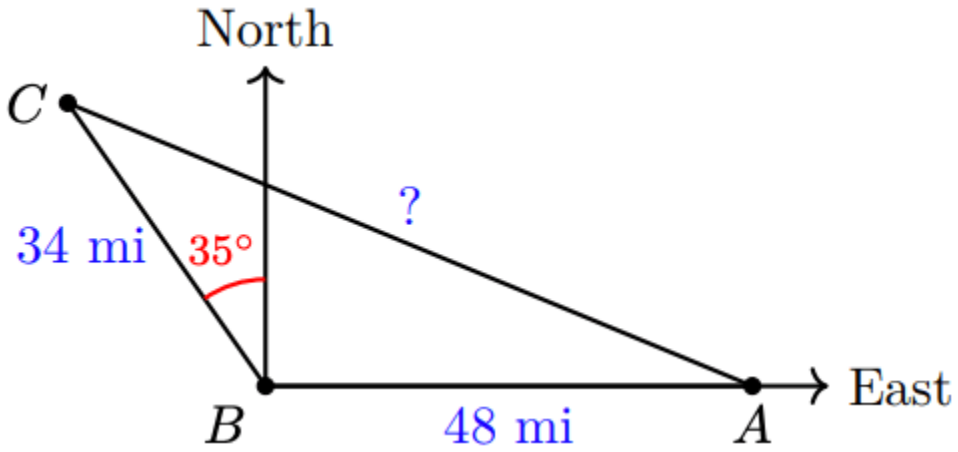


Figure 7.1.1

Recall that we call this type of triangle an oblique triangle. Later in this course, we learn how to solve oblique triangles, but first, we must be able to find the values of the trigonometric functions for obtuse angles.

Reference Angles

We now introduce an essential tool for evaluating trigonometric functions and solving trigonometric equations involving non-acute angles.

Definition: Reference Angle

The **reference angle** for an angle θ given in standard form is the acute angle formed between the terminal side of θ and the x-axis. By convention, the reference angle is always considered positive (even if it opens in a clockwise direction). It is denoted with the "hat" notation, $\hat{\theta}$.

Any acute angle θ is the reference angle for four angles between 0° and 360°, one in each quadrant. The figure below shows the

four angles in standard position whose reference angle is 35°. Note that each angle is found by measuring 35° from the x-axis in the appropriate quadrant and that the four angles together make a "bow-tie" shape.

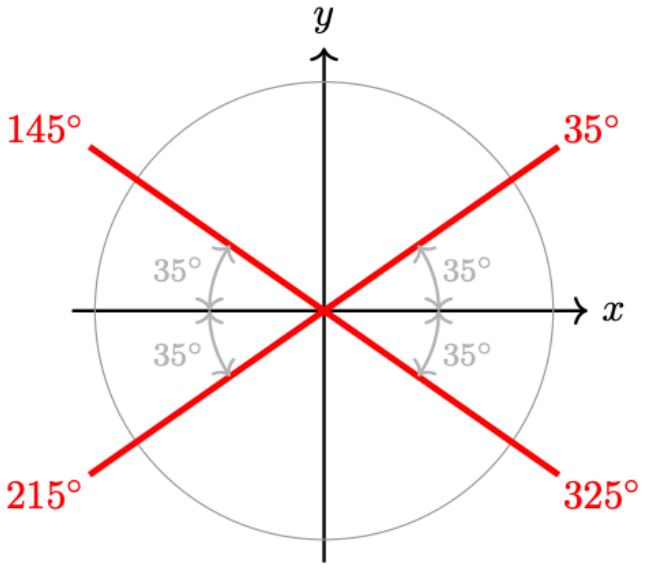


Figure 7.1.2

Interactive Element: One Reference Angle, Four Angles (the "Bow-Tie")

The following Interactive Element might help you visualize why a single acute reference angle belongs to four different angles in standard position, one in each quadrant.

Interact: Drag the first-quadrant point Q to change the shared reference angle.

Observation: All four reference triangles stay congruent — each is a mirror image of the first-quadrant one across the axes — so the four angles share the same reference angle and their trig values agree up to sign.

Example 7.1.1

Find and label the reference angle for the given angle.

a. $A = 125^\circ$
b. $B = 240^\circ$
c. $\alpha = 310^\circ$
d. $\beta = -315^\circ$
e. $t = 820^\circ$

Solutions available at the end of this section.

While you should strive for understanding in mathematics over memorization, some students initially find the following guidelines helpful until they get used to reference angles. Trust me when I say you want to use the following guideline only for a while—it's too bulky.

Let θ be an angle in standard position.

1. If $0^\circ < \theta < 360^\circ$, let $t = \theta$.
2. If θ is not between 0° and 360°, then let t be the angle coterminal to θ that *is* between 0° and 360°.

$$\text{If} \quad t \text{ terminates in QI}, \quad \text{then} \quad \hat{\theta} = t$$

$$\text{If} \quad t \text{ terminates in QII}, \quad \text{then} \quad \hat{\theta} = 180^\circ - t$$

$$\text{If} \quad t \text{ terminates in QIII}, \quad \text{then} \quad \hat{\theta} = t - 180^\circ$$

$$\text{If} \quad t \text{ terminates in QIV}, \quad \text{then} \quad \hat{\theta} = 360^\circ - t$$

Caution: Reference Angles are Acute

By definition, reference angles are acute angles. This means they are *not* 0° or 90°. In fact, quadrantal angles do not have reference angles.

Checkpoint 7.1.1

Draw the angle and state its reference angle.

a. $\alpha = 80^\circ$
b. $\beta = 130^\circ$
c. $\kappa = 225^\circ$
d. $\gamma = 330^\circ$
e. $\lambda = -125^\circ$

Answers available at the end of this section.

Evaluating Trigonometric Functions Using Reference Angles

We can use reference angles to help us evaluate a trigonometric function at a non-acute angle θ. To do so, we construct a **reference triangle** for θ.

Definition: Reference Triangle

A **reference triangle** is a right triangle formed by drawing a perpendicular from a point on the terminal side of an angle (in standard position) to the x-axis.

A reference triangle can easily be drawn by choosing a point P on the terminal side of θ, and then drawing a line from P perpendicular to the x-axis.

The figure below shows angles θ in all four quadrants and the reference angle, $\hat{\theta}$, for each. Study the figures, and make sure you understand how to find the reference angle in each quadrant.

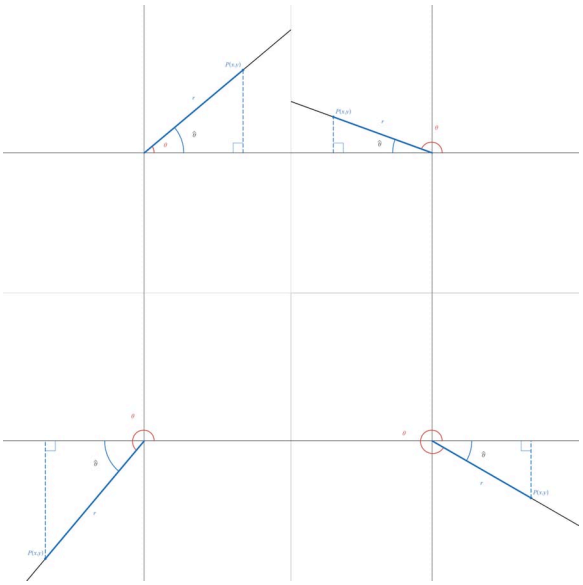


Figure 7.1.10**:** *Reference triangles in all four quadrants*

Summary

These observations may help you remember the formulas:

- The right triangle formed in this way always lies between the terminal side and the x-axis.
- The positive acute angle formed between the terminal side and the x-axis is the reference angle, and the right triangle is the reference triangle.

Interactive Element: Reference Angle and Reference Triangle

The following Interactive Element might help you visualize how the reference triangle and reference angle change as the terminal side sweeps through the four quadrants.

Interact: Drag the point P on the terminal side into each quadrant.

Observation: The shaded right triangle always sits between the terminal side and the x-axis, and the acute angle at the origin is the reference angle $\hat{\theta}$. The legs keep the lengths $|x|$ and $|y|$ in every quadrant, so the trig values of θ match those of $\hat{\theta}$ except for a sign set by the quadrant.

Activity

The following activity reveals the usefulness of reference angles and reference triangles when evaluating trigonometric functions of non-acute angles.

Activity: Reference Angles

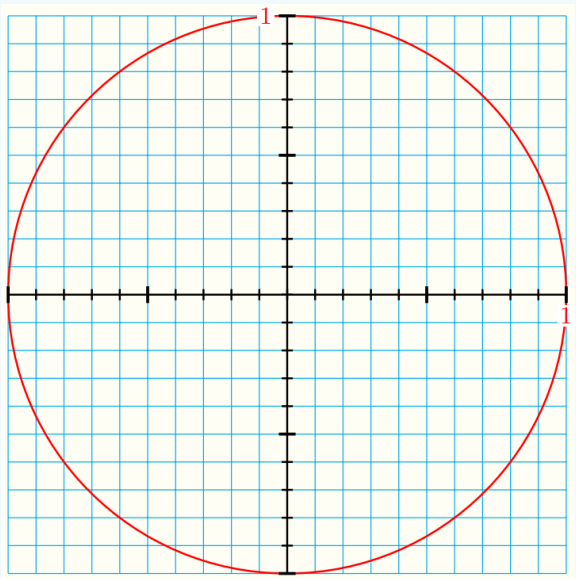


Figure 7.1.11

a. Use a protractor to draw an angle of 56° in standard position. Draw its reference triangle.
b. Use your calculator to find the sine and cosine of 56°, rounded to two decimal places. Label the sides of the reference triangle with their lengths.
c. What are the coordinates of the point P where your angle intersects the unit circle?
d. Draw the reflection of your reference triangle across the y-axis so that you have a congruent triangle in the second quadrant.
e. You now have the reference triangle for a second-quadrant angle in standard position. What is that angle?
f. Use your calculator to find the sine and cosine of your new angle. Label the coordinates of the point Q where the angle intersects the unit circle.
g. Draw the reflection of your triangle from part (d) across the x-axis so that you have a congruent triangle in the third quadrant.
h. You now have the reference triangle for a third-quadrant angle in standard position. What is that angle?
i. Use your calculator to find the sine and cosine of your new angle. Label the coordinates of the point R where the angle intersects the unit circle.
j. Draw the reflection of your triangle from part (g) across the y-axis so that you have a congruent triangle in the fourth quadrant.
k. You now have the reference triangle for a fourth-quadrant angle in standard position. What is that angle?
l. Use your calculator to find the sine and cosine of your new angle. Label the coordinates of the point where the angle intersects the unit circle.
m. **Generalize:** All four of your angles have the same reference angle, 56°. For each quadrant, write a formula for the angle whose reference angle is θ.
    i. Quadrant I:
    ii. Quadrant II:
    iii. Quadrant III:
    iv. Quadrant IV:

Reference Angle Theorem

In all four versions of Figure 7.1.10, coordinate trigonometry states that

$$\begin{aligned} \sin(\theta) &= \frac{y}{r} & \csc(\theta) &= \frac{r}{y} \\ \cos(\theta) &= \frac{x}{r} & \sec(\theta) &= \frac{r}{x} \\ \tan(\theta) &= \frac{y}{x} & \cot(\theta) &= \frac{x}{y} \end{aligned}$$

However, it is also true that in all four versions of Figure 7.1.10, the side of the reference triangle opposite $\hat{\theta}$ has length $|y|$, the side of the reference triangle adjacent to $\hat{\theta}$ has length $|x|$, and the hypotenuse of the reference triangle has a length of r. Therefore, by right triangle trigonometry,

$$\begin{aligned} \sin\left(\hat{\theta}\right) &= \frac{|y|}{r} & \csc\left(\hat{\theta}\right) &= \frac{r}{|y|} \\ \cos\left(\hat{\theta}\right) &= \frac{|x|}{r} & \sec\left(\hat{\theta}\right) &= \frac{r}{|x|} \\ \tan\left(\hat{\theta}\right) &= \frac{|y|}{|x|} & \cot\left(\hat{\theta}\right) &= \frac{|x|}{|y|} \end{aligned}$$

Thus, the values of the trigonometric functions of the angle θ match the values of the trigonometric functions of $\hat{\theta}$, except, possibly, for a difference in sign. The sign of the trigonometric function's value can be determined by the quadrant in which the terminal side of the angle lies. This result is summarized in the following theorem.

Theorem: Reference Angle Theorem

The value of a trigonometric function of any angle is equal to that of the function at its reference angle, except for sign. The quadrant determines the sign of the function.

In the previous chapter, we used a calculator to approximate the values of the trigonometric functions at *acute* angles. Your calculator can easily find the value of a trigonometric function at *any* angle. We showcase this in the following example.

Example 7.1.2

a. Find the reference angle for 200°.
b. Sketch 200° and its reference angle in standard position, along with the reference triangle. Verify that both angles have the same trigonometric values, up to sign.

Solutions available at the end of this section.

Note that the results in Example 7.1.2 align with what we said previously—the sign of the trigonometric function's value can be determined by the quadrant in which the terminal side of the angle lies. Thus, to compute $\sin(200^\circ)$, we could note that $\theta^{\text{int}}\text{QIII}$ and the reference angle is 20°. Since the sine is negative in QIII, we get

Double exponent: use braces to clarify

From this point forward in Trigonometry, the following Trigonometry Mantra will be helpful.

> Trigonometry is about two things—reference angles and quadrants.

While this mantra oversimplifies the subject, most of Trigonometry is easier once you know the reference angle and quadrant for a given angle.

Checkpoint 7.1.2

a. Find the reference angle for 285°.

b. Sketch 285° and its reference angle in standard position, along with the reference triangle. Verify that both angles have the same trigonometric ratios, up to sign.

Answers available at the end of this section.

When combined with the Reference Angle Theorem, the following corollary improves the speed with which we can evaluate trigonometric functions.

Corollary: Rotational Symmetry

For any integer n,

$$\begin{aligned} \sin(\theta + 360^\circ n) &= \sin(\theta) & \csc(\theta + 360^\circ n) &= \csc(\theta) \\ \cos(\theta + 360^\circ n) &= \cos(\theta) & \sec(\theta + 360^\circ n) &= \sec(\theta) \\ \tan(\theta + 360^\circ n) &= \tan(\theta) & \cot(\theta + 360^\circ n) &= \cot(\theta) \end{aligned}$$

This result is from our previous discussion that the value of a trigonometric function is the same for all angles coterminal with the original given angle.

An Additional Note About the Tangent

As we will investigate later, it turns out that

$$\tan(\theta + 180^\circ n) = \tan(\theta) \quad \text{and} \quad \cot(\theta + 180^\circ n) = \cot(\theta)$$

for any integer n; however, this is not needed for our current discussion.

Let's see how this corollary, along with focusing on reference angles and quadrants, speeds up the evaluation of trigonometric functions whose reference angles are the special angles.

Example 7.1.3

Find the exact value of each trigonometric function.

a. $\cos(225^\circ)$
b. $\cot(120^\circ)$
c. $\tan(-210^\circ)$
d. $\sin(690^\circ)$

Solutions available at the end of this section.

Checkpoint 7.1.3

Find the exact value of each trigonometric function (without touching a calculator).

a. $\sin(210^\circ)$
b. $\sec(-780^\circ)$
c. $\sin(405^\circ)$

Answers available at the end of this section.

Finding Non-Acute Angles

We now turn our attention back to finding angles given values of trigonometric functions. Last chapter, we were able to find θ such that $\tan(\theta) = 0.5662$. At that time, we used the TAN^{-1} button on our calculator to find $\tan^{-1}(0.5662) \approx 29.5185^\circ$. At that time, however, we were only working with right triangle trigonometry, and the angles we were interested in were all acute. When working with non-acute angles, the inverse trigonometric functions can get tricky.

To avoid the trickiness associated with inverse trigonometric functions (and to give us more time with Trigonometry before diving into the more profound theory of the inverse trigonometric functions), it's best to think of them as returning *reference* angles. Since reference angles are always acute, this interpretation works with what we did in the last chapter; however, as we have seen in this section, we are now dealing with angles that are not necessarily acute.

> For now, think of the inverse trigonometric functions as returning reference angles.

We must evaluate the inverse trigonometric function at a *positive* value to return a reference angle. The reason for this cannot be fully justified at this time. Still, we *can* say that, for example, $\sin^{-1}(-0.31)$ returns an angle θ so that $\sin(\theta) = -0.31$; however, we know that the values of *all* trigonometric functions are *positive* in QI. Since we want the returned angle to be a reference angle, it must be acute (and, therefore, in the first quadrant). Hence, there is no possibility for a reference angle to be returned if the argument of our inverse trigonometric function is negative.

If we agree that inverse trigonometric functions evaluated at positive numbers return reference angles, we can ask for solutions to equations involving trigonometric functions.

For example, find θ if

$$\sin(\theta) = -0.31,$$

where $\theta^{\text{int}}\text{QIII}$.

First, it should make sense that θ could be in the third quadrant because the sine at θ is negative, and the sine is negative in the third quadrant (as well as in the fourth quadrant). Remember, Trigonometry is about two things—reference angles and quadrants. We have already been given a quadrant. We evaluate the inverse sine function at $|-0.31| = 0.31$ to find the reference angle. Hence, the reference angle is

$$\hat{\theta} = \sin^{-1}(|-0.31|) = \sin^{-1}(0.31) \approx 18.1^\circ.$$

The angle in QIII with reference angle $\hat{\theta} \approx 18.1^\circ$ is $\theta = 180^\circ + \hat{\theta} \approx 180^\circ + 18.1^\circ = 198.1^\circ$.

Summary

To find the angle θ so that the trigonometric function of θ is a given ratio:

1. **Reference Angle.** Use the inverse trigonometric function of the absolute value of the given ratio to determine the reference angle, $\hat{\theta}$.
2. **Quadrant.** Use the given information within the problem to "lock down" the quadrant in which θ terminates.
3. **Actual Angle.** Use the reference angle and quadrant from (1) and (2) to determine θ.

The second step in the summary above assumes you are given extra information to help determine the quadrant in which θ terminates. While this will be true in this section, we will eventually shed that requirement. We will begin to find *all* angles that make a given trigonometric equation valid.

Example 7.1.4

Find θ to the nearest tenth of a degree, if $0^\circ < \theta < 360^\circ$.

a. $\cos(\theta) = -0.8457$, where $\theta^{\text{int}}\text{QIII}$.
b. $\tan(\theta) = -1.9122$, where $\sin(\theta) < 0$.

c. $\csc(\theta) = 2.1174$, where $\cos(\theta) < 0$.

Solutions available at the end of this section.

Checkpoint 7.1.4

a. Solve $\sec(\theta) = -1.8$, where $90^\circ < \theta < 180^\circ$. Round your answer to the nearest tenth of a degree.
b. Find the sine of θ given that $\tan(\theta) = -2$, $\cos(\theta) > 0$, and $0^\circ < \theta < 360^\circ$. Round your answer to four decimal places.

Answers available at the end of this section.

Of course, if given a "special" trigonometric ratio, you should not reach for a calculator.

Example 7.1.5

Find α if $\cos(\alpha) = -\frac{\sqrt{3}}{2}$ and $\alpha \overset{\text{int}}{\text{QII}}$, where $0^\circ < \alpha < 360^\circ$.

Solution available at the end of this section.

Checkpoint 7.1.5

Solve $\tan(\beta) = -\sqrt{3}$ given $0^\circ < \beta < 360^\circ$ and $\cos(\beta) > 0$.

Answer available at the end of this section.

Miscellaneous Topics: Thinking About Mathematics (Part 1)

1. What is it about math that you feel is difficult?
2. Would you think it an odd statement if someone told you that they suck at reading and could barely read a comic book? Why do you think people are more apt to proclaim that they suck at math?

Hidden Resources: Microsoft Office 365

Los Rios students have free access to Microsoft Office 365 for as long as they remain enrolled! This is an incredible deal! A warning is necessary - do not use the included OneDrive storage as it gets deleted each semester. Other than that minor issue, this is a fantastic resource. Check out the details on this CRC webpage.

Answers

Section Preview

In this section, we return to considering *any* angle (not just acute angles) — related to the *special angles* or not (although the focus is primarily on angles related to special angles). This is where we tie Coordinate Trigonometry to Right Triangle Trigonometry via the use of reference triangles and reference angles. We increase our calculator usage in two ways:

- We finally have the need to compute trigonometric functions at any angle (e.g., -137°). While we can do so by direct calculator usage, we should not miss the opportunity to show that, for example, $\sin(-137^\circ)$ is nearly the same as the sine of the reference angle for -137°. This foreshadows how we will be working with Trigonometry from this point forward.
- We need to find *non-acute* angles satisfying given trigonometric ratios (e.g., for what θ is $\sin(\theta) = -0.682$?).

Section Preview

- **Solving Equations**
  - Solving Linear Equations — One or Two-Steps: This material often requires solving equations of the form $-360^\circ + \hat{\beta} = -315^\circ$.

Section Preview

- Identify the reference angle for an angle given in standard position.
- Evaluate a trigonometric function at a non-acute angle using the reference angle.
- Use technology to approximate the value of a trigonometric function at a non-acute angle.
- Determine an angle given the quadrant in which the angle terminates and the value of a trigonometric function.

Example 7.1.1

a. We begin by sketching the angle and its terminal side.

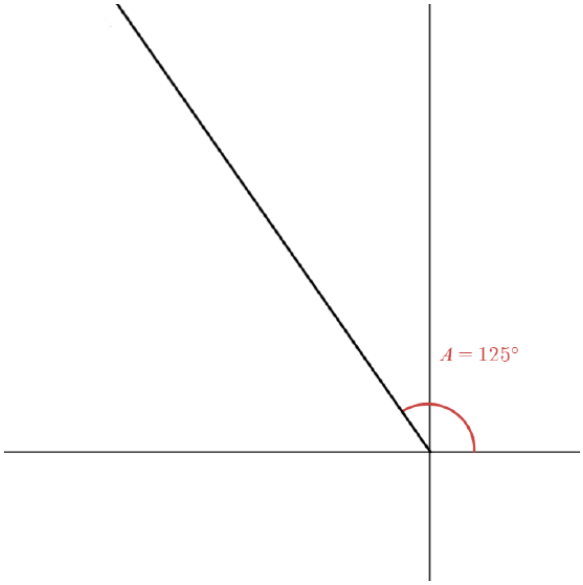


Figure 7.1.3

Since the terminal side of A is in QII, the smallest angle that gets us directly to the x-axis can be found by imagining we rotated to the negative x-axis $(180°)$ and then backed off by $\hat{A}$ to get to $A = 125°$. Mathematically, this is $180° - 125° = 55°$. Hence, the reference angle for A is $\hat{A} = 55°$.

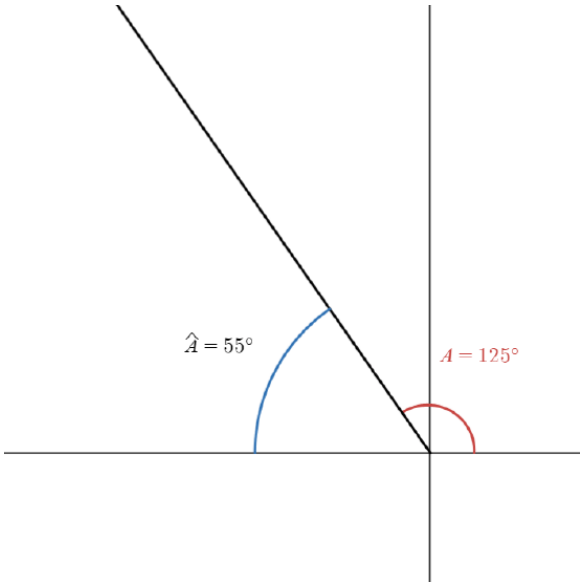


Figure 7.1.4

b. Again, sketch the original angle and its terminal side. B^{int}QIII, so the smallest angle to the x-axis can be found by imagining we rotated to the negative x-axis $(180°)$ and then added the reference angle $\hat{B}$ to get to $B = 240°$. That is, $180° + \hat{B} = 240°$. Therefore, $\hat{B} = 240° - 180° = 60°$ (see Figure 7.1.5).

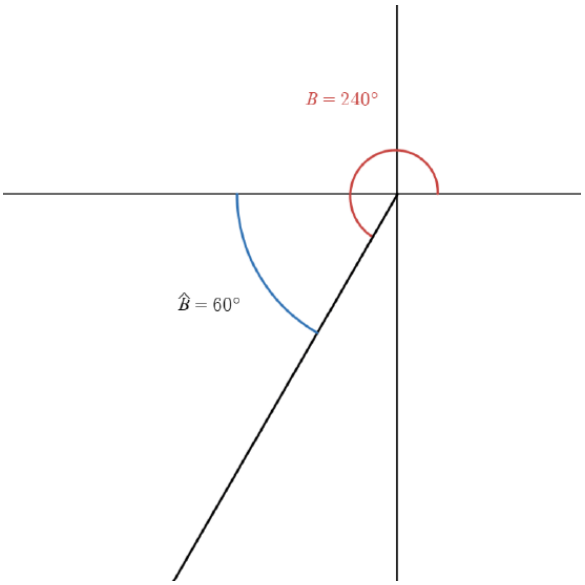


Figure 7.1.5

c. A quick sketch shows that $\alpha^{\text{int}}\text{QIV}$. To find the reference angle, we imagine rotating in a full circle $(360°)$ and backing off by the angle $\hat{\alpha}$ to arrive at $310°$. Therefore, $360° - \hat{\alpha} = 310°$. Hence, $360° - 310° = 50° = \hat{\alpha}$ (see Figure 7.1.6).

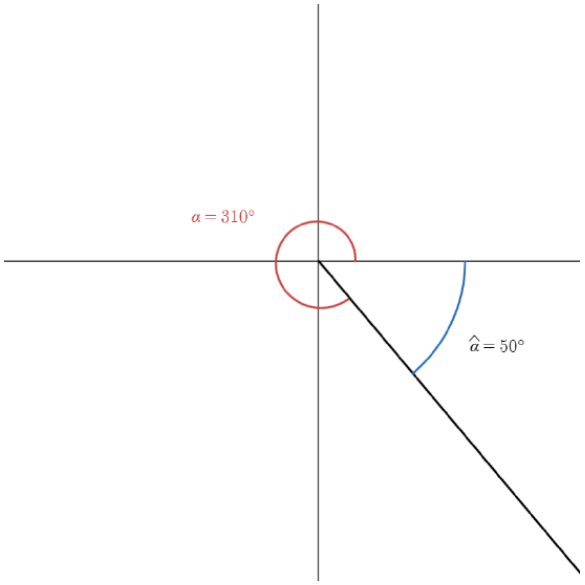


Figure 7.1.6

d. β being a negative angle doesn't change the tactic too much. We sketch $\beta = -315°$ and its terminal side. Since β terminates in QI, we can imagine finding the reference angle by rotating a full revolution in the negative direction $(-360°)$ and backing off by $\hat{\beta}$ to arrive at $\beta = -315°$. "Backing off" in this instance means "adding back" $\hat{\beta}$. Therefore, we want to compute $-360° + \hat{\beta} = -315°$. Solving this equation yields $\hat{\beta} = 45°$ (see Figure 7.1.7).

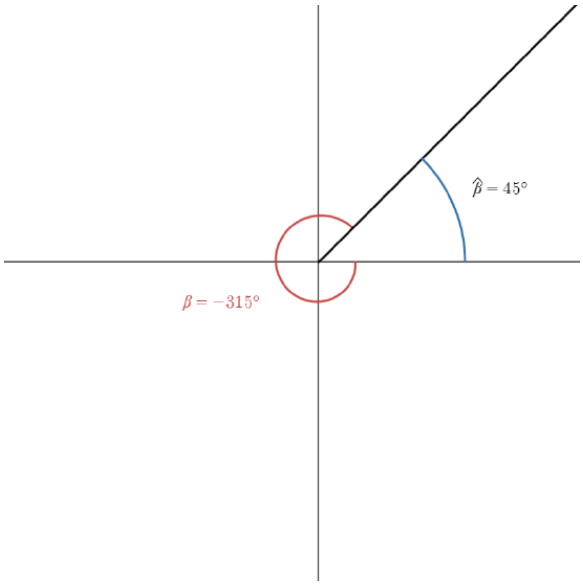


Figure 7.1.7

e. When an angle is outside of the interval $(-360°, 360°)$, a little extra thought is involved (but not much more). Sketching the angle and its terminal side, it's easy to notice that we can find a coterminal angle to $t = 820°$ that is between $0°$ and $360°$. Subtracting $720°$, we arrive at the coterminal angle $100°$ (see Figure 7.1.8).

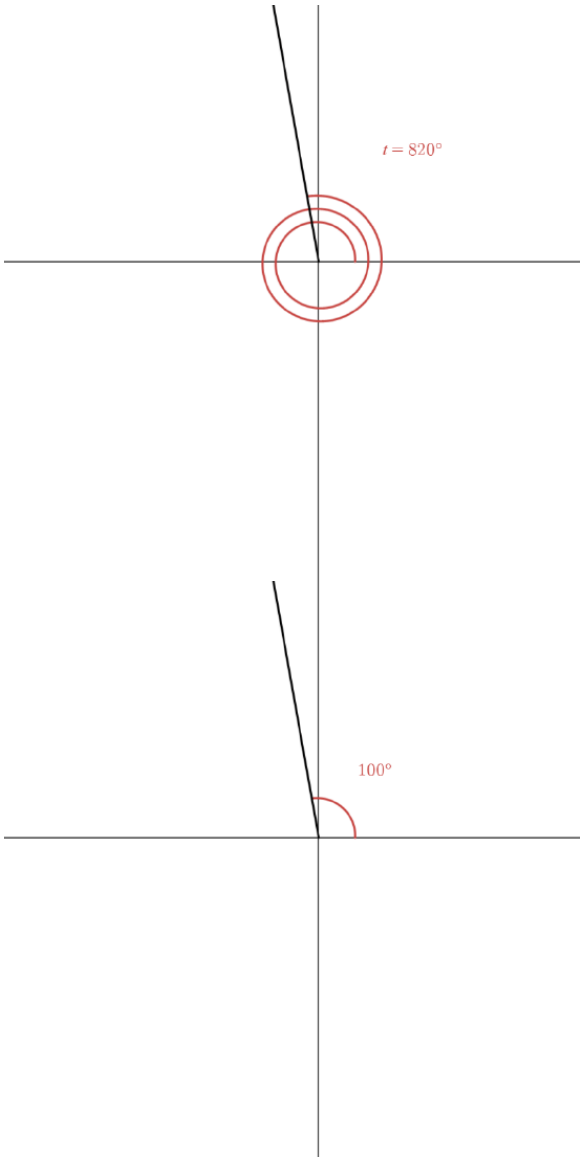


Figure 7.1.8

We are now back to the simpler case of a quadrant II angle. To find the reference angle, we imagine we rotated to the negative x-axis (180°) and then backed off by $\hat{t}$ to arrive at 100°. Mathematically, this means

$$180^\circ - \hat{t} = 100^\circ \implies \hat{t} = 80^\circ.$$

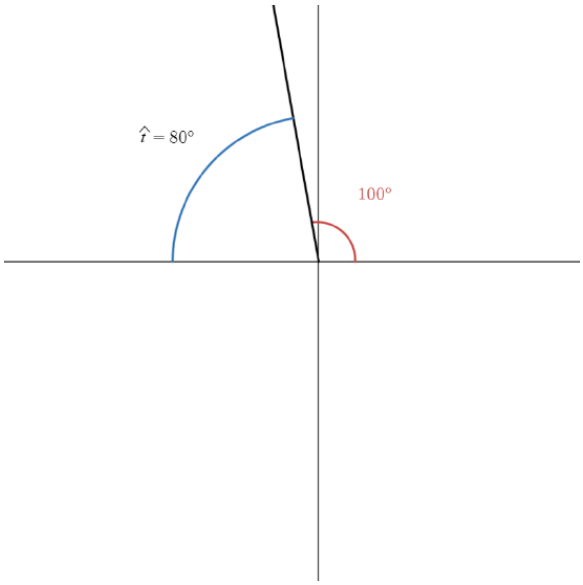


Figure 7.1.9

Checkpoint 7.1.1

a. $\hat{\alpha} = 80^\circ$
b. $\hat{\beta} = 50^\circ$
c. $\hat{\kappa} = 45^\circ$
d. $\hat{\gamma} = 30^\circ$
e. $\hat{\lambda} = 55^\circ$

Example 7.1.2

a. In standard position, an angle of 200° lies in the third quadrant. Therefore, we rotate to 180° and add the reference to get to 200°. Therefore, the reference angle is

$$200^\circ - 180^\circ = 20^\circ = \hat{\theta}.$$

b. Both angles are shown in Figure 7.1.12.

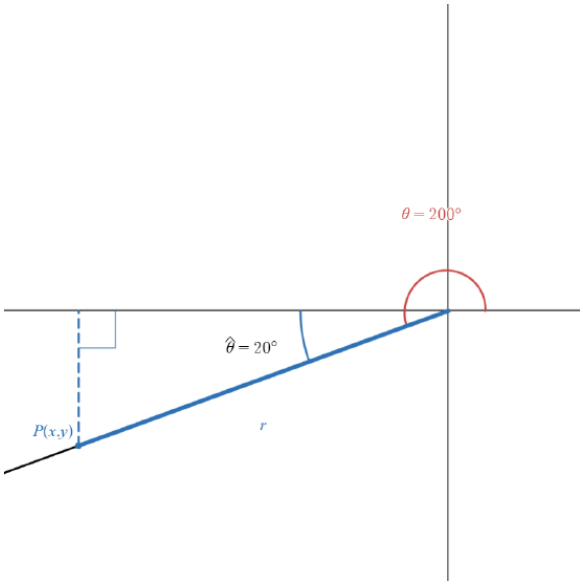


Figure 7.1.12

You can use your calculator to verify the following values.

$$\begin{aligned} \sin(20^\circ) &\approx 0.3420 & \sin(200^\circ) &\approx -0.3420 \\ \cos(20^\circ) &\approx 0.9397 & \cos(200^\circ) &\approx -0.9397 \\ \tan(20^\circ) &\approx 0.3640 & \tan(200^\circ) &\approx 0.3640 \end{aligned}$$

Checkpoint 7.1.2

a. 75°
b. $\sin(285^\circ) = -\sin(75^\circ) \approx -0.9659, \cos(285^\circ) = \cos(75^\circ) \approx 0.2588, \tan(285^\circ) = -\tan(75^\circ) \approx -3.7321$

Example 7.1.3

a. We quickly note that Double exponent: use braces to clarify and the reference angle is $\hat{\theta} = 45^\circ$. Thus,

$$\begin{aligned} \cos(225^\circ) &= \boxed{?}\cos(45^\circ) && \text{(Reference Angle Theorem)} \\ &= -\cos(45^\circ) && \text{(cosine is negative in QIII)} \\ &= -\frac{1}{\sqrt{2}} \end{aligned}$$

b. Double exponent: use braces to clarify and the reference angle is $\hat{\theta} = 60^\circ$. Thus,

$$\begin{aligned} \cot(120^\circ) &= \boxed{?}\cot(60^\circ) && \text{(Reference Angle Theorem)} \\ &= -\cot(60^\circ) && \text{(cotangent is negative in QII)} \\ &= -\frac{1}{\sqrt{3}} \end{aligned}$$

c. While we could solve this problem by rotating clockwise by 210°, let's use the corollary.

Double exponent: use braces to clarify

d. Double exponent: use braces to clarify

Checkpoint 7.1.3

a. $-\frac{1}{2}$
b. 2
c. $\frac{1}{\sqrt{2}}$

Example 7.1.4

a. **Reference Angle.** We start by finding the reference angle using the absolute value of the given ratio.

$$\hat{\theta} = \cos^{-1}(|-0.8457|) = \cos^{-1}(0.8457) \approx 32.3^\circ$$

Quadrant. Luckily, we were explicitly given the quadrant in which θ terminates as QIII.
Actual Angle. Thus,

$$\theta = 180^\circ + \hat{\theta} \approx 180^\circ + 32.3^\circ = 212.3^\circ.$$

b. **Reference Angle.**

$$\hat{\theta} = \tan^{-1}(|-1.9122|) = \tan^{-1}(1.9122) \approx 62.4^\circ$$

Quadrant. This example requires a little extra thought than part (a). We are given that both the angle's sine and tangent are negative. The sine is negative in quadrants III and IV, while the tangent is negative in quadrants II and IV. The quadrant where both of these conditions are true is quadrant IV.
Actual Angle. Hence,

$$\theta = 360^\circ - \hat{\theta} \approx 360^\circ - 62.4^\circ = 297.6^\circ.$$

c. **Reference Angle.** Since our calculator (or other technology) does not have an inverse cosecant button, we begin by rewriting the given equation using the Ratio Identities.

$$\csc(\theta) = 2.1174 \implies \frac{1}{\sin(\theta)} = 2.1174 \implies \frac{1}{2.1174} = \sin(\theta)$$

Therefore,

$$\hat{\theta} = \sin^{-1}\left(\frac{1}{2.1174}\right) \approx 28.2^\circ.$$

Quadrant. The cosecant is positive in the same quadrants the sine is positive—namely, QI and QII. The cosine is negative in QII and QIII. Therefore, the quadrant where θ terminates will be the shared quadrant, which is QII.
Actual Angle. Thus,

$$\theta = 180^\circ - \hat{\theta} \approx 180^\circ - 28.2^\circ = 151.8^\circ.$$

Checkpoint 7.1.4

a. $\theta \approx 123.7^\circ$
b. $\sin(\theta) \approx -0.8944$

Example 7.1.5

Reference Angle. It is crucial that you recognize the special ratios. Sketching the 30°-60°-90° triangle, we see that the reference angle for which $\cos(\alpha) = \left|-\frac{\sqrt{3}}{2}\right| = \frac{\sqrt{3}}{2}$ is $\hat{\alpha} = 30^\circ$.
Quadrant. α^{int}QII
Actual Angle. Thus,

$$\alpha = 180^\circ - \hat{\alpha} = 180^\circ - 30^\circ = 150^\circ.$$

Checkpoint 7.1.5

$\beta = 300^\circ$

This page titled 7.1: Reference Angles is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

7.1.1: Resources and Key Concepts

Resources

Videos

- Reference Angle
  - Angles - Reference Angles

Key Concepts

Definitions

- **Reference Angle:** For an angle θ in standard position, the reference angle, denoted $\hat{\theta}$, is the positive acute angle formed between the terminal side of θ and the x-axis.
- **Reference Triangle:** A right triangle formed by drawing a perpendicular line from a point on the terminal side of an angle in standard position to the x-axis.

Theorems

- **Reference Angle Theorem:** The value of a trigonometric function of any angle is equal to the value of that same function at its reference angle, except possibly for a difference in sign. The sign is determined by the quadrant in which the angle's terminal side lies.
- **Corollary: Rotational Symmetry:** For any integer n, the value of a trigonometric function is the same for all coterminal angles. For example, $\sin(\theta + 360^\circ n) = \sin(\theta)$.

Common Mistakes

- **Reference Angles are Acute:** By definition, reference angles are always acute (strictly between 0° and 90°). Quadrantal angles do not have reference angles.
- **Measuring from the y-axis:** A reference angle is always measured from the terminal side to the x-axis, never to the y-axis.
- **Negative Reference Angles:** Reference angles are always considered positive, even if the rotation to the x-axis is clockwise.
- **Using Negative Inputs for Inverse Functions:** When using an inverse trigonometric function to find a reference angle, the input must be the positive (absolute) value of the trigonometric ratio.

This page titled 7.1.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

7.1.2: Homework

Reading Questions

1. What is a reference angle?
2. How is a reference triangle for an angle θ constructed?
3. Are reference angles ever negative or greater than 90°?
4. What is the reference angle for 150°?
5. What is the reference angle for 300°?
6. Explain the Reference Angle Theorem in your own words.
7. What two pieces of information are needed to evaluate a trigonometric function at a non-acute angle without a calculator?
8. If you are trying to find an angle θ in Quadrant III where $\sin(\theta) = -0.5$, what is the first step in finding the reference angle using a calculator?
9. Once you find the reference angle $\hat{\theta}$, how do you find the actual angle θ if it terminates in Quadrant II?
10. Once you find the reference angle $\hat{\theta}$, how do you find the actual angle θ if it terminates in Quadrant IV?

Homework

Vocabulary Check

1. The ___ for θ is the positive acute angle formed between the terminal side of θ and the x-axis.

Concept Check

2. Camille wants an obtuse angle, θ, that satisfies $\sin(\theta) = 0.3$. Ivan presses some buttons on his calculator and reports that $\theta \approx 17.46^\circ$. Explain Ivan's error and give a correct approximation of θ accurate to two decimal places.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

3. To construct a reference triangle for an angle, we must choose a point P on the terminal side of θ and draw a line from point P perpendicular to the y-axis.

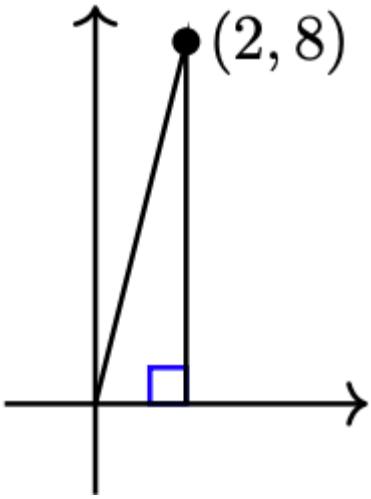
4. The trigonometric ratios of any angle are equal to the ratios of its reference angle, except for sign. The sign of the ratio is determined by the quadrant.

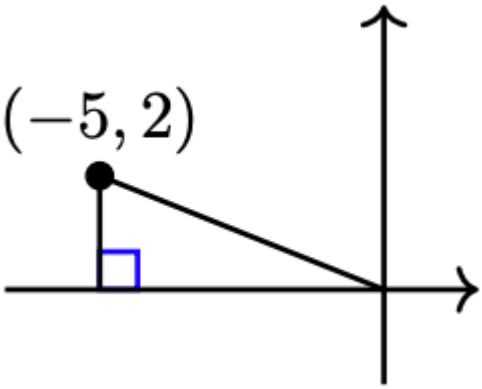
Basic Skills

5. Give the lengths of the legs and the hypotenuse of each right triangle.

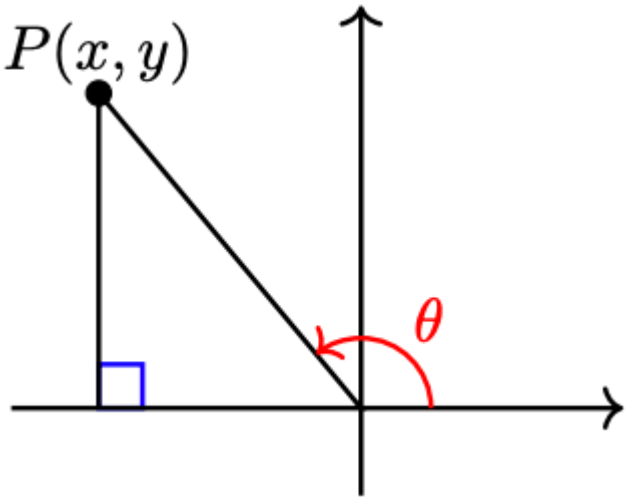
    a.

b.





6. Explain why the length of the horizontal leg of the right triangle is $-x$.



For the following exercises, find the reference angle for the given angle. Make a sketch showing the angle, the reference angle, and the reference triangle.

7. $\theta = 100^\circ$

8. $A = 125^\circ$

9. (MyOpenMath) $\beta = 216^\circ$
10. $\theta = 194^\circ\ 15'$
11. $\alpha = 242^\circ$
12. $\phi = 297^\circ$
13. $\lambda = 137^\circ\ 50'$
14. $\theta = 336^\circ$

For the following exercises, find three angles between 90° and 360° having the given reference angle, and sketch all four angles on the same grid.

15. 15°
16. 26°
17. 40°
18. 50°
19. 68°
20. 75°

For the following exercises, find the exact value of the trigonometric function using reference angles and quadrants (do not use a calculator).

21. $\cos(120^\circ)$
22. $\sin(135^\circ)$
23. $\tan(150^\circ)$
24. $\sec(225^\circ)$
25. $\sin(240^\circ)$
26. $\cot(210^\circ)$
27. $\csc(330^\circ)$
28. $\cos(315^\circ)$
29. $\sin(300^\circ)$
30. $\cos(-150^\circ)$
31. $\sin(-210^\circ)$
32. $\sec(-315^\circ)$
33. $\sin(585^\circ)$
34. $\cos(840^\circ)$
35. $\tan(-585^\circ)$

For the following exercises, use the values below to find the value of the trigonometric function. Do not use a calculator!

$\cos(23^\circ) \approx 0.9205 \quad \sin(46^\circ) \approx 0.7193 \quad \tan(78^\circ) \approx 4.7046$

36. $\cos(157^\circ)$

37. $\sin(226^\circ)$

38. $\csc(314^\circ)$

39. $\sec(203^\circ)$

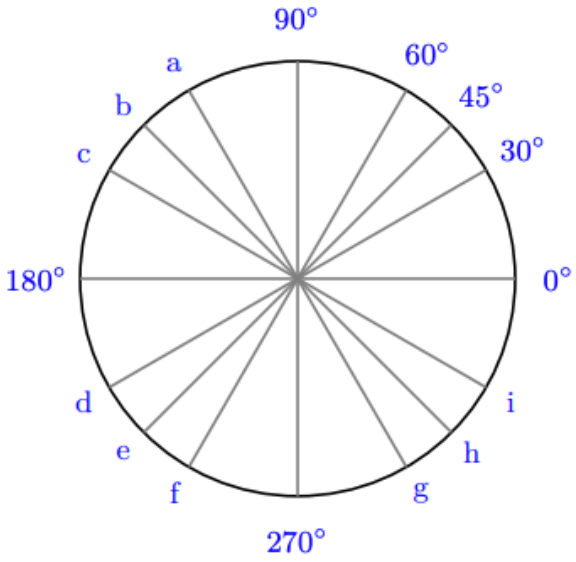
40. $\tan(258^\circ)$

41. $\cot(282^\circ)$

42. $\sin(-134^\circ)$

43. $\cos(-383^\circ)$

44. On the circle in the figure, all angles are shown in standard position. Find the measure in degrees of the angles labeled (a) - (i) and state their reference angles.



45. a. Draw three angles, one in each quadrant except the first, whose reference angle is 60°.

    b. Find exact values for the sine, cosine, and tangent of each angle in part (a).

46. a. Draw three angles, one in each quadrant except the first, whose reference angle is 30°.

    b. Find exact values for the sine, cosine, and tangent of each angle in part (a).

47. a. Draw three angles, one in each quadrant except the first, whose reference angle is 45°.

    b. Find exact values for the sine, cosine, and tangent of each angle in part (a).

48. Complete the table with exact values.

θ	30°	45°	60°	120°	135°	150°	210°	225°	240°	300°	315°	330°
$\sin(\theta)$												
$\cos(\theta)$												

$\tan(\theta)$												
$\csc(\theta)$												
$\sec(\theta)$												
$\cot(\theta)$												

For the following exercises, find a second angle between 0° and 360° with the given trigonometric ratio.

49. $\sin(75^\circ)$

50. $\cos(32^\circ)$

51. $\tan(84^\circ)$

52. $\csc(16^\circ)$

53. $\sec(47^\circ)$

54. $\cot(56^\circ)$

For the following exercises, use the given information to find an approximation of θ to the nearest degree if $0^\circ \leq \theta < 360^\circ$.

55. $\tan(\theta) = 8.1443$ with $\theta^{\text{int}}\text{QIII}$

56. $\sin(\theta) = 0.7880$ with $\theta^{\text{int}}\text{QII}$

57. $\cos(\theta) = 0.9205$ with $\theta^{\text{int}}\text{QIV}$

58. $\tan(\theta) = -3.4874$ with $\theta^{\text{int}}\text{QII}$

59. $\sin(\theta) = -0.9962$ with $\theta^{\text{int}}\text{QIV}$

60. $\cos(\theta) = -0.0349$ with $\theta^{\text{int}}\text{QII}$

61. $\sin(\theta) = 0.8381$ where $\cos(\theta) > 0$

62. $\tan(\theta) = -2.1728$ where $\sin(\theta) > 0$

63. $\cos(\theta) = -0.2155$ where $\tan(\theta) > 0$

64. $\sin(\theta) = -0.1179$ where $\tan(\theta) < 0$

65. $\cos(\theta) = 0.1818$ where $\csc(\theta) < 0$

66. $\tan(\theta) = 1.7531$ where $\sec(\theta) < 0$

67. $\csc(\theta) = 2.8549$ where $\cos(\theta) < 0$

68. $\sec(\theta) = -1.3888$ where $\csc(\theta) > 0$

69. $\cot(\theta) = -0.9411$ where $\cos(\theta) < 0$

For the following exercises, use the given information to find the exact value of θ if $0^\circ \leq \theta < 360^\circ$. Do not use a calculator!

70. $\tan(\theta) = \sqrt{3}$ with $\theta^{\text{int}}\text{QIII}$

71. $\sin(\theta) = \frac{1}{2}$ with $\theta^{\text{int}}\text{QII}$

72. $\cos(\theta) = \frac{1}{\sqrt{2}}$ with $\theta^{\text{int}}\text{QIV}$

73. $\tan(\theta) = -1$ with $\theta^{\text{int}}\text{QII}$

74. $\sin(\theta) = -\frac{\sqrt{3}}{2}$ with $\theta^{\text{int}}\text{QIV}$

75. $\cos(\theta) = -\frac{1}{2}$ with $\theta^{\text{int}}\text{QII}$

76. $\sin(\theta) = \frac{1}{\sqrt{2}}$ where $\cos(\theta) > 0$

77. $\tan(\theta) = -\frac{1}{\sqrt{3}}$ where $\sin(\theta) > 0$

78. $\cos(\theta) = -\frac{\sqrt{3}}{2}$ where $\tan(\theta) > 0$

79. $\sin(\theta) = -\frac{1}{2}$ where $\tan(\theta) < 0$

80. $\cos(\theta) = \frac{1}{2}$ where $\csc(\theta) < 0$

81. $\tan(\theta) = 1$ where $\sec(\theta) < 0$

82. $\csc(\theta) = 2$ where $\cos(\theta) < 0$

83. $\sec(\theta) = -\frac{2}{\sqrt{3}}$ where $\csc(\theta) > 0$

84. $\cot(\theta) = -\sqrt{3}$ where $\cos(\theta) < 0$

Synthesis Questions

For the following exercises, find exact values for all solutions between 0° and 360°.

85. $\cos(\theta) = -\cos(24^\circ)$

86. $\tan(\theta) = -\tan(9^\circ)$

87. $\sin(\theta) = -\sin(66^\circ)$

88. $\cos(\theta) = -\cos(78^\circ)$

89. $\tan(\theta) = -\tan(31^\circ)$

90. $\sin(\theta) = -\sin(42^\circ)$

91. Later we will be able to show that $\sin(18^\circ) = \frac{\sqrt{5}-1}{4}$. What is the exact value of $\sin(162^\circ)$?

92. Later we will be able to show that $\cos(36^\circ) = \frac{\sqrt{5}+1}{4}$. What is the exact value of $\cos(144^\circ)$?

For the following exercises,
(a) Sketch an angle in standard position with the given properties.
(b) Find $\cos(\theta)$, $\sin(\theta)$, and $\tan(\theta)$.
(c) Find the angle θ, rounded to tenths of a degree.

93. The point (–5, 12) is on the terminal side.

94. The point (12, 9) is on the terminal side.

95. $\cos(\theta) = -0.8$

96. $\cos(\theta) = \frac{5}{13}$

97. $\cos(\theta) = \frac{3}{11}$

98. $\cos(\theta) = -\frac{5}{6}$

99. $\tan(\theta) = -\frac{1}{6}$

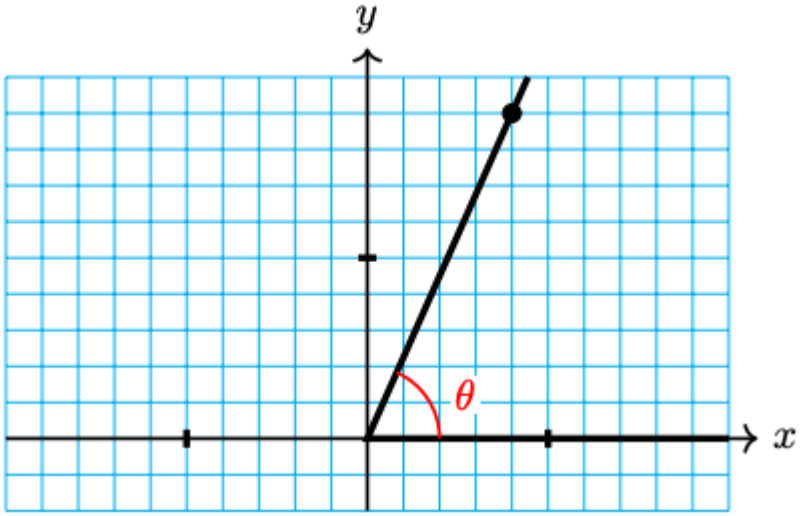
100. $\tan(\theta) = \frac{9}{5}$

101. $\tan(\theta) = 4$

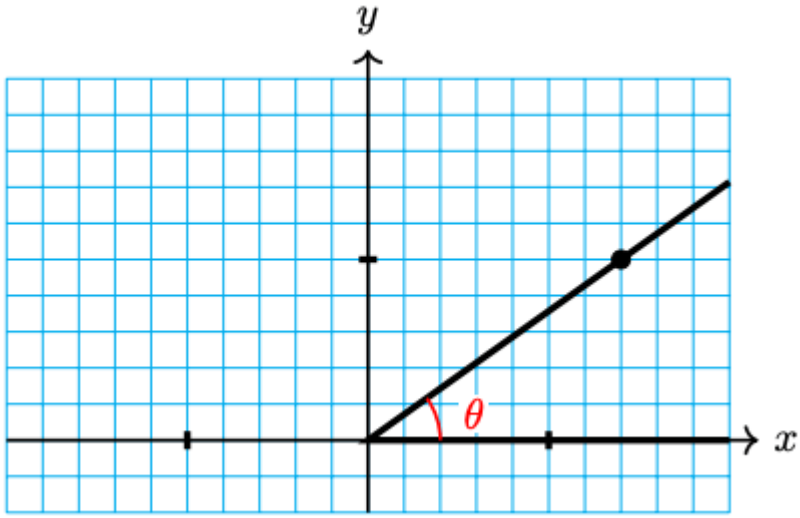
102. $\tan(\theta) = -1$

For the following exercises,
(a) Find the sine and cosine of the angle.
(b) Sketch the supplement of the angle in standard position. (Use congruent triangles.)
(c) Find the sine and cosine of the supplement.
(d) Find the angle and its supplement, rounded to the nearest degree.

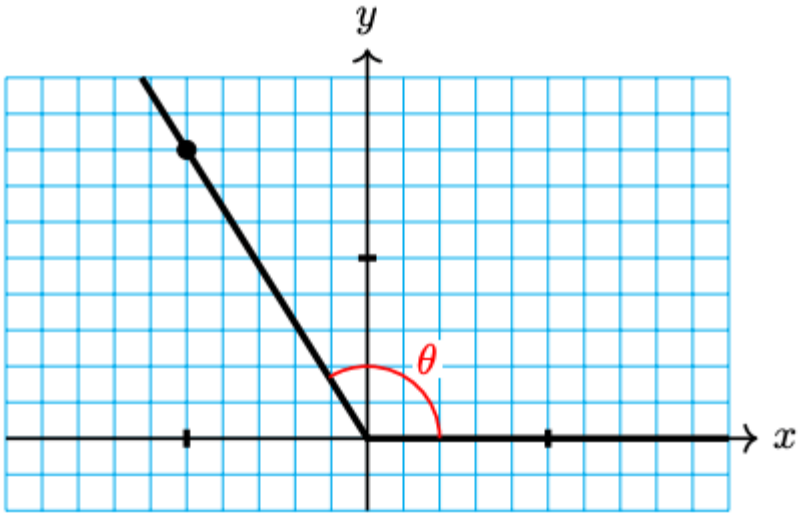
103.



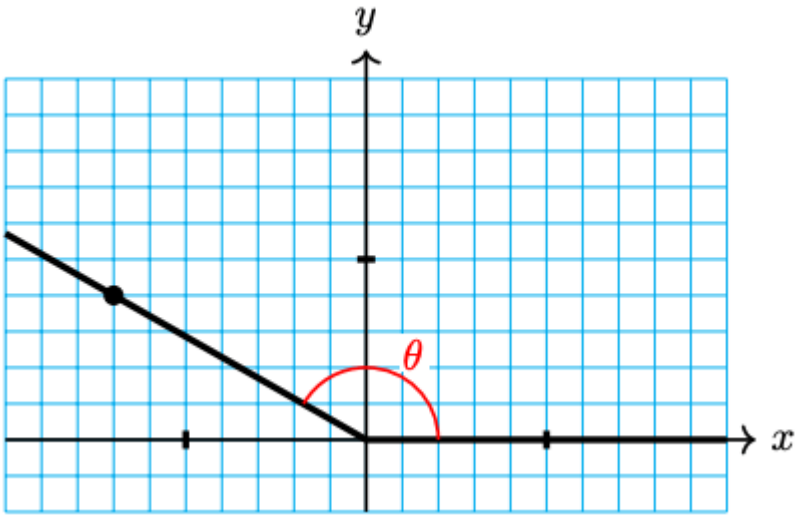
104.



105.



96.



For the following exercises, refer to the following table of values.

θ	15°	25°	65°	75°	105°	115°	155°	165°
$\cos(\theta)$								
$\sin(\theta)$								
$\tan(\theta)$								

97. Use your calculator to fill in the table. Round values to four decimal places.

98. For each angle θ in the table, the angle $180^\circ - \theta$ is also in the table.

a. What is true about $\sin\theta$ and $\sin(180^\circ - \theta)$?

b. What is true about $\cos\theta$ and $\cos(180^\circ - \theta)$?

c. What is true about $\tan\theta$ and $\tan(180^\circ - \theta)$?

The following exercises preview the concept of solving trigonometric equations (a topic reserved for later in the course). For each equation, find two different angles between 0° and 360° that satisfy the equation. Round your answers to the nearest 0.1°.

9. $\sin(\theta) = 0.7$

10. $\sin(\theta) = 0.1$

11. $\dfrac{\sin(\theta)}{6} = 0.14$

12. $\dfrac{5}{\sin(\theta)} = 6$

13. $4.8 = \dfrac{3.2}{\sin(\theta)}$

14. $1.5 = \dfrac{\sin(\theta)}{0.3}$

For the following exercises, fill in the blanks with complements or supplements.

15. If $\sin(57^\circ) = q$, then $\sin(\_\_\_\_) = q$, and $\cos(\_\_\_\_)$ and $\cos(\_\_\_\_)$ both equal q.

16. If $\sin(18^\circ) = w$, then $\sin(\_\_\_\_) = w$, $\cos(\_\_\_\_) = w$, and $\cos(\_\_\_\_) = -w$.

17. If $\cos(74^\circ) = m$, then $\cos(\_\_\_\_) = -m$, and $\sin(\_\_\_\_)$ and $\sin(\_\_\_\_)$ both equal m.

18. If $\cos(36^\circ) = t$, then $\cos(\_\_\_\_) = -t$, and $\sin(\_\_\_\_)$ and $\sin(\_\_\_\_)$ both equal t.

19. a. Sketch the line $y = \dfrac{3}{4}x$.

    b. A point on the line with a positive x-coordinate.

    c. The line $y = \frac{3}{4}x$ makes an angle with the positive x-axis. What is that angle?

    d. Repeat parts (a) through (c) for the line $y = \frac{-3}{4}x$, except find a point with a *negative* x-coordinate.

20. a. Sketch the line $y = \frac{5}{3}x$.

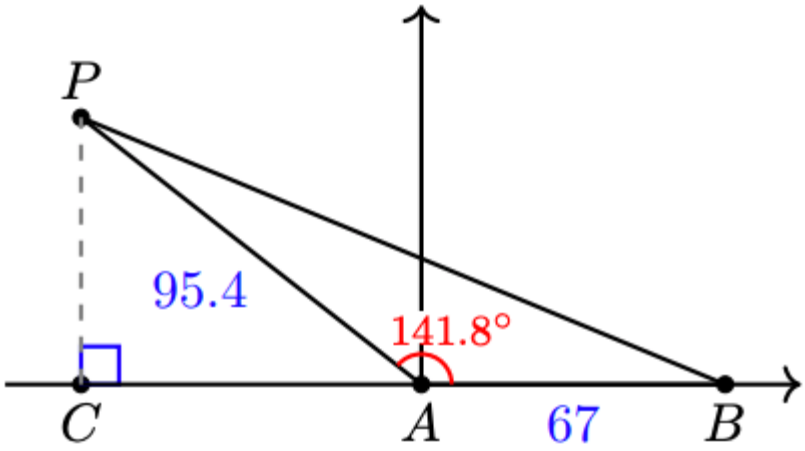
    b. Find a point on the line with a positive x-coordinate.

    c. The line $y = \frac{5}{3}x$ makes an angle with the positive x-axis. What is that angle?

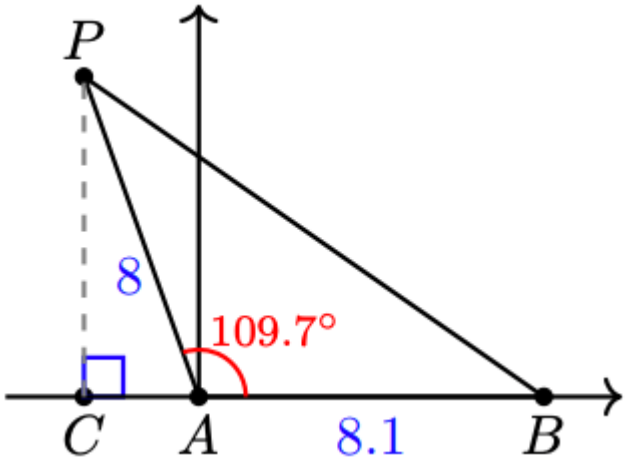
    d. Repeat parts (a) through (c) for the line $y = \frac{-5}{3}x$, except find a point with a *negative* x-coordinate.

For the following exercises,
(a) Find the coordinates of point P. Round to two decimal places.
(b) Find the sides BC and PC of $\triangle PCB$.
(c) Find side PB.

21.



22.



Challenge Problems

23. Why are the sines of supplementary angles equal, but the cosines are not? What about the tangents of supplementary angles?

This page titled 7.1.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

7.2: Radian Measure, Arc Length, and Sector Area

Note to the Instructor (click to expand)

This section introduces radian measure and, as a natural consequence of the definition, arc length. All of the previous work we did evaluating trigonometric functions (including using reference angles and quadrants) is quickly repeated using radian measure. We finish this section with sector areas.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Find the radian measure of the central angle given the radius and arc length.
- Convert an angle from degrees to radians or vice versa.
- Evaluate a trigonometric function using radians.
- Identify the reference angle for a given angle measured in radians.
- Solve an application involving arc length.
- Calculate the area of a sector formed by a central angle.
- Solve an application involving sector area.

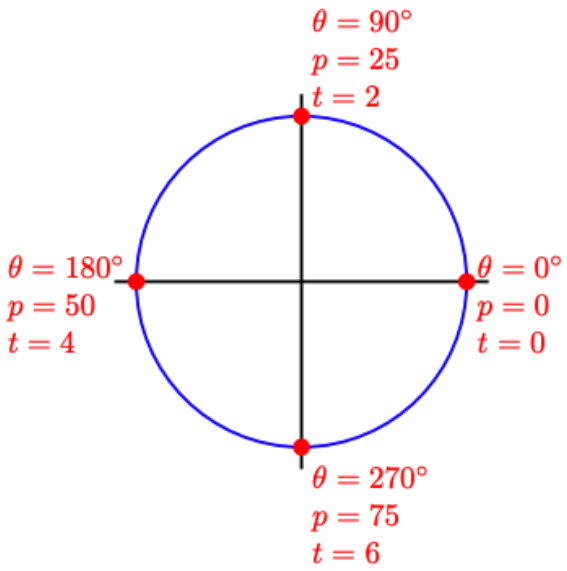
Hawk A.I. Section-Specific Tutor

EST. 1970 COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Imagine riding on a Ferris wheel with a radius of 100 feet, where each rotation takes eight minutes. We can use angles in a standard position to describe your location as you travel around the wheel. Figure $7.2.1$ shows the locations indicated by $\theta = 0°$, $90°$, $180°$, and $270°$.

Figure 7.2.1

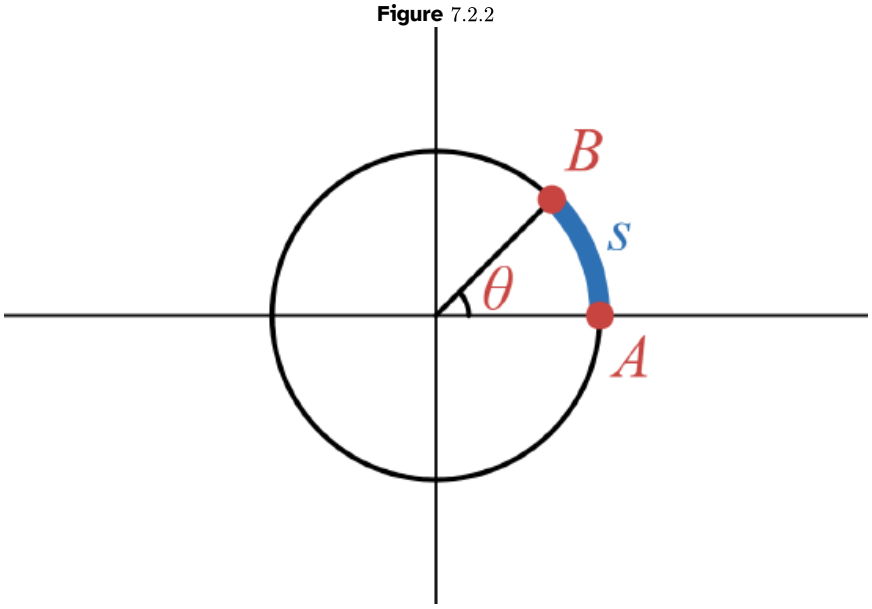


However, degrees are not the only way to specify a location on a circle. We could use a percent of one complete revolution[1] and label the same locations by $p = 0$, $p = 25$, $p = 50$, and $p = 75$. Another option is to use the time elapsed so that, for this example, we would have $t = 0$, $t = 2$, $t = 4$, and $t = 6$ minutes (see Figure 7.2.1).

Another helpful method uses distance traveled, or arc length, along the circle. In Figure 7.2.1, how far have you traveled around the Ferris wheel at each of the locations shown, assuming you hopped onto the ride at $\theta = 0°$?

Before we consider that question, let's agree on some vocabulary. To help us understand the vocabulary, let's consider Figure 7.2.2 .

Figure 7.2.2



We begin by naming a portion of the edge of a circle.

Definition: Arc and Arc Length

An **arc** along a circle is a portion of the circumference of a circle. It's the curved line connecting two points on the circle's edge. The length of this curve is called the **arc length**.

In Figure 7.2.2, the arc is in blue and its length, which we labeled as s, is called the arc length.

It should be natural to think that any arc along the circumference of a circle is related to an angle "opening" at the center of the circle. This angle (which is labeled as θ in Figure 7.2.2) gets its own name.

Definition: Central Angle

An angle with vertex at the center of a circle is called a **central angle** of the circle.

Thus, for the given arc in Figure 7.2.2, the related central angle is θ. If the arc represents a distance traveled, we sometimes refer to such an angle as the **angle of displacement**.

Finally, we need to introduce the conventional language used throughout most of Mathematics when it comes to relating the central angle to the arc.

Definition: Subtend

A central angle whose initial and terminal sides meet the endpoints of an arc is said to **subtend** (or **span**) the arc.

This definition means that the angle θ in Figure 7.2.2 subtends an arc of length s.

Arc Length

Recall that the circumference of a circle is given by the formula

$$C = 2\pi r. \tag{7.2.1}$$

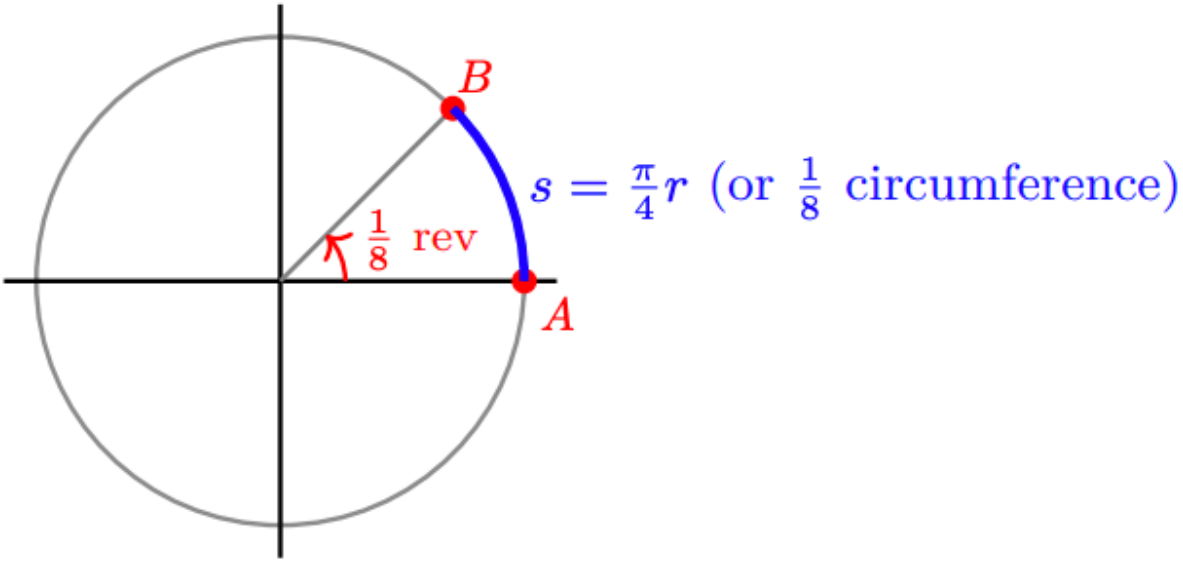
Another way to say this is that the circumference of a circle is proportional to its radius - increasing the radius of the circle increases the circumference and vice versa.

According to Equation 7.2.1, if we walk around the entire circumference of a circle, the distance we travel is the product of 2π and the radius, or *approximately* 6.28 times the radius (by the way, just like π, 2π gets its own special name - the Greek letter tau. That is, we define $\tau = 2\pi$). If we walk only part of the way around the circle, then the distance we travel depends also on the angle of displacement.

For example, an angle of 45° is $\frac{1}{8}$ of a complete revolution, so the arc length, s, from point A to point B in Figure 7.2.3 is $\frac{1}{8}$ of the circumference. Therefore,

$$s = \frac{1}{8}(2\pi r) = \frac{\pi}{4}r.$$

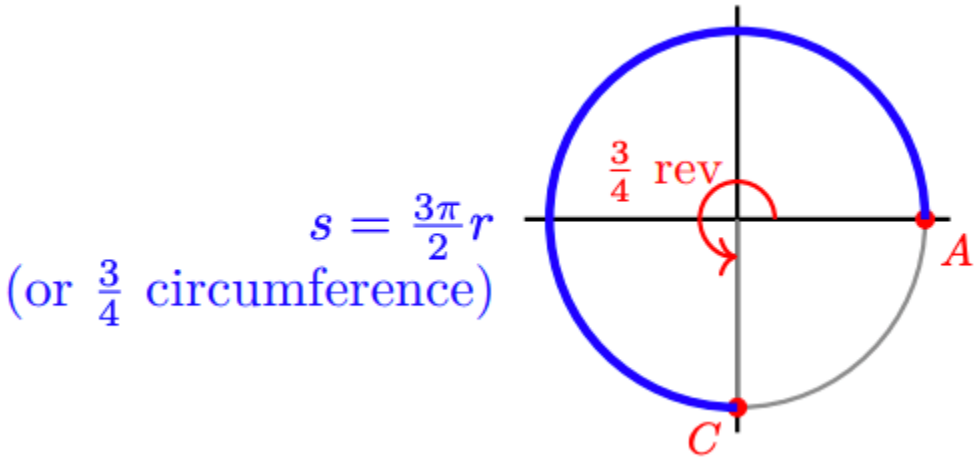
Figure 7.2.3



Similarly, the angle of displacement from point A to point C in Figure 7.2.4 is $\frac{3}{4}$ of a complete revolution, so the arc length along the circle from A to C, shown below, is

$$s = \frac{3}{4}(2\pi r) = \frac{3\pi}{2}r.$$

Figure 7.2.4



In general, the length of the arc spanned by an angle is proportional to the size of the angle for a given circle.

Theorem: Arc Length Formula (portion of a circle version)

The length of the arc, s, in a circle of radius r, spanned by a fraction of a revolution, p, is

$$s = p \cdot (2\pi r).$$

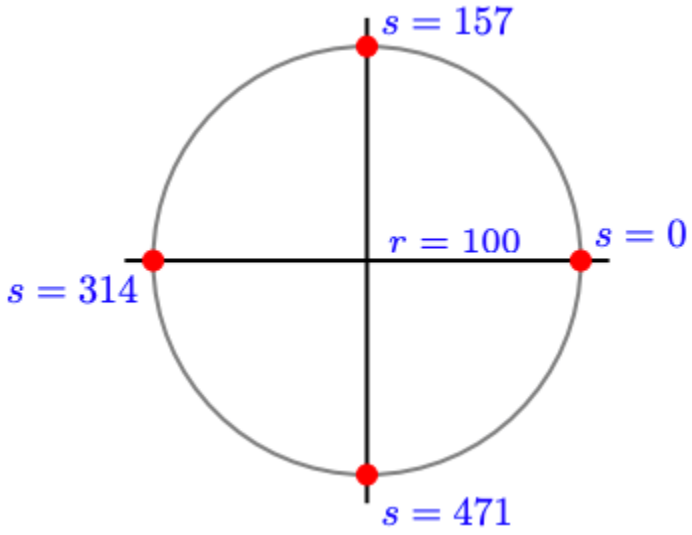
The Ferris wheel in the introduction has circumference

$$C = 2\pi(100) \approx 628 \text{ feet},$$

so in half a revolution, you travel approximately 314 feet around the edge. In one-quarter revolution, you travel approximately 157 feet.

To indicate the same four locations from Figure 7.2.1 on the wheel by distance traveled, we would use $s = 0$, $s \approx 157$, $s \approx 314$, and $s \approx 471$, as shown below.

Figure 7.2.5



Example 7.2.1

What length of arc is spanned by an angle of 120° on a circle of radius 12 centimeters?

Solution available at the end of this section.

Checkpoint 7.2.1

How far have you traveled around the edge of a Ferris wheel of radius 100 feet when you have turned through an angle of 150°?

Answer available at the end of this section.

Measuring Angles in Radians

If you think about measuring arc length, you will see that the degree measure of the spanning angle is not as important as the fraction of one revolution it covers. This observation suggests a new unit of measurement for angles, one that is better suited to calculations involving arc length. We'll make one change in our formula for arc length from

$$\textbf{arc length} = (\textbf{fraction of one revolution}) \cdot (2\pi r)$$

to

$$\textbf{arc length} = (\textbf{fraction of one revolution} \cdot 2\pi) \cdot r.$$

You might be asking yourself why we made such a silly change since these two formulas state the same thing (just with the parentheses in different spots).

The subtle difference between these two interpretations of arc length leads to an entirely new angular measurement system. We'll call the quantity in parentheses, (fraction of one revolution $\times 2\pi$), the **radian measure** of the angle that spans the arc.

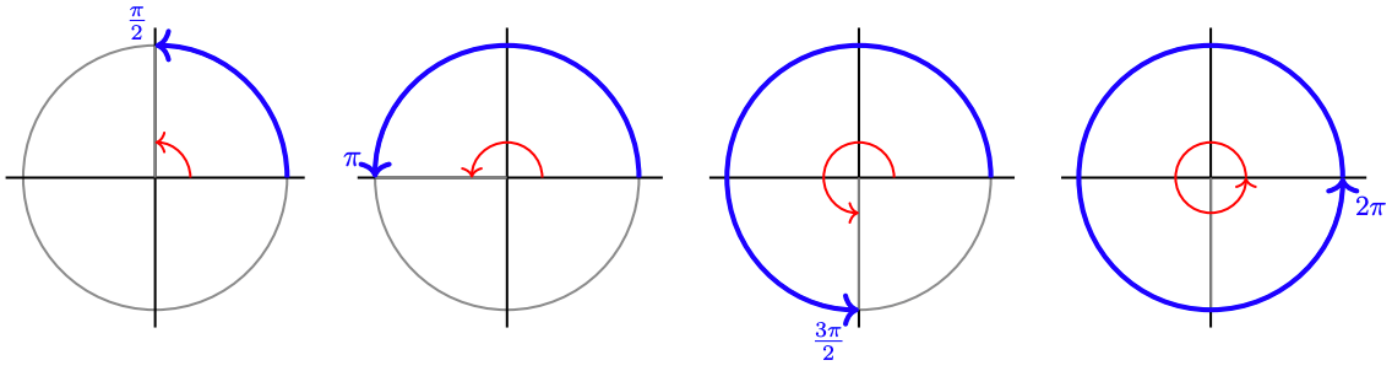
Definition: Radian Measure (portion of a circle version)

The **radian measure** of an angle is given by

$$\text{fraction of one revolution } \times 2\pi.$$

For example, one complete revolution (360°) is equal to 2π radians, and one-quarter revolution (90°) is equal to $\frac{1}{4}(2\pi) = \frac{\pi}{2}$ radians. Figure 7.2.7 shows the radian measure of the quadrantal angles.

Figure 7.2.7



Note: Radian Measure is Universal

Unlike degree measure, which depends on a human-created construct dependent on either the number of days for the Earth to rotate around the sun or the base-60 numbering system used by the Babylonians (depending on which historical viewpoint of the creation of the degree measurement system you believe), radian measure makes sense on *any* planet and within *any* numbering system. That is, if an alien species were to arrive on Earth, radian measures for angles are likely something we would both have in common.

Example 7.2.2

What is the radian measure of an angle of 120°?

Solution available at the end of this section.

Checkpoint 7.2.2

What fraction of a revolution is π radians? How many degrees is that?

Answer available at the end of this section.

Radian measure does not have to be expressed in multiples of π. Remember that $\pi \approx 3.14$, so one complete revolution is approximately 6.28 radians, and one-quarter revolution is $\frac{1}{4}(2\pi) = \frac{\pi}{2}$, or about 1.57 radians. Figure 7.2.8 shows decimal approximations for the quadrantal angles.

Figure 7.2.8

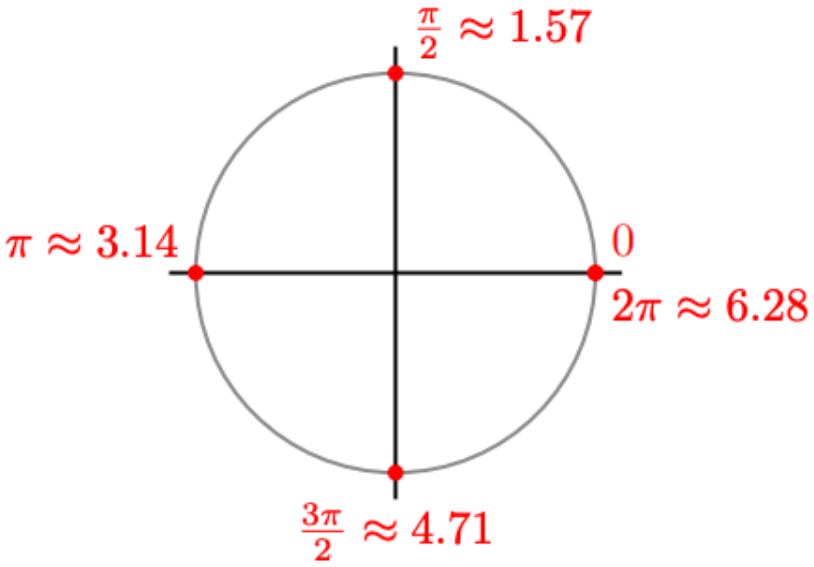


Table 7.2.1**:** *Quadrantal Angles in Both Degrees and Radians*

Degrees	Radians: Exact Values	Radians: Decimal Approximations
0°	0	0
90°	$\frac{\pi}{2}$	1.57
180°	π	3.14
270°	$\frac{3\pi}{2}$	4.71
360°	2π	6.28

Memorize the Exact Radian Values of Common Angles

Because they are "benchmarks" for comparing angles, you should memorize the exact values of the angles in the previous table in radians!

Example 7.2.3

In which quadrant would you find an angle of 2 radians? An angle of 5 radians?

Solution available at the end of this section.

Checkpoint 7.2.3

Draw a circle centered at the origin and sketch (in standard position) angles of approximately 3 radians, 4 radians, and 6 radians.

Answer available at the end of this section.

It turns out that measuring angles in radians is helpful for many applications besides calculating arc lengths, so we need to start thinking in radians. To help that process, we'll first learn to convert between degrees and radians.

Converting Between Degrees and Radians

It is not difficult to convert the measure of an angle in degrees to its measure in radians or vice versa. One complete revolution is equal to 2π radians or to 360°, so

$$2\pi \text{ radians } = 360^\circ.$$

Dividing both sides of this equation by 2π radians or by 360° gives us an **angle measure conversion theorem**:

Theorem: Angle Measure Conversion Theorem

Assuming the unitless measure of radians,

$$2\pi = 360^\circ$$

$$\Longrightarrow \quad 1 = \frac{180^\circ}{\pi} \quad \text{and} \quad \frac{\pi}{180} = 1^\circ$$

(dividing both sides by 2π radians) (dividing both sides instead by 360)

Example 7.2.4

a. Convert 3 radians to degrees.
b. Convert 3 degrees to radians.

Solutions available at the end of this section.

Technically speaking, a radian is what is known as a **unitless measure**. This is because, from the definition of radian measure, a radian is the product of a fraction of a revolution (no units) and 2π (also, no units). Therefore, it is acceptable to answer Example $7.2.4b$ with "approximately 0.05" instead of "approximately 0.05 radians." Moreover, there is no need to include the word "radian" when the context of the situation implies the constant is an angular measure (and the angular measure is not in degrees). For example, the solution to Example $7.2.4a$ could have been written as

$$\begin{aligned} 3 &= 3 \times (1) \\ &= 3 \times \left(\frac{180^\circ}{\pi}\right) \\ &= \frac{540^\circ}{\pi} \\ &\approx 171.9^\circ \end{aligned}$$

Checkpoint 7.2.4

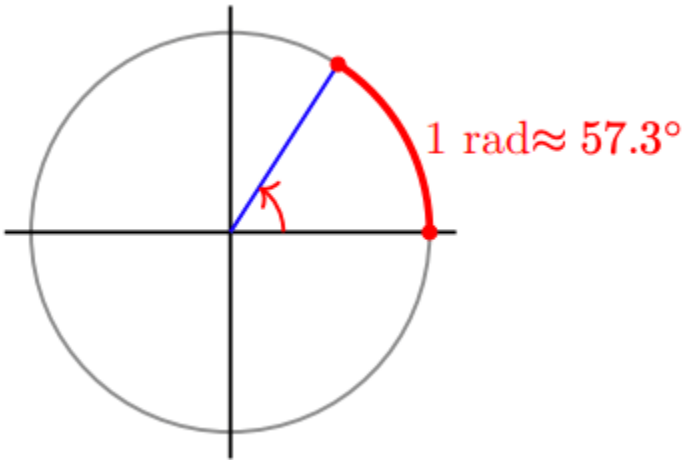
a. Convert 60° to radians. Give both an exact answer and an approximation to three decimal places.
b. Convert $\frac{3\pi}{4}$ to degrees.

Answers available at the end of this section.

From our angle conversion theorem, we also learn that

$$1 \text{ radian } = \frac{180^\circ}{\pi} \approx 57.3^\circ.$$

Figure 7.2.10

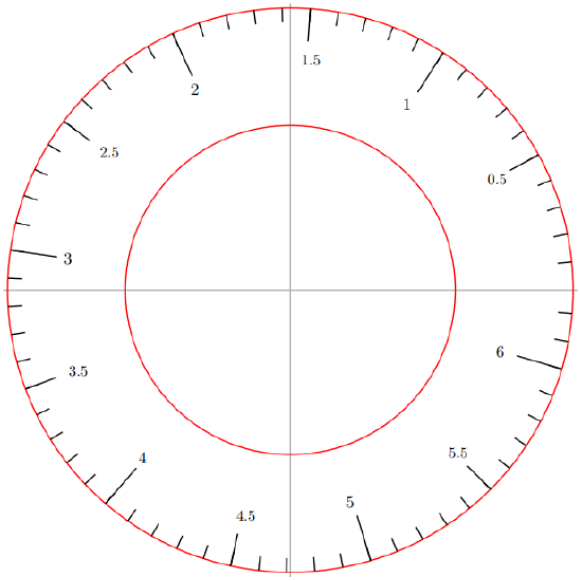


So while 1° is a relatively small angle, 1 radian is much larger (nearly 60°, in fact). However, this is reasonable because there are only a little more than 6 radians in an entire revolution. An angle of 1 radian is shown above.

We'll soon see that, for many applications, it is easier to work entirely in radians. In fact, in the study of Calculus, you will find that radian measure *must* be used instead of degree measure.

For reference, Figure 7.2.11 shows a radian protractor.

Figure 7.2.11



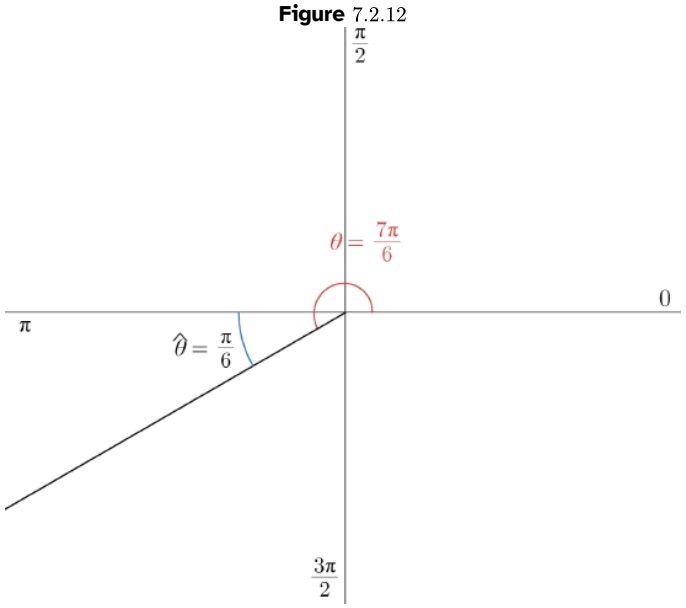
Quadrants and Reference Angles in Radians

Radian measure takes a bit to get used to, but you will soon find it easier to work with than degree measure. Understanding the acute special angles and quadrantal angles in radian measure, we now ask how to quickly determine in which quadrant an angle terminates. We also want to know the reference angle for a given angle in radian measure. We will focus on angles of the form $\frac{n\pi}{d}$ for this discussion.[2]

A beneficial habit to get into when given an angle in radian measure of the form $\frac{n\pi}{d}$ is to think in multiples of the denominator, d. This is because $\frac{d\pi}{d} = \pi$, $\frac{2d\pi}{d} = 2\pi$, $\frac{3d\pi}{d} = 3\pi$, and so on. If you sketch these angles, they terminate on either the positive or negative x-axis - this will be useful.

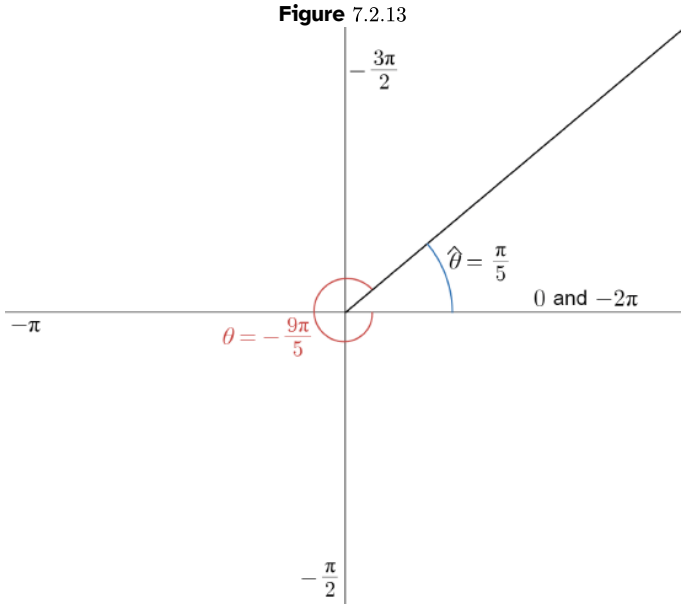
For example, if asked to find the quadrant in which $\theta = \frac{7\pi}{6}$ terminates, we consider that θ is *slightly* more than $\frac{6\pi}{6} = \pi$ (which is equivalent to $180°$). In fact, it is only $\frac{1\pi}{6}$ more than $\frac{6\pi}{6} = \pi$. Thus, to get to θ, we rotate *past* π and into quadrant three by the small amount of $\frac{\pi}{6}$ radians. This small extra angle of rotation is the reference angle for θ. That is, the reference angle for $\theta = \frac{7\pi}{6}$ is $\hat{\theta} = \frac{\pi}{6}$ (see Figure 7.2.12).

Figure 7.2.12



This concept of using multiples of the denominator need not be limited to positive angles. For example, if $\theta = -\frac{9\pi}{5}$, we imagine rotating in the negative direction to $-\frac{5\pi}{5} = -\pi$; however, this is nowhere near $-\frac{9\pi}{5}$. Therefore, we continue until we get to $-\frac{10\pi}{5} = -2\pi$. At this point, we will have rotated back to the positive x-axis, but we will have rotated *slightly* too far. Therefore, we back off (in the positive direction) by $\frac{1\pi}{5}$ to arrive at $-\frac{9\pi}{5}$, which terminates in the first quadrant. Hence, $\theta = -\frac{9\pi}{5}$ ^{int}QI with reference angle $\hat{\theta} = \frac{\pi}{5}$ (see Figure 7.2.13).

Figure 7.2.13



Example 7.2.5

State the quadrant in which $\beta = \frac{5\pi}{3}$ terminates, state its reference angle and sketch.

Solution available at the end of this section.

Checkpoint 7.2.5

State the quadrant in which $\alpha = -\frac{17\pi}{16}$ terminates, state its reference angle, and sketch.

Answer available at the end of this section.

Arc Length Formula

We now return to our calculation of arc length and see the first instance in which measuring angles in radians is useful. To calculate an arc length, we need only multiply the circle's radius by the radian measure of the spanning angle, θ. Look again at our formula for arc length:

$$\text{arc length } = (\text{fraction of one revolution } \times 2\pi) \cdot r$$

The quantity in parentheses, "fraction of one revolution $\times 2\pi$," is the measure of the spanning angle in radians. Thus, if θ is measured in radians, we have the following formula for arc length, s.

Theorem: Arc Length Formula (radian version)

The arc length, s, subtended by an angle θ (in radians) in a circle of radius r is

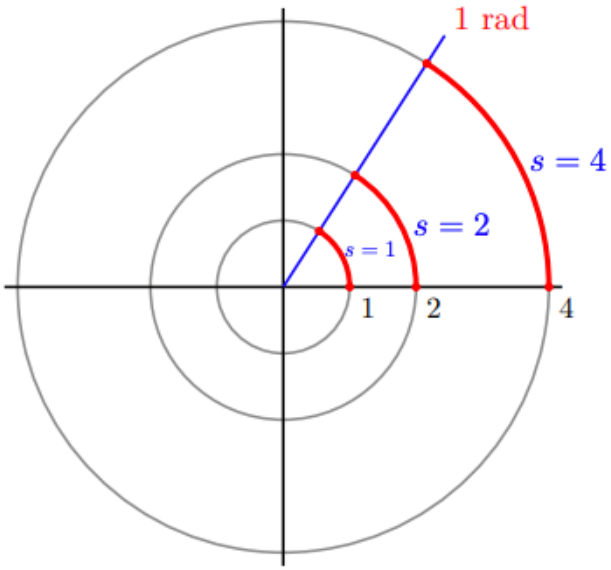
$$s = r\theta.$$

Caution: The Radian Requirement

It is crucial to remember that the arc length formula, $s = r\theta$, is only valid if θ is in radians.

Thus, there is a special relationship between arc length and radians. An angle of 1 radian spans an arc on a circle equal to the radius of the circle, as shown in Figure 7.2.15. Likewise, the length of any arc is just the product of the measure of its spanning angle in radians and the radius of the circle.

Figure 7.2.15



So, for instance, we can use the formula to find the arc length spanned by an angle of 2.5 radians on a circle of radius six inches:

$$s = r\theta = (6\,\text{inches})\,(2.5) = 15\ \text{inches}.$$

Note that we didn't use the word "radians" when replacing θ with 2.5. This is because, as we mentioned previously, radians is a unitless measure. Therefore, when performing **unit analysis** (which is the act of including units when performing computations to understand the meaning of the result of a series of computations), we leave the word "radians" off the angle.

We can also use the formula in the form $\theta = \frac{s}{r}$ to find an angle that spans a given arc. For example, an arc length equal to one radius determines a central
angle of one radian, or approximately $57.3°$.

In the following example, we use the arc length formula to compute a change in latitude on the Earth's surface.

Example 7.2.6

The Earth's radius is roughly 3,960 miles. If you travel 500 miles due north, how many degrees of latitude will you traverse? (Latitude is measured in degrees north or south of the equator.)

Solution available at the end of this section.

Checkpoint 7.2.6

The distance around the face of a large clock from 2 to 3 is five feet. What is the radius of the clock?

Answer available at the end of this section.

Example 7.2.7

The London Eye, located on the South Bank of the River Thames in London, is Europe's tallest Ferris wheel. In fact, when it first opened in 2000, it was the *world*'s tallest Ferris wheel. It has a diameter of 120 meters. Interestingly, the London Eye rotates slowly enough that it never stops - riders easily step on and off at the wheel's bottom platform while the wheel is in motion.

Figure 7.2.17



Loi boards the London Eye to take in an elevated city view. If Loi has moved 80 meters along the circumference of the wheel, how many degrees has the London Eye rotated?

Solution available at the end of this section.

Checkpoint 7.2.7

A central angle in a circle of radius 6 cm cuts off an arc of length 24 cm. What is the radian measure of the angle? What is the degree measure?

Answer available at the end of this section.

Trigonometric Functions of Angles in Radians

Measuring angles in radians has other applications besides calculating arc length. One such use is evaluating trigonometric functions of angles in radians. Many applications in Trigonometry (and beyond) use radians for input values rather than degrees. For this reason, it is crucial to know the trigonometric values for the special angles, which you learned in degrees previously, when the angles are given in radians. Fortunately, the value of a trigonometric function is the same whether the angle is measured in radians or degrees.

For example, $\frac{\pi}{3}$ radians is the same as 60°, because $\frac{\pi}{3} \cdot \frac{180^\circ}{\pi} = 60^\circ$, so

$$\sin\left(\frac{\pi}{3}\right) = \sin\left(60^\circ\right) = \frac{\sqrt{3}}{2}$$

Table 7.2.2 shows the radian measures for the special angles and the quadrantal angles, as well as the exact values of the sine, cosine, and tangent at these angles.

Table 7.2.2

θ in Degrees	θ in Radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	undefined
180°	π	0	-1	0
270°	$\frac{3\pi}{2}$	-1	0	undefined
360°	2π	0	1	0

You should memorize these function values and be able to use them to find trigonometric values for the special angles in all four quadrants, as they will appear frequently throughout the rest of Trigonometry.

Example 7.2.8

Compute the value of $\csc\left(\frac{\pi}{3}\right)$.

Solution available at the end of this section.

Checkpoint 7.2.8

Compute the value of $\cot\left(\frac{\pi}{4}\right)$.

Answer available at the end of this section.

As before, we use reference angles to define the trigonometric functions for angles greater than $90°$.

Example 7.2.9

Give exact values for the following.

a. $\tan\left(\frac{2\pi}{3}\right)$
b. $\cos\left(\frac{5\pi}{4}\right)$

Solutions available at the end of this section.

Checkpoint 7.2.9

Give exact values for the following.

a. $\sin\left(\frac{5\pi}{6}\right)$
b. $\tan\left(\frac{7\pi}{4}\right)$

Answers available at the end of this section.

Areas of Sectors

Now that we have paired angles with real numbers via radian measure, a world of applications awaits us.

We recently discovered the relationship between the length, s, of the arc subtended by an angle θ (in radians) in a circle of radius r to be

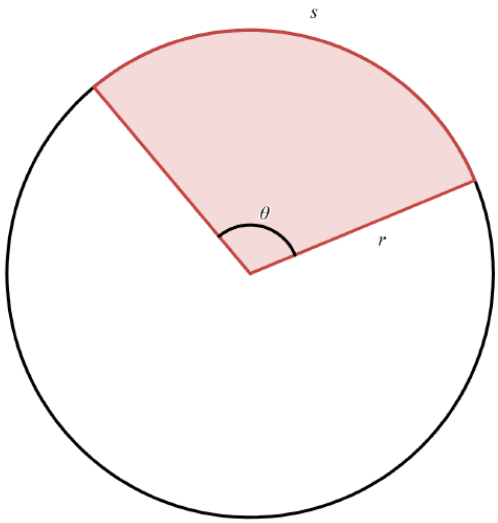
$$s = r\theta.$$

We now investigate the area of the **sector** formed by this central angle θ.

Definition: Sector (of a circle)

The region of a circle bounded by the initial and terminal sides of a central angle θ and the arc subtended by θ is called a **sector of the circle** (or, simply, **sector**).

Figure 7.2.20**:** *The area of a sector of a circle of radius r subtended by the angle θ*



To find the area of a sector, recall that the *total area* of a circle of radius r is πr^2; however, the shaded region in Figure \(7.2.20 \) is not the entire circle. In fact, it is a specific *portion* of this circle. In terms of angular measure, we have only shaded the region covered by an angle rotated through θ out of 2π radians of a full circle. That is, the portion of the circle we have shaded is

$$A = \frac{\theta}{2\pi} \text{ of the area of the full circle} = \frac{\theta}{2\cancel{\pi}} \cdot \cancel{\pi} r^2 = \frac{1}{2} r^2 \theta.$$

This proves the following theorem.

Theorem: Area of a Sector

If θ (in radians) is the central angle in a circle of radius r, then the area of the sector subtended by θ is

$$A = \frac{1}{2} r^2 \theta.$$

Finding the areas of sectors becomes increasingly essential as you move into Calculus (specifically, Integral Calculus). Be sure to spend the time now to master this relatively straightforward concept.

Example 7.2.10

Find the area of the sector subtended by a central angle of 120° in a circle of radius 4.

Solution available at the end of this section.

Checkpoint 7.2.10

If the area of a sector of a circle having radius 12 meters is 100 square meters, find the measure of the central angle (in radians).

Answer available at the end of this section.

Example 7.2.11: The Midnight Sun Golf Challenge

In certain parts of the world near the Arctic, such as Norway, the sun doesn't set during part of the summer. This phenomenon is known as the Midnight Sun. A golf course in Norway takes advantage of this by hosting a special event: The Midnight Sun Golf Challenge. During this event, golfers can play as many holes as they want throughout the night.

The golf course has a unique challenge on their signature par 3 hole, which features a circular green with a radius of 15 meters. If a golfer lands their ball within a particular marked sector of the green, where the central angle is 30 degrees, they win a prize. To determine the likelihood of winning, calculate the area of the sector where golfers must land their ball.

Solution available at the end of this section.

Checkpoint 7.2.11

A circular park has a sector fenced off for dogs. The sign posted by Sacramento County states that dogs can enjoy 6,000 square feet of play area; however, Paige finds this claim rather dubious. Unafraid about how foolish she might look, Paige takes a very large tape measure to the park and measures the radius of the park to be 120 feet. She then measures the length of fencing along the arc of the dog park area to be roughly 84 feet. Is the sign claiming that dogs have 6,000 square feet of play area correct?

Answer available at the end of this section.

Footnotes

[1] The word "revolution" will be used extensively from this point forward. Some texts use "rotation."

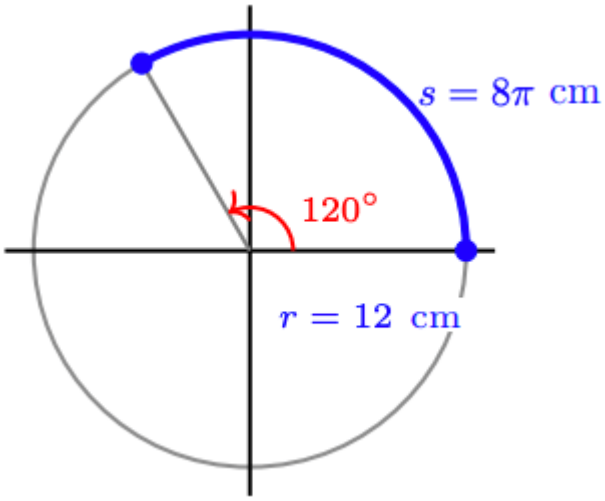
[2] Angles in radians need not have a π attached to them, but it does help when determining the quadrant in which the angle terminates. Therefore, we restrict ourselves to angles of the form $\frac{n\pi}{d}$ for this short discussion.

Answers

Example 7.2.1

Because $\frac{120^\circ}{360^\circ} = \frac{1}{3}$, an angle of 120° is $\frac{1}{3}$ of a complete revolution, as shown below.

Figure 7.2.6



Using the arc length formula with $r = 12$, we find that

$$s = \frac{1}{3}(2\pi \cdot 12) = \frac{2\pi}{3} \cdot 12 = 8\pi \text{cm}$$

or about 25.1 cm.

Checkpoint 7.2.1

Approximately 261.8 ft

Example 7.2.2

An angle of 120° is $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of a complete revolution, as we saw in the previous example. Thus, an angle of 120° has a radian measure of $\frac{1}{3}(2\pi)$, or $\frac{2\pi}{3}$.

Checkpoint 7.2.2

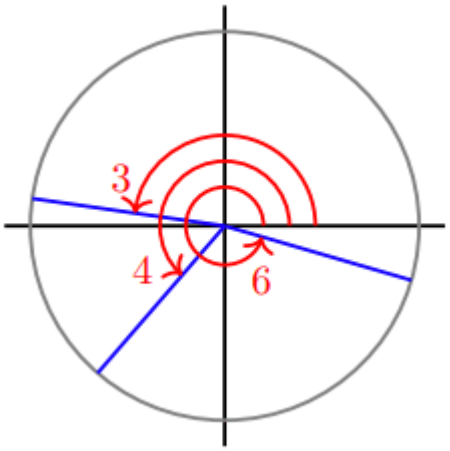
Half a revolution, 180°

Example 7.2.3

Look at Figure 7.2.5. The second quadrant includes angles between $\frac{\pi}{2}$ and π, or approximately 1.57 and 3.14 radians, so 2 radians lies in the second quadrant. An angle of 5 radians is between 4.71 and 6.28, or between $\frac{3\pi}{2}$ and 2π radians, so it lies in the fourth quadrant.

Checkpoint 7.2.3

Figure 7.2.9



Example 7.2.4

a.

$$\begin{aligned} 3 \text{ radians} &= 3 \text{ radians} \times (1) \\ &= 3 \text{ } \cancel{\text{radians}} \times \left(\frac{180^\circ}{\pi \text{ } \cancel{\text{radians}}} \right) \\ &= \frac{540^\circ}{\pi} \\ &\approx 171.9^\circ \end{aligned}$$

b.

$$\begin{aligned} 3^\circ &= 3^\circ \times (1) \\ &= 3^{\cancel{\circ}} \times \left(\frac{\pi \text{ radians}}{180^{\cancel{\circ}}} \right) \\ &= \frac{\pi}{60} \text{ radians} \\ &\approx 0.05 \text{ radians} \end{aligned}$$

Checkpoint 7.2.4

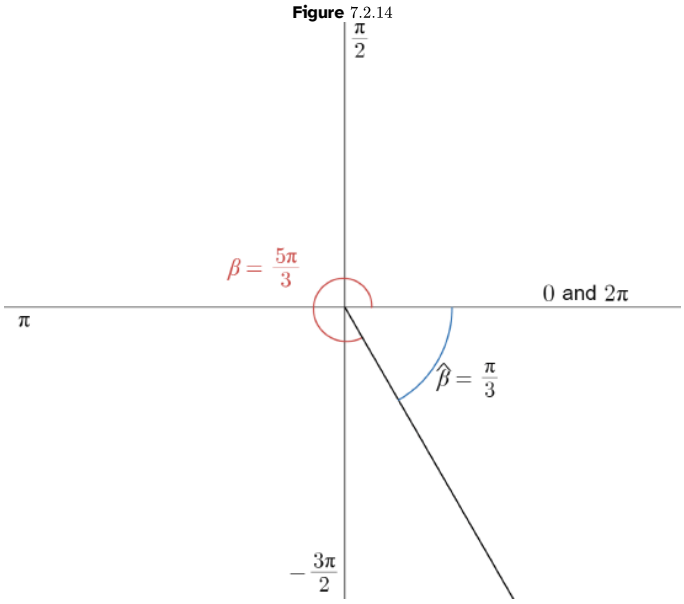
a. $\frac{\pi}{3} \approx 1.047$

b. 135°

Example 7.2.5

$\beta = \frac{5\pi}{3}$ is almost all the way to $\frac{6\pi}{3} = 2\pi$, but we have to back off by $\frac{\pi}{3}$. Hence, $\frac{5\pi}{3}$ [int]QIV and $\hat{\beta} = \frac{\pi}{3}$ (see Figure 7.2.14).

Figure 7.2.14



Checkpoint 7.2.5

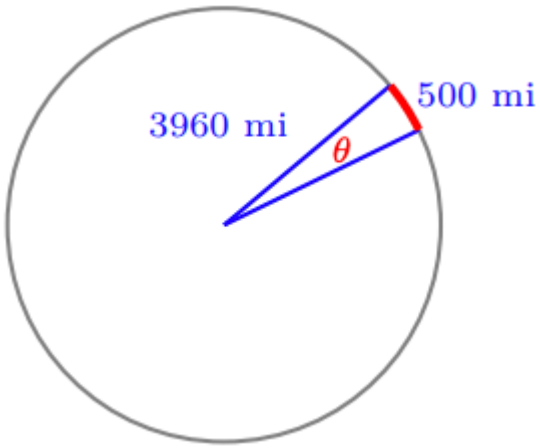
$\alpha^{\text{int}}\text{QII}$ and $\hat{\alpha} = \frac{\pi}{16}$

Example 7.2.6

We think of the distance of 500 miles as an arc length on the surface of the Earth, as shown below. Substituting $s = 500$ and $r = 3960$ into the arc length formula gives

$$\begin{aligned} 500\,\text{miles} &= 3960\,\text{miles} \times \theta \\ \Longrightarrow \quad \frac{500\ \cancel{\text{miles}}}{3960\ \cancel{\text{miles}}} &= \theta \\ \Longrightarrow \quad \theta &\approx 0.1263 \text{ radians} \end{aligned}$$

Figure 7.2.16



To convert the angle measure to degrees, we multiply by the conversion factor $\frac{180^\circ}{\pi}$ to get

$$0.1263\left(\frac{180^\circ}{\pi}\right) \approx 7.23^\circ$$

Your latitude has changed by about 7.23°.

Checkpoint 7.2.6

9.55 feet

Example 7.2.7

Parsing this question into language associated with our recent material, we have a circle of diameter 120 meters, which means the radius of the circle is 60 meters. Loi is moving through an arc of length 80 meters along the edge of this circle. Our task is to find the central angle that subtends this arc. In the formula

$$s = r\theta,$$

$s = 80$ meters and $r = 60$ meters. Therefore,

$$80\,\text{meters} = (60\,\text{meters})\,\theta \implies \frac{80\ \cancel{\text{meters}}}{60\ \cancel{\text{meters}}} = \frac{4}{3} = \theta.$$

However, this angle is in *radians*. We must convert this to degrees.

$$\frac{4}{3} \times \frac{180^\circ}{\pi} \approx 76.4^\circ.$$

Hence, Loi moves through an approximate angle of 76.4°.

Checkpoint 7.2.7

$4 \text{ radians} \approx 229^\circ$

Example 7.2.8

Our identities still hold - working in radians does not change that fact. Therefore, by the Reciprocal Identities,

$$\csc\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}.$$

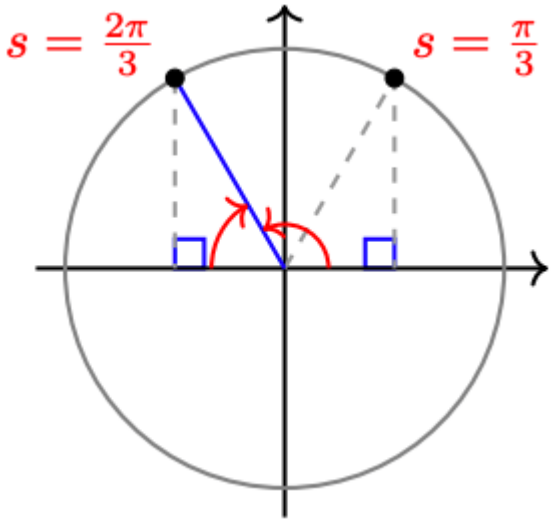
Checkpoint 7.2.8

1

Example 7.2.9

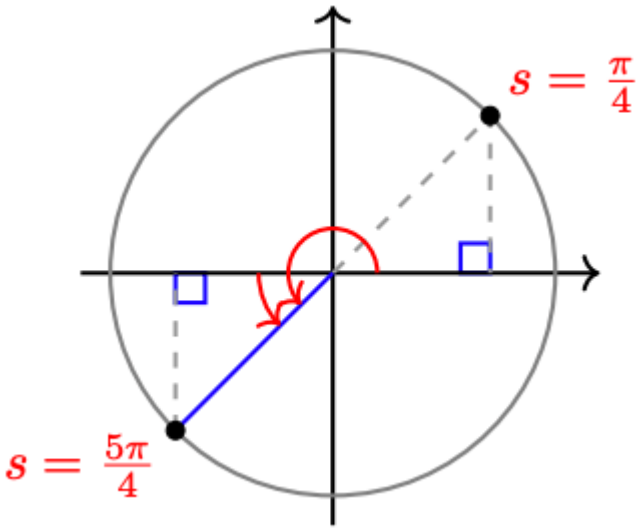
a. The reference angle for $\theta = \frac{2\pi}{3}$ is $\hat{\theta} = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$, and the tangent is negative in the second quadrant. (See the figure below.) Thus, $\tan\left(\frac{2\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$.

Figure 7.2.18



b The reference angle for $\theta = \frac{5\pi}{4}$ is $\hat{\theta} = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$, and the cosine is negative in the third quadrant, so $\cos\left(\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

Figure 7.2.19



Checkpoint 7.2.9

a. $\frac{1}{2}$
b. -1

Example 7.2.10

The most common mistake when using the formula for the area of a sector is not working in radian measure. Since $120^\circ = \frac{2\pi}{3}$, we find the area of the sector to be

$$A = \frac{1}{2} \cdot (4)^2 \left(\frac{2\pi}{3}\right) = \frac{16\pi}{3}.$$

Checkpoint 7.2.10

$\frac{25}{18}$ radians

Example 7.2.11**: The Midnight Sun Golf Challenge**

We are given $\theta = 30^\circ = \frac{\pi}{6}$ radians and $r = 15$ meters. Therefore, the area of the sector is

$$A = \frac{1}{2} \cdot (15)^2 \frac{\pi}{6} = \frac{225\pi}{12} \approx 59 \,\text{square meters}$$

As an aside, the probability that the golfer's ball would land in this sector (given the ball lands on the green and that all areas are equally likely for the ball to land in) would be the ratio of this sector area to the area of the entire green, which is

$$\frac{A_{\text{sector}}}{A_{\text{green}}} = \frac{225\pi/12}{\pi(15)^2} = \frac{1}{12} \approx 0.083 = 8.3\%.$$

This should make sense because 30° is one-twelfth of a full rotation of a circle.

Checkpoint 7.2.11

No. The given measurements yield a central angle of 0.7 radians. Thus, the area is actually

$$A = \frac{1}{2}(120)^2(0.7) = 5,040 \,\text{square feet}.$$

This page titled 7.2: Radian Measure, Arc Length, and Sector Area is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

7.2.1: Resources and Key Concepts

Resources

Videos

- Radian Measure
  - Angles - Angle Measure: Radians
- Arc Length
  - Angles - Arc Length
  - Angle Measure - Arc Length
- Degree to Radian Conversion
  - Angles - Relating Degrees to Radians
- Reference Angles in Radians
  - Angles - Reference Angles
- Area of a Sector
  - Angle Measure - Area of a Sector

Key Concepts

Definitions

- **Arc and Arc Length:** An arc is the portion of the circumference of a circle between two points on its edge. The arc length is the distance along this curved portion.
- **Central Angle:** An angle whose vertex is at the center of a circle.
- **Subtend:** A central angle is said to subtend (or span) an arc if its sides pass through the endpoints of the arc.
- **Radian Measure:** The radian measure of an angle is defined as the product of its fraction of a complete revolution and 2π.
- **Sector (of a circle):** The region of a circle that is bounded by a central angle's initial and terminal sides and the arc it subtends.

Theorems

- **Angle Measure Conversion Theorem:** The relationship between radians and degrees can be used to create conversion factors: $1 = \frac{180^\circ}{\pi \text{ radians}}$ and $1 = \frac{\pi \text{ radians}}{180^\circ}$.
- **Arc Length Formula:** On a circle of radius r, the length s of an arc spanned by a central angle θ is given by $s = r\theta$, provided that θ is measured in radians.
- **Area of a Sector:** If θ (in radians) is the central angle in a circle of radius r, then the area of the sector subtended by θ is $A = \frac{1}{2}r^2\theta$.

Common Mistakes

- **The Radian Requirement:** The arc length formula $s = r\theta$ is only valid if the angle θ is measured in radians. Using an angle in degrees will produce an incorrect result.
- **Confusing Radians and Degrees:** Be mindful of the units for a given angle. An angle of 1 radian is approximately 57.3°, whereas an angle of 1° is very small. Do not use the two interchangeably.
- **Thinking Radians Must Include π:** While many common angles in radians are expressed as multiples of π (e.g., $\frac{\pi}{4}$), an angle can be measured in radians without π (e.g., 2 radians).
- **Using Degrees in Formulas:** The formulas for the area of a sector ($A = \frac{1}{2}r^2\theta$) and the relationship between linear and angular velocity ($v = r\omega$) both require the angle θ and the angular velocity ω to be in radian-based units. Using degrees will lead to incorrect answers.

This page titled 7.2.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

7.2.2: Homework

Reading Questions

1. What is the difference between an arc and an arc length?
2. What defines a central angle?
3. How is the radian measure of an angle defined based on a fraction of a revolution?
4. How many radians are in one full revolution (360°)?
5. What is the formula for arc length, s, if the central angle θ is measured in radians and the radius is r?
6. What is the most important requirement for the angle θ when using the arc length formula $s = r\theta$?
7. What is the conversion factor used to convert an angle from degrees to radians?
8. What is the conversion factor used to convert an angle from radians to degrees?
9. In which quadrant does an angle of 5 radians terminate?
10. What is the reference angle for $\theta = \frac{5\pi}{3}$?
11. What is a sector of a circle?
12. What is the formula for the area of a sector, and what unit must the central angle be in for the formula to be valid?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, use the appropriate conversion factor to convert units.

1. $\frac{1 \text{ mile}}{1.609 \text{ kilometers}} = 1$
   a. 10 miles = _____ km
   b. 50 km = _____ miles
2. $\frac{1 \text{ acre}}{0.405 \text{ hectare}} = 1$
   a. 40 acres = _____ hectares
   b. 5 hectares = _____ acres
3. $\frac{1 \text{ horsepower}}{746 \text{ watts}} = 1$
   a. 250 horsepower = _____ watts
   b. 1000 watts = _____ horsepower
4. $\frac{1 \text{ troy ounce}}{480 \text{ grains}} = 1$
   a. 0.5 troy oz = _____ grains
   b. 100 grains = _____ troy oz

For the following exercises, determine how many degrees are in each fraction of one complete revolution.

5. $\frac{1}{4}$
6. $\frac{1}{5}$
7. $\frac{1}{6}$
8. $\frac{1}{8}$

Answers available at the end of this section.

Homework

Vocabulary Check

1. Unlike degree measure, radian measure is a ___ measure (i.e., radian measure does not require a measurement symbol).
2. An angle with vertex at the center of the circle is called a(n) ___.
3. A central angle whose sides meet the endpoints of an arc is said to ___ the arc. Another way to say this is the angle ___ the arc.
4. The ___ length is the distance along the edge of a circle subtended by a central angle θ.
5. The product of 2π and the fraction of one revolution cut out by the angle θ is named the ___ measure of the angle.
6. The region bounded by the initial and terminal sides of a central angle θ and the edge of the circle is called a ___ of a circle.

Concept Check

7. The length of a circular arc depends on what two variables?
8. Define the radian measure of an angle.
9. What is the conversion factor from radians to degrees?
10. On a unit circle, the length of an arc is equal to what other quantity?
11. Write each statement as a mathematical formula.
    a. The length of the arc, s, is equal to:
    $$(\text{the fraction of a revolution}) \times (\text{circumference of the circle})$$
    b. The area of the sector, A, is equal to:
    $$(\text{the fraction of a revolution}) \times (\text{area of the circle})$$

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

12. The distance we travel around a circle of diameter d is proportional to the angle of displacement. In fact,
$$\text{arc length } = (\text{fraction of one revolution}) \cdot (\pi d)$$
13. We measure angles in radians when we work with arc length.

14. An arc length equal to one radius determines a central angle of one degree.

15. An angle given in radian measure must be written as $k\pi$, where k is a real number.

16. $\dfrac{180^\circ}{\pi \text{ radians}} = 1$

17. On a circle of radius r, the length s of an arc spanned by an angle θ is $s = r\theta$.

18. To compute the area of a sector, we multiply the given angle by half of the square of the circle's radius.

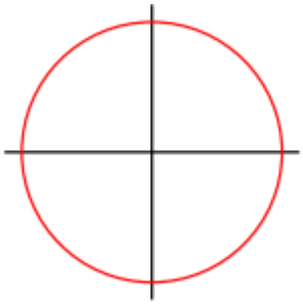
Basic Skills

19.

Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
Degrees									

a. Convert each angle to degrees.

b. Sketch each angle on a circle like this, and label in radians.

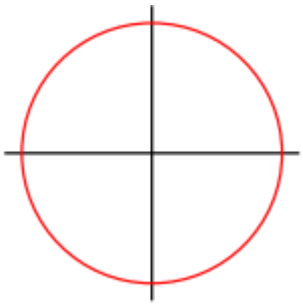


20.

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
Degrees													

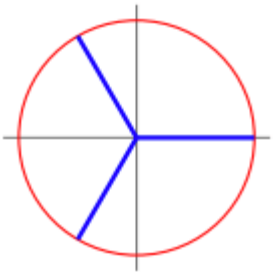
a. Convert each angle to degrees.

b. Sketch each angle on a circle like this, and label in radians.



For the following exercises, express each fraction of one complete rotation in degrees and radians.

21.



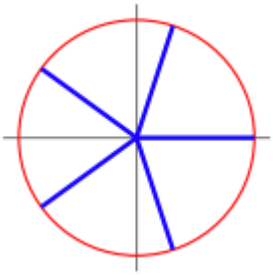
a. $\frac{1}{3}$

b. $\frac{2}{3}$

c. $\frac{4}{3}$

d. $\frac{5}{3}$

22.



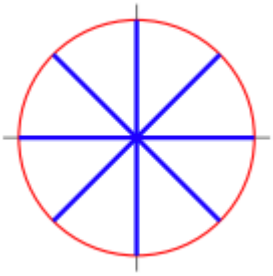
a. $\frac{1}{5}$

b. $\frac{2}{5}$

c. $\frac{3}{5}$

d. $\frac{4}{5}$

23.



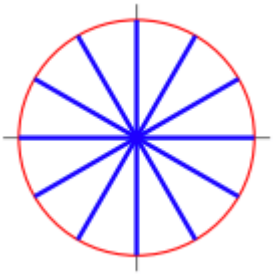
a. $\frac{1}{8}$

b. $\frac{3}{8}$

c. $\frac{5}{8}$

d. $\frac{7}{8}$

24.



a. $\frac{1}{12}$

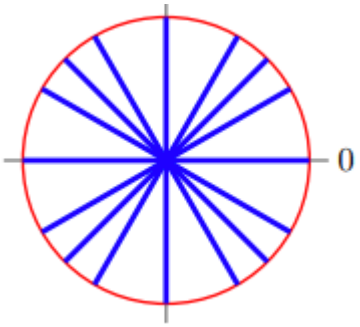
b. $\frac{1}{6}$

c. $\frac{5}{12}$

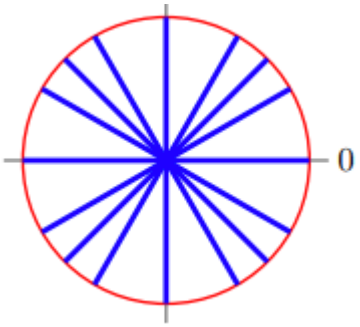
d. $\frac{5}{6}$

For the following exercises, label each angle in standard position with radian measure (assume the angles shown are special angles).

25. Rotate counter-clockwise from 0.



26. Rotate clockwise from 0.



For the following exercises, give a decimal approximation to hundredths for each angle in radians.

27. a. $\frac{\pi}{6}$

b. $\frac{5\pi}{6}$

c. $\frac{7\pi}{6}$

d. $\frac{11\pi}{6}$

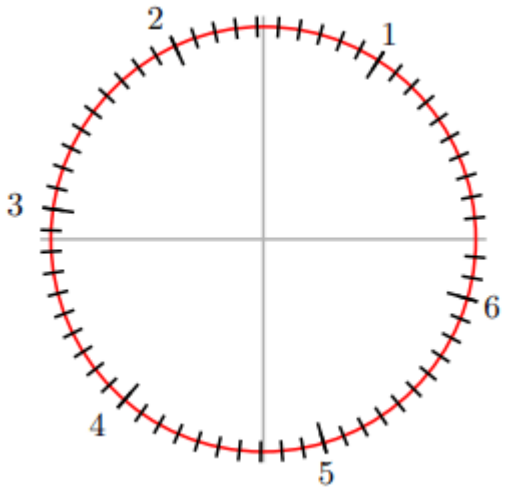
28. a. $\frac{\pi}{4}$

b. $\frac{\pi}{4}$

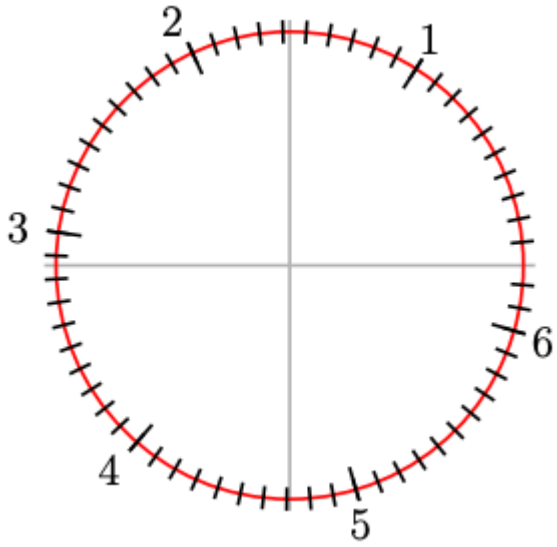
c. $\frac{5\pi}{4}$

d. $\frac{7\pi}{4}$

29. Locate and label each angle from Problem 24 on the unit circle below. (The circle is marked off in tenths of a radian.)



30. Locate and label each angle from Problem 25 on the unit circle below. (The circle is marked off in tenths of a radian.)



For the following exercises, choose the best decimal approximation for each angle in radians from the list below. Do not use a calculator; use the fact that π is a little greater than 3.

$$0.52, 0.79, 2.09, 2.36, 2.62, 3.67, 5.24, 5.50$$

31. $\frac{2\pi}{3}$

32. $\frac{\pi}{4}$

33. $\frac{5\pi}{6}$

34. $\frac{5\pi}{3}$

35. $\frac{\pi}{6}$

36. $\frac{7\pi}{4}$

37. $\frac{3\pi}{4}$

38. $\frac{7\pi}{6}$

For the following exercises, state in which quadrant each angle lies.

39. a. $\frac{\pi}{4}$

    b. $\frac{3\pi}{4}$

    c. $\frac{5\pi}{4}$

    d. $\frac{7\pi}{4}$

40. a. $\frac{\pi}{6}$

    b. $\frac{5\pi}{6}$

    c. $\frac{7\pi}{6}$

    d. $\frac{11\pi}{6}$

41. a. $\frac{\pi}{3}$

    b. $\frac{2\pi}{3}$

    c. $\frac{4\pi}{3}$

    d. $\frac{5\pi}{3}$

42. a. $\frac{\pi}{5}$

b. $\frac{8\pi}{7}$

c. $\frac{25\pi}{13}$

d. $\frac{47\pi}{48}$

For the following exercises, complete the table.

13. (MyOpenMath)

Degrees	18°	60°	90°
Radians			

14. (MyOpenMath)

Radians	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
Degrees			

15. (MyOpenMath)

Radians	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$
Degrees			

16.

Radians	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
Degrees			

For the following exercises, convert to radians. Write the answer in exact form and then give approximations rounded to the nearest hundredths.

17. (MyOpenMath)

a. 120°

b. 420°

c. -180°

d. -225°

18. (MyOpenMath)

a. 75°

b. 236°

c. -327°

19. a. 138°

b. 194°

c. 342°

For the following exercises, convert to degrees. Round answers to the nearest tenth.

50. (MyOpenMath)

    a. $-\frac{11\pi}{15}$

    b. $\frac{2\pi}{7}$

    c. $-\frac{6\pi}{5}$

51. a. 0.8

    b. 3.5

    c. 5.1

52. a. 1.1

    b. 2.6

    c. 4.6

For the following exercises, state the quadrant in which the angle terminates and give the reference angle (in radians).

53. $\frac{\pi}{3}$

54. (MyOpenMath) $\frac{3\pi}{4}$

55. $\frac{9\pi}{8}$

56. $\frac{21\pi}{22}$

57. (MyOpenMath) $-\frac{5\pi}{6}$

58. (MyOpenMath) $-\frac{5\pi}{3}$

59. (MyOpenMath) $\frac{11\pi}{4}$

60. $-\frac{15\pi}{7}$

61. 2.4

62. -3.1

For the following exercises, find an angle in each quadrant, rounded to tenths, with the same reference angle as the angle given in radians.

63. 5.8

64. 2.9

65. 3.7

66. 5.1

67. 1.8

68. 4.4

For the following exercises, use the arc length formula to answer the questions. Round answers to hundredths.

69. (MyOpenMath) Find the arc length spanned by an angle of 2 radians on a circle of radius 2 miles.

70. Find the arc length spanned by an angle of 80° on a circle of radius 4 inches.

71. Find the arc length spanned by an angle of 200° on a circle of radius 18 feet.

72. Find the radius of a circle if an angle of 250° spans an arc length of 18 meters.

73. Find the radius of a circle if an angle of 20° spans an arc length of 0.5 kilometers.

74. (MyOpenMath) Find the angle, in radians, subtended by an arc length of 7 feet on a circle of radius 10 feet.

75. (MyOpenMath) Find the angle, in degrees, subtended by an arc length of 7 feet on a circle of radius 6 feet.

76. Find the angle, in radians, subtended by an arc length of 28 centimeters on a circle of diameter 20 centimeters.

77. Find the angle, in radians, subtended by an arc length of 1.6 yards on a circle of diameter 2 yards.

For the following exercises, find the exact value of the trigonometric function. Do not use a calculator!

78. $\sin\left(\frac{\pi}{3}\right)$

79. $\cos\left(\frac{\pi}{6}\right)$

80. $\tan\left(\frac{\pi}{4}\right)$

81. $\csc\left(\frac{\pi}{4}\right)$

82. $2\sec\left(\frac{\pi}{3}\right)$

83. $\cot\left(\frac{\pi}{6}\right)$

84. $-\sin\left(\frac{5\pi}{6}\right)$

85. $\cos\left(\frac{4\pi}{3}\right)$

86. $\tan\left(\frac{11\pi}{6}\right)$

87. $3\csc\left(-\frac{4\pi}{3}\right)$

88. $\sec\left(-\frac{3\pi}{4}\right)$

89. $-\pi\cot\left(-\frac{13\pi}{6}\right)$

90. $\sin\left(\frac{\pi}{2}\right)$

91. $\cos(\pi)$

92. $\tan\left(\frac{3\pi}{2}\right)$

For the following exercises, θ is the central angle in a circle of radius r. In each case, find the area of the sector.

93. $\theta = \frac{2\pi}{3}, \qquad r = 7$ yards

94. $\theta = \frac{7\pi}{6}, \quad r = \sqrt{2}$ m

95. $\theta = \frac{6\pi}{7}, \quad r = \sqrt{3}$ miles

96. $\theta = 3, \quad r = \frac{1}{2}$

97. $\theta = 5, \quad r = 4.1$ cm

98. $\theta = 60^\circ, \quad r = 2$ feet

99. $\theta = 330^\circ, \quad r = \pi$ inches

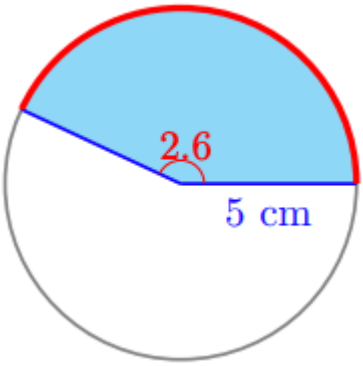
100. $\theta = 32^\circ, \quad r = 8$ parsecs

101. (MyOpenMath) $\theta = 212^\circ, \quad r = 0.88$ leagues

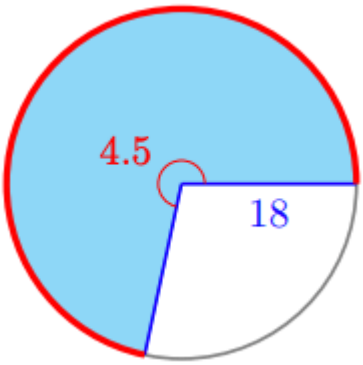
For the following exercises, find the area of the shaded sector.

102.

a.



b.



Determining Areas of Sectors

103. An arc of length 8 feet is cut off by a central angle of $2\pi/3$ radians. Find the area of the resulting sector.

104. A central angle of 4 radians cuts off an arc of length 16 meters. Find the area of the resulting sector.

105. (MyOpenMath) Calculate the area of a sector with a radius of 6 cm and a central angle of 30°.

106. A sector has a radius of 4 meters and a central angle of 45°. Find its area.

107. (MyOpenMath) If the radius of a sector is 7.4 inches and the central angle is 225°, what is the area of the sector?

98. Determine the area of a sector with a radius of 8 feet and central angle $2\pi/3$.

99. Calculate the area of a sector with a 3-meter radius and central angle $5\pi/6$.

Determining the Radius of a Sector

100. If the area of a sector is 25 cm^2 and the central angle is $45°$, find the radius.

101. Determine the radius of a sector with an area of 30 m^2 and a central angle of 60 degrees.

102. A sector has an area of 14 in^2 and central angle $\pi/6$. What is its radius?

103. What is the radius of a sector with an area of 78 cm^2 and a central angle of 120 degrees?

104. What is the radius of a sector if the area is 100 cm^2 and the central angle is 250 degrees?

Determining the Central Angle of a Sector

105. Find the radian measure of the central angle of a sector with a radius of 5 cm and an area of 20 cm^2.

106. Determine the radian measure of the central angle if a sector has a radius of 3 meters and an area of 15 m^2.

107. What is the degree measure of the central angle of a sector with a radius of 7 inches and an area of 24 in^2?

108. Calculate the degree measure of the central angle for a sector with a radius of 10 feet and an area of 50 ft^2.

109. Calculate the degree measure of the central angle for a sector with a radius of 11 cm and an area of 30.25 cm^2.

Synthesis Questions

For the following exercises, find an angle between 0 and 2π that is coterminal with the given angle.

110. (MyOpenMath) $\frac{29\pi}{10}$

For the following exercises, find the exact value of the expression for $x = \frac{\pi}{6}$.

111. $\sin(2x)$

112. $\cos(3x)$

113. $\frac{1}{3}\sin(5x)$

114. $6 - \sin(x)$

115. $\frac{2}{3} + \tan\left(\frac{3}{2}x\right)$

116. $\sin(\pi - x)$

117. $\cos(\pi + 2x)$

118. $3\sin\left(2x + \frac{\pi}{3}\right)$

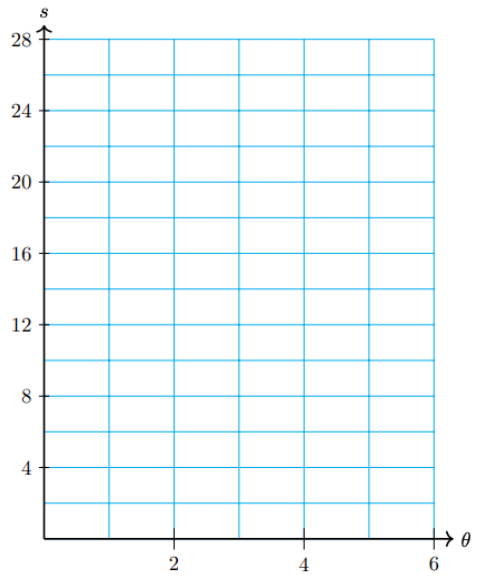
119. $-2 + \frac{3}{4}\cos\left(7x - \frac{\pi}{2}\right)$

120. a. Sketch a circle of radius 4 units, and mark the positions of 1, 2, 3, 4, 5, and 6 radians on the circle.

b. On a circle of radius 4 feet, find the arc length determined by each angle in radians.

θ	1	2	3	4	5	6
s						

c. Graph θ against s. What is the slope of the graph?



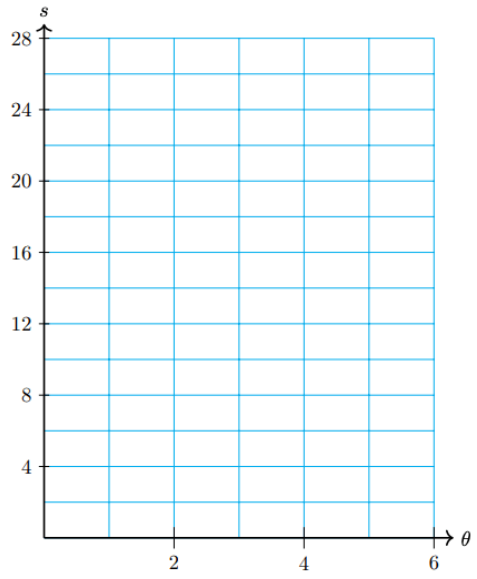
d. If you double the angle, what happens to the arc length? What happens if you triple θ?

31. a. Sketch several concentric circles with increasing radii, and draw an angle of 2 radians through all of them.

b. Find the arc length determined by an angle of 2 radians on circles of given radius.

r	1	2	3	4	5	6
s						

c. Graph s against r. What is the slope of the graph?



d. If you double the radius r, what happens to the arc length? What happens if you triple r?

32. Recall that to calculate a fraction of a revolution in degrees, we divide the angle by $360°$. For example, $90°$ is $\frac{1}{4}$ of a revolution because $\frac{90°}{360°} = \frac{1}{4}$.

a. Write an expression that gives the fraction of a revolution for an angle θ in radians.

b. Use your expression to calculate what fraction of a revolution is represented by each of the following angles: $\theta = \frac{3\pi}{4}$, $\theta = \frac{5\pi}{3}$, $\theta = \frac{7\pi}{6}$.

33. Calculate the radius of a sector whose area is one-fifth of the area of a circle with a radius of 10 cm.

34. Determine the radius of a sector if its area is equivalent to half the area of a circle with a radius of 14 meters.

Applications

For the following exercises, use the arc length formula to answer the questions.

35. **Big Ben.**

a. Through how many radians does the minute hand of a clock sweep between 9:05 pm and 9:30 pm?

b. The dial of Big Ben's clock in London is 23 feet in diameter. How long is the arc traced by the minute hand between $9:05\text{pm}$ and $9:30\text{pm}$?

36. **Floral Clock.** The largest clock ever constructed was the Floral Clock in the garden of the 1904 World's Fair in St. Louis. The hour hand was 50 feet long, the minute hand was 75 feet long, and the radius of the clock's face was 112 feet.

a. If you started at the 12 and walked 500 feet clockwise around the clock face, through how many radians would you walk?

b. If you started your walk at noon, how long would it take the minute hand to reach your position? How far did the tip of the minute hand move in its arc?

37. **The Foucault Pendulum.** In 1851 Jean-Bernard Foucault demonstrated the rotation of the earth with a pendulum installed in the Pantheon in Paris. Foucault's pendulum consisted of a cannonball suspended on a 67 meter wire, and it swept out an arc of 8 meters on each swing. Through what angle did the pendulum swing? Give your answer in radians and then in degrees, rounded to the nearest hundredth.

38. **Linear Distance.** A wheel with radius 40 centimeters is rolled a distance of 1000 centimeters on a flat surface. Through what angle has the wheel rotated? Give your answer in radians and then in degrees, rounded to one decimal place.

39. **Distance Between Cities.** Cosumnes River College (Sacramento, California) and Stony Brook University (Stony Brook, New York) are 2,544 miles apart along the surface of the Earth. Assuming that the radius of the Earth is 3,963 miles, find the radian measure of the central angle having vertex at the center of the Earth that has Cosumnes River College on one side and Stony Brook University on the other.

40. **Land Allocation.** A farmer plans to allocate a portion of his circular field for a special crop. The field has a radius of 100 meters. If he uses a 60-degree sector for this crop, calculate the area that will be cultivated.

41. **Clock Face Design.** A designer is creating a new clock face where each hour marker is represented by a different colored sector. The clock has a radius of 12 cm. Calculate the area of one hour sector assuming a simple 12-hour clock.

42. **Architectural Design.** An architect designs a circular window with decorative sectors. Each sector has a central angle of 45 degrees and the window has a diameter of 2 meters. Calculate the area of one decorative sector.

43. **City Planning.** A city planner needs to design a roundabout garden that occupies a 38-degree sector of a circle with a radius of 20 meters. Calculate the area available for planting.

44. **Pizza Slicing.** A pizza place offers a unique pizza slice shaped as a sector. The pizza has a diameter of 40 cm. If the central

angle for each slice is 20 degrees, find the area of one slice of pizza.

45. **Sports Fields.** A sports complex includes a sector-shaped running track portion used for sprinting drills. If the track follows a 40-degree sector of a circle with a radius of 100 meters, calculate the area of this track section.

46. **Public Park Design.** A public park is designed with a circular area for a water fountain, surrounded by flower beds laid out in sectors. If the circular area has a radius of 15 meters and each sector for the flower beds is 30 degrees, find the area of one flower bed.

47. **Non-Uniform Rainfall.** A circular region with a radius of 20 km receives rainfall. However, the rain is not uniformly distributed. If a 120-degree sector receives twice the amount of rain per square kilometer as the rest of the region, calculate the total rainfall if the sector received 600 mm.

48. **Engineering Design.** An engineer is designing a gear in the shape of a sector. The gear must mesh with another gear with an area three times larger but with the same arc length. If the radius of the smaller gear is 15 cm, determine the central angle of the sector for the smaller gear.

49. **Advanced Pizza Slicing.** A circular pizza is sliced such that each consecutive slice has a central angle 10 degrees greater than the previous one, starting from 10 degrees. Calculate the area of the 5th slice if the pizza has a radius of 30 cm.

50. **Sector-based Cooling Zones.** A circular mechanical part is designed with cooling zones in the form of sectors. Each zone dissipates heat differently. If one zone with a 60-degree angle dissipates heat at 100 watts per square meter and the radius of the part is 30 cm, find the total heat dissipation for this zone.
(*Hint:* Heat dissipation is the total number of watts of heat released in a given area.)

Challenge Problems

51. If the radius of a sector is doubled while the central angle remains the same, by what factor does the area increase?

Answers

Skills Refresher

1. a. 16.09 km
   b. 31.08 mi
2. a. 16.2 hectares
   b. 12.35 acres
3. a. 186,500 watts
   b. 1.34 horsepower
4. a. 240 grains
   b. 0.21 troy oz
5. 90°
6. 72°
7. 60°
8. 45°

This page titled 7.2.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

7.3: The Trigonometric Functions - Unit Circle Definition

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We now venture into the third and final definition of the trigonometric functions. Many students have already encountered this definition from their high school Pre-Calculus course; however, there are three reasons I have held off on introducing the trigonometric functions via this definition until now:

1. In my experience, learning the unit circle definition of the trigonometric functions before the right triangle definition leads to more memorization and a less accurate understanding of the trigonometric functions.
2. Unit circle trigonometry is theoretically more challenging to grasp. It has two interpretations—one in terms of angles and the other in terms of arc lengths (and, as such, most students who learn it before right triangle trigonometry tend to have a weak grasp of the subject).
3. Unit circle trigonometry relies on radian measure.

Connecting the Sine and Cosine Functions to the Unit Circle

The left image in Figure $7.3.1$ below shows an angle of 30° in standard position in a circle of radius 2. The hypotenuse of its reference triangle is the radius of the circle, so the legs of the triangle have lengths 1 and $\sqrt{3}$. The coordinates of the point P where the terminal side meets the circle are thus $(\sqrt{3}, 1)$.

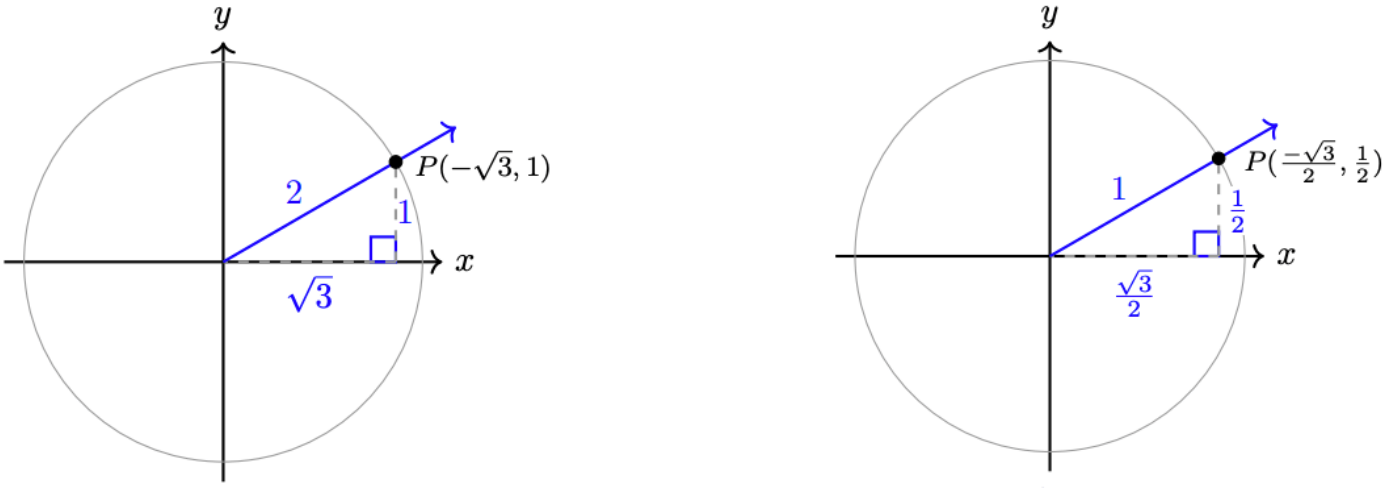


Figure $7.3.1$

The right image in Figure $7.3.1$ is a unit circle. In this figure, the hypotenuse of the reference triangle for 30° has length 1. What are the coordinates of the point P where the terminal side meets the circle? Each side of this triangle is half the length of the sides of the similar triangle in the left image, so the coordinates of P are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

You should recognize the coordinates of the point P as $(\cos(30°), \sin(30°))$. In fact, from the coordinate definition of the trigonometric functions, we should not be surprised to see that, given the point (x, y) on the unit circle on the terminal side of the angle θ,

$$\cos(\theta) = \frac{x}{r} = \frac{x}{1} = x \quad \text{and} \quad \sin(\theta) = \frac{y}{r} = \frac{y}{1} = y$$

We have discovered an important property of unit circles.

Lemma

Let θ be an angle in standard position, and suppose the terminal side of θ intersects the unit circle at the point $P(x, y)$. Then

$$\cos(\theta) = x \quad \text{and} \quad \sin(\theta) = y.$$

While this lemma is not entirely profound, it will lead to a more significant concept. Before we get there, let's use the lemma in an example.

Example 7.3.1

Find the coordinates of point P on the unit circle below.

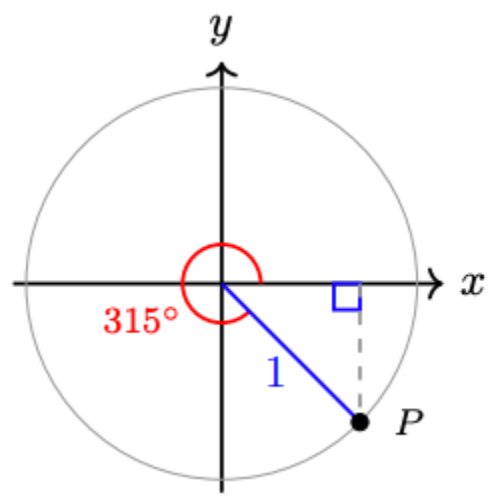


Figure 7.3.2

Solution available at the end of this section.

Checkpoint 7.3.1

Find the sine, cosine, and tangent of the angle φ shown below.

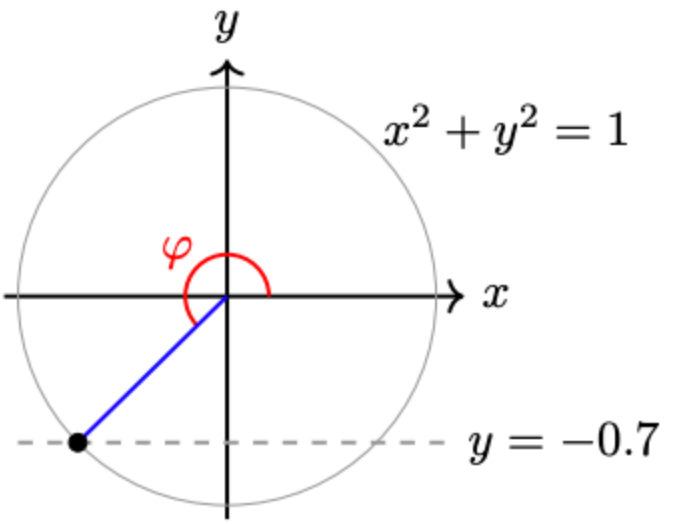


Figure 7.3.3

Answer available at the end of this section.

A Reminder of Radian Measure and Arc Length

For the remainder of this chapter, we will focus on using radians instead of degrees. This change will greatly increase the utility of the trigonometric functions. To illustrate this, and as a refresher of previously learned material, recall that the radius of a unit circle is 1. Therefore, the arc length formula becomes $s = r\theta = (1)\theta = \theta$. Thus, on a *unit circle*, an arc of length 1 determines a central angle of 1 radian. Moreover, the measure of an angle in radians is equal to the length of the arc it spans. In other words, if we walk along the edge of a circle of radius 1, each time we travel through an arc length equivalent to one radius, we have rotated through an angle of one radian.

Example 7.3.2

You are walking on a trail around a circular pond with a radius of one mile. You have hiked for four miles. Sketch your location now.

Solution available at the end of this section.

Checkpoint 7.3.2

An ant walks around the rim of a circular birdbath of a radius of 1 foot. How far has the ant walked when it has turned through an angle of 210°?

Answer available at the end of this section.

Trigonometric Functions of Real Numbers and the Unit Circle Definition

We now venture into a more theoretical area of Trigonometry. For this discussion, we will denote the arc length of a circle subtended by the angle θ with the letter t instead of s (for no better reason than tradition).

Consider the unit circle shown in Figure 7.3.5 below, and the angle θ determined by the arc of length t. The radian measure of θ is the same as the length t of the arc. For example, in this figure $\theta = t = 2$.

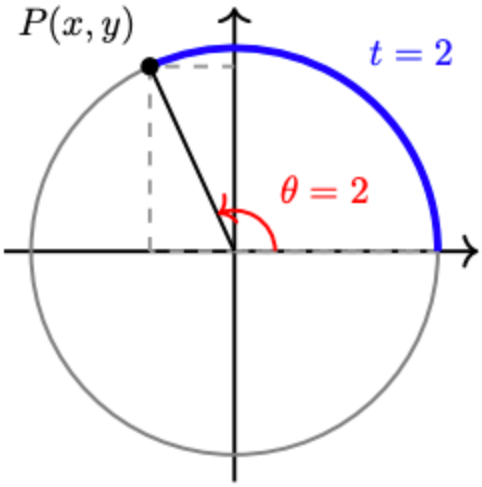


Figure 7.3.5**:** *Traversing the unit circle*

Therefore, as long as we restrict our conversation to the unit circle, instead of thinking of the sine or cosine as a function of an *angle*, θ, we can change our perspective and think of these as functions of the *length of an arc*, t, along the unit circle. Since the arc length is not an angle but instead a simple real number, we are evaluating the trigonometric function at a real number.

This brings us to our third and final definition of the trigonometric functions.

Definition: Trigonometric Functions (Unit Circle Definition)

Let $P(x, y)$ be a point on the unit circle, and let t be the arc length from the point $(1, 0)$ to P along the circumference of the unit circle. The trigonometric functions of the real number t (also called the **circular functions** in this context) are defined as follows:



We call the trigonometric functions of arc length the circular functions because they are defined by the coordinates of points on a unit circle.

We can think of the definition this way: to find the sine or cosine of a real number t, we draw an arc of length t on a unit circle and then find the sine or cosine of the angle θ determined by the arc.

Interactive Element: Tracing an Arc of Length t

The following Interactive Element might help you visualize how the coordinates of the terminal point of an arc of length t are exactly $\cos(t)$ and $\sin(t)$.

Interact: Drag the red point P around the unit circle. The highlighted arc from $(1, 0)$ to P has length t.

Observation: The coordinates of P are always $(\cos(t), \sin(t))$, and the length of the highlighted arc equals the radian value t. Notice that both coordinates stay between -1 and 1, no matter where P is.

Note: The Difference is All in Your Head

In practice, there is no difference between finding the value of a trigonometric function at the number 2 and the value at the angle of 2 radians. In each case, we set the calculator to Radian mode and evaluate, for example,

$$\mathrm{COS}\,2 \approx -0.4161 \quad \text{and} \quad \mathrm{SIN}\,2 \approx 0.9093.$$

So, in some sense, a trigonometric function of a radian is the same as a trigonometric function of a real number. This result is so important that it bears repeating:

> To find the value of a trigonometric function of a real number t, we find the value of the trigonometric function of the angle t radians.

Example 7.3.3

Let $t = \frac{2\pi}{3}$ be the length of the arc along the unit circle from the point $(1, 0)$ to $P(x, y)$. Evaluate each of the following functions.

a. $\sin(t)$
b. $\cot(t)$

Solutions available at the end of this section.

Checkpoint 7.3.3

Evaluate $\sec(t)$, where $t = \frac{5\pi}{4}$ is the arc length along the unit circle from the point $(1, 0)$.

Answer available at the end of this section.

Example 7.3.4

Find the coordinates of the terminal point, P, of an arc of length t starting at $(1, 0)$ on a unit circle.

a. $t = \frac{\pi}{2}$
b. $t = \frac{5\pi}{6}$

Solutions available at the end of this section.

Checkpoint 7.3.4

Find the coordinates of the terminal point, P, of an arc of length t starting at $(1, 0)$ on a unit circle.

a. $t = \pi$
b. $t = \frac{\pi}{3}$

Answers available at the end of this section.

Using a Calculator to Compute the Values of Trigonometric Functions

Before moving forward, let's talk about using your calculator's Radian mode. In the previous section, we evaluated the trigonometric functions at the special radian angles (i.e., multiples of $\pi/6$, $\pi/4$, and $\pi/3$). We now focus on evaluating the trigonometric functions at non-special radian angles (and, therefore, non-special real numbers).

Note: Change to Radian Mode

If you use a scientific or graphing calculator, refer to your instruction manual on setting it to radian mode. The scientific calculator in Desmos is set to degree mode by default, so if you are using that technology, you need to click the "DEG" toggle so that "RAD" is showing; however, the *graphing* calculator in Desmos defaults to radian mode.

Setting the calculator to radian mode allows us to avoid converting from radians to degrees to evaluate trigonometric functions. First, change the calculator setting from Degree mode to Radian mode. Then enter

$$\text{SIN}\,\pi \div 3$$

and the calculator will return 0.8660254038. You can check that this number is a decimal approximation for $\frac{\sqrt{3}}{2}$.

Example 7.3.5

Use your calculator to find the sine and cosine of the following angles in radians. Round your answers to four decimal places.

a. $\theta = \dfrac{7\pi}{4}$
b. $\theta = 3.5$

Solutions available at the end of this section.

Checkpoint 7.3.5

Use your calculator to find the tangents of the following real numbers. Round your answers to four decimal places.

a. $t = \frac{5\pi}{12}$
b. $t = 5.2$

Answers available at the end of this section.

We can now use the trigonometric functions to model periodic behavior as functions of *time* instead of angles. Indeed, we now have the ability to use any variable as the argument of a trigonometric function.

Example 7.3.6

The sunset time in Stockholm, Sweden, on the nth day of the year can be modeled by

$$T = 3.11 \sin(0.017n - 1.38) + 6.03$$

where T is given in hours after noon. Find the sunset time on January 1 (day $n = 1$) and on July 1 (day $n = 182$).

Solution available at the end of this section.

Checkpoint 7.3.6

Variable stars are important in astronomy because they are used to estimate distances. Their magnitude, or brightness, varies periodically and can be modeled by trigonometric functions. The star T Herculis reached its minimum magnitude on December 27, 2004, and t days later its magnitude is approximately

$$M = 10.2 - 2.2 \cos(0.038t)$$

a. What was the magnitude of T Herculis on December 27, 2004?
b. What was the magnitude of T Herculis on December 27, 2006 (730 days later)?

Answers available at the end of this section.

Example 7.3.7

Use the graph of the unit circle shown below to estimate $\cos(2.5)$, $\sin(2.5)$, and $\tan(2.5)$.

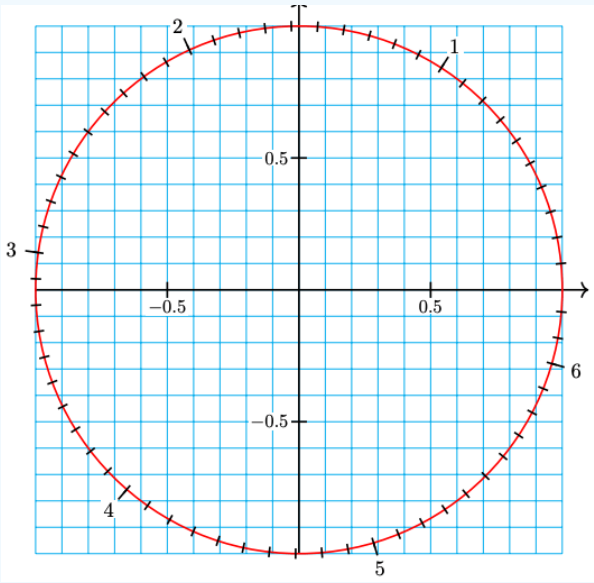


Figure 7.3.8

Solution available at the end of this section.

Checkpoint 7.3.7

Use the graph of the unit circle in the previous example to estimate $\cos(4.2)$, $\sin(4.2)$, and $\tan(4.2)$.

Answer available at the end of this section.

Activities

Activity 7.3.1: Special Angles in Radians

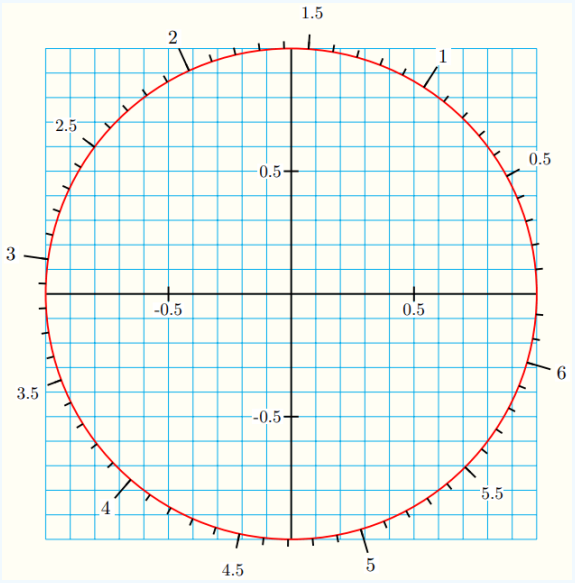


Figure 7.3.9

1. Use the unit circle to estimate the sine, cosine, and tangent of each arc of given length.
   a. 0.6
   b. 2.3
   c. 3.5
   d. 5.3
2. Use the unit circle to estimate two solutions to each equation.
   a. $\cos(t) = 0.3$
   b. $\sin(t) = 0.7$
3. Sketch the angle on the unit circle. Find the reference angle in radians, rounded to two decimal places, and sketch the reference triangle.
   a. 1.8
   b. 5.2
   c. 3.7
4. Give a decimal approximation to two places for each angle, then the degree measure of each.

Radians	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
Decimal Approx.													
Degrees													

Radians	π	$\frac{13\pi}{12}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{17\pi}{12}$	$\frac{3\pi}{2}$	$\frac{19\pi}{12}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	$\frac{23\pi}{12}$	2π
Decimal Approx.													
Degrees													

5. On the unit circle above, plot the endpoint of each arc in standard position.
   a. $\frac{\pi}{3}$
   b. $\frac{7\pi}{6}$
   c. $\frac{7\pi}{4}$

We have already introduced the idea that, given an arc length t along the unit circle, $\cos(t)$ is the x-value of the point where the arc length terminates and $\sin(t)$ is the y-value; however, there are other interesting geometric interpretations of the circular functions of real number. To understand these interpretations better, the following activity should be explored independently or during class.

Activity 7.3.2: Geometric Interpretations of the Circular Functions

Part 1

Figure 7.3.10 shows a unit circle centered at O. The line through AC is tangent to the circle at A. (Recall that a tangent to a circle is perpendicular to the radius that meets it.)

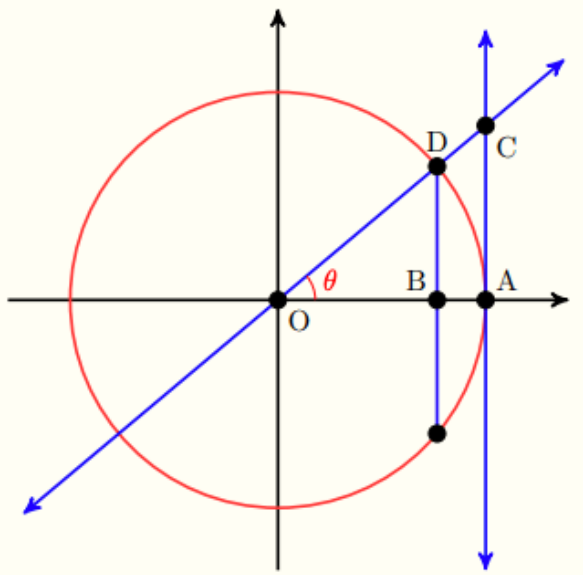


Figure 7.3.10

a. What is the radius of the circle?
b. Find line segments whose lengths are $\cos(\theta)$ and $\sin(\theta)$.
c. Explain why $\triangle OAC$ is similar to $\triangle OBD$.
d. Find a line segment in $\triangle OAC$ whose length is $\tan(\theta)$. Explain why your choice is correct.
e. Find a line segment in $\triangle OAC$ whose length is $\sec(\theta)$. Explain why your choice is correct.

Part 2

Figure 7.3.11 is another unit circle, with tangent BD.

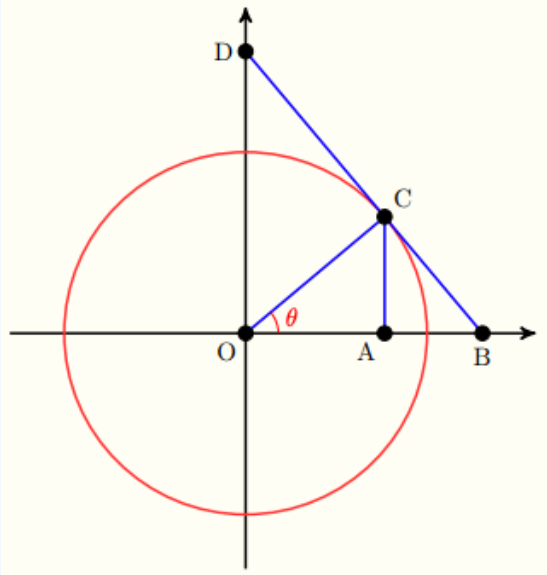


Figure 7.3.11

a. What are the measures of the angles $\angle OCD$ and $\angle OCB$?
b. Explain why $\triangle OCB$ is similar to $\triangle OAC$.
c. Find a line segment in $\triangle OCB$ whose length is $\tan(\theta)$. Explain why your choice is correct.
d. Find a line segment in $\triangle OCB$ whose length is $\sec(\theta)$. Explain why your choice is correct.
e. Explain why $\triangle OCB$ is similar to $\triangle DCO$.
f. Find a line segment in $\triangle DCO$ whose length is $\csc(\theta)$. Explain why your choice is correct.

Domain and Range of the Circular Functions

By interpreting the trigonometric functions in terms of the unit circle definition (as circular functions), we can discuss their domains and ranges.

Interactive Element: Geometric Interpretations of the Circular Functions

Interact: Move the point D to help you investigate the circular functions geometrically.

If t is the length of the arc along the unit circle, then we can see that, no matter how large or small, negative or positive, we make t, the arc length will terminate at a point $P(x, y)$ on the unit circle. Since $x = \cos(t)$ and $y = \sin(t)$, the cosine and sine functions are defined for all real numbers.

Moreover, whenever $x = 0$, we know that $\sec(t) = \frac{1}{x}$ and $\tan(t) = \frac{y}{x}$ are both undefined. We know the values of x on the unit circle become 0 precisely when $t = \pm\frac{\pi}{2}$ or $\pm\frac{3\pi}{2}$ or $\pm\frac{5\pi}{2}$. In general, these two functions are undefined at $t = \frac{\pi}{2} + k\pi$, where k is an integer. A similar argument reveals the cosecant and cotangent functions to be undefined when $t = k\pi$ (again, where k is an integer). The following theorem summarizes these results.

Theorem: Domains of the Circular Functions

Let t be a real number and k be an integer. The domains of the circular functions are

$\sin(t)$ and $\cos(t)$	All real numbers, or $(-\infty, \infty)$
$\tan(t)$ and $\sec(t)$	All real numbers except $t = \frac{\pi}{2} + k\pi$
$\cot(t)$ and $\csc(t)$	All real numbers except $t = k\pi$

While there is a way to write the domains of tangent, cotangent, secant, and cosecant in interval notation, it's a bit complex.

The activity, Geometric Interpretations of the Circular Functions, can be beneficial in visualizing the ranges of the circular functions. For example, Figure 7.3.10 shows us that the values of the cosine and sine functions (which are the x and y coordinates of points along the unit circle) are always between -1 and 1, inclusive. Thus, the range of these functions is $[-1, 1]$.

Since the values of the tangent function can be interpreted as the slope of the ray emanating from the origin through the point $P(x, y)$ on the unit circle, we can easily see that the tangent can get as large (both positively and negatively) as we want. This is because the slopes of these lines can get as steep as we want (and they can even be zero). Hence, the range of the tangent function is $(-\infty, \infty)$.

The following theorem states the ranges of all of the circular functions. While not incredibly useful now, they will become increasingly important as we move forward, and we should introduce them early so you can get used to them.

Theorem: Ranges of the Circular Functions

Let t be a real number. The ranges of the circular functions are

$\sin(t)$ and $\cos(t)$	$[-1, 1]$
$\tan(t)$ and $\cot(t)$	$(-\infty, \infty)$
$\sec(t)$ and $\csc(t)$	$(-\infty, -1] \cup [1, \infty)$

Example 7.3.8

Is it possible for $\sec(t) = -\frac{1}{2}$?

Solution available at the end of this section.

Checkpoint 7.3.8

For each function, state a value the function could return and a value it could not.

a. $\cos(t)$
b. $\csc(t)$

Answers available at the end of this section.

Success in Mathematics: Why Study Mathematics? (Part 2)

Tracy Moody graduated in 2006 with a bachelor's degree in mathematics. She then went to graduate school, studying to become a physical therapist. She writes,

> "Many people cannot seem to fathom how math and physical therapy overlap. Well, one of the obvious things is that physical therapists use statistics to see if new treatments work. However, the main thing I bring as a mathematician to my graduate studies is my ability to problem solve. A degree in mathematics taught me how to look at a problem from several angles. Furthermore, one of the main classes that carried over into my career choice is, believe it or not, real analysis.

I can remember working with my fellow classmates on problems for hours. However, through this process, I learned the importance of perseverance, and I learned how to communicate the problem and solutions in several ways."

Source: www.maa.org, Career Profiles.

1. Do you think studying mathematics has helped you become a better problem solver? Explain.
2. Do you think studying mathematics has helped you learn to "persevere," to keep working on a difficult problem even though you are struggling? Explain.

Success in College: Getting Help

Learning math can be a struggle. Sometimes your professor seems to be talking in a foreign language. Other students look like they get it. The professor keeps going. The stuff you don't understand keeps piling up.

Struggling in a college math class is expected. You must learn to work through your confusion. Your professor expects you to ask questions in class. If needed, you must also get help outside of class. Visit your professor during office hours. Ask if appointments are available outside of office hours. Use the math tutors on campus. It is your responsibility to get the help you need. You must work, ask questions, practice, and think until you get it. You have to take the time to get extra help.

Students who let frustration get to them fail. They put off getting help. They sometimes skip class. They forget that college is different than many high schools. In college, you can fail. If you do not learn enough, you cannot go on.

When successful students are confused, they get help. They work until they understand. You should expect to be confused at times. It would be best if you planned on needing extra time and energy to deal with it.

Here are some questions that may help you succeed in this class (and possibly many other classes as well):

1. Where is your professor's office located?
2. Your professor might keep regular office hours when they are available for help. When are your professor's office hours?

If you are unsure about the answers to either of these questions, feel free to ask your professor - that's what they are there for!

Answers

Section Preview

We wrap up our trio of trig definitions with the Unit Circle definition. I deliberately chose to introduce this definition last because we (human beings) tend to stick with the first or second way we learn something. Since the Unit Circle definition of the trigonometric functions is a completely memorizable definition/method, I consider it the worst to introduce early. Most of your time in this section will be spent on convincing the student that trigonometric functions can also be defined in terms of arc length along the unit circle (hence, the phrase "circular functions"). While I do not spend an insane amount of time on this material, understanding the definition presented in this section (along with its relationship to the area of the sector with arc length t) is important for the development of the hyperbolic functions in Calculus I.

Section Preview

- **Sets and Numbers**
  - Intervals and Interval Notation: The domains of the circular functions are expressed using interval notation (e.g., $(-\infty, \infty)$) and set exclusion (e.g., "all real numbers except $t = \frac{\pi}{2} + k\pi$"). The ranges include unions like $(-\infty, -1] \cup [1, \infty)$. Fluency with interval notation is genuinely required to read and use these theorems.

Section Preview

- Evaluate a trigonometric function using the unit circle.
- Find the value of a trigonometric function given a point on the unit circle.
- Use a calculator to approximate the value of a trigonometric function for an angle in radians.
- Use the unit circle to answer a conceptual question about a trigonometric function.

Example 7.3.1

The coordinates of P are given by

$$x = \cos(315^\circ), \quad y = \sin(315^\circ).$$

The reference angle for 315° is $360^\circ - 315^\circ = 45^\circ$, so the trigonometric functions of 315° are the same as the functions for 45°, up to sign.

Because 315° is in the fourth quadrant, we have $\cos(315^\circ) = \frac{1}{\sqrt{2}}$ and $\sin(315^\circ) = -\frac{1}{\sqrt{2}}$. Thus, the coordinates of P are $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Checkpoint 7.3.1

$\sin(\varphi) = -0.7000, \cos(\varphi) \approx -0.7141, \tan(\varphi) \approx 0.9802$

Example 7.3.2

The pond is a unit circle, so you have traversed an angle in radians equal to the arc length traveled, 4 miles. An angle of 4 radians is in the middle of the third quadrant relative to your starting point, more than halfway but less than three-quarters around the pond.

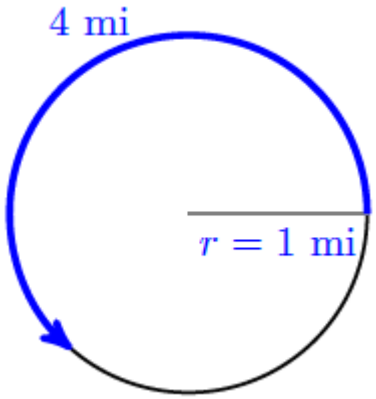


Figure 7.3.4

Checkpoint 7.3.2

3.67 ft

Example 7.3.3

a. Since $t = \frac{2\pi}{3}$, we are being asked to evaluate $\sin\left(\frac{2\pi}{3}\right)$. As stated before, a trigonometric function's value at a real number (an arc length along the unit circle) and the equivalent radian angle are the same. Thus,

$$\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

b. As in part (a), we just evaluate

$$\cot\left(\frac{2\pi}{3}\right) = -\cot\left(\frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}}.$$

Checkpoint 7.3.3

$-\sqrt{2}$

Example 7.3.4

a. On a unit circle, the coordinates of P are $\left(\cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right)\right)$, so

$$x = \cos\left(\frac{\pi}{2}\right) = 0 \quad \text{and} \quad y = \sin\left(\frac{\pi}{2}\right) = 1$$

The coordinates of P are $(0, 1)$, as shown below.

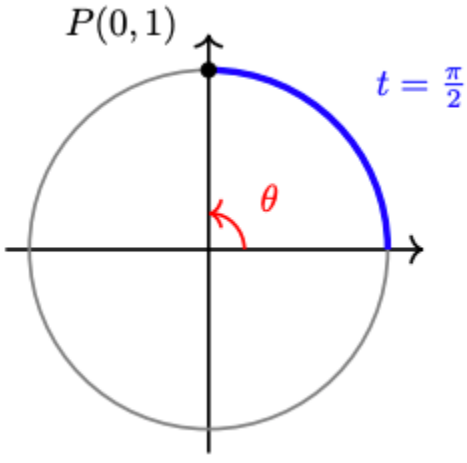


Figure 7.3.6

(We could also observe that an arc of length $\frac{\pi}{2}$ is one-quarter of a unit circle, so the point P sits at the twelve o'clock position on the circle.)

b. The coordinates of P are $\left(\cos\left(\frac{5\pi}{6}\right), \sin\left(\frac{5\pi}{6}\right)\right)$, so

$$x = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$y = \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$$

Thus, the coordinates of P are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, as shown below.

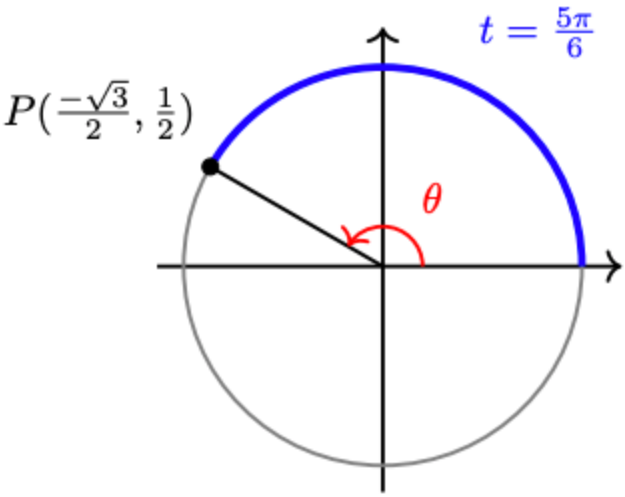


Figure 7.3.7

Checkpoint 7.3.4

a. $(-1, 0)$
b. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Example 7.3.5

a. With your calculator in radian mode, enter

$$\text{COS}\, 7\,\pi \div 4$$

and

$$\text{SIN}\, 7\,\pi \div 4$$

to find

$$\cos\left(\frac{7\pi}{4}\right) \approx 0.7071067812$$

$$\sin\left(\frac{7\pi}{4}\right) \approx -0.7071067812$$

Rounding to four decimal places gives $\cos\left(\frac{7\pi}{4}\right) \approx 0.7071$ and $\sin\left(\frac{7\pi}{4}\right) \approx -0.7071$.

b. Your calculator will give you the values

$$\cos(3.5) \approx -0.9364566873$$

$$\sin(3.5) \approx -0.3507832277$$

Rounding to four places, we have $\cos(3.5) \approx -0.9365$ and $\sin(3.5) \approx -0.3508$. Note that 3.5 radians is a third-quadrant angle, so the signs of the trigonometric values make sense.

Checkpoint 7.3.5

a. 3.7321
b. -1.8856

Example 7.3.6

Evaluate the function for $n = 1$ to find

$$T = 3.11 \sin\left(0.017(1) - 1.38\right) + 6.03 \approx 2.99.$$

On January 1, sunset in Stockholm occurs about 2.99 hours after noon, or at 2:59 pm.

Evaluate the function at $n = 182$ to find

$$T = 3.11 \sin\left(0.017(182) - 1.38\right) + 6.03 \approx 9.11.$$

On July 1, sunset occurs about 9.11 hours after noon, or at 9:07 pm. (Actually, 10:07 pm, because of daylight saving time.)

Checkpoint 7.3.6

a. 8
b. 12.09

Example 7.3.7

The circle is scaled in units of 0.1 radians, and an arc of 2.5 radians in standard position has its terminal point, P, in the second quadrant. The coordinates of P are approximately $(-0.8, 0.6)$, so we have

$$\cos(2.5) \approx -0.8 \quad \text{and} \quad \sin(2.5) \approx 0.6.$$

To find $\tan(2.5)$, we calculate $\frac{y}{x}$.

$$\tan(2.5) = \frac{y}{x} \approx \frac{0.6}{-0.8} = -0.75.$$

Checkpoint 7.3.7

Approximately -0.49, -0.87, and 1.78, respectively.

Example $7.3.8$

Since the range of the secant function is $(-\infty, -1] \cup [1, \infty)$, and $-\frac{1}{2}$ is not in this interval, it is *not* possible for $\sec(t) = -\frac{1}{2}$.

Checkpoint $7.3.8$

a. 2 is not a possible output, but 1/2 is.
b. 1/2 is not a possible output, but 2 is.

This page titled 7.3: The Trigonometric Functions - Unit Circle Definition is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

7.3.1: Resources and Key Concepts

Resources

Videos

- The Unit Circle
  - The Unit Circle - The Unit Circle
- Unit Circle Definition of the Trigonometric Functions
  - Unit Circle - Trigonometric Functions: A Unit Circle Approach
  - Unit Circle - Evaluating Trigonometric Functions of Non-Acute, Special Angles
  - Unit Circle - Evaluating Trigonometric Functions at Axial Angles
  - The Unit Circle - Special Triangles
  - The Unit Circle - The Reference Number
  - Trigonometry Mini-Lecture: Graph the Unit Circle with a TI Graphing Calculator & Find Trig Values

Key Concepts

Definitions

- **Trigonometric Functions (Unit Circle Definition):** Let t be a real number representing the arc length from the point (1, 0) to a point $P(x, y)$ on the unit circle. The trigonometric functions of t, also called circular functions, are defined by the coordinates of P. For example, $\cos(t) = x$ and $\sin(t) = y$.

Theorems

- **Lemma:** Let θ be an angle in standard position, and suppose the terminal side of θ intersects the unit circle at the point $P(x, y)$. Then $\cos(\theta) = x$ and $\sin(\theta) = y$.
- **Domains of the Circular Functions:** Let t be a real number and k be an integer. The domains of the circular functions are:
  - $\sin(t)$ and $\cos(t)$: All real numbers, or $(-\infty, \infty)$
  - $\tan(t)$ and $\sec(t)$: All real numbers except $t = \frac{\pi}{2} + k\pi$
  - $\cot(t)$ and $\csc(t)$: All real numbers except $t = k\pi$
- **Ranges of the Circular Functions:** Let t be a real number. The ranges of the circular functions are:
  - $\sin(t)$ and $\cos(t)$: $[-1, 1]$
  - $\tan(t)$ and $\cot(t)$: $(-\infty, \infty)$
  - $\sec(t)$ and $\csc(t)$: $(-\infty, -1] \cup [1, \infty)$

Common Mistakes

- **Using the Wrong Calculator Mode:** When working with the unit circle definition, angles and arc lengths are measured in radians. You must ensure your calculator is set to Radian mode, not Degree mode, to get correct approximations.
- **Confusing Angle Measure with Arc Length:** While the numerical value of an angle in radians is equal to the arc length it subtends on a unit circle, it is important to remember they are conceptually different. An angle is an amount of rotation, while an arc length is a distance.

This page titled 7.3.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

7.3.2: Homework

Reading Questions

1. What is the radius of a unit circle?
2. If the terminal side of an angle θ intersects the unit circle at a point $P(x, y)$, what do the x- and y-coordinates represent?
3. On a unit circle, what is the relationship between the radian measure of a central angle and the length of the arc it subtends?
4. When trigonometric functions are defined based on the arc length on a unit circle, what are they sometimes called?
5. To find the value of a trigonometric function for a real number t, what do you do?
6. What calculator mode should be used when evaluating trigonometric functions for angles given in radians or for real numbers?
7. What is the domain of the sine function?
8. What is the domain of the tangent function?
9. What is the range of the cosine function?
10. What is the range of the secant function?
11. Is it possible for $\sin(t)$ to equal 2? Why or why not?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, simplify.

1. a. $\frac{2}{3}+\frac{1}{6}$

   b. $\frac{2x}{3}+\frac{x}{6}$

2. a. $\frac{3}{4}-\frac{5}{8}$

   b. $\frac{3n}{4}-\frac{5n}{8}$

3. a. $2-\frac{3}{4}$

   b. $2b-\frac{3b}{4}$

4. a. $1+\frac{3}{8}$

   b. $m+\frac{3m}{8}$

5. a. $\frac{5}{6}-\frac{7}{6}$

   b. $\frac{5q}{6}-\frac{7q}{6}$

6. a. $\frac{2}{3}-\frac{5}{3}$

b. $\frac{2p}{3} - \frac{5p}{3}$

Answers available at the end of this section.

Homework

Vocabulary Check

1. A circle of radius 1 having center at the origin is called a(n) ___.
2. When considered as functions of arc length, we call the trigonometric functions ___ functions because they are defined by the coordinates of points on a unit circle.

Concept Check

3. Explain why $\cos\theta \leq 1$ for any angle θ.
4. How many solutions are there to the equation $\cos(\theta) = 0.4$ between 0° and 360°?
5. On a unit circle, the ___ measure of the central angle and the ___ length it cuts off are the same value.
6. List the trigonometric functions having a domain of all real numbers.
7. List the trigonometric functions having a range of all real numbers.
8. Sketch a figure on a unit circle to illustrate each equation.
   a. $\sin\frac{3\pi}{4} = 0.7071$
   b. $\cos 2.5 = -0.8011$
9. Write down the multiples of $\frac{\pi}{12}$ from 0 to 2π. Reduce each fraction.
10. Sketch arcs in standard position on a unit circle with the following lengths.
$$s = \frac{\pi}{2}, \quad s = \frac{3\pi}{4}, \quad s = \frac{\pi}{3}, \quad s = \frac{2\pi}{3}$$

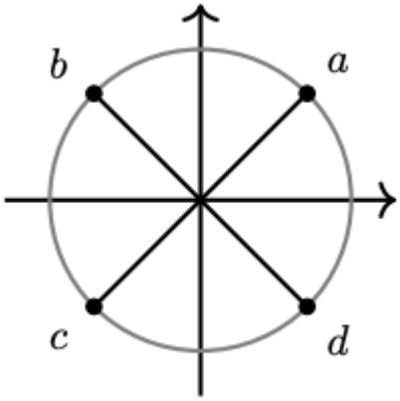
True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

11. Let $P(x, y)$ be a point on the unit circle determined by an arc of length t from the point $(1, 0)$ to P. Then the coordinates of P are given by
$$x = \sin(t), \quad y = \cos(t).$$
12. If $\beta < \alpha$, then $\sin(\beta) < \sin(\alpha)$.
13. The circular functions are functions of real numbers. As such, we can set our calculator to *either* degree *or* radian mode to get approximations of these functions.
14. The domain of the tangent function includes all real numbers *except* for $\frac{\pi}{2}$.
15. The range of the sine and cosine functions is -1 to 1, inclusive.

Basic Skills

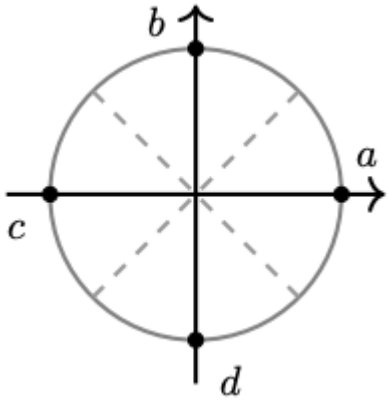
For the following exercises, each point on the unit circle is the terminal point of a special angle in standard position. Give exact values for the radian measure, t, of the angle, and the coordinates (x, y) of the point.

16.



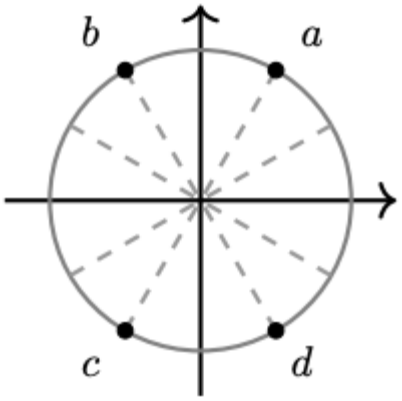
	a	b	c	d
t				
x				
y				

17.



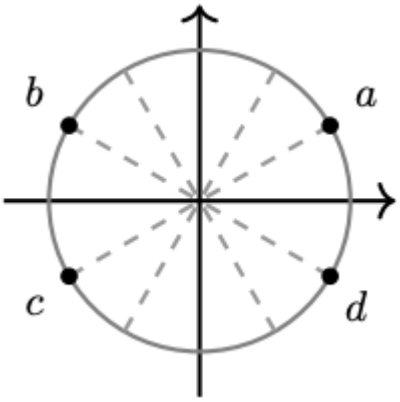
	a	b	c	d
t				
x				
y				

18.



	a	b	c	d
t				
x				
y				

19.



	a	b	c	d
t				
x				
y				

20. Complete the table.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{2\pi}{3}$			
$\frac{3\pi}{4}$			
$\frac{5\pi}{6}$			

21. Complete the table.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$

$\frac{7\pi}{6}$			
$\frac{5\pi}{4}$			
$\frac{4\pi}{3}$			

22. Complete the table.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{5\pi}{3}$			
$\frac{7\pi}{4}$			
$\frac{11\pi}{6}$			

For the following exercises, evaluate each function.

23. $\sin(45^\circ)$

24. $\cos(150^\circ)$

25. $\cot(270^\circ)$

26. $\tan(240^\circ)$

27. $\sec(315^\circ)$

28. $\csc(210^\circ)$

29. $\cos\left(\frac{\pi}{6}\right)$

30. $\sin\left(\frac{3\pi}{2}\right)$

31. $\tan\left(\frac{2\pi}{3}\right)$

32. $\sec\left(\frac{7\pi}{6}\right)$

33. $\csc\left(\frac{5\pi}{4}\right)$

34. $\cot\left(\frac{4\pi}{3}\right)$

For the following exercises, use the unit circle to find all values of θ between 0 and 2π for which the given equation is true.

35. $\sin(\theta) = \frac{\sqrt{3}}{2}$

36. $\cos(\theta) = -\frac{\sqrt{3}}{2}$

37. $\tan(\theta) = -1$

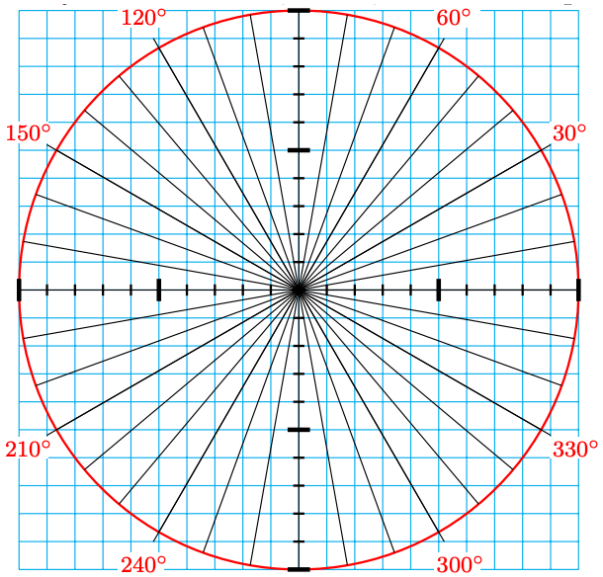
38. $\csc(\theta) = -2$

39. $\sec(\theta) = 1$

40. $\cot(\theta) = \sqrt{3}$

41. Estimate the x-coordinate of the points on the unit circle designated by each angle and complete the table. (Hint: Use

symmetry in the second, third, and fourth quadrants.)



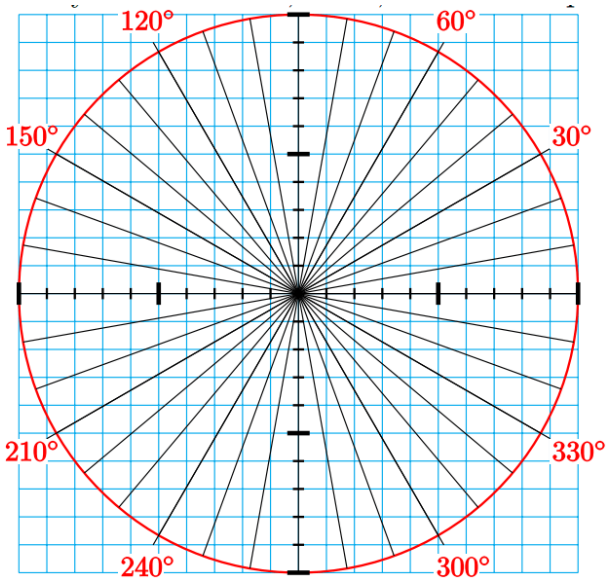
Angle	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
x-coordinate										

Angle	100°	110°	120°	130°	140°	150°	160°	170°	180°
x-coordinate									

Angle	190°	200°	210°	220°	230°	240°	250°	260°	270°
x-coordinate									

Angle	280°	290°	300°	310°	320°	330°	340°	350°	360°
x-coordinate									

42. Estimate the y-coordinate of the points on the unit circle designated by each angle and complete the table. (Hint: Use symmetry in the second, third, and fourth quadrants.)



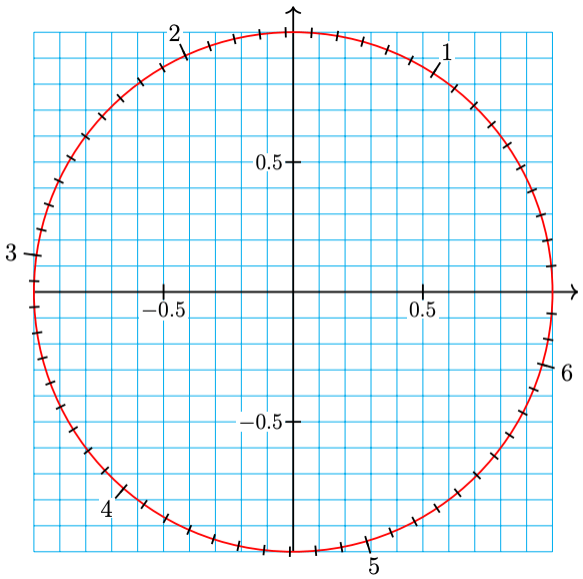
Angle	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
x-coordinate										

Angle	100°	110°	120°	130°	140°	150°	160°	170°	180°
x-coordinate									

Angle	190°	200°	210°	220°	230°	240°	250°	260°	270°
x-coordinate									

Angle	280°	290°	300°	310°	320°	330°	340°	350°	360°
x-coordinate									

For the following exercises, use the unit circle to estimate the sine, cosine, and tangent of each arc of given length.



13. a. 0.4

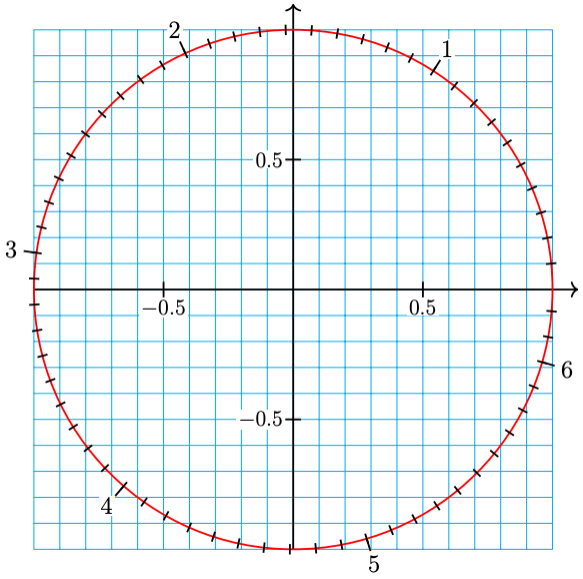
b. 1.2
c. 2

14. a. 0.8
b. 2.6
c. 4

15. a. 2.8
b. 3.5
c. 5

16. a. 3
b. 4.3
c. 5.5

For the following exercises, use the unit circle to estimate two numbers with the given trigonometric value.



47. $\cos t = 0.3$

48. $\sin t = 0.1$

49. $\sin t = -0.7$

50. $\cos t = -0.6$

51. $\tan t = \dfrac{-4}{9}$

52. $\tan t = \dfrac{8}{6}$

Each of the following exercises describes an arc in standard position on the unit circle. In which quadrant does the terminal point of the arc lie?

53. $\sin s > 0, \quad \cos s < 0$

54. $\sin s < 0, \quad \cos s > 0$

55. $\cos t < 0, \quad \tan t < 0$

56. $\cos t > 0, \quad \tan t < 0$

57. $\sin x < 0, \quad \tan x > 0$

58. $\sin x > 0, \quad \tan x < 0$

For the following exercises, without a calculator, decide whether the quantity is positive or negative.

59. $\cos 2.7$

60. $\sin 4.1$

61. $\tan 3.8$

62. $\tan 5.4$

63. $\sin 2.2$

64. $\cos 4.9$

Starting at $(1, 0)$, you walk s units around a unit circle. For the following exercises, sketch a unit circle showing your position. What are your coordinates?

65. $s = 2.5$

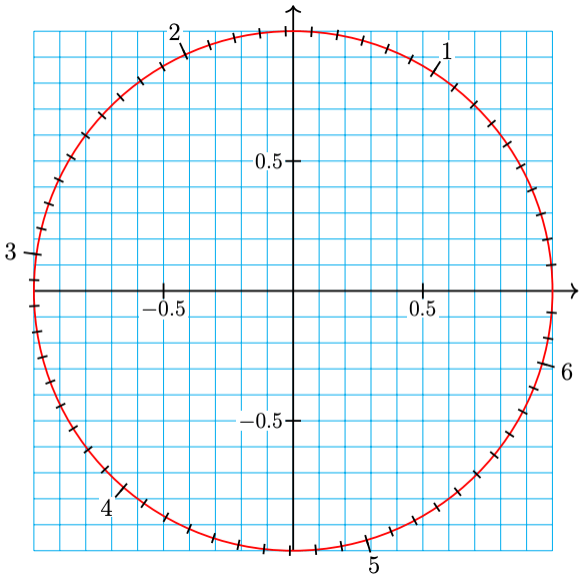
66. $s = 4.3$

67. $s = 8.5$

68. $s = 11$

Synthesis Questions

For the following exercises, place the trigonometric values in order from smallest to largest. Use the figure to help you, but try not to use a calculator!



69. $\sin 0.5, \quad \sin 1.5, \quad \sin 2.5, \quad \sin 3.5$

70. $\cos 1.6, \quad \cos 2.6, \quad \cos 3.6, \quad \cos 5.6$

71. $\cos 2, \quad \cos 3, \quad \cos 4, \quad \cos 5$

72. $\sin 2.8, \quad \sin 3.8, \quad \sin 4.8, \quad \sin 5.8$

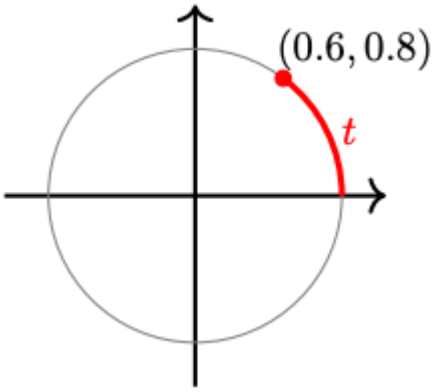
For the following exercises, the figure shows an arc of length t, and the coordinates of its terminal point. Find the terminal point of each related arc given below, and give its sine, cosine, and tangent.

a $2\pi - t$

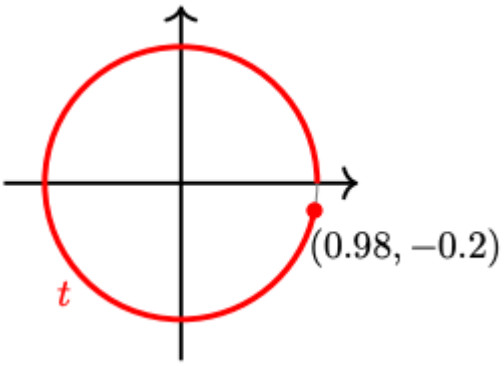
b $\pi - t$

c $\pi + t$

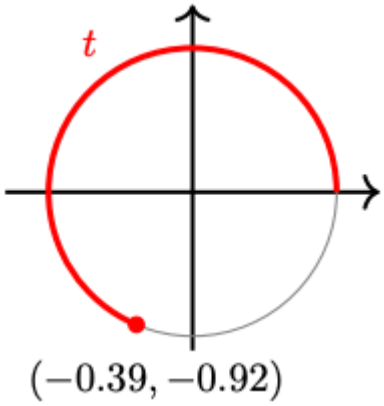
73.



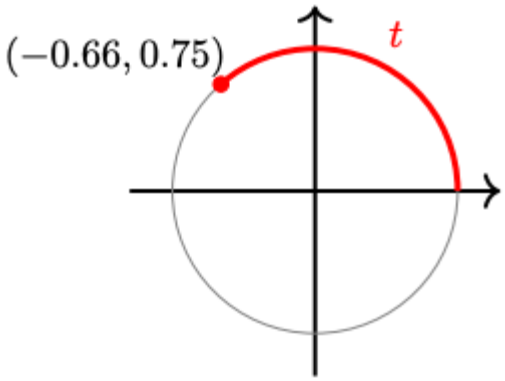
74.



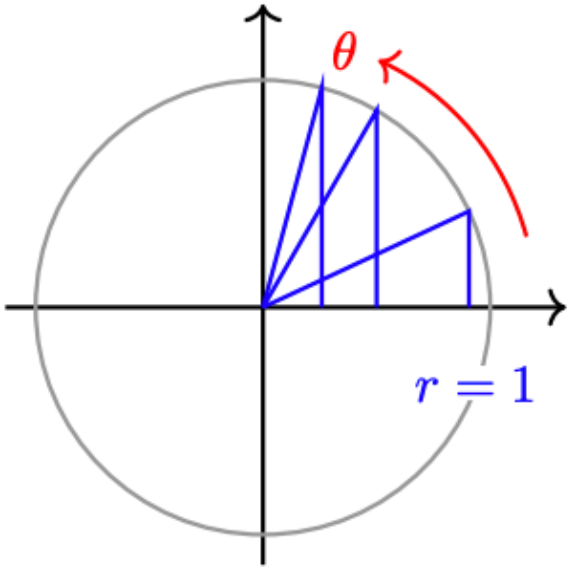
75.



76.



77. Use the figure to help you fill in the blanks.



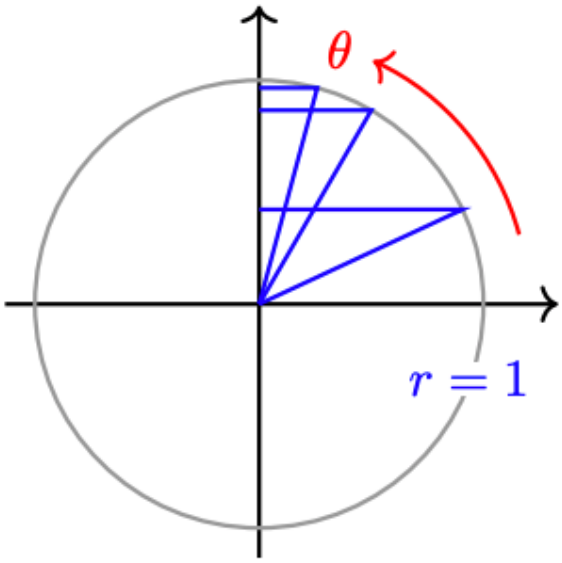
a. As θ increases from 0° to 90°, $f(\theta) = \sin\theta$ _____ from _____ to _____.

b. As θ increases from 90° to 180°, $f(\theta) = \sin\theta$ _____ from _____ to _____.

c. As θ increases from 180° to 270°, $f(\theta) = \sin\theta$ _____ from _____ to _____.

d. As θ increases from 270° to 360°, $f(\theta) = \sin\theta$ _____ from _____ to _____.

78. Use the figure to help you fill in the blanks.



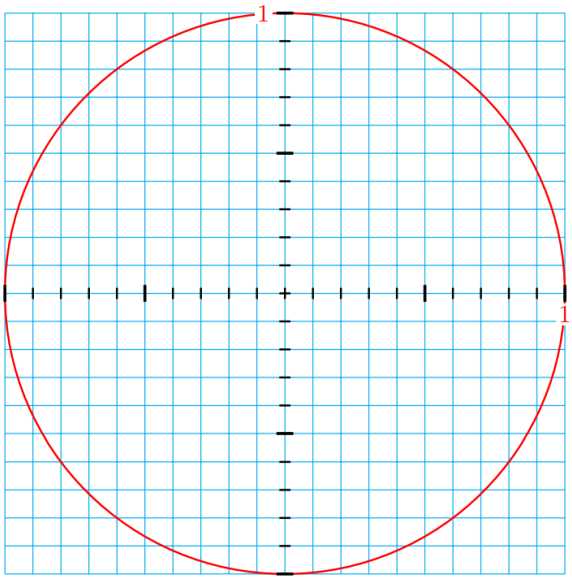
a. As θ increases from 0° to 90°, $f(\theta) = \cos\theta$ _____ from _____ to _____.

b. As θ increases from 90° to 180°, $f(\theta) = \cos\theta$ _____ from _____ to _____.

c. As θ increases from 180° to 270°, $f(\theta) = \cos\theta$ _____ from _____ to _____.

d. As θ increases from 270° to 360°, $f(\theta) = \cos\theta$ _____ from _____ to _____.

For the following exercises, use the grid provided below.



79. Draw two different angles α and β in standard position whose sine is 0.6.

a. Use a protractor to measure α and β.

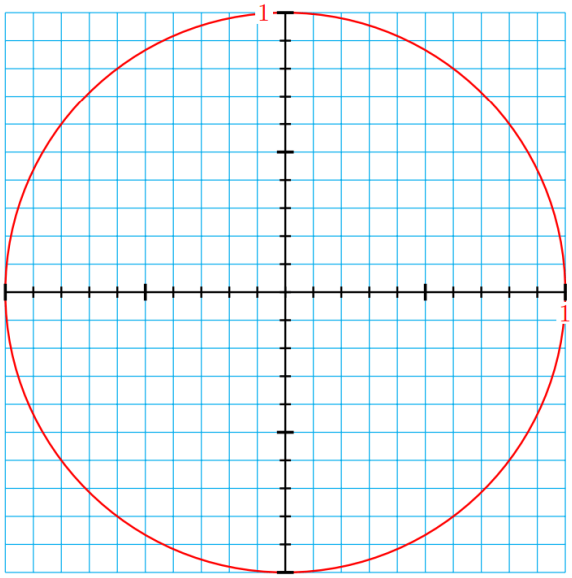
b. Find the reference angles for both α and β. Draw in the reference triangles.

80. Draw two different angles θ and ϕ in standard position whose sine is -0.8.

a. Use a protractor to measure θ and ϕ.

b. Find the reference angles for both θ and ϕ. Draw in the reference triangles.

For the following exercises, use the grid provided below.



81. Draw two different angles α and β in standard position whose cosine is 0.3.

a. Use a protractor to measure α and β.

b. Find the reference angles for both α and β. Draw in the reference triangles.

82. Draw two different angles θ and ϕ in standard position whose cosine is –0.4.

a. Use a protractor to measure θ and ϕ.

b. Find the reference angles for both θ and ϕ. Draw in the reference triangles.

For the following exercises, find all angles between 0° and 360° for which the statement is true.

83. a. $\cos\theta = -1$

b. $\sin\theta = -1$

c. $\tan\theta = -1$

84. a. Find two angles, $0 \le \theta \le 360^\circ$, with $\sin\theta = 0$.

b. Find two angles, $0 \le \theta \le 360^\circ$, with $\cos\theta = 0$.

For the following exercises, find two points on the unit circle with the given coordinate. Sketch the approximate location of the points on the circle. (Hint: what is the equation for the unit circle?)

85. $x = 0.2$

86. $x = -0.6$

87. $y = -0.35$

88. $y = 0.7$

89. $x = \dfrac{-\sqrt{3}}{2}$

90. $y = \dfrac{1}{\sqrt{2}}$

For the following exercises, find the angle in radians between 0 and 2π determined by the terminal point on the unit circle. Round your answer to hundredths.

91. (−0.1782, 0.9840)

92. (−0.8968, −0.4425)

93. (0.8855, −0.4646)

94. (0.9801, 0.1987)

95. (−0.7659, −0.6430)

96. (0.9602, −0.2794)

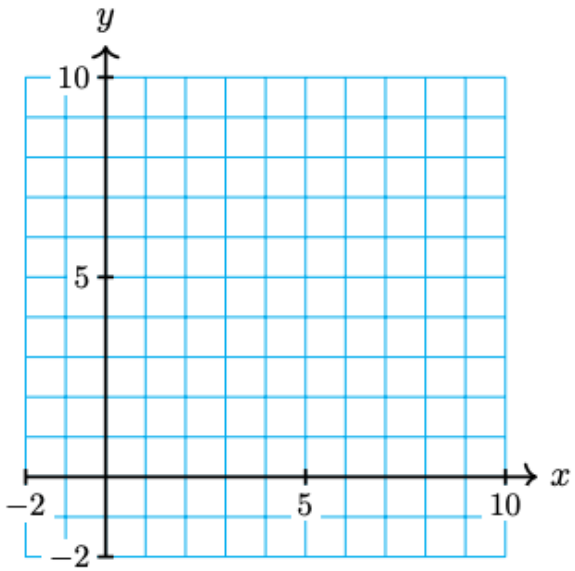
97. a. Sketch a unit circle and the line $y = x$. Find the coordinates of the two points where the line and the circle intersect.

    b. State your answers to part (a) using trigonometric functions.

98. a. Sketch a unit circle and the line $y = -x$. Find the coordinates of the two points where the line and the circle intersect.

    b. State your answers to part (a) using trigonometric functions.

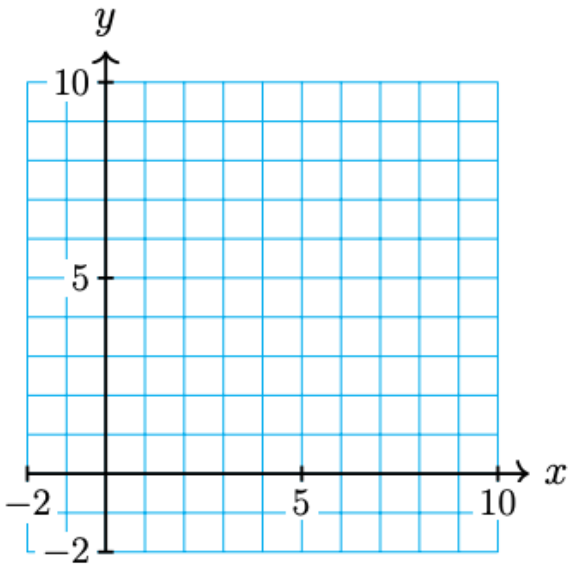
99.



    a. Sketch a line that passes through the origin and the point (8,3). What is the slope of the line?

    b. What is the angle of inclination of the line in radians, measured from the positive x-axis?

100.



a. Sketch a line that passes through the origin and the point (3,8). What is the slope of the line?

b. What is the angle of inclination of the line in radians, measured from the positive x-axis?

For the following exercises, find an equation for the line with the given angle of inclination passing through the given point.

91. $\alpha = \frac{\pi}{3}, (4, 2)$

92. $\alpha = \frac{5\pi}{6}, (-6, 3)$

93. $\alpha = 2.4, (5, -8)$

94. $\alpha = 0.6, (-2, -3)$

95. If $\sec(\theta) = \frac{x}{4}$ for $0 < \theta < \frac{\pi}{2}$, find an expression for $\ln|\sec(\theta) + \tan(\theta)|$ in terms of x.

Applications

96. The sunrise time in Wellington, New Zealand, on the nth day of the year can be modeled by

$$S = 1.93\sin(0.016n - 1.13) + 6.14$$

where S is given in hours after midnight. Find the sunrise time on January 1 (day $n = 1$), April 1 (day $n = 91$), July 1 (day $n = 182$), and on October 1 (day $n = 274$).

97. The variable star RT Cygni reached its maximum magnitude on May 22, 2004, and t days later its magnitude is given by

$$M = 9.55 - 2.25\cos(0.033t)$$

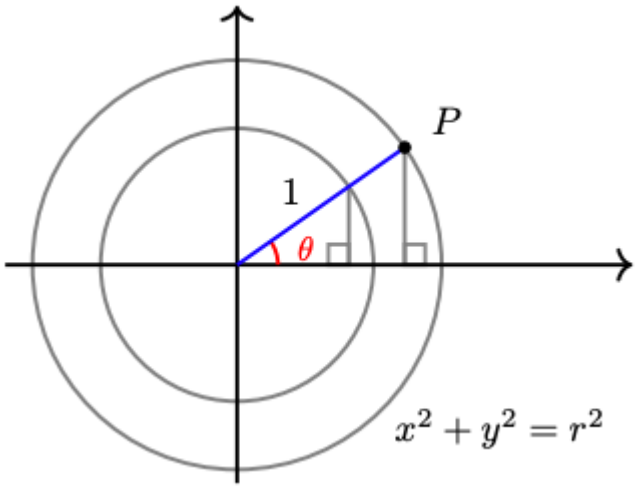
Find the magnitude of RT Cygni on days $t = 0$, $t = 48$, $t = 95$, $t = 142$, and $t = 190$. (Note that smaller values of M denote brighter magnitudes.)

98. City Park features a circular jogging track with a radius of 1 mile centered on the open-air bandstand. You start jogging on the track 1 mile due east of the bandstand and proceed counterclockwise. What are your coordinates relative to the bandstand when you jogged five miles?

99. Silver Reservoir is a circular man-made lake with a radius of 1 kilometer. If you start at the easternmost point on the reservoir and walk counterclockwise for 4 kilometers, how far south of your initial position are you?

Use the results of Problem 110 for Problems 111 - 113.

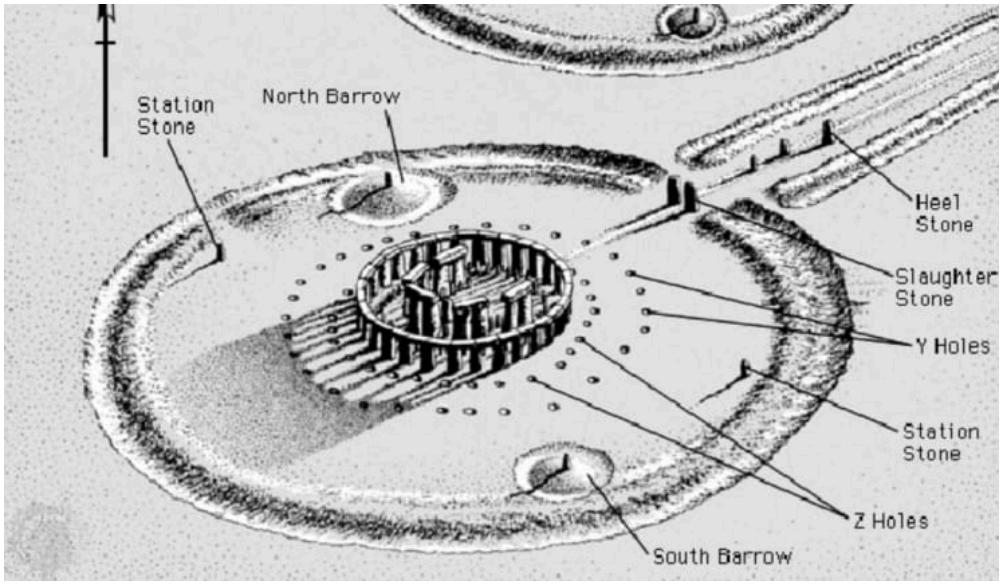
10. Use similar triangles to show that the coordinates of a point P determined by angle θ on a circle of radius r are $x = r\cos\theta, y = r\sin\theta$. (See the figure below.)



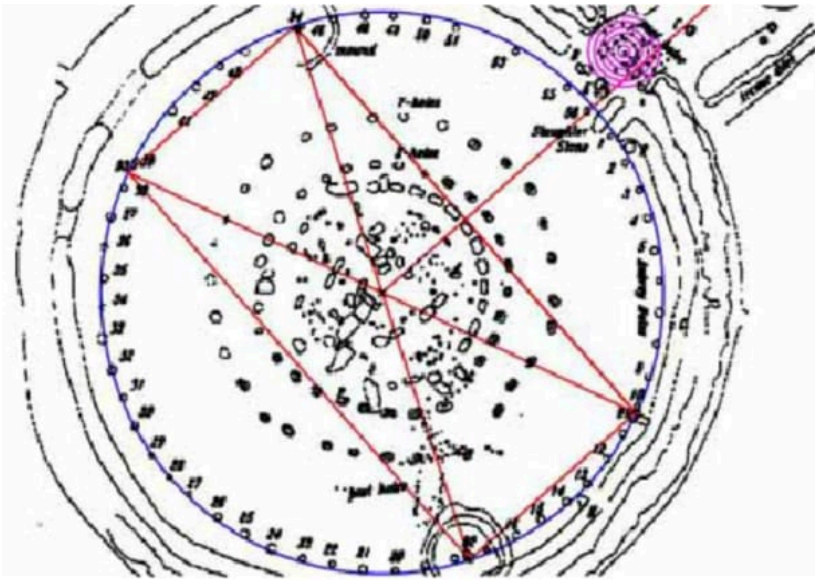
11. The Astrodome in Houston has a diameter of 710 feet. If you start at the easternmost point and walk counterclockwise around its perimeter for 250 feet, how far north of your starting point are you?

12. The Barringer meteor crater near Winslow, Arizona, is 1182 meters in diameter. You start at the easternmost point on the crater's rim and walk counterclockwise around the edge. After walking for 1 kilometer, what is your position relative to your starting point?

13. One of the most intriguing features of Stonehenge is the position of the four Station Stones.



They form the corners of a rectangle inscribed in the Aubrey Circle on the perimeter of the henge, which has a diameter of 288 feet. A line from the center of the circle and perpendicular to the long edge of the rectangle points through the Slaughter Stones at the entrance of the henge and out to the Heel Stone. If you stood in the circle's center on the summer solstice, you would see the sun rise directly over the Heel Stone.

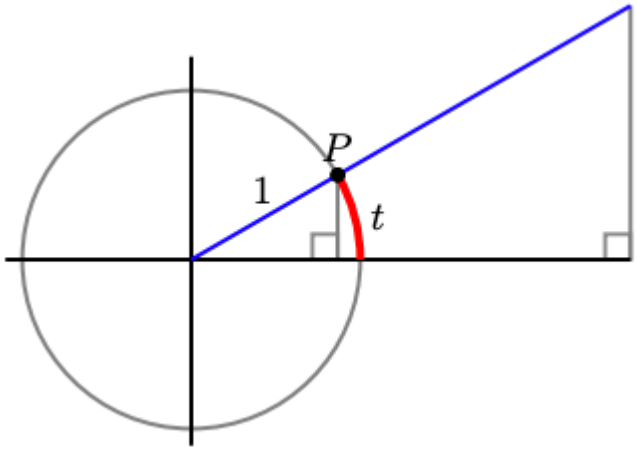


a. The sun rises 48.6° east of north on the summer solstice at Stonehenge. If the positive y-axis points north, find the coordinates of the henge entrance relative to its center.

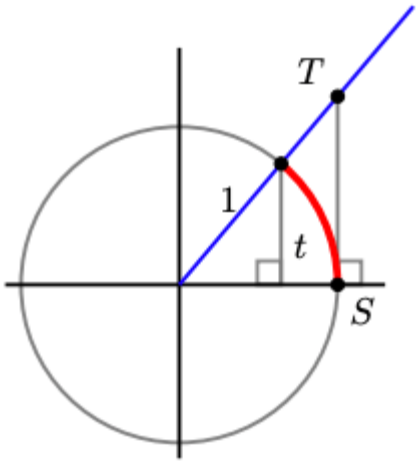
b. The northernmost station stone is located 66.6° of arc counterclockwise from the entrance. Find its coordinates relative to the center of the henge.

Challenge Problems

14. Use similar triangles to show that the coordinates of point P on the unit circle shown below are $(\cos t, \sin t)$.



15. Use similar triangles to show that $ST = \tan t$.



Answers

Skills Refresher

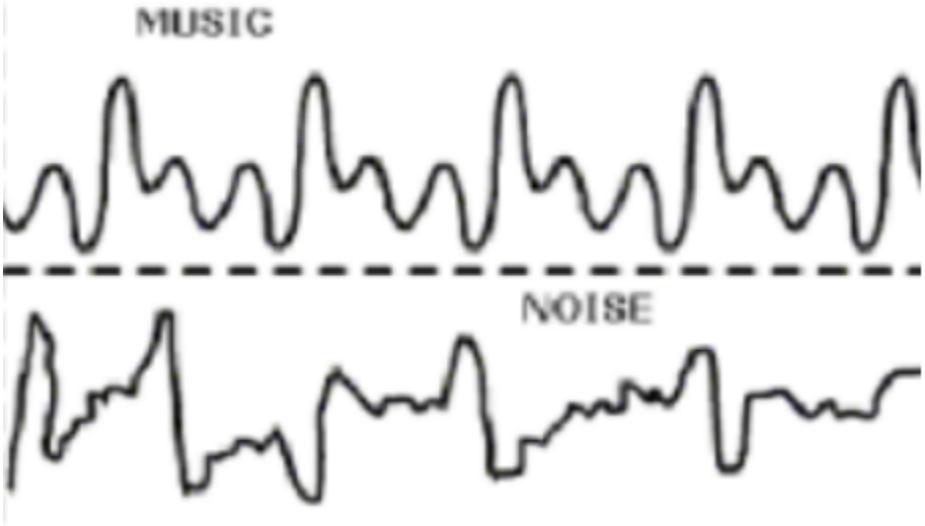
1. a $\frac{5}{6}$, b $\frac{5x}{6}$
2. a $\frac{1}{8}$, b $\frac{n}{8}$
3. a $\frac{5}{4}$, b $\frac{5b}{4}$
4. a $\frac{11}{8}$, b $\frac{11m}{8}$
5. a $\frac{-1}{6}$, b $\frac{-q}{6}$
6. a -1, b $-p$

This page titled 7.3.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

8: GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

Musical instruments produce sound by the vibration of a string on a violin, guitar, or piano, or a column of air in a brass or woodwind instrument. The vibration causes a periodic variation in air pressure that is heard as sound. Periodic vibrations create a pleasing, musical sound, while random vibrations sound like noise.

Figure 1

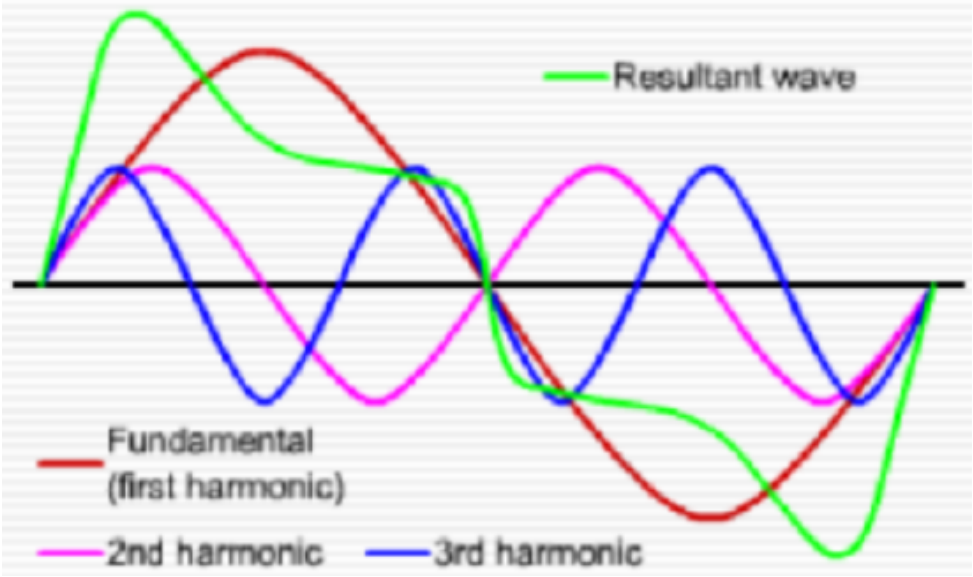


There are three distinguishing characteristics of musical notes: loudness, pitch, and timbre.

- The **loudness** of a note is measured by the magnitude, or amplitude, of the changes in air pressure.
- The **pitch** of the note is determined by its frequency, the number of times its basic pattern is repeated each second. For example, a note with pitch 440 hertz repeats its pattern 440 times per second. Humans can hear frequencies that range roughly from 20 to 18,000 hertz.
- The **timbre** of the note allows us to tell a piano note from a guitar note with the same loudness and pitch.

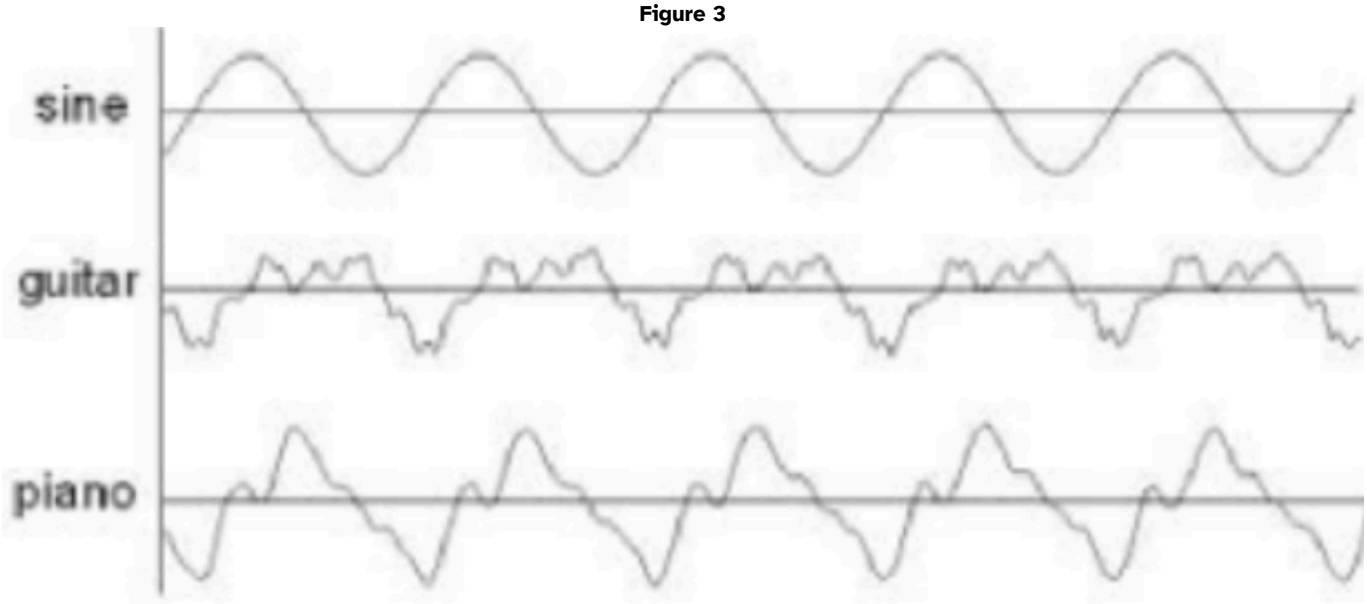
The simplest pressure vibration is produced by a tuning fork. The graph of its pressure function resembles a sine curve, and the corresponding sound is called a **pure tone**. When a musical instrument produces a pure tone, not only is its frequency produced, but integer multiples of the **fundamental frequency**, called **harmonics**, are produced as well. Sine curves of different frequencies combine to create the timbre of the note.

Figure 2



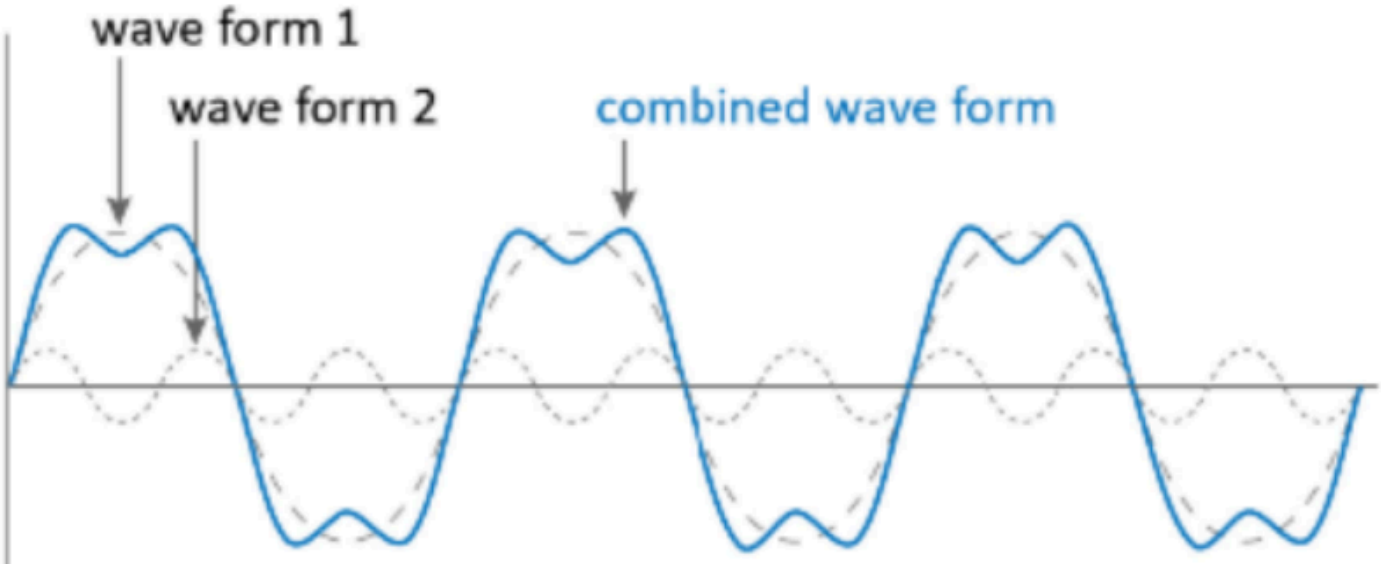
If two musical instruments play the same note, the notes have the same pitch, but they sound different because the amplitudes of each of the harmonics is different for the two instruments.

Figure 3



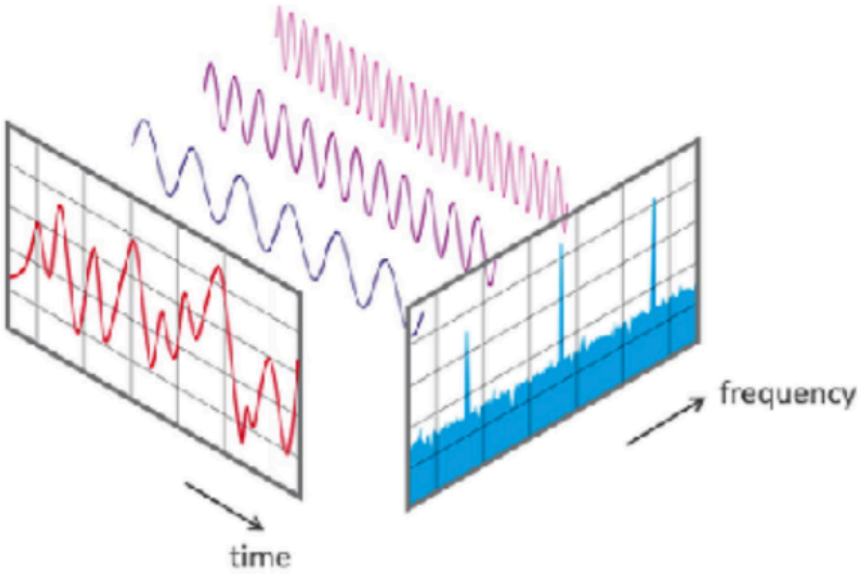
The French mathematician Joseph Fourier, who lived from 1768 to 1830, discovered that any periodic wave can be written as the sum of a number of sines and cosines with differing amplitudes.

Figure 4



Fourier created a method for determining the frequencies and amplitudes of the simpler waves that make up a more complicated periodic function. Applying the Fourier transform to a sampled musical note reveals which component frequencies are present in the note. The same sound can then be re-synthesized by including those frequency components.

Figure 5



Today, Fourier analysis is the foundation of signal processing, not only for audio waves, but for radio waves, light waves, seismic waves, and even images. By analyzing a compound waveform, the "noise" components can be isolated and removed in order to smooth out the signal.

Fourier analysis is also used in X-ray crystallography to reconstruct a crystal structure from its diffraction pattern, and in nuclear magnetic resonance spectroscopy to determine the mass of ions from the frequency of their motion in a magnetic field.

8.1: Base Graphs of the Fundamental Trigonometric Functions

8.1.1: Resources and Key Concepts

8.1.2: Homework

8.2: Non-Rigid Transformations

8.2.1: Resources and Key Concepts

8.2.2: Homework

8.3: Rigid Transformations

8.3.1: Resources and Key Concepts

8.3.2: Homework

8.4: Graphs of the Remaining Trigonometric Functions

8.4.1: Resources and Key Concepts

8.4.2: Homework

This page titled 8: Graphs of the Trigonometric Functions is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Katherine Yoshiwara & Roy Simpson.

8.1: Base Graphs of the Fundamental Trigonometric Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

When you were learning Algebra, you were introduced to variables, expressions involving variables, equations involving variables, and graphs of those equations (in that order). In Trigonometry, this order is slightly modified. We have learned about trigonometric functions and manipulated expressions involving those functions; however, before solving equations involving those functions, it's best to be able to visualize their graphs.

This section focuses on the base graphs of the **fundamental trigonometric functions**.

Definition: Fundamental Trigonometric Functions

The sine, cosine, and tangent are called the **fundamental trigonometric functions**.

The graphs of the remaining trigonometric functions will be introduced after we have a few sections to get used to the graphs of the fundamental trigonometric functions.

Motivating Periodic Functions of Angles

Imagine that you are riding on a Ferris wheel. As the wheel turns, your height above the ground increases and decreases. This pattern repeats each time the Ferris wheel makes a complete rotation. This type of motion is called **periodic motion**.

Definition: Periodic Motion

When a body repeats its movement along a path in a regular and predictable manner over a specific time interval, its movement is called **periodic motion**.

In Mathematics, we seek to model periodic motion with a function.

Definition: Periodic Function

A mathematical function modeling periodic motion is called a **periodic function**. A periodic function repeats its values at regular intervals, meaning its graph has a pattern that repeats itself over a specific horizontal distance or time interval.

We use periodic functions to model phenomena that exhibit cyclical behavior, such as the height of tides, seasonal growth

patterns in plants and animals, radio waves, and planetary motion.

Let's create a mathematical model for a ride on a Ferris wheel. Suppose the Ferris wheel has a radius of 100 feet and rotates counterclockwise. Our model will be a function that describes the rider's height above the ground as they ride the wheel. To graph the Ferris wheel function, we must first specify the input and output variables and then choose a coordinate system to display their values.

While developing the model, let's place the origin at the center of the Ferris wheel (see Figure 8.1.1 below).

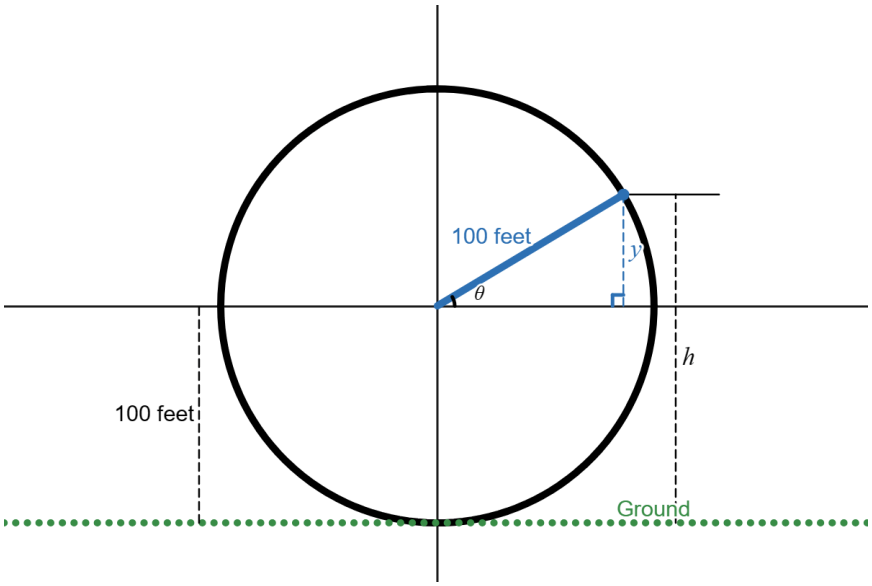


Figure 8.1.1**:** *A graphical representation of a 100-foot-radius Ferris wheel centered at the origin.*

While this is not the most realistic placement for us to imagine (it makes us think that the bottom of the Ferris wheel is below ground level), this temporary placement will reveal an important characteristic of the rider's motion. The line from the origin to the rider's position on the wheel makes an angle, θ, with the horizontal. This angle will be the independent variable for the function. The height of the rider, h, is also a variable and is related to the y-coordinate of the rider's position. Figure 8.1.1 shows that $h = y + 100$ because the center of the wheel is 100 feet above the ground.

Let's focus on the rider's height as a function of angle. Therefore, we'll first graph y as the dependent variable instead of h. As the angle θ increases from 0° to 90°, the rider's y-coordinate increases from 0 to 100. At that point, they are at the top of the wheel. Then, as θ increases from 90° to 180°, the rider's y-coordinate decreases from 100 back to 0 (see Figure 8.1.2).

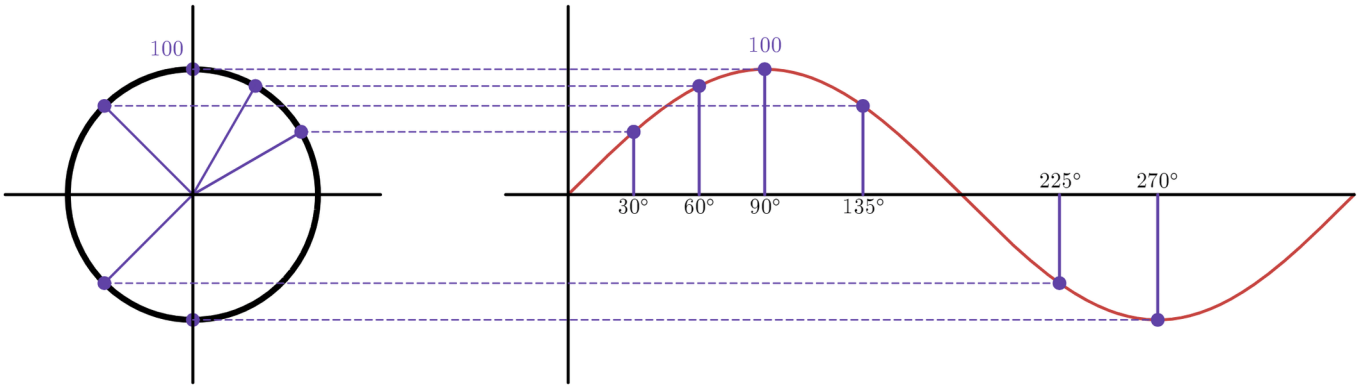


Figure 8.1.2**:** *"Unwinding" the circle to get a visual model of the height function.*

Finally, as θ increases from 180° to 360°, the rider's y-coordinate decreases from 0 to -100 and then increases from -100 back to 0. At that point, the rider has made one complete rotation on the Ferris wheel. If she goes around again, θ increases from 360° to 720°, and the graph of her y-coordinate will repeat the pattern of the first rotation. Again, Figure 8.1.2 shows how the rider's y

-coordinate is plotted as a function of the angle θ.

Look back at Figure 8.1.1 and notice from the reference triangle that $\sin(\theta) = \frac{y}{100}$. Therefore, $y = 100\sin(\theta)$. For example, when $\theta = 30^\circ$, the y-coordinate of the rider is

$$y = 100\sin(30^\circ) = 100\left(\frac{1}{2}\right) = 50$$

and the rider's height above the ground is $h = y + 100 = 150$ feet. In general, then, h is given as a function of θ by

$$h = y + 100 = 100\sin(\theta) + 100.$$

This is our model for a rider's height on a Ferris wheel as a function of angle.

The graph on the right side of Figure 8.1.2 is called a **sine curve**.

Definition: Sine Curve

A smooth, wave-like curve that oscillates (moves up and down) periodically and which can be derived from either the sine or cosine function is called a **sine curve**. A curve of this nature is often called **sinusoidal**,

While it might seem odd that we use the blanket phrase "sine curve" when describing a graph derived from the sine *or* the cosine, this consolidation will make sense shortly.

An Aside: Degrees or Radians?

Before we go any further, we need to make one thing clear:

> Figure 8.1.2 is the first and the last time you will see a graph of the sine (or any other trigonometric) function in terms of degrees.

While there are several reasons that we endeavor only to graph trigonometric functions using radians, the two most impactful are the relation to the unit circle and scale. When we derived the unit circle definition of the trigonometric functions, we did so using radian measure. If we redraw Figure 8.1.2 using radians as the angular measure (and the unit circle instead of a circle of radius 100), we get Figure 8.1.3.

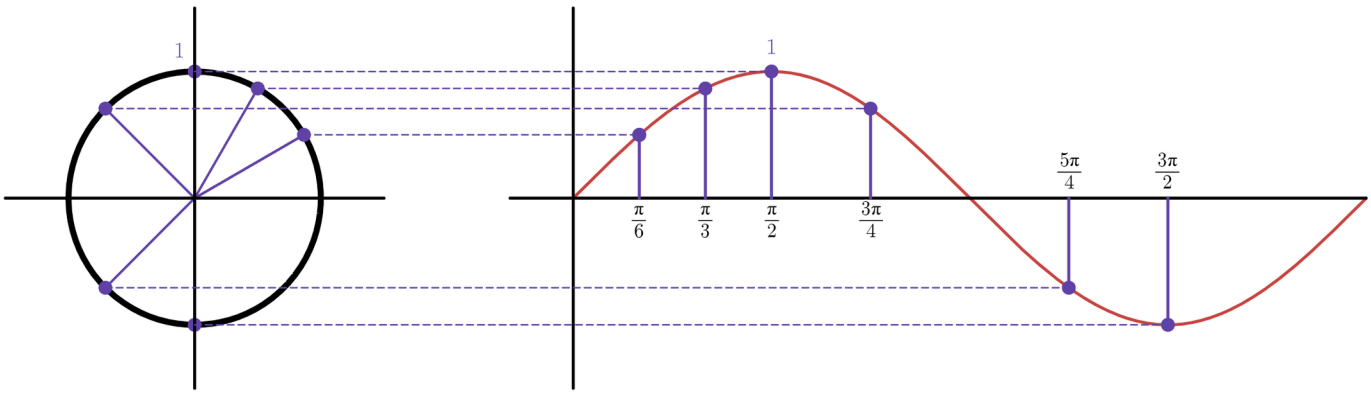


Figure 8.1.3: *Drawing the sine curve using the unit circle.*

Note that the scale on the vertical axis (namely, that the largest and smallest y-values are 1 and -1, respectively) and the scale on the horizontal axis (from $x = 0$ to $x = 2\pi$) are more closely aligned than had we kept the angular measure in degrees.

Graphing the Sine Function

There are two main ways students initially learn to graph the sine function—by plotting points or by tracing the y-values along the

unit circle. The latter is what we did in Figure 8.1.3. The former is not the ideal method, but it's a good idea to showcase it right now so we can link the computed values of the sine function to points on a graph.

Point-Plotting Approach

Up to this point, we have mostly let sine be a function of the angle θ; however, we recently discovered (via our discussion of circular functions) that the trigonometric functions are also functions of real numbers. Therefore, we should feel fine working with $f(x) = \sin(x)$. The independent variable, x, could represent an angle, the arc length along a circle, or time—our application determines the context.

Example 8.1.1

Sketch a portion of the graph of $y = \sin(x)$ on the interval $[0, 2\pi]$ by making a table of values.

Solution available at the end of this section.

Checkpoint 8.1.1

Sketch a portion of the graph of $y = \sin(x)$ on the interval $\left[-\frac{3\pi}{2}, \frac{\pi}{2}\right]$ by making a table of values.

Answer available at the end of this section.

There are several issues with plotting points to obtain the sine function graph. The first is the amount of time spent computing values. We will be graphing the sine curve throughout the rest of our mathematical journey, and if we need to plot points every time we graph, we will be wasting time and effort. The second issue has to do with "connecting the dots." Those who have no idea of the curvature of the sine graph end up connecting the points with lines and arrive at something like Figure 8.1.6.

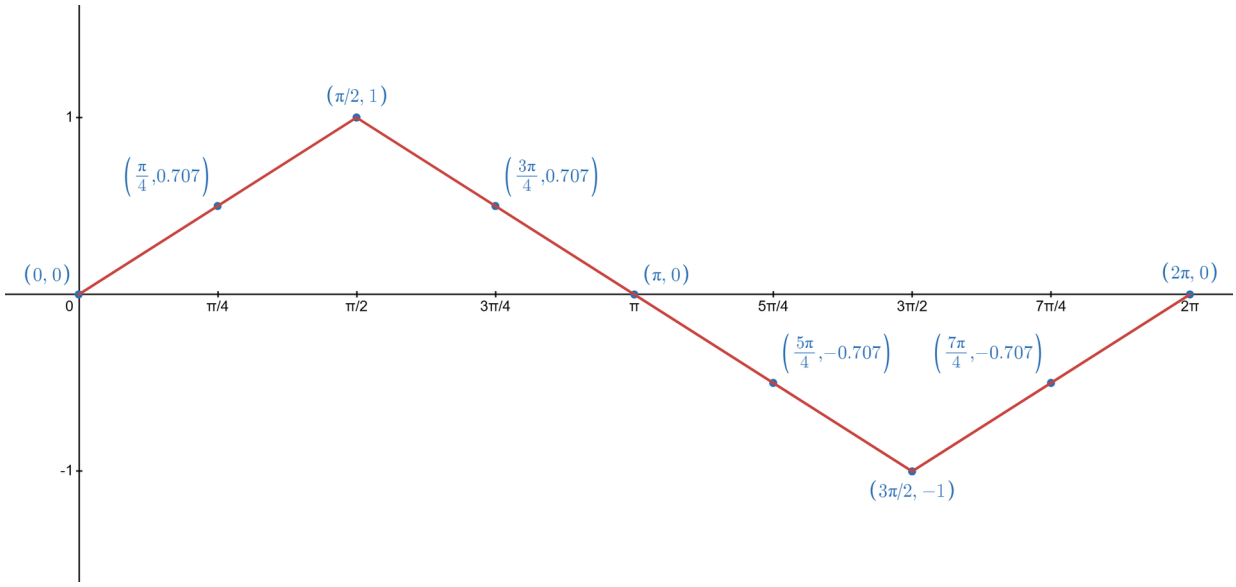


Figure 8.1.6: *NEVER GRAPH A TRIGONOMETRIC FUNCTION USING LINES*

I hope you can see the difference between the correct graph of the sine function from Example 8.1.1 and the incorrect graph represented in Figure 8.1.6.

The third issue with graphing using point-plotting involves building a natural understanding of the sine curve. Many features, like domain and range, slip out of the student's grasp, who consistently requires the "connect the dots" method when graphing trigonometric functions.

Finally, the fourth issue occurs when needing to extend the graph of the sine function beyond the interval $[0, 2\pi]$. As Checkpoint $8.1.1$ shows, and as we shall see in more concrete form shortly, the sine curve oscillates in both directions (as $x \to \infty$ and as $x \to -\infty$). If we needed to plot points to extend the graph, we would be plotting and computing for a long time.

With all these considerations in mind, we will avoid using point-plotting when graphing trigonometric functions (other than introducing the base graphs of the trigonometric functions).

Unit Circle Approach

We have already motivated the sine curve using the unit circle in Figure $8.1.3$; however, let's spend a few more moments solidifying our understanding of the relationship between the unit circle and the sine curve.

Recall, $\sin(t)$ is the value of the y-coordinate for the point P on the unit circle, where t is simultaneously the arc length from $(1, 0)$ to the point P and the angle subtended by this arc. The following Interactive Element shows how the sine curve is created by tracing these y-coordinates as the angle (or arc length) t increases from 0 to 2π (if you only have access to the PDF of this textbook, then please go to the following link: https://www.desmos.com/calculator/aq5r4ka4dd)

Interactive Element: Developing the Graph of the Sine Function via the Unit Circle

Observation: As the angle (or arc length) t increases, the y-coordinate increases to 1, then decreases to -1, then increases back to $y = 0$.

We now can easily "extend" the graph of the sine function by allowing t to go beyond 2π (https://www.desmos.com/calculator/imbgdopjtz).

Interactive Element: Extending the Graph of the Sine Function

Basic Features of the Sine Curve

Now that we know how to graph the base sine curve, we can dive into the details of this function.

Domain and Range

We now have enough knowledge to graph the *entire* sine curve. Figure $8.1.7$ shows the general sine curve along with a single cycle (in blue) on the interval $[0, 2\pi]$.

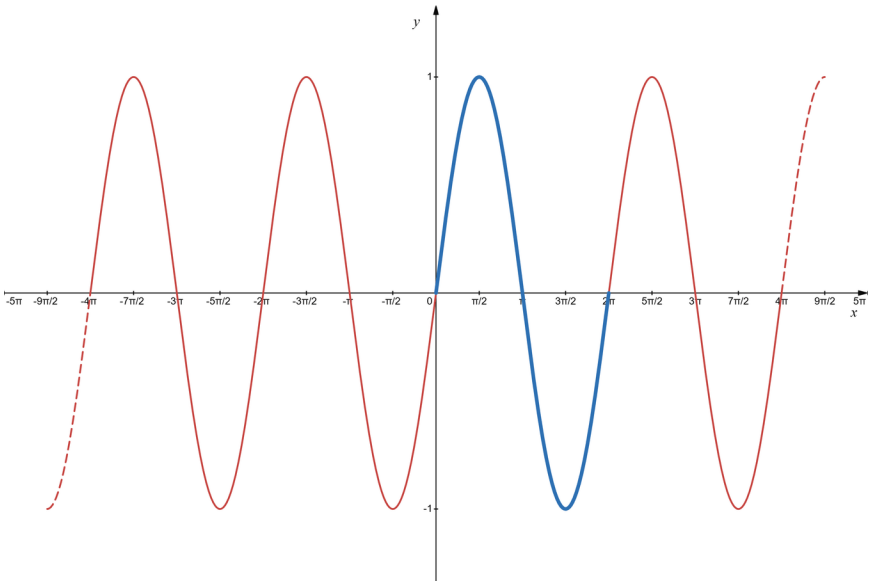


Figure 8.1.7**:** $y = \sin(x)$

Since the sine curve oscillates forever in both directions, it should be evident that the sine function has an output for every input. Therefore, the domain of the sine function is all real numbers. Moreover, we can see from the graph that the values of the sine function oscillate between -1 and 1 (inclusive). Hence, the range must be $[-1, 1]$. We summarize these results in the following theorem.

Theorem: Domain and Range of the Sine Function

The domain of $f(x) = \sin(x)$ is $(-\infty, \infty)$ and the range is $[-1, 1]$.

Caution: Is it $\sin(x)$ or $\sin(t)$?

It is common for those learning Trigonometry to need clarification about the meaning of the argument of the sine function at this point. When we speak of the unit circle, we are speaking of the graph of all solutions, (x, y), to the equation

$$x^2 + y^2 = 1.$$

As was mentioned in a previous section, the x-coordinate of a point on the unit circle is defined to be $\cos(t)$ and the y-coordinate is defined to be $\sin(t)$, where t is simultaneously the arc length along the unit circle to the point in question and the angle subtended by that arc length.

Unfortunately, we will now be graphing $f(x) = \sin(x)$. This value of x differs from the related x-coordinate on the unit circle! You can (and should) think of the x in $f(x) = \sin(x)$ as a "dummy variable." If you prefer to replace it with θ or t, please feel free to do so; however, we will continue to write sine as a function of x to get you used to this often confusing convention.

Period

Another observation is that the sine function repeats its pattern every 2π units along the x-axis.

Definition: Period

For any function $f(x)$, the smallest positive number k for which

$$f(x + k) = f(x)$$

for all x in the domain of f is called the **period** of $f(x)$.

If we take any interval of length 2π along the x-axis, we will see that the corresponding section of the sine curve over that interval can be "copied-and-pasted" an infinite number of times to create the entire sine curve. For example, while not convenient, we could consider the interval $[-5, -5+2\pi]$ along the x-axis, as shown in Figure $8.1.8$ below.

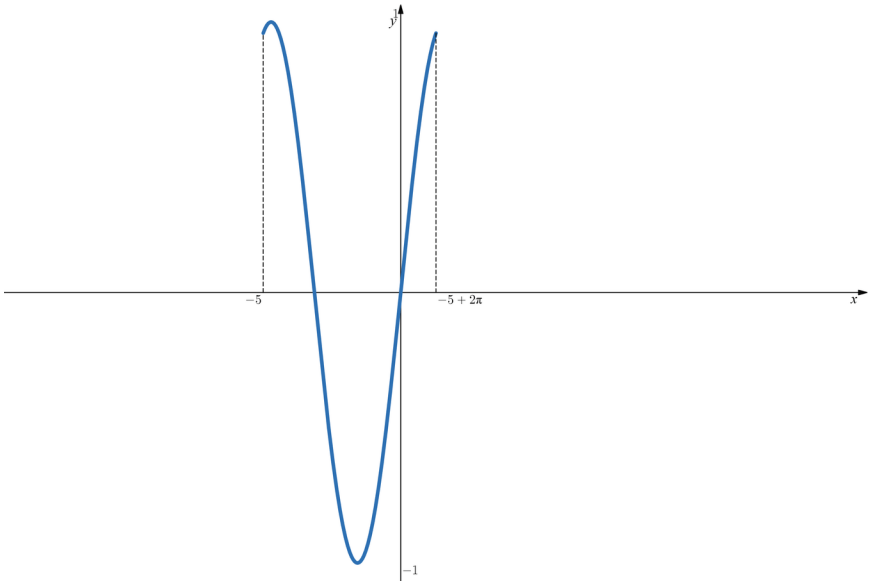


Figure $8.1.8$**:** *The sine curve on the interval* $[-5, -5+2\pi]$

"Copying-and-pasting" that segment an infinite number of times in both directions, we rebuild the entire sine curve.

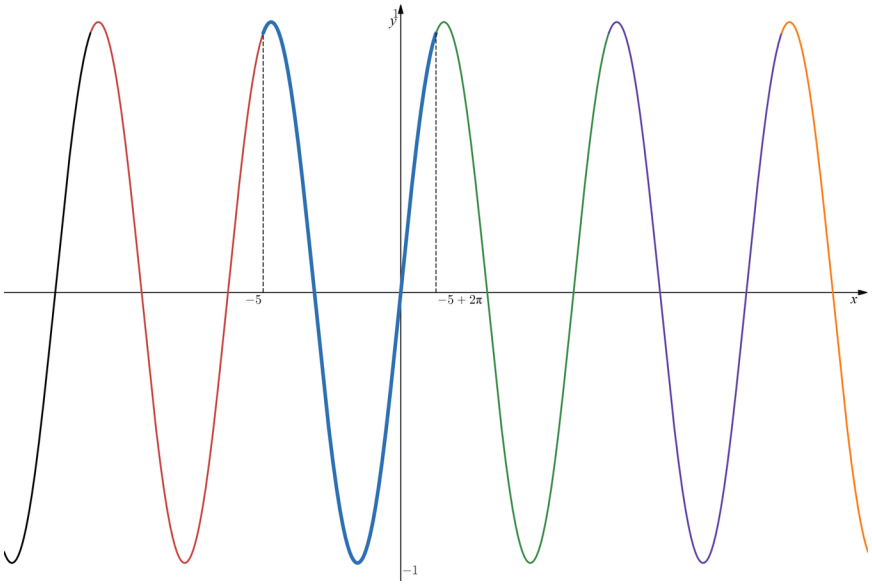


Figure $8.1.9$

Moreover, if we had selected a smaller interval along the x-axis, say $\left[-5, -5+\frac{3\pi}{2}\right]$, we would not have captured enough of the behavior of the sine curve to appropriately recreate the entire curve by "copying-and-pasting" the smaller section.

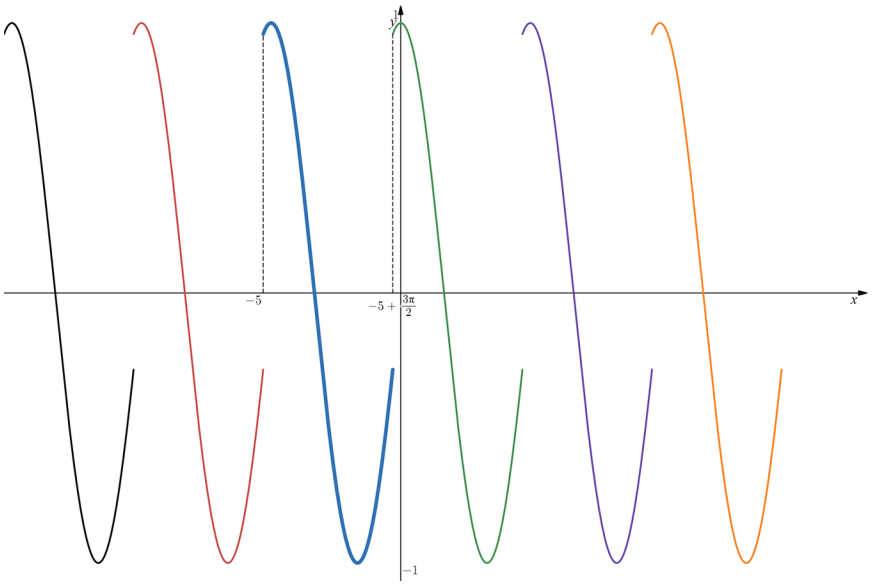


Figure 8.1.10

Hence, by the definition of the period of a function, the following theorem should make sense.

Theorem: Period of the Sine Function

The period of $f(x) = \sin(x)$ is 2π. This is also known as the **natural period** of the sine function.

While the graph of the sine function (and all of the other trigonometric functions) extends in both directions forever (see Figure 8.1.7), the section of the graph starting at $x = 0$ and ending at $x = 2\pi$ is going to be very important as we move forward.

Definition: Principal Cycle (of a trigonometric function)

The **principal cycle** of a trigonometric function is the interval starting at $x = 0$ and terminating at $x = p$, where p is the period of the trigonometric function.

Thus, by definition, we get the following theorem.

Theorem: Principal Cycle of the Sine Function

The principal cycle of the sine function is $[0, 2\pi]$.

It is important to note that there are other periodic functions besides the sine function.

Example 8.1.2

Which of the functions shown below are periodic? If the function is periodic, give its period.

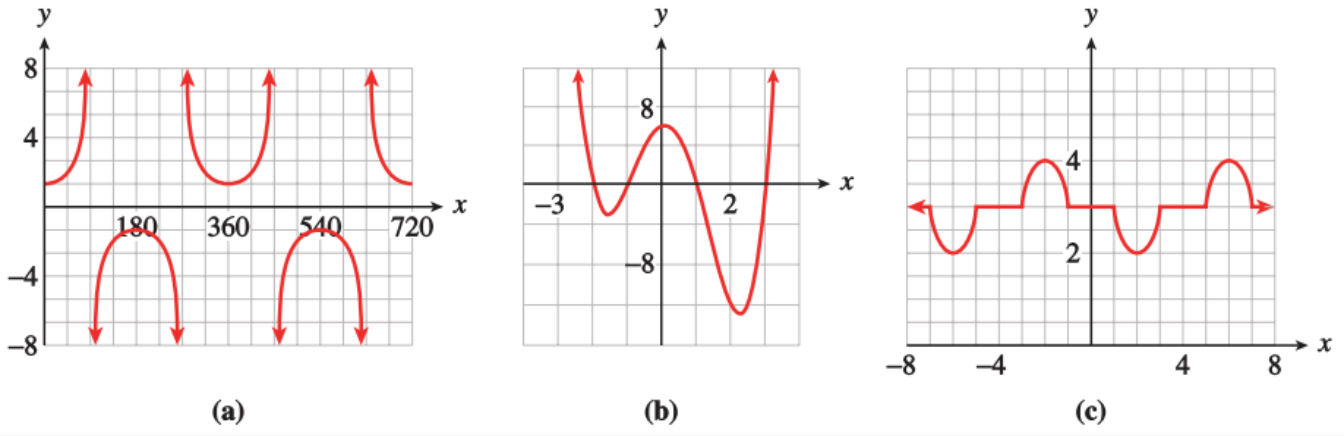


Figure 8.1.11

Solutions available at the end of this section.

Checkpoint 8.1.2

Which of the functions shown below are periodic? If the function is periodic, give its period.

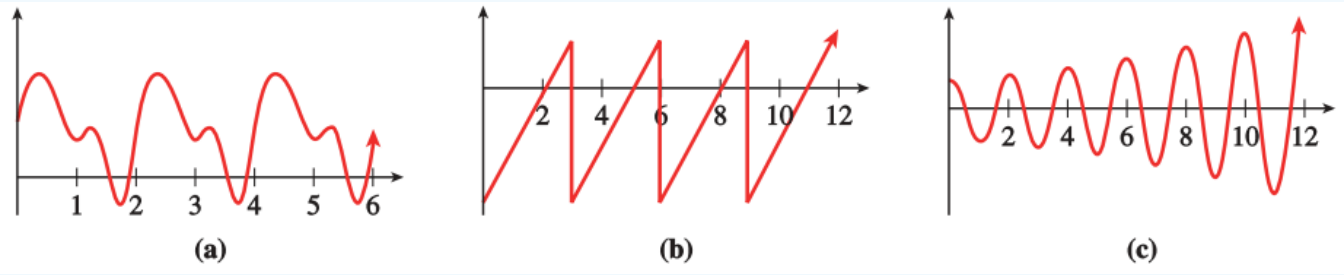


Figure 8.1.12

Answers available at the end of this section.

Example 8.1.3

A patient receives regular doses of medication to maintain a certain level of the drug in his body. After each dose, the patient's body eliminates a certain percentage of the medication before the next dose. The graph shows the amount of the drug, in milliliters, in the patient's body as a function of time in hours.

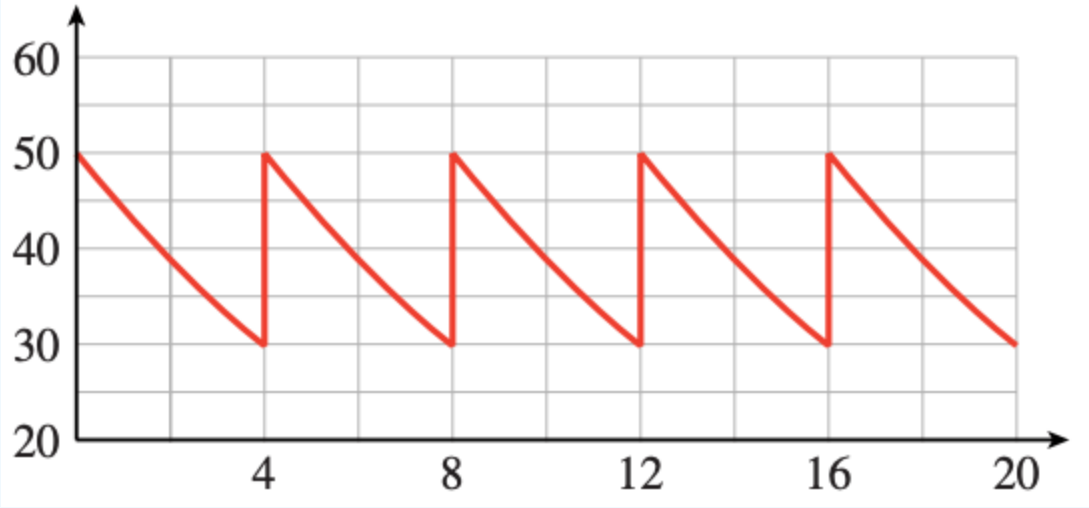


Figure 8.1.13

a. How much of the medication is administered with each dose?
b. How often is the medication administered?
c. What percent of the drug is eliminated from the body between doses?

Solutions available at the end of this section.

Checkpoint 8.1.3

You are sitting on your front porch late one evening, and you see a light coming down the road tracing out the path shown below, with distances in inches. You realize you are seeing a bicycle light fixed on a bike's front wheel.

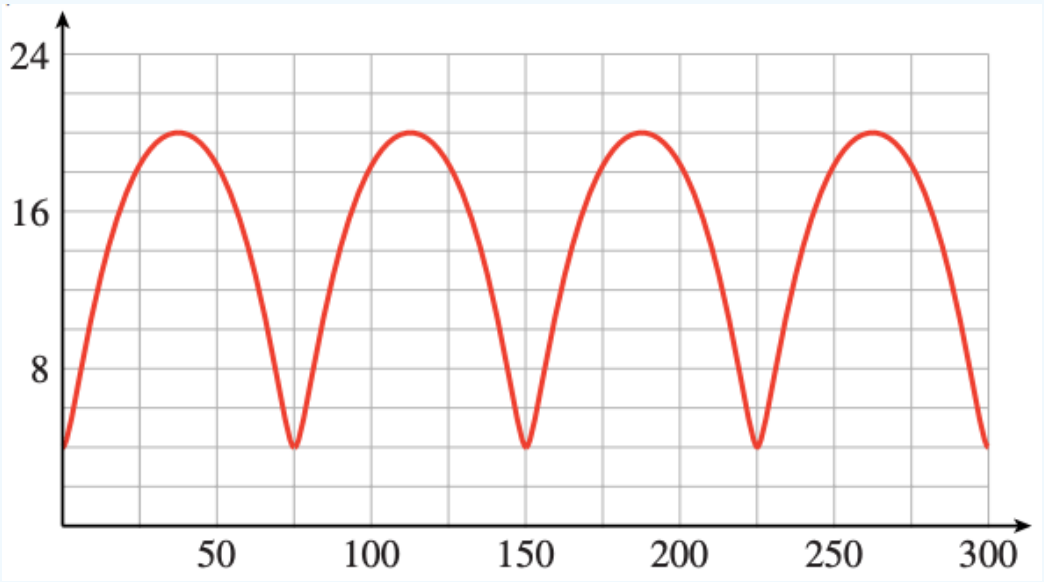


Figure 8.1.14

a. Approximately what is the period of the graph?
b. How far above the ground is the light?
c. What is the diameter of the bicycle wheel?

Answers available at the end of this section.

Amplitude

The sine curve (and, as we shall soon see, the cosine curve) has a feature almost exclusively unique among the graphs of all other functions—**amplitude**.

Definition: Amplitude (of a function)

Let M and m be a function's greatest and least values, respectively. The **amplitude** of the graph of the function is defined to be

$$\text{Amplitude} = \frac{1}{2}|M - m|.$$

That is, the amplitude is half the distance between the highest and lowest values of the function.

Example 8.1.4

Determine the amplitude of each function.

a.

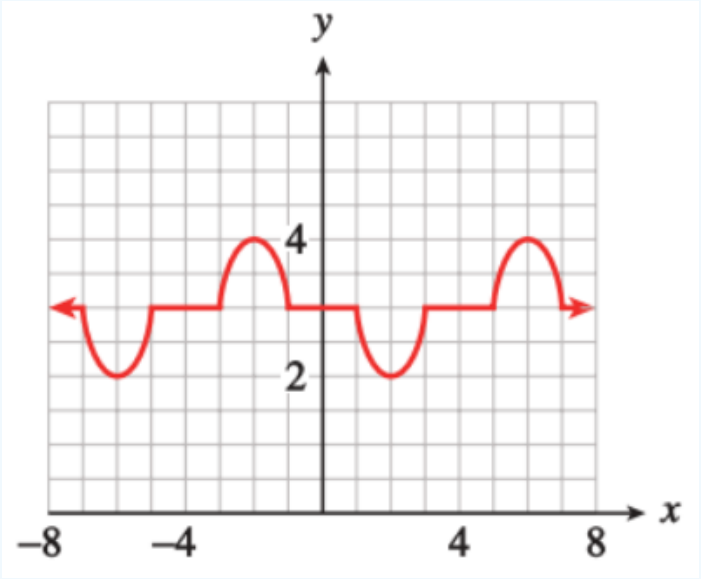


Figure 8.1.15

b.

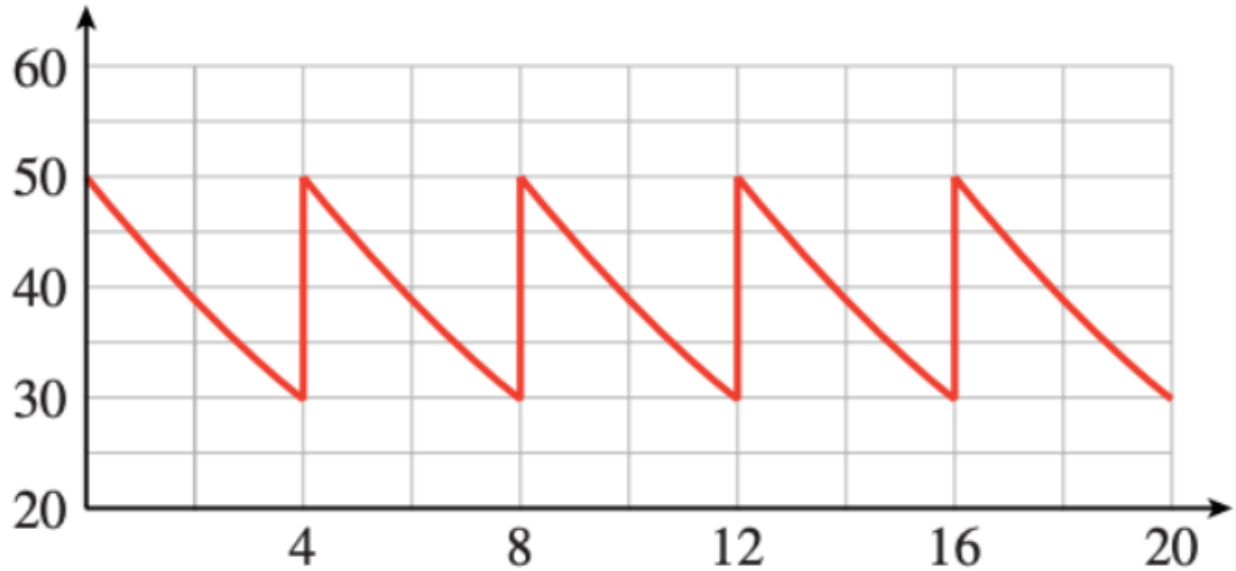


Figure 8.1.16

Solutions available at the end of this section.

Checkpoint 8.1.4

Determine the amplitude and period of the function.

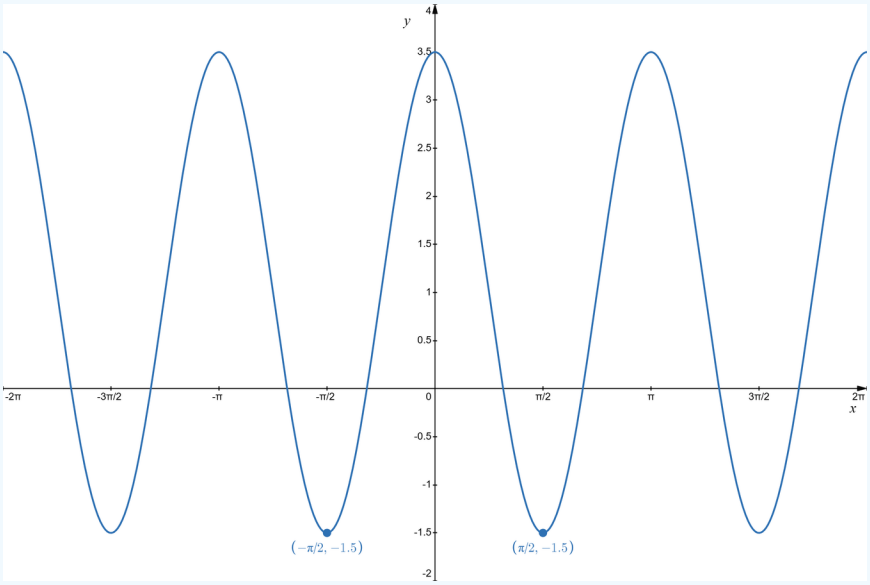


Figure 8.1.17

Answer available at the end of this section.

The concept of amplitude is generally reserved for the sine and cosine functions; however, it occasionally gets referenced when working with other periodic functions.

Theorem: Amplitude of the Sine Function

The amplitude of $f(x) = \sin(x)$ is 1.

Proof available at the end of this section.

Zeros

We know that the graph of $f(x) = \sin(x)$ crosses the x-axis when $\sin(x) = 0$; however, using the intuitive visualization of the sine function being the y-coordinates of points along the edge of the unit circle, we can see (from Figure $8.1.3$, Example $8.1.1$, or either of the Interactive Elements above) that this occurs when $x = 0, \pi, 2\pi, -\pi, -2\pi, \ldots$. The following theorem provides a much better way to write this.

Theorem: Zeros of the Sine Function

The zeros of $f(x) = \sin(x)$ occur at $x = \pi k$, where $k \in^{\text{int}} \mathbb{Z}$.

As a reminder, the notation, $\in^{\text{int}}$, means "belongs to" and the notation, $\mathbb{Z}$, means "the set of all integers." Therefore, as long as $k \in^{\text{int}} \mathbb{Z}$, $x = \pi k$ is shorthand for

$$x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \cdots.$$

We will be using this notation extensively from this point forward in Mathematics.

Graphing the Cosine Function

Now that we have set the foundation for all the concepts we need, our discussion of the cosine function graph will be shorter than that of the sine function. We begin using the worst method for graphing a function—point-plotting.

Point-Plotting Approach

Example $8.1.5$

Sketch a portion of the graph of $y = \cos(x)$ on the interval $[0, 2\pi]$ by making a table of values.

Solution available at the end of this section.

Checkpoint $8.1.5$

Sketch a portion of the graph of $y = \cos(x)$ on the interval $\left[-\frac{3\pi}{2}, \frac{\pi}{2}\right]$ by making a table of values.

Answer available at the end of this section.

Unit Circle Approach

Relating the cosine curve to the unit circle is similar to what we did with the sine curve; however, we must remember that $\cos(t)$ is the x-coordinate of the points on the unit circle. This makes the visual a little more challenging, but you can play with the following Interactive Element (https://www.desmos.com/calculator/rr51fjmchw) to help cement the concept.

Interactive Element: Developing the Graph of the Cosine Function via the Unit Circle

Observation: As the angle (or arc length) t increases, the x-coordinate decreases from 1 to -1, then increases from -1 back to 1.

As before, we show the general cosine curve along with a single cycle (in blue) on the interval $[0, 2\pi]$ in Figure $8.1.20$ below.

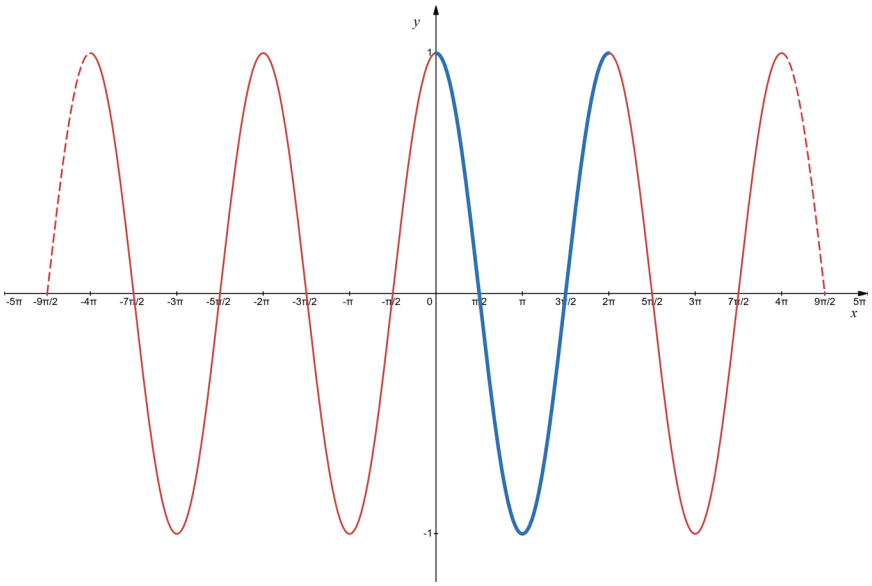


Figure $8.1.20$: $y = \cos(x)$

It is no coincidence that the graph in Figure $8.1.20$ has many characteristics resembling the graph of the sine function (Figure $8.1.7$). The sine and cosine functions are highly related, so their graphs are similar (but not the same). We will explore this similarity later.

Domain, Range, Period, Amplitude, and Zeros

As with the graph of the sine function, the graph of the cosine function reveals the domain and range of the cosine, its period, and the amplitude.

Theorem: Domain and Range of the Cosine Function

The domain of $f(x) = \cos(x)$ is $(-\infty, \infty)$ and the range is $[-1, 1]$.

Theorem: Period of the Cosine Function

The period of $f(x) = \cos(x)$ is 2π. Thus, the natural period of the cosine function is 2π.

Theorem: Principal Cycle of the Cosine Function

The principal cycle of the cosine function is $[0, 2\pi]$.

Theorem: Amplitude of the Cosine Function

The amplitude of $f(x) = \cos(x)$ is 1.

Finally, we know that the graph of $f(x) = \cos(x)$ crosses the x-axis when $\cos(x) = 0$. This happens when the x-coordinates of points along the edge of the unit circle become 0. This occurs when the angular argument of the cosine function is $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$. The following theorem summarizes this result.

Theorem: Zeros of the Cosine Function

The zeros of $f(x) = \cos(x)$ occur at $x = \frac{\pi}{2} + \pi k$, where $k \in \mathbb{Z}$.

There are other ways you could write this set of zeros. For example,

$$x = \frac{\pi}{2} + \pi k = \frac{\pi}{2} + \frac{2\pi k}{2} = \frac{\pi + 2\pi k}{2} = \frac{\pi}{2}(1 + 2k) = (2k+1)\frac{\pi}{2}.$$

I find that students remember the form given in the theorem the best.

Graphing the Tangent Function

The tangent function is the more abstract graph of the fundamental trigonometric functions. Therefore, we will spend much time deriving the graph and discussing how it is generated. In the end, students will gravitate to either the conceptual approach (which uses the graphs of the sine and cosine to help develop the tangent curve) or the unit circle approach (which uses the slope of the terminal side of the angle). Understanding both approaches to graphing the tangent curve is best because both will be needed in future mathematics courses.

Before we get to those "headier" approaches to graphing the tangent function, we start, as we did with the sine curve, by plotting points.

Point-Plotting Approach

As with the sine and cosine functions, we begin by making a table of values and plotting points. If we decide to use the same "test points" to build our table of values, something odd happens.



The fact that the tangent function is undefined at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ should not be surprising. By the Ratio Identity, $\tan(x) = \frac{\sin(x)}{\cos(x)}$, and we have just mentioned that the cosine has zeros at $x = \frac{\pi}{2} + \pi k$. That is, the tangent function encounters division by zero at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.

What does this mean about the graph of the tangent function? If we decide to plot the points in our list where the tangent *is* defined, all we would get is the following graph.

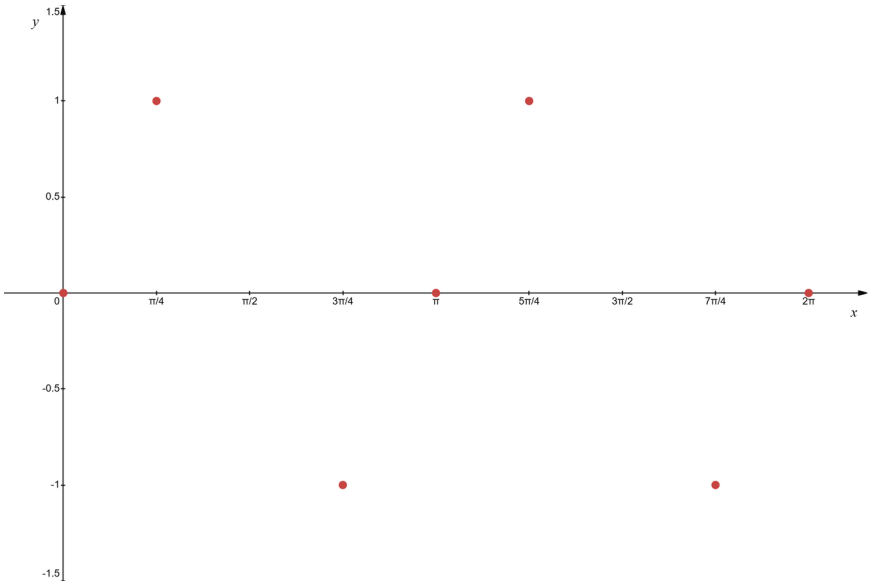


Figure 8.1.21

We are now faced with the question, "How do we connect these points?"

This is one of the many weaknesses of graphing via point-plotting—we are often left guessing how the graph behaves between the points.

In these situations, most people plot more points to see if the true behavior of the graph reveals itself. If we make our table more robust by including every special angle from 0 to 2π, we get the following:



However, as detailed and exhaustive as this might seem, the graphical result is only slightly better.

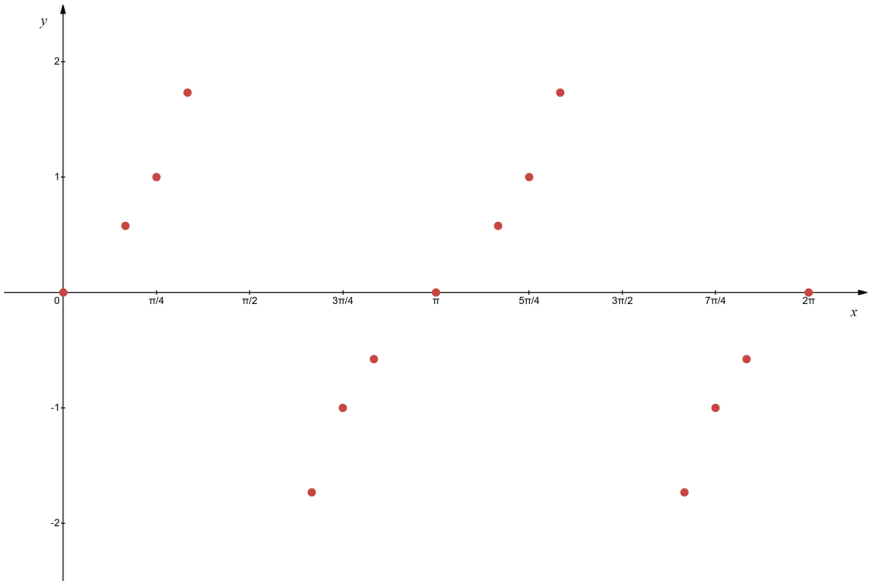


Figure 8.1.22

While Figure 8.1.22 begins to reveal some of the patterns of the tangent graph, it's not quite revealing everything we need.

Therefore, we must throw in the proverbial towel and try a different method of graphing the tangent function.

Unit Circle Approach

From the unit circle definition of the trigonometric functions, if $P(x, y)$ is a point on the unit circle, then $\tan(\theta) = \frac{y}{x}$, where θ is the angle from the positive x-axis to the ray connecting the origin to P (see Figure 8.1.23).

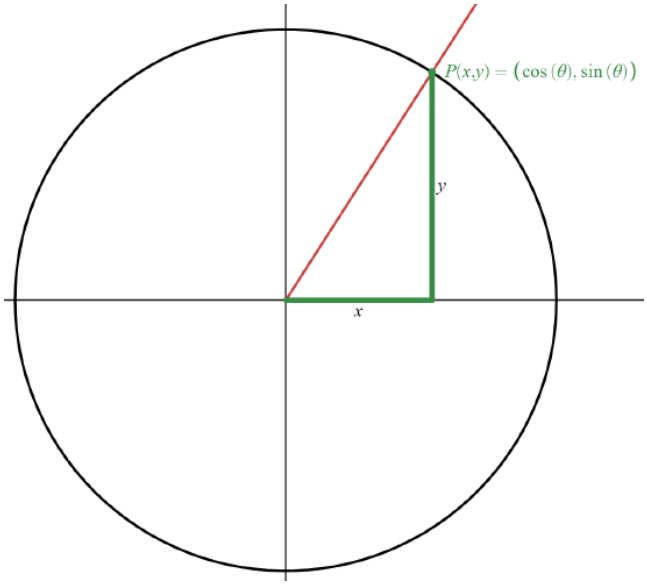


Figure 8.1.23

Therefore, $\tan(\theta)$ is the slope of the ray connecting the origin to P. When $\theta = 0$, this slope is 0. As θ increases from 0, the slope of this ray also increases; however, as θ approaches $\frac{\pi}{2}$, something interesting happens—the slope of the ray increases without bound. Eventually, at $\theta = \frac{\pi}{2}$, the ray becomes vertical; however, we know from Algebra that vertical lines have undefined slopes. As such, this results in a vertical asymptote in our tangent function graph. The following Interactive Element (https://www.desmos.com/calculator/tv71jqbwil) displays the resulting curve. I recommend you spend a while truly trying to understand what is going on here.

Interactive Element

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Desmos Interactive online

Conceptual Approach

To truly understand what is going on with the tangent function graph, we need to understand the nature of a ratio when its denominator is approaching zero. Consider the values of the three functions, $f(x) = \frac{1}{x}$, $g(x) = \frac{13}{x}$, and $h(x) = -\frac{2}{x^2}$, as the values of x approach 0.

$f(x) = \frac{1}{x}$ as $x \to 0^+$	$g(x) = \frac{13}{x}$ as $x \to 0^-$	$h(x) = -\frac{2}{x^2}$ as $x \to 0^-$
$\frac{1}{0.1} = 10$	$\frac{13}{-0.1} = -130$	$-\frac{2}{(-0.1)^2} = -200$
$\frac{1}{0.01} = 100$	$\frac{13}{-0.01} = -1300$	$-\frac{2}{(-0.01)^2} = -20000$

$f(x)=\frac{1}{x}$ as $x\to 0^+$	$g(x)=\frac{13}{x}$ as $x\to 0^-$	$h(x)=-\frac{2}{x^2}$ as $x\to 0^-$
$\frac{1}{0.001}=1000$	$\frac{13}{-0.001}=-13000$	$-\frac{2}{(-0.001)^2}=-2000000$
$\frac{1}{0.0001}=10000$	$\frac{13}{-0.0001}=-130000$	$-\frac{2}{(-0.0001)^2}=-200000000$

Before we continue, it's important to clarify the notations 0^+ and 0^-.

Definition

The notation $x\to a^+$ is read as "x is approaching a from the right side of a," and it means that x is strictly greater than a and is getting closer and closer to a.

The notation $x\to a^-$ is read as "x is approaching a from the left side of a," and it means that x is strictly less than a and is getting closer and closer to a.

In both cases, x will never take on the value of a.

Hence, $x\to 0^+$ means that x is getting close to 0, but is still greater than 0, and $x\to 0^-$ means x is getting close to 0, but is still less than 0.

In all three functions from the table, the denominators are approaching zero. In the first and third columns, the denominators approach 0^+ (zero from the right). In the second column, the denominators approach 0^- (zero from the left). Notice that, in all three cases, the smaller the denominator gets, the larger the value of the fraction becomes. Understanding this property is critical to conceptually understanding vertical asymptotes. I have summarized this result below.

If the numerator is approaching a **nonzero** constant and the denominator is approaching (but never taking on the value of) 0, the ratio $\frac{N(x)}{D(x)}$ approaches either $+\infty$ or $-\infty$. The sign of the result is determined by comparing the signs of the numerator and denominator of the ratio of functions as x is approaching a.

Thus, a vertical asymptote for a function results from what happens as its numerator approaches some nonzero constant while its denominator approaches (but does not become) 0.

We have already mentioned that the tangent function is undefined when cosine is 0 and that this occurs precisely when $x=\frac{\pi}{2}+k\pi$, where $k^{\text{int}}\mathbb{Z}$. If we use the knowledge from the conversation above, we will be able to determine if the tangent has vertical asymptotes at these values (and the behavior of the tangent curve near these vertical asymptotes).

Let's focus our efforts on what happens as the x-values approach $\frac{\pi}{2}$. Since $\tan(x)=\frac{\sin(x)}{\cos(x)}$, we start by sketching both the sine and cosine functions near $x=\frac{\pi}{2}$.

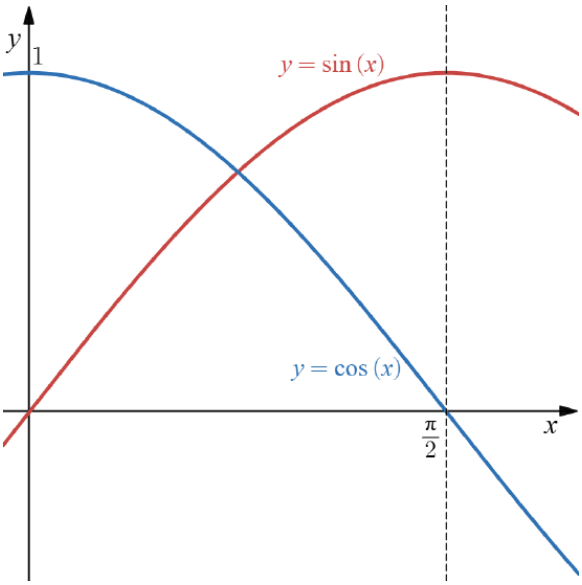


Figure 8.1.24

Figure 8.1.25 shows the values of the sine as $x \to \frac{\pi}{2}^-$.

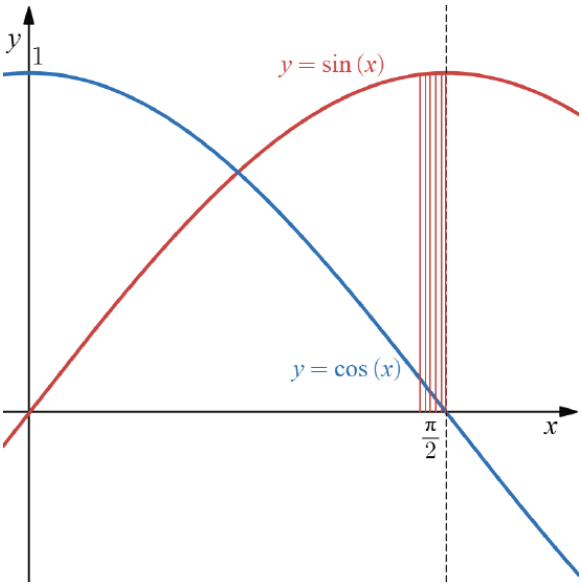


Figure 8.1.25**:** *The values of the sine function as* $x \to \frac{\pi}{2}^-$

Likewise, Figure 8.1.26 shows the values of cosine as $x \to \frac{\pi}{2}^-$.

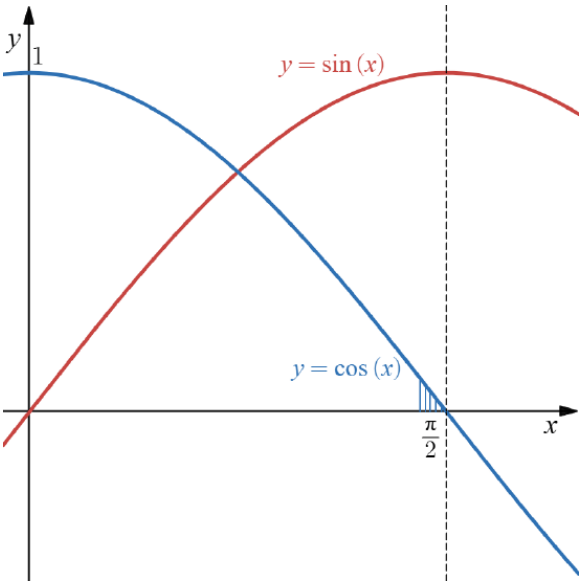


Figure 8.1.26**:** *The values of the cosine function as* $x \to \frac{\pi}{2}^-$

From a graphical investigation, as x approaches $\frac{\pi}{2}$ from the left,

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \text{ approaches } \frac{1}{0^+} \text{ which, in turn, approaches } +\infty. \tag{8.1.1}$$

We now consider what happens as $x \to \frac{\pi}{2}^+$. Consider Figures 8.1.27 and 8.1.28.

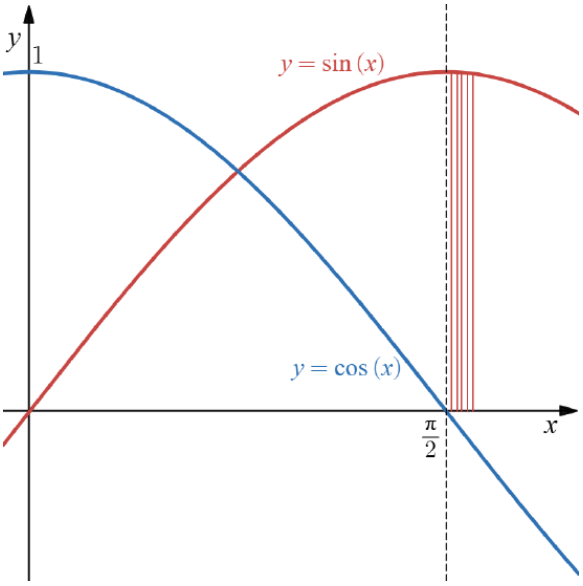


Figure 8.1.27**:** *The values of the sine function as* $x \to \frac{\pi}{2}^+$

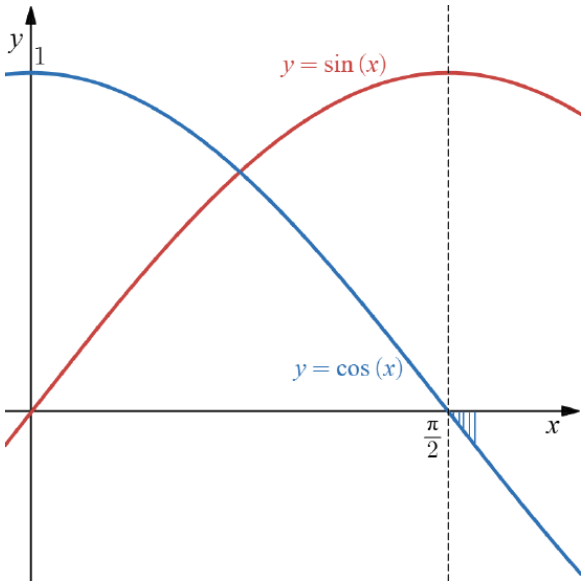


Figure 8.1.28**:** *The values of the cosine function as* $x \to \frac{\pi}{2}^+$

In this case, we can see the sine function is approaching 1, but the cosine is approaching 0 from the negative side of 0. Hence,

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \text{ approaches } \frac{1}{0^-} \text{ which, in turn, approaches } -\infty. \tag{8.1.2}$$

Combining our point-plotting in Figure 8.1.22 with the results from 8.1.1 and 8.1.2, we get the following graph.

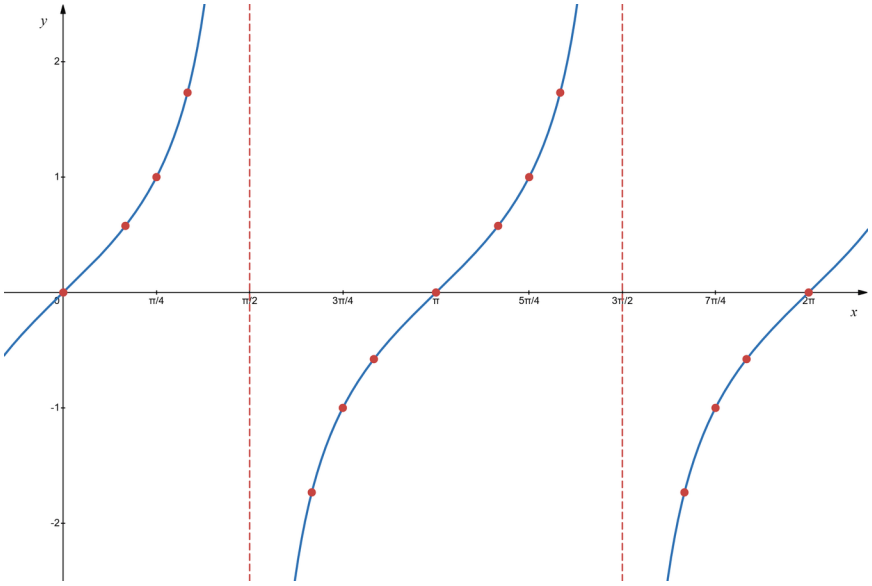


Figure 8.1.29

Notice that we took the liberty to draw in the vertical asymptote at $x = \frac{3\pi}{2}$.

We now have enough knowledge to graph the *entire* tangent curve. Figure 8.1.30 shows the general tangent curve along with a single cycle (in blue) on the interval $[0, \pi]$.

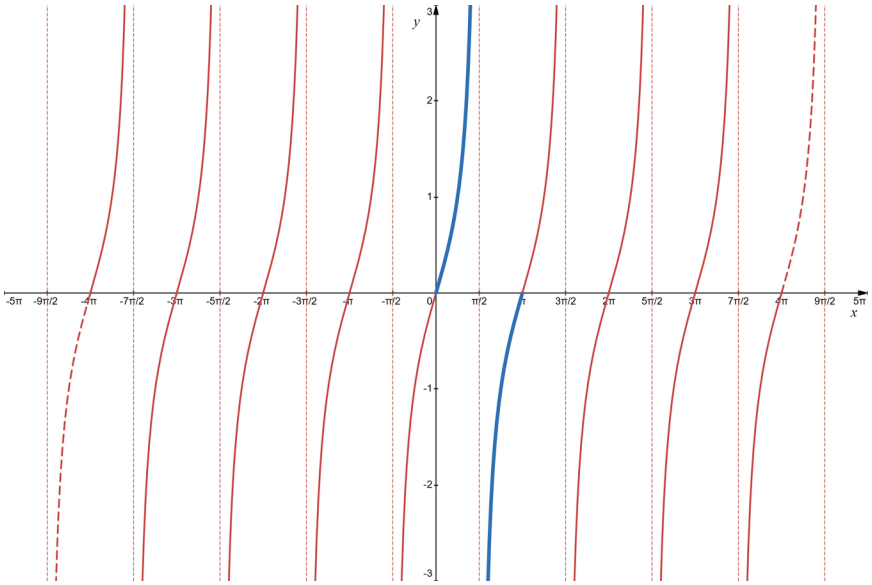


Figure 8.1.30**:** $y = \tan(x)$

Domain, Range, Period, and Zeros

There are a few essential features to mention about the tangent curve. First, the pattern repeats every π units. Therefore, the period of the tangent function is π. The domain is missing the values of x for which the cosine is zero, and the range is all real numbers. Finally, the zeros of the tangent function occur when the sine function has zeros. All of these facts are summarized below.

Theorem: Domain and Range of the Tangent Function

The domain of the tangent function is $\{x \mid x \neq \frac{\pi}{2} + \pi k\}$, where $k^{\text{int}}\mathbb{Z}$. The range is $(-\infty, \infty)$.

Theorem: Period of the Tangent Function

The period of $f(x) = \tan(x)$ is π. Thus, the natural period of the tangent function is π.

Theorem: Principal Cycle of the Tangent Function

The principal cycle of the tangent function is $[0, \pi]$.

Theorem: Zeros of the Tangent Function

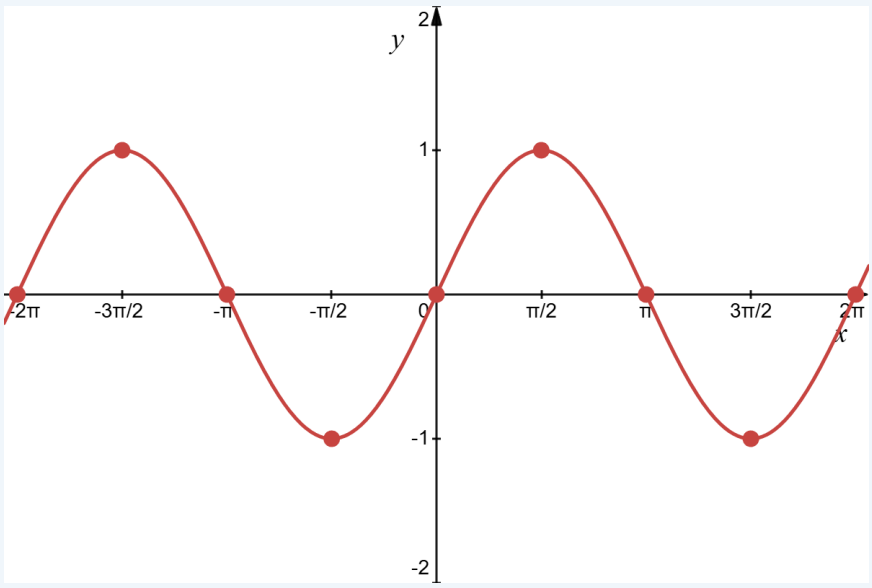
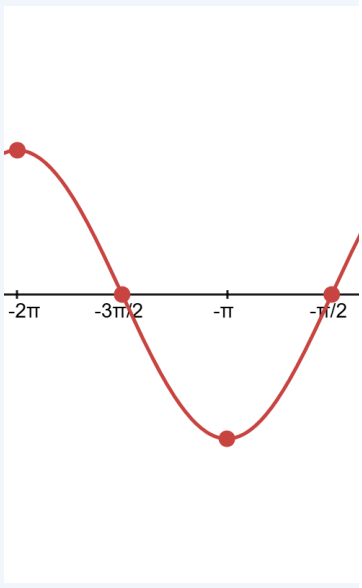
The zeros of the tangent function occur at $x = \pi k$, where $k^{\text{int}}\mathbb{Z}$.

In Figure 8.1.30, we highlighted the single period on the interval $[0, \pi]$. Most textbooks use the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$; however, our choice of highlighted interval will serve an important purpose in the next section.

Also, note that we didn't state the amplitude of the tangent function. This is because the tangent function does not have an amplitude. To have an amplitude, a function must have a greatest and least value; however, the range of the tangent is $(-\infty, \infty)$. Therefore, it is **unbounded** (i.e., it does not have a greatest or least value). As such, it cannot have an amplitude.

Summary of the Graphs of the Fundamental Trigonometric Functions

Before we continue, let's summarize what we have discovered about the graphs of the fundamental trigonometric functions.

	Sine Function	C
Graph		
Domain	$(-\infty, \infty)$	
Range	$[-1, 1]$	
Amplitude	1	
Natural Period	2π	
Principal Cycle	$[0, 2\pi]$	
Zeros	$x = \pi k,\ k^{\text{int}}\mathbb{Z}$	x

Finally, when asked to graph the sine or cosine functions, we only need to know their values at the quadrantal angles (these correspond to the "dots" on the graphs of the sine and cosine curves above). This concept will lead to a graphing method for transformations of trigonometric functions in our next section.

Symmetry Identities

Now that we have the base graphs of the fundamental trigonometric functions, it's natural to discuss their graphs' symmetric behavior.

Recall from Algebra the following definition.

Definition: Even and Odd Functions

The function f is called an **even function** if for every input x in the domain of f

$$f(-x) = f(x).$$

The function f is called an **odd function** if for every input x in the domain of f

$$f(-x) = -f(x).$$

Figure 8.1.31 is the key to proving the symmetric behavior of the trigonometric functions.

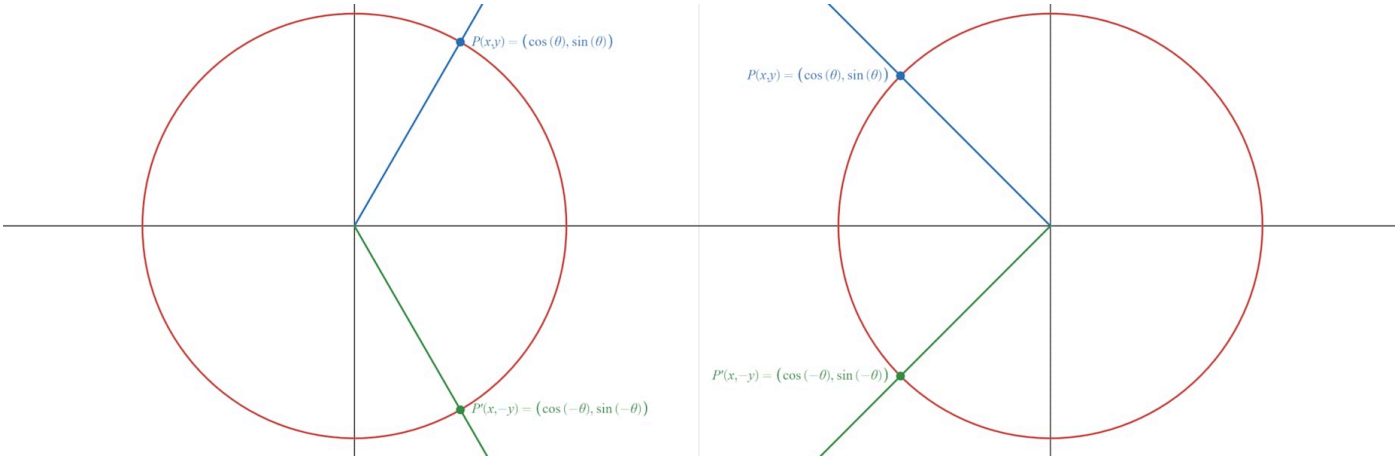


Figure 8.1.31

Note that, in both graphs in Figure 8.1.31, the point P on the terminal side of θ is a reflection across the x-axis of the point P' on the terminal side of $-\theta$. Hence, each point's x-coordinate remains the same, but the y-coordinates are opposite in sign. Therefore, no matter what quadrant in which θ terminates,

$$x = \cos(\theta) \qquad \text{and} \qquad x = \cos(-\theta).$$

Thus,

$$\cos(\theta) = \cos(-\theta),$$

and so cosine is an even function.

Moreover,

$$y = \sin(\theta) \qquad \text{and} \qquad -y = \sin(-\theta).$$

The equation on the right side can be solved for y to become $y = -\sin(-\theta)$. Hence, $\sin(\theta) = -\sin(-\theta)$. Stated more clearly,

$$\sin(-\theta) = -\sin(\theta).$$

Thus, sine is an odd function.

From these two results, the proofs of the symmetric behaviors of the remaining trigonometric functions are much easier. For example,

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\frac{\sin(\theta)}{\cos(\theta)} = -\tan(\theta).$$

Thus, tangent is an odd function. The rest of the proofs are left as homework exercises.

These results are summarized in the following theorem.

Theorem: Symmetry Identities

The sine, cosecant, tangent, and cotangent are odd functions. The cosine and secant are even functions.

Of course, we could have guessed that the sine and tangent are odd functions because their graphs are symmetric about the origin. Similarly, we could have guessed the cosine is an even function because its graph is symmetric about the y-axis.

Advice: Memorize These Identities

The Symmetry Identities are needed throughout the rest of Trigonometry and Calculus. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Example 8.1.6

Is the function even, odd, or neither?

$$f(x) = x \sin\left(x^3 - x\right) + x^2$$

Solution available at the end of this section.

Checkpoint 8.1.6

Determine if the function is even, odd, or neither.

$$g(x) = x^3 \cos(x) - \tan(x)$$

Answer available at the end of this section.

Finally, we can take advantage of the symmetric behavior of a trigonometric function to speed up evaluations.

Example 8.1.7

Evaluate each function using symmetry.

a. $\sec\left(-\frac{5\pi}{6}\right)$
b. $\sin\left(-\frac{5\pi}{4}\right)$

Solutions available at the end of this section.

Checkpoint 8.1.7

Evaluate $\cot\left(-\frac{2\pi}{3}\right)$ using symmetry.

Answer available at the end of this section.

Evaluating trigonometric functions using symmetry can be thought of as an alternative, not a replacement, to how we have been evaluating them up to this point. I recommend practicing using symmetry for evaluation because it can speed up computations.

Activity

Activity: Solving Basic Trigonometric Equations

1.

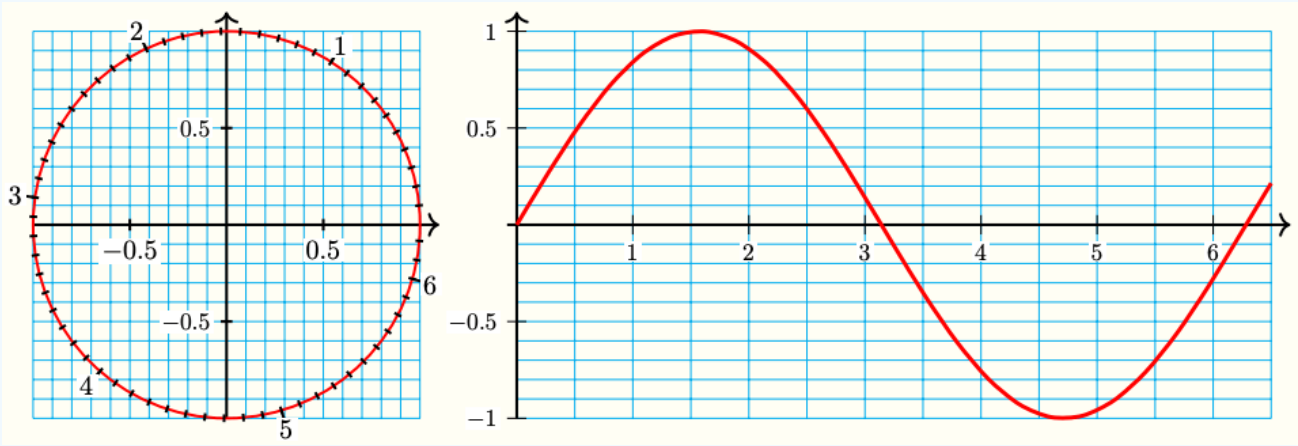


Figure 8.1.33

a. Use the graph of $y = \sin x$ to estimate two solutions of the equation $\sin(x) = 0.65$. Show your solutions on the graph.
b. Use the unit circle to estimate two solutions of the equation $\sin(x) = 0.65$. Show your solutions on the circle.
c. Use the graph of $y = \sin x$ to estimate two solutions of the equation $\sin(x) = -0.2$. Show your solutions on the graph.
d. Use the unit circle to estimate two solutions of the equation $\sin(x) = -0.2$. Show your solutions on the circle.

2.

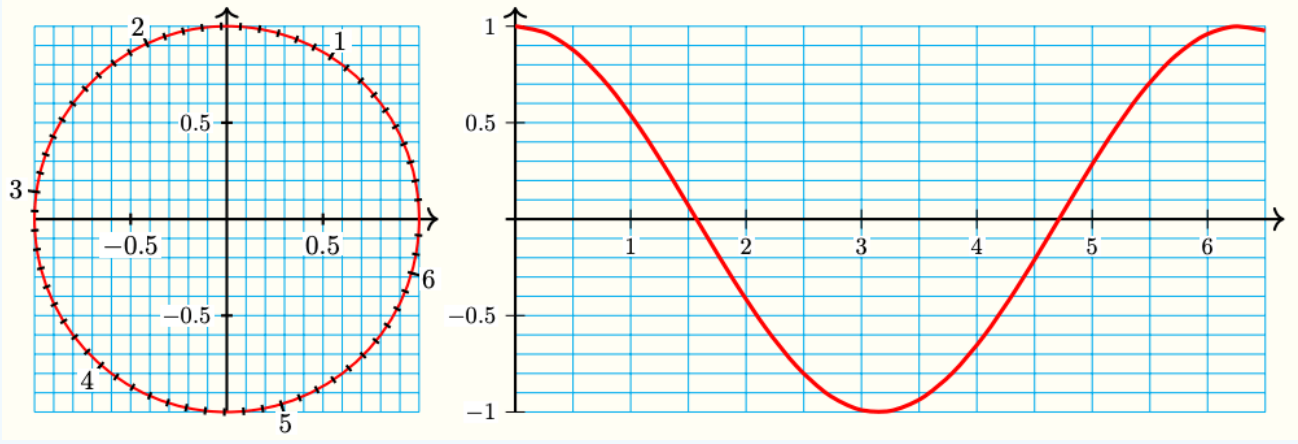


Figure 8.1.34

a. Use the graph of $y = \cos x$ to estimate two solutions of the equation $\cos(x) = 0.15$. Show your solutions on the graph.
b. Use the unit circle to estimate two solutions of the equation $\cos(x) = 0.15$. Show your solutions on the circle.
c. Use the graph of $y = \cos x$ to estimate two solutions of the equation $\cos(x) = -0.4$. Show your solutions on the graph.
d. Use the unit circle to estimate two solutions of the equation $\cos(x) = -0.4$. Show your solutions on the circle.

Solving Trigonometric Equations Using Graphs

Now that we have a basic understanding of the graphs of the fundamental trigonometric functions, we can begin to visualize the solutions to basic trigonometric equations.

Example 8.1.8

Use a graph to find all solutions of $\cos(t) = -0.62$ between 0 and 2π.

Answer available at the end of this section.

Caution: Graphs Are a Terrible Method for Solving Equations

Example $8.1.8$ showcases the issue with using a graph to solve *any* equation—you get very rough approximations. In truth, the solutions to $\cos(t) = -0.62$ on the interval $[0, 2\pi]$ are approximately 2.2395 and 4.0436 (rounded to four decimal places). There is virtually no way to gain this level of accuracy using a hand-drawn graph.

An interesting observation comes from the fact that the sine and cosine curves oscillate forever in both directions. Had we not restricted our answers in Example $8.1.8$ to lie between 0 and 2π, we would have encountered an infinite number of times where the cosine curve intersects the line $y = -0.62$. Thus, the equation $\cos(t) = -0.62$ has an infinite number of solutions when considering all real numbers instead of restricting ourselves to the interval $[0, 2\pi]$.

Checkpoint $8.1.8$

Use a graph to find the two angles between 0 and 2π that satisfy $\sin(t) = -0.85$.

Answer available at the end of this section.

Time-Sensitive Advice: Drop Deadline

Within the Los Rios Community College District, the last day you can drop a class is the Monday (or, sometimes, the Tuesday) of the thirteenth week of the semester. After this date, you will no longer be able to drop a class for that semester and the only grades you can receive in the classes you remain in are A, B, C, D, or F.

Will you still get a "W?"

Dropping any class between the third week and this final drop deadline will result in a "W" notation on your record for that class.

What does this mean?

Once you withdraw (W) or fail (D or F) a course three times within the Los Rios Community College District (CRC, SCC, FLC, or ARC), you are never allowed to take that course again at any of the colleges within the district. Thus, if you wanted to continue through the math sequence, you would need to take this course outside the Los Rios Community College District.

Success in College: Memorization Learning Preferences

> Students may be inclined to approach their courses in one of three ways. Those with a reproducing orientation tend to take a surface approach to learning, relying on rote memorization and mechanical formula substitution and making little or no effort to understand the material being taught. Those with a meaning orientation tend to adopt a deep approach, probing and questioning and exploring the limits of applicability of new material. Those with an achieving orientation tend to use a strategic approach, doing whatever is necessary to get the highest grade they can, taking a surface approach if that suffices and a deep approach when necessary.
>
> Source: R. Felder, R. Brent , Journal of Engineering Education, 94 (1), 57-72 (2005).

1. Students have different motivations for going to math class. What do you think the main motivation is for students who have a reproducing orientation and who take a surface approach?
2. What do you think is the main motivation for students who have a meaning orientation and who take a deep approach?

Answers

Section Preview

I chose to only introduce the graphs of the sine, cosine, and tangent because these are the foundational graphs for the remainder of trigonometry. The graphs of the remaining trigonometric functions are held off until the very end of the chapter. Additionally, this section introduces the symmetry of these functions. Therefore, we can spend some time doing proofs involving symmetry and these functions.

Section Preview

- **Sets and Numbers**
  - Set-Builder Notation: Needed for the domain of tangent.
- **Rational Functions**
  - Asymptotes: Needed to understand what a vertical asymptote is before encountering it in the tangent graph.

Section Preview

- Sketch the base graphs of the fundamental trigonometric functions.
- Determine whether the graph of an equation involving trigonometric functions is even, odd, or neither.
- Evaluate a trigonometric function using the Symmetry Identities.

Example 8.1.1

To graph $f(x) = \sin(x)$ without relating to the unit circle, we need to list out several values of the sine function.



Plotting these values and connecting them with a smooth curve, we get the following:

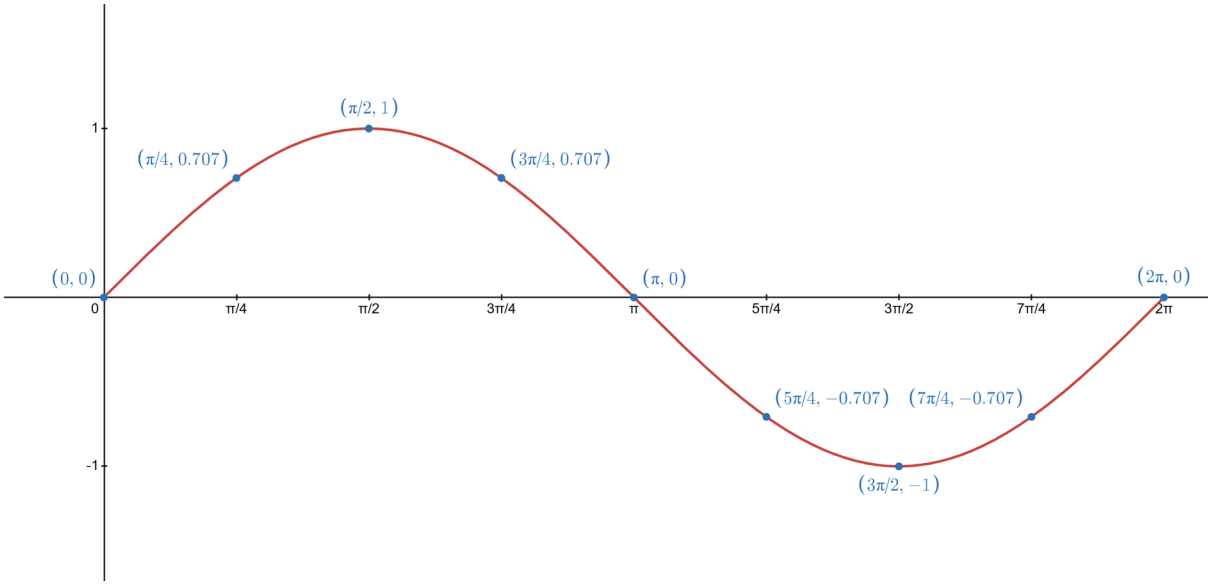


Figure 8.1.4

Checkpoint 8.1.1

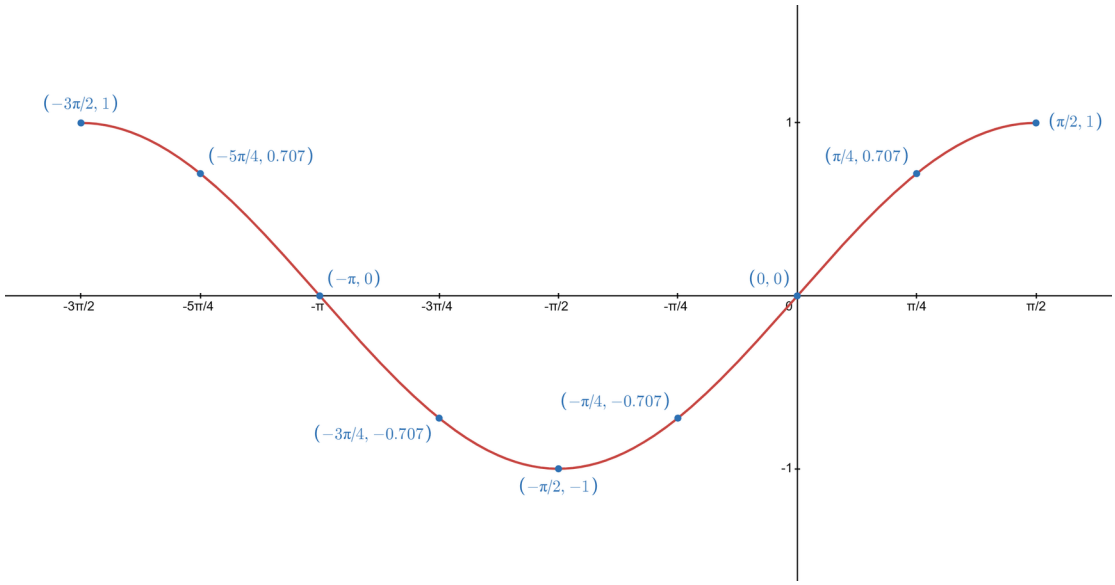


Figure 8.1.5

Example 8.1.2

a. This graph is periodic with period 360.
b. This graph is not periodic.
c. This graph is periodic with period 8.

Checkpoint 8.1.2

a. Period: 2
b. Period: 3
c. Not periodic

Example 8.1.3

a. The medication level increases from 30 ml to 50 ml at each cycle of the graph, so 20 ml of medication is administered at each dose.
b. The medication level peaks sharply every four hours after administering each new dose.
c. The medication level declines by 20 ml between doses, or $\frac{20}{50} = 0.6$, or 60%.

Checkpoint 8.1.3

a. 75 in
b. 4 in
c. 24 in

Example 8.1.4

a. The maximum value for the graph of this function is $M = 4$ and the minimum is $m = 2$. Therefore, the amplitude of the graph is

$$\text{Amplitude} = \frac{1}{2}|4 - 2| = 1.$$

b. The maximum value is $M = 50$ and the minimum is $m = 30$. Thus,

$$\text{Amplitude} = \frac{1}{2}|50 - 30| = 10.$$

Checkpoint 8.1.4

Period: π
Amplitude: 2.5

Theorem: Amplitude of the Sine Function

The maximum value of $f(x) = \sin(x)$ is $M = 1$ and the minimum value is $m = -1$. Hence,

$$\text{Amplitude} = \frac{1}{2}|1 - (-1)| = 1.$$

Example 8.1.5



Plotting these values and connecting them with a smooth curve, we get the following:

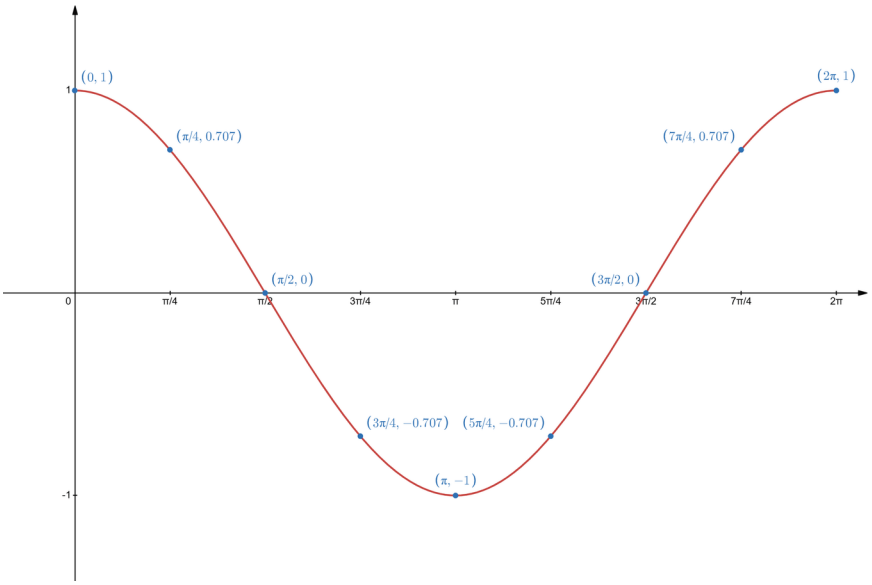


Figure 8.1.18

Checkpoint 8.1.5

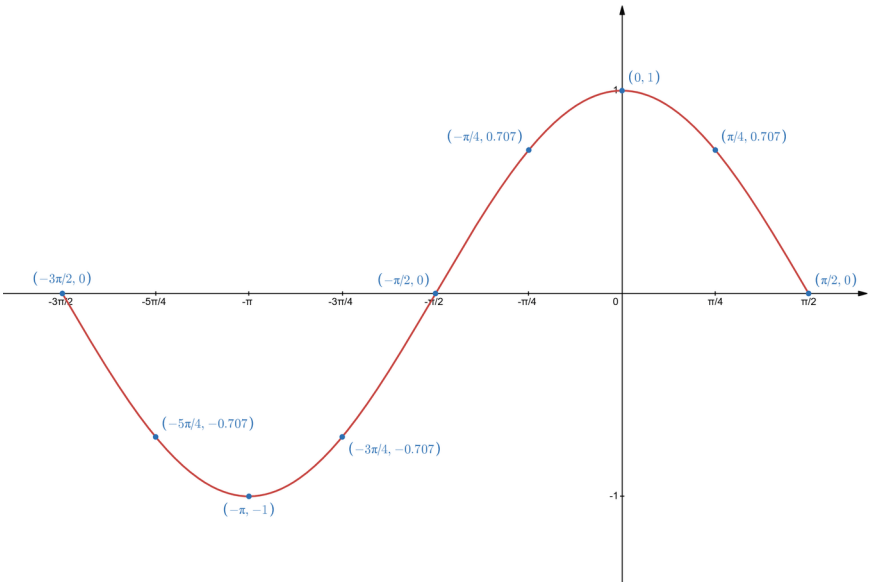


Figure 8.1.19

Example 8.1.6

$$\begin{aligned} f(-x) &= (-x)\sin\left((-x)^3-(-x)\right)+(-x)^2 \\ &= -x\sin\left(-x^3+x\right)+x^2 \\ &= -x\sin\left(-(x^3-x)\right)+x^2 \\ &= x\sin\left(x^3-x\right)+x^2 \qquad \text{(sine is an odd function)} \\ &= f(x) \end{aligned}$$

Thus, f is an even function.

Checkpoint 8.1.6

Odd

Example 8.1.7

a.

$$\begin{aligned}
\sec\left(-\frac{5\pi}{6}\right) &= \sec\left(\frac{5\pi}{6}\right) && \text{(secant is even)}\\
&= \frac{1}{\cos\left(\frac{5\pi}{6}\right)} && \text{(Reciprocal Identity)}\\
&= \frac{1}{-\cos\left(\frac{\pi}{6}\right)} && \text{(reference angle and quadrant)}\\
&= -\frac{1}{\sqrt{3}/2}\\
&= -\frac{2}{\sqrt{3}}
\end{aligned}$$

b.

$$\begin{aligned}
\sin\left(-\frac{5\pi}{4}\right) &= -\sin\left(\frac{5\pi}{4}\right) && \text{(sine is odd)}\\
&= \sin\left(\frac{\pi}{4}\right) && \text{(reference angle and quadrant)}\\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

Checkpoint 8.1.7

$\frac{1}{\sqrt{3}}$

Example 8.1.8

Figure 8.1.32 shows a graph of $y = \cos(t)$, with t in radians, and the horizontal line $y = -0.62$. The solutions of the equation are the t-coordinates of the intersection points, and these *appear* to be about $t = 2.25$ and slightly over 4.0, perhaps 4.05. You can check that these values roughly satisfy the equation; it is difficult to read the graph with any greater accuracy.

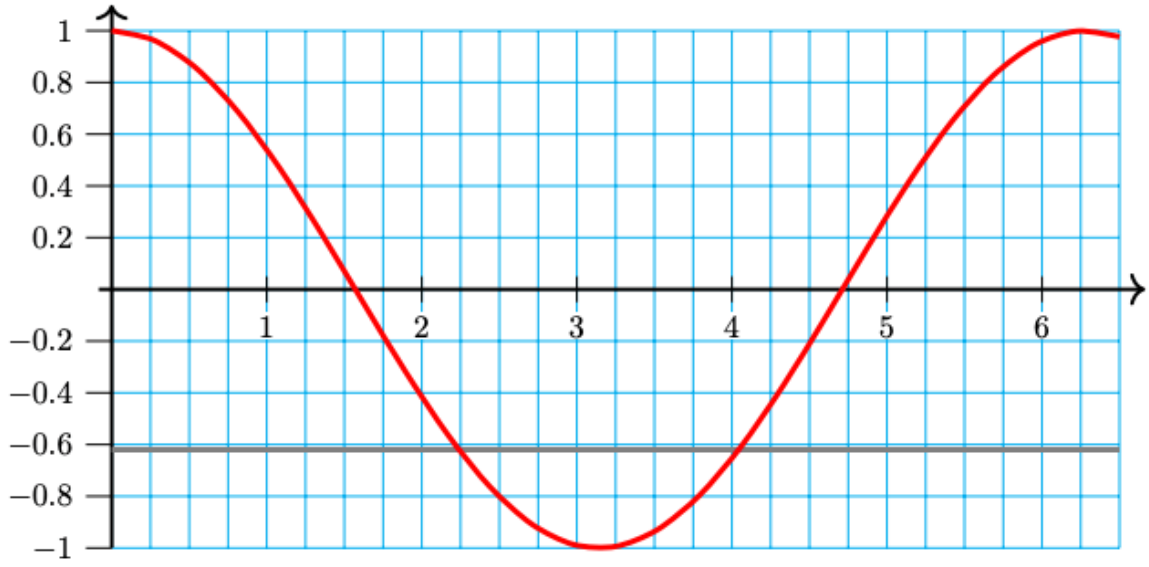


Figure 8.1.32

Checkpoint 8.1.8

Approximately 4.14 and 5.25

This page titled 8.1: Base Graphs of the Fundamental Trigonometric Functions is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

8.1.1: Resources and Key Concepts

Resources

Videos

- Graphs of the Sine and Cosine Functions
  - Trigonometric Graphs - Review of Graphs of Sine and Cosine Functions
- Graphs of the Tangent Function
  - Graphs of the Other Trigonometric Functions - Deriving the Base Graphs for Tangent and Cotangent

Key Concepts

Definitions

- **Fundamental Trigonometric Functions:** The sine, cosine, and tangent functions.
- **Periodic Motion:** Movement that repeats along a path in a regular and predictable manner over a specific time interval.
- **Periodic Function:** A function that models periodic motion by repeating its values at regular intervals.
- **Sine Curve (Sinusoidal):** A smooth, wave-like curve that oscillates periodically and can be derived from either the sine or cosine function.
- **Period:** For a function $f(x)$, the period is the smallest positive number k such that $f(x+k)=f(x)$ for all x in the domain of f.
- **Principal Cycle (of a trigonometric function):** The portion of the graph of a trigonometric function on the interval starting at $x=0$ and ending at $x=p$, where p is the period of the function.
- **Amplitude (of a function):** Half the distance between the function's maximum (M) and minimum (m) values, calculated as $\frac{1}{2}|M-m|$.
- **Even Function:** A function f is even if $f(-x)=f(x)$ for every x in its domain. Its graph is symmetric with respect to the y-axis.
- **Odd Function:** A function f is odd if $f(-x)=-f(x)$ for every x in its domain. Its graph is symmetric with respect to the origin.

Theorems

- **Domain and Range of the Sine Function:** The domain of $f(x)=\sin(x)$ is $(-\infty,\infty)$ and the range is $[-1,1]$.
- **Period of the Sine Function:** The period of $f(x)=\sin(x)$ is 2π.
- **Principal Cycle of the Sine Function:** The principal cycle of the sine function is $[0,2\pi]$.
- **Amplitude of the Sine Function:** The amplitude of $f(x)=\sin(x)$ is 1.
- **Zeros of the Sine Function:** The zeros of $f(x)=\sin(x)$ occur at $x=\pi k$, where $k^{\text{int}}\mathbb{Z}$.
- **Domain and Range of the Cosine Function:** The domain of $f(x)=\cos(x)$ is $(-\infty,\infty)$ and the range is $[-1,1]$.
- **Period of the Cosine Function:** The period of $f(x)=\cos(x)$ is 2π.
- **Principal Cycle of the Cosine Function:** The principal cycle of the cosine function is $[0,2\pi]$.
- **Amplitude of the Cosine Function:** The amplitude of $f(x)=\cos(x)$ is 1.
- **Zeros of the Cosine Function:** The zeros of $f(x)=\cos(x)$ occur at $x=\frac{\pi}{2}+\pi k$, where $k^{\text{int}}\mathbb{Z}$.
- **Domain and Range of the Tangent Function:** The domain of the tangent function is $\{x|x\neq\frac{\pi}{2}+\pi k\}$, where $k^{\text{int}}\mathbb{Z}$. The range is $(-\infty,\infty)$.
- **Period of the Tangent Function:** The period of $f(x)=\tan(x)$ is π.
- **Principal Cycle of the Tangent Function:** The principal cycle of the tangent function is $[0,\pi]$.
- **Zeros of the Tangent Function:** The zeros of the tangent function occur at $x=\pi k$, where $k^{\text{int}}\mathbb{Z}$.
- **Symmetry Identities:** The sine, cosecant, tangent, and cotangent functions are odd. The cosine and secant functions are even.

Common Mistakes

- **Graphing in Degrees:** From this point forward, graphs of trigonometric functions will use radians for the horizontal axis, not degrees. This is critical for maintaining proper scale and for applications in calculus.
- **Connecting Plotted Points with Straight Lines:** The graphs of trigonometric functions are smooth curves. Never connect plotted points with straight line segments.
- **Confusing the Argument of a Function with a Coordinate:** The input variable for a function like $y = \sin(x)$ is a dummy variable representing an angle or real number. It should not be confused with the x-coordinate of a point on the unit circle.
- **Using Graphs to Solve Equations:** Using a graph to find the solution to an equation provides only a rough approximation and is not a valid method for finding exact solutions.
- **Using Graphs as Proofs:** A graph cannot be used to prove an equation is an identity, as it may be deceiving. However, a graph can be used to show an equation is *not* an identity if the graphs of the two sides are different.

This page titled 8.1.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

8.1.2: Homework

Reading Questions

1. What are the three fundamental trigonometric functions?
2. What is a periodic function, and what does its "period" represent?
3. What is the period of the sine function? What is the period of the tangent function?
4. What is the "amplitude" of a function?
5. What is the amplitude of the cosine function? Why does the tangent function not have an amplitude?
6. What is the domain and range of the sine function?
7. What is the domain and range of the tangent function?
8. What is the "principal cycle" of the sine function?
9. At what values of x does the graph of $y = \cos(x)$ have x-intercepts?
10. At what values of x does the graph of $y = \tan(x)$ have vertical asymptotes?
11. What does it mean for a function to be "even"? Which of the fundamental trigonometric functions are even?
12. What does it mean for a function to be "odd"? Which of the fundamental trigonometric functions are odd?

Homework

Vocabulary Check

1. Which trigonometric functions are called the *fundamental* trigonometric functions?
2. The smallest distance (along the x-axis) for which a function starts a repeating pattern (that continues forever) is called the ___ of the function.
3. The ___ for a function is found by taking one-half the sum of the function's maximum and minimum values.
4. If the graph of a function is symmetric about the origin, we say that the function is ___.

Concept Check

5. List the values from 0 to 2π by multiples of $\frac{\pi}{4}$.
6. List the values from 0 to 2π by multiples of $\frac{\pi}{6}$.
7. List several ways in which the graph of $y = \tan(\theta)$ is different from the graphs of $y = \sin(\theta)$ and $y = \cos(\theta)$.
8. Sketch two examples of a function with period 8: one that is sinusoidal and one that is not.

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

9. The amplitude of a function is the vertical distance between the maximum and minimum values (if these exist).
10. $\tan(\theta + 180^\circ) = \tan\theta$ for any value of θ.
11. If $f(x + p) = f(x)$ for all x, then the constant p is called the period of the function.
12. The tangent function has period π. It is undefined at odd multiples of $\frac{\pi}{2}$ and is increasing on each interval of its domain.
13. Functions that have graphs shaped like sines or cosines are called sinusoidal.
14. The cosine graph looks just like the sine graph except flipped upside down.

15. The period of a sine or cosine function is the distance between horizontal intercepts.

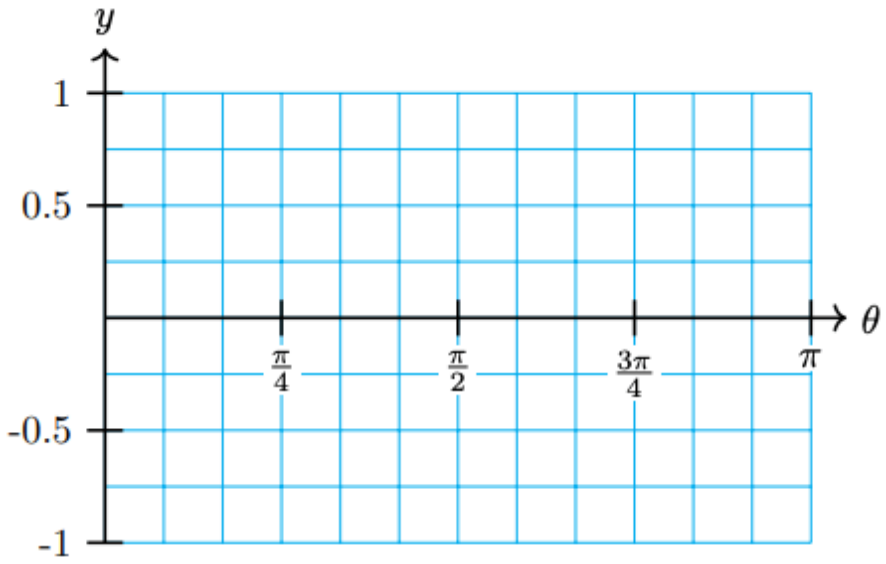
16. Periodic functions are used to model phenomena that exhibit cyclical behavior.

Basic Skills

17. a. Use your calculator to complete the table of values. Round values to hundredths.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$\cos\theta$													

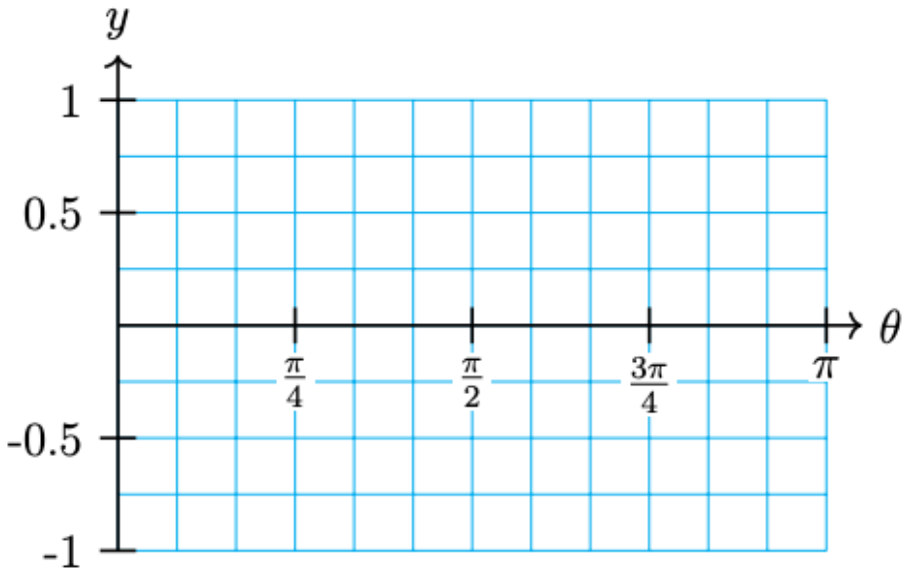
b. Sketch a graph of $y = \cos\theta$ on the grid.



18. a. Use your calculator to complete the table of values. Round values to hundredths.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$\sin\theta$												

b. Sketch a graph of $y = \sin\theta$ on the grid.



19. a. Sketch a graph of $y = \cos x$ using the guide points in the table below.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$					

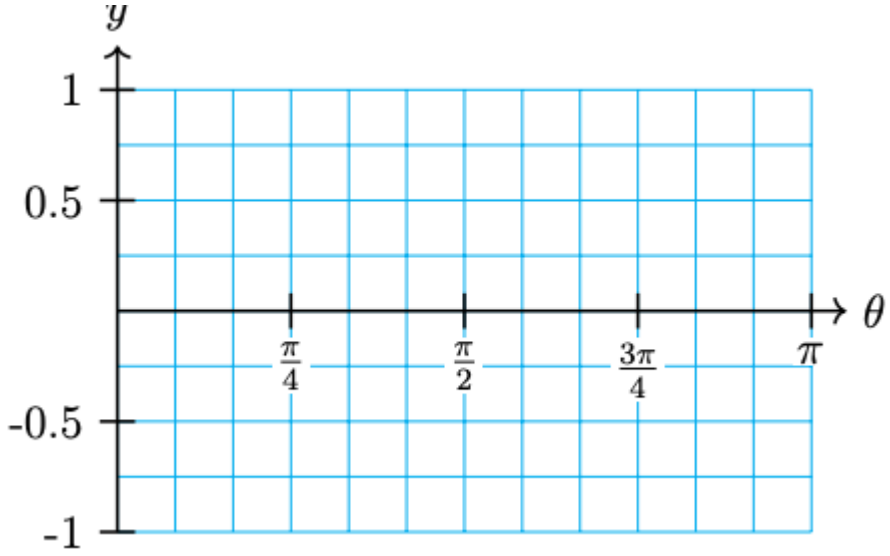
b. State the domain and range of the function $y = \cos x$.

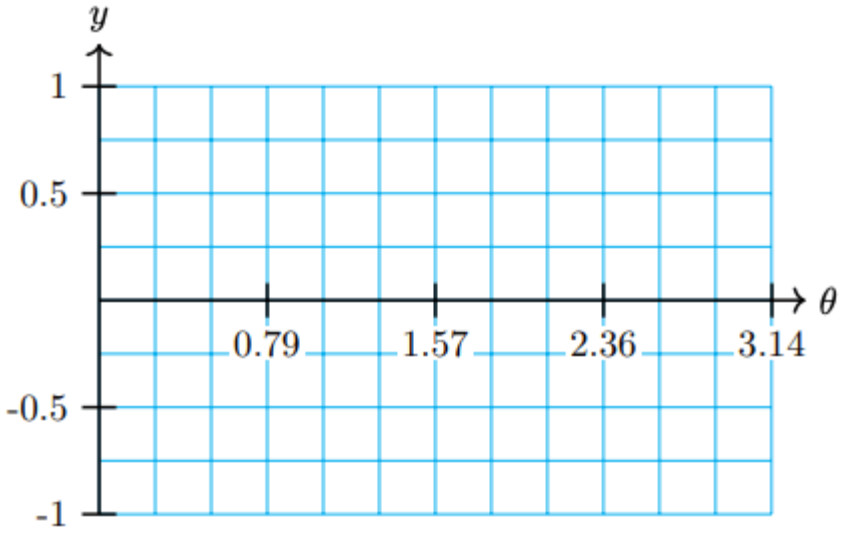
20. a. Sketch a graph of $y = \tan x$ using the guide points in the table below.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$\cos x$					

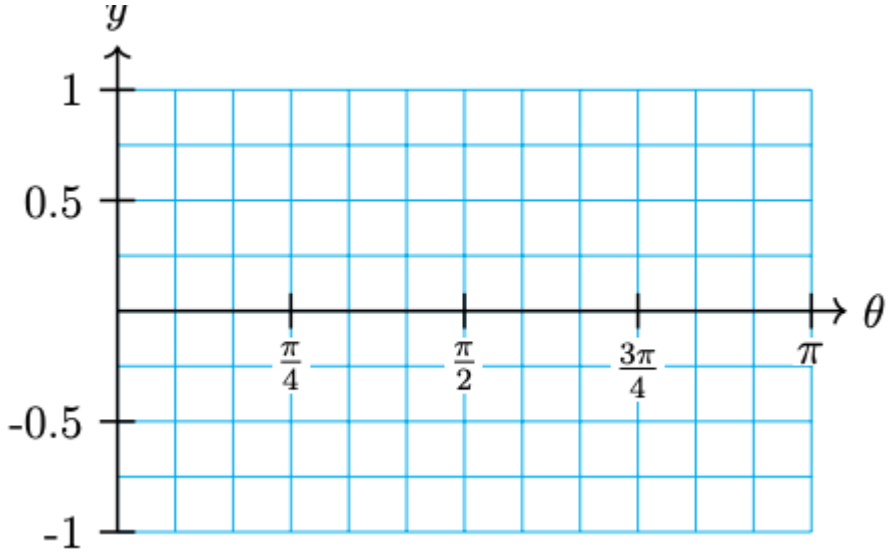
b. State the domain and range of the function $y = \tan x$.

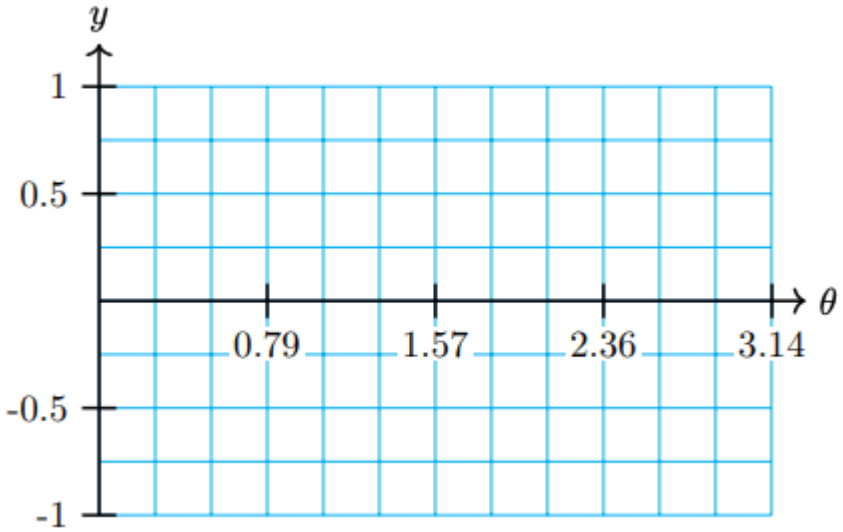
21. Sketch a graph of $y = \sin x$ on each grid.





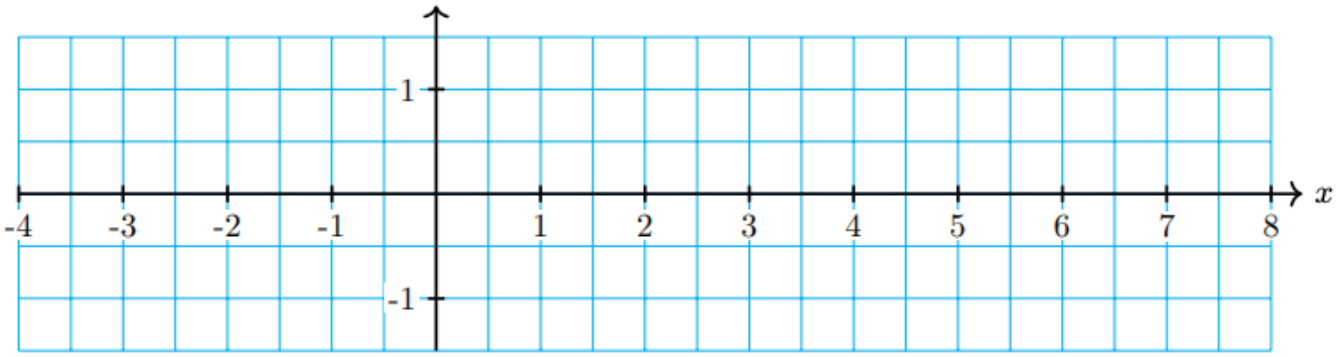
22. Sketch a graph of $y = \cos x$ on each grid.





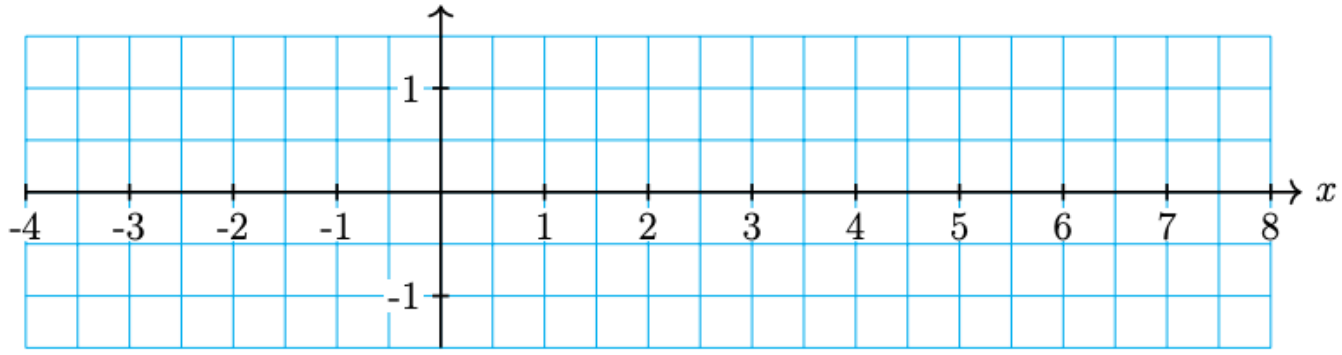
23. a. Sketch a graph of $f(x) = \sin x$ where x is a real number.

b. State the domain and range of $f(x) = \sin x$.



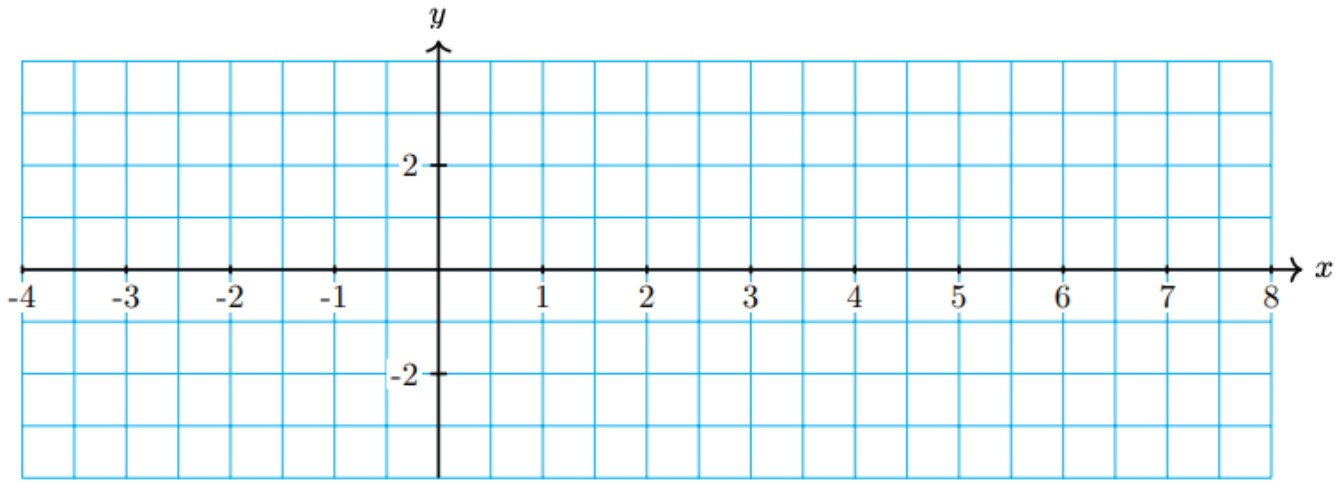
24. a. Sketch a graph of $g(x) = \cos x$ where x is a real number.

b. State the domain and range of $g(x) = \cos x$.

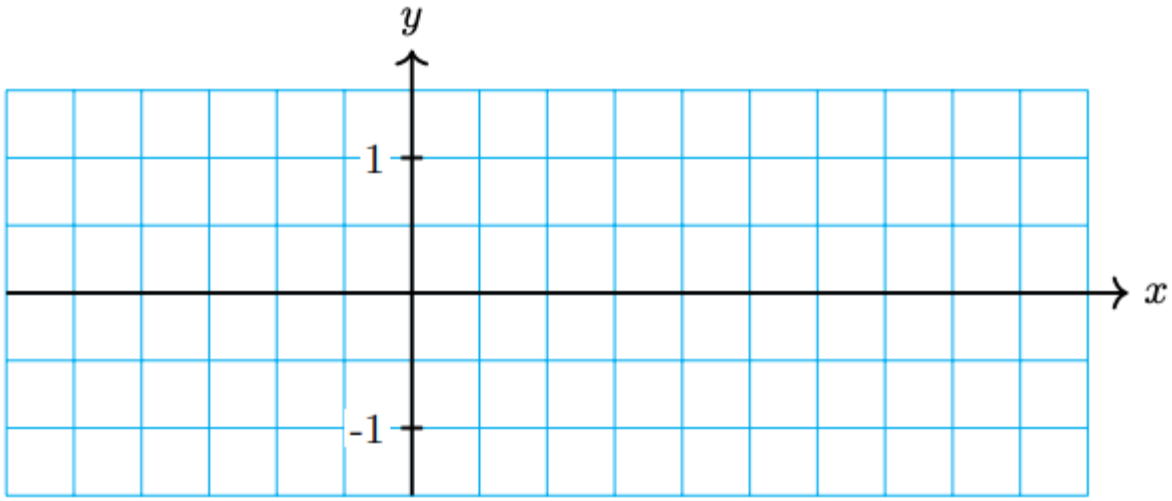


25. a. Sketch a graph of $h(x) = \tan x$ where x is a real number.

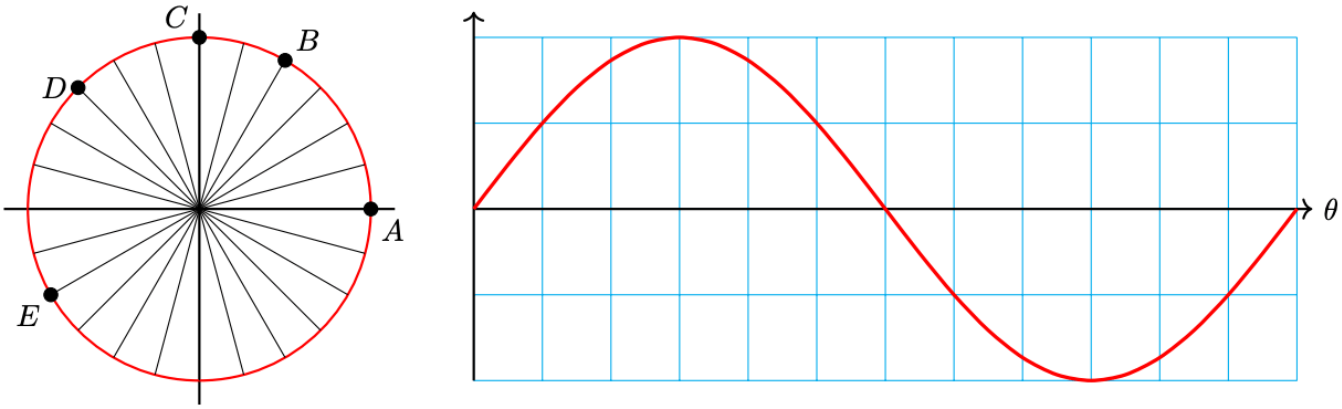
b. State the domain and range of $h(x) = \tan x$.



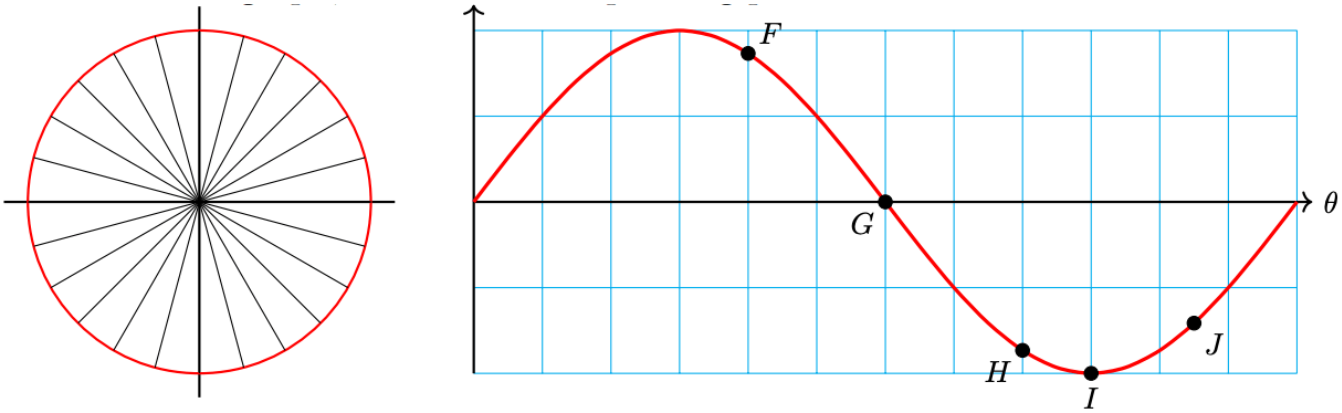
26. Sketch $f(x) = \cos x$ and $g(x) = \sin x$ on the same grid.



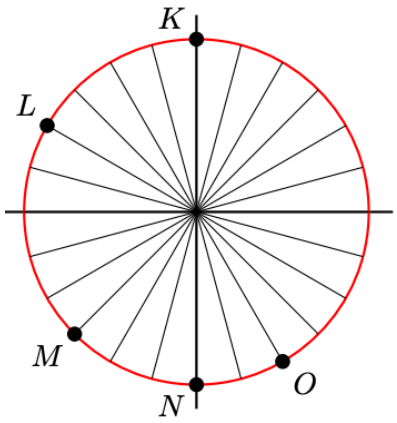
27. The graph shows your height as a function of angle as you ride the Ferris wheel. For each location A - E on the Ferris wheel, mark the corresponding point on the graph.

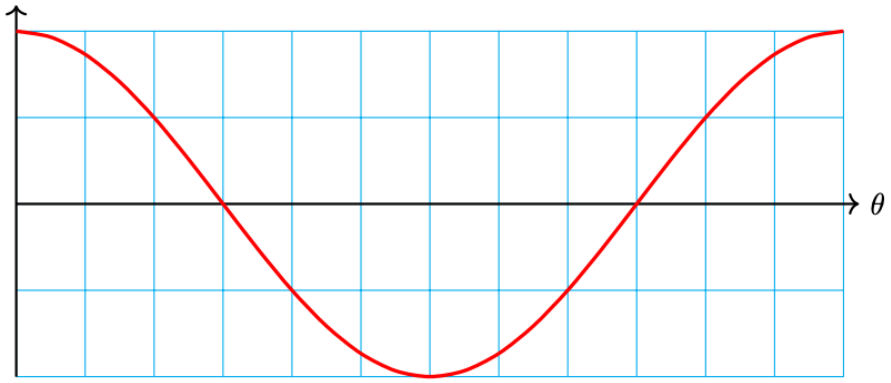


28. The graph shows your height as a function of angle as you ride the Ferris wheel. For each location F - J on the graph, mark the corresponding point on the Ferris wheel.

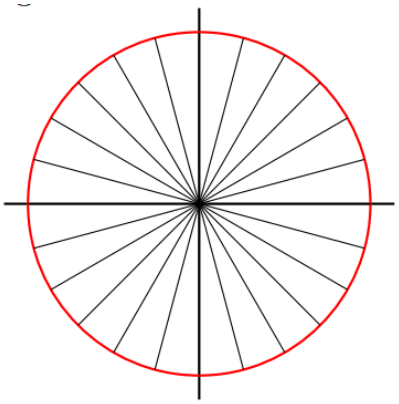


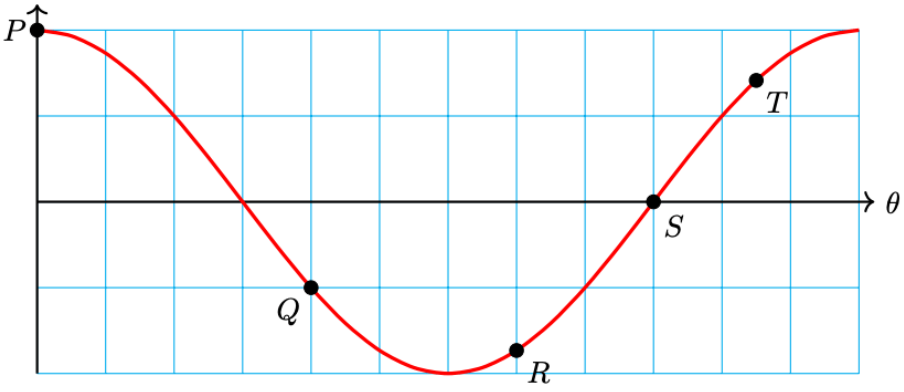
29. The graph shows the horizontal displacement of your foot from the center of the chain gear as you pedal a bicycle. For each location K - O on the chain gear, mark the corresponding point on the graph.



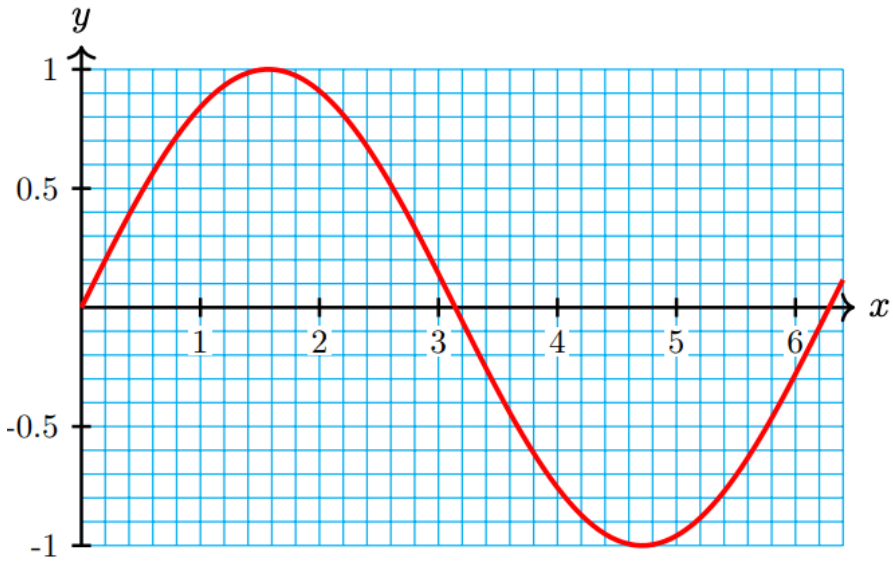


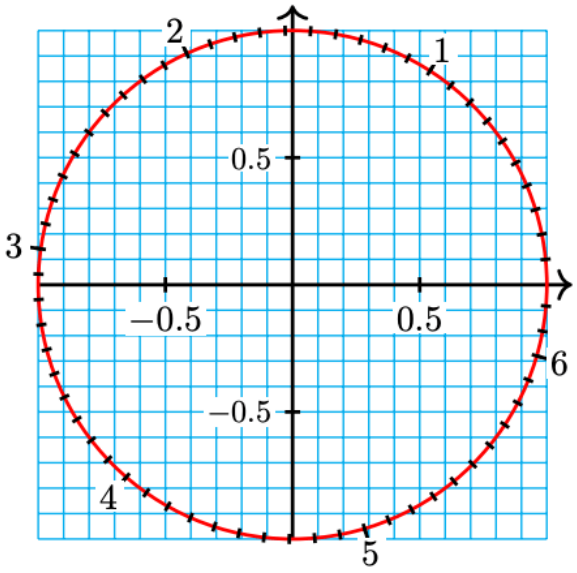
30. The graph shows the horizontal displacement of your foot from the center of the chain gear as you pedal a bicycle. For each location P - T on the graph, mark the corresponding point on the chain gear.





For the following exercises, use the figures below. Show your solutions on the graphs.





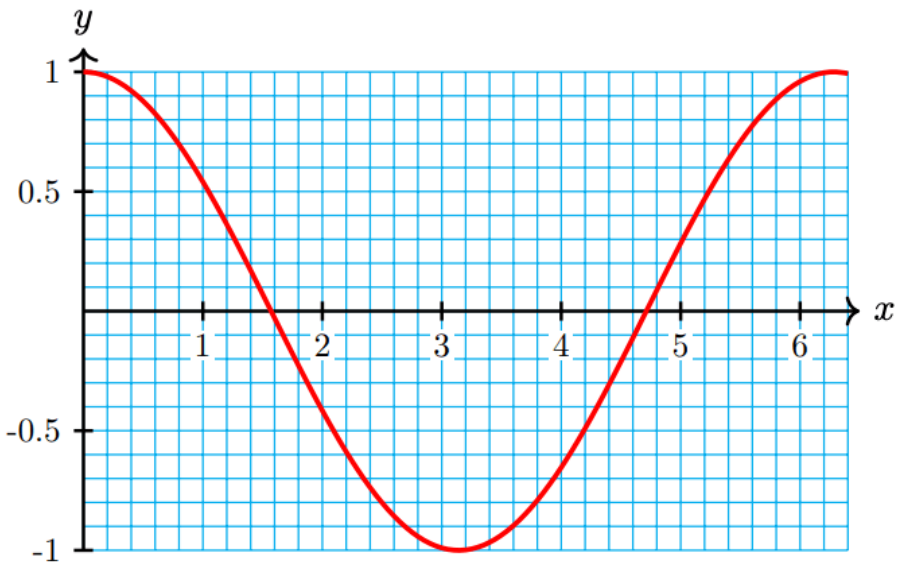
31. a. Use the graph of $y = \sin x$ to estimate two solutions of the equation $\sin x = 0.65$.

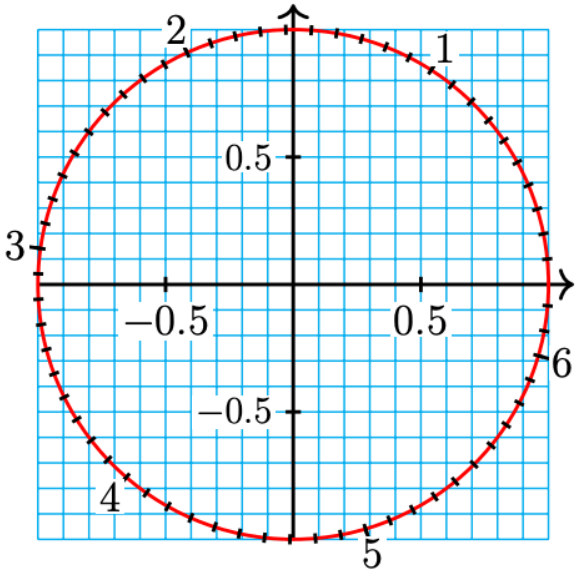
    b. Use the unit circle to estimate two solutions of the equation $\sin x = 0.35$.

32. a. Use the graph of $y = \sin x$ to estimate two solutions of the equation $\sin x = -0.2$.

    b. Use the unit circle to estimate two solutions of the equation $\sin x = -0.6$.

For the following exercises, use the figures below. Show your solutions on the graphs.





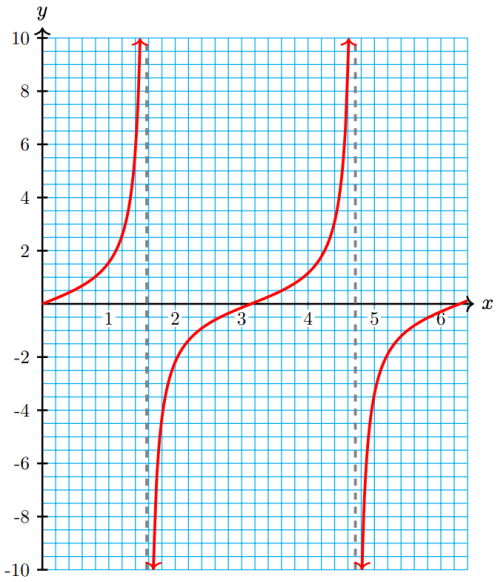
33. a. Use the graph of $y = \cos x$ to estimate two solutions of the equation $\cos x = -0.4$.

    b. Use the unit circle to estimate two solutions of the equation $\cos x = -0.8$.

34. a. Use the graph of $y = \cos x$ to estimate two solutions of the equation $\cos x = 0.15$.

    b. Use the unit circle to estimate two solutions of the equation $\cos x = 0.55$.

For the following exercises, use the graph of $y = \tan x$ to estimate two solutions to the equation.



35. $\tan x = 4$

36. $\tan x = 7$

37. $\tan x = -0.5$

38. $\tan x = -2.5$

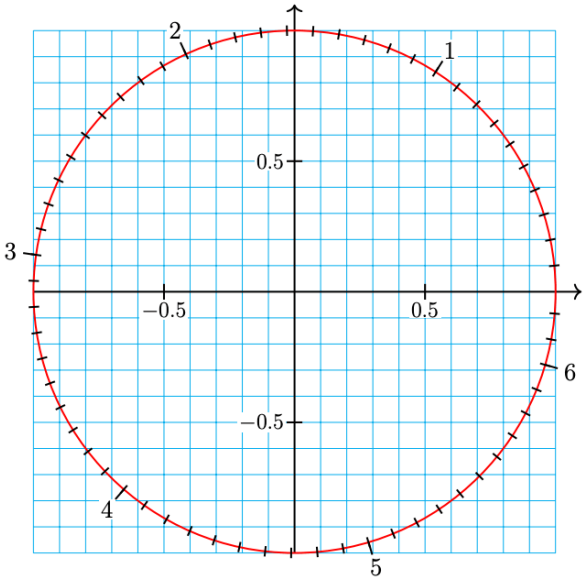
39. $\sin x = 5.5 \cos x$

40. $\sin x = -4.5 \cos x$

41. $-2 \sin x = 3 \cos x$

42. $2 \sin x = \cos x$

For the following exercises, find all solutions between 0 and 2π. Round to two decimal places. Sketch your solutions on a unit circle.



43. $\cos t = 0.74$

44. $\sin t = 0.58$

45. $\tan t = 1.6$

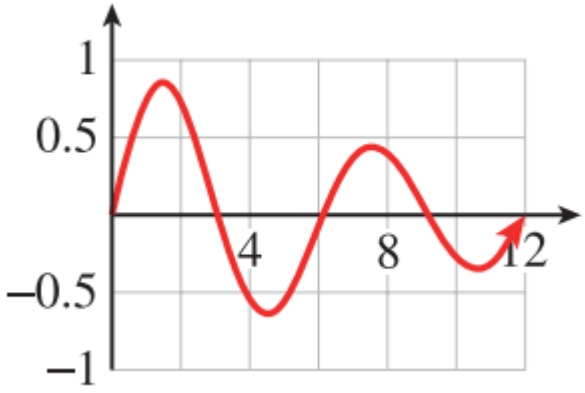
46. $\tan x = -0.6$

47. $\sin x = -0.72$

48. $\cos x = -0.48$

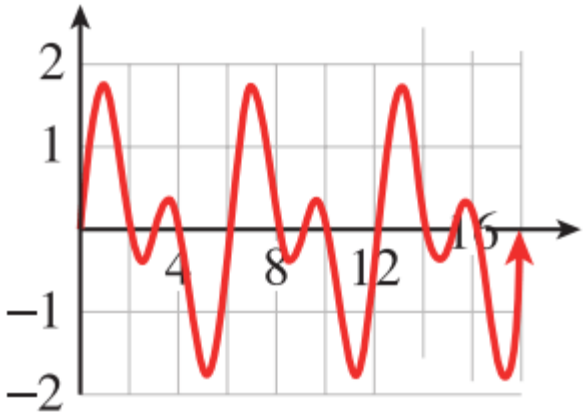
Which of the graphs in the following exercises are periodic? If the graph is periodic, give its period.

49.

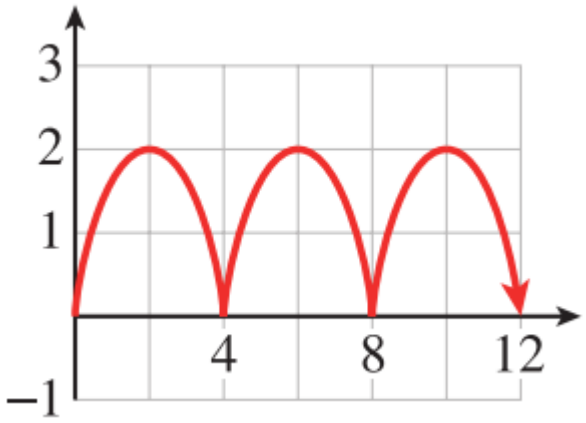


50.

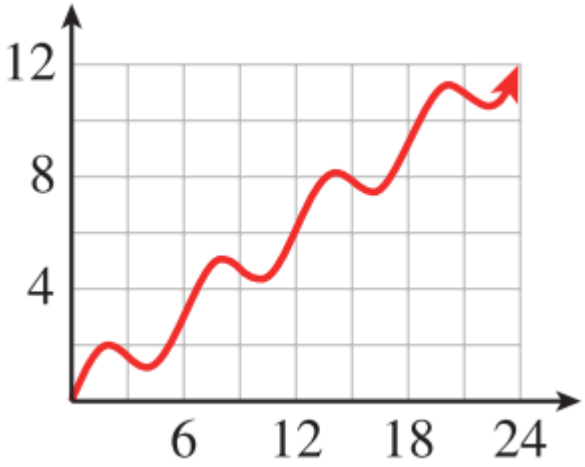
51.

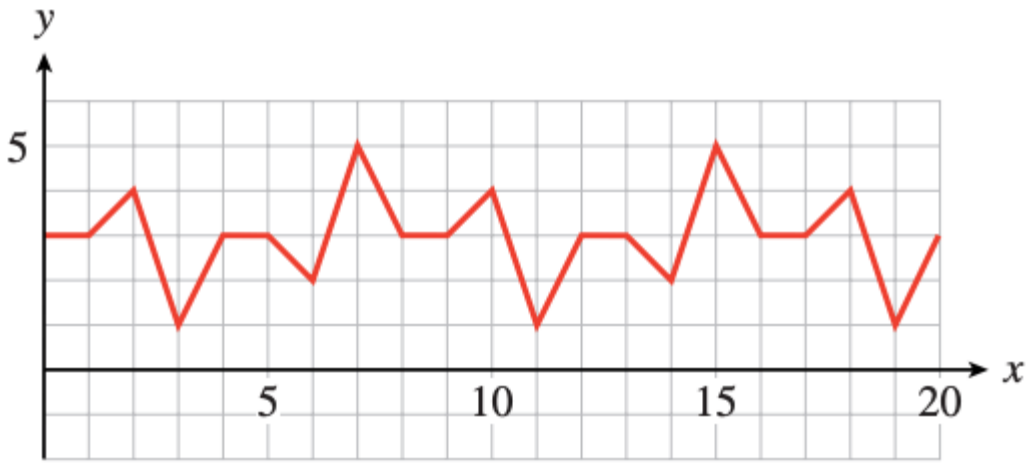


52.



53. a. Is the function shown periodic? If so, what is its period? If not, explain why not.





b. Compute the difference between the maximum and minimum function values. Sketch in the midline of the graph.

c. Find the smallest positive value of k for which $f(x) = f(x + k)$ for all x.

d. Find the smallest positive values of a and b for which $f(b) - f(a)$ is a maximum.

For the following exercises, use symmetry to help in evaluating the given trigonometric function.

54. $\cos(-30^\circ)$

55. $\sin\left(-\frac{5\pi}{6}\right)$

56. $\tan(-135^\circ)$

57. $\sec\left(-\frac{5\pi}{3}\right)$

58. $\csc(-315^\circ)$

59. $\cot\left(-\frac{7\pi}{6}\right)$

Synthesis Questions

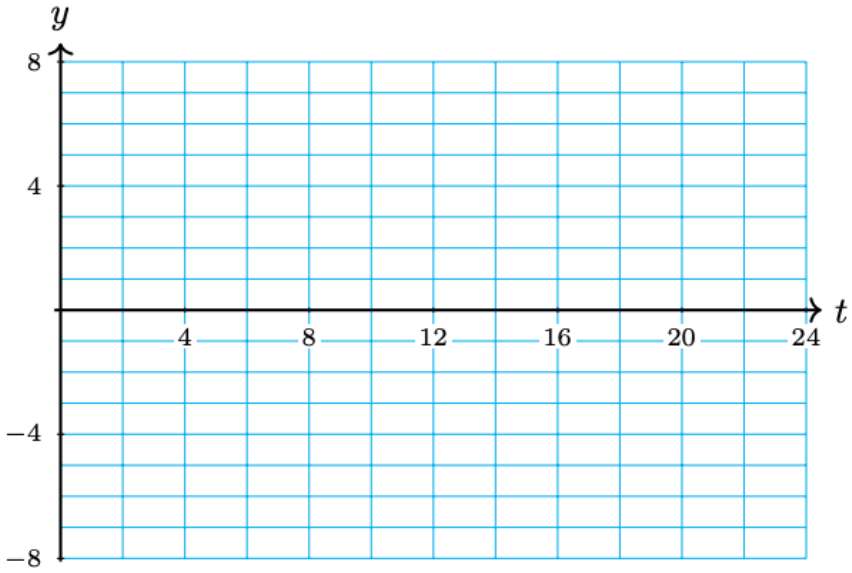
60. An ant is walking clockwise around the face of a sundial in the garden. The sundial is a circle with a 12-inch diameter, and the ant makes one circuit of the sundial in 24 seconds.

a. Sketch the sundial and a coordinate system with its origin at the center of the dial. Draw the y-axis to align with the gnomon (pointer) of the sundial.

b. Suppose we start timing the ant when it is at the tip of the gnomon. Complete the table showing the ant's location as an angle in standard position and its y-coordinate at that time.

t	0	2	4	6	8	10	12	14	16	18	20	22	24
θ	90°	60°											
$y = f(t)$	6	$3\sqrt{3}$											

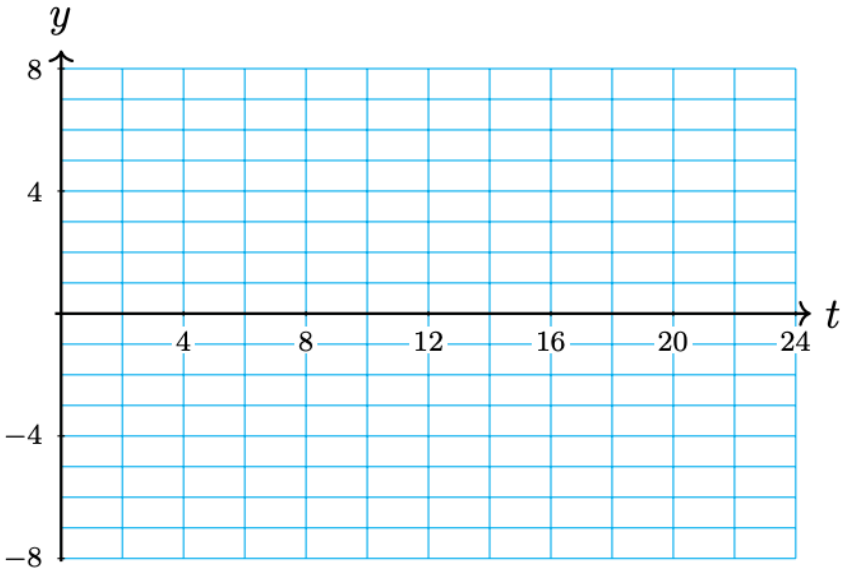
c. Sketch a graph of the ant's y-coordinate as a function of time.



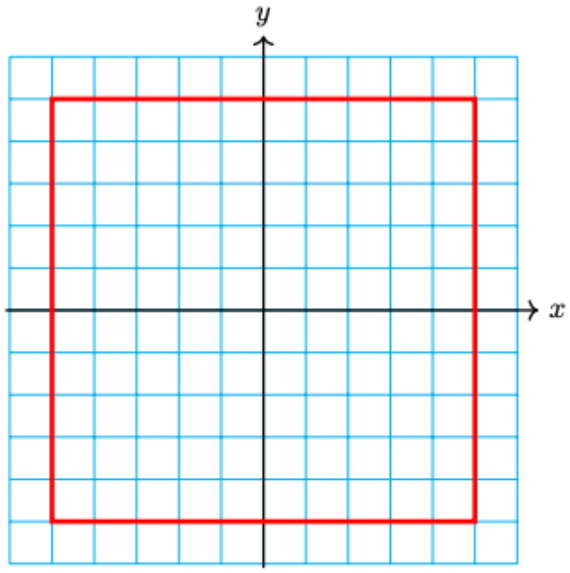
d. If the ant makes a second trip around the sundial, what will the graph look like from $t = 24$ to $t = 48$? Explain the statement $f(t + 24) = f(t)$.

61. Repeat the previous exercise, but make a table and graph of the ant's x-coordinate as a function of time.

t	0	2	4	6	8	10	12	14	16	18	20	22	24
θ	90°	60°											
$x = g(t)$	0	3											



62. Jorge is standing at the point $(5, 0)$ on the square shown. If he walks around the square in the counter-clockwise direction, his y-coordinate is a function of the distance, d, he has walked.



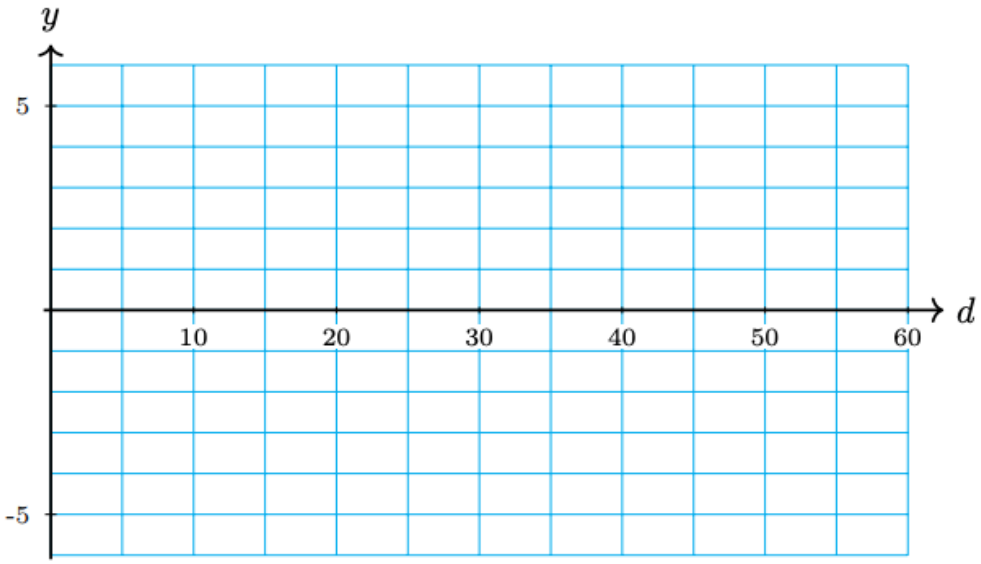
a. What is the y-coordinate of Jorge's position when he has walked 2 units? 5 units? 8 units? (Start filling in the table in part (c).)

b. After Jorge reaches the upper right corner of the square, he will turn left and start walking along the top of the square. What is his y-coordinate from $d = 5$ to $d = 15$?

c. Fill in the rest of the table with the y-coordinates of Jorge's position when he has walked a distance d.

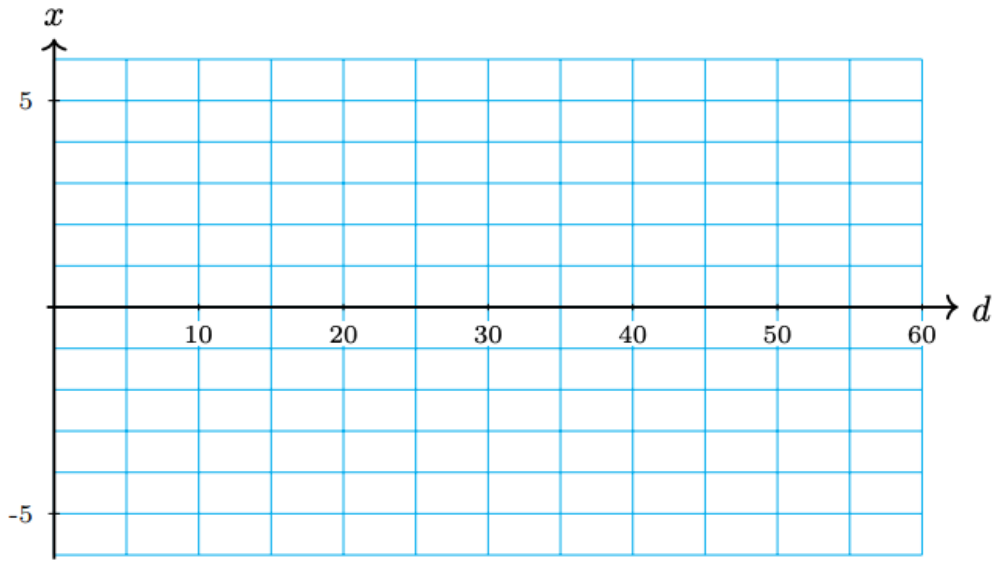
d	0	2	5	8	10	12	15	18	20	22	25	28	30	32	35	38	40
y																	

d. Plot a graph of Jorge's coordinate as a function of the distance he has walked.



53. Repeat the previous exercise, except consider Jorge's x-coordinate as a function of the distance he has walked.

d	0	2	5	8	10	12	15	18	20	22	25	28	30	32	35	38	40
x																	



a. Refer to the previous exercise. Suppose you know where Jorge is on the square at some given time. Can you predict where he will be after he has walked another 40 units?

b. Let $y = f(d)$. What does your answer say about $f(d)$ and $f(d + 40)$ for any positive value of d?

c. How does the graph of f for $0 \leq d \leq 40$ compare with the graph for $40 \leq d \leq 80$?

d. Describe how the graph of f would continue for $0 \leq d \leq 400$.

64. Refer to the previous exercise. Sketch a graph of Jorge's x-coordinate, $x = g(d)$, for $0 \leq d \leq 120$.

Proofs

For the following exercises, you may assume that sine is an odd function and cosine is an even function.

65. Prove that tangent is an odd function.

66. Prove that secant is an even function.

67. Prove that cosecant is an odd function.

68. Prove that cotangent is an odd function.

For the following exercises, prove the identity.

69. $\cot(-\theta)\sin(-\theta) = \cos(\theta)$

70. $\dfrac{\cos(-x)\sec(x)}{\tan(-x)} = -\cot(x)$

71. $\sin(-y)\sec(-y) = \tan(-y)$

72. $\sin(-\theta)\sec(-\theta)\cot(-\theta) = 1$

73. $1 = 2\cos(x)\sec(-x) + \tan(-y)\cot(y)$

74. $\dfrac{\tan(-\gamma)}{\sin(\gamma)} = -\sec(\gamma)$

75. $\tan(A) - \dfrac{1}{\tan(-A)} = \dfrac{\sec(-A)}{\sin(A)}$

76. $\sec(z) + \tan(z)\sin(-z) = \cos(-z)$

77. $\sin(-\phi)[\csc(-\phi) + \sec(-\phi)] = 1 + \tan(-\phi)$

Applications

For the following exercises, sketch a periodic function that models the situation.

78. At a ski slope, the lift chairs take 5 minutes to travel from the bottom, at an elevation of 3000 feet, to the top, at elevation 4000 feet. The cable supporting the ski lift chairs is a loop turning on pulleys at a constant speed. At the top and bottom, the chairs are at a constant elevation for a few seconds to allow skiers to get on and off.

    a. Sketch a graph of $h(t)$, the height of one chair at time t. Show at least two complete trips.

    b. What is the period of $h(t)$?

79. The heater in Wyatt's house doesn't have a thermostat; it runs on a timer. It uses 300 watts when it is running. Wyatt sets the heater to run from 6 am to noon, and again from 4 pm to 10 pm in the evening.

    a. Sketch a graph of $P(t)$, the power drawn by the heater as a function of time. Show at least two days of heater use.

    b. What is the period of $P(t)$?

80. Piyali adds water to her fish pond once a week to keep the depth at 30 centimeters. During the week, the water evaporates at a constant rate of 0.5 centimeters per day.

    a. Sketch a graph of $D(t)$, the depth of the water as a function of time. Show at least two weeks.

    b. What is the period of $D(t)$?

81. Roy's miniature schnauzer, Sophie, is very energetic and bounces excitedly at dinner time. Sophie can jump 30 inches high, and each jump takes him 0.8 seconds.

    a. Sketch a graph of Sophie's height, $D(t)$, as a function of time. Show at least two jumps.

    b. What is the period of $h(t)$?

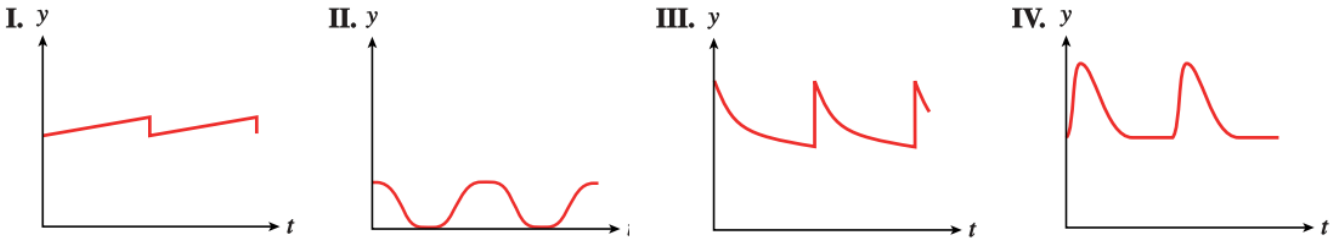
82. Each situation describes a periodic function. Match each situation with the appropriate graph.

    a. When the heart contracts, blood pressure in the arteries rises rapidly to a peak (systolic blood pressure) and then falls off quickly to a minimum (diastolic blood pressure). Blood pressure is a function of time.

    b. After an injection is given to a patient, the amount of the drug present in his bloodstream decreases over time. The patient receives injections at regular intervals to restore the drug level to the prescribed level. The amount of the drug present is a function of time.

    c. The monorail shuttle train between the north and south terminals at Gatwick Airport departs from the south terminal every 12 minutes. The distance from the train to the south terminal is a function of time.

    d. Ron gets a haircut every two weeks. The length of his hair is a function of time.



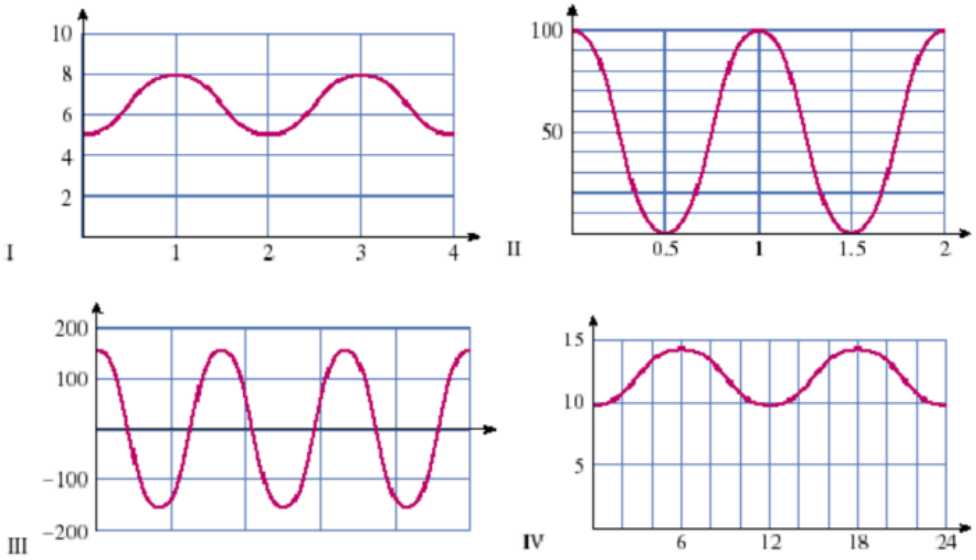
33. Match each of the following situations with an appropriate graph below.

a. The number of hours of daylight in Salt Lake City varies from a minimum of 9.6 hours on the winter solstice to a maximum of 14.4 hours on the summer solstice.

b. A weight is 6.5 feet above the floor, suspended from the ceiling by a spring. The weight is pulled down to 5 feet above the floor and released, rising past 6.5 feet in 0.5 seconds before attaining its maximum height of feet. Neglecting the effects of friction, the height of the weight will continue to oscillate between its minimum and maximum height.

c. The voltage used in U.S. electrical current changes from 155V to –155V and back 60 times each second.

d. Although the moon is spherical, what we see from earth looks like a disk, sometimes only partly visible. The percentage of the moon's disk that is visible varies between 0 (at new moon) to 100 (at full moon).



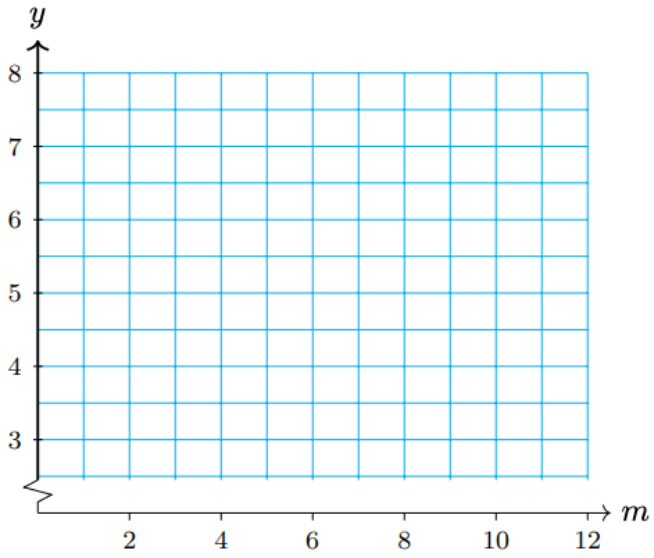
34. The table shows sunrise and sunset times in Los Angeles on the fifteenth of each month.

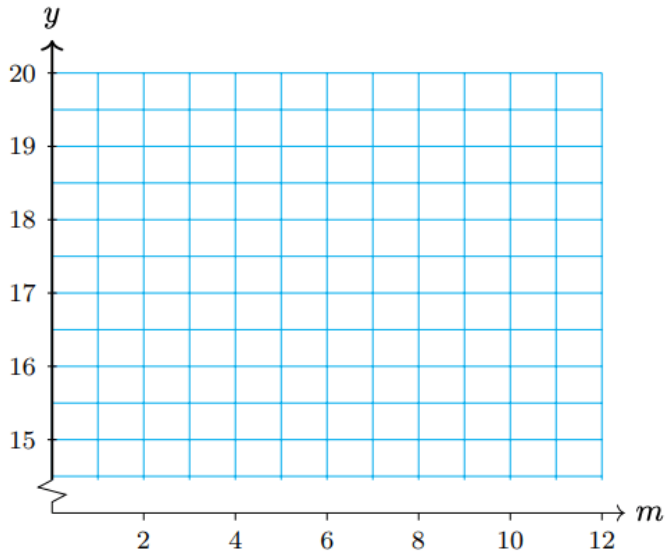
Month	Oct	Nov	Dec	Jan	Feb	Mar
Sunrise	5:58	6:26	6:51	6:59	6:39	6:04
Sunset	17:20	16:50	16:45	17:07	17:37	18:01

Month	Apr	May	Jun	Jul	Aug	Sep
Sunrise	5:22	4:52	4:42	4:43	5:15	5:37
Sunset	18:25	18:48	19:07	19:05	18:40	18:00

a. Use the left-hand grid to plot the sunrise times and sketch a sinusoidal graph through the points.

b. Use the right-hand grid to plot the sunset times and sketch a sinusoidal graph through the points.



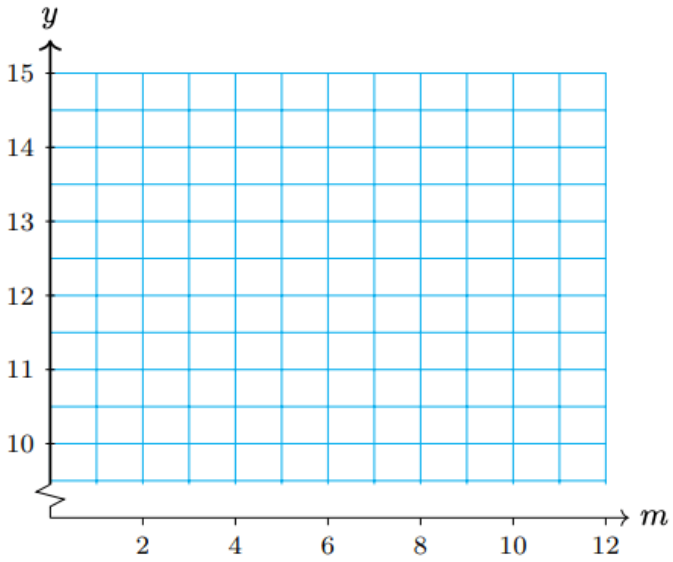


35. a. Use the data from Problem 65 to complete the table with the hours of sunlight in Los Angeles on the fifteenth of each month.

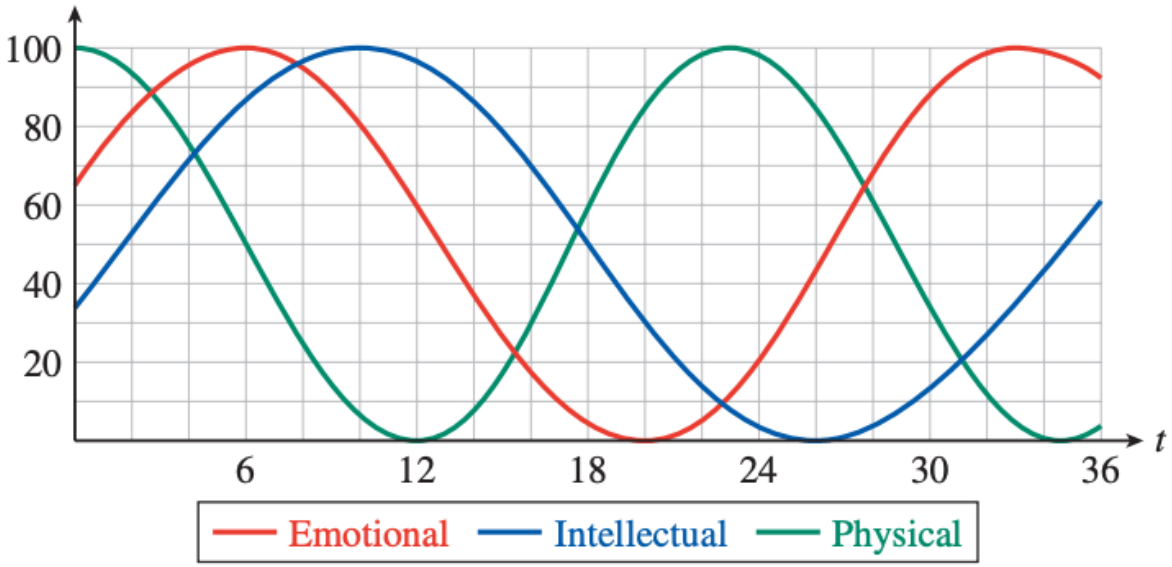
Month	Oct	Nov	Dec	Jan	Feb	Mar
Hours of Daylight						

Month	Apr	May	Jun	Jul	Aug	Sep
Hours of Daylight						

b. Plot the daylight hours and sketch a sinusoidal graph through the points.



36. Many people who believe in astrology also believe in biorhythms. The graph shows an individual's three biorhythms, physical, emotional, and intellectual, for 36 days, from $t = 0$ on September 30 to November 5.

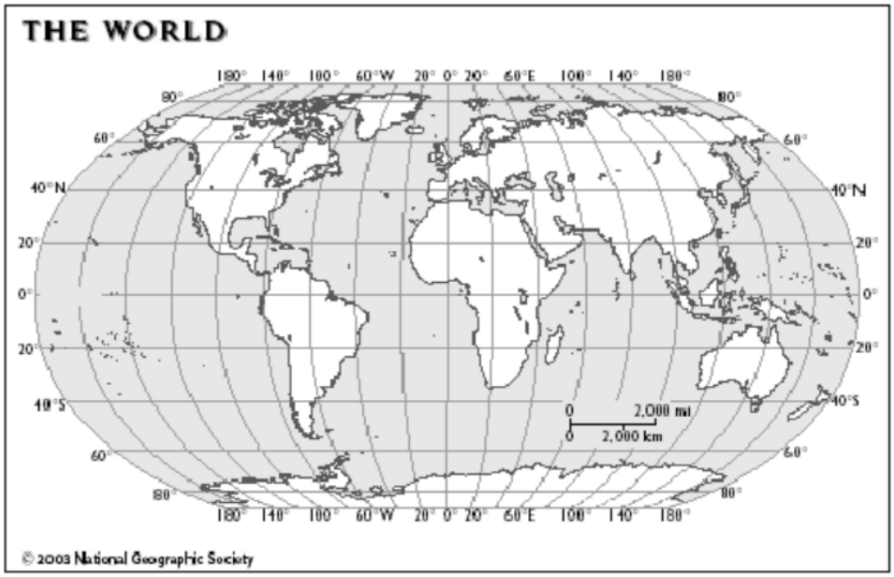


a. Find the dates of highest and lowest activity for each biorhythm during the month of October.

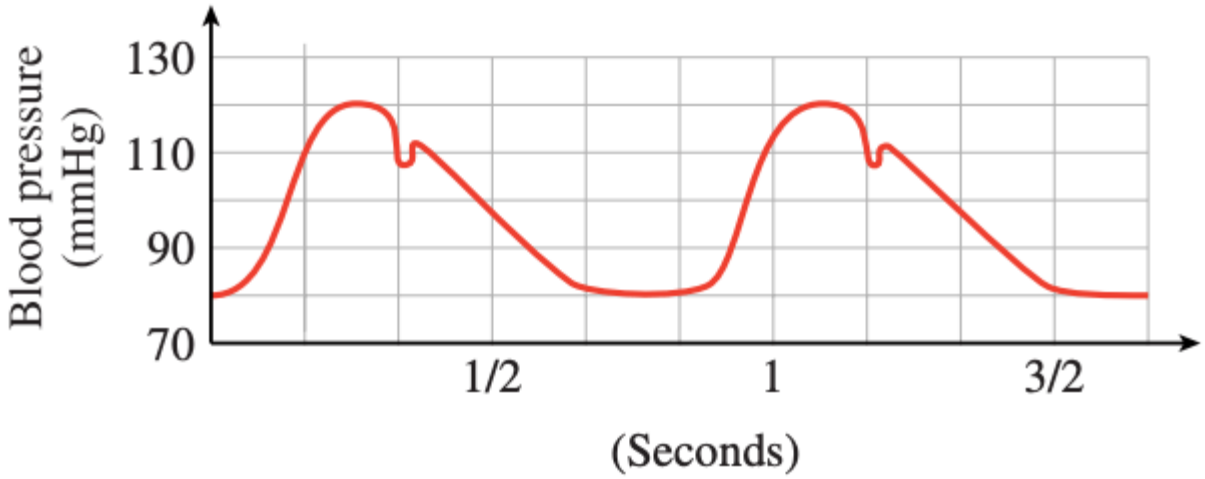
b. Find the period of each biorhythm in days.

c. On the day of your birth, all three biorhythms are at their maximum. How old will you be before all three are again at the maximum level?

37. The path of a satellite orbiting above the earth makes a sinusoidal graph on a map of the earth, with its midline at the equator. On the map on the next page, sketch a graph for a satellite that orbits the earth every 90 minutes, and strays no farther than 4000 km from the equator. (One degree of latitude is equal to 111 kilometers.) The satellite passes over the spot 0° latitude and 0° longitude at time $t = 0$. Label a scale on the equator to serve as a time axis for your graph.



38. The graph shows arterial blood pressure, measured in millimeters of mercury (mmHg), as a function of time.

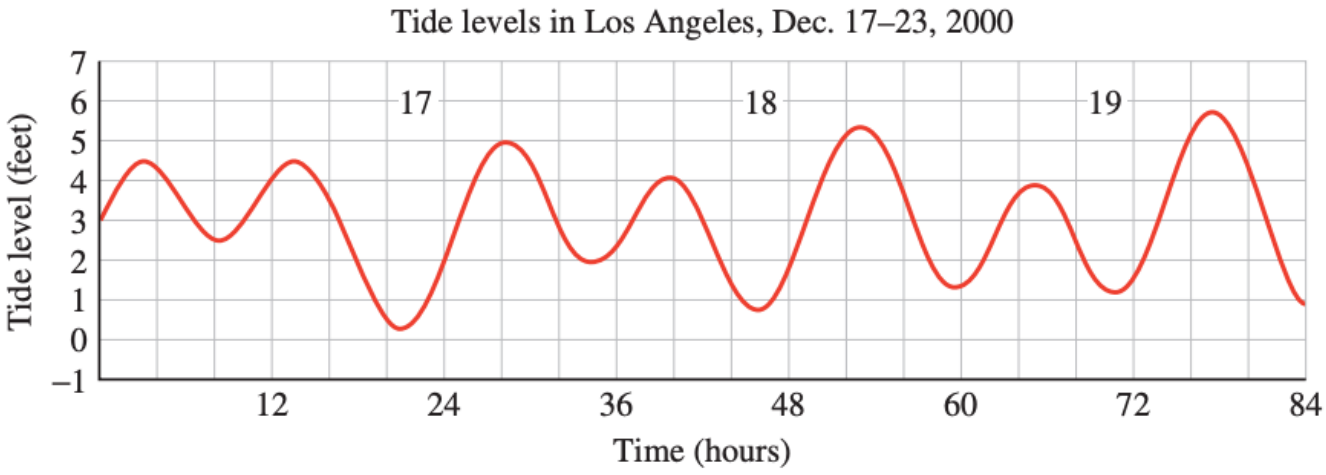


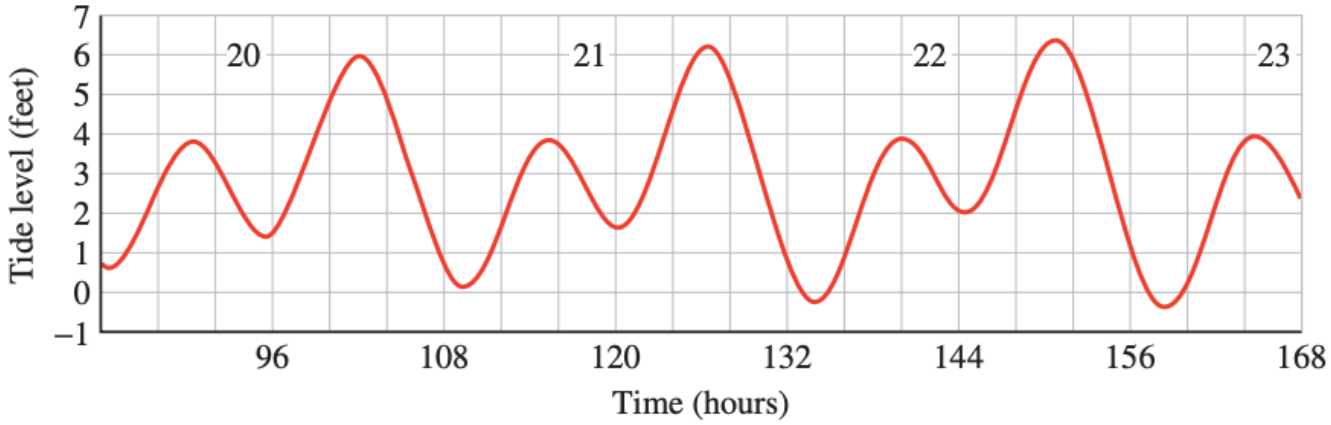
a. What are the maximum (systolic) and minimum (diastolic) pressures? The pulse pressure is the difference of systolic and diastolic pressures. What is the pulse pressure?

b. The mean arterial pressure is the diastolic pressure plus one-third of the pulse pressure. Calculate the mean arterial pressure, and draw a horizontal line on the graph at that pressure.

c. The blood pressure graph repeats its cycle with each heartbeat. What is the heart rate, in beats per minute, of the person whose blood pressure is shown in the graph?

39. Here is a tide chart for Los Angeles for the week of December 17–23, 2000. The horizontal axis shows time in hours, with $t = 12$ corresponding to noon on December 17. The vertical axis shows the height of the tide in feet above mean sea level.





a. High tides occurred at 3:07 am and 2:08 pm on December 17, and low tides at 8:41 am and 9:02 pm. Estimate the heights of the high and low tides on that day.

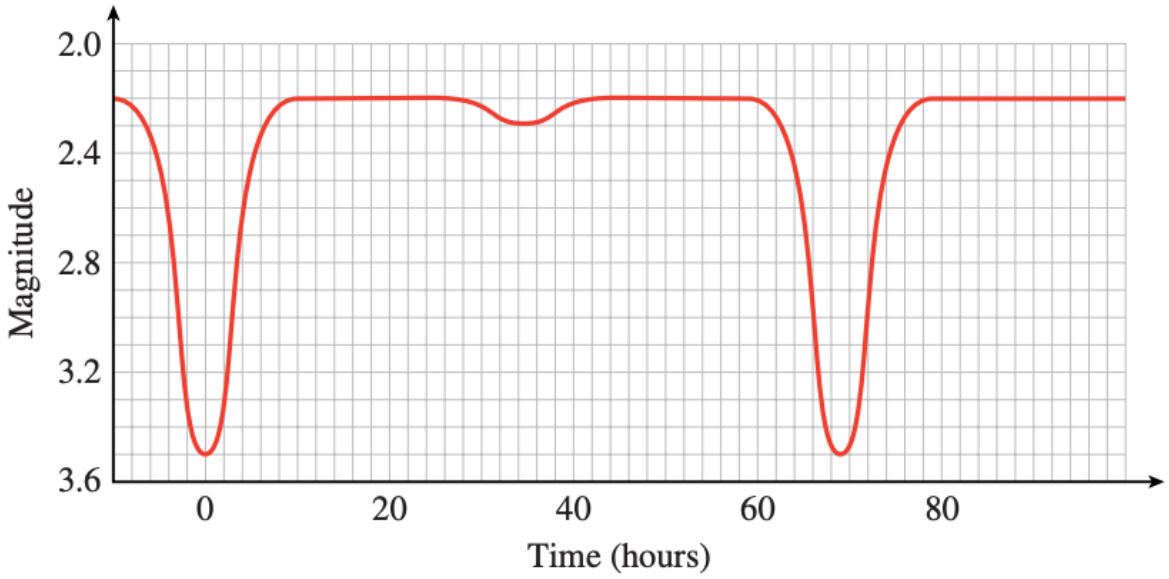
b. Is tide height a periodic function of time? Use the information from part (a) to justify your answer.

c. Make a table showing approximate times and heights for the high tides throughout the week. Make a similar table for the low tides.

d. Describe the trend in the heights of the high tides over the week. Describe the trend in the heights of the low tides.

e. What is the largest height difference between consecutive high and low tides during the week shown? When does it occur?

20. The apparent magnitude of a star is a measure of its brightness as seen from earth. Smaller values of apparent magnitude correspond to brighter stars. The graph below, called a light curve, shows the apparent magnitude of the star Algol as a function of time. Algol is actually a system of two stars, a bright principal star and its dimmer companion, in orbit around each other. As each star passes in front of the other it eclipses some of the light that reaches earth.

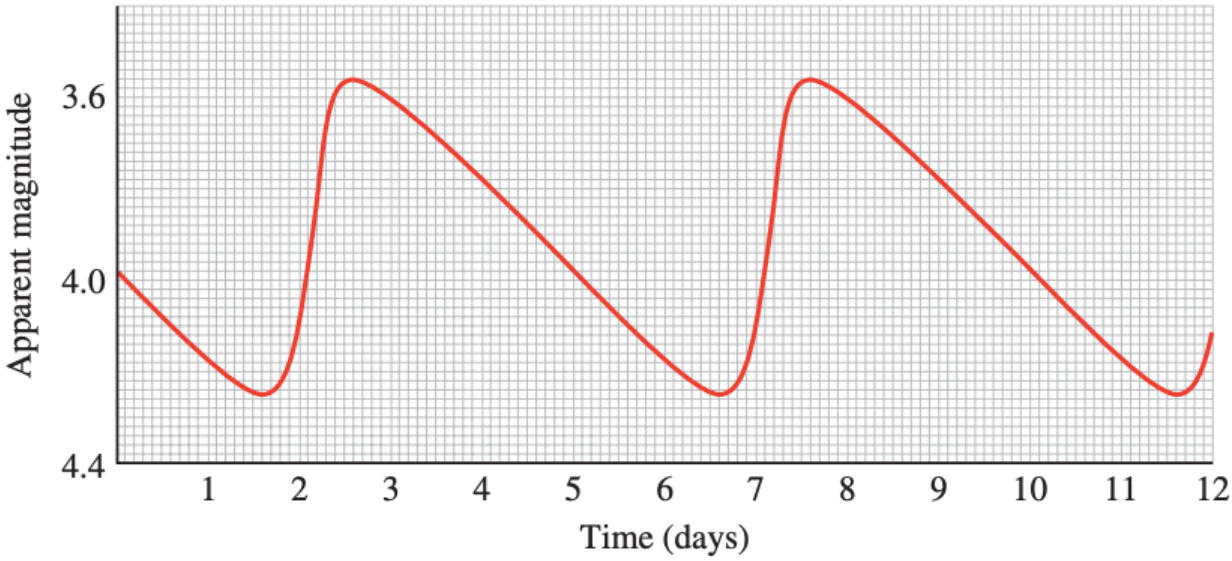


a. The light curve is periodic. What is its period?

b. What is the range of apparent magnitudes of the Algol system?

c. Explain the large and small dips in the light curve. What is happening to cause the dips?

91. Some stars, called Cepheid variable stars, appear to pulse, getting brighter and dimmer periodically. The graph shows the light curve for the star Delta Cephei.



a. What is the period of the graph?

b. What is the range of apparent magnitudes for Delta Cephei?

This page titled 8.1.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

8.2: Non-Rigid Transformations

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The material in the next few sections relies heavily on your prerequisite graphing skills from Algebra. As such, we will try our best to "backfill" some of that information here. We begin by reintroducing the definition of a **transformation** of the graph of a function.

Definition: Transformation (of the graph of a function)

A **transformation** of the graph of a function is an alteration of the graph's orientation (by reflections across axes), scale, and position through various operations.

What you may *not* have learned in your Algebra course is that these transformations can be categorized into two distinct groups—**rigid** and **non-rigid transformations**.

Definition: Rigid Transformation

A **rigid transformation** of a graph is a transformation that changes the position of the graph without changing its shape or size.

You likely remember rigid transformations as **shifts**. As a reminder of the basic idea of a rigid transformation, let's consider the adorable "base graph" of my dog, $y = \text{Sophie}\,(x)$ in Figure $8.2.1$.

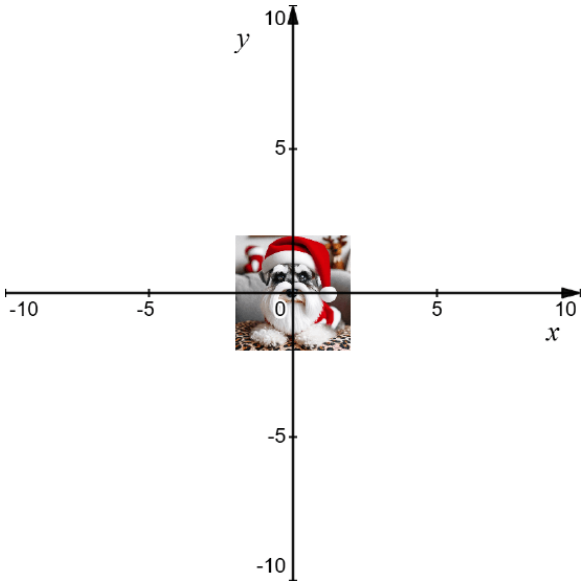


Figure 8.2.1**:** *The base graph of* $y = \text{Sophie}\,(x)$

Note: This is *not* a function

Technically, this is not a function (Sophie doesn't pass the Vertical Line Test); however, using an image usually conveys the point of transformations to students more efficiently than traditional functions.

From Algebra, we know that if we wanted to shift the graph of Sophie 2 units to the right and 3 units down, the resulting function would be

$$g(x) = \text{Sophie}\,(x-2) - 3.$$

The orientation and scale of the graph of this new "function" (seen in Figure 8.2.2) would be the same as the base graph of the "Sophie" function, but the position is different.

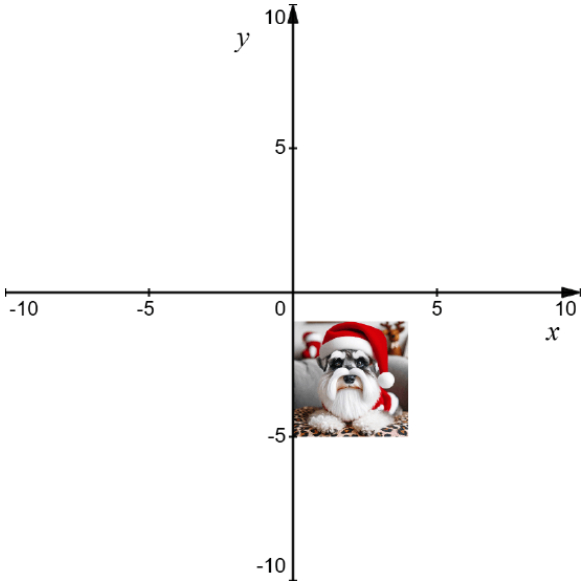


Figure 8.2.2**:** $g(x) = \text{Sophie}\,(x-2) - 3$

In general, given a function $f(x)$, we can obtain the graph of $g(x) = f(x + C) + D$ by *subtracting* C from all of the x-coordinates of the graph of $f(x)$ and *adding* D to all the y-coordinates.

Definition: Non-rigid Transformation

A **non-rigid transformation** of a graph if a transformation that retains the graph's original center position but scales the graph vertically or horizontally, or changes the graph's orientation by reflecting the graph about a line.

For example, if we took the base graph of our "Sophie function" from Figure $8.2.1$ and decided to scale it vertically (also known as stretching) by a factor of 2, we would need to multiply all the y-coordinates of the points in the base graph by 2. From Algebra, we know this is achieved by multiplying the function by 2 to get $h(x) = 2\ \text{Sophie}\,(x)$. The graph of this new image would look like the following:

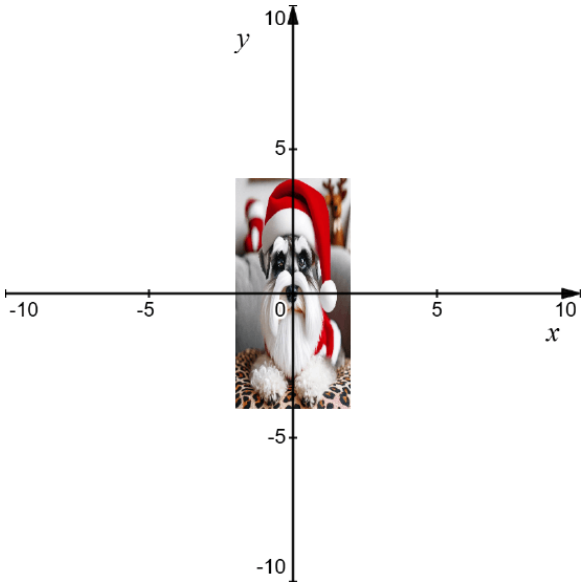


Figure $8.2.3$**:** $h(x) = 2\ \text{Sophie}\,(x)$

We also learned (again, from Algebra) to combine transformations to get a new function from an old one. For example, we could take $\text{Sophie}\,(x)$, scale it horizontally by a factor of $1/2$ (also known as compressing), reflect it about the y-axis, shift it 3 units left, and shift it 4 units up. The result of all of these manipulations is the function $k(x) = \text{Sophie}\,(-2\,(x+3)) + 4$, and it looks like the following image:

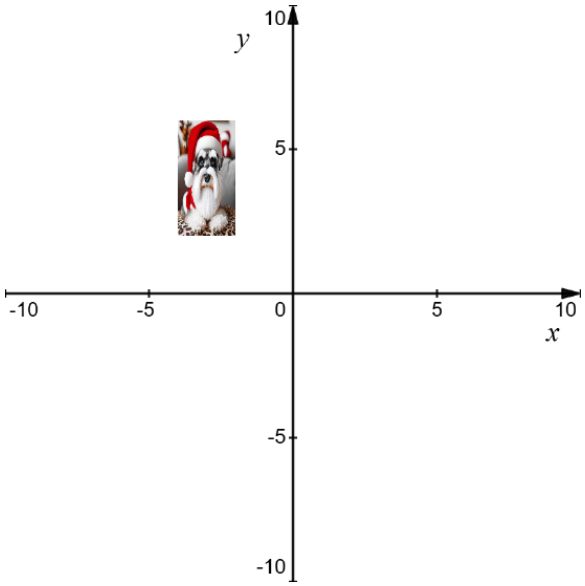


Figure $8.2.4$**:** $k(x) = \text{Sophie}\,(-2\,(x+3)) + 4$

In this section, we will perform these same transformations on trigonometric functions; however, because these functions are periodic, we can only capture a small section of their graphs. The process we introduce to apply these transformations and the names we define during this development will be used throughout the rest of Trigonometry and Calculus.

Our ultimate goal is to graph a trigonometric function of the form

$$f(x) = A\,\mathrm{trig}\,(Bx + C) + D, \tag{8.2.1}$$

where "trig" could be any of the trigonometric functions. The unknown values, A, B, C, and D, are called **parameters**.

Definition: Parameter

A **parameter** is a variable that influences the behavior or output of a mathematical object (like a function or equation) but is held constant within a specific context or scenario. It's a quantity that helps define or constrain a system, function, or dataset.

We will review what happens to the graph of the base trigonometric function when we adjust these parameters.

Amplitude and the Graph of $y = A\ \mathrm{trig}(x)$

We start our investigation of the effect of the parameters in Equation 8.2.1 by setting $B = 1$, $C = 0$, and $D = 0$. Therefore, we are interested in the effect of A on the graph of

$$f(x) = A\,\mathrm{trig}\,(x)\,.$$

If $A > 0$, then from Algebra, we know that the graph of $y = Ag(x)$ is a vertical stretch (if $A > 1$) or compression (if $0 < A < 1$) of the graph of g; however, does this still hold in Trigonometry? The following example might lead us to the answer.

Example 8.2.1

Make a table of values and sketch the graph of each function. Compare your graph to the function's base graph.

a. $f(x) = 3\sin(x)$
b. $g(x) = \frac{1}{5}\cos(x)$
c. $h(x) = \pi\tan(x)$

Solutions available at the end of this section.

Note: Abusive Language

In Example 8.2.1b, I state that " the graph of $g(x) = \frac{1}{5}\cos(x)$ is a vertical compression of the base graph by a factor of $1/5$." Some authors (and instructors) will say, "$g(x)$ is a compression of the graph of $y = \cos(x)$ by a factor of 5." I have always found this language a bit confusing. Therefore, I will adhere to the convention in this textbook of stating "a vertical compression by a factor of $\frac{1}{5}$." The fact that the word "compression" is involved gives the context that the y-values of the original function are being "squished" to make them smaller than before.

Example 8.2.1 verifies that our experience in Algebra still holds in Trigonometry; however, we artificially restricted the values of A to be positive. What happens if $A < 0$? Again, from Algebra, we know the graph of $y = -g(x)$ would be a reflection of the graph of the function g about the x-axis. Does this still hold in Trigonometry?

Checkpoint 8.2.1

Make a table of values and sketch the graph of each function. Compare your graph to the function's base graph.

a. $f(x) = -2\cos(x)$
b. $g(x) = -\frac{1}{2}\sin(x)$

Answers available at the end of this section.

The following theorem summarizes these results.

Theorem: Vertical Scaling

The graph of

$$f(x) = A\,\text{trig}(x)$$

vertically scales the graph of $y = \text{trig}(x)$ by a factor of $|A|$.

- If $|A| > 1$, it is a vertical stretch by a factor of $|A|$.
- If $0 < |A| < 1$, it is a vertical compression by a factor of $|A|$.

Additionally, if $A < 0$, the graph is reflected about the horizontal axis.

In either case, the y-coordinates of points on the graph of $y = \text{trig}(x)$ are multiplied by A to become the new points on the graph of $f(x) = A\ \text{trig}(x)$.

Note: Vertical Scaling Changes the Range

Except for the tangent function (whose range is all real numbers), the range of a trigonometric function will change under vertical scaling. For example, in Example $8.2.1b$ the range of $g(x) = \frac{1}{5}\cos(x)$ is $\left[-\frac{1}{5}, \frac{1}{5}\right]$.

When we introduced the definition of amplitude, we also mentioned that while the *base* sine and cosine functions have amplitudes of 1, the tangent does not have an amplitude because its range is unbounded. We now introduce the amplitude of *any* sinusoidal function.

Example $8.2.2$

Compute the amplitude of each function.

a. $f(x) = 3\sin(x)$
b. $g(x) = \frac{1}{5}\cos(x)$

Solutions available at the end of this section.

Example $8.2.2$ provides an observation that will allow us to state the amplitude of a sinusoidal function quickly.

Theorem: Amplitude of a Sinusoidal Function

The amplitude of $f(x) = A\cos(x)$ or $g(x) = A\sin(x)$ is $|A|$.

Checkpoint $8.2.2$

State the amplitude of each function.

a. $f(x) = -2\cos(x)$

b. $g(x) = -\frac{1}{2}\sin(x)$
c. $h(x) = 12\tan(x)$

Answers available at the end of this section.

If you are using the digital version of this textbook, feel free to use the following Interactive Element to help solidify the concept of amplitude and the effect of A on the graph of $f(x) = A\sin(x)$.

Interactive Element: Investigating the Effects of A on the Graph of $y = A\sin(x)$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Desmos Interactive online

Interaction: Adjust the slider for A to see the effect on the graph of $y = A\sin(x)$.

Period

The next parameter in Equation 8.2.1 is B. Very technically, you should have already encountered graphs of functions of the form $y = g(Bx)$ in Algebra, but I find that many people are confounded by what is truly happening and, very honestly, you don't see the true effect of that transformation until Trigonometry. Therefore, we are going to cover it in detail here. To determine this parameter's effects on the trigonometric function's graph, we take an example and build a table of values.

Suppose we are asked to graph $y = \cos(2x)$. By building a table of values using the quadrantal angles, we get the following:

[redacted]

Graphing the base cosine curve along with these points yields Figure 8.2.10.

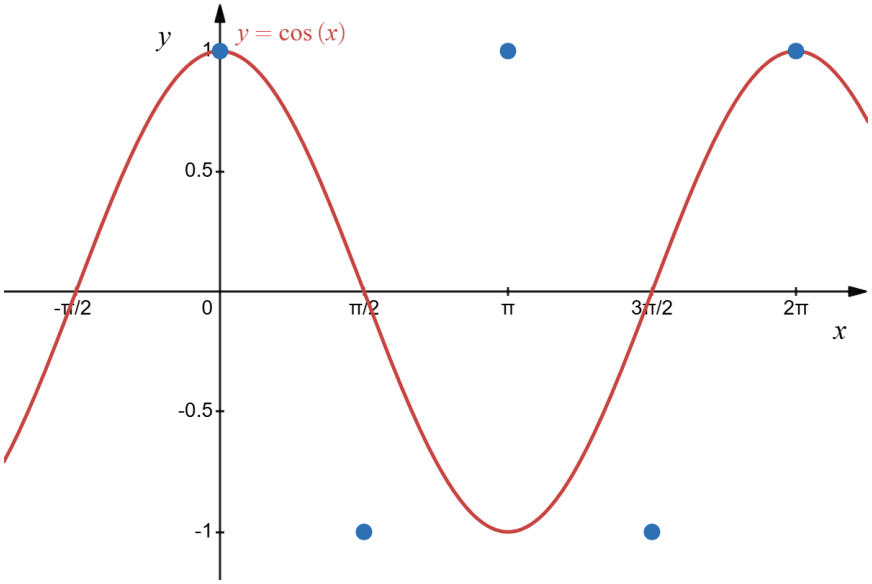


Figure 8.2.10

The issue should be readily apparent—we need more points to plot. The quadrantal angles provide too coarse of a picture. Let's

add some values to our table.

[redacted]

Plotting these additional points (and connecting them with a nice sinusoidal curve), we get the following figure.

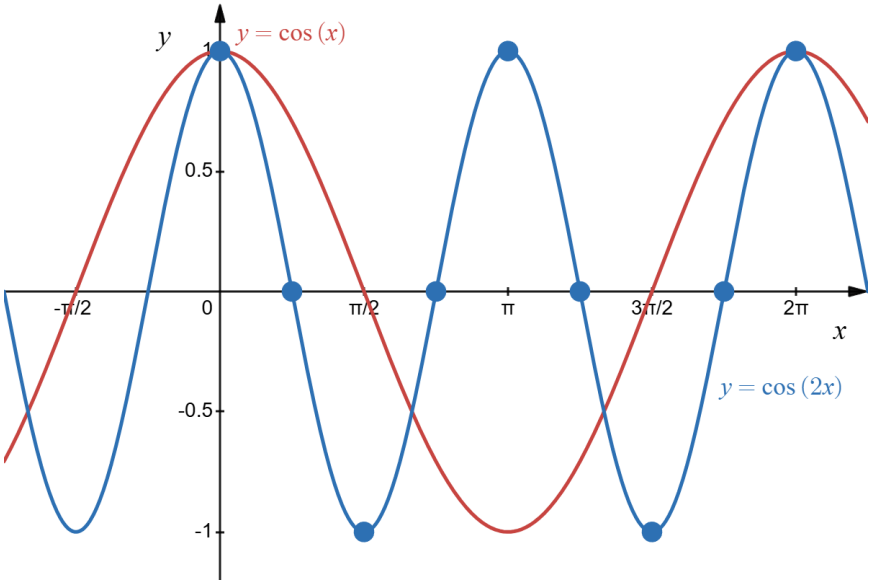


Figure 8.2.11

Before moving on, we need to mention a few things about Figure 8.2.11. First, recall that the principal cycle of the cosine function is $[0, 2\pi]$. This means that standard cosine function goes through a single cycle on that interval; however, $y = \cos(2x)$ goes through *two* complete cycles on the same interval. This is a beautiful way of thinking about the effect of B on the graph of $y = A\ \text{trig}\,(Bx)$. B tells us how many "cycles" of the graph get squeezed into the principal cycle of the base function. Another interpretation is that B tells us how many times faster the curve is "oscillating" than the base curve. In this case, $y = \cos(2x)$ oscillates twice as fast as $y = \cos(x)$. This being said, we can safely conjecture that B modifies the function's period (how long it takes to complete one cycle).

Note: "Oscillating" doesn't work for the tangent

I placed "oscillating" in quotes because this interpretation doesn't work for the tangent function. That is, $y = \tan(2x)$ doesn't oscillate twice as fast as the base tangent function because the tangent doesn't oscillate.

The second item worth mentioning is that you will usually only be asked to graph a single cycle of a trigonometric function, and this single cycle will be based on the principal cycle. Hand-drawn (and computer-generated) graphs look crowded when more than one cycle is sketched. In general, your instructor will know you just copied a computer-generated graph of a given trigonometric function if you include several cycles.

This brings us to our third and most important observation. For any trigonometric function, we only need five special x-values to obtain an accurate sketch of a single cycle (again, based on the principal cycle). For simplicity, we label these x_0, x_1, x_2, x_3, and x_4. These are

- x_0**:** the beginning x-value
- x_4**:** the ending x-value
- x_2**:** the value of x directly between the beginning and ending x-values
- x_1 **and** x_3**:** the two values of x between the beginning and middle, and middle and ending x-values

As they say, a picture is worth a thousand words, so let's visualize this using $y = \cos(2x)$.

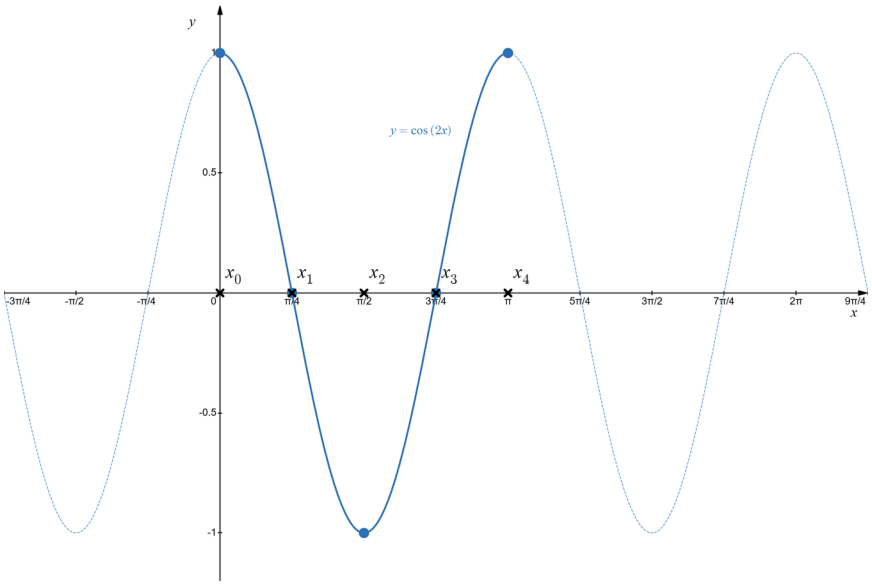


Figure 8.2.12

Each of the x-values in Figure 8.2.12 is called a **key number** of a single graph cycle.

Definition: Key Numbers

The **key numbers** of a cycle for a trigonometric function are the x-values of the of the initial, final, middle, and quarter-points in the principal cycle. These can be found by partitioning the cycle into four equally-spaced subintervals.

To showcase how the key numbers help graph trigonometric functions, suppose we are asked to graph the principal cycle of the base cosine function, $y = \cos(\theta)$. Again, the principal cycle for the cosine function is $0 \leq \theta \leq 2\pi$. Partitioning this interval into four equally-spaced subintervals requires taking the length of this interval (2π) and dividing by four to get

$$\frac{2\pi}{4} = \frac{\pi}{2}.$$

We will call the length of this subinterval the **step size** for this function.

Definition: Step Size

The distance between any two consecutive key numbers is called the **step size**.

To graph, we mark the first x-value in the *principal cycle*, which is $x = 0$. Hence, $x_0 = 0$. From there, we "mark by steps" until we reach the end of the single cycle (see the figure below).

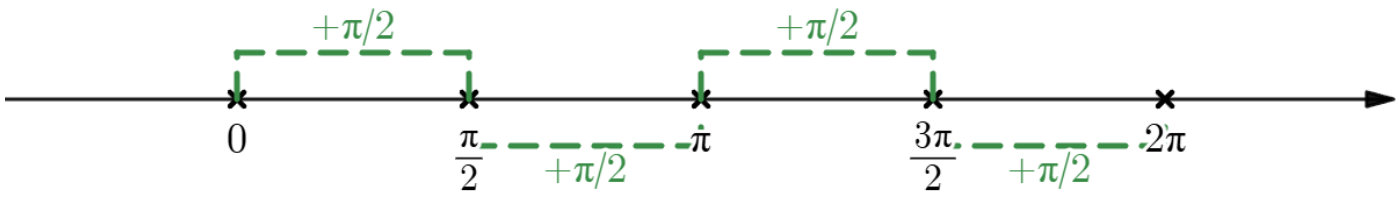


Figure 8.2.13

Since we know the basic shape of the cosine graph, we can complete the graph using these key numbers.

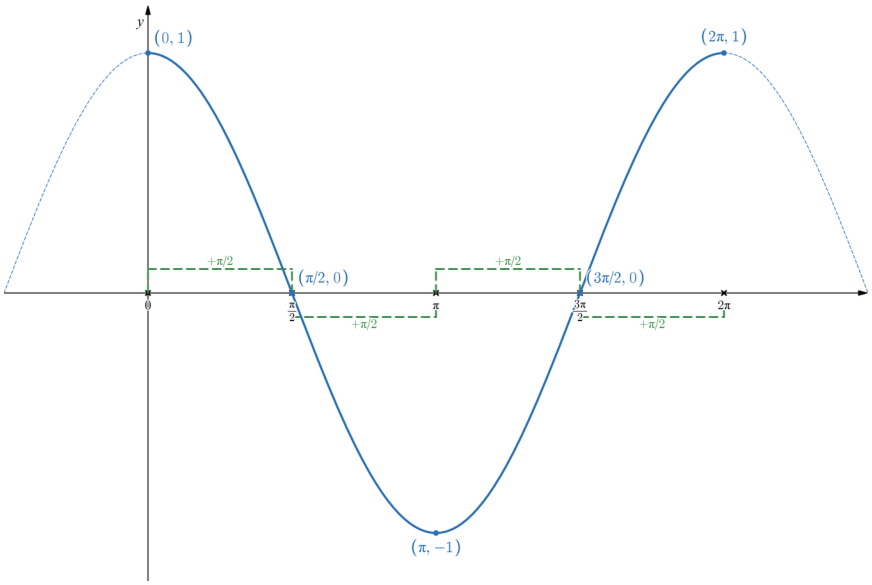


Figure 8.2.14

Before going into an example, let's amend a definition we created in the previous section. We know that the *natural period* of the cosine function (and sine function) is 2π; however, what is the period of the modified function $y = \cos(2x)$?

Recall that a single cycle of $y = \cos(\theta)$ occurs on $0 \leq \theta \leq 2\pi$. Replacing θ with $2x$, we have $0 \leq 2x \leq 2\pi$. Dividing all three sides of this inequality by 2, we get $0 \leq x \leq \pi$. That is, $y = \cos(2x)$ goes through a single cycle on the interval $[0, \pi]$. Hence, the period of $y = \cos(2x)$ is π. Generalizing this observation leads us to the following theorem.

Theorem: Period of a Trigonometric Function

Let p be the natural period of a trigonometric function. Then the period of $y = A\ \text{trig}(Bx)$ is $\frac{p}{B}$.

Thus, the period of $y = A\cos(Bx)$ is $2\pi/B$, and the period of $y = A\tan(Bx)$ is π/B (remember, the natural period of the tangent is π).

While we are at it, let's create a formula to help us compute the step size.

Theorem: Step Size Formula

The step size for a trigonometric function is $\frac{\text{Period}}{4}$.

The step size simplifies the process of finding the key numbers. By starting with the first key number, x_0, we add the step size to arrive at the second key number, x_1. To this, we add the step size to get the third key number, x_2. And so on.

We are overdue for some examples.

Example 8.2.3

Sketch a single cycle of each function, stating its amplitude (if applicable), period, and step size.

a. $f(x) = \cos(2x)$
b. $g(x) = -3\sin\left(\frac{x}{3}\right)$
c. $y = \frac{1}{5}\tan(-4x)$

Solutions available at the end of this section.

Did you notice I didn't bother simplifying fractions when I labeled the key numbers in Examples $8.2.3b$ and $8.2.3c$? This was purposeful and is a technique we will use moving forward.

Note: Changes in the Period Affect the Domain of the Tangent

While the domain of the sine and cosine functions remain all real numbers no matter the period, the tangent's domain changes when we change the period. For example, in Example $8.2.3c$ the domain of the tangent does not include $\frac{\pi}{8}$ (whereas, in the base tangent function, $\frac{\pi}{8}$ is in the domain). This is because a change in the period of a function causes **horizontal scaling** (stretching or compressing along the horizontal axis). When this scaling occurs, any vertical asymptotes the function might have also move.

Checkpoint $8.2.3$

Sketch a single cycle of $y = -\frac{1}{2}\cos(7x)$, stating its amplitude, period, and step size.

Answer available at the end of this section.

As before, the following Interactive Element (accessible through the digital version of the textbook) can help to solidify the concept of period and the effect of B on the graph of $f(x) = A\sin(Bx)$.

Interactive Element: Investigating the Effects of B on the Graph of $y = A\sin(Bx)$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Desmos Interactive online

Interaction: Adjust the slider for A and B to see the effect on the graph of $y = A\sin(Bx)$.

Summary of Graphing Trigonometric Functions Using Non-Rigid Transformations

Summary of Graphing $f(x) = \overline{A}\ \text{trig}\left(\overline{B}x\right)$

To graph the function

$$f(x) = \overline{A}\,\text{trig}\left(\overline{B}x\right):$$

Precondition Stage

We begin by cleaning up the trigonometric function and ensuring that the coefficient of x is positive. This is done using the Symmetry Identities. Once the Symmetry Identities have been used, your new function will be of the form

$$f(x) = A\text{trig}(Bx).$$

Information Stage

We now identify the following information (in the order listed):

- $\text{Amplitude} = |A|$
- $\text{Period} = \dfrac{\text{Natural Period}}{B}$
- $\text{Step Size} = \dfrac{\text{Period}}{4}$

where the natural period is 2π if given the sine or cosine functions, and π if given the tangent function.

Graphing Stage

We then use each of these *in reverse order* to obtain a sketch of the trigonometric function.

- Draw a horizontal line, mark the initial key number as 0, and mark the remaining key numbers by successively adding the **step size** to the previous key number.
- Double-check that the distance from the initial key number to the final key number matches the **period**, then sketch one cycle of the base trigonometric function, making sure to account for the sign of A and drawing any vertical asymptotes (if the given function is the tangent).
- Draw the vertical axis at $x = 0$ and use the **amplitude** to either
  - label the maximum and minimum values of the trigonometric function on the vertical axis (if the given function is sinusoidal), or
  - label the remaining key points on the graph (if the given function is the tangent).

To be clear, let's do a final example. This time, however, let's practice extending the graph as well.

Example 8.2.4

Sketch the graph of each function over the given interval.

a. $g(x) = -\dfrac{1}{2}\cos\left(-\dfrac{2}{5}x\right)$ for $-10\pi \le x \le 10\pi$
b. $h(x) = 3\tan(6x)$ for $-\frac{\pi}{12} \le x \le \frac{\pi}{2}$

Solutions available at the end of this section.

Finding Equations Given Graphs

We wrap up this section by inverting our process. Instead of graphing a function given its equation, we find the equation for a function given its graph. This process is similar to what we introduced in the summary above.

Example 8.2.5

Find the equation of the graphed function.

a.

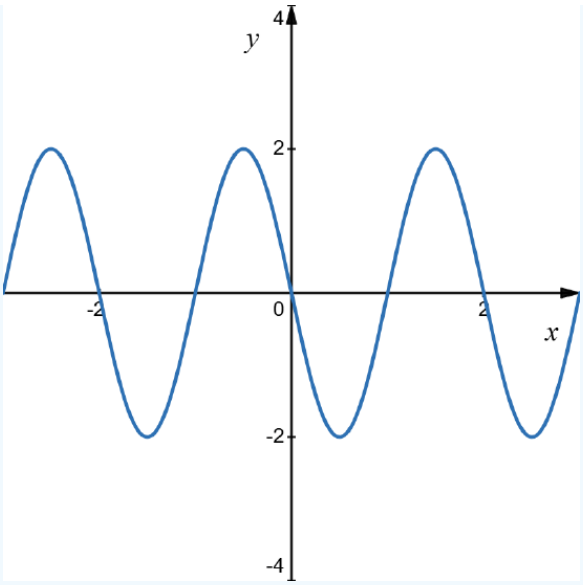


Figure 8.2.31

b.

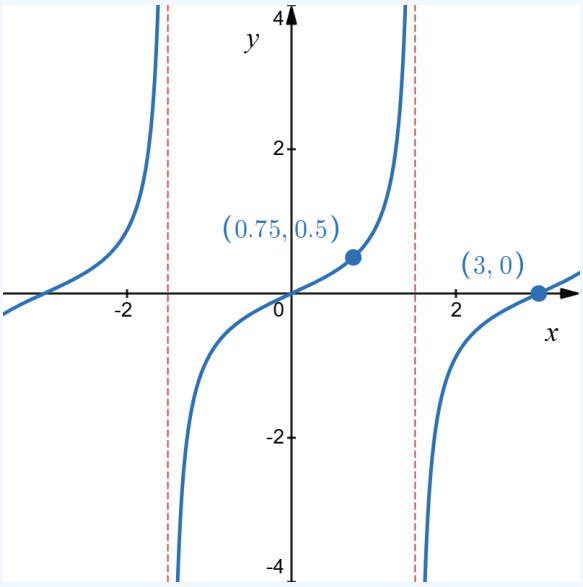


Figure 8.2.32

Solutions available at the end of this section.

Checkpoint 8.2.5

Find the equation of the function graphed below.

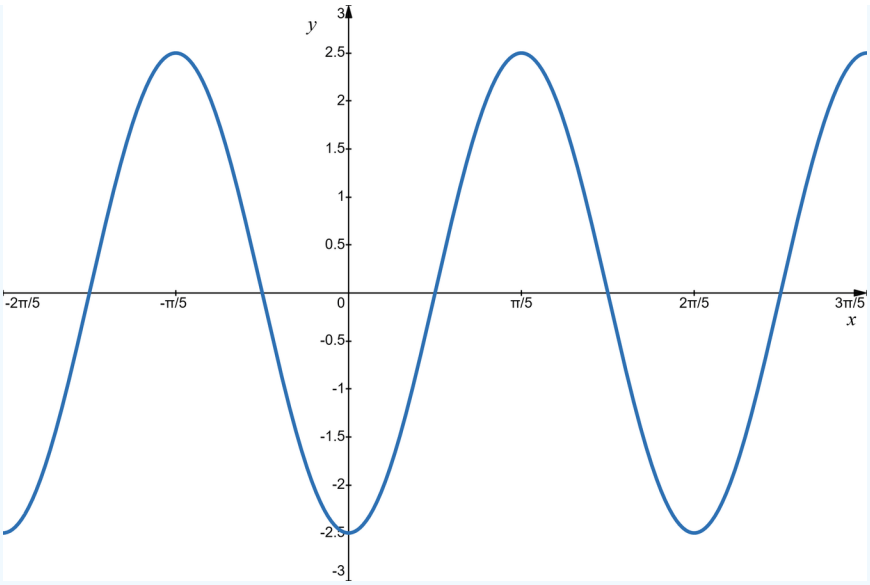


Figure 8.2.33

Answer available at the end of this section.

Success in College: Dealing With Failure (Part 3)

Sometimes unexpected events happen during a term. You get sick and are in bed for a week. Someone that you need to help care for gets sick. You are in a car accident and lose your transportation. A daycare provider quits. A relationship falls apart. Your boss at work needs you to increase your hours. Because this happens, you don't have time to keep up with your courses. You miss class. You fail one or more tests.

1. A student in this situation decides to withdraw from the math class. She reasons that it is better to concentrate on passing her other classes. She will re-enroll in math the next term. Are there any adverse consequences of making this choice? Explain.
2. A student in this situation decides to keep coming to class and trying to do the homework. He plans to take the course over the next term but wants to learn as much as he can now. This should make the course easier next term. Are there any adverse consequences of making this choice? Explain.

Success in Mathematics: Math Anxiety (Part 2)

I'm going to keep this short (for a change). Please take this math anxiety test to help you better understand your math anxiety. While at that website, check out the hints on overcoming math anxiety.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. A suggestion for reducing math anxiety is to come to class prepared. If you are prepared, then you are ready to be successful. When you are successful, you start learning that you do not have to be anxious. What do you think it means to be prepared for your math class?
2. A suggestion for reducing math anxiety is to practice math every day. How many days a week do you do math? How long, on average, do you spend doing math on those days?

Answers

Section Preview

The choice to cover non-rigid transformations (amplitude and period) before the rigid transformations is purposeful. Traditionally, when performing transformations,

it is best to save the rigid transformations for the end.

Section Preview

- **Graphing Functions**
  - Transformations — Non-Rigid Transformations: Students need to already know what a vertical stretch/compression of a general function looks like before the section can apply those ideas to trig functions.

Section Preview

- Determine the amplitude of a sinusoidal function.
- Determine the period of a sine, cosine, or tangent function.
- Graph a sine, cosine, or tangent function having a non-standard amplitude or period.
- Determine the equation of the graph of a trigonometric function.

Example 8.2.1

a. Now that we know the basic shape of the sine function, we only need to make a table of the quadrantal angles.

[redacted]

Plotting these points and graphing the base sine function, we get the following:

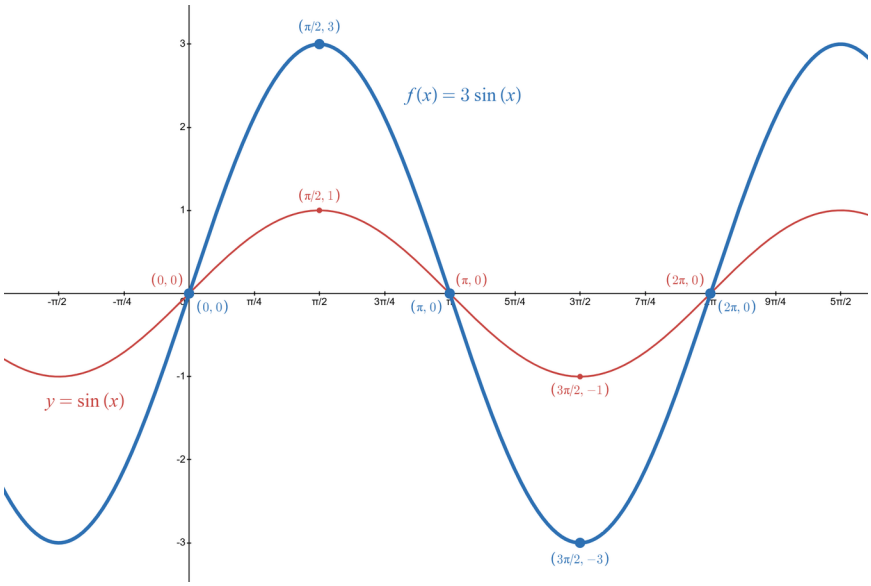


Figure 8.2.5

We can immediately see that the graph of $f(x) = 3\sin(x)$ is a vertical stretch of the base graph by a factor of 3. That is, all y-values of $f(x) = 3\sin(x)$ are three times that of the y-values for $y = \sin(x)$. Judging from this single example, it *looks like* the role of A in $f(x) = A\ \text{trig}(x)$ is to multiply the y-values of the base trigonometric function by A. However, looks can be deceiving, and this is *not* a proof. Let's do the second example and see if our insight holds.

b. Again, we use our knowledge of the basic shape of the cosine graph, along with a table of quadrantal angles.

[redacted]

Plotting these points and graphing the base cosine function, we get the following:

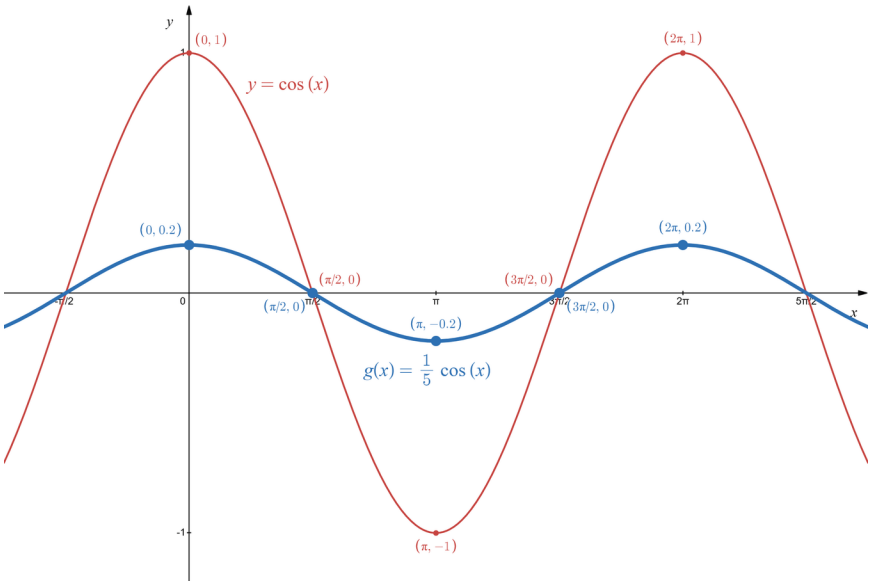


Figure 8.2.6

As before, we see that the graph of $g(x) = \frac{1}{5}\cos(x)$ is a vertical compression of the base graph by a factor of $1/5$. That is, all y-values of $g(x) = \frac{1}{5}\cos(x)$ are one-fifth of the y-values for $y = \cos(x)$.

c. The tangent function always requires special consideration because it is undefined at certain quadrantal angles. Therefore, our approach from parts (a) and (b) needs to be modified. Again, this is where having a strong grasp of the base graph for the tangent function comes into play. Since the natural period of the tangent is π, we will plot coordinates for $x = 0, \pi/4, 3\pi/4$, and π (completely avoiding the domain issue by skipping over $x = \pi/2$).

[redacted]

Plotting these points and graphing the base tangent function, we get the following:

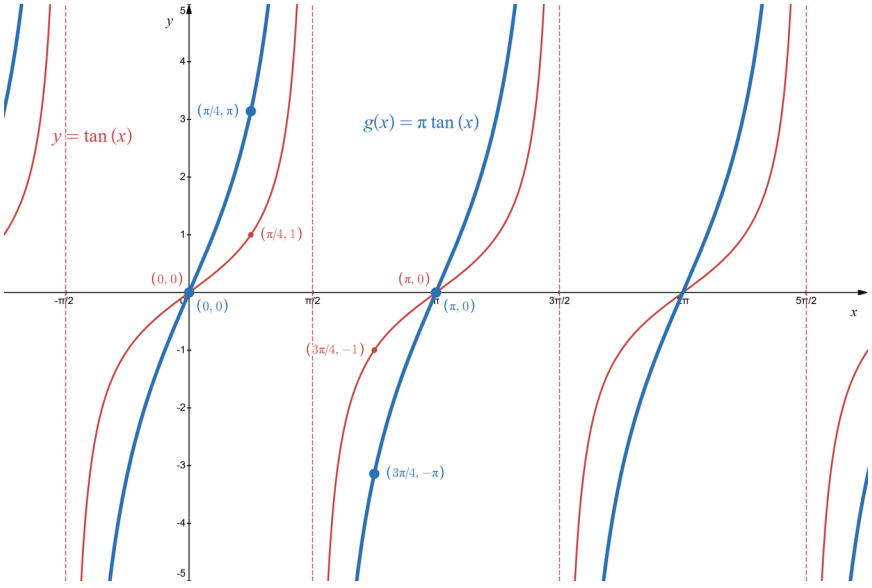


Figure 8.2.7

Again, $h(x) = \pi\tan(x)$ is a vertical stretch of the base graph by a factor of π.

Checkpoint 8.2.1

a.

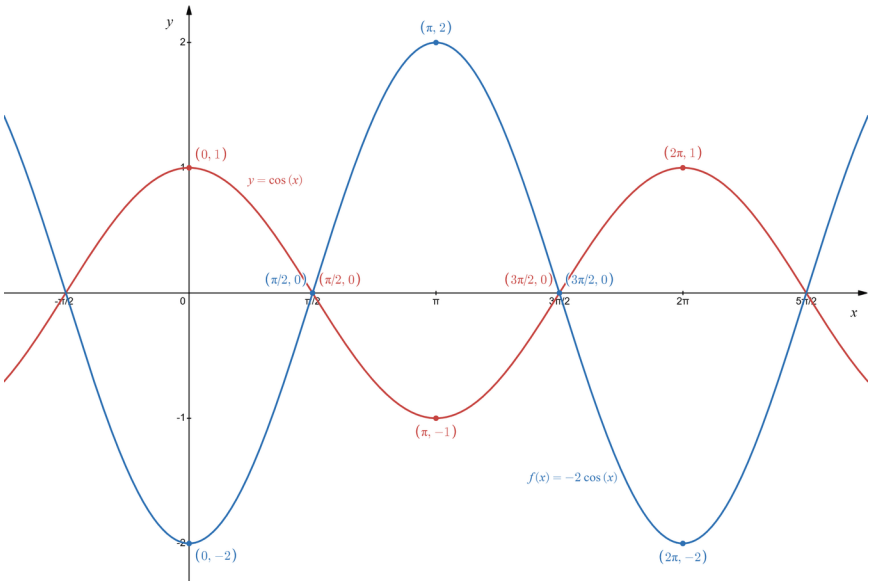


Figure 8.2.8

The coefficient of -2 multiplies all y-values of the base graph by -2. This effectively causes a vertical stretch (by a factor of 2) and a reflection about the x -axis.

b.

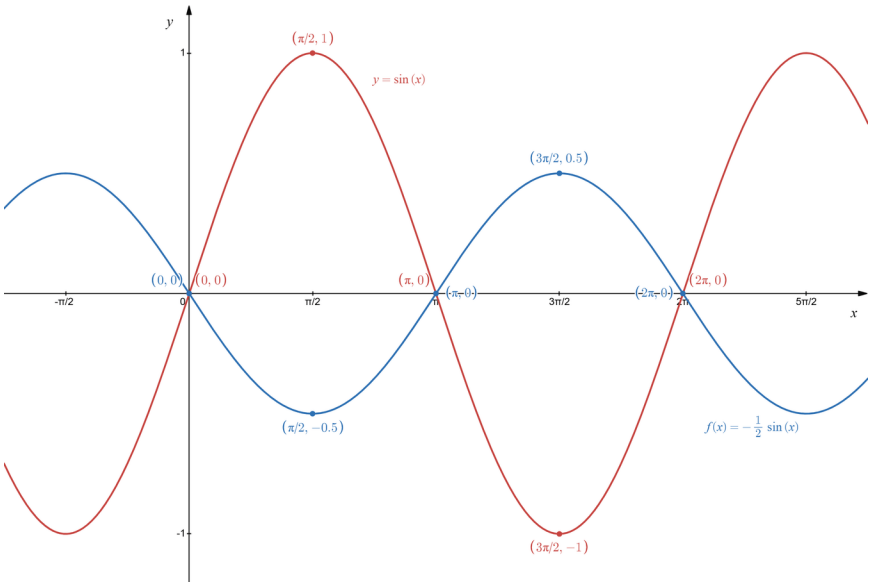


Figure 8.2.9

The coefficient of $-1/2$ multiplies all y-values of the base graph by $-1/2$, effectively compressing the graph vertically by a factor of $1/2$ and reflecting it about the x-axis.

Example 8.2.2

a. From Example 8.2.1a, we see the greatest and least values of the graph are 3 and -3, respectively. Thus, the amplitude of $f(x) = 3\sin(x)$ is

$$\text{Amplitude} = \frac{1}{2}|3-(-3)| = \frac{1}{2}|6| = 3.$$

b. From Example 8.2.1b, we see the greatest and least values of the graph are $1/5$ and $-1/5$, respectively. Thus, the amplitude of $g(x) = \frac{1}{5}\cos(x)$ is

$$\text{Amplitude} = \frac{1}{2}\left|\frac{1}{5}-\left(-\frac{1}{5}\right)\right| = \frac{1}{2}\left|\frac{2}{5}\right| = \frac{1}{5}.$$

Checkpoint 8.2.2

a. 2
b. $\frac{1}{2}$
c. Trick question... the tangent function does not have an amplitude.

Example 8.2.3

a. The process we illustrate here (and that we build on moving forward) is a bit algorithmic but very helpful in graphing trigonometric functions. It's based on the "template" trigonometric function

$$y = A\,\text{trig}\,(Bx + C) + D.$$

In this section, $C = D = 0$, so we are only dealing with functions of the form

$$y = A\,\text{trig}\,(Bx)\,.$$

Specifically, for this problem, we have $f(x) = 1\cos(2x)$. We read this from left to right.

$$\begin{array}{rcl} A = 1 \implies \text{Amplitude} & = & 1 \\ B = 2 \implies \text{Period} & = & \dfrac{\text{natural period}}{B} = \dfrac{2\pi}{2} = \pi \\ \text{Step Size} & = & \dfrac{\text{Period}}{4} = \dfrac{\pi}{4} \end{array}$$

We can now sketch an accurate graph of a single cycle. The initial x-value in the principal cycle of all trigonometric functions is $x = 0$. To find the next key number, we add the step size. See the figure below.

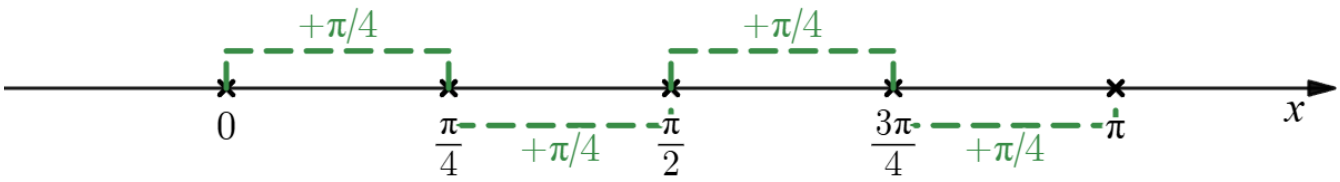


Figure 8.2.15

The length of this interval is π, which matches the period, so we are on track for success. Since the initial key number is at $x = 0$, we now know where the y-axis is, so let's draw that in.

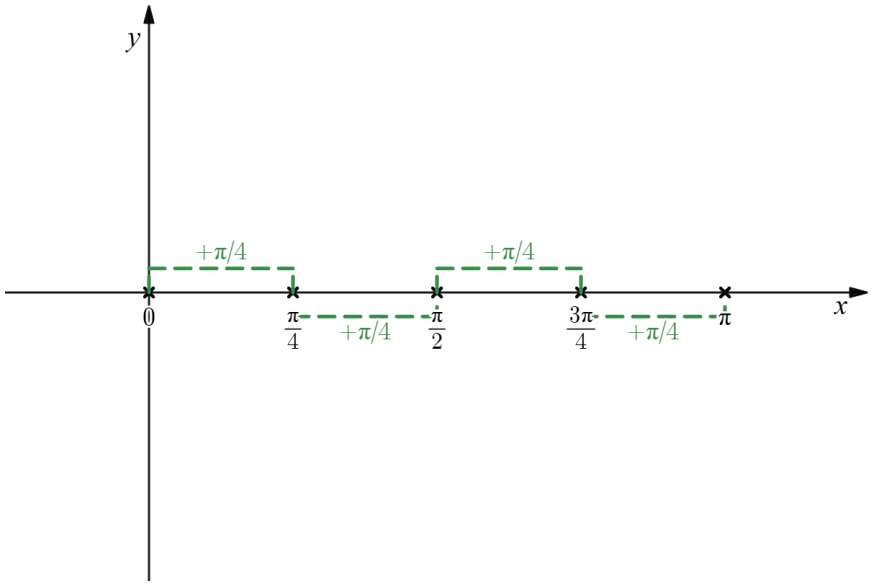


Figure 8.2.16

The function we are graphing is the *positive* cosine, so we can fill in the basic shape of a single cycle.

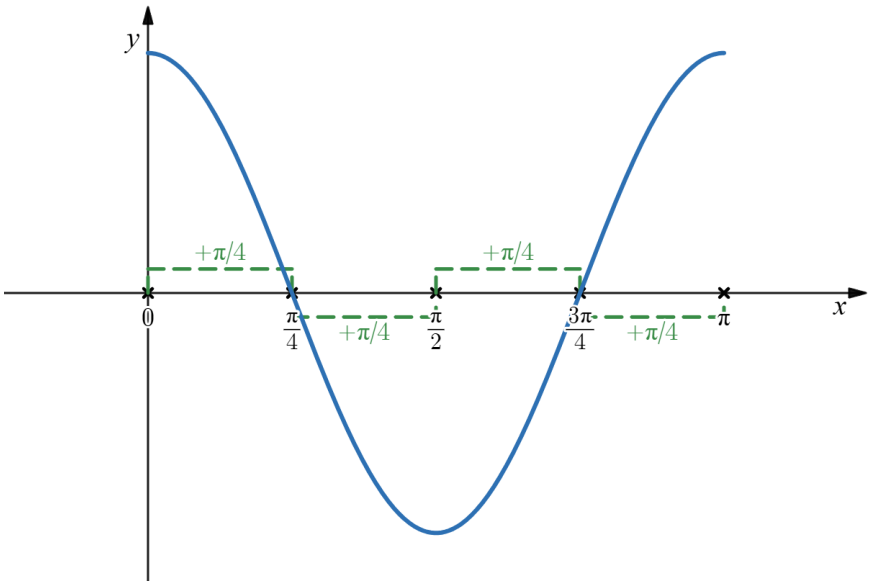


Figure 8.2.17

Finally, we use the amplitude to label the y-axis. Since the amplitude is 1, the maximum value is 1 and the minimum value of the function is -1.

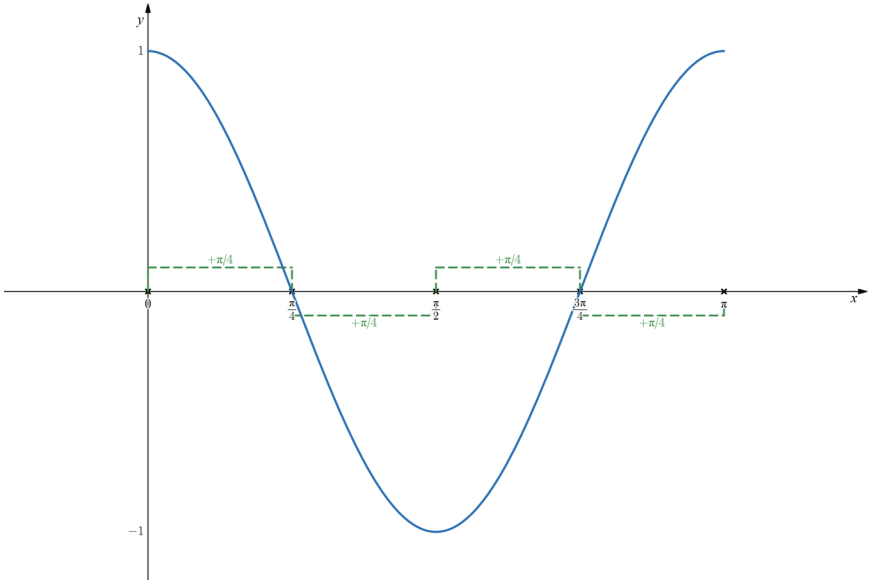


Figure 8.2.18

I know I said that, unless asked, you should refrain from graphing more than a single cycle, however, it is nice to let your reader know that this pattern is cyclic. We showcase this by extending the graph for a short distance in both directions.

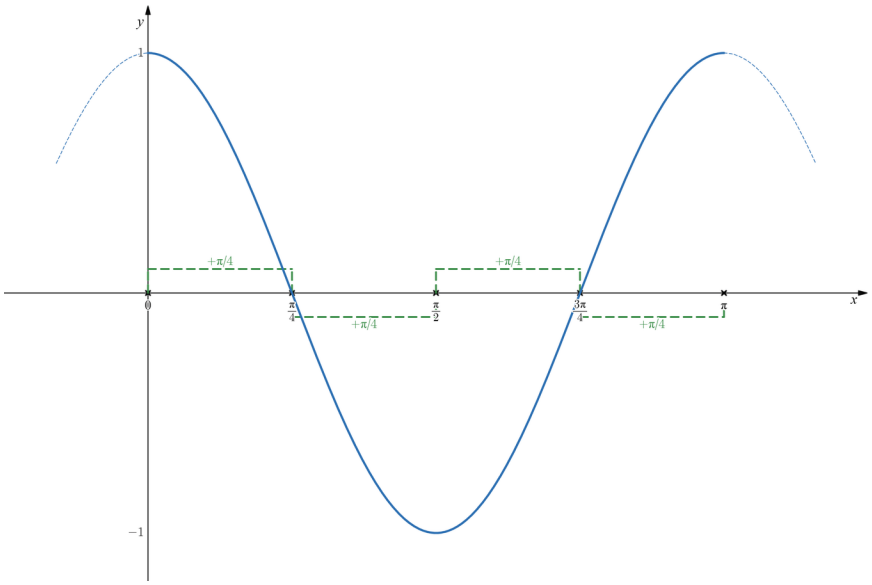


Figure 8.2.19

b. We have $g(x) = -3\sin\left(\frac{x}{3}\right) = -3\sin\left(\frac{1}{3}x\right)$. Reading this from left-to-right, we get the following.

A	$=$	-3	$\Longrightarrow$	Amplitude	$=$	3
B	$=$	$\frac{1}{3}$	$\Longrightarrow$	Period	$=$	$\frac{\text{natural period}}{B} = \frac{2\pi}{1/3} = 6\pi$
				Step Size	$=$	$\frac{\text{Period}}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$

Remembering that the initial x-value for the principal cycle is $x = 0$, we find all of the key numbers and label the x-axis.

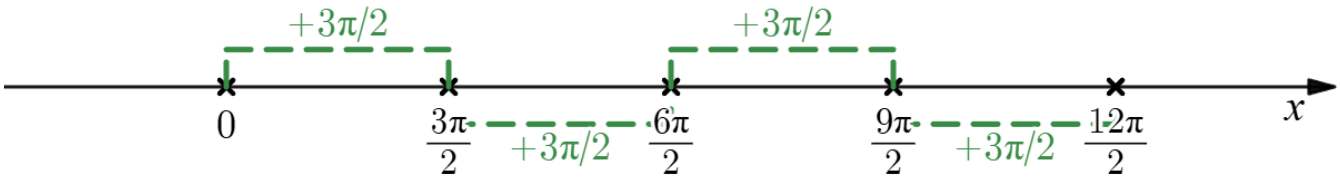


Figure 8.2.20

Again, we see the length of this interval is $\frac{12\pi}{2} = 6\pi$, which is the period. Therefore, we must be doing something right! Placing the y-axis appropriately, we get the following.

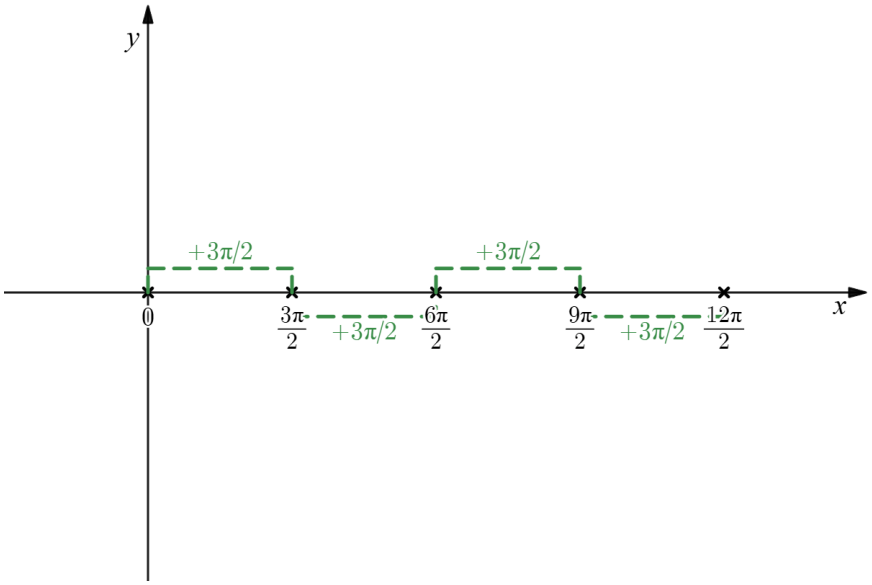


Figure 8.2.21

Here is where paying attention to the original function is essential (technically, paying attention is always important)—the amplitude of the function is 3, but

the sine function has been reflected about the x-axis because $A < 0$. Therefore, we label the y-axis with the maximum and minimum values of 3 and -3 but draw our sine function reflected about the x-axis.

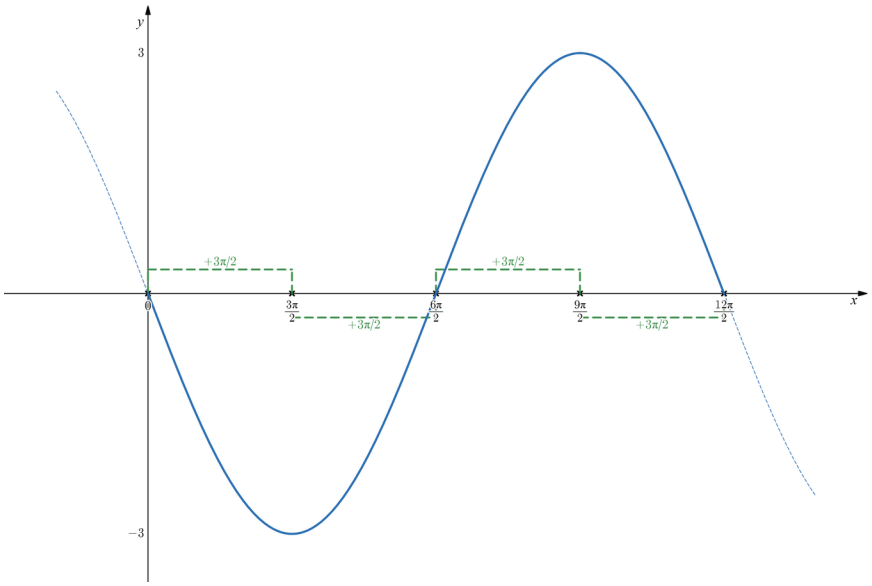


Figure 8.2.22

c. The tangent function might seem like the oddball of the trigonometric functions. Still, the process of sketching a single cycle remains the same, except that the tangent does not have an amplitude. One difference you should notice with this specific example is that the coefficient on x is *negative*. This is where remembering the Symmetry Identities pays off. Since tangent is an odd function,

$$y = \frac{1}{5}\tan(-4x) = -\frac{1}{5}\tan(4x).$$

Now that we have taken care of the negative argument in the function, we get to work. It is essential to remember that the natural period of the tangent function is π.

$$\begin{array}{rcl} B = 4 \implies \text{Period} & = & \dfrac{\text{natural period}}{B} = \dfrac{\pi}{4} \\ \text{Step Size} & = & \dfrac{\text{Period}}{4} = \dfrac{\pi/4}{4} = \dfrac{\pi}{16} \end{array}$$

Again, the initial x-value in the principal cycle is $x = 0$. To find the next key number, we add the step size. See the figure below.

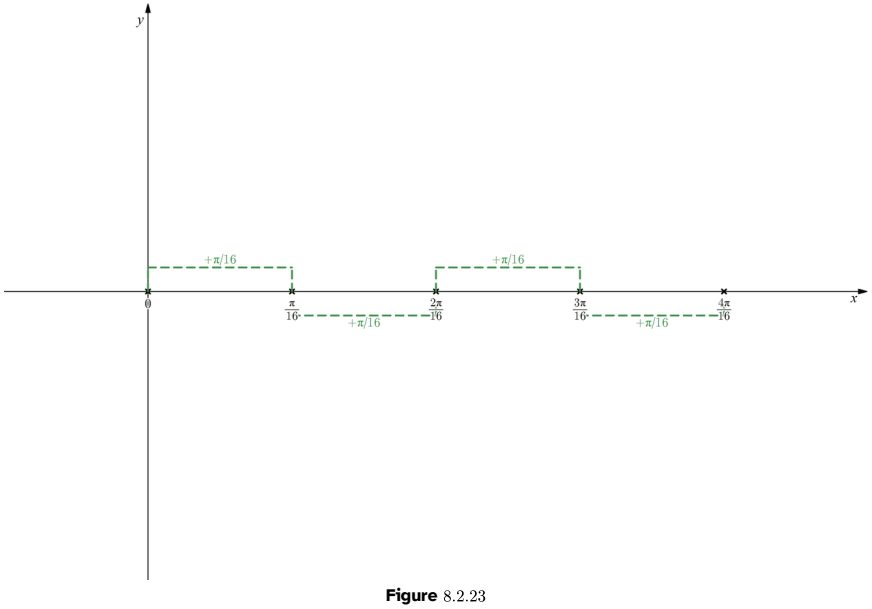


Figure 8.2.23

We know what the base graph of the tangent looks like, so we know the beginning and ending key numbers (0 and $4\pi/16$) are going to be the zeros of the function and the middle key number ($2\pi/16$) is the vertical asymptote; however, what do we do with $\pi/16$ and $3\pi/16$? Up to this point, we have not bothered with the coefficient of the function, $-1/5$. Here is where it comes into play. If we were to evaluate the given function at $\pi/16$, we would get

$$-\frac{1}{5}\tan\left(4\left(\frac{\pi}{16}\right)\right) = -\frac{1}{5}\tan\left(\frac{\pi}{4}\right) = -\frac{1}{5}(1) = -\frac{1}{5}.$$

That is, the excess key numbers, $\pi/16$ and $3\pi/16$, give us two convenient points to plot on the graph, and the value at these points is $\pm A$. Sketching a tangent function that has been reflected about the x-axis and plotting those two points, we get the following graph.

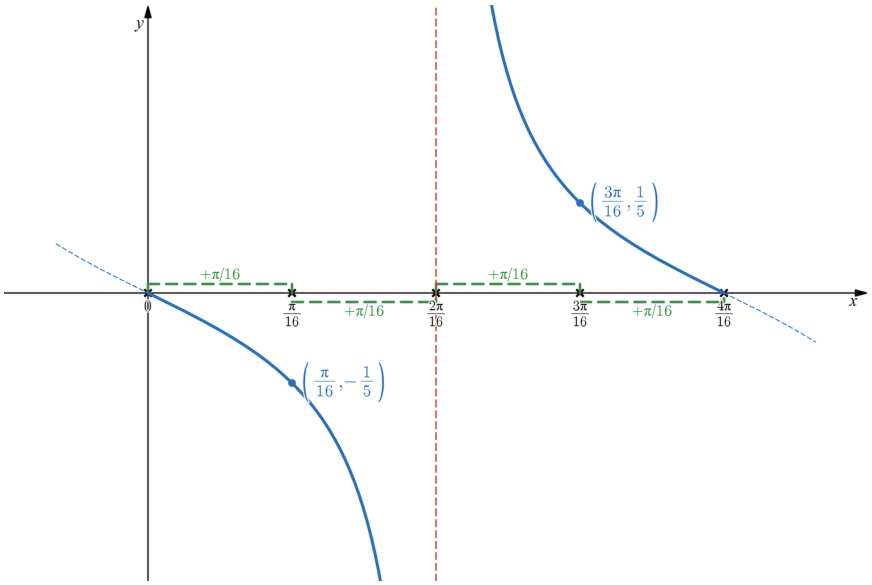
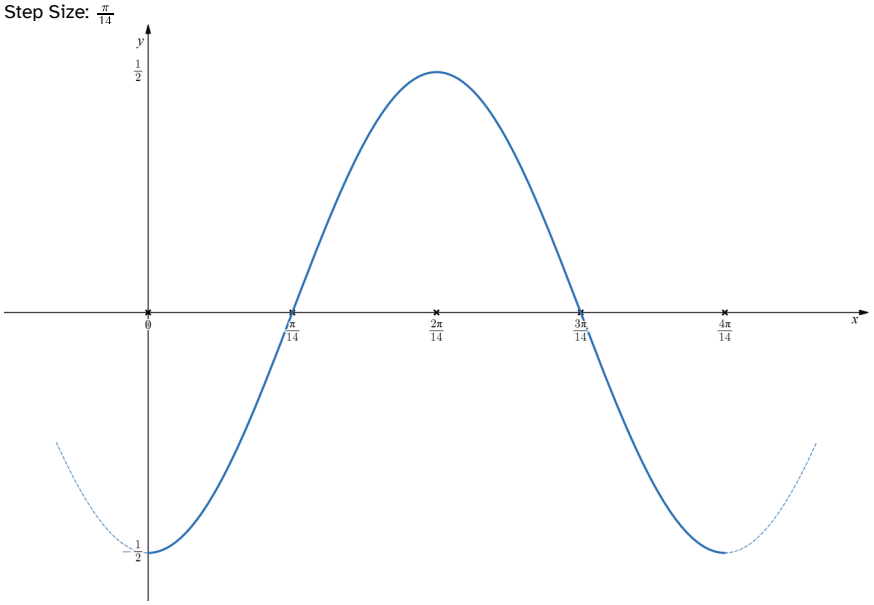


Figure 8.2.24

Checkpoint 8.2.3

Amplitude: $\frac{1}{2}$
Period: $\frac{2\pi}{7}$
Step Size: $\frac{\pi}{14}$



Example 8.2.4

a. **Precondition Stage:** Since the coefficient of x is negative, we use the Symmetry Identities to rewrite the given function. Cosine is an even function. Therefore, $\cos(-\theta) = \cos(\theta)$. Hence, $g(x)$ can be rewritten as

$$g(x) = -\frac{1}{2}\cos\left(\frac{2}{5}x\right).$$

Information Stage: Since the given function is sinusoidal, it has an amplitude, and its natural period is 2π.

$$\begin{array}{rcl} A = -\frac{1}{2} & \Longrightarrow & \text{Amplitude} = |A| = \frac{1}{2} \\ B = \frac{2}{5} & \Longrightarrow & \text{Period} = \frac{\text{Natural Period}}{B} = \frac{2\pi}{2/5} = 10\pi \\ & \Longrightarrow & \text{Step Size} = \frac{\text{Period}}{4} = \frac{10\pi}{4} = \frac{5\pi}{2} \end{array}$$

Graphing Stage: Drawing a horizontal line (the x-axis), marking the initial key number ($x = 0$), and marking the remaining key numbers using the step size, we get the following.

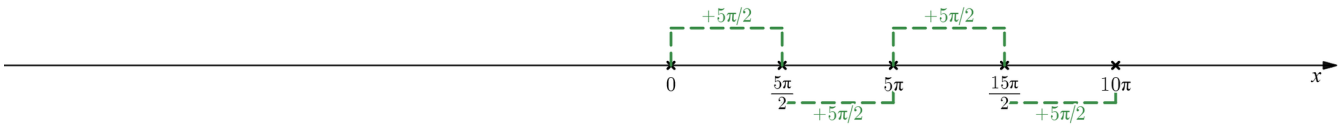


Figure 8.2.25

Notice that I left quite a bit of open space on the left half of the number line. This is because we will eventually extend the single cycle of our cosine function so that our graph covers the interval $[-10\pi, 10\pi]$. We take a moment to make sure the distance from the initial key number ($x = 0$) to the final key number ($x = 10\pi$) is the same as the period. In this case, the distance is

$$|10\pi - 0| = 10\pi = \text{Period},$$

so we are on the right track. We now sketch a single cycle of the *reflected* cosine (because $A < 0$).

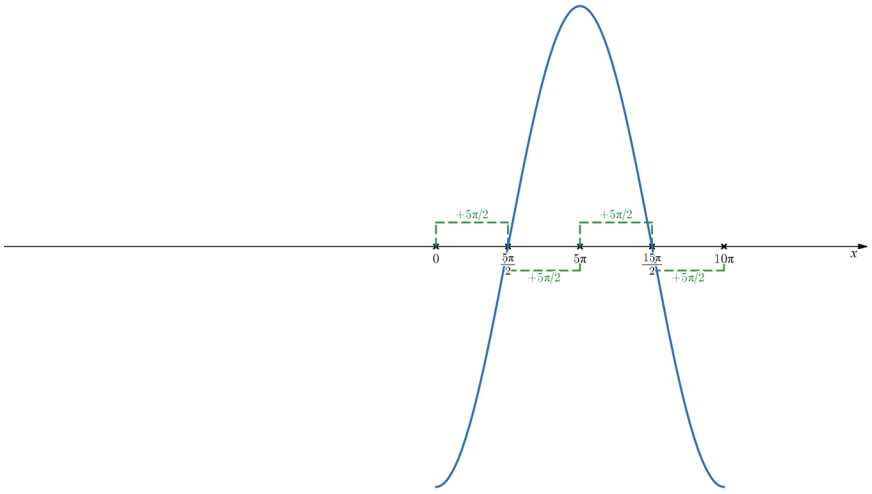


Figure 8.2.26

Finally, we draw in the y-axis at $x = 0$, label the maximum and minimum values using the amplitude, and extend the graph to cover the interval $[-10\pi, 10\pi]$.

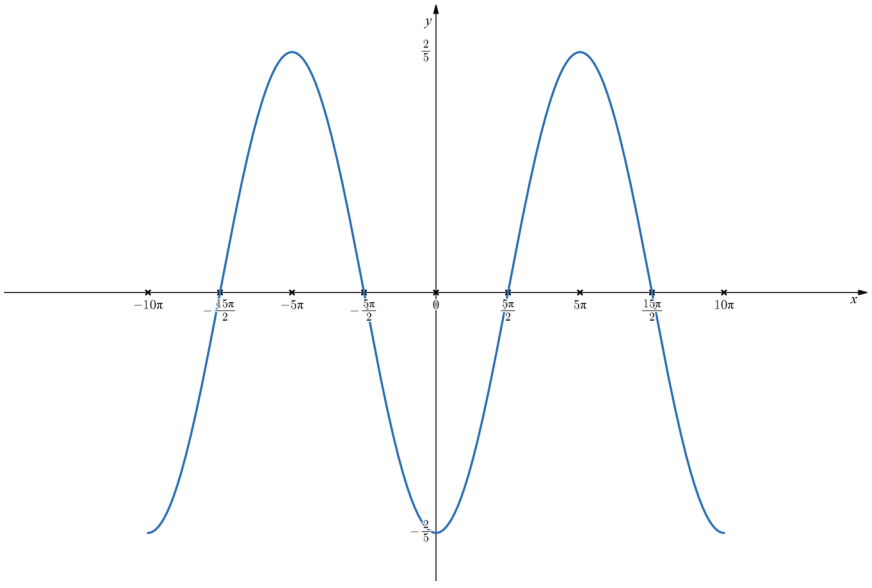


Figure 8.2.27

b. **Precondition Stage:** Since the coefficient of x is already positive, we do not need to clean up the trigonometric function.
Information Stage: Since h is a tangent function, it does not have an amplitude, and its natural period is π.

$$\begin{array}{rcl} B = 6 & \Longrightarrow & \text{Period} = \dfrac{\text{Natural Period}}{B} = \dfrac{\pi}{6} \\ & \Longrightarrow & \text{Step Size} = \dfrac{\text{Period}}{4} = \dfrac{\pi/6}{4} = \dfrac{\pi}{24} \end{array}$$

Graphing Stage: Starting at $x = 0$ and marking the key points, we get the following number line (which has been extended because of the interval over which we are asked to graph).



Figure 8.2.28

Note that the distance between the beginning key number and the ending key number is

$$\left|\frac{4\pi}{24} - 0\right| = \frac{4\pi}{24} = \frac{\pi}{6} = \text{Period}.$$

Thus, we are good to go! We now graph a single cycle of the tangent function. Remember that there is a vertical asymptote in the middle of the cycle (at $x = 2\pi/24$).

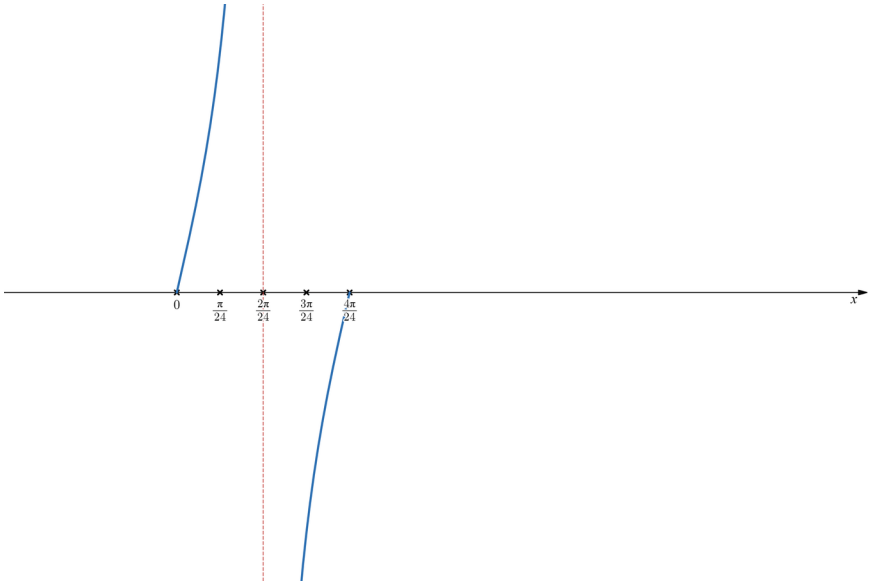


Figure 8.2.29

From here, we draw the y-axis at $x = 0$, label the points on the curve at $x = \pi/24$ and $x = 3\pi/24$, and extend the graph to cover the requested interval.

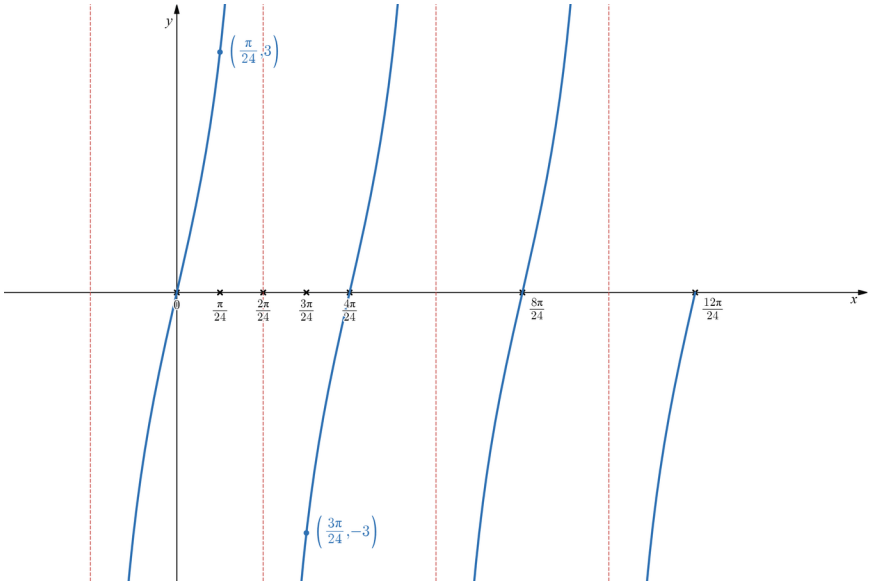


Figure 8.2.30

Example 8.2.5

a. Our first task is to identify the base function. The shape of this function is sinusoidal. Moreover, because it goes through the origin, it is a sine function (although, technically, it can be written as a cosine function as well). However, it has been reflected about the x-axis, so we know that $A < 0$. We can also see that the amplitude is 2. Thus, $A = -2$ so that

$$f(x) = -2\sin(Bx).$$

All we need to do is identify B. We know that B is *associated* with the period of the function, but it is not *equal* to the period. The relationship between B and the period is

$$\text{Period} = \frac{\text{Natural Period}}{B}.$$

However, the natural period of the sine function is 2π, so

$$\text{Period} = \frac{2\pi}{B} \implies B = \frac{2\pi}{\text{Period}}.$$

The graph shows that the function completes one full cycle every 2 units. Hence, the period is 2. Substituting that into our formula, we get

$$B = \frac{2\pi}{2} = \pi.$$

Thus, our function is

$$f(x) = -2\sin(\pi x).$$

b. The base function here is the tangent. So we know we are dealing with a function of the form

$$g(x) = A\tan(Bx).$$

The value of A will be a bit trickier to get here because the tangent does not have an amplitude; however, all is not lost. We instead start by finding B. As in part (a),

$$\text{Period} = \frac{\text{Natural Period}}{B}$$

and we know the natural period of the tangent function is π. Therefore,

$$\text{Period} = \frac{\pi}{B} \implies B = \frac{\pi}{\text{Period}}.$$

We can see that a single cycle gets completed in 3 units; thus, the period of this function is 3. Hence,

$$B = \frac{\pi}{3}.$$

Now we have

$$g(x) = A\tan\left(\frac{\pi}{3}x\right).$$

To find the value of A, we recall that any point on the graph of an equation must also satisfy the equation. In particular, because the point $\left(\frac{3}{4}, \frac{1}{2}\right)$ lies on this tangent curve, when $x = \frac{3}{4}$, it must be the case that $y = \frac{1}{2}$. That is,

$$\frac{1}{2} = A\tan\left(\frac{\pi}{3}\cdot\frac{3}{4}\right) \implies \frac{1}{2} = A\tan\left(\frac{\pi}{4}\right) \implies \frac{1}{2} = A\cdot 1 \implies A = \frac{1}{2}.$$

Thus, the equation of this tangent function is

$$g(x) = \frac{1}{2}\tan\left(\frac{\pi}{3}x\right).$$

Checkpoint 8.2.5

$f(x) = -\frac{5}{2}\cos(5x)$

This page titled 8.2: Non-Rigid Transformations is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

8.2.1: Resources and Key Concepts

Resources

Prerequisite Topics

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Graphing Functions**
  - Transformations: This section is entirely about applying non-rigid transformations (stretches, compressions, and reflections) to the base graphs of trigonometric functions. A prior understanding of how parameters alter a base graph is essential.

Videos

- Transformations
  - Trigonometric Graphs - Review of Graphs of Sine and Cosine Functions
  - Trigonometric Graphs - Review of Identifying the Equation for a Sinusoidal Function from a Graph
  - Graphs of the Other Trigonometric Functions - Graphing Tangent and Cotangent Functions
  - Graphs of the Other Trigonometric Functions - Identifying Equations of Trig Functions from Graphs

Key Concepts

Definitions

- **Transformation (of the graph of a function):** An alteration of a graph's orientation, scale, and/or position through various mathematical operations.
- **Rigid Transformation:** A transformation that changes a graph's position (e.g., shifts) but keeps its original orientation and scale.
- **Non-rigid Transformation:** A transformation that keeps the graph's original center position but changes its scale (stretching or compressing) or orientation (reflecting).
- **Parameter:** A variable that influences the output or behavior of a function but is held constant within a specific context. In this section, A and B in $y = A\text{trig}(Bx)$ are parameters.
- **Key Numbers:** The x-values of the initial, final, middle, and quarter-points of a single cycle of a trigonometric function's graph.
- **Step Size:** The horizontal distance between any two consecutive key numbers.

Theorems

- **Vertical Scaling:** The graph of $f(x) = A\text{trig}(x)$ vertically scales the graph of $y = \text{trig}(x)$ by a factor of $|A|$. If $|A| > 1$, it is a vertical stretch. If $0 < |A| < 1$, it is a vertical compression. If $A < 0$, the graph is also reflected about the horizontal axis.
- **Amplitude of a Sinusoidal Function:** The amplitude of $f(x) = A\cos(x)$ or $g(x) = A\sin(x)$ is $|A|$.
- **Period of a Trigonometric Function:** If p is the natural period of a trigonometric function, then the period of $y = A\text{trig}(Bx)$ is $\frac{p}{|B|}$. For sine and cosine, the period is $\frac{2\pi}{|B|}$. For tangent, the period is $\frac{\pi}{|B|}$.
- **Step Size Formula:** The step size for graphing a trigonometric function is given by the formula $\frac{\text{Period}}{4}$.

Common Mistakes

- **Confusing Period and the Parameter B:** The period of a function like $y = \sin(Bx)$ is not B. The period is calculated by

dividing the natural period of the base function by $|B|$.

- **Using the Wrong Natural Period:** The natural period of sine and cosine is 2π, while the natural period of tangent is π. Using the wrong natural period when calculating the new period is a common error.
- **Ignoring a Negative Coefficient on the Angle:** When graphing a function like $y = \tan(-4x)$, you must first use the Symmetry Identities to handle the negative. Since tangent is an odd function, $y = \tan(-4x) = -\tan(4x)$. Graphing without this first step will lead to an incorrect orientation.
- **Forgetting Tangent Has No Amplitude:** The concept of amplitude applies to sinusoidal functions (sine and cosine) because they have maximum and minimum values. The tangent function is unbounded, so it does not have an amplitude.

This page titled 8.2.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

8.2.2: Homework

Reading Questions

1. What is the difference between a rigid and a non-rigid transformation?
2. In the function $y = A\sin(Bx)$, what effect does the parameter A have on the graph? What happens if A is negative?
3. In the function $y = A\cos(Bx)$, what effect does the parameter B have on the graph?
4. What is the amplitude of the function $y = -5\cos(x)$?
5. What is the period of the function $y = \sin(3x)$?
6. What is the period of the function $y = \tan(2x)$?
7. What are the five "key numbers" used to sketch one cycle of a trigonometric graph?
8. How do you calculate the "step size" for a graph?
9. When graphing $y = \cos(-2x)$, what property of the cosine function allows you to rewrite it first?
10. Why does the tangent function not have an amplitude?

Homework

Vocabulary Check

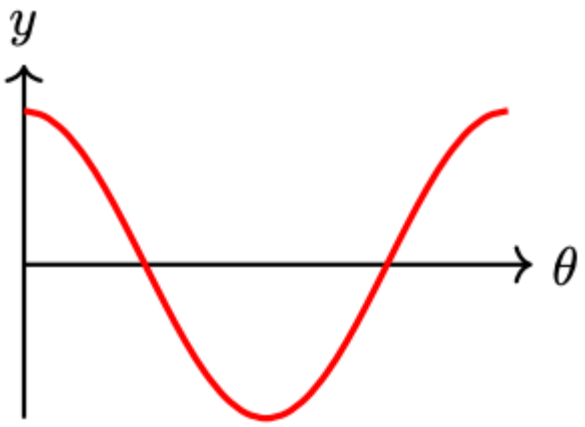
1. To ___ the graph of a function, we mean to stretch or compress the graph horizontally or vertically.
2. A ___ transformation keeps the function's original scale and orientation but changes its position.
3. A ___ transformation keeps the function's original central position but changes its scale or orientation.
4. If a graph is symmetric about the y-axis, we say it is an ___ function.
5. An___ function is one whose graph is symmetric about the origin.
6. If $A > 1$, then the graph of $y = A\sin(Bx)$ is ___ vertically by a factor of ___.
7. If $0 < A < 1$, then the graph of $y = A\tan(Bx)$ is ___ vertically by a factor of ___.

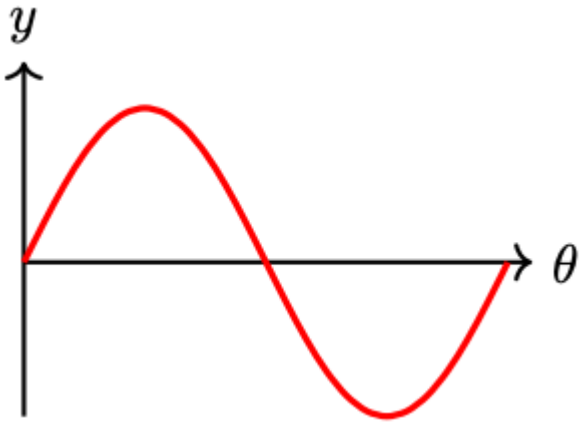
Concept Check

8. What is the range of $f(x) = A\sin(Bx)$?
9. What is the effect of $A < 0$ on the graph of the function $g(x) = A\tan(Bx)$?
10. What is the period of $h(x) = A\cos(Bx)$?
11. What is the period of $k(x) = A\tan(Bx)$?
12. When graphing $y = \cos(-Bx)$, we first rewrite the function as $y =$ ___ because cosine is an ___ function.
13. When graphing $y = \tan(-Bx)$, we first rewrite the function as $y =$ ___ because tangent is an ___ function.
14. One of these graphs is $y = A\sin k\theta$, and the other is $y = A\cos k\theta$. Explain how you know which is which.

    a.

b.





True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

15. The natural period of the cosine function is 2π.

16. The natural period of the tangent function is 2π.

17. Dividing the period by 4 gives us the step size.

18. Dividing the period of the function by 4 partitions a single cycle of the trigonometric function into four subintervals.

19. There are four key numbers.

Basic Skills

20. Write an equation for a sine function with amplitude 6.

21. Write an equation for a cosine function with amplitude $\frac{1}{2}$.

22. Write an equation for a sine function with period $\frac{\pi}{2}$.

23. Write an equation for a cosine function with period 4π.

For the following exercises, state the amplitude (if applicable) and period of the graph.

24. $y = -\cos(4x)$

25. $y = -\sin(3x)$

26. $y = -5\sin\left(\frac{x}{3}\right)$

27. $y = 6\cos\left(\frac{x}{2}\right)$

28. $y = \tan(2x)$

29. $y = \tan(4x)$

For the following exercises,
(i) Graph the function.
(ii) State the amplitude and period of the function.

30. $y = 3\cos(\theta)$

31. $y = 4\sin(\theta)$

32. $y = \cos(3\theta)$

33. $y = \sin(2\theta)$

For the following exercises, sketch one cycle of each graph by hand and label scales on the axes.

34. a. $y = \cos x$
    b. $y = \cos 4x$
    c. $y = -\cos x$

35. a. $y = \sin x$
    b. $y = \sin 3x$
    c. $y = -\sin 3x$

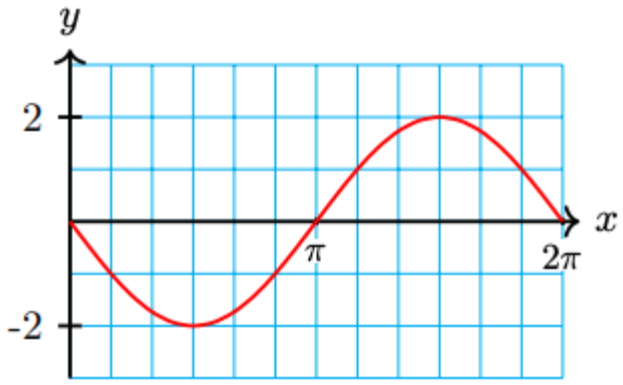
36. a. $y = \sin x$
    b. $y = \sin\frac{x}{3}$
    c. $y = -5\sin\frac{x}{3}$

37. a. $y = \cos x$
    b. $y = \cos\frac{x}{2}$
    c. $y = 6\cos\frac{x}{2}$

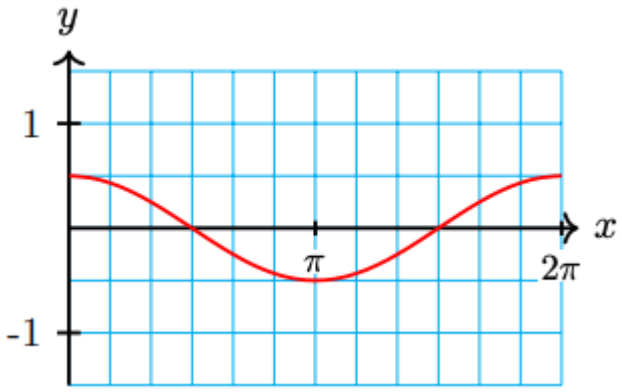
For the following exercises, write an equation for the graph using sine or cosine.

38.

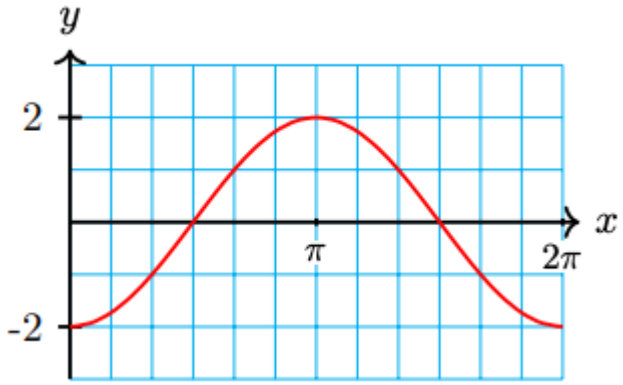
39.



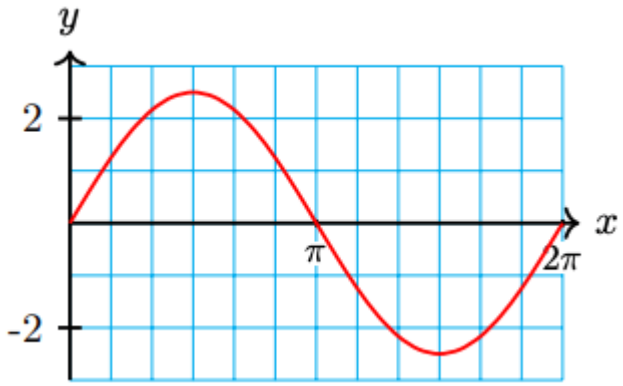
40.

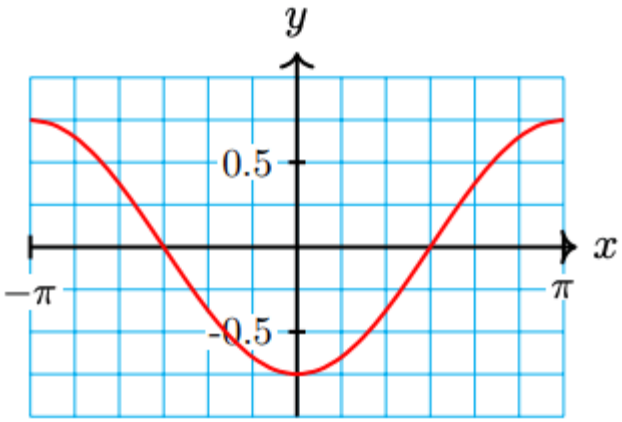


41.

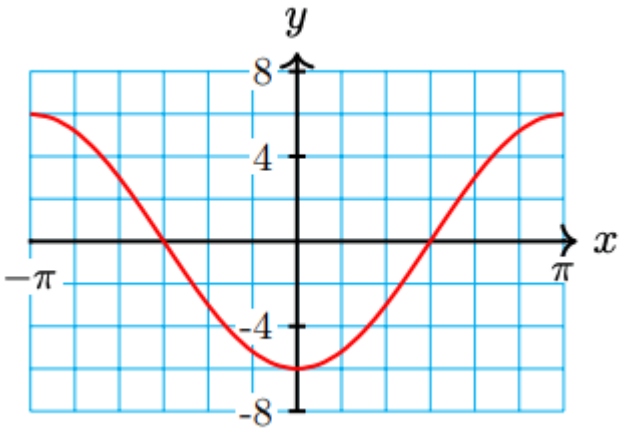


42.



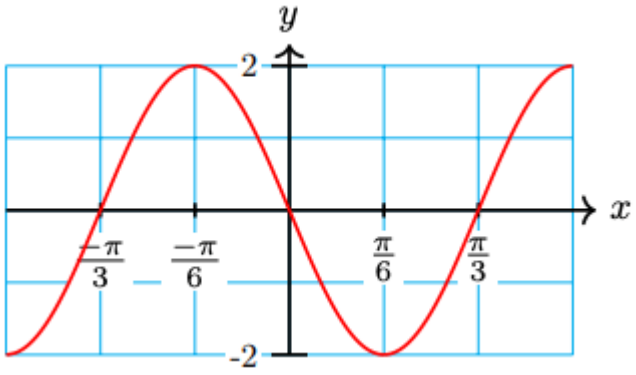


13.

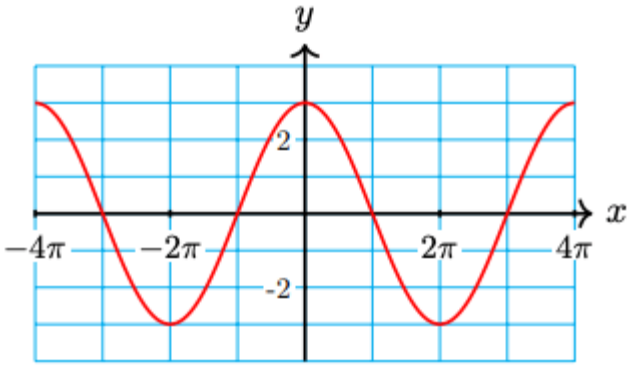


For the following exercises,
(i) State the graph's amplitude and period.
(ii) Write an equation for the graph using sine or cosine.

14.



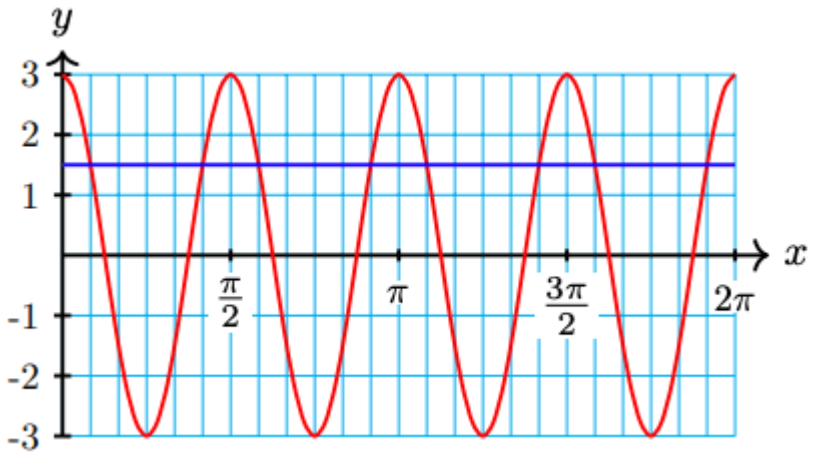
15.



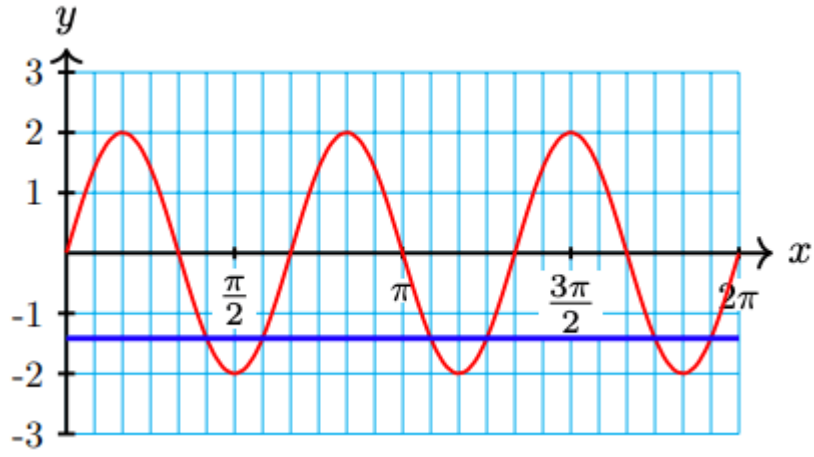
Synthesis Questions

For the following exercises, use the graph to find all solutions between 0 and 2π.

16. $3\cos(4x) = 1.5$



17. $2\sin(3x) = -\sqrt{2}$



For the following exercises,
(i) Use technology to graph the function for $0 \le x \le 2\pi$.
(ii) Use the graph from part (i) to approximate all solutions between 0 and 2π. Round your answers to hundredths.

18. a. $f(x) = 3\sin(2x)$

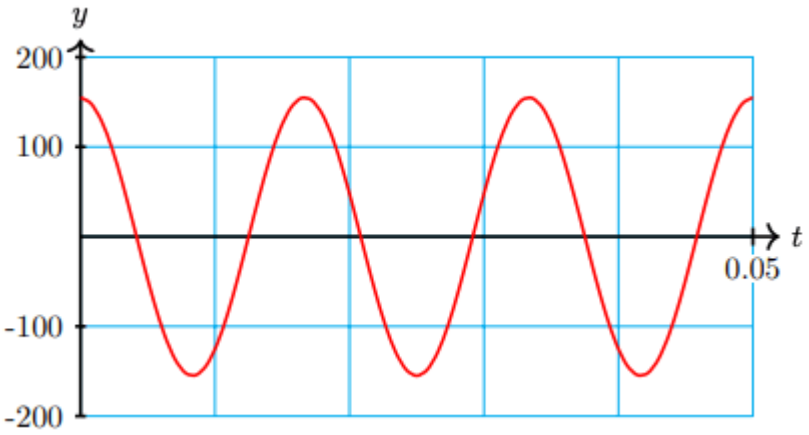
b. $3\sin(2x) = -1.5$

49. a. $g(x) = -2\cos(3x)$

b. $-2\cos(3x) = 1$

Applications

50. **Electric Current.** The following graph illustrates that the voltage used in U.S. electrical current changes from 155V to -155V and back 60 times each second. Write an equation for this graph.



51. **Electric Current.** If I represents the intensity of the current in an alternating circuit and t represents time, then

$$I(t) = 50\sin(120\pi t),$$

where I is measured in amperes and t is measured in seconds. Find the maximum value of I and the time it takes for I to go through one complete cycle.

52. **Sound Waves.** The oscillations in air pressure representing the sound wave for a tone at the standard pitch of C can be modeled by the equation

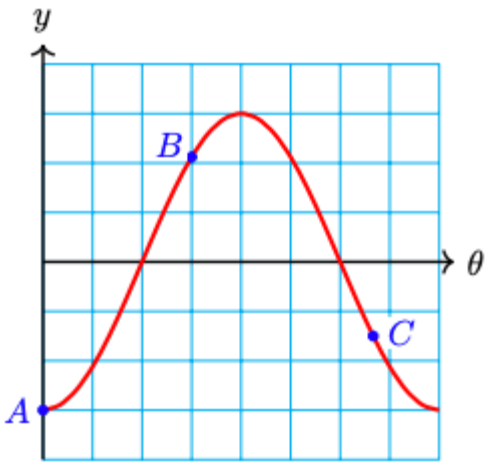
$$S(t) = 0.05\sin(524\pi),$$

where S is the sound pressure in pascals after t seconds. Sketch a single cycle of the graph of this function.

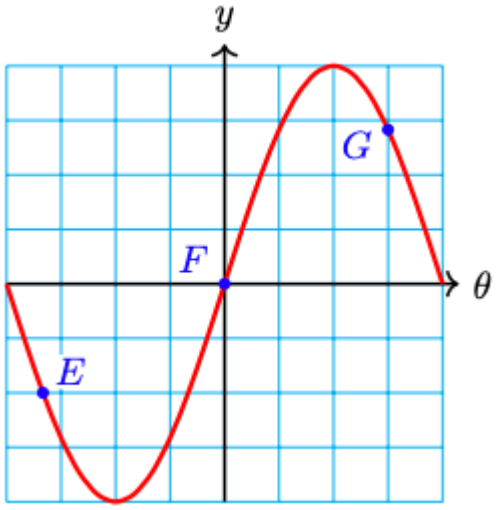
Challenge Problems

For the following exercises, give the coordinates of the points on the graph.

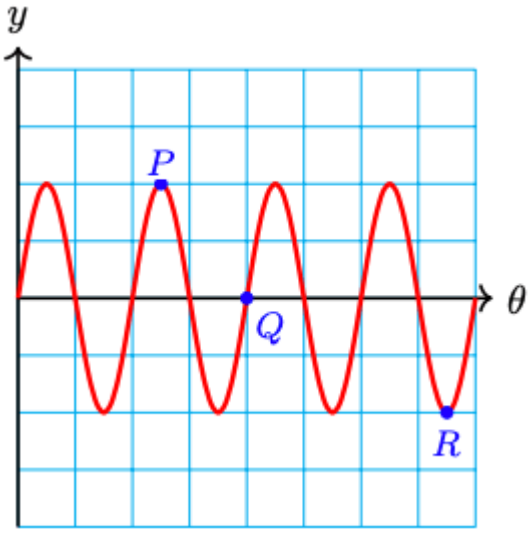
53. $f(\theta) = -3\cos(\theta)$



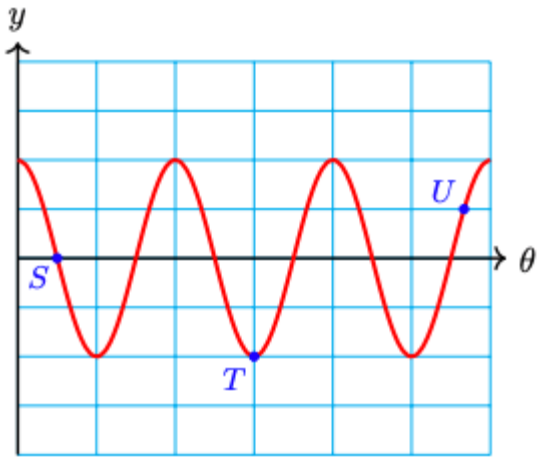
54. $f(\theta) = 4\sin(\theta)$



55. $f(\theta) = \sin(4\theta)$



56. $f(\theta) = \cos(3\theta)$



This page titled 8.2.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

8.3: Rigid Transformations

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

In the previous section, we stated that our ultimate goal (when it comes to graphing the trigonometric functions) is to graph any trigonometric function of the form

$$f(x) = A\,\mathrm{trig}\,(Bx + C) + D. \tag{8.3.1}$$

Now that we know the effect of the parameters A and B on the graph of the fundamental trigonometric functions, it's time to turn our attention to the parameters C and D.

Phase Shifts (also known as Horizontal Shifts)

Continuing with alphabetical order, let's focus on the effect of the parameter C; however, let's make things simple (for now). Recall (from Algebra), the graph of the function $y = g(x + C)$ is a horizontal shift of the graph of g. It moves the graph to the *left* C units if $C > 0$, and *right* C units if $C < 0$. Most students in Algebra accept this as truth and memorize the result; however, as you are taking a college-level Trigonometry course, it might serve you well to understand *why* this is true.

Click here to delve into why the graph of $y = g(x + C)$ is a horizontal shift of the graph of $g(x)$

Let $g(x)$ be a function. Then each input leads to a single output. Let's grab an *arbitrary* value from the domain of g and call it x_1 (you can call it "happy face" or "star," but x_1 is easier to write). Again, since g is a function, $g(x_1)$ must return a single value. Let's call this output y_1.

So far, all we have said is that g is a function and that $g(x_1) = y_1$; however, this last statement means the point $(x_1, g(x_1))$ is on the graph of g.

Now, your friend comes along and says, "Hey, I need to graph $g(x + C)$. All I have been given is that C is positive."

You pause and think for a moment. Turning to your friend, you say, "I don't know what that graph looks like, but I know that when I plug in x_1 to g, I get out y_1."

Your friend smirks and says, "Okay, let's plug x_1 into $g(x + C)$ to find its output!"

You watch him work for a few moments until he gets stuck. He ends up with $g(x_1 + C)$, and you look at him quizzically, "I don't know what the value of g is at $x_1 + C$. I only know its value at x_1."

Then it hits you!

"You did it wrong! You want to evaluate $g(x_1)$. You need to let the argument of your function, which is $x + C$, be x_1."

You start doing the math,

$$x + C = x_1 \implies x = x_1 - C.$$

You pause... thinking.

"So, to get the argument of $g(x+C)$ to be x_1, we need to plug in $x_1 - C$ for x."

Thinking even harder, you have the epiphany.

"With the original function, we plugged in x_1 and got out y_1. With $g(x+C)$ we needed to plug in $x_1 - C$ to get $g(x_1 - C + C) = g(x_1) = y_1$. That is, the original point, x_1, gets shifted to the left C units!"

Since our point, $(x_1, g(x_1))$ was arbitrary, all points on the graph of g experience the same shift. Hence, the graph of $g(x+C)$, where $C > 0$, is the graph of g shifted C units to the left.

Example 8.3.1

Make a table of values and sketch a graph of $y = -2\cos\left(x + \frac{\pi}{3}\right)$.

Solution available at the end of this section.

Checkpoint 8.3.1

Complete the table and sketch a graph of $y = -3\sin\left(x - \frac{\pi}{6}\right)$. (Do you expect the graph to be shifted to the left or the right, compared to the graph of $y = -3\sin(x)$?)

x	$x - \frac{\pi}{6}$	$\sin\left(x - \frac{\pi}{6}\right)$	$y = -3\sin\left(x - \frac{\pi}{6}\right)$
	0		
	$\frac{\pi}{2}$		
	π		
	$\frac{3\pi}{2}$		
	2π		

Answer available at the end of this section.

Example 8.3.1 and Checkpoint 8.3.1 should convince you that our previous experience from Algebra with horizontal shifts of functions still works with trigonometric functions; however, let's solidify that concept to drive the point home.

We know that all the fundamental trigonometric functions have principal cycles occurring on the interval $0 \le \theta \le p$, where θ is the argument and p is the function's natural period. This means that the modified function, $f(x) = \text{trig}(x + C)$, goes through that same cycle when $0 \le x + C \le p$. Subtracting C from all three sides, we discover that this modified trigonometric function goes through the same single cycle on the interval $-C \le x \le p - C$. That is, the starting point for the principal cycle has been shifted from 0 to $-C$. This shift in the starting point from the principal cycle is known as the **phase shift** of the trigonometric function. This definition is so important that we should codify it.

Definition: Phase Shift

The **phase shift** of a trigonometric function is where the original x-value from the principal cycle (which is $x = 0$) has been shifted.

Moreover, we just proved the following result.

Lemma: Phase Shift Formula for $y = \text{trig}(x + C)$

The phase shift for a trigonometric function of the form $y = \text{trig}(x + C)$ is $-C$.

Thus, the phase shift for the function in Example $8.3.1$ is $-C = -\frac{\pi}{3}$ and the phase shift for the function in Checkpoint $8.3.1$ is $-C = -\left(-\frac{\pi}{6}\right) = \frac{\pi}{6}$.

If you're wondering why we called that previous statement a lemma instead of a theorem, it is because a lemma is a stepping stone to a bigger theorem, and the bigger theorem (which we will get to shortly) is going to supercede that lemma. Before we get there, however, we still have some work to do.

We now can expertly graph a trigonometric function of the form

$$f(x) = A\,\text{trig}\,(x + C)\,,$$

but what happens when the coefficient of x is not 1? That is, if we are given

$$f(x) = A\,\text{trig}\,(Bx + C)\,,$$

is the phase shift still $-C$?

The answer to this question is an adamant **no** (and this is why we only wrote that last conjecture as a lemma).

Using our previous derivation, the modified function, $f(x) = A\ \text{trig}\,(Bx + C)$, goes through its principal cycle when $0 \leq Bx + C \leq p$, where p is the natural period of the trigonometric function. Subtracting C from all three sides, we get $-C \leq Bx \leq p - C$. *Assuming* $B > 0$, we can divide all three sides by B to find that this modified trigonometric function goes through the same single cycle on the interval $-\frac{C}{B} \leq x \leq \frac{p-C}{B}$. That is, the starting point for the principal cycle has been shifted from 0 to $-\frac{C}{B}$. Therefore, the phase shift is $-\frac{C}{B}$. The following theorem summarizes all of this into a nice, clean statement.

Theorem: Phase Shift Formula

The phase shift of the function $y = A\ \text{trig}\,(Bx + C)$ is $-\frac{C}{B}$.

If $B < 0$, we use the Symmetry Identities to force it to become positive before attempting to find the phase shift.

Caution: C is Not Always the Phase Shift

When considering the function

$$f(x) = A\,\text{trig}\,(Bx + C)\,,$$

it is crucial to remember that B is ***not*** the period and C is ***not necessarily*** the phase shift (although, it does adopt the name *phase angle* in Physics).

- $\text{Period} = \dfrac{\text{natural period}}{B}$
- $\text{Phase Shift} = -\dfrac{C}{B}$

Example $8.3.2$

State the phase shift of each function.

a. $y = 2\tan\,(6x - \pi)$
b. $y = \sqrt{5}\sin\left(-\frac{\pi}{2}x + \frac{\pi}{4}\right)$

Solutions available at the end of this section.

Checkpoint 8.3.2

State the phase shift of $y = -2\cos\left(2x - \frac{\pi}{3}\right)$

Answer available at the end of this section.

Some instructors (including myself) introduce an alternative way of manipulating the argument of the trigonometric function $f(x) = A\ \text{trig}\,(Bx + C)$. If you factor B from all terms in the argument, your function turns into

$$f(x) = A\,\text{trig}\left(B\left(x + \frac{C}{B}\right)\right).$$

In this form, the phase shift reveals itself because $\frac{C}{B} = -\left(-\frac{C}{B}\right)$ and so

$$f(x) = A\,\text{trig}\left(B\left(x - \left(-\frac{C}{B}\right)\right)\right).$$

We will include this in an example or two, just in case your instructor likes this form.

We now modify our graphing tactic from the previous section to include this new transformation.

Summary of Graphing $f(x) = \overline{A}\ \text{trig}\left(\overline{B}x + \overline{C}\right)$

To graph the function

$$f(x) = \overline{A}\,\text{trig}\left(\overline{B}x + \overline{C}\right):$$

Precondition Stage

We begin by cleaning up the trigonometric function and ensuring that the coefficient of x is positive. This is done using the Symmetry Identities. Again, as a warning, if $\overline{B} < 0$, you will factor a negative off both terms in the argument of the trigonometric function. Once the Symmetry Identities have been used, your new function will be of the form

$$f(x) = A\,\text{trig}\,(Bx + C).$$

Information Stage

We now identify the following information (in the order listed):

- $\text{Amplitude} = |A|$
- $\text{Period} = \dfrac{\text{Natural Period}}{B}$
- $\text{Step Size} = \dfrac{\text{Period}}{4}$
- $\text{Phase Shift} = -\dfrac{C}{B}$

where the natural period is 2π if given the sine or cosine functions, and π if given the tangent function.

Graphing Stage

We then use each of these *in reverse order* to obtain a sketch of the trigonometric function.

- Draw a horizontal line and mark the initial key number as the **phase shift**.
- Mark the remaining key numbers by successively adding the **step size** to the previous key number.
- Double-check that the distance from the initial key number to the final key number matches the **period**, then sketch one cycle of the base trigonometric function, making sure to account for the sign of A and drawing any vertical asymptotes (if the given function is the tangent).
- Draw the vertical axis at $x = 0$ and use the **amplitude** to either
  - label the maximum and minimum values of the trigonometric function on the vertical axis (if the given function is sinusoidal), or

- label the remaining key points on the graph (if the given function is the tangent).

Example 8.3.3

Graph a single cycle of the given trigonometric function, labeling key numbers, the y-axis, stating the period, step size, amplitude, and phase shift.

a. $y = 2\tan(6x - \pi)$
b. $y = \sqrt{5}\sin\left(-\frac{\pi}{2}x + \frac{\pi}{4}\right)$

Solutions available at the end of this section.

Checkpoint 8.3.3

Graph $y = -2\cos\left(2x + \frac{\pi}{3}\right)$ and state the period, amplitude, step size, and phase shift.

Answer available at the end of this section.

As we have done previously, you can use the following Interactive Element to help visualize the effect of C on the graph of the sine function.

Interactive Element: Investigating the Effects of C on the Graph of $y = A\ \sin(Bx + C)$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Desmos Interactive online

Interaction: Adjust the slider for A, B, and C to see the effect on the graph of $y = A\ \sin(Bx + C)$.

Vertical Shifts and the Midline

We finish our discussion of transformations with the parameter D in the function

$$f(x) = A\,\text{trig}\,(Bx + C) + D.$$

Of all the transformations from your Algebra class, this is the one most students are pretty good with. Recall, the graph of $y = g(x) + D$ is the graph of g shifted up (if $D > 0$) or down (if $D < 0$) $|D|$ units. That is, the "+ D" is just adding D to all the y-values of the original function.

The only new concept here is the idea of a **midline** for a trigonometric function.

Definition: Midline

The **midline** of a trigonometric function is a horizontal line that represents the average value of the function. For sinusoidal functions, it is situated halfway between the maximum and minimum points. The midline for all base graphs of the trigonometric functions is $y = 0$.

The midline is essentially the center line around which the graph of the trigonometric function oscillates (for the case of the

sinusoidal curves). Once a vertical shift occurs, this "hidden" axis appears, and it is handy when graphing trigonometric functions.

For example, consider the function

$$y = -2\sin(\pi x) + 3.$$

From the previous section, we should be able to quickly sketch *most* of this function, but the "$+3$" at the end is new. We know the base function is the sine, the fact that the coefficient of the sine is negative means the base graph has been reflected about the x-axis, the amplitude (because the graph is sinusoidal) is $|-2| = 2$, the period is $\frac{2\pi}{B} = \frac{2\pi}{\pi} = 2$, and the step size is $\frac{\text{Period}}{4} = \frac{2}{4} = \frac{1}{2}$. If this is all we had, then the graph would look like the following:

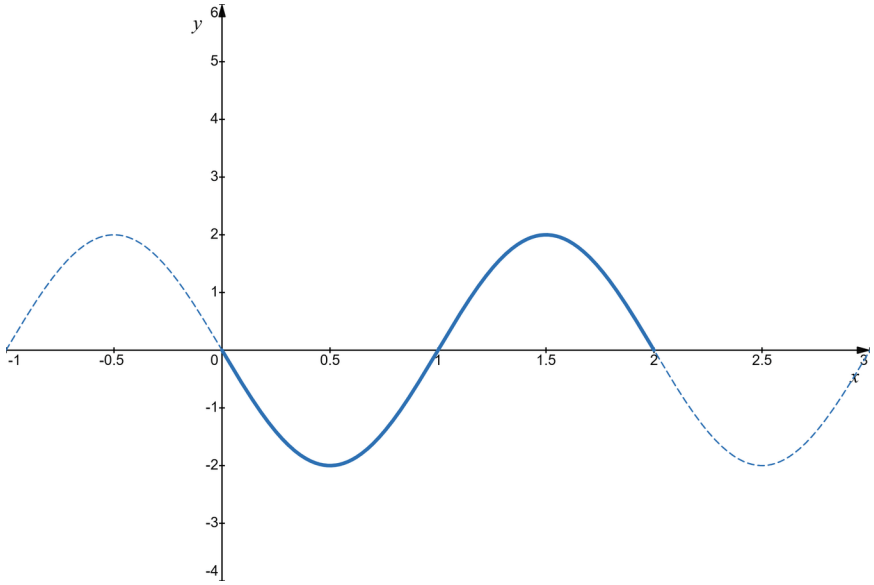


Figure $8.3.10$**:** $y = -2\sin(\pi x)$

Since our given function is $y = -2\sin(\pi x) + 3$, we need to add three to all of the outputs in Figure $8.3.10$. As we redraw our graph and shift up three units, it is helpful to move up the hidden midline.

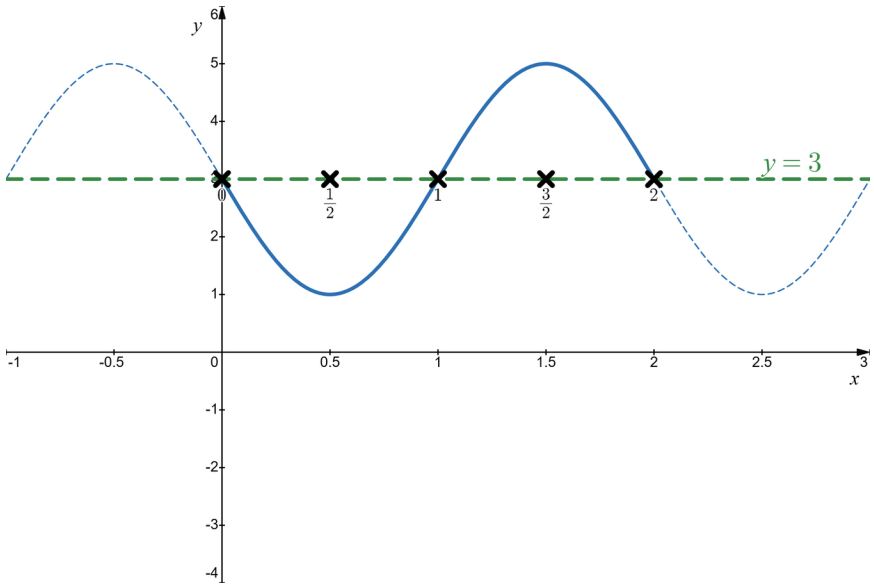


Figure $8.3.11$**:** $y = -2\sin(\pi x) + 3$

Let's update our graphing tactic a final time.

Summary of Graphing $f(x) = \overline{A}\ \text{trig}\left(\overline{B}x + \overline{C}\right) + D$

To graph the function

$$f(x) = \overline{A}\,\text{trig}\left(\overline{B}x + \overline{C}\right) + D:$$

Precondition Stage

- If the trigonometric function is not written in the order listed above (e.g. if you are given $y = 3 - 2\sin(5 - x)$), rewrite it so it matches the form above—this is critical!
- Clean up the trigonometric function using the Symmetric Identities to ensure the coefficient of x is positive. Again, as a warning, if $\overline{B} < 0$, you will factor a negative off both terms in the argument of the trigonometric function. Once the Symmetry Identities have been used, your new function will be of the form

$$f(x) = A\,\text{trig}\,(Bx + C) + D.$$

Information Stage

We now identify the following information (in the order listed):

- $\text{Amplitude} = |A|$
- $\text{Period} = \dfrac{\text{Natural Period}}{B}$
- $\text{Step Size} = \dfrac{\text{Period}}{4}$
- $\text{Phase Shift} = -\dfrac{C}{B}$
- $\text{Vertical Shift} = D$

where the natural period is 2π if given the sine or cosine functions, and π if given the tangent function.

Graphing Stage

We then use each of these *in reverse order* to obtain a sketch of the trigonometric function.

- Draw a horizontal line and label it $y = D$. This is your **vertical shift**.
- Mark the initial key number as the **phase shift**.
- Mark the remaining key numbers by successively adding the **step size** to the previous key number.
- Double-check that the distance from the initial key number to the final key number matches the **period**, then sketch one cycle of the base trigonometric function, making sure to account for the sign of A and drawing any vertical asymptotes (if the given function is the tangent).
- Draw the vertical axis at $x = 0$ and use the **amplitude** to either
  - label the maximum and minimum values of the trigonometric function on the vertical axis (if the given function is sinusoidal), or
  - label the remaining key points on the graph (if the given function is the tangent).

Example 8.3.4

Graph each function. State the amplitude (if applicable), period, step size, phase shift, and vertical shift. Your graph should have five key points (or four key points and an asymptote) labeled, as well as the midline.

a. $y = -5 + 5\sin\left(3\pi x + \frac{\pi}{2}\right)$
b. $y = \frac{1}{3}\tan(2x - 1) + 1$

Solutions available at the end of this section.

Checkpoint 8.3.4

Graph the function. State the amplitude (if applicable), period, step size, phase shift, and vertical shift. Your graph should have five key points (or four key points and an asymptote) labeled, as well as the midline.

$$f(x) = 3 - \frac{7}{5}\tan\left(2x - \frac{\pi}{2}\right)$$

Answer available at the end of this section.

Interactive Element: Investigating the Effects of D on the Graph of $y = A \sin(Bx + C) + D$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Desmos Interactive online

Interaction: Adjust the sliders for each parameter to see the effects on the graph of $y = A\sin(Bx + C) + D$.

Finding Equations Given Graphs

Finding the equation of a trigonometric function given its graph is only slightly more complex when there has been a phase shift or vertical shift. The key to success with these problems is identifying two points on the curve. This is *usually* enough to find the equation.

Example 8.3.5

Find the equation of the graphed function.

a.

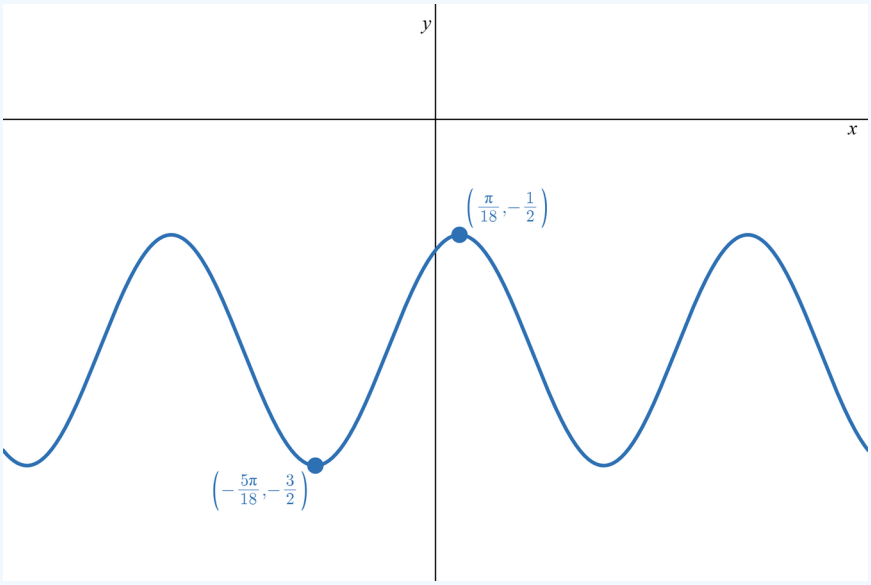


Figure 8.3.22

b.

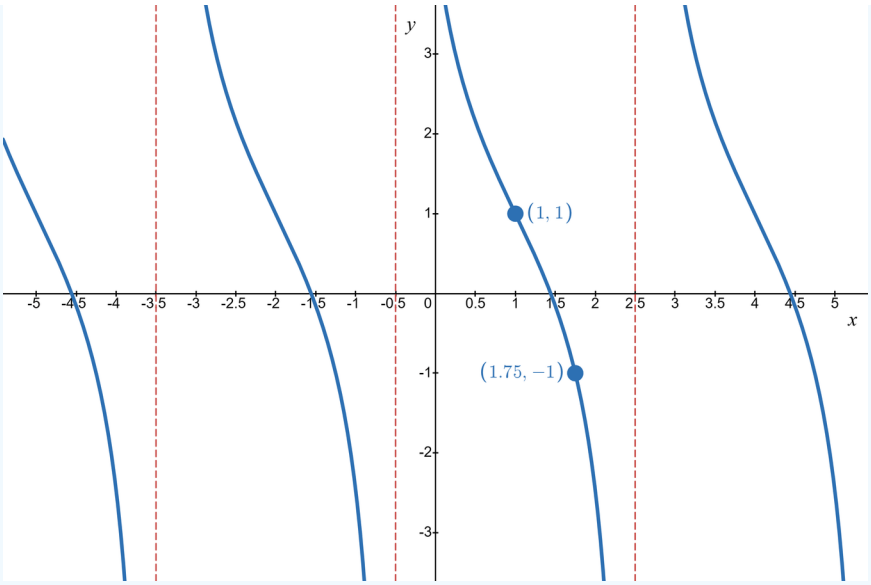


Figure 8.3.23

Solutions available at the end of this section.

Checkpoint 8.3.5

Find the equation of the graphed function.

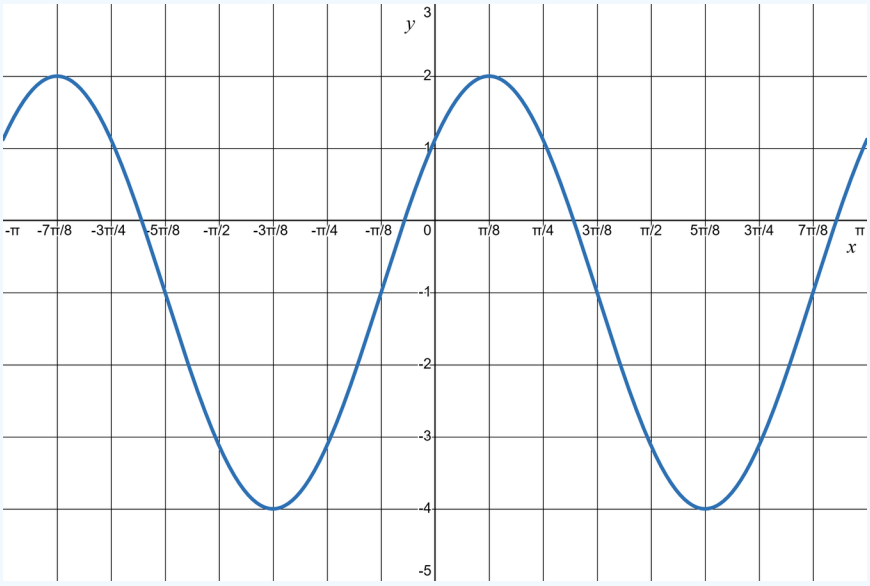


Figure 8.3.24

Answer available at the end of this section.

Revisiting the Cofunction Identities

Recall the Cofunction Identities (which we rewrite here in radian measure).

Theorem: Cofunction Identities (radians)

The trigonometric function of an angle is equal to the cofunction evaluated at the complement of the angle. That is,

$$\sin(\theta) = \cos\left(\frac{\pi}{2}-\theta\right) \quad \text{and} \quad \cos(\theta) = \sin\left(\frac{\pi}{2}-\theta\right)$$
$$\sec(\theta) = \csc\left(\frac{\pi}{2}-\theta\right) \quad \text{and} \quad \csc(\theta) = \sec\left(\frac{\pi}{2}-\theta\right)$$
$$\tan(\theta) = \cot\left(\frac{\pi}{2}-\theta\right) \quad \text{and} \quad \cot(\theta) = \tan\left(\frac{\pi}{2}-\theta\right)$$

At the time, we could only prove these identities for acute angles. We are still not at a point where we can prove these for all angles, but we can now *visualize* these identities. For example, let's consider the function $y = \cos\left(\frac{\pi}{2}-x\right)$. The Cofunction Identities tell us that this should be the same as $y = \sin(x)$. Let's manipulate that cosine function a little.

$$\begin{aligned} y &= \cos\left(\tfrac{\pi}{2}-x\right) \\ &= \cos\left(-\left(-\tfrac{\pi}{2}+x\right)\right) \\ &= \cos\left(-\left(x-\tfrac{\pi}{2}\right)\right) \\ &= \cos\left(x-\tfrac{\pi}{2}\right) \qquad \text{(cosine is an even function)} \end{aligned}$$

So, that cosine function is just like a regular cosine function, but with a phase shift of $\frac{\pi}{2}$. Graphing this, we get the following:

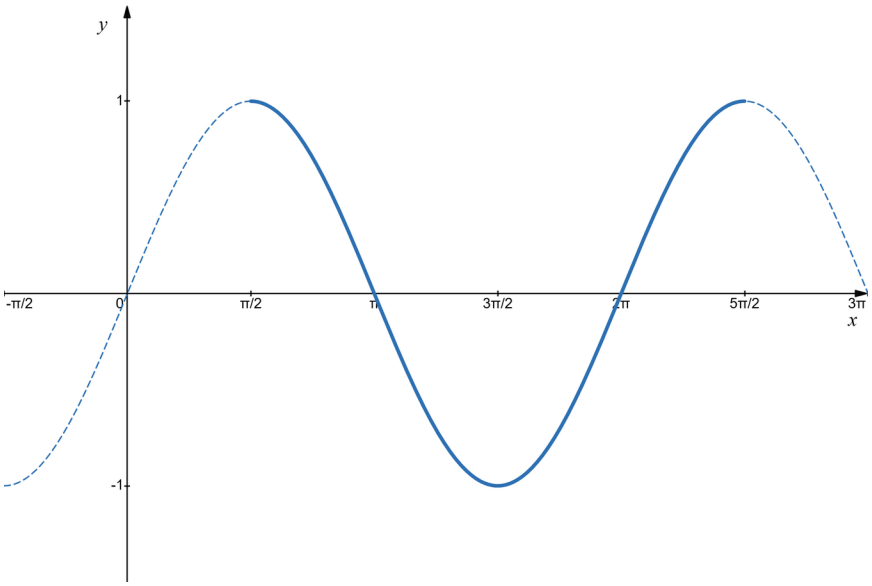


Figure 8.3.25

However, this is *exactly* the graph of $y = \sin(x)$. Therefore, we feel confident in stating that $\cos\left(\frac{\pi}{2}-x\right) = \sin(x)$. You can use the same technique to convince yourself that the other Cofunction Identities are true (go ahead, it's a great exercise!).

Introduction to Sinusoidal Models

Many natural phenomena can be modeled with sinusoidal functions. In this subsection, we will work with applications where we have either been given the explicit sinusoidal model or the graph that models a situation. In the next section, we will learn to build sinusoidal models from language.

Example 8.3.6

The formula

$$D(t) = 12.25 - 5.25\cos\left(\frac{\pi}{6}t\right)$$

gives the number of hours of daylight in Glasgow, Scotland, as a function of time, in months after January 1.

a. Graph the function by hand.
b. State the midline, amplitude, and period of the graph.
c. How many hours of daylight does Glasgow enjoy on the longest day of the year?
d. In which months does Glasgow experience less than 8 hours of light daily?

Solutions available at the end of this section.

Checkpoint 8.3.6

The pistons in an automobile engine move up and down in the cylinders. If t is in milliseconds, the distance from the top of the piston to the top of the cylinder is given in centimeters by

$$D = 7 + 6\sin(4\pi t)$$

a. Graph the function by hand.
b. State the midline, amplitude, and period of the graph.
c. Find the largest and the smallest clearance between the piston and the top of the cylinder.

Answer available at the end of this section.

Example 8.3.7

Sunspots are dark regions on the Sun first observed by Galileo in 1610. The number of sunspots is not constant but varies with a period of approximately 11 years, called the solar cycle. Solar activity is directly related to this cycle and can cause disturbances in the Earth's upper atmosphere. Planning for satellite orbits and space missions requires knowledge of solar activity levels years in advance.

The figure below shows sunspot data for the last solar cycle, from July 1985 through June 1996, and a sinusoidal function that models the data.

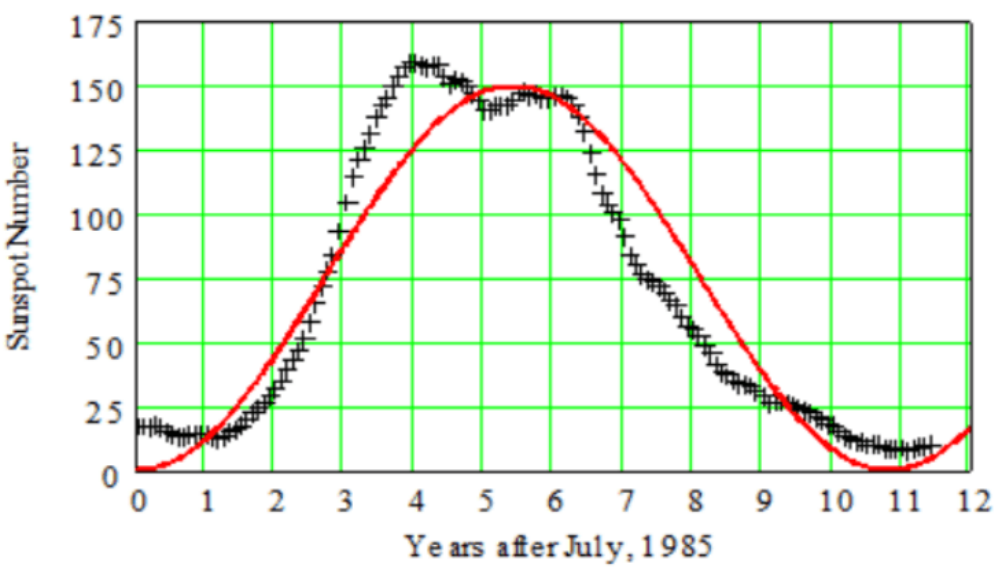


Figure 8.3.28

The period of the solar cycle shown is 10.8 years. Use this information and the graph to write a formula for the model.

Solution available at the end of this section.

Checkpoint 8.3.7

The figure below shows the sunspot data for the solar cycle that began in July 1996, and a curve of best fit calculated by NASA. This curve is not sinusoidal, but has a similar shape.

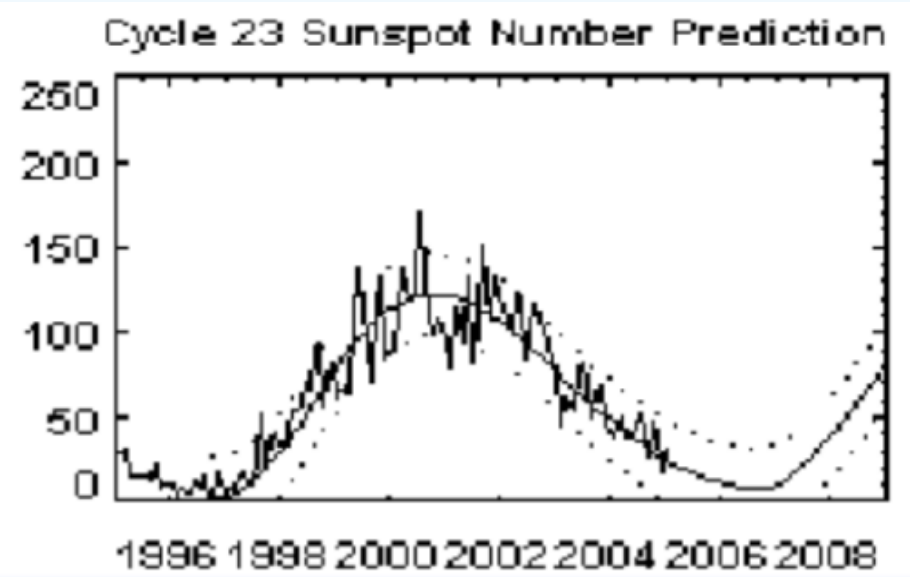


Figure 8.3.29

a. The minimum sunspot number occurred in October 1996. Use the graph to estimate the period, midline, and amplitude of a sinusoidal function that approximates the data.
b. Write a function, $s(t)$, that approximates the data, where t is the number of years since July 1996.
c. Use your function to predict the sunspot number in January 2025.

Answer available at the end of this section.

Miscellaneous Topics: Thinking About Mathematics (Part 2)

A significant concern in mathematics education is the trend that students rely on calculators more. An argument for calculator usage is that, in "real life," a person will always have access to a computer or a calculator. Therefore, forcing students to do problems "by hand" is useless.

1. Give an argument against high calculator usage in math classes.
2. Do you think that your professor relies too much on calculator usage in class? What about their use of technology? Would you like more or less?

Hidden Resources: Adobe Creative Cloud

The Adobe Creative Cloud is a collection of Adobe software products. In addition to offering the products, the Creative Cloud also offers cloud storage.

Apps Included in the Adobe Creative Suite are:

- Design Apps: Photoshop, InCopy, Spark, Illustrator, Adobe XD, Dimension, InDesign
- Web & UX Apps: Adobe XD, Spark, Dreamweaver, Animate, InCopy, InDesign, Photoshop
- Photography Apps: Photoshop, Lightroom
- Video & Audio Apps: Premiere Pro, Premiere Rush, After Effects, Spark, Audition

Typically, the Adobe Creative Suite costs $55/month, but Los Rios students have access to Adobe Creative Cloud for $39.99 per semester or $79.99 per year. Check out this page for more details.

Answers

Section Preview

While the rigid transformations are introduced in this section, the majority of the instruction will likely be focused on graphing trigonometric functions requiring *all* transformations. There isn't another section on graphing transformations of the fundamental trigonometric functions, so make sure students really have practice graphing lots of transformations here.

Section Preview

- **Graphing Functions**
  - Transformations — Rigid Transformations: The opening explicitly says "Recall, the graph of $y = g(x + C)$ is a horizontal shift of the graph of g." The rigid transformation framework — horizontal and vertical shifts of a general function — is the direct prior-course prerequisite.

Section Preview

- Graph fundamental trigonometric functions having vertical transformations.
- Graph fundamental trigonometric functions having horizontal transformations.
- Graph fundamental trigonometric functions having combinations of all transformations.
- Given the graph, write the corresponding trigonometric function.
- Evaluate given sinusoidal models for applications.
- Build a model for an application based on the graph.

Example 8.3.1

From Algebra, we expect the graph to be shifted $\frac{\pi}{3}$ units to the left compared to the graph of $y = \cos(x)$.

As we commonly try to do, we want the argument of $\cos\left(x + \frac{\pi}{3}\right)$ (which is the expression $x + \frac{\pi}{3}$) to be quadrantal angles.

x	$x + \frac{\pi}{3}$	$\cos\left(x + \frac{\pi}{3}\right)$	$y = 2\cos\left(x + \frac{\pi}{3}\right)$
	0		
	$\frac{\pi}{2}$		
	π		
	$\frac{3\pi}{2}$		
	2π		

The second column of this table is what we desire $x+\frac{\pi}{3}$ to be. For $x+\frac{\pi}{3}$ to become 0, we solve

$$x+\frac{\pi}{3}=0 \implies x=-\frac{\pi}{3}.$$

Likewise, for $x+\frac{\pi}{3}$ to become $\frac{\pi}{2}$, we solve

$$x+\frac{\pi}{3}=\frac{\pi}{2} \implies x=\frac{\pi}{6}.$$

We continue this process to fill out the first column of the table.

x	$x+\frac{\pi}{3}$	$\cos\left(x+\frac{\pi}{3}\right)$	$y=2\cos\left(x+\frac{\pi}{3}\right)$
$-\frac{\pi}{3}$	0		
$\frac{\pi}{6}$	$\frac{\pi}{2}$		
$\frac{2\pi}{3}$	π		
$\frac{7\pi}{6}$	$\frac{3\pi}{2}$		
$\frac{5\pi}{3}$	2π		

Finally, we fill out the third and fourth columns using the second column as the arguments.

x	$x+\frac{\pi}{3}$	$\cos\left(x+\frac{\pi}{3}\right)$	$y=2\cos\left(x+\frac{\pi}{3}\right)$
$-\frac{\pi}{3}$	0	1	-2
$\frac{\pi}{6}$	$\frac{\pi}{2}$	0	0
$\frac{2\pi}{3}$	π	-1	2
$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	0	0
$\frac{5\pi}{3}$	2π	1	-2

We now plot the x-values (in the first column) and their corresponding function values (in the fourth column) to make the graph. One cycle of the graph is shown below. As we expected, the graph is shifted $\frac{\pi}{3}$ units to the left compared to the graph of $y=-2\cos(x)$.

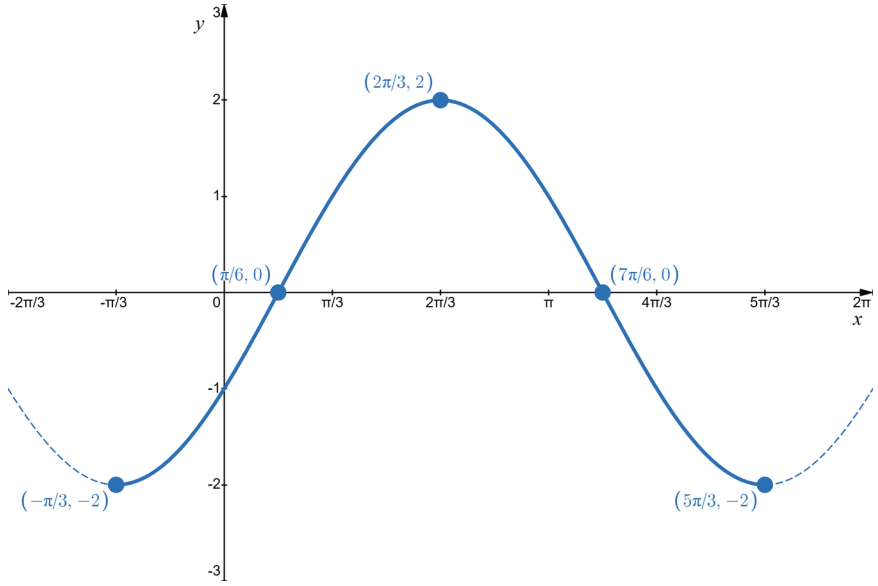


Figure 8.3.1

Checkpoint 8.3.1

x	$x-\frac{\pi}{6}$	$\sin\left(x-\frac{\pi}{6}\right)$	$y=-3\sin\left(x-\frac{\pi}{6}\right)$
$\frac{\pi}{6}$	0	0	0
$\frac{2\pi}{3}$	$\frac{\pi}{2}$	1	-3
$\frac{7\pi}{6}$	π	0	0
$\frac{5\pi}{3}$	$\frac{3\pi}{2}$	-1	3
$\frac{13\pi}{6}$	2π	0	0

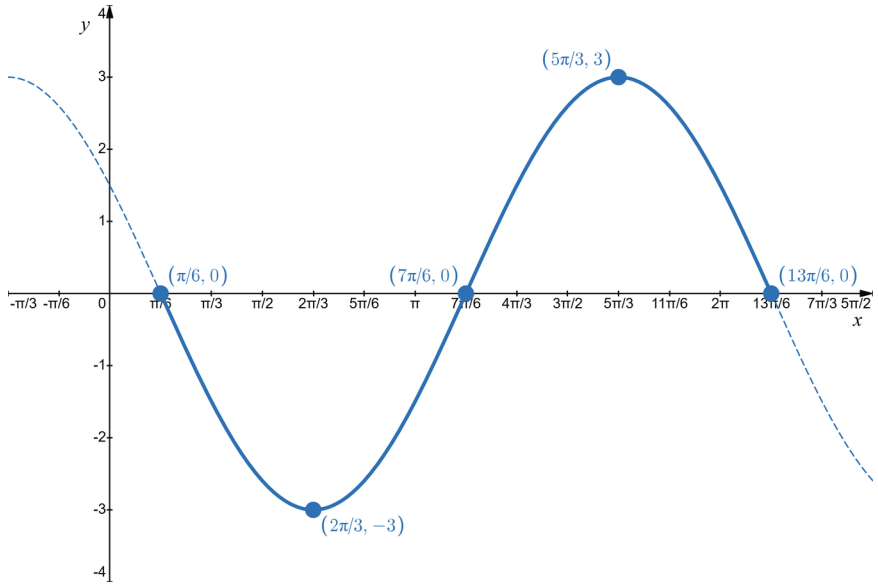


Figure 8.3.2

Example 8.3.2

a. In this case, $B=6$ and $C=-\pi$. Hence, the phase shift for this function is

$$-\frac{C}{B}=-\frac{-\pi}{6}=\frac{\pi}{6}.$$

b. As in the previous section, we must use the Symmetry Identities when $B<0$. When using these identities, be sure to factor the negative off *both* B and C.

$$\begin{aligned} y &= \sqrt{5}\sin\left(-\frac{\pi}{2}x+\frac{\pi}{4}\right) \\ &= \sqrt{5}\sin\left(-\left(\frac{\pi}{2}x-\frac{\pi}{4}\right)\right) \\ &= -\sqrt{5}\sin\left(\frac{\pi}{2}x-\frac{\pi}{4}\right) \qquad \text{(sine is an odd function)} \end{aligned}$$

Hence, the phase shift is

$$-\frac{C}{B}=-\frac{-\pi/4}{\pi/2}=\frac{1}{2}.$$

Checkpoint 8.3.2

$\frac{\pi}{6}$

Example 8.3.3

a. **Precondition Stage:** Since $B=6>0$, there is nothing to precondition; however, if we wanted to factor the 6 from both terms in the argument, we would get

$$y=2\tan\left(6\left(x-\frac{\pi}{6}\right)\right),$$

which reveals the phase shift to be $\frac{\pi}{6}$.
Information Stage: This is not a sinusoidal function, so we will not have an amplitude, and the natural period is π.

$$\begin{array}{rcl} B = 6 & \Longrightarrow & \text{Period} = \dfrac{\text{Natural Period}}{B} = \dfrac{\pi}{6} \\ & \Longrightarrow & \text{Step Size} = \dfrac{\text{Period}}{4} = \dfrac{\pi/6}{4} = \dfrac{\pi}{24} \\ C = -\pi & \Longrightarrow & \text{Phase Shift} = -\dfrac{C}{B} = -\dfrac{-\pi}{6} = \dfrac{\pi}{6} = \dfrac{4\pi}{24} \end{array}$$

It's a good idea to get the phase shift and the step size to have the same denominator, as you will add the step size to the phase shift in the graphing stage.
Graphing Stage: Graphing a number line (the x-axis), we plot the phase shift and, from there, mark the remaining key numbers using the step size.

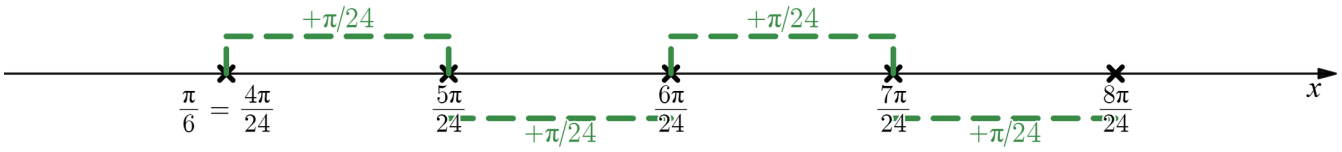


Figure 8.3.3

The distance between the initial key number and the final key number is

$$\left|\frac{8\pi}{24} - \frac{4\pi}{24}\right| = \frac{4\pi}{24} = \frac{\pi}{6} = \text{Period}$$

so we have done well so far. We now sketch a single cycle of the unreflected tangent function, drawing in the vertical asymptote halfway through the cycle.

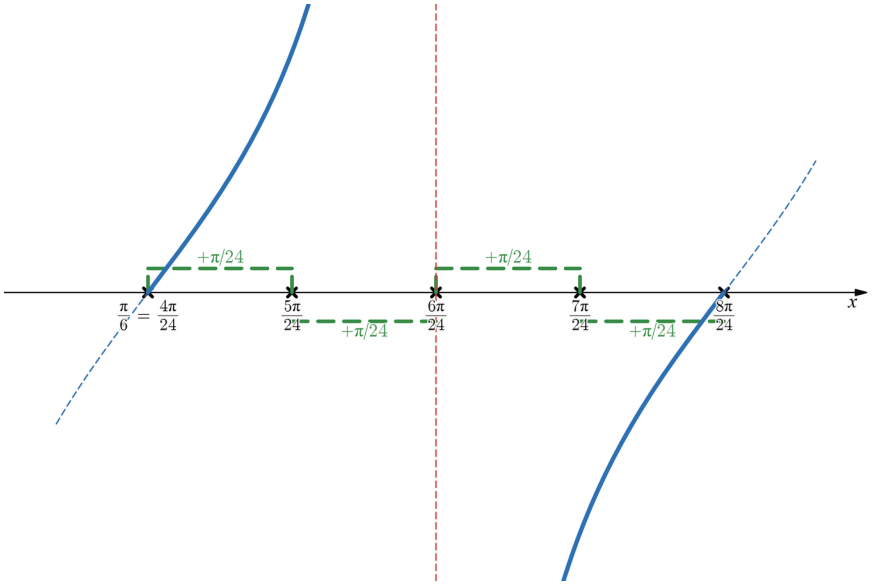


Figure 8.3.4

We can see that the initial key number is positive and, therefore, to the right of the y-axis. Consequently, we draw in the y-axis where we think it should be. Simultaneously, let's plot the points corresponding to the key numbers $\frac{5\pi}{24}$ and $\frac{7\pi}{24}$.

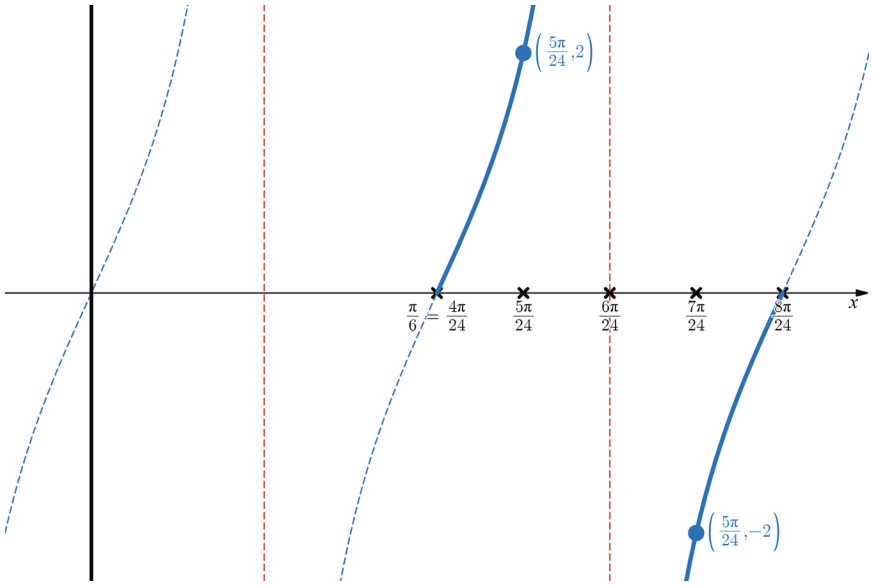


Figure 8.3.5

This completes one full cycle of the graph of the given function (although we decided to show some extra cycles, it's not a requirement).

b. **Precondition Stage:** Since $B = -\frac{\pi}{2} < 0$, we need to factor a negative from both terms in the argument of the function.

$$\begin{aligned} y &= \sqrt{5}\sin\left(-\frac{\pi}{2}x + \frac{\pi}{4}\right) \\ &= \sqrt{5}\sin\left(-\left(\frac{\pi}{2}x - \frac{\pi}{4}\right)\right) \\ &= -\sqrt{5}\sin\left(\frac{\pi}{2}x - \frac{\pi}{4}\right) \qquad \text{(sine is an odd function)} \end{aligned}$$

Moreover, if your instructor prefers the factored form, we could factor the $\frac{\pi}{2}$ from both terms in the argument to get

$$y = -\sqrt{5}\sin\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right).$$

Doing this reveals the phase shift to be $\frac{1}{2}$.

Information Stage: This is a sinusoidal function, so we will have an amplitude, and the natural period is 2π.

A	$=$	$-\sqrt{5}$	$\Longrightarrow$	Amplitude	$=$	$\lvert A\rvert = \lvert-\sqrt{5}\rvert = \sqrt{5}$
B	$=$	$\frac{\pi}{2}$	$\Longrightarrow$	Period	$=$	$\frac{\text{Natural Period}}{B} = \frac{2\pi}{\pi/2} = 4$
			$\Longrightarrow$	Step Size	$=$	$\frac{\text{Period}}{4} = \frac{4}{4} = 1$
C	$=$	$-\frac{\pi}{4}$	$\Longrightarrow$	Phase Shift	$=$	$-\frac{C}{B} = -\frac{-\pi/4}{\pi/2} = \frac{1}{2}$

Graphing Stage: It's important to remember that we are graphing the "cleaned" function

$$y = -\sqrt{5}\sin\left(\frac{\pi}{2}x - \frac{\pi}{4}\right).$$

Graphing the x-axis, plotting the phase shift, and marking the subsequent key numbers by increasing by the step size, we get the following:

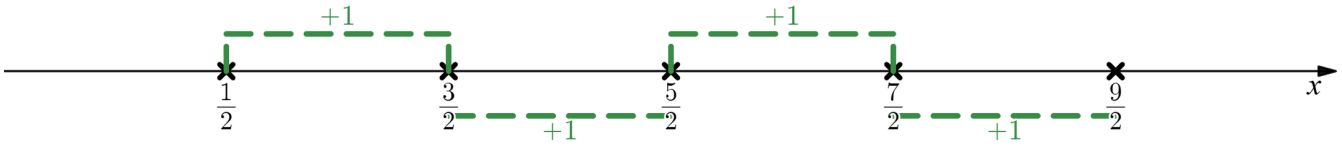


Figure 8.3.6

We then graph the *reflected* sine function.

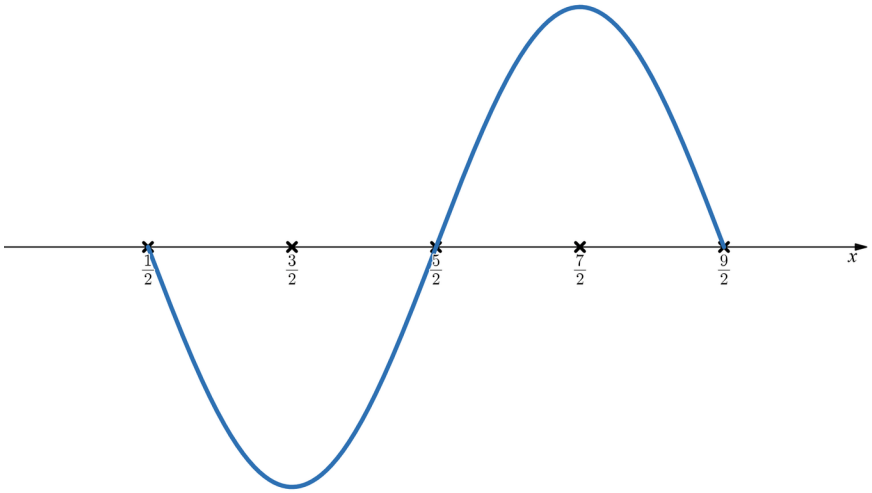


Figure 8.3.7

Finally, we place the y-axis appropriately, label it using the amplitude, and label the maximum and minimum points on the sine curve (this is something we didn't do before, but some instructors expect it, so we want to showcase good behavior).

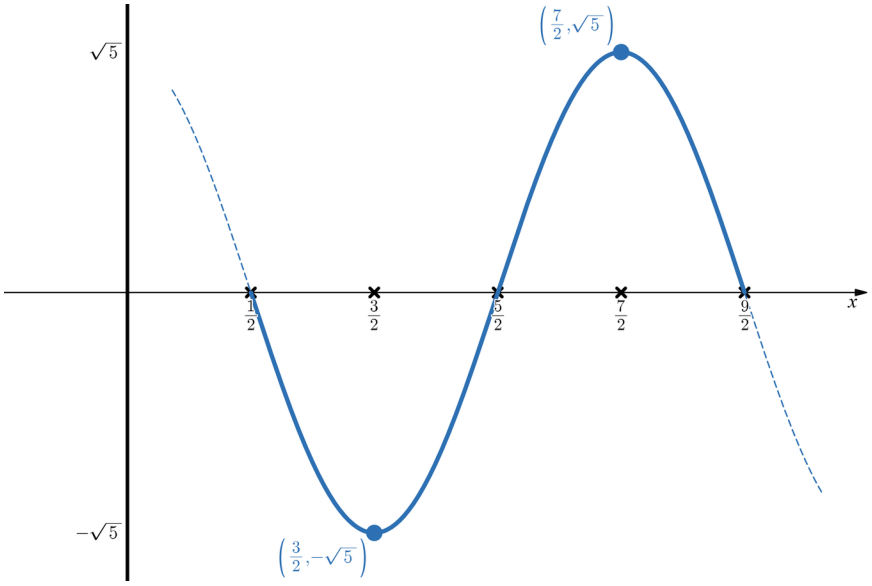


Figure 8.3.8

Checkpoint 8.3.3

Amplitude: 2
Period: π
Step Size: $\frac{\pi}{4}$
Phase Shift: $-\frac{\pi}{6}$

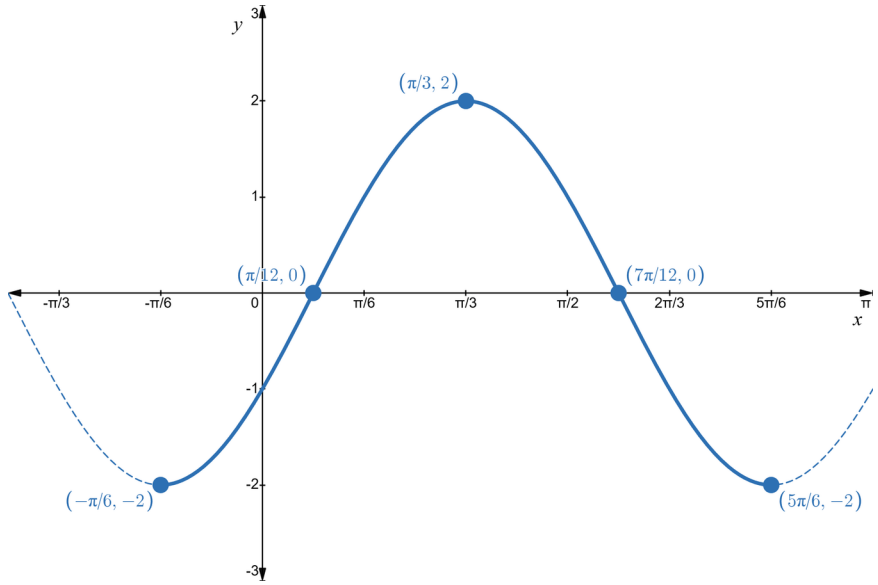


Figure 8.3.9

Example 8.3.4

a. **Precondition Stage:** Rewriting the given function into proper form, we get

$$y = 5\sin\left(3\pi x + \frac{\pi}{2}\right) - 5.$$

Information Stage:

A	$=$	5	$\Longrightarrow$	Amplitude	$=$	5	
B	$=$	3π	$\Longrightarrow$	Period	$=$	$\frac{\text{Natural Period}}{B} = \frac{2\pi}{3\pi} = \frac{2}{3}$	
			$\Longrightarrow$	Step Size	$=$	$\frac{\text{Period}}{4} = \frac{2/3}{4} = \frac{1}{6}$	
C	$=$	$\frac{\pi}{2}$	$\Longrightarrow$	Phase Shift	$=$	$-\frac{C}{B} = -\frac{\pi/2}{3\pi} = -\frac{1}{6}$	
D	$=$	-5	$\Longrightarrow$	Vertical Shift	$=$	-5	

That last line tells us the midline is $y = -5$.
Graphing Stage: We start by graphing a horizontal line and labeling it $y = -5$.



Figure 8.3.12

We then plot five equally-spaced points, labeling the first using the phase shift, and adding the step size to reach each subsequent point.

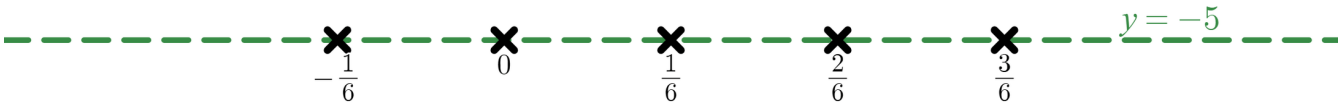


Figure 8.3.13

Double-checking the length of the interval, we see that

$$\left|\frac{3}{6} - \left(-\frac{1}{6}\right)\right| = \frac{4}{6} = \frac{2}{3} = \text{Period}.$$

So far, so good! The function we want to graph is an *unreflected* sine function, so let's use these key numbers to sketch.

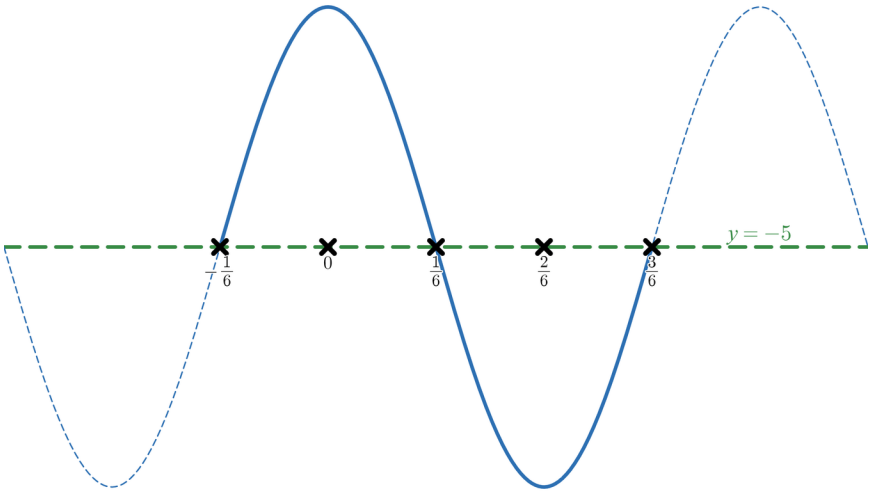


Figure 8.3.14

From here, we can see where $x = 0$ is, so we know where the y-axis goes.

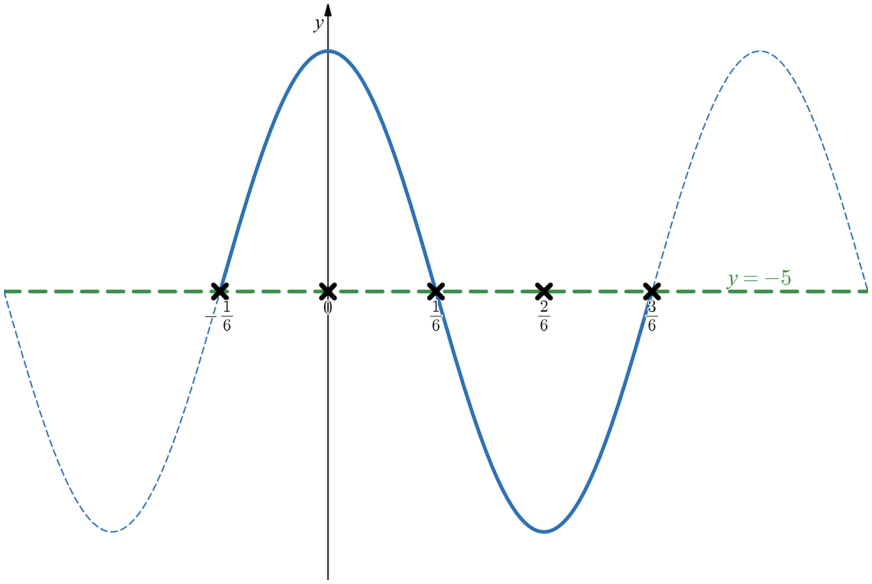


Figure 8.3.15

Since the midline is at $y = -5$ and the amplitude is 5, we can label the y-axis by adding and subtracting the amplitude from the midline.

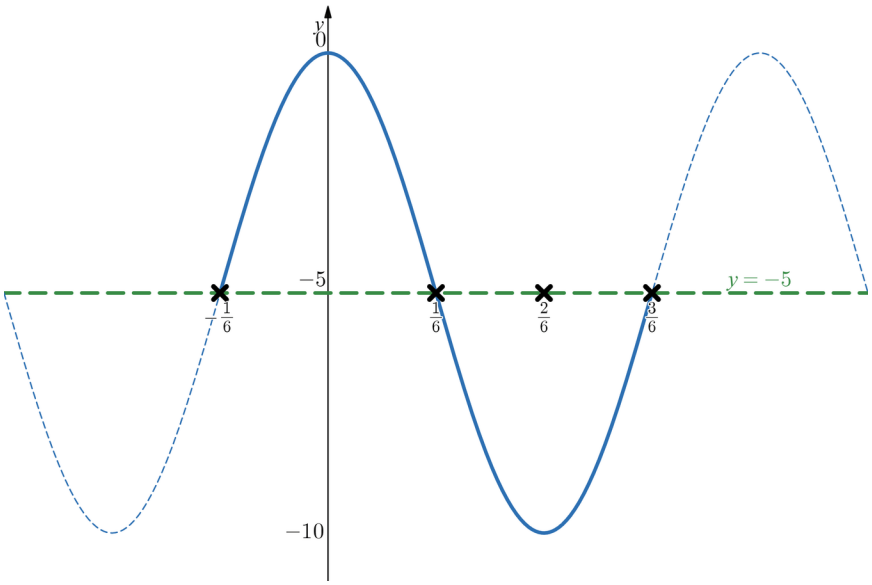


Figure 8.3.16

Finally, now that we see where $y = 0$, we can draw the x-axis.

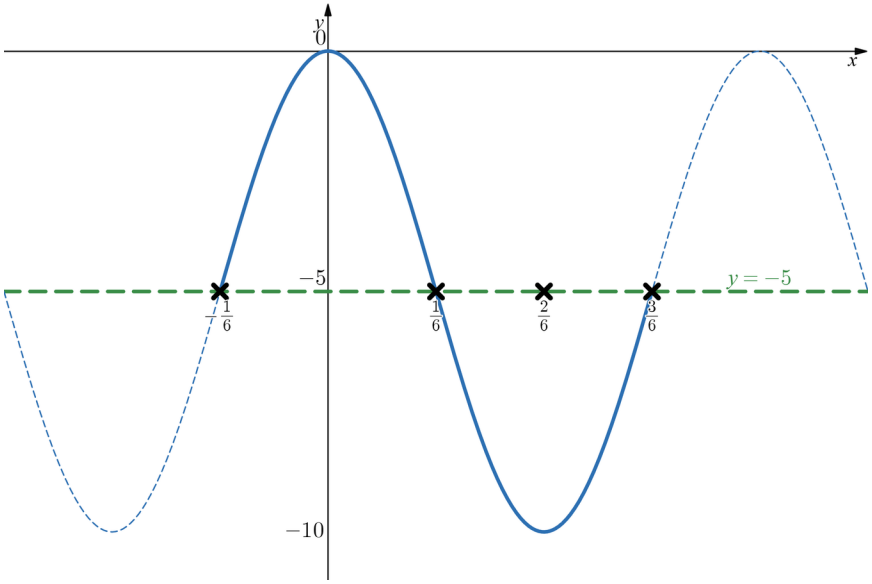


Figure 8.3.17

That is a complete cycle of the graph of $y = -5 + 5\sin\left(3\pi x + \frac{\pi}{2}\right)$.

b. **Precondition Stage:** There is nothing to precondition here.
Information Stage:

B	$=$	2	$\Longrightarrow$	Period	$=$	$\dfrac{\text{Natural Period}}{B} = \dfrac{\pi}{2}$
			$\Longrightarrow$	Step Size	$=$	$\dfrac{\text{Period}}{4} = \dfrac{\pi/2}{4} = \dfrac{\pi}{8}$
C	$=$	-1	$\Longrightarrow$	Phase Shift	$=$	$-\dfrac{C}{B} = -\dfrac{-1}{2} = \dfrac{1}{2}$
D	$=$	1	$\Longrightarrow$	Vertical Shift	$=$	1

This is a case where the phase shift ($\frac{1}{2}$) and the step size ($\frac{\pi}{8}$) are not easily added together. Please pay close attention to how we handle this during the graphing stage.
Graphing Stage: Graph the midline, plot five equally-spaced points, call the first point the phase shift, and label the remaining using the step size.

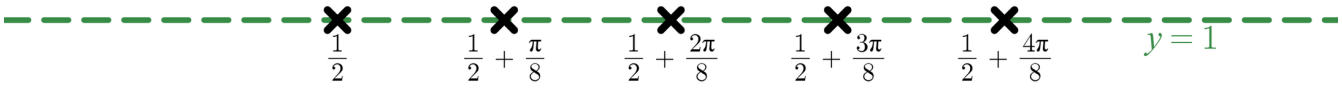


Figure 8.3.18

We now graph the *unreflected* tangent function and graph the vertical asymptote in the middle of the cycle. Moreover, we place the y-axis.

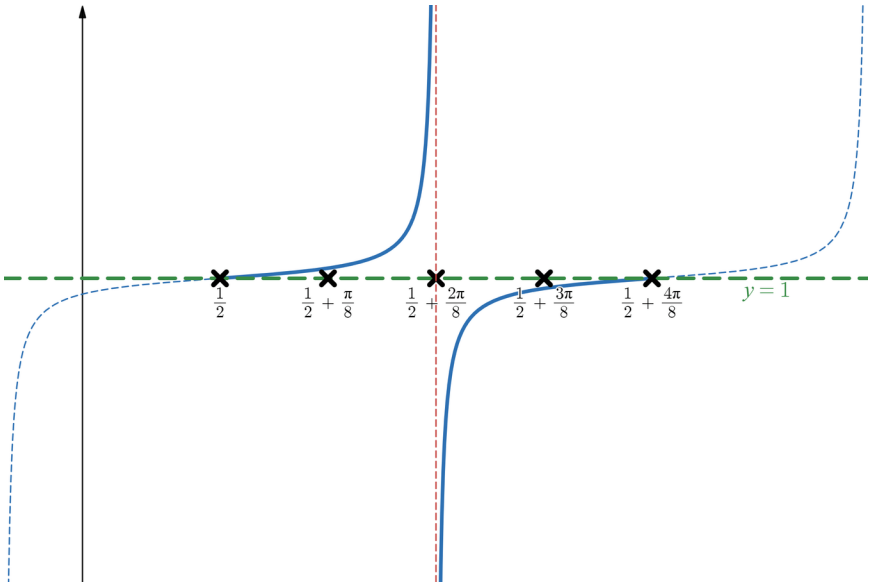


Figure 8.3.19

We can see that two of our key points are "labeled" ($(\frac{1}{2},1)$ and $(\frac{1}{2}+\frac{4\pi}{8},1)$) and we have the asymptote labeled ($x=\frac{1}{2}+\frac{2\pi}{8}$); however, we have two extra key numbers that need to be used. Despite the tangent function not having an amplitude, $|A|=\frac{1}{3}$ still serves a purpose. It tells us how far away from the midline the points corresponding to those unlabeled key numbers should be. That is, the y-value associated with $x=\frac{1}{2}+\frac{\pi}{8}$ is $1+\frac{1}{3}=\frac{4}{3}$. Likewise, the y-value for $x=\frac{1}{2}+\frac{3\pi}{8}$ is $1-\frac{1}{3}=\frac{2}{3}$. We label these points and draw in the x-axis (appropriately).

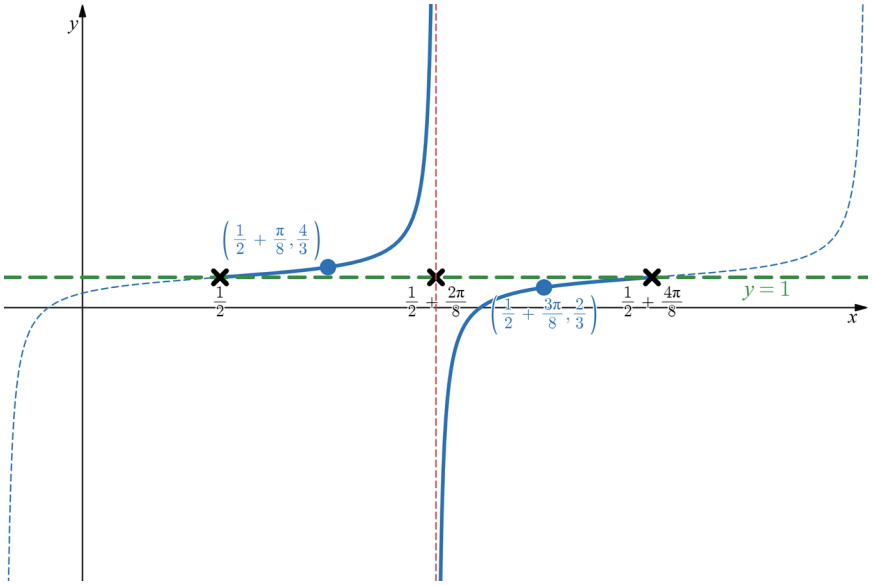


Figure 8.3.20

Checkpoint 8.3.4

Period: $\frac{\pi}{2}$
Step Size: $\frac{\pi}{8}$
Phase Shift: $\frac{\pi}{4}$
Vertical Shift: 3

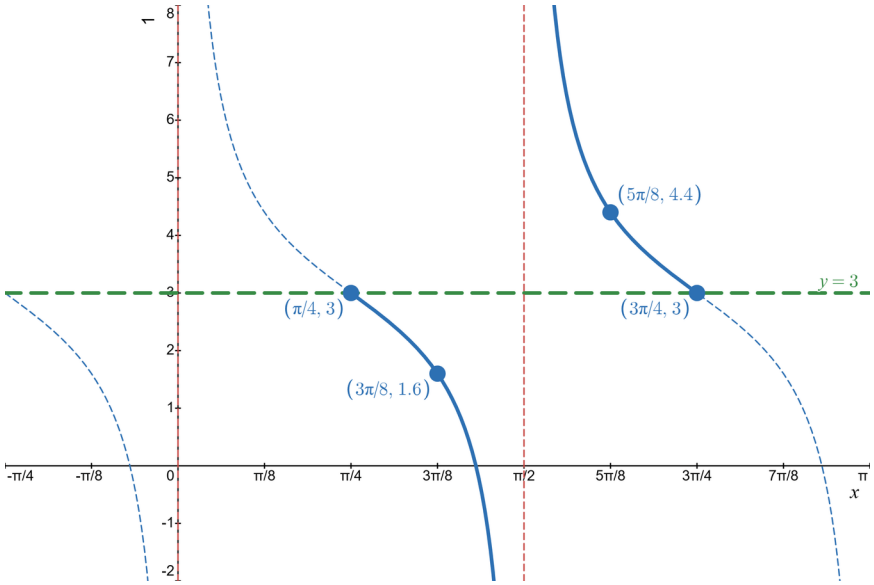


Figure 8.3.21

Example 8.3.5

a. When rebuilding an equation from a graph, it's best to figure out the information using the order with which we graph. That is, try to identify the vertical shift, phase shift, period (which might require finding the step size), and amplitude (if applicable). This is more of a "rule of thumb," so there might be times you do things out of order.

Vertical Shift: Since the maximum and minimum of this sinusoidal function are $-\frac{1}{2}$ and $-\frac{3}{2}$, respectively, the midline must sit directly between these two values. That is, the vertical shift is the *average* of these.

$$\text{Vertical Shift} = D = \frac{-\frac{1}{2} + \left(-\frac{3}{2}\right)}{2} = -1.$$

Phase Shift: Given the graph of a sinusoidal function and asked to find the equation, you have *a lot* of options. For example, the given graph could be interpreted as a sine function shifted slightly to the left (or slightly to the right and reflected about $y = -1$); however, try to use the given information to make the process easier. We will assume the leftmost point we were given, $\left(-\frac{5\pi}{18}, -\frac{3}{2}\right)$, is the phase shift. Thus,

$$\text{Phase Shift} = -\frac{C}{B} = -\frac{5\pi}{18}. \tag{8.3.2}$$

While this does not immediately give us B or C, it does tell us that we are dealing with a reflected cosine function. We will use Equation 8.3.2 in a moment.

Period or Step Size: Assuming the cycle begins at $x = -\frac{5\pi}{18}$, we can see that the curve is *halfway* through a full cycle by $x = \frac{\pi}{18}$. That is, half the period is $\left|-\frac{5\pi}{18} - \frac{\pi}{18}\right| = \frac{6\pi}{18} = \frac{\pi}{3}$. Therefore, the full period is $\frac{2\pi}{3}$, and so

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{3} \implies B = 3.$$

We can now go back to Equation 8.3.2 and let $B = 3$ to get

$$-\frac{C}{3} = -\frac{5\pi}{18} \implies C = \frac{5\pi}{6}.$$

Amplitude: The amplitude is the distance from the midline to the maximum (or minimum). In this case, the amplitude is

$$\text{Amplitude} = |A| = \frac{1}{2}.$$

Putting this altogether, and remembering that the function is a *reflected* cosine function, we get

$$f(x) = -\frac{1}{2}\cos\left(3x + \frac{5\pi}{6}\right) - 1.$$

It's *crucial* for me to state that there are an infinite number of correct answers when it comes to stating the equation of a trigonometric function. If checking your work against a friend's work and if your friend has a different function, you should graph both equations to see if they are equivalent.

b. **Vertical Shift:** If you imagine a horizontal midline for this tangent-like curve, it would be at $y = 1$. Thus,

$$D = 1.$$

Phase Shift: Remember that the principal cycle of the tangent function starts at the midline (x-axis). Thus, we will assume the first point, $(1, 1)$, is where the cycle of this shifted function starts. This means the function is a reflected tangent function and

$$\text{Phase Shift} = -\frac{C}{B} = 1. \tag{8.3.3}$$

Period or Step Size: The tangent is a nice function to deal with sometimes because we can see the period from the vertical asymptotes. The distance between these matches the period of the given tangent function. Therefore,

$$\text{Period} = \frac{\pi}{B} = |2.5 - (-0.5)| = 3 \implies B = \frac{\pi}{3}.$$

Plugging this back into Equation 8.3.3, we get

$$-\frac{C}{\pi/3} = 1 \implies C = -\frac{\pi}{3}.$$

Amplitude: Since we are dealing with a tangent function, there is no amplitude, but we still need to determine the coefficient of the tangent function. Luckily, we were given a second point, $(\frac{7}{4}, -1)$. So far we have

$$g(x) = A\tan(Bx + C) + D = A\tan\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) + 1.$$

Letting the output be -1 and $x = \frac{7}{4}$, we get

$$\begin{aligned} -1 &= A\tan\left(\frac{7\pi}{12} - \frac{\pi}{3}\right) + 1 \\ \implies -2 &= A\tan\left(\frac{7\pi}{12} - \frac{4\pi}{12}\right) \\ \implies -2 &= A\tan\left(\frac{3\pi}{12}\right) \\ \implies -2 &= A\tan\left(\frac{\pi}{4}\right) \\ \implies -2 &= A(1) \\ \implies -2 &= A \end{aligned}$$

Hence,

$$g(x) = -2\tan\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) + 1.$$

Checkpoint 8.3.5

$y = 3\sin(2x + \frac{\pi}{4}) - 1$

Example 8.3.6

a. We begin by rewriting the function as

$$D(t) = -5.25\cos\left(\frac{\pi}{6}t\right) + 12.25.$$

The amplitude is $A = 5.25$, the period is $\frac{2\pi}{\pi/6} = 12$. Therefore, the step size is 3. Since $C = 0$, there is no phase shift. Finally, the vertical shift is 12.25. Graphing the reflected cosine starting at $t = 0$, stepping by 3, with an amplitude of 5.25, we get the following graph.

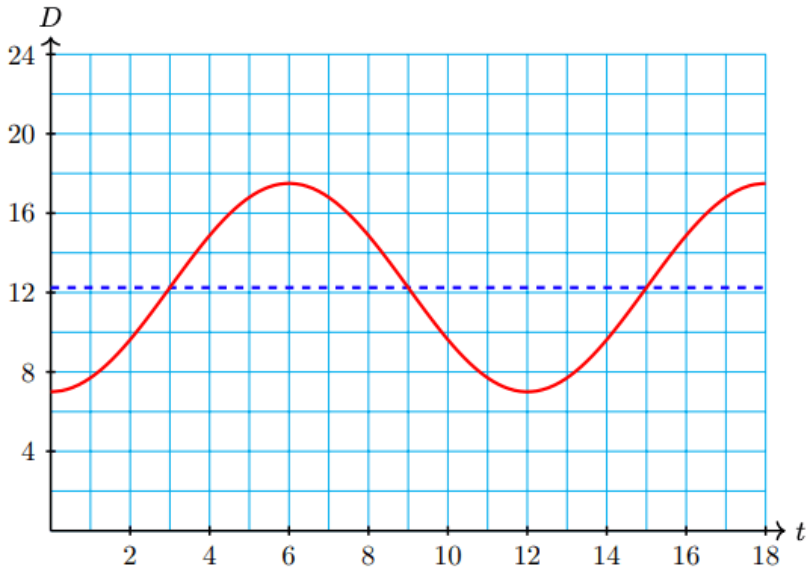


Figure 8.3.26

b. The midline of the graph is $D = 12.25$ hours, the amplitude is 5.25 hours, and the period is 12 months.
c. On the longest day of the year, there are $12.25 + 5.25 = 17.5$ hours of daylight in Glasgow.
d. From the graph, we see that $D(t) < 8$ for t between 0 and 2, or in January and February. $D(t)$ is also less than 8 for a short time between $t = 11$ and $t = 12$ at the end of December.

Checkpoint 8.3.6

a.

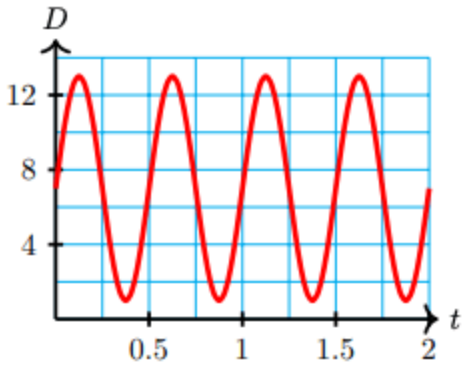


Figure 8.3.27

b. Midline: $D = 7$, amplitude: 6, period: 0.5 millisec
c. Largest: 13 cm, smallest: 1 cm

Example 8.3.7

The graph has the shape of $y = -\cos(x)$. It appears to have midline $y = 75$ and amplitude 75, so $|A| = 75$ and $A = -75$. To find B, we compute

$$\text{Period} = \frac{2\pi}{B} \implies B = \frac{2\pi}{\text{Period}} = \frac{2\pi}{10.8} \approx 0.58.$$

Thus, a formula for the model is

$$y = -75\cos(0.58x + C) + 75.$$

If we are trying to fit the red curve, the phase shift would be 0 and so $C = 0$. Hence,

$$y = -75\cos(0.58x) + 75.$$

Checkpoint 8.3.7

a. Period: 10 years, midline: $y = 61.5$, amplitude: 53
b. $s(t) = 61.5 - 53\cos\left(\frac{\pi}{5}t - \frac{\pi}{20}\right)$
c. ≈ 18.6

This page titled 8.3: Rigid Transformations is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

8.3.1: Resources and Key Concepts

Resources

Videos

- Graphing Trigonometric Functions
  - Graphs of the Other Trigonometric Functions - Graphing Tangent and Cotangent Functions
- Sinusoidal Models
  - Trigonometric Graphs - Technology and Graphs of Trigonometric Functions (Part 1)
  - Graphs of the Other Trigonometric Functions - An Application Involving the Tangent Function
- Finding Equations from Graphs
  - Trigonometric Graphs - Review of Identifying the Equation for a Sinusoidal Function from a Graph

Key Concepts

Definitions

- **Phase Shift:** The horizontal shift of a trigonometric function. It represents the x-value to which the starting point of the principal cycle (originally at $x = 0$) has been moved.
- **Midline:** A horizontal line that represents the average value of a trigonometric function. For sinusoidal functions, it is the line situated halfway between the maximum and minimum points.

Theorems

- **Lemma: Phase Shift Formula for y = trig(x + C):** The phase shift for a trigonometric function of the form $y = \text{trig}(x + C)$ is $-C$.
- **Phase Shift Formula:** The phase shift of the function $y = A\text{trig}(Bx + C)$ is $-\frac{C}{B}$.
- **Cofunction Identities (radians):** The trigonometric function of an angle is equal to the cofunction evaluated at the complement of the angle. For example, $\sin(\theta) = \cos(\frac{\pi}{2} - \theta)$.

Common Mistakes

- **Confusing C with the Phase Shift:** In the function $y = A\text{trig}(Bx + C)$, the phase shift is $-\frac{C}{B}$, not simply $-C$. The value of B affects the horizontal shift.
- **Ignoring Negative B-values:** Before determining the period or phase shift, if the parameter B is negative, you must first use the Symmetry Identities to make the coefficient of x positive. Forgetting this step will result in an incorrect phase shift.
- **Incorrectly Identifying the Midline:** The midline is the average of the maximum and minimum values, not simply the lowest point of the graph. It is the horizontal line the graph oscillates around.

This page titled 8.3.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

8.3.2: Homework

Reading Questions

1. What is a phase shift and what does it represent on a graph?
2. What is the midline of a trigonometric function?
3. For the general function $f(x) = A\text{trig}(Bx + C) + D$, which parameter determines the midline?
4. What is the formula for the phase shift of the function $y = A\sin(Bx + C)$?
5. How can you find the midline of a sinusoidal graph if you know its maximum and minimum values?
6. When graphing a function like $y = \sin(x + \frac{\pi}{3})$, does the graph shift to the left or to the right?
7. Before finding the phase shift for a function like $y = \cos(-2x + \pi)$, what is the necessary first step?
8. The text shows that the graph of $y = \cos(\frac{\pi}{2} - x)$ is identical to the graph of $y = \sin(x)$. What transformation of the cosine graph produces the sine graph?
9. When finding the equation of a sinusoidal function from its graph, why is it helpful to identify the vertical shift first?
10. When finding the period of a tangent function from its graph, what graphical feature can you measure the distance between?

Homework

Vocabulary Check

1. The ___ tells us the x-value of where the original initial number in the principal cycle has moved.
2. Given the function $f(x) = A\ \text{trig}\,(Bx + C) + D$, the graph of the equation $y = D$ is called the ___ of the trigonometric function.
3. Given the function $f(x) = A\ \text{trig}\,(Bx + C) + D$, D represents the ___.

Concept Check

4. Write an equation for a cosine function with midline -5.
5. Write an equation for a sine function with midline 2.
6. Which of the following functions are the same? Explain your answer.
$$f(x) = \cos\left(2x - \frac{\pi}{3}\right), \quad g(x) = (\cos 2x) - \frac{\pi}{3}, \quad h(x) = \cos 2x - \cos\frac{\pi}{3}$$
7. Which of the following functions are the same? Explain your answer.
$$f(x) = \sin\left(2x - \frac{\pi}{3}\right), \quad g(x) = \sin\left[2\left(x - \frac{\pi}{6}\right)\right], \quad h(x) = 2\sin\left(x - \frac{\pi}{6}\right)$$

True or False? For the following exercises, determine if the statement is true or false. If true, cite the definition or theorem stated in the text supporting your claim. If false, explain why it is false and, if possible, correct the statement.

8. The phase shift is also known as the horizontal shift.
9. The amplitude of $f(x) = -3\tan(2x - 1)$ is 3.
10. The phase shift of the function $f(x) = -3\tan(2x - 1)$ is 1.
11. The phase shift of the function $f(x) = -3\tan(2x - 1)$ is -1.
12. The functions $y = 2\cos(\pi x - 4)$ and $y = -3\sin\left(\pi\left(x - \frac{4}{\pi}\right)\right)$ have the same phase shift.

Basic Skills

For the following exercises, calculate the phase shift.

13. $y = -2\sin\left(3x - \frac{\pi}{4}\right)$

14. $y = 6\cos\left(2x + \frac{3\pi}{4}\right)$

15. $y = 12\cos\left(\frac{t}{3} + 0.8\right)$

16. $y = -5\sin\left(\frac{t}{4} - 0.6\right)$

For the following exercises, state the amplitude, period, and midline of the graph.

17. $y = -3 + 2\sin(x)$

18. $y = 4 - 3\cos(x)$

19. $y = 1 - \cos(\pi x)$

20. $y = 2 + \sin(2\pi x)$

For the following exercises,
(a) Sketch the graphs of f and g on the same axes.
(b) State the phase shift from f to g.
(c) Find all values of x for which $g(x) = 1$ for $-\pi \le x \le \pi$.
(d) Find all values of x for which $g(x) = 0$ for $-\pi \le x \le \pi$.

21. $f(x) = \sin(x)$ and $g(x) = \sin\left(x - \frac{\pi}{3}\right)$.

22. $f(x) = \cos(x)$ and $g(x) = \cos\left(x + \frac{\pi}{3}\right)$.

23. $f(x) = \tan(x)$ and $g(x) = \tan\left(x + \frac{\pi}{4}\right)$.

24. $f(x) = \tan(x)$ and $g(x) = \tan\left(x - \frac{\pi}{2}\right)$.

For the following exercises,
(a) What are the amplitude and phase shift of the given function?
(b) Sketch the graph.
(c) Find all values of x for which $f(x) = 0$ for $0 \le x \le 2\pi$.

25. $f(x) = -2\cos\left(x + \frac{\pi}{6}\right)$

26. $f(x) = -6\sin\left(x - \frac{2\pi}{3}\right)$

27. $f(x) = \frac{1}{2}\tan\left(x + \frac{\pi}{5}\right)$

28. $f(x) = -\frac{3}{7}\cos\left(x - \frac{\pi}{4}\right)$

For the following exercises,
(a) What are the period and phase shift of the given function?
(b) Sketch the graph.
(c) Find all values of x for which $f(x) = 1$ for $0 \le x \le 2\pi$.
(d) Find all values of x for which $f(x) = 0$ for $0 \le x \le 2\pi$.

29. $f(x) = \cos\left(2x - \frac{\pi}{3}\right)$

30. $f(x) = \sin\left(3x + \frac{\pi}{2}\right)$

31. $f(x) = \sin\left(\pi x + \frac{\pi}{3}\right)$

32. $f(x) = \cos\left(\pi x - \frac{\pi}{3}\right)$

For the following exercises,
(a) What are the midline, period, phase shift, and amplitude of the given function?
(b) Sketch the graph.
(c) Find all values of x for which $f(x) = 1$ for $0 \le x \le 2\pi$.
(d) Find all values of x for which $f(x) = k$ for $0 \le x \le 2\pi$, where k is stated within the problem.

33. $y = 3\sin\left(\frac{x}{2} - \frac{\pi}{6}\right) + 4,\ k = 4$

34. $y = 2\cos\left(\frac{x}{3} - \frac{\pi}{4}\right) - 1,\ k = -1$

For the following exercises,
(a) State the amplitude.
(b) State the period.
(c) State the step size.
(d) State the phase shift.
(e) State the vertical shift.
(f) Use this information to create an accurate graph of a single cycle of the trigonometric function. Be sure to label any asymptotes and all key points.

35. $y = 2 - 5\cos(2t)$

36. $y = -2 + 4\sin(3t)$

37. $y = 1 + 3\cos\left(\frac{t}{2}\right)$

38. $y = -2 - 3\sin\left(\frac{t}{4}\right)$

39. $y = 4 + 2\tan(3x)$

40. $y = 3 - \tan\left(\frac{x}{4}\right)$

41. $y = 1 - 2\tan\left(\frac{x}{3}\right)$

42. $y = 2\sin(2x - \pi) + 1$

43. $y = -1 + 3\cos(2x + \pi)$

44. $y = -2 - \tan\left(3x - \frac{\pi}{2}\right)$

45. $y = \frac{3}{2} - \frac{1}{2}\cos(3x + \pi)$

46. $y = -\frac{1}{3}\sin\left(\pi x + \frac{3\pi}{2}\right) + 2$

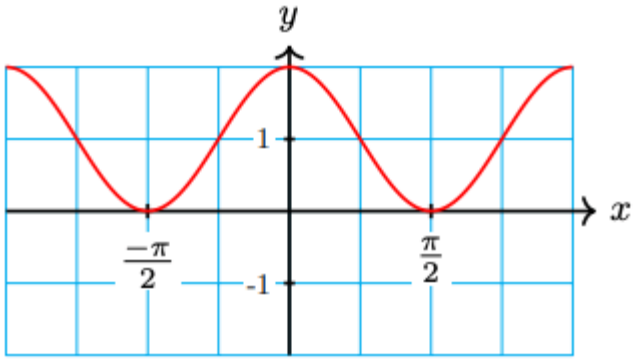
47. $y = -3\tan(2x - 1) + 1$

48. $y = -4\cos(\pi x - 3) - 4$

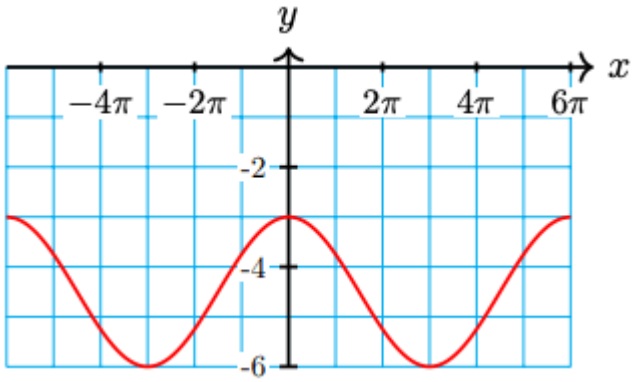
For the following exercises, state the equation of the function.

49.

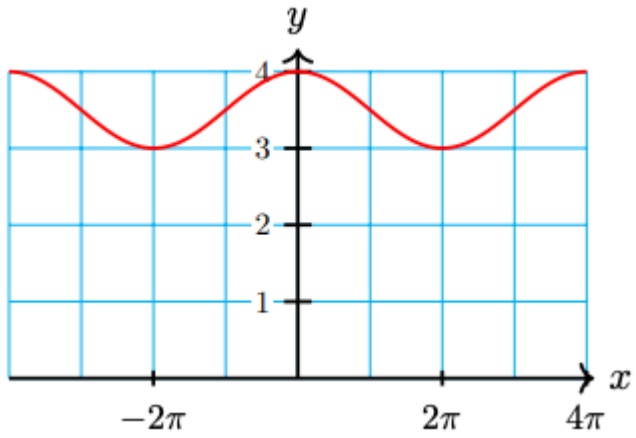
50.


y
1
x
$\frac{-\pi}{2}$
-1
$\frac{\pi}{2}$

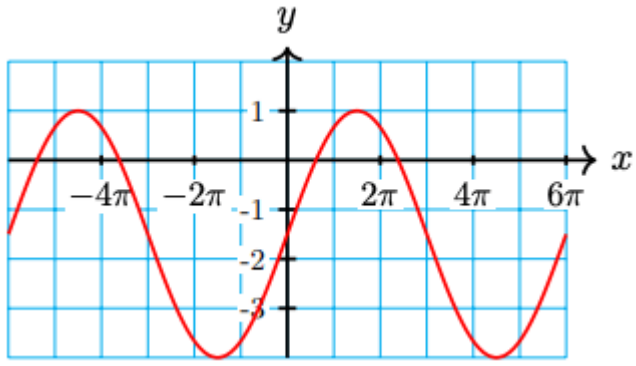
51.


y
x
-4π
-2π
2π
4π
6π
-2
-4
-6

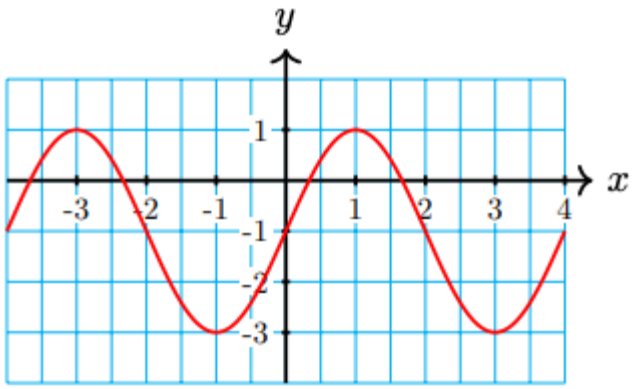
52.


y
4
3
2
1
x
-2π
2π
4π

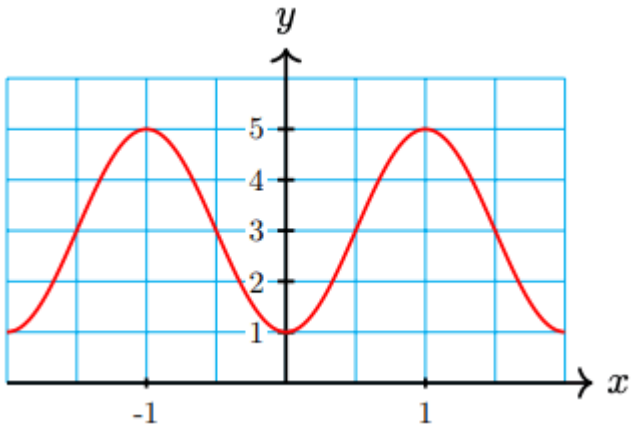
53.


y
1
x
-4π
-2π
-1
2π
4π
6π
-2
-3

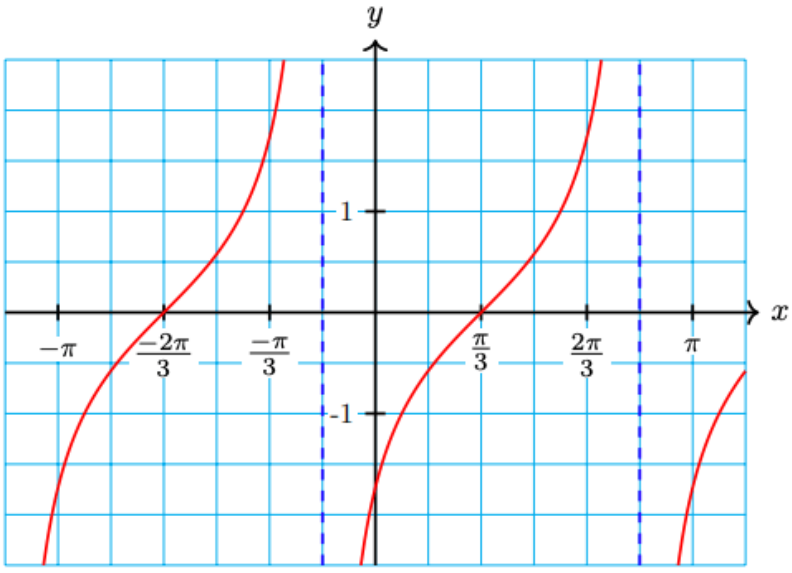
54.

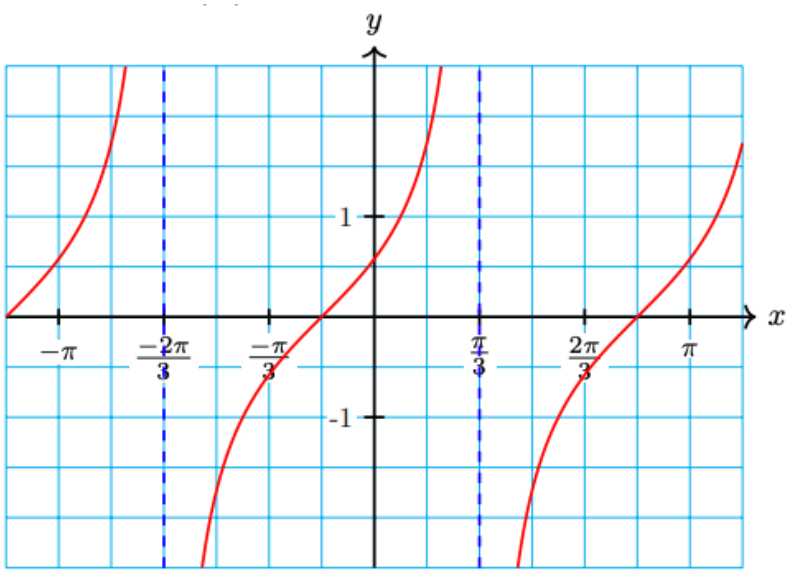

y
x
1
-1
-2
-3
-3
-2
-1
1
2
3
4

55.


y
x
5
4
3
2
1
-1
1

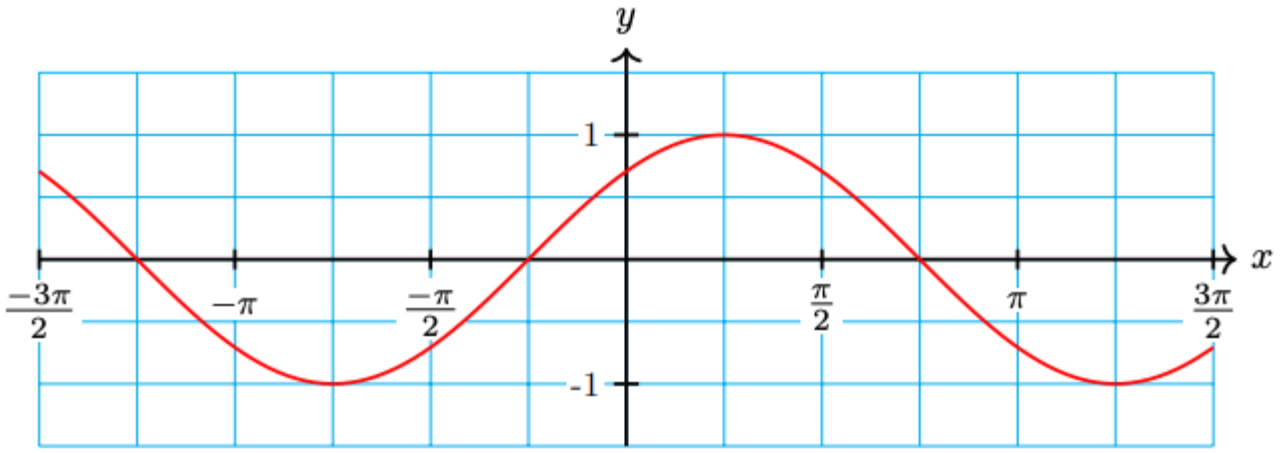
56.


y
x
1
-1
$-\pi$
$\frac{-2\pi}{3}$
$\frac{-\pi}{3}$
$\frac{\pi}{3}$
$\frac{2\pi}{3}$
π

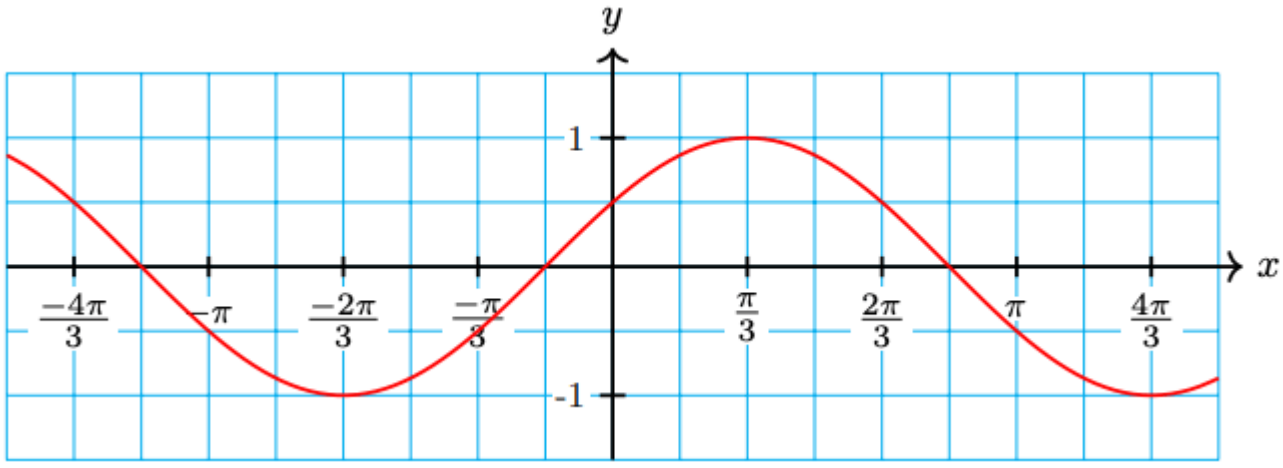


For the following exercises, find a formula for the graph of the given function in terms of
(a) sine
(b) cosine

57.

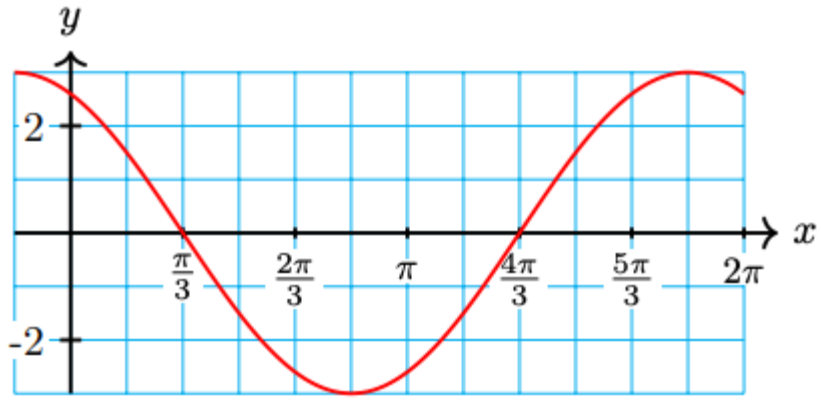


58.

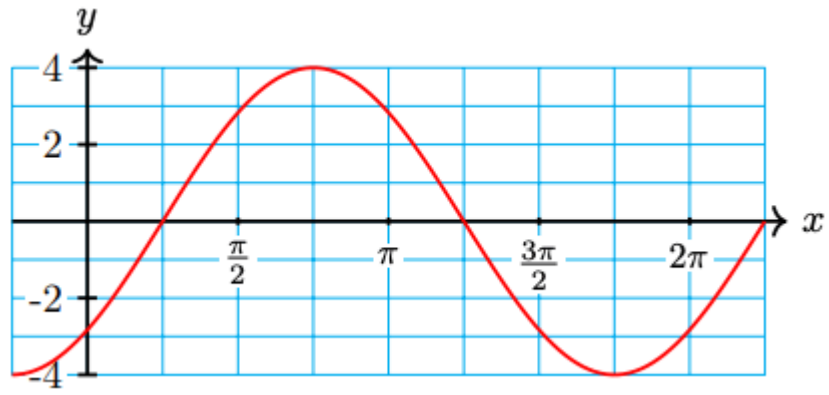


59.

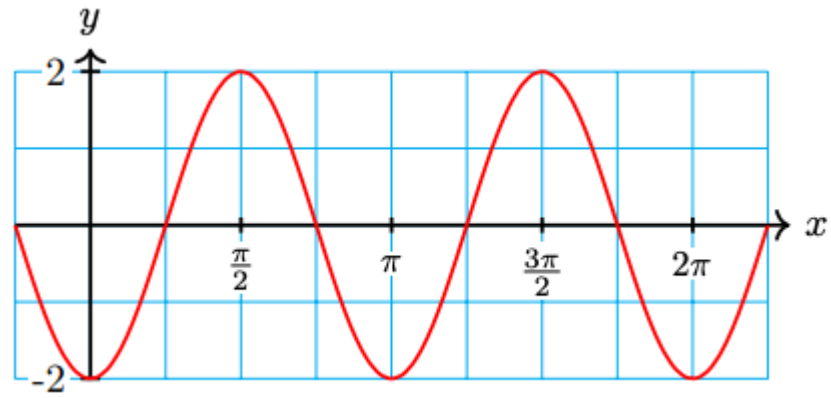
60.

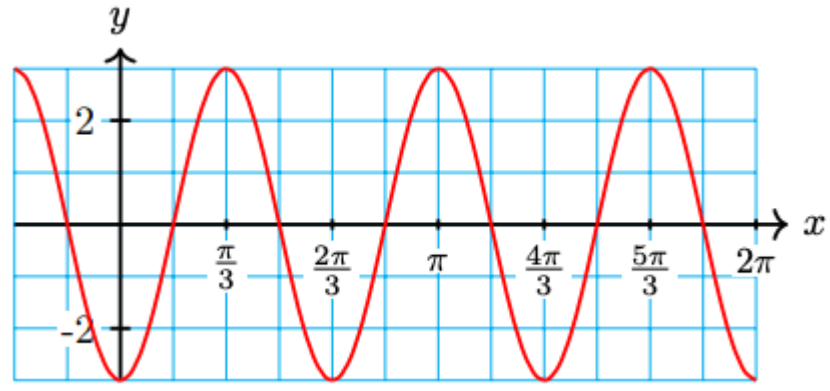

y
2
-2
x
$\frac{\pi}{3}$
$\frac{2\pi}{3}$
π
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$
2π

61.

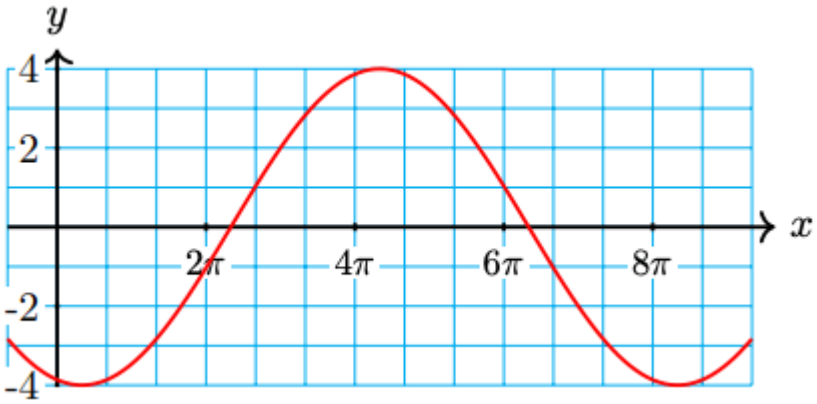

y
4
2
-2
-4
x
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

62.

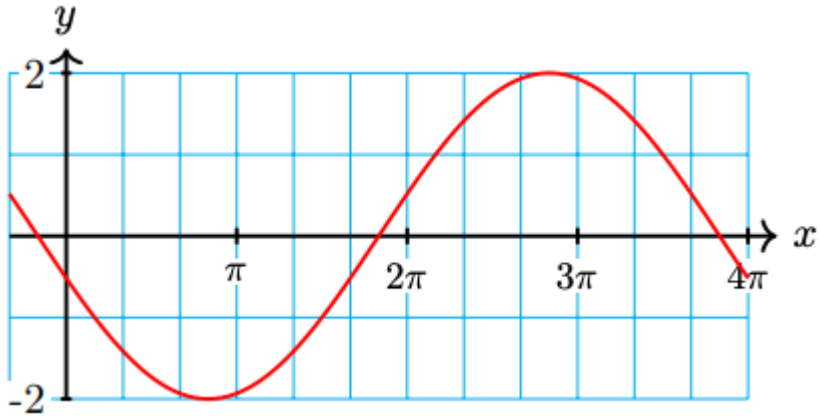

y
2
-2
x
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π


y
2
-2
x
$\frac{\pi}{3}$
$\frac{2\pi}{3}$
π
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$
2π

53.



54.



For the following exercises, find and graph a single cycle of a sinusoidal function with the given properties.

55. Amplitude of 2, a period of 3, and is shifted 4 units to the left and 5 units upwards compared with the sine function.

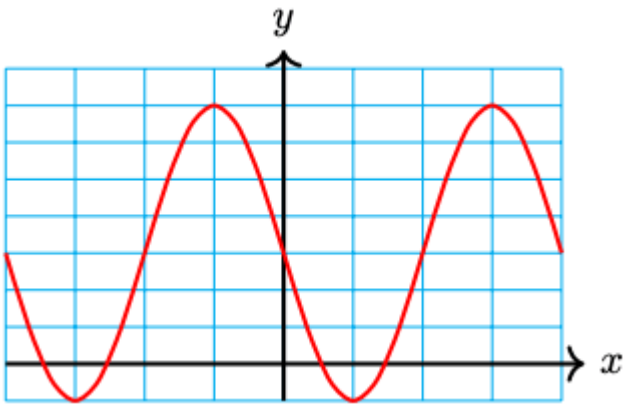
56. Amplitude of 3, a period of 24, and is shifted 2 units to the right and 4 units upwards compared with the cosine function.

57. Amplitude of 5, a period of 360, its midline at $y = 12$, and passes through (0,7).

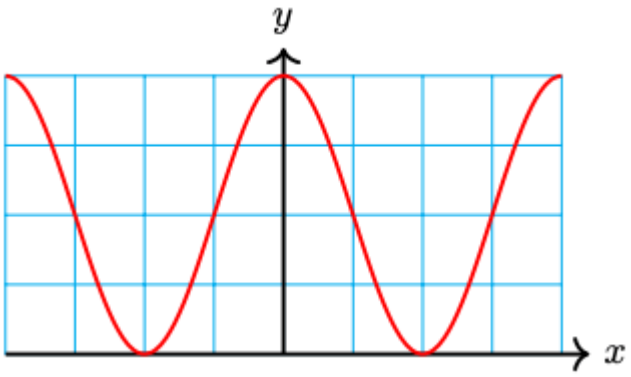
58. Amplitude of 50, a period of 30, its midline at $y = 50$, and passes through (0,100).

For the following exercises, label the scales on the axes for the graph.

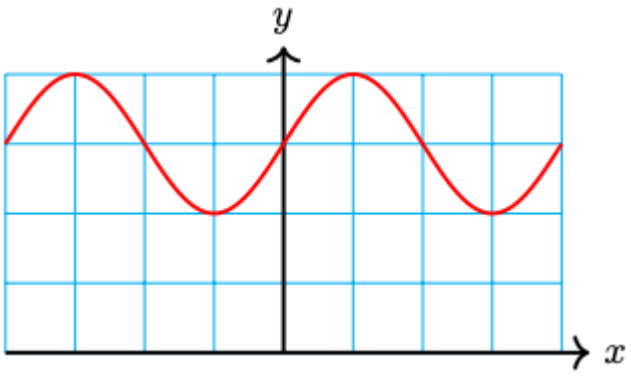
59. $y = 3 - 4\sin(2x)$



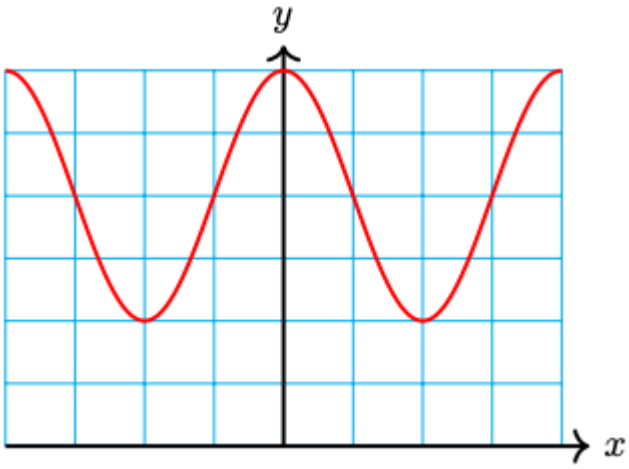
70. $y = 2\cos(5x) + 2$



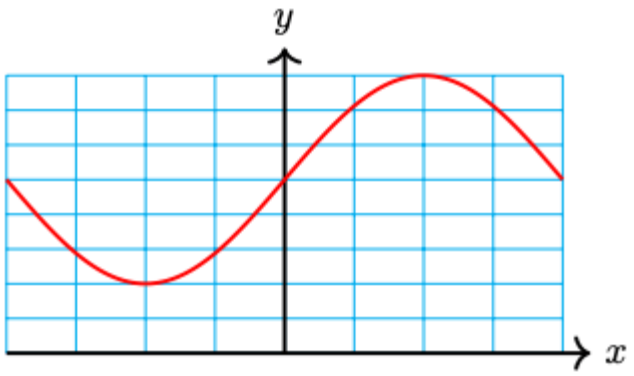
71. $y = \frac{1}{2}\sin(3x) + \frac{3}{2}$



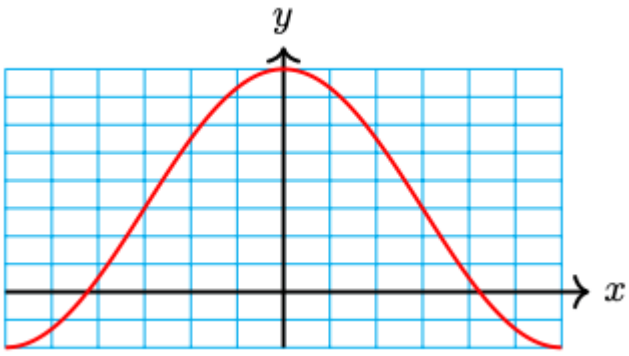
72. $y = \frac{2}{5}\cos(6x) + \frac{4}{5}$



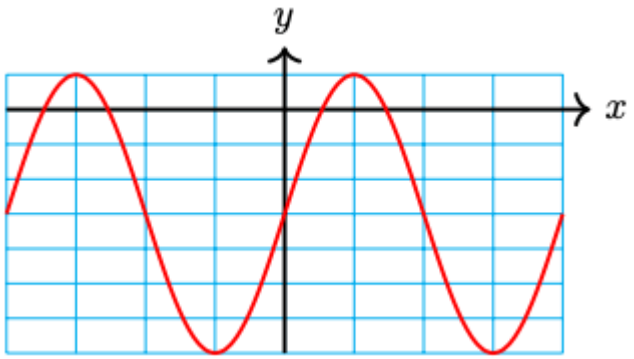
73. $50 - 30\sin\left(\frac{x}{4}\right)$



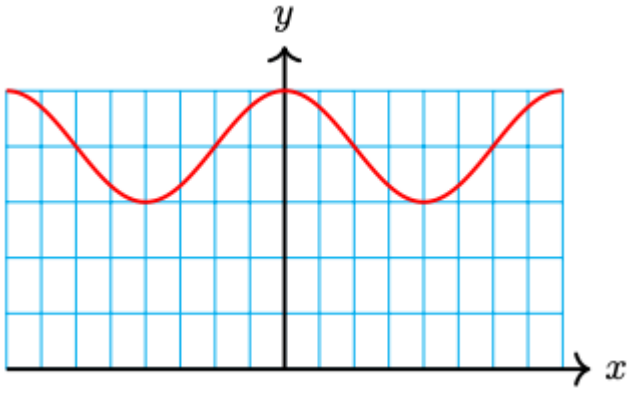
74. $25\cos\left(\frac{x}{3}\right)+15$



75. $y=4\sin(\pi x)-3$



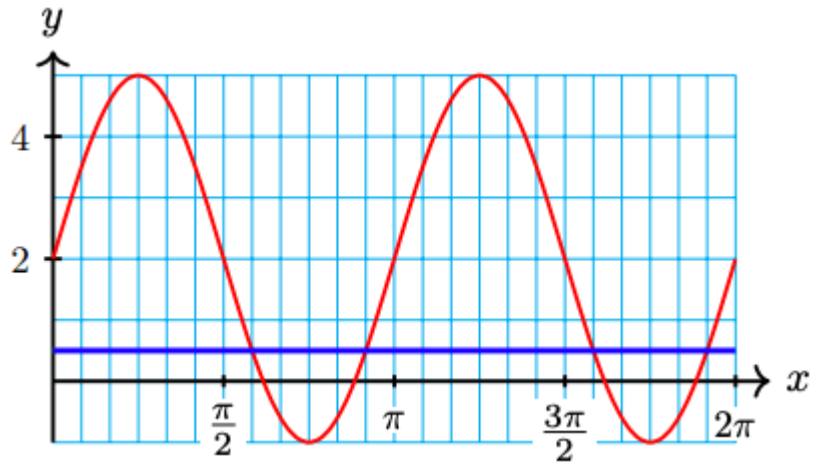
76. $y=\frac{1}{2}\cos\left(\frac{\pi x}{2}\right)+2$



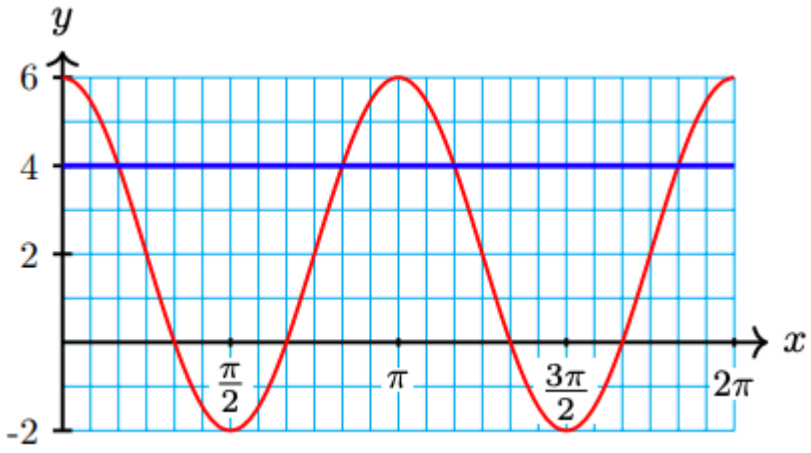
Synthesis Questions

For the following exercises, use the graph to find all solutions between 0 and 2π.

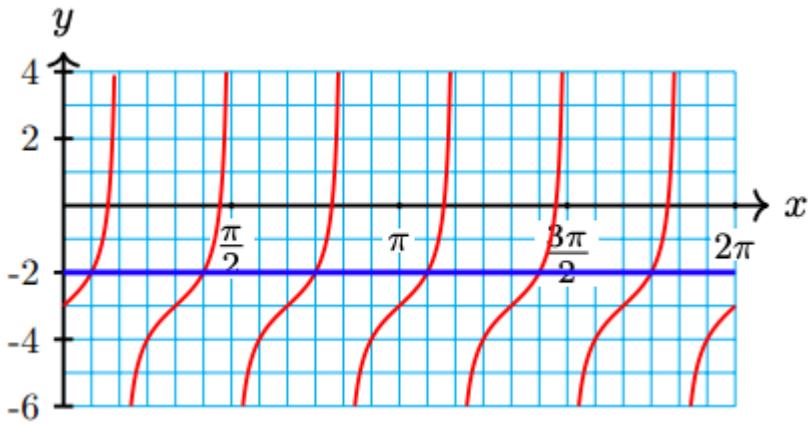
77. $2 + 3\sin(2x) = 0.5$



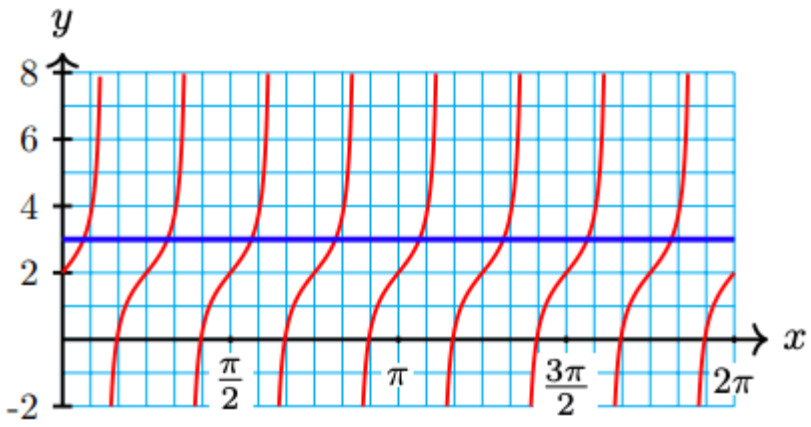
78. $2 + 4\cos(2x) = 4$



79. $-3 + \tan(3x) = -2$



80. $2 + \tan(4x) = 3$



For the following exercises,
(i) Use a calculator to graph the function for $0 \le x \le 2\pi$.
(ii) Use the intersect feature to find all solutions between 0 and 2π. Round your answers to hundredths.

81. a. $h(x) = 2 - 4\cos\left(\frac{x}{4}\right)$

    b. $2 - 4\cos\left(\frac{x}{4}\right) = 0$

82. a. $H(x) = 3 + 2\sin\left(\frac{x}{2}\right)$

    b. $3 + 2\sin\left(\frac{x}{2}\right) = 5$

83. a. $G(x) = -1 + 3\cos(3x)$

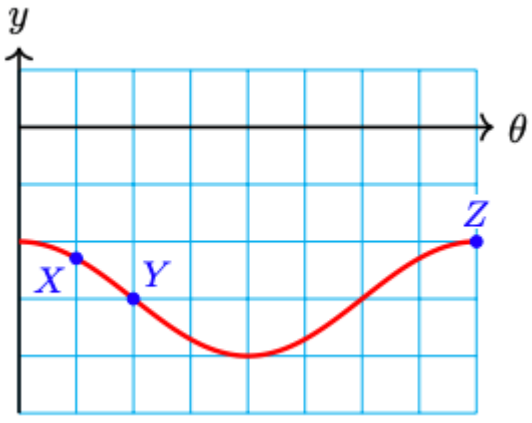
    b. $-1 + 3\cos(3x) = 1$

84. a. $F(x) = 4 - 3\sin(2x)$

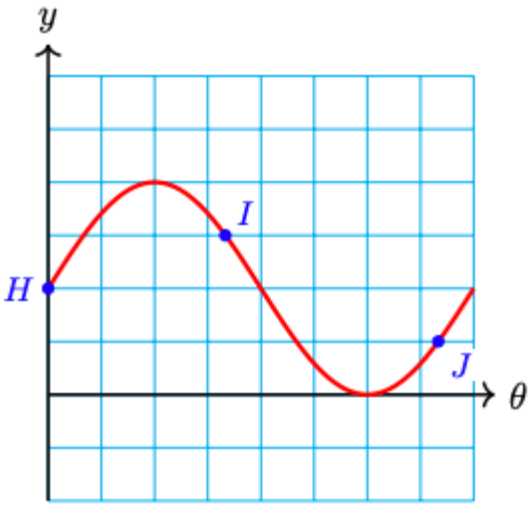
    b. $4 - 3\sin(2x) = 2.5$

For the following exercises, give the coordinates of the points on the graph.

85. $f(\theta) = -3 + \cos\theta$



86. $f(\theta) = 1 + \sin\theta$



Applications

37. The tide in Yorktown is approximated by the function

$$h(t) = 1.4 - 1.4\cos(0.51t)$$

measured in feet above low tide, where t is the number of hours since the last low tide.

a. What are the midline, period, and amplitude?

b. Graph $h(t)$ for two periods, labeling the points that correspond to high tide and low tide.

c. If the last low tide occurred at 5:00 am, predict when the next high and low tides will occur.

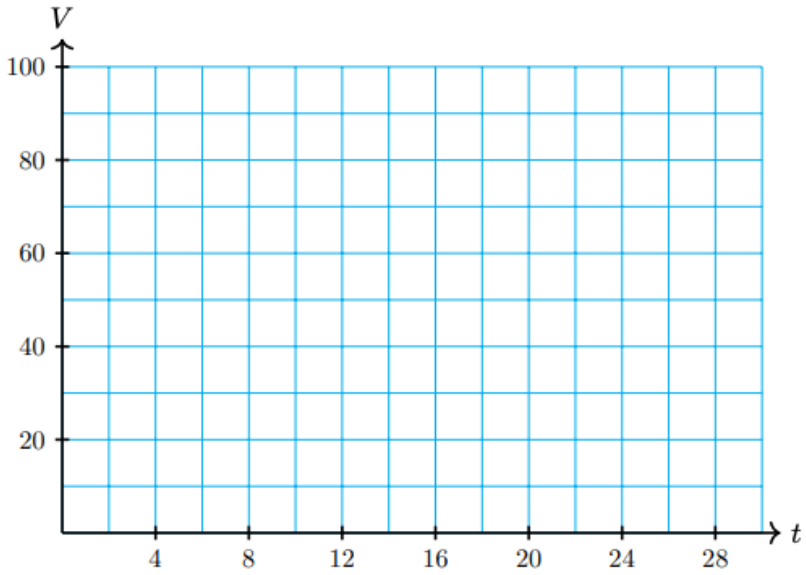
38. When observed from Earth, the moon looks like a disk that is partially visible and partially in shadow. The percentage of the disk that is visible can be approximated by

$$V(t) = 50\cos(0.21t) + 50$$

where t is the number of days since the last full moon.

a. Graph $V(t)$ in the window

$$X_{\min} = 0, X_{\max} = 30$$
$$Y_{\min} = 0, Y_{\max} = 120$$



b. Sketch the graph on the grid.

c. Label on your graph the points that correspond to full moon, half moon, and new moon. (New moon occurs when the part of the moon receiving sunlight is facing directly away from the Earth.)

d. At what times during the lunar month is 25% of the moon visible? Mark those points on your graph.

e. During which days is less than 50% of the moon visible? Mark the corresponding points on your graph.

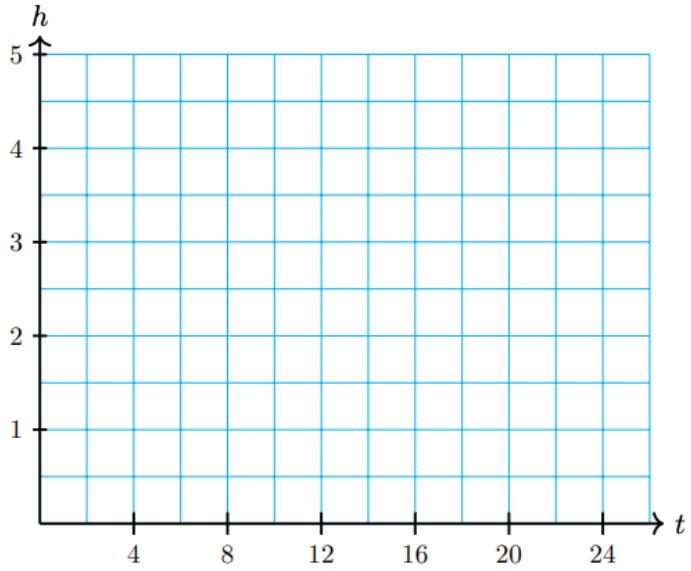
39. The tide in Malibu is approximated by the function

$$h(t) = 2.5 - 2.5\cos(0.5t)$$

measured in feet above low tide, where t is the number of hours since the last low tide.

a. Graph $h(t)$ in the window

$$X_{\min} = 0, X_{\max} = 25$$
$$Y_{\min} = 0, Y_{\max} = 6$$



b. Sketch the graph on the grid.

c. Label on your graph the points that correspond to high tide and low tide.

d. How high is high tide, and at what times does it occur?

e. At what times during the 25-hour period is the tide 4 feet above low tide? Mark those points on your graph.

f. Kathie walks along the beach only when the tide is below 1 foot. Find the intervals on your graph when the tide is below 1 foot.

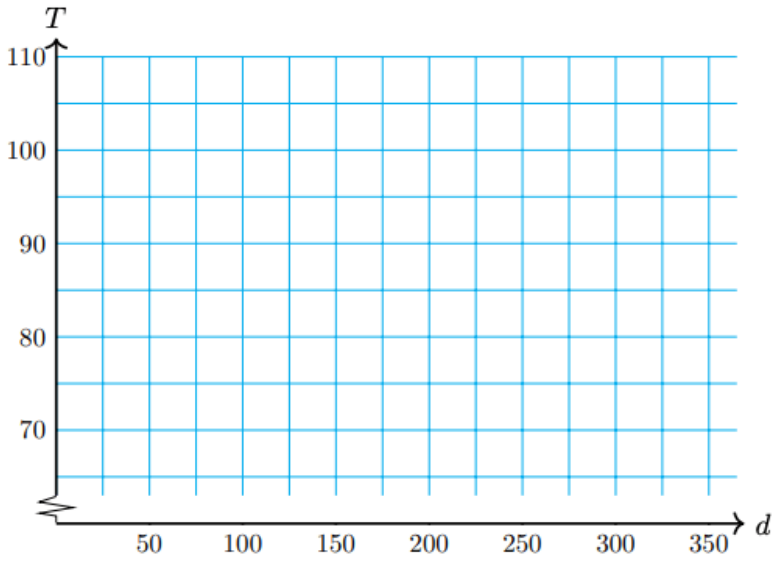
90. The average daily high temperature in the town of Beardsley, Arizona is approximated by the function

$$T(d) = 85.5 - 19.5\cos(0.0175d - 0.436)$$

where the temperature is measured in degrees Fahrenheit, and d is the number of days since January 1.

a. Graph $T(d)$ in the window

$$X_{\min} = 0, X_{\max} = 365$$
$$Y_{\min} = 0, Y_{\max} = 110$$



b. Sketch the graph on the grid.

c. Label on your graph the points that correspond to highest and lowest average temperature.

d. What is the hottest day and what is its average temperature? What is the day with the lowest average temperature, and what is that temperature?

e. At what times during the year are average high temperatures above $90°$? Mark those points on your graph.

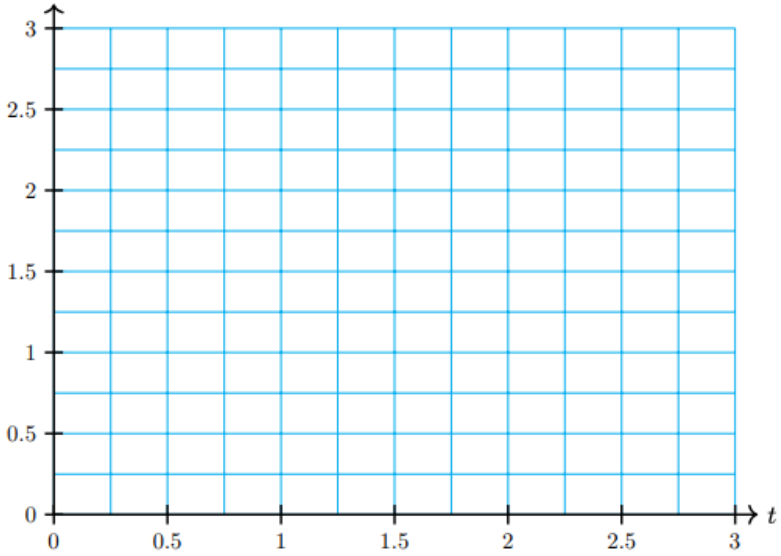
91. A weight is suspended from the ceiling on a spring. The weight is pushed straight up, compressing the spring, then released. The height of the weight above the ground is given by the function

$$H(t) = 2 + 0.5\cos(6t)$$

where the height is measured in meters, and t is the number of seconds since the mass was released.

a. Graph $H(t)$ in the window

$$X_{\min} = 0, X_{\max} = 3$$
$$Y_{\min} = 0, Y_{\max} = 3$$



b. Sketch the graph on the grid.

c. Label on your graph the points that correspond to highest and lowest positions of the mass.

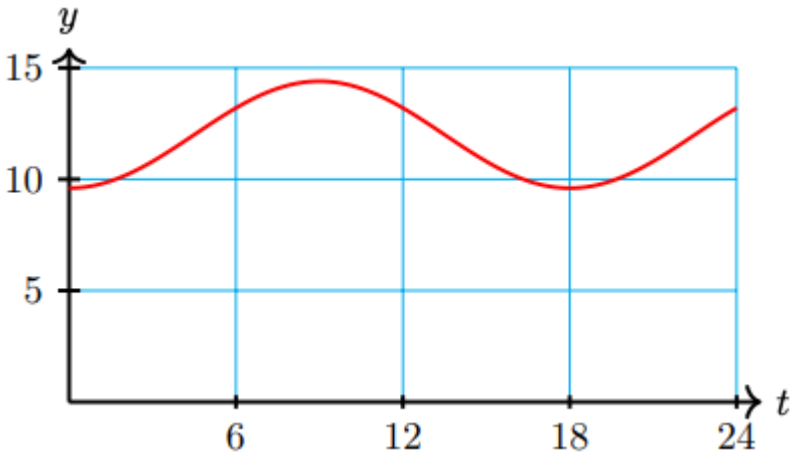
d. How high is highest point, and when is that height attained during the first 3 seconds?

e. How high is lowest point, and when is that height attained during the first 3 seconds?

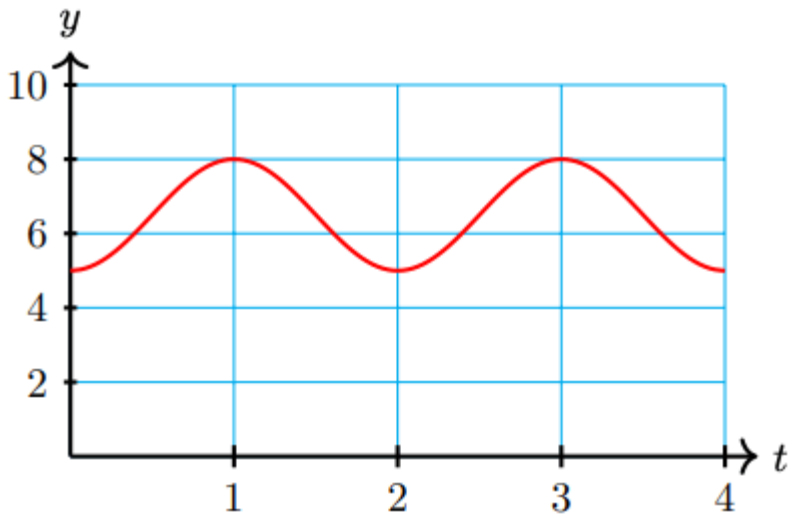
f. Find the intervals during the first 3 seconds when the mass is less than 2 meters above the ground.

For the following exercises, write an equation for the sinusoidal function whose graph is shown.

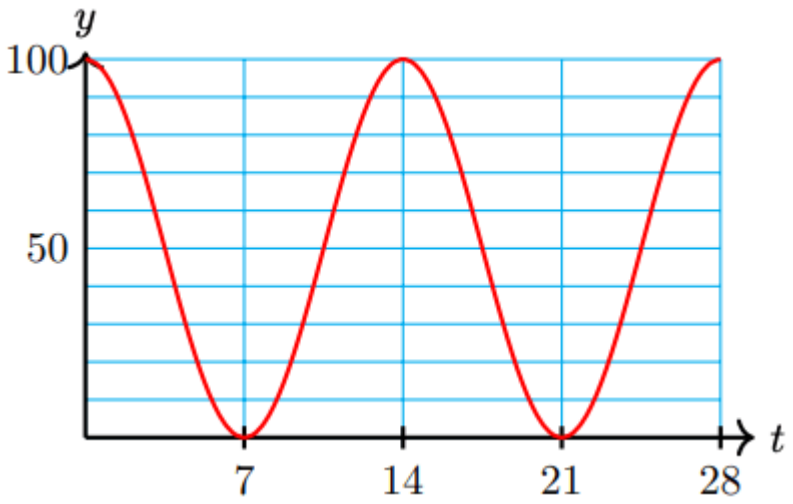
92. The number of hours of daylight in Salt Lake City varies from a minimum of 9.6 hours on the winter solstice to a maximum of 14.4 hours on the summer solstice. Time is measured in months, starting at the winter solstice.



93. A weight is 6.5 feet above the floor, suspended from the ceiling by a spring. The weight is pulled down to 5 feet above the floor and released, rising past 6.5 feet in 0.5 seconds before attaining its maximum height of feet. The weight oscillates between its minimum and maximum height.



94. Although the moon is spherical, what we see from Earth looks like a disk, sometimes only partly visible. The percentage of the moon's disk that is visible varies between 0 (at new moon) to 100 (at full moon), over a 28-day cycle.



This page titled 8.3.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

8.4: Graphs of the Remaining Trigonometric Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

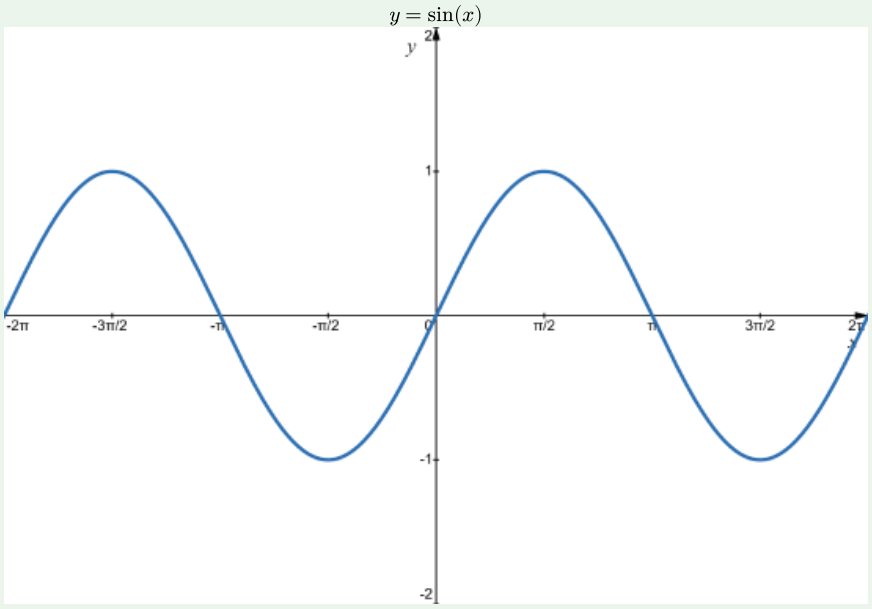
The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

A Summary of the Graphs of the Fundamental Trigonometric Functions

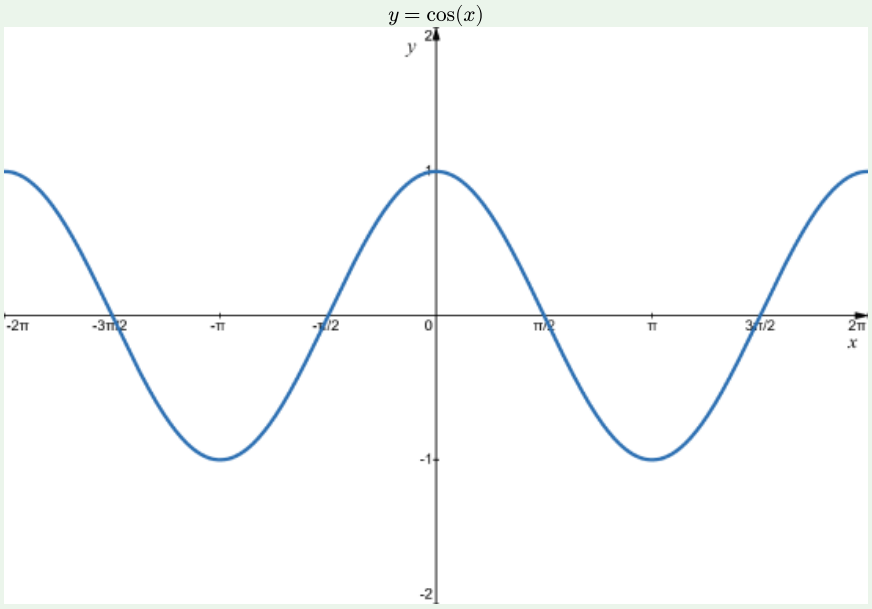
Before we go too far, let's summarize what we know about the fundamental trigonometric functions.

Theorem: Properties of the Graph of the Sine Function



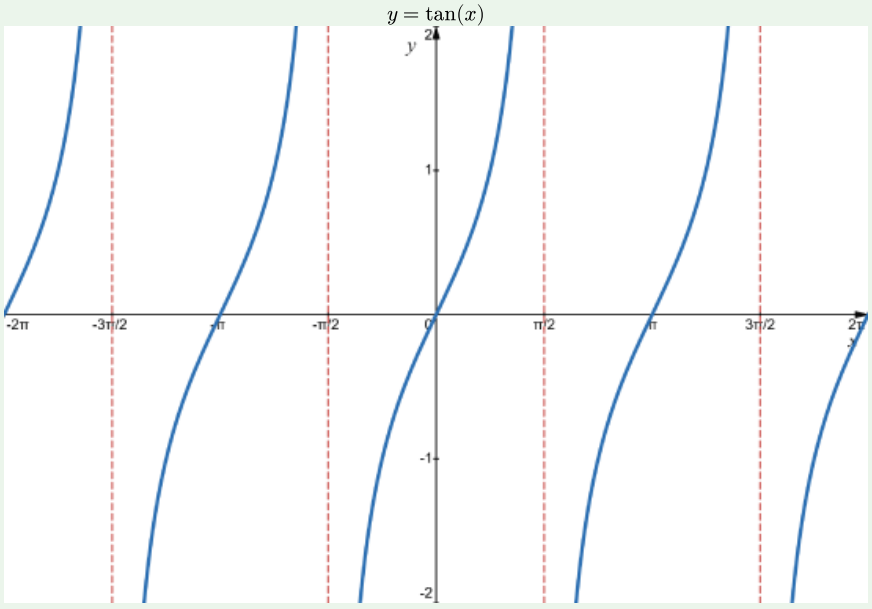
- **Domain:** $(-\infty, \infty)$
- **Range:** $[-1, 1]$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** 1
- **Zeros:** $x = k\pi$, where $k^{\text{int}}\mathbb{Z}$

Theorem: Properties of the Graph of the Cosine Function



- **Domain:** $(-\infty, \infty)$
- **Range:** $[-1, 1]$
- **Symmetry:** Even (symmetric about the y-axis)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** 1
- **Zeros:** $x = \frac{\pi}{2} + k\pi$, where $k^{\text{int}}\mathbb{Z}$

Theorem: Properties of the Graph of the Tangent Function



- **Domain:** All $x \neq \frac{\pi}{2} + k\pi$, where $k^{\text{int}}\mathbb{Z}$
- **Range:** $(-\infty, \infty)$
- **Symmetry:** Odd (symmetric about the origin)

- **Principal Cycle:** $[0, \pi]$
- **Natural Period:** π
- **Amplitude:** N/A
- **Zeros:** $x = k\pi$, where $k \in \mathbb{Z}$

The principal cycle of the tangent is $[0, \pi]$, even though the tangent is undefined halfway through the cycle. It's just the cycle we use as the "standard" for our base graph, and it is okay that the tangent is undefined at some point within the cycle. The principal cycle is *not* the domain.

An Interlude—Vertical Asymptotes

We now turn our attention to the graphs of the remaining trigonometric functions—$\csc(x)$, $\sec(x)$, and $\cot(x)$. In each of these cases, we must understand the conceptual approach to graphing we developed in when we introduced the graphs of the fundamental trigonometric functions.

Recall, if given a fraction where the numerator is held constant and the denominator is *approaching* zero (but not equal to zero), then the value of the fraction is approaching either $+\infty$ or $-\infty$. To drive the point home, let's demonstrate this behavior both numerically and graphically.

For the numerical demonstration, you can create any rational function you want where the numerator is a constant and the denominator is an algebraic expression. All we require is that the denominator approaches zero as the values of x approach some value. For example, suppose we choose the numerator to be 7 and the denominator to be $6x^2 + 11x + 3$. Then the function we are working with is

$$R(x) = \frac{7}{6x^2 + 11x + 3}.$$

We now want the values of x to approach some number that causes division by zero. To reveal what causes division by zero for this function, we factor the denominator as follows:

$$R(x) = \frac{7}{(3x + 1)(2x + 3)}.$$

We can see this function would have division by zero at $x = -\frac{1}{3}$ and $x = -\frac{3}{2}$.

Let's build two tables of values—the first to see what happens to the function as $x \to -\frac{3}{2}^-$, and the second to see what happens as $x \to -\frac{3}{2}^+$. For convenience, we will rewrite $-\frac{3}{2}$ in its decimal form, -1.5.





In the first table, as x is approaching -1.5 from the left, the denominator of $R(x)$ is approaching zero from the right. Simultaneously, the function values are becoming larger and larger positive numbers. That is,

$$\text{As } x \to -\frac{3}{2}^-,\ R(x) \to \infty.$$

Similarly, the second table demonstrates that, as x is approaching -1.5 from the right, the denominator of $R(x)$ approaches zero from the left. This causes the function values to become larger and larger negative numbers. We would say,

$$\text{As } x \to -\frac{3}{2}^+,\ R(x) \to -\infty.$$

For the graphical demonstration, we quickly sketch the graph of $R(x)$ around $x = -\frac{3}{2}$.

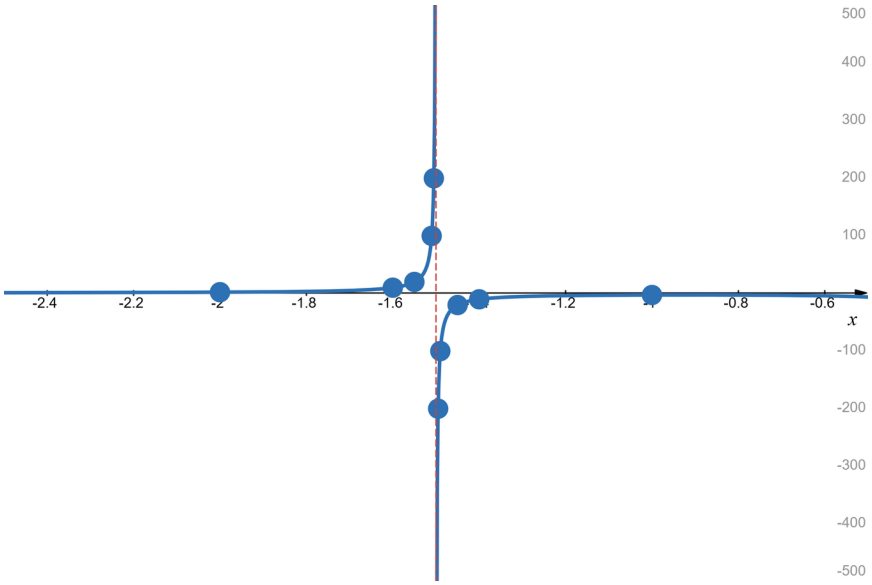


Figure 8.4.1

As you can see from the graph, the y-values of the function (also called the function values) become more and more positive as the x-values get closer and closer to -1.5 from the left. Likewise, the function values become increasingly more negative as the x -values approach -1.5 from the right.

The moral of the story can be stated as follows:

> If the numerator is a nonzero constant and the denominator is tending to (but not becoming) zero, the ratio
>
> $$\frac{\text{numerator}}{\text{denominator}}$$
>
> approaches either $+\infty$ or $-\infty$. The sign can be determined by comparing the signs of the numerator and denominator as the denominator approaches zero. Graphically, this means there is a vertical asymptote at this value of x.

Graphing the Cosecant Function

We know that $\csc(x) = \frac{1}{\sin(x)}$. Therefore, by the preceding discussion, as the denominator tends to zero, the overall ratio (which is the cosecant) will tend to $+\infty$ or $-\infty$. The zeros of the sine function occur at $x = k\pi$, where $k \in \mathbb{Z}$ (see Figure 8.4.2). Let's start by focusing on the zero at $x = 0$.

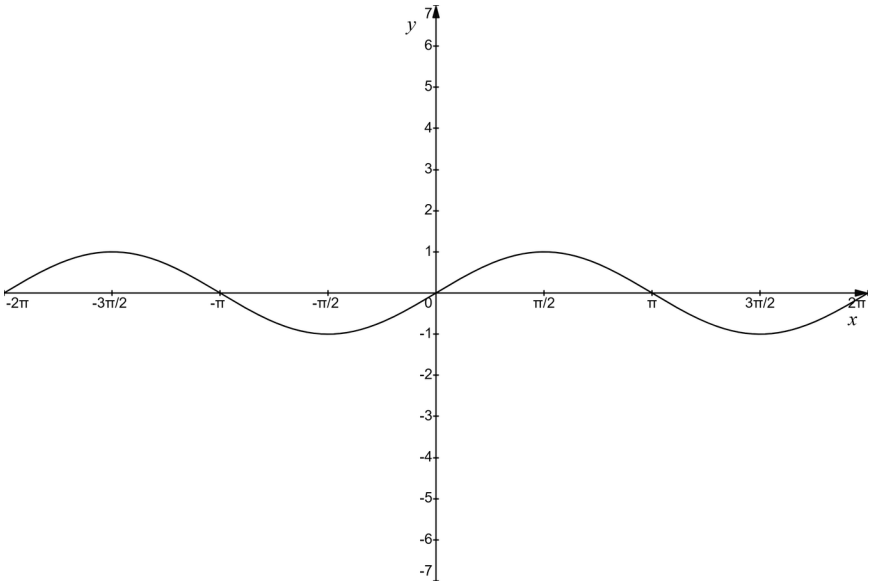


Figure 8.4.2**:** $y = \sin(x)$

From Figure 8.4.2, we can see as $x \to 0^-$, the y-values of $\sin(x)$ are approaching 0^-. Therefore,

$$\text{as } x \to 0^-,\ \frac{1}{\sin(x)} \to \frac{1}{0^-} \to -\infty.$$

It is *negative* infinity because the numerator is *positive*, but the denominator is negative (albeit, very close to zero). Similarly, as $x \to 0^+$, the y-values of $\sin(x)$ are approaching 0^+. Therefore,

$$\text{as } x \to 0^+,\ \frac{1}{\sin(x)} \to \frac{1}{0^+} \to +\infty = \infty.$$

Again, we know the result is *positive* infinity because the numerator is positive and the denominator is positive. Sketching this behavior, we get the following:

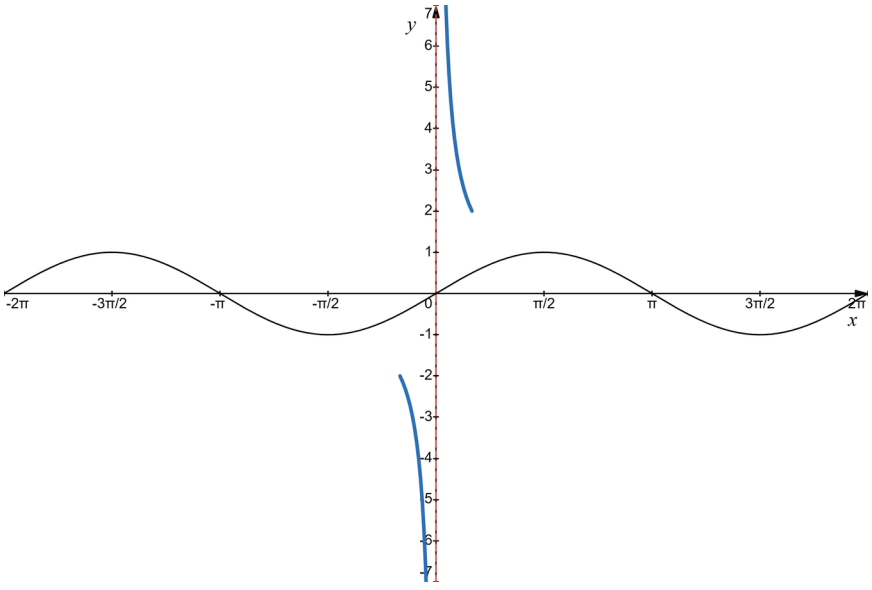


Figure 8.4.3**:** *Beginning the sketch of* $y = \csc(x)$

It's hard to see in Figure 8.4.3, but we drew a vertical asymptote at $x = 0$ (the y-axis). A similar investigation shows that

$$\text{as } x \to \pi^-,\ \frac{1}{\sin(x)} \to \frac{1}{0^+} \to \infty$$

and

$$\text{as } x \to \pi^+, \ \frac{1}{\sin(x)} \to \frac{1}{0^-} \to -\infty.$$

Using this to fill in a little more of the graph, we get the following:

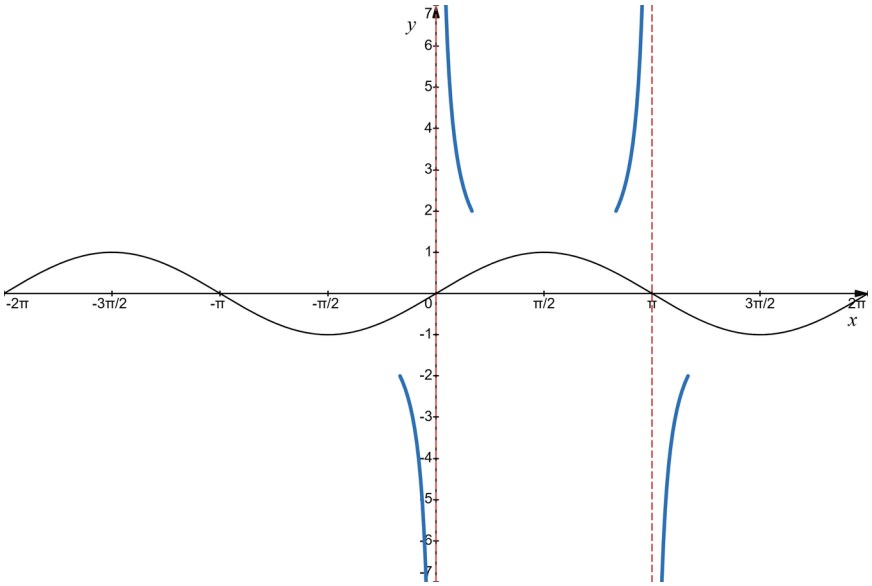


Figure 8.4.4**:** *Getting closer to the graph of* $y = \csc(x)$

From this, we expect the cosecant function to have vertical asymptotes at these zeros of $\sin(x)$ (due to the division by zero). Furthermore, we expect the cosecant to be positive when its reciprocal (the sine) is positive, and negative when the reciprocal is negative. This gets us very close to the base graph of the cosecant.

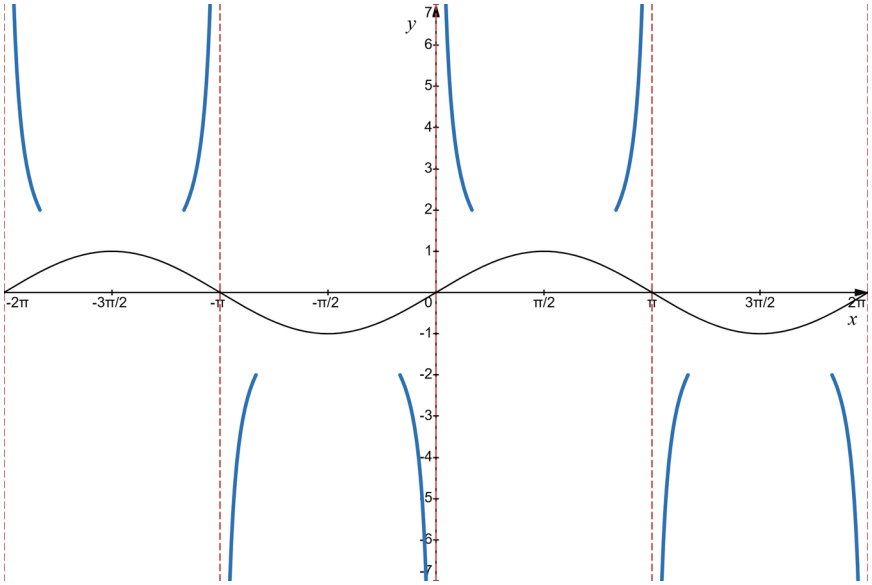
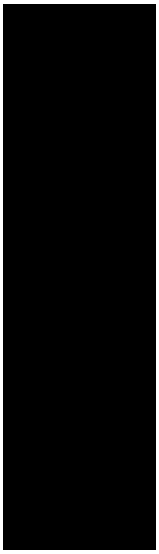


Figure 8.4.5**:** *Almost there*

To finish out the graph of the base cosecant function, we will build a table of values for the interval $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ and use this to extrapolate the features for the rest of the graph.



Plotting these values, we get the following:

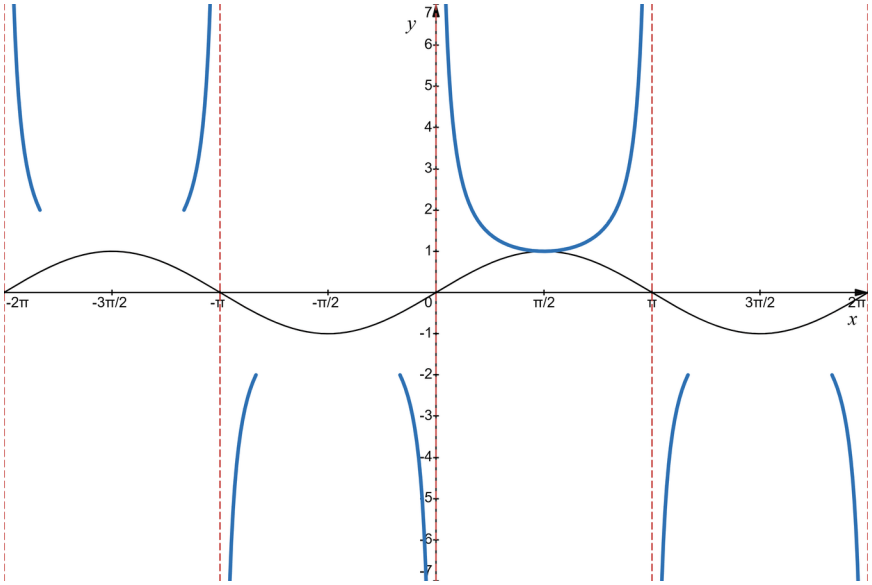


Figure 8.4.6**:** *So close!*

We can now see the pattern for the entire graph of the cosecant. Filling in the blanks, we get our final graph.

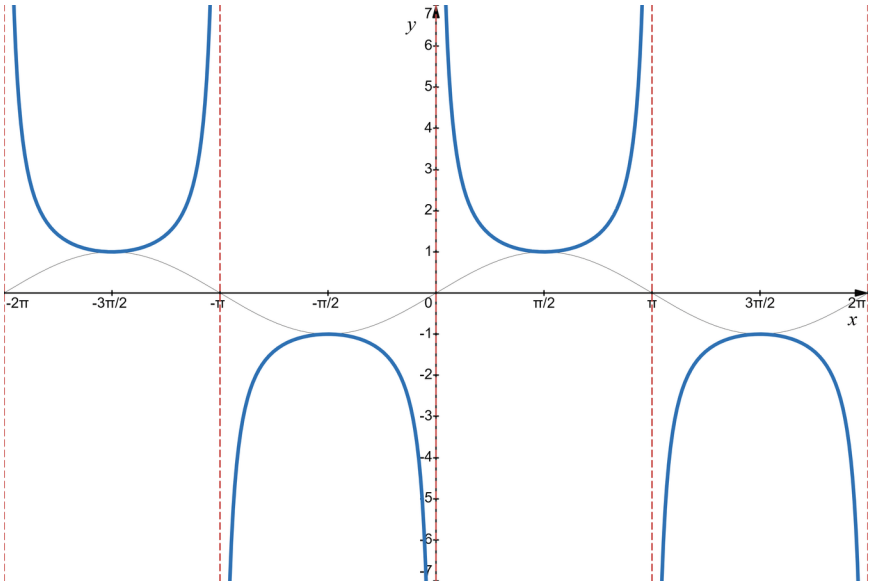
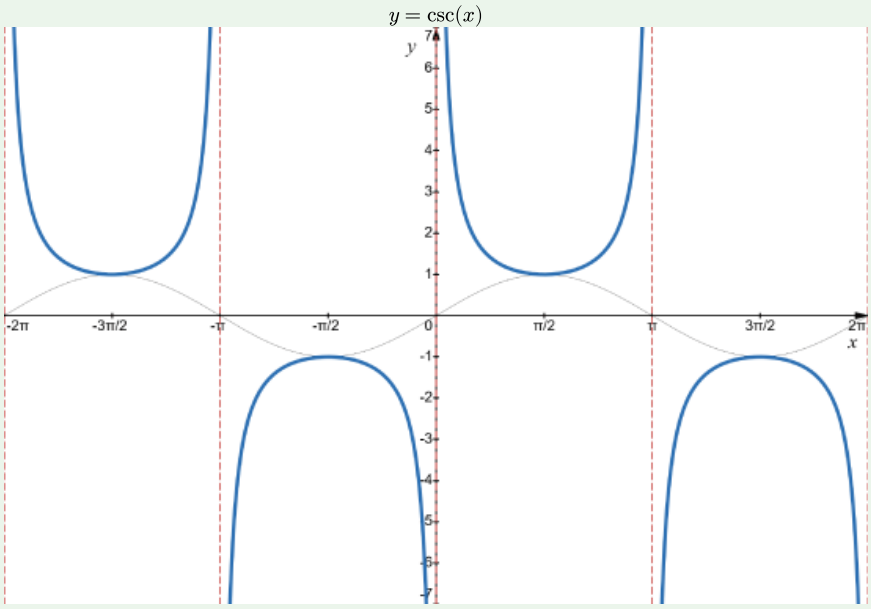


Figure 8.4.7**:** *The base graph of the cosecant,* $y = \csc(x)$

I have left a very lightly sketched sine function in this graph. This is because when you are graphing the cosecant, you will inevitably need to start by graphing its reciprocal.

Just as we did with the fundamental trigonometric functions, we now summarize the properties of the cosecant function (with the help of its graph as a visual aid).

Theorem: Properties of the Graph of the Cosecant Function

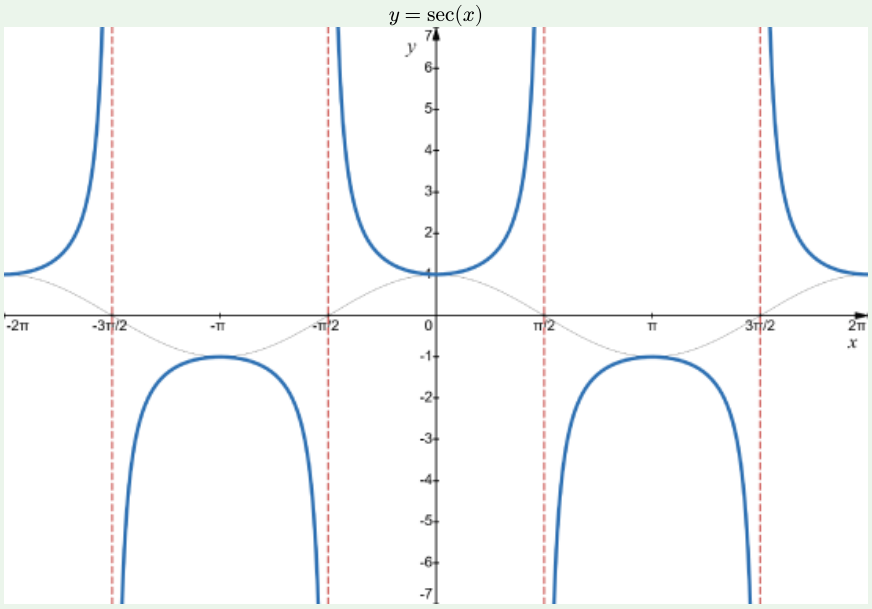


- **Domain:** All $x \neq k\pi$, where $k^{\text{int}}\mathbb{Z}$
- **Range:** $(-\infty, -1] \cup [1, \infty)$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** N/A
- **Zeros:** None

Graphing the Secant Function

The derivation of the graph of the secant function is similar to that of the cosecant function. It is left for you as an exercise.

Theorem: Properties of the Graph of the Secant Function



- **Domain:** All $x \neq \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$
- **Range:** $(-\infty, -1] \cup [1, \infty)$
- **Symmetry:** Even (symmetric about the y-axis)
- **Principal Cycle:** $[0, 2\pi]$
- **Natural Period:** 2π
- **Amplitude:** N/A
- **Zeros:** None

As with the graph of the cosecant, we usually lightly graph the reciprocal of the secant (which is the cosine) to help us in graphing the secant.

Graphing the Cotangent Function

As with the cosecant and the secant, to graph the cotangent function, we will use its reciprocal. That is, because $\cot(x) = \frac{1}{\tan(x)}$, we will first graph the tangent and discuss what happens as the tangent function tends to zero.

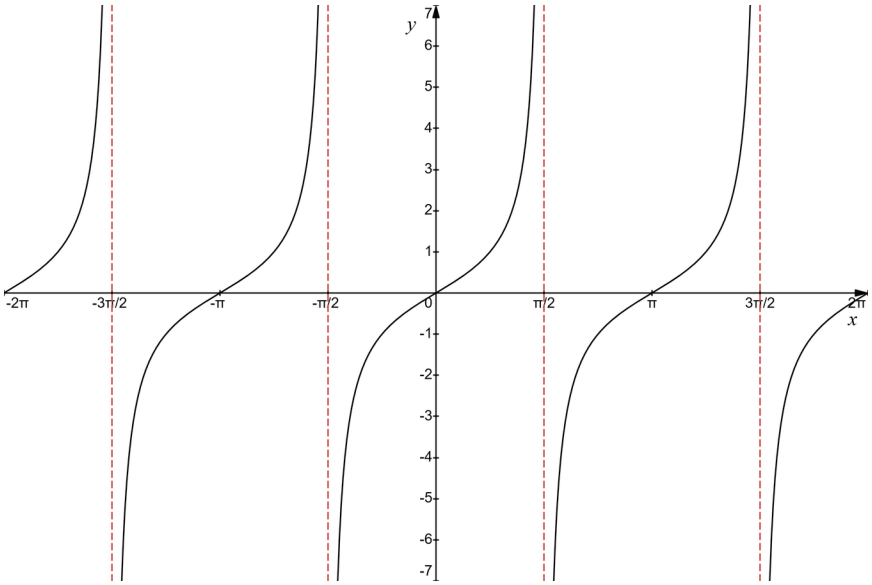


Figure 8.4.8**:** $y = \tan(x)$

Figure 8.4.8 shows the base graph of the tangent function. Using the same arguments as before, we can get the behavior of the reciprocal function, $\frac{1}{\tan(x)} = \cot(x)$, near the zeros of the tangent.

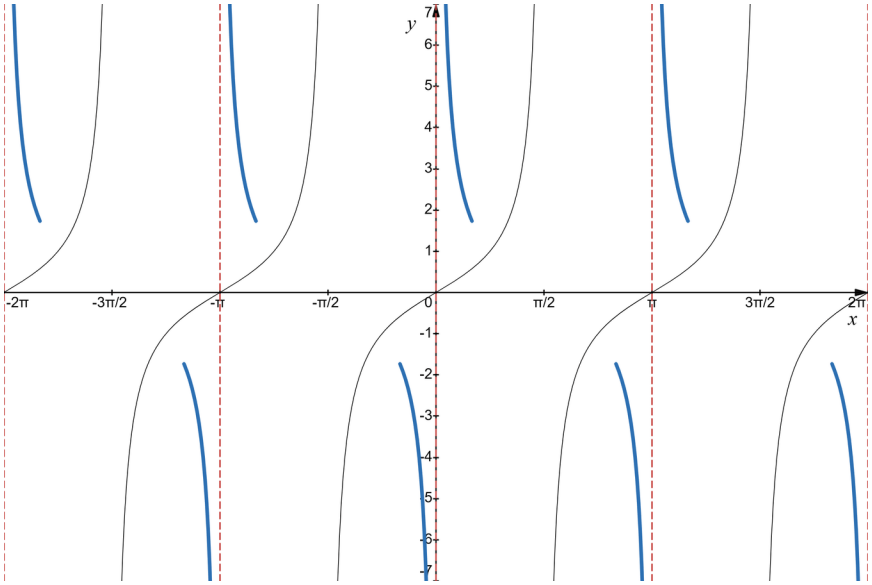


Figure 8.4.9**:** *Almost done graphing the base graph of the cotangent*

Notice in Figure 8.4.9 that the vertical asymptotes are not in the same places as the vertical asymptotes for the tangent. This is because the tangent is undefined when the cosine function is zero (at $x = \frac{\pi}{2} + k\pi$) and the cotangent is undefined when the sine is zero (at $x = k\pi$). Evaluating the cotangent function for a few special values between $x = 0$ and $x = \pi$ should give us the pattern of those missing middle regions.



Plotting these points, filling out the rest of the curve with the same pattern, and erasing the underlying tangent graph, we get the following:

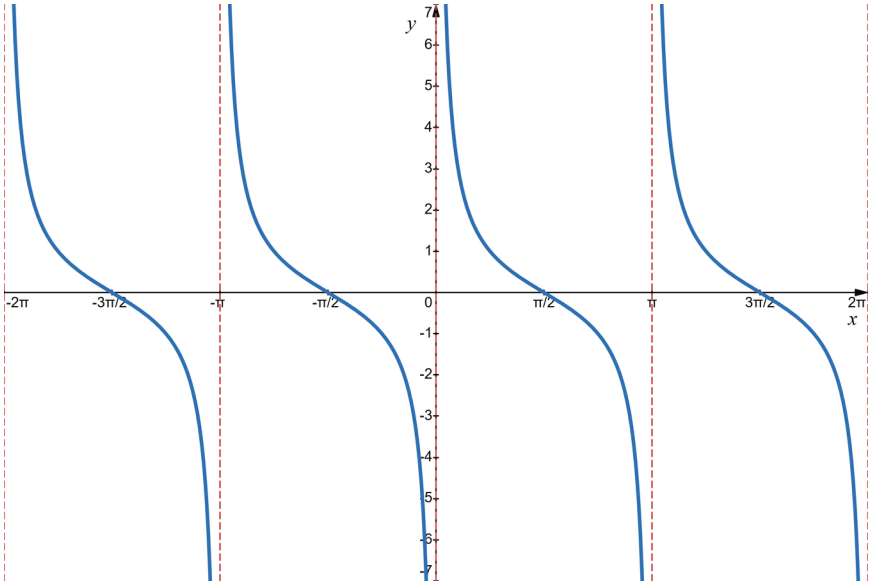
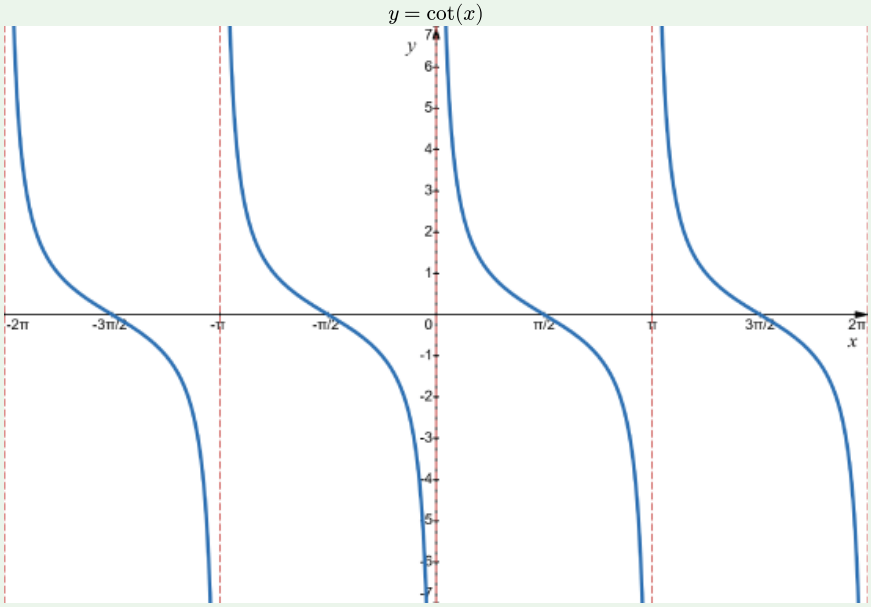


Figure 8.4.10**:** *The base graph of the cotangent,* $y = \cot(x)$

As before, we list all of the properties in a convenient theorem.

Theorem: Properties of the Graph of the Cotangent Function

$y = \cot(x)$



- **Domain:** All $x \neq k\pi$, where $k^{\text{int}}\mathbb{Z}$
- **Range:** $(-\infty, \infty)$
- **Symmetry:** Odd (symmetric about the origin)
- **Principal Cycle:** $[0, \pi]$
- **Natural Period:** π
- **Amplitude:** N/A
- **Zeros:** $x = \frac{\pi}{2} + k\pi$, where $k^{\text{int}}\mathbb{Z}$

Transformations of the Reciprocal Functions

Graphing transformations of the reciprocal functions is the same as graphing the transformations of the fundamental trigonometric functions. In fact, to graph transformations of the cosecant and secant functions, we first fully graph the transformations of their related fundamental functions. After that, graphing the transformed cosecant and secant functions is a somewhat trivial move. Let's see this in action.

Example 8.4.1

Graph a single cycle of each function. State all key characteristics of the graph (including domain, range, period, phase shift, etc.).

a. $f(x) = -1.2 + 2.8 \sec\left(-3x - \frac{\pi}{2}\right)$
b. $g(x) = 2 \cot(2x - 2) + 1$

Solutions available at the end of this section.

Checkpoint 8.4.1

Graph the function. Clearly state all key information (including domain and range).

$$y = 3 - 4 \csc\left(-\pi x + \frac{2}{3}\pi\right)$$

Answer available at the end of this section.

Finding Equations Given Graphs

Finding the equation of a reciprocal function given its graph requires a little bit more work than with the fundamental trigonometric functions.

Example 8.4.2

Find the equation of the graphed function.

a.

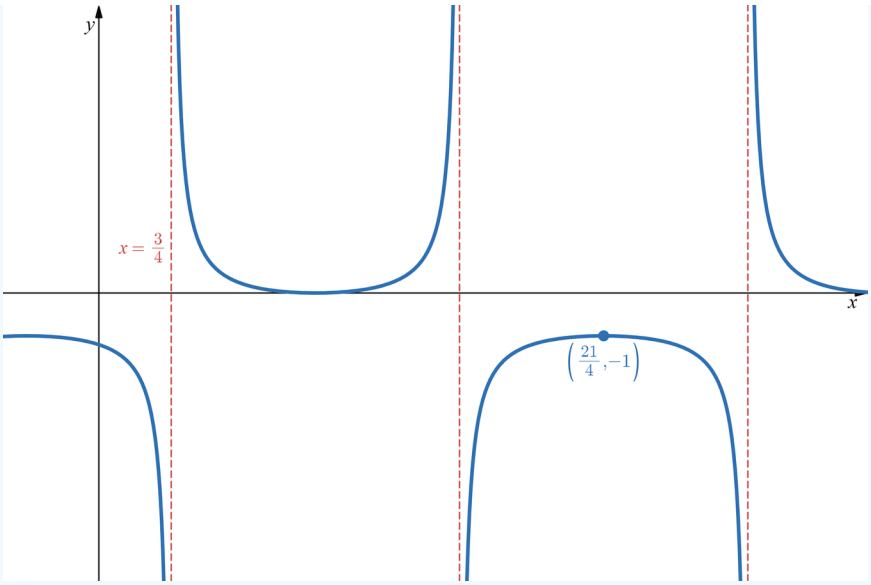


Figure 8.4.19

b.

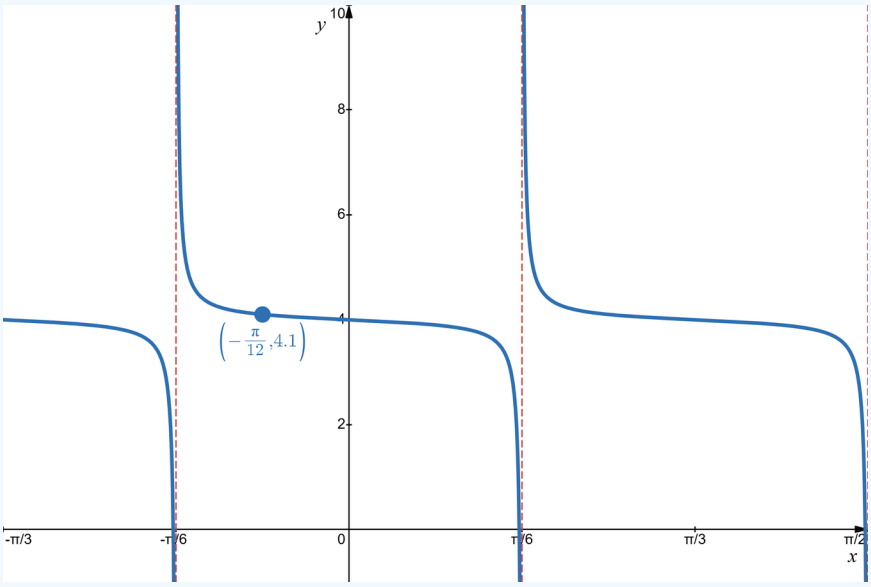


Figure 8.4.20

Solutions available at the end of this section.

Checkpoint 8.4.2

Find the equation of the graphed function.

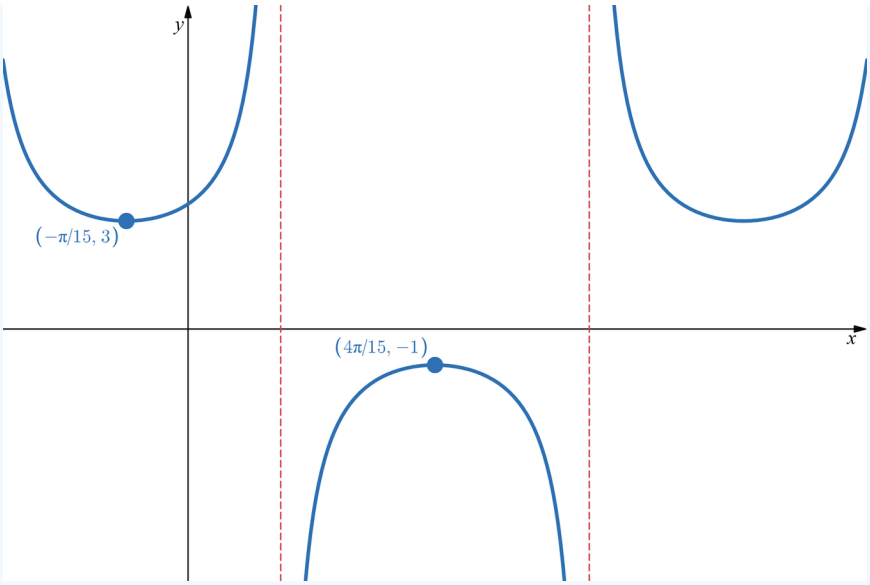


Figure 8.4.22

Answer available at the end of this section.

Success in Mathematics: Math Anxiety (Part 3)

For some math students, anxiety changes their ability to show what they know. They have physical and emotional symptoms. You are responsible for recognizing if math anxiety is interfering with your success. Do you need to make some changes to reduce your math anxiety? Do you need help to do so?

Researchers have identified various events that seem to result in math anxiety. Perhaps you had negative experiences learning math. For you, learning math meant feeling dumb or embarrassed. Perhaps a teacher or parent who struggled with math has passed on their fears to you. Or you believe that your gender or ethnic heritage means you cannot do the math.

Some students learn to cope on their own with their math anxiety. They use relaxation techniques. They practice problems over and over. At test time, their anxiety does not block them from doing the work. Others find help from counselors at their college.

Math anxiety is common among college students. It is your responsibility to think about it and, if needed, get help so that you can cope with your fears.

Math anxiety affects students in physical ways such as increasing blood pressure and heart rate, breathing faster, excess sweating, dry mouth, restlessness, or nausea. Emotionally, it can cause fear, anger, nervousness, worry, and feelings of hopelessness. Students may have trouble concentrating, sitting still, reading, or keeping their legs from bouncing. They may feel desperate - like they need to run away or cry.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Using a scale of 1 to 10, with 10 describing overwhelming anxiety that makes it extremely difficult to do math, rate your math anxiety. How does it affect you?
2. Most students have some anxiety about their classes. They worry about giving speeches, taking tests, writing a good paper, or completing all their work. Worrying can be motivating. It can help us get going and to get things done. Or, worrying can be a bad thing. It paralyzes us. In general, how does worrying affect you?

Applications of Mathematics: Poker

So, where does mathematics apply in poker? Poker players can use mathematics in many different situations; however, it is most commonly used when a player is on a draw, such as a flush or straight draw. If an opponent makes a bet and you are on a draw, you are faced with the decision of whether or not you should call to try and complete the draw by the next card or fold and let your opponent take the pot. In this situation, a player with good knowledge of poker mathematics will always know whether or not to call, while a player who does not know mathematics will be unsure.

Players unfamiliar with poker odds will make an educated guess on whether or not they should call. If the bet is large, they may feel it is too expensive to try and catch the right card, but if the bet is small, they will be more inclined to call. On the other hand, a player that can use mathematics correctly will be able to work out the pot odds they are getting on the hand and act accordingly. Pot odds consider the amount your opponent has bet in relation to the pot size and the likelihood of completing your draw to inform you about whether or not you should call or fold.

1. Can you think of three other games that rely explicitly or implicitly on mathematics? Explain.
2. Oddly enough, most of us are superstitious about numbers. Some people even play their "favorite" numbers when they play the lottery. Suppose you pick your favorite numbers, and I pick the numbers 1, 2, 3, 4, 5, and 6. Who has better odds of winning? Why are you still likely to win more if the odds are the same?

Answers

Section Preview

The graphs of the rest of the trigonometric functions are covered in this section. Try your best to avoid treating this material like an afterthought. The graphs of the secant and cosecant spring up in odd places in Calculus, so the students need to be aware that this material, while held off for the end of the chapter, is critical to understand.

Section Preview

- Graph the cosecant, secant, and cotangent functions, along with transformations of these functions.
- Determine the equation of a trigonometric function given its graph.

Example 8.4.1

a. Just as we have done previously, we rewrite it in the better form

$$f(x) = 2.8\sec\left(-3x - \frac{\pi}{2}\right) - 1.2$$

Using the fact that secant is an even function, we clean this up a little more to the following:

$$f(x) = 2.8\sec\left(-\left(3x + \frac{\pi}{2}\right)\right) - 1.2 = 2.8\sec\left(3x + \frac{\pi}{2}\right) - 1.2$$

Now that we are finished "preconditioning" the function, it's time we discussed how we are going to graph it. With the secant and cosecant, we perform the intermediate step of graphing their reciprocal functions, the cosine and sine, respectively. So, let's graph

$$y = 2.8\cos\left(3x + \frac{\pi}{2}\right) - 1.2.$$

We list out all the essential information.

A	$=$	2.8	$\Longrightarrow$	Amplitude	$=$	2.8
B	$=$	3	$\Longrightarrow$	Period	$=$	$\dfrac{\text{Natural Period}}{B} = \dfrac{2\pi}{3}$
			$\Longrightarrow$	Step Size	$=$	$\dfrac{\text{Period}}{4} = \dfrac{2\pi/3}{4} = \dfrac{\pi}{6}$
C	$=$	$\dfrac{\pi}{2}$	$\Longrightarrow$	Phase Shift	$=$	$-\dfrac{C}{B} = -\dfrac{\pi/2}{3} = -\dfrac{\pi}{6}$
D	$=$	-1.2	$\Longrightarrow$	Vertical Shift	$=$	-1.2

As usual, we graph and label the midline, label the initial number as the phase shift, and step to the other key numbers using the step size.

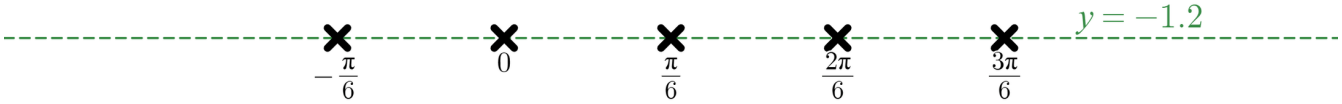


Figure 8.4.11

Note that the distance between the first and last key number is

$$\left|\frac{3\pi}{6} - \left(-\frac{\pi}{6}\right)\right| = \frac{2\pi}{3} = \text{Period},$$

so we have done something right! We then graph the non-reflected cosine, draw in and label the y-axis, and use that to place the x-axis.

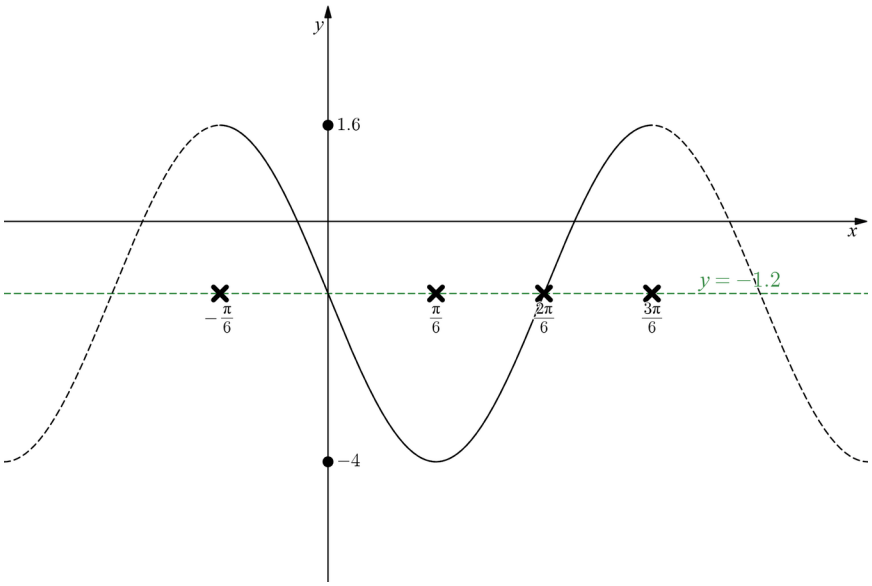


Figure 8.4.12

Now comes the new part. Wherever the cosine function crosses the midline, the original function is undefined. This gives us our vertical asymptotes at $x = 0$, $x = \pm\frac{2\pi}{6} = \frac{\pi}{3}$, $x = \pm\frac{4\pi}{6} = \frac{2\pi}{3}$, and so on. Graphing these vertical asymptotes allows us to graph the secant.

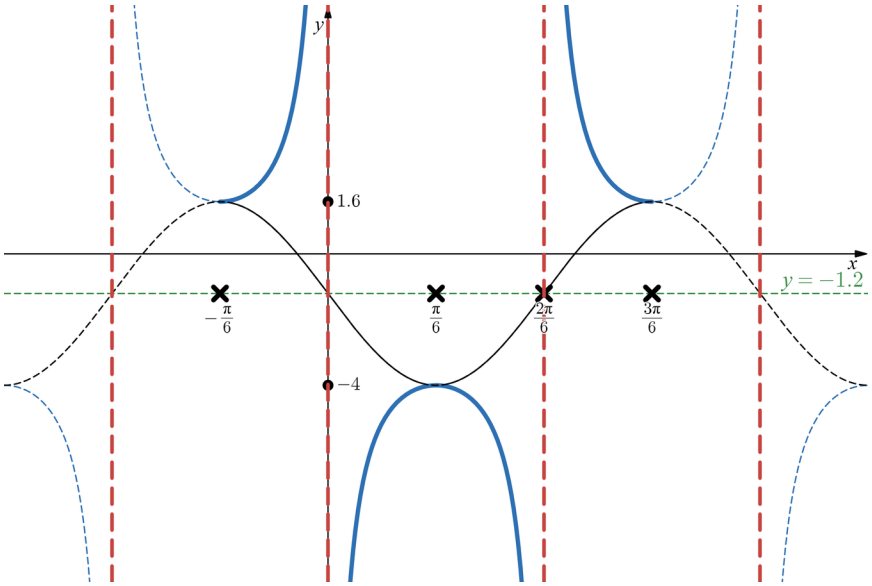


Figure 8.4.13

Notice that I didn't remove the sketch of the cosine. This is perfectly fine (and some instructors expect you to leave it). Finally, we should state as much about this function as we can. The domain (by looking at our spectacular graph) is all $x \neq \frac{\pi}{3}k$, where $k \in \mathbb{Z}$. The range (again, from the graph) is $(-\infty, -4] \cup [1.6, \infty)$. The only thing that we cannot state at this point are the zeros of this function (luckily, I didn't ask for them).

b. Before we start, I need to be clear and state that the secant and cosecant functions are the only ones where we must first graph their reciprocal functions. The base graph for the cotangent is simple enough to handle without bothering with the tangent.

Now, there isn't any preconditioning necessary for this function, so we go ahead and jump right into the other stages. Just like the tangent, the cotangent does not have an amplitude and its natural period is π.

$$\begin{array}{rcrcrcl}
B & = & 2 & \Longrightarrow & \text{Period} & = & \dfrac{\text{Natural Period}}{B} = \dfrac{\pi}{2} \\
 & & & \Longrightarrow & \text{Step Size} & = & \dfrac{\text{Period}}{4} = \dfrac{\pi/2}{4} = \dfrac{\pi}{8} \\
C & = & -2 & \Longrightarrow & \text{Phase Shift} & = & -\dfrac{C}{B} = -\dfrac{-2}{2} = 1 \\
D & = & 1 & \Longrightarrow & \text{Vertical Shift} & = & 1
\end{array}$$

Graphing and labeling the midline, plotting the phase shift as the first key number, and stepping to each subsequent key number using the step size, we get the following:

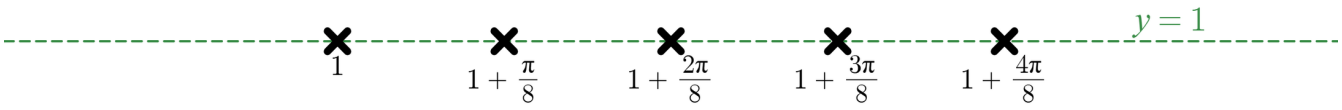


Figure 8.4.14

Because the phase shift was a rational number and the step size was an irrational number, we could not easily add them. Leaving them in the form above is preferred. If you look at the base graph for the cotangent, its principal cycle occurs from $x = 0$ to $x = \pi$. This cycle starts and ends with a vertical asymptote, decreases from left to right, and crosses the midline halfway through. That's precisely what we are going to do here.

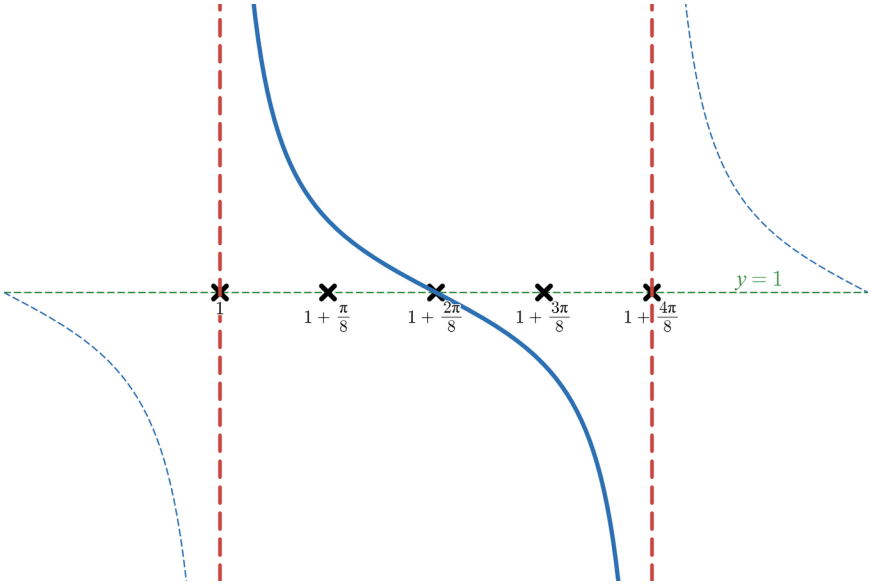


Figure 8.4.15

Now we place the y-axis one unit to the left of the phase shift and the x-axis one unit down from the midline. How do I know how far one unit is? It's more of an art of approximation.

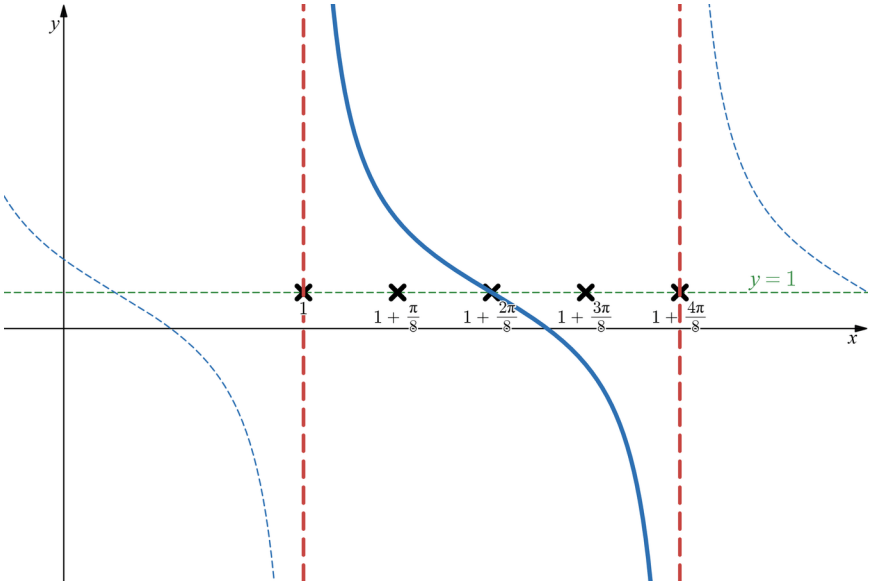


Figure 8.4.16

Finally, we *should* label the two points on the curve at $x = 1 + \frac{\pi}{8}$ and $x = 1 + \frac{3\pi}{8}$. The good news is that the y-values of these points vary from the midline by $|A|$. Thus, the corresponding y-values are $y = 1 + 2 = 3$ and $y = 1 - 2 = -1$.

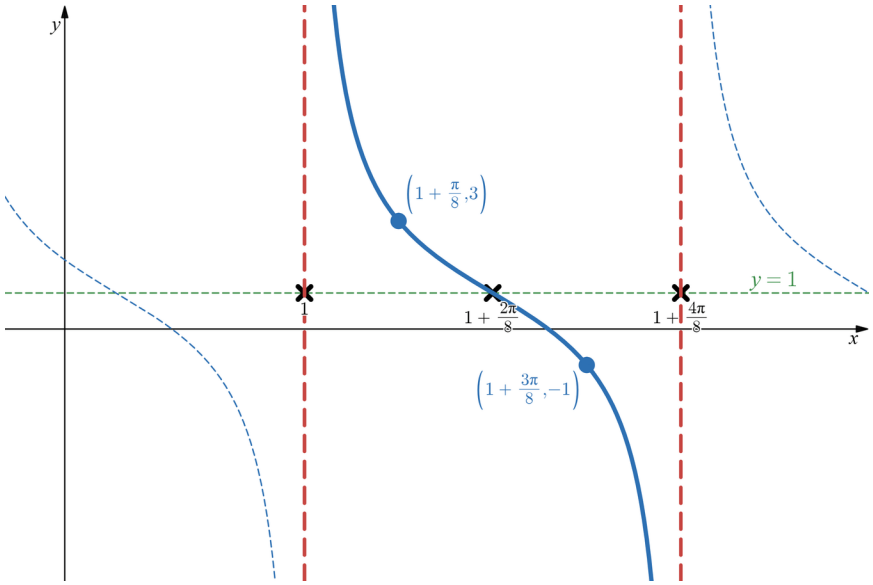


Figure 8.4.17

The domain of the function is all $x \neq 1+\frac{\pi}{2}k$, where $k^{\text{int}}\mathbb{Z}$. The range is all real numbers, $\mathbb{R}$.

Checkpoint 8.4.1

Amplitude of the related sine function: 4
Period: 2
Step Size: $\frac{1}{2}$
Phase Shift: $\frac{2}{3}$
Vertical Shift: 3
Domain: All $x \neq \frac{2}{3}+k$, where $k^{\text{int}}\mathbb{Z}$
Range: $(-\infty,1]\cup[7,\infty)$

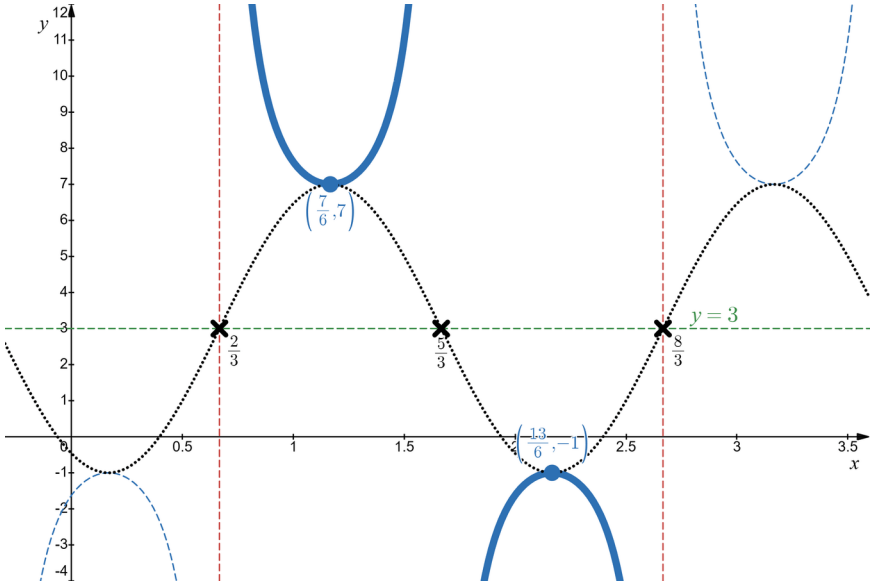


Figure 8.4.18

Example 8.4.2

a. As with the sine and cosine functions, graphs of the cosecant and secant are so similar that you could use either as an equation for the given graph (they would just be shifts of one another). A good rule of thumb when building trigonometric equations from graphs is to read the graph from left to right. The piece of information (whether a point or an equation of an asymptote) you have been given will set the beginning of the cycle for the trigonometric function. Therefore, in this case, we will assume the cycle of the trigonometric function starts at $x=\frac{3}{4}$. With the cosecant and secant, it's beneficial to draw a sinusoid "helper" function. Looking at the given graph, we can see that the midline should be at $y=-\frac{1}{2}$. Drawing this in allows us to sketch our "helper" function.

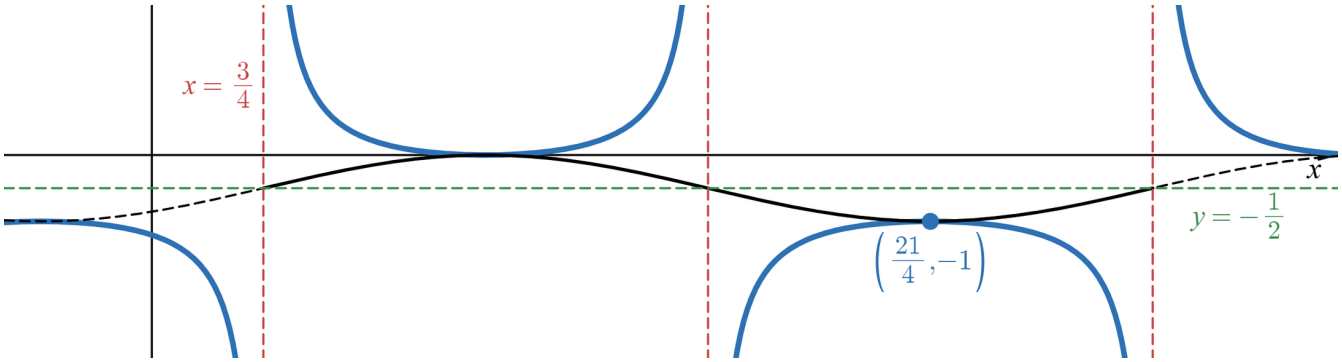


Figure 8.4.21

We now find the equation of the "helper" function. It is definitely a sine function starting at $x = \frac{3}{4}$, so the phase shift is

$$-\frac{C}{B} = \frac{3}{4}.$$

The midline is $y = -\frac{1}{2}$, so the vertical shift is

$$D = -\frac{1}{2}.$$

Additionally, the amplitude is the distance between the midline and the maximum or minimum values of the sinusoid. Since the maximum is $y = 0$ and the midline is $y = -\frac{1}{2}$, the amplitude is $\frac{1}{2}$. Moreover, since this sine function is not reflected,

$$A = \frac{1}{2}.$$

The only thing left to find is the period. This would give us an equation to find B, which in turn would allow us to find C. The cycle begins at $x = \frac{3}{4}$ and three-quarters of the way through the cycle, we are given the point $\left(\frac{21}{4}, -1\right)$. That is, the distance between $\frac{3}{4}$ and $\frac{21}{4}$ is three-quarters of the period.

$$\begin{aligned} \frac{3}{4} \text{ of the Period} &= \left|\frac{3}{4} - \frac{21}{4}\right| \\ &= \left|-\frac{18}{4}\right| \\ &= \frac{9}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Period} &= \left(\frac{9}{2}\right)\left(\frac{4}{3}\right) \\ &= 6 \end{aligned}$$

Since the period is $\frac{2\pi}{B}$,

$$\frac{2\pi}{B} = 6 \implies \frac{2\pi}{6} = B \implies \frac{\pi}{3} = B.$$

Earlier we stated that $-\frac{C}{B} = \frac{3}{4}$, so

$$-\frac{C}{\pi/3} = \frac{3}{4} \implies C = -\frac{\pi}{4}.$$

Thus, the "helper" function is

$$y = A\sin(Bx + C) + D = \frac{1}{2}\sin\left(\frac{\pi}{3}x - \frac{\pi}{4}\right) - \frac{1}{2}.$$

Finally, the *actual* function is the reciprocal of the sine. That is, the function in the initial graph is

$$y = \frac{1}{2}\csc\left(\frac{\pi}{3}x - \frac{\pi}{4}\right) - \frac{1}{2}.$$

b. Again, you *could* write the equation for this in terms of the (reflected) tangent function; however, we will write our equation in terms of the cotangent. Sometimes, you have to "eyeball" the features of a graph. In this example, the midline *looks* to be $y = 4$, so we will say $D = 4$. The phase shift could be $-\frac{\pi}{6}$ or $+\frac{\pi}{6}$, but the extra point they gave us is in the interval $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$, so we will choose our phase shift to be $-\frac{\pi}{6}$. This means

$$-\frac{C}{B} = -\frac{\pi}{6}.$$

Furthermore, the period is easy to spot on this graph. From vertical asymptote to vertical asymptote, the distance is $\frac{2\pi}{6} = \frac{\pi}{3}$. Therefore, the period is $\frac{\pi}{3}$ and, as such,

$$\frac{\pi}{B} = \frac{\pi}{3} \implies B = 3.$$

With this, we can find C.

$$-\frac{C}{B} = -\frac{\pi}{6} \implies \frac{C}{3} = \frac{\pi}{6} \implies C = \frac{\pi}{2}.$$

We are close to the finish line. Our function is

$$y = A\cot\left(3x + \frac{\pi}{2}\right) + 4.$$

Now we use the given data point to help us find the value of A.

$$\begin{aligned} 4.1 &= A\cot\left(3\left(-\frac{\pi}{12}\right)+\frac{\pi}{2}\right)+4 \\ \Longrightarrow \quad 0.1 &= A\cot\left(-\frac{\pi}{4}+\frac{\pi}{2}\right) \\ \Longrightarrow \quad 0.1 &= A\cot\left(\frac{\pi}{4}\right) \\ \Longrightarrow \quad 0.1 &= A\,(1) \\ \Longrightarrow \quad 0.1 &= A \end{aligned}$$

Thus, the function is

$$y = 0.1\cot\left(3x+\frac{\pi}{2}\right)+4.$$

Checkpoint 8.4.2

$y = 2\sec\left(3x+\frac{\pi}{5}\right)+1$

This page titled 8.4: Graphs of the Remaining Trigonometric Functions is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

8.4.1: Resources and Key Concepts

Resources

Videos

- Graphs of Cosecant and Secant
  - Graphs of the Other Trigonometric Functions - Deriving the Base Graphs for Secant and Cosecant
  - Graphs of the Other Trigonometric Functions - Graphing Secant and Cosecant Functions
- Graph of Cotangent
  - Graphs of the Other Trigonometric Functions - Deriving the Base Graphs for Tangent and Cotangent
  - Graphs of the Other Trigonometric Functions - Graphing Tangent and Cotangent Functions

Key Concepts

Theorems

- **Properties of the Graph of the Cosecant Function:**
  - Domain: All $x \neq k\pi$, where $k^{\text{int}}\mathbb{Z}$
  - Range: $(-\infty, -1] \cup [1, \infty)$
  - Symmetry: Odd (symmetric about the origin)
  - Principal Cycle: $[0, 2\pi]$
  - Natural Period: 2π
  - Amplitude: N/A
  - Zeros: None
- **Properties of the Graph of the Secant Function:**
  - Domain: All $x \neq \frac{\pi}{2} + k\pi$, where $k^{\text{int}}\mathbb{Z}$
  - Range: $(-\infty, -1] \cup [1, \infty)$
  - Symmetry: Even (symmetric about the y-axis)
  - Principal Cycle: $[0, 2\pi]$
  - Natural Period: 2π
  - Amplitude: N/A
  - Zeros: None
- **Properties of the Graph of the Cotangent Function:**
  - Domain: All $x \neq k\pi$, where $k^{\text{int}}\mathbb{Z}$
  - Range: $(-\infty, \infty)$
  - Symmetry: Odd (symmetric about the origin)
  - Principal Cycle: $[0, \pi]$
  - Natural Period: π
  - Amplitude: N/A
  - Zeros: $x = \frac{\pi}{2} + k\pi$, where $k^{\text{int}}\mathbb{Z}$

Common Mistakes

- **Forgetting the Reciprocal Relationship:** The easiest way to graph the cosecant and secant functions is to first lightly sketch the graph of their reciprocal functions (sine and cosine, respectively). The key features of the reciprocal graph (zeros, maximums, minimums) determine the key features of the cosecant/secant graph (asymptotes, minimums, maximums).
- **Misplacing Asymptotes:** The vertical asymptotes for $y = \csc(x)$ and $y = \cot(x)$ occur where $\sin(x) = 0$. The vertical asymptotes for $y = \sec(x)$ occur where $\cos(x) = 0$. Confusing these will lead to incorrect graphs.
- **Incorrectly Transforming Asymptotes:** When graphing a transformed function like $y = A\sec(Bx + C) + D$, the vertical asymptotes are located where the "helper" function, $y = A\cos(Bx + C) + D$, crosses its midline, $y = D$.

This page titled 8.4.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

8.4.2: Homework

Reading Questions

1. How is the graph of the cosecant function related to the graph of the sine function?
2. Where do the vertical asymptotes of the secant function occur?
3. What is the natural period of the cotangent function?
4. What is the domain of the cosecant function?
5. What is the range of the secant function?
6. Which of the reciprocal trigonometric functions are odd? Which are even?
7. What is the recommended first step when graphing a transformed cosecant or secant function?
8. Why do the cosecant, secant, and cotangent functions not have an amplitude?
9. The zeros of the sine function correspond to what features on the graph of the cosecant function?
10. The local maximums on the graph of the sine function correspond to what features on the graph of the cosecant function?

Homework

Concept Check

1. Describe the general shape of the graph of the cosecant function.
2. What is the relationship between the sine function and the cosecant function? How does this relationship affect their graphs?
3. Describe the general shape of the graph of the secant function.
4. What is the relationship between the cosine function and the secant function? How does this relationship affect their graphs?
5. Describe the general shape of the graph of the cotangent function.
6. What is the relationship between the tangent function and the cotangent function? How does this relationship affect their graphs?

Basic Skills

7. Determine the period, phase shift, and midline of the function $y = 3\sec(2x - \pi) + 1$.
8. Determine the period, phase shift, and midline of the function $y = 2\csc(x + \frac{\pi}{4}) - 3$.
9. Determine the period, phase shift, and midline of the function $y = -\cot\left(\frac{1}{2}x - \frac{\pi}{3}\right) + 4$.
10. Determine the period, phase shift, and midline of the function $y = 4\sec\left(\frac{1}{3}x + \frac{\pi}{6}\right) - 2$.
11. Determine the period, phase shift, and midline of the function $y = -2\csc(3x - \pi) + 1$.
12. Determine the period, phase shift, and midline of the function $y = 3\cot\left(4x + \frac{\pi}{2}\right) - 5$.
13. Graph $y = \sec(x - \frac{\pi}{2}) + 2$.
14. Graph $y = 2\csc\left(\frac{\pi}{2}x\right) - 1$.
15. Graph $y = -\cot\left(x + \frac{\pi}{4}\right) + 3$.
16. Graph $y = 3\sec(2x) - 4$.

17. Graph $y = -\csc\left(\frac{1}{2}x - \frac{\pi}{3}\right) + 2$.

18. Graph $y = \cot\left(\frac{3}{2}x + \frac{\pi}{6}\right) - 1$.

19. Graph $y = -2\sec\left(\frac{1}{3}x - \frac{\pi}{4}\right) + 5$.

20. Graph $y = 4\csc(3x) - 3$.

21. Graph $y = \cot(2x) + 4$.

22. Graph $y = -\sec\left(\frac{2}{3}x + \frac{\pi}{2}\right) - 2$.

23. Graph $y = \csc\left(\frac{\pi}{3}x - \frac{\pi}{6}\right) + 1$.

24. Graph $y = -3\cot\left(\frac{1}{2}x + \frac{\pi}{3}\right) + 2$.

For the following exercises, assume that the related sinusoidal function has an amplitude of 1.

25. Given the graph of a secant function that has a period of π, a phase shift of $\frac{\pi}{4}$ to the right, and a vertical shift up by 2 units, determine the equation of the graph.

26. Given the graph of a cosecant function that has a period of 2π, a phase shift of $\frac{\pi}{6}$ to the left, and a vertical shift down by 3 units, determine the equation of the graph.

27. Given the graph of a reflected secant function that has a period of $\frac{\pi}{2}$, a phase shift of $\frac{\pi}{2}$ to the left, and a vertical shift down by 4 units, determine the equation of the graph.

28. Given the graph of a reflected cosecant function that has a period of 4π, a phase shift of $\frac{\pi}{4}$ to the right, and a vertical shift up by 5 units, determine the equation of the graph.

29. Given the graph of a secant function with a period of 2π, a vertical shift up by 1 unit, and where the related sinusoidal function has an amplitude of 3, determine the equation of the graph.

30. Given the graph of a cosecant function with a period of π, a vertical shift down by 4 units, and where the related sinusoidal function has an amplitude of 2, determine the equation of the graph.

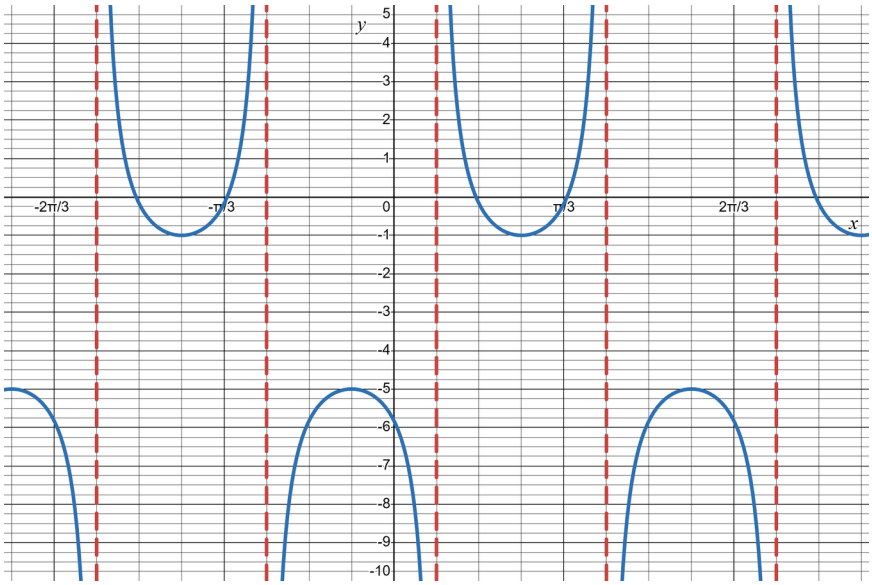
31. Given the graph of a reflected secant function with a period of $\frac{\pi}{3}$, a phase shift of $\frac{\pi}{2}$ to the left, a vertical shift down by 3 units, and where the related sinusoidal function has an amplitude of 1.2, determine the equation of the graph.

32. Given the graph of a reflected cosecant function with a period of $\frac{\pi}{4}$, a phase shift of $\frac{\pi}{6}$ to the right, a vertical shift up by 2 units, and where the related sinusoidal function has an amplitude of 2, determine the equation of the graph.

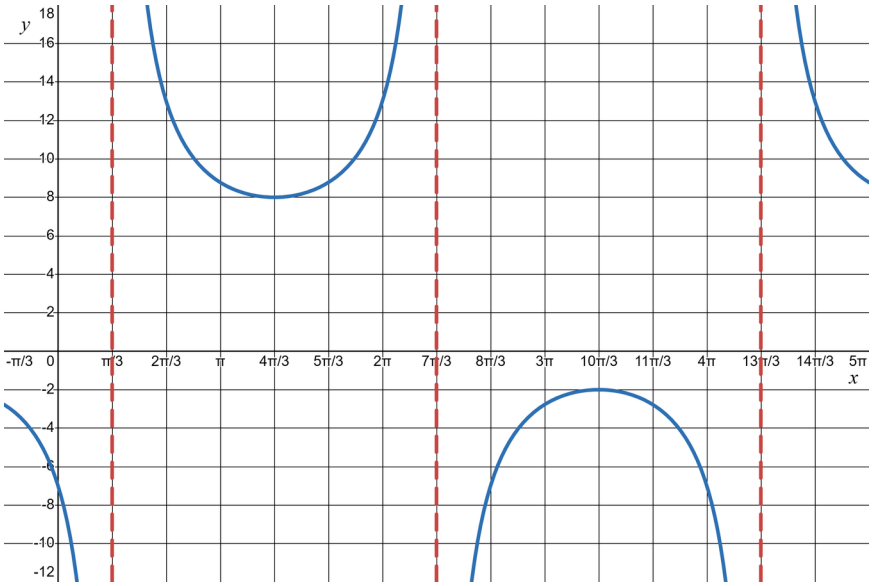
For the following exercises, find the equation of the function in terms of cosecant, secant, or cotangent.

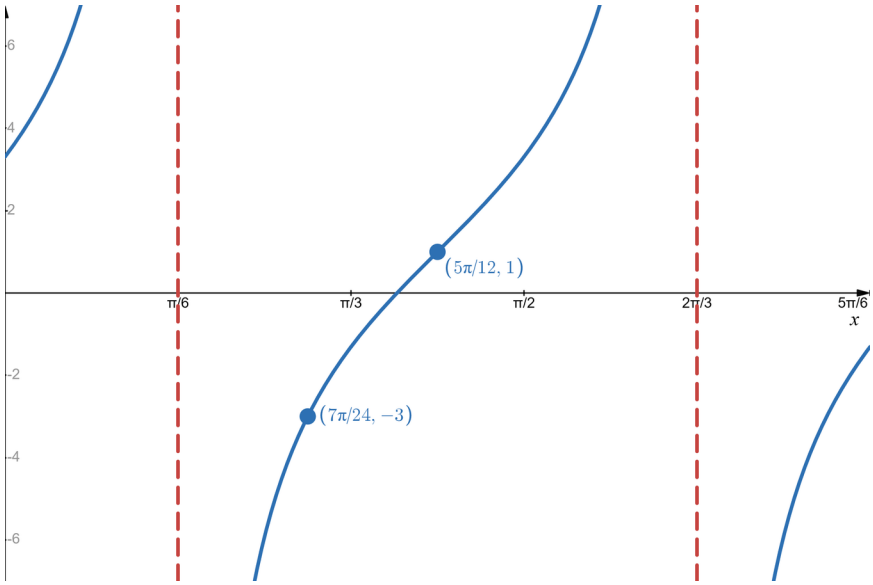
33.

34.


y
x
-2π/3
-π/3
π/3
2π/3
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10

35.


y
x
18
16
14
12
10
8
6
4
2
-2
-4
-6
-8
-10
-12
-π/3
0
π/3
2π/3
π
4π/3
5π/3
2π
7π/3
8π/3
3π
10π/3
11π/3
4π
13π/3
14π/3
5π



Synthesis Questions

36. Given the function $y = 3\csc(2x - \frac{\pi}{2}) + 4$, find all asymptotes within the interval $[0, 2\pi]$. Explain the process of identifying these asymptotes.

37. Given the function $y = 2\sec\left(\frac{1}{2}x - \frac{\pi}{3}\right) - 1$, find all asymptotes within the interval $[-2\pi, 2\pi]$. Describe the method used to determine these asymptotes.

38. For the function $y = -3\csc(4x + \frac{\pi}{4}) + 2$, determine all asymptotes within the interval $[0, \pi]$. Explain your reasoning and the steps taken to find these asymptotes.

39. Given the function $y = \sec(3x - \frac{\pi}{6}) - 2$, find the equations of all vertical asymptotes within the interval $[-\pi, \pi]$. Describe the procedure used to identify these asymptotes.

40. For the function $y = -\csc\left(\frac{3}{2}x + \frac{\pi}{3}\right) + 5$, find the equations of all vertical asymptotes within the interval $[-\pi, \pi]$. Detail the steps and logic used to locate these asymptotes.

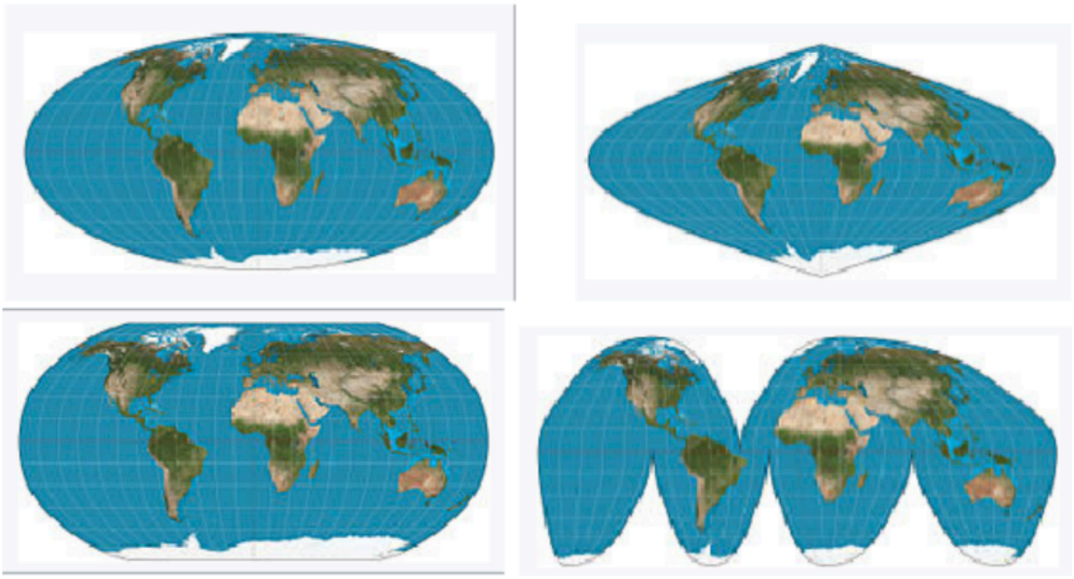
41. Consider the function $y = 4\sec\left(2x - \frac{\pi}{2}\right) + 1$. Identify all vertical asymptotes within the interval $[0, 2\pi]$ and explain how you determined their locations.

This page titled 8.4.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

9: ANALYTIC TRIGONOMETRY

Mapmakers have always faced an unavoidable challenge: It is impossible to translate the surface of a sphere onto a flat map without some form of distortion. Over the years, a variety of map projections have been developed to suit different uses.

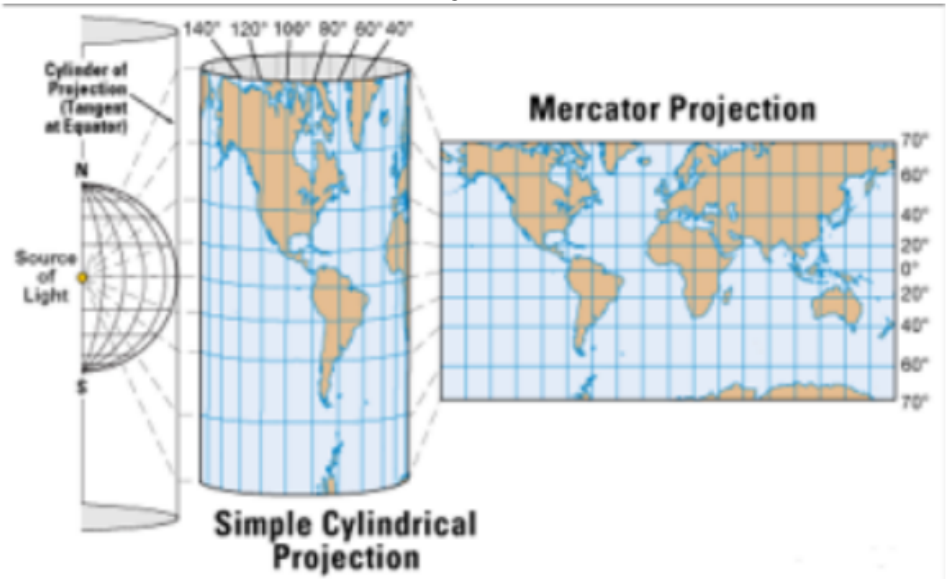
Figure 1



The sixteenth century was an age of discovery, when explorers and merchants began sailing to distant and previously unknown lands. But at that time there was no reliable technology for navigation. Although more regions of the world were being mapped more accurately, a flat map by itself was not enough to help a sailor in the middle of the ocean.

In 1569, the Flemish cartographer Gerardus Mercator published a new map using what is known as a cylindrical projection. To imagine how a Mercator projection works, picture shining a light through a glass globe onto a piece of paper rolled into a cylinder and wrapped around the globe. The cylinder is tangent to the globe at its equator.

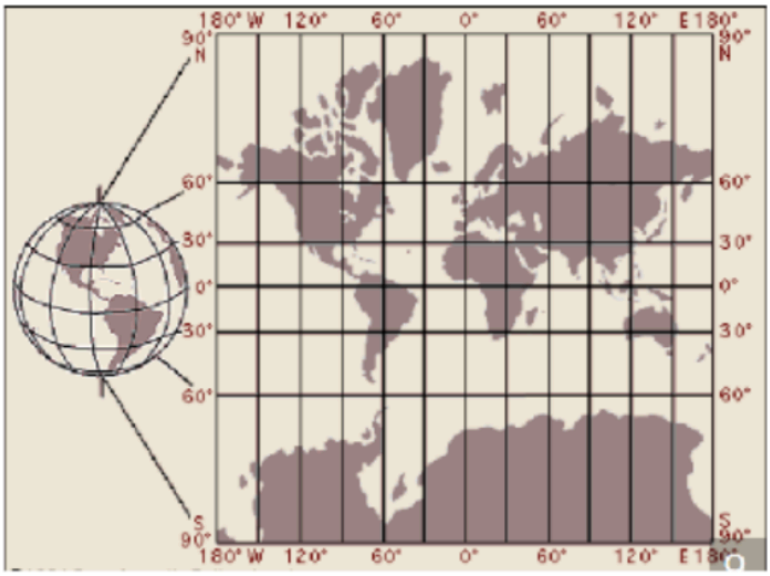
Figure 2



Notice how the latitude lines are farther apart the farther you get from the Equator. This projection distorts the size of objects as

the latitude increases, so that Greenland and Antarctica appear much larger than they actually are.

Figure 3



But the Mercator projection map is ideally suited for navigation, because any straight line on the map is a line of constant true bearing. If a navigator measures the bearing on the map from his location to his destination, he can set his ship's compass for the same bearing and maintain that course.

However, the Mercator projection does not preserve distances. On a globe, circles of latitude (also known as parallels) get smaller as they move away from the Equator towards the poles. Thus, in the Mercator projection, when a globe is "unwrapped" on to a rectangular map, the parallels need to be stretched to the length of the Equator. Mercator had to increase the scale of his map gradually as it moved away from the equator, so that the latitude lines appear equal in length to the equator.

The horizontal scale factor at any latitude must be inversely proportional to lengths on that latitude. Because the radius of the circle of latitude θ is $R\cos\theta$, the corresponding parallel on the map must be stretched by a factor of $\frac{1}{\cos\theta}$. And because the secant is the reciprocal of the cosine, the scale factor is proportional to the secant of the latitude.

A variety of "equal area" projections have been developed in modern times, but the Mercator projection is still widely used, in classrooms and atlases. For many people, it represents our image of the world.

Figure 4



9.1: Basic Trigonometric Identities and Proof Techniques

9.1.1: Resources and Key Concepts

9.1.2: Homework

9.2: Sum and Difference Identities

9.2.1: Resources and Key Concepts

9.2.2: Homework

9.3: Double-Angle Identities

9.3.1: Resources and Key Concepts

9.3.2: Homework

This page titled 9: Analytic Trigonometry is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Katherine Yoshiwara & Roy Simpson.

9.1: Basic Trigonometric Identities and Proof Techniques

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We already introduced the concept of an identity early on in this textbook. At the time, we had only recently been introduced to the trigonometric functions; however, we had enough knowledge to understand and use the Reciprocal, Ratio, and Pythagorean Identities. Since then, we have encountered two other sets of identities—the Symmetry Identities and the Cofunction Identities. For convenience, the following tables summarize all of these basic identities and some of their more common equivalent forms.

Reciprocal Identities	
Basic Identity	***Alternate Form***
$\sin(\theta) = \dfrac{1}{\csc(\theta)}$	$\csc(\theta) = \dfrac{1}{\sin(\theta)}$
$\cos(\theta) = \dfrac{1}{\sec(\theta)}$	$\sec(\theta) = \dfrac{1}{\cos(\theta)}$
$\tan(\theta) = \dfrac{1}{\cot(\theta)}$	$\cot(\theta) = \dfrac{1}{\tan(\theta)}$

Ratio Identities	
Basic Identity	***Alternate Form***
$\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$

Pythagorean Identities	
Basic Identity	***Alternate Forms***
$\cos^2(\theta) + \sin^2(\theta) = 1$	$\cos^2(\theta) = 1 - \sin^2(\theta)$ and $\cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$
	$\sin^2(\theta) = 1 - \cos^2(\theta)$ and $\sin(\theta) = \pm\sqrt{1 - \cos^2(\theta)}$
$1 + \tan^2(\theta) = \sec^2(\theta)$	$\tan^2(\theta) = \sec^2(\theta) - 1$
$\cot^2(\theta) + 1 = \csc^2(\theta)$	$\cot^2(\theta) = \csc^2(\theta) - 1$

Symmetry Identities	
Basic Identity	***Alternate Form***
$\sin(-\theta) = -\sin(\theta)$	$\csc(-\theta) = -\csc(\theta)$
$\cos(-\theta) = \cos(\theta)$	$\sec(-\theta) = \sec(\theta)$
$\tan(-\theta) = -\tan(\theta)$	$\cot(-\theta) = -\cot(\theta)$

Cofunction Identities	
Basic Identity	***Alternate Form***

Cofunction Identities	
$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$
$\sec(\theta) = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc(\theta) = \sec\left(\frac{\pi}{2} - \theta\right)$
$\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot(\theta) = \tan\left(\frac{\pi}{2} - \theta\right)$

Students who focus on memorization over understanding tend to panic at this point. They look at the preceding tables as huge lists of things to memorize. While it is true that you will have to recall these identities without looking them up or having them on a note card, it is crucial you understand most of these identities can be derived from the basic definitions of Trigonometry—this will not only shorten the list of what you feel you need to memorize, it will also give you a stronger foundation for success in Mathematics.

For example, the Pythagorean Identities are built from the unit circle definition of the trigonometric functions, the Ratio Identities, and the Reciprocal Identities. The unit circle definition tells us that any point, (x, y), on the unit circle can be represented as $(\cos(\theta), \sin(\theta))$, where θ is the angle in standard position whose terminal side contains the point (x, y). Since the equation of the unit circle is $x^2 + y^2 = 1$, this gives us the main Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$. The alternate forms come from algebraic manipulations.

If we divide both sides of the main Pythagorean Identity by $\cos^2(\theta)$, we get

$$\frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} \quad \Longrightarrow \quad \frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \sec^2(\theta) \quad \Longrightarrow \quad 1 + \tan^2(\theta) = \sec^2(\theta).$$

A similar approach proves $\cot^2(\theta) + 1 = \csc^2(\theta)$.

The Symmetry Identities can be recalled very quickly from our graphical understanding of the trigonometric functions—cosine and its reciprocal are the only two trigonometric functions whose graphs are symmetric about the y-axis, so they are even functions. The rest of the trigonometric functions have graphs symmetric about the origin (so they are odd functions).

The Cofunction Identities are the only set of identities that we have not proved so far; however, we will remedy that in the next section.

Proving Identities

We have used identities to manipulate trigonometric expressions into different forms and to prove other identities. The remainder of this section reinforces the latter skill (which, in turn, will strengthen the former).

We start with a tactic to help us prove trigonometric identities.

How To... Prove a Trigonometric Identity

1. **Complex to Simple:** It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. **Algebra Before Trigonometry:** Follow the Mathematical Mantra. Look for opportunities to factor expressions, square a binomial, or add fractions.
3. **Trigonometric Identities:** Always look for fundamental trigonometric identities - these can often be used as a key to proving a given statement. As we move forward, we will learn new identities to add to our list of fundamental identities.
4. **Keep an "Eye on the Prize:"** Always take note of the side you are not currently working on. It might contain clues as to what you should do next.
5. **Switch to Sines and Cosines:** If these steps do not yield the desired result, try converting all terms to sines and cosines.
6. **Switch Sides:** If you get stuck manipulating one side of the statement, move to the other side and work on that for a while.

It's important to note that the preceding "How To" is a "rule of thumb." As with most rules of thumb, there are exceptions to what to do and when to do it. For example, you might be given a statement to prove where *both* sides appear to be equal in complexity. In this case, you just choose a side to start with based on other factors. Likewise, you might find that switching

trigonometric functions to sines and cosines makes things too messy. You might look for fundamental identities involving the reciprocal functions in these cases.

In the end, becoming proficient in proving trigonometric identities is all about practice, and the method is entirely heuristic. Despite having a tactic for tackling proofs of trigonometric identities, most of your understanding and mastery will come from proving them on your own.

Example 9.1.1

Verify $\tan(\theta)\cos(\theta) = \sin(\theta)$.

Solution available at the end of this section.

In the solution to Example 9.1.1, we created a table where the first column contained the left-hand side (and its manipulations) and the second column contained the right-hand side (and, if we needed, it would have had manipulations of the right-hand side as well). The vertical bar between these two sides is purposeful. When trying to prove an identity, you are trying to show, through logical, deductive steps, that one side of the identity is equivalent to the other side. Until, or unless, you have proven these two sides are equivalent, you cannot assume they are the same. Therefore, you cannot treat it as an equation. This means you cannot add, subtract, multiply, divide, or square both sides because you assume the equation is true once you do so.

To be clear, when proving the identity in Example 9.1.1, it would be illegal for us to do the following

$$\begin{aligned} & & \mathbf{LHS} &= \mathbf{RHS} \\ \Longrightarrow & & \tan(\theta)\cos(\theta) &= \sin(\theta) \\ \Longrightarrow & & \left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos(\theta) &= \sin(\theta) \\ \Longrightarrow & \cancel{\cos(\theta)} \cdot \left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cos(\theta) &= \cos(\theta)\cdot\sin(\theta) \\ \Longrightarrow & & \sin(\theta)\cos(\theta) &= \sin(\theta)\cos(\theta) \end{aligned}$$

Doing this assumes that the two statements (the left side and the right side) are equal from the beginning.

Caution: Do not cross the line

When proving any identity (whether in Trigonometry or other Mathematics courses), treat each side separately and do not perform operations *between* sides. Do not add, subtract, multiply, or divide both sides by the same expression. Moreover, do not square both sides or perform any other operations between sides simultaneously.

Checkpoint 9.1.1

Verify the identity $\csc(\theta)\cos(\theta)\tan(\theta) = 1$.

Example 9.1.2

Prove the identity.

a. $[1+\sin(x)][1+\sin(-x)] = \cos^2(x)$
b. $\sin^2(\theta) = \frac{\sec^2(\theta)-1}{\sec^2(\theta)}$
c. $\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} = \cos(\theta)+\sin(-\theta)$
d. $[1-\cos^2(x)][1+\cot^2(x)] = 1$

e. $\frac{\cos(\theta)}{1+\sin(\theta)} = \frac{1-\sin(\theta)}{\cos(\theta)}$

Solutions available at the end of this section.

Example $9.1.2d$ is a great example of a proof where making a different choice from the start can change the proof dramatically. For example, most people new to proofs would likely have distributed the expressions on the left side to get

	LHS	**RHS**
	$\left[1-\cos^2(x)\right]\left[1+\cot^2(x)\right]$	1
$=$	$1+\cot^2(x)-\cos^2(x)-\cos^2(x)\cot^2(x)$	(distribution)
$=$	$1-\cos^2(x)+\cot^2(x)-\cos^2(x)\cot^2(x)$	(Commutative Property of Addition)
$=$	$\sin^2(x)+\cot^2(x)-\cos^2(x)\cot^2(x)$	(Pythagorean Identity)
$=$	$\sin^2(x)+\cot^2(x)\left[1-\cos^2(x)\right]$	(factor)
$=$	$\sin^2(x)+\cot^2(x)\sin^2(x)$	(Pythagorean Identity)
$=$	$\sin^2(x)+\frac{\cos^2(x)}{\sin^2(x)}\sin^2(x)$	(Ratio Identity)
$=$	$\sin^2(x)+\frac{\cos^2(x)}{\cancel{\sin^2(x)}}\cancel{\sin^2(x)}$	(cancel like factors)
$=$	$\sin^2(x)+\cos^2(x)$	
$=$	1	(Pythagorean Identity)
$=$	**RHS**	

The moral of the story here is that there are often many ways to prove an identity. With practice, you begin to see the fastest routes through the proof.

Checkpoint $9.1.2$

Prove the identity.

a. $\frac{\cot(\theta)}{\csc(\theta)} = \cos(\theta)$
b. $\frac{\sin^2(\theta)-1}{\tan(\theta)\sin(\theta)-\tan(\theta)} = \frac{\sin(\theta)+1}{\tan(\theta)}$

Hidden Resources: Kanopy

Did you know that as a Los Rios Community College student, you have free access to Kanopy?

What is Kanopy?

According to Wikipedia, Kanopy "is an on-demand streaming video platform for public libraries and universities that offers films and documentaries."

Why would you be interested?

There are a lot of great movies, documentaries, and educational videos on this service - and it's free for you because you are a student of the Los Rios Community College District.

Is it as good as Hulu or Netflix?

No, but it does have a lot of the same titles.

Why aren't any of the Great Courses available to us?

Not being able to access the Great Courses is a disappointment. That was one of the options Los Rios chose not to purchase; however, drown your sorrows in What We Do in the Shadows instead!

How do I sign up?

All you need to do is go to https://losrios.kanopy.com and start typing Los Rios when asked for your institution. That should be it.

Success in College: Environment and Learning Preferences

Throughout your academic career, you have completed many exams. While you might not have thought about it, you should be able to identify a strong correlation between the environment you study in and your performance on exams.

Learning preferences describe the situations in which you best learn math. Are you a visual, auditory, or kinesthetic learner? Should you take notes as you listen? Should you record your class? Should you trace the graphs in your book as you study?

Think about whether you learn better by working by yourself or with others. Do you prefer to work in quiet or with music in the background? Do you like to sit at a desk or recline in an easy chair? Do you learn best at a particular time of the day? When you study, does it help if you can move around? All of these are part of your learning preferences.

If you know your learning preferences, you can try to adjust your studying and class behavior to improve your work. It is your responsibility to think about and, if needed, make changes in how you are learning and studying.

Think about the following list. When you study, do you prefer...

- a quiet room, or background noise (music or TV)
- bright light shining on your work, or dim lighting
- sitting at a desk or table, or being relaxed on a couch or a bed
- eating or drinking while you learn, or only eating or drinking during breaks
- working hard for a reasonably long time or taking frequent breaks

1. Do you work in conditions that match these preferences when you do your math homework?
2. Do you work in conditions that match these preferences when you study for math tests?

Answers

Section Preview

This section might seem awkward to you and your students if you don't buy into the purpose of its existence. The point of this material is to clarify what identities and contradictions are. Use this time to summarize the identities the students already know and focus on proof tactics. **Be very clear that performing mathematical operations "between sides of the equality" is illegal!** Doing such tom-foolery is assuming that the identity is true to begin with.

Section Preview

- Prove a trigonometric identity.

Example 9.1.1

We will start on the left side, as it is the more complicated side:

	LHS		RHS
	$\tan(\theta)\cos(\theta)$		$\sin(\theta)$
$=$	$\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos(\theta)$	(Ratio Identity)	
$=$	$\sin(\theta)$		
$=$	**RHS**		

Example 9.1.2

a. Working on the left side of the equation, we have

	LHS		RHS
	$[1+\sin(x)][1+\sin(-x)]$		$\cos^2(x)$
$=$	$[1+\sin(x)][1-\sin(x)]$	(Symmetry Identity)	
$=$	$1-\sin^2(x)$	(distribution)	
$=$	$\cos^2(x)$	(Pythagorean Identity)	
$=$	**RHS**		

b. This is a case where the right side is more complex. Let's start there.

	LHS		RHS
	$\sin^2(\theta)$		$\frac{\sec^2(\theta)-1}{\sec^2(\theta)}$
$=$	$\frac{\sec^2(\theta)}{\sec^2(\theta)}-\frac{1}{\sec^2(\theta)}$	(arithmetic)	
$=$	$1-\frac{1}{\sec^2(\theta)}$	(arithmetic)	
$=$	$1-\cos^2(\theta)$	(Reciprocal Identity)	
$=$	$\sin^2(\theta)$	(Pythagorean Identity)	
$=$	**LHS**		

c. It should be obvious that the left side is the more complicated side in this case; however, notice that both sides have negative angular arguments for some of their trigonometric functions. Let's start by cleaning those up on both sides.

	LHS		RHS
	$\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)}$		$\cos(\theta)+\sin(-\theta)$
		$=$	$\cos(\theta)-\sin(\theta)$ (Symmetry Identity)
$=$	$\frac{[\sin(-\theta)]^2-[\cos(-\theta)]^2}{\sin(-\theta)-\cos(-\theta)}$	(it's best to rewrite powers first)	
$=$	$\frac{[-\sin(\theta)]^2-[\cos(\theta)]^2}{-\sin(\theta)-\cos(\theta)}$	(Symmetry Identities)	
$=$	$\frac{[\sin(\theta)]^2-[\cos(\theta)]^2}{-\sin(\theta)-\cos(\theta)}$	(squaring the negative)	
$=$	$-\frac{[\sin(\theta)]^2-[\cos(\theta)]^2}{\sin(\theta)+\cos(\theta)}$	(factoring the negative from the denominator)	
$=$	$-\frac{[\sin(\theta)-\cos(\theta)][\sin(\theta)+\cos(\theta)]}{\sin(\theta)+\cos(\theta)}$	(Difference of Squares)	
$=$	$-\frac{[\sin(\theta)-\cos(\theta)]\ \cancel{[\sin(\theta)+\cos(\theta)]}}{\cancel{\sin(\theta)+\cos(\theta)}}$	(canceling like factors)	
$=$	$-[\sin(\theta)-\cos(\theta)]$		
$=$	$-\sin(\theta)+\cos(\theta)$	(distribution)	
$=$	$\cos(\theta)-\sin(\theta)$	(Commutative Property of Addition)	
$=$	**RHS**		

d. Starting with the left side, we get the following:

	LHS		RHS
	$[1-\cos^2(x)]\,[1+\cot^2(x)]$		1
$=$	$[1-\cos^2(x)]\left[1+\dfrac{\cos^2(x)}{\sin^2(x)}\right]$	(rewriting in terms of sines and cosines)	
$=$	$[1-\cos^2(x)]\left[\dfrac{\sin^2(x)}{\sin^2(x)}+\dfrac{\cos^2(x)}{\sin^2(x)}\right]$	(getting a common denominator)	
$=$	$[1-\cos^2(x)]\left[\dfrac{\sin^2(x)+\cos^2(x)}{\sin^2(x)}\right]$	(adding fractions)	
$=$	$[1-\cos^2(x)]\left[\dfrac{1}{\sin^2(x)}\right]$	(Pythagorean Identity)	
$=$	$[\sin^2(x)]\left[\dfrac{1}{\sin^2(x)}\right]$	(Pythagorean Identity)	
$=$	$\dfrac{\sin^2(x)}{\sin^2(x)}$		
$=$	1		
$=$	**RHS**		

e. Both sides of this identity are equally complex. Let's start with the left side and see how far we get. This example illustrates our need to "keep our eyes on the prize." If you want to somehow transform the left side into the right, we will eventually need to get a $1-\sin(\theta)$ in the numerator of the left side. That tells us where to start with this proof.

	LHS		RHS
	$\dfrac{\cos(\theta)}{1+\sin(\theta)}$		$\dfrac{1-\sin(\theta)}{\cos(\theta)}$
$=$	$\dfrac{\cos(\theta)\,[1-\sin(\theta)]}{1-\sin^2(\theta)}$	(distribution)	
$=$	$\dfrac{\cos(\theta)\,[1-\sin(\theta)]}{\cos^2(\theta)}$	(Pythagorean Identity)	
$=$	$\dfrac{\cancel{\cos(\theta)}\,[1-\sin(\theta)]}{\cos^{\cancel{2}}(\theta)}$	(cancel like factors)	
$=$	$\dfrac{1-\sin(\theta)}{\cos(\theta)}$		
$=$	**RHS**		

This page titled 9.1: Basic Trigonometric Identities and Proof Techniques is shared under a CC BY-SA 3.0 license and was authored, remixed, and/or curated by Roy Simpson.

9.1.1: Resources and Key Concepts

Resources

Videos

- Proof Techniques
  - Fundamental Identities - Tactics for Proving Trigonometric Identities
  - A Review of the Basic Trigonometric Identities - Proving Trigonometric Identities (Part 2)

Key Concepts

Common Mistakes

- **Do Not Cross the Line:** When proving an identity, you must treat each side of the equation separately. Do not add, subtract, multiply, divide, or perform any other operation on both sides simultaneously, as this assumes the identity is already true.
- **Giving Up on One Side:** If you get stuck while manipulating one side of an identity, you should try working with the other side to see if you can meet in the middle or gain insight into the necessary steps.
- **Not Using Sines and Cosines:** When other algebraic or identity-based strategies fail, a reliable technique is to convert all trigonometric functions in the expression into their sine and cosine equivalents.
- **Building Instead of Simplifying:** It is almost always easier to start with the more complex side of an identity and simplify it to match the other side, rather than trying to build up the simpler side.

This page titled 9.1.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

9.1.2: Homework

Reading Questions

1. What are the five main families of basic trigonometric identities summarized in this section?
2. What is the first step recommended in the "How To" guide for proving a trigonometric identity?
3. When proving an identity, why is it considered "illegal" to perform an operation on both sides of the equals sign?
4. If you are stuck while trying to prove an identity, what are two strategies the text suggests trying?
5. What does it mean to "keep an eye on the prize" when proving an identity?
6. In the proof of $[1+\sin(x)][1+\sin(-x)]=\cos^2(x)$, which two fundamental identities were used?
7. What is the general strategy for proving an identity if you are unsure where to begin?
8. Explain why it is generally easier to start with the more complex side of an identity.

Homework

Concept Check

1. We showed that $\sin(\theta)=\pm\sqrt{1-\cos^2(\theta)}$ for all θ. Give an example of an angle θ such that $\sin(\theta)=-\sqrt{1-\cos^2(\theta)}$.
2. We showed that $\cos(\theta)=\pm\sqrt{1-\sin^2(\theta)}$ for all θ. Give an example of an angle θ such that $\cos(\theta)=-\sqrt{1-\sin^2(\theta)}$.
3. A student claims that $\cos(\theta)+\sin(\theta)=1$ is an identity because $\cos(0)+\sin(0)=1+0=0$. How would you respond to this student?

Proofs

For the following exercises, prove the identity. Assume all quantities are defined.

4. $\tan(\theta)+\tan(-\theta)=0$
5. $\cos^2(\theta)-\sin^2(\theta)=2\cos^2(\theta)-1$
6. $\cos^2(\theta)-\sin^2(\theta)=1-2\sin^2(\theta)$
7. $\dfrac{\sin^2(\theta)}{1-\sin^2(\theta)}=\tan^2(\theta)$
8. $\sin^4(\theta)-\cos^4(\theta)=\sin^2(\theta)-\cos^2(\theta)$
9. $\sin^2(x)+\tan^2(x)+\cos^2(x)=\sec^2(x)$
10. $\cos^4(\theta)-\sin^4(\theta)=1-2\sin^2(\theta)$
11. $\dfrac{\sin(x)+\cos(x)}{\cos(x)}=\tan(x)+1$
12. $\dfrac{\cos^2(\theta)}{1+\sin(\theta)}=1-\sin(\theta)$
13. $\dfrac{\sin^2(x)}{1+\cos(x)}=1-\cos(x)$
14. $\dfrac{\cos(x)}{1+\sin(x)}=\sec(x)-\tan(x)$

15. $\dfrac{\sin(x)-\cos(x)}{\sin(x)+\cos(x)}=\dfrac{\tan(x)-1}{1+\tan(x)}$

16. $\dfrac{1-\tan(\theta)}{1+\tan(\theta)}=\dfrac{\cot(\theta)-1}{\cot(\theta)+1}$

17. $\dfrac{1}{\tan(x)+\cot(x)}=\sin(x)\cdot\cos(x)$

18. $\dfrac{\tan(x)+\sec(x)-1}{\tan(x)-\sec(x)+1}=\dfrac{1+\sin(x)}{\cos(x)}$

19. $\dfrac{1+\sin(x)}{\cos(x)}=\sec(x)+\tan(x)$

20. $\dfrac{\sin(x)-\cos(x)}{\cos(-x)-\sin(-x)}=\dfrac{-1-\tan(-x)}{1+\tan(x)}$

21. $\dfrac{\sin(-x)}{1+\cos(x)}=\dfrac{\cos(-x)-1}{\sin(x)}$

22. $\dfrac{\sec(x)-1}{\sec(x)+1}=\dfrac{\sin^2(x)}{(\cos(x)+1)^2}$

23. $\dfrac{\sec(x)-1-\tan(-x)}{\tan(x)-\sec(-x)+1}=\dfrac{1-\sin(-x)}{\cos(x)}$

24. $\dfrac{\cos(-x)}{1-\sin(-x)}=\sec(-x)+\tan(-x)$

25. $\dfrac{1-\sin(-x)}{\cos(x)}=\sec(-x)-\tan(-x)$

26. $\sin(3\pi-2\theta)=-\sin(2\theta-3\pi)$

27. $\dfrac{\cos(x)}{1-\sin(x)}=\sec(x)+\tan(x)$

28. $\dfrac{1+\cos\left(\frac{\pi}{2}-x\right)}{\cos(x)}=\csc\left(\frac{\pi}{2}-x\right)+\tan(x)$

29. $\dfrac{1-\cos(x)}{\cos(x)}=\dfrac{\sin^2(x)}{\cos(x)(1+\cos(x))}$

30. $\dfrac{\sin^2(x)}{\cos(x)}=\tan(x)\cdot\sin(x)$

31. $\dfrac{\sin(-x)-\cos(x)}{\cos(x)}=\tan(-x)-1$

32. $\dfrac{1-2\cos^2(\theta)}{\sin(\theta)\cos(\theta)}=\tan(\theta)-\cot(\theta)$

33. $\cos\left(-\dfrac{\pi}{4}-5t\right)=\cos\left(5t+\dfrac{\pi}{4}\right)$

34. $\dfrac{\cos\left(\frac{\pi}{2}-x\right)}{\cos(x)+1}=\dfrac{1-\sin\left(\frac{\pi}{2}-x\right)}{\sin(x)}$

35. $\dfrac{1-\tan^2(\theta)}{1-\cot^2(\theta)}=1-\sec^2(\theta)$

36. $\dfrac{\cos(-x)}{1+\sin(-x)} = \sec(x) - \tan(-x)$

37. $\dfrac{1+\cos(x)}{\sin(x)} = \csc(x) + \cot(x)$

38. $\dfrac{1-\sin^2(x)}{\cos(x)} = \cos(x)$

39. $\dfrac{1-\sin(x)}{1+\sin(x)} = \dfrac{\sec(x)-\tan(x)}{\sec(x)+\tan(x)}$

40. $\dfrac{\sec(x)-1}{\sec(x)+1} = \dfrac{1-\cos(x)}{1+\cos(x)}$

41. $\dfrac{1+\sin(x)}{\cos(x)} = \dfrac{\cos(x)}{1-\sin(x)}$

42. $\tan(-t^2+1) = -\tan(t^2-1)$

43. $\dfrac{1-\cos\left(\frac{\pi}{2}-x\right)}{\cos(x)} = \csc\left(\frac{\pi}{2}-x\right) - \tan(x)$

44. $\dfrac{\sin(x)-\sin\left(\dfrac{\pi}{2}-x\right)}{\sin(x)+\cos(x)} = \dfrac{\cot\left(\dfrac{\pi}{2}-x\right)-1}{1+\tan(x)}$

45. $\dfrac{\cot\left(\frac{\pi}{2}-B\right)+\sec(B)-1}{\tan(B)-\csc\left(\frac{\pi}{2}-B\right)+1} = \dfrac{1-\sin(-B)}{\cos(B)}$

46. $\dfrac{\sin(\phi)}{1+\cos(\phi)} = \dfrac{1-\cos(\phi)}{\sin(\phi)}$

47. $\dfrac{\cos^2(\theta)}{1-\sin(\theta)} = 1+\sin(\theta)$

48. $\csc(-\theta-5) = -\csc(\theta+5)$

49. $\dfrac{\sin\left(\dfrac{\pi}{2}-\alpha\right)}{1+\sin(\alpha)} = \csc\left(\dfrac{\pi}{2}-\alpha\right) - \tan(\alpha)$

50. $\dfrac{1+\sin(-\theta)}{1+\sin(\theta)} = \dfrac{\sec(-\theta)+\tan(-\theta)}{\sec(-\theta)+\tan(\theta)}$

51. $\dfrac{1+\cos(-A)}{\sin(-A)} = -\csc(A) + \cot(-A)$

52. $\dfrac{\cos(-\phi)-1}{\cos(\phi)} = \dfrac{\sin^2(-\phi)}{\cos(\phi)(1+\cos(\phi))}$

53. $\dfrac{\sin\left(\frac{\pi}{2}-\theta\right)}{1-\sin(\theta)} = \csc\left(\frac{\pi}{2}-\theta\right) + \cot\left(\frac{\pi}{2}-\theta\right)$

54. $\dfrac{\sec(-\alpha)-1}{\sec(\alpha)+1} = \dfrac{\sin^2(-\alpha)}{(\cos(-\alpha)+1)^2}$

55. $\sec(-6t) = \sec(6t)$

56. $-\dfrac{\cos^2(-A)}{1+\sin(-A)} = -1+\sin(-A)$

57. $\dfrac{\tan(\theta)+\tan(\phi)}{\cot(\theta)+\cot(\phi)}=\tan(\theta)\tan(\phi)$

58. $\dfrac{\sin^2(-\alpha)}{\cos(-\alpha)}=-\tan(\alpha)\cdot\sin(-\alpha)$

59. $\dfrac{\sec(y)-1}{\sec(y)+1}=\dfrac{1-\cos(-y)}{1+\cos(-y)}$

60. $\dfrac{\sin^2\left(\frac{\pi}{2}-A\right)}{1-\sin(A)}=\dfrac{\cos^2(A)}{1-\cos\left(\frac{\pi}{2}-A\right)}$

61. $\dfrac{1+\sin\left(\frac{\pi}{2}-B\right)}{\sin(B)}=\sec\left(\frac{\pi}{2}-B\right)+\cot(B)$

62. $\dfrac{\sin\left(\frac{\pi}{2}-\phi\right)-1}{\sin\left(\frac{\pi}{2}-\phi\right)}=\dfrac{\sin^2(\phi)}{\sin\left(\frac{\pi}{2}-\phi\right)\left(1+\sin\left(\frac{\pi}{2}-\phi\right)\right)}$

63. $\cot(9-7\theta)=-\cot(7\theta-9)$

64. $\dfrac{\csc\left(\frac{\pi}{2}-B\right)-1}{\sec(B)+1}=\dfrac{\cos^2\left(\frac{\pi}{2}-B\right)}{\left(\sin\left(\frac{\pi}{2}-B\right)+1\right)^2}$

65. $\dfrac{1-\cos(\alpha)}{\sin^2(\alpha)}=\dfrac{1}{1+\cos(\alpha)}$

Challenge Problems

66. Suppose that you are given a system of two equations of the following form:

$$\begin{aligned} A\cos(\phi) &= B\nu_1-B\nu_2\cos(\theta) \\ A\sin(\phi) &= B\nu_2\sin(\theta) \end{aligned}$$

Show that

$$A^2=B^2\left(\nu_1^2+\nu_2^2-2\nu_1\nu_2\cos(\theta)\right).$$

67. Suppose that a point with coordinates

$$(x,y)=\left(a\left(\cos(\psi)-\epsilon\right),a\sqrt{1-\epsilon^2}\sin(\psi)\right)$$

is a distance $r>0$ from the origin, where $a>0$ and $0<\epsilon<1$. Use $r^2=x^2+y^2$ to show that $r=a\left(1-\epsilon\cos(\psi)\right)$. (Note: These coordinates arise in the study of elliptical orbits of planets.)

68. Show that each trigonometric function can be written solely in terms of the sine function.

This page titled 9.1.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

9.2: Sum and Difference Identities

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We have come a long way in Trigonometry. We have defined the trigonometric functions (in terms of angles, ratios of sides of right triangles, and real numbers), proved trigonometric identities with them, used them to solve applications, graphed them, and evaluated them by hand at the special angles. If we needed to evaluate a trigonometric function at a non-special angle, however, we must use technology. But, is this always the case?

For example, if we know the sine of 27°, can we find the sine of $2\,(27^\circ) = 54^\circ$ without using a calculator? Or, if we know $\cos(\alpha)$ and $\cos(\beta)$, can we calculate $\cos(\alpha+\beta)$?

The Sum and Difference Identities for the Cosine

All the identities that relate the trigonometric ratios of different angles are derived from the Sum and Difference Identities (which really should be named the "Sum and Difference of Angles Identities," but that's too long to write). Let's see why we need these formulas.

Is it true that

$$\cos(\alpha+\beta) = \cos(\alpha) + \cos(\beta)?$$

We can test this hypothesis by evaluating both expressions for some specific values of α and β, say $\alpha = 45^\circ$ and $\beta = 30^\circ$, as shown below.

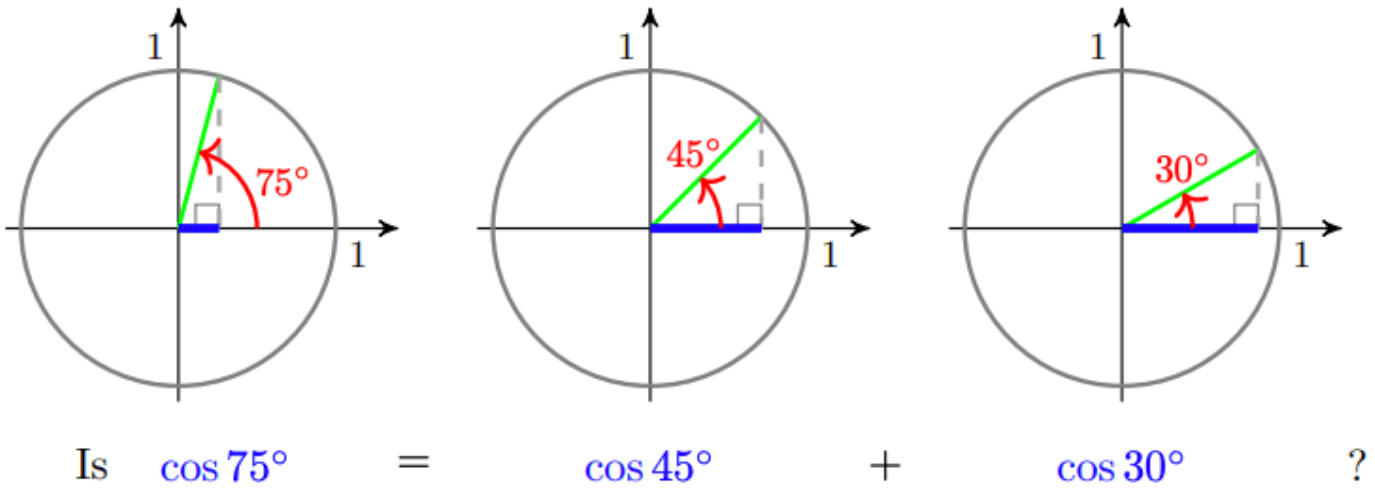


Figure 9.2.1

Taking a unit circle approach, you should be able to see that $\cos(75^\circ)$ (which is the x-value of the point on the terminal side of 75°) is in fact smaller than either $\cos(45^\circ)$ or $\cos(30^\circ)$. Therefore, it cannot be true that $\cos(75^\circ)$ is equal to $\cos(45^\circ) + \cos(30^\circ)$.

Example 9.2.1

Verify that $\cos(45^\circ + 30^\circ)$ is not equal to $\cos(45^\circ) + \cos(30^\circ)$.

Solution available at the end of this section.

Checkpoint 9.2.1

Show that $\sin(60^\circ + 30^\circ)$ is not equal to $\sin(60^\circ) + \sin(30^\circ)$.

Answer available at the end of this section.

Caution: You Cannot "Split" a Trigonometric Function Across its Arguments

Because the values of the expressions in the Example 9.2.1 and Checkpoint 9.2.1 are different, it is *not* true that $\cos(\alpha + \beta)$ is equal to $\cos(\alpha) + \cos(\beta)$ for all angles α and β, or that $\sin(\alpha + \beta)$ is equal to $\sin(\alpha) + \sin(\beta)$.

This is a general property of almost all functions. That is, if f is a function, it is generally the case that

$$f(a \pm b) \neq f(a) \pm f(b).$$

It turns out that there is a relationship between the trigonometric ratios for $\alpha \pm \beta$ and the trigonometric ratios of α and β.

Theorem: Sum and Difference Identities for the Cosine

For angles α and β,

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta).$$

Proof available at the end of this section.

The $\pm$ and $\mp$ signs on the left and right side of this identity are always opposite.

Advice: Memorize These Identities

The Sum and Difference Identities for the cosine are needed throughout the rest of Trigonometry, Calculus, and Differential Equations. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Notice that to find the cosine of $\alpha + \beta$ we must know (or be able to find) both trigonometric ratios for both and α and β.

The Sum and Difference Identities can be used to find exact values for trigonometric ratios of various angles.

Example 9.2.2

Find an exact value for $\cos(105^\circ)$.

Solution available at the end of this section.

Checkpoint 9.2.2

Suppose that $\sin(\theta) = \frac{-5}{13}$ and $\cos(\theta) = \frac{12}{13}$. Find an exact value for $\cos\left(\frac{\pi}{4} + \theta\right)$.

Answer available at the end of this section.

Proving the Cofunction Identities

We have been using the Cofunction Identities for quite a while, but have yet to see a proof. This is usually unforgivable in Mathematics; however, I wanted you to have some experience with these identities and, at the time, we didn't have the "mathematical technology" (read as, the knowledge) to prove them—now, we do. Let's prove $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$.

$$\begin{aligned} \cos\left(\frac{\pi}{2} - \theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta) + \sin\left(\frac{\pi}{2}\right)\sin(\theta) && \text{(Difference Identity for the Cosine)} \\ &= (0)\cos(\theta) + (1)\sin(\theta) && \text{(evaluating trig functions)} \\ &= \sin(\theta) \end{aligned}$$

The remaining Cofunction Identities are left for you to prove as exercises.

The Sum and Difference Identities for the Sine

Theorem: Sum and Difference Identities for the Sine

For angles α and β,

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta).$$

Proof available at the end of this section.

Notice that the sine function "keeps its sign." That is, if we are given $\alpha + \beta$, the result will be a sum of functions. If given $\alpha - \beta$, we will have a difference of functions.

Advice: Memorize These Identities

The Sum and Difference Identities for the sine are needed throughout the rest of Trigonometry, Calculus, and Differential Equations. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated.

Example 9.2.3

Suppose that $\sin(\theta) = 0.6$ and $\cos(\theta) = -0.8$. Find an exact value for $\sin\left(\theta + \frac{2\pi}{3}\right)$.

Solution available at the end of this section.

Example 9.2.4

Find the exact value of $\sin\left(\frac{19\pi}{12}\right)$.

Solution available at the end of this section.

Checkpoint 9.2.4

Evaluate $\sin\left(\frac{7\pi}{12}\right)$ exactly.

Answer available at the end of this section.

Example 9.2.5

If α is a Quadrant II angle with $\sin(\alpha) = \frac{5}{13}$, and β is a Quadrant III angle with $\tan(\beta) = 2$, find $\sin(\alpha - \beta)$.

Solution available at the end of this section.

Checkpoint 9.2.5

If α is a Quadrant II angle with $\sin(\alpha) = \frac{3}{5}$, and β is a Quadrant III angle with $\cos(\beta) = -\frac{5}{13}$, find $\cos(\alpha - \beta)$.

Answer available at the end of this section.

The Sum and Difference Identities for the Tangent

There are also sum and difference formulas for the tangent. The proof for the tangent versions involves using the Quotient Identities to rewrite the tangent in terms of the sine and cosine, which we leave for you in the homework area.

Theorem: Sum and Difference Identities for the Tangent

For angles α and β where the following formula is defined,

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

There is a subtle restriction in the last theorem that the two before it didn't have. Since the tangent function does not have a domain consisting of all real numbers, the Sum and Difference Identities for the tangent have exceptions whenever α or β are odd multiples of $\frac{\pi}{2}$; however, this theorem is rarely used and, as such, this exception is not going to be encountered (at least, not at this level).

Example 9.2.6

Find an exact value for $\tan(75^\circ)$.

Solution available at the end of this section.

Checkpoint 9.2.6

Evaluate $\tan\left(\frac{\pi}{12}\right)$ exactly.

Answer available at the end of this section.

Revisiting Proofs of Identities

Every time we introduce a new set of major identities (like the Sum and Difference Identities), we get to practice proofs again. In some sense, proofs begin to get much easier once we learn identities like the Sum and Difference Identities. This is because one of the first things we should look for when trying to prove an identity is the existence of other identities within the claim. That is, we look for Pythagorean Identities, Symmetry Identities, and so on. Now we have a new set of identities to look for.

Example 9.2.7

Prove the following identity.

$$\frac{\sin(\alpha-\beta)}{\cos(\alpha)\cos(\beta)} = \tan(\alpha) - \tan(\beta)$$

Solution available at the end of this section.

Checkpoint 9.2.7

Prove.

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta)$$

Revisiting Graphs of Trigonometric Functions

The Sum and Difference Identities allow us to expand the set of trigonometric functions we can graph. Up to this point, we have only been able to graph functions of the form

$$f(x) = A\,\mathrm{trig}\,(Bx+C) + D.$$

However, now we can graph certain *combinations* of trigonometric functions. Specifically, those having the form of the Sum and Difference Identities.

Example 9.2.8

Graph the function.

$$f(x) = \frac{1}{2}\cos(2x)\cos(3x) - \frac{1}{2}\sin(2x)\sin(3x)$$

Solution available at the end of this section.

Checkpoint 9.2.8

Graph the function.

$$g(x) = -2\sin(\pi x)\cos\left(\frac{\pi}{2}x\right) + 2\sin\left(\frac{\pi}{2}x\right)\cos(\pi x)$$

Answer available at the end of this section.

Success in College: Learning From Tests (Part 3)

1. Look at your last test from this class. How many points did you miss because you could not remember something or did not know how to do the problem?

2. Knowing what you know now, do you think you could do better on this exam?

Success in Mathematics: Sequential Versus Global Learners

Sequential learners tend to think in a linear manner and are able to function with only partial understanding of material they have been taught. Global learners think in a systems-oriented manner, and may have trouble applying new material until they fully understand it and see how it relates to material they already know about and understand. Once they grasp the big picture, however, their holistic perspective enables them to see innovative solutions to problems that sequential learners might take much longer to reach.

Source: R. Felder, R. Brent , Journal of Engineering Education, 94 (1), 57-72 (2005).

1. When learning math, do you think you are more of a sequential learner or a global learner?
2. Students often have different learning preferences for different subjects. When learning history or another subject in the humanities, do you think you are more a sequential learner or a global learner?

Answers

Section Preview

Do not succumb to laziness! Prove one of the Sum and Difference Identities. It's important that students see the logic necessary to derive such foundational theorems.

Section Preview

- Use the Sum and Difference Identities to find the exact value of a trigonometric function.
- Prove an equation is an identity using the Sum and Difference Identities.
- Simplify an expression using the Sum and Difference Identities.
- Use the Sum and Difference Identities to graph an equation.

Example $9.2.1$

We know that $\cos(45^\circ) = \frac{1}{\sqrt{2}} \approx 0.7071$ and $\cos(30^\circ) = \frac{\sqrt{3}}{2} \approx 0.8660$. Thus, $\cos(45^\circ) + \cos(30^\circ) \approx 1.5731$. For $\cos(45^\circ + 30^\circ) = \cos(75^\circ)$, we use a calculator to get the approximation 0.2588. Therefore, the two expressions are not equal.

Checkpoint $9.2.1$

$\sin(60^\circ + 30^\circ) = 1$, but $\sin(60^\circ) + \sin(30^\circ) = \frac{\sqrt{3}}{2} + \frac{1}{2}$

Theorem: Sum and Difference Identities for the Cosine

Let α and β be angles in standard position and let α_0 and β_0 be the angles coterminal to α and β, respectively, each of which measure between 0 and 2π radians. Moreover, without loss of generality, assume that $\alpha_0 > \beta_0$ (if this assumption were not true, we would just relabel the variables accordingly).

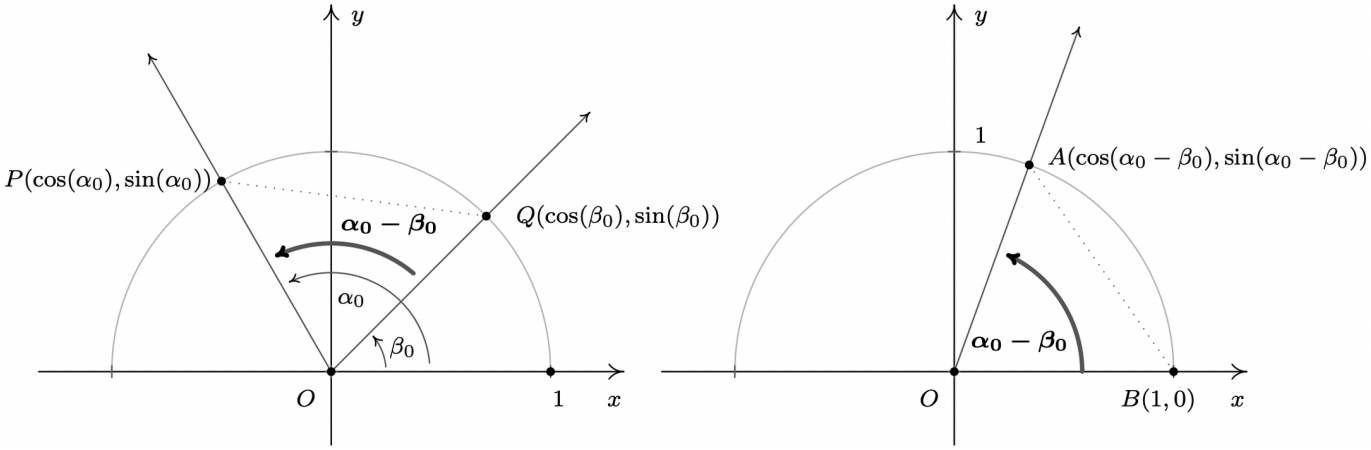
Since α_0 is coterminal with α, $\alpha = \alpha_0 + 2\pi m$ for some $m^{\text{int}}\mathbb{Z}$. Likewise, $\beta = \beta_0 + 2\pi n$ for some $n^{\text{int}}\mathbb{Z}$. Therefore,

$$\cos(\alpha - \beta) = \cos(\alpha_0 + 2\pi m - (\beta_0 + 2\pi n)) = \cos(\alpha_0 - \beta_0 + 2\pi(m - n)).$$

Since m and n are both integers, $m - n$ is also an integer. Moreover, cosine has a period of 2π means that $\cos(\alpha_0 - \beta_0 + 2\pi(m - n)) = \cos(\alpha_0 - \beta_0)$. Thus,

$$\cos(\alpha - \beta) = \cos(\alpha_0 - \beta_0).$$

All of this is to reduce our work to the angles α_0 and β_0. We now draw and label the points P and Q on the unit circle which are on the terminal sides of α_0 and β_0, respectively.



The image on the right side of this figure is a rotation of the image on the left clockwise by β_0 radians. Since the angles $\angle POQ$ and $\angle AOB$ are congruent, the distance between P and Q is equal to the distance between A and B. The Distance Formula yields

$$\begin{aligned}
&& \text{Disatnce between } P \text{ and } Q &= \text{Distance between } A \text{ and } B && \\
\Longrightarrow\quad & \sqrt{(\cos(\alpha_0)-\cos(\beta_0))^2+(\sin(\alpha_0)-\sin(\beta_0))^2} &= \sqrt{(\cos(\alpha_0-\beta_0)-1)^2+(\sin(\alpha_0-\beta_0)-0)^2} && \text{(Distance Formula)} \\
\Longrightarrow\quad & (\cos(\alpha_0)-\cos(\beta_0))^2+(\sin(\alpha_0)-\sin(\beta_0))^2 &= (\cos(\alpha_0-\beta_0)-1)^2+(\sin(\alpha_0-\beta_0)-0)^2 && \text{(squaring both sides)} \\
\Longrightarrow\quad & \begin{array}{c}\cos^2(\alpha_0)-2\cos(\alpha_0)\cos(\beta_0)+\cos^2(\beta_0)\\ +\sin^2(\alpha_0)-2\sin(\alpha_0)\sin(\beta_0)+\sin^2(\beta_0)\end{array} &= \begin{array}{c}\cos^2(\alpha_0-\beta_0)-2\cos(\alpha_0-\beta_0)+1\\ +\sin^2(\alpha_0-\beta_0)\end{array} && \text{(distributing)} \\
\Longrightarrow\quad & \begin{array}{c}\cos^2(\alpha_0)+\sin^2(\alpha_0)\\ -2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)\\ +\cos^2(\beta_0)+\sin^2(\beta_0)\end{array} &= \begin{array}{c}\cos^2(\alpha_0-\beta_0)+\sin^2(\alpha_0-\beta_0)\\ -2\cos(\alpha_0-\beta_0)\\ +1\end{array} && \text{(Commutative Property of Addition)} \\
\Longrightarrow\quad & \begin{array}{c}1\\ -2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0)\\ +1\end{array} &= \begin{array}{c}1\\ -2\cos(\alpha_0-\beta_0)\\ +1\end{array} && \text{(Pythagorean Identity)} \\
\Longrightarrow\quad & 2-2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0) &= 2-2\cos(\alpha_0-\beta_0) && \text{(simplifying)} \\
\Longrightarrow\quad & -2\cos(\alpha_0)\cos(\beta_0)-2\sin(\alpha_0)\sin(\beta_0) &= -2\cos(\alpha_0-\beta_0) && \text{(subtracting 2 from both sides)} \\
\Longrightarrow\quad & \cos(\alpha_0)\cos(\beta_0)+\sin(\alpha_0)\sin(\beta_0) &= \cos(\alpha_0-\beta_0) && \text{(dividing both sides by } -2)
\end{aligned}$$

Combining this result with the fact that $\cos(\alpha-\beta)=\cos(\alpha_0-\beta_0)$, we have the identity

$$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta).$$

To get the sum identity for cosine, we use this result along with the Symmetry Identities.

$$\cos(\alpha+\beta)=\cos(\alpha-(-\beta))=\cos(\alpha)\cos(-\beta)+\sin(\alpha)\sin(-\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$

Hence,

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).$$

Example 9.2.2

We can write 105° as the sum of two special angles: $105^\circ=60^\circ+45^\circ$. Now apply the sum of angles identity for the cosine.

$$\begin{aligned}
\cos(60^\circ+45^\circ) &= \cos(60^\circ)\cos(45^\circ)-\sin(60^\circ)\sin(45^\circ)\\
&= \frac{1}{2}\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}\\
&= \frac{\sqrt{2}-\sqrt{6}}{4}
\end{aligned}$$

Thus, $\cos 105^\circ=\frac{\sqrt{2}-\sqrt{6}}{4}$. You can check that your calculator gives the same decimal approximation of about -0.2588 for both $\cos 105^\circ$ and $\frac{\sqrt{2}-\sqrt{6}}{4}$.

Checkpoint 9.2.2

$$\frac{17}{13\sqrt{2}}=\frac{17\sqrt{2}}{26}$$

Theorem: Sum and Difference Identities for the Sine

To prove the Sum and Difference Identities for the sine, we make use of the Cofunction Identities.

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\left(\tfrac{\pi}{2}-(\alpha+\beta)\right) && \text{(Cofunction Identities)}\\
&= \cos\left(\left(\tfrac{\pi}{2}-\alpha\right)-\beta\right) && \text{(Associative Property of Addition)}\\
&= \cos\left(\tfrac{\pi}{2}-\alpha\right)\cos(\beta)+\sin\left(\tfrac{\pi}{2}-\alpha\right)\sin(\beta) && \text{(Difference Identity for Cosine)}\\
&= \sin(\alpha)\cos(\beta)+\sin\left(\tfrac{\pi}{2}-\alpha\right)\sin(\beta) && \text{(Cofunction Identities)}\\
&= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) && \text{(Cofunction Identities)}
\end{aligned}$$

Thus, we get the identity.

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta).$$

A similar proof shows that

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta).$$

Example 9.2.3

Recall that $\sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}$ and $\cos\left(\frac{2\pi}{3}\right)=\frac{-1}{2}$. Substituting these values into the sum formula for sine, we find

$$\begin{aligned}\sin\left(\theta+\frac{2\pi}{3}\right) &= \sin(\theta)\cos\left(\frac{2\pi}{3}\right)+\cos(\theta)\sin\left(\frac{2\pi}{3}\right)\\ &= 0.6\left(\frac{-1}{2}\right)+(-0.8)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{3}{5}\cdot\frac{-1}{2}+\frac{-4}{5}\cdot\frac{\sqrt{3}}{2}\\ &= \frac{-3-4\sqrt{3}}{10}\end{aligned}$$

Example 9.2.4

We need to write the angle $\frac{19\pi}{12}$ as a sum or difference of common angles. The denominator of 12 suggests a combination of angles with denominators 3 and 4. One such combination is $\frac{19\pi}{12}=\frac{16\pi}{12}+\frac{3\pi}{12}=\frac{4\pi}{3}+\frac{\pi}{4}$. We get

$$\begin{aligned}\sin\left(\frac{19\pi}{12}\right) &= \sin\left(\frac{4\pi}{3}+\frac{\pi}{4}\right)\\ &= \sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)\\ &= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Checkpoint 9.2.4

$$\frac{\sqrt{3}+1}{2\sqrt{2}}$$

Example 9.2.5

This is one of my favorite styles of problems. There are two main ways to find the solution—using identities or using right triangle trigonometry. I will cover both methods here, but I must confess—I highly prefer the right triangle approach.

Using Identities

In order to find $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$, we need to find $\cos(\alpha)$ and both $\cos(\beta)$ and $\sin(\beta)$. To find $\cos(\alpha)$, we use the Pythagorean Identity $\cos^2(\alpha)+\sin^2(\alpha)=1$. Since $\sin(\alpha)=\frac{5}{13}$, we have $\cos^2(\alpha)+\left(\frac{5}{13}\right)^2=1$, or $\cos(\alpha)=\pm\frac{12}{13}$. Since α is a Quadrant II angle, $\cos(\alpha)=-\frac{12}{13}$. We now set about finding $\cos(\beta)$ and $\sin(\beta)$. We have several ways to proceed, but the Pythagorean Identity $1+\tan^2(\beta)=\sec^2(\beta)$ is a quick way to get $\sec(\beta)$, and hence, $\cos(\beta)$. With $\tan(\beta)=2$, we get $1+2^2=\sec^2(\beta)$ so that $\sec(\beta)=\pm\sqrt{5}$. Since β is a Quadrant III angle, we choose $\sec(\beta)=-\sqrt{5}$ so $\cos(\beta)=\frac{1}{\sec(\beta)}=\frac{1}{-\sqrt{5}}=-\frac{\sqrt{5}}{5}$. We now need to determine $\sin(\beta)$. We could use The Pythagorean Identity $\cos^2(\beta)+\sin^2(\beta)=1$, but we opt instead to use a quotient identity. From $\tan(\beta)=\frac{\sin(\beta)}{\cos(\beta)}$, we have $\sin(\beta)=\tan(\beta)\cos(\beta)$ so we get $\sin(\beta)=(2)\left(-\frac{\sqrt{5}}{5}\right)=-\frac{2\sqrt{5}}{5}$. We now have all the pieces needed to find $\sin(\alpha-\beta)$:

$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\\ &= \left(\frac{5}{13}\right)\left(-\frac{\sqrt{5}}{5}\right)-\left(-\frac{12}{13}\right)\left(-\frac{2\sqrt{5}}{5}\right)\\ &= -\frac{29\sqrt{5}}{65}\end{aligned}$$

Using Right Triangle Trigonometry

For this method, we remind ourselves that Trigonometry is mostly about quadrants and reference angles. We draw α^{int}QII and β^{int}QIII and their corresponding reference angles.

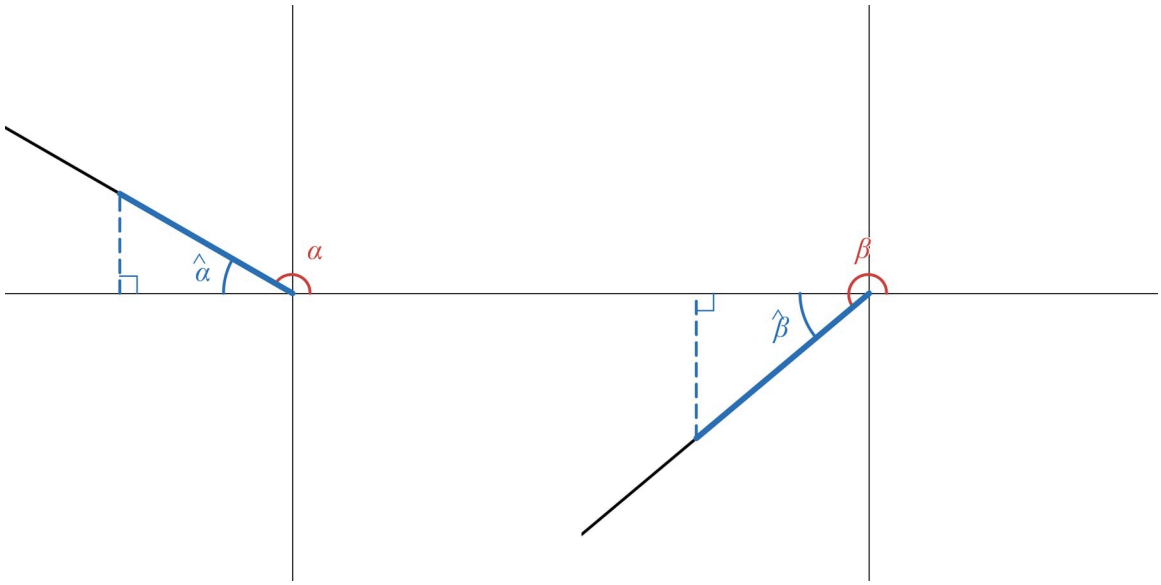


Figure 9.2.2

Neither of these graphs are drawn to scale, nor do they need to be. Since $\sin(\alpha) = \frac{5}{13}$, we wisely choose the terminal point so that its y-value is 5 and its distance to the origin is 13. Looking at the reference triangle, this should remind you that sine is opposite over hypotenuse. Likewise, since $\tan(\beta) = 2 = \frac{2}{1}$, we choose the terminal point of β to have a y-value of -2 (remember, β^{int}QIII) and an x-value of -1. Again, the tangent is opposite over adjacent, so the tangent of $\hat{\beta}$ is $\frac{-2}{-1} = 2$.

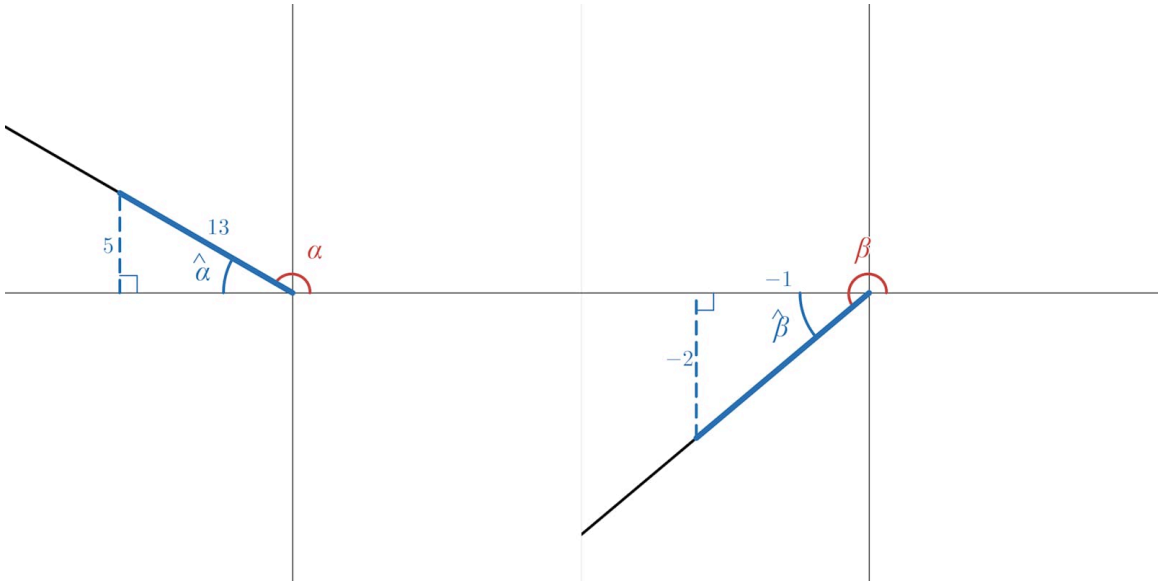


Figure 9.2.3

Finally, we use the Pythagorean Theorem to help fill in the missing sides (taking into account the quadrants).

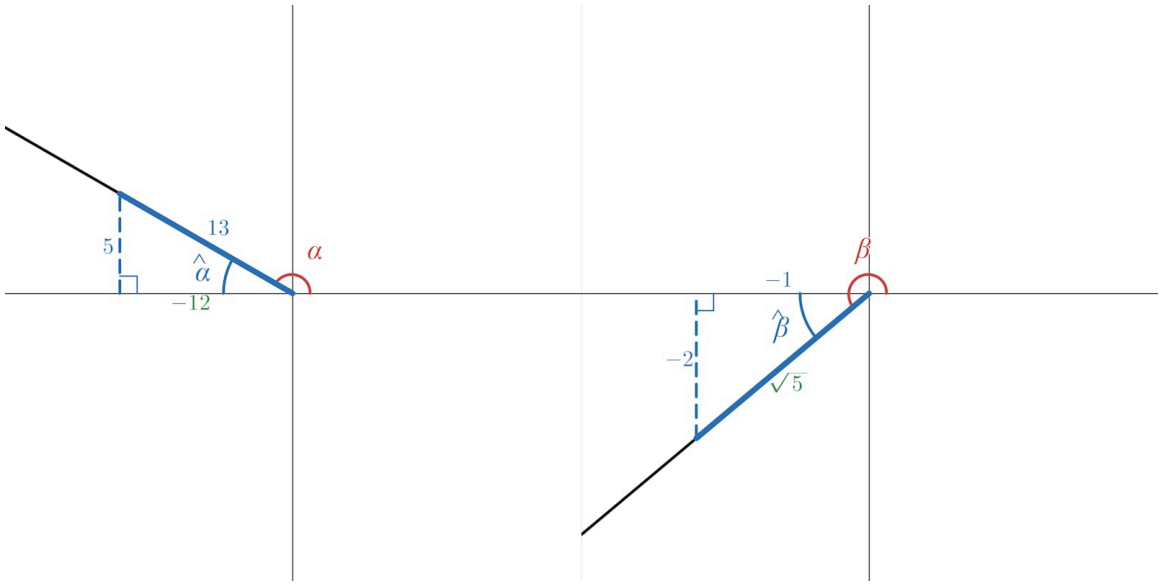


Figure 9.2.4

From these figures, we read off the information we need.

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)=\frac{5}{13}\cdot\frac{-1}{\sqrt{5}}-\frac{-12}{13}\cdot\frac{-2}{\sqrt{5}}=\frac{-5+(-24)}{13\sqrt{5}}=-\frac{29}{13\sqrt{5}}.$$

Checkpoint 9.2.5

$-\frac{56}{65}$

Example 9.2.6

We observe that $75^\circ=45^\circ+30^\circ$, so $\tan(75^\circ)=\tan(45^\circ+30^\circ)$. We can apply the sum formula for tangent.

$$\begin{aligned}\tan(45^\circ+30^\circ) &= \frac{\tan(45^\circ)+\tan(30^\circ)}{1-\tan(45^\circ)\tan(30^\circ)}\\ &= \frac{1+\frac{1}{\sqrt{3}}}{1-1\left(\frac{1}{\sqrt{3}}\right)}\cdot\frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\end{aligned}$$

Checkpoint 9.2.6

$\frac{\sqrt{3}-1}{\sqrt{3}+1}$

Example 9.2.7

The left side is the more complex side, so we start there. Also, we immediately spot our newly-discovered Sum and Difference Identities, so we know where to begin.

$$\begin{aligned}\frac{\sin(\alpha-\beta)}{\cos(\alpha)\cos(\beta)} &= \frac{\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)} && \text{(Sum and Difference Identities)}\\ &= \frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)} && \text{(property of fractions)}\\ &= \frac{\sin(\alpha)\,\cancel{\cos(\beta)}}{\cos(\alpha)\,\cancel{\cos(\beta)}}-\frac{\cancel{\cos(\alpha)}\,\sin(\beta)}{\cancel{\cos(\alpha)}\,\cos(\beta)} && \text{(cancel like factors)}\\ &= \frac{\sin(\alpha)}{\cos(\alpha)}-\frac{\sin(\beta)}{\cos(\beta)}\\ &= \tan(\alpha)-\tan(\beta) && \text{(Ratio Identities)}\end{aligned}$$

Example 9.2.8

Prior to this moment, there would have been no possibility for us to graph this function by hand; however, we now know, from the Sum and Difference Identities, that

$$f(x)=\frac{1}{2}\cos(2x)\cos(3x)-\frac{1}{2}\sin(2x)\sin(3x)=\frac{1}{2}\left[\cos(2x)\cos(3x)-\sin(2x)\sin(3x)\right]=\frac{1}{2}\cos(2x+3x)=\frac{1}{2}\cos(5x).$$

The amplitude of this cosine function is $\frac{1}{2}$, the period is $\frac{2\pi}{5}$, the step size is $\frac{\pi}{10}$, the phase shift is 0, and the vertical shift is 0. Putting this together, we get the following graph.

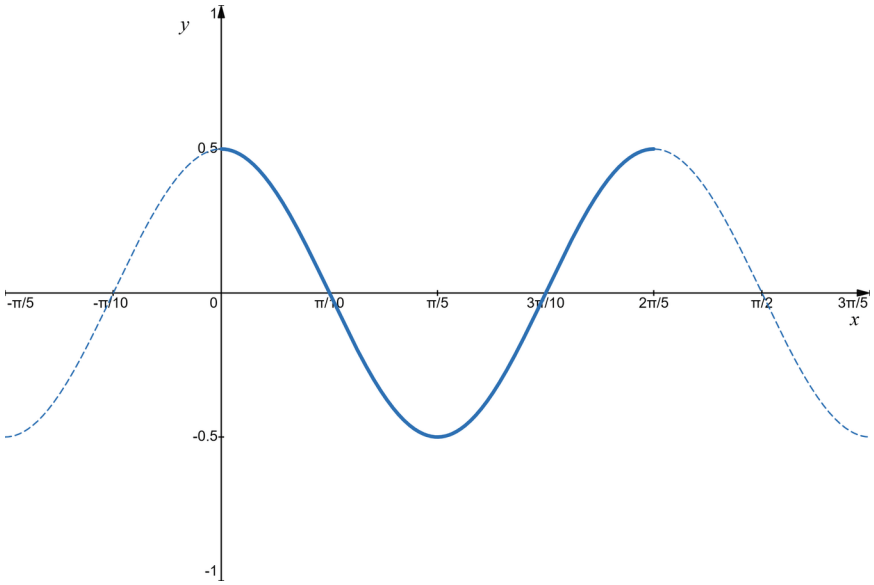


Figure 9.2.5

Checkpoint 9.2.8

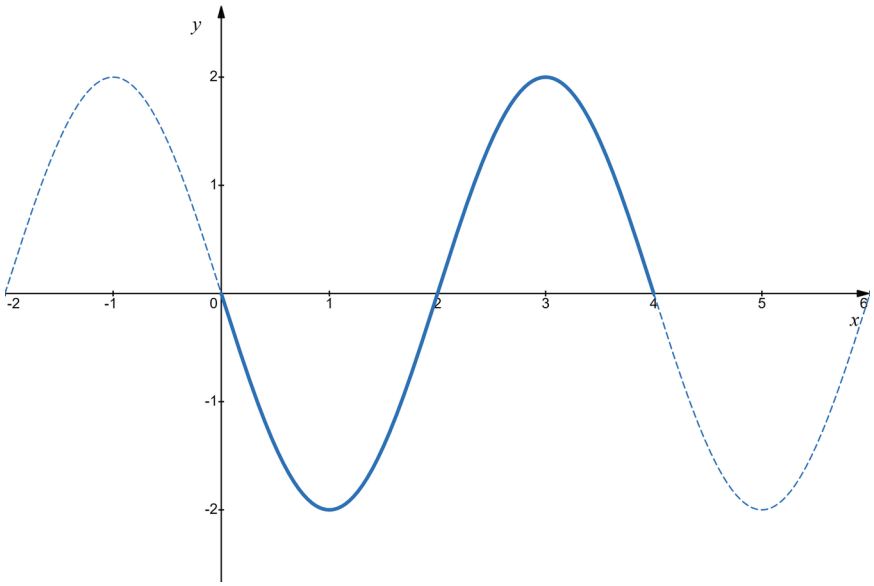


Figure 9.2.6

This page titled 9.2: Sum and Difference Identities is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

9.2.1: Resources and Key Concepts

Resources

Videos

- Sum and Difference Identities
  - Sum and Difference Identities - The Sum and Difference Identities: Theorem and Proof
  - Sum and Difference Identities - Evaluating Trigonometric Functions of Non-Special Angles
  - Sum and Difference Identities - Using the Sum and Difference Identities
  - Sum and Difference Identities - Proving Trigonometric Identities (Part 3)

Key Concepts

Theorems

- **Sum and Difference Identities for the Cosine:** For any angles α and β:
  - $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$
  - $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$
- **Sum and Difference Identities for the Sine:** For any angles α and β:
  - $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$
  - $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$
- **Sum and Difference Identities for the Tangent:** For any angles α and β where the functions are defined:
  - $\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$
  - $\tan(\alpha-\beta)=\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

Common Mistakes

- **You Cannot "Split" a Trigonometric Function Across its Arguments:** A very common mistake is to assume that a trigonometric function distributes over a sum or difference. For example, $\cos(\alpha+\beta)$ is not equal to $\cos(\alpha)+\cos(\beta)$. The Sum and Difference Identities must be used instead.
- **Confusing the Signs in the Identities:** The signs in the Sum and Difference Identities can be tricky. For the cosine identities, the sign in the middle of the expression is the opposite of the operation in the argument (e.g., $\cos(\alpha+\beta)$ has a minus sign). For the sine identities, the sign in the middle matches the operation in the argument (e.g., $\sin(\alpha+\beta)$ has a plus sign).

This page titled 9.2.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

9.2.2: Homework

Reading Questions

1. Is $\cos(A+B)$ equivalent to $\cos(A)+\cos(B)$? Explain why or why not.
2. What is the Sum Identity for the cosine function?
3. What is the Difference Identity for the sine function?
4. How can the Sum and Difference Identities be used to find an exact value for $\cos(105^\circ)$?
5. This section proves the Cofunction Identity $\cos(\frac{\pi}{2}-\theta)=\sin(\theta)$. What primary identity was used to begin this proof?
6. What is the Sum Identity for the tangent function?
7. If you need to find $\sin(\alpha-\beta)$ and you are only given $\sin(\alpha)$ and $\tan(\beta)$ (along with their quadrants), what other trigonometric values must you find first?
8. How can the Sum and Difference Identities be used to simplify an expression like $\cos(2x)\cos(3x)-\sin(2x)\sin(3x)$ into a single trigonometric function?

Skills Refresher

Skills Refresher

For the following exercises,
(a) Compute $f(2), f(3)$, and $f(2+3)$.
(b) Determine if it is true that $f(a+b)=f(a)+f(b)$ whenever the function values are defined.

1. $f(x)=3x+2$
2. $f(x)=x^2$
3. $f(x)=\sqrt{x}$
4. $f(x)=\dfrac{1}{x}$
5. $f(x)=|x|$
6. $f(x)=2^x$

Answers available at the end of this section.

Homework

Concept Check

1. Explain why $f(a+b)=f(a)+f(b)$ is not a valid application of the Distributive Law.
2. Nhat says that $\sin\left(\theta+\frac{\pi}{6}\right)=\frac{1}{2}+\sin(\theta)$. Is she correct? Explain.
3. Piyali says that $\tan\left(\theta+\frac{\pi}{4}\right)=\frac{1+\tan(\theta)}{1-\tan(\theta)}$. Is she correct? Explain.

Basic Skills

4. Show that $\cos(45^\circ + 45^\circ)$ is not equal to $\cos(45^\circ) + \cos(45^\circ)$.
5. Show that $\tan(60^\circ - 30^\circ)$ is not equal to $\tan(60^\circ) - \tan(30^\circ)$.
6. Use your calculator to verify that $\tan(87^\circ - 29^\circ)$ is not equal to $\tan(87^\circ) - \tan(29^\circ)$.
7. Use your calculator to verify that $\cos(52^\circ + 64^\circ)$ is not equal to $\cos(52^\circ) + \cos(64^\circ)$.
8. Use graphs to show that $\sin\left(x - \frac{\pi}{6}\right)$ is not equivalent to $\sin(x) - \sin\left(\frac{\pi}{6}\right)$.
9. Use graphs to show that $\tan\left(x + \frac{\pi}{4}\right)$ is not equivalent to $\tan(x) + \tan\left(\frac{\pi}{4}\right)$.

For the following exercises, use the Sum and Difference Identities to find the exact value. You may have need of the Ratio, Reciprocal, or Symmetry Identities as well.

10. $\cos(15^\circ)$
11. $\sin(15^\circ)$
12. $\tan(15^\circ)$
13. $\cos(75^\circ)$
14. $\cos(165^\circ)$
15. $\sin(165^\circ)$
16. $\sec(165^\circ)$
17. $\sin(105^\circ)$
18. $\csc(195^\circ)$
19. $\cot(255^\circ)$
20. $\tan(375^\circ)$
21. $\cos\left(\frac{13\pi}{12}\right)$
22. $\sin\left(\frac{11\pi}{12}\right)$
23. $\tan\left(\frac{13\pi}{12}\right)$
24. $\cos\left(\frac{7\pi}{12}\right)$
25. $\tan\left(\frac{17\pi}{12}\right)$
26. $\sin\left(\frac{\pi}{12}\right)$
27. $\cot\left(\frac{11\pi}{12}\right)$
28. $\csc\left(\frac{5\pi}{12}\right)$

29. $\sec\left(-\frac{\pi}{12}\right)$

30. $\sin\left(\frac{29\pi}{12}\right)$

For the following exercises, find the exact value of the expression without using a calculator.

31. $\cos(-10^\circ)\cos(35^\circ)+\sin(-10^\circ)\sin(35^\circ)$

32. $\cos\left(\frac{7\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)-\sin\left(\frac{7\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)$

33. $\sin\left(\frac{7\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{7\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)$

34. $\sin(80^\circ)\cos(55^\circ)+\cos(80^\circ)\sin(55^\circ)$

For the following exercises, find exact values for the trigonometric ratios. (Do not use a calculator!)

35. Suppose $\cos(\alpha)=\frac{3}{5}, \sin(\alpha)=\frac{4}{5}, \cos(\beta)=\frac{5}{13}$, and $\sin(\beta)=\frac{-12}{13}$. Evaluate the following.

    a. $\cos(\alpha+\beta)$

    b. $\sin(\alpha+\beta)$

    c. $\tan(\alpha+\beta)$

36. Suppose $\cos(\alpha)=\frac{-2}{3}, \sin(\alpha)=\frac{\sqrt{5}}{3}, \cos(\beta)=\frac{\sqrt{3}}{2}$, and $\sin(\beta)=\frac{-1}{2}$. Evaluate the following.

    a. $\cos(\alpha-\beta)$

    b. $\sin(\alpha-\beta)$

    c. $\tan(\alpha-\beta)$

37. If $\tan(t)=\frac{3}{4}$ and $\tan(s)=\frac{-7}{24}$, find exact values for the following.

    a. $\tan(s+t)$

    b. $\tan(s-t)$

38. If $\tan(x)=-3$ and $\tan(y)=-5$, find exact values for the following.

    a. $\tan(x+y)$

    b. $\tan(x-y)$

39. Suppose $\cos(\theta)=\frac{15}{17}$ and $\sin(\phi)=\frac{3}{5}$, where θ and ϕ are in quadrant I. Evaluate the following.

    a. $\cos(\theta+\phi)$

    b. $\tan(\theta-\phi)$

40. Suppose $\cos(\theta)=\frac{15}{17}$, where θ is in quadrant IV, and $\sin(\phi)=\frac{3}{5}$, where ϕ is in quadrant II. Evaluate the following.

    a. $\sin(\theta-\phi)$

    b. $\tan(\theta+\phi)$

41. If $\sin(\alpha) = \frac{12}{13}, \frac{\pi}{2} < \alpha < \pi$, and $\cos(\beta) = \frac{-3}{5}, \pi < \beta < \frac{3\pi}{2}$, find exact values for the following.

    a. $\sin(\alpha + \beta)$

    b. $\cos(\alpha + \beta)$

    c. $\tan(\alpha + \beta)$

    d. Sketch the angles α, β and $\alpha + \beta$.

42. If $\cos(\alpha) = \frac{3}{8}, \frac{3\pi}{2} < \alpha < 2\pi$, and $\sin(\beta) = \frac{-1}{4}, \pi < \beta < \frac{3\pi}{2}$, find exact values for the following.

    a. $\sin(\alpha - \beta)$

    b. $\cos(\alpha - \beta)$

    c. $\tan(\alpha - \beta)$

    d. Sketch the angles α, β and $\alpha - \beta$.

43. If α is a Quadrant IV angle with $\cos(\alpha) = \frac{\sqrt{5}}{5}$, and $\sin(\beta) = \frac{\sqrt{10}}{10}$, where $\frac{\pi}{2} < \beta < \pi$, find

    a. $\cos(\alpha + \beta)$

    b. $\sin(\alpha + \beta)$

    c. $\tan(\alpha + \beta)$

    d. $\cos(\alpha - \beta)$

    e. $\sin(\alpha - \beta)$

    f. $\tan(\alpha - \beta)$

44. If $\csc(\alpha) = 3$, where $0 < \alpha < \frac{\pi}{2}$, and β is a Quadrant II angle with $\tan(\beta) = -7$, find

    a. $\cos(\alpha + \beta)$

    b. $\sin(\alpha + \beta)$

    c. $\tan(\alpha + \beta)$

    d. $\cos(\alpha - \beta)$

    e. $\sin(\alpha - \beta)$

    f. $\tan(\alpha - \beta)$

45. If $\sin(\alpha) = \frac{3}{5}$, where $0 < \alpha < \frac{\pi}{2}$, and $\cos(\beta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \beta < 2\pi$, find

    a. $\sin(\alpha + \beta)$

    b. $\cos(\alpha - \beta)$

    c. $\tan(\alpha - \beta)$

46. If $\sec(\alpha) = -\frac{5}{3}$, where $\frac{\pi}{2} < \alpha < \pi$, and $\tan(\beta) = \frac{24}{7}$, where $\pi < \beta < \frac{3\pi}{2}$, find

a. $\csc(\alpha - \beta)$

b. $\sec(\alpha + \beta)$

c. $\cot(\alpha + \beta)$

47. If $\sin(\theta) = 0.2$ and $\cos(\theta) > 0$, find $\sin\left(\theta + \frac{\pi}{3}\right)$.

48. If $\cos(\theta) = 0.6$ and $\sin(\theta) < 0$, find $\cos\left(\theta + \frac{3\pi}{4}\right)$.

For the following exercises, use the Sum and Difference Identities to expand each expression. Simplify your answers completely.

49. $\sin(\theta - 270^\circ)$

50. $\cos(270^\circ + \theta)$

51. $\cos\left(t + \frac{\pi}{6}\right)$

52. $\sin\left(t - \frac{2\pi}{3}\right)$

53. $\tan\left(\beta - \frac{\pi}{6}\right)$

54. $\tan\left(\phi + \frac{\pi}{4}\right)$

55. Use the Sum Identities for sine and cosine to derive a formula for each expression. Then use graphs to verify your formula.

a. $\cos(\theta + 90^\circ)$

b. $\sin(\theta + 90^\circ)$

56. Use the Sum Identities for sine and cosine to derive a formula for each expression. Then use graphs to verify your formula.

a. $\cos(\theta + \pi)$

b. $\sin(\theta + \pi)$

57. Use the Difference Identities for sine and cosine to derive formulas for the following.

a. $\cos\left(\theta - \frac{\pi}{2}\right)$

b. $\sin\left(\theta - \frac{\pi}{2}\right)$

For the following exercises, graph one cycle of the function by hand.

58. $f(x) = \sin(4x)\cos(3x) - \cos(4x)\sin(3x)$

59. $f(x) = \sin(x)\cos(2x) + \cos(2x)\sin(x)$

60. $f(x) = \frac{1}{5}\cos(5x)\cos(2x) + \frac{1}{5}\sin(5x)\sin(2x)$

61. $f(x) = 2\cos(x)\cos(2x) - 2\sin(x)\sin(2x)$

Synthesis Questions

For the following exercises, simplify the function using the Sum and Difference Identities.

62. $f(x) = \sin\left(x + \frac{\pi}{2}\right)$

53. $f(x) = \cos\left(x - \frac{\pi}{4}\right)$

54. $f(x) = \tan(\pi - x)$

55. $f(x) = \sin\left(\frac{\pi}{6} - x\right)$

56. $f(x) = \cos\left(x + \frac{11\pi}{12}\right)$

57. $f(x) = \cos\left(\frac{2\pi}{3} - x\right)$

For the following exercises,
(a) Use graphs to decide if the equation is an identity.
(b) If the equation is not an identity, find a value of the variable that makes the equation false.

58. $\sin\left(\frac{\pi}{2} + \beta\right) = 1 + \sin(\beta)$

59. $\cos\left(\frac{\pi}{3} - \beta\right) = \cos\left(\beta - \frac{\pi}{3}\right)$

70. $\sin(A + 180^\circ) = -\sin(A)$

71. $\cos\left(\phi + \frac{\pi}{3}\right) = \frac{1}{2} + \cos(\phi)$

72. $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\sin(x) + \cos(x))$

73. $2\cos\left(x - \frac{\pi}{6}\right) = \sin(x) + \sqrt{3}\cos(x)$

74. $\sin\left(x - \frac{\pi}{3}\right) + \cos\left(x + \frac{\pi}{6}\right) = 0$

75. a. Use a graphing device to draw the graph of $f(x) = \sin(x) + \cos(x)$ using $-\pi \le x \le 2\pi$ and $-2 \le y \le 2$. Does the graph of this function appear to be a sinusoid? If so, approximate the amplitude and phase shift of the sinusoid. What is the period of this sinusoid.

    b. Use one of the sum identities to rewrite the expression $\sin\left(x + \frac{\pi}{4}\right)$. Then use the values of $\sin\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{\pi}{4}\right)$ to further rewrite the expression.

    c. Use the result from part (b) to show that the function $f(x) = \sin(x) + \cos(x)$ is indeed a sinusoidal function. What is its amplitude, phase shift, and period?

76. a. Use a graphing device to draw the graph of $g(x) = \sin(x) + \sqrt{3}\cos(x)$ using $-\pi \le x \le 2\pi$ and $-2.5 \le y \le 2.5$. Does the graph of this function appear to be a sinusoid? If so, approximate the amplitude and phase shift of the sinusoid. What is the period of this sinusoid.

    b. Use one of the sum identities to rewrite the expression $\sin\left(x + \frac{\pi}{3}\right)$. Then use the values of $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{3}\right)$ to further rewrite the expression.

    c. Use the result from part (b) to show that the function $g(x) = \sin(x) + \sqrt{3}\cos(x)$ is indeed a sinusoidal function. What is its amplitude, phase shift, and period?

Proofs

77. Use the Difference Identities for sine and cosine to prove the statement.

    a. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$

b. $\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$

For the following exercises, prove the identity.

78. $\cos(\theta-\pi)=-\cos(\theta)$

79. $\sin(\pi-\theta)=\sin(\theta)$

80. $\tan\left(\theta+\frac{\pi}{2}\right)=-\cot(\theta)$

81. $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin(\alpha)\cos(\beta)$

82. $\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos(\alpha)\sin(\beta)$

83. $\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos(\alpha)\cos(\beta)$

84. $\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin(\alpha)\sin(\beta)$

85. $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{1+\cot(\alpha)\tan(\beta)}{1-\cot(\alpha)\tan(\beta)}$

86. $\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\frac{1-\tan(\alpha)\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

87. $\frac{\tan(\alpha+\beta)}{\tan(\alpha-\beta)}=\frac{\sin(\alpha)\cos(\alpha)+\sin(\beta)\cos(\beta)}{\sin(\alpha)\cos(\alpha)-\sin(\beta)\cos(\beta)}$

88. $\frac{\sin(t+h)-\sin(t)}{h}=\cos(t)\left(\frac{\sin(h)}{h}\right)+\sin(t)\left(\frac{\cos(h)-1}{h}\right)$

89. $\frac{\cos(t+h)-\cos(t)}{h}=\cos(t)\left(\frac{\cos(h)-1}{h}\right)-\sin(t)\left(\frac{\sin(h)}{h}\right)$

90. $\frac{\tan(t+h)-\tan(t)}{h}=\left(\frac{\tan(h)}{h}\right)\left(\frac{\sec^2(t)}{1-\tan(t)\tan(h)}\right)$

91. $\cot(A+B)=\frac{\cot(A)\cot(B)-1}{\cot(A)+\cot(B)}$

92. $\cot(A-B)=\frac{\cot(A)\cot(B)+1}{\cot(B)-\cot(A)}$

93. $\tan(\theta+45^\circ)=\frac{1+\tan(\theta)}{1-\tan(\theta)}$

94. $\frac{\cos(A+B)}{\sin(A)\cos(B)}=\cot(A)-\tan(B)$

95. $\cot(A)+\cot(B)=\frac{\sin(A+B)}{\sin(A)\sin(B)}$

96. $\frac{\sin(A-B)}{\sin(A+B)}=\frac{\cot(B)-\cot(A)}{\cot(B)+\cot(A)}$

97. $\cos(A+B+C)=\cos(A)\cos(B)\cos(C)-\cos(A)\sin(B)\sin(C)-\sin(A)\cos(B)\sin(C)-\sin(A)\sin(B)\cos(C)$

98. $\tan(A+B+C)=\frac{\tan(A)+\tan(B)+\tan(C)-\tan(A)\tan(B)\tan(C)}{1-\tan(B)\tan(C)-\tan(A)\tan(C)-\tan(A)\tan(B)}$

99. Verify the Cofunction Identities for tangent, secant, cosecant and cotangent.

90. Verify the Difference Identities for sine and tangent.

Challenge Problems

91. Prove the identity

$$\sin(\theta) + \cos(\theta) = \sqrt{2}\sin(\theta + 45^\circ).$$

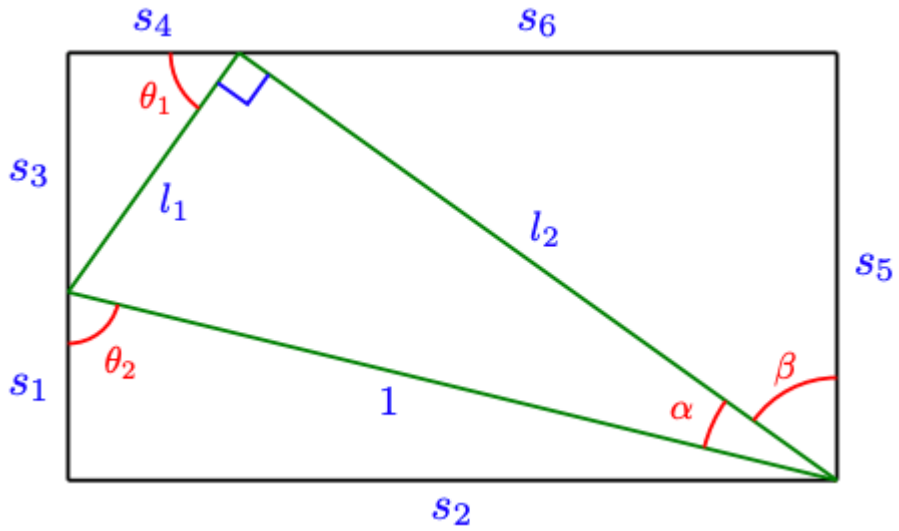
Explain why this shows that

$$-\sqrt{2} \leq \sin(\theta) + \cos(\theta) \leq \sqrt{2}$$

for all angles θ. For which θ between 0° and 360° would $\sin(\theta) + \cos(\theta)$ be the largest?

For the following exercises, verify the Addition and Subtraction Identities for acute angles.

92. The figure below shows a right triangle inscribed in a rectangle.



a. Label the legs l_1 and l_2 of the right triangle with their lengths.

b. Explain why $\theta_1 = \beta$ and $\theta_2 = \alpha + \beta$. Label the diagram with these angles.

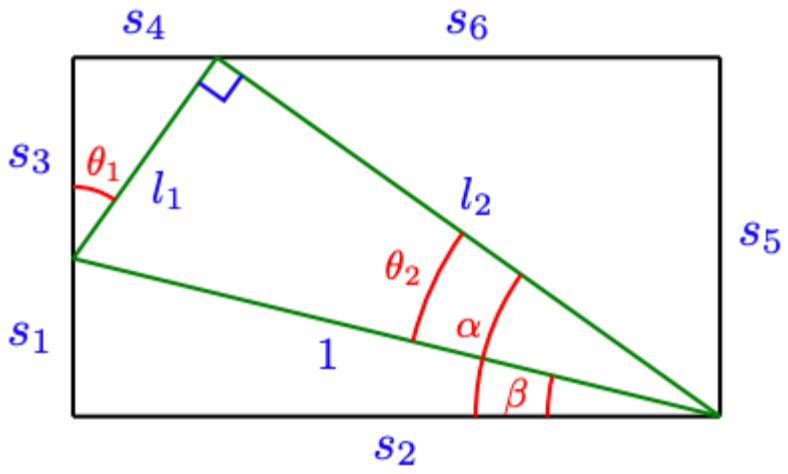
c. Label the legs s_1 and s_2 of the bottom triangle with their lengths.

d. Label the legs s_3 and s_4 of the top left triangle with their lengths.

e. Label the legs s_5 and s_6 of the top right triangle with their lengths.

f. Use the fact that the opposite sides of a rectangle are equal to state the addition formulas for sine and cosine.

93. The figure below shows a right triangle inscribed in a rectangle.



a. Label the legs l_1 and l_2 of the right triangle with their lengths.

b. Explain why $\theta_1 = \beta$ and $\theta_2 = \alpha - \beta$. Label the diagram with these angles.

c. Label the legs s_1 and s_2 of the bottom triangle with their lengths.

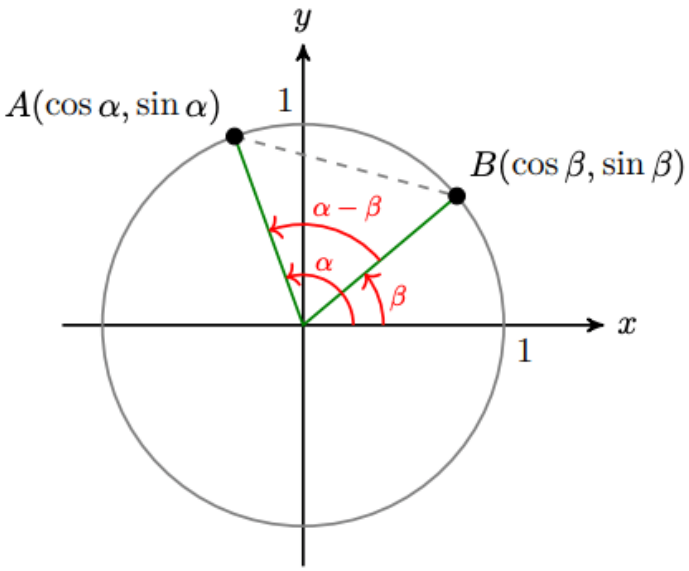
d. Label the legs s_3 and s_4 of the top left triangle with their lengths.

e. Label the legs s_5 and s_6 of the top right triangle with their lengths.

f. Use the fact that the opposite sides of a rectangle are equal to state the subtraction formulas for sine and cosine.

94. Follow the steps to prove the difference of angles formula for cosine,

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$



a. Write an expression for $(AB)^2$, the square of the distance between the points A and B, using the Law of Cosines for $\triangle AOB$.

b. Write another expression for $(AB)^2$ using the Distance Formula and the coordinates of A and B.

c. Equate the two expressions for $(AB)^2$ you obtained in (a) and (b). Simplify the equation to obtain $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

95. For any positive angles A, B, and C such that $A+B+C=90^{\circ}$, show that

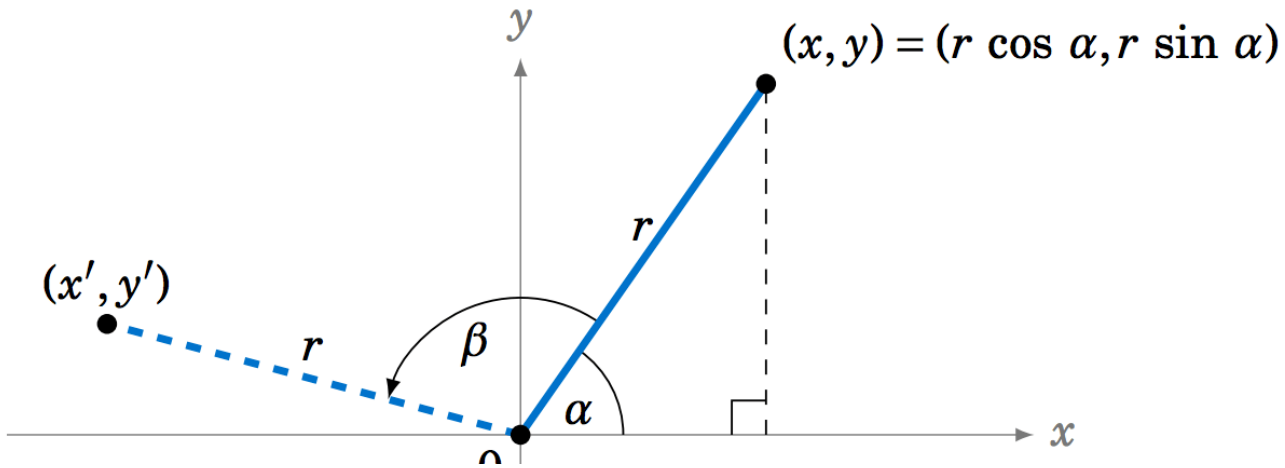
$$\tan(A)\tan(B)+\tan(B)\tan(C)+\tan(C)\tan(A)=1.$$

96. Prove the identity

$$\sin(A+B)\cos(B)-\cos(A+B)\sin(B)=\sin(A).$$

Note that the right side depends only on A, while the left side depends on both A and B.

97. A line segment of length $r>0$ from the origin to the point (x,y) makes an angle α with the positive x-axis, so that $(x,y)=(r\cos(\alpha), r\sin(\alpha))$, as in the figure below. What are the endpoint's new coordinates (x',y') after a counterclockwise rotation by an angle β? Your answer should be in terms of r, α, and β.



Answers

Skills Refresher

1. a. 8, 11, 17
   b. Not true
2. a. 4, 9, 25
   b. Not true
3. a. $\sqrt{2}, \sqrt{3}, \sqrt{5}$
   b. Not true
4. a. $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$
   b. Not true
5. a. 2, 3, 5
   b. Not true
6. a. 4, 8, 32
   b. Not true

This page titled 9.2.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

9.3: Double-Angle Identities

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The Double Angle Identities

Several other useful identities can be derived from the Sum and Difference Identities. The Double-Angle Identities are a special case of the Sum Identities, where $\alpha = \beta$.

Theorem: Double-Angle Identities

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\
&= 2\cos^2(\theta) - 1 \\
&= 1 - 2\sin^2(\theta) \\
\tan(2\theta) &= \frac{2\tan(\theta)}{1 - \tan^2(\theta)}
\end{aligned}$$

Proof available at the end of this section.

Advice: Memorize The Sine and Cosine Versions of These Identities

The Double-Angle Identities for the sine and cosine are needed throughout the rest of Trigonometry, Calculus, and Differential Equations. While not as prevalent as the Pythagorean Theorem or the Quadratic Formula, the need to commit them to memory and their importance cannot be overstated. The tangent version is less common and can be derived from the sine and cosine versions if needed.

You can also justify the identities to yourself by graphing both sides of the formula to see that the graphs are identical.

Caution: Don't Factor Out of Functions!

The first thing we can learn from the Double-Angle Identities is that $\sin(2\theta)$ is not equal to $2\sin(\theta)$! You can check this very easily by choosing a value for θ, say 45°. Then

$$\sin(2\theta) = \sin(90^\circ) = 1$$

but

$$2\sin(\theta) = 2\sin(45^\circ) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

and $\sqrt{2} \neq 1$. Similarly, $\cos(2\theta) \neq 2\cos(\theta)$ and $\tan(2\theta) \neq 2\tan(\theta)$.

Finding Exact Values of Trigonometric Functions Involving Double Angles

We now reach a point in Trigonometry where we are told the value of a trigonometric function of θ, but we are interested in the value of a trigonometric function of 2θ. In these instances, the Double-Angle Identities (which, again, are special sub-cases of the Sum and Difference Identities) are what we need.

Example 9.3.1

Find $\sin(2\theta)$ for the angle θ shown in Figure 9.3.1.

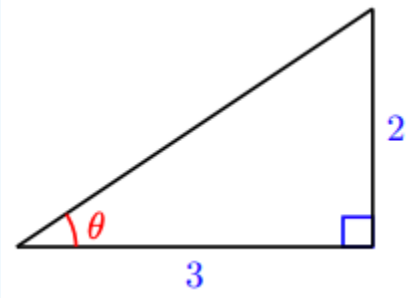


Figure 9.3.1

Solution available at the end of this section.

Checkpoint 9.3.1

Find $\cos(2\theta)$ and $\tan(2\theta)$ for the angle θ shown in Example 9.3.1.

Answer available at the end of this section.

Example 9.3.2

Given that $\cot(\theta) = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$
b. $\cos(2\theta)$
c. $\tan(2\theta)$

Solutions available at the end of this section.

A note about Example $9.3.2c$ is in order. As we have seen throughout Trigonometry, several ways to approach a problem usually exist. It's easy to get so intent on using one method with a problem that we blind ourselves to simpler approaches. For example, we could have used the Ratio Identities to get

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{-24/25}{-7/25} = \frac{24}{7}.$$

Checkpoint 9.3.2

Given $\sin(\alpha) = \frac{5}{8}$, with α in quadrant I, find $\cos(2\alpha)$.

Answer available at the end of this section.

We can work with algebraic expressions instead of numerical values for the trigonometric ratios.

Example 9.3.3

Use Figure $9.3.3$ to express $\cos(2\phi)$ in terms of a.

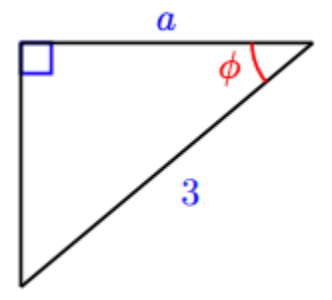


Figure $9.3.3$

Solution available at the end of this section.

Checkpoint 9.3.3

For the triangle in Example $9.3.3$, find expressions for $\sin(2\phi)$ and $\tan(2\phi)$.

Answer available at the end of this section.

Example 9.3.4

Find an expression for $\cos(2\beta)$ if you know that $\cos(\beta) = \frac{x}{4}$.

Solution available at the end of this section.

Checkpoint 9.3.4

Find an expression for $\cos(2\alpha)$ if you know that $\sin(\alpha) = \frac{6}{w}$.

Answer available at the end of this section.

Revisiting Proofs of Identities

Establishing identities using the Double-Angle Identities is performed using the same steps we used previously.

Example 9.3.5

Verify the following identity.

$$1 + \sin(2\theta) = (\sin(\theta) + \cos(\theta))^2$$

Solution available at the end of this section.

The process in Example 9.3.5 is not complicated, as long as we recall the perfect square formula from algebra:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2,$$

where $a = \sin(\theta)$ and $b = \cos(\theta)$. Part of being successful in Mathematics is the ability to recognize patterns. While the terms or symbols may change, the Algebra remains consistent.

Checkpoint 9.3.5

Verify the identity.

$$\cos^4(\theta) - \sin^4(\theta) = \cos(2\theta).$$

Answer available at the end of this section.

Example 9.3.6

Prove the identity.

$$\tan(2\theta) = \frac{2}{\cot(\theta) - \tan(\theta)}$$

Solution available at the end of this section.

In Example 9.3.6, if we had chosen to work on the right side, we would have worked backward to arrive at the equivalency. While this is possible (and not too tricky), I like taking advantage of obvious identities. There are usually several paths to a desired result when using the identities to simplify a trigonometric expression or solve a trigonometric equation.

Checkpoint 9.3.6

Prove the identity.

$$\cos(2\theta)\cos\theta = \cos^3(\theta) - \cos(\theta)\sin^2(\theta)$$

Answer available at the end of this section.

Revisiting Graphs of Trigonometric Functions

As with the previous section, we use this opportunity to expand the types of trigonometric functions we can graph.

Example 9.3.7

Graph the function.

$$f(x) = -8\cos^2(3x) + 4$$

Solution available at the end of this section.

Checkpoint 9.3.7

Graph the function.

$$g(x) = \frac{6\tan\left(\frac{\pi}{4}x\right)}{1 - \tan^2\left(\frac{\pi}{4}x\right)}$$

Answer available at the end of this section.

Miscellaneous Topics: What Are Your Top Degree Choices?

I have seen all the statistics regarding the correlation between income and education. In general, the higher your education, the higher your income.

1. Name three benefits, besides money, that a higher degree can give you.
2. What are your top three choices for degrees (e.g., English, Mathematics, Psychology)?

Success in College: Affirmations

Do you ever tell yourself (or others) that you're not good at math? This is called negative self-talk. The more you talk that way, the more likely your subconscious will believe it - and you will do poorly in math.

You can counteract years of negative self-talk by telling yourself with conviction that you're good at math. It might seem strange to state that something that hasn't happened yet is true, but it works! Such statements are called affirmations.

There are four guiding principles for getting the most out of affirmations:

I. Say affirmations that imply that the desired event is currently happening. For example, say, "I am good at algebra," not "I will be good at algebra."
II. Say affirmations that you are making progress toward the desired result. For example, say, "I am good at trigonometry, and I continue to improve at it."
III. Say affirmations in the positive rather than the negative. For example, say "I attend each class," not "I don't cut classes."
IV. Say affirmations with conviction.

1. Do you practice affirmations? If so, how?
2. After reading this, will you try to practice affirmations? Where else in life could affirmations be put to work?

Answers

Section Preview

There is not much I can clarify here that you don't already know. I suppose the only thing that I should repeat (from the text) is that the Sum and Difference Identities, and the Double-Angle Identities must be memorized (at least for the sine and cosine). Do *not* fall into the trap of allowing students to have these on a note card during exams — they will not have such luxury in Calculus.

Section Preview

- **Arithmetic**
  - Simplifying Fractions and Mixed Numbers: The three forms of $\cos(2\theta)$ allow selection based on what information is given, with the Pythagorean Identity substitutions requiring basic arithmetic and fraction simplification.
- **Solving Equations**
  - Solving Quadratic Equations by Factoring: Solving for a trig function value when another is given (Examples 1–4) requires solving a linear or quadratic equation in the trig function — particularly solving quadratic equations by factoring when the double-angle substitution produces a quadratic.

Section Preview

- **Arithmetic**
  - Simplifying Fractions and Mixed Numbers: The three forms of $\cos(2\theta)$ allow selection based on what information is given, with the Pythagorean Identity substitutions requiring basic arithmetic and fraction simplification.
- **Solving Equations**
  - Solving Quadratic Equations by Factoring: Solving for a trig function value when another is given (Examples 1–4) requires solving a linear or quadratic equation in the trig function — particularly solving quadratic equations by factoring when the double-angle substitution produces a quadratic.

Section Preview

- Use a Double-Angle Identity to find the exact value of a trigonometric function.
- Prove an identity using a Double-Angle Identity.
- Use a Double-Angle Identity to graph an equation.

Theorem: Double-Angle Identities

Deriving the Double-Angle Identity for sine begins with the Sum Identity,

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta).$$

If we let $\alpha=\beta=\theta$, then we have

$$\begin{aligned}\sin(\theta+\theta) &= \sin(\theta)\cos(\theta)+\cos(\theta)\sin(\theta)\\ \Longrightarrow\quad \sin(2\theta) &= 2\sin(\theta)\cos(\theta)\end{aligned}$$

Deriving the Double-Angle Identity for cosine gives us three options. First, starting from the Sum Identity,

$$\cos(\alpha+\beta)=\cos\ (\alpha)\ \cos\ (\beta)-\sin\ (\alpha)\ \sin\ (\beta),$$

and letting $\alpha=\beta=\theta$, we have

$$\begin{aligned}\cos(\theta+\theta) &= \cos(\theta)\cos(\theta)-\sin(\theta)\sin(\theta)\\ \Longrightarrow\quad \cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta)\end{aligned}$$

Using the Pythagorean Identities, we can expand this Double-Angle Identity for cosine and get two more variations. The first variation is

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta)\\ &= (1-\sin^2(\theta))-\sin^2(\theta)\\ &= 1-2\sin^2(\theta)\end{aligned}$$

The second variation is

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta)\\ &= \cos^2(\theta)-(1-\cos^2(\theta))\\ &= 2\cos^2(\theta)-1\end{aligned}$$

Similarly, to derive the Double-Angle Identity for the tangent, replacing $\alpha=\beta=\theta$ in the sum formula gives

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\\ \Longrightarrow\quad \tan(\theta+\theta) &= \frac{\tan(\theta)+\tan(\theta)}{1-\tan(\theta)\tan(\theta)}\\ \Longrightarrow\quad \tan(2\theta) &= \frac{2\tan(\theta)}{1-\tan^2(\theta)}\end{aligned}$$

Example 9.3.1

We start by using the Pythagorean Theorem to find the hypotenuse of the triangle.

$$c^2=2^2+3^2=13.$$

So $c=\sqrt{13}$. Thus, $\cos(\theta)=\frac{3}{\sqrt{13}}$ and $\sin(\theta)=\frac{2}{\sqrt{13}}$. Now we can use these values in the Double-Angle Identity to find $\sin(2\theta)$.

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\ &= 2\left(\frac{2}{\sqrt{13}}\right)\left(\frac{3}{\sqrt{13}}\right)\\ &= \frac{12}{13}\end{aligned}$$

Checkpoint 9.3.1

$\cos(2\theta)=\dfrac{5}{13}, \quad \tan(2\theta)=\dfrac{12}{5}$

Example 9.3.2

If we draw a triangle to reflect the information given, we can find the values needed to help us. We are given $\cot(\theta)=-\frac{3}{4}$, such that θ is in quadrant II. The

cotangent of an angle is equal to the adjacent side over the opposite side, and because θ is in the second quadrant, the adjacent side is on the x-axis and is negative. We use the Pythagorean Theorem to find the length of the hypotenuse:

$$\begin{aligned} (-4)^2 + (3)^2 &= c^2 \\ \Longrightarrow \quad 16 + 9 &= c^2 \\ \Longrightarrow \quad 25 &= c^2 \\ \Longrightarrow \quad c &= 5 \end{aligned}$$

Now we can draw the situation appropriately.

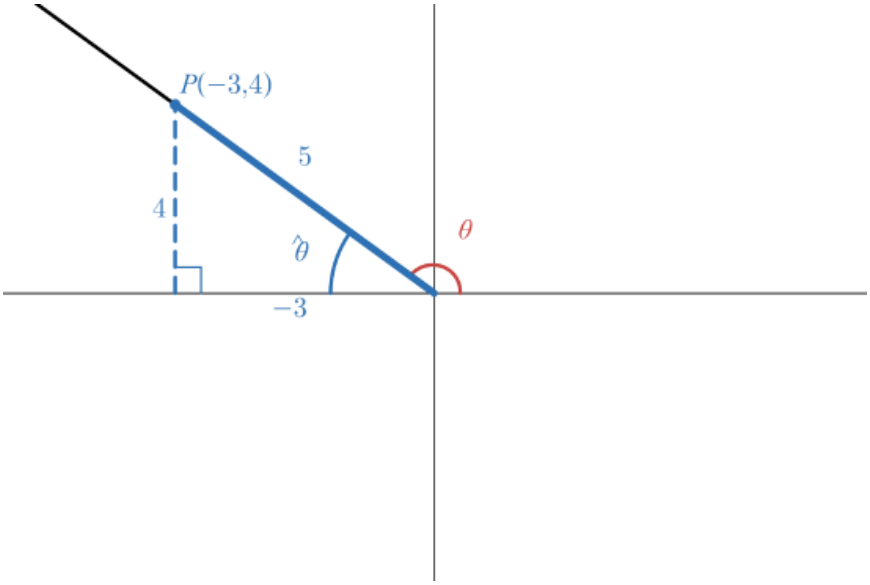


Figure 9.3.2**:** *Not drawn to scale!*

When drawn like this, you can use the coordinate definition of the trigonometric functions or the right triangle definition using $\hat{\theta}$ and the fact that θ terminates in QII — it's your choice.

a. Let's begin by writing the Double-Angle Identity for sine.

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

We see that we need to find $\sin(\theta)$ and $\cos(\theta)$. Based on our drawing, we see that the hypotenuse equals 5, so $\sin(\theta) = \frac{4}{5}$, and $\cos(\theta) = -\frac{3}{5}$. Substitute these values into the equation and simplify. Thus,

$$\begin{aligned} \sin(2\theta) &= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) \\ &= -\frac{24}{25} \end{aligned}$$

b. Write the Double-Angle Identity for cosine.

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

Again, substitute the values of the sine and cosine into the equation and simplify.

$$\begin{aligned} \cos(2\theta) &= \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 \\ &= \frac{9}{25} - \frac{16}{25} \\ &= -\frac{7}{25} \end{aligned}$$

c. Write the Double-Angle Identity for tangent.

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

In this formula, we need the tangent. Using the Reciprocal Identity, we have $\tan(\theta) = -\frac{4}{3}$. Substitute this value into the equation, and simplify.

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2}\\ &= \frac{-\frac{8}{3}}{1-\frac{16}{9}}\\ &= -\frac{8}{3}\left(-\frac{9}{7}\right)\\ &= \frac{24}{7}\end{aligned}$$

Checkpoint 9.3.2

$\cos(2\alpha) = \frac{7}{32}$

Example 9.3.3

We use the Pythagorean Theorem to find an expression for the third side of the triangle.

$$\begin{aligned}a^2+b^2 &= 3^2\\ \Longrightarrow \qquad b &= \sqrt{9-a^2}\end{aligned}$$

Now we can write expressions for the sine and cosine of ϕ.

$$\cos(\phi) = \frac{a}{3} \quad \text{and} \quad \sin(\phi) = \frac{\sqrt{9-a^2}}{3}$$

Finally, we substitute these expressions into the Double-Angle Identity.

$$\begin{aligned}\cos(2\phi) &= \cos^2(\phi) - \sin^2(\phi)\\ &= \left(\frac{a}{3}\right)^2 - \left(\frac{\sqrt{9-a^2}}{3}\right)^2\\ &= \frac{a^2}{9} - \frac{9-a^2}{9}\\ &= \frac{2a^2-9}{9}\end{aligned}$$

Checkpoint 9.3.3

$\sin(2\phi) = \frac{2a\sqrt{9-a^2}}{9}, \quad \tan(2\phi) = \frac{2a\sqrt{9-a^2}}{2a^2-9}$

Example 9.3.4

We use the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$.

$$\begin{aligned}\cos(2\beta) &= 2\cos^2(\beta) - 1\\ &= 2\left(\frac{x}{4}\right)^2 - 1\\ &= \frac{2x^2}{16} - 1\\ &= \frac{2x^2-16}{16}\\ &= \frac{x^2-8}{8}\end{aligned}$$

Checkpoint 9.3.4

$\frac{w^2-76}{w^2}$

Example 9.3.5

We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}\text{RHS} &= (\sin(\theta)+\cos(\theta))^2\\ &= \sin^2(\theta) + 2\sin(\theta)\cos(\theta) + \cos^2(\theta) && \text{(distribution)}\\ &= (\sin^2(\theta)+\cos^2(\theta)) + 2\sin(\theta)\cos(\theta) && \text{(Commutative and Associative Properties of Addition)}\\ &= 1 + 2\sin(\theta)\cos(\theta) && \text{(Pythagorean Identities)}\\ &= 1 + \sin(2\theta) && \text{(Double-Angle Identities)}\\ &= \text{LHS}\end{aligned}$$

Checkpoint 9.3.5

$\cos^4(\theta) - \sin^4(\theta) = (\cos^2(\theta) - \sin^2(\theta))(\cos^2(\theta) + \sin^2(\theta)) = \cos(2\theta)\cdot 1 = \cos(2\theta)$

Example 9.3.6

Despite the right side looking more complicated, the obvious Double-Angle Identity on the left side tells us that we should at least start on that side and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned}
\text{LHS} &= \tan(2\theta) \\
&= \frac{2\tan(\theta)}{1-\tan^2(\theta)} && \text{(Double-Angle Identity)} \\
&= \frac{2\tan(\theta)\left(\frac{1}{\tan(\theta)}\right)}{(1-\tan^2(\theta))\left(\frac{1}{\tan(\theta)}\right)} && \text{(multiply by a term that results in desired numerator)} \\
&= \frac{2}{\frac{1}{\tan(\theta)}-\frac{\tan^2(\theta)}{\tan(\theta)}} \\
&= \frac{2}{\cot(\theta)-\tan(\theta)} && \text{(Reciprocal Identity)} \\
&= \text{RHS}
\end{aligned}$$

Checkpoint 9.3.6

$\cos(2\theta)\cos(\theta) = (\cos^2(\theta)-\sin^2(\theta))\cos(\theta) = \cos^3(\theta)-\cos(\theta)\sin^2(\theta)$

Example 9.3.7

Careful observation and a little pre-factoring will reveal that this function contains a Double-Angle Identity.

$$\begin{aligned}
f(x) &= -8\cos^2(3x)+4 \\
&= -4(2\cos^2(3x)-1) && \text{(factoring out the GCF)} \\
&= -4\cos(6x) && \text{(Double-Angle Identity)}
\end{aligned}$$

This is a reflected cosine function with amplitude 4, period $\frac{\pi}{3}$, step size $\frac{\pi}{12}$, and no phase shift or vertical shift.

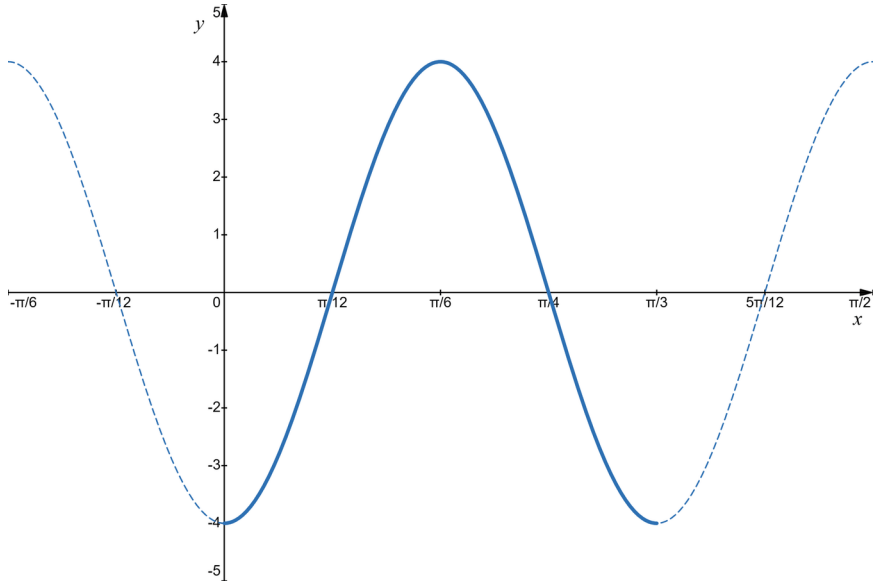


Figure 9.3.4

Checkpoint 9.3.7

$g(x) = 3\tan\left(\frac{\pi}{2}x\right)$.

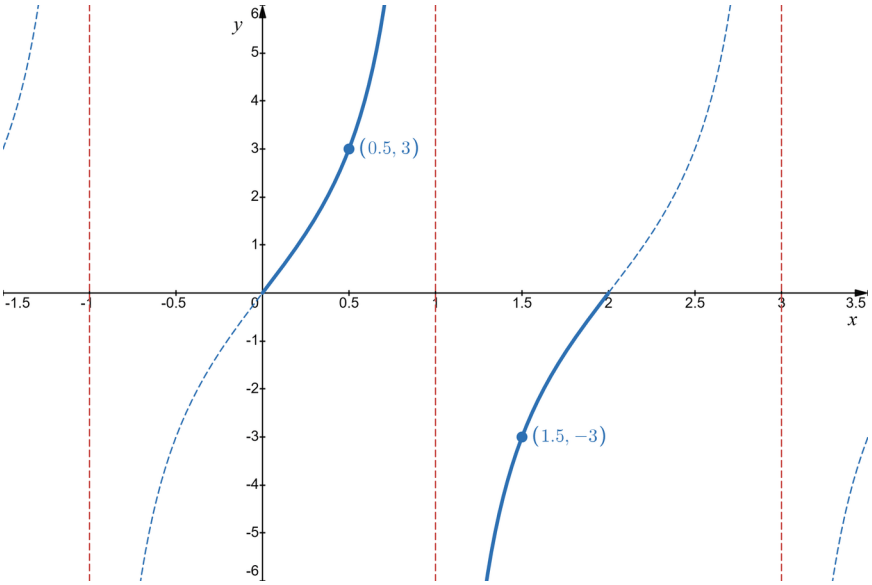


Figure 9.3.5

This page titled 9.3: Double-Angle Identities is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

9.3.1: Resources and Key Concepts

Resources

Videos

- Double-Angle Identities
  - Double-Angle Identities - Theorem and Proof
  - Double-Angle Identities - Using the Double-Angle Identities
  - Double-Angle Identities - Proving Trigonometric Identities (Part 4)

Key Concepts

Theorems

- **Double-Angle Identities:**
  - $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
  - $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
  - $\cos(2\theta) = 2\cos^2(\theta) - 1$
  - $\cos(2\theta) = 1 - 2\sin^2(\theta)$
  - $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

Common Mistakes

- **Don't Factor Out of Functions:** A common mistake is to treat a trigonometric function as if it distributes over its argument. For example, $\sin(2\theta)$ is not equal to $2\sin(\theta)$. The Double-Angle Identities must be used to correctly evaluate or expand such expressions.
- **Ignoring Simpler Approaches:** It is easy to get so focused on using one specific method that you miss a simpler path. For example, after finding $\sin(2\theta)$ and $\cos(2\theta)$, the easiest way to find $\tan(2\theta)$ is often to use the Ratio Identity rather than the Double-Angle Identity for tangent.

This page titled 9.3.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

9.3.2: Homework

Reading Questions

1. How are the Double-Angle Identities derived from the Sum and Difference Identities?
2. What is the Double-Angle Identity for $\sin(2\theta)$?
3. List the three different forms of the Double-Angle Identity for $\cos(2\theta)$.
4. If you are given the value of $\cos(\theta)$ and need to find $\cos(2\theta)$, which of the three forms of the cosine double-angle identity would be the most direct to use?
5. Provide a numerical example that demonstrates that $\sin(2\theta) \neq 2\sin(\theta)$.
6. If you need to find the value of $\sin(2\theta)$, what two pieces of information about θ must you know?
7. How can a Double-Angle Identity be used to help graph a function like $f(x) = -8\cos^2(3x) + 4$?
8. When proving an identity, what should you look for if you see an expression like $\cos(2x)$?

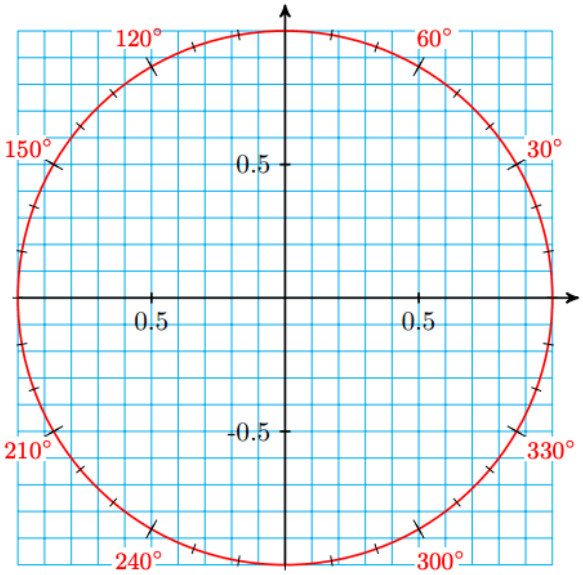
Homework

Concept Check

1. Provide an example to show that doubling an angle does not double its sine or cosine.

Basic Skills

For the following exercises, use the unit circle to estimate trigonometric values. Then verify with your calculator.



2. Does $\sin(2 \cdot 90^\circ) = 2\sin(90^\circ)$?
3. Does $\cos(2 \cdot 25^\circ) = 2\cos(25^\circ)$?
4. Does $\tan(2 \cdot 70^\circ) = 2\tan(70^\circ)$?
5. Does $\tan(2 \cdot 100^\circ) = 2\tan(100^\circ)$?

For the following exercises, verify that each statement is true

6. $\sin(90^\circ) = 2\sin(45^\circ)\cos(45^\circ)$

7. $\sin(60^\circ) = 2\sin(30^\circ)\cos(30^\circ)$

8. $\cos(60^\circ) = \cos^2(30^\circ) - \sin^2(30^\circ)$

9. $\tan(60^\circ) = \dfrac{2\tan(30^\circ)}{1-\tan^2(30^\circ)}$

For the following exercises, determine if the statement is true or false. Explain your answer.

10. If $\cos(\alpha) = 0.32$, then $\cos(2\alpha) = 0.64$.

11. If $\sin(\phi) = r$, then $\sin(2\phi) = 2r$.

For the following exercises, use the Double Angle Identities to simplify the expression. Leave your answer in exact form.

12. $2\sin(34^\circ)\cos(34^\circ)$

13. $\cos^2\left(\dfrac{\pi}{10}\right) - \sin^2\left(\dfrac{\pi}{10}\right)$

14. $1 - 2\sin^2\left(\dfrac{\pi}{16}\right)$

15. $2\cos^2(18^\circ) - 1$

16. $\cos^2(3\theta) - \sin^2(3\theta)$

17. $2\sin(2\alpha)\cos(2\alpha)$

18. $2\sin(5t)\cos(5t)$

19. $\cos^2(4w) - \sin^2(4w)$

20. $\dfrac{2\tan(64^\circ)}{1-\tan^2(64^\circ)}$

21. $\dfrac{2\tan\left(\frac{\pi}{5}\right)}{1-\tan^2\left(\frac{\pi}{5}\right)}$

22. $2\cos^2(2\beta) - 1$

23. $1 - 2\sin^2(6s)$

24. $\cos^2(28^\circ) - \sin^2(28^\circ)$

25. $2\cos^2(37^\circ) - 1$

26. $1 - 2\sin^2(17^\circ)$

27. $\cos^2(37^\circ) - \sin^2(37^\circ)$

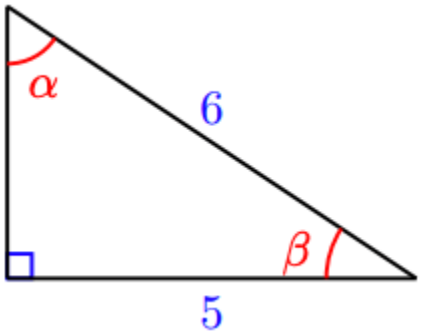
28. $\cos^2(9x) - \sin^2(9x)$

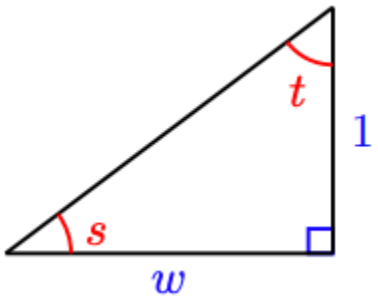
29. $\cos^2(6x) - \sin^2(6x)$

30. $4\sin(8x)\cos(8x)$

31. $6\sin(5x)\cos(5x)$

For the following exercises, use the figures to find the trigonometric ratios.





32. a. $\sin(\alpha)$

    b. $\cos(\alpha)$

    c. $\tan(\alpha)$

    d. $\sin(2\alpha)$

    e. $\cos(2\alpha)$

    f. $\tan(2\alpha)$

33. a. $\sin(\beta)$

    b. $\cos(\beta)$

    c. $\tan(\beta)$

    d. $\sin(2\beta)$

    e. $\cos(2\beta)$

    f. $\tan(2\beta)$

34. a. $\sin(s)$

    b. $\cos(s)$

    c. $\tan(s)$

    d. $\sin(2s)$

    e. $\cos(2s)$

    f. $\tan(2s)$

35. a. $\sin(t)$

    b. $\cos(t)$

    c. $\tan(t)$

    d. $\sin(2t)$

    e. $\cos(2t)$

f. $\tan(2t)$

36. Suppose $\cos(\theta) = \frac{12}{13}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Compute exact values for the following.

a. $\sin(\theta)$

b. $\sin(2\theta)$

c. $\cos(2\theta)$

d. $\tan(2\theta)$

e. Sketch the angles θ and 2θ.

37. Suppose $\sin(\phi) = \frac{5}{6}$ and $\frac{\pi}{2} < \phi < \pi$. Compute exact values for the following.

a. $\cos(\phi)$

b. $\sin(2\phi)$

c. $\cos(2\phi)$

d. $\tan(2\phi)$

e. Sketch the angles ϕ and 2ϕ.

38. If $\tan(u) = -4$ and \(270^{\circ}

a. $\tan(2u)$

b. $\cos(2u)$

c. $\sin(2u)$

39. If $\tan(v) = \frac{2}{3}$ and \(180^{\circ}

a. $\tan(2v)$

b. $\cos(2v)$

c. $\sin(2v)$

For the following exercises, use the given information about θ to find the exact values of $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

40. $\sin(\theta) = -\frac{7}{25}$ where $\frac{3\pi}{2} < \theta < 2\pi$

41. $\cos(\theta) = \frac{28}{53}$ where $0 < \theta < \frac{\pi}{2}$

42. $\tan(\theta) = \frac{12}{5}$ where $\pi < \theta < \frac{3\pi}{2}$

43. $\csc(\theta) = 4$ where $\frac{\pi}{2} < \theta < \pi$

44. $\cos(\theta) = \frac{3}{5}$ where $0 < \theta < \frac{\pi}{2}$

45. $\sin(\theta) = -\frac{4}{5}$ where $\pi < \theta < \frac{3\pi}{2}$

46. $\cos(\theta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \theta < 2\pi$

47. $\sin(\theta) = \frac{5}{13}$ where $\frac{\pi}{2} < \theta < \pi$

48. $\sec(\theta) = \sqrt{5}$ where $\frac{3\pi}{2} < \theta < 2\pi$

49. $\tan(\theta) = -2$ where $\frac{\pi}{2} < \theta < \pi$

50. Suppose θ is a Quadrant I angle with $\sin(\theta) = x$. Verify the following formulas

    a. $\cos(\theta) = \sqrt{1 - x^2}$

    b. $\sin(2\theta) = 2x\sqrt{1 - x^2}$

    c. $\cos(2\theta) = 1 - 2x^2$

51. Suppose θ is a Quadrant I angle with $\tan(\theta) = x$. Verify the following formulas

    a. $\cos(\theta) = \frac{1}{\sqrt{x^2 + 1}}$

    b. $\sin(\theta) = \frac{x}{\sqrt{x^2 + 1}}$

    c. $\sin(2\theta) = \frac{2x}{x^2 + 1}$

    d. $\cos(2\theta) = \frac{1 - x^2}{x^2 + 1}$

52. If $\sin(\theta) = \frac{x}{2}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\cos(2\theta)$ in terms of x.

53. If $\tan(\theta) = \frac{x}{7}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

For the following exercises, graph one cycle of the given function by hand.

54. $f(x) = 4\sin(3x)\cos(3x)$

55. $g(x) = \frac{2}{3}\cos^2\left(\frac{\pi}{3}x\right) - \frac{1}{2}$

56. $h(x) = -\frac{5}{7}\sin\left(\frac{1}{3}x\right)\cos\left(\frac{1}{3}x\right)$

57. $k(x) = \frac{3\tan\left(\frac{1}{2}x\right)}{2 - 2\tan^2\left(\frac{1}{2}x\right)}$

58. $m(x) = 2\cos^2(5x) - 2\sin^2(5x)$

Proofs

For the following exercises, prove the given identity for any right triangle $\triangle ABC$, where C is a right angle.

59. $\sin(A - B) = \cos(2B)$

60. $\cos(A - B) = \sin(2A)$

61. $\sin(2A) = \frac{2ab}{c^2}$

62. $\cos(2A) = \frac{b^2 - a^2}{c^2}$

53. $\tan(2A) = \dfrac{2ab}{b^2 - a^2}$

54. Prove the Double-Angle Identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

55. Prove the Double-Angle Identity $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.

For the following exercises,
(a) Use graphs to decide if the equation is an identity.
(b) If the equation is not an identity, find a value of the variable that makes the equation false.

56. $\cos(4\theta) = 4\cos(\theta)$

57. $\cos(2x) = (\cos(x) + \sin(x))(\cos(x) - \sin(x))$

For the following exercises, prove the identity. Assume all quantities are defined.

58. $(\cos(\theta) + \sin(\theta))^2 = 1 + \sin(2\theta)$

59. $(\cos(\theta) - \sin(\theta))^2 = 1 - \sin(2\theta)$

60. $\dfrac{\sin(2\theta)}{\sin(\theta)} - \dfrac{\cos(2\theta)}{\cos(\theta)} = \sec(\theta)$

61. $\dfrac{\sin(3\theta)}{\sin(\theta)} - \dfrac{\cos(3\theta)}{\cos(\theta)} = 2$

62. $\tan(2\theta) = \dfrac{1}{1 - \tan(\theta)} - \dfrac{1}{1 + \tan(\theta)}$

63. $\csc(2\theta) = \dfrac{\cot(\theta) + \tan(\theta)}{2}$

64. $(\sin^2 x - 1)^2 = \cos(2x) + \sin^4 x$

65. $\sin(2x) = \dfrac{2\tan(x)}{1 + \tan^2(x)}$

66. $\tan(2x) = \dfrac{2\sin(x)\cos(x)}{2\cos^2(x) - 1}$

67. $\cot(x) - \tan(x) = 2\cot(2x)$

68. $\dfrac{\sin(2\theta)}{1 + \cos(2\theta)} = \tan(\theta)$

69. $\cos(2\alpha) = \dfrac{1 - \tan^2(\alpha)}{1 + \tan^2(\alpha)}$

70. $\dfrac{1 + \cos(2t)}{\sin(2t) - \cos(t)} = \dfrac{2\cos(t)}{2\sin(t) - 1}$

71. $\sin(3x) = 3\sin(x)\cos^2(x) - \sin^3(x)$

72. $\cos(3x) = \cos^3(x) - 3\sin^2(x)\cos(x)$

73. $8\sin^4(\theta) = \cos(4\theta) - 4\cos(2\theta) + 3$

74. $8\cos^4(\theta) = \cos(4\theta) + 4\cos(2\theta) + 3$

75. $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$

36. $\sin(4\theta) = 4\sin(\theta)\cos^3(\theta) - 4\sin^3(\theta)\cos(\theta)$

37. $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$

38. $\cos(8\theta) = 128\cos^8(\theta) - 256\cos^6(\theta) + 160\cos^4(\theta) - 32\cos^2(\theta) + 1$

39. $\sec(2\theta) = \dfrac{\cos(\theta)}{\cos(\theta) + \sin(\theta)} + \dfrac{\sin(\theta)}{\cos(\theta) - \sin(\theta)}$

40. $\dfrac{1}{\cos(\theta) - \sin(\theta)} + \dfrac{1}{\cos(\theta) + \sin(\theta)} = \dfrac{2\cos(\theta)}{\cos(2\theta)}$

41. $\dfrac{1}{\cos(\theta) - \sin(\theta)} - \dfrac{1}{\cos(\theta) + \sin(\theta)} = \dfrac{2\sin(\theta)}{\cos(2\theta)}$

42. $\dfrac{\cos^2(\psi)}{\cos^2(\theta)} = \dfrac{1 + \cos(2\psi)}{1 + \cos(2\theta)}$

Challenge Problems

43. Some trigonometry textbooks used to claim (incorrectly) that $\sin(\theta) + \cos(\theta) = \sqrt{1 + \sin(2\theta)}$ was an identity. Give an example of a specific angle θ that would make that equation false. Is $\sin(\theta) + \cos(\theta) = \pm\sqrt{1 + \sin(2\theta)}$ an identity? Justify your answer.

44. In general, what is the largest value that $\sin(\theta)\cos(\theta)$ can take? Justify your answer.

This page titled 9.3.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

10: TRIGONOMETRIC EQUATIONS

10.1: The Fundamental Inverse Trigonometric Functions

10.1.1: Resources and Key Concepts

10.1.2: Homework

10.2: The Remaining Inverse Trigonometric Functions

10.2.1: Resources and Key Concepts

10.2.2: Homework

10.3: Solving Trigonometric Equations - Algebraic Techniques

10.3.1: Resources and Key Concepts

10.3.2: Homework

10.4: Solving Trigonometric Equations Using Identities

10.4.1: Resources and Key Concepts

10.4.2: Homework

This page titled 10: Trigonometric Equations is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

- Chapter 5: Equations and Identities by Katherine Yoshiwara is licensed GNU FDL. Original source: https://yoshiwarabooks.org/trig/trig.html.

10.1: The Fundamental Inverse Trigonometric Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Core Prerequisites (i.e., "Understanding the Lesson"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are *required* to understand the core concepts in this lesson.

(click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We previously introduced the idea of finding an *acute* angle given the value of a trigonometric function. We learned to use the calculator keys $\mathrm{SIN}^{-1}, \mathrm{COS}^{-1}$, and TAN^{-1} to find approximate values of θ when we know either $\sin(\theta), \cos(\theta)$, or $\tan(\theta)$. For example, if we know that $\cos(\theta) = 0.3$, then the *acute* angle θ is approximated by

$$\theta = \cos^{-1}(0.3) \approx 1.2661 \text{ radians}.$$

If you consider the initial piece of information, namely that $\cos(\theta) = 0.3$, this implies that we can use the $\mathrm{SIN}^{-1}, \mathrm{COS}^{-1}$, and TAN^{-1} keys to solve trigonometric equations (as long as the angle we are interested in is acute), just as we use square roots to solve quadratic equations.

Up to this point, that's all we needed to know about the inverse trigonometric functions; however, we will soon be solving more complex trigonometric equations, requiring a deeper discussion of them.

Does Every Function Have an Inverse?

Consider the function $F(x) = x^2 - 4$. First, we'll find a formula for the inverse. We let $y = F(x)$ and solve for x

$$\begin{aligned} y &= x^2 - 4 \\ \Longrightarrow \quad y + 4 &= x^2 \\ \Longrightarrow \quad \pm\sqrt{y+4} &= x \end{aligned}$$

We then interchange x and y to get $y = \pm\sqrt{x+4}$.

The graphs of F and its inverse, $F^{-1}(x) = \pm\sqrt{x+4}$, are shown in Figure $10.1.1$. You can see that although F is a function, its inverse is not because the graph of the inverse does not pass the Vertical Line Test.

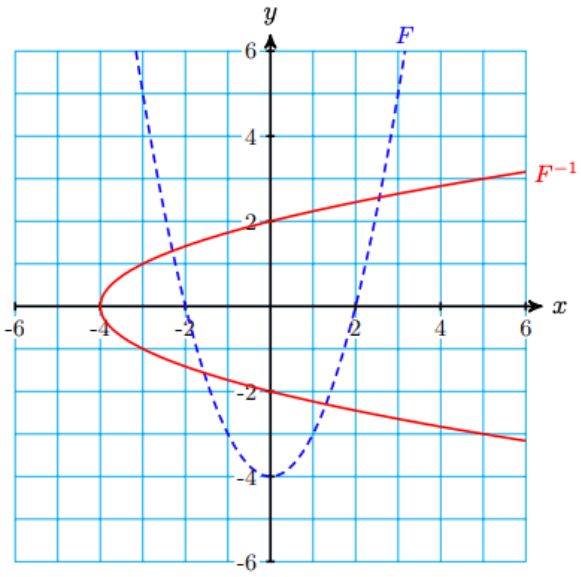


Figure 10.1.1

From Algebra, for a function, f, to have an inverse that is also a function, we require f to pass the Horizontal Line Test. Recall that a function that passes the Horizontal Line Test is called one-to-one.

At this point, I want to be *very* clear about a few points:

1. You can always find the inverse of a function graphically. All you need to do is go through each point on the graph of the function and swap the x- and y-coordinates. In doing so, the domain of your new graph is the old function's range, and the new graph's range is the old function's domain.
2. It might be impossible to find the equation representing the inverse. For example, $f(x) = e^x - 3x^2 + \sin(x)$ is definitely a function, but solving $y = e^x - 3x^2 + \sin(x)$ for x is not going to happen.
3. The inverse of a function need not be a function itself. We have just seen that $F(x) = x^2 - 4$ is an example of a function with an inverse that is *not* a function.
4. If a function f passes the Horizontal Line Test, then the function is called one-to-one and its inverse is also a function that we call the inverse function of f.

Example 10.1.1

a. Which of the graphs below represent functions?
b. Which of the functions are one-to-one?
c. Which of these functions have inverse functions?

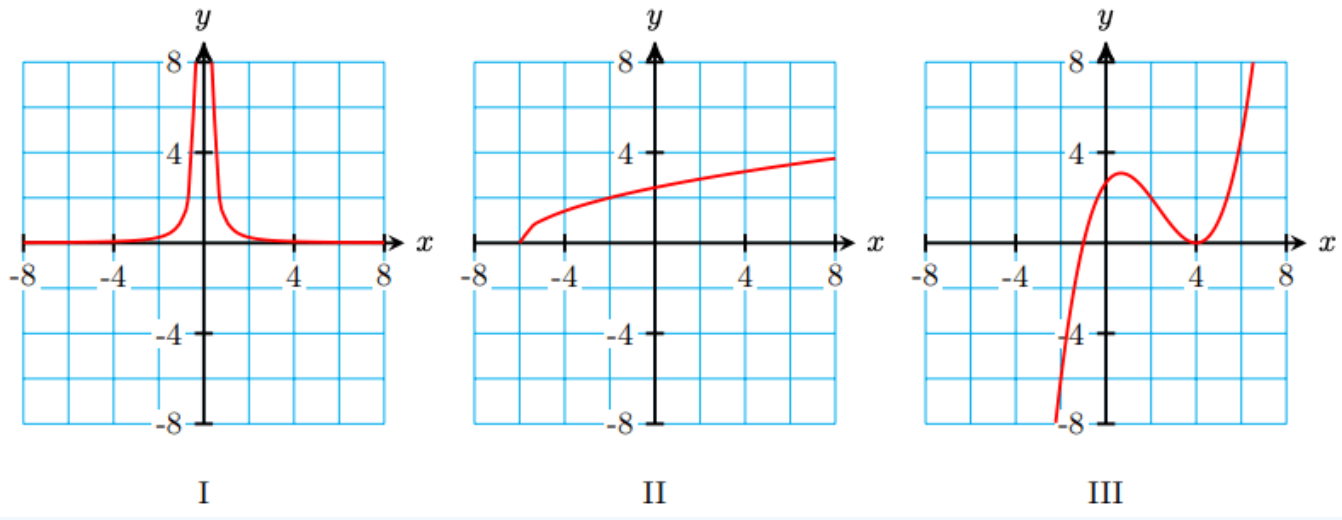


Figure 10.1.2

Solutions available at the end of this section.

Checkpoint 10.1.1

Which of the functions below are one-to-one?

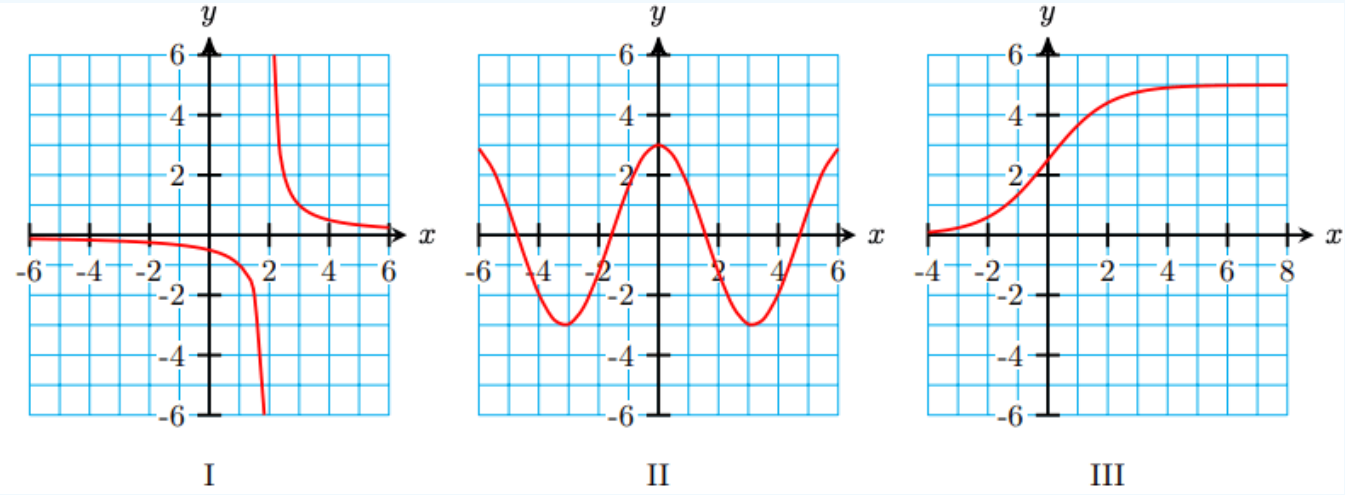


Figure 10.1.3

Answer available at the end of this section.

Restricting the Domain: The Inverse Sine Function

Sometimes, it is so crucial that the inverse be a function that we are willing to sacrifice part of the original function to achieve this result.

Look again at the graph of $F(x) = x^2 - 4$. If we use only nonnegative x-values for the domain, we create a new function,

$$f(x) = x^2 - 4, \quad x \geq 0.$$

The graph of this new function is shown as a dashed curve in Figure 10.1.4.

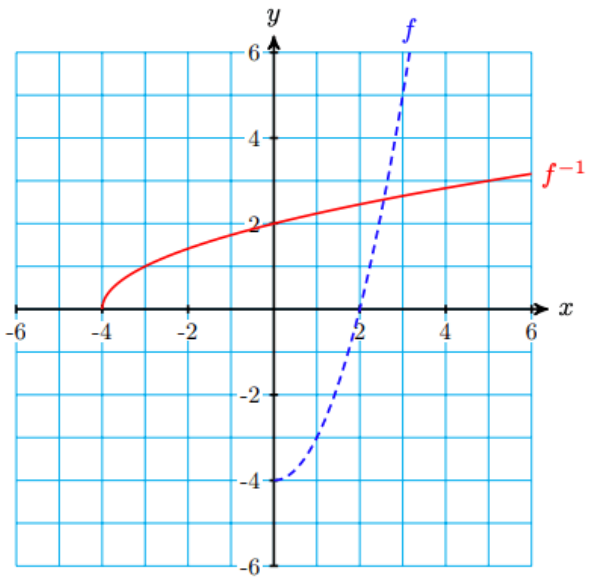


Figure 10.1.4

The new function is one-to-one, and its inverse, $f^{-1}(x) = \sqrt{x+4}$, is also a function.

We say that we have **restricted the domain** of the original function, and we will use this technique to define inverse functions for the trigonometric functions.

Theorem: Forcing Invertibility by Restricting Domain

If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.

The sine function is *not* one-to-one; there are many angles that have the same sine value. In order to define its inverse function, we must restrict the domain of the sine to an interval on which the y-values do not repeat. But there are many candidates for such an interval; which one do we choose?

The most useful interval starts at $x = 0$ and moves as far as we can in either direction until the y-values begin to repeat. By doing so, we obtain the restricted domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, as shown in Figure 10.1.5. This piece of the function includes all of the original range values, from -1 to 1.

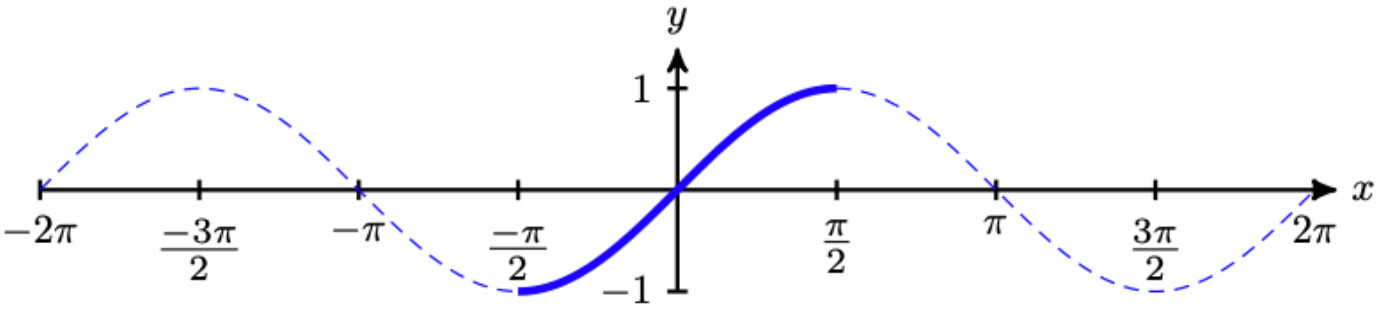


Figure 10.1.5

Because the sine function is one-to-one in this domain, its inverse is a function.

The graph of the inverse sine function, $y = \sin^{-1}(x)$, is shown in Figure 10.1.6. Its domain is the same as the range of the sine function, namely $-1 \leq x \leq 1$, and its range is our restricted domain for the sine, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

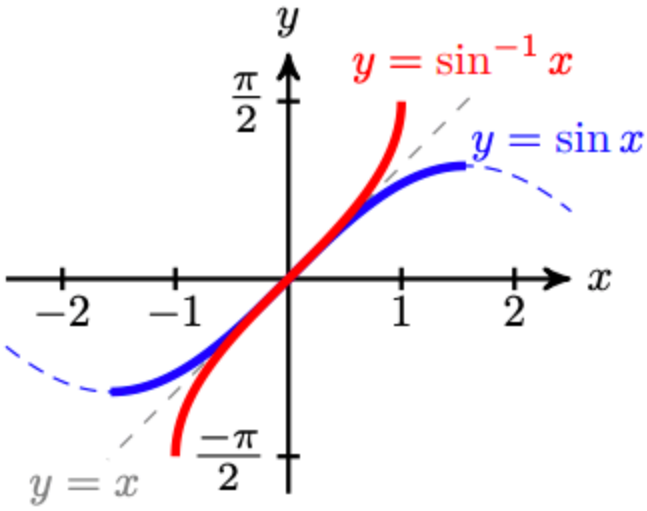


Figure 10.1.6

Definition: The Inverse Sine Function

The function $f(x) = \sin^{-1}(x)$ is defined as follows:

$$\sin^{-1}(x) = \theta \quad \text{if and only if} \quad \sin(\theta) = x \quad \text{and} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

In other words, $\sin^{-1}(x)$ is the angle in radians between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, whose sine is x. There are many angles with a given sine value x, but only one of these angles can be $\sin^{-1}(x)$. This is why the calculator's SIN^{-1} key only returns angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Prior to this moment, we have only considered inverse trigonometric functions returning acute (and, therefore, positive) angles; however, with this new definition (based upon domain-restriction of the sine function), we have something more useful:

- If $x = \sin(\theta)$ is positive, the inverse sine function delivers a first quadrant angle, $0 \leq \theta \leq \frac{\pi}{2}$. This should make sense because all trigonometric functions are positive in the first quadrant.
- If $x = \sin(\theta)$ is negative, the inverse sine function delivers a fourth quadrant angle, $-\frac{\pi}{2} \leq \theta \leq 0$.

As the next example illustrates, when dealing with inverse trigonometric functions, we focus on finding the *quadrant* and the *reference angle* for the function.

Example 10.1.2

Simplify each expression without using a calculator.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
b. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$
c. $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$
d. $\sin^{-1}(\sin(\pi))$
e. $\sin\left(\sin^{-1}(0.2)\right)$

Solutions available at the end of this section.

The process we performed in Example 10.1.2 is crucial to understand. It reinforces the statement, "Trigonometry is mostly about quadrants and reference angles."

When evaluating $\sin^{-1}(r)$, where r is some number (often a ratio), we let the inverse equal θ (this is to remind us that we are looking for an angle). That is, we begin by writing $\sin^{-1}(r) = \theta$. In *most* cases, we are looking for a *reference angle*, $\hat{\theta}$, such that

$\sin\left(\hat{\theta}\right) = |r|$. We use the sign of r to inform us as to which *quadrant* θ is in. Once we have figured out these two details, we easily state the value of θ (remembering the requirement that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$).

Example $10.1.2d$ showcases the exception to the "find the reference angle and quadrant" rule. If r is 0, 1, or -1, then we want an angle such that $\sin(\theta)$ is equal to that value; however, these values are from quadrantal angles (0, $\frac{\pi}{2}$, and $-\frac{\pi}{2}$, respectively). As such, we will not have a reference angle to find.

Finally, while Example $10.1.2e$ illustrates that $\sin\left(\sin^{-1}(r)\right) = r$ as long as $r^{\text{int}}[-1,1]$, Examples $10.1.2c$ and $10.1.2d$ give us the following caution.

Caution: $\sin^{-1}(\sin(\theta))$

$\sin^{-1}(\sin(\theta))$ is not necessarily θ.

The following theorem codifies the properties of the inverse sine function.

Theorem: Properties of the Inverse Sine Function

Properties of $f(x) = \sin^{-1}(x)$:

- $\theta = \sin^{-1}(x)$ if and only if $\sin(\theta) = x$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- Domain: $[-1,1]$
- Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\sin\left(\sin^{-1}(x)\right) = x$ for all $x^{\text{int}}[-1,1]$
- $\sin^{-1}(\sin(x)) = x$ provided $x^{\text{int}}[-\frac{\pi}{2}, \frac{\pi}{2}]$

Checkpoint $10.1.2$

Simplify each expression without using a calculator.

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$
b. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
c. $\sin\left(\sin^{-1}(-0.389)\right)$

Answers available at the end of this section.

The Inverse Cosine and Inverse Tangent Functions

The cosine and tangent functions are also periodic, so, just as with the sine function, to define their inverse functions we must restrict their domains to intervals where they are one-to-one. The graph of cosine is shown in Figure $10.1.7$.

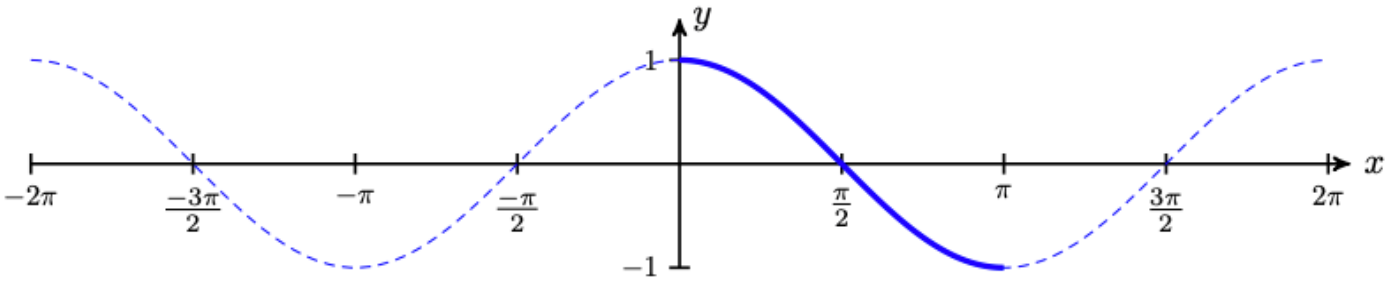


Figure $10.1.7$

Once again, the choice of these intervals is arbitrary. Suppose we start at $\theta = 0$ on the cosine graph. In that case, we can move in only one direction, either right or left, without encountering repeated y-values. As shown in Figure $10.1.7$, we choose to move in the positive direction to obtain the interval $0 \leq \theta \leq \pi$. In this domain, the inverse of cosine is a function (see Figure $10.1.8$).

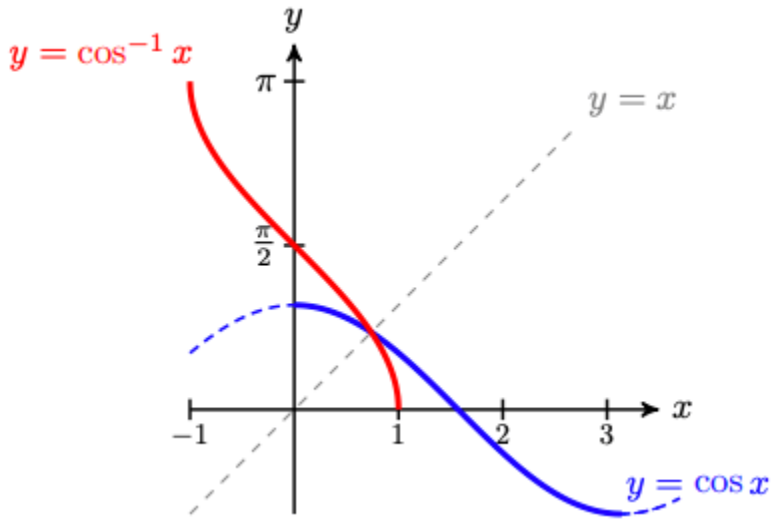


Figure $10.1.8$

Definition: Inverse Cosine Function

The function $f(x) = \cos^{-1}(x)$ is defined as follows:

$$\cos^{-1}(x) = \theta \quad \text{if and only if} \quad \cos(\theta) = x \quad \text{and} \quad 0 \leq \theta \leq \pi.$$

The range of the inverse cosine function is $0 \leq y \leq \pi$, so it delivers angles in the first and second quadrants. (Compare to the inverse sine, whose outputs are angles in the first or fourth quadrants.)

Theorem: Properties of the Inverse Cosine Function

Properties of $f(x) = \cos^{-1}(x)$:

- $\theta = \cos^{-1}(x)$ if and only if $\cos(\theta) = x$ where $0 \leq \theta \leq \pi$
- Domain: $[-1, 1]$
- Range: $[0, \pi]$
- $\cos\left(\cos^{-1}(x)\right) = x$ for all x in $[-1, 1]$
- $\cos^{-1}(\cos(x)) = x$ provided x in $[0, \pi]$

Example $10.1.3$

Simplify each expression without using a calculator.

a. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
b. $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)$

Solutions available at the end of this section.

Let's take a moment to highlight an important takeaway from Example $10.1.3b$. First, we know that the inverse trigonometric functions *return* angles. What part (b) of $10.1.3$ shows is that, given $\cos^{-1}(\cos(\alpha)) = \theta$, the reference angle for both θ and α are the same. That is, we had $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right) = \theta$. The reference angle for $\frac{7\pi}{4}$ is $\frac{\pi}{4}$, which is exactly the reference angle $\hat{\theta}$.

This same principle holds for $\sin^{-1}(\sin(\alpha)) = \theta$. Therefore, given $\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$, we know this will return an angle θ such that $\hat{\theta} = \frac{\pi}{3}$.

The reason for this common reference angle stems from the definition of the inverse trigonometric function,

$$\sin^{-1}(\sin(\alpha)) = \theta \quad \text{if and only if} \quad \sin(\theta) = \sin(\alpha).$$

Therefore, $\hat{\alpha}$ must equal $\hat{\theta}$.

Checkpoint 10.1.3

Simplify $\cos^{-1}(-1)$ without using a calculator.

Answer available at the end of this section.

Finally, consider the graph of the tangent function. To choose a convenient interval on which the tangent is one-to-one, we start at $x = 0$ and move as far as we can in either direction along the x-axis. In this way, we obtain one cycle of the graph on the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, as shown in Figure 10.1.9.

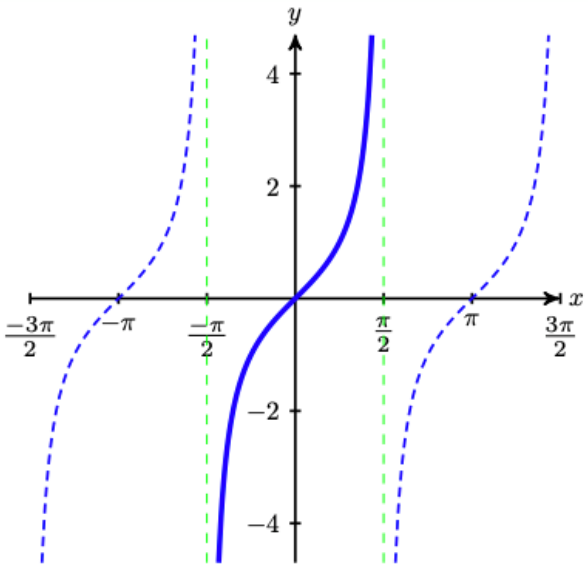


Figure 10.1.9

The range of the tangent on that interval includes all real numbers. Consequently, the domain of the inverse tangent function consists of all real numbers, and its range is the interval \(-\frac{\pi}{2}

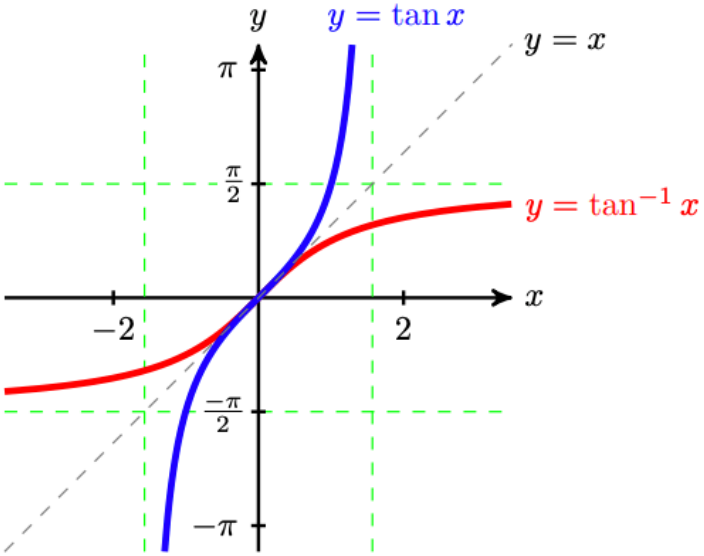


Figure 10.1.10

Note: The Inverse Tangent Graph

While you don't see the inverse sine and inverse cosine graphs in Mathematics beyond this point, the graph of the inverse tangent function is vital for Calculus concepts. This is a good one to be able to derive quickly.

Definition: The Inverse Tangent Function

The function $f(x) = \tan^{-1}(x)$ is defined as follows:

$$\tan^{-1}(x) = \theta \quad \text{if and only if} \quad \tan(\theta) = x \qquad \text{and} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Theorem: Properties of the Inverse Tangent Function

Properties of $f(x) = \tan^{-1}(x)$:

- $\theta = \tan^{-1}(x)$ if and only if $\tan(\theta) = x$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- Domain: $(-\infty, \infty)$
- Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$
- As $x \to -\infty$, $\tan^{-1}(x) \to -\frac{\pi}{2}$, and as $x \to \infty$, $\tan^{-1}(x) \to \frac{\pi}{2}$
- $\tan\left(\tan^{-1}(x)\right) = x$ for all $x \in \mathbb{R}$
- $\tan^{-1}\left(\tan(x)\right) = x$ provided $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Example 10.1.4

Simplify each expression without using a calculator.

a. $\tan^{-1}(-\sqrt{3})$
b. $\tan^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right)$

Solutions available at the end of this section.

Checkpoint 10.1.4

Simplify $\tan^{-1}(-1)$ without using a calculator.

Answer available at the end of this section.

Summary of Inverse Trigonometric Functions

Summarizing the inverse trigonometric functions and their properties before moving forward is helpful.

Summary: Inverse Trigonometric Functions

Key Concepts

Trigonometric functions take angles as arguments and return ratios. Inverse trigonometric functions take ratios as arguments and return (restricted) angles. That is,

$$\text{trig}\,(\text{angle}) = \text{ratio} \quad \text{and} \quad \text{trig}^{-1}\,(\text{ratio}) = \text{angle}.$$

To drive this point home, the rest of this summary uses r for "ratio" and θ for the angle.

Basic Definitions

$$\begin{aligned}
\sin^{-1}(r) = \theta \quad &\text{if and only if} \quad \sin(\theta) = r, \quad \text{where} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\
\cos^{-1}(r) = \theta \quad &\text{if and only if} \quad \cos(\theta) = r, \quad \text{where} \quad 0 \le \theta \le \pi \\
\tan^{-1}(r) = \theta \quad &\text{if and only if} \quad \tan(\theta) = r, \quad \text{where} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}
\end{aligned}$$

Domains and Ranges

Function	Domain	Range	Function	Domain	Range
$r = \sin(\theta)$	$\theta^{\text{int}}\,(-\infty, \infty)$	$r^{\text{int}}\,[-1, 1]$	$\theta = \sin^{-1}(r)$	$r^{\text{int}}\,[-1, 1]$	$\theta^{\text{int}}\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$r = \cos(\theta)$	$\theta^{\text{int}}\,(-\infty, \infty)$	$r^{\text{int}}\,[-1, 1]$	$\theta = \cos^{-1}(r)$	$r^{\text{int}}\,[-1, 1]$	$\theta^{\text{int}}\,[0, \pi]$
$r = \tan(\theta)$	$\theta \neq \frac{\pi}{2} + \pi k$	$r^{\text{int}}\,(-\infty, \infty)$	$\theta = \tan^{-1}(r)$	$r^{\text{int}}\,(-\infty, \infty)$	$\theta^{\text{int}}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Invertibility Properties

$$\begin{aligned}
\sin\left(\sin^{-1}(x)\right) &= x \quad \text{for } -1 \le x \le 1 \\
\cos\left(\cos^{-1}(x)\right) &= x \quad \text{for } -1 \le x \le 1 \\
\tan\left(\tan^{-1}(x)\right) &= x \quad \text{for } -\infty < x < \infty \\
\sin^{-1}\left(\sin(x)\right) &= x \quad \text{only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\
\cos^{-1}\left(\cos(x)\right) &= x \quad \text{only for } 0 \le x \le \pi \\
\tan^{-1}\left(\tan(x)\right) &= x \quad \text{only for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{aligned}$$

Advice for Use

When asked to find the value of an inverse trigonometric function, $\text{trig}^{-1}(r)$, we start by recalling that the inverse trigonometric functions return angles. Therefore, we write

$$\text{trig}^{-1}(r) = \theta.$$

Our goal is to find *an appropriately restricted angle*, θ, such that

$$\text{trig}(\theta) = r.$$

A good piece of advice is to find the *reference angle*, $\hat{\theta}$, by determining the value of

$$\text{trig}^{-1}(|r|) = \hat{\theta}$$

(Recall that *all* inverse trigonometric functions return angles in quadrant I if the argument of the inverse is positive.)

Once you have the reference angle, you can use the sign of r along with the knowledge of where your inverse trigonometric function returns angles to determine the value of $\theta = \text{trig}^{-1}(r)$.

The Invertibility Properties are tricky for most students. Just remember that the inverse trigonometric functions return restricted angles. That's why, for example, $\cos^{-1}(\cos(x))$ is not necessarily x.

Using a Calculator to Evaluate Inverse Trigonometric Functions

We have been working with the inverse trigonometric buttons on your calculator for quite some time; however, it is important to clarify how they are used once you encounter negative ratios as arguments of inverse trigonometric functions.

Example 10.1.5

Evaluate each of the following using a calculator. Your answers should be in radians.

a. $\sin^{-1}(0.97)$
b. $\cos^{-1}(-0.561)$
c. $\tan^{-1}(-3)$

Solutions available at the end of this section.

I want to be clear, your calculator would have returned the answers to Example $10.1.5b$ and $10.1.5c$ if you had just entered $\text{COS}^{-1}\ -0.561$ and $\text{TAN}^{-1}\ -3$, respectively; however, as we move forward, your calculator will become less reliable when it comes to giving you the answer you're looking for. This is why I recommend evaluating the inverse trigonometric functions at their arguments' absolute value and treating the result as a reference angle. You honestly cannot go wrong if you do this.

Checkpoint 10.1.5

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

Answer available at the end of this section.

Alternate Notations

The inverse sine function, $\sin^{-1}(x)$, is also called the **arcsine function** and denoted by $\arcsin(x)$. (This terminology reminds us that the output of the inverse sine function is an angle, or the arc on a unit circle determined by that angle, as shown in Figure 10.1.11.)

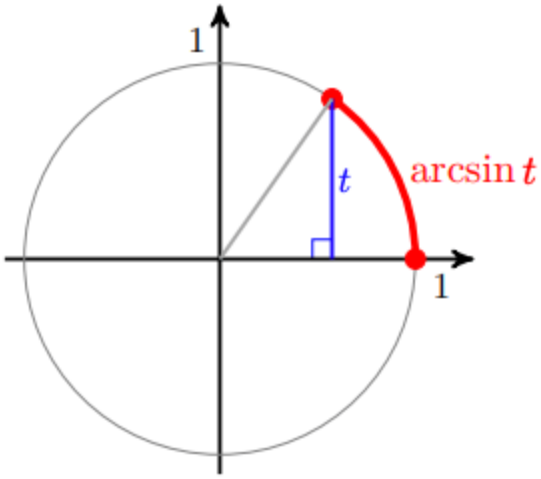


Figure 10.1.11

Similarly, the inverse cosine function is sometimes denoted by $\arccos(x)$, and the inverse tangent function by $\arctan(x)$.

Example 10.1.6

Simplify each expression.

a. $\cos\left(\arccos\left(\frac{1}{3}\right)\right)$
b. $\arcsin\left(\sin\left(\frac{3\pi}{2}\right)\right)$

Solutions available at the end of this section.

Checkpoint 10.1.6

Simplify each expression.

a. $\tan\left(\arctan\left(-\frac{5}{2}\right)\right)$
b. $\arctan\left(\tan\left(-\frac{5\pi}{6}\right)\right)$

Answers available at the end of this section.

Note: Do Arc Functions Return Angles?

Up to this point, we have been very clear that the inverse trigonometric functions return angles; however, very technically, the arcfunctions return *real numbers*—not angles. This is why many instructors prefer you to answer in radians when dealing with arcfunctions rather than degrees. Fortunately, this is a subtlety that you don't have to worry about. You can assume the arcfunctions, like the inverse trigonometric functions, return angles if that helps you remember them.

Simplifying Expressions Involving Different Trigonometric Functions

The key to simplifying expressions involving inverse trigonometric functions is to remember that the inverse sine, cosine, or tangent of a number can be treated as an angle. It can often clarify the computations if we assign a name such as θ or ϕ to the inverse trigonometric value. Additionally, drawing triangles on the Cartesian coordinate system can be immensely helpful.

Example 10.1.7

Evaluate $\sin\left(\cos^{-1}\left(\frac{3}{4}\right)\right)$.

Solution available at the end of this section.

Checkpoint 10.1.7

Evaluate $\cos\left(\tan^{-1}\left(\frac{2}{3}\right)\right)$.

Answer available at the end of this section.

We can verify the results of the previous example using a calculator, but the same technique can be applied to simplify similar expressions involving variables.

Example 10.1.8

Simplify $\tan\left(\sin^{-1}(x)\right)$, assuming that $-1 < x < 0$.

Solution available at the end of this section.

The answer to Example 10.1.8, $\frac{x}{\sqrt{1-x^2}}$, *looks* positive, but we know this cannot be the case because the tangent should be negative in QIV. Looks can be deceiving! While the denominator is definitely positive, the numerator, x, is negative. Therefore, the answer is, in fact, negative!

Checkpoint 10.1.8

Simplify $\sin\left(\tan^{-1}(z)\right)$, assuming that $z \geq 0$.

Answer available at the end of this section.

Simplifying Expressions Involving Identities

Now that we have learned all of the major identities in Trigonometry, we can use them with the inverse trigonometric functions. In doing so, we will demonstrate our fluency with inverse trigonometric functions and identities.

Example 10.1.9

a. Rewrite the following as an algebraic expression of x and state the domain on which the equivalence is valid.

$$f(x) = \cos\left(2\arcsin(x)\right)$$

b. Find the exact value of the expression.

$$\sin\left(\cos^{-1}\left(-\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{5}\right)\right)$$

Solutions available at the end of this section.

Checkpoint 10.1.9

Find the exact value of the expression.

$$\sin\left(2\cos^{-1}\left(-\frac{1}{3}\right)\right)$$

Answer available at the end of this section.

Modeling with Inverse Functions

The inverse trigonometric functions are used to model situations in which an angle is described in terms of one of its trigonometric ratios.

Example 10.1.10

The bottom of a 3-meter-tall tapestry on a chateau wall is at your eye level. The angle θ subtended vertically by the tapestry changes as you approach the wall.

a. Express your distance from the wall, d, as a function of θ.
b. Express θ as a function of d.

Solutions available at the end of this section.

In Example 10.1.10, we didn't have to worry about the quadrant in which θ belongs. Most (but not all) applications share this quality. If we are looking at the top of a tapestry on a wall, and if the bottom of the tapestry is at eye level, no matter how close or how far we are away from the tapestry, the angle of inclination from eye level to the top of the tapestry will always be acute (i.e., a quadrant I angle).

Checkpoint 10.1.10

The tapestry from the previous example includes a 2-meter-tall unicorn with its feet at the bottom of the tapestry.

a. Express α, the angle subtended vertically by the unicorn, as a function of d, your distance to the tapestry.
b. Express β, the angle subtended by the portion of the tapestry above the unicorn, as a function of d. (See the figure below).

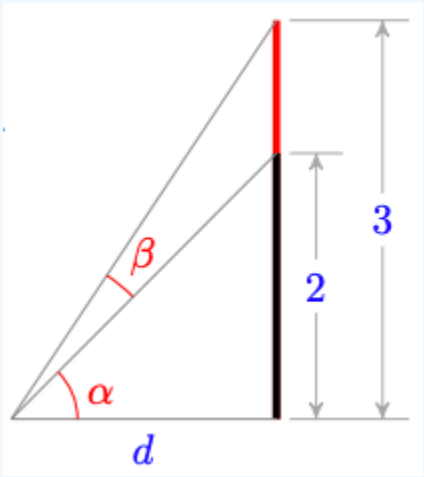


Figure 10.1.16

Answers available at the end of this section.

Success in Mathematics: Check Your Work

Checking your work is essential, but most students want to complete it and walk away from it. The fastest way to check your work in algebra is to plug your answers into the original problem. In trigonometry and calculus, this method may be a bit difficult. However, you can always check your solutions with graphs on the graphing calculator! It's a great idea.

1. Do you usually check your work?
2. Have you used the graphing calculator to check your work, or do you usually check "by hand?"

Miscellaneous Topics: Teachers and Role Models

Parents, teachers, and other role models influence our attitudes about math. What they say and do can change our view of the importance of math, how hard math is, and whether we are good at math. Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Describe the attitudes towards math of your parent, teacher (not me, of course), or another personal role model.
2. Describe how the opinion of this role model has influenced your attitudes about math.

Answers

Section Preview

When writing parts of this text, I was trying my best to stick with the mantra, "Don't introduce it, unless they are going to use it." While we did introduce the inverse trigonometric functions back when we covered Right Triangle Trigonometry, it was strictly from a "button-mashing on a calculator" perspective for *positive* ratios. Hence, from a student perspective, those pesky buttons on their calculator (or the arc-functions in Desmos) only returned angles between 0 and 90° (or, when we revisited them after learning about radian measure, between 0 and $\frac{\pi}{2}$). We had another visit of the inverse trigonometric functions when we discussed reference angles. At that time, I was very careful to tip-toe around the truth of inverse trigonometric functions and just told the reader to evaluate inverse trigonometric functions at absolute values of the given ratios. In doing so, I mentioned that this would return a reference angle. This section dives into the deep waters and discusses what inverse trigonometric functions truly are for the big three (sine, cosine, and tangent). We need this theory before diving into solving trigonometric equations.

Section Preview

- **An Overview of Functions**
  - Inverse Functions: This entire section assumes students understand the concept of an inverse function (and how to find the inverse of a function algebraically and graphically).
  - Restricting Domains and Forcing Invertibility: Since the trigonometric functions are not naturally invertible, we must use domain restrictions to help us create invertible versions of these functions. Students should have some experience with this already.

Section Preview

- **An Overview of Functions**
  - Function Composition and Domain: Function composition is needed for the composition evaluations in Examples 7–9.

Section Preview

- Find the exact value of an inverse trigonometric function.
- Use technology to approximate the value of an inverse trigonometric function.
- Evaluate compositions involving trigonometric functions and inverse trigonometric functions.
- Simplify expressions involving trigonometric functions and inverse trigonometric functions.

Example $10.1.1$

a. All three graphs pass the Vertical Line Test, so all three represent functions.
b. Only the function represented by Graph II passes the Horizontal Line Test, so it is the only one-to-one function.
c. Only function II is one-to-one, so it is the only function that has an inverse function.

Checkpoint $10.1.1$

I and III

Example $10.1.2$

a. By definition, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\theta$ is an angle whose sine is $-\frac{\sqrt{3}}{2}$, where $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$. That is, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\theta$ means $\sin(\theta)=-\frac{\sqrt{3}}{2}$ and $\theta^{\text{int}}\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. While you might be able to quickly state the value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ right now, we are going to take our time showcasing the thought process so that we can work with more complicated problems.

   Quadrant: Because $\sin(\theta)=-\frac{\sqrt{3}}{2}$ is negative, we further restrict θ to the interval $\left[-\frac{\pi}{2},0\right]$. In fact, since the value of the sine function is not ±1 or 0, we know θ is not a quadrantal angle, so $\theta^{\text{int}}\left(-\frac{\pi}{2},0\right)$. Hence, $\theta^{\text{int}}\text{QIV}$.

   Reference Angle: Since $\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$, the reference angle for θ is $\hat{\theta}=\frac{\pi}{3}$.

Now that we have the quadrant and reference angle, we can state the value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. $\theta^{\text{int}}\left(-\frac{\pi}{2},0\right)$ with a reference angle of $\hat{\theta}=\frac{\pi}{3}$ means

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}.$$

b. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\theta$ means $\sin(\theta)=-\frac{1}{\sqrt{2}}$ and $\theta^{\text{int}}\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Quadrant: Since the sine is negative, we need $\theta^{\text{int}}\left(-\frac{\pi}{2},0\right)$.

Reference Angle: We know the reference angle for $\sin(\theta)=-\frac{1}{\sqrt{2}}$ is $\hat{\theta}=\frac{\pi}{4}$.

A reference angle of $\frac{\pi}{4}$ in the interval $\left(-\frac{\pi}{2},0\right)$ means

$$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)=-\frac{\pi}{4}.$$

c. As promised, the method we used in parts (a) and (b) can be used for more complicated problems (like this one).

$\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)=\theta$ means $\sin(\theta)=\sin\left(\frac{5\pi}{6}\right)$ where $\theta^{\text{int}}\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. A lot of people would say that θ must therefore be $\frac{5\pi}{6}$; however, $\frac{5\pi}{6}$ is not in the interval between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Quadrants and reference angles to the rescue!

Quadrant: Since $\sin(\theta)=\sin\left(\frac{5\pi}{6}\right)$ is positive (because $\frac{5\pi}{6}^{\text{int}}$QII), $\theta^{\text{int}}\left(0,\frac{\pi}{2}\right)$. Hence, θ^{int}QI.

Reference Angle: Since $\sin(\theta)=\sin\left(\frac{5\pi}{6}\right)$, θ must have the same reference angle as $\frac{5\pi}{6}$. That is, $\hat{\theta}=\frac{\pi}{6}$.

So, we know θ^{int}QI with a reference angle of $\frac{\pi}{6}$. Hence,

$$\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)=\frac{\pi}{6}.$$

d. $\sin^{-1}(\sin(\pi))=\theta$ means $\sin(\theta)=\sin(\pi)$, where $\theta^{\text{int}}\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

This is one of those moments where quadrants and reference angles will not work. This is because π is a quadrantal angle. In these cases, we simply evaluate $\sin(\pi)=0$. Therefore, $\sin^{-1}(\sin(\pi))=\sin^{-1}(0)$. We want an angle whose sine is 0, and which lies in the interval $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. This angle is 0. Therefore, $\sin^{-1}(\sin(\pi))=0$.

e. We finally reach an example where the inverse trigonometric function is *the argument* of another function. Our approach will be the same as before—we start by focusing on the inverse trigonometric function.

$\sin^{-1}(0.2)=\theta$ means $\sin(\theta)=0.2$, where $\theta^{\text{int}}\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Believe it or not, we have actually found the value of $\sin\left(\sin^{-1}(0.2)\right)$. Reread the statement above, and then follow the next line:

$$\sin\left(\sin^{-1}(0.2)\right)=\sin(\theta)=0.2.$$

This illustrates the fact that

$$\sin\left(\sin^{-1}(r)\right)=r$$

as long as $r^{\text{int}}[-1,1]$.

Checkpoint 10.1.2

a. $\frac{\pi}{3}$
b. $\frac{\pi}{3}$
c. -0.389

Example 10.1.3

a. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\theta$ means $\cos(\theta)=-\frac{\sqrt{3}}{2}$, where $\theta^{\text{int}}[0,\pi]$.

Quadrant: Since $\cos(\theta)=-\frac{\sqrt{3}}{2}$ is negative, we further restrict θ so that $\theta^{\text{int}}\left(\frac{\pi}{2},\pi\right)$. Thus, θ^{int}QII.

Reference Angle: To find the reference angle, we ask ourselves what acute angle, $\hat{\theta}$, results in

$$\cos\left(\hat{\theta}\right)=\left|-\frac{\sqrt{3}}{2}\right|=\frac{\sqrt{3}}{2}.$$

From our special triangles, we know this is $\hat{\theta}=\frac{\pi}{6}$.

The angle in the second quadrant with this reference angle is $\theta=\frac{5\pi}{6}$, so

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}.$$

b. $\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)=\theta$ means $\cos(\theta)=\cos\left(\frac{7\pi}{4}\right)$, where $\theta^{\text{int}}[0,\pi]$.

Quadrant: Because $\frac{7\pi}{4}^{\text{int}}$QIV, we know that $\cos(\theta)=\cos\left(\frac{7\pi}{4}\right)$ is positive. Therefore, $\theta^{\text{int}}\left(0,\frac{\pi}{2}\right)$.

Reference Angle: Since $\cos(\theta)=\cos\left(\frac{7\pi}{4}\right)$, the reference angle for θ must be equal to the reference angle for $\frac{7\pi}{4}$. That is, $\hat{\theta}=\frac{\pi}{4}$.

We have a reference angle of $\frac{\pi}{4}$ in Quadrant I, this means

$$\cos^{-1}\left(\cos\left(\frac{7\pi}{4}\right)\right)=\frac{\pi}{4}.$$

Checkpoint 10.1.3

π

Example 10.1.4

a. $\tan^{-1}(-\sqrt{3})=\theta$ means $\tan(\theta)=-\sqrt{3}$, where $\theta^{\text{int}}\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Quadrant: Since $\tan(\theta)$ is negative, we further restrict the angle to $\theta^{\text{int}}\left(-\frac{\pi}{2},0\right)$, or Quadrant IV.

Reference Angle: Since $\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$, $\hat{\theta}=\frac{\pi}{3}$.

Combining our results, we get

$$\tan^{-1}(-\sqrt{3})=-\frac{\pi}{3}.$$

b. $\tan^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right)=\theta$ means $\tan(\theta)=\tan\left(\frac{7\pi}{4}\right)$, where $\theta^{\text{int}}\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Quadrant: Because $\frac{7\pi}{4}$ is in Quadrant IV, $\tan\left(\frac{7\pi}{4}\right)$ is negative. Hence, $\tan(\theta)$ is negative. As such, we further restrict the angle to $\theta^{\text{int}}\left(-\frac{\pi}{2},0\right)$.

Reference Angle: Since $\tan(\theta)=\tan\left(\frac{7\pi}{4}\right)$, both arguments must share the same reference angle. Therefore, $\hat{\theta}=\frac{\pi}{4}$.

Thus,

$$\tan^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right)=-\frac{\pi}{4}.$$

Checkpoint 10.1.4

$-\frac{\pi}{4}$

Example 10.1.5

a. This is a straightforward use of the calculator. At this point, you should be familiar enough with this to know we enter $\text{SIN}^{-1}\ 0.97$ in the calculator to get approximately 1.3252 radians; however, we are going to use this to link back to the *quadrant* and *reference angle* concept we have been building throughout this section.

$\sin^{-1}(0.97)=\theta$ means $\sin(\theta)=0.97$, where $\theta^{\text{int}}\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Quadrant: Since the sine is positive, we further restrict our angle to $\theta^{\text{int}}\left(0,\frac{\pi}{2}\right)$, or Quadrant I.

Reference Angle: The reference angle can be found by using your calculator (in radian mode) to evaluate $\sin^{-1}(|0.97|)\approx 1.3252$.

So our angle is in Quadrant I with a reference angle of $\hat{\theta}\approx 1.3252$. This means,

$$\sin^{-1}(0.97)\approx 1.3252.$$

b. $\cos^{-1}(-0.561)=\theta$ means $\cos(\theta)=-0.561$, where $\theta^{\text{int}}[0,\pi]$.

Quadrant: Since $\cos(\theta)$ is negative, we can guarantee that $\theta^{\text{int}}\text{QII}$ (more specifically, $\theta^{\text{int}}\left(\frac{\pi}{2},\pi\right)$).

Reference Angle: To find the *reference angle* for θ, we compute $\hat{\theta}=\cos^{-1}(|-0.561|)=\cos^{-1}(0.561)\approx 0.9752$ radians.

Now that we have a quadrant (II) and a reference angle (0.9752), we find the actual angle by subtracting the reference angle from π. Thus,

$$\cos^{-1}(-0.561)\approx\pi-0.9752\approx 2.1664.$$

c. $\tan^{-1}(-3)=\theta$ means $\tan(\theta)=-3$, where $\theta^{\text{int}}\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Quadrant: Since the tangent is negative, we can restrict this interval to $\theta^{\text{int}}\left(-\frac{\pi}{2},0\right)$, or Quadrant IV.

Reference Angle: The reference angle is

$$\hat{\theta}=\tan^{-1}(|-3|)=\tan^{-1}(3)\approx 1.2490.$$

Thus,

$$\tan^{-1}(-3)\approx -1.2490.$$

Checkpoint 10.1.5

1.9823 radians or 113.578°

Example 10.1.6

a. $\arccos\left(\frac{1}{3}\right)=\theta$ means $\cos(\theta)=\frac{1}{3}$, where $0\le\theta\le\pi$. Since $\frac{1}{3}^{\text{int}}[-1,1]$, we can use the Invertibility Property to state

$$\cos\left(\arccos\left(\frac{1}{3}\right)\right)=\cos(\theta)=\frac{1}{3}.$$

b. $\arcsin\left(\sin\left(\frac{3\pi}{2}\right)\right)=\theta$ means $\sin(\theta)=\sin\left(\frac{3\pi}{2}\right)$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Noticing that $\frac{3\pi}{2}$ is a quadrantal angle, we go ahead and simplify the right side of that last equation.

$$\sin(\theta) = \sin\left(\frac{3\pi}{2}\right) = -1$$

Therefore, we require an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that the sine of that angle is -1. This would be $-\frac{\pi}{2}$. Thus,

$$\arcsin\left(\sin\left(\frac{3\pi}{2}\right)\right) = -\frac{\pi}{2}.$$

Checkpoint 10.1.6

a. $-\frac{5}{2}$

b. $\frac{\pi}{6}$

Example 10.1.7

Notice this is a mixture of trigonometric functions. Our process changes only slightly from before.

$\cos^{-1}\left(\frac{3}{4}\right) = \theta$ means $\cos(\theta) = \frac{3}{4}$, where $\theta^{\text{int}}\,[0, \pi]$.

Quadrant: Since the cosine of θ is positive, we know $\theta^{\text{int}}\text{QI}$. *Because we have a mixture of trigonometric functions*, it's best to draw a reference triangle in QI.

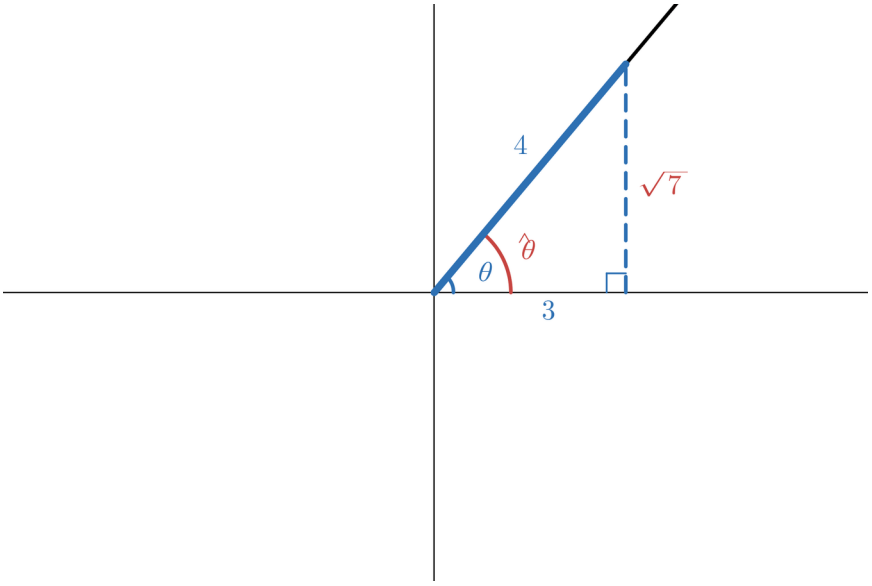


Figure 10.1.12

We now can use this reference triangle to answer the question (i.e., there is no need to find the reference angle).

$\sin\left(\cos^{-1}\left(\frac{3}{4}\right)\right)$ simplifies to $\sin(\theta)$. Using the Pythagorean Theorem and the definition for sine, we get

$$\sin\left(\cos^{-1}\left(\frac{3}{4}\right)\right) = \sin(\theta) = \frac{\sqrt{7}}{4}.$$

Checkpoint 10.1.7

$\frac{3}{\sqrt{13}}$

Example 10.1.8

Let $\theta = \sin^{-1}(x)$, so that $\sin(\theta) = x = \frac{x}{1}$, and the expression $\tan\left(\sin^{-1}(x)\right) = \tan(\theta)$. Since $-1 < x < 0$, sine is negative. Therefore, the inverse sine returns an angle in QIV. Specifically, $-\frac{\pi}{2} \leq \theta \leq 0$. Drawing a reference triangle in QIV, we get

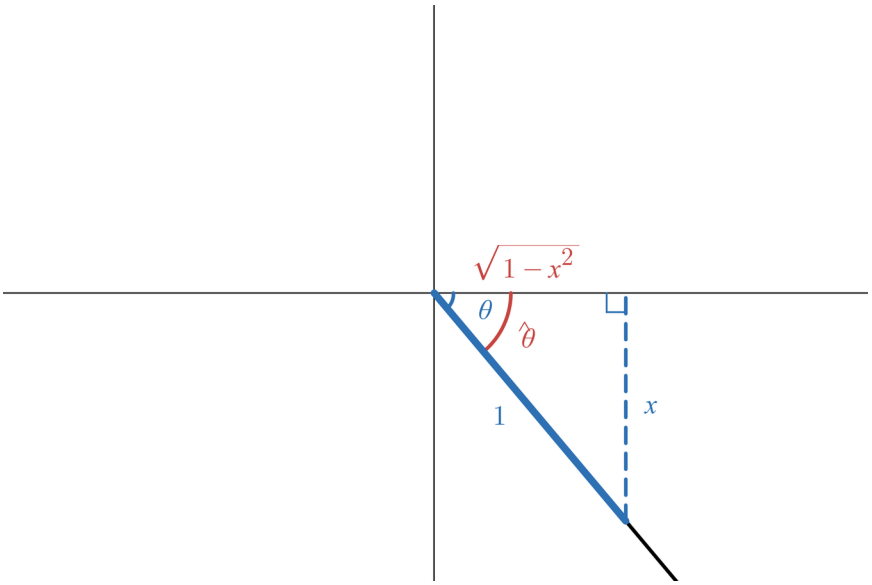


Figure 10.1.13

Now we use the fact that $\tan(\theta)$ is opposite over adjacent to get

$$\tan\left(\sin^{-1}(x)\right) = \tan(\theta) = \frac{x}{\sqrt{1-x^2}}.$$

Checkpoint 10.1.8

$$\frac{z}{\sqrt{1+z^2}}$$

Example 10.1.9

a. We start by writing $\theta = \arcsin(x)$ so that θ lies in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin(\theta) = x$. We want to express $\cos(2\arcsin(x)) = \cos(2\theta)$ in terms of x. Since $\cos(2\theta)$ is defined everywhere, we get no additional restrictions on θ. We have three choices for rewriting $\cos(2\theta)$:

$$\cos^2(\theta) - \sin^2(\theta), \quad 2\cos^2(\theta) - 1, \quad \text{and} \quad 1 - 2\sin^2(\theta).$$

Since we know $x = \sin(\theta)$, it is easiest to use the last form:

$$\cos(2\arcsin(x)) = \cos(2\theta) = 1 - 2\sin^2(\theta) = 1 - 2x^2.$$

To find the restrictions on x, we consider our substitution, $\theta = \arcsin(x)$. Since $\arcsin(x)$ is defined only for $-1 \le x \le 1$, the function, $f(x) = \cos(2\arcsin(x)) = 1 - 2x^2$, is valid only on $[-1, 1]$.

b. Let $\alpha = \cos^{-1}(-\frac{1}{3})$ and $\beta = \sin^{-1}(\frac{2}{5})$. Then we can rewrite the expression as

$$\sin\left(\cos^{-1}\left(-\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{5}\right)\right) = \sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),$$

where

$$\cos(\alpha) = -\frac{1}{3}, \quad 0 \le \alpha \le \pi$$

and

$$\sin(\beta) = \frac{2}{5}, \quad -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}.$$

Since $\cos(\alpha)$ is negative, we immediately know α^{int}QII. Moreover, $\sin(\beta)$ being positive means that β^{int}QI. Drawing reference triangles, we get the following figures:

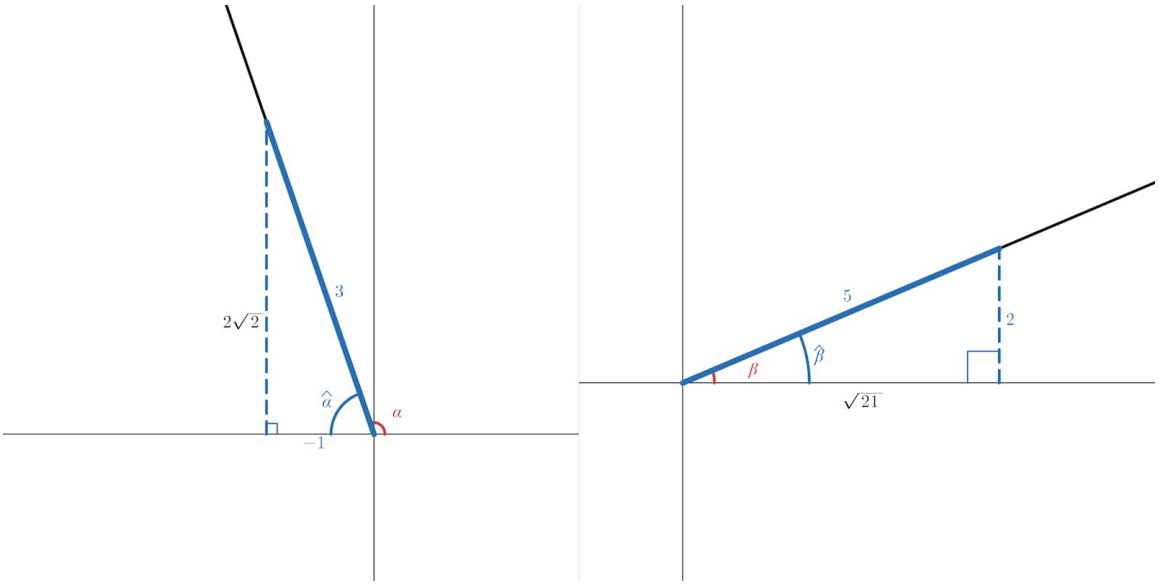


Figure 10.1.14

We now evaluate the equivalent expression, $\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$, using these reference triangles.

$$\begin{aligned}\sin\left(\cos^{-1}\left(-\frac{1}{3}\right)+\sin^{-1}\left(\frac{2}{5}\right)\right) &= \sin(\alpha+\beta)\\ &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\\ &= \left(\frac{2\sqrt{2}}{3}\right)\left(\frac{\sqrt{21}}{5}\right)+\left(-\frac{1}{3}\right)\left(\frac{2}{5}\right)\\ &= \frac{2\sqrt{42}}{15}-\frac{2}{15}\\ &= \frac{2\sqrt{42}-2}{15}\end{aligned}$$

Checkpoint 10.1.9

$$-\frac{4\sqrt{2}}{9}$$

Example 10.1.10

a. As shown below, we sketch the triangle formed by the tapestry and the lines of sight to its bottom and top. From the triangle we see that $\tan(\theta)=\frac{3}{d}$, so $d=\frac{3}{\tan(\theta)}$.

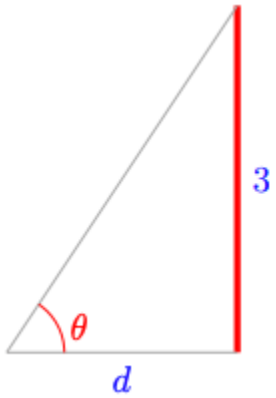


Figure 10.1.15

b. Because $\tan(\theta)=\frac{3}{d}$, $\theta=\tan^{-1}\left(\frac{3}{d}\right)$.

Checkpoint 10.1.10

a. $\alpha=\tan^{-1}\left(\frac{2}{d}\right)$

b. $\beta=\tan^{-1}\left(\frac{3}{d}\right)-\tan^{-1}\left(\frac{2}{d}\right)$

This page titled 10.1: The Fundamental Inverse Trigonometric Functions is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

10.1.1: Resources and Key Concepts

Resources

Prerequisite Topics

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **An Overview of Functions**
  - Function Composition and Domain: This skill is necessary for evaluating and simplifying composite functions involving trigonometric and inverse trigonometric functions, such as $\sin(\arccos(x))$.
  - Inverse Functions: This section is built on the concept of inverse functions. Finding a formula for an inverse, using the Horizontal Line Test to determine if a function has an inverse function, and understanding the relationship between the domain and range of a function and its inverse are all critical prerequisite skills. These are reviewed in the "Skills Refresher" exercises.
  - Restricting Domains and Forcing Invertibility: This section's primary theoretical basis is the idea of restricting the domains of the non-invertible trigonometric functions to create new functions that are one-to-one and thus have inverse functions.

Videos

- Inverse Trigonometric Functions
  - Implicit Differentiation - Derivatives of Inverse Trig Functions: Theorem, Proof, and Example
  - Inverse Trigonometric Functions - Defining the Inverse Trigonometric Functions
  - Inverse Trigonometric Functions - Evaluating Inverse Trigonometric Functions
  - Inverse Trigonometric Functions - Applications of Right Triangle Trigonometry (Part 2)

Key Concepts

Definitions

- **The Inverse Sine Function:** The function $f(x) = \sin^{-1}(x)$ (or $\arcsin(x)$) is defined as $\sin^{-1}(x) = \theta$ if and only if $\sin(\theta) = x$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
- **The Inverse Cosine Function:** The function $f(x) = \cos^{-1}(x)$ (or $\arccos(x)$) is defined as $\cos^{-1}(x) = \theta$ if and only if $\cos(\theta) = x$ and $0 \leq \theta \leq \pi$.
- **The Inverse Tangent Function:** The function $f(x) = \tan^{-1}(x)$ (or $\arctan(x)$) is defined as $\tan^{-1}(x) = \theta$ if and only if $\tan(\theta) = x$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Theorems

- **Lemma:** Let θ be an angle in standard position, and suppose the terminal side of θ intersects the unit circle at the point $P(x, y)$. Then $\cos(\theta) = x$ and $\sin(\theta) = y$.
- **Forcing Invertibility by Restricting Domain:** If we restrict the domain of a non-invertible function so that the resulting function becomes one-to-one, then this new function will have an inverse.
- **Properties of the Inverse Sine Function:** For $f(x) = \sin^{-1}(x)$:
  - Domain: $[-1, 1]$
  - Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
  - $\sin(\sin^{-1}(x)) = x$ for all x in $[-1, 1]$
  - $\sin^{-1}(\sin(x)) = x$ provided x is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

- **Properties of the Inverse Cosine Function:** For $f(x) = \cos^{-1}(x)$:
  - Domain: $[-1, 1]$
  - Range: $[0, \pi]$
  - $\cos(\cos^{-1}(x)) = x$ for all x in $[-1, 1]$
  - $\cos^{-1}(\cos(x)) = x$ provided x is in $[0, \pi]$
- **Properties of the Inverse Tangent Function:** For $f(x) = \tan^{-1}(x)$:
  - Domain: $(-\infty, \infty)$
  - Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$
  - $\tan(\tan^{-1}(x)) = x$ for all x in $\mathbb{R}$
  - $\tan^{-1}(\tan(x)) = x$ provided x is in $(-\frac{\pi}{2}, \frac{\pi}{2})$

Common Mistakes

- **Inverse Composition Property:** Be cautious with compositions like $\sin^{-1}(\sin(\theta))$. This expression is not necessarily equal to θ. It is only equal to θ if θ is within the restricted range of the inverse sine function, $[-\frac{\pi}{2}, \frac{\pi}{2}]$. A similar caution applies to inverse cosine and inverse tangent.
- **Inverse Trigonometric Functions are NOT Reciprocals:** The notation $\sin^{-1}(x)$ denotes the inverse sine function, not the reciprocal of sine. The reciprocal of $\sin(x)$ is $\csc(x)$.
- **Forgetting Domain and Range Restrictions:** The inverse trigonometric functions have specific, restricted ranges. For example, the output of $\arccos(x)$ must be an angle between 0 and π, inclusive. Forgetting these restrictions is a common source of error.

This page titled 10.1.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

10.1.2: Homework

Reading Questions

1. What graphical test is used to determine if a function is one-to-one?
2. Why is it necessary to restrict the domain of the sine function before we can define its inverse function?
3. What is the restricted domain of the cosine function that is used to define the inverse cosine function?
4. What is the domain of the function $y = \arcsin(x)$?
5. What is the range of the function $y = \arccos(x)$?
6. What is the range of the function $y = \arctan(x)$?
7. The expression $\sin^{-1}(0.5)$ represents an angle. In which quadrant does this angle lie?
8. The expression $\cos^{-1}(-0.5)$ represents an angle. In which quadrant does this angle lie?
9. Is the statement $\sin^{-1}(\sin(3\pi)) = 3\pi$ true or false? Explain your reasoning.
10. Is the statement $\sin(\sin^{-1}(0.5)) = 0.5$ true or false? Explain your reasoning.
11. What is the key difference between the meaning of $\sin^{-1}(x)$ and $(\sin(x))^{-1}$?
12. When simplifying an expression like $\cos(\arctan(x))$, what is a helpful first step?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, find the inverse of the function, if it exists.

1. $f(x) = 2x^2 - 16x + 1$
2. $g(x) = (x-2)^3$
3. $f(x) = \dfrac{3x-1}{5x-1}$

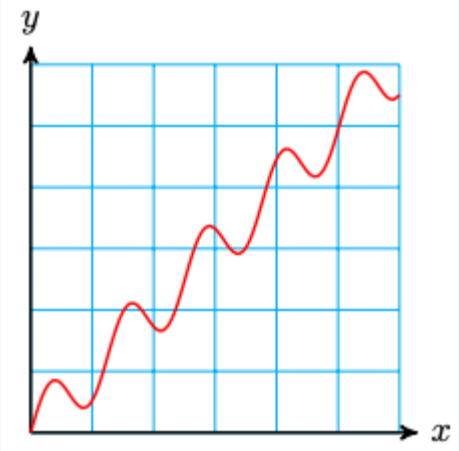
For the following exercises,
(a) Find a formula for the inverse function.
(b) State the domain and range of the inverse function.
(c) Graph the function and its inverse on the same grid.

4. $f(x) = \frac{1}{2}x - 4$
5. $g(x) = 3x + 6$
6. $F(x) = 2 + \frac{1}{x}$
7. $G(x) = \frac{1}{x+5}$
8. $h(x) = \sqrt{x+2}$
9. $H(x) = 3 + \sqrt[3]{x}$

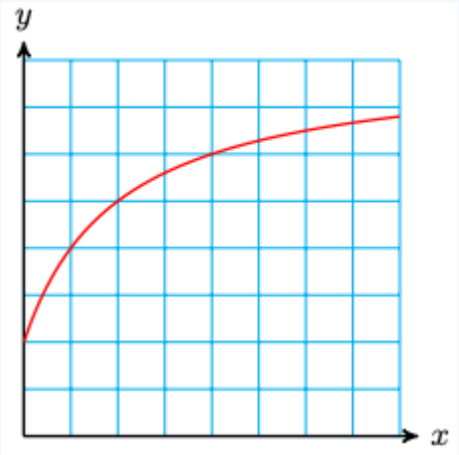
For the following exercises, which functions have an inverse function? Explain your answer.

10.

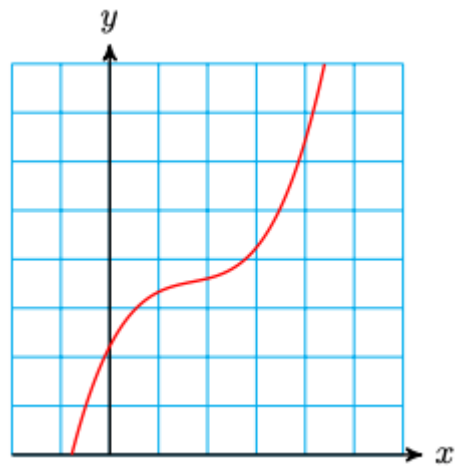
11.

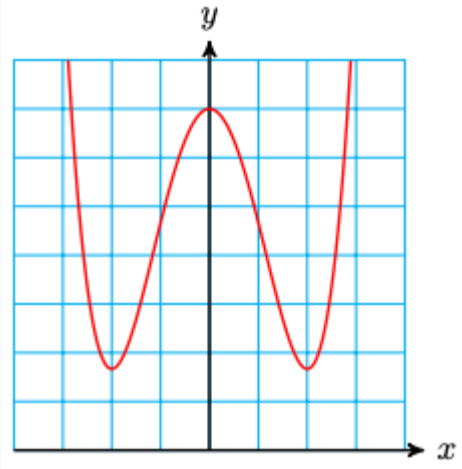


12.



13.





For the following exercises, use technology to graph the function and decide if it has an inverse function.

14. $f(x) = \sin(2x) - \cos(x)$
15. $g(x) = 4e^{-(x/4)^2}$
16. $G(x) = \sqrt{25 - x^2}$
17. $F(x) = \ln(x^3 + 8)$

Answers available at the end of this section.

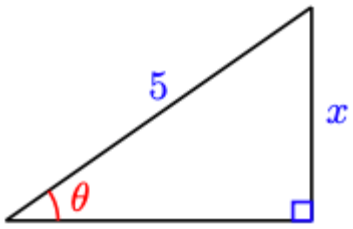
Homework

Concept Check

1. The following is a table of values defining a function f. Make a table of values for f^{-1}.

x	-3	-2	0	1	4
$f(x)$	6	3	1	0	-1

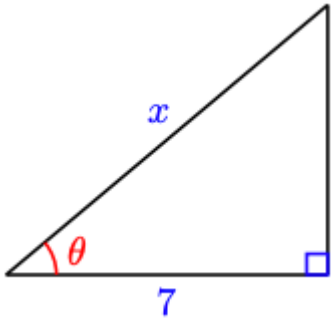
2. What does it mean for a function to be one-to-one? Give an example.
3. Why do we restrict the domains of the trigonometric functions when we define their inverse functions?
4. Which of the following expressions is undefined? Why?
   a. $\cos^{-1}(0)$
   b. $\arctan(-2)$
   c. $\arcsin(-2)$
5. Use your calculator to evaluate $\sin(118°)$, then evaluate $\sin^{-1}$ of your answer. Explain the result.
6.



a. Write x as a function of θ.

b. Write θ as a function of x.

7.



a. Write x as a function of θ.

b. Write θ as a function of x.

Basic Skills

For the following exercises,
(a) Evaluate each pair of angles to the nearest 0.1 radian, and show that they are supplements.
(b) Sketch both angles.
(c) Find the sine of each angle.

8. $\theta = \cos^{-1}\left(\frac{3}{4}\right), \phi = \cos^{-1}\left(-\frac{3}{4}\right)$

9. $\theta = \cos^{-1}\left(\frac{1}{5}\right), \phi = \cos^{-1}\left(-\frac{1}{5}\right)$

10. $\theta = \cos^{-1}(0.1525), \quad \phi = \cos^{-1}(-0.1525)$

11. $\theta = \cos^{-1}(0.6825), \quad \phi = \cos^{-1}(-0.6825)$

For the following exercises, use a calculator to evaluate. Round your answers to the nearest tenth of a radian.

12. $\sin^{-1}(0.2838)$

13. $\tan^{-1}(4.8972)$

14. $\cos^{-1}(0.6894)$

15. $\arccos(-0.8134)$

16. $\arctan(-1.2765)$

17. $\arcsin(-07493)$

For the following exercises, give exact values.

18. $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

19. $\tan^{-1}(-1)$

20. $\sin^{-1}\left(-\frac{1}{2}\right)$

21. $\arccos\left(\frac{\sqrt{3}}{2}\right)$

22. $\arctan\left(\frac{1}{\sqrt{3}}\right)$

23. $\arcsin(-1)$

24. $\arccos(-1)$

25. $\arctan\left(\sqrt{3}\right)$

26. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$

27. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

28. $\arctan\left(-\sqrt{3}\right)$

29. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

30. $\arccos(0)$

31. $\arctan(0)$

32. $\arcsin(0)$

33. $\arccos\left(\frac{1}{2}\right)$

34. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

35. $\arcsin\left(\frac{1}{2}\right)$

36. $\arccos\left(\frac{\sqrt{2}}{2}\right)$

37. $\arctan(1)$

38. $\arcsin\left(\frac{\sqrt{2}}{2}\right)$

39. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

40. $\arcsin(1)$

41. $\arccos(1)$

For the following exercises, without using a calculator, find the exact value or state that it is undefined.

42. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$

43. $\sin\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right)$

44. $\sin\left(\arcsin\left(\frac{3}{5}\right)\right)$

45. $\sin\left(\arcsin\left(-0.42\right)\right)$

46. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$

47. $\cos\left(\arccos\left(\frac{\sqrt{2}}{2}\right)\right)$

48. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$

49. $\cos\left(\arccos\left(\frac{5}{13}\right)\right)$

50. $\cos\left(\arccos\left(-0.998\right)\right)$

51. $\cos\left(\arccos\left(\pi\right)\right)$

52. $\tan\left(\arctan\left(-1\right)\right)$

53. $\tan\left(\arctan\left(\sqrt{3}\right)\right)$

54. $\tan\left(\arctan\left(\frac{5}{12}\right)\right)$

55. $\tan\left(\arctan\left(0.965\right)\right)$

56. $\tan\left(\arctan\left(3\pi\right)\right)$

For the following exercises, find the exact value without using a calculator.

57. $\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right)$

58. $\tan\left(\cos^{-1}\left(\frac{3}{4}\right)\right)$

59. $\cos\left[\tan^{-1}(-2)\right]$

60. $\sin\left[\tan^{-1}\left(-\frac{3}{\sqrt{5}}\right)\right]$

61. $\sin\left[\cos^{-1}\left(-\frac{2\sqrt{6}}{7}\right)\right]$

62. $\cos\left[\sin^{-1}\left(-\frac{2}{7}\right)\right]$

63. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$

64. $\sin\left(\arccos\left(\frac{3}{5}\right)\right)$

65. $\sin\left(\arctan\left(-2\right)\right)$

66. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right)$

67. $\cos\left(\arctan\left(\sqrt{7}\right)\right)$

68. $\tan\left(\arcsin\left(-\frac{2\sqrt{5}}{5}\right)\right)$

69. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right)$

70. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right)$

71. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

72. $\cot\left(\arctan\left(0.25\right)\right)$

73. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

74. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right)$

75. $\sec\left(\arctan\left(10\right)\right)$

76. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right)$

77. $\csc\left(\arctan\left(-\frac{2}{3}\right)\right)$

For the following exercises, simplify the expression and state the domain on which the equivalence is valid.

78. $\tan\left(\cos^{-1}(x)\right)$

79. $\cos\left(\sin^{-1}(h)\right)$

80. $\sin\left(\arccos\left(x\right)\right)$

81. $\cos\left(\arctan\left(x\right)\right)$

82. $\tan\left(\arcsin\left(x\right)\right)$

83. $\sec\left(\arctan\left(x\right)\right)$

84. $\csc\left(\arccos\left(x\right)\right)$

85. $\sin\left(\tan^{-1}(2t)\right)$

86. $\tan\left(\sin^{-1}(3b)\right)$

For the following exercises, find the exact value or state that it is undefined.

87. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right)$

88. $\arcsin\left(\sin\left(-\frac{\pi}{3}\right)\right)$

89. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$

90. $\arcsin\left(\sin\left(\frac{11\pi}{6}\right)\right)$

91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$

92. $\arccos\left(\cos\left(\frac{\pi}{4}\right)\right)$

93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$

94. $\arccos\left(\cos\left(\frac{3\pi}{2}\right)\right)$

95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right)$

96. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right)$

97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$

98. $\arctan\left(\tan\left(-\frac{\pi}{4}\right)\right)$

99. $\arctan\left(\tan\left(\pi\right)\right)$

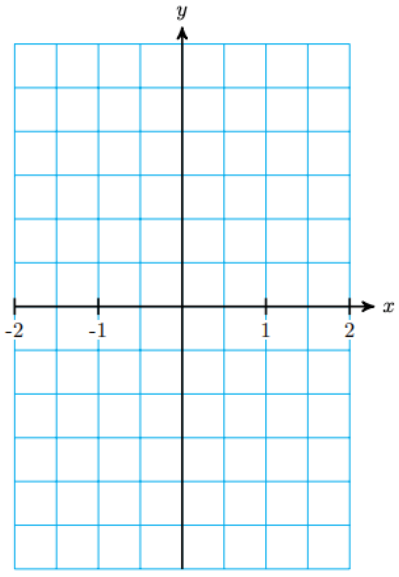
100. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$

101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right)$

For the following exercises, complete the table of values and sketch the function.

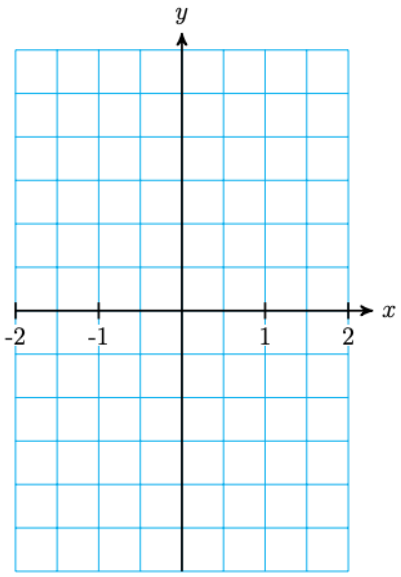
102.

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos^{-1} x$									



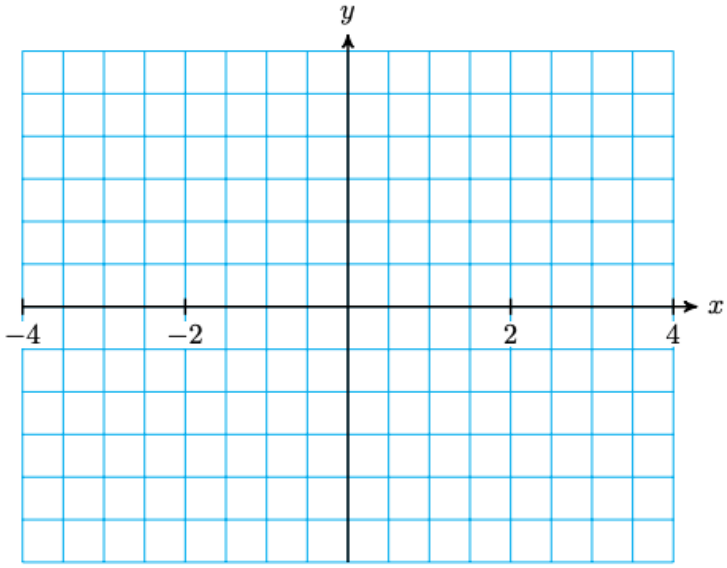
103.

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sin^{-1} x$									



94.

x	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$\tan^{-1} x$							



95. Use a graphing calculator to answer each of the following questions. Then explain the results.

a. Does $\cos^{-1}(x) = \frac{1}{\cos(x)}$?

b. Does $\sin^{-1}(x) = \frac{1}{\sin(x)}$?

c. Does $\tan^{-1}(x) = \frac{1}{\tan(x)}$?

96. a. Sketch a graph of $y = \cos^{-1}(x)$, and label the scales on the axes.

b. Use transformations to sketch graphs of $y = 2\cos^{-1}(x)$ and $y = \cos^{-1}(2x)$.

c. Does $2\cos^{-1}(x) = \cos^{-1}(2x)$?

97. a. Sketch a graph of $y = \sin^{-1}(x)$, and label the scales on the axes.

b. Use transformations to sketch graphs of $y = \frac{1}{2}\sin^{-1}(x)$ and $y = \cos^{-1}\left(\frac{1}{2}x\right)$.

c. Does $\frac{1}{2}\sin^{-1}(x) = \cos^{-1}\left(\frac{1}{2}x\right)$?

98. a. Sketch a graph of $y = \tan^{-1}(x)$, and label the scales on the axes.

b. Use your calculator to graph $y = \frac{\sin^{-1}(x)}{\cos^{-1}(x)}$ on a suitable domain.

c. Does $\tan^{-1}(x) = \frac{\sin^{-1}(x)}{\cos^{-1}(x)}$?

99. a. Use your calculator to sketch $y = \sqrt[3]{x}$ and $\tan^{-1}(x)$ on $[-10, 10]$.

b. Describe the similarities and differences in the two graphs.

For the following exercises, find the exact value for the expression.

100. $\sin\left(\arcsin\left(\frac{5}{13}\right) + \frac{\pi}{4}\right)$

101. $\tan\left(\arctan(3) + \arccos\left(-\frac{3}{5}\right)\right)$

102. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right)$

103. $\sin\left(2\arctan\left(2\right)\right)$

104. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right)$

105. $\sin\left(\frac{\arctan(2)}{2}\right)$

106. $\sin\left(2\tan^{-1}(4)\right)$

107. $\cos\left(2\sin^{-1}\left(\frac{5}{13}\right)\right)$

108. $\tan\left(2\cos^{-1}\left(\frac{3}{4}\right)\right)$

109. $\sin\left(2\cos^{-1}\left(-\frac{4}{5}\right)\right)$

110. $\tan\left(2\sin^{-1}\left(\frac{1}{3}\right)\right)$

111. $\cos\left(2\tan^{-1}\left(\frac{3}{2}\right)\right)$

112. Let $\alpha = \cos^{-1}\left(-\frac{4}{5}\right), \beta = \sin^{-1}\left(\frac{5}{13}\right)$. Find exact values for the following.

a. $\cos(\alpha + \beta)$

b. $\sin(\alpha + \beta)$

c. $\cos(\alpha - \beta)$

d. $\sin(\alpha - \beta)$

113. Let $\alpha = \sin^{-1}\left(-\frac{15}{17}\right), \beta = \tan^{-1}\left(\frac{4}{3}\right)$. Find exact values for the following.

a. $\cos(\alpha + \beta)$

b. $\sin(\alpha + \beta)$

c. $\cos(\alpha - \beta)$

d. $\sin(\alpha - \beta)$

24. Find an exact value for $\sin\left(\tan^{-1}\left(\frac{3}{4}\right) + \sin^{-1}\left(-\frac{4}{5}\right)\right)$.

25. Find an exact value for $\cos\left(\tan^{-1}\left(\frac{5}{12}\right) + \sin^{-1}\left(-\frac{3}{5}\right)\right)$.

For the following exercises, rewrite the expression as an algebraic expression without trigonometric functions.

26. $\sin\left(2\tan^{-1}(x)\right)$

27. $\cos\left(2\sin^{-1}(x)\right)$

28. $\sin\left(2\cos^{-1}(w)\right)$

29. $\cos\left(2\tan^{-1}(w)\right)$

30. $\sin(\arccos(2x))$

31. $\sin\left(\arccos\left(\frac{x}{5}\right)\right)$

32. $\cos\left(\arcsin\left(\frac{x}{2}\right)\right)$

33. $\cos\left(\arctan\left(3x\right)\right)$

34. $\sin(2\arcsin(7x))$

35. $\sin\left(2\arcsin\left(\frac{x\sqrt{3}}{3}\right)\right)$

36. $\cos(2\arcsin(4x))$

37. $\sec(\arctan(2x))\tan(\arctan(2x))$

38. $\sin\left(\arcsin(x) + \arccos(x)\right)$

39. $\cos\left(\arcsin(x) + \arctan(x)\right)$

40. $\tan\left(2\arcsin(x)\right)$

41. $\sin\left(\frac{1}{2}\arctan(x)\right)$

42. If $x = 5\sin(\theta), 0^\circ < \theta < 90^\circ$, express $\sin(2\theta)$ and $\cos(2\theta)$ in terms of x.

43. If $x - 1 = 2\cos(\theta), 0^\circ < \theta < 90^\circ$, express $\sin(2\theta)$ and $\cos(2\theta)$ in terms of x.

44. If $x = 3\tan(\theta)$, write $\theta + \frac{1}{4}\sin(2\theta)$ in terms of x.

45. If $x = 5\cos(\theta)$, write $\frac{\theta}{2} - \cos(2\theta)$ in terms of x.

46. If $\sin(\theta) = \dfrac{x}{2}$ for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, find an expression for $\theta + \sin(2\theta)$ in terms of x.

47. If $\tan(\theta) = \dfrac{x}{7}$ for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, find an expression for $\dfrac{1}{2}\theta - \dfrac{1}{2}\sin(2\theta)$ in terms of x.

48. If $\sec(\theta) = \dfrac{x}{4}$ for $0 < \theta < \dfrac{\pi}{2}$, find an expression for $4\tan(\theta) - 4\theta$ in terms of x.

49. a. For what values of x is the function $f(x) = \sin(\arcsin(x))$ defined?

b. Is $\sin(\arcsin(x)) = x$ for all x where it is defined? If not, for what values of x is the equation false?

c. For what values of x is the function $g(x) = \arcsin(\sin(x))$ defined?

d. Is $\arcsin(\sin(x)) = x$ for all x where it is defined? If not, for what values of x is the equation false?

50. a. For what values of x is the function $f(x) = \cos(\arccos(x))$ defined?

b. Is $\cos(\arccos(x)) = x$ for all x where it is defined? If not, for what values of x is the equation false?

c. For what values of x is the function $g(x) = \arccos(\cos(x))$ defined?

d. Is $\arccos(\cos(x)) = x$ for all x where it is defined? If not, for what values of x is the equation false?

Synthesis Questions

For the following exercises, state the domain of the function.

51. $f(x) = \arcsin(5x)$

52. $f(x) = \arccos\left(\frac{3x-1}{2}\right)$

53. $f(x) = \arcsin\left(2x^2\right)$

54. $f(x) = \arccos\left(\frac{1}{x^2-4}\right)$

55. $f(x) = \arctan(4x)$

Applications

For the following exercises, sketch a figure to help you model each problem.

56. Ivan is watching the launch of a satellite at Cape Canaveral. The viewing area is 500 yards from the launch site. The angle of elevation, θ, of Ivan's line of sight increases as the booster rocket rises.

a. Write a formula for the height, h, of the rocket as a function of θ.

b. Write a formula for θ as a function of h.

c. Evaluate the formula in part (b) for $h = 1000$, and interpret the result.

57. Camille's house lies under the flight path from the city airport, and commercial airliners pass overhead at an altitude of 15,000 feet. As Camille watches an airplane recede, its angle of elevation, θ, decreases.

a. Write a formula for the horizontal distance, d, to the airplane as a function of θ.

b. Write a formula for θ as a function of d.

c. Evaluate the formula in part (b) for $d = 20,000$, and interpret the result.

58. While driving along the interstate, you approach an enormous 50-foot-wide billboard that sits just beside the road. Your viewing angle, θ, increases as you get closer to the billboard.

a. Write a formula for your distance, d, from the billboard as a function of θ.

b. Write a formula for θ as a function of d.

c. Evaluate the formula in part (b) for $d = 200$, and interpret the result.

59. Sang is walking along the bank of a straight river toward a 200-meter-long bridge over the river. Let θ be the angle subtended horizontally by Sang's view of the bridge.

    a. Write a formula for Sang's distance from the bridge, d, as a function of θ.

    b. Write a formula for θ as a function of d.

    c. Evaluate the formula in part (b) for $d = 500$, and interpret the result.

60. Wyatt is viewing a 4-meter tall painting whose base is 1 meter above his eye level.

    a. Write a formula for α, the angle subtended from Wyatt's eye level to the bottom of the painting, when he stands x meters from the wall.

    b. Write a formula for β, the angle subtended by the painting, in terms of x.

    c. Evaluate the formula in part (b) for $x = 5$, and interpret the result.

61. A 5-foot-tall mirror is positioned so that its bottom is 1.5 feet below Kim's eye level.

    a. Write a formula for α, the angle subtended by the section of the mirror below Kim's eye level, when she stands x feet from the mirror.

    b. Write a formula for θ, the angle subtended by the entire mirror, in terms of x.

    c. Evaluate the formula in part (b) for $x = 10$, and interpret the result.

Challenge Problems

62. Use your calculator to graph $y = \sin^{-1}(x) + \cos^{-1}(x)$.

    a. State the domain and range of the graph.

    b. Explain why the graph looks as it does.

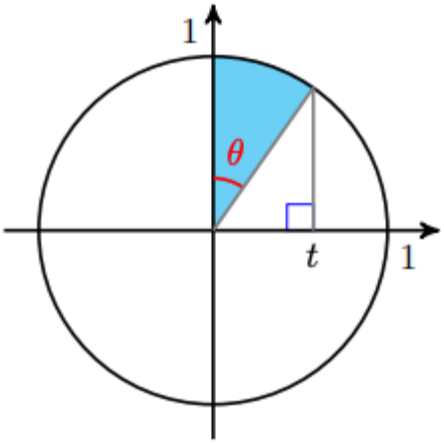
63. Use your calculator to graph $y = \tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$.

    a. State the domain and range of the graph.

    b. Explain why the graph looks as it does.

64. Show that $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$ for $-1 \leq x \leq 1$.

65. Use the figure of a unit circle to answer the following.

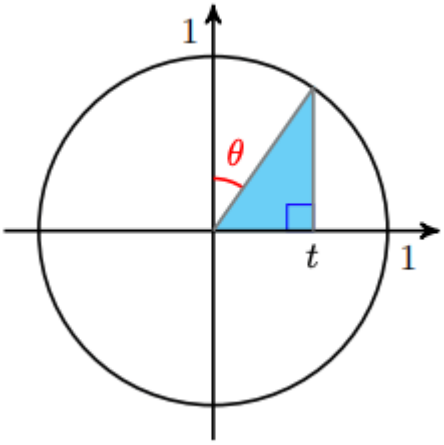


a. Write an expression for the area of the shaded sector in terms of θ.

b. How are θ and t related in the figure? (Hint: Write an expression for $\sin(\theta)$.)

c. Combine your answers to (a) and (b) to write an expression for the area of the sector in terms of t.

56. Use the figure of a unit circle to answer the following.



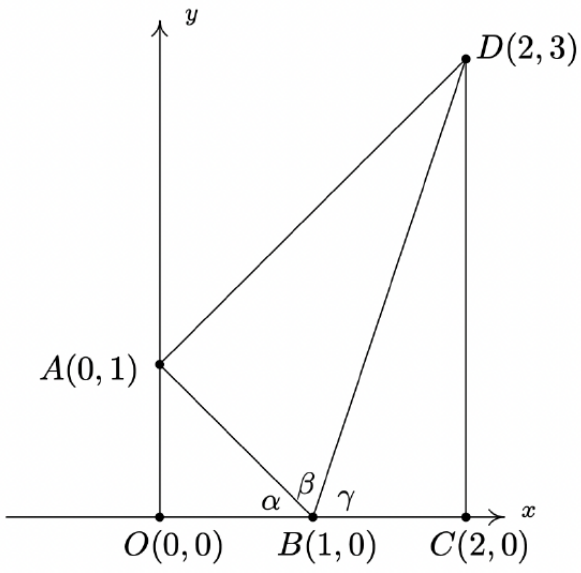
a. Write an expression for the height of the shaded triangle in terms of t. (Hint: Use the Pythagorean Theorem.)

b. Write an expression for the area of the triangle in terms of t.

c. Combine your answers to (b) and to the previous problem to write an expression for the area bounded above by the upper semicircle, below by the x-axis, on the left by the y-axis, and on the right by $x = t$, when $0 \leq t \leq 1$.

57. Use the following picture and the series of exercises on the next page to show that

$$\arctan(1) + \arctan(2) + \arctan(3) = \pi$$



a. Clearly $\triangle AOB$ and $\triangle BCD$ are right triangles because the line through O and A and the line through C and D are perpendicular to the x-axis. Use the distance formula to show that $\triangle BAD$ is also a right triangle (with $\angle BAD$ being the right angle) by showing that the sides of the triangle satisfy the Pythagorean Theorem.

b. Use $\triangle AOB$ to show that $\alpha = \arctan(1)$

c. Use $\triangle BAD$ to show that $\beta = \arctan(2)$

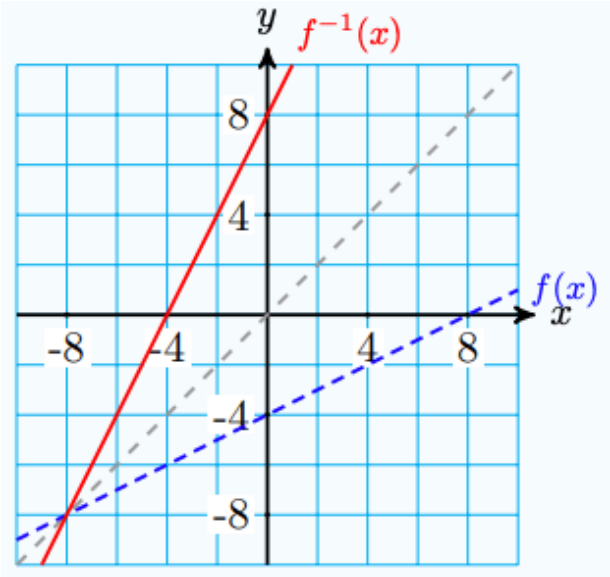
d. Use $\triangle BCD$ to show that $\gamma = \arctan(3)$

e. Use the fact that O, B and C all lie on the x-axis to conclude that $\alpha + \beta + \gamma = \pi$. Thus $\arctan(1) + \arctan(2) + \arctan(3) = \pi$.

Answers

Skills Refresher

1. No inverse exists for this function.
2. $g^{-1}(x) = \sqrt[3]{x} + 2$
3. $\dfrac{x-1}{5x-3}$
4.
   a. $f^{-1}(x) = 2x + 8$
   b. Dom: $(-\infty, \infty)$ Rge: $(-\infty, \infty)$
   c.

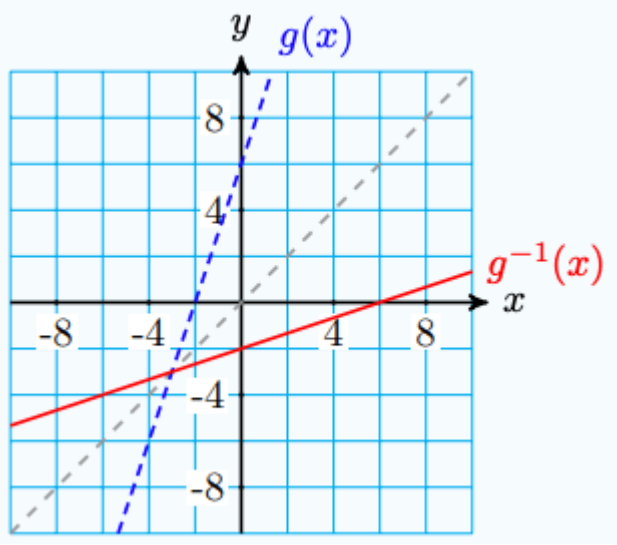


5.

a. $g^{-1}(x) = \frac{1}{3}x - 2$

b. Dom: $(-\infty, \infty)$ Rge: $(-\infty, \infty)$

c.

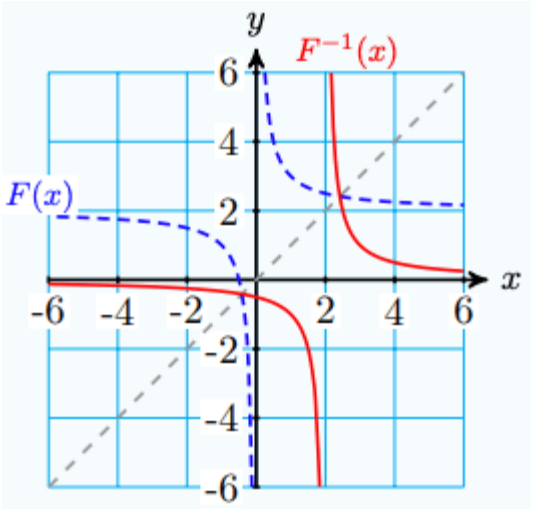


6.

a. $F^{-1}(x) = \frac{1}{x-2}$

b. Dom: $x \neq 0$ Rge: $y \neq 2$

c.

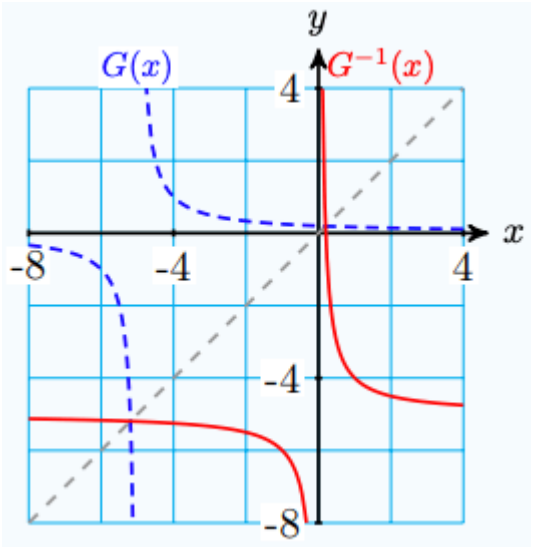


7.

a. $G^{-1}(x) = \frac{1}{x} - 5$

b. Dom: $x \neq -5$ Rge: $y \neq 0$

c.

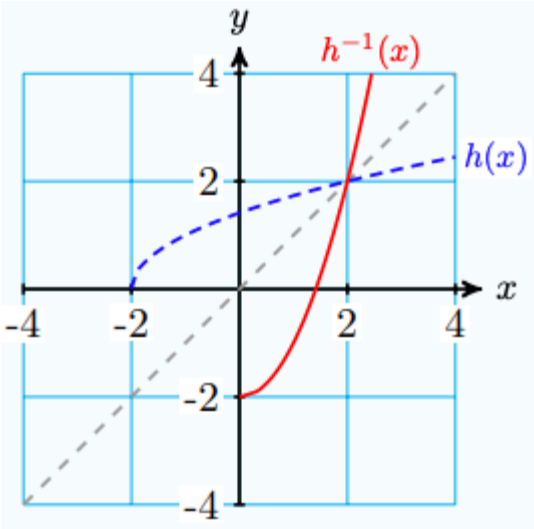


8.

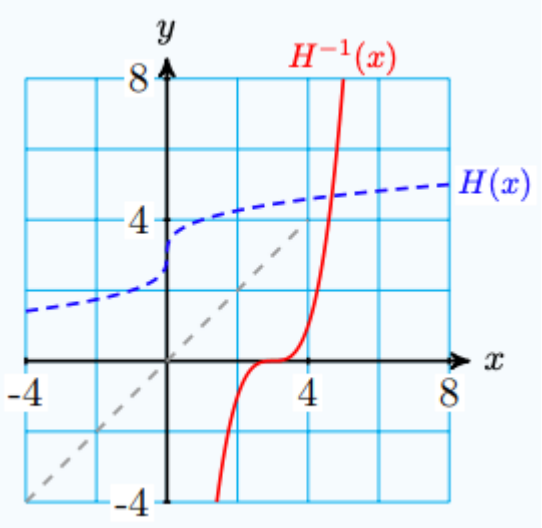
a. $h^{-1}(x) = x^2 - 2$

b. Dom: $x \geq 0$ Rge: $y \geq -2$

c.



9.
   a. $H^{-1}(x) = (x-3)^3$
   b. Dom: $(-\infty, \infty)$ Rge: $(-\infty, \infty)$
   c.



10. No. This graph does not pass the Horizontal Line Test.
11. Yes. This graph passes the Horizontal Line Test.
12. Yes. This graph passes the Horizontal Line Test.
13. No. This graph does not pass the Horizontal Line Test.
14. No.
15. No.
16. No.
17. Yes.

This page titled 10.1.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

10.2: The Remaining Inverse Trigonometric Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

In the previous section, we completely ignored the inverses of the secant, cosecant, and cotangent. This is not because they are unimportant—in fact, the inverses of the secant and cosecant are essential parts of Integral Calculus (also known as Calculus II). We have delayed their introduction because it's best to master the inverses of the fundamental trigonometric functions before jumping into the remaining inverse trigonometric functions.

The Inverse Cotangent Function

Like the rest of the trigonometric functions, the cotangent is not naturally one-to-one. Therefore, we restrict the domain of $g(x) = \cot(x)$ to its principal cycle on $(0, \pi)$ to obtain a one-to-one segment of the cotangent. The vertical asymptotes, $x = 0$ and $x = \pi$, of the graph of $g(x) = \cot(x)$ become the horizontal asymptotes, $y = 0$ and $y = \pi$, of the graph of $g^{-1}(x) = \operatorname{arccot}(x)$ (which, recall, is a reflection of the graph of $y = \cot(x)$ about the line $y = x$).

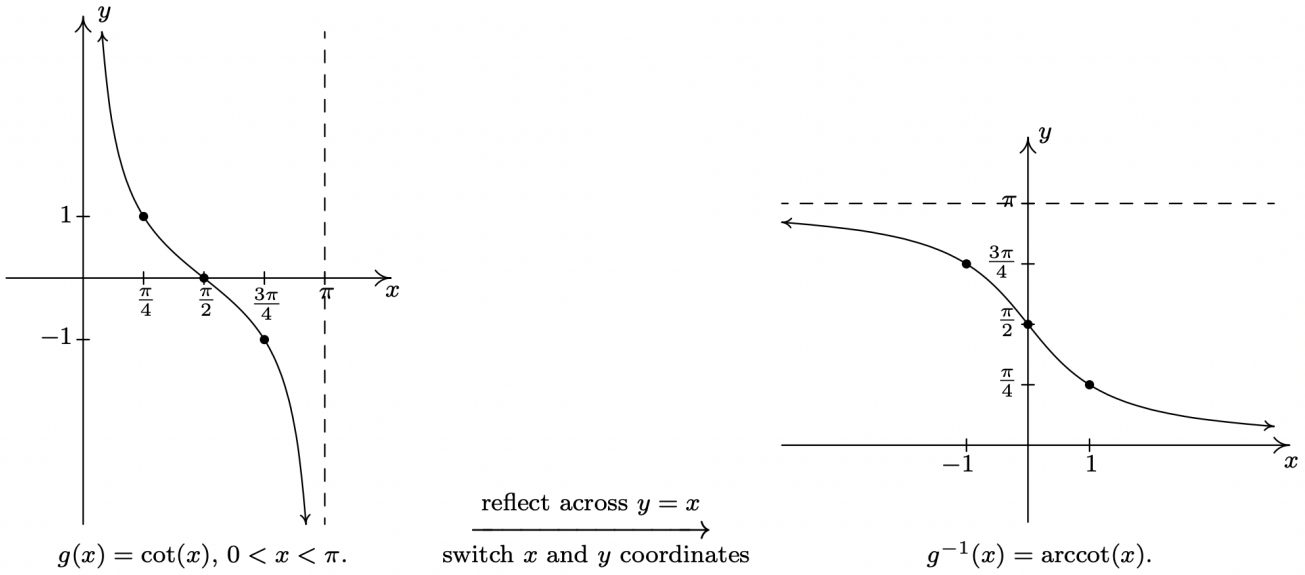


Figure 10.2.1

The definition of the arccotangent follows the same pattern as the definitions of our other inverse trigonometric functions.

Definition: Inverse Cotangent Function

The function $f(x) = \cot^{-1}(x)$ is defined as follows:

$$\cot^{-1}(x) = \theta \quad \text{if and only if} \quad \cot(\theta) = x \quad \text{and} \quad 0 < \theta < \pi.$$

Again, I want to be clear: Very technically, the function $\cot^{-1}(x)$ returns an *angle* and $\operatorname{arccot}(x)$ returns a *real number*. This is true for all inverse trigonometric functions. Therefore, it is valid for $\cot^{-1}(x)$ to return an angle in degree measure or radian measure;

however, it is technically not valid for $\operatorname{arccot}(x)$ to return a number in degree measure (as this is an angle). Since radians is a unitless measure, a radian is just a real number, so $\operatorname{arccot}(x)$ can return a number in radian measure. The moral of the story is that you are always safe to work in radian measure when dealing with inverse trigonometric functions, but working in degree measure is a little risky.

Theorem: Properties of the Inverse Cotangent Function

Properties of $f(x) = \cot^{-1}(x)$

- $\theta = \cot^{-1}(x)$ if and only if $\cot(\theta) = x$ where $0 < \theta < \pi$
- Domain: $(-\infty, \infty)$
- Range: $(0, \pi)$
- As $x \to -\infty$, $\cot^{-1}(x) \to \pi^-$, and as $x \to \infty$, $\cot^{-1}(x) \to 0^+$
- $\cot(\cot^{-1}(x)) = x$ for all $x \in \mathbb{R}$
- $\cot^{-1}(\cot(x)) = x$ provided $0 < x < \pi$

Example 10.2.1

a. Find the exact values of the following.
   i. $\operatorname{arccot}(-\sqrt{3})$
   ii. $\cot(\operatorname{arccot}(-5))$

b. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

$$\cos(\operatorname{arccot}(2x))$$

Solutions available at the end of this section.

Checkpoint 10.2.1

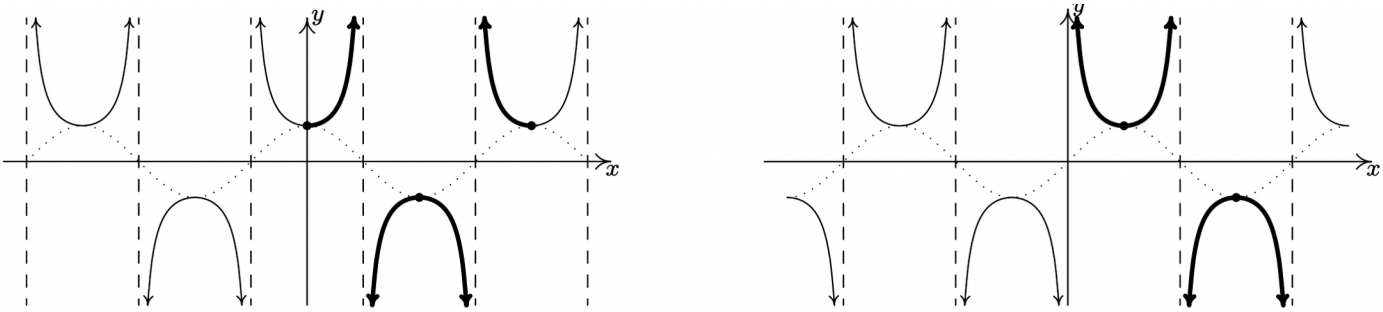
Find the exact value of the expression.

$$\sin\left(\cot^{-1}\left(-\frac{5}{2}\right)\right)$$

Answer available at the end of this section.

The Inverse Secant and Cosecant Functions

The last two functions to invert are secant and cosecant. The principal cycle for each is highlighted in the figure below.



The graph of $y = \sec(x)$. The graph of $y = \csc(x)$.

Figure 10.2.3

It is clear from the graph of secant that we cannot find one single, continuous piece of its graph which covers its entire range of

$(-\infty, -1] \cup [1, \infty)$, and which restricts the domain of the function so that it is one-to-one. The same is true for cosecant. Thus, in order to define the arcsecant and arccosecant functions, we must settle for a piecewise approach wherein we choose one piece to cover the top of the range, namely $[1, \infty)$, and another piece to cover the bottom, namely $(-\infty, -1]$. To make the definitions of the inverse secant and cosecant functions align with our previous work, we restrict their domains to coincide with the restrictions on cosine and sine, respectively.

It is critical to note that there are two ways to restrict the domains of the secant and cosecant functions. The method shown in this text aligns with all other texts in Trigonometry and Precalculus; however, there is a *much* better choice of domain restriction for these functions that makes Calculus (specifically, Integral Calculus) work much more smoothly. Despite this, we give the traditional definition here because it aligns with all of our previous work.

When defining the arccosine, we restricted the cosine to the interval $[0, \pi]$; however, such a restriction needs to be modified for the secant because the secant is undefined at $x = \frac{\pi}{2}$. Thus, we restrict the domain of the secant to $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ to help us define the arcsecant.

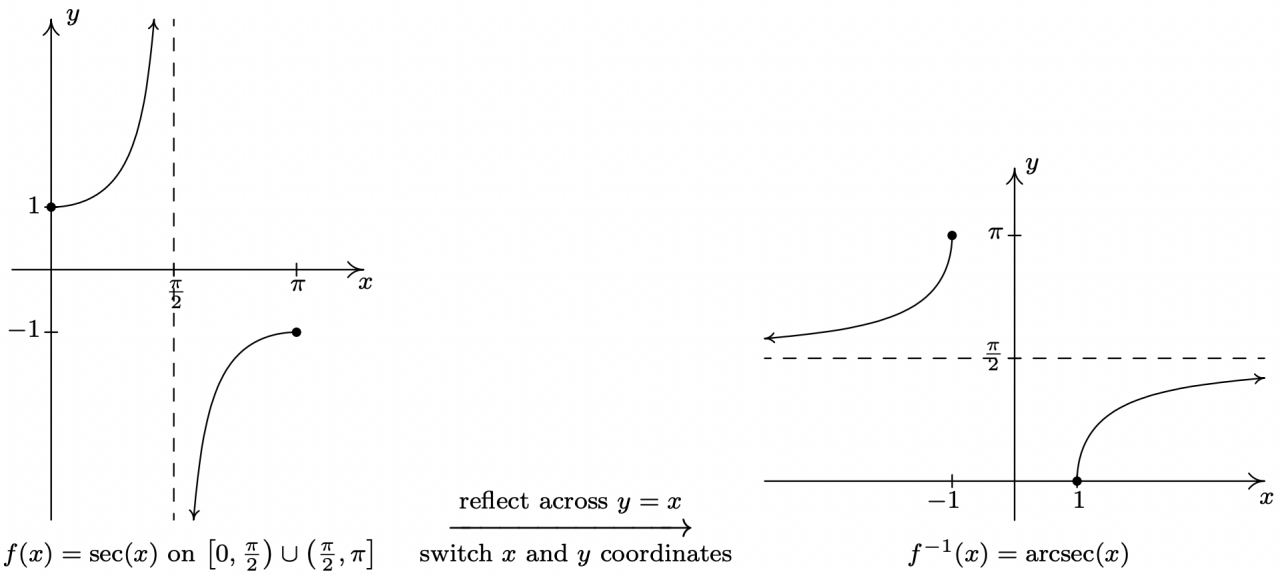


Figure 10.2.4

This gives us the following definition.

Definition: The Inverse Secant Function

The function $f(x) = \sec^{-1}(x)$ is defined as follows:

$$\sec^{-1}(x) = \theta \quad \text{if and only if} \quad \sec(\theta) = x \quad \text{and} \quad \theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right].$$

Likewise, we restricted the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ when defining the arcsine, but such a restriction will not work for the arccosecant because the cosecant is undefined at $x = 0$. Therefore, we restrict $g(x) = \csc(x)$ to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ to define the arccosecant.

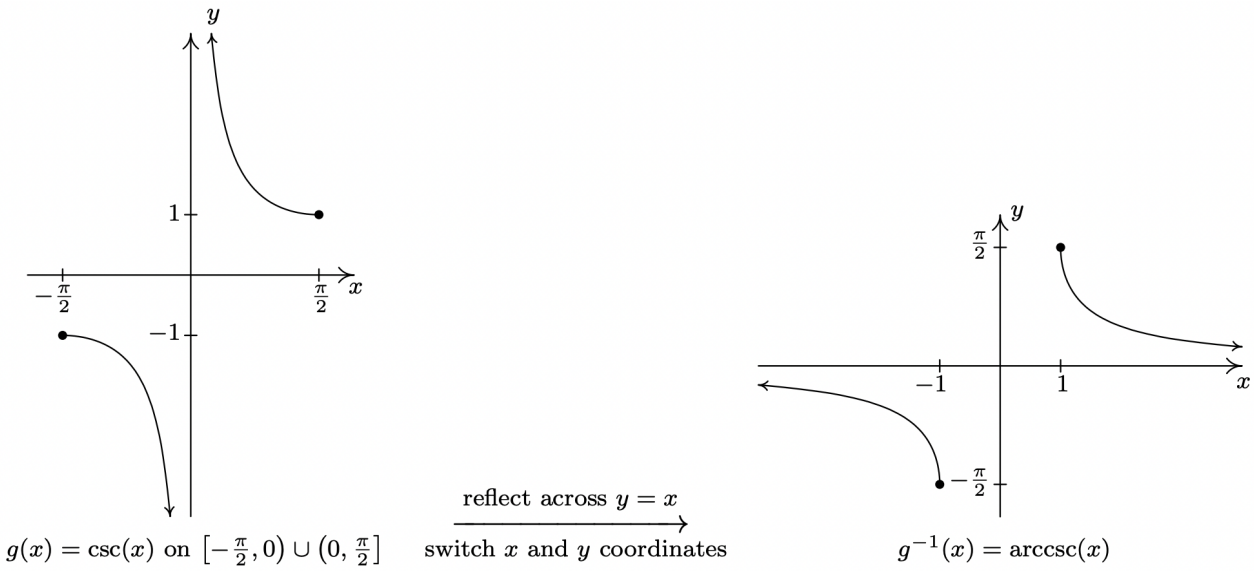


Figure 10.2.5

Note that, for both arcsecant and arccosecant, the domain is $(-\infty,-1]\cup[1,\infty)$. We can rewrite this using set-builder notation as $\{x \mid |x| \geq 1\}$. This is often done in Calculus textbooks, so we include it here for completeness.

Definition: The Inverse Cosecant Function

The function $f(x)=\csc^{-1}(x)$ is defined as follows:

$$\csc^{-1}(x)=\theta \quad \text{if and only if} \quad \csc(\theta)=x \quad \text{and} \quad \theta^{\text{int}}\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right].$$

Using these definitions, we get the following properties of the arcsecant and arccosecant functions.

Theorem: Properties of the Inverse Secant and Inverse Cosecant Functions

Properties of $f(x)=\operatorname{arcsec}(x)$

- $\sec^{-1}(x)=\theta$ if and only if $\sec(\theta)=x$ where $\theta^{\text{int}}\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$
- Domain: $\{x:|x|\geq 1\}=(-\infty,-1]\cup[1,\infty)$
- Range: $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$
- As $x\to-\infty$, $\operatorname{arcsec}(x)\to\frac{\pi}{2}^{+}$, and as $x\to\infty$, $\operatorname{arcsec}(x)\to\frac{\pi}{2}^{-}$
- $\sec(\operatorname{arcsec}(x))=x$ provided $|x|\geq 1$
- $\operatorname{arcsec}(\sec(x))=x$ provided $0\leq x<\frac{\pi}{2}$ or $\frac{\pi}{2}<x\leq\pi$

Properties of $f(x)=\operatorname{arccsc}(x)$

- $\csc^{-1}(x)=\theta$ if and only if $\csc(\theta)=x$ where $\theta^{\text{int}}\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
- Domain: $\{x:|x|\geq 1\}=(-\infty,-1]\cup[1,\infty)$
- Range: $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
- As $x\to-\infty$, $\operatorname{arccsc}(x)\to 0^{-}$, and as $x\to\infty$, $\operatorname{arccsc}(x)\to 0^{+}$
- $\csc(\operatorname{arccsc}(x))=x$ provided $|x|\geq 1$
- $\operatorname{arccsc}(\csc(x))=x$ provided $-\frac{\pi}{2}\leq x<0$ or $0<x\leq\frac{\pi}{2}$
- Additionally, arccosecant is an odd function.

Example 10.2.2

a. Find the exact values of the following.
   i. $\sec^{-1}(2)$
   ii. $\operatorname{arccsc}(-2)$
   iii. $\operatorname{arcsec}\left(\sec\left(\frac{8\pi}{7}\right)\right)$
   iv. $\cot\left(\operatorname{arccsc}\left(-3\right)\right)$

b. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.
   i. $\tan(\operatorname{arcsec}(x))$
   ii. $\cos(\operatorname{arccsc}(4x))$

Solutions available at the end of this section.

Checkpoint 10.2.2

Rewrite the function as one that does not involve trigonometric functions and state the domain.

$$f(x) = \sin\left(\sec^{-1}\left(2x\right)\right)$$

(Your answer might be a piecewise function or it might involve absolute values.)

Answer available at the end of this section.

Calculators and the Non-Fundamental Inverse Trigonometric Functions

In the sections to come, we will need to approximate the values of the inverse trigonometric functions. We have already discussed how to use technology to approximate values of the arcsine, arccosine, and arctangent functions; however, if we are asked for an approximation of the arccotangent, arcsecant, or arccosecant, how is this to be done when our calculators do not have these functions?

Example 10.2.3

a. Use a calculator to approximate the following values to four decimal places.
   i. $\cot^{-1}(2)$
   ii. $\operatorname{arcsec}(5)$
   iii. $\operatorname{arccot}(-2)$
   iv. $\csc^{-1}\left(-\frac{3}{2}\right)$

b. Find the domain and range of

$$f(x) = \frac{\pi}{2} - \sec^{-1}\left(\frac{x}{5}\right).$$

Solutions available at the end of this section.

Checkpoint 10.2.3

Use a calculator to approximate $\operatorname{arcsec}\left(-\frac{7}{3}\right)$.

Answer available at the end of this section.

The inverse trigonometric functions are typically found in applications whenever the measure of an angle is required. The following example presents one such scenario.

Example 10.2.4

The roof on the house below has a "$6/12$ pitch." This means that when viewed from the side, the roof line has a rise of 6 feet over a run of 12 feet. Find the angle of inclination from the bottom of the roof to the top of the roof. Express your answer in decimal degrees, rounded to the nearest hundredth of a degree.



Figure 10.2.8

Solution available at the end of this section.

Success in College: The Classroom and Learning Preferences

Read the strategies suggested for your learning preferences below.

Visual Learners

Strategies for visual learners include the following:

- take detailed notes during class
- reorganize and rewrite notes after class
- rework important examples
- watch the videos linked to your textbook
- use flashcards to study vocabulary

Auditory Learners

Strategies for auditory learners include the following:

- record your class, take brief notes of the big ideas and examples, and listen to the recording after class
- restate the main ideas from the lecture aloud to yourself
- talk to yourself as you do your homework, explaining each step to yourself
- study vocabulary by repeating definitions (listen to a recording of the words and definitions)

Kinesthetic Learners

Strategies for kinesthetic learners include the following:

- record the lectures, take brief notes of the big ideas and examples, and listen to the recording after class
- draw pictures, when possible
- with your finger, trace diagrams and graphs - don't just look at them
- imagine symbols as three-dimensional objects, or even cartoon characters
- study vocabulary as you exercise or walk to your car

- walk around your room as you repeat definitions

1. Do you use any of these strategies currently? If not, should you be?
2. Can you suggest any other ideas that I should add to this list?

Success in Mathematics: Why Study Mathematics? (Part 4)

Dr. Robert Cabin teaches biology in New York State. He writes,

> "... but many students are disillusioned by and uninterested in academics. They are on our campuses because they think they need a degree to get a good job, because their parents expect them to go to college, or because they just don't know what else to do."
>
> *Source: Chronicle of Higher Education, October 2003.*

1. Dr. Cabin states that many students are uninterested in academics. What do you think he means by academics? Explain.
2. Are you interested in academics? Explain.

Answers

Section Preview

For those who don't know, the Stewart Calculus textbook and, subsequently, Cosumnes River College's OER Calculus textbook use a different range definition for the secant and cosecant functions than introduced in this section. The redefinition that occurs in Calculus is to make some developments in Calculus II work more smoothly; however, I chose to stick with the conventional definitions in this text so students going to other campuses (or for those who are using the online homework) don't have to worry about things.

Section Preview

- Find the exact value of a non-fundamental inverse trigonometric function.
- Use technology to approximate the value of a non-fundamental inverse trigonometric function.
- Evaluate compositions involving trigonometric functions and non-fundamental inverse trigonometric functions.
- Simplify expressions involving trigonometric functions and non-fundamental inverse trigonometric functions.

Example 10.2.1

a. i. $\operatorname{arccot}(-\sqrt{3}) = \theta$ means $\cot(\theta) = -\sqrt{3}$, where $\theta^{\text{int}}\ (0, \pi)$.

Quadrant: Since the cotangent is negative, we further restrict the angle to $\theta^{\text{int}}\ (\frac{\pi}{2}, \pi)$, or Quadrant II.

Reference Angle: The reference angle for the cotangent that gives $\cot\left(\hat{\theta}\right) = \sqrt{3}$ is $\hat{\theta} = \frac{\pi}{6}$.

Given the quadrant and reference angle, we get

$$\operatorname{arccot}\left(-\sqrt{3}\right) = \frac{5\pi}{6}.$$

ii. It is helpful to remember that $\operatorname{trig}\left(\operatorname{trig}^{-1}(x)\right) = x$ for all of the trigonometric functions as long as x is in the domain of trig^{-1} (which is the range of trig). Therefore, $\cot(\operatorname{arccot}(-5)) = \cot\left(\cot^{-1}(-5)\right) = -5$ because -5 is definitely in the domain of the inverse cotangent.

Another way to think about this is to let $\theta = \operatorname{arccot}(-5)$. Then θ belongs to the interval $(0, \pi)$ and $\cot(\theta) = -5$. Hence, $\cot(\operatorname{arccot}(-5)) = \cot(\theta) = -5$.

b. To get started, we let $\theta = \operatorname{arccot}(2x)$ so that $\cot(\theta) = 2x$ where $0 < \theta < \pi$. We then sketch two reference triangles, one in QI and the other in QII.

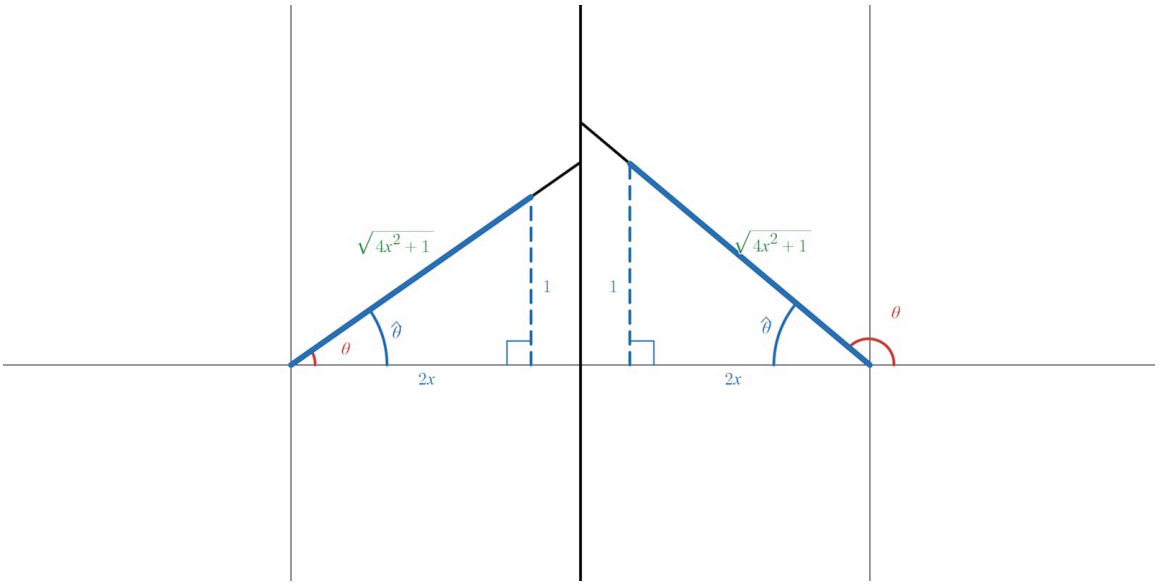


Figure 10.2.2

In terms of θ,

$$\cos(\operatorname{arccot}(2x)) = \cos(\theta).$$

Therefore, we can use our reference triangles to evaluate $\cos(\theta)$. Looking at the reference triangle in QI, $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$. Likewise, the reference triangle from the QII sketch yields $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$. Hence, it appears that

$$\cos(\operatorname{arccot}(2x)) = \cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}.$$

Being good mathematicians, we take a moment to reflect and make sure our logic is solid. What might give us pause is the fact that our second sketch, where θ^{int}QII, leads to cosine being negative. Still, our result, $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$, looks to be positive; however, looks can be deceiving!

The expression, $2x$, in the second sketch, *is*, in fact, negative. The negativity of it is "hidden" within the variable x. That is, if the x-value of the point on the upper-left vertex of that reference triangle is -4, then $2x = -4 \implies x = -2$. Hence, when θ^{int}QII, $\cos(\theta) = \frac{2x}{\sqrt{4x^2+1}}$ is negative (which is what we want).

We now concern ourselves with the domain of the function. We know that the domain of the arccotangent is all real numbers. Hence, the domain of $\operatorname{arccot}(2x)$ is also all real numbers. Moreover, the range of the arccotangent is $(0, \pi)$. Since the outputs of $\operatorname{arccot}(2x)$ are the inputs of $\cos(\operatorname{arccot}(2x))$ and the cosine can take any value as an input, the domain remains unchanged. That is, the domain of $y = \cos(\operatorname{arccot}(2x))$ is $(-\infty, \infty)$.

Checkpoint 10.2.1

$$\frac{2}{\sqrt{29}}$$

Example 10.2.2

a. i. Let $\theta = \sec^{-1}(2)$. Then $\sec(\theta) = 2$, where $\theta^{\text{int}} \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. Of these two intervals, the only one where secant is positive is $\left[0, \frac{\pi}{2}\right)$. Thus, θ^{int}QI. Moreover, $\sec\left(\hat{\theta}\right) = 2$ implies $\cos\left(\hat{\theta}\right) = \frac{1}{2}$. Hence, $\hat{\theta} = \frac{\pi}{3}$. Therefore,

$$\sec^{-1}(2) = \theta = \frac{\pi}{3}.$$

ii. We now start to streamline our arguments.

$$\begin{aligned}
& \operatorname{arccsc}(-2) = \theta && \text{(inverse trigonometric functions return angles,)} \\
\implies\ & \csc(\theta) = -2, \quad \text{where } \theta^{\text{int}} \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] && \text{(but the angle is restricted)} \\
\implies\ & \theta \overset{\text{int}}{=} \left[-\frac{\pi}{2}, 0\right) && \text{(cosecant is negative)} \\
\text{and}\ & \hat{\theta} = \frac{\pi}{6} \\
\implies\ & \operatorname{arccsc}(-2) = -\frac{\pi}{6} && \left(\text{reference angle of } \frac{\pi}{6} \text{ in the interval } \left[-\frac{\pi}{2}, 0\right)\right)
\end{aligned}$$

iii. Our choice of non-standard angle for this example is purposeful. It's to force you to think about the process instead of focusing solely on computing.

$$\begin{aligned}
& \operatorname{arcsec}\left(\sec\left(\frac{8\pi}{7}\right)\right) = \theta, \quad \text{where } \theta^{\text{int}} \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] && \text{(inverse trigonometric functions return restricted angles)} \\
\implies\ & \sec(\theta) = \sec\left(\frac{8\pi}{7}\right)
\end{aligned}$$

Therefore, the reference angle for θ matches the reference angle for $\frac{8\pi}{7}$. That is,

$$\hat{\theta} = \frac{\pi}{7}.$$

We now need to determine in which quadrant θ belongs. Since $\frac{8\pi}{7}{}^{\text{int}}$QIII, $\sec\left(\frac{8\pi}{7}\right)$ is negative. Moreover, since $\sec(\theta) = \sec\left(\frac{8\pi}{7}\right)$, $\sec(\theta)$ must be negative. Hence, $\theta^{\text{int}}\left(\frac{\pi}{2}, \pi\right]$. Thus, $\theta = \frac{6\pi}{7}$. This means that

$$\operatorname{arcsec}\left(\sec\left(\frac{8\pi}{7}\right)\right) = \theta = \frac{6\pi}{7}.$$

iv. As usual, we begin by addressing the fact that the inverse trigonometric function is an angle.

$$\begin{aligned} \operatorname{arccsc}(-3) &= \theta, \quad \text{where } \theta^{\text{int}}\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \quad \text{(inverse trigonometric functions return restricted angles)} \\ \implies \quad \csc(\theta) &= -3 \end{aligned}$$

However, we are interested in

$$\cot\left(\operatorname{arccsc}(-3)\right) = \cot(\theta).$$

Since the cosecant is negative, we know that $\theta^{\text{int}}\left[-\frac{\pi}{2}, 0\right)$. Let us sketch a reference triangle in QIV to help us determine $\cot(\theta)$.

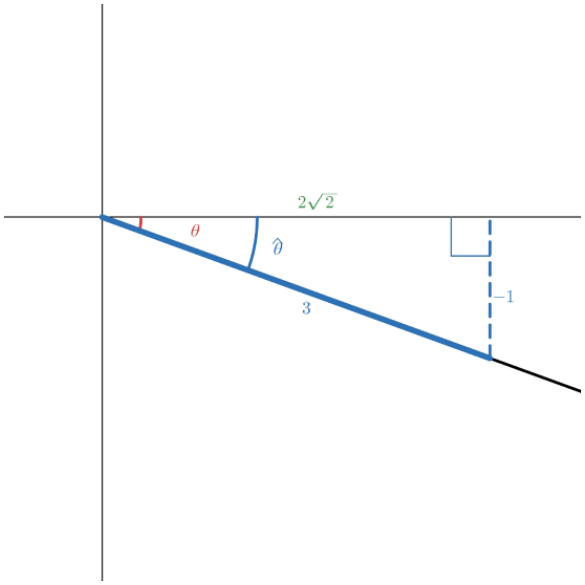


Figure 10.2.6

From the reference triangle, we see that

$$\cot\left(\operatorname{arccsc}(-3)\right) = \cot(\theta) = -2\sqrt{2}.$$

b. i. As per usual, let $\theta = \operatorname{arcsec}(x)$ so that $\sec(\theta) = x$, where we restrict θ to the interval $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. We then sketch two reference triangles, one for $\theta^{\text{int}}\left[0, \frac{\pi}{2}\right)$ and the other for $\theta^{\text{int}}\left(\frac{\pi}{2}, \pi\right]$.

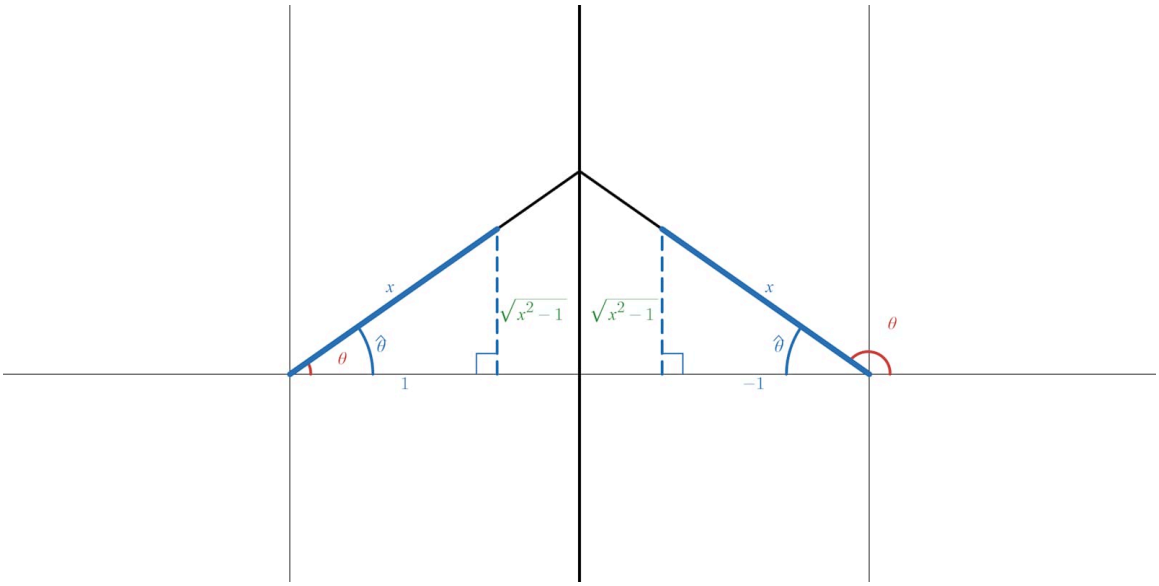


Figure 10.2.7

We now use these reference triangles to help us determine the value of $\tan(\operatorname{arcsec}(x)) = \tan(\theta)$. In this case, we run into a very interesting problem. In Quadrant I, $\tan(\theta) = \sqrt{x^2-1}$; however, in Quadrant II, $\tan(\theta) = -\sqrt{x^2-1}$. Pausing for a moment, let's think about this before going further. Does this make sense?

We know tangent is supposed to be positive in Quadrant I. Since $\sec(\theta) = x$, we know either $x \geq 1$ or $x \leq -1$. Luckily, $\sqrt{x^2-1}$ is definitely positive for these values of x (unless $x = \pm 1$, but that would mean $\tan(\theta) = 0 \implies \theta = 0$ which is not in Quadrant I). So the statement, $\tan(\theta) = \sqrt{x^2-1}$ when $\theta^{\text{int}}\left(0, \frac{\pi}{2}\right)$ makes sense.

In Quadrant II, the tangent is supposed to be negative. It's easy to see that $-\sqrt{x^2-1}$ is negative whenever $\theta^{\text{int}}\left(\frac{\pi}{2}, \pi\right)$. Thus, the statement, $\tan(\theta) = -\sqrt{x^2-1}$ when $\theta^{\text{int}}\left(\frac{\pi}{2}, \pi\right)$ also makes sense. Therefore,

$$\tan(\theta) = \begin{cases} \sqrt{x^2-1}, & \text{if } 0 \leq \theta < \dfrac{\pi}{2} \\ -\sqrt{x^2-1}, & \text{if } \dfrac{\pi}{2} < \theta \leq \pi \end{cases}$$

Now we need to determine what these conditions on θ mean for x. Since $x = \sec(\theta)$, when $0 \leq \theta < \frac{\pi}{2}$, $x \geq 1$, and when $\frac{\pi}{2} < \theta \leq \pi$, $x \leq -1$. Since we encountered no further restrictions on θ, the equivalence below holds for all x in $(-\infty, -1] \cup [1, \infty)$.

$$\tan(\operatorname{arcsec}(x)) = \begin{cases} \sqrt{x^2-1}, & \text{if } x \geq 1 \\ -\sqrt{x^2-1}, & \text{if } x \leq -1 \end{cases}$$

ii. While I always use the graphing tactic, you do not have to think geometrically to work with inverse trigonometric functions. You can rely solely on identities.

For example, to simplify $\cos(\operatorname{arccsc}(4x))$, we start by letting $\theta = \operatorname{arccsc}(4x)$. Then $\csc(\theta) = 4x$ for $\theta^{\text{int}}\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

We now set about finding an expression for $\cos(\operatorname{arccsc}(4x)) = \cos(\theta)$. Since $\cos(\theta)$ is defined for all θ, we do not encounter any additional restrictions on θ. From $\csc(\theta) = 4x$, we get $\sin(\theta) = \frac{1}{4x}$, so to find $\cos(\theta)$, we can make use of the Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$. Substituting $\sin(\theta) = \frac{1}{4x}$ gives

$$\cos^2(\theta) + \left(\frac{1}{4x}\right)^2 = 1.$$

Solving, we get

$$\cos(\theta) = \pm\sqrt{\frac{16x^2-1}{16x^2}} = \pm\frac{\sqrt{16x^2-1}}{4|x|}.$$

Since $\theta^{\text{int}}\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, we know $\cos(\theta) \geq 0$. Therefore, we choose

$$\cos(\theta) = \frac{\sqrt{16x^2-1}}{4|x|}.$$

(The absolute values here are necessary, since x could be negative.) To find the values for which this equivalence is valid, we look back at our original substitution, $\theta = \operatorname{arccsc}(4x)$. Since the domain of $\operatorname{arccsc}(x)$ requires its argument x to satisfy $|x| \geq 1$, the domain of $\operatorname{arccsc}(4x)$ requires $|4x| \geq 1$. This implies $x \leq -\frac{1}{4}$ or $x \geq \frac{1}{4}$. Since we had no additional restrictions on θ, the equivalence

$$\cos(\operatorname{arccsc}(4x)) = \frac{\sqrt{16x^2-1}}{4|x|}$$

holds for all x in $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$.

Checkpoint 10.2.2

There are two ways to write this function:

$$f(x) = \frac{\sqrt{4x^2-1}}{2|x|},$$

or

$$f(x) = \begin{cases} \dfrac{\sqrt{4x^2-1}}{2x}, & \text{if } x \geq \dfrac{1}{2} \\ -\dfrac{\sqrt{4x^2-1}}{2x}, & \text{if } x < -\dfrac{1}{2} \end{cases}$$

The domain of this function is all $x^{\text{int}}\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$.

Example 10.2.3

a. i. $\cot^{-1}(2) = \theta$ means $\cot(\theta) = 2$, where $\theta^{\text{int}}(0, \pi)$.

Quadrant: Since the cotangent is positive, we know $\theta^{\text{int}}\left(0, \frac{\pi}{2}\right)$.

Reference Angle: 2 is not a special ratio for the cotangent, so we will need technology to help us find the reference angle; however, our calculators do not have an inverse cotangent button. What do we do? The answer is quite simple—we use the Reciprocal Identities. Since $\cot(\theta) = 2$, $\tan(\theta) = \frac{1}{2}$. To compute the reference angle, we evaluate

$$\hat{\theta} = \tan^{-1}\left(\left|\frac{1}{2}\right|\right) = \tan^{-1}\left(\frac{1}{2}\right) \approx 0.4636.$$

Now that we have a reference angle (≈ 0.4636) and a quadrant (QI), we get

$$\cot^{-1}(2) \approx 0.4636.$$

ii. $\operatorname{arcsec}(5) = \theta$ means $\sec(\theta) = 5$, where $\theta^{\text{int}}\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

Quadrant: Since the secant is positive, we further restrict the angle to $\theta^{\text{int}}\left(0, \frac{\pi}{2}\right)$ (Quadrant I).

Reference Angle: Again, since our technology does not have an inverse secant button, we use the Reciprocal Identities.

$$\sec(\theta) = 5 \implies \cos(\theta) = \frac{1}{5}.$$

Now we use a calculator to compute

$$\hat{\theta} = \cos^{-1}\left(\left|\frac{1}{5}\right|\right) = \cos^{-1}\left(\frac{1}{5}\right) \approx 1.3694.$$

Using our reference angle (≈ 1.3694) and quadrant (QI), we get

$$\operatorname{arcsec}(5) \approx 1.3694.$$

iii. $\operatorname{arccot}(-2) = \theta$ means $\cot(\theta) = -2$, where $\theta^{\text{int}}(0, \pi)$.

Quadrant: Since $\cot(\theta) < 0$, we know $\frac{\pi}{2} < \theta < \pi$. That is, θ^{int}QII.

Reference Angle: Once again, we do not have an appropriate inverse trigonometric function on the calculator, so we use the Reciprocal Identities.

$$\cot(\theta) = -2 \implies \tan(\theta) = -\frac{1}{2}.$$

Thus, we compute the reference angle as follows:

$$\hat{\theta} = \tan^{-1}\left(\left|-\frac{1}{2}\right|\right) = \tan^{-1}\left(\frac{1}{2}\right) \approx 0.4636.$$

We have a reference angle (≈ 0.4636) and a quadrant (QII). To find the actual angle, we subtract our reference angle from π to get

$$\operatorname{arccot}(-2) = \theta = \pi - \hat{\theta} = \pi - \arctan\left(\frac{1}{2}\right) \approx 2.6779.$$

iv. $\csc^{-1}\left(-\frac{3}{2}\right) = \theta$ means $\csc(\theta) = -\frac{3}{2}$, where $\theta^{\text{int}}\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

Quadrant: Since $\csc(\theta) < 0$, we further restrict the angle to the interval $\theta^{\text{int}}\left[-\frac{\pi}{2}, 0\right)$ (Quadrant IV).

Reference Angle: We do not have an appropriate inverse trigonometric button on our calculator for the cosecant. Therefore, we must use the Reciprocal Identities.

$$\csc(\theta) = -\frac{3}{2} \implies \sin(\theta) = -\frac{2}{3}.$$

Hence,

$$\hat{\theta} = \sin^{-1}\left(\left|-\frac{2}{3}\right|\right) = \sin^{-1}\left(\frac{2}{3}\right) \approx 0.7297.$$

Now that we have a reference angle (≈ 0.7297) and a quadrant (QIV), we get

$$\csc^{-1}\left(-\frac{3}{2}\right) \approx -0.7297.$$

b. The domain of the arcsecant is normally $(-\infty, -1] \cup [1, \infty)$. That is, if we were given $\sec^{-1}(r)$, we know that allowable values of r are $r^{\text{int}}(-\infty, -1] \cup [1, \infty)$. In this case, however, the argument of the arcsecant is $\frac{x}{5}$. Hence, $\frac{x}{5}^{\text{int}}(-\infty, -1] \cup [1, \infty)$. Rewriting this in interval notation will be helpful.

$$\begin{aligned} & \frac{x}{5} \le -1 \quad \text{or} \quad \frac{x}{5} \ge 1 \\ \implies \quad & x \le -5 \quad \text{or} \quad x \ge 5 \end{aligned}$$

Thus, the domain of f is $(-\infty, -5] \cup [5, \infty)$.

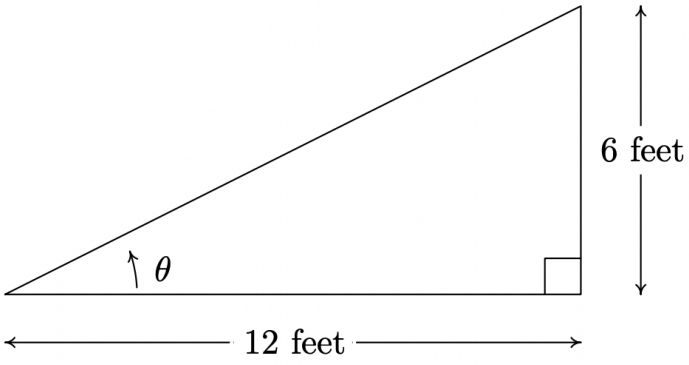
We know the range of the arcsecant is normally $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. Given the function $f(x) = \frac{\pi}{2} - \sec^{-1}\left(\frac{x}{5}\right)$, the only transformation that affects the range is the addition of $\frac{\pi}{2}$ to the output values. Thus, the range of f is $[\frac{\pi}{2}, \pi) \cup (\pi, \frac{3\pi}{2}]$.

Checkpoint 10.2.3

≈ 2.0137

Example 10.2.4

If we divide the side view of the house down the middle, we find that the roof line forms the hypotenuse of a right triangle with legs of length 6 feet and 12 feet. We find the angle of inclination, labeled θ below, satisfies $\tan(\theta) = \frac{6}{12} = \frac{1}{2}$. Since θ is an acute angle, we can use the arctangent function and we find $\theta = \arctan\left(\frac{1}{2}\right)$ radians $\approx 26.56°$.



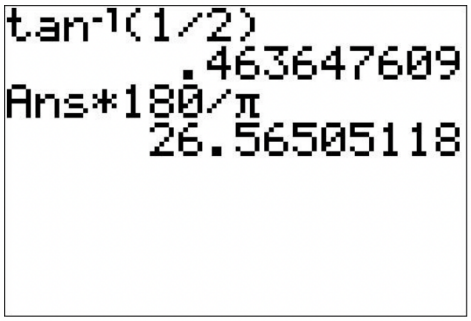


Figure 10.2.9

This page titled 10.2: The Remaining Inverse Trigonometric Functions is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

10.2.1: Resources and Key Concepts

Resources

Videos

- Inverse Trigonometric Functions
  - Inverse Trigonometric Functions - Defining the Inverse Trigonometric Functions
  - Inverse Trigonometric Functions - Evaluating Inverse Trigonometric Functions

Key Concepts

Definitions

- **Inverse Cotangent Function:** The function $f(x) = \cot^{-1}(x)$ or $\operatorname{arccot}(x)$ is defined as $\operatorname{arccot}(x) = \theta$ if and only if $\cot(\theta) = x$ and $0 < \theta < \pi$.
- **The Inverse Secant Function:** The function $f(x) = \sec^{-1}(x)$ or $\operatorname{arcsec}(x)$ is defined as $\operatorname{arcsec}(x) = \theta$ if and only if $\sec(\theta) = x$ and $\theta \overset{\text{int}}{} [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
- **The Inverse Cosecant Function:** The function $f(x) = \csc^{-1}(x)$ or $\operatorname{arccsc}(x)$ is defined as $\operatorname{arccsc}(x) = \theta$ if and only if $\csc(\theta) = x$ and $\theta \overset{\text{int}}{} [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.

Theorems

- **Properties of the Inverse Cotangent Function:**
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \pi)$
  - As $x \to -\infty$, $\operatorname{arccot}(x) \to \pi^-$; as $x \to \infty$, $\operatorname{arccot}(x) \to 0^+$
  - $\cot(\operatorname{arccot}(x)) = x$ for all real numbers x
  - $\operatorname{arccot}(\cot(x)) = x$ provided $0 < x < \pi$
- **Properties of the Inverse Secant and Inverse Cosecant Functions:**
  - **arcsec(x):** Domain is $(-\infty, -1] \cup [1, \infty)$; Range is $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
  - **arccsc(x):** Domain is $(-\infty, -1] \cup [1, \infty)$; Range is $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.
  - $\sec(\operatorname{arcsec}(x)) = x$ provided $|x| \geq 1$
  - $\csc(\operatorname{arccsc}(x)) = x$ provided $|x| \geq 1$
  - $\operatorname{arcsec}(\sec(x)) = x$ provided $x \overset{\text{int}}{} [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
  - $\operatorname{arccsc}(\csc(x)) = x$ provided $x \overset{\text{int}}{} [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$
  - Arccosecant is an odd function.

Common Mistakes

- **Forgetting Domain and Range Restrictions:** The inverse reciprocal trigonometric functions have very specific ranges. For example, $\operatorname{arccot}(x)$ only returns angles in Quadrants I and II, while $\operatorname{arcsec}(x)$ returns angles in Quadrants I and II (excluding $\frac{\pi}{2}$). Forgetting these restrictions is a primary source of error, especially when simplifying compositions like $\operatorname{arcsec}(\sec(\frac{8\pi}{7}))$.
- **Calculator Usage:** Calculators do not have dedicated buttons for arccot, arcsec, or arccsc. To evaluate these, you must use the Reciprocal Identities first. For example, to find $\operatorname{arccot}(2)$, you must evaluate $\arctan(\frac{1}{2})$.
- **Ignoring the Sign of x in Algebraic Expressions:** When simplifying an expression like $\tan(\operatorname{arcsec}(x))$, the result may depend on whether x is positive or negative, as this determines the quadrant of the angle $\theta = \operatorname{arcsec}(x)$. The result can be a piecewise function.

This page titled 10.2.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

10.2.2: Homework

Reading Questions

1. Why must the domain of the cotangent function be restricted to define its inverse?
2. What is the restricted domain (principal cycle) used to define the inverse cotangent function?
3. What is the domain and range of the $\operatorname{arccot}(x)$ function?
4. Why is the domain of the $\operatorname{arcsec}(x)$ and $\operatorname{arccsc}(x)$ functions the set $(-\infty,-1]\cup[1,\infty)$?
5. What is the range of the $\operatorname{arcsec}(x)$ function? Why is $\frac{\pi}{2}$ excluded?
6. How would you use a calculator to find the value of $\operatorname{arccot}(5)$?
7. Is the statement $\operatorname{arcsec}(\sec(x))=x$ true for all values of x? Explain.
8. When simplifying an expression like $\sin(\operatorname{arccot}(x))$, what is a helpful first step?
9. What is the key difference between the range of $\arccos(x)$ and $\operatorname{arcsec}(x)$?

Homework

Basic Skills

For the following exercises, find the exact value.

1. $\operatorname{arccot}\left(\sqrt{3}\right)$
2. $\operatorname{arccot}\left(\frac{1}{\sqrt{3}}\right)$
3. $\operatorname{arcsec}\left(\sqrt{2}\right)$
4. $\operatorname{arccsc}\left(\sqrt{2}\right)$
5. $\operatorname{arcsec}\left(\frac{2}{\sqrt{3}}\right)$
6. $\operatorname{arccsc}\left(\frac{2}{\sqrt{3}}\right)$
7. $\operatorname{arcsec}(2)$
8. $\operatorname{arccsc}(2)$
9. $\operatorname{arccot}(1)$
10. $\operatorname{arcsec}(1)$
11. $\operatorname{arccsc}(1)$
12. $\operatorname{arccot}(0)$
13. $\operatorname{arccot}\left(-\sqrt{3}\right)$
14. $\operatorname{arccot}\left(-\frac{1}{\sqrt{3}}\right)$
15. $\operatorname{arcsec}(-2)$

16. $\operatorname{arccsc}(-2)$

17. $\operatorname{arcsec}\left(-\sqrt{2}\right)$

18. $\operatorname{arccsc}\left(-\sqrt{2}\right)$

19. $\operatorname{arcsec}\left(-\frac{2}{\sqrt{3}}\right)$

20. $\operatorname{arccsc}\left(-\frac{2}{\sqrt{3}}\right)$

21. $\operatorname{arcsec}(-1)$

22. $\operatorname{arccot}(-1)$

23. $\operatorname{arccsc}(-1)$

For the following exercises, find the exact value or state that it is undefined.

24. $\cot(\operatorname{arccot}(1))$

25. $\cot\left(\operatorname{arccot}\left(-\sqrt{3}\right)\right)$

26. $\cot\left(\operatorname{arccot}\left(-\frac{7}{24}\right)\right)$

27. $\cot(\operatorname{arccot}(-0.001))$

28. $\cot\left(\operatorname{arccot}\left(\frac{17\pi}{4}\right)\right)$

29. $\sec(\operatorname{arcsec}(2))$

30. $\sec(\operatorname{arcsec}(-1))$

31. $\sec\left(\operatorname{arcsec}\left(\frac{1}{2}\right)\right)$

32. $\sec(\operatorname{arcsec}(0.75))$

33. $\sec(\operatorname{arcsec}(117\pi))$

34. $\csc\left(\operatorname{arccsc}\left(\sqrt{2}\right)\right)$

35. $\csc\left(\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right)$

36. $\csc\left(\operatorname{arccsc}\left(\frac{\sqrt{2}}{2}\right)\right)$

37. $\csc(\operatorname{arccsc}(1.0001))$

38. $\csc\left(\operatorname{arccsc}\left(\frac{\pi}{4}\right)\right)$

For the following exercises, find the exact value or state that it is undefined.

39. $\operatorname{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$

40. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right)$

41. $\operatorname{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right)$

42. $\operatorname{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$

43. $\operatorname{arccot}\left(\cot\left(\frac{\pi}{2}\right)\right)$

44. $\operatorname{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$

45. $\operatorname{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$

46. $\operatorname{arccot}\left(\cot\left(\pi\right)\right)$

47. $\operatorname{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right)$

48. $\operatorname{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$

49. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$

50. $\operatorname{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$

51. $\operatorname{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$

52. $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$

53. $\operatorname{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$

54. $\operatorname{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$

55. $\operatorname{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

For the following exercises, find the exact value or state that it is undefined.

56. $\sin\left(\operatorname{arccot}\left(\sqrt{5}\right)\right)$

57. $\sin\left(\operatorname{arccsc}\left(-3\right)\right)$

58. $\cos\left(\operatorname{arccot}\left(3\right)\right)$

59. $\cos\left(\operatorname{arcsec}\left(5\right)\right)$

60. $\tan\left(\operatorname{arcsec}\left(\frac{5}{3}\right)\right)$

61. $\tan\left(\operatorname{arccot}\left(12\right)\right)$

62. $\cot\left(\operatorname{arccsc}\left(\sqrt{5}\right)\right)$

63. $\sec\left(\operatorname{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right)$

64. $\csc\left(\operatorname{arccot}\left(9\right)\right)$

Synthesis Questions

For the following exercises, find the exact value or state that it is undefined.

65. $\cos\left(\operatorname{arcsec}(3)+\arctan(2)\right)$

66. $\sin\left(2\operatorname{arccsc}\left(\frac{13}{5}\right)\right)$

67. $\cos\left(2\operatorname{arcsec}\left(\frac{25}{7}\right)\right)$

68. $\cos\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right)$

For the following exercises, find the domain of the given function. Write your answers in interval notation.

69. $f(x)=\operatorname{arccot}\left(\frac{2x}{x^2-9}\right)$

70. $f(x)=\arctan(\ln(2x-1))$

71. $f(x)=\operatorname{arccot}(\sqrt{2x-1})$

72. $f(x)=\operatorname{arcsec}(12x)$

73. $f(x)=\operatorname{arccsc}(x+5)$

74. $f(x)=\operatorname{arcsec}\left(\frac{x^3}{8}\right)$

75. $f(x)=\operatorname{arccsc}\left(e^{2x}\right)$

Applications

76. A guy wire 1000 feet long is attached to the top of a tower. When pulled taut, it touches level ground 360 feet from the tower's base. What angle does the wire make with the ground? Express your answer using degree measure rounded to one decimal place.

77. At Cliffs of Insanity Point, The Great Sasquatch Canyon is 7117 feet deep. From that point, a fire is seen at a location known to be 10 miles away from the base of the sheer canyon wall. What angle of depression is made by the line of sight from the canyon edge to the fire? Express your answer using degree measure rounded to one decimal place.

78. Shelving is being built at the Utility Muffin Research Library, which is to be 14 inches deep. An 18-inch rod will be attached to the wall and the underside of the shelf at its edge away from the wall, forming a right triangle under the shelf to support it. What angle, to the nearest degree, will the rod make with the wall?

79. A parasailor is being pulled by a boat on Lake Ippizuti. The cable is 300 feet long, and the parasailor is 100 feet above the surface of the water. What is the angle of elevation from the boat to the parasailor? Express your answer using degree measure rounded to one decimal place.

80. A tag-and-release program to study the Sasquatch population of the eponymous Sasquatch National Park has begun. From a 200-foot-tall tower, a ranger spots a Sasquatch lumbering through the wilderness directly toward the tower. Let θ denote the angle of depression from the top of the tower to a point on the ground. If the range of the rifle with a tranquilizer dart is 300 feet, find the smallest value of θ for which the corresponding point on the ground is in range of the rifle. Round your answer to the nearest hundredth of a degree.

Challenge Problems

81. Show that $\operatorname{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ for $|x| \geq 1$.

82. Show that $\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ for $|x| \geq 1$.

This page titled 10.2.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

10.3: Solving Trigonometric Equations - Algebraic Techniques

Note to the Instructor - Start Simple

The equations in this section include everything *except for* those requiring trigonometric identities. Hence, students need to recall how to solve linear, quadratic, and rational equations. Moreover, they will need to recall what it means to factor out a GCF and the implications of dividing both sides of an equation by an algebraic expression (something I just tell students to not do, but to each their own).

Learning Objectives

- Solve a simple trigonometric equation.
- Use techniques from algebra to solve a trigonometric equation.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

The "Non-Uniqueness" of Solutions to Trigonometric Equations

Throughout Algebra, you constantly looked for "the" solution to an equation. For example, you might have been asked to solve

$$3x - \frac{1}{2} = 10 \quad \text{or} \quad 2e^{2x-1} = 5 \quad \text{or} \quad \sqrt{5x-1} = 3.$$

In each of these cases, the equation has a single solution; however, this was not the extent of your "equation solving" in Algebra. You also solved equations like

$$x^2 + 5x + 6 = 0 \quad \text{and} \quad \log_2\left(5x^2 - 1\right) = 2.$$

These last two equations each have two solutions. We now explore equations that often have an infinite number of solutions.

Definition: Trigonometric Equation

A **trigonometric equation** is an equation that involves trigonometric functions of a variable, typically an angle.

For example,

$$2\sec(x) - 1 = 0, \quad \sin^2(x) = 1, \quad \text{and} \quad 2\cos^2(x) = \sin(x) + 1$$

are all trigonometric equations.

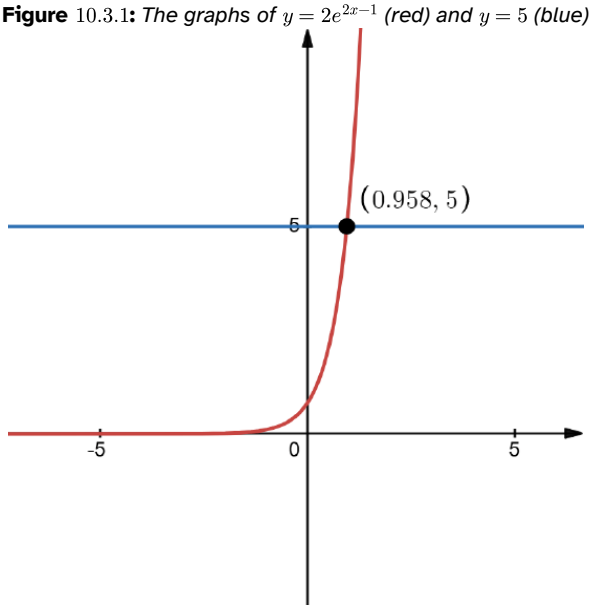
Before we discuss how to solve trigonometric equations, it's best to recall the graphical meaning of solutions of equations.

In Algebra, when you were asked to solve an equation like $2e^{2x-1} = 5$, you learned all the mathematical manipulations - divide by

5, take the natural logarithm of both sides, add 1, then divide by 2. This results in the solution $x = \frac{1}{2}\ln\left(\frac{5}{2}\right) + \frac{1}{2}$; however, before you learned how to procedurally work to an answer to such an equation, you learned a way to interpret this solution graphically.

When asked to solve an equation, one interpretation is that you are being asked to find the x-values for which the graph of the left side of the equation meets the graph of the right side. For example, when asked to solve $2e^{2x-1} = 5$, we could view the graphs of $y = 2e^{2x-1}$ and $y = 5$ to visualize the solution.

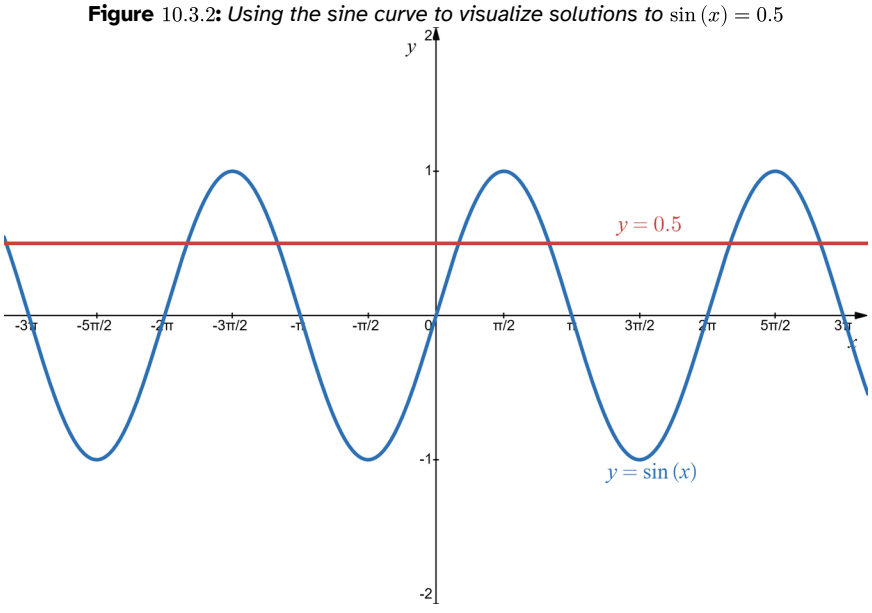
Figure $10.3.1$**:** *The graphs of* $y = 2e^{2x-1}$ *(red) and* $y = 5$ *(blue)*



We can see in Figure $10.3.1$ that these graphs meet at $x \approx 0.958 \approx \frac{1}{2}\ln\left(\frac{5}{2}\right) + \frac{1}{2}$.

Although not the best way (or even a good way) to solve an equation, the visual we get from the example above justifies why trigonometric equations often have infinite solutions. For example, Figure $10.3.2$ illustrates that $\sin(x) = 0.5$ has five solutions on the interval $\left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$. These solutions correspond to the points on the unit circle where $y = 0.5$. Since the sine function oscillates in both directions forever, we can easily imagine that the sine wave intersects the line $y = 0.5$ an infinite number of times.

Figure $10.3.2$**:** *Using the sine curve to visualize solutions to* $\sin(x) = 0.5$



Despite knowing there are an infinite number of solutions to $\sin(x) = 0.5$, we know from our recent discussions of the inverse trigonometric functions that $x = \sin^{-1}(0.5)$ will only give us one of these solutions. Therefore, we must create a method to obtain *all* solutions to a given trigonometric equation.

Solving Basic Trigonometric Equations

The keys to finding all solutions to a trigonometric equation can be summarized in two fundamental concepts from Trigonometry: quadrants and reference angles. We will begin with an example to illustrate our thought process.

Example 10.3.1

Find all degree solutions to the equation

$$\sin(\theta) = -0.6428,$$

and state the solutions occurring on the interval $[0°, 360°]$. Round your answers to the nearest tenth of a degree.

Solution available at the end of this section.

A few notes about Example 10.3.1 are in order. First, our equation had solutions derived from two different quadrants. This should not be a surprise when you consider Figure 10.3.2. The sine curve should attain a y-value of -0.6428 twice through any single cycle.

Second, since the given ratio, -0.6428, was not a special ratio, we had to grab a calculator to get an approximation of the reference angle by way of the arcsine. Moreover, we had to change our calculator to degree mode. You must remain aware of the angular measurement system in which you are being asked to work. You will frequently switch between degree and radian mode when solving trigonometric equations.

Third, we were told to round to the nearest tenth of a degree. If you are using online homework and need to answer with an approximation, pay attention to the level of rounding they request. If they don't state it explicitly, assume four decimal places (as this works well for most online homework systems).

Checkpoint 10.3.1

Solve the equation $\sin(\theta) = -0.4$ for angles between $0°$ and $360°$. Round your answers to the nearest tenth of a degree.

Answer available at the end of this section.

Example 10.3.2

Find all solutions to the given equation, then state the solutions occurring on the interval $[0, 2\pi]$. Round your answers to the nearest tenth.

a. $\sin(x) = \frac{1}{2}$
b. $\cos(x) = -0.6$
c. $\tan(x) = -\sqrt{3}$
d. $\sec(x) = 3.7$

Solutions available at the end of this section.

Note: Combining Solutions for the Tangent

We found the set of all solutions to Example 10.3.2c to be

$$x = \frac{2\pi}{3} + 2\pi k$$

and

$$x = \frac{5\pi}{3} + 2\pi k.$$

However, the tangent and cotangent are special functions in the world of Trigonometry in that their natural periods are π. That is, when finding their solutions, we can add π instead of 2π.

For example, when solving $\tan(x) = -\sqrt{3}$, we could have written our solution associated with Quadrant II as

$$x = \frac{2\pi}{3} + \pi k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

In doing so, we capture all of the Quadrant IV solutions as well because, when $k = 1$, $x = \frac{2\pi}{3} + \pi = \frac{5\pi}{3}$. This is precisely our Quadrant IV solution!

Thus, from this point forward, when solving trigonometric equations involving the tangent and cotangent functions, we will use this strategy to "combine" our solutions into a single statement.

Checkpoint 10.3.2

Find exact solutions to the given equation, and then state the solutions occurring on the interval $[0, 2\pi]$.

$$\cos(x) = -\frac{1}{\sqrt{2}}$$

Answer available at the end of this section.

Solving Trigonometric Equations Requiring Algebraic Techniques

The first Ferris wheel was built for the Chicago World's Fair in 1893. It had a diameter of 250 feet and could carry 2160 people in 36 carriages. From the top of the wheel, passengers could see into four states. After loading all the passengers, the wheel made one revolution in nine minutes.

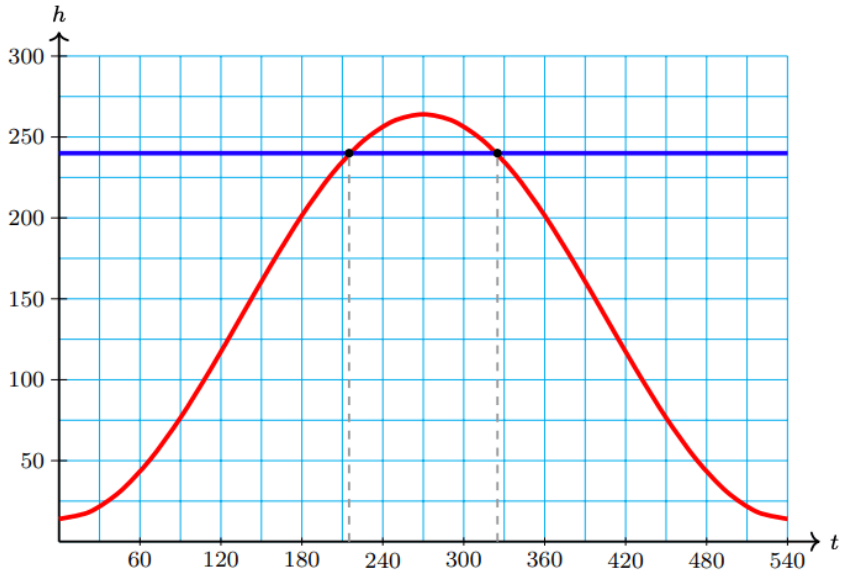
If you are in the bottom carriage of the Ferris wheel at the start of its revolution, your height after t seconds is given by

$$h = f(t) = 139 - 125\cos\left(\frac{2t}{3}\right).$$

For how long are you more than 240 feet above the ground?

Figure 10.3.3 shows a graph of the height function and a horizontal line at $h = 240$.

Figure 10.3.3



The graph shows that $h = 240$ at approximately 215 seconds and 325 seconds into the ride. Your height is more than 240 feet between those two times, or for about 110 seconds. To find a more precise solution, we can use algebraic methods. As an example, we'll solve the slightly simpler equation

$$139 - 125\cos(\theta) = 240.$$

We'll look for all solutions for θ between 0° and 360°. We begin by isolating the trigonometric ratio on one side of the equation.

$$\begin{aligned} & 139 - 125\cos(\theta) = 240 \\ \Longrightarrow \quad & -125\cos(\theta) = 101 && \text{(subtract 139 from both sides)} \\ \Longrightarrow \quad & \cos(\theta) = -0.808 && \text{(divide both sides by } -125\text{)} \end{aligned}$$

We have solved equations like this one before: we use the inverse cosine to solve for θ. Remember that there are two angles between 0° and 360° that have a cosine of -0.808, one in the second quadrant and one in the third quadrant. The reference angle is

$$\hat{\theta} = \cos^{-1}(|-0.808|) = \cos^{-1}(0.808) \approx 36.09901626^\circ.$$

Thus, the solution (rounded to the nearest tenth of a degree) in the second quadrant is

$$\theta = 180^\circ - \hat{\theta} \approx 143.9^\circ$$

and the solution in the third quadrant is

$$\theta = 180^\circ + \hat{\theta} \approx 216.1^\circ.$$

It's important to note that we have not discussed how to solve the original equation, $240 = 139 - 125\cos\left(\frac{2t}{3}\right)$. Instead, we considered the slightly simpler version, $240 = 139 - 125\cos(\theta)$. We will explore the original version a little bit later in this section.

In general, our old techniques of solving algebraic equations for an unknown still hold when solving trigonometric equations.

Example 10.3.3

Solve the equation $4\csc(\theta) - 1 = 7$ for $0^\circ \le \theta \le 360^\circ$.

Solution available at the end of this section.

Checkpoint 10.3.3

Solve the equation $5\tan(\theta)+7=2$ for $0\le\theta\le 2\pi$.

Answer available at the end of this section.

Example 10.3.3 and Checkpoint 10.3.3 illustrate an **implied angle measure**. That is, in Example 10.3.3, the statement, "for $0^\circ\le\theta\le 360^\circ$" implies our answer should be in degrees. Likewise, the statement, "for $0\le\theta\le 2\pi$," in Checkpoint 10.3.3, implies our answer should be in radians.

Example 10.3.4

Solve the equation $3\tan(\beta)+1=-8$ for $0\le\beta\le 2\pi$. Round your solutions to three decimal places.

Solution available at the end of this section.

It should not be surprising in Example 10.3.4 that the solutions are π radians apart. As we mentioned previously, the solutions of an equation $\tan(\theta)=k$ always differ by π radians (or 180°). Once we have found one solution, we can add π to find the other solution between 0 and 2π.

Checkpoint 10.3.4

Solve the equation $3\sec(C)-6=-2$, for $0^\circ\le C\le 360^\circ$. Round your solutions to three decimal places.

Answer available at the end of this section.

Example 10.3.5

Solve $\dfrac{\sin(\beta)}{5}-3=1$ for $0\le\beta\le 2\pi$

Solution available at the end of this section.

Checkpoint 10.3.5

Solve $2\sec(\theta)-5=-4$ for $0^\circ\le\theta\le 360^\circ$.

Answer available at the end of this section.

Example 10.3.5 and Checkpoint 10.3.5 show us that not all trigonometric equations have solutions. Just as you did in Algebra, you should pause after each step to check the result for realism. Forming this into a habit can save you time and frustration in the future.

The equations in Examples 10.3.3 through 10.3.5 are **linear-in-form**. That is, if you substitute a variable, say t, for the trigonometric function, the equation would look like a linear equation in each case. The initial steps to solve the equation mimic those of a linear equation.

For example, the equation in Example 10.3.4 could be written as $3t+1=-8$. Solving for t, we get $t=-3$; however, t was a substitution for $\tan(\beta)$. Thus, $\tan(\beta)=-3$.

This tactic of simplifying the look of an equation using a substitution is beneficial as you move forward in Mathematics. While we will not use it in every example, Example 10.3.6 (below) shows how it can be helpful.

Example 10.3.6

Solve $4\tan^2(\theta) + 3 = 15$ for $0 \leq \theta \leq 2\pi$.

Solution available at the end of this section.

Checkpoint 10.3.6

Solve the equation $2\sin^2(\theta) = 1$ for $0^\circ \leq \theta \leq 360^\circ$.

Answer available at the end of this section.

Example 10.3.6 is **quadratic-in-form** because, with the substitution $t = \tan(\theta)$, it can be written as the quadratic equation $4t^2 + 3 = 15$.

Definition: Quadratic-in-Form

An equation in the variable x is **quadratic-in-form** if it can be rewritten as

$$au^2 + bu + c = 0,$$

where a, b, and c are constants with $a \neq 0$, and u represents an algebraic expression in x.

From this point forward in Mathematics, you will be using your techniques from Algebra to solve trigonometric equations that are quadratic-in-form. If you are rusty with that material, I highly recommend reviewing it.

Example 10.3.7

Solve $2\cos^2(\theta) - \cos(\theta) - 1 = 0$ for $0 \leq \theta \leq 2\pi$.

Solution available at the end of this section.

Checkpoint 10.3.7

Solve $9\sin^2(\theta) - 3\sin(\theta) = 2$ for $0^\circ \leq \theta \leq 360^\circ$. Round your answers to the nearest degree.

Answer available at the end of this section.

Solving Trigonometric Equations with Non-Standard Arguments

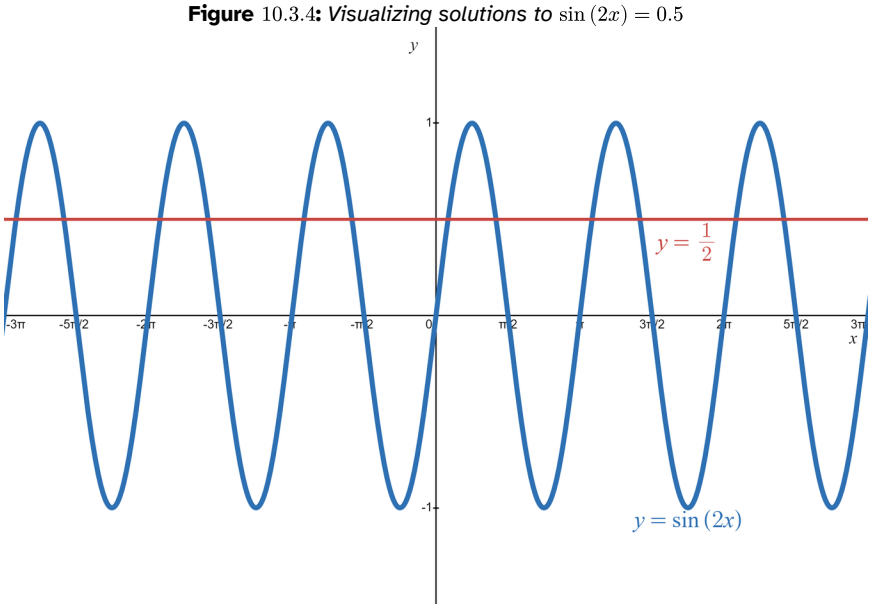
If n is an integer, what can we say about the solutions of the equation

$$\sin(nx) = k?$$

For example, how many solutions are there for the equation $\sin(2x) = 0.5$? Truthfully, we already know there will be an infinite number of solutions; however, on the interval from 0 to 2π, how many solutions will this equation have?

As was stated earlier in the text, the coefficient of x can be thought of as the number of oscillations over an interval of length equal to the natural period of the function. That is, $\sin(2x)$ goes through two full oscillations on any interval of length 2π. Compare Figure 10.3.4 (below) to Figure 10.3.2.

Figure 10.3.4**:** *Visualizing solutions to* $\sin(2x) = 0.5$



You can see there are twice as many solutions on the same interval as that of $\sin(x) = 0.5$. This visual helps us wrap our brains around the fact that the number of solutions to such equations over an interval like $[0, 2\pi]$ might be more (or less) than two.

Example 10.3.8

Find all solutions of $\sin(2x) = 0.5$ between 0 and 2π.

Solution available at the end of this section.

Notice in Example 10.3.8, that we had *twice* the number of solutions we would typically expect. Do you see why?

Checkpoint 10.3.8

a. Sketch a graph of $y = \cos(2x)$ for $0 \leq \theta \leq 2\pi$.
b. Find exact values for all solutions of $\cos(2x) = -\frac{1}{\sqrt{2}}$ between 0 and 2π.

Answer available at the end of this section.

As we observed earlier, equations involving the tangent function are easier to solve because there is only one solution in each cycle of the graph. Once we have found one solution, we can find all the others by adding multiples of the period.

Example 10.3.9

Find all solutions of $\tan(2t) = -1$ between 0 and 2π.

Solution available at the end of this section.

Checkpoint 10.3.9

Find all solutions of $\tan(2x) = \sqrt{3}$ between 0 and 2π.

Answer available at the end of this section.

Applications

Trigonometric equations often arise in the study of periodic models.

Example 10.3.10

A piston is pumping vertically at a rate of 1000 cycles per second. The distance between its lowest and highest position is 16 centimeters.

a. Suppose the piston is at its midline position at $t = 0$ and moving downwards. Find a formula for the sinusoidal function $h(t)$ that gives the piston's height.
b. Use the formula to find the first two times the piston is 14 centimeters above its lowest position.

Solutions available at the end of this section.

Checkpoint 10.3.10

A lever on an oil well is pumping vertically at a rate of 10 cycles per minute. The distance between its lowest position at the ground and its highest position is 1.8 meters.

a. Suppose the lever is at its midline position at $t = 0$ and moving downwards. Find a formula for the sinusoidal function $h(t)$ that gives the lever's height.
b. Find the first two times the lever is 1 meter above its lowest position.

Answer available at the end of this section.

Answers

Example 10.3.1

Since -0.6428 is not a special ratio, we will need to use technology to help us find the solution; however, technology should be reserved for the last step in any mathematical procedure.
Quadrants: Since $\sin(\theta)$ is negative, we know that θ is either in QIII or QIV.
Reference Angle: To find the reference angle, we compute

$$\sin^{-1}(|-0.6428|) = \sin^{-1}(0.6428) \approx 40.0^\circ.$$

So we have a reference angle, $\hat{\theta} \approx 40.0^\circ$, and *two* quadrants, QIII and QIV.
Solutions Associated with Quadrant III: The solution in QIII having reference angle $\hat{\theta} \approx 40.0^\circ$ is

$$\theta = 180^\circ + \hat{\theta} \approx 220.0^\circ.$$

Since $\sin(\theta) = \sin(\theta + 360^\circ k)$ (where $k^{\text{int}}\mathbb{Z}$), any angle coterminal to 220.0° will also lead to a solution of our equation. Thus, the general solution associated with Quadrant III is

$$\theta \approx 220.0^\circ + 360^\circ k, \quad \text{where } k^{\text{int}}\mathbb{Z}.$$

Solutions Associated with Quadrant IV: Taking a lesson from the previous paragraph, the solution in QIV having reference angle $\hat{\theta} \approx 40.0^\circ$ is

$$\theta = 360^\circ - \hat{\theta} + 360^\circ k \approx 320.0^\circ + 360^\circ k, \quad \text{where } k^{\text{int}}\mathbb{Z}.$$

Now that we have all solutions, let's restrict them to only the solutions occurring on the interval $[0^\circ, 360^\circ]$. To do this, we consider both sets of infinite solutions,

$$\begin{aligned} \theta &\approx 220.0^\circ + 360^\circ k \\ \theta &\approx 320.0^\circ + 360^\circ k, \end{aligned}$$

and we let $k = 0$ (any other value of k will result in angles outside of the requested interval). Hence, the solutions on $[0^\circ, 360^\circ]$ are approximately 220.0° and 320.0°.

Checkpoint 10.3.1

$203.6^\circ, 336.4^\circ$

Example 10.3.2

a. Since $\frac{1}{2}$ is a special ratio for the sine function, we do not bother (and should not bother) grabbing a calculator.
Quadrants: Since $\sin(x) = \frac{1}{2}$ is positive, we know that x is either in QI or QII.
Reference Angle: From our special triangles, we know that $\hat{x} = \frac{\pi}{6}$.
Solutions Associated with Quadrant I: The angle in Quadrant I with reference angle $\frac{\pi}{6}$ is $x = \frac{\pi}{6}$. Any angle coterminal with this will also be a solution to the

given equation, so our set of solutions associated with Quadrant I is

$$x = \frac{\pi}{6} + 2\pi k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

Solutions Associated with Quadrant II: The angle in Quadrant II with reference angle $\frac{\pi}{6}$ is $\frac{5\pi}{6}$. Thus,

$$x = \frac{5\pi}{6} + 2\pi k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

Finally, the solutions in the interval $[0, 2\pi]$ are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

b. -0.6 is not a special ratio for cosine, so we will eventually need to reach for technology to solve this equation.
Quadrants: We are given a negative cosine function, and we know this can only occur if the argument of the cosine terminates in either QII or QIII.
Reference Angle: To find the reference angle associated with the equation $\cos(x) = -0.6$, we approximate $\cos^{-1}(|-0.6|) = \cos^{-1}(0.6)$. It's important to note that we are looking for radian solutions to this equation, so we need to switch our calculator into radian mode.

$$\hat{x} = \cos^{-1}(0.6) \approx 0.927295218$$

Notice that I wrote down all of the digits to the approximation the calculator returned. This is because we should save our rounding until the very end of the problem. We want to work with as many digits as possible but *display* a solution rounded to the nearest tenth to the reader.
Solutions Associated with Quadrant II: The angle in Quadrant II having reference angle $\hat{x}$ is $\pi - \hat{x}$. Again, we list all coterminal angles as well.

$$x = \pi - \hat{x} + 2\pi k \approx 2.2 + 2\pi k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

Solutions Associated with Quadrant III: The angle in Quadrant III having reference angle $\hat{x}$ is $\pi + \hat{x}$. Therefore,

$$x = \pi + \hat{x} + 2\pi k \approx 4.1 + 2\pi k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

The solutions restricted to the interval $[0, 2\pi]$ are $x \approx 2.2$ and $x \approx 4.1$.

c. $\frac{\sqrt{3}}{1}$ is definitely a special ratio for the tangent function, so we will abstain from calculator usage here.
Quadrants: The given tangent function is negative, so we know the argument of the function terminates in either QII or QIV.
Reference Angle: From our special triangles, $\hat{x} = \frac{\pi}{3}$.
Solutions Associated with Quadrant II: The angle in Quadrant II having reference angle $\hat{x}$ is $\pi - \hat{x}$. Therefore,

$$x = \pi - \hat{x} + 2\pi k = \pi - \frac{\pi}{3} + 2\pi k = \frac{2\pi}{3} + 2\pi k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

Solutions Associated with Quadrant IV: The angle in Quadrant IV having reference angle $\hat{x}$ is $2\pi - \hat{x}$. Therefore,

$$x = 2\pi - \hat{x} + 2\pi k = 2\pi - \frac{\pi}{3} + 2\pi k = \frac{5\pi}{3} + 2\pi k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

The solutions restricted to the interval $[0, 2\pi]$ are $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$.

d. 3.7 is not a special ratio for the secant, so we know we will need a calculator at some point.
Quadrants: The given secant function is positive. Therefore, the argument terminates in either QI or QIV.
Reference Angle: Since our calculator doesn't have an inverse secant button, we rewrite $\sec(x) = 3.7$ using the Reciprocal Identities as

$$\cos(x) = \frac{1}{3.7}.$$

Therefore,

$$\hat{x} = \cos^{-1}\left(\left|\frac{1}{3.7}\right|\right) = \cos^{-1}\left(\frac{1}{3.7}\right) \approx 1.297122589.$$

Solutions Associated with Quadrant I: The angle in Quadrant I having reference angle $\hat{x}$ is $\hat{x}$. Thus,

$$x = \hat{x} + 2\pi k \approx 1.3 + 2\pi k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

Solutions Associated with Quadrant IV: The angle in Quadrant IV having reference angle $\hat{x}$ is $2\pi - \hat{x}$. Therefore,

$$x = 2\pi - \hat{x} + 2\pi k \approx 5.0 + 2\pi k, \text{ where } k^{\text{int}}\mathbb{Z}.$$

The solutions restricted to the interval $[0, 2\pi]$ are $x \approx 1.3$ and $x \approx 5.0$.

Checkpoint 10.3.2

$$\begin{aligned} \text{Solutions associated with QII}: \quad x &= \frac{3\pi}{4} + 2\pi k, \text{ where } k^{\text{int}}\mathbb{Z} \\ \text{Solutions associated with QIII}: \quad x &= \frac{5\pi}{4} + 2\pi k \text{ where } k^{\text{int}}\mathbb{Z} \end{aligned}$$

On $[0, 2\pi]$, the solutions are $x = \frac{3\pi}{4}$ and $x = \frac{5\pi}{4}$.

Example 10.3.3

We begin by isolating the trigonometric ratio.

$$\begin{aligned} 4\csc(\theta) - 1 &= 7 \\ \Longrightarrow \quad 4\csc(\theta) &= 8 && \text{(add 1 to both sides)} \\ \Longrightarrow \quad \csc(\theta) &= 2 && \text{(divide both sides by 4)} \\ \Longrightarrow \quad \sin(\theta) &= \frac{1}{2} && \text{(Reciprocal Identity)} \end{aligned}$$

Quadrants: Sine is positive in QI and QII.

Reference Angle: $\frac{1}{2}$ is a special ratio for the sine function. Using our knowledge of the special triangles, we get the reference angle $\hat{\theta} = 30^\circ$.
Now that we know the angles terminate in QI and QII with reference angle $\hat{\theta} = 30^\circ$, we can confidently state the solutions are $\theta = 30^\circ$ and $\theta = 150^\circ$.

Checkpoint 10.3.3
$\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$

Example 10.3.4
First, we isolate the trigonometric ratio.

$$\begin{aligned} 3\tan(\beta) + 1 &= -8 \\ \Longrightarrow \quad 3\tan(\beta) &= -9 \qquad \text{(subtract 1 from both sides)} \\ \Longrightarrow \quad \tan(\beta) &= -3 \qquad \text{(divide both sides by 3)} \end{aligned}$$

There are two angles with tangent -3, one in the second quadrant and one in the fourth quadrant. The reference angle is $\hat{\beta} = \tan^{-1}(3) \approx 1.249045772$. Thus, we get the following:

$$\begin{aligned} \text{Solution Associated with QII}: \quad \beta &= \pi - \hat{\beta} \approx 1.893 \\ \text{Solution Associated with QIV}: \quad \beta &= 2\pi - \hat{\beta} \approx 5.034 \end{aligned}$$

Checkpoint 10.3.4
$C = 41.410^\circ$ or $C = 318.590^\circ$

Example 10.3.5
We begin by isolating $\sin(\beta)$.

$$\begin{aligned} \frac{\sin(\beta)}{5} - 3 &= 1 \\ \Longrightarrow \quad \frac{\sin(\beta)}{5} &= 4 \qquad \text{(add 3 to both sides)} \\ \Longrightarrow \quad \sin(\beta) &= 20 \qquad \text{(multiply both sides by 5)} \end{aligned}$$

Because $\sin(\beta)$ is never greater than 1, there is no angle β whose sine is 20. The equation has no solution.

Checkpoint 10.3.5
No solution.

Example 10.3.6
We start with the substitution $t = \tan(\theta)$ and solve $4t^2 + 3 = 15$ for t.

$$\begin{aligned} 4t^2 + 3 &= 15 \\ \Longrightarrow \quad 4t^2 &= 12 \qquad \text{(subtract 3 from both sides)} \\ \Longrightarrow \quad t^2 &= 3 \qquad \text{(divide both sides by 4)} \\ \Longrightarrow \quad t &= \pm\sqrt{3} \qquad \text{(Extraction of Roots)} \end{aligned}$$

Resubstituting $\tan(\theta)$ for t, we get

$$\tan(\theta) = \pm\sqrt{3}.$$

There are two angles between 0 and 2π with tangent $\sqrt{3}$ and two angles with tangent $-\sqrt{3}$, making four solutions to this equation - one in each quadrant! Moreover, we know the reference angle such that $\tan\left(\hat{\theta}\right) = \sqrt{3}$ is $\hat{\theta} = \frac{\pi}{3}$.

Recall that tangent is the special function with period π. Therefore, we only need to list the solutions associated with QI and QII and add integer multiples of π to get all other solutions.

$$\begin{aligned} \text{Solutions associated with QI}: \quad \theta &= \frac{\pi}{3} + \pi k, \text{ where } k^{\text{int}}\mathbb{Z} \\ \text{Solutions associated with QII}: \quad \theta &= \pi - \hat{\theta} + \pi k = \pi - \frac{\pi}{3} + \pi k = \frac{2\pi}{3} + \pi k, \text{ where } k^{\text{int}}\mathbb{Z} \end{aligned}$$

To restrict this infinite set of solutions to those in the interval $[0, 2\pi]$, we let $k = 0$ and $k = 1$ to get

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ and } \frac{5\pi}{3}.$$

Checkpoint 10.3.6
$\theta = 45^\circ, \theta = 135^\circ, \theta = 225^\circ$, or $\theta = 315^\circ$

Example 10.3.7
Compare this equation to the algebraic equation $2t^2 - t - 1 = 0$. Our equation has the same algebraic form but with t replaced by $\cos(\theta)$.
We can solve the algebraic equation by factoring $2t^2 - t - 1$ as $(2t + 1)(t - 1)$, and we'll use the same strategy on the trigonometric equation.

$$\begin{aligned} 2\cos^2(\theta) - \cos(\theta) - 1 &= 0 \\ \Longrightarrow \quad (2\cos(\theta) + 1)(\cos(\theta) - 1) &= 0 \qquad \text{(factor)} \end{aligned}$$

By the Zero Factor Property, this means

$$\begin{aligned} 2\cos(\theta) + 1 &= 0 \qquad \text{or} \qquad \cos(\theta) - 1 = 0 \\ \Longrightarrow \quad \cos(\theta) &= -\frac{1}{2} \qquad \text{or} \qquad \cos(\theta) = 1 \end{aligned}$$

Now we solve each equation for θ.

For $\cos(\theta) = -\frac{1}{2}$: We know cosine is negative in QII and QIII and that $\hat{\theta} = \frac{\pi}{3}$. Thus,

$$\text{Solutions associated with QII}: \quad \theta = \pi - \hat{\theta} + 2\pi k = \pi - \frac{\pi}{3} + 2\pi k = \frac{2\pi}{3} + 2\pi k$$

$$\text{Solutions associated with QIII}: \quad \theta = \pi + \hat{\theta} + 2\pi k = \pi + \frac{\pi}{3} + 2\pi k = \frac{4\pi}{3} + 2\pi k$$

For $\cos(\theta) = 1$: This will result in a quadrantal angle. Visualizing the graph of the cosine, we can (hopefully) see that the cosine attains a value of 1 at $x = 0$ and every integer multiple of 2π after that. That is,

$$\theta = 0 + 2\pi k = 2\pi k.$$

Combining all of these solution sets and restricting our answers to those between 0 and 2π, we get

$$\theta = 0,\ \theta = \frac{2\pi}{3},\ \theta = \frac{4\pi}{3},\ \text{or}\ \theta = 2\pi.$$

Checkpoint 10.3.7

$\theta = 42^\circ, \theta = 138^\circ, \theta = 199^\circ$, or $\theta = 341^\circ$

Example 10.3.8

Treat the argument as if it were a single variable. The sine is positive, meaning the angle, $2x$, must terminate in QI or QII. Additionally, we know that the reference angle is $\widehat{2x} = \frac{\pi}{6}$. At this point, let's pause to make sure we understand what just happened. I am treating the argument of the sine, which is $2x$, as a single variable - sort of like calling it θ. Once I find a reference angle, I have found it for that single variable, which is $2x$. Like we traditionally do, we use the "hat" notation to denote the reference angle; however, in this case, the reference angle is for $2x$ - not x. Therefore, our reference angle notation is $\widehat{2x}$.

Continuing as usual,

$$\text{Solutions associated with QI}: \quad 2x = \frac{\pi}{6} + 2\pi k$$

$$\text{Solutions associated with QII}: \quad 2x = \frac{5\pi}{6} + 2\pi k$$

where $k^{\text{int}}\mathbb{Z}$. We finally arrive at the reason why I write "Solutions **associated with**" rather than "Solutions **in**." Our goal is to solve for x. This requires us to divide both equations by 2 (including dividing the $2\pi k$). Therefore, we get

$$\text{Solutions associated with QI}: \quad x = \frac{\pi}{12} + \pi k$$

$$\text{Solutions associated with QII}: \quad x = \frac{5\pi}{12} + \pi k$$

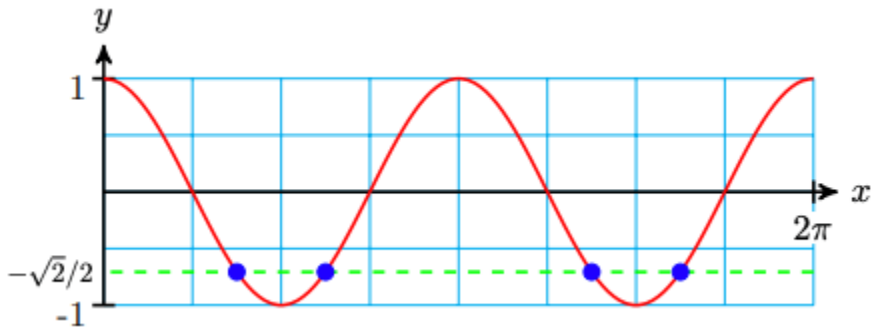
where $k^{\text{int}}\mathbb{Z}$. Do you notice that the "Solutions associated with QII" doesn't start with a Quadrant II angle? That's perfectly fine! The solution angles have been adjusted by the coefficient of x so they might not belong to the original quadrant they were derived from. This is why I use the words "associated with."

We now need to restrict our answers to the interval $[0, 2\pi]$. Allowing $k = 0$, we get $x = \frac{\pi}{12}$ and $x = \frac{5\pi}{12}$ - both of which are in the requested interval. Trying $k = 1$, we get $x = \frac{13\pi}{12}$ and $x = \frac{17\pi}{12}$. Again, both of these solutions are still within the requested interval. Trying $k = 2$, however, gives $x = \frac{25\pi}{12}$ and $x = \frac{29\pi}{12}$. Neither of these are in the requested interval. Thus, the solutions on $[0, 2\pi]$ are

$$x = \frac{\pi}{12},\ x = \frac{5\pi}{12},\ \frac{13\pi}{12},\ \text{or}\ x = \frac{17\pi}{12}.$$

Checkpoint 10.3.8

a.



b. $\frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$

Example 10.3.9

Again, treating the argument, $2t$, as a single variable, we know the tangent is negative in QII and QIV; however, the tangent is that special case (along with the cotangent) where we can say the second solution is just π plus the first solution. So, we only need to concentrate on the solution in the second quadrant. The reference angle is $\widehat{2t} = \frac{\pi}{4}$. This is all we need!

$$\text{Solutions associated with QII}: \quad 2t = \frac{3\pi}{4} + \pi k$$

where $k^{\text{int}}\mathbb{Z}$. We then divide both sides of the equation by 2 to get

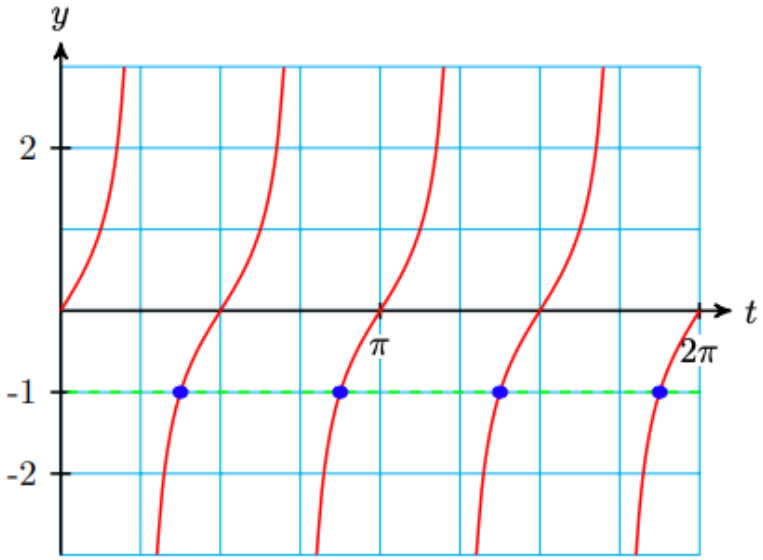
$$\text{Solutions associated with QII}: \quad t = \frac{3\pi}{8} + \frac{\pi}{2}k$$

where $k \in \mathbb{Z}$. Just so you can see a different approach, let's build a table to get all the solutions between 0 and 2π.



If we move to $k = 4$, the solutions will be outside of the requested interval. Thus, our solutions are $\frac{3\pi}{8}$, $\frac{7\pi}{8}$, $\frac{11\pi}{8}$ and $\frac{15\pi}{8}$. The following figure gives a visual of these solutions.

Figure 10.3.5



Checkpoint 10.3.9

$\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

Example 10.3.10

a. Because the piston starts at its midline and moves down, we will write $h(y)$ as a sine function:

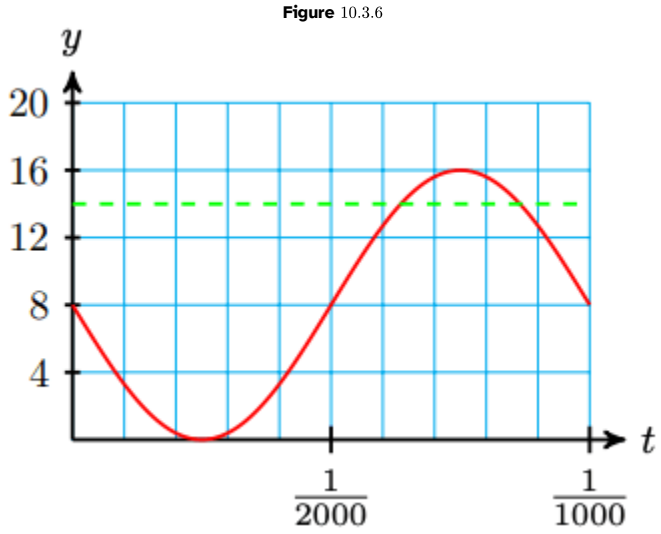
$$h(t) = -A\sin(Bt) + D.$$

The amplitude is $A = \frac{16}{2} = 8$, and the midline is $D = 8$. The period is $\frac{1}{1000}$, so $\frac{B}{1000} = 2\pi$, and $B = 2000\pi$. Thus,

$$h(t) = -8\sin(2000\pi t) + 8.$$

The graph is shown in Figure 10.3.6.

Figure 10.3.6



b. With $h(t) = 14$, we have

$$\begin{aligned} -8\sin(2000\pi t) + 8 &= 14 \\ \implies -8\sin(2000\pi t) &= 6 \qquad \text{(subtract 8 from both sides)} \\ \implies \sin(2000\pi t) &= -\frac{3}{4} \qquad \text{(divide both sides by } -8\text{)} \end{aligned}$$

Sine is negative in QIII and QIV. Moreover, we can compute the reference angle as follows:

$$\widehat{2000\pi t} = \sin^{-1}\left(\left|-\frac{3}{4}\right|\right) = \sin^{-1}\left(\frac{3}{4}\right) \approx 0.848062079.$$

We want the first two *positive* solutions to this equation.

$$\text{Solutions associated with QIII}: \quad 2000\pi t = \pi + \sin^{-1}\left(\frac{3}{4}\right) + 2\pi k \implies t = \frac{1}{2000} + \frac{\sin^{-1}\left(\frac{3}{4}\right)}{2000\pi} + \frac{1}{1000}k$$

$$\text{Solutions associated with QIV}: \quad 2000\pi t = 2\pi - \sin^{-1}\left(\frac{3}{4}\right) + 2\pi k \implies t = \frac{1}{1000} - \frac{\sin^{-1}\left(\frac{3}{4}\right)}{2000\pi} + \frac{1}{1000}k$$

where $k \in \mathbb{Z}$. Building a quick table, we get the following:



From our table, we can see the first two times the piston is at a height of 14 centimeters are approximately $t = 0.000865$ and $t = 0.000635$ seconds.

Checkpoint 10.3.10

a. $h(t) = -0.9\sin(20\pi t) + 0.9$
b. 0.0518 minutes and 0.9823 minutes

This page titled 10.3: Solving Trigonometric Equations - Algebraic Techniques is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

10.3.1: Resources and Key Concepts

Resources

Prerequisite Topics

This page lists the prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Solving Equations**
  - Solving Quadratic Equations: This section requires solving equations that are quadratic-in-form. The "Skills Refresher" also includes basic quadratic equations.
  - Solving Equations Quadratic-in-Form: This section uses this skill frequently.
  - Solving Rational Equations: This skill is needed for the "Skills Refresher" exercises.

Videos

- Solving Trigonometric Equations
  - Solving Trigonometric Equations - The Concepts Behind Solving Trigonometric Equations
  - Solving Trigonometric Equations - Basic Equations

Key Concepts

Definitions

- **Trigonometric Equation:** An equation that involves one or more trigonometric functions of a variable.
- **Quadratic-in-Form:** An equation that can be written in the form $a[f(x)]^2 + b[f(x)] + c = 0$, where $f(x)$ is some function.

Common Mistakes

- **Forgetting to Find All Solutions:** Trigonometric equations typically have an infinite number of solutions due to the periodic nature of the functions. Forgetting to add multiples of the period ($+2\pi k$ or πk) will result in an incomplete solution set.
- **Stopping After the Inverse Function:** Using an inverse trigonometric function on a calculator provides only one solution (the principal value). You must use that angle as a reference angle to find all solutions in the required quadrants.
- **Incorrectly Handling Non-Standard Arguments:** When solving an equation like $\sin(2x) = k$, the solutions found are for the argument $2x$. A common mistake is forgetting the final step of dividing by 2 to solve for x.
- **Not Checking for Extraneous or Invalid Solutions:** It is important to check for realism. For example, an equation that simplifies to $\sin(\beta) = 20$ has no solution, since the range of the sine function is $[-1, 1]$.
- **Algebraic Errors:** Common algebraic mistakes, such as incorrectly applying the zero-factor principle or dividing by a variable that could be zero, can lead to incorrect or incomplete solutions.

This page titled 10.3.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

10.3.2: Homework

Reading Questions

1. Why do trigonometric equations typically have an infinite number of solutions?
2. What are the two fundamental concepts from trigonometry that are the keys to finding all solutions to a trigonometric equation?
3. If you are solving $\cos(\theta) = -0.6$, in which two quadrants will the solutions lie?
4. After using a calculator to find a reference angle, how do you find the general solution for all angles coterminal with that solution?
5. What does it mean for a trigonometric equation to be "quadratic-in-form"?
6. What algebraic technique is used to solve an equation like $2\cos^2(\theta) - \cos(\theta) - 1 = 0$?
7. When solving an equation involving $\tan(x)$ or $\cot(x)$, how does finding the general solution differ from finding it for sine or cosine?
8. When solving an equation with a non-standard argument, such as $\sin(2x) = 0.5$, what should you treat as the "angle" when finding the reference angle?
9. How does the coefficient on the variable, like the '2' in $\sin(2x)$, affect the number of solutions within the interval $[0, 2\pi]$?
10. Why does an equation like $\sin(x) = 5$ have no solution?

Skills Refresher

Review the following skills you will need for this section.

Skills Refresher

For the following exercises, solve the equation.

1. $(2x - 6)^2 = 4$
2. $\sqrt{2x - 6} = 4$
3. $\dfrac{1}{2x - 6} = 4$
4. $|2x - 6| = 4$
5. $\log 2x - 6 = 4$
6. $2^{2x-6} = 4$

For the following exercises, the "solution" contains an error. Find the error, and supply a correct solution.

7.
$$\begin{aligned} & 3x^2 - 5x &&= 0 \\ \implies \quad & 3x^2 &&= 5x \\ \implies \quad & 3x &&= 5 \\ \implies \quad & x &&= \frac{5}{3} \end{aligned}$$

8.

$$\begin{aligned} 4x^2 - 5x &= 12 \\ \implies x^2 &= 3 \\ \implies x &= \sqrt{3} \end{aligned}$$

9.

$$\begin{aligned} (2x+1)^2 &= 4 \\ \implies 2x+1 &= 2 \\ \implies 2x &= 1 \\ \implies x &= \frac{1}{2} \end{aligned}$$

10.

$$\begin{aligned} 2x^2 - 4x &= 3 \\ \implies 2x(x-2) &= 3 \\ \implies 2x = 3 \quad &\text{or} \quad x - 2 = 3 \\ \implies x = \frac{3}{2} \quad &\text{or} \quad x = 5 \end{aligned}$$

Answers available at the end of this section.

Homework

Concept Check

1. How many solutions between 0° and 360° does the equation $\cos(\theta) = k$ have for each value of k between -1 and 0?
2. How many solutions between 0° and 360° does the equation $\cos(\theta) = k$ have for each value of k greater than 1?
3. How many solutions between 0° and 360° does the equation $\sin^2(\theta) = k$ have for each value of k between -1 and 0?
4. If $x = 0.68$ is one solution of the equation $\cos(x) = c$, what is other? Illustrate on a unit circle.
5. If $t = 2.45$ is one solution of the equation $\sin(t) = k$, what is the other? Illustrate on a unit circle.
6. If $\theta = 1.73$ is one solution of the equation $\tan(\theta) = C$, what is the other?
7. Explain why the equation $\cos(nx) = k, \quad 0 < k < 1$, has $2n$ solutions between 0 and 2π.
8. How many solutions are there to the equation $\cos(\theta) = 0.4$? How many solutions are there between 0 and 2π?
9. Find all angles between 0° and 360° for which the statement is true.
   a. $\cos(\theta) = -1$
   b. $\sin(\theta) = -1$
   c. $\tan(\theta) = -1$
10. a. Find two angles, $0 \le \theta \le 360^\circ$, with $\sin(\theta) = 0$.
    b. Find two angles, $0 \le \theta \le 360^\circ$, with $\cos(\theta) = 0$.

11. a. Find two angles, $0 \le \theta \le 360^\circ$, with $\sin(\theta) = \cos(\theta)$.

    b. Find two angles, $0 \le \theta \le 360^\circ$, with $\sin(\theta) = -\cos(\theta)$.

Basic Skills

For the following exercises,
(a) Evaluate the expression at the given values of the variable.
(b) Give one solution to the equation.

12. a. $\sin(\theta) + \cos(\theta)\,; \quad \theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ$

    b. $\sin(\theta) + \cos(\theta) = \sqrt{2}$

13. a. $\sin^2(\alpha) - \cos(\alpha)\,; \quad \alpha = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

    b. $\sin^2(\alpha) - \cos(\alpha) = 1$

14. a. $\sin(\beta) + 2\cos^2(\beta)\,; \quad \beta = 210^\circ, 225^\circ, 240^\circ, 270^\circ$

    b. $\sin(\beta) + 2\cos^2(\beta) = -1$

15. a. $3\cos^2(\phi) - \sin^2(\phi)\,; \quad \phi = \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{7\pi}{4}, \frac{11\pi}{6}$

    b. $3\cos^2(\phi) - \sin^2(\phi) = 2$

For the following exercises, find all degree solutions (in exact form) and then state the solutions for $0^\circ \le \theta \le 360^\circ$.

16. $7\sin(\theta) + 11 = 11$

17. $3 = 5 - 4\cos(\theta)$

18. $3\tan(\theta) = \sqrt{3}$

19. $8\csc(\theta) + 5 = -11$

20. $9\sec(\theta) + 15 = 6$

21. $6\cot(\theta) + 21 = 15$

22. $0 = \sqrt{2} + 2\sin(\theta)$

23. $\sqrt{3}\cos(\theta) = -\frac{3}{2}$

24. $\cos^2(\theta) - 1 = 0$

25. $1 - \sin^2(\theta) = 0$

26. $3\csc^2(\theta) - 4 = 0$

27. $0 = 4 - 2\sec^2(\theta)$

28. $1 - \tan^2(\theta) = 0$

29. $0 = 2\cot^2(\theta) - 6$

30. $2\sin^2(\theta) + \sin(\theta) - 1 = 0$

31. $4 - \cos(\theta) = 4\cos^2(\theta) + 3\cos(\theta) + 5$

32. $2\sin^2(\theta) - 11\sin(\theta) = -5$

33. $2 = \csc^2(\theta) - \csc(\theta)$

34. $\sec^2(\theta) + 3\sec(\theta) + 2 = 0$

35. $\sin(4x) = -1$

36. $\cos(3t) = 0$

37. $5\tan(2q) = 0$

38. $6\sin(4w) = -3\sqrt{2}$

39. $4\cos(3\phi) = -2$

For the following exercises, find all radian solutions (in exact form) and then state the solutions for $0 \leq \theta \leq 2\pi$.

40. $\sqrt{2}\sin(\theta) + 9 = 8$

41. $\tan(x) - 2\sin(x)\tan(x) = 0$

42. $13 = 13 - 4\cos(\theta)$

43. $4\tan(\theta) - 2 = -6$

44. $8\csc(\theta) + 5 = -3$

45. $2\cos^2(t) + \cos(t) = 1$

46. $\sec(x)\sin(x) - 2\sin(x) = 0$

47. $-6\sec(\theta) + 4 = -8$

48. $3\cot(\theta) + \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

49. $0 = 4\sqrt{2} + 8\sin(\theta)$

50. $\sqrt{3}\cos(\theta) = -\frac{3}{2}$

51. $\sin^2(\theta) - 1 = 0$

52. $1 - \cos^2(\theta) = 0$

53. $4\sec^2(\theta) - 3 = 0$

54. $0 = 1.5 - 0.75\csc^2(\theta)$

55. $1 - \cot^2(\theta) = 0$

56. $0 = -6 + 18\tan^2(\theta)$

57. $\tan^5(x) = \tan(x)$

58. $2\sin^2(\theta) + 3\sin(\theta) = -1$

59. $3 = 6\cos^2(\theta) - 3\cos(\theta)$

60. $8\sin^2(x) + 6\sin(x) + 1 = 0$

61. $\tan^2(x) = -1 + 2\tan(-x)$

52. $2\cos^2(\theta) + 7\cos(\theta) + 3 = 0$

53. $-2 = \csc^2(\theta) - 3\csc(\theta)$

54. $\sec^2(\theta) - 2 = -\sec(\theta)$

55. $\sqrt{3}\tan(2\alpha) = 3$

56. $2\sin(2\beta) = 1$

57. $-6\cos(2\theta) = 6$

58. $3\tan(3w) = \sqrt{3}$

59. $2\tan(3u) = -2$

70. $\frac{5}{7}\tan(\pi x) + 11 = 11$

For the following exercises, solve the equation for $0^\circ \le \theta \le 360^\circ$. Round your answers to two decimal places.

71. $4 = 6\sin(\theta)$

72. $3 = 5\cos(\theta)$

73. $\frac{1}{2}\tan(\theta) - 1 = -3$

74. $7\csc(\theta) + 2 = -16$

75. $3\sec(\theta) - 2 = 5$

76. $2 = 5 - \frac{1}{3}\cot(\theta)$

77. $6\cos^2(\theta) = 2$

78. $2 - 7\sin^2(\phi) = 1$

79. $-4\tan^2(\theta) + 5 = -4$

80. $8 - 2\sec^2(\theta) = -2$

81. $3 = 9 - 2\csc^2(\theta)$

82. $9\cot^2(\theta) - 9 = -5$

83. $5\sin^2(\theta) + \sin(\theta) = 0$

84. $10\cos^2(\theta) - 7\cos(\theta) + 1 = 0$

85. $4\tan^2(\theta) = \tan(\theta)$

86. $\cot^2(\theta) - 2\cot(\theta) = 15$

87. $5\sin(3\theta) - 3 = -4$

88. $9\cos(2\theta) + 1 = 6$

89. $8\tan(4t) + 1 = -11$

90. $6\csc(2r) + 2 = 13$

For the following exercises, solve the equation for $0 \le \theta \le 2\pi$. Round your answers to two decimal places.

91. $5\sin(\theta) - 4 = -6$

92. $-4 + 3\cos(\theta) = -2$

93. $-2\tan(\theta) + 5 = 6$

94. $-4 = 5 - \csc(\theta)$

95. $3\sec(\theta) - 2 = -10$

96. $\frac{1}{3}\cot(\theta) + 1 = -2$

97. $5\sin^2(\theta) = 2$

98. $3 - 6\cos^2(\phi) = -1$

99. $4 - 4\tan^2(\theta) = -5$

100. $4 - \sec^2(\theta) = -1$

101. $-1 + 2\csc^2(\theta) = 5$

102. $\frac{9}{2}\cot^2(\theta) - 1 = 1$

103. $\sin^2(\theta) + 8\sin(\theta) + 7 = 0$

104. $\cos^2(\theta) - 4\cos(\theta) + 3 = 0$

105. $\tan^2(\theta) - 5\tan(\theta) + 6 = 0$

106. $6\cot^2(\theta) - \cot(\theta) - 1 = 0$

107. $150\sin(3s) = 27$

108. $7\cos(2t) - 3 = 2$

109. $3 = 3\tan(4x) + 4$

110. $2 - 8\sec(3t) = -10$

Proofs

111. What is the angle in the first quadrant with reference angle θ? What is the angle in the third quadrant with reference angle θ ? What is the difference of these angles? Explain how this fact shows that the solutions of $\tan(\theta) = k$, for $k > 0$, differ by π.

112. What is the angle in the second quadrant with reference angle θ? What is the angle in the fourth quadrant with reference angle θ? What is the difference of these angles? Explain how this fact shows that the solutions $\tan(\theta) = k$, for $k < 0$, differ by π.

Applications

113. A sinusoidal function can approximate the population of deer in Marquette County over a typical year. The population reached a maximum of 50,000 deer on September 1st and a minimum of 42,000 on March 1st.

    a. Write a formula for the function $P(t)$ that gives the deer population on the first of each month, where $t = 0$ is September 1.

    b. When is the deer population 45,000? Give exact expressions and approximations rounded to two decimal places.

c. Graph your function over one period, and label the points that correspond to a deer population of 45,000. Is the population greater or less than 45,000 between the two solutions?

14. The percent of the moon visible from Earth is a sinusoidal function ranging from 0% to 100%, with a period of 29.5 days.

    a. Write a formula for the function $f(t)$ that gives the percent of the moon that is visible, if a new moon (0% visible) occurs at $t = 0$ days.

    b. When is 25% of the moon visible? Give approximations rounded to two decimal places.

    c. Graph your function over one period, and label the points corresponding to a quarter moon. Is more or less than 25% of the moon visible between the two solutions you found in part (b)?

15. A Ferris wheel has a diameter of 20 meters and completes one revolution every 60 seconds. Jorge is at the lowest position of the Ferris wheel, 1 meter above the ground, at $t = 0$ seconds.

    a. Write a formula for the function $h(t)$ that gives Jorge's altitude in meters after t seconds.

    b. When is Jorge at an altitude of 18 meters during his first revolution? Give exact expressions and approximations rounded to two decimal places.

    c. Graph your function over one period, labeling the points that correspond to an altitude of 18 meters. Is Jorge above or below 18 meters between the two solutions you found in part (b)?

16. High tides occur every 12.2 hours at Point Lookout. The depth of the water at the end of Roy's dock is 2.6 meters at high tide and 1.8 meters at low tide.

    a. Write a formula for the function $d(t)$ that gives the depth of the water t hours after last night's high tide.

    b. When is the water at the end of the dock 2 meters deep? Give exact approximations rounded to two decimal places.

    c. Graph your function over one period, labeling the points that correspond to a depth of 2 meters. Is the water depth greater or less than 2 meters between the two solutions you found in part (b)?

Answers

Skills Refresher

1. 2, 4
2. 11
3. $\frac{25}{8}$
4. 1, 5
5. 5003
6. 4
7. We can't divide by x. $x = 0, \frac{5}{3}$
8. 3 has two square roots. $x = \pm\sqrt{3}$
9. 4 has two square roots. $x = \frac{-3}{2}, \frac{1}{2}$
10. One side must be zero to use the zero-factor principle. No real solutions.

This page titled 10.3.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

10.4: Solving Trigonometric Equations Using Identities

Note to the Instructor - The Remaining Types of Equations

The equations in this section require students to use trigonometric identities. Additionally, students will learn how to rewrite a sum or difference of sine and cosine as a single sinusoidal function. If you don't recall, this is a necessary technique for Differential Equations.

Learning Objectives

- Use an identity to solve a trigonometric equation.
- Rewrite a sum or difference of sine and cosine as a single sinusoidal function.
- Use technology to approximate solutions to equations involving trigonometric functions.



Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

Having a solid foundation of how trigonometric equations are solved, we now integrate our knowledge of trigonometric identities to further expand our equation-solving abilities.

Solving Equations Involving Different Trigonometric Functions Having the Same Arguments

The equations encountered in the previous section were ideal in that each featured only one trigonometric function, and all occurrences of that function within the equation shared the same argument. That is, we only encountered equations like the following:

$$\begin{aligned} \sin(\theta) &= -0.6428 \\ \tan(2x) &= \sqrt{3} \\ 4\tan^2(\theta) + 3 &= 15 \\ 2\cos^2(\theta) - \cos(\theta) - 1 &= 0 \end{aligned}$$

Each of these equations features only one trigonometric function. In the fourth equation, that trigonometric function (the cosine) occurs twice, but in both cases, the arguments are identical (θ).

How would we solve a trigonometric equation if it contains two *different* trigonometric functions? For example, if we were tasked with solving

$$\sec^2(x) = \tan(x) + 3,$$

what would we do?

Ultimately, the answer is that we use a mixture of the trigonometric identities learned in this course and algebraic manipulations learned in Algebra. The worst thing I could do at this point is throw a bunch of examples at you showing how to solve specific equations - that approach will make the *single* equation-solving tactic seem like a *dozen* different tactics. To avoid this, let's start

by reviewing how we solved trigonometric equations in the previous section and codify this into a tactic.

When we were given an equation of the form

$$A\,\text{trig}\,(Bx+C)+D=E,$$

we first isolated the trigonometric function (using techniques from Algebra) to a form like

$$\text{trig}\,(Bx+C)=c,$$

where c is a real number. I also suggested a "substitution" technique if the argument of the trigonometric function wasn't a simple variable like θ or x. Let's suppose we use that technique to let $u=Bx+C$. Then our equation would simplify down to

$$\text{trig}\,(u)=c.$$

At this point, we used the sign of c to help us determine the quadrants and reference angle associated with our solution. Once those two pieces of information were found, we stated the answers.

This review leads to the following tactic for solving "basic" trigonometric equations.

How To... Solve a "Basic" Trigonometric Equation

Given *any* trigonometric equation of the form

$$\text{trig}(u)=c,$$

where "trig" could be $\sin,\cos,\tan,\csc,\sec,$ or $\cot$, do the following:

1. If the value of c is such that u is a quadrantal angle, skip to Step 5.
2. Use the sign of c to determine in which quadrants u terminates.
3. Find the reference angle, $\hat{u}$.
4. Using the reference angle from Step 3, write the solutions *associated* with the quadrants listed in Step 2. If possible, compress the solution set to a single equation (this is most often the case with equations involving the tangent or the cotangent functions).
5. If u is a function of another variable, solve the equations in Step 4 (or Step 1, if the angle is a quadrantal angle) for the "true" variable.
6. Check for extraneous solutions!

Therefore, our ultimate goal when solving *any* trigonometric equation is to get it into the form $\text{trig}\,(u)=c$.

Example 10.4.1

Solve the following equations and list the solutions which lie in the interval $[0,2\pi)$.

a. $3\cos(x)\sin^2(x)=\sin^2(x)$
b. $\sec^2(x)=\tan(x)+3$
c. $\cos^2(A)-\sin^2(A)-\sin(A)=1$

Solutions available at the end of this section.

Advice: Squares on Trigonometric Functions Are Clues

In Examples $10.4.1b$ and $10.4.1c$, we used a Pythagorean Identity to transform the square of a trigonometric function into a different trigonometric function. This should convince you that squares on trigonometric functions are often desirable because they allow us to rewrite an expression in an equivalent form using a different trigonometric function.

Checkpoint 10.4.1

Solve each equation on the indicated interval.

a. $1-\sin^2\theta+\cos\theta=0$ for $0^\circ\le\theta<360^\circ$
b. $2\tan(\theta)\sin(\theta)+\tan(\theta)=1+2\sin(\theta)$ for $\theta^{\text{int}}\,[0,2\pi)$

Answers available at the end of this section.

Solving Equations Involving Trigonometric Functions Having Different Angular Frequencies

Suppose an equation contains more than one trigonometric function (whether the same or not) but with different angular frequencies. In that case, we will need to use the Double-Angle Identities to help simplify the equation into the form $\text{trig}\,(u)=c$.

Example 10.4.2

Solve the following equations and list the solutions in the interval $[0,2\pi)$.

a. $\cos(2x)=3\cos(x)-2$
b. $\sin(2x)=\sqrt{3}\cos(x)$
c. $\sin(x)\cos\left(\frac{x}{2}\right)+\cos(x)\sin\left(\frac{x}{2}\right)=1$

Solutions available at the end of this section.

Instructors will tell you to "check your solutions" after solving an equation. While this is always great advice, let's be realistic - most people don't check their solutions. With that sense of realism in mind, I want to clarify something from Algebra and tell you when you *must* check the validity of your solutions for the risk of them being wrong. There are three general cases when you *must* check solutions after solving an equation:

1. when the equation contains a function with a restricted domain,
2. when you multiply both sides of an equation by a variable expression, and
3. when you apply an even power (e.g., a square) to both sides of an equation.

When solving trigonometric equations, you can run into any of these issues. In Example $10.4.2e$, we ended up squaring both sides of the equation in the process of finding the solution (see the step with "**CHECK SOLUTIONS!**"). When doing so, you risk introducing "false" solutions - called **extraneous solutions**. Indeed, that is precisely what happened!

Checkpoint 10.4.2

Find the solutions to the equation on requested the interval.

$$\tan(\theta)+\cot(\theta)=4\sin(2\theta)\quad\text{on}\quad[0,2\pi)$$

Answers available at the end of this section.

Solving Other Trigonometric Equations

At any given step while solving an equation, the tactic I have continually modeled throughout this material is to do the Arithmetic first (i.e., performing simple operations on constants). Once this is done, you move on to see if there is any Algebra to be done (e.g., combining like terms, factoring, etc.). Once you have exhausted all possibilities to use Arithmetic and Algebra, then (and only then) you move on to looking for identities or concepts from Trigonometry.

But what happens when you exhaust all three and you're stuck?

In truth, this means one of two things - either the equation you were given involves a combination of trigonometric functions and non-trigonometric functions (in which case you have no choice but to use technology), or you missed some not-so-obvious Algebra or Trigonometry. There is no way to cover all possible "oddball" cases. Still, the following example covers a case that comes up often enough that we should investigate it.

Example 10.4.3

Solve the equation and list the solutions that lie in the interval $[0, 2\pi)$.

$$\cos(x) = \sin(x)$$

Solutions available at the end of this section.

Another option with Example 10.4.3 is to start by dividing both sides by $\cos(x)$. Doing so, we get

$$1 = \tan(x).$$

This is *much* easier to solve, however, by dividing by the cosine, we have introduced a possible issue - what if $\cos(x) = 0$?

A little thought goes a long way here. If $\cos(x) = 0$, then $x = \frac{\pi}{2} + \pi k$. If we go back to the original equation and plug in $\frac{\pi}{2}$, we would get

$$\begin{aligned} \sin\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}\right) \\ \Longrightarrow \qquad 1 &= 0 \end{aligned}$$

Since this is obviously not true, $x = \frac{\pi}{2}$ is not a solution to the equation. It is a simple task to verify that $x = \frac{3\pi}{2}$ is also not a solution. Therefore, we don't have to worry about $\cos(x) = 0$, and we can go back to solving

$$\tan(x) = 1.$$

Tangent is positive in QI and QIII, and $\hat{x} = \frac{\pi}{4}$. Hence, the solutions are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Footnotes

[1] Recall from Algebra that when you divide both sides of an equation by a variable expression, you are assuming the expression you are dividing by is nonzero. Therefore, in the given equation, unless you can guarantee that $\sin^2(x) \neq 0$, you cannot divide both sides of the equation by $\sin^2(x)$.

[2] Technically, we can use the Double-Angle Identities, however, they would not be very helpful. This is because, for example,

$$\cos(3x) = \cos\left(2\left(\frac{3}{2}x\right)\right) = 2\cos^2\left(\frac{3}{2}x\right) - 1,$$

and this looks *much* worse than the original expression.

[3] Notice that I am speeding up descriptions as we move through. That is, the amount of detail I describe once a topic is introduced becomes less and less. This is purposeful. You need to be fluent in this material so it becomes second nature.

[4] Remember, when you square both sides of an equation, you run the risk of introducing extraneous solutions. Therefore, you must *always* check your solutions when you perform this action.

Answers

Example 10.4.1

a. We resist the temptation to divide both sides of $3\cos(x)\sin^2(x) = \sin^2(x)$ by $\sin^2(x)$.[1] Instead, we look at our desired destination - $\text{trig}(u) = c$. One thing is certain about that desired form: We need all trigonometric functions on one side. Let's start there.

$$\begin{aligned} & & 3\cos(x)\sin^2(x) &= \sin^2(x) & \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & 3\cos(x)\sin^2(x) - \sin^2(x) &= 0 & \text{(move all variable expressions to one side)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & & \sin^2(x)\left[3\cos(x) - 1\right] &= 0 & \text{(factor out the GCF)} \end{aligned}$$

We now use the Zero Factor Property from Algebra to break the equation into two different (and simpler) equations.

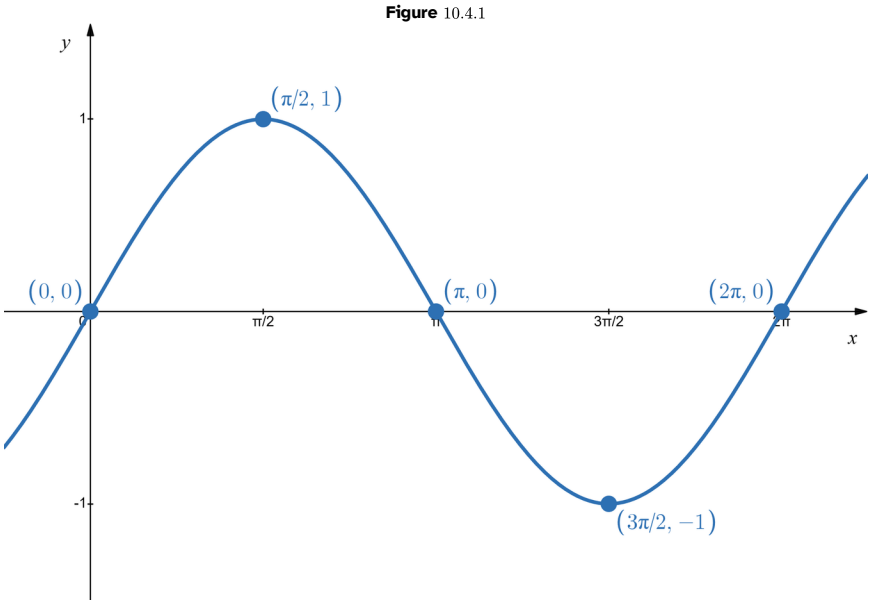
$$\begin{aligned} & \sin^2(x) = 0 \quad \text{or} \quad 3\cos(x) - 1 = 0 \\ \Longrightarrow \quad & \sin(x) = 0 \quad \text{or} \quad \cos(x) = \frac{1}{3} \end{aligned}$$

These equations match the form $\text{trig}(u) = c$, so solving them will mirror what we did in the previous section.

For $\sin(x) = 0$

The right side of $\sin(x) = 0$ implies a quadrantal angle.

Figure 10.4.1



From the graph of $y = \sin(x)$, we know the solution to this equation is $x = \pi k$, with $x = 0$ and $x = \pi$ being the two solutions which lie in $[0, 2\pi)$.

For $\cos(x) = \frac{1}{3}$

Since the cosine is positive, its argument is an angle that terminates in QI or QIV. Additionally, the ratio, $\frac{1}{3}$, is not a special ratio, so we will eventually need the use of technology. For now, we note that the reference angle is $\hat{x} = \cos^{-1}\left(\left|\frac{1}{3}\right|\right) = \cos^{-1}\left(\frac{1}{3}\right)$. Hence, the two families of solutions are

$$\begin{aligned} \text{Solutions associated with QI}: \quad x &= \cos^{-1}\left(\frac{1}{3}\right) + 2\pi k \\ \text{Solutions associated with QIV}: \quad x &= 2\pi - \cos^{-1}\left(\frac{1}{3}\right) + 2\pi k \end{aligned}$$

where $k \overset{\text{int}}{\in} \mathbb{Z}$. We find the two solutions which lie in $[0, 2\pi)$ to be

$$x = \cos^{-1}\left(\frac{1}{3}\right) \quad \text{and} \quad x = 2\pi - \cos^{-1}\left(\frac{1}{3}\right).$$

Thus, the solutions to the original equation on the interval $[0, 2\pi)$ are

$$x = 0,\ x = \cos^{-1}\left(\frac{1}{3}\right),\ x = \pi,\ \text{and}\ x = 2\pi - \cos^{-1}\left(\frac{1}{3}\right).$$

Grabbing a calculator (and rounding to four decimal places), we get

$$x = 0,\ x = \pi,\ x \approx 1.2310,\ \text{and}\ x \approx 5.0522.$$

b. Again, our ultimate goal is to get an equation (or equations) equivalent to $\text{trig}(u) = c$. At the very least, this means we need to move all the trigonometric functions to one side of the equation.

$$\begin{aligned} & \sec^2(x) &= \tan(x) + 3 & \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} & \sec^2(x) - \tan(x) &= 3 & \text{(move all variable expressions to one side)} \end{aligned}$$

Again, we have a mixture of different trigonometric functions on the left side, but this time, Algebra (specifically, factoring) is not going to help (do you see why?). Instead, we look for any trigonometric identities that might be of use. The $\sec^2(x)$ in an equation with $\tan(x)$ screams "Pythagorean Identity!"

$$\begin{aligned} \sec^2(x) - \tan(x) &= 3 \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} \quad \tan^2(x) + 1 - \tan(x) &= 3 && \text{(Pythagorean Identities)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad \tan^2(x) - \tan(x) - 2 &= 0 && \text{(subtract 3 from both sides)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad u^2 - u - 2 &= 0 && \text{(substitute } u = \tan(x)\text{)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad (u+1)(u-2) &= 0 && \text{(factor)} \end{aligned}$$

This gives $u = -1$ or $u = 2$. Since $u = \tan(x)$, we have $\tan(x) = -1$ or $\tan(x) = 2$.

For $\tan(x) = -1$

Since the tangent is negative, we know the argument terminates in either QII or QIV. Moreover, the reference angle is $\frac{\pi}{4}$. Hence,

$$\text{Solutions associated with QII and IV}: \quad x = \frac{3\pi}{4} + \pi k$$

where $k^{\text{int}}\mathbb{Z}$. Notice that we used the fact that the period of the tangent is π to compress our *two* solution sets into *one*. Letting $k = 0$ and then $k = 1$, we get the solutions $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$, respectively. Any further, and our values for x will go beyond the requested interval.

For $\tan(x) = 2$

The tangent is positive in QI and QIII. We should recognize that the right side is *not* a special ratio for the tangent function. Therefore, we set the reference angle to $\hat{x} = \tan^{-1}(|2|) = \tan^{-1}(2)$. Thus,

$$\text{Solutions associated with QI and III}: \quad x = \tan^{-1}(2) + \pi k$$

where $k^{\text{int}}\mathbb{Z}$. On $[0, 2\pi)$, $x = \tan^{-1}(2)$ and $x = \pi + \tan^{-1}(2)$ are the solutions.

Hence, the solution set to the original equation on the interval $[0, 2\pi)$ is

$$x^{\text{int}}\left\{\tan^{-1}(2),\ \frac{3\pi}{4},\ \pi + \tan^{-1}(2),\ \frac{7\pi}{4}\right\},$$

where $\tan^{-1}(2) \approx 1.1071$ and $\pi + \tan^{-1}(2) \approx 4.2487$.

c. Since this equation involves both the sine and cosine functions, we will need to either perform some Algebra or look for trigonometric identities to get to a point where we have a simple equation (or equations) like $\text{trig}(u) = c$. Factoring out the GCF is out of the question because the three terms on the left side of

$$\cos^2(A) - \sin^2(A) - \sin(A) = 1$$

do not share a common factor. Therefore, we immediately look for obvious trigonometric identities. We know by this point that any time we encounter the square of a trigonometric function, we can rewrite it using a Pythagorean Identity. In this case, we have *two* terms containing squares of trigonometric functions. We will rewrite $\cos^2(A)$ in terms of $\sin^2(A)$ so our equation will only have terms involving the sine function.

$$\begin{aligned} \cos^2(A) - \sin^2(A) - \sin(A) &= 1 \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} \quad 1 - \sin^2(A) - \sin^2(A) - \sin(A) &= 1 && \text{(Pythagorean Identity)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad 1 - 2\sin^2(A) - \sin(A) &= 1 && \text{(combine like terms)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad -2\sin^2(A) - \sin(A) &= 0 && \text{(subtract 1 from both sides)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad 2\sin^2(A) + \sin(A) &= 0 && \text{(multiply both sides by } -1\text{)} \\ \xRightarrow{\text{Arithmetic}\to\text{Algebra}} \quad \sin(A)[2\sin(A) + 1] &= 0 && \text{(factor out the GCF)} \end{aligned}$$

Thus, by the Zero Factor Property,

$$\sin(A) = 0 \quad \text{or} \quad \sin(A) = -\frac{1}{2}.$$

For $\sin(A) = 0$

From part (a), we know the solutions on $[0, 2\pi)$ are $x = 0$ and $x = \pi$.

For $\sin(A) = -\frac{1}{2}$

The sine is negative in QIII and QIV, and the reference angle is $\frac{\pi}{6}$. Therefore, the solutions on $[0, 2\pi)$ are $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$.

Thus, the solution set to this equation on the interval $[0, 2\pi)$ is

$$x^{\text{int}}\left\{0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}.$$

Checkpoint 10.4.1

a. $90^\circ, 180^\circ, 270^\circ$

b. $\frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4},\ \frac{11\pi}{4}$

Example 10.4.2

a. In the equation $\cos(2x) = 3\cos(x) - 2$, we have the same circular function, namely cosine, on both sides but the arguments differ. Using the Double-Angle Identity

$$\cos(2x) = 2\cos^2(x) - 1,$$

we obtain a "quadratic in disguise" and proceed as we have done in the past.

$$\begin{aligned}
& & \cos(2x) &= 3\cos(x) - 2 & \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} & & 2\cos^2(x) - 1 &= 3\cos(x) - 2 & \text{(Double-Angle Identity)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & 2\cos^2(x) - 3\cos(x) + 1 &= 0 & \text{(gathering terms on the left side)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & 2u^2 - 3u + 1 &= 0 & (u-\text{Substitution}) \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & (2u-1)(u-1) &= 0 & \text{(factor)}
\end{aligned}$$

This gives $u = \frac{1}{2}$ or $u = 1$. Since $u = \cos(x)$, we get $\cos(x) = \frac{1}{2}$ or $\cos(x) = 1$.

For $\cos(x) = \frac{1}{2}$

Cosine is positive in QI and QIV and the reference angle is $\hat{x} = \frac{\pi}{3}$. Therefore, we have the following:

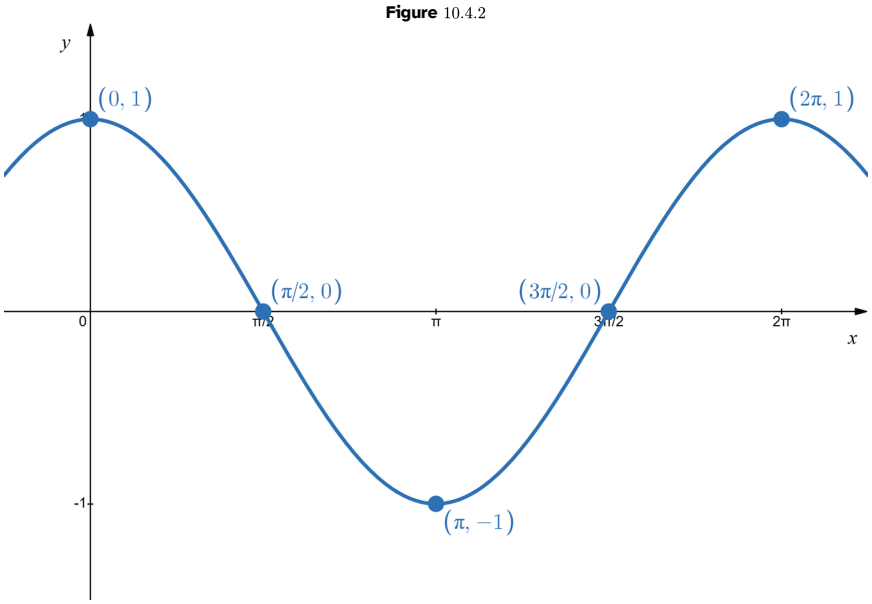
$$\begin{aligned}
\text{Solutions associated with QI:} \quad x &= \tfrac{\pi}{3} + 2\pi k \\
\text{Solutions associated with QIV:} \quad x &= \tfrac{5\pi}{3} + 2\pi k,
\end{aligned}$$

where $k \overset{\text{int}}{\in} \mathbb{Z}$.

For $\cos(x) = 1$

This implies x is a quadrantal angle. Quickly sketching the cosine curve,

Figure 10.4.2



we see that $\cos(x) = 1$ when $x = 2\pi k$.

Thus, the answers which lie in $[0, 2\pi)$ are $x = 0$, $x = \frac{\pi}{3}$, and $x = \frac{5\pi}{3}$.

b. Here is where I shorten the discussion and rely on your ability to follow the logic we have been using up to this point.

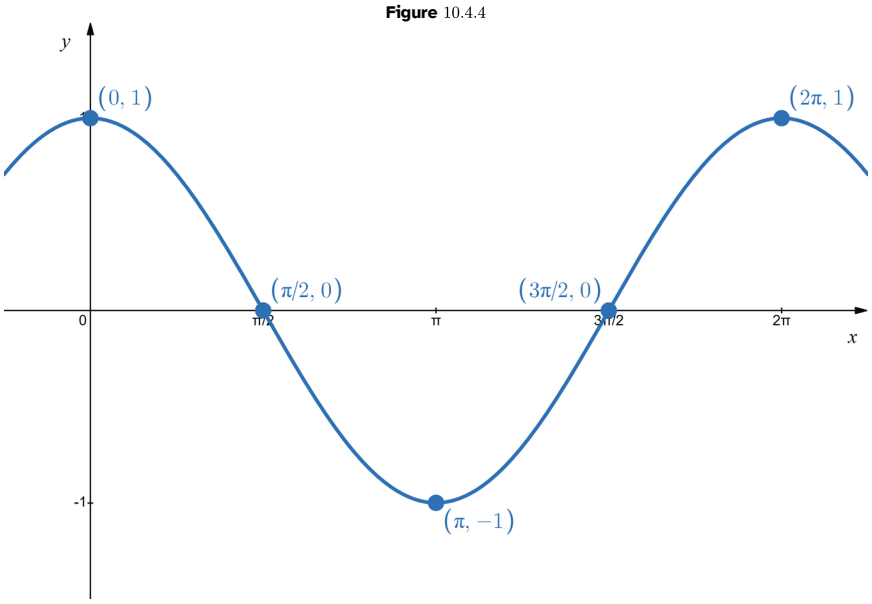
$$\begin{aligned}
& & \sin(2x) &= \sqrt{3}\cos(x) & \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & \sin(2x) - \sqrt{3}\cos(x) &= 0 & \text{(move variable terms to one side)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} & & 2\sin(x)\cos(x) - \sqrt{3}\cos(x) &= 0 & \text{(Double-Angle Identity)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & \cos(x)\left(2\sin(x) - \sqrt{3}\right) &= 0 & \text{(factor out the GCF)}
\end{aligned}$$

From this, we get $\cos(x) = 0$ or $\sin(x) = \frac{\sqrt{3}}{2}$.

For $\cos(x) = 0$

Since 0 is a quadrantal ratio for the cosine, we quickly sketch the cosine function.

Figure 10.4.4



We see the solutions are $x = \frac{\pi}{2} + \pi k$ for $k^{\text{int}}\mathbb{Z}$. On this interval $[0, 2\pi)$, this gives the restricted solutions $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

For $\sin(x) = \frac{\sqrt{3}}{2}$

The sine is positive in QI and QII, and the reference angle for sine with the ratio $\frac{\sqrt{3}}{2}$ is $\hat{x} = \frac{\pi}{3}$. Thus, we get the solutions $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{2\pi}{3} + 2\pi k$ for $k^{\text{int}}\mathbb{Z}$ [3].

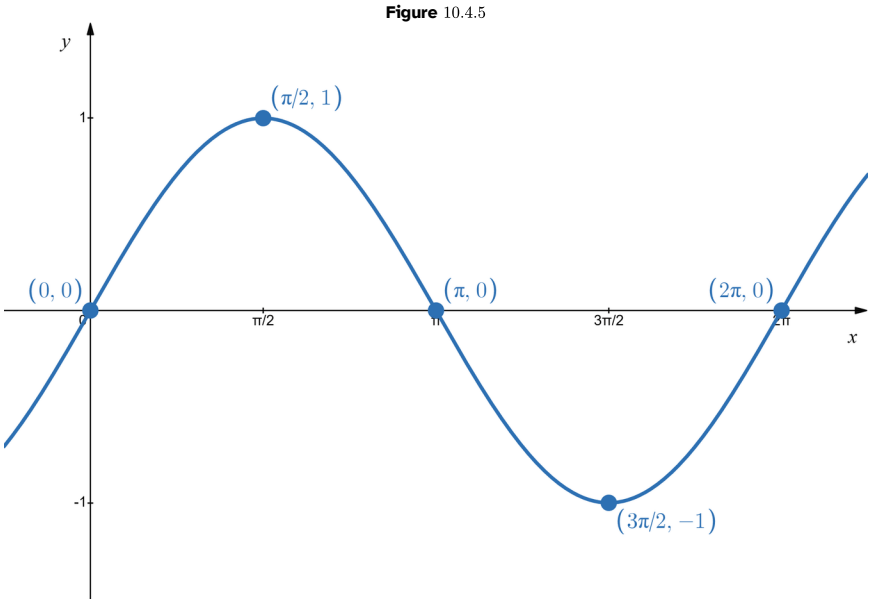
The answers which lie in $[0, 2\pi)$ are $x = \frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

c. There seems to be no quick way to get the trigonometric functions or their arguments to match in the equation $\sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) = 1$; however, if we stare at it long enough, we realize that the left side is the expanded form of the Sum of Angles Identity for the sine.

$$\begin{aligned} \sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) &= 1 \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} \quad \sin\left(x + \frac{x}{2}\right) &= 1 \qquad \text{(Sum of Angles Identity)} \\ \xrightarrow{\text{Arithmetic}\to\text{Algebra}} \quad \sin\left(\frac{3}{2}x\right) &= 1 \qquad \text{(combine like terms)} \end{aligned}$$

Since 1 is a quadrantal ratio for the sine, we can quickly sketch $y = \sin(\theta)$

Figure 10.4.5



to see that $\sin(\theta) = 1$ when $\theta = \frac{\pi}{2} + 2\pi k$, where $k^{\text{int}}\mathbb{Z}$. Therefore,

$$\frac{3}{2}x = \frac{\pi}{2} + 2\pi k \implies x = \frac{\pi}{3} + \frac{4\pi}{3}k.$$

Two of these solutions lie in $[0, 2\pi)$: $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

Checkpoint 10.4.2

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

Example 10.4.3

The angular frequencies are the same, but the circular functions are different. This has been fine up to this point because all the previous examples involving two different circular functions had at least one of these functions being squared. Squares of trigonometric functions are blessings in disguise, as we can always try to change them using the Pythagorean Identities. In this case, there are no squares. So what do we do? Since we would like squares, why don't we square both sides?[4]

$$\begin{aligned}
& & \cos(x) &= \sin(x) & \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & \cos^2(x) &= \sin^2(x) & \text{(square both sides} \\
& & & & \textbf{CHECK SOLUTIONS!)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}\to\text{Trigonometry}} & & 1-\sin^2(x) &= \sin^2(x) & \text{(Pythagorean Identity)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & 1 &= 2\sin^2(x) & \text{(move all variable terms to one side)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & \frac{1}{2} &= \sin^2(x) & \text{(divide both sides by 2)} \\
\xRightarrow{\text{Arithmetic}\to\text{Algebra}} & & \pm\frac{1}{\sqrt{2}} &= \sin(x) & \text{(Extraction of Roots)}
\end{aligned}$$

Therefore, $\hat{x} = \frac{\pi}{4}$ and we need to check *all* quadrants.

$$\begin{aligned}
\text{Solutions associated with QI:} \quad & x = \hat{x} + 2\pi k = \frac{\pi}{4} + 2\pi k \\
\text{Solutions associated with QII:} \quad & x = \pi - \hat{x} + 2\pi k = \pi - \frac{\pi}{4} + 2\pi k = \frac{3\pi}{4} + 2\pi k \\
\text{Solutions associated with QIII:} \quad & x = \pi + \hat{x} + 2\pi k = \pi + \frac{\pi}{4} + 2\pi k = \frac{5\pi}{4} + 2\pi k \\
\text{Solutions associated with QIV:} \quad & x = 2\pi - \hat{x} + 2\pi k = 2\pi - \frac{\pi}{4} + 2\pi k = \frac{7\pi}{4} + 2\pi k
\end{aligned}$$

where $k^{\text{int}}\mathbb{Z}$. This is a case where, rather than compressing these solutions into two families (which we *could* do), it's easier to check the four possibilities separately. In all four cases, the reference angles are $\hat{x} = \frac{\pi}{4}$. Therefore, the sine and cosine functions become $\pm\frac{1}{\sqrt{2}}$. We just require the signs to be the same.

In QI, $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right)$. In QIII, $\cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} = \sin\left(\frac{5\pi}{4}\right)$; however, we can throw out the other two possibilities because the signs of the functions are opposite in both QII and QIV. Thus, the solution set on the interval $[0, 2\pi)$ is $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

This page titled 10.4: Solving Trigonometric Equations Using Identities is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

10.4.1: Resources and Key Concepts

Resources

Videos

- Solving Trigonometric Equations
  - Solving Trigonometric Equations - Equations Requiring the Use of Identities
  - Solving Trigonometric Equations - Multiple-Angle Equations
  - Solving Trigonometric Equations - Equations Requiring Technology and a Warning

Key Concepts

Common Mistakes

- **Dividing by a Variable Expression:** Do not divide both sides of an equation by an expression that could be equal to zero, such as $\sin^2(x)$. This can lead to losing solutions. Instead, move all terms to one side and factor.
- **Forgetting to Check for Extraneous Solutions:** When you square both sides of an equation, you risk introducing extraneous solutions. You must check all potential solutions in the original equation to ensure they are valid.
- **Ignoring Simpler Paths:** It is easy to get focused on one solution method. Always look for the simplest approach. For example, after finding $\sin(2\theta)$ and $\cos(2\theta)$, it's often easier to find $\tan(2\theta)$ using the Ratio Identity rather than a more complex identity.

This page titled 10.4.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

10.4.2: Homework

Reading Questions

1. What is the ultimate goal when solving a trigonometric equation, according to the text?
2. When an equation contains a squared trigonometric function, what type of identity is often useful for simplifying it?
3. When solving an equation like $3\cos(x)\sin^2(x) = \sin^2(x)$, what is the recommended first step instead of dividing by $\sin^2(x)$?
4. What algebraic principle allows you to solve a factored trigonometric equation like $\sin(A)[2\sin(A) + 1] = 0$?
5. If an equation contains trigonometric functions with different arguments, like $\cos(2x)$ and $\cos(x)$, what is the general strategy?
6. Under what circumstances must you check your solutions for extraneous solutions?

Homework

Concept Check

1. Will there always be solutions to trigonometric function equations? If not, describe an equation that would not have a solution. Explain why or why not.

2. When solving a trigonometric equation involving more than one trigonometric function, do we always want to try to rewrite it so it is expressed in one trigonometric function? Why or why not?

Basic Skills

For the following exercises, solve the equation, giving exact solutions on the interval $[0, 2\pi)$.

3. $\sin(2\theta) + \sqrt{2}\cos(\theta) = 0$
4. $\sin(2\alpha)\sin(\alpha) = \cos(\alpha)$
5. $\cos(2t) - 5\cos(t) + 3 = 0$
6. $\cos(2x) + 3\sin(x) = 2$
7. $\tan(2\beta) + 2\sin(\beta) = 0$
8. $\tan(2z) - 2\cos(z) = 0$
9. $3\cos(\phi) - \sin\left(\frac{\pi}{2} - \phi\right) = \sqrt{3}$
10. $\sin(w) + \cos\left(\frac{\pi}{2} - w\right) = 1$
11. $\sin(2\phi)\cos(\phi) + \cos(2\phi)\sin(\phi) = 1$
12. $\cos(\theta)\cos(3\theta) + \sin(\theta)\sin(3\theta) = \frac{\sqrt{2}}{2}$
13. $\sin(x) = \cos(x)$
14. $\sin(2x) = \sin(x)$
15. $\sin(2x) = \cos(x)$
16. $2\tan^2(t) = 3\sec(t)$
17. $\cos(2x) = \sin(x)$

18. $\cos(2x) = \cos(x)$

19. $\cos(2x) = 2 - 5\cos(x)$

20. $3\cos(2x) + \cos(x) + 2 = 0$

21. $\cos(2x) = 5\sin(x) - 2$

22. $3\cos(2x) = \sin(x) + 2$

23. $2\sec^2(x) = 3 - \tan(x)$

24. $\tan^2(x) = 1 - \sec(x)$

25. $\cot^2(x) = 3\csc(x) - 3$

26. $\sec(x) = 2\csc(x)$

27. $\cos(x)\csc(x)\cot(x) = 6 - \cot^2(x)$

28. $\sin(2x) = \tan(x)$

29. $\cot^4(x) = 4\csc^2(x) - 7$

30. $\cos(2x) + \csc^2(x) = 0$

31. $\tan^3(x) = 3\tan(x)$

32. $\cos(6x) - \cos(3x) = 0$

33. $\sin^2(x) - \cos^2(x) - \sin(x) = 0$

34. $\tan^2(x) = \frac{3}{2}\sec(x)$

35. $2\sin(x)\cos(x) - \sin(x) + 2\cos(x) - 1 = 0$

36. $\cos^3(x) = -\cos(x)$

37. $\tan(2x) - 2\cos(x) = 0$

38. $\csc^3(x) + \csc^2(x) = 4\csc(x) + 4$

39. $2\tan(x) = -1 - \tan^2(x)$

40. $\tan(x) = \sec(x)$

41. $\sin(6x)\cos(x) = -\cos(6x)\sin(x)$

42. $\sin(3x)\cos(x) = \cos(3x)\sin(x)$

43. $\cos(2x)\cos(x) + \sin(2x)\sin(x) = 1$

44. $\cos(5x)\cos(3x) - \sin(5x)\sin(3x) = \frac{\sqrt{3}}{2}$

45. $9\cos(2\theta) = 9\cos^2(\theta) - 4$

46. $\tan(x) = \cos(x)$

For the following exercises, solve the equation for $0^\circ \leq \theta \leq 360^\circ$. Round angles to three decimal places if necessary.

47. $\cos(\theta) - \sin^2(\theta) + 1 = 0$

48. $4\sin(\theta)+2\cos^2(\theta)-3=-1$

49. $1-\sin(\theta)-2\cos^2(\theta)=0$

50. $3\cos^2(\theta)-\sin^2(\theta)=2$

51. $2\cos(\theta)\tan(\theta)+1=0$

52. $\cos(\theta)-\sin(\theta)=0$

53. $\frac{1}{3}\cos(\theta)=\sin(\theta)$

54. $5\sin(C)=2\cos(C)$

For the following exercises, find all radian solutions to the equation.

55. $\sin(3x)\cos(6x)-\cos(3x)\sin(6x)=-0.9$

56. $\sin(6x)\cos(11x)-\cos(6x)\sin(11x)=-0.1$

57. $\cos(2x)\cos(x)+\sin(2x)\sin(x)=1$

58. $\cos(5x)\cos(3x)-\sin(5x)\sin(3x)=\frac{\sqrt{3}}{2}$

59. $6\sin(2t)+9\sin(t)=0$

For the following exercises, solve the equation for $0^\circ \le x < 360^\circ$.

60. $9\sin^2(\theta)-6\sin(\theta)=1$

61. $4\cos^2(\theta)+4\cos(\theta)=1$

62. $\sec^2(\alpha)-2\sec(\alpha)-3=0$

63. $\csc^2(\beta)+4\csc(\beta)-10=0$

64. $\csc^2(x)+4\csc(x)-7=0$

65. $3\cot^2(x)-3\cot(x)-1=0$

66. $2\sin^2(x)=1-\cos(x)$

67. $\cos^2(\alpha)+4=2\sin(\alpha)-3$

68. $\cos^2(\beta)-3\sin(\beta)+2\sin^2(\beta)=0$

69. $\sin^2(\theta)=2\cos(\theta)+3\cos^2(\theta)$

70. $\sec^2(x)=2\tan(x)+4$

71. $3\tan^2(x)=\sec(x)+2$

72. $\cos(\alpha)+1=2\cos(2\alpha)$

73. $\cos(2x)-3\sin(x)-2=0$

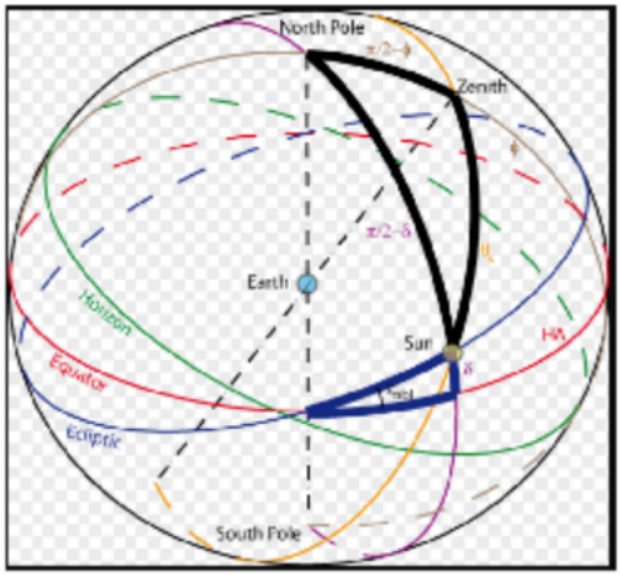
74. $\csc^2(\theta)=\cot(\theta)+5$

75. $\csc(\theta)+5=2\cot^2(\theta)+2$

This page titled 10.4.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

11: NON-RIGHT TRIANGLE TRIGONOMETRY

Figure 1

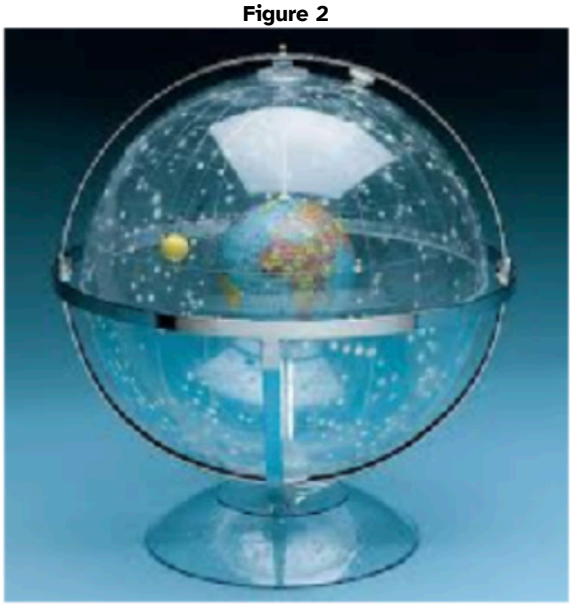


The first science developed by humans is probably Astronomy. Before the invention of clocks and calendars, early people looked to the night sky to help them keep track of time. What is the best time to plant crops, and when will they ripen? On what day exactly do important religious festivals fall?

By tracking the motions of the stars, early astronomers could identify the summer and winter solstices and the equinoxes. The rising and setting of certain stars marked the hours of the night.

If we think of the stars as traveling on a dome above the Earth, we create the **celestial sphere**. Actually, of course, the Earth itself rotates among the stars, but for calculating the motions of heavenly objects, this model works very well.

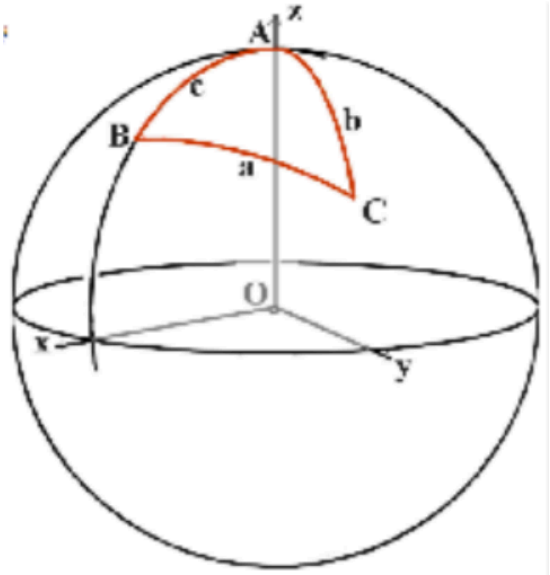
Figure 2



Babylonian astronomers kept detailed records on the motion of the planets, and were able to predict solar and lunar eclipses. All of this required familiarity with angular distances measured on the celestial sphere.

To find angles and distances on this imaginary sphere, astronomers invented techniques that are now part of **spherical trigonometry**. The laws of sines and cosines were first stated in this context, in a slightly different form than the laws for plane trigonometry.

Figure 3



On a sphere, a **great-circle** lies in a plane passing through the sphere's center. It gives the shortest distance between any two points on a sphere, and is the analogue of a straight line on a plane. A spherical angle is formed where two such arcs intersect, and a spherical triangle is made up of three arcs of great circles.

The spherical law of sines was first introduced in Europe in 1464 by Johann Muller, also known as Regiomontus, who wrote:

> "You, who wish to study great and wondrous things, who wonder about the movement of the stars, must read these theorems about triangles. ... For no one can bypass the science of triangles and reach a satisfying knowledge of the stars."

11.1: The Law of Sines

11.1.1: Resources and Key Concepts

11.1.2: Homework

11.2: The Law of Cosines

11.2.1: Resources and Key Concepts

11.2.2: Homework

This page titled 11: Non-Right Triangle Trigonometry is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Katherine Yoshiwara & Roy Simpson.

11.1: The Law of Sines

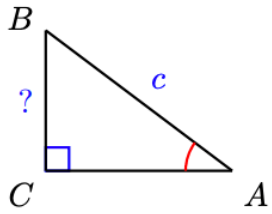
Section Preview

Note to the Instructor (click to expand) available at the end of this section.

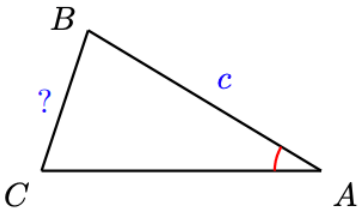
The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

We have learned to use trigonometric ratios to solve right triangles. However, the ratios are only valid for the sides of right triangles. Can we find unknown sides or angles in an oblique triangle?



We can find a if we know A and c (and $C = 90°$).



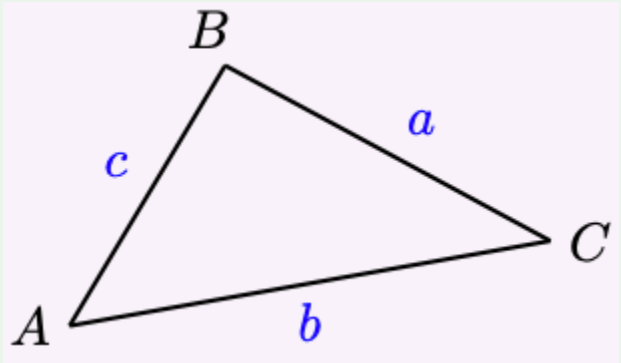
Can find a if we know A and c and C?

Figure 11.1.1

In this section and the next, we find relationships between the sides and angles of oblique triangles. These relationships are called the Law of Sines and the Law of Cosines. We use what we already know about right triangles to derive these new rules.

The Law of Sines

Theorem: Law of Sines



If the angles of a triangle are A, B, and C, and the opposite sides are a, b, and c, respectively, then

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c},$$

or equivalently,

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

Proof available at the end of this section.

The Law of Sines is true for any triangle, whether acute, right, or obtuse. In the following example, we use the Law of Sines to find a distance.

Example 11.1.1

Two observers on the shore sight a ship at an unknown distance from the shoreline. The observers are 400 yards apart at points A and B, and they each measure the angle from the shoreline to the ship, as shown below. How far is the ship from the observer at A?

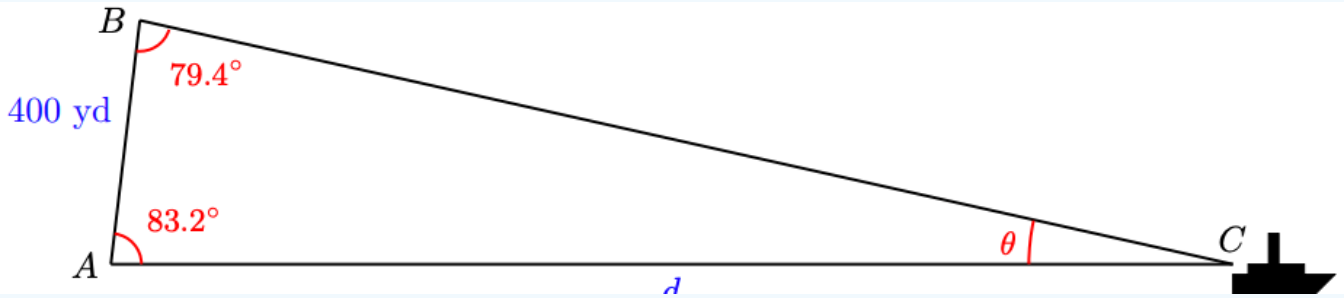


Figure 11.1.2

Solution available at the end of this section.

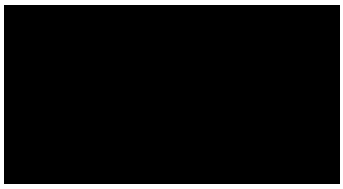
In Example 11.1.1, we saved all calculator work for the last step. That is, we did not approximate the values of $\sin(79.4^\circ)$ and $\sin(17.4^\circ)$, and then use those approximations to get a further approximation—doing so would propagate errors.

Advice: Save Calculator Work for the Last Step

As much as possible, only reach for technology in any mathematical work at the very last moment.

Another piece of advice is to collect information into tables when working through an application. This habit is especially helpful with the current material (and the material in the next section).

In Example 11.1.1, we could have collected the information into a table where the columns are the angles and their opposite side lengths. When missing information, we place a temporary question mark. Therefore, we would have started with the table



Looking at that first column, we realize these are the interior angles of a triangle and, as such, should sum to 180°. Hence,

$$\angle C = 180^\circ - 83.2^\circ - 79.4^\circ = 17.4^\circ.$$

Thus, our table becomes



That checkmark (✓) is not an accident. Recall, the Law of Sines tells us

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

If we have the sine of an angle and the length of its opposite side, then we know the value of these ratios.

That is, once we have a complete row in our table, we can *try* to use the Law of Sines (I say "try" because sometimes solving a triangle using the Law of Sines will not work. We will see those cases momentarily). Since our goal is to find the side length opposite $\angle B$ (which, for the sake of building the table, I call b), we work with the equation

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$

We substitute the values from the table to get

$$\frac{b}{\sin(79.4^\circ)} = \frac{400}{\sin(17.4^\circ)}$$

and finish as we did in Example 11.1.1.

Checkpoint 11.1.1

Delbert and Francine are 40 feet apart on one side of a river. They make angle measurements to a pine tree on the opposite shore, as shown below. What is the distance from Francine to the pine tree?

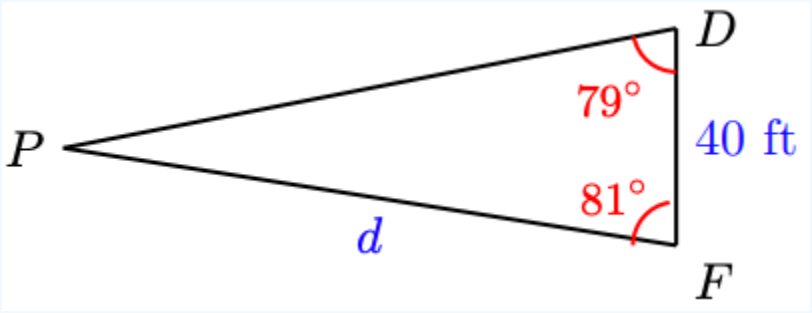


Figure 11.1.3

Answer available at the end of this section.

Solving Triangles with the Law of Sines

To apply the Law of Sines to find a side, we must know one angle of the triangle and its opposite side (either a and A, or b and B, or c and C), and one other angle. Then, we can find the side opposite that angle.

Example 11.1.2

In the triangle shown below, $A = 37^\circ$, $B = 54^\circ$, and $a = 11$.

a. Find b.
b. Solve the triangle.

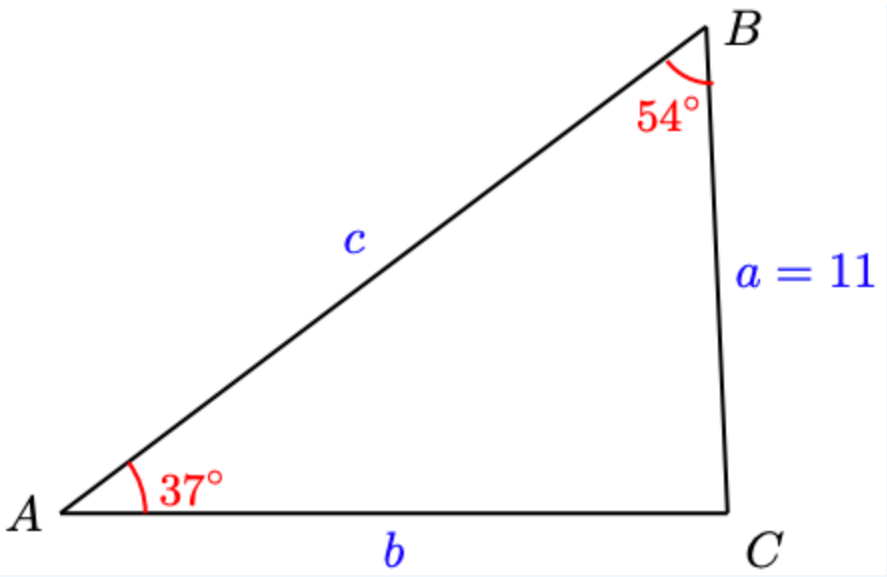


Figure 11.1.4

Solutions available at the end of this section.

Checkpoint 11.1.2

In the triangle below, $A = 65^\circ$, $C = 42^\circ$, and $c = 16$. Solve the triangle.

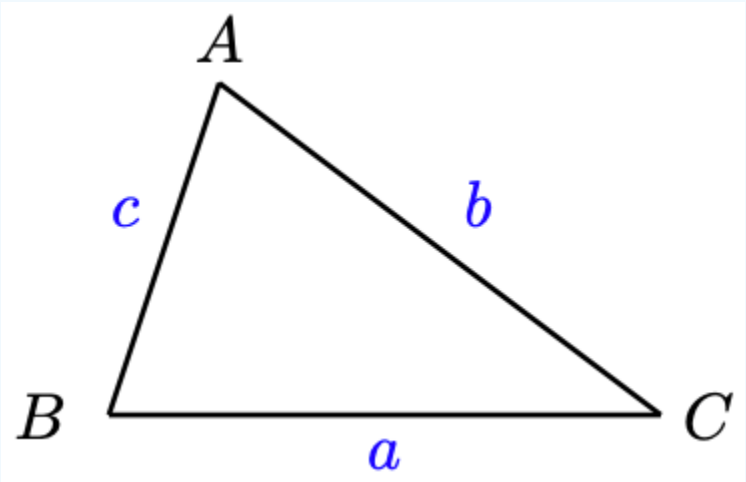


Figure 11.1.5

Answer available at the end of this section.

Finding an Angle

We can also use the Law of Sines to find an unknown angle of a triangle. We must know two sides of the triangle and the angle opposite one of them.

Example 11.1.3

In the triangle below, $B = 55°$, $a = 5$, and $b = 11$. Solve the triangle.

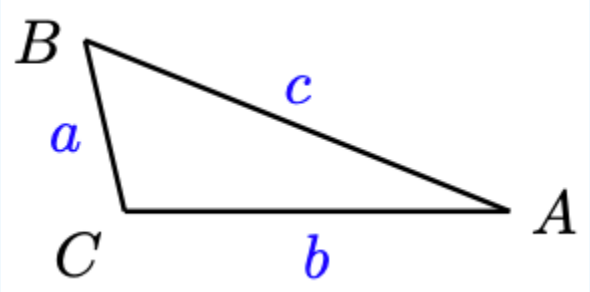


Figure 11.1.6

Solution available at the end of this section.

Caution: Do Not Use Rounded Values to Calculate

Example 11.1.3 has a lot going on; however, the simplest thing to cover at this point is proper etiquette when it comes to calculations.

While I am *displaying* the approximations in the tables within Example 11.1.3, any computations relying on these are done using the exact or stored values in the calculator. For example, when I do any computation relying upon A, I either use $\sin^{-1}\left(\frac{5\sin(55°)}{11}\right)$ or the unrounded approximation stored in my calculator. This is done to avoid the propagation of rounding errors. All modern calculators can store results, and I suggest you learn how to use that feature. For example, on the calculator I use (the TI-30XS), there is a button labeled $\boxed{\text{sto}\rightarrow}$ and another button labeled $\boxed{\text{X}^{\text{yzt}}_{\text{abc}}}$. To store the result of $\sin^{-1}\left(\frac{5\sin(55°)}{11}\right)$, I use the following button sequence:

$$\boxed{\text{sto}\rightarrow} \quad \boxed{\text{X}^{\text{yzt}}_{\text{abc}}} \text{ (tapping this a few times until } a \text{ is displayed)} \quad \boxed{\text{enter}}$$

When I need to access that stored value, I tap $\boxed{\text{X}^{\text{yzt}}_{\text{abc}}}$ until I get the variable I stored it as, and then hit $\boxed{\text{enter}}$.

Checkpoint 11.1.3

Sketch a triangle with $C = 93°$, $a = 7$, and $c = 11$.

a. Use the Law of Sines to find another angle of the triangle.
b. Solve the triangle, and label your sketch with the results.

Answers available at the end of this section.

In Examples 11.1.1 and 11.1.2, we used the Law of Sines to find missing side lengths, but we didn't need the Law of Sines to find angles. In Example 11.1.3 (and Checkpoint 11.1.3), on the other hand, we had no choice but to use the Law of Sines to find a missing angle—this is where we need to be careful when using the Law of Sines.

Note: Unpacking Law of Sines Subtleties

It's imperative to recognize that all angles within a triangle must be positive but less than $180°$. Thus, the sine of *any* angle within a triangle will always be positive. Since the side lengths are also positive, the arcsine we use to find the missing angle will always return an acute angle (i.e., an angle between $0°$ and $90°$).

"Why do we need to be aware of this?"

When using the Law of Sines to find a missing angle in a triangle, there could be two such triangles satisfying the given conditions—one containing the acute returned by the arcsine, and the other containing its supplement.

"How do we know when we have two triangles for a given set of information?"

When using the Law of Sines to find a missing angle, we consider the angle returned by the arcsine to be a *reference angle*. To help with this discussion, let's call this reference angle $\hat{\theta}$.

$\hat{\theta}$ gives two possibilities for the true value of the angle we seek. The first is $\theta = \hat{\theta}$. The second is its supplement, $\theta' = 180^\circ - \hat{\theta}$ (the "prime" notation is traditionally used for the supplementary angle). While θ is the solution that gives one triangle, θ' is a candidate for a second value (and, therefore, a second triangle). This is because the sine is positive and, therefore, its argument could also be between 90° and 180°.

With our candidate angles, θ and θ', in hand, we then check whether the other given angle works with each of these computed values. Sometimes, the provided information works with *both* candidate values; however, at the very least, the acute angle always works.

To illustrate this concept, let's look back at Example 11.1.3. We found $\widehat{A} = \sin^{-1}\left(\frac{5\sin(55^\circ)}{11}\right) \approx 21.9^\circ$. Thus, we immediately knew $A \approx 21.9^\circ$ worked and would yield a triangle; however, what about the supplement of this angle?

In QII, our alternate option for A is $A' = 180^\circ - \widehat{A} \approx 158.1^\circ$; however, if we tried to fit this obtuse angle into a triangle, we would compute the final missing angle to be

$$C = 180^\circ - A - B \approx 180^\circ - 158.1^\circ - 55^\circ = -33.1^\circ.$$

This is impossible because angles within a triangle are never negative. Thus, we only have one triangle (given by the acute angle $A \approx 21.9^\circ$).

Again, sometimes both angles produce (different) triangles, and sometimes only the acute angle will work. You should always check whether both angles provide solutions.

Because there can be more than one solution for a triangle when using the Law of Sines to find a missing angle, we call this situation the **ambiguous case** for the Law of Sines.

Example 11.1.4

Solve the triangle in which $B = 14.4^\circ$, $a = 8$, and $b = 3$. Sketch the resulting triangle.

Solution available at the end of this section.

Checkpoint 11.1.4

Suppose that $C = 29.7^\circ$, $b = 8$, and $c = 5$.

a. Find two possible values for angle B.
b. Solve the triangle for both values of B, and sketch both solutions.

Answers available at the end of this section.

Example 11.1.5

Solve the triangle in which $C = 73^\circ$, $a = 15$, and $c = 2$. Sketch your result.

Solution available at the end of this section.

Example $11.1.5$ illustrates the final possibility when using the Law of Sines to find a missing angle. In summary, when using the Law of Sines to find a missing angle within a triangle, one of three possibilities occurs:

1. There is a single triangle satisfying the given information. In this case, the angle will directly result from the arcsine when using the Law of Sines.
2. Two triangles satisfy the given information. In this case, one triangle will be the direct result of the arcsine, and the second triangle will contain the supplement of that angle.
3. There is no such triangle satisfying the given information. In this case, your calculator will return a domain error when computing the arcsine.

Checkpoint $11.1.5$

You are told that, in a triangle, $B = 30^\circ$, $b = 7$, and $c = 15$. Without a calculator, explain how you know no such triangle satisfies the given information.

Answer available at the end of this section.

Visualizing the Law of Sines Cases

Up to this point, we have computationally justified that the Law of Sines results in one, two, or no triangles for a given set of information; however, the visual of why this is true is often more impactful.

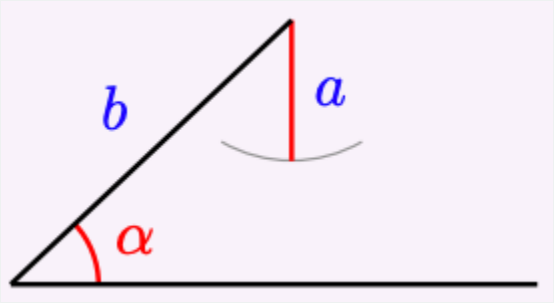
If we know two sides a and b of a triangle and the acute angle α opposite one of them, there may be one solution, two solutions, or no solution, depending on the size of a in relation to b and α, as shown below.

The Law of Sines - Ambiguous Case

In each of the following cases, $h = b\sin(\alpha)$ is the altitude of the triangle. Therefore, $b > h$.

The "No Solution" Case

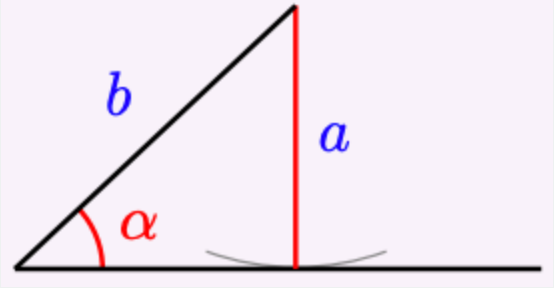
Condition: $a < h$



Reason: a is shorter than the altitude of the (non-existent) triangle. Therefore, it is impossible for side a to intersect the base of the "triangle."

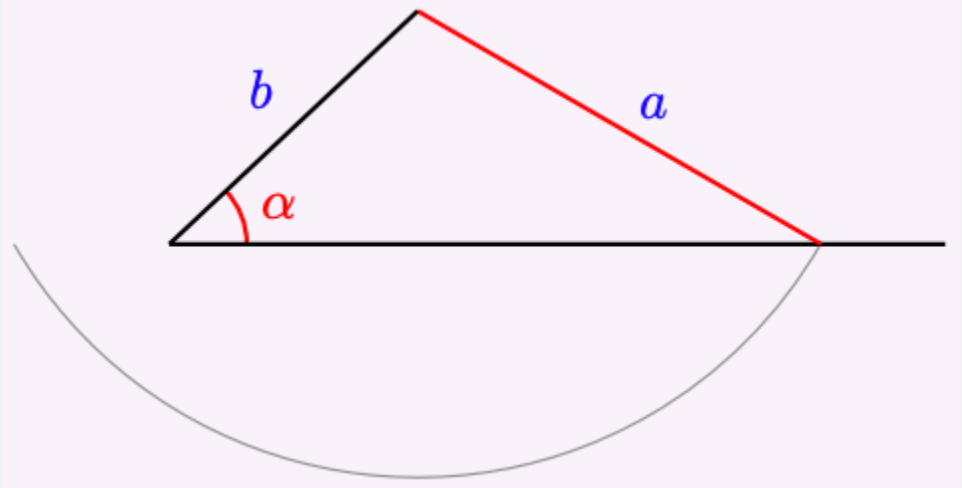
The "One Solution" Cases

Condition #1: $a = h$



Reason: a is exactly the right length to make a right triangle.

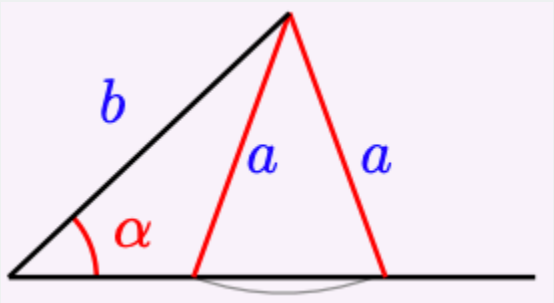
Condition #2: $a > b > h$



Reason: If $a > b$, then one triangle is created as side a swings downward to intersect the base; however, as that side continues to swing, it is too long to strike the base again. Hence, there can be only one triangle.

The "Two Solutions" Case

Condition: $b > a > h$



Reason: Since $a > h$, side a will strike the base a it swings downward, but before becoming vertical. Moreover, because $a < b$, side a will strike the base once more as it continues to swing clockwise.

Applications Requiring the Law of Sines

The Law of Sines is used quite a bit in applications within Trigonometry. While I could not cover all possible applications, I have chosen two that commonly occur.

Example 11.1.6

A vertical communications tower is located at the top of a steep hill, as shown below. The angle of inclination of the hill is 56°. A guy wire is to be attached to the top of the tower and the ground 175 yards downhill from the base of the tower. The angle formed by the guy wire is 26°. Find the length of the cable required for the guy wire. Round your answer to the nearest yard.

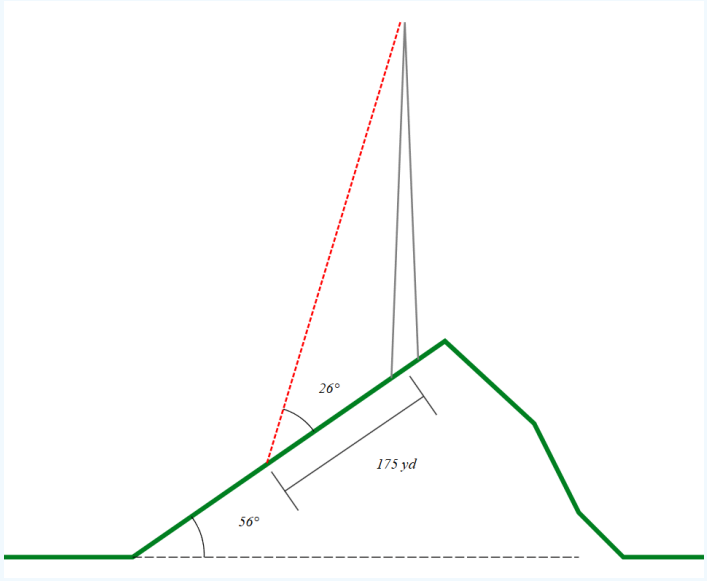


Figure 11.1.11

Solution available at the end of this section.

Checkpoint 11.1.6

Lap and Piyali are on the north side of a river that runs east to west. Lap is standing 225 meters due west of Piyali. They spot a treasure chest on the south shore of the river. For Lap, the angle between the chest and Piyali is 81°, and for Piyali, the angle between the chest and Lap is 58° (see Figure 11.1.14 below). How far is Lap from the chest? Round your answer to the nearest meter.

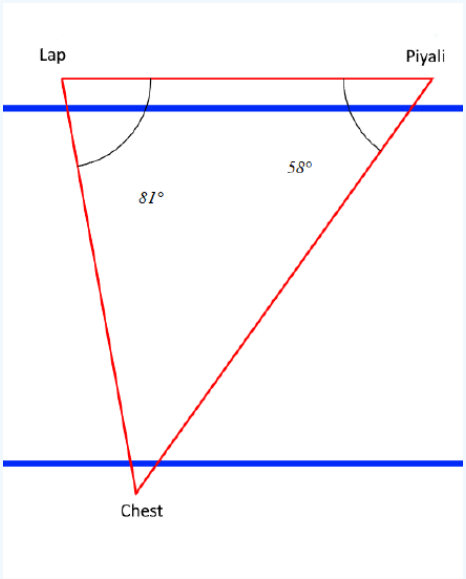


Figure 11.1.14

Answers available at the end of this section.

The following example (and the subsequent Checkpoint) illustrates that the Law of Sines can be used when you might otherwise use Right Triangle Trigonometry.

Example 11.1.7

Ron wants to measure the height of a castle controlled by hostile forces. When he is as close as he can get to the castle, the angle of elevation to the top of the wall is 18.5°. He then retreats 20 yards and measures the angle of elevation again; this time, it is 15.9°. How tall is the castle?

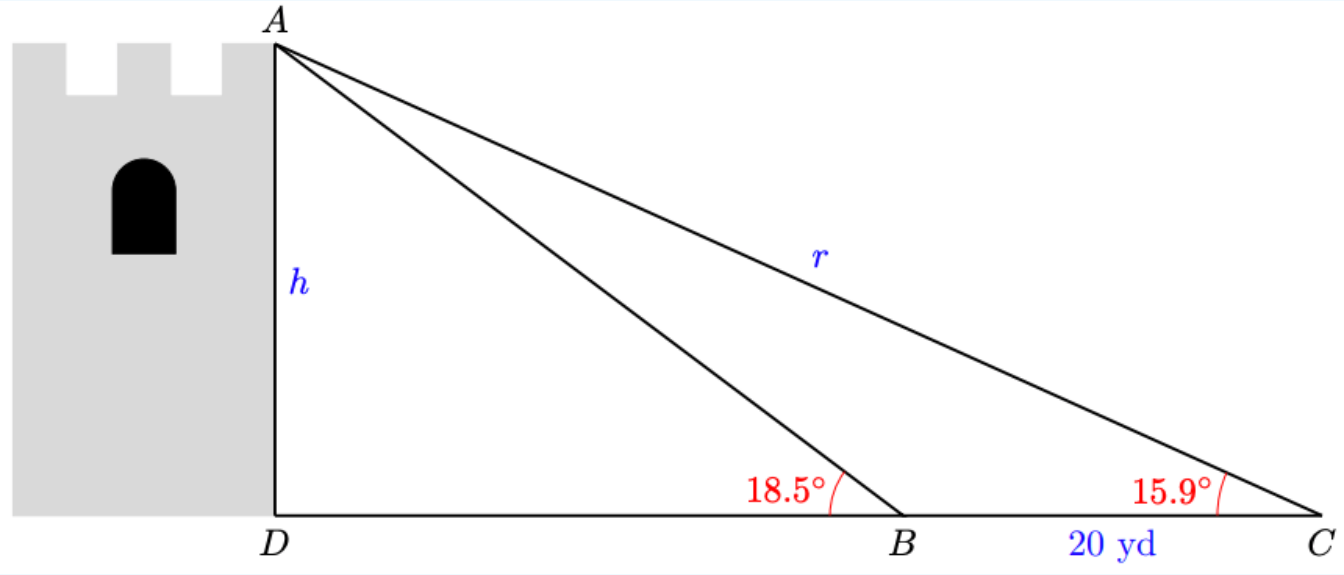


Figure 11.1.15

Solution available at the end of this section.

Checkpoint 11.1.7

Solve the problem in the previous example again, but instead of finding r, find the length AB, and then use $\triangle ABD$ to find h.

Answer available at the end of this section.

If you look at a nearby object and alternately close your left and right eyes, the object seems to jump in position. This apparent change occurs because your eyes view the object from two positions spaced several centimeters apart. If the object at point O is straight ahead of one eye, it appears to be at some angle p away from the line of sight of the other eye. The angle p is called the **parallax** of the object.

Use Figure 11.1.17 to see that p is also the angle between the directions to your two eyes when viewed from point O. (What fact from Geometry justifies this statement?)

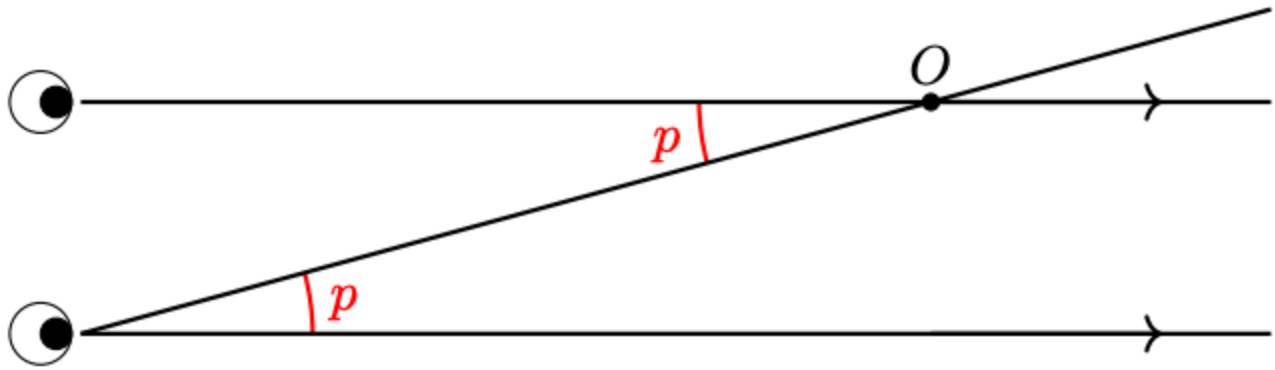


Figure 11.1.17

Astronomers use parallax to determine the distance from Earth to stars and other celestial objects. Two observers on Earth at a known distance apart both measure the direction to the star. The difference in angle between those two directions is the parallax.

Example 11.1.8

Astronomers 1000 kilometers apart observe an asteroid with a parallax of $0.001°$. How far is the asteroid from Earth?

Solution available at the end of this section.

Checkpoint 11.1.8

Two observers 800 kilometers apart observe an object with a parallax of $0.0005°$. How far is the object from Earth?

Answer available at the end of this section.

Answers

Section Preview

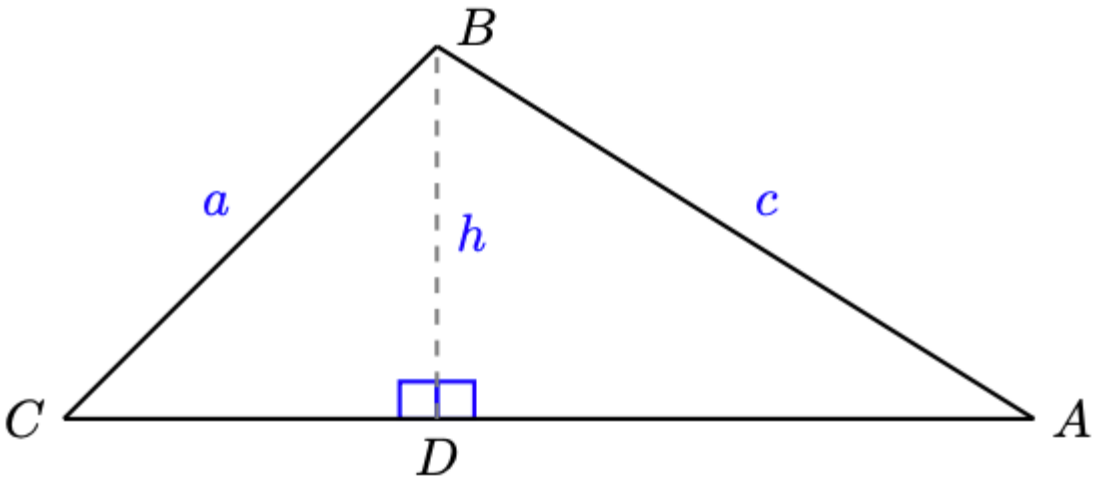
This is a very thorough section of all cases for the Law of Sines and their applications. Please pay close attention to my tabular method. While it is terribly algorithmic, students tend to do much better with it than other methods I have tried.

Section Preview

- Use the Law of Sines to find a missing side in an oblique triangle.
- Solve an application using the Law of Sines.
- Use the Law of Sines to find all possible solutions for the ambiguous case.
- Solve an application involving the ambiguous case.

Theorem: Law of Sines

Consider the oblique triangle in the figure below.



By drawing in the altitude h of the triangle, we create two right triangles, $\triangle BCD$ and $\triangle ABD$. We can write expressions in terms of h for $\sin(A)$ and for $\sin(C)$.

Looking at $\triangle BCD$, we see that

$$\frac{h}{a} = \sin(C).$$

Looking at $\triangle ABD$, we see that

$$\frac{h}{c} = \sin(A).$$

Now we solve each of these equations for h:

$$\frac{h}{a} = \sin(C) \quad \Longrightarrow \quad h = a\sin(C)$$

$$\text{and} \quad \frac{h}{c} = \sin(A) \quad \Longrightarrow \quad h = c\sin(A)$$

Therefore,

$$\begin{aligned} a\sin(C) &= c\sin(A) \\ \Longrightarrow \quad \frac{\sin(C)}{c} &= \frac{\sin(A)}{a} \qquad (\text{divide both sides by } ac) \end{aligned}$$

We have derived a relationship between the angles A and C and their opposite sides, a and c. If we know any three of these quantities, we can find the fourth.

In a similar way, by drawing in the altitude from the vertex C, we can show that

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}.$$

Putting both results together, we have proved the Law of Sines.

Example 11.1.1

First, note that $\triangle ABC$ is not a right triangle, so we cannot use the trigonometric ratios directly to find the sides of this triangle.

Let d be the unknown distance opposite $\angle B = 79.4^\circ$. To use the Law of Sines, we must know another angle and the side opposite that angle. Luckily, we know that side $c = 400$, and we can compute the angle at the ship, $\angle C$.

$$\angle C = 180^\circ - (79.4^\circ + 83.2^\circ) = 17.4^\circ.$$

Now we apply the Law of Sines, using angles B and C.

$$\begin{aligned} & \frac{b}{\sin(B)} = \frac{c}{\sin(C)} && (\text{Law of Sines}) \\ \Longrightarrow \quad & \frac{d}{\sin(79.4^\circ)} = \frac{400}{\sin(17.4^\circ)} && (\text{substitute}) \\ \Longrightarrow \quad & d = \frac{400\sin(79.4^\circ)}{\sin(17.4^\circ)} && (\text{multiply both sides by } \sin(79.4^\circ)) \\ \Longrightarrow \quad & d \approx 1315 \end{aligned}$$

The ship is about 1315 yards from the observer at A.

Checkpoint 11.1.1

About 114.8 feet

Example 11.1.2

We begin by building our table of given information (using question marks where we aren't explicitly given information).



We already have a complete row! This means we can immediately try using the Law of Sines; however, I cannot resist finding the value of C when it is so easy to compute.

$$C = 180^\circ - 54^\circ - 37^\circ = 89^\circ.$$

Hence, our table becomes



a. We use the Law of Sines with a and angle A to find b.

$$\begin{aligned} & \frac{a}{\sin(A)} = \frac{b}{\sin(B)} && \text{(Law of Sines)} \\ \Longrightarrow \quad & \frac{11}{\sin(37^\circ)} = \frac{b}{\sin(54^\circ)} && \text{(substitute)} \\ \Longrightarrow \quad & \frac{11\sin(54^\circ)}{\sin(37^\circ)} = b && \text{(multiply both sides by } \sin(54^\circ)) \end{aligned}$$

Evaluating this expression with a calculator, we find that $b \approx 14.79$. We fill this in our table for display purposes only (we will not use the rounded value for computing).



b. At this point, we are very close to having the triangle solved. We only need the length of side c. Using the given side, a, is safer than using the value we calculated for b.

$$\begin{aligned} & \frac{a}{\sin(A)} = \frac{c}{\sin(C)} && \text{(Law of Sines)} \\ \Longrightarrow \quad & \frac{11}{\sin(37^\circ)} = \frac{c}{\sin(89^\circ)} && \text{(substitute)} \\ \Longrightarrow \quad & \frac{11\sin(89^\circ)}{\sin(37^\circ)} = c && \text{(multiply both sides by } \sin(89^\circ)) \end{aligned}$$

So $c \approx 18.28$. Now our table is complete, and we have solved the triangle.



Checkpoint 11.1.2

$B = 73^\circ, b \approx 22.87, a \approx 21.68$

Example 11.1.3

Again, let's start by filling out a table.



This is the first time where we cannot simply find the remaining angles using the fact that the sum of angles in a triangle is 180°. Luckily, we still have a complete row so we can use the Law of Sines, and we will do so to find $\sin(A)$.

$$\begin{aligned} \frac{\sin(A)}{a} &= \frac{\sin(B)}{b} && \text{(Law of Sines)} \\ \Rightarrow \quad \frac{\sin(A)}{5} &= \frac{\sin(55^\circ)}{11} && \text{(substitute)} \\ \Rightarrow \quad \sin(A) &= \frac{5\sin(55^\circ)}{11} && \text{(multiply both sides by 5)} \\ \Rightarrow \quad \widehat{A} &= \sin^{-1}\left(\frac{5\sin(55^\circ)}{11}\right) && \text{(finding the reference angle -} \\ & && \text{see } \textbf{Unpacking Law of Sines Subtleties}\text{)} \end{aligned}$$

We have a *reference* angle, and we know that each angle in a triangle must be positive but less than 180°. Therefore, A is either in QI or QII. In all honesty, since A is an angle within a triangle and *not* an angle in standard position within a prescribed coordinate system, it is incorrect to say that A^{int}QI or A^{int}QII; however, I am okay with the "looseness" of this language.

Here's where a little extra logic comes into play. Since the length of side b is greater than the length of side a, it must be the case that $B > A$. Since $B = 55^\circ$, $0 < A < 55^\circ$. So

$$A = \sin^{-1}\left(\frac{5\sin(55^\circ)}{11}\right) \approx 21.9^\circ.$$

Let's update our table.



Now that we know two angles, we can find the third.

$$C = 180^\circ - B - A \approx 103.1^\circ.$$

Again, we update the table.



Finally, we use the Law of Sines to find side c.

$$\begin{aligned} \frac{b}{\sin(B)} &= \frac{c}{\sin(C)} && \text{(Law of Sines)} \\ \Rightarrow \quad \frac{b\sin(C)}{\sin(B)} &= c && \text{(multiplying both sides by } \sin(C)\text{)} \end{aligned}$$

Substituting in the values of b, B, and C (stored in the calculator) and evaluating this expression with a calculator gives $c \approx 13.1$. Thus, our final table is the following:



Checkpoint 11.1.3

a. $A \approx 39.5^\circ$
b. $b \approx 8.13, B \approx 47.5^\circ$

Example 11.1.4

We start, as usual, by building a table.



Other than the numbers, this looks like it's going to mimic what we did in Example 11.1.3.

$$\begin{aligned} \frac{\sin(A)}{a} &= \frac{\sin(B)}{b} && \text{(Law of Sines)} \\ \Rightarrow \quad \sin(A) &= \frac{a\sin(B)}{b} && \text{(multiplying both sides by } a\text{)} \\ \Rightarrow \quad \widehat{A} &= \sin^{-1}\left(\frac{a\sin(B)}{b}\right) && \text{(finding the reference angle)} \end{aligned}$$

Grabbing a calculator, substituting in the known values of B, a, and b, we get $\widehat{A} \approx 41.5^\circ$ (again, the exact value is stored in the calculator to be used for other calculations). As stated previously, the acute angle will always work because if $A \approx 41.5^\circ$, then $C = 180^\circ - A - B \approx 124.1^\circ$; however, before throwing all of this into a table, let's check to see if the supplementary (obtuse) angle works as well. Let

$$A' = 180^\circ - A \approx 138.5^\circ.$$

Then

$$C' = 180^\circ - A' - B \approx 27.1^\circ.$$

Since this is not a negative angle, this creates another workable triangle!

Thus, if $A \approx 41.5^\circ$, there is room in the triangle for the final, third angle $C \approx 124.1^\circ$, and if $A' \approx 138.5^\circ$, there is room in another triangle for the final, third angle C' $\approx 27.1^\circ$. In these ambiguous cases, we build *two* tables of information, one for the acute angle and one for the obtuse angle.




In either case, we need to compute the final missing sides, c and c', using the Law of Sines. Remember, when using the Law of Sines, we want to use as much of the given information as possible because it's cleaner than our approximated information. Moreover, we use the non-rounded values and save rounding for the end.

$$\begin{aligned} \frac{c}{\sin(C)} &= \frac{b}{\sin(B)} && \text{(Law of Sines)} \\ \Rightarrow \quad c &= \frac{b\sin(C)}{\sin(B)} && \text{(multiply both sides by } \sin(C)) \end{aligned}$$

This formula is the same for c' (just replace C with C'). Hence, we grab a calculator, substitute the necessary values, and get our approximations (listed in the tables below).




The following figure shows both triangles.

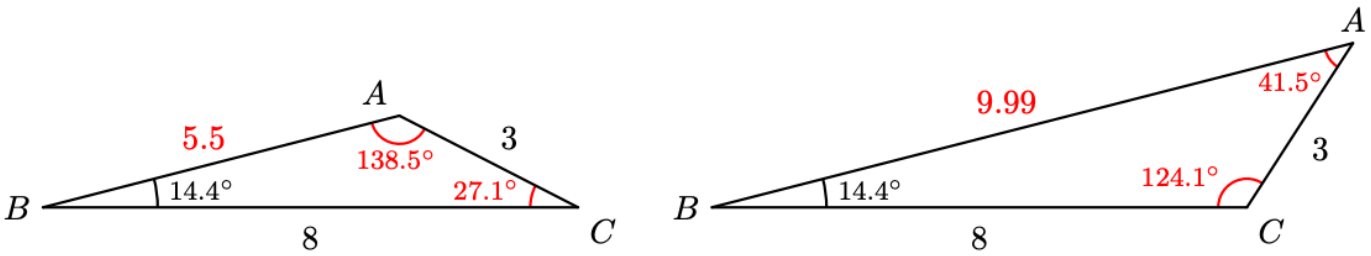


Figure 11.1.7

Checkpoint 11.1.4

a. $B = 52.4^\circ$ or $B = 127.6^\circ$
b. $A = 97.9^\circ, a = 10$

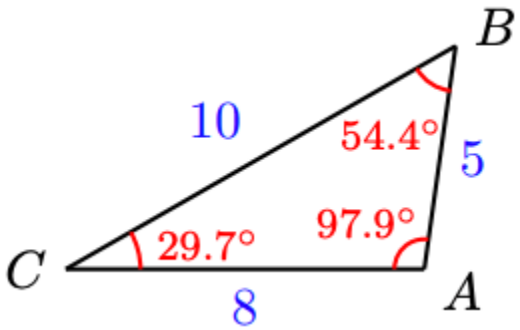


Figure 11.1.8

or $A = 22.7^\circ, a = 3.9$

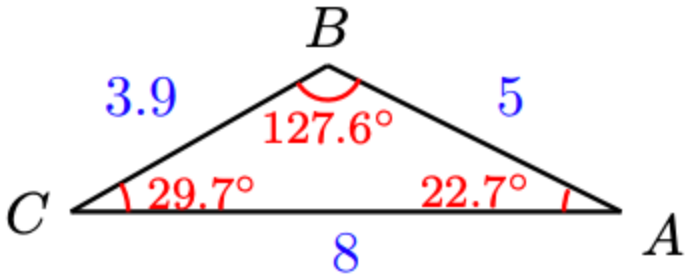


Figure 11.1.9

Example 11.1.5

Building a table, we get the following:



Seeing that we have a complete row, we use that to help us find the missing angle from the first row (angle A).

$$\frac{\sin(A)}{a} = \frac{\sin(C)}{c} \qquad \text{(Law of Sines)}$$

$$\Longrightarrow \quad \sin(A) = \frac{a\sin(C)}{c} \qquad \text{(multiplying both sides by } a\text{)}$$

$$\Longrightarrow \quad \widehat{A} = \sin^{-1}\left(\frac{a\sin(C)}{c}\right) \qquad \text{(finding the reference angle)}$$

Grabbing a calculator, substituting in the known values of C, a, and c, we find the calculator returns an error!

What happened?

Consider the following value:

$$\frac{a\sin(C)}{c} = \frac{15\sin(73^\circ)}{2} \approx 7.2$$

This is what $\sin(A)$ is supposed to equal; however, we know that the sine function is always between -1 and 1.

What does this mean?

This means there is no such triangle where $C = 73^\circ$, $a = 15$, and $c = 2$. We can see this from a sketch.

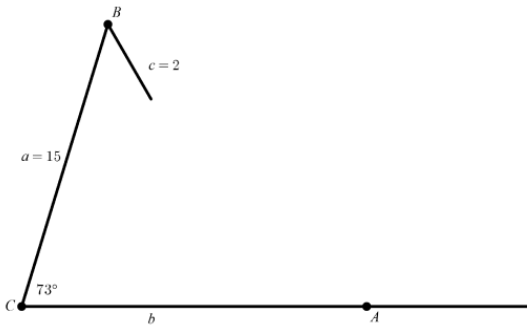


Figure 11.1.10

The angle C is large, but the length of its opposite side is relatively small (when compared to side length a). If we imagine side c being able to swing freely, we can see it will never intersect side b. Therefore, forming a triangle with the given information is impossible.

Checkpoint 11.1.5

While using the Law of Sines to find a missing angle in a triangle, you must compute

$$\sin^{-1}\left(\frac{15\sin(30^\circ)}{7}\right) = \sin^{-1}\left(\frac{15}{14}\right).$$

This means that $\sin(C) = \frac{15}{14}$, which is not possible.

Example 11.1.6

While it's nice that we have been given a sketch of the situation, I find it's best to redraw the sketch without the "fluff," labeling vertices in uppercase and side lengths in lowercase (as we have been doing this entire time).

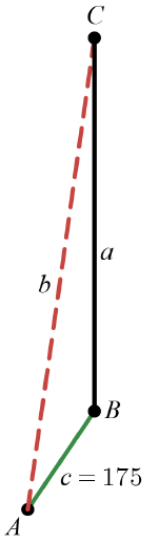


Figure 11.1.12

We now build our table.



We do not have a complete row in this table, so we might think the Law of Sines is not going to be useable; however, looking at the provided sketch, we see one additional piece of information that we have not used.

The green line (side c) makes an angle of 56° with the horizontal. If we draw a horizontal line at vertex B, we can label the angle between the horizontal and the extended green line as 56° (see Figure 11.1.13 below). Since the communications tower (side a) is vertical, the angle between the extended green line and the tower is the complement of 56° (again, see Figure 11.1.13 below). This is 34°.

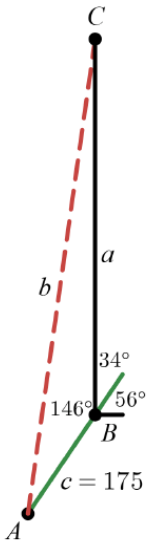


Figure 11.1.13

Finally, we can see that the angle within the triangle at vertex B is the supplement of this 34° angle. Hence, $B = 180^\circ - 34^\circ = 146^\circ$. Let's go ahead and place this information in our table.



Again, we do not have a complete row; however, now that we have two angles, we can easily find the third.

$$C = 180^\circ - A - B = 8^\circ$$

Filling in the table with this information, we get the following:



Finally, we have a complete row in the table, which tells us that the Law of Sines is now available. We are being asked to find side length b. Therefore,

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \qquad \text{(Law of Sines)}$$

$$\Longrightarrow \quad b = \frac{c\sin(B)}{\sin(C)} \qquad \text{(multiplying both sides by } \sin(B)\text{)}$$

Substituting in the values of B, C, and c, we get

$$b = \frac{175\sin(146^\circ)}{\sin(8^\circ)} \approx 703.$$

Hence, the length of the guy wire is approximately 703 yards.

Checkpoint 11.1.6

291 meters.

Example 11.1.7

Notice that h is one side of the right triangle $\triangle ADC$. If we find its hypotenuse, labeled r in Figure 11.1.15, we can use the sine ratio to find h. To find r, we consider a second triangle, $\triangle ABC$, as shown in Figure 11.1.16.

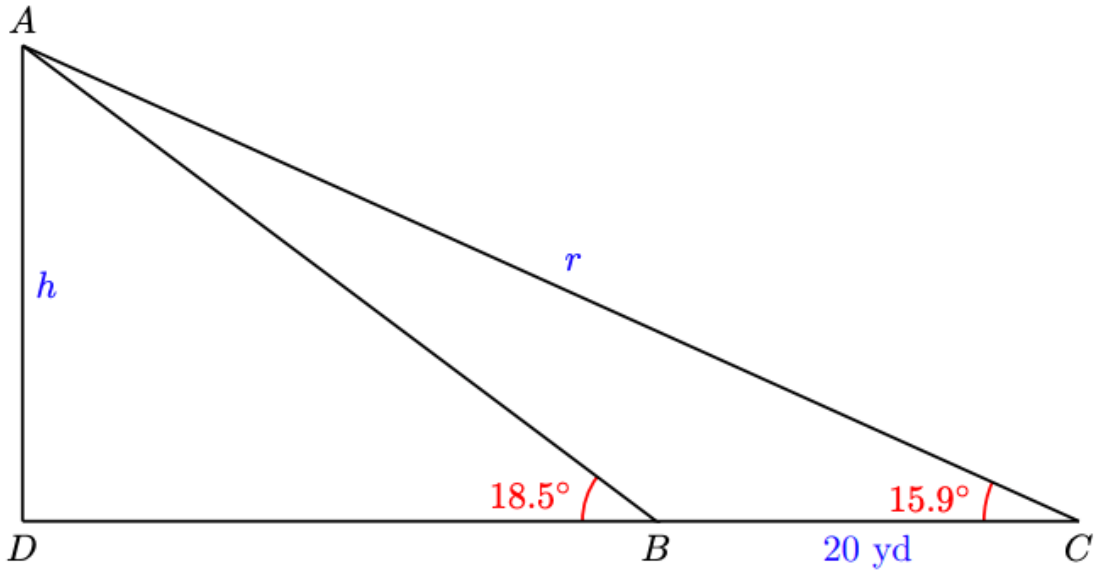


Figure 11.1.16

In this triangle, we know side $BC = 20$ and would like to find side $AC = r$. We can use the Law of Sines to find r, but first, we must calculate the other angles of the triangle.

Now, the angle opposite r, $\angle ABC$, is the complement of 18.5°, so

$$\angle ABC = 180^\circ - 18.5^\circ = 161.5^\circ$$

The angle opposite the 20-yard side, $\angle BAC$, is

$$\angle BAC = 180^\circ - (161.5^\circ + 15.9^\circ) = 2.6^\circ$$

Now we can apply the Law of Sines to find r. We have

$$\frac{r}{\sin(161.5^\circ)} = \frac{20}{\sin(2.6^\circ)} \qquad \text{(Law of Sines)}$$

$$\Longrightarrow \quad r = \frac{20\sin(161.5^\circ)}{\sin(2.6^\circ)} \qquad \text{(multiplying both sides by } \sin(161.5^\circ)\text{)}$$

Finally, using the right triangle $\triangle ADC$, we can write

$$\begin{aligned} \frac{h}{r} &= \sin(15.9^\circ) && \text{(right triangle definition of the sine)} \\ \Longrightarrow \quad h &= r\sin(15.9^\circ) && \text{(multiplying both sides by } r\text{)} \\ &= \frac{20\sin(161.5^\circ)}{\sin(2.6^\circ)} \cdot \sin(15.9^\circ) \\ &\approx 38.33 \end{aligned}$$

The castle is about 38.33 yards tall.

Checkpoint 11.1.7

$AB = 120.79$, the castle is about 38.33 yards tall.

Example 11.1.8

We let x represent the distance to the asteroid. The asteroid and the two observers make an isosceles triangle with a base of approximately 1000 km and equal sides of length x, as shown below.

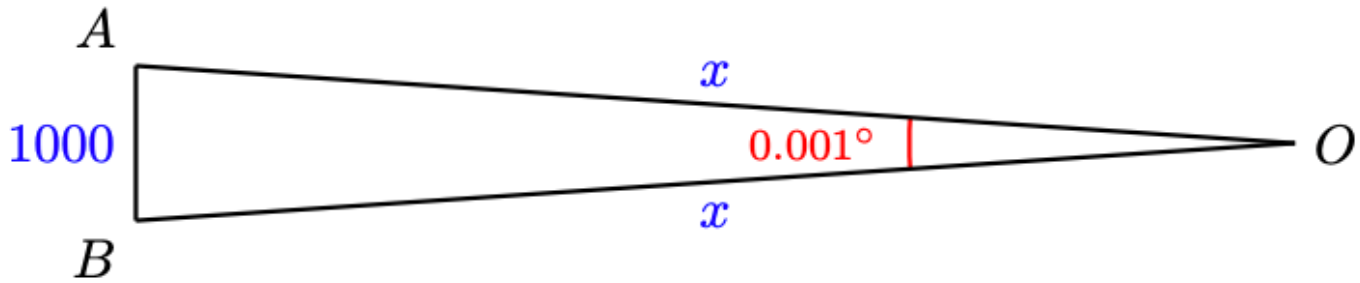


Figure 11.1.18

The base angles of the triangle are both $\frac{180^\circ - 0.001^\circ}{2} = 89.999^\circ$. Thus,

$$\begin{aligned} \frac{x}{\sin(89.999^\circ)} &= \frac{1000}{\sin(0.001^\circ)} && \text{(Law of Sines)} \\ \Longrightarrow \quad x &= \frac{1000\sin(89.999^\circ)}{\sin(0.001^\circ)} && \text{(multiplying both sides by } \sin(89.999^\circ)\text{)} \\ \Longrightarrow \quad x &\approx 57,000,000 \end{aligned}$$

The asteroid is about 57 million kilometers from Earth (roughly one-third of the distance to the Sun).

Checkpoint 11.1.8

About $91,673,247$ km

This page titled 11.1: The Law of Sines is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

11.1.1: Resources and Key Concepts

Resources

Videos

- Law of Sines
  - Non-Right Triangles - Law of Sines: Theorem and Proof
  - Non-Right Triangles - Law of Sines: A Conceptual Investigation of the Ambiguous Case
  - Non-Right Triangles - Law of Sines: Using the Law of Sines to Solve Triangles
  - Non-Right Triangles - Law of Sines: Applications of Non-Right Triangle Trigonometry (Part 1)
  - The Law of Sines - Review of the Law of Sines
  - The Law of Sines - Review of Applications of Non-Right Triangle Trigonometry (Part 1)

Key Concepts

Definitions

- **Ambiguous Case:** A situation that arises when using the Law of Sines to find a missing angle, where the given information (typically two sides and a non-included angle, or SSA) could result in one triangle, two different triangles, or no triangle at all.

Theorems

- **Law of Sines:** If the angles of a triangle are A, B, and C, and the opposite sides are a, b, and c, respectively, then

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \tag{11.1.1.1}$$

  or equivalently,

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \tag{11.1.1.2}$$

- **The Law of Sines - Ambiguous Case (SSA):** Given two sides a and b and an acute angle α opposite side a, let $h = b\sin(\alpha)$ be the altitude of the triangle.
  - **No Solution:** If $a < h$, no triangle can be formed.
  - **One Solution:** If $a = h$ (a right triangle) or if $a > b$, one triangle can be formed.
  - **Two Solutions:** If $h < a < b$, two distinct triangles can be formed.

Common Mistakes

- **Do Not Use Rounded Values to Calculate:** When solving a triangle, use the exact or stored values from your calculator for all intermediate steps. Using rounded approximations can propagate errors and lead to an inaccurate final answer.
- **Forgetting the Ambiguous Case:** When using the Law of Sines to solve for a missing angle (the SSA case), remember that there might be two possible triangles. The arcsine function on a calculator will only return an acute angle; you must also check its supplement as a potential second solution for the angle.
- **Misidentifying the Number of Solutions:** In the ambiguous case, failing to compare the side opposite the given angle to the altitude and the other given side can lead to incorrectly concluding there is one triangle when there are two or none, or vice versa.

This page titled 11.1.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

11.1.2: Homework

Reading Questions

1. What is an oblique triangle?
2. State the Law of Sines.
3. To use the Law of Sines to solve for a missing side or angle, what information must you already have about the triangle?
4. What does it mean to "solve a triangle"?
5. When using the Law of Sines to find an unknown angle, why must you be careful?
6. What is the "ambiguous case" for the Law of Sines? What given information (e.g., SSS, ASA, SSA) leads to this case?
7. If you use the Law of Sines to find that a missing angle θ could be $40°$, what is the other possible value for θ?
8. How do you determine if both possible angles in the ambiguous case lead to valid triangles?
9. If your calculator returns an error when you try to use the arcsin function to find a missing angle, what does this imply about the triangle?
10. Why is it better to use the given values in calculations rather than values you have already calculated and rounded?

Homework

Concept Check

1. Can we use the Law of Sines to solve a right triangle?

2. Explain why we cannot use the Law of Sines to solve the triangle with $a = 8$, $b = c$, and $C = 35°$.

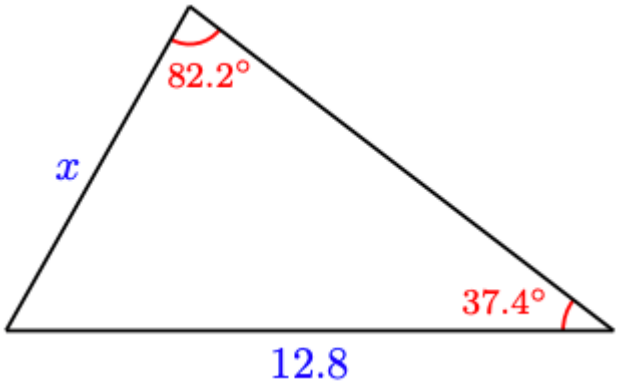
3. Linda says, "I'm thinking of an angle whose sine is 0.3420 (rounded to four decimal places)." Matt says, "The angle must be $20°$ (rounded to the nearest degree)." Is he correct? Why or why not?

4. Sketch two possible triangles with $A = 25°$, $b = 18$, and $a = 10$.

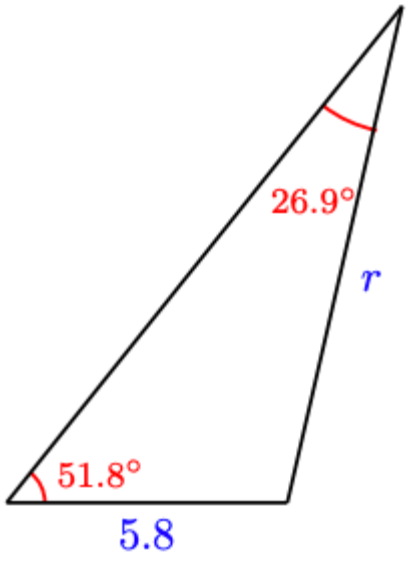
5. Try to sketch a triangle with $A = 65°$, $b = 18$, and $a = 10$. What went wrong?

Basic Skills

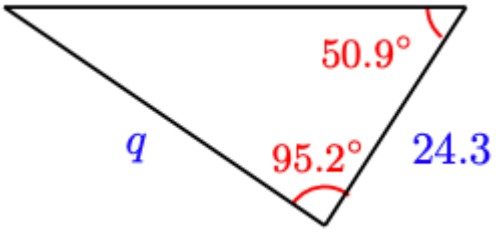
For the following exercises, find the indicated side. Round to two decimal places.

6. 

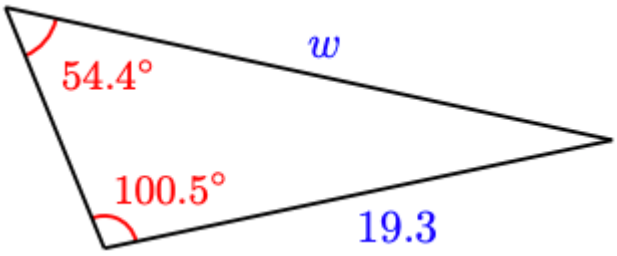
7.



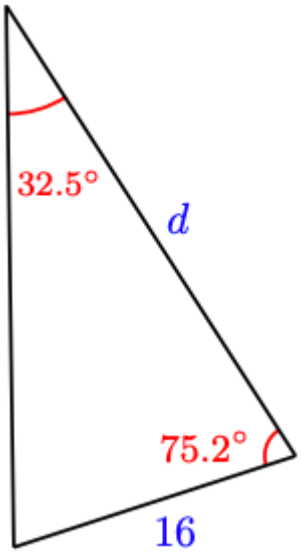
8.

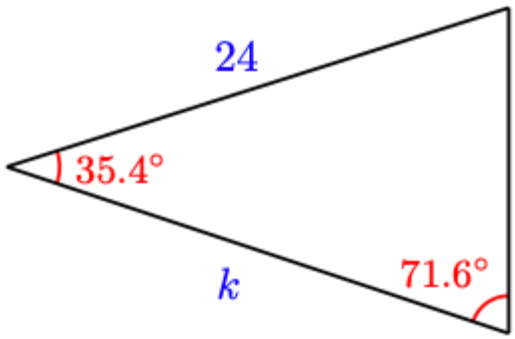


9.



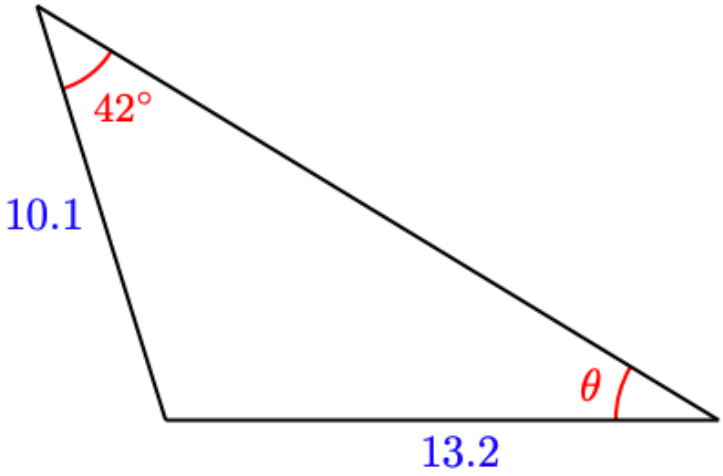
10.



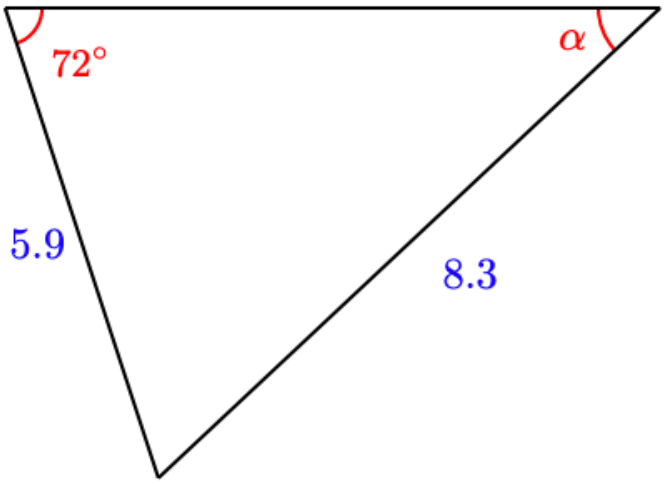


11.

For the following exercises, find the indicated angle. Round to two decimal places.

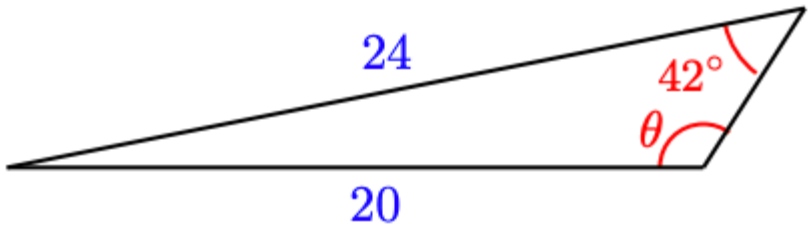


12. (θ is acute.)



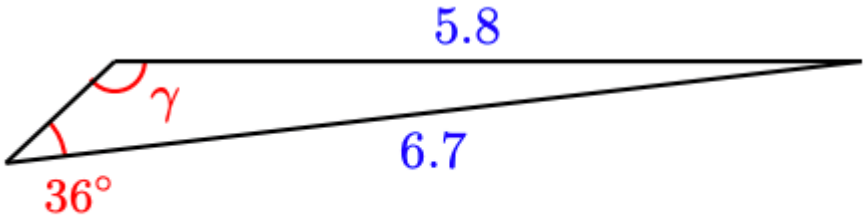
13. (α is acute.)

14.



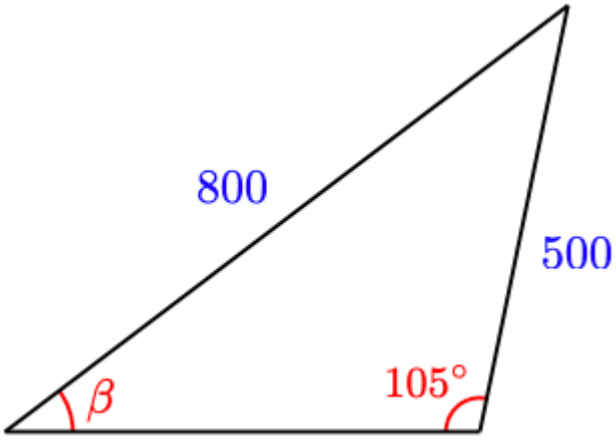
(θ is acute.)

15.

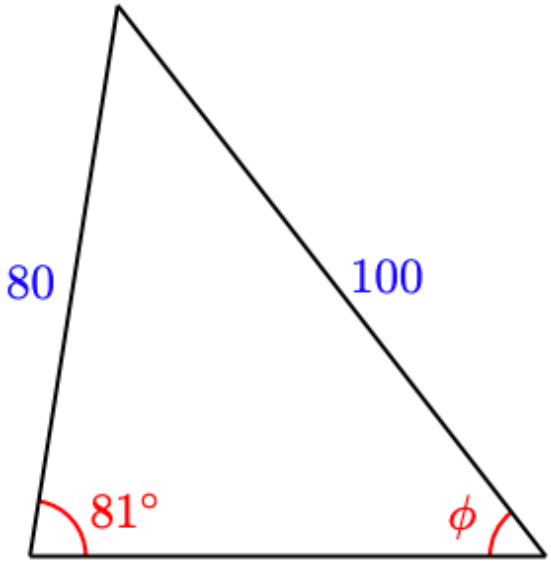


(γ is acute.)

16.



17.



For the following exercises, sketch the triangle and solve. Round answers to two decimal places.

18. $b = 7$, $A = 23^\circ$, $B = 42^\circ$

19. $c = 34$, $A = 53^\circ$, $C = 26^\circ$

20. $a = 1.8$, $c = 2.1$, $C = 44^\circ$

21. $b = 8.5$, $c = 6.8$, $B = 23^\circ$

22. $c = 75$, $A = 35^\circ$, $B = 46^\circ$

23. $a = 94$, $B = 29^\circ$, $C = 84^\circ$

24. In $\triangle ABC$, $\angle A = 30^\circ$ and $c = 12$. How many triangles are possible for each of the following lengths for side a? Sketch the solutions in each case.

    a. $a = 6$

    b. $a = 4$

    c. $a = 9$

    d. $a = 15$

For the following exercises, find the remaining angles of the triangle (or triangles, if this is the ambiguous case of the Law of Sines). Round answers to two decimal places.

25. $a = 66$, $c = 43$, $C = 25^\circ$

26. $b = 10$, $c = 14$, $B = 20^\circ$

27. $b = 100$, $c = 80$, $B = 49^\circ$

28. $b = 4.7$, $c = 6.3$, $C = 54^\circ$

29. a. Sketch a triangle with $A = 25^\circ$, $B = 35^\circ$, and $b = 16$.

    b. Use the Law of Sines to find a.

    c. Use the Law of Sines to find c.

    d. Find c without using the Law of Sines. (Hint: Sketch the altitude, h, from C to make two right triangles. Find h, then use h to find c.)

Synthesis Questions

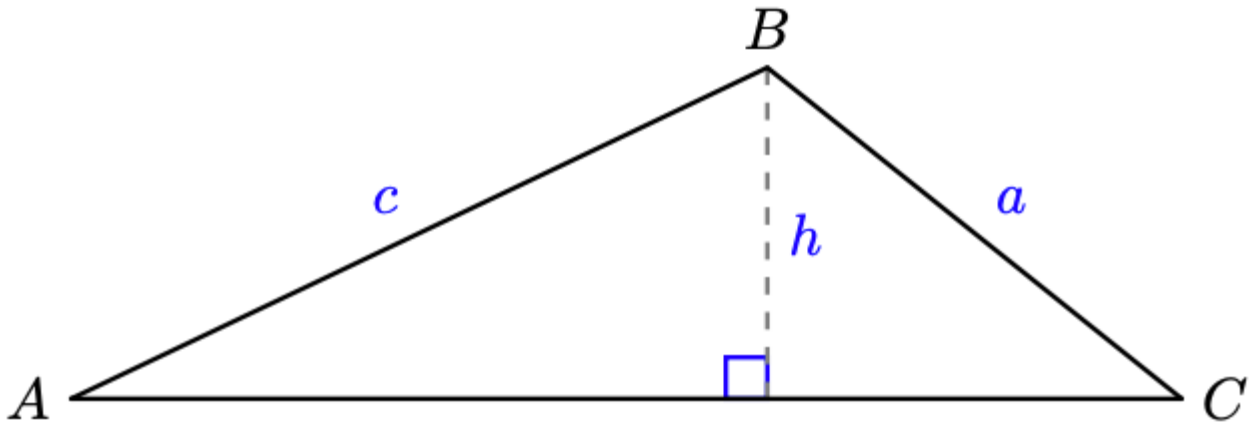
30. In this problem, we show that there are two different triangles $\triangle ABC$ with $A = 30^\circ$, $a = 2$, and $c = 3$.

    a. Use a protractor to draw an angle $A = 30^\circ$. Mark point B on one side of the angle so that $\overline{AB}$ is 3 inches long.

    b. Locate two distinct points on the other side of the angle that are each 2 inches from point B. These points are both possible locations for point C.

    c. Use the Law of Sines to find two distinct possible measures for $\angle C$.

31. Consider the triangle $\triangle ABC$ shown below.



a. Express the length of the altitude in terms of $\angle A$ and c.

b. Suppose we keep $\angle A$ and side c fixed but allow a to vary in length. What is the smallest value a can have and still be long enough to make a triangle?

c. What are the largest and smallest values that a can have in order to produce two distinct triangles $\triangle ABC$ (without changing $\angle A$ and side c)

32. For the triangle in the previous problem, suppose $A = 40^\circ$ and $c = 8$.

a. Sketch and solve the triangle if $a = 12$.

b. Sketch and solve the triangle if $a = 6$.

c. Sketch and solve the triangle if $a = 4$.

d. For what value of a is c the hypotenuse of a right triangle?

33. For the triangle in Problem 31, suppose $A = 70^\circ$ and $c = 20$.

a. For what value of a is the triangle a right triangle?

b. For what values of a are there two solutions for the triangle?

c. For what values of a is there one obtuse solution for the triangle?

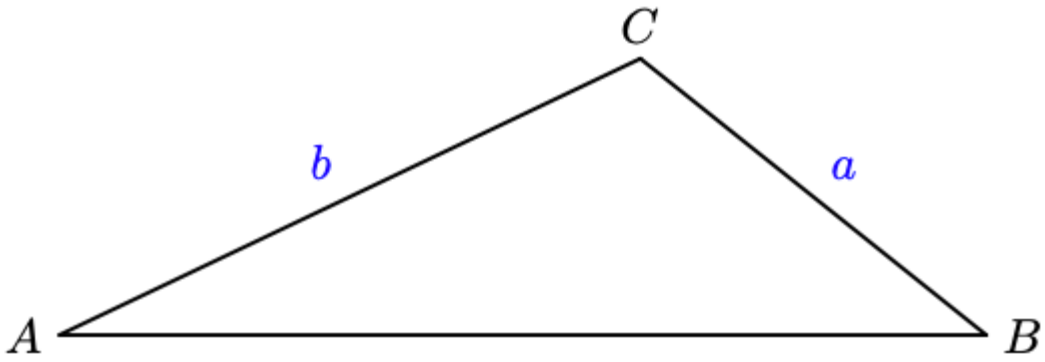
d. For what value of a is there no solution?

34. a. Sketch a triangle with $A = 75^\circ$, $a = 15$, and $b = 6$.

b. Use the Law of Sines to find c.

c. Find c without using the Law of Sines.

35. Here is a method for solving certain oblique triangles by dividing them into two right triangles. In the triangle shown, we know two angles, A and B, and the side opposite one of them, say a. We would like to find side b.



a. Draw the altitude h from angle C.

b. Write an expression for b in terms of h and angle A.

c. Write an expression for h in terms of angle B.

d. Substitute your expression for h into your expression for b.

e. Which of the following is equivalent to the formula you wrote in part (d)?

i. $a\sin(A) = b\sin(B)$

ii. $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$

iii. $\frac{a}{\sin(B)} = \frac{b}{\sin(A)}$

Proofs

For the following exercises, you will prove the Law of Sines using the trigonometric formula for the area of a triangle. We will derive this formul later. For now, you can use the fact that the area of triangle $\triangle ABC$ is $\frac{1}{2}ab\sin(\theta)$, where θ is the angle between sides a and b.

36. Sketch a triangle with angles A, B, and C, and opposite sides of lengths a, b, and c, respectively.

a. Write the area of the triangle in terms of a, b, and angle C.

b. Write the area of the triangle in terms of a, c, and angle B.

c. Write the area of the triangle in terms of b, c, and angle A.

37. Equate the three different expressions from Problem 36 for the area of the triangle. Multiply through by $\frac{2}{abc}$ and simplify to deduce the Law of Sines.

Applications

For the following exercises,
(a) Sketch and label a triangle to illustrate the problem.
(b) Solve the problem.

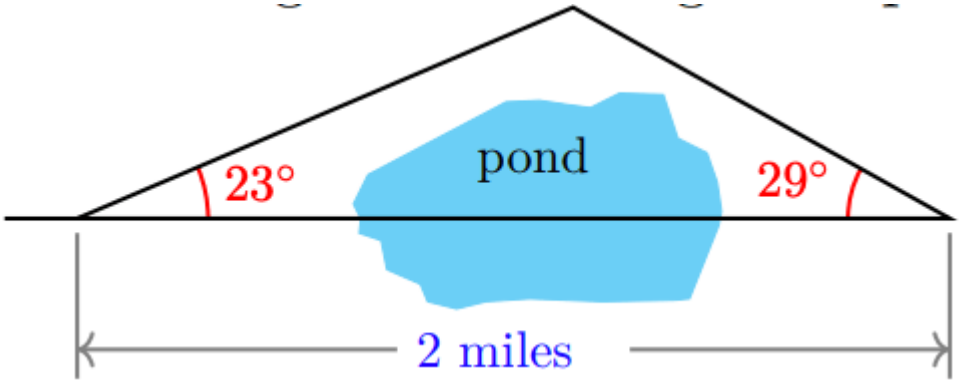
38. Phuong wants to know the height of a cliff on the other side of a ravine. The angle of elevation from the edge of the ravine to the cliff top is 84.6°. When she moves 30 feet back from the ravine, the angle of elevation is 82.5°. How tall is the cliff?

39. Nam wants to know the height of a tree in the median strip of a highway. The angle of elevation from the highway shoulder to the treetop is 43.5°. When he moves 10 feet farther away from the tree, the angle of elevation is 37.2°. How tall is the tree?

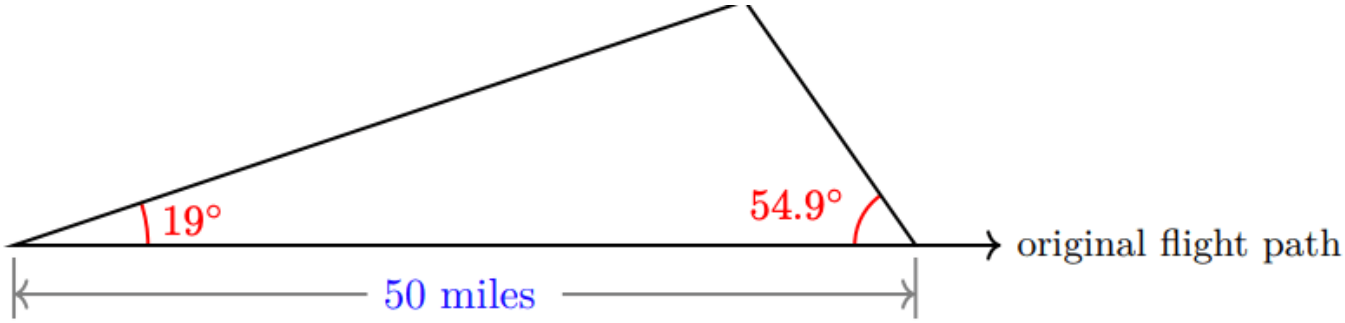
10. Mike and Loi are 10 kilometers apart, observing a satellite that passes directly over their heads. At a moment when the satellite is between them, Loi measures its angle of elevation as 84.6°, and Mike measures an angle of 87°. How far is the satellite from Mike?

11. Paige rows her kayak due east. When she began, she spotted a lighthouse 2000 meters in the distance at an angle of 14° south of east. After traveling for an hour, the lighthouse was at an angle of 83° south of east. How far did Paige travel, and what was her average speed?

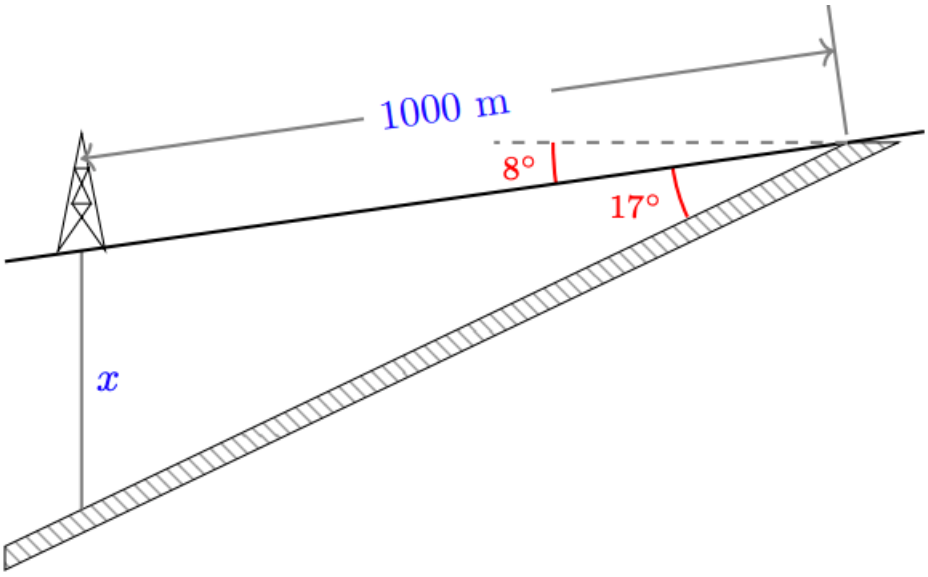
12. Ivan is hiking along a straight path but needs to detour around a large pond. He turns 23° from his path until clear of the pond, then walks back to his original path, intercepting it at an angle of 29° and at a distance of 2 miles from where he had left the path. How far did Ivan walk in each of the two segments of his detour, and how much farther did his detour require compared with a straight line through the pond?



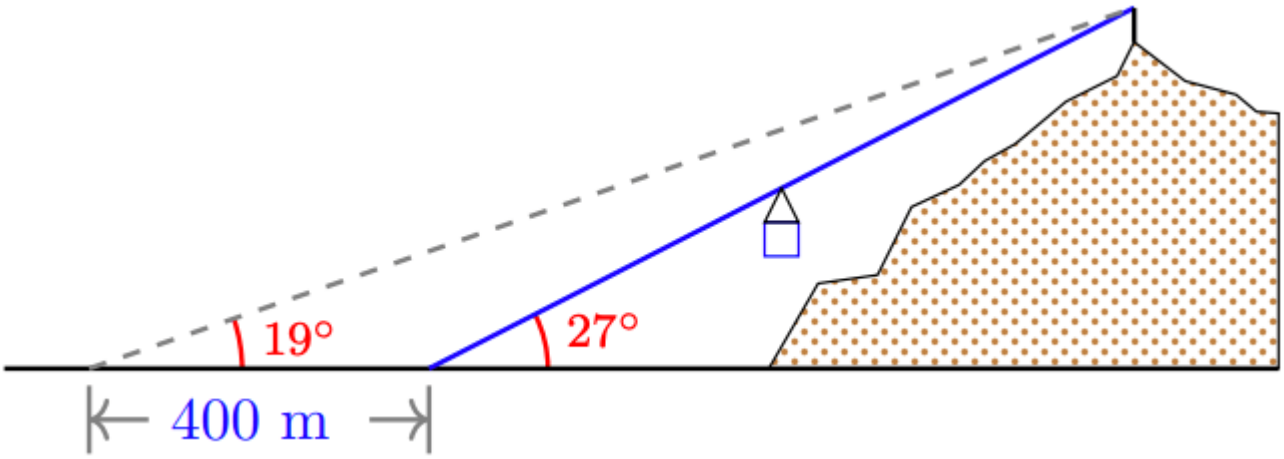
13. Wyatt is flying to Monterey but must change course to avoid a storm. He flies 19° off from his original direction until he clears the storm, then turns again to return to his original flight path, intercepting it at an angle of 54.9° and at a distance of 50 miles from where he had left it. How much farther did his detour require compared with his original course?



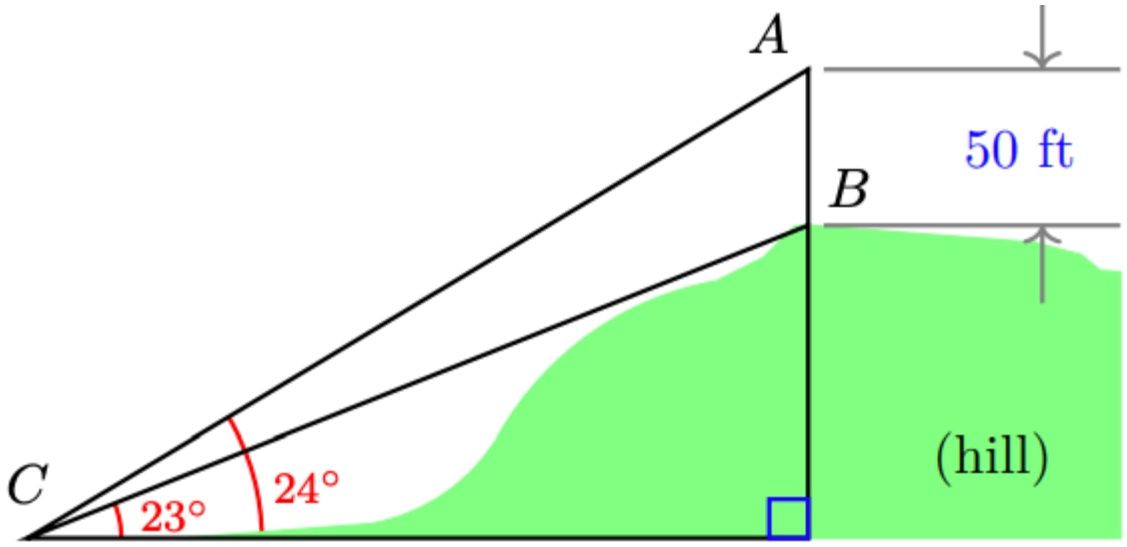
14. Geologists find an outcropping from an underground rock formation that usually indicates the presence of oil. The outcropping is on a hillside, and the formation dips another 17° from the surface. If an oil well is placed 1000 meters downhill from the outcropping, how far will they drill before reaching the formation?



15. A proposed ski lift will rise from a point near the base of the slope with an angle of 27°. At a distance of 400 meters further from the hill, the angle of elevation to the top of the ski lift is 19°. How long is the ski lift?



16. Nhat wants to measure the height of a hill. She first plants a 50-foot-tall antenna at the hill's peak. Then, she descends the hill and finds a point where she can see the antenna's top and bottom. The angle of elevation to the bottom of the antenna is 23°, and the angle of elevation to the top of the antenna is 24°.



a. Find $\angle ACB$.

b. Find $\angle CAB$ at the top of the antenna.

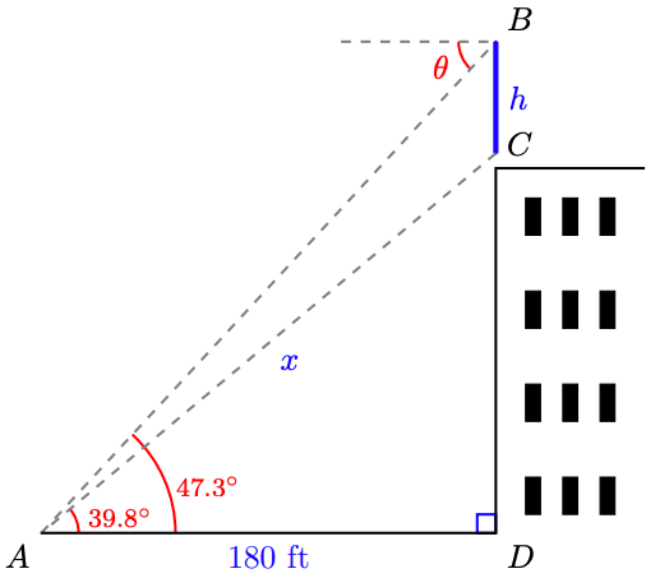
c. How long is BC, the distance from the bottom of the antenna to C?

d. How tall is the hill?

17. Jorge and Camille are 1000 yards apart. The angle Jorge sees between Camille and a particular tree is 38°. If the tree is 800 yards from Camille, how far is it from Jorge? (There are two possible answers.)

18. From the lookout point on Fabrick Rock, Sang can not only see the famous "Crooked Spire" in Chesterfield, which is 8 miles away but also the red phone box in the village of Alton. Chesterfield and Alton are 7 miles apart. Fabrick Rock has a plaque that shows directions to famous sites, and from the plaque, Sang determines that the angle between the lines to the spire and the phone box measures 19°. How far is Fabrick Rock from the phone box? (There are two possible answers.)

19. A billboard of California's gubernatorial candidate, Min, is located on the roof of a building. At a distance of 180 feet from the building, the angles of elevation to the bottom and top of the billboard are respectively 39.8° and 47.3°. How tall is the billboard?



For the following exercises, compute the distances in Astronomical Units. Then convert to kilometers, using the fact that 1 AU $\approx 1.5 \times 108$ km.

50. When observed from opposite sides of Earth's orbit, the star Alpha Centauri has a parallax of $0.76''$. How far from the Sun is Alpha Centauri?

51. How far from the Sun is Barnard's star, which has a parallax of $1.1''$ when observed at opposite ends of Earth's orbit?

52. How far from the Sun is Tau Ceti, which has a parallax of $0.55''$ when observed from opposite ends of Earth's orbit?

53. How far from the Sun is Sirius, which has a parallax of $0.75''$ when observed from opposite ends of Earth's orbit?

This page titled 11.1.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

11.2: The Law of Cosines

Section Preview

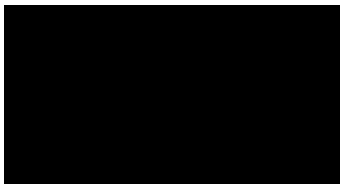
Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

The Law of Cosines

When learning the Law of Sines, we used a table to help organize information and determine which unknown to find first. For example, suppose the information in the following table was what we were given.



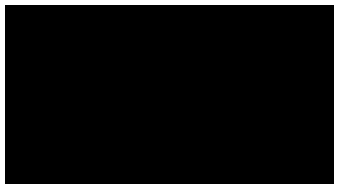
I mentioned in the previous section that a complete row in this table meant we could use the Law of Sines. Therefore, we can see that, given two angles and a side opposite one of those angles, we can use the Law of Sines to solve the triangle.

Suppose, instead, the given information looked as follows:

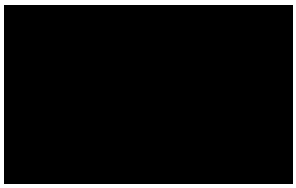


Again, we have a complete row, so we can use the Law of Sines. Therefore, given two sides and an angle opposite one of those sides, we could use the Law of Sines to solve the triangle (if one exists).

In contrast, given any other combination of sides or angles, the Law of Sines is not helpful. If you consider what is required for this to happen, we just need a table with no complete rows. The two major examples are the following:



or



Notice that there is not a complete row within either of these tables and, as of now, we have no simple way of completing a row. Not having an angle-side pairing presents an issue that the Law of Sines cannot surmount (unless you were given two angles, in which case you can find the third and proceed to use the Law of Sines).

To solve a triangle when we know two sides and the included angle, we will need a generalization of the Pythagorean Theorem.

In a right triangle, with $C = 90°$, the Pythagorean Theorem tells us that

$$c^2 = a^2 + b^2.$$

If we allow angle C to vary (rather than being locked in at a right angle) but keep a and b the same length, the side c will grow or

shrink, depending on whether we increase or decrease the angle C, as shown below.

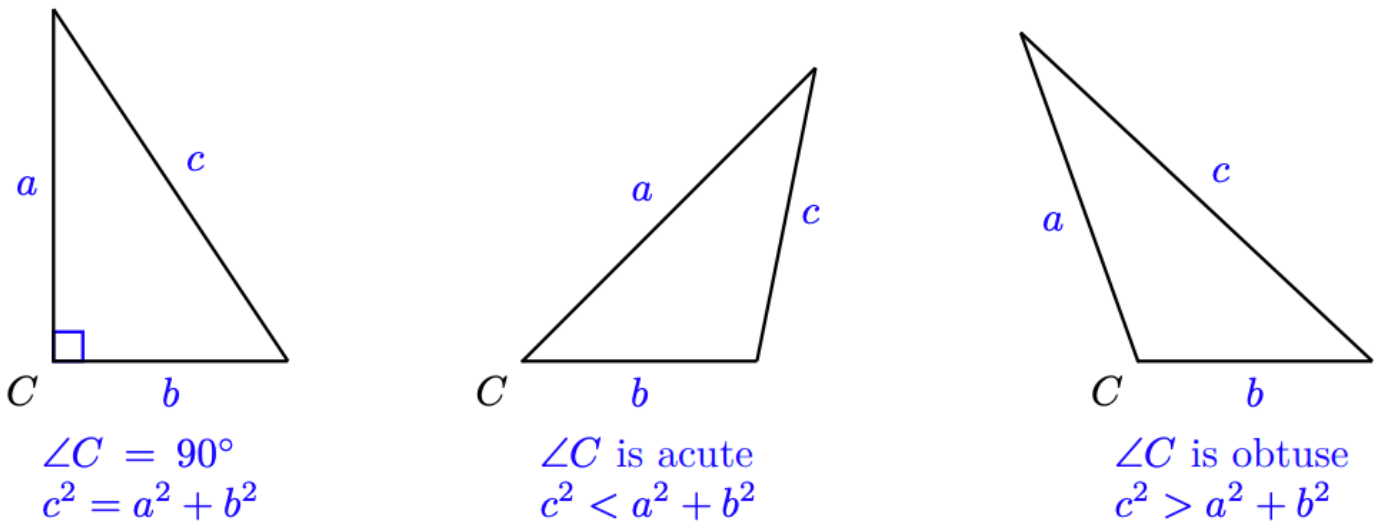


Figure 11.2.1

The Pythagorean Theorem is a special case of a more general law that applies to *all* triangles, no matter the size of angle C. This law is known as the **Law of Cosines**.

Theorem: Law of Cosines

If the angles of a triangle are A, B, and C, and the opposite sides are respectively a, b, and c, then

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos(A) \\ b^2 &= a^2 + c^2 - 2ac\cos(B) \\ c^2 &= a^2 + b^2 - 2ab\cos(C) \end{aligned}$$

Proof available at the end of this section.

The Law of Cosines is also known as the **Generalized Pythagorean Theorem**.

Rather than committing all three forms to memory, I find it much easier to think of these as modifications of the Pythagorean Theorem. We have a sum of squares of sides *almost* equal to the square of a quasi-hypotenuse. The adjustment we make to the sum of squares side is to subtract twice the product of the non-quasi-hypotenuse sides and the cosine of the angle opposite the quasi-hypotenuse.

Now that I have written that out, I can see that it might be confusing to most people, so let me say it symbolically. If s_1 and s_2 are playing the roles of the non-quasi-hypotenuse sides, and h is playing the role of the quasi-hypotenuse, then

$$s_1^2 + s_2^2 - 2s_1 s_2 \cos(H) = h^2,$$

where H is the angle opposite side h.

Finding a Side

Example 11.2.1

The town of Avery lies 48 miles due east of Baker, and Clio is 34 miles from Baker, in the direction $35°$ west of north. How far is it from Avery to Clio?

Solution available at the end of this section.

Caution: Follow the Order of Operations

When simplifying the Law of Cosines, follow the Order of Operations carefully. In Example 11.2.1, the right side of the equation

$$b^2 = 34^2 + 48^2 - 2(34)(48)\cos(125^\circ)$$

has three terms, and simplifies to

$$\begin{aligned} b^2 &= 1156 + 2304 - 3264\cos(125^\circ) \\ \Longrightarrow \quad b^2 &\approx 3460 - 3264(-0.573567364\ldots) \end{aligned}$$

Note that 3264 is the coefficient of $\cos(125^\circ)$, so it would be incorrect to subtract 3264 from 3460. Using a graphing calculator, you can enter the right side of the equation exactly as it is written.

Checkpoint 11.2.1

In $\triangle ABC$, $a = 11$, $c = 23$, and $B = 87^\circ$. Find b, and round your answer to two decimal places.

Answer available at the end of this section.

Finding an Angle

We can also use the Law of Cosines to find an angle when we know all three sides of a triangle. In the following example, pay close attention to the algebraic steps used to solve the equation.

Example 11.2.2

In the triangle below, $a = 6$, $b = 7$, and $c = 11$. Find angle C.

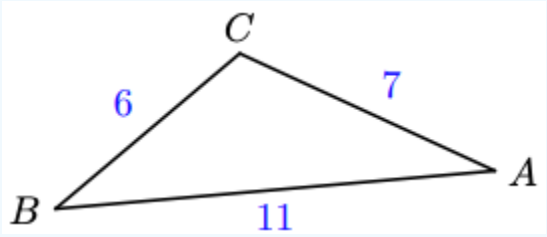


Figure 11.2.3

Solution available at the end of this section.

Checkpoint 11.2.2

In $\triangle ABC$, $a = 5.3$, $b = 4.7$, and $c = 6.1$. Find angle B, and round your answer to two decimal places.

Answer available at the end of this section.

Once we have calculated one of the angles in a triangle, we can use either the Law of Sines or the Law of Cosines to find a second angle. Here is how we would use the Law of Sines to find angle A in Example 11.2.2.

$$\begin{aligned} \frac{\sin(A)}{a} &= \frac{\sin(C)}{c} && \text{(Law of Sines)} \\ \Longrightarrow \quad \sin(A) &= \frac{a\sin(C)}{c} && \text{(multiplying both sides by } a\text{)} \\ \Longrightarrow \quad \widehat{A} &= \sin^{-1}\left(\frac{a\sin(C)}{c}\right) && \text{(finding the reference angle)} \end{aligned}$$

Substituting in the given values, we get $\widehat{A} \approx 29.5^\circ$. Since A is the shortest side of the triangle, A must be acute. Therefore, $A \approx 29.5^\circ$. Finally,

$$B = 180^\circ - A - C \approx 35.1^\circ.$$

Alternatively, we can use the Law of Cosines to find angle A.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos(A) && \text{(Law of Cosines)} \\ \Longrightarrow \quad a^2 - b^2 - c^2 &= -2bc\cos(A) && \text{(subtracting } b^2 \text{ and } c^2 \text{ from both sides)} \\ \Longrightarrow \quad \frac{a^2 - b^2 - c^2}{-2bc} &= \cos(A) && \text{(dividing both sides by } -2bc\text{)} \\ \Longrightarrow \quad \cos^{-1}\left(\left|\frac{a^2 - b^2 - c^2}{-2bc}\right|\right) &= \widehat{A} && \text{(finding the reference angle)} \end{aligned}$$

Substituting the given values into this result, we get the reference angle of $\widehat{A} \approx 29.5^\circ$. If we had computed the value of $\frac{a^2-b^2-c^2}{-2bc}$, we would have found that this fraction is positive. Since the arccosine returns in QI or QII, and, of those two quadrants, the cosine is positive in QI, our actual angle is $A \approx 29.5^\circ$.

From this conversation about finding a second angle once you have used the Law of Cosines to find an initial angle, I hope you agree that the amount of computational work is minimized when using the Law of Sines. You can choose either method to find a second angle, but I will always choose the Law of Sines over the Law of Cosines.

Advice: Choose Your Angle Wisely!

When you have used the Law of Cosines to find the first missing angle within a triangle, you can use the Law of Sines or the Law of Cosines to find the next unknown angle. If using the Law of Sines, however, *always* choose to find the angle opposite the smallest remaining side. This is because the angle opposite the smallest remaining side is *guaranteed* to be acute. Hence, the reference angle returned when using the Law of Sines will be the actual angle!

Applications Involving the Law of Cosines

The applications of the Law of Cosines are countless, more so than the Law of Sines; however, they all somewhat follow the structure of the following example.

Example 11.2.3

A researcher wishes to determine the width of a vernal pond, as drawn below. From a point P, he finds the distance to the eastern-most point of the pond to be 950 feet, while the distance to the western-most point of the pond from P is 1000 feet (obviously, the perspective of the provided image is such that due north is roughly facing the 4 o'clock position). If the angle between the two lines of sight is 60°, find the width of the pond.

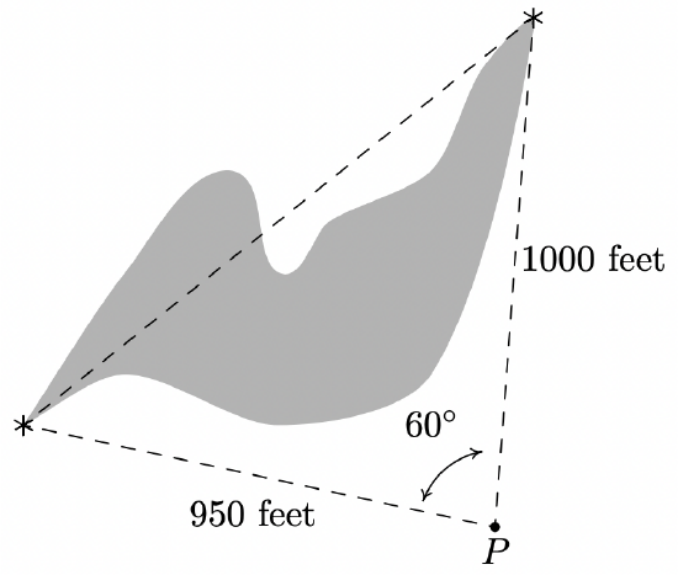


Figure 11.2.4

Solution available at the end of this section.

Checkpoint 11.2.3

The hour hand on Kim's antique clock is 4 inches long and the minute hand is 5.5 inches long. Find the distance between the ends of the hands when the clock reads four o'clock. Round your answer to the nearest hundredth of an inch.

Answer available at the end of this section.

Revisiting Navigation Problems

In a previous section, we were introduced to navigation problems (heading and bearing problems). At that time, however, we were only able to work with these styles of problems when they resulted in right triangles. We now have enough Trigonometry "under our belts" to approach these problems in *any* situation.

Example 11.2.4

The sailing club leaves the marina on a heading of 15° and sails for 18 miles. They then change course, and after traveling for 12 miles on a heading of 35°, they experience engine trouble and radio for help. The marina sends a speed boat to rescue them. How far should the speed boat go, and on what heading?

Solution available at the end of this section.

Checkpoint 11.2.4

Roy wants to fly from Anchorage to Nome, Alaska, a distance of 540 miles on a heading of 303°. After flying for some time, he discovers that his heading is in error and is actually flying at a heading of 313°. Roy corrects his flight plan and changes course when he is 200 miles from Anchorage. What is his new heading, and how far is he from Nome?

Answer available at the end of this section.

Success in College: Dealing With Failure (Part 4)

1. A first-year student fails her math class. She hates math. Her advisor wants her to retake the course the next term. The student wants to wait a year or so before trying again. Discuss the advantages and disadvantages of waiting to retake this math course.
2. Some colleges restrict the number of times a student can repeat a course. Describe any restrictions on repeating a course at your school.

Answers

Section Preview

This section is a very traditional treatment of the Law of Cosines and applications involving the Law of Cosines. You will likely spend a good chunk of time on navigation problems, as these are often difficult for students to master.

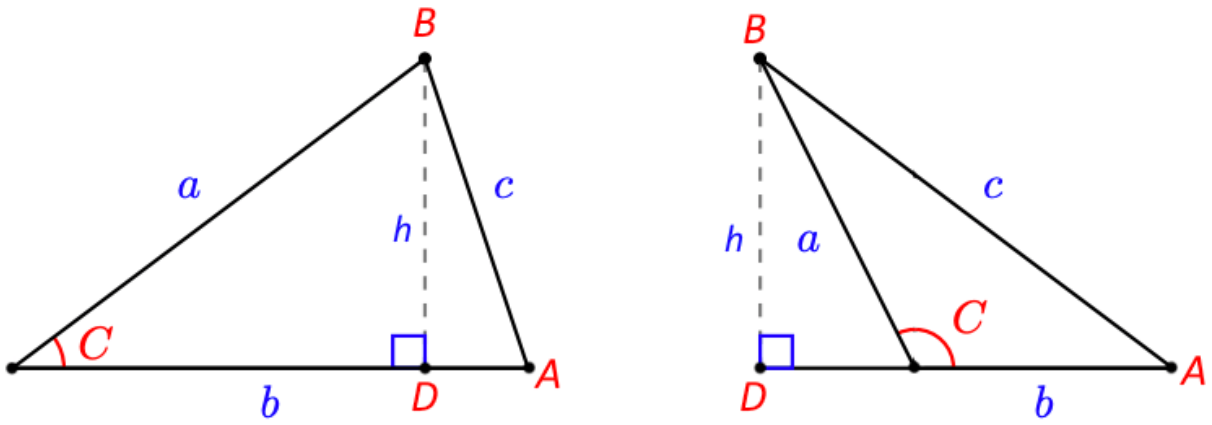
Section Preview

- Use the Law of Cosines to find a missing side or a missing angle in an oblique triangle.
- Use the Law of Sines to find the remaining angle once the Law of Cosines has been used to find one angle in an oblique triangle.
- Use the Law of Cosines to solve an application involving heading or bearing.
- Use the Law of Cosines to solve an application.

Theorem: Law of Cosines

Start with general triangles as seen in the figures below.

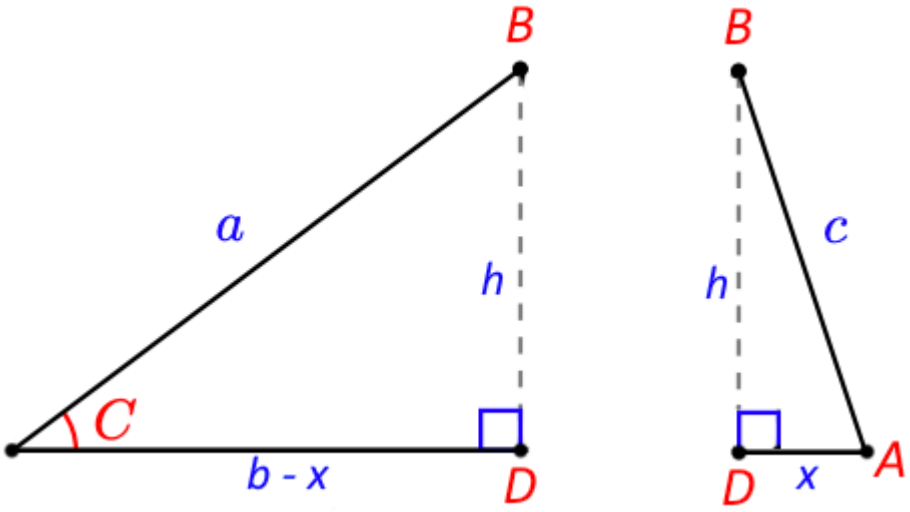
Two general triangles, one acute (left) and one obtuse (right), with altitudes labeled as h



We will perform the proof for the acute triangle (the proof associated with the obtuse triangle is similar).

The acute triangle is comprised of two right triangles - $\triangle ABD$ and $\triangle BCD$. Let x be the length of the line segment $\overline{AD}$. Then the length of the segment $\overline{CD}$ is $b - x$. We show these facts in the following figure, splitting the general right triangle into its two right triangle pieces.

"Splitting" the acute triangle into two right triangles



With the leftmost triangle, we have the following:

$$\begin{aligned} (b-x)^2+h^2 &= a^2 && \text{(Pythagorean Theorem)} \\ \implies b^2-2bx+x^2+h^2 &= a^2 && \text{(distributing)} \end{aligned}$$

With the rightmost triangle, we have

$$x^2+h^2=c^2 \quad \text{(Pythagorean Theorem)}$$

Therefore,

$$b^2-2bx+x^2+h^2=a^2 \quad \text{and} \quad x^2+h^2=c^2.$$

Replace x^2+h^2 in the first equation with c^2. Doing so yields the following:

$$b^2-2bx+c^2=a^2 \implies b^2+c^2-2bx=a^2.$$

From the rightmost triangle in the figure above, we see that $\cos(A)=\frac{x}{c}$, and so $c\ \cos(A)=x$. Hence, we have developed the following relationship between the sides of *any* triangle.

$$\begin{aligned} b^2+c^2-2bx &= a^2 \\ \implies b^2+c^2-2b(c\cos(A)) &= a^2 && \text{(substitution)} \\ \implies b^2+c^2-2bc\cos(A) &= a^2 \end{aligned}$$

When A is a right angle, $\cos(A)=\cos(90^\circ)=0$, so the equation reduces to the Pythagorean Theorem.

We can write similar equations involving the angles B or C. In all cases, the angle within the cosine is opposite the side playing the role of the "hypotenuse" (really, a **quasi-hypotenuse**).

Example 11.2.1

We start with a simple sketch.

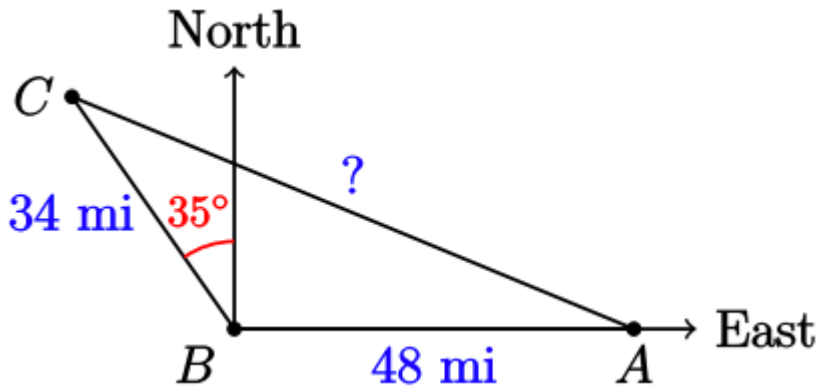


Figure 11.2.2

The angle $\angle ABC=35^\circ+90^\circ=125^\circ$. Thus, in $\triangle ABC$ we have the following:



Since we do not have a complete row, we must use the Law of Cosines. We are interested in knowing the length of side b, so that will be our quasi-hypotenuse.

$$\begin{aligned} b^2 &= a^2+c^2-2ac\cos(B) && \text{(Law of Cosines)} \\ \implies b^2 &= 34^2+48^2-2(34)(48)\cos(125^\circ) && \text{(substitute)} \\ \implies b^2 &= 3460-3264\cos(125^\circ) \\ \implies b^2 &\approx 5332.153 \\ \implies b &\approx 73.02 \end{aligned}$$

Therefore, Avery is about 73 miles from Clio.

Checkpoint 11.2.1

24.97

Example 11.2.2

Our table looks like



Since there is no way to get a complete row, we must use the Law of Cosines. Moreover, because we are asked to find C, we choose the version of the Law of Cosines that uses angle C.

$$\begin{aligned} & c^2 = a^2 + b^2 - 2ab\cos(C) && \text{(Law of Cosines)} \\ \Longrightarrow\quad & 11^2 = 6^2 + 7^2 - 2(6)(7)\cos(C) && \text{(substitute)} \\ \Longrightarrow\quad & 121 = 36 + 49 - 84\cos(C) && \text{(apply powers and compute the product)} \\ \Longrightarrow\quad & 36 = -84\cos(C) && \text{(subtract 85 from both sides)} \\ \Longrightarrow\quad & -\frac{3}{7} = \cos(C) && \text{(divide both sides by } -84) \\ \Longrightarrow\quad & \widehat{C} = \cos^{-1}\left(\frac{3}{7}\right) && \text{(finding the reference angle)} \\ \Longrightarrow\quad & C \approx 115.4^\circ && \text{(arccosine is either in QI or QII.} \\ & && \text{Since the cosine is negative, } C \text{ is QII.} \\ & && \text{Thus, } C = 180^\circ - \widehat{C}\Big) \end{aligned}$$

Angle C is roughly 115.4°.

Checkpoint 11.2.2

48.07°

Example 11.2.3

Let p be the side length opposite angle P, $a = 1000$, and $b = 950$. Then our table looks like the following:



No complete rows (and no hope of getting a complete row using simple arithmetic) means we are using the Law of Cosines. We want to find length p.

$$\begin{aligned} & p^2 = a^2 + b^2 - 2ab\cos(P) && \text{(Law of Cosines)} \\ \Longrightarrow\quad & p = \sqrt{a^2 + b^2 - 2ab\cos(P)} && \text{(Extraction of Roots)} \end{aligned}$$

Substituting in the given values, we get

$$p = \sqrt{950^2 + 1000^2 - 2(950)(1000)\cos(60^\circ)} = \sqrt{952{,}500} \approx 976.$$

Thus, the pond is roughly 976 feet wide.

Checkpoint 11.2.3

8.26 inches.

Example 11.2.4

As we did when first introduced to heading problems, we sketch the situation, placing small axes at each pivot point.

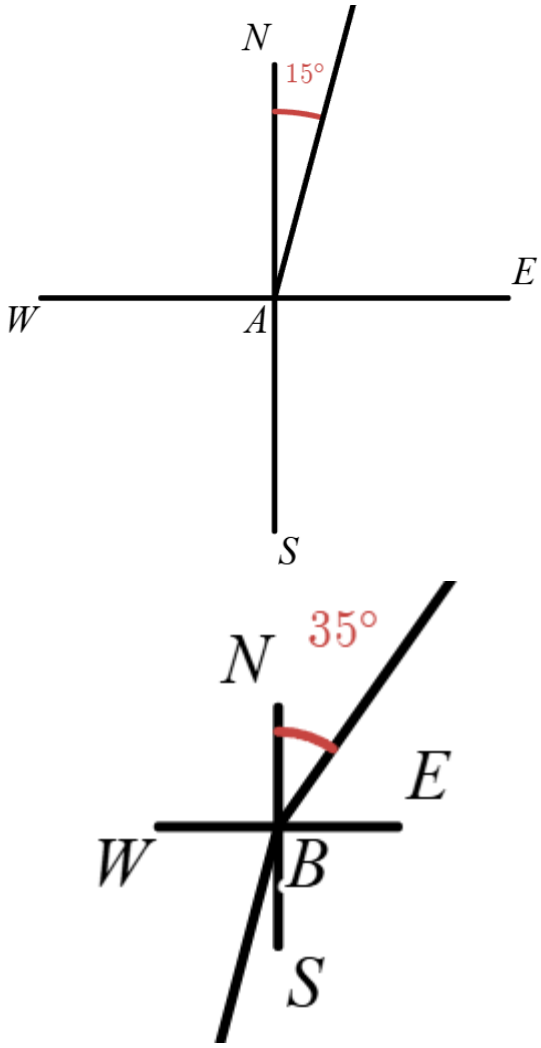


Figure 11.2.5

This results in the following figure.

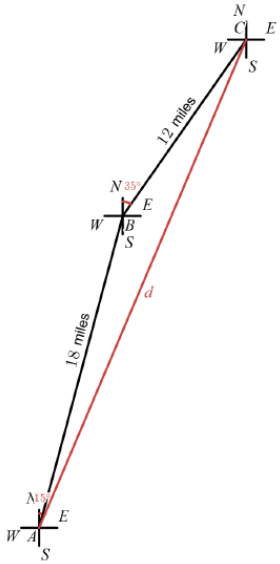


Figure 11.2.6

We want to find the distance d and the angle $\angle BAC$ shown in the figure. We use alternate interior angles to get further information about the angles within $\triangle ABC$.

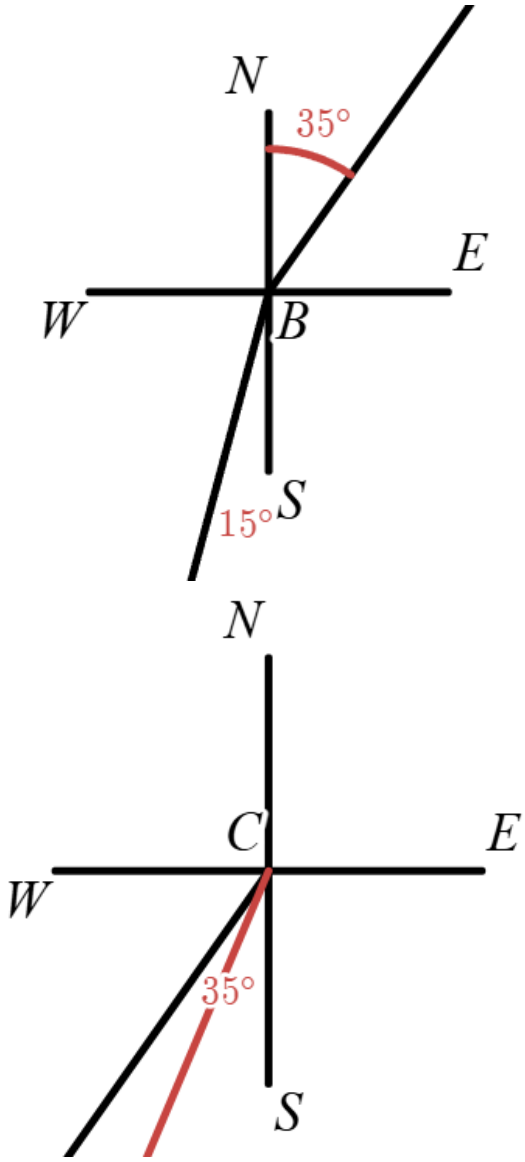


Figure 11.2.7

This results in the final, clean sketch.

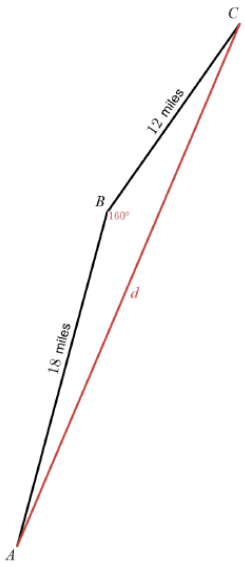


Figure 11.2.8

In $\triangle ABC$, we calculated the angle at point B where the sailing club changed course as follows:

$$B = 180^\circ - 35^\circ + 15^\circ = 160^\circ.$$

We now build our table by using the Law of Cosines to find b and $\angle A$.



Since there is not a complete row, and there is no way to get a complete row without Trigonometry, we use the Law of Cosines to find the length of side b (accidentally labeled as side d in the figure above).

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos(B) && \text{(Law of Cosines)} \\ \Longrightarrow \quad b &= \sqrt{a^2 + c^2 - 2ac\cos(B)} && \text{(Extraction of Roots)} \end{aligned}$$

Substituting in the given values, we get $b \approx 29.56$ miles. Taking the time to add this to our table, we get the following:



Now that we have a complete row, we can use the Law of Sines to find one of the remaining angles. The advice given previously was to find the angle opposite the smallest remaining side. This would be A (which is, in fact, the angle we need to find).

$$\begin{aligned} \frac{\sin(A)}{a} &= \frac{\sin(B)}{b} && \text{(Law of Sines)} \\ \Longrightarrow \quad \sin(A) &= \frac{a\sin(B)}{b} && \text{(multiplying both sides by } a\text{)} \\ \Longrightarrow \quad \widehat{A} &= \sin^{-1}\left(\frac{a\sin(B)}{b}\right) && \text{(finding the reference angle)} \end{aligned}$$

Substituting in the nonrounded values for a, b, and B, we get $\widehat{A} \approx 8.0^\circ$. Since A is opposite the smallest remaining side, A must be acute. Therefore, $A \approx 8.0^\circ$.

We finish this problem with a bit of extra logic. We know the speed boat must travel $b \approx 29.56$ miles; however, the heading of the speed boat is not 8.0°. This is because they must rotate through 15° and then an additional 8.0° to face the sailing club's boat. Thus, their heading is 23.0°.

Checkpoint 11.2.4

297.2°, 344.8 miles

This page titled 11.2: The Law of Cosines is shared under a CC BY-SA 1.2 license and was authored, remixed, and/or curated by Roy Simpson.

11.2.1: Resources and Key Concepts

Resources

Videos

- Law of Cosines
  - Non-Right Triangles - Law of Cosines: Theorem and Proof
  - Non-Right Triangles - Law of Cosines: Using the Law of Cosines to Solve Triangles
  - Non-Right Triangles - Law of Cosines: Applications of Non-Right Triangle Trigonometry (Part 2)
  - The Law of Cosines - Review of the Law of Cosines
  - The Law of Cosines - Applications of Non-Right Triangle Trigonometry (Part 2)
  - The Law of Cosines - Applications of Non-Right Triangle Trigonometry (Part 3): Navigation Problems

Key Concepts

Theorems

- **Law of Cosines:** If the angles of a triangle are A, B, and C, and the opposite sides are respectively a, b, and c, then:
  - $a^2 = b^2 + c^2 - 2bc\cos(A)$
  - $b^2 = a^2 + c^2 - 2ac\cos(B)$
  - $c^2 = a^2 + b^2 - 2ab\cos(C)$

Common Mistakes

- **Order of Operations:** When simplifying the Law of Cosines, follow the order of operations carefully. In an expression like $b^2 + c^2 - 2bc\cos(A)$, the term $2bc\cos(A)$ must be calculated before subtracting from the sum of squares. Do not subtract $2bc$ from $b^2 + c^2$ first.
- **Incorrectly Applying Law of Sines After Law of Cosines:** After using the Law of Cosines to find an angle, you can use the Law of Sines to find a second angle. To avoid the ambiguous case, you should always choose to find the angle opposite the smallest remaining side, as this angle is guaranteed to be acute.

This page titled 11.2.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

11.2.2: Homework

Reading Questions

1. The Law of Cosines is a generalization of what famous theorem?
2. Under what two conditions (e.g., SSS, ASA, etc.) is the Law of Cosines useful for solving a triangle when the Law of Sines is not?
3. State one of the three versions of the Law of Cosines.
4. When using the Law of Cosines to find a missing side, what information must you have?
5. When using the Law of Cosines to find a missing angle, what information must you have?
6. What is a common mistake regarding the order of operations when simplifying the right side of the Law of Cosines formula?
7. After using the Law of Cosines to find one angle in a triangle, what is the recommended method for finding a second angle?
8. When using the Law of Sines after using the Law of Cosines, which angle should you choose to find next to avoid the ambiguous case? Why?

Homework

Concept Check

1. The Law of Cosines is a generalization of what theorem?

2. If you know all three sides of a triangle and one angle, what might be the advantage of using the Law of Cosines to find another angle, instead of the Law of Sines?

3. Piyali is solving a triangle in which $a = 20$, $b = 16$, and $A = 26^\circ$. She finds that $\sin(B) \approx 0.3507$. How does she know that $B \approx 20.5^\circ$, and not 159.5°?

Basic Skills

4. a. Simplify $5^2 + 7^2 - 2(5)(7)\cos(\theta)$.

   b. Evaluate the expression in part (a) for $\theta = 29^\circ$.

   c. Evaluate the expression in part (a) for $\theta = 151^\circ$.

5. a. Simplify $26.1^2 + 32.5^2 - 2(26.1)(32.5)\cos(\phi)$.

   b. Evaluate the expression in part (a) for $\phi = 64^\circ$.

   c. Evaluate the expression in part (a) for $\phi = 116^\circ$.

6. a. Solve $b^2 = a^2 + c^2 - 2ac\cos(\beta)$ for $\cos(\beta)$.

   b. For the equation in part (a), find $\cos(\beta)$ if $a = 5$, $b = 11$, and $c = 8$.

7. a. Solve $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$ for $\cos(\alpha)$.

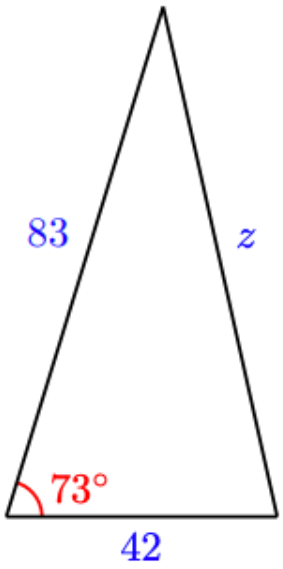
   b. For the equation in part (a), find $\cos(\alpha)$ if $a = 4.6$, $b = 7.2$, and $c = 9.4$.

For the following exercises, find the indicated side. Round to two decimal places.

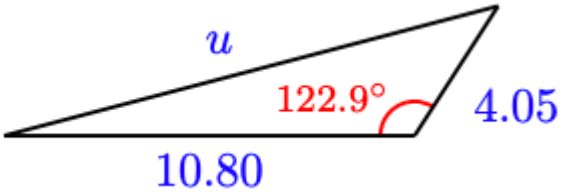
8.



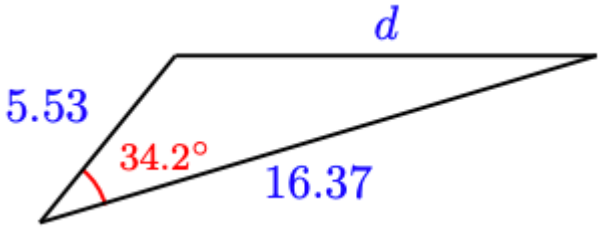
9.



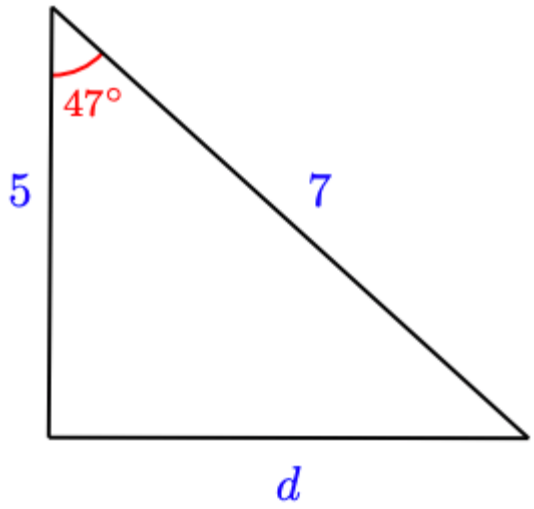
10.



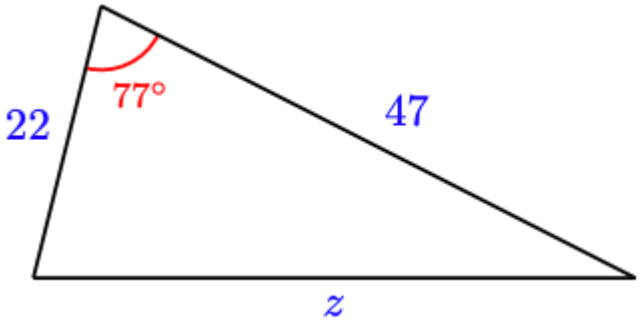
11.



12.

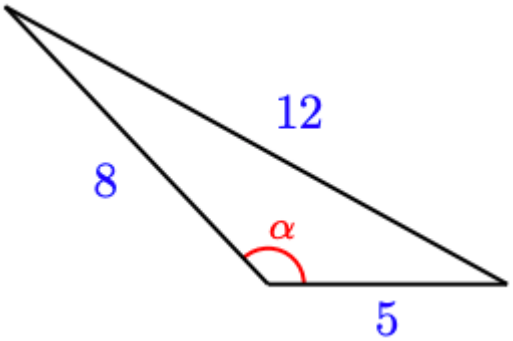


13.

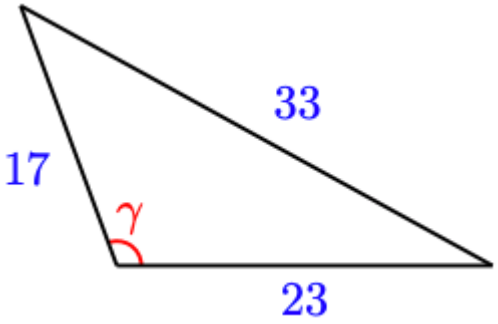


For the following exercises, find the indicated angle. Round to two decimal places.

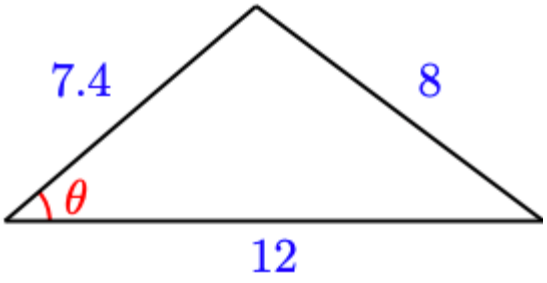
14.



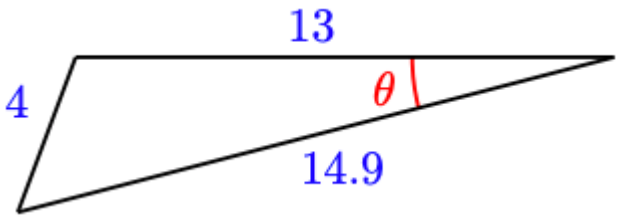
15.



16.



17.



For the following exercises, find the angles of the triangle. Round answers to two decimal places.

18. $a = 23, b = 14, c = 18$

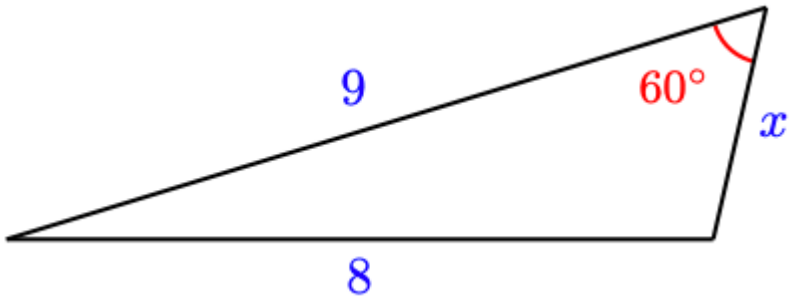
19. $a = 18, b = 25, c = 19$

20. $a = 16.3, b = 28.1, c = 19.4$

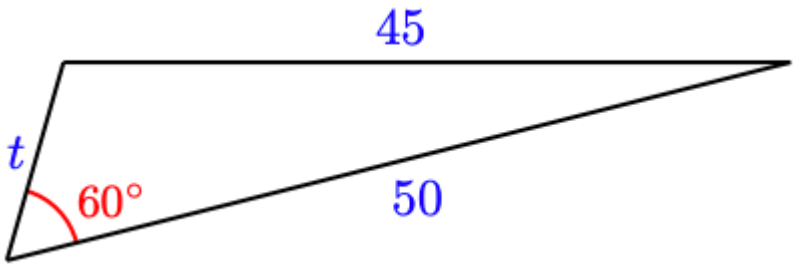
21. $a = 82.3, b = 22.5, c = 66.8$

For the following exercises, find the unknown side. Round your answers to two decimal places.

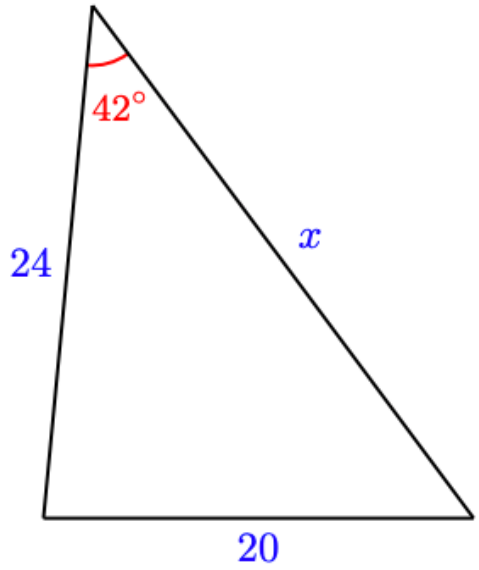
22.



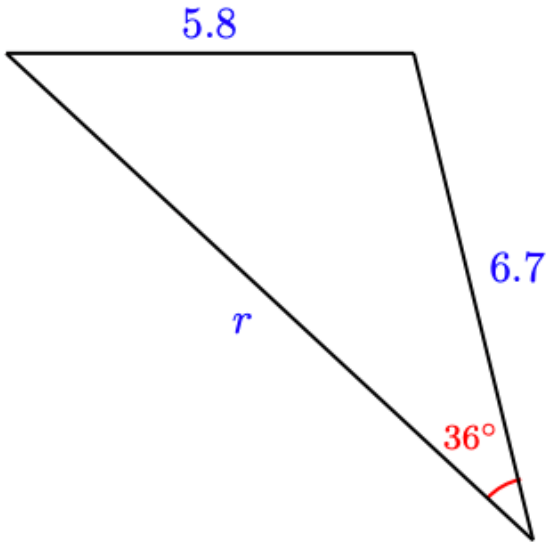
23.



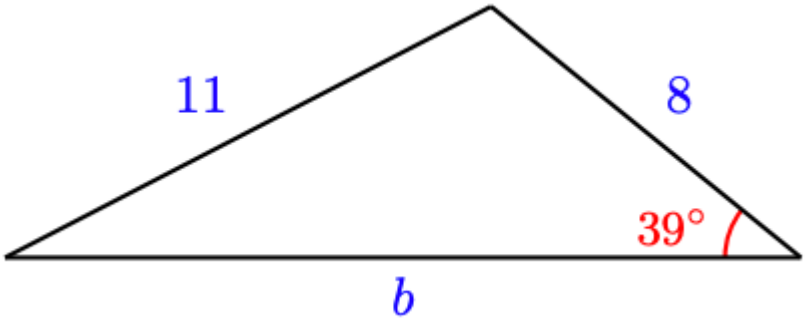
24.



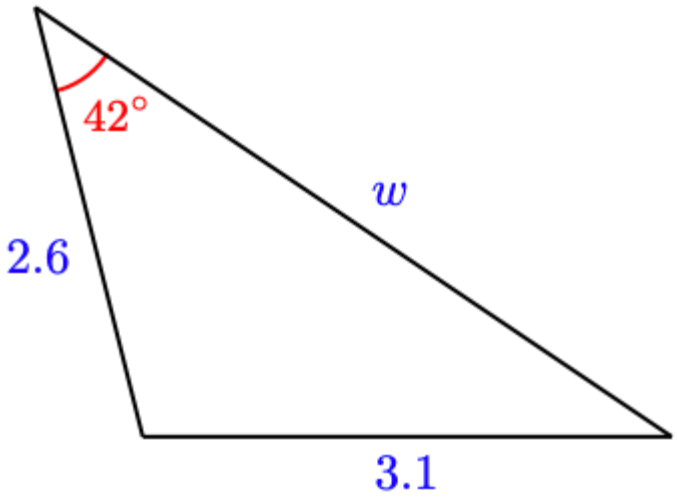
25.



26.

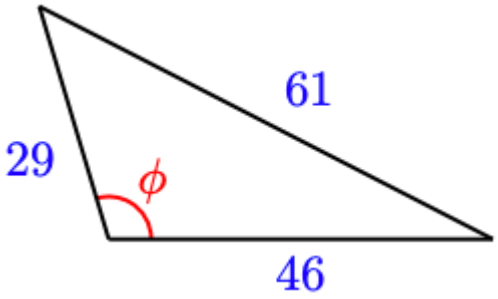


27.

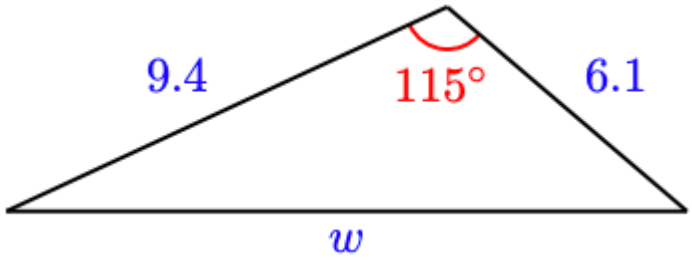


For the following exercises, which law should you use to find the labeled unknown value, the Law of Sines or the Law of Cosines? Write an equation you can solve to find the unknown value. For Problems 32 - 35, you may need two steps to find the unknown value.

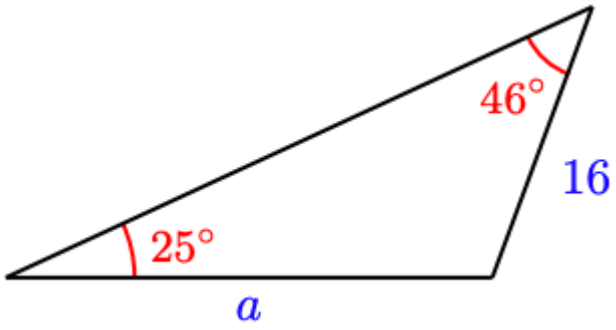
28.



29.



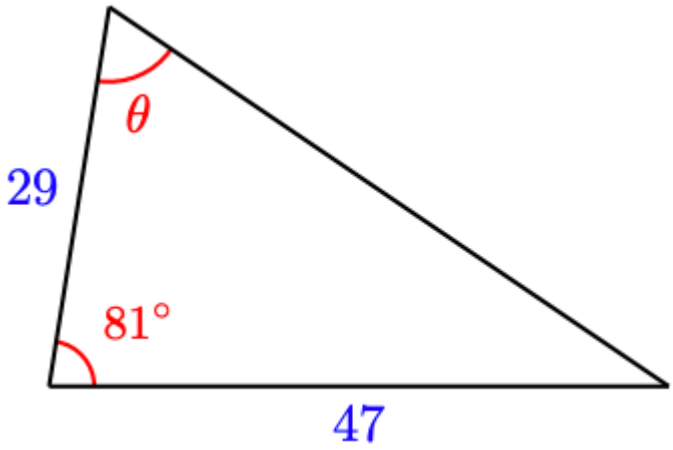
30.



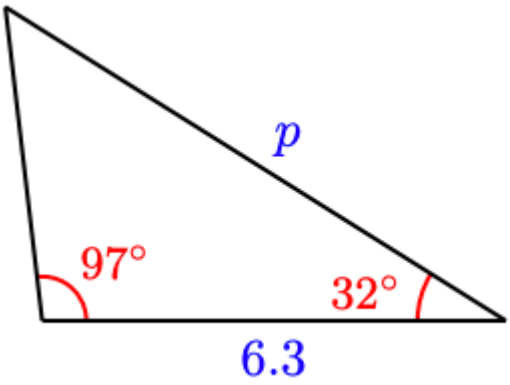
31.



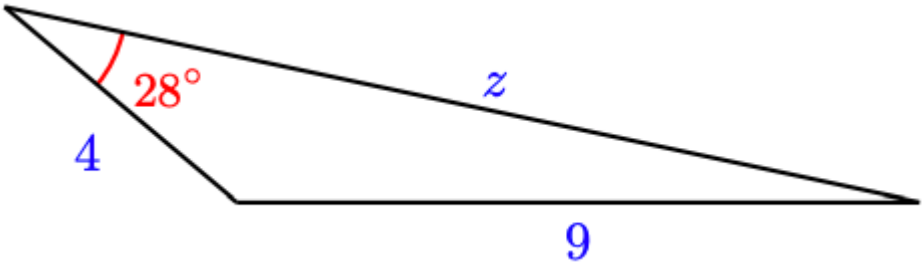
32.



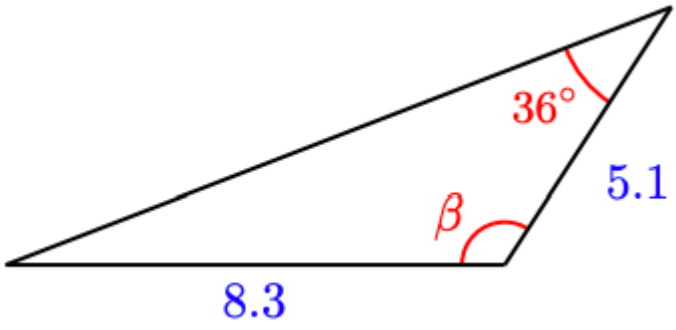
33.



34.



35.



For the following exercises,

(a) Sketch and label the triangle.
(b) Solve the triangle. Round answers to two decimal places.

36. $B = 47^\circ$, $a = 23$, $c = 17$

37. $C = 32^\circ$, $a = 14$, $b = 18$

38. $a = 8$, $b = 7$, $c = 9$

39. $a = 23$, $b = 34$, $c = 45$

40. $b = 72$, $c = 98$, $B = 38^\circ$

41. $a = 28$, $c = 41$, $A = 27^\circ$

42. $c = 5.7$, $A = 59^\circ$, $B = 82^\circ$

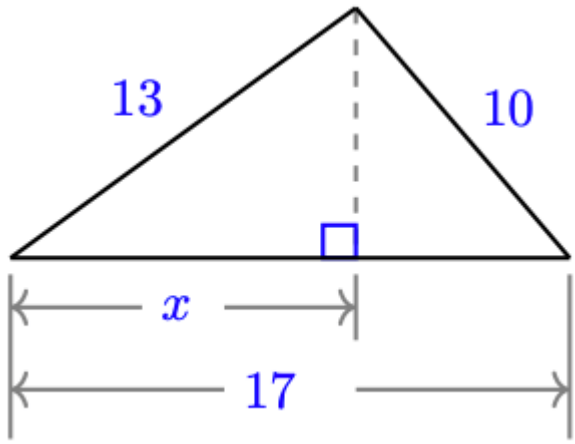
43. $b = 82$, $A = 11^\circ$, $C = 42^\circ$

Synthesis Questions

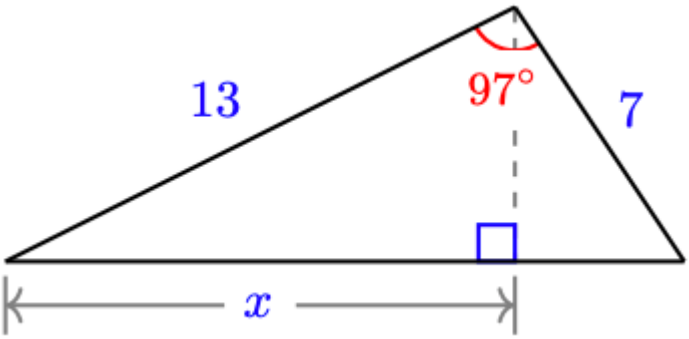
44. The sides of a parallelogram are 10 inches and 8 inches, and form an angle of 130°. Find the lengths of the diagonals of the parallelogram.

For the following exercises, find x, the distance from one vertex to the foot of the altitude.

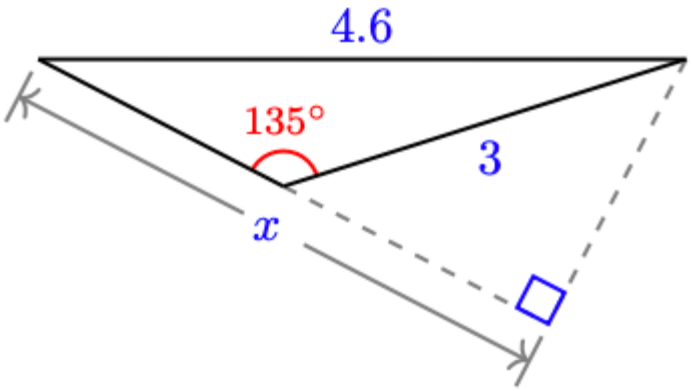
45.



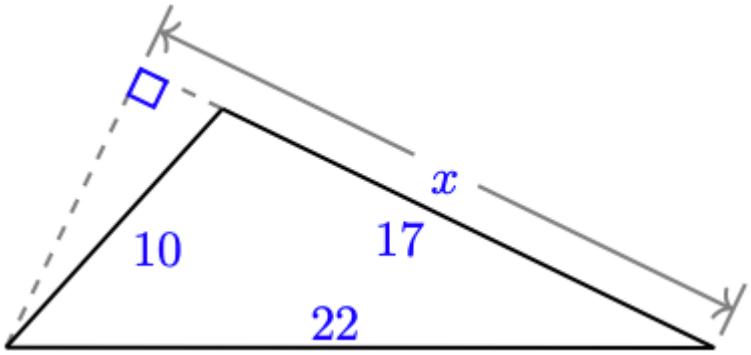
46.



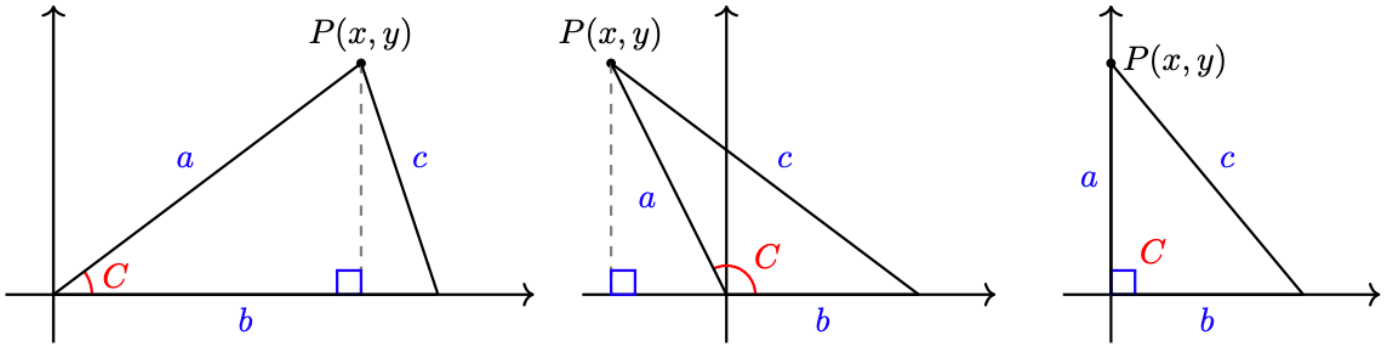
47.



48.



Proofs



49. a. Copy the three figures above showing the three possibilities for an angle C in a triangle: C is acute, obtuse, or a right angle. For each figure, explain why it is true that $c^2 = (b - x)^2 + y^2$, then rewrite the right side to get $c^2 = (x^2 + y^2) + b^2 - 2bx$.

b. For each figure, explain why it is true that $x^2 + y^2 = a^2$.

c. For all three figures, a is the distance from the origin to the point (x, y). Use the definition of cosine to write $\cos(C)$ in terms of a and x, then solve your equation for x.

d. Start with the last equation from (a), and substitute expressions from (b) and (c) to conclude one case of the Law of Cosines.

50. Demonstrate the other two cases of the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

and

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

(Hint: See the previous problem and switch the roles of a and c, etc.)

51. Use the Law of Cosines to prove the projection laws:

$$\begin{aligned} a &= b\cos(C) + c\cos(B) \\ b &= c\cos(A) + a\cos(C) \\ c &= a\cos(B) + b\cos(A) \end{aligned}$$

Illustrate with a sketch. (Hint: Add together two of the versions of the Law of Cosines.)

52. If $\triangle ABC$ is isosceles with $a = b$, show that $c^2 = 2a^2(1 - \cos(C))$.

53. Use the Law of Cosines to prove:

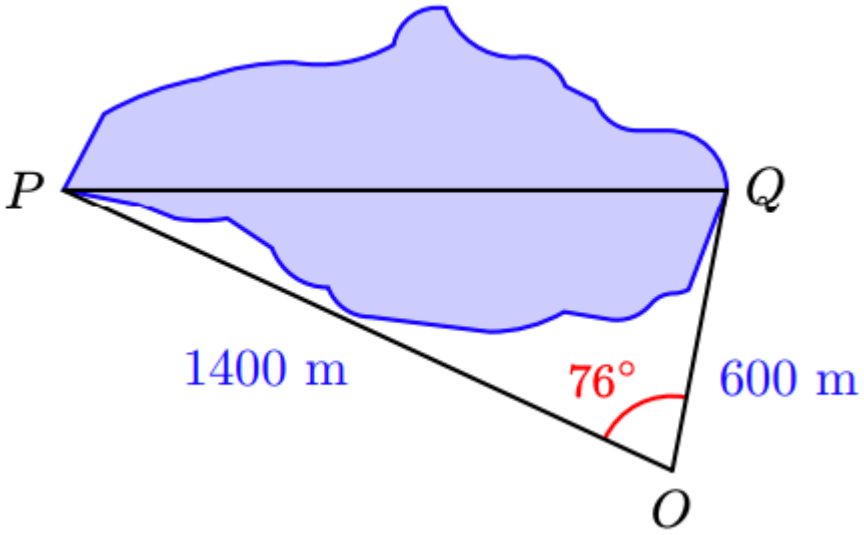
$$\begin{aligned} 1 + \cos(A) &= \frac{(a+b+c)(-a+b+c)}{2bc} \\ 1 - \cos(A) &= \frac{(a-b+c)(-a+b+c)}{2bc} \end{aligned}$$

54. Prove that

$$\frac{\cos(A)}{a} + \frac{\cos(B)}{b} + \frac{\cos(C)}{c} = \frac{a^2+b^2+c^2}{2abc}$$

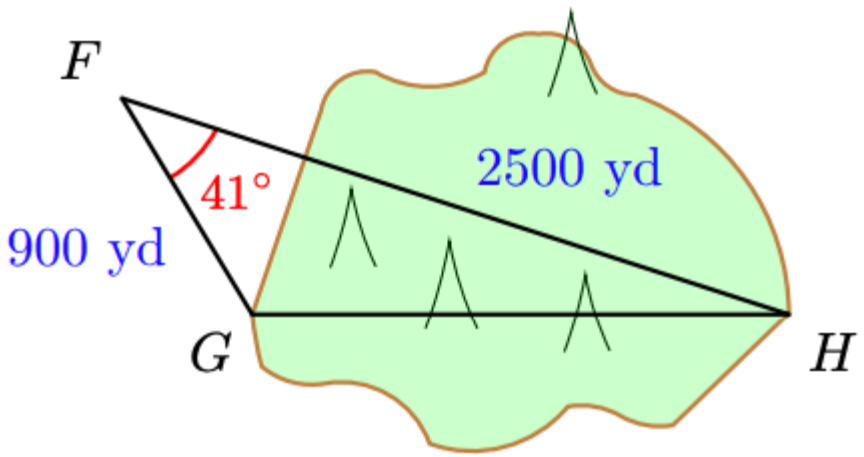
Applications

55. A surveyor would like to know the distance, PQ, across a small lake, as shown in the figure. She stands at point O and measures the angle between the lines of sight to points P and Q at 76°. She also finds $OP = 1400$ meters and $OQ = 600$ meters. Calculate the distance PQ.



56. A geologist wants to measure the diameter of a crater. From her camp, it is 4 miles to the northern-most point of the crater and 2 miles to the southern-most point. If the angle between the two lines of sight is 117°, what is the diameter of the crater? Round your answer to the nearest hundredth of a mile.

57. Highway engineers plan to drill a tunnel through Boney Mountain from G to H, as shown in the figure. The angle at point F is 41°, and the distances to G and H are 900 yards and 2500 yards, respectively. How long will the tunnel be?



58. From the Pedimaxus International Airport a tour helicopter can fly to Cliffs of Insanity Point by following a bearing of $\mathrm{N}\ 8.2^\circ\ \mathrm{E}$ for 192 miles and it can fly to Bigfoot Falls by following a bearing of $\mathrm{S}\ 68.5^\circ\ \mathrm{E}$ for 207 miles. Find the distance between Cliffs of Insanity Point and Bigfoot Falls. Round your answer to the nearest mile.

59. Cliffs of Insanity Point and Bigfoot Falls from Problem 58 above both lie on a straight stretch of the Great Sasquatch Canyon. What bearing would the tour helicopter need to follow to go directly from Bigfoot Falls to Cliffs of Insanity Point? Round your angle to the nearest tenth of a degree.

60. Two pilots leave an airport at the same time. One pilot flies at a heading of 3° and at a speed of 320 miles per hour, the other flies in the heading 157° at a speed of 406 miles per hour. How far apart are the two pilots after 3 hours? What is the heading from the first plane to the second plane at that time?

61. Two boats leave port at the same time. One boat sails due west at a speed of 17 miles per hour, the other moves in the heading 42° at a speed of 23 miles per hour. How far apart are the two boats after 2 hours? What is the heading from the first boat to the second boat at that time?

62. Linda wants to fly directly south from Indianapolis to Cancun, Mexico, a distance of 1290 miles. However, to avoid bad weather, she flies for 400 miles on a heading of 162°. What is the heading to Cancun from that location, and how far is it?

63. Lap sails 8 miles from Key West, Florida on a heading of 140°. He then changes course and sails for 10 miles due east. What is the heading back to Key West from that point, and how far is it?

64. A naturalist sets off on a hike from a lodge on a bearing of $\mathrm{S}\ 80^\circ\ \mathrm{W}$. After 1.5 miles, she changes her bearing to $\mathrm{S}\ 17^\circ\ \mathrm{W}$ and continues hiking for 3 miles. Find her distance from the lodge at this point. Round your answer to the nearest hundredth of a mile. What bearing should she follow to return to the lodge? Round your angle to the nearest degree.

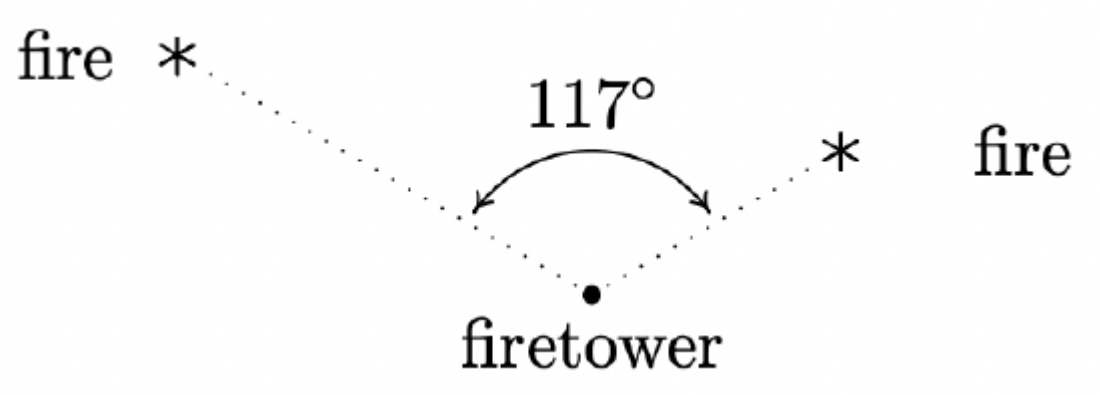
65. The HMS Sasquatch leaves port on a bearing of $\mathrm{N}\ 23^\circ\ \mathrm{E}$ and travels for 5 miles. It then changes course and follows a bearing of $\mathrm{S}\ 41^\circ\ \mathrm{E}$ for 2 miles. How far is it from port? Round your answer to the nearest hundredth of a mile. What is its bearing to port? Round your angle to the nearest degree.

66. The SS Bigfoot leaves a harbor bound for Nessie Island which is 300 miles away at a bearing of $\mathrm{N}\ 32^\circ\ \mathrm{E}$. A storm moves in and after 100 miles, the captain of the Bigfoot finds he has drifted off course. If his bearing to the harbor is now $\mathrm{S}\ 70^\circ\ \mathrm{W}$, how far is the SS Bigfoot from Nessie Island? Round your answer to the nearest hundredth of a mile. What course should the captain set to head to the island? Round your angle to the nearest tenth of a degree.

67. The phone company wants to erect a cell tower on a steep hill inclined 26° to the horizontal. The installation crew plans to run a guy wire from a point on the ground 20 feet uphill from the base of the tower and attach it to the tower at a height of 100 feet. How long should the guy wire be?

68. Sandstone Peak rises 3500 above the desert. The Park Service plans to run an aerial tramway up the north face, which is inclined at an angle of 68° to the horizontal. The base station will be located 500 feet from the foot of Sandstone Peak. Ignoring any slack in the cable, how long should it be?

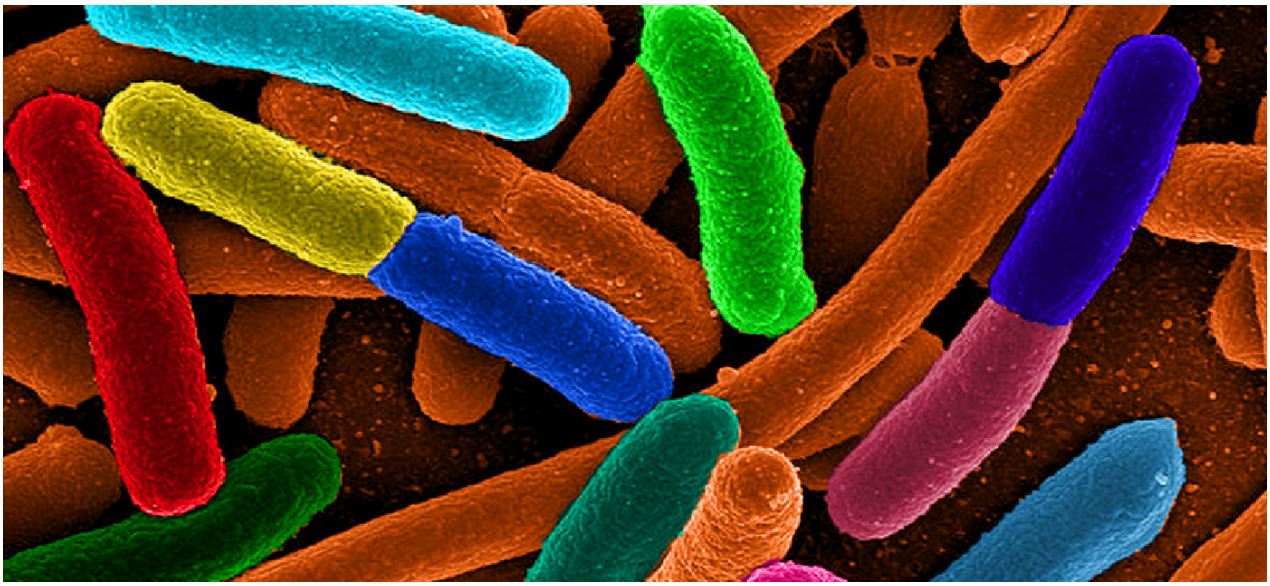
59. From a point 300 feet above level ground in a firetower, a ranger spots two fires in the Yeti National Forest. The angle of depression made by the line of sight from the ranger to the first fire is 2.5° and the angle of depression made by line of sight from the ranger to the second fire is 1.3°. The angle formed by the two lines of sight is 117°. Find the distance between the two fires. Round your answer to the nearest foot. (Hint: In order to use the 117° angle between the lines of sight, you will first need to use right angle Trigonometry to find the lengths of the lines of sight. This will allow you to apply the Law of Cosines.)



This page titled 11.2.2: Homework is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

12: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Electron micrograph of E.Coli bacteria (credit: "Mattosaurus," Wikimedia Commons)



Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years.

For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. The table below shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

12.1: Exponential Functions

12.1.1: Resources and Key Concepts

12.1.2: Homework

12.2: Graphs of Exponential Functions

12.2.1: Resources and Key Concepts

12.2.2: Homework

12.3: Logarithmic Functions

12.3.1: Resources and Key Concepts

12.3.2: Homework

12.4: Graphs of Logarithmic Functions

12.4.1: Resources and Key Concepts

12.4.2: Homework

12.5: Laws of Logarithms

12.5.1: Resources and Key Concepts

12.5.2: Homework

12.6: Exponential and Logarithmic Equations

12.6.1: Resources and Key Concepts

12.6.2: Homework

12.7: Exponential and Logarithmic Models

12.7.1: Resources and Key Concepts

12.7.2: Homework

This page titled 12: Exponential and Logarithmic Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by OpenStax via source content that was edited to the style and standards of the LibreTexts platform.

- 4: Exponential and Logarithmic Functions by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/precalculus.

12.1: Exponential Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

When the original iteration of this textbook was written (2013), India was the second most populous country in the world with a population of about 1.25 billion people. At that time, the population was growing at a rate of about 1.2% each year. If that rate continued, the population of India would have exceeded China's population by the year 2031. Interestingly, the rate of growth for India's population actually sped up! In 2025, India's population was 1.42 billion while China's was 1.41 billion. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at **exponential functions**, which model this kind of rapid growth.

Identifying Exponential Functions

When exploring linear growth back in Algebra, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x) = 3x + 4$, the slope tells us the output increases by 3 each time the input increases by 1. We call this the **Slope Addition Property** and it is unique to linear functions. If you never knew of this property from your old Algebra class, then it's best we codify it here.

Theorem: Slope Addition Property

For a linear function with slope m, every increase of 1 in input results in adding m to the function value. This is called the **slope addition property**.

The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, 2.5% of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continued, vegans would have made up 10% of the U.S. population in 2015, 40% in 2019, and 80% in 2021.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of exponential growth, let us contrast exponential growth with linear growth. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs. See Table $12.1.1$.

Table $12.1.1$

x	$f(x)=2^x$	$g(x)=2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12
7	128	14
8	256	16
9	512	18
10	1024	20

From Table $12.1.1$ we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** means the original value from the range increases by the *same percentage* over equal increments in the domain.
- **Linear growth** means the original value from the range increases by the *same amount* over equal increments in the domain (again, this is called the Slope Addition Property).

Apparently, the difference between "the same percentage" and "the same amount" is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. If we agree to call this multiplier the **base**, then this aspect that is unique to exponential functions is called the **Base Multiplier Property**.

Theorem: Base Multiplier Property

For an exponential function with base b, every increase of 1 in input results in multiplying the previous function value by b. This is called the **base multiplier property**.

So, how do we define exponential functions?

Definition: Exponential Function

An **exponential function** is a function with the form

$$f(x)=ab^x,$$

where a is a non-zero real number called the **initial value** and b is any positive real number, called the **base**, such that $b\neq 1$. This function is also referred to as the **exponential function base** b.

Let's look at the function $f(x)=2^x$ from our example. We will create a table (Table $12.1.2$) to determine the corresponding outputs over an interval in the domain from -3 to 3.

Table 12.1.2

x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	$2^{-3}=\frac{1}{8}$	$2^{-2}=\frac{1}{4}$	$2^{-1}=\frac{1}{2}$	$2^0=1$	$2^1=2$	$2^2=4$	$2^3=8$

Let us examine the graph of f by plotting the ordered pairs we observe on the table in Figure 12.1.1, and then make a few observations.

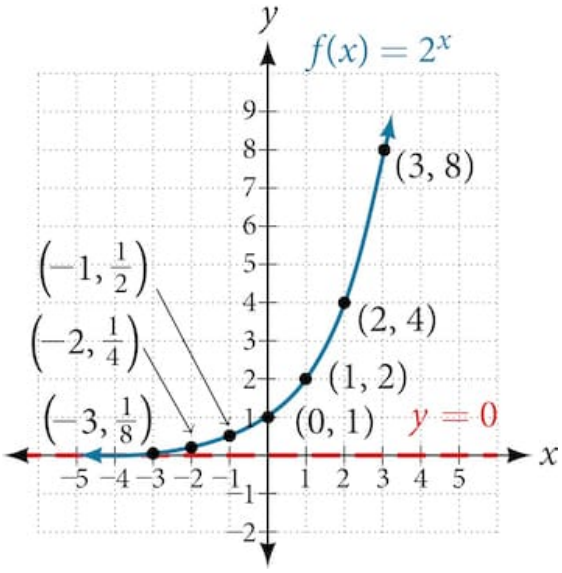


Figure 12.1.1

From the graph, it appears that the following are some key characteristics of $f(x)=2^x$:

- the domain is $(-\infty,\infty)$,
- the range is $(0,\infty)$,
- as $x\to\infty$, $f(x)\to\infty$,
- as $x\to-\infty$, $f(x)\to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the x-axis because base two raised to any exponent never has the result of zero,
- $y=0$ appears to be the horizontal asymptote, and
- the y-intercept is 1.

In fact, these characteristics are not unique to the exponential function base 2.

Theorem: Properties of the Exponential Function

Let $f(x)=ab^x$, with $a>0$, $b>0$, and $b\neq 1$:

- The domain of f is all real numbers.
- The range of f is all positive real numbers.
- The y-intercept is $(0,a)$.
- The graph of f includes the point $(1,ab)$.
- The horizontal asymptote is $y=0$.
- If $b>1$,
  - f is increasing,
  - as $x\to-\infty$, $f(x)\to 0$, and
  - as $x\to\infty$, $f(x)\to\infty$.
- If $0<b<1$,
  - $f(x)$ is decreasing,

- as $x \to -\infty$, $f(x) \to \infty$, and
- as $x \to \infty$, $f(x) \to 0$.

We state this theorem assuming $a > 0$ for sake of simplicity. If $a < 0$, then we can visualize the graphical change in f as a reflection about the x-axis. Moreover, we require $a \neq 0$. Otherwise, $f(x) = 0 \cdot b^x = 0$, and this is the horizontal line along the x-axis. Hence, it is not exponentially increasing nor decreasing.

The domain of an exponential function is all real numbers because we can evaluate b^{power} for any power we desire. The range is all positive real numbers because, as long as $b > 0$, $b^{\text{power}} > 0$.

If $f(x) = ab^x$, where $b \neq 0$, then $f(0) = a \cdot b^0 = a \cdot 1 = a$ (remember, we cannot determine the value of 0^0, so we cannot let $b = 0$). Furthermore, $f(1) = a \cdot b^1 = ab$. Thus, the points $(0, 1)$ and $(1, ab)$ are on the graph of the exponential function.

Example 12.1.1: Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution available at the end of this section.

Checkpoint 12.1.1

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1. Why do we limit the base b to positive values?

Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{1/2} = \sqrt{-9}$, which is not a real number. This should convince us that allowing b to be negative would cause too much of a headache. Therefore, it is convention to restrict the base of exponential functions to positive values only.

Why do we limit the base to positive values other than 1?

Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x. Since this is a constant, it is neither exponentially increasing nor exponentially decreasing. As such, we further restrict the base of exponential functions to not include 1.

To evaluate an exponential function with the form $f(x) = b^x$, we simply substitute x with the given value, and calculate the resulting power. For example, let $f(x) = 2^x$. What is $f(3)$?

$$\begin{aligned} & f(x) &= 2^x & \\ \implies & f(3) &= 2^3 & \text{(substituting)} \\ & &= 8 & \text{(simplifying)} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the Order of Operations. For example, let $f(x) = 30(2)^x$. What is $f(3)$?

$$\begin{aligned} & f(x) &=& 30(2)^x & \\ \implies & f(3) &=& 30(2)^3 & \text{(substituting)} \\ & &=& 30(8) & \text{(exponents before multiplication)} \\ & &=& 240 & \text{(multiplication)} \end{aligned}$$

Note that if the Order of Operations were not followed, the result would be incorrect:

$$f(3) = 30(2)^3 \neq 60^3 = 216,000.$$

Example 12.1.2: Evaluating Exponential Functions

Let $f(x) = 5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

Solution available at the end of this section.

Checkpoint 12.1.2

Let $f(x) = 8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term "exponential growth" is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

Definition: Exponential Growth and Decay

A function that models **exponential growth** or **exponential decay** grows or decays by a rate proportional to the amount present. For any real number t and any positive real numbers P_0 and b such that $b \neq 1$, an exponential growth or decay function has the form

$$P(t) = P_0\, b^t,$$

where

- P_0 is the **initial** or **starting value** of the function, and
- b is the **growth** or **decay factor**
  - if $b > 1$, it is a growth factor or **growth multiplier** per unit t
  - if $0 < b < 1$, it is a decay factor or **decay multiplier** per unit t.

Note that this definition mirrors the definition of an exponential function, but with f replaced by P, a replaced by P_0, and x replaced by t. The traditional notation introduced with the original definition (using $f(x)$ and a) only differs from this new notation in letters. Most applications involving exponential functions will use time (t) as the independent variable and are modeling populations (thus, the choice of function name, $P(t)$). Moreover, P_0 (read as, "P naught") stands for the initial population size (or the time-zero population size).

To differentiate between linear and exponential functions, let's consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A(x) = 100 + 50x$. Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x) = 100(1 + 0.5)^x$.

A few years of growth for these companies are illustrated in Table 12.1.3.

Table 12.1.3

Year, x	Stores, Company A	Stores, Company B
0	$100 + 50(0) = 100$	$100(1 + 0.5)^0 = 100$
1	$100 + 50(1) = 150$	$100(1 + 0.5)^1 = 150$
2	$100 + 50(2) = 200$	$100(1 + 0.5)^2 = 225$
3	$100 + 50(3) = 250$	$100(1 + 0.5)^3 = 337.5$
x	$A(x) = 100 + 50x$	$B(x) = 100(1 + 0.5)^x$

The graphs comparing the number of stores for each company over a five-year period are shown in Figure 12.1.2. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.

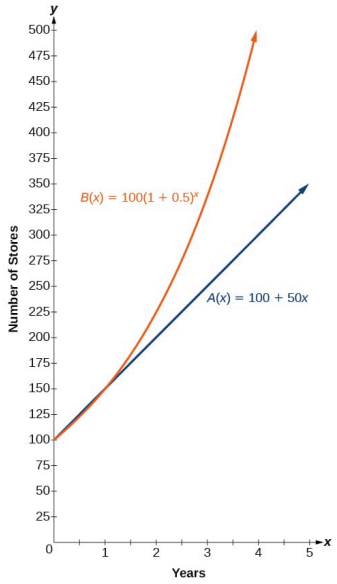


Figure 12.1.2**:** *The graph shows the numbers of stores Companies A and B opened over a five-year period.*

Notice that the *realistic* domain for both functions is $[0, \infty)$, and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Interactive Element: Exponential vs. Linear Growth

The following Interactive Element might help you visualize how exponential growth overtakes linear growth.

Interact: Drag the green point, which is Company B's number of stores at year 1, up and down.

Observation: Both companies start at 100 stores, but once Company B's growth factor rises above 1, its curve bends upward and overtakes Company A's straight line — pulling further ahead every year after.

Now we will turn our attention to the function representing the number of stores for Company B, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the **growth rate**, and $1 + 0.5 = 1.5$ represents the **growth factor**. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$, where 100 is the initial value, 1.5 is the base, and x is the exponent.

Example 12.1.3: Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, and using this model, what will the population of India be in 2031?

Solution available at the end of this section.

Checkpoint 12.1.3

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about 0.6%. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in Example 12.1.3?

Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a and b, and evaluate the function.

How to... find the equation of an exponential function given two points

1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

Example 12.1.4: Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an exponential function $N(t)$ representing the population (N) of deer over time t.

Solution available at the end of this section.

Checkpoint 12.1.4

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

Example 12.1.5: Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

Solution available at the end of this section.

Checkpoint 12.1.5

Given the two points $(1, 3)$ and $(2, 4.5)$, find the equation of the exponential function that passes through these two points.

Q&A

Do two points always determine a unique exponential function?

Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in x, which in many real world cases involves time.

Example 12.1.6: Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in Figure 12.1.5.

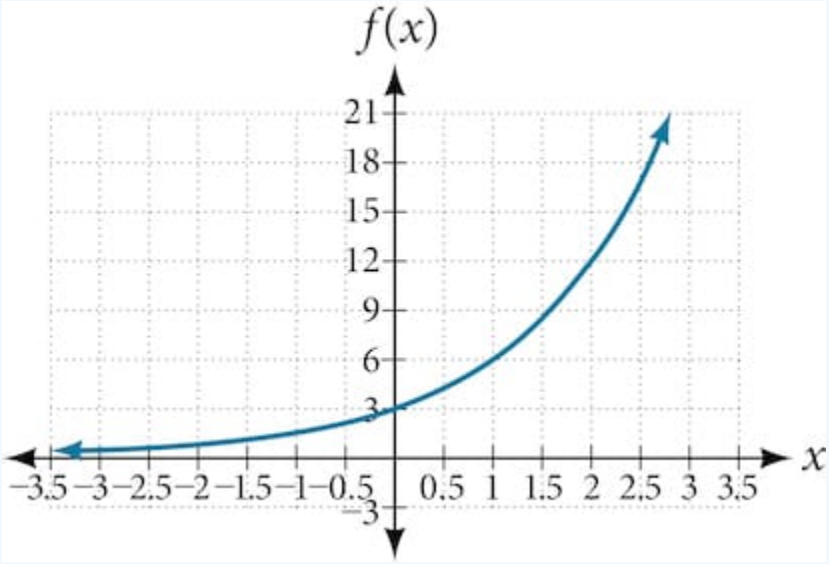


Figure 12.1.5

Solution available at the end of this section.

Checkpoint 12.1.6

Find an equation for the exponential function graphed in Figure 12.1.6.

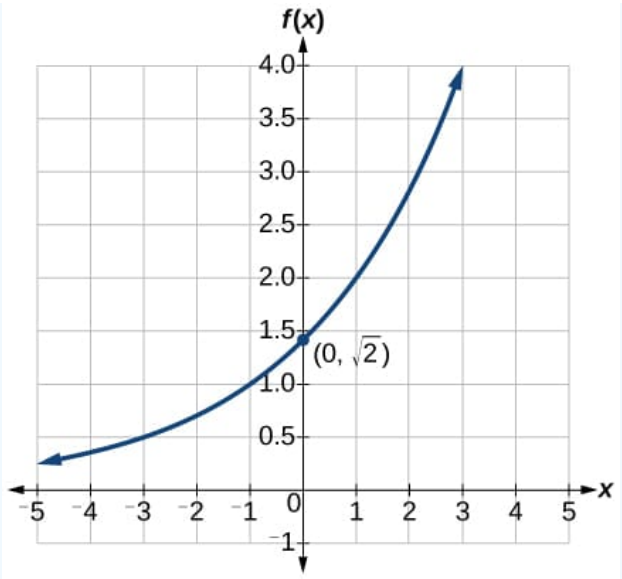


Figure 12.1.6

Compound Interest

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, earn interest that is compounded.

Definition: Compounding

Compounding refers to the process of earning interest not only on the initial investment (**principal**) but also on the accumulated interest itself.

To convey to the customer how much their money is earning annually (per year), investors cite the **annual percentage rate** (**APR**).

Definition: Annual Percentage Rate

The **annual percentage rate** (**APR**) of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account.

The term nominal is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the **effective interest rate** ends up being *greater* than the nominal rate! This is a powerful tool for investing.

Definition: Effective Rate

The **effective interest rate** (**EIR**), **effective annual interest rate**, **annual equivalent rate** (**AER**) or simply **effective rate** is the percentage of interest on a loan or financial product if compound interest accumulates in periods different than a year. Another, very popular, name for the effective rate is the **annual percentage yield** (**APY**).

Typical compounding periods (other than once per year) are:

- semiannual (twice per year, at six-month intervals)
- quarterly (once per three months)
- monthly

- daily

Other compounding periods exist (e.g., weekly), but they are not as common.

Developing a formula to predict the effects of compound interest on an investment takes a little bit of patience and insight. Let's assume we have $1,000 and invest it for one year at a bank that pays 10% interest.

If the bank is only paying **simple interest** (which means it is compounding annually), then at the end of the year we would have

$$1000 + 10\% \text{ of } 1000 = 1000 + 0.10 \times 1000 = 1000 + 100 = 1100.$$

Our money has grown from $1,000 to $1,100, earning us $100 in interest!

What would happen if the bank was paying 10% APR, but compounding semiannually? Since they are compounding twice per year, at the end of six months they pay 5% on what we invested. Then, at the end of the second six months, they pay another 5% *of what is now in our account*! That is, the second compounding period earns interest not only on what we invested, but also on the interest we received during the first six months.

$$\begin{aligned} \text{End of First Six Months:} \quad & 1000 + 5\% \text{ of } 1000 = 1000 + 0.05 \times 1000 = 1000 + 50 = 1050 \\ \text{End of Second Six Months:} \quad & 1050 + 5\% \text{ of } 1050 = 1050 + 0.05 \times 1050 = 1050 + 52.50 = 1102.50 \end{aligned}$$

We end the year with $1,102.50. We earned more interest than before!

Let's show one more case before stating the observation. In this case, let's assume that the bank compounds interest quarterly. Moreover, instead of simplifying our computations, we will use some Algebra to showcase a pattern.

$$
\begin{aligned}
\text{End of First Quarter:} \quad 1000 + \frac{10\%}{4} \text{ of } 1000 &= 1000 + \frac{0.10}{4} \times 1000 \\
&= 1000\left(1 + \frac{0.10}{4}\right) && \text{(factoring out 1000)} \\
&= P_1 && \text{(labeling this amount as } P_1\text{)} \\
\text{End of Second Quarter:} \quad P_1 + \frac{10\%}{4} \text{ of } P_1 &= P_1 + \frac{0.10}{4} \times P_1 && \text{(earning interest on the previous quarte} \\
&= P_1\left(1 + \frac{0.10}{4}\right) && \text{(factoring out } P_1\text{)} \\
&= 1000\left(1 + \frac{0.10}{4}\right)\left(1 + \frac{0.10}{4}\right) && \text{(substituting)} \\
&= 1000\left(1 + \frac{0.10}{4}\right)^2 && \text{(simplifying)} \\
&= P_2 && \text{(labeling this amount as } P_2\text{)} \\
\text{End of ThirdQuarter:} \quad P_2 + \frac{10\%}{4} \text{ of } P_2 &= P_2 + \frac{0.10}{4} \times P_2 && \text{(earning interest on the previous quarte} \\
&= P_2\left(1 + \frac{0.10}{4}\right) && \text{(factoring out } P_2\text{)} \\
&= 1000\left(1 + \frac{0.10}{4}\right)^2\left(1 + \frac{0.10}{4}\right) && \text{(substituting)} \\
&= 1000\left(1 + \frac{0.10}{4}\right)^3 && \text{(simplifying)} \\
&= P_3 && \text{(labeling this amount as } P_3\text{)} \\
\text{End of FourthQuarter:} \quad P_3 + \frac{10\%}{4} \text{ of } P_3 &= P_3 + \frac{0.10}{4} \times P_3 && \text{(earning interest on the previous quarte} \\
&= P_3\left(1 + \frac{0.10}{4}\right) && \text{(factoring out } P_3\text{)} \\
&= 1000\left(1 + \frac{0.10}{4}\right)^3\left(1 + \frac{0.10}{4}\right) && \text{(substituting)} \\
&= 1000\left(1 + \frac{0.10}{4}\right)^4 && \text{(simplifying)}
\end{aligned}
$$

We can grab a calculator to compute this final value to be approximately \$1,103.81. This is even more than our previous amount!

Now that we have taken our time, hopefully we can see a pattern. If we replace the interest rate (10%) with r, the number of compounding periods (4) with n, and the principal invested (\$1000) with P_0, we have the amount of money at the end of a single year as

$$A = P_0\left(1 + \frac{r}{n}\right)^n.$$

Allowing our money to sit over t years gives us the **compound interest formula**.

Theorem: Compound Interest Formula

The amount an investment of P_0 dollars is worth after t years earning an annual percentage rate of $r \times 100\%$ compounding n times per year is given by

$$A(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}.$$

P_0 is called the **principal**, or, more generally, the **present value**.

For example, observe Table 12.1.4, which shows the result of investing $1,000 at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Table 12.1.4

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

Example 12.1.7: Calculating Compound Interest

If we invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution available at the end of this section.

Checkpoint 12.1.7

An initial investment of $100,000 at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

Example 12.1.8: Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $40,000 over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Solution available at the end of this section.

Checkpoint 12.1.8

Refer to Example 12.1.8. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

Evaluating Functions with Base e

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. Table 12.1.5 shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of $1 invested at 100% interest for 1 year, compounded at various frequencies, listed in Table 12.1.5.

Table 12.1.5

Frequency	$A(n) = \left(1 + \frac{1}{n}\right)^n$	Value
Annually	$\left(1 + \frac{1}{1}\right)^1$	$2

Frequency	$A(n) = \left(1 + \frac{1}{n}\right)^n$	Value
Semiannually	$\left(1 + \frac{1}{2}\right)^2$	\$2.25
Quarterly	$\left(1 + \frac{1}{4}\right)^4$	\$2.441406
Monthly	$\left(1 + \frac{1}{12}\right)^{12}$	\$2.613035
Daily	$\left(1 + \frac{1}{365}\right)^{365}$	\$2.714567
Hourly	$\left(1 + \frac{1}{8760}\right)^{8760}$	\$2.718127
Once per minute	$\left(1 + \frac{1}{525600}\right)^{525600}$	\$2.718279
Once per second	$\left(1 + \frac{1}{31536000}\right)^{31536000}$	\$2.718282

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1 + \frac{1}{n}\right)^n$ approaches a number used so frequently in Mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating.

Definition: e

The letter e represents the irrational number

$$\left(1 + \frac{1}{n}\right)^n, \text{ as } n \text{ increases without bound.}$$

It is approximately 2.718281828459045. This is also known as **Euler's number**, **Euler's constant**, and the **natural exponential base**.

The letter e is used as a base for many real-world exponential models. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Example 12.1.9: Using a Calculator to Find Powers of e

Calculate $e^{3.14}$. Round to five decimal places.

Solution available at the end of this section.

Checkpoint 12.1.9

Use a calculator to find $e^{-0.5}$. Round to five decimal places.

Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, e is used as the base for exponential functions. Exponential models that use e as the base are called **continuous growth or decay models**. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

Definition: Continuous Growth/Decay

For all real numbers t, and all positive numbers a and r, **continuous exponential growth** (or, simply **continuous growth**) or **continuous exponential decay** (**continuous decay**) is represented by the formula

$$P(t) = P_0 e^{rt},$$

where r is called the **continuous growth** (or **decay**) **rate** per unit time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

Example 12.1.10: Calculating Continuous Growth

A person invested $1,000 in an account earning a nominal 10% per year compounded continuously. How much was in the account at the end of one year?

Solution available at the end of this section.

Checkpoint 12.1.10

A person invests $100,000 at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

Example 12.1.11: Calculating Continuous Decay

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Solution available at the end of this section.

Checkpoint 12.1.11

Using the data in Example 12.1.11, how much radon-222 will remain after one year?

Time-Sensitive Advice: Drop Deadline

Within the Los Rios Community College District, the last day you can drop a class is the Monday (or, sometimes, the Tuesday) of the thirteenth week of the semester. After this date, you will no longer be able to drop a class for that semester and the only grades you can receive in the classes you remain in are A, B, C, D, or F.

Will you still get a "W?"

Dropping any class between the third week and this final drop deadline will result in a "W" notation on your record for that class.

What does this mean?

Once you withdraw (W) or fail (D or F) a course three times within the Los Rios Community College District (CRC, SCC, FLC, or ARC), you are never allowed to take that course again at any of the colleges within the district. Thus, if you wanted to continue through the math sequence, you would need to take this course outside the Los Rios Community College District.

Success in Mathematics: Why Study Mathematics? (Part 3)

Matthew Jannsen graduated with a degree in mathematics in 2000. He went to law school and is now an attorney. He writes,

> "At this stage of my career I spend a majority of my time conducting legal research and drafting legal memoranda. Most people assume that mathematics would not be the most appropriate preparation for a person who makes their living

engaging in these activities. I would disagree. In fact, I would suggest that my mathematics background has been one of my greatest assets throughout my law school and legal careers and will continue to help me develop as a lawyer throughout my career. In order to succeed as a math major, I always had to engage in logical thought... I constantly rely on the logical thinking skills that I developed as a math major in executing my tasks as an attorney and I am grateful for my mathematics education."

Source: http://www.maa.org/careers/janssen.html

1. Describe, in detail, what you think it means to "think logically."
2. Is logical thought a vital skill in the career that you hope to have? Explain.

Answers

Section Preview

After all this time, we finally get to a chapter where we can assume that students have either never seen the material or have so little experience with it that it essentially needs to be built from the ground up. In this section, we cover the basics of exponential functions—essentially everything excpetion for graphing them (that is reserved for the next section).

Section Preview

- **Arithmetic**
  - Simplifying Fractions and Mixed Numbers: Example 4 reduces $\frac{180}{80}$ to $\frac{9}{4}$ before solving for the base.
  - Percents: Converting a stated rate to decimal form (e.g., $3\% \to 0.03$; a 50% increase $\to$ factor 1.5) is required throughout the growth and compound-interest examples (Examples 3, 7, 8).

Section Preview

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base e.

Example $12.1.1$: Identifying Exponential Functions

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base b of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

Example $12.1.2$: Evaluating Exponential Functions

Follow the Order of Operations. Be sure to pay attention to the parentheses.

$$\begin{aligned} & f(x) = 5(3)^{x+1} \\ \implies & f(2) = 5(3)^{2+1} && \text{(substituting)} \\ & = 5(3)^3 && \text{(simplifying)} \\ & = 5(27) && \text{(simplifying)} \\ & = 135 && \text{(simplifying)} \end{aligned}$$

Example $12.1.3$: Evaluating a Real-World Exponential Model

To estimate the population in 2031, we evaluate the model for $t = 18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549.$$

There will be about 1.549 billion people in India in the year 2031.

Example $12.1.4$: Writing an Exponential Model When the Initial Value Is Known

We let our independent variable t be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: $(0, 80)$ and $(6, 180)$. Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find b:

$$\begin{aligned} & N(t) = 80b^t \\ \implies & 180 = 80b^6 && \text{(substituting)} \\ \implies & \frac{9}{4} = b^6 && \text{(dividing both sides by 80)} \\ \implies & \left(\frac{9}{4}\right)^{1/6} = b && \left(\text{raising both sides to the } \tfrac{1}{6} \text{ power}\right) \\ \implies & b \approx 1.1447 && \text{(rounding to four decimal places)} \end{aligned}$$

NOTES: When we took the sixth root of both sides, we did not bother writing $b = \pm\left(\frac{9}{4}\right)^{1/6}$. This is because the base of an exponential function is never negative. Moreover, *do not round any intermediate calculations. Doing so leads to error propagation.*

The exponential model for the population of deer is $N(t) \approx 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs get large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in Figure $12.1.3$ passes through the initial points

given in the problem, $(0, 80)$ and $(6, 180)$. We can also see that the domain for the function is $[0, \infty)$, and the range for the function is $[80, \infty)$.

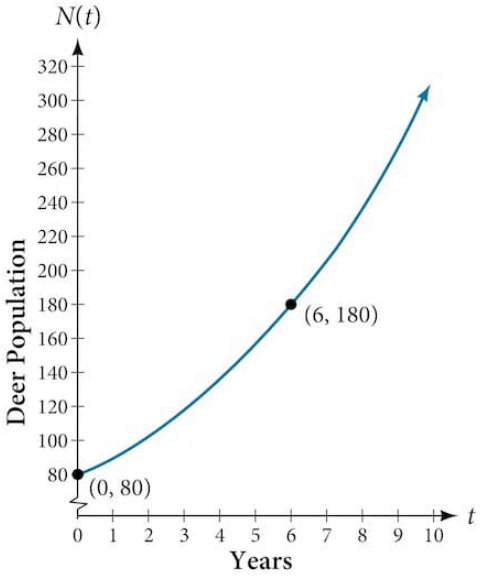


Figure 12.1.3**:** *Graph showing the population of deer over time,* $N(t) \approx 80(1.1447)^t$*, t years after 2006*

Example 12.1.5**: Writing an Exponential Model When the Initial Value is Not Known**

Because we don't have the initial value, we substitute both points into an equation of the form $f(x) = a\ b^x$, and then solve the system for a and b.

- Substituting $(-2, 6)$ gives $6 = a\ b^{-2}$
- Substituting $(2, 1)$ gives $1 = a\ b^2$

Use the first equation to solve for a in terms of b:

$$\begin{aligned} & 6 = a\,b^{-2} \\ \Longrightarrow\quad & 6b^2 = a \qquad \text{(multiplying both sides by } b^2\text{)} \end{aligned}$$

Substitute a in the second equation, and solve for b:

$$\begin{aligned} & 1 = a\,b^2 \\ \Longrightarrow\quad & 1 = 6b^2b^2 \qquad \text{(substituting } a = 6b^2\text{)} \\ \Longrightarrow\quad & 1 = 6b^4 \qquad \text{(Laws of Exponents)} \\ \Longrightarrow\quad & \frac{1}{6} = b^4 \qquad \text{(dividing both sides by 6)} \\ \Longrightarrow\quad & \left(\frac{1}{6}\right)^{1/4} = b \qquad \text{(raise both sides to the } \tfrac{1}{4} \text{ power)} \end{aligned}$$

Now that we have b, we go back to the first equation to find a:

$$a = 6b^2 = 6\left(\left(\frac{1}{6}\right)^{1/4}\right)^2 = 6\left(\frac{1}{6}\right)^{2/4} = 6\left(\frac{1}{6}\right)^{1/2} = \frac{6}{6^{1/2}} = 6^{1-1/2} = 6^{1/2} = \sqrt{6}.$$

Thus, the equation is

$$f(x) = \sqrt{6}\left(\left(\frac{1}{6}\right)^{1/4}\right)^x = \sqrt{6}\left(\frac{1}{6}\right)^{x/4}.$$

We can graph our model to check our work. Notice that the graph in Figure 12.1.4 passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

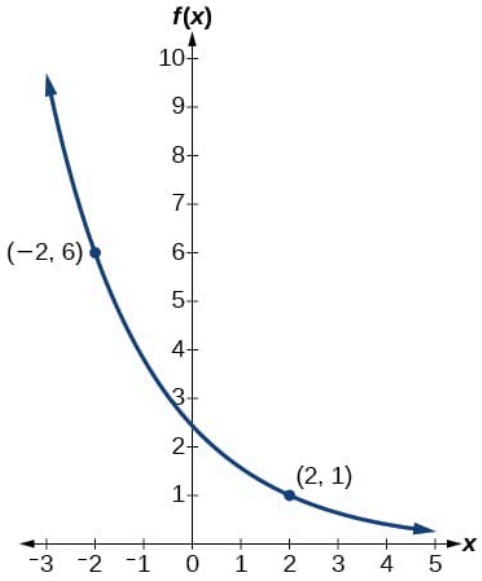


Figure 12.1.4**:** *The graph of* $f(x) = \sqrt{6}\left(\frac{1}{6}\right)^{x/4}$ *models exponential decay.*

Example 12.1.6**: Writing an Exponential Function Given Its Graph**

We can choose the y-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

$$\begin{aligned} y &= ab^x && \text{(writing the general form)} \\ \implies y &= 3b^x && \text{(substituting } a = 3\text{)} \\ \implies 12 &= 3b^2 && \text{(substituting } x = 2 \text{ and } y = 12\text{)} \\ \implies 4 &= b^2 && \text{(dividing both sides by 3)} \\ \implies \pm 2 &= b && \text{(Extraction of Roots)} \\ \implies 2 &= b && \text{(exponential bases are positive)} \end{aligned}$$

Substitute a and b into the standard form to yield the equation $f(x) = 3(2)^x$.

Example 12.1.7**: Calculating Compound Interest**

Because we are starting with $3,000, $P = 3000$. Our interest rate is 3%, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$.

$$\begin{aligned} A(t) &= P\left(1 + \frac{r}{n}\right)^{nt} \\ \implies A(10) &= 3000\left(1 + \frac{0.03}{4}\right)^{4 \cdot 10} && \text{(substituting known values)} \\ &\approx \$4,045.05 && \text{(rounding to two decimal places)} \end{aligned}$$

The account will be worth about $4,045.05 in 10 years.

Example 12.1.8**: Using the Compound Interest Formula to Solve for the Principal**

The nominal interest rate is 6%, so $r = 0.06$. Interest is compounded twice a year, so $n = 2$.

We want to find the initial investment, P, needed so that the value of the account will be worth $40,000 in 18 years. Substitute the given values into the compound interest formula, and solve for P.

$$\begin{aligned} A(t) &= P\left(1 + \frac{r}{n}\right)^{nt} \\ \implies 40,000 &= P\left(1 + \frac{0.06}{2}\right)^{2 \cdot 18} && \text{(substituting known values)} \\ \implies 40,000 &= P\,(1.03)^{36} && \text{(simplifying)} \\ \implies \frac{40,000}{1.03^{36}} &= P && \text{(isolating } P\text{)} \\ \implies P &\approx \$13,801 && \text{(rounding to the nearest dollar)} \end{aligned}$$

Lily will need to invest $13,801 to have $40,000 in 18 years.

Example 12.1.9**: Using a Calculator to Find Powers of** e

On a calculator, press the button labeled $[e^x]$. The window shows $e^($. Type 3.14 and then close parenthesis, $)$. Press $[\text{ENTER}]$. Rounding to five decimal places, $e^{3.14} \approx 23.10387$.

Caution: Many scientific calculators have an Exp button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

Example 12.1.10**: Calculating Continuous Growth**

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was $1,000, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

$$\begin{aligned} A(t) &= Pe^{rt} \\ &= 1000e^{0.1} && \text{(substituting known values)} \\ &\approx 1105.17 && \text{(rounding to the nearest hundredth)} \end{aligned}$$

The account is worth $1,105.17 after one year.

Example 12.1.11**: Calculating Continuous Decay**

Since the substance is decaying, the rate, 17.3%, is negative. So, $r = -0.173$. The initial amount of radon-222 was 100 mg, so $a = 100$. We use the continuous decay formula to find the value after $t = 3$ days:

$$\begin{aligned} A(t) &= ae^{rt} \\ &= 100e^{-0.173(3)} && \text{(substituting known values)} \\ &\approx 59.5115 && \text{(rounding to four decimal places)} \end{aligned}$$

So approximately 59.5115 mg of radon-222 will remain.

This page titled 12.1: Exponential Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

12.1.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Arithmetic**
  - Percents: Understanding how to convert percentages to decimals (e.g., 9% to 0.09) is crucial for working with growth/decay rates in exponential functions and compound interest formulas.
- **Systems**
  - Solving 2x2 Systems of Linear Equations: This skill is needed to find equations of exponential functions going through two given points.

Videos

- Exponential functions
  - Intermediate Algebra - Exponential Functions: The Concepts
  - Exponential Functions - A Brief Review of Exponential Functions
- Exponential growth and decay
  - Exponential Functions - Types of Exponential Growth and Decay
- Finding equation of exponential function
  - Exponential Functions - Finding Equations of Exponential Functions
- Compound interest formula
  - Exponential Functions - Applying the Compound Interest Formula
- Natural exponential base e
  - Exponential Functions - Investigating Continuous Growth
  - Exponential Functions - Calculus and the Natural Base
- Applications of exponential functions
  - Exponential Functions - Applications Involving Exponential Functions
  - Intermediate Algebra - Exponential Functions: Creating from Applications

Key Concepts

Definitions

- **Percent Change**: A change based on a percent of the original amount.
- **Exponential Growth**: An increase based on a constant multiplicative rate of change over equal increments of time, that is, a percent increase of the original amount over time.
- **Exponential Decay**: A decrease based on a constant multiplicative rate of change over equal increments of time, that is, a percent decrease of the original amount over time.
- **Exponential Function**: A function with the form $f(x) = ab^x$, where a is a non-zero real number called the initial value and b is any positive real number, called the base, such that $b \neq 1$.
- **Initial Value (of an exponential function)**: In the form $f(x) = ab^x$, a is the initial value (the value of the function when $x = 0$).
- **Base (of an exponential function)**: In the form $f(x) = ab^x$, b is the base. It must be a positive real number not equal to 1.
- **Exponential Growth/Decay Function (alternative form)**: A function that models exponential growth or decay can be written as $P(t) = P_0 b^t$, where P_0 is the initial value, and b is the growth/decay factor. If $b > 1$, it's a growth factor. If $0 < b < 1$, it's a decay factor.

- **Compounding**: The process of earning interest not only on the initial investment (principal) but also on the accumulated interest itself.
- **Annual Percentage Rate (APR) / Nominal Rate**: The yearly interest rate earned by an investment account before considering the effect of compounding frequency.
- **Effective Rate / Annual Percentage Yield (APY)**: The percentage of interest on a loan or financial product if compound interest accumulates in periods different than a year; it reflects the actual annual rate earned after compounding.
- **The number e (Euler's number / natural exponential base)**: An irrational number approximately equal to 2.71828, which arises as the limit of $(1+\frac{1}{n})^n$ as n increases without bound. It is used as the base for continuous growth/decay models.
- **Continuous Growth/Decay**: Exponential growth or decay represented by the formula $P(t) = P_0 e^{rt}$, where P_0 is the initial value, r is the continuous growth rate (if $r > 0$) or continuous decay rate (if $r < 0$) per unit time, and t is time.
- **Continuous Growth/Decay Rate**: The parameter r in the continuous growth/decay formula $P(t) = P_0 e^{rt}$.

Theorems

- **Slope Addition Property**: For a linear function with slope m, every increase of 1 in input results in adding m to the function value.
- **Base Multiplier Property**: For an exponential function with base b, every increase of 1 in input results in multiplying the previous function value by b.
- **Properties of the Exponential Function $f(x) = ab^x$, $a > 0$**:
  - Domain: All real numbers.
  - Range: All positive real numbers $(0, \infty)$.
  - y-intercept: $(0, a)$.
  - Includes point $(1, ab)$.
  - Horizontal asymptote: $y = 0$.
  - If $b > 1$, f is increasing; as $x \to -\infty, f(x) \to 0$; as $x \to \infty, f(x) \to \infty$.
  - If $0 < b < 1$, f is decreasing; as $x \to -\infty, f(x) \to \infty$; as $x \to \infty, f(x) \to 0$.
- **Compound Interest Formula**: The amount $A(t)$ an investment of P_0 dollars is worth after t years earning an annual percentage rate of $r \times 100\%$ compounding n times per year is given by $A(t) = P_0\left(1+\frac{r}{n}\right)^{nt}$.

Common Mistakes

- **Confusing Exponential Growth with Linear Growth**: Mistaking a situation with a constant additive change for exponential growth, or vice-versa. Exponential growth involves a constant multiplicative factor (percent change).
- **Incorrect Base for Exponential Functions**: Using a base $b \leq 0$ or $b = 1$ for an exponential function $f(x) = ab^x$. The base must be positive and not equal to 1.
- **Order of Operations Errors in Evaluation**: When evaluating exponential functions like $f(x) = ab^{x+c} + d$, performing operations in the wrong order (e.g., multiplying by a before evaluating the exponent).
- **Incorrectly Using the Continuous Compounding Formula**: Applying $A = P_0 e^{rt}$ when interest is compounded discretely (e.g., monthly, quarterly), or using the discrete compounding formula for continuous compounding.
- **Sign Errors with Decay Rates**: For continuous decay $P(t) = P_0 e^{rt}$, forgetting that the rate r must be negative.

This page titled 12.1.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

12.1.2: Homework

Reading Questions

1. Explain the difference between linear growth and exponential growth.
2. What is the general form of an exponential function? Define the "initial value" and the "base," including any restrictions on the base.
3. Describe the key characteristics of the graph of $f(x) = ab^x$ when $a > 0$ and $b > 1$ (domain, range, intercepts, asymptotes, increasing/decreasing behavior).
4. How does the graph of $f(x) = ab^x$ change if $0 < b < 1$ compared to when $b > 1$?
5. If you are given two points, how can you find the formula for an exponential function $f(x) = ab^x$ that passes through them? What if one of the points is the y-intercept $(0, a)$?
6. What is the compound interest formula? Define each variable in the formula.
7. What is the number e? Why is it significant in the context of exponential functions?
8. What is the formula for continuous exponential growth or decay? Define each variable.
9. If a population is decaying continuously at a rate of 5% per year, what would be the value of r in the formula $P(t) = P_0e^{rt}$?
10. What is the difference between the annual percentage rate (APR) and the annual percentage yield (APY)?

Homework

Verbal

11. Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.
12. Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.
13. The Oxford Dictionary defines the word **nominal** as a value that is "stated or expressed but not necessarily corresponding exactly to the real value." Develop a reasonable argument for why the term **nominal rate** is used to describe the annual percentage rate of an investment account that compounds interest.

Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.

14. The average annual population increase of a pack of wolves is 25.
15. A population of bacteria decreases by a factor of $\frac{1}{8}$ every 24 hours.
16. The value of a coin collection has increased by 3.25% annually over the last 20 years.
17. For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.
18. The height of a projectile at time t is represented by the function $h(t) = -4.9t^2 + 18t + 40$.

For the following exercises, consider this scenario: For each year t, the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)

19. Which forest's population is growing at a faster rate?
20. Which forest had a greater number of trees initially? By how many?
21. Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?
22. Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100 years? By how many?
23. Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.

24. $y = 300(1-t)^5$
25. $y = 220(1.06)^x$
26. $y = 16.5(1.025)^{1/x}$
27. $y = 11,701(0.97)^t$

For the following exercises, find the formula for an exponential function that passes through the two points given.

28. $(0, 6)$ and $(3, 750)$
29. $(0, 2000)$ and $(2, 20)$
30. $\left(-1, \frac{3}{2}\right)$ and $(3, 24)$
31. $(-2, 6)$ and $(3, 1)$
32. $(3, 1)$ and $(5, 4)$

For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

33.

x	1	2	3	4
$f(x)$	70	40	10	-20

34.

x	1	2	3	4
$h(x)$	70	49	34.3	24.01

35.

x	1	2	3	4
$m(x)$	80	61	42.9	25.61

36.

x	1	2	3	4
$f(x)$	10	20	40	80

37.

x	1	2	3	4
$g(x)$	-3.25	2	7.25	12.5

For the following exercises, use the compound interest formula, $A(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$.

38. After a certain number of years, the value of an investment account is represented by the equation $A = 10,250\left(1 + \frac{0.04}{12}\right)^{120}$. What is the value of the account?
39. What was the initial deposit made to the account in the previous exercise?
40. How many years had the account from the previous exercise been accumulating interest?
41. An account is opened with an initial deposit of \$6,500 and earns 3.6% interest compounded semi-annually. What will the account be worth in 20 years?
42. How much more would the account in the previous exercise have been worth if the interest were compounding weekly?
43. Solve the compound interest formula for the principal, P_0.

44. Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $\$14,472.74$ after earning 5.5% interest compounded monthly for 5 years. (Round to the nearest dollar.)
45. How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?
46. Use properties of rational exponents to solve the compound interest formula for the interest rate, r.
47. Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $9,000 and was worth $13,373.53 after 10 years.
48. Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $5,500, and was worth $38,455 after 30 years.

For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.

49. $y = 3742e^{0.75t}$
50. $y = 150e^{3.25/t}$
51. $y = 2.25e^{-2t}$
52. Suppose an investment account is opened with an initial deposit of $\$12,000$ earning 7.2% interest compounded continuously. How much will the account be worth after 30 years?
53. How much less would the account from the previous exercise be worth after 30 years if it were compounded monthly instead?

Numeric

For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.

54. $f(x) = 2(5)^x$, for $f(-3)$
55. $f(x) = -4^{2x+3}$, for $f(-1)$
56. $f(x) = e^x$, for $f(3)$
57. $f(x) = -2e^{x-1}$, for $f(-1)$
58. $f(x) = 2.7(4)^{-x+1} + 1.5$, for $f(-2)$
59. $f(x) = 1.2e^{2x} - 0.3$, for $f(3)$
60. $f(x) = -\frac{3}{2}(3)^{-x} + \frac{3}{2}$, for $f(2)$

Technology

For the following exercises, use graphing technology to find the equation of an exponential function given the points on the curve.

61. $(0, 3)$ and $(3, 375)$
62. $(3, 222.62)$ and $(10, 77.456)$
63. $(20, 29.495)$ and $(150, 730.89)$
64. $(5, 2.909)$ and $(13, 0.005)$
65. $(11, 310.035)$ and $(25, 356365.2)$

Extensions

66. The annual percentage yield (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $\text{APY} = \left(1 + \frac{r}{12}\right)^{12} - 1$.
67. Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds n times per year.
68. Recall that an exponential function is any equation written in the form $f(x) = a \cdot b^x$ such that a and b are positive numbers and $b \neq 1$. Any positive number b can be written as $b = e^n$ for some value of n. Use this fact to rewrite the formula for an exponential function that uses the number e as a base.
69. In an exponential decay function, the base of the exponent is a value between 0 and 1. Thus, for some number $b > 1$, the exponential decay function can be written as $f(x) = a \cdot \left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b = e^n$, to show that an exponential decay function takes the form $f(x) = ae^{-nx}$ for some positive number n.
70. The formula for the amount A in an investment account with a nominal interest rate r at any time t is given by $A(t) = P_0e^{rt}$, where P_0 is the amount of principal initially deposited into an account that compounds continuously. Prove that the

percentage of interest earned to principal at any time t can be calculated with the formula $I(t) = e^{rt} - 1$.

Real-World Applications

71. The fox population in a certain region has an annual growth rate of 9% per year. In the year 2012, there were 23,900 fox counted in the area. What is the fox population predicted to be in the year 2020?
72. A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?
73. In the year 1985, a house was valued at $110,000. By the year 2005, the value had appreciated to $145,000. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?
74. A car was valued at $38,000 in the year 2007. By 2013, the value had depreciated to $11,000. If the car's value continues to drop by the same percentage, what will it be worth by 2017?
75. Jamal wants to save $54,000 for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?
76. Kyoko has $10,000 that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $15,000 by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (Hint: solve the compound interest formula for the interest rate.)
77. Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was $13,500. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?
78. An investment account with an annual interest rate of 7% was opened with an initial deposit of $4,000. Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

This page titled 12.1.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

12.2: Graphs of Exponential Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events.

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table $12.2.1$ change as the input increases by 1.

Table $12.2.1$

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the **constant ratio**. In fact, for any exponential function with the form $f(x) = ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a. If this sounds familiar, it's because we gave this property a name in the previous section—the Base Multiplier Property.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

Figure $12.2.1$ shows the exponential growth function $f(x) = 2^x$.

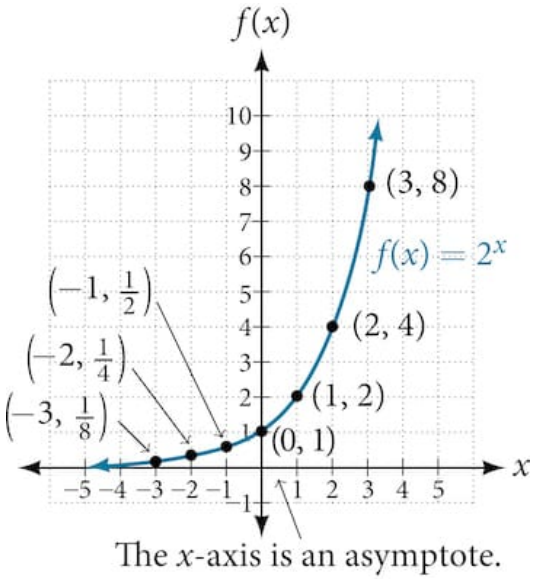


Figure 12.2.1**:** *Notice that the graph gets close to the x-axis, but never touches it.*

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table 12.2.2 change as the input increases by 1.

Table 12.2.2

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

Figure 12.2.2 shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

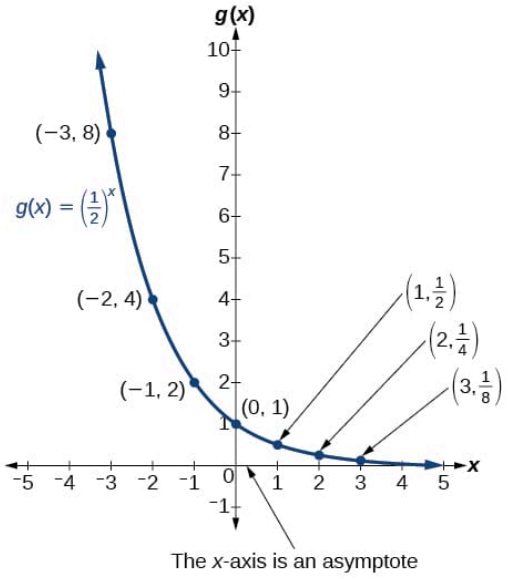


Figure 12.2.2

The domain of $g(x) = \left(\frac{1}{2}\right)^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

Interactive Element: Growth, Decay, and the Base

The following Interactive Element might help you visualize how the base b alone decides whether $f(x) = b^x$ grows or decays.

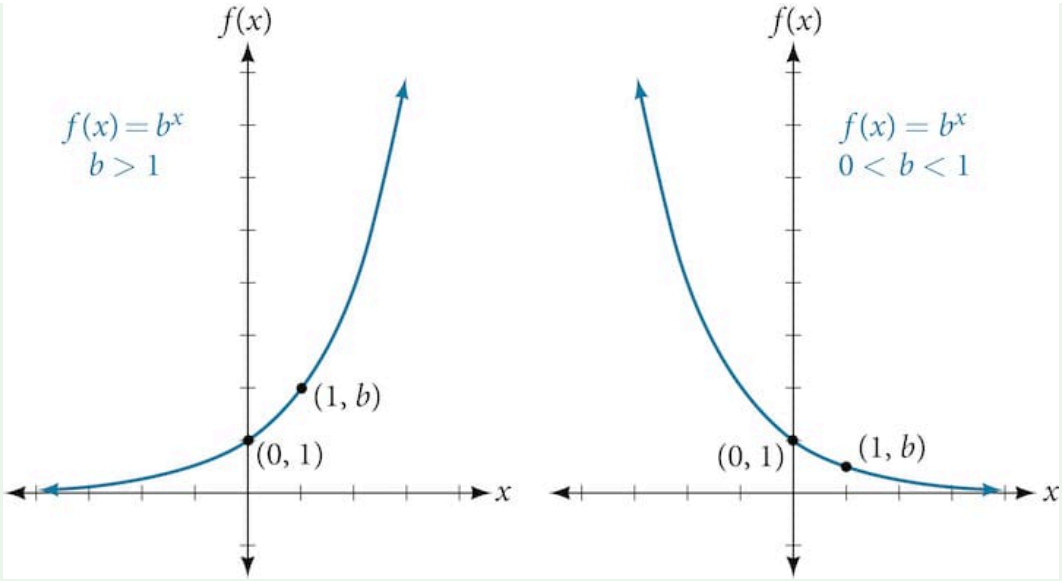
Interact: Drag the red point up and down — its height is the base b.

Observation: When $b > 1$ the curve grows; when $0 < b < 1$ it decays; at $b = 1$ it flattens to $y = 1$. The point $(0, 1)$ never moves.

Theorem: Properties of the Graphs of Exponential Functions

An exponential function with the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$, has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x-intercept: none
- y-intercept: $(0, 1)$
- Secondary key point: $(1, b)$
- increasing if $b > 1$
- decreasing if $0 < b < 1$



Example 12.2.1: Sketching the Graph of an Exponential Function of the Form $f(x) = b^x$

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 12.2.1

Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other toolkit functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the function $f(x) = b^x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in Figure 12.2.4 .

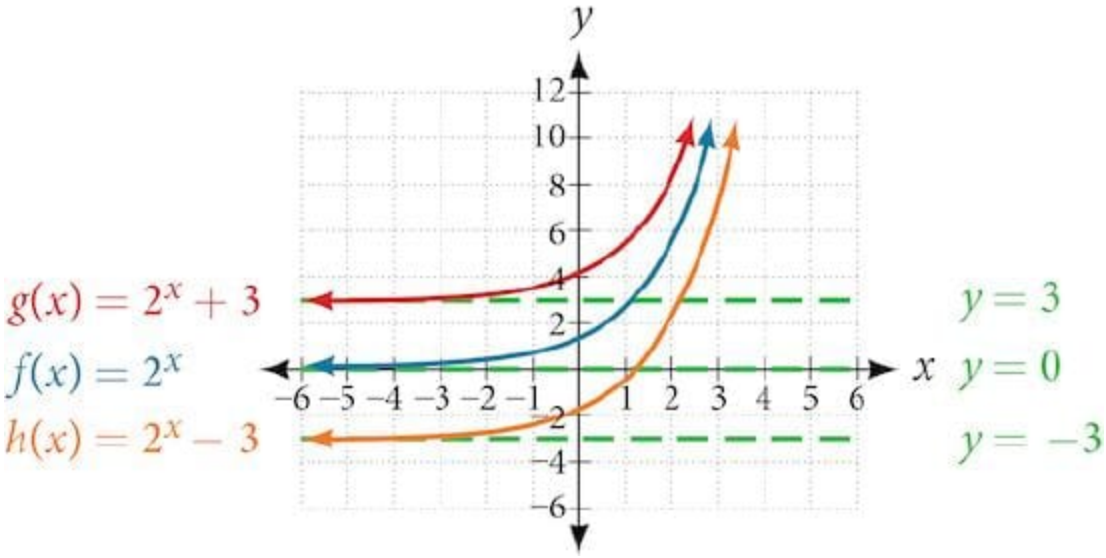


Figure 12.2.4

Interactive Element: Vertical Shift and the Asymptote

The following Interactive Element might help you visualize how a vertical shift d moves both the curve and its horizontal asymptote.

Interact: Drag the red point on the dashed line up and down to change d.

Observation: The curve and its horizontal asymptote $y = d$ always move together, so the range is (d, ∞).

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The y-intercept shifts up 3 units to $(0, 4)$.
  - The asymptote shifts up 3 units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The y-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. Both horizontal shifts are shown in Figure 12.2.5.

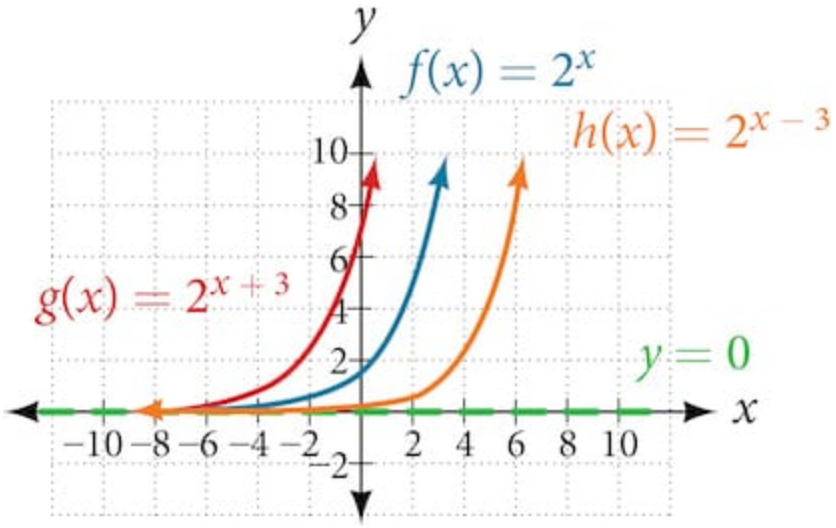


Figure 12.2.5

Interactive Element: Horizontal Shift

The following Interactive Element might help you visualize why shifting an exponential sideways also changes its y-intercept.

Interact: Drag the red point (it rides the curve at height 1) left and right.

Observation: The curve slides sideways and the asymptote stays at $y = 0$, but the orange y-intercept moves — a shift of c acts like multiplying by 2^c.

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$, remains unchanged.
- The asymptote, $y = 0$, remains unchanged.
- The y-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$, the y-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)2^x$, so the initial value of the function is 8.
  - When the function is shifted right 3 units to $h(x) = 2^{x-3}$, the y-intercept becomes $(0, \frac{1}{8})$. Again, see that $2^{x-3} = \left(\frac{1}{8}\right)2^x$, so the initial value of the function is $\frac{1}{8}$.

Example 12.2.2: Graphing a Shift of an Exponential Function

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 12.2.2

Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

Example 12.2.3: Approximating the Solution of an Exponential Equation

Solve $42 = 1.2(5)^x + 2.8$ using graphing technology. Round to the nearest thousandth.

Solution available at the end of this section.

Checkpoint 12.2.3

Solve $4 = 7.85(1.15)^x - 2.27$ graphically. Round to the nearest thousandth.

Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a stretch or compression occurs when we multiply the function $f(x) = b^x$ by a constant $|a| > 0$. For example, if we begin by graphing the function $f(x) = 2^x$, we can then graph the stretch, using $a = 3$, to get $g(x) = 3(2)^x$ as shown on the left in Figure 12.2.7, and the compression, using $a = \frac{1}{3}$, to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure 12.2.7.

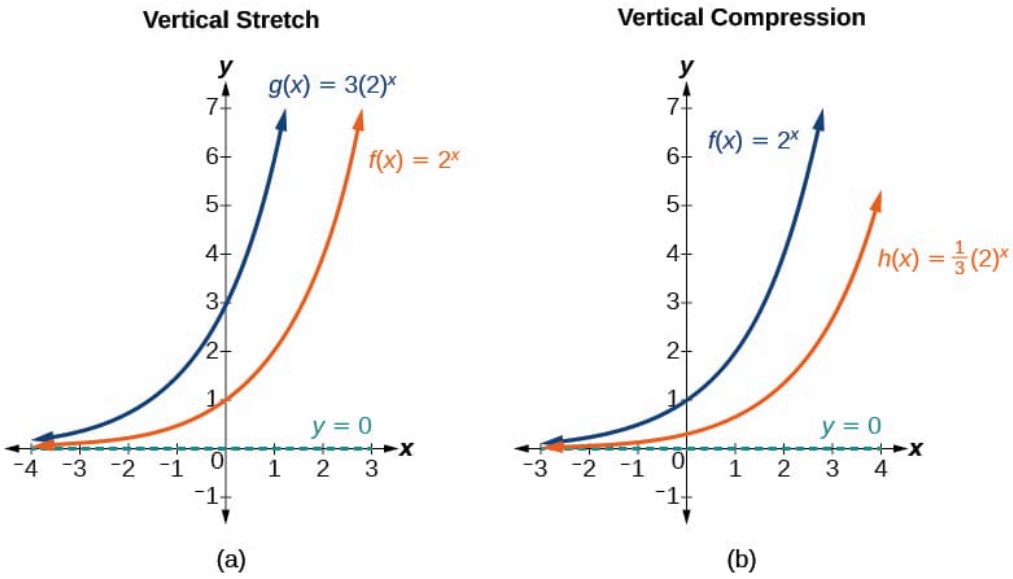


Figure 12.2.7: *(a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.*

Interactive Element: The Leading Coefficient a

The following Interactive Element might help you visualize how the coefficient a in $f(x) = a\ b^x$ stretches, compresses, or flips the graph.

Interact: Drag the red y-intercept point up and down — its height is a.

Observation: The curve always crosses at $(0, a)$; $|a| > 1$ stretches, $0 < |a| < 1$ compresses, and $a < 0$ reflects it about the x -axis.

Example 12.2.4: Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 12.2.4

Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the x-axis or the y-axis. When we multiply the function $f(x) = b^x$ by -1, we get a reflection about the x-axis. When we multiply the input by -1, we get a reflection about the y-axis. For example, if we begin by graphing the function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection about the x-axis, $g(x) = -2^x$, is shown on the left side of Figure 12.2.9, and the reflection about the y-axis $h(x) = 2^{-x}$, is shown on the right side of Figure 12.2.9.

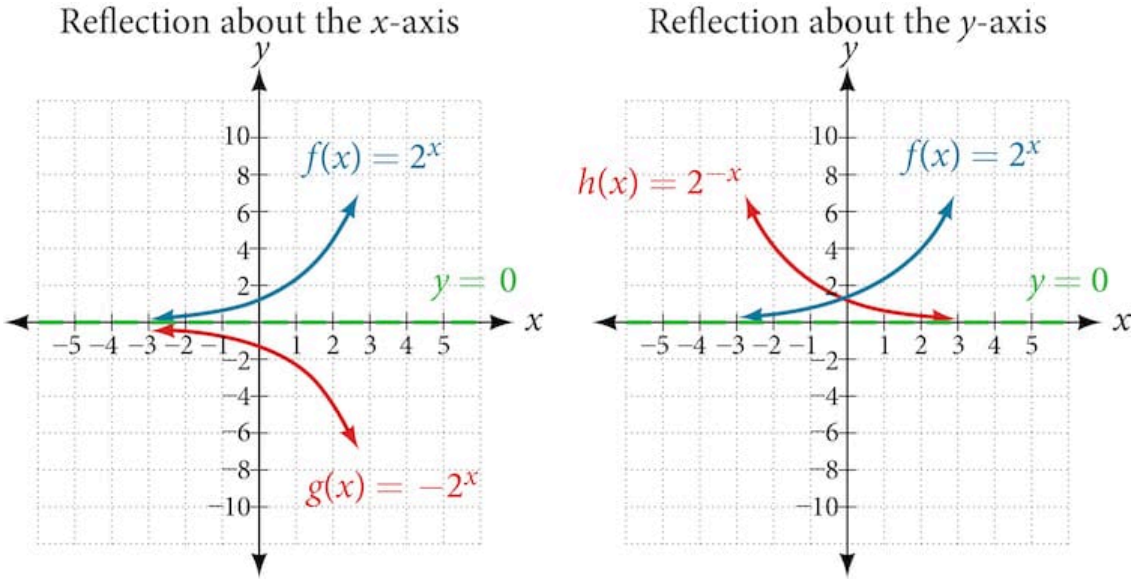


Figure 12.2.9: *(a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $h(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the y-axis.*

Example 12.2.5: Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = \left(\frac{1}{4}\right)^x$ about the x-axis. State its domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 12.2.5

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the y-axis. State its domain, range, and asymptote.

Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in Table 12.2.6 to arrive at the general equation for translating exponential functions.

Table $12.2.6$***: Transformations of the Parent Function*** $f(x) = b^x$

Transformation	Form
Shift • Horizontally c units to the left (if $c > 0$) • Vertically d units up (if $d > 0$)	$f(x) = b^{x+c} + d$
Stretch and Compress • Stretch if $\|a\| > 1$ • Compression if $0 < \|a\| < 1$	$f(x) = ab^x$
Reflect about the x-axis	$f(x) = -b^x$
Reflect about the y-axis	$f(x) = b^{-x} = \left(\frac{1}{b}\right)^x$
General equation for all transformations	$f(x) = ab^{x+c} + d$

Example $12.2.6$: Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the y-axis, and then shifted up 4 units.

Solution available at the end of this section.

Checkpoint $12.2.6$

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the x-axis and then shifted down 2 units.

Success in College: Dealing With Failure (Part 2)

A student gets a test back from the professor. The test score is 55%. The professor has written at the top of the exam, "Make an appointment to talk to me about this."

1. Why do you think that the professor wants to talk to the student?
2. Making an appointment to talk to your professor is usually an option in college. You can choose not to do this. Describe the reasons someone may use for making the appointment. Describe the reasons someone may use for not making the appointment.

Miscellaneous Topics: Thinking About Mathematics (Part 3)

When I would get frustrated in school, I often found it good to reflect upon some of my favorite teachers and think about how they would approach the subject.

- Professor Simpson, Cosumnes River College

1. Who was your favorite teacher, and what subject did they teach?
2. Can you make any suggestions I can incorporate into the classroom that worked for that teacher?

Answers

Section Preview

This is a standard treatment of the graphs of exponential functions. Give some extra time for the base e exponential as it is incredibly important in Calculus.

Section Preview

- Graph exponential functions.
- Graph exponential functions using transformations.

Example $12.2.1$**: Sketching the Graph of an Exponential Function of the Form** $f(x) = b^x$

Before graphing, identify the behavior and create a table of points for the graph.

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in Table $12.2.3$.

Table $12.2.3$

x	-3	-2	-1	0	1	2	3
$f(x) = 0.25^x$	64	16	4	1	0.25	0.0625	0.015625

- Plot the y-intercept, $(0, 1)$, along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in Figure $12.2.3$.

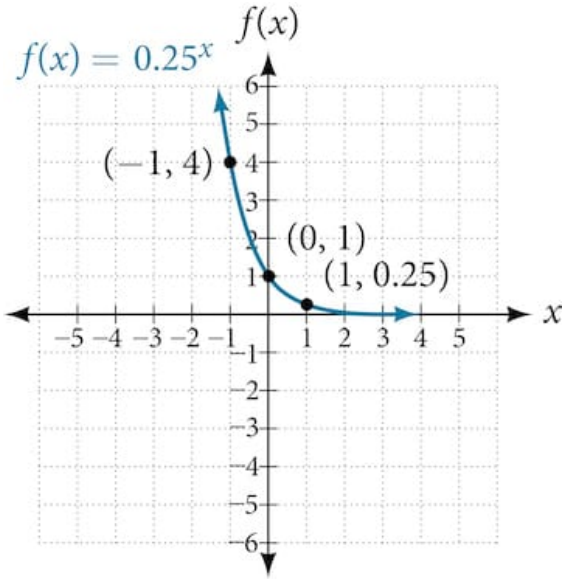


Figure $12.2.3$

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Example $12.2.2$**: Graphing a Shift of an Exponential Function**

We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d = -3$.

- Draw the horizontal asymptote $y = d$, so draw $y = -3$.
- Identify the shift as $(-c, d)$, so the shift is $(-1, -3)$.
- Shift the graph of $f(x) = b^x$ left 1 unit and down 3 units.

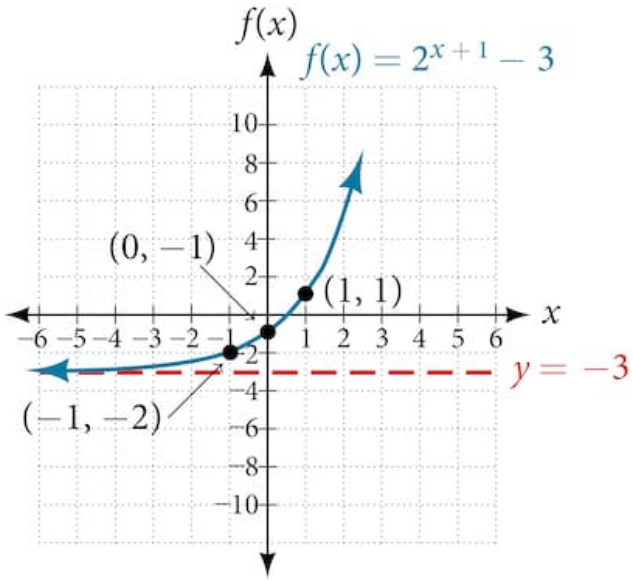


Figure 12.2.6

The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

Example 12.2.3**: Approximating the Solution of an Exponential Equation**

If you use Desmos, then you would go to the graphing calculator and type in $f(x) = 1.2(5)^x + 2.8$. On the next line, type $y = 42$. For a window, use the values -3 to 3 for x and -5 to 55 for y. If you click where the two graphs intersect, Desmos will display the point $(2.1662, 42)$. Therefore, the solution, which is the value of x where these graphs intersect, is approximately 2.166.

Example 12.2.4**: Graphing the Stretch of an Exponential Function**

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the x-axis as x increases.
- Since $a = 4$, the graph of $f(x) = \left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.
- Create a table of points as shown in Table 12.2.4.

Table 12.2.4

x	-3	-2	-1	0	1	2	3
$f(x) = 4\left(\frac{1}{2}\right)^x$	32	16	8	4	2	1	0.5

- Plot the y-intercept, $(0, 4)$, along with two other points. We can use $(-1, 8)$ and $(1, 2)$.

Draw a smooth curve connecting the points, as shown in Figure 12.2.8.

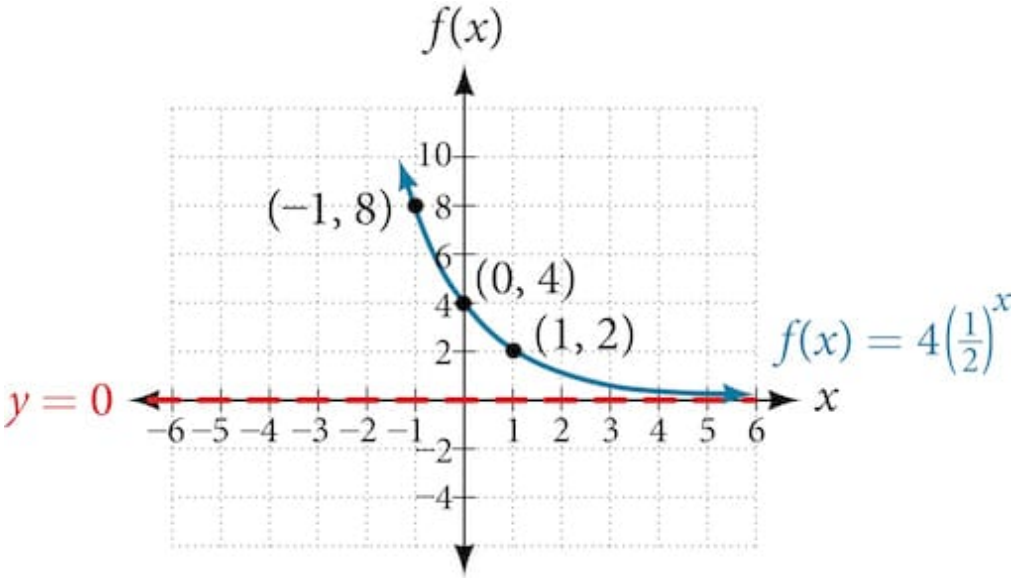


Figure 12.2.8

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Example 12.2.5**: Writing and Graphing the Reflection of an Exponential Function**

Since we want to reflect the parent function $f(x) = \left(\frac{1}{4}\right)^x$ about the x-axis, we multiply $f(x)$ by -1 to get, $g(x) = -\left(\frac{1}{4}\right)^x$. Next we create a table of points as in Table 12.2.5.

Table 12.2.5

x	-3	-2	-1	0	1	2	3
$g(x) = -\left(\frac{1}{4}\right)^x$	-64	-16	-4	-1	-0.25	-0.0625	-0.0156

Plot the y-intercept, $(0, -1)$, along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

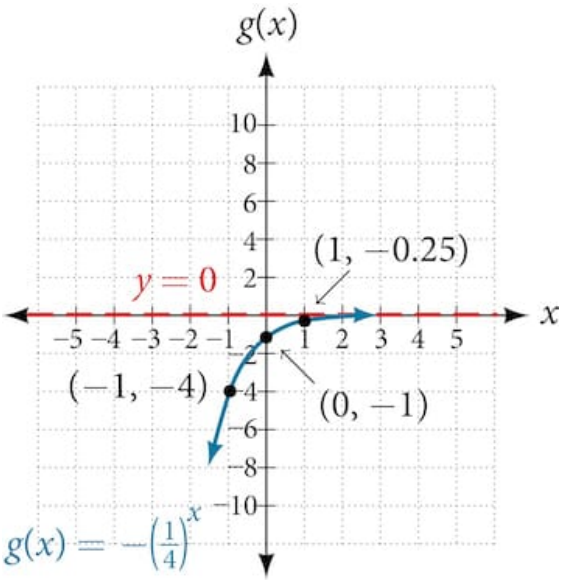


Figure 12.2.10

The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

Example 12.2.6**: Writing a Function from a Description**

We want to find an equation of the general form $f(x) = ab^{x+c} + d$. We use the description provided to find a, b, c, and d.

- We are given the parent function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of 2, so $a = 2$.
- The function is reflected about the y-axis. We replace x with $-x$ to get: e^{-x}.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,

$$f(x) = ab^{x+c} + d = 2e^{-x+0} + 4 = 2e^{-x} + 4.$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

This page titled 12.2: Graphs of Exponential Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

12.2.1: Resources and Key Concepts

Resources

Videos

- Graphs of exponential functions
  - Graphs of Exponential Functions - Graphing Exponential Functions
  - Exponential Functions - Advanced Topics in Exponential Functions
  - Intermediate Algebra - Exponential Functions: Graphs of Exponential Functions

Key Concepts

Definitions

- **Constant Ratio (for exponential functions)**: For an exponential function $f(x) = ab^x$, the base b is the constant ratio by which the output is multiplied when the input increases by 1.

Theorems

- **Properties of the Graphs of Exponential Functions** $f(x) = b^x$ **(where** $b > 0, b \neq 1$**)**:
  - One-to-one function.
  - Horizontal asymptote: $y = 0$.
  - Domain: $(-\infty, \infty)$.
  - Range: $(0, \infty)$.
  - x-intercept: none.
  - y-intercept: $(0, 1)$.
  - Increasing if $b > 1$.
  - Decreasing if $0 < b < 1$.

Common Mistakes

- **Incorrect Order of Transformations**: Applying shifts, reflections, and stretches/compressions in the wrong order, which can lead to an incorrect final graph. For example, applying a vertical shift before a vertical stretch will result in the shift also being stretched.
- **Errors in Horizontal Shifts**: Confusing the direction of horizontal shifts (e.g., $f(x + c)$ shifts left, $f(x - c)$ shifts right for $c > 0$).
- **Misinterpreting Stretches vs. Shifts**: Confusing the effect of multiplying the function/input (stretch/compression) with adding to the function/input (shift).
- **Incorrectly Identifying the Asymptote after Transformations**: Forgetting that vertical shifts change the horizontal asymptote, while horizontal shifts, reflections, and vertical stretches/compressions (by a non-zero factor) do not change the location of the horizontal asymptote $y = 0$ unless there's also a vertical shift.
- **Sign Errors with Reflections**: Incorrectly applying reflections; $-f(x)$ reflects across the x-axis, and $f(-x)$ reflects across the y-axis.
- **Errors in Graphing Key Points after Transformation**: Incorrectly transforming key points like the y-intercept or the point $(1, b)$ from the parent function.

This page titled 12.2.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

12.2.2: Homework

Reading Questions

1. What is the "constant ratio" for an exponential function of the form $f(x) = ab^x$, and what does it signify about the function's values as the input increases by 1?
2. Describe the domain, range, and horizontal asymptote of the basic exponential function $f(x) = b^x$ where $b > 0$ and $b \neq 1$.
3. How does a vertical shift by a constant d (i.e., $g(x) = b^x + d$) affect the domain, range, and horizontal asymptote of $f(x) = b^x$?
4. How does a horizontal shift by a constant c (i.e., $g(x) = b^{x+c}$) affect the domain, range, and horizontal asymptote of $f(x) = b^x$?
5. Explain how the graph of $g(x) = a \cdot b^x$ is related to the graph of $f(x) = b^x$ if $|a| > 1$. What if $0 < |a| < 1$?
6. How is the graph of $g(x) = -b^x$ related to the graph of $f(x) = b^x$?
7. How is the graph of $g(x) = b^{-x}$ related to the graph of $f(x) = b^x$?
8. When graphing $f(x) = ab^{x+c} + d$, what is a general order in which transformations can be applied to the parent graph $y = b^x$?
9. If the graph of $f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the y-axis, and then shifted up 4 units, what is the equation of the new function?
10. What is the y-intercept of the function $f(x) = 4(\frac{1}{2})^x$?
11. What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?
12. What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

Homework

Algebraic

13. The graph of $f(x) = 3^x$ is reflected about the y-axis and stretched vertically by a factor of 4. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.
14. The graph of $f(x) = \left(\frac{1}{2}\right)^{-x}$ is reflected about the y-axis and compressed vertically by a factor of $\frac{1}{5}$. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.
15. The graph of $f(x) = 10^x$ is reflected about the x-axis and shifted upward 7 units. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.
16. The graph of $f(x) = (1.68)^x$ is shifted right 3 units, stretched vertically by a factor of 2, reflected about the x-axis, and then shifted downward 3 units. What is the equation of the new function, $g(x)$? State its y-intercept (to the nearest thousandth), domain, and range.
17. The graph of $f(x) = -\frac{1}{2}\left(\frac{1}{4}\right)^{x-2} + 4$ is shifted downward 4 units, and then shifted left 2 units, stretched vertically by a factor of 4, and reflected about the x-axis. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

Graphical

For the following exercises, graph the function and its reflection about the y-axis on the same axes, and give the y-intercept.

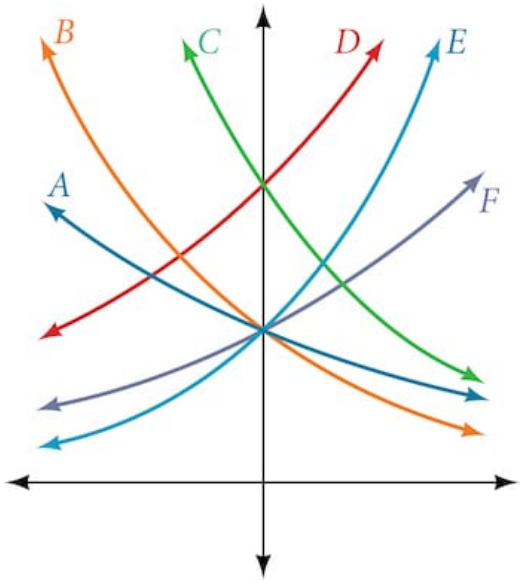
18. $f(x) = 3\left(\frac{1}{2}\right)^x$
19. $g(x) = -2(0.25)^x$
20. $h(x) = 6(1.75)^{-x}$

For the following exercises, graph each set of functions on the same axes.

21. $f(x) = 3\left(\frac{1}{4}\right)^x$, $g(x) = 3(2)^x$, and $h(x) = 3(4)^x$
22. $f(x) = \frac{1}{4}(3)^x$, $g(x) = 2(3)^x$, and $h(x) = 4(3)^x$

For the following exercises, match each function with one of the graphs in Figure $12.2.2.1$.

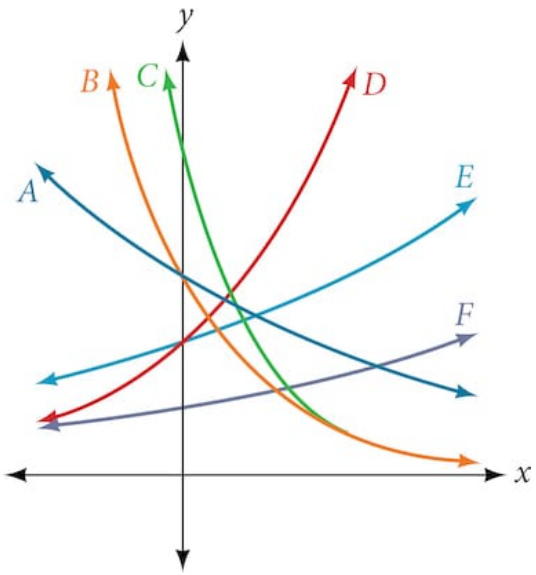
Figure 12.2.2.1



23. $f(x) = 2(0.69)^x$
24. $f(x) = 2(1.28)^x$
25. $f(x) = 2(0.81)^x$
26. $f(x) = 4(1.28)^x$
27. $f(x) = 2(1.59)^x$
28. $f(x) = 4(0.69)^x$

For the following exercises, use the graphs shown in Figure 12.2.2.2**. All have the form** $f(x) = ab^x$**.**

Figure 12.2.2.2



29. Which graph has the largest value for b?
30. Which graph has the smallest value for b?
31. Which graph has the largest value for a?
32. Which graph has the smallest value for a?

For the following exercises, graph the function and its reflection about the x**-axis on the same axes.**

33. $f(x) = \frac{1}{2}(4)^x$
34. $f(x) = 3(0.75)^x - 1$
35. $f(x) = -4(2)^x + 2$

For the following exercises, graph the transformation of $f(x) = 2^x$. Give the horizontal asymptote, the domain, and the range.

36. $f(x) = 2^{-x}$
37. $h(x) = 2^x + 3$
38. $f(x) = 2^{x-2}$

For the following exercises, describe the end behavior of the graphs of the functions.

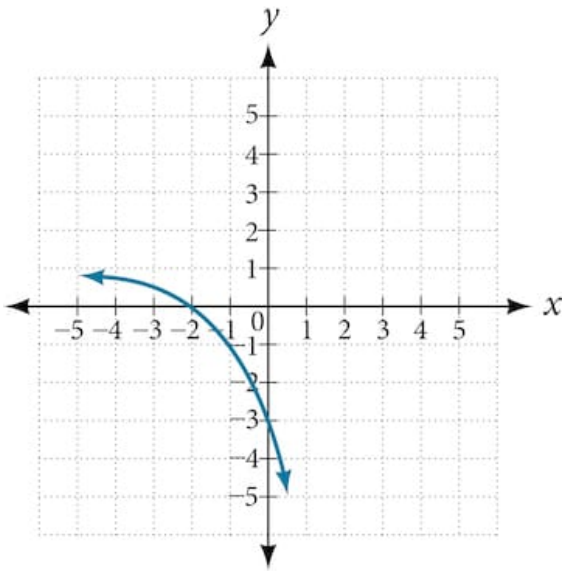
39. $f(x) = -5(4)^x - 1$
40. $f(x) = 3\left(\frac{1}{2}\right)^x - 2$
41. $f(x) = 3(4)^{-x} + 2$

For the following exercises, start with the graph of $f(x) = 4^x$. Then write a function that results from the given transformation.

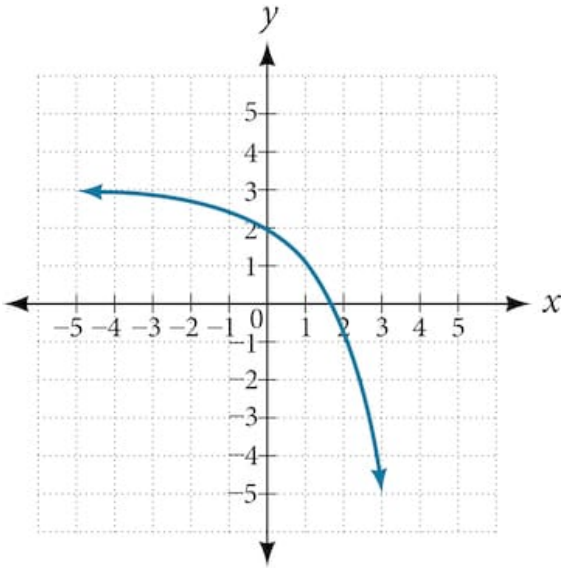
42. Shift $f(x)$ 4 units upward
43. Shift $f(x)$ 3 units downward
44. Shift $f(x)$ 2 units left
45. Shift $f(x)$ 5 units right
46. Reflect $f(x)$ about the x-axis
47. Reflect $f(x)$ about the y-axis

For the following exercises, each graph is a transformation of $y = 2^x$. Write an equation describing the transformation.

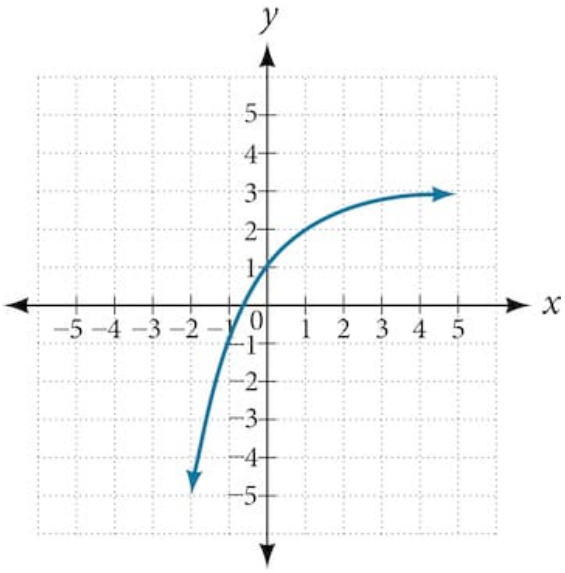
48.



49.



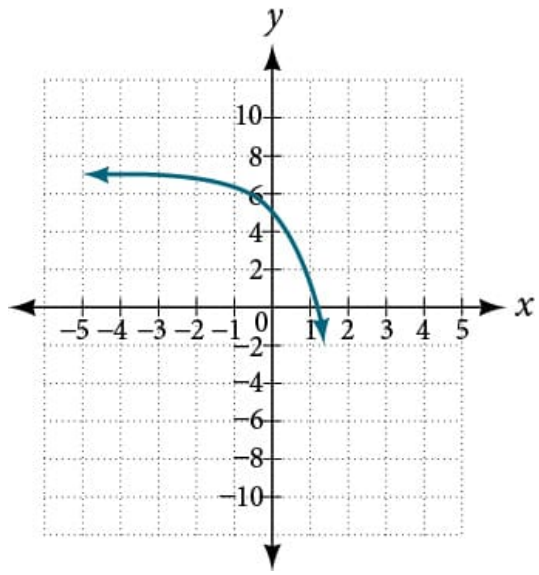
50.

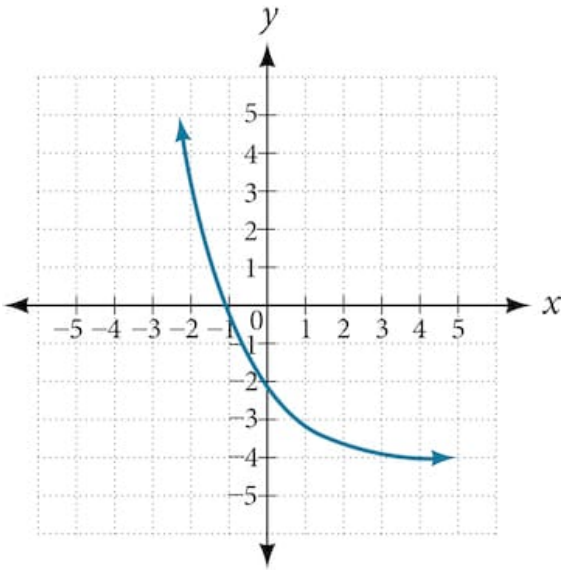


For the following exercises, find an exponential equation for the graph.

51.

52.





Numeric

For the following exercises, evaluate the exponential functions for the indicated value of x.

53. $g(x) = \frac{1}{3}(7)^{x-2}$ for $g(6)$.
54. $f(x) = 4(2)^{x-1} - 2$ for $f(5)$.
55. $h(x) = -\frac{1}{2}\left(\frac{1}{2}\right)^{x} + 6$ for $h(-7)$.

Technology

For the following exercises, use graphing technology to approximate the solutions of the equation. Round to the nearest thousandth.

56. $-50 = -\left(\frac{1}{2}\right)^{-x}$
57. $116 = \frac{1}{4}\left(\frac{1}{8}\right)^{x}$
58. $12 = 2(3)^{x} + 1$
59. $5 = 3\left(\frac{1}{2}\right)^{x-1} - 2$

60. $-30 = -4(2)^{x+2} + 2$

Extensions

61. Explore and discuss the graphs of $F(x) = b^x$ and $G(x) = \left(\frac{1}{b}\right)^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b}\right)^x$ for any real number $b > 0$.
62. Prove the conjecture made in the previous exercise.
63. Explore and discuss the graphs of $f(x) = 4^x$, $g(x) = 4^{x-2}$, and $h(x) = \left(\frac{1}{16}\right)4^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b^n}\right)b^x$ for any real number n and real number $b > 0$.
64. Prove the conjecture made in the previous exercise.

This page titled 12.2.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

12.3: Logarithmic Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure 12.3.1**:** *Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce)*

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings, like those shown in Figure 12.3.1. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the **Richter Scale**. The Haitian earthquake registered a 7.0 on the Richter Scale whereas the Japanese earthquake registered a 9.0.

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is $10^{8-4} = 10^4 = 10,000$ times as great! In this section, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in Figure 12.3.2, to better estimate the solution.

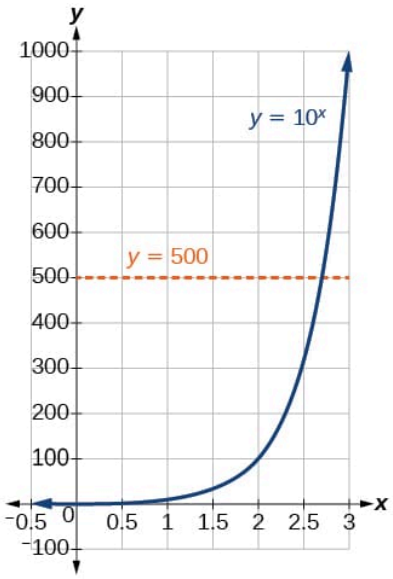


Figure 12.3.2

Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure 12.3.2 passes the Horizontal Line Test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a **logarithmic function** of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

Interactive Element: A Logarithm Is an Exponent

The following Interactive Element might help you see that a common logarithm is simply the exponent you put on 10 to reach a given number.

Interact: Drag the red point along the curve $y = 10^x$; its x-coordinate is the exponent and its height is the output. Drag it up until the height reaches the line $y = 500$.

Observation: At every position the x-coordinate equals $\log$ of the height. When the height is 500, the exponent is about 2.7, so $\log(500) \approx 2.7$ — a logarithm is just an exponent.

Definition: Logarithmic Function

The **base** b **logarithmic function** is defined as follows:

$$y = \log_b(x) \text{ is equivalent to } b^y = x,$$

where, $x > 0$, $b > 0$, and $b \neq 1$. We read $\log_b(x)$ as, "the logarithm with base b of x," or the "log base b of x."

It is critical to recognize that logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2(32) = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \leftrightarrow b^y = x, b > 0, b \neq 1.$$

The symbol, $\leftrightarrow$, means "**if and only if**." Since b is the base of an exponential function, b is always positive.



Figure 12.3.3

Note: Lax Notation

Because the logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. While this "lax" notation has been adopted as acceptable by most mathematicians and scientists, we will stick with parentheses as often as possible to reinforce the concept that the logarithm is a function.

Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:

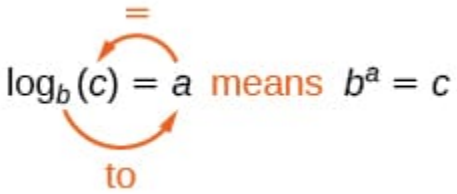


Figure 12.3.4

Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log_b(x)$ and $y = b^x$ are inverse functions. This leads us to quickly develop some fundamental theory about logarithmic functions.

Theorem: Domain and Range of the Logarithmic Function

The domain of the logarithmic function $f(x) = \log_b(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

Proof available at the end of this section.

In fact, it is due to the inverse nature of the logarithmic and exponential functions that $x > 0$, $b > 0$, and $b \neq 1$ are restrictions for the logarithmic function. Since $y = \log_b(x)$ is the same as $b^y = x$, and we know the base of an exponential function must be positive, $b > 0$. Moreover, when defining the exponential function, we took great care to state that the base could not be 1. Thus, $b \neq 1$. Finally, exponential functions only return positive values. Hence, $x > 0$.

Q&A

Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

Example 12.3.1: Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$
b. $\log_3(9) = 2$

Solutions available at the end of this section.

Checkpoint 12.3.1

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1,000,000) = 6$
b. $\log_5(25) = 2$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b, exponent x, and output y. Then we write $x = \log_b(y)$.

Example 12.3.2: Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \frac{1}{10,000}$

Solutions available at the end of this section.

Checkpoint 12.3.2

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \frac{1}{2}$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2(8)$. We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2(8) = 3$.

Now consider simplifying $\log_7(49)$ and $\log_3(27)$ mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know $7^2 = 49$. Therefore, $\log_7(49) = 2$
- We ask, "To what exponent must 3 be raised in order to get 27?" We know $3^3 = 27$. Therefore, $\log_3(27) = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{2/3}\left(\frac{4}{9}\right)$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$?" We know $2^2 = 4$ and $3^2 = 9$, so $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Therefore, $\log_{2/3}\left(\frac{4}{9}\right) = 2$.

Example 12.3.3: Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

Solution available at the end of this section.

Checkpoint 12.3.3

Solve $y = \log_{121}(11)$ without using a calculator.

Example 12.3.4: Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

Solution available at the end of this section.

Checkpoint 12.3.4

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base-10 logarithm a **common logarithm**.

Definition: Common Logarithm

A **common logarithm** is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$.

The common logarithm otherwise follows the standard definition of a logarithmic function: For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x.$$

Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars, decibel levels of sounds, and the pH of acids and bases also use common logarithms.

Example 12.3.5: Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution available at the end of this section.

Checkpoint 12.3.5

Evaluate $y = \log(1,000,000)$.

Example 12.3.6: Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution available at the end of this section.

We can perform a quick "logic check" on our answer to Example 12.3.6. Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100) = 2$ and $\log(1000) = 3$.

Checkpoint 12.3.6

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Example 12.3.7: Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution available at the end of this section.

Checkpoint 12.3.7

The amount of energy released from one earthquake was 8,500 times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in Calculus and some scientific applications; they are called **natural logarithms**.

Definition: Natural Logarithm

A **natural logarithm** is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$.

The natural logarithm otherwise follows the standard definition of a logarithmic function: For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x.$$

Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln(1) = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

Example 12.3.8: Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Solution available at the end of this section.

Checkpoint 12.3.8

Evaluate $\ln(-500)$.

Success in College: Self-Motivation Learning Preferences

Choose the statements that best describe you.

I come to class because I don't want to miss anything.	or	I come to class because it is required.
I would probably do less work in this class if I were graded pass/fail instead of a letter grade.	or	I would probably do the same amount of work in this class if I was graded pass/fail instead of with a letter grade.
I need frequent homework assignments to keep me working.	or	I do homework frequently to know what is going on in class.

1. Do you consider the choices above to be good or bad for your education?
2. Some professors provide detailed schedules with assignments and test dates at the beginning of the term. Others announce the homework assignments at each class meeting and test days during the term. Which do you prefer?

Success in Mathematics: Math Anxiety (Part 3)

For some math students, anxiety changes their ability to show what they know. They have physical and emotional symptoms. You are responsible for recognizing if math anxiety is interfering with your success. Do you need to make some changes to reduce your math anxiety? Do you need help to do so?

Researchers have identified various events that seem to result in math anxiety. Perhaps you had negative experiences learning math. For you, learning math meant feeling dumb or embarrassed. Perhaps a teacher or parent who struggled with math has passed on their fears to you. Or you believe that your gender or ethnic heritage means you cannot do the math.

Some students learn to cope on their own with their math anxiety. They use relaxation techniques. They practice problems over and over. At test time, their anxiety does not block them from doing the work. Others find help from counselors at their college.

Math anxiety is common among college students. It is your responsibility to think about it and, if needed, get help so that you can cope with your fears.

Math anxiety affects students in physical ways such as increasing blood pressure and heart rate, breathing faster, excess sweating, dry mouth, restlessness, or nausea. Emotionally, it can cause fear, anger, nervousness, worry, and feelings of hopelessness. Students may have trouble concentrating, sitting still, reading, or keeping their legs from bouncing. They may feel desperate - like they need to run away or cry.

Here are some questions about this topic that may help you succeed in this class (and possibly many other classes as well):

1. Using a scale of 1 to 10, with 10 describing overwhelming anxiety that makes it extremely difficult to do math, rate your math anxiety. How does it affect you?
2. Most students have some anxiety about their classes. They worry about giving speeches, taking tests, writing a good paper, or completing all their work. Worrying can be motivating. It can help us get going and to get things done. Or, worrying can be a bad thing. It paralyzes us. In general, how does worrying affect you?

Answers

Section Preview

The structure of this section mimics that of the introduction to exponential functions. It is important to note that the properties of logarithms are held off for a later section.

Section Preview

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

Theorem: Domain and Range of the Logarithmic Function

Let $y = \log_b(x)$. Since the logarithm base b is the inverse of the exponential function base b, the domain of the logarithm is the range of the exponential function. This is precisely $(0, \infty)$. Moreover, the range of the logarithm is the domain of the exponential function, which is $(-\infty, \infty)$.

Example 12.3.1: Converting from Logarithmic Form to Exponential Form

First, identify the values of b, y, and x. Then, write the equation in the form $b^y = x$.

a. Here, $b = 6$, $y = \frac{1}{2}$, and $x = \sqrt{6}$. Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to
$$6^{1/2} = \sqrt{6}.$$
b. Here, $b = 3$, $y = 2$, and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9$.

Example 12.3.2: Converting from Exponential Form to Logarithmic Form

First, identify the values of b, y, and x. Then, write the equation in the form $x = \log_b(y)$.

a. Here, $b = 2$, $x = 3$, and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.
b. Here, $b = 5$, $x = 2$, and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.
c. Here, $b = 10$, $x = -4$, and $y = \frac{1}{10,000}$. Therefore, the equation $10^{-4} = \frac{1}{10,000}$ is equivalent to $\log_{10}\left(\frac{1}{10,000}\right) = -4$.

Example 12.3.3: Solving Logarithms Mentally

First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know
$$4^3 = 64.$$
Therefore,
$$\log_4(64) = 3.$$

Example 12.3.4: Evaluating the Logarithm of a Reciprocal

First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

We know $3^3 = 27$, but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write
$$3^{-3} = \frac{1}{3^3} = \frac{1}{27}.$$
Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

Example 12.3.5: Finding the Value of a Common Logarithm Mentally

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know
$$10^3 = 1000.$$
Therefore, $\log(1000) = 3$.

Example 12.3.6: Finding the Value of a Common Logarithm Using a Calculator

- Press **[LOG]**.
- Enter 321, followed by **)**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

Example 12.3.7: Rewriting and Solving a Real-World Exponential Model

We begin by rewriting the exponential equation in logarithmic form.

$$\begin{aligned} & 10^x = 500 \\ \Longrightarrow\ & \log(500) = x && \text{(definition of the common logarithm)} \\ \Longrightarrow\ & x \approx 2.699 && \text{(using a calculator)} \end{aligned}$$

The difference in Richter Scale magnitudes was about 2.699.

Example 12.3.8: Evaluating a Natural Logarithm Using a Calculator

- Press **[LN]**.
- Enter 500, followed by **)**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$.

This page titled 12.3: Logarithmic Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

12.3.1: Resources and Key Concepts

Resources

Videos

- Logarithmic functions
  - Intermediate Algebra - Logarithmic Functions: Concepts and Basic Use
  - Intermediate Algebra - Logarithms: Logarithmic Functions
  - Logarithmic Functions - A Focus on Logarithmic Functions
- Converting between logarithmic to exponential forms
  - Logarithmic Functions - Converting Between Logarithmic and Exponential Forms
- Evaluating logarithms
  - Intermediate Algebra - Logarithmic Functions: Evaluating Logarithms

Key Concepts

Definitions

- **Logarithmic Function (base b)**: The function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. It is defined as: $y = \log_b(x)$ is equivalent to $b^y = x$, where $x > 0$, $b > 0$, and $b \neq 1$. The base b logarithm of a number x is the exponent by which we must raise b to get x.
- **Common Logarithm**: A logarithm with base 10. It is written as $\log(x)$, which is shorthand for $\log_{10}(x)$.
- **Natural Logarithm**: A logarithm with base e (Euler's number). It is written as $\ln(x)$, which is shorthand for $\log_e(x)$.

Theorems

- **Domain and Range of the Logarithmic Function**: The domain of the logarithmic function $f(x) = \log_b(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

Common Mistakes

- **Incorrect Conversion Between Logarithmic and Exponential Form**: Misplacing the base, argument, or exponent when converting between $y = \log_b(x)$ and $b^y = x$.
- **Taking the Logarithm of a Non-Positive Number**: Attempting to evaluate the logarithm of zero or a negative number (e.g., $\ln(-500)$ in Checkpoint 4.3.8). The argument of a logarithm must be positive.
- **Errors in Evaluating Basic Logarithms**: Difficulty in mentally evaluating logarithms by not recognizing them as exponents (e.g., for $\log_2(8)$, thinking "2 to what power is 8?").
- **Calculator Errors**: Incorrectly inputting expressions, especially with parentheses, when using a calculator.
- **Assuming $\log_b(0) = 1$ or similar misconceptions**: The logarithm of 0 is undefined. $\log_b(1) = 0$ because $b^0 = 1$.

This page titled 12.3.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

12.3.2: Homework

Reading Questions

1. What is a base b logarithm? Explain its relationship to an exponential function.
2. How do you convert the logarithmic equation $\log_b(x) = y$ into exponential form?
3. How do you convert the exponential equation $b^y = x$ into logarithmic form?
4. What is the domain and range of the logarithmic function $f(x) = \log_b(x)$?
5. Can you take the logarithm of a negative number or zero? Explain why or why not based on the properties of exponential functions.
6. What is a "common logarithm"? What is its base, and how is it typically notated?
7. What is a "natural logarithm"? What is its base, and how is it typically notated?
8. How would you mentally evaluate $\log_5(25)$?
9. If $10^x = 500$, how can you rewrite this equation using a common logarithm to solve for x?
10. What is the value of $\ln(1)$? What is the value of $\log(1)$? Why?
11. What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b(x) = y$ for $b > 0$, $b \neq 1$.
12. How is the logarithmic function $f(x) = \log_b(x)$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?
13. How can the logarithmic equation $\log_b(x) = y$ be solved for x using the properties of exponents?
14. Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?
15. Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Homework

Algebraic

For the following exercises, rewrite each equation in exponential form.

16. $\log_4(q) = m$
17. $\log_a(b) = c$
18. $\log_{16}(y) = x$
19. $\log_x(64) = y$
20. $\log_y(x) = -11$
21. $\log_{15}(a) = b$
22. $\log_y(137) = x$
23. $\log_{13}(142) = a$
24. $\log(v) = t$
25. $\ln(w) = n$

For the following exercises, rewrite each equation in logarithmic form.

26. $4^x = y$
27. $c^d = k$
28. $m^{-7} = n$
29. $19^x = y$
30. $x^{-10/13} = y$
31. $n^4 = 103$
32. $\left(\frac{7}{5}\right)^m = n$
33. $y^x = \frac{39}{100}$
34. $10^a = b$

35. $e^k = h$

For the following exercises, solve for x by converting the logarithmic equation to exponential form.

36. $\log_3(x) = 2$
37. $\log_2(x) = -3$
38. $\log_5(x) = 2$
39. $\log_3(x) = 3$
40. $\log_2(x) = 6$
41. $\log_9(x) = \frac{1}{2}$
42. $\log_{18}(x) = 2$
43. $\log_6(x) = -3$
44. $\log(x) = 3$
45. $\ln(x) = 2$

For the following exercises, use the definition of common and natural logarithms to simplify.

46. $\log(100^8)$
47. $10^{\log(32)}$
48. $2\log(0.0001)$
49. $e^{\ln(1.06)}$
50. $\ln(e^{-5.03})$
51. $e^{\ln(10.125)} + 4$

Numeric

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

52. $\log_3\left(\frac{1}{27}\right)$
53. $\log_6(\sqrt{6})$
54. $\log_2\left(\frac{1}{8}\right) + 4$
55. $6\log_8(4)$

For the following exercises, evaluate the common logarithmic expression without using a calculator.

56. $\log(10,000)$
57. $\log(0.001)$
58. $\log(1) + 7$
59. $2\log(100^{-3})$

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

60. $\ln(e^{1/3})$
61. $\ln(1)$
62. $\ln(e^{-0.225}) - 3$
63. $25\ln(e^{2/5})$

Technology

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

64. $\log(0.04)$
65. $\ln(15)$
66. $\ln\left(\frac{4}{5}\right)$
67. $\log(\sqrt{2})$
68. $\ln(\sqrt{2})$

Extensions

69. Is $x = 0$ in the domain of the function $f(x) = \log(x)$? If so, what is the value of the function when $x = 0$? Verify the result.
70. Is $f(x) = 0$ in the range of the function $f(x) = \log(x)$? If so, for what value of x? Verify the result.
71. Is there a number x such that $\ln(x) = 2$? If so, what is that number? Verify the result.
72. Is the following true: $\frac{\log_3(27)}{\log_4(\frac{1}{64})} = -1$? Verify the result.
73. Is the following true: $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$? Verify the result.

Real-World Applications

74. The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where f is the "f-stop" setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?
75. Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?
76. The intensity levels I of two earthquakes measured on a seismograph can be compared by the formula

$$\log\left(\frac{I_1}{I_2}\right) = M_1 - M_2,$$

where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0. How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.

This page titled 12.3.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

12.4: Graphs of Logarithmic Functions

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

When we investigated graphs of exponential functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest \$2500 in an account that offers an annual interest rate of 5%, compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? Figure $12.4.1$ shows this point on the logarithmic graph.

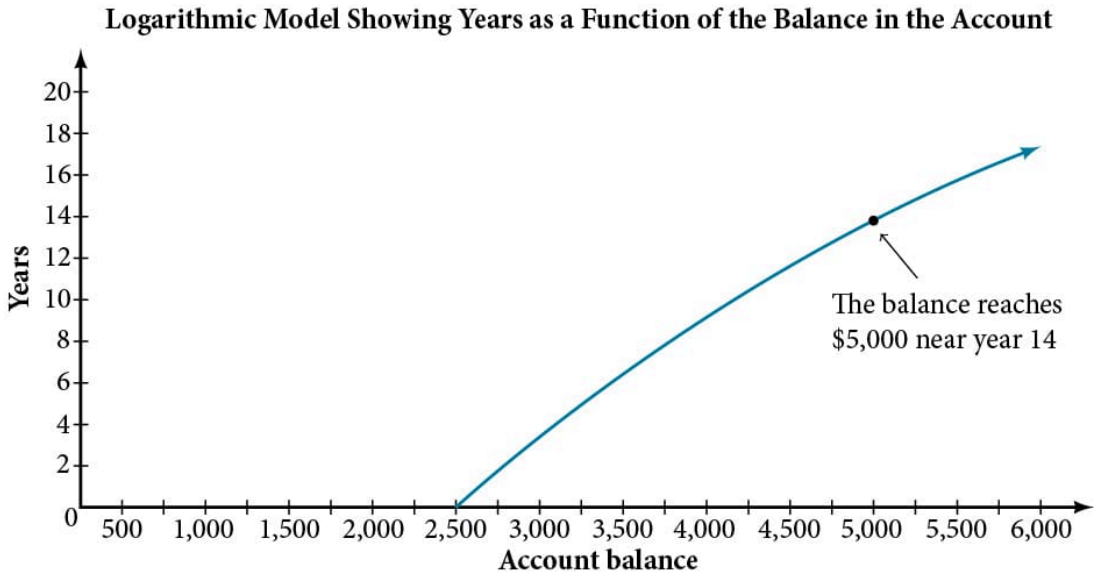


Figure $12.4.1$

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where the domain of y is $(-\infty, \infty)$ and the range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Theorem: Domain and Range of the Logarithmic Function

The domain of the logarithmic function $f(x) = \log_b(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

Proof available at the end of this section.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

We have seen that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be strictly greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$, is greater than zero. To find the domain, we set up an inequality and solve for x:

$$\begin{aligned} & 2x - 3 > 0 \\ \Rightarrow \quad & 2x > 3 && \text{(adding 3 to both sides)} \\ \Rightarrow \quad & x > 1.5 && \text{(dividing both sides by 2)} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

Example 12.4.1: Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x) = \log_2(x + 3)$?

Solution available at the end of this section.

Checkpoint 12.4.1

What is the domain of $f(x) = \log_5(x - 2) + 1$?

Example 12.4.2: Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x) = \log(5 - 2x)$?

Solution available at the end of this section.

Checkpoint 12.4.2

What is the domain of $f(x) = \log(x - 5) + 2$?

Of course, there is no need to restrict the argument of a logarithm to linear expressions. We can rely on the graphing skills we built in previous chapters to find the domain of the following example.

Example 12.4.3: Identifying the Domain of a Logarithm Composed with a Nonlinear Function

State the domain of the function.

$$g(x) = -2\log_5\left(\frac{2x^3 - 3x^2 - 3x + 2}{3x^3 + 375}\right)$$

Solution available at the end of this section.

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the parent function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in Table 12.4.1.

Table 12.4.1

x	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from Table 12.4.1, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See Table 12.4.2.

Table 12.4.2

$f(x) = 2^x$	$\left(-3, \frac{1}{8}\right)$	$\left(-2, \frac{1}{4}\right)$	$\left(-1, \frac{1}{2}\right)$	$(0, 1)$	$(1, 2)$	$(2, 4)$	$(3, 8)$
$g(x) = \log_2(x)$	$\left(\frac{1}{8}, -3\right)$	$\left(\frac{1}{4}, -2\right)$	$\left(\frac{1}{2}, -1\right)$	$(1, 0)$	$(2, 1)$	$(4, 2)$	$(8, 3)$

As we'd expect, the x- and y-coordinates are reversed for the inverse functions. Figure 12.4.3 shows the graph of f and g.

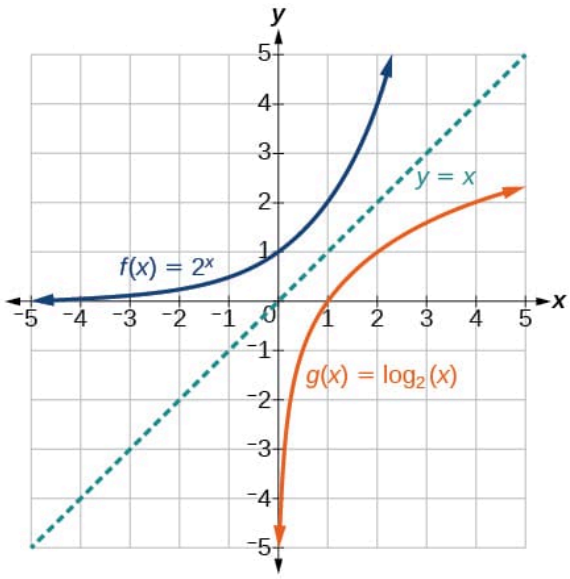


Figure 12.4.3**:** *Notice that the graphs of* $f(x) = 2^x$ *and* $g(x) = \log_2(x)$ *are reflections about the line* $y = x$.

Observe the following from the graph:

- $f(x) = 2^x$ has a y-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an x- intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

Interactive Element: A Logarithm Is the Mirror of Its Exponential

The following Interactive Element might help you visualize why $y = \log_2(x)$ is the reflection of $y = 2^x$ across the line $y = x$.

Interact: Drag the point along the curve $y = 2^x$.

Observation: Its mirror image across $y = x$ always lands on $y = \log_2(x)$, and the coordinates swap: $(t,\ 2^t)$ becomes $(2^t,\ t)$.

This leads to the following theorem.

Theorem: Properties of the Graphs of Logarithmic Functions

A logarithmic function with the form $f(x) = \log_b(x)$, where $b > 0$ and $b \neq 1$, has these characteristics:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- x-intercept: $(1, 0)$
- Secondary key point: $(b, 1)$
- y-intercept: none
- increasing if $b > 1$
- decreasing if \(0

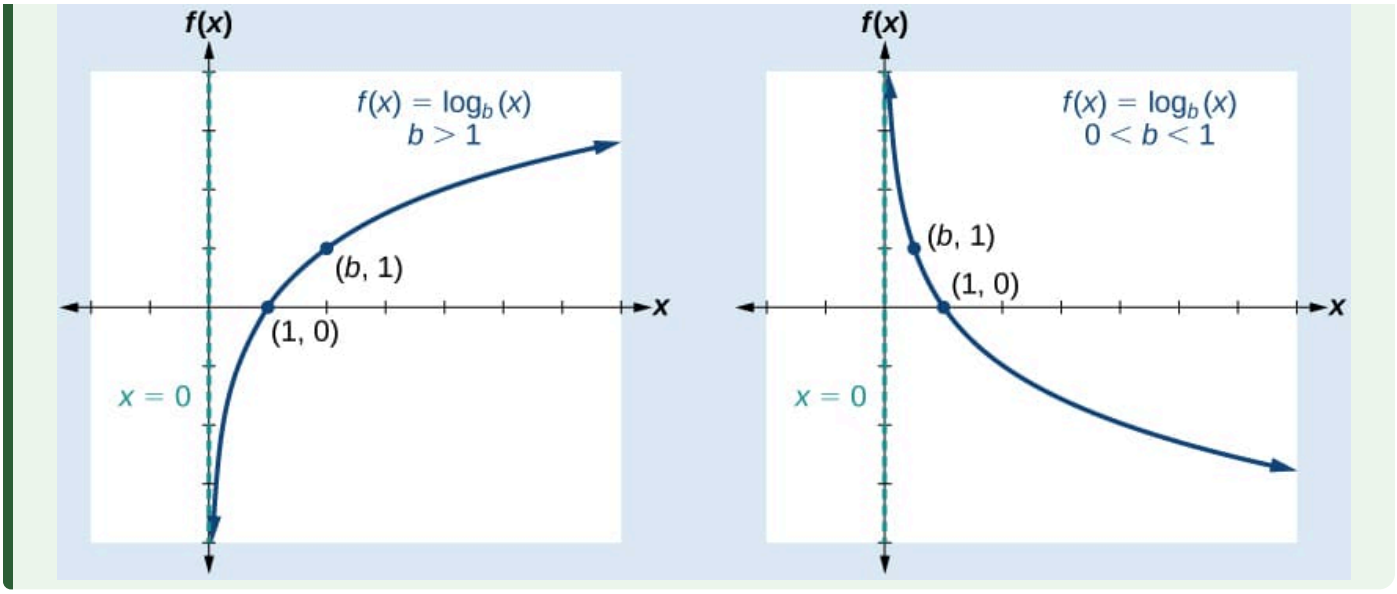


Figure 12.4.4 shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (*Note:* recall that the function $\ln(x)$ has base $e \approx 2.718$).

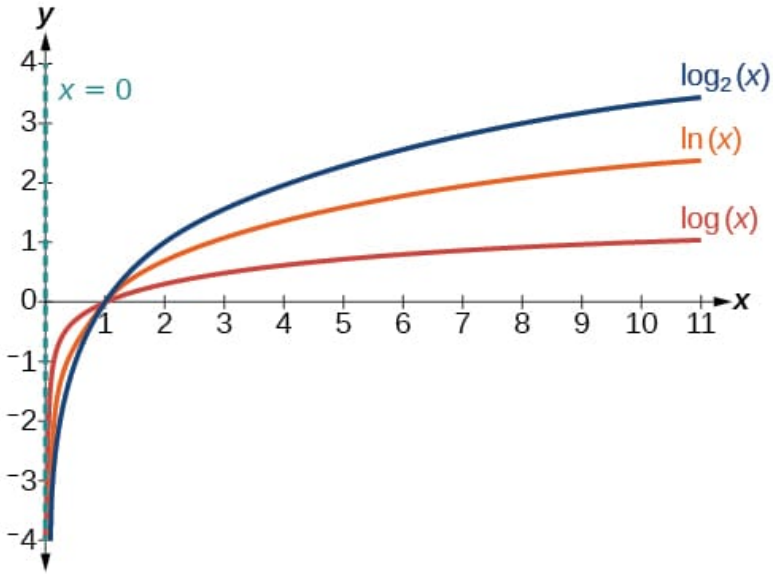


Figure 12.4.4**:** *The graphs of three logarithmic functions with different bases, all greater than 1.*

Example 12.4.4: Graphing a Logarithmic Function with the Form $f(x) = \log_b(x)$.

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 12.4.4

Graph $f(x) = \log_{1/5}(x)$. State the domain, range, and asymptote.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift of $f(x) = \log_b(x)$

When a constant c is added to the input of the parent function $f(x) = \log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 12.4.6.

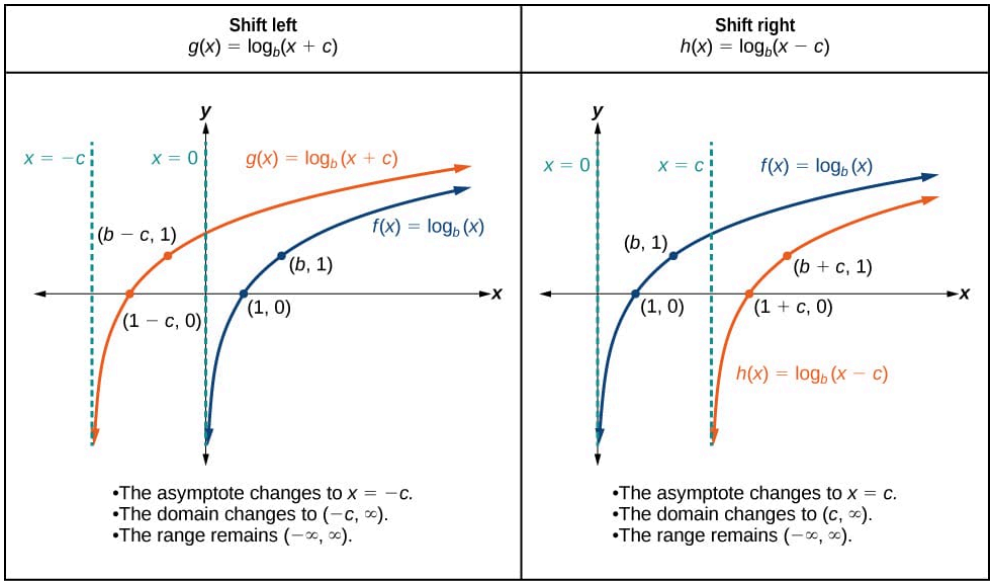


Figure 12.4.6

Example 12.4.5: Graphing a Horizontal Shift of the Parent Function $y = \log_b(x)$

Sketch the horizontal shift $f(x) = \log_3(x - 2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 12.4.5

Sketch a graph of $f(x) = \log_3(x + 4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Graphing a Vertical Shift of $y = \log_b(x)$

When a constant d is added to the parent function $f(x) = \log_b(x)$, the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$. See Figure 12.4.8.

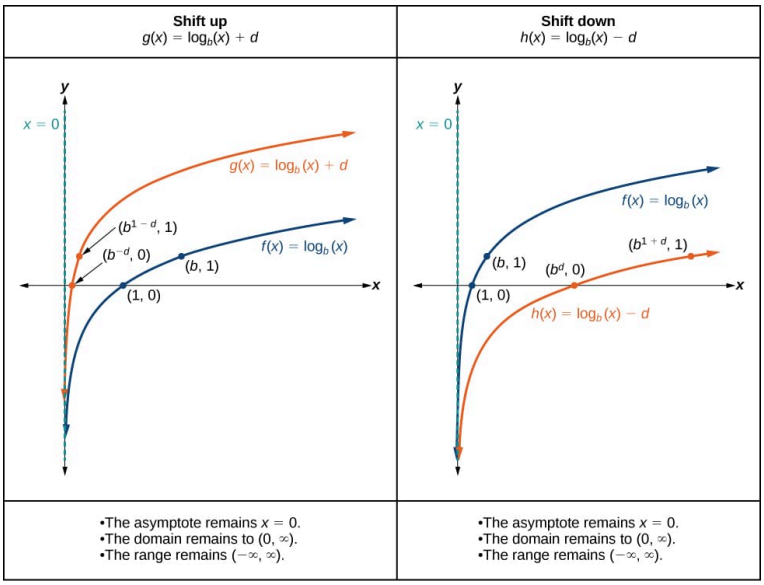


Figure 12.4.8

Example 12.4.6: Graphing a Vertical Shift of the Parent Function $y = \log_b(x)$

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 12.4.6

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Graphing Stretches and Compressions of $y = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a > 1$ and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$. See Figure 12.4.10.

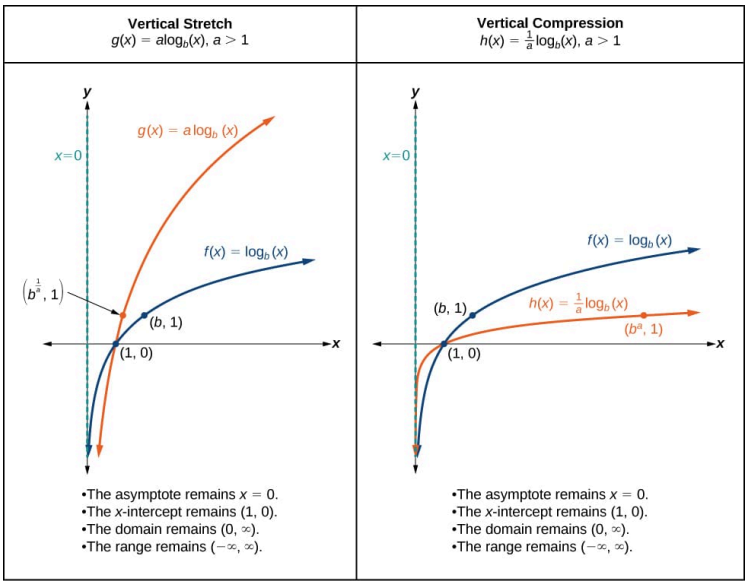


Figure 12.4.10

Example 12.4.7: Graphing a Stretch or Compression of the Parent Function $y = \log_b(x)$

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 12.4.7

Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Example 12.4.8: Combining a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x + 2)$. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 12.4.8

Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.

Graphing Reflections of $f(x) = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by -1, the result is a reflection about the x-axis. When the *input* is multiplied by -1, the result is a reflection about the y-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the x-axis, $g(x) = -\log_b(x)$ and the reflection about the y-axis, $h(x) = \log_b(-x)$.

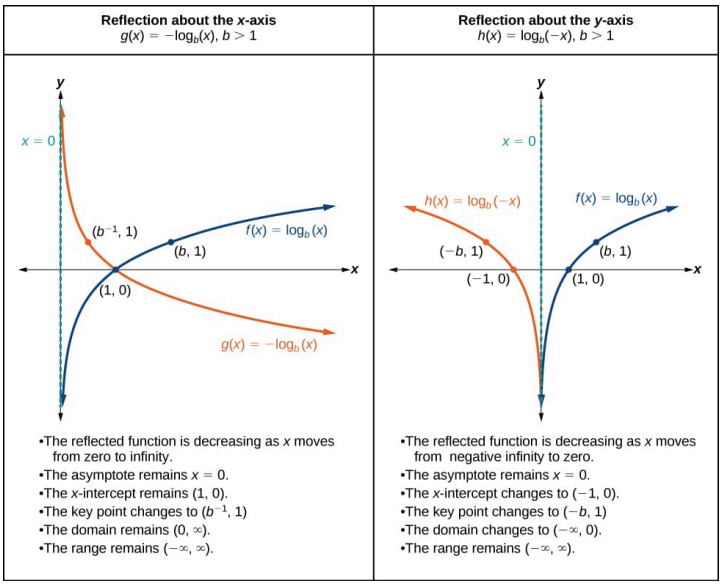


Figure 12.4.13

Example 12.4.9: Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution available at the end of this section.

Checkpoint 12.4.9

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

Example 12.4.10: Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x) + 1 = -2\ln(x - 1)$ using graphing technology. Round to the nearest thousandth.

Solution available at the end of this section.

Checkpoint 12.4.10

Solve $5\log(x + 2) = 4 - \log(x)$ using graphing technology. Round to the nearest thousandth.

Summarizing Transformations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in Table 12.4.3 to arrive at the general equation for translating exponential functions.

Table 12.4.3

Transformations of the Parent Function $y = \log_b(x)$

Transformation	Form
Shift • Horizontally c units to the left (if $c > 0$) • Vertically d units up (if $d > 0$)	$y = \log_b(x + c) + d$
Stretch and Compress • Stretch if $\lvert a\rvert > 1$ • Compression if $\lvert a\rvert < 1$	$y = a\log_b(x)$
Reflect about the x-axis	$y = -\log_b(x)$
Reflect about the y-axis	$y = \log_b(-x)$
General equation for all transformations	$y = a\log_b(x + c) + d$

Translations of Logarithmic Functions

All translations of the parent logarithmic function, $y = \log_b(x)$, have the form

$$f(x) = a\log_b(x + c) + d$$

where the parent function, $y = \log_b(x)$, $b > 1$, is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $\lvert a\rvert$ if $\lvert a\rvert > 0$.
- compressed vertically by a factor of $\lvert a\rvert$ if $0 < \lvert a\rvert < 1$.
- reflected about the x-axis when $a < 0$.

For $f(x) = \log_b(-x)$, the graph of the parent function is reflected about the y-axis.

Interactive Element: Transforming the Logarithmic Parent Function

The following Interactive Element might help you visualize how the constants in $a\log_2(x + c) + d$ reshape the graph, and which one moves the vertical asymptote.

Interact: Drag the orange anchor point to shift the graph, and drag the green handle up or down to change a.

Observation: Moving the anchor left or right slides the vertical asymptote along with the curve, but moving it up or down—or changing a—leaves the asymptote where it is. Only the horizontal shift relocates the asymptote.

Example 12.4.11: Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2\log_3(x + 4) + 5$?

Solution available at the end of this section.

In Example 12.4.11, the coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to $x = -4$.

Checkpoint 12.4.11

What is the vertical asymptote of $f(x) = 3 + \ln(x - 1)$?

Example 12.4.12: Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in Figure 12.4.15.

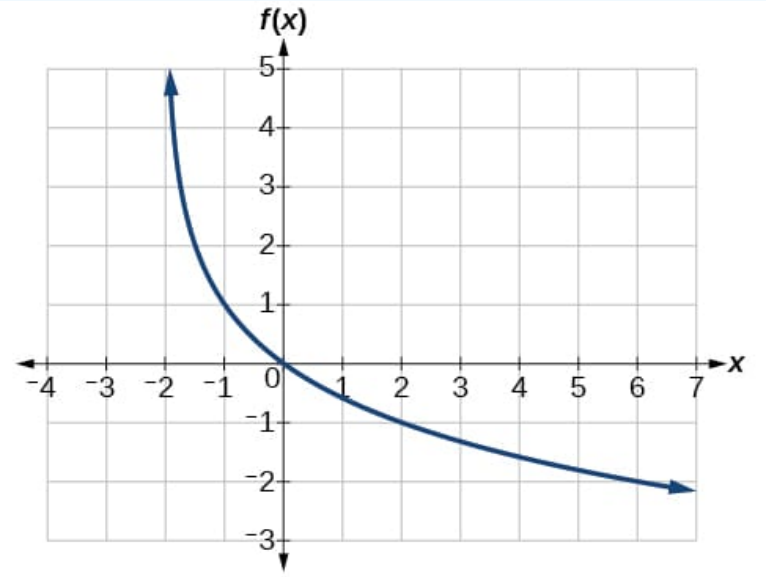


Figure 12.4.15

Solution available at the end of this section.

Checkpoint 12.4.12

Give the equation of the natural logarithm graphed in Figure 12.4.16.

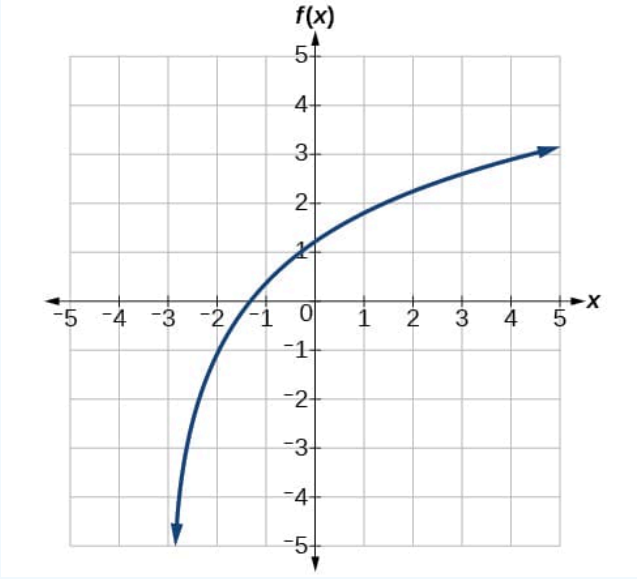


Figure 12.4.16

Q&A

Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?

Yes, if we know the function is a general logarithmic function. For example, look at the graph in Figure 12.4.16. The graph approaches $x = -3$ (or thereabouts) more and more closely, so $x = -3$ is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right, $\{x \mid x > -3\}$. The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is that as $x \to -3^+$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.

More Complex Transformations

Just as we encountered early in this course, logarithmic functions can also have horizontal stretches or compressions. Since these more robust transformations occur in Calculus, we spend a few moments reminding ourselves how to deal with graphing such types of functions.

Example 12.4.13: Graphing a More Complex Logarithmic Function

Sketch a graph of the function

$$h(x) = 2\ln\left(-\frac{1}{4}x + 1\right) - 1.$$

Solution available at the end of this section.

Revisiting Graphs of Absolute Value Functions

Finally, while you might not know it yet, absolute values and natural logarithms will appear together quite often in Calculus. It's best that we spend an example reminding ourselves of the effect of the absolute value on a function.

Example 12.4.14

Graph

$$k(x) = \left|2\ln\left(-\frac{1}{4}x + 1\right) - 1\right|.$$

Solution available at the end of this section.

Applications of Mathematics: Poker

So, where does mathematics apply in poker? Poker players can use mathematics in many different situations; however, it is most commonly used when a player is on a draw, such as a flush or straight draw. If an opponent makes a bet and you are on a draw, you are faced with the decision of whether or not you should call to try and complete the draw by the next card or fold and let your opponent take the pot. In this situation, a player with good knowledge of poker mathematics will always know whether or not to call, while a player who does not know mathematics will be unsure.

Players unfamiliar with poker odds will make an educated guess on whether or not they should call. If the bet is large, they may feel it is too expensive to try and catch the right card, but if the bet is small, they will be more inclined to call. On the other hand, a player that can use mathematics correctly will be able to work out the pot odds they are getting on the hand and act accordingly. Pot odds consider the amount your opponent has bet in relation to the pot size and the likelihood of completing your draw to inform you about whether or not you should call or fold.

1. Can you think of three other games that rely explicitly or implicitly on mathematics? Explain.
2. Oddly enough, most of us are superstitious about numbers. Some people even play their "favorite" numbers when they play the lottery. Suppose you pick your favorite numbers, and I pick the numbers 1, 2, 3, 4, 5, and 6. Who has better odds of winning? Why are you still likely to win more if the odds are the same?

Hidden Resources: Kanopy

Did you know that as a Los Rios Community College student, you have free access to Kanopy?

What is Kanopy?

According to Wikipedia, Kanopy "is an on-demand streaming video platform for public libraries and universities that offers films and documentaries."

Why would you be interested?

There are a lot of great movies, documentaries, and educational videos on this service - and it's free for you because you are a student of the Los Rios Community College District.

Is it as good as Hulu or Netflix?

No, but it does have a lot of the same titles.

Why aren't any of the Great Courses available to us?

Not being able to access the Great Courses is a disappointment. That was one of the options Los Rios chose not to purchase; however, drown your sorrows in What We Do in the Shadows instead!

How do I sign up?

All you need to do is go to https://losrios.kanopy.com and start typing Los Rios when asked for your institution. That should be it.

Answers

Section Preview

This section covers graphing logarithmic functions—**DO NOT SKIP THIS MATERIAL!** Knowing how to graph a logarithmic function is critical in Calculus.

Section Preview

- **Factoring Techniques**
  - Factoring Sums and Differences of Cubes: Example 3 factors $x^3 + 125$ as a sum of cubes to locate the denominator's real zero and vertical asymptote at $x = -5$.
- **Absolute Value Functions**
  - Graphing the Composition of a Function and the Absolute Value Function: Example 14 graphs $|h(x)|$ by reflecting the negative-y portion of $h(x)$ about the x-axis while leaving the rest unchanged.

Section Preview

- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

Theorem: Domain and Range of the Logarithmic Function

Let $y = \log_b(x)$. Since the logarithm base b is the inverse of the exponential function base b, the domain of the logarithm is the range of the exponential function. This is precisely $(0, \infty)$. Moreover, the range of the logarithm is the domain of the exponential function, which is $(-\infty, \infty)$.

Example $12.4.1$**: Identifying the Domain of a Logarithmic Shift**

The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

$$x + 3 > 0 \implies x > -3.$$

The domain of $f(x) = \log_2(x + 3)$ is $(-3, \infty)$.

Example $12.4.2$**: Identifying the Domain of a Logarithmic Shift and Reflection**

The logarithmic function is defined only when the input is positive, so this function is defined when $5-2x>0$. Solving this inequality,

$$\begin{aligned} 5-2x &> 0 \\ \Longrightarrow \quad -2x &> -5 \qquad \text{(subtracting 5 from both sides)} \\ \Longrightarrow \quad x &< \frac{5}{2} \qquad \text{(dividing both sides by } -2 \text{ and switching the inequality)} \end{aligned}$$

The domain of $f(x)=\log(5-2x)$ is $\left(-\infty,\frac{5}{2}\right)$.

Example 12.4.3**: Identifying the Domain of a Logarithm Composed with a Nonlinear Function**

Since the argument of a logarithm must be positive, answering this question requires us to solve

$$\frac{2x^3-3x^2-3x+2}{3x^3+375}>0.$$

From our previous work, we have two ways to solve this rational inequality—sign charts or a quick graph. As you probably will guess, I am in favor of a quick sketch. In either case, we will need to factor the numerator and denominator of the rational expression.

Factoring the denominator is the easier task, so we start there.

$$\begin{aligned} 3x^3+375 &= 3\left(x^3+125\right) && \text{(factoring out the GCF)} \\ &= 3(x+5)\left(x^2-5x+25\right) && \text{(factoring the sum of cubes)} \end{aligned}$$

Hence, the rational inequality is

$$\frac{2x^3-3x^2-3x+2}{3(x+5)\left(x^2-5x+25\right)}>0,$$

where the trinomial in the denominator is prime (i.e., it does not have real zeros).

Turning our attention to the numerator, we are forced to use the Rational Zeros Theorem. The candidate roots are ±1, ±2, and $\pm\frac{1}{2}$. Starting with 1, we get

$$\begin{array}{r|cccc} \underline{1} & 2 & -3 & -3 & 2 \\ & & 2 & -1 & -4 \\ \hline & 2 & -1 & -4 & -2 \end{array}$$

Since the remainder is nonzero, $(x-1)$ is not a factor. Let's move to -1.

$$\begin{array}{r|cccc} \underline{-1} & 2 & -3 & -3 & 2 \\ & & -2 & 5 & -2 \\ \hline & 2 & -5 & 2 & 0 \end{array}$$

The remainder is zero. Therefore, $(x-(-1))=(x+1)$ is a factor. Moreover, the remaining factor is $2x^2-5x+2=(2x-1)(x-2)$. Thus, the rational inequality we need to solve can be rewritten as

$$\frac{(x+1)(2x-1)(x-2)}{3(x+5)(x^2-5x+25)}>0.$$

The intercepts of the function on the left side are $(-1,0)$, $\left(\frac{1}{2},0\right)$, $(2,0)$, and $\left(0,\frac{2}{375}\right)$. The vertical asymptote is $x=-5$ and the horizontal asymptote is $y=\frac{2}{3}$. Noting that the multiplicities of the x-intercepts are all 1 and the multiplicity of the vertical asymptote is odd, we quickly sketch the following graph.

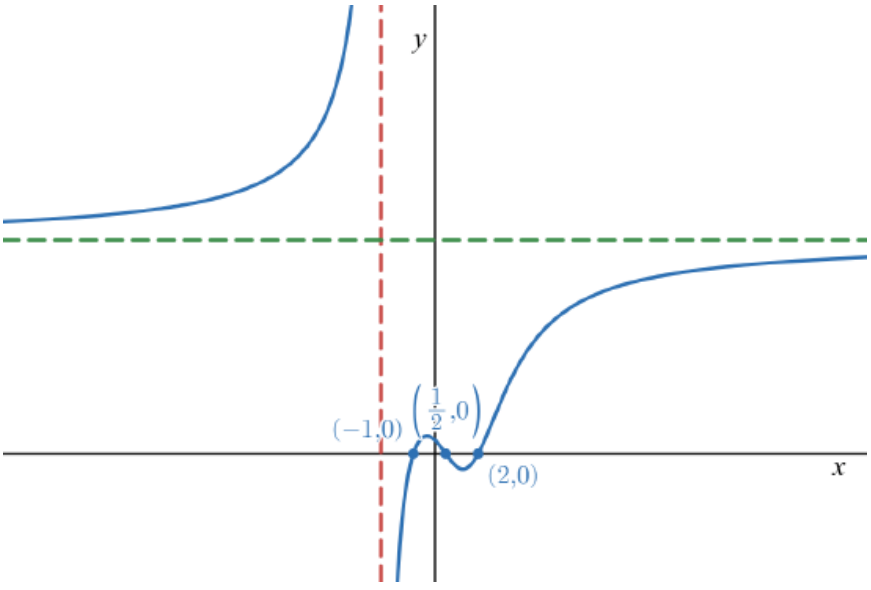


Figure 12.4.2

We can see this graph is above the x-axis on the interval $(-\infty,-5)\cup\left(-1,\frac{1}{2}\right)\cup(2,\infty)$. Hence, this is the domain of the logarithmic function.

Example 12.4.4**: Graphing a Logarithmic Function with the Form** $f(x)=\log_b(x)$**.**

Before graphing, identify the behavior and key points for the graph.

- Since $b=5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x=0$, and the right tail will increase slowly without bound.

- The x-intercept is $(-1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see Figure $12.4.5$).

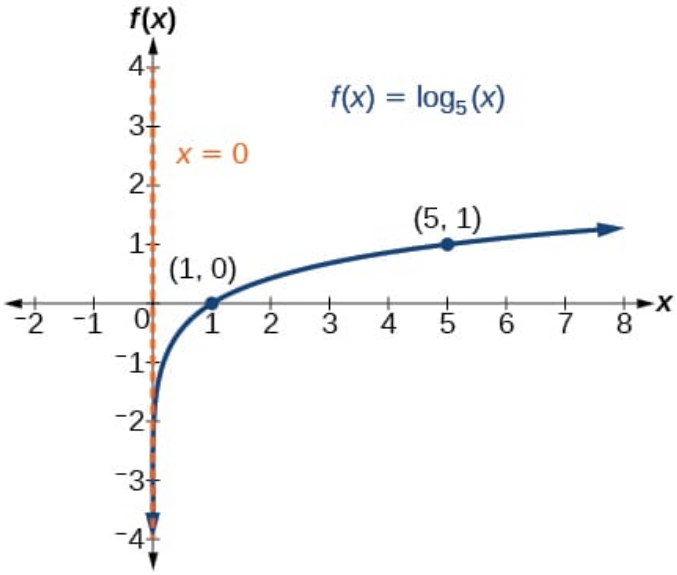


Figure $12.4.5$

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Example $12.4.5$**: Graphing a Horizontal Shift of the Parent Function** $y = \log_b(x)$

Since the function is $f(x) = \log_3(x - 2)$, we notice $x + (-2) = x - 2$.

Thus $c = -2$, so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the parent function, $(\frac{1}{3}, -1)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by adding 2 to the x coordinates.

Label the points $(\frac{7}{3}, -1)$, $(3, 0)$, and $(5, 1)$.

The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.

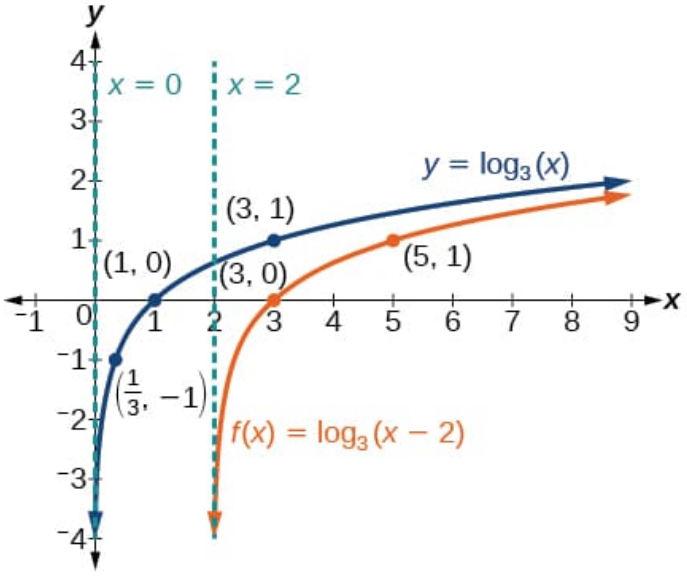


Figure $12.4.7$

Example $12.4.6$**: Graphing a Vertical Shift of the Parent Function** $y = \log_b(x)$

Since the function is $f(x) = \log_3(x) - 2$, we will notice $d = -2$. Thus $d < 0$. This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $(\frac{1}{3}, -1)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by subtracting 2 from the y coordinates.

Label the points $(\frac{1}{3}, -3)$, $(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

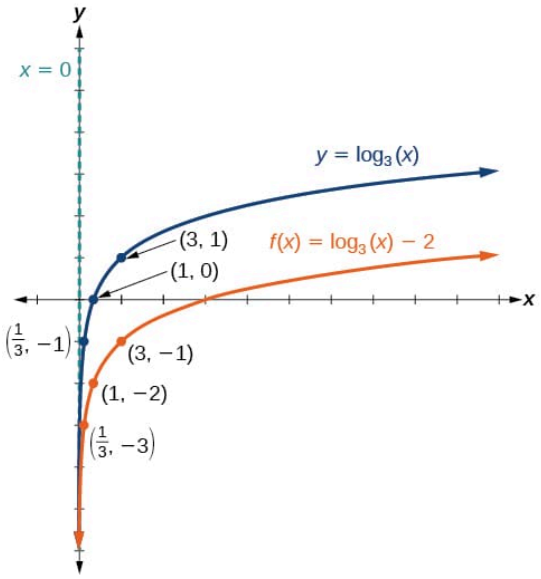


Figure 12.4.9

Example 12.4.7**: Graphing a Stretch or Compression of the Parent Function** $y = \log_b(x)$

Since the function is $f(x) = 2\log_4(x)$, we will notice $a = 2$. This means we will stretch the function $f(x) = \log_4(x)$ by a factor of 2.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $(\frac{1}{4}, -1)$, $(-1, 0)$, and $(4, 1)$.

The new coordinates are found by multiplying the y coordinates by 2.

Label the points $(\frac{1}{4}, -2)$, $(-1, 0)$, and $(4, 2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$. See Figure 12.4.11.

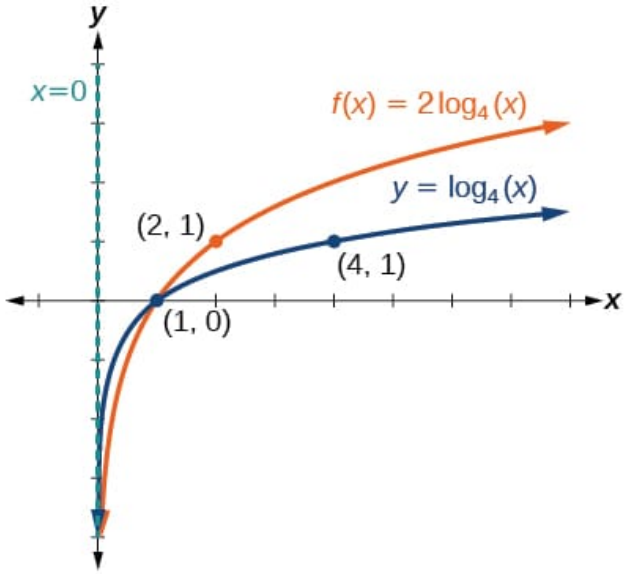


Figure 12.4.11

Example 12.4.8**: Combining a Shift and a Stretch**

Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in Figure 12.4.12. The vertical asymptote will be shifted to $x = -2$. The x-intercept will be $(-1, 0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5)$. We chose $x = 8$ as the x-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.

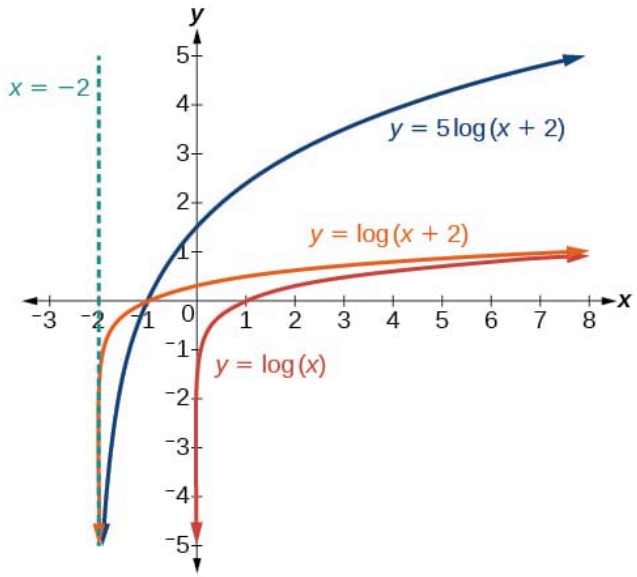


Figure 12.4.12

The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -2$.

Example 12.4.9**: Graphing a Reflection of a Logarithmic Function**

Before graphing $f(x) = \log(-x)$, identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by -1, f is a reflection of the parent graph about the y-axis. Thus, $f(x) = \log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The x-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

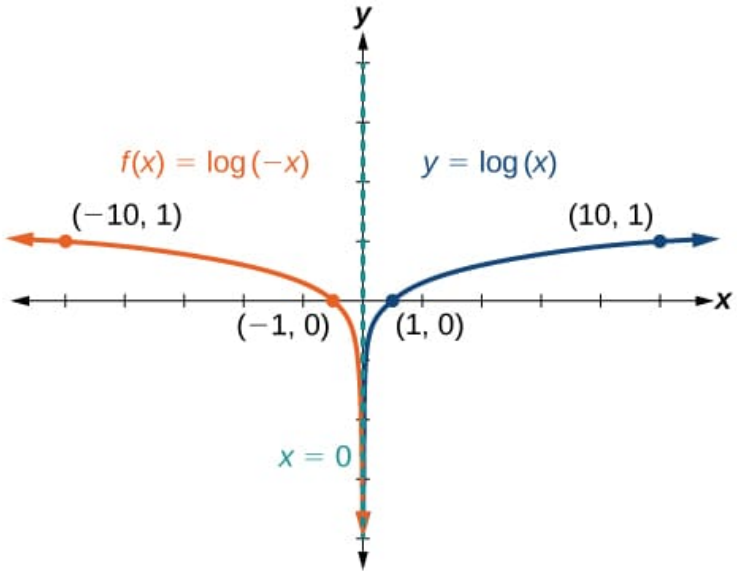


Figure 12.4.14

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Example 12.4.10**: Approximating the Solution of a Logarithmic Equation**

In the graphing calculator portion of Desmos, we enter $y = 4\ln(x) + 1$ on one line and $y = -2\ln(x - 1)$ on the next line. For a window, use the values 0 to 5 for x and -10 to 10 for y. The graphs should intersect somewhere a little to right of $x = 1$. Clicking on their intersection point, we see the x-coordinate of the point of intersection is displayed as 1.3385. So, to the nearest thousandth, $x \approx 1.339$.

Example 12.4.11**: Finding the Vertical Asymptote of a Logarithm Graph**

The vertical asymptote is at $x = -4$.

Example 12.4.12**: Finding the Equation from a Graph**

This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

$$f(x) = -a\log(x + 2) + k$$

It appears the graph passes through the points $(-1, 1)$ and $(2, -1)$. Substituting $(-1, 1)$,

$$\begin{aligned} & 1 = -a\log(-1+2)+k && \text{(substituting)} \\ \Longrightarrow\ & 1 = -a\log(1)+k && \text{(simplifying)} \\ \Longrightarrow\ & 1 = -a(0)+k && (\log(1)=0) \\ \Longrightarrow\ & 1 = k && (\log(1)=0) \end{aligned}$$

Next, substituting in $(2,-1)$,

$$\begin{aligned} & -1 = -a\log(2+2)+1 && \text{(substituting)} \\ \Longrightarrow\ & -1 = -a\log(4)+1 && \text{(simplifying)} \\ \Longrightarrow\ & -2 = -a\log(4) && \text{(subtracting 1 from both sides)} \\ \Longrightarrow\ & -\frac{2}{\log(4)} = a && \text{(dividing both sides by } -\log(4)) \end{aligned}$$

This gives us the equation $f(x)=-\frac{2}{\log(4)}\log(x+2)+1$.

We can verify this answer by comparing the function values in Table 12.4.4 with the points on the graph in Figure 12.4.15.

Table 12.4.4

x	-1	0	1	2	3
$f(x)$	1	0	-0.58496	-1	-1.3219
x	4	5	6	7	8
$f(x)$	-1.5850	-1.8074	-2	-2.1699	-2.3219

Example 12.4.13**: Graphing a More Complex Logarithmic Function**

As we have done previously, we begin by factoring the argument of the function.

$$h(x)=2\ln\left(-\frac{1}{4}(x-4)\right)-1.$$

Starting with the parent graph (which contains the anchor points $(1,0)$ and $(e,1)$), we read our function from left to right:

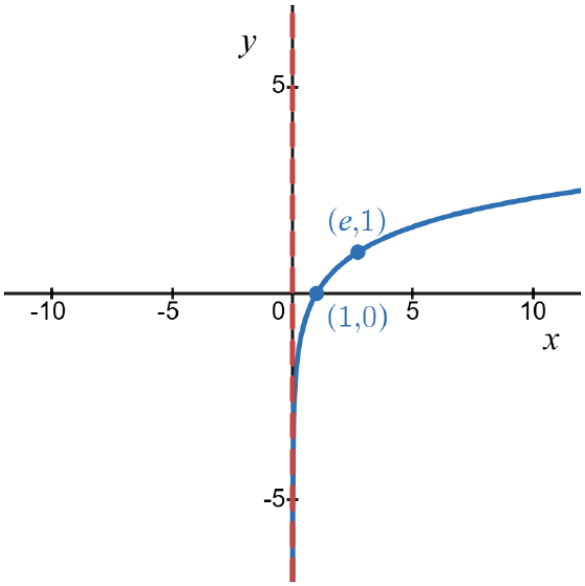


Figure 12.4.17

- The parent graph, $y=\ln(x)$, has been **stretched vertically** by a factor of 2. Thus, we multiply the y-coordinates of our anchor points by 2 to get $(1,0)$ and $(e,2)$ to get the graph of $y=2\ln(x)$.

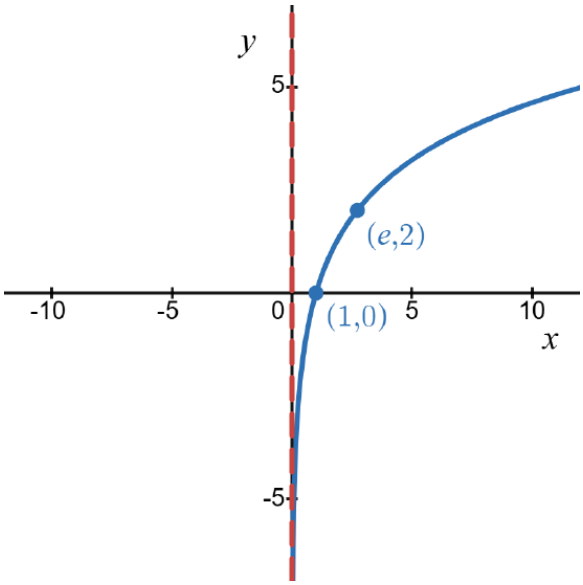


Figure 12.4.18

- The graph of $y = 2\ln(x)$ has been **reflected horizontally** about the y-axis and **stretched horizontally** by a factor of 4. Therefore, we multiply the x-coordinates of our anchor points by -4 to get $(-4, 0)$ and $(-4e, 2)$ to get the graph of $y = 2\ln\left(-\frac{1}{4}x\right)$.

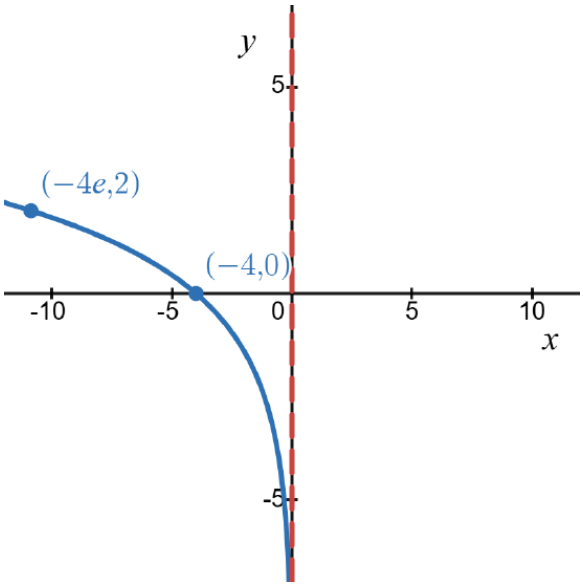


Figure 12.4.19

- The graph of $y = 2\ln\left(-\frac{1}{4}x\right)$ gets **shifted horizontally** 4 units to the right. In this horizontal shift, the vertical asymptote also shifts. Hence, we add 4 to all the x-coordinates of our anchor points and to the vertical asymptote. Thus, the graph of $y = 2\ln\left(-\frac{1}{4}(x-4)\right)$ contains the points $(0, 0)$ and $(-4e + 4, 2)$, and the vertical asymptote is at $x = 4$.

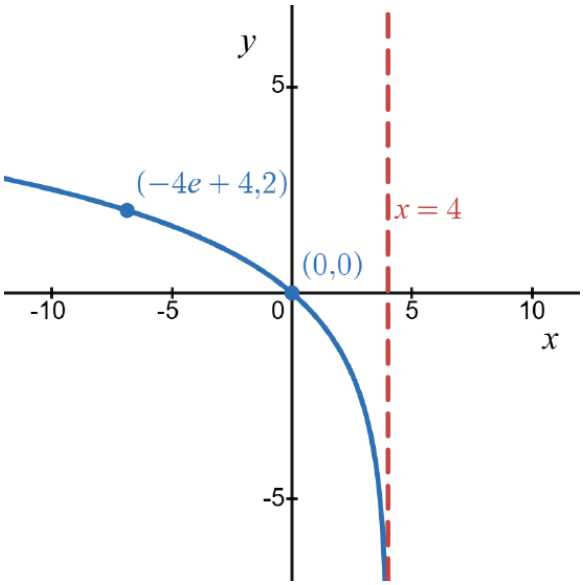


Figure 12.4.20

- The graph of $y = 2\ln\left(-\frac{1}{4}(x-4)\right)$ gets **shifted vertically** 1 unit down to arrive at our final graph of the function $h(x) = 2\ln\left(-\frac{1}{4}(x-4)-1\right)$. The anchor points get moved to $(0,-1)$ and $(-4e+4,1)$. The final graph is shown below.

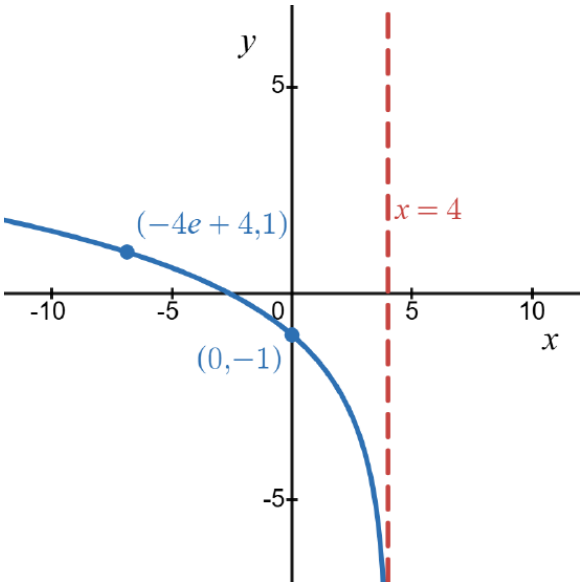


Figure 12.4.21

Example 12.4.14

If this function looks familiar, it is because it is the absolute value of the function $h(x)$ from Example 12.4.13. The easiest way to graph the absolute value of a function is to recall that the absolute value keeps non-negative values the same, but forces negative values to become positive. Graphically, this means that any coordinate on the graph of $h(x)$ with negative y-values will be reflected about the x-axis to their positive values. All other coordinates remain unchanged.

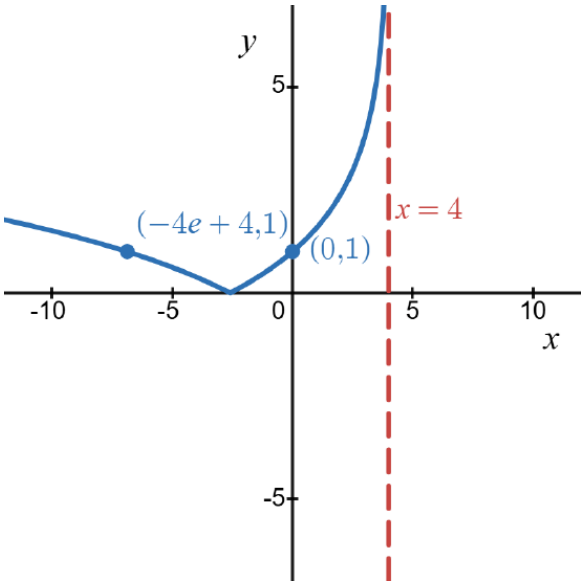


Figure 12.4.22

This page titled 12.4: Graphs of Logarithmic Functions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

12.4.1: Resources and Key Concepts

Resources

Videos

- Graphs of logarithmic functions
  - Graphs of Logarithmic Functions - Graphing Logarithmic Functions
  - Logarithmic Functions - Graphs of Logarithmic Functions
  - Intermediate Algebra - Logarithms: Graphing Logarithmic Functions
  - Intermediate Algebra - Logarithms: Graphing Logarithmic Functions Using Transformations
- Domain of logarithmic function
  - Graphs of Logarithmic Functions - Finding the Domain of a Logarithmic Function
- Logarithms and You
  - Logarithmic Functions - Logarithms and You

Key Concepts

Theorems

- **Domain and Range of the Logarithmic Function** $f(x) = \log_b(x)$: The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
- **Properties of the Graphs of Logarithmic Functions** $f(x) = \log_b(x)$ **(where** $b > 0, b \neq 1$**)**:
  - One-to-one function.
  - Vertical asymptote: $x = 0$.
  - Domain: $(0, \infty)$.
  - Range: $(-\infty, \infty)$.
  - x-intercept: $(1, 0)$.
  - Secondary key point: $(b, 1)$.
  - y-intercept: none.
  - Increasing if $b > 1$.
  - Decreasing if $0 < b < 1$.

Common Mistakes

- **Incorrectly Identifying the Domain of Transformed Logarithmic Functions**: The argument of the logarithm must always be positive. For $f(x) = \log_b(expression)$, one must solve $expression > 0$. Errors occur if this inequality is set up or solved incorrectly, especially with reflections or multiple transformations.
- **Errors in Horizontal Shifts and Vertical Asymptote Location**: Confusing the direction of horizontal shifts (e.g., $\log_b(x + c)$ shifts left, $\log_b(x - c)$ shifts right for $c > 0$) and consequently misplacing the vertical asymptote. The vertical asymptote shifts with the horizontal shift.
- **Order of Transformations**: Applying shifts, reflections, and stretches/compressions in the wrong order, particularly when combining horizontal transformations (e.g., in $h(x) = 2\ln(-\frac{1}{4}x + 1) - 1$, factoring the argument is key: $-\frac{1}{4}(x - 4)$).
- **Misinterpreting Stretches vs. Shifts**: Confusing the effect of multiplying the function/input (stretch/compression) with adding to the function/input (shift).
- **Incorrectly Identifying the Range of Logarithmic Functions**: The range of any untransformed or vertically stretched/compressed/shifted logarithmic function $y = a\log_b(expression) + d$ is always $(-\infty, \infty)$. Horizontal transformations do not affect the range.
- **Sign Errors with Reflections**: Incorrectly applying reflections; $-f(x)$ reflects across the x-axis, and $f(-x)$ reflects across the y-axis. This also impacts the domain for $f(-x)$.
- **Graphing Absolute Value of Logarithmic Function**: Incorrectly reflecting parts of the graph. Only the portions of the original logarithmic graph that are *below* the x-axis should be reflected *above* the x-axis. Portions already above remain unchanged.

This page titled 12.4.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

12.4.2: Homework

Reading Questions

1. What is the domain of the basic logarithmic function $f(x) = \log_b(x)$? What is its range?
2. What is the equation of the vertical asymptote for the basic logarithmic function $f(x) = \log_b(x)$?
3. How does a horizontal shift by a constant c (i.e., $g(x) = \log_b(x + c)$) affect the domain and vertical asymptote of $f(x) = \log_b(x)$?
4. How does a vertical shift by a constant d (i.e., $g(x) = \log_b(x) + d$) affect the domain, range, and vertical asymptote of $f(x) = \log_b(x)$?
5. Explain how the graph of $g(x) = a \log_b(x)$ is related to the graph of $f(x) = \log_b(x)$ if $|a| > 1$. What if $0 < |a| < 1$?
6. How is the graph of $g(x) = -\log_b(x)$ related to the graph of $f(x) = \log_b(x)$?
7. How is the graph of $g(x) = \log_b(-x)$ related to the graph of $f(x) = \log_b(x)$? How does this affect the domain?
8. When graphing $f(x) = a \log_b(B(x - h)) + k$, what is a general order in which transformations can be applied to the parent graph $y = \log_b(x)$? Why is factoring the argument $B(x - h)$ important?
9. What are the two "anchor" points typically used when graphing transformations of $y = \log_b(x)$?
10. How do you graph the function $k(x) = |h(x)|$ if you know the graph of $h(x) = 2\ln(-\frac{1}{4}x + 1) - 1$?
11. The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?
12. What type(s) of transformation(s), if any, affect the range of a logarithmic function?
13. What type(s) of transformation(s), if any, affect the domain of a logarithmic function?
14. Consider the general logarithmic function $f(x) = \log_b(x)$. Why can't x be zero?
15. Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

Homework

Algebraic

For the following exercises, state the domain and range of the function.

16. $f(x) = \log_3(x + 4)$
17. $h(x) = \ln\left(\frac{1}{2} - x\right)$
18. $g(x) = \log_5(2x + 9) - 2$
19. $h(x) = \ln(4x + 17) - 5$
20. $f(x) = \log_2(12 - 3x) - 3$

For the following exercises, state the domain and the vertical asymptote of the function.

21. $f(x) = \log_b(x - 5)$
22. $g(x) = \ln(3 - x)$
23. $f(x) = \log(3x + 1)$
24. $f(x) = 3\log(-x) + 2$
25. $g(x) = -\ln(3x + 9) - 7$

For the following exercises, state the domain, vertical asymptote, and end behavior of the function.

26. $f(x) = \ln(2 - x)$
27. $f(x) = \log\left(x - \frac{3}{7}\right)$
28. $h(x) = -\log(3x - 4) + 3$
29. $g(x) = \ln(2x + 6) - 5$
30. $f(x) = \log_3(15 - 5x) + 6$

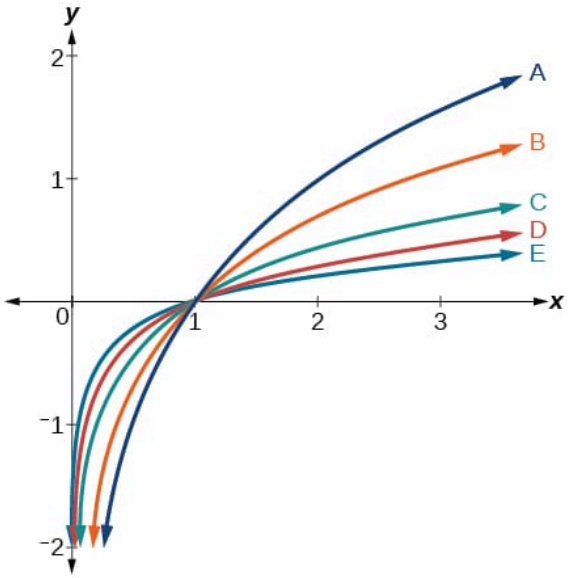
For the following exercises, state the domain, range, and x- and y-intercepts, if they exist. If they do not exist, write DNE.

31. $h(x) = \log_4(x-1)+1$
32. $f(x) = \log(5x+10)+3$
33. $g(x) = \ln(-x)-2$
34. $f(x) = \log_2(x+2)-5$
35. $h(x) = 3\ln(x)-9$

Graphical

For the following exercises, match each function in Figure 12.4.2.1 with the letter corresponding to its graph.

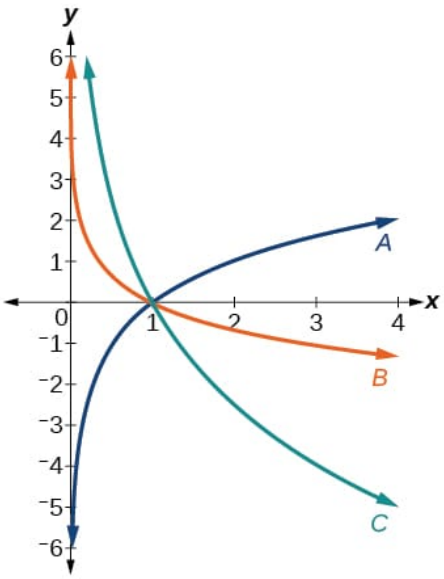
Figure 12.4.2.1



36. $d(x) = \log(x)$
37. $f(x) = \ln(x)$
38. $g(x) = \log_2(x)$
39. $h(x) = \log_5(x)$
40. $j(x) = \log_{25}(x)$

For the following exercises, match each function in Figure 12.4.2.2 with the letter corresponding to its graph.

Figure 12.4.2.2



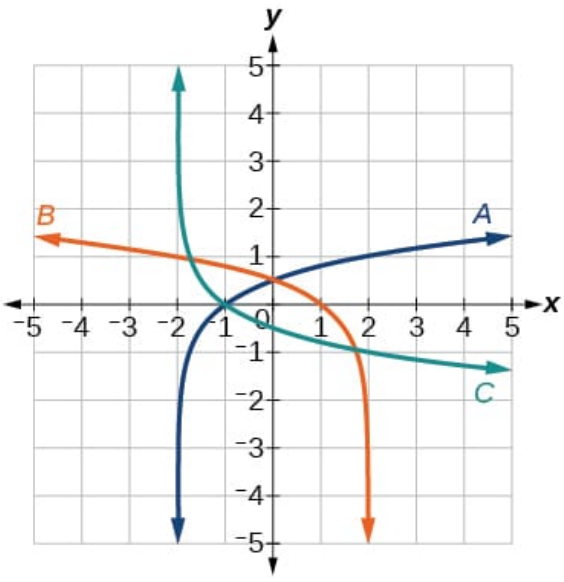
41. $f(x) = \log_{1/3}(x)$
42. $g(x) = \log_2(x)$
43. $h(x) = \log_{3/4}(x)$

For the following exercises, sketch the graphs of each pair of functions on the same axis.

44. $f(x) = \log(x)$ and $g(x) = 10^x$
45. $f(x) = \log(x)$ and $g(x) = \log_{1/2}(x)$
46. $f(x) = \log_4(x)$ and $g(x) = \ln(x)$
47. $f(x) = e^x$ and $g(x) = \ln(x)$

For the following exercises, match each function in Figure 12.4.2.3 **with the letter corresponding to its graph.**

Figure 12.4.2.3



48. $f(x) = \log_4(-x + 2)$
49. $g(x) = -\log_4(x + 2)$

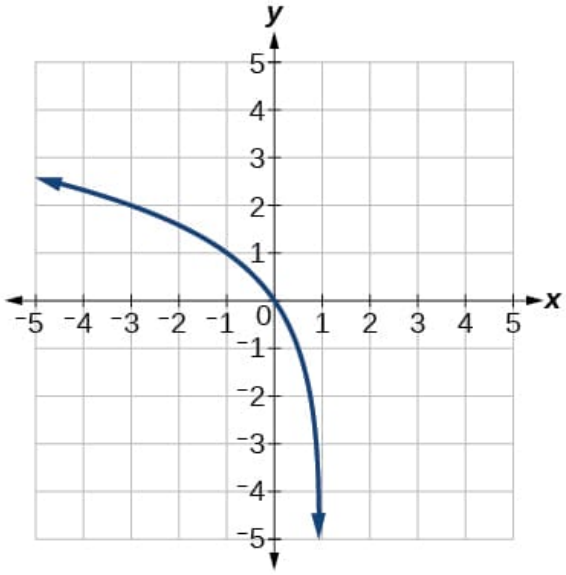
50. $h(x) = \log_4(x+2)$

For the following exercises, sketch the graph of the indicated function.

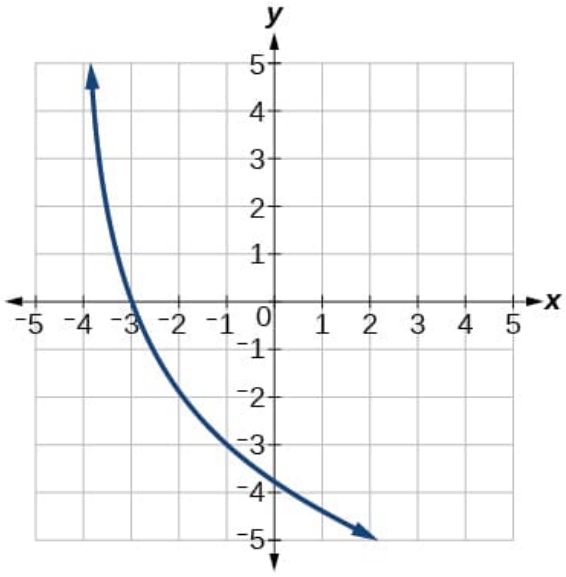
51. $f(x) = \log_2(x+2)$
52. $f(x) = 2\log(x)$
53. $f(x) = \ln(-x)$
54. $g(x) = \log(4x+16)+4$
55. $g(x) = \log(6-3x)+1$
56. $h(x) = -\frac{1}{2}\ln(x+1)-3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

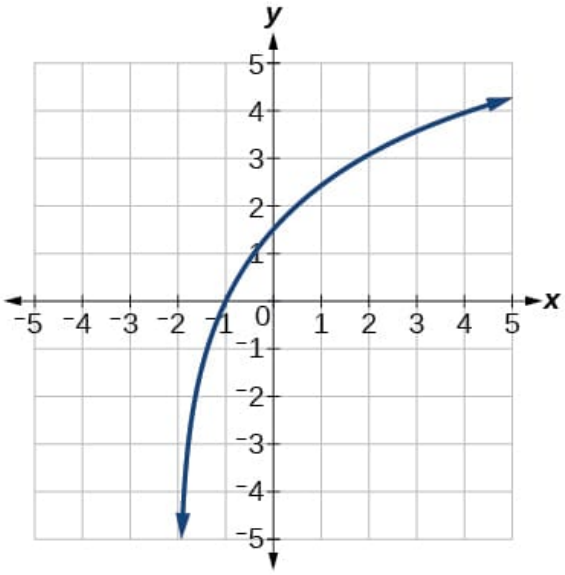
57. Use $y = \log_2(x)$ as the parent function.



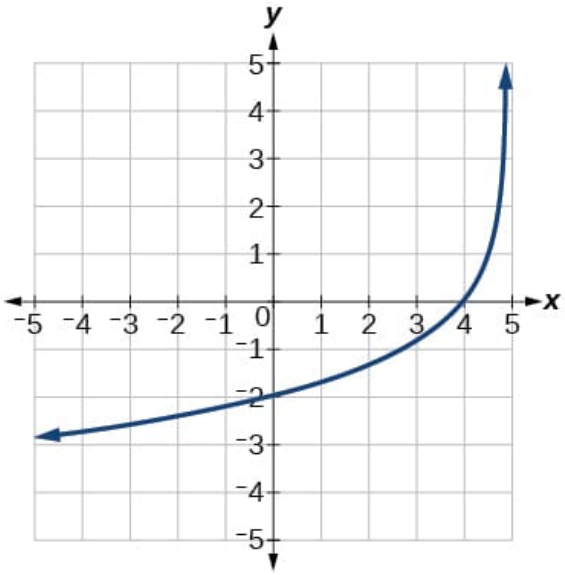
58. Use $f(x) = \log_3(x)$ as the parent function.



59. Use $f(x) = \log_4(x)$ as the parent function.



60. Use $f(x) = \log_5(x)$ as the parent function.



Technology

For the following exercises, use graphing technology to find approximate solutions to each equation.

61. $\log(x-1)+2 = \ln(x-1)+2$
62. $\log(2x-3)+2 = -\log(2x-3)+5$
63. $\ln(x-2) = -\ln(x+1)$
64. $2\ln(5x+1) = \frac{1}{2}\ln(-5x)+1$
65. $\frac{1}{3}\log(1-x) = \log(x+1)+\frac{1}{3}$

Extensions

66. Let b be any positive real number such that $b \neq 1$. What must $\log_b 1$ be equal to? Verify the result.
67. Explore and discuss the graphs of $f(x) = \log_{1/2}(x)$ and $g(x) = -\log_2(x)$. Make a conjecture based on the result.
68. Prove the conjecture made in the previous exercise.
69. What is the domain of the function $f(x) = \ln\left(\frac{x+2}{x-4}\right)$? Discuss the result.
70. Use properties of exponents to find the x-intercepts of the function $f(x) = \log(x^2+4x+4)$ algebraically. Show the steps for

solving, and then verify the result by graphing the function.

This page titled 12.4.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

12.5: Laws of Logarithms

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure 12.5.1**:** *The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan)*

In chemistry, pH is used as a measure of the acidity or alkalinity of a substance. The pH scale runs from 0 to 14. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 (at 25° C)
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where H^+, is the concentration of hydrogen ion in the solution

$$\text{pH} = -\log([H^+]) = \log\left(\frac{1}{[H^+]}\right).$$

The equivalence of $-\log([H^+])$ and $\log\left(\frac{1}{[H^+]}\right)$, is one of the Laws of Logarithms we will examine in this section.

Using the Product Law for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar laws to

exponents. Some important laws of logarithms are given here. First, the following "Base Laws" are easy to prove.

Theorem: Laws of Logarithms - Base Laws

Let b be a positive real number with $b \neq 1$. Then

$$\log_b(1) = 0 \quad \text{and} \quad \log_b(b) = 1.$$

Proofs available at the end of this section.

For example, $\log_5 1 = 0$, since $5^0 = 1$, and $\log_5 5 = 1$, since $5^1 = 5$.

Next, we have the Inverse Laws.

Theorem: Laws of Logarithms - Inverse Laws

Let b be a positive real number with $b \neq 1$. Then

$$\log_b\left(b^x\right) = x \quad \text{for every real number } x,$$

and

$$b^{\log_b(x)} = x \quad \text{for every } x > 0.$$

In particular, the functions $f(x) = b^x$ and $f^{-1}(x) = \log_b(x)$ are inverses of each other.

Proofs available at the end of this section.

To illustrate these laws, we imagine evaluating $\log(100)$. We can rewrite the logarithm as $\log_{10}(10^2)$, and then apply the Inverse Law $\log_b(b^x) = x$, to get $\log_{10}(10^2) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the Inverse Law $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the One-to-One Law.

Theorem: Laws of Logarithms - One-to-One Law

Let b, M, and N be positive real numbers with $b \neq 1$. Then

$$\log_b(M) = \log_b(N) \text{ if and only if } M = N.$$

Proof available at the end of this section.

We can use the One-to-One Law to solve the equation $\log_3(3x) = \log_3(2x + 5)$, for x. Since the bases are the same, we can apply the One-to-One Law by setting the arguments equal and solving for x:

$$\begin{aligned} 3x &= 2x + 5 && \text{(setting the arguments equal to each other)} \\ \implies x &= 5 && \text{(subtracting } 2x \text{ from both sides)} \end{aligned}$$

But what about the equation $\log_3(3x) + \log_3(2x + 5) = 2$? The One-to-One Law does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *Product Law of Exponents* to combine the product of powers by adding exponents: $x^a x^b = x^{a+b}$. We have a similar law for logarithms, called the **Product Law for Logarithms**.

Theorem: Laws of Logarithms - Product Law

Let b, M, and N be positive real numbers with $b \neq 1$. Then

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

That is, the logarithm of a product is the sum of the logarithms of its factors.

Proof available at the end of this section.

Note that repeated applications of the Product Law for Logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the Product Law for Logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b(w) + \log_b(x) + \log_b(y) + \log_b(z).$$

Example 12.5.1: Using the Product Law for Logarithms

Expand $\log_3(30x(3x+4))$.

Solution available at the end of this section.

Checkpoint 12.5.1

Expand $\log_b(8k)$.

Using the Quotient Law for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *Quotient Law of Exponents* to combine the quotient of exponents by subtracting: $\frac{x^a}{x^b} = x^{a-b}$. The **Quotient Law for Logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the Product Law, we can use the Inverse Law to derive the Quotient Law.

Theorem: Laws of Logarithms - Quotient Law

Let b, M, and N be positive real numbers with $b \neq 1$. Then

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

That is, the logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator.

Proof available at the end of this section.

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{aligned}
\log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{(factoring out the GCF from numerator and denominator)} \\
&= \log\left(\frac{2x\ \cancel{(x+3)}}{3\ \cancel{(x+3)}}\right) && \text{(canceling like factors)} \\
&= \log\left(\frac{2x}{3}\right) && \text{(simplifying)}
\end{aligned}$$

Next we apply the Quotient Law by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the Product Law.

$$\log\left(\frac{2x}{3}\right) = \log(2x) - \log(3) = \log(2) + \log(x) - \log(3).$$

Caution: Assumptions Abound

When using the Product and Quotient Laws of Logarithms, we are *assuming* (often falsely) that the component logarithms are well-defined. That is, when expanding $\log\left(\frac{2x^2+6x}{3x+9}\right)$ to arrive at $\log(2) + \log(x) - \log(3)$, we are assuming

1. $x > 0$, and
2. $x + 3 \neq 0$ (remember, this is the factor we canceled in the process of simplifying the logarithm).

These assumptions are likely *not correct*! However, when *practicing* the Laws of Logarithms, we allow ourselves the "training wheels" of assuming everything is going to work out. When we start to solve equations involving logarithms, on the other hand, we remove those "training wheels" and become concerned with the possibility that the component logarithms are not well-defined.

Example 12.5.2: Using the Quotient Law for Logarithms

Expand

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right).$$

Solution available at the end of this section.

Let's restate our "Caution" from above. There are exceptions to consider in Example 12.5.2. First, because denominators must never be zero, the expression in Example 12.5.2 is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

Checkpoint 12.5.2

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

Using the Power Law for Logarithms

We've explored the Product Law and the Quotient Law, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\begin{aligned} \log_b(x^2) &= \log_b(x \cdot x) && \text{(Laws of Exponents)} \\ &= \log_b x + \log_b x && \text{(Laws of Logarithms: Product Law)} \\ &= 2\log_b x && \text{(combining like terms)} \end{aligned}$$

Notice that we used the Product Law for Logarithms to find a solution for the example above. By doing so, we have derived the **Power Law for Logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \quad \sqrt{3} = 3^{1/2} \quad \frac{1}{e} = e^{-1}.$$

Theorem: Laws of Logarithms - Power Law

Let b and M be positive real numbers with $b \neq 1$, and let p be any real number. Then

$$\log_b(M^p) = p\log_b(M).$$

Proof available at the end of this section.

Example 12.5.3: Expanding a Logarithm with Powers

Expand $\log_2(x^5)$.

Solution available at the end of this section.

Checkpoint 12.5.3

Expand $\ln(x^2)$.

Example 12.5.4: Rewriting an Expression as a Power before Using the Power Law

Expand $\log_3(25)$, using the Power Law for Logs.

Solution available at the end of this section.

Checkpoint 12.5.4

Expand $\ln\left(\frac{1}{x^2}\right)$.

Example 12.5.5: Using the Power Law in Reverse

Rewrite $4\ln(x)$, using the Power Law for logs to a single logarithm with a leading coefficient of 1.

Solution available at the end of this section.

Checkpoint 12.5.5

Rewrite $2\log_3 4$, using the Power Law for logs to a single logarithm with a leading coefficient of 1.

Expanding Logarithmic Expressions

Taken together, the laws introduced in this section so far are collectively called the "Laws of Logarithms." Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the Power Law to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the Quotient Law for Logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b(AC^{-1}) \\ &= \log_b(A)+\log_b(C^{-1}) \\ &= \log_b A+(-1)\log_b C \\ &= \log_b A-\log_b C\end{aligned}$$

We can also apply the Product Law to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Example 12.5.6: Expanding Logarithms Using Product, Quotient, and Power Laws

Rewrite $\ln\left(\frac{x^4y}{7}\right)$, as a sum or difference of logs.

Solution available at the end of this section.

Checkpoint 12.5.6

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

Example 12.5.7: Using the Power Law for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log(\sqrt{x})$.

Solution available at the end of this section.

Checkpoint 12.5.7

Expand $\ln(\sqrt[3]{x^2})$.

Q&A

Can we expand $\ln(x^2+y^2)$?

No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

Example 12.5.8: Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{2x-1}\right)$.

Solution available at the end of this section.

Checkpoint 12.5.8

Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{x^2-9}\right)$.

Condensing Logarithmic Expressions

We can use the Laws of Logarithms to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that *the logarithms must have the same base to be combined*. We will learn later how to change the base of any logarithm before condensing. Because students sometimes make mistakes on the order with which they apply the Laws of Logarithms, we provide the following "How To" as a beginner's reference guide.

How To

Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1. Apply the Power Law first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the Product Law. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the Quotient Law last. Rewrite differences of logarithms as the logarithm of a quotient.

Example 12.5.9: Using the Product and Quotient Laws to Combine Logarithms

Write $\log_3(5)+\log_3(8)-\log_3(2)$ as a single logarithm.

Solution available at the end of this section.

Checkpoint 12.5.9

Condense $\log 3-\log 4+\log 5-\log 6$.

Example 12.5.10: Condensing Complex Logarithmic Expressions

Condense $\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2((x+3)^2)$.

Solution available at the end of this section.

Checkpoint 12.5.10

Rewrite $\log(5)+0.5\log(x)-\log(7x-1)+3\log(x-1)$ as a single logarithm.

Example 12.5.11: Rewriting as a Single Logarithm

Rewrite $2\log x-4\log(x+5)+\frac{1}{x}\log(3x+5)$ as a single logarithm.

Solution available at the end of this section.

Checkpoint 12.5.11

Condense $4(3\log(x)+\log(x+5)-\log(2x+3))$.

Example 12.5.12: Applying of the Laws of Logs

Recall that, in chemistry, $\text{pH}=-\log[H^+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution available at the end of this section.

Checkpoint 12.5.12

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e, we use the **Change-of-Base Formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

Theorem: Change-of-Base Formula

Let b, M, and n be positive real nubmers with $n\neq 1$ and $b\neq 1$. Then

$$\log_b(M)=\frac{\log_n(M)}{\log_n(b)}.$$

In particular, rewriting in terms of the natural logarithm or the common (base 10) logarithm gives

$$\log_b(x)=\frac{\ln(x)}{\ln(b)}=\frac{\log(x)}{\log(b)}.$$

Proof available at the end of this section.

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\log_5 36=\frac{\log(36)}{\log(5)}\approx 2.2266.$$

Example 12.5.13: Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5(3)$ to a quotient of natural logarithms.

Solution available at the end of this section.

Checkpoint 12.5.13

Change $\log_{0.5}(8)$ to a quotient of natural logarithms.

Q&A

Can we change common logarithms to natural logarithms?

Yes. Remember that $\log 9$, means $\log_{10} 9$. So, $\log 9 = \frac{\ln 9}{\ln 10}$.

Example 12.5.14: Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$, using the Change-of-Base Formula with a calculator.

Solution available at the end of this section.

Checkpoint 12.5.14

Evaluate $\log_5(100)$, using the Change-of-Base Formula.

Success in College: Environment and Learning Preferences

Throughout your academic career, you have completed many exams. While you might not have thought about it, you should be able to identify a strong correlation between the environment you study in and your performance on exams.

Learning preferences describe the situations in which you best learn math. Are you a visual, auditory, or kinesthetic learner? Should you take notes as you listen? Should you record your class? Should you trace the graphs in your book as you study?

Think about whether you learn better by working by yourself or with others. Do you prefer to work in quiet or with music in the background? Do you like to sit at a desk or recline in an easy chair? Do you learn best at a particular time of the day? When you study, does it help if you can move around? All of these are part of your learning preferences.

If you know your learning preferences, you can try to adjust your studying and class behavior to improve your work. It is your responsibility to think about and, if needed, make changes in how you are learning and studying.

Think about the following list. When you study, do you prefer...

- a quiet room, or background noise (music or TV)
- bright light shining on your work, or dim lighting
- sitting at a desk or table, or being relaxed on a couch or a bed
- eating or drinking while you learn, or only eating or drinking during breaks
- working hard for a reasonably long time or taking frequent breaks

1. Do you work in conditions that match these preferences when you do your math homework?
2. Do you work in conditions that match these preferences when you study for math tests?

Success in Mathematics: Sequential Versus Global Learners

> Sequential learners tend to think in a linear manner and are able to function with only partial understanding of material they have been taught. Global learners think in a systems-oriented manner, and may have trouble applying new material until they fully understand it and see how it relates to material they already know about and understand. Once they grasp the big picture, however, their holistic perspective enables them to see innovative solutions to problems that sequential learners might take much longer to reach.
>
> Source: R. Felder, R. Brent , Journal of Engineering Education, 94 (1), 57-72 (2005).

1. When learning math, do you think you are more of a sequential learner or a global learner?
2. Students often have different learning preferences for different subjects. When learning history or another subject in the humanities, do you think you are more a sequential learner or a global learner?

Answers

Section Preview

This is the typical "Laws of Logarithms" material that you would find in any College Algebra textbook.

Section Preview

- Use the Product Law for logarithms.
- Use the Quotient Law for logarithms.
- Use the Power Law for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the Change-of-Base Formula for logarithms.

Theorem: Laws of Logarithms - Base Laws

By definition of a logarithm, $\log_b(1)$ is the exponent, p, for which $b^p = 1$. Therefore, $p = 0$. As such,

$$\log_b(1) = 0.$$

Similarly, $\log_b(b)$ is the exponent, q, such that $b^q = b$. Since $b = b^1$, this means that $b^q = b^1$. In other words, $q = 1$. Hence,

$$\log_b(b) = 1.$$

Theorem: Laws of Logarithms - Inverse Laws

Consider $\log_b(b^x)$. By the definition of a logarithm, this is the exponent, p, such that $b^p = b^x$. That is, $x = p$. Thus, $\log_b(b^x) = x$.

Now consider $b^{\log_b(x)}$, where $x > 0$. Recall that $\log_b(x)$ is the exponent, q, such that $b^q = x$. That is,

$$b^{\log_b(x)} = b^q = x.$$

Theorem: Laws of Logarithms - One-to-One Law

From the definition of a logarithm, if we let $\log_b(M) = m$ and $\log_b(N) = n$, then $b^m = M$ and $b^n = N$.

( ): If $\log_b(M) = \log_b(N)$, then $m = n$. However, if $m = n$, then $b^m = b^n$. Therefore, $M = N$.

( ): If $M = N$, then $b^m = b^n$. However, by laws of exponents, this means $m = n$. Hence, $\log_b(M) = \log_b(N)$.

Theorem: Laws of Logarithms - Product Law

Let $m = \log_b(M)$, and $n = \log_b(N)$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{(substituting)} \\ &= \log_b(b^{m+n}) && \text{(Laws of Exponents: Product Rule)} \\ &= m + n && \text{(Laws of Logarithms: Inverse Law)} \\ &= \log_b(M) + \log_b(N) && \text{(substituting)} \end{aligned}$$

Example 12.5.1: Using the Product Law for Logarithms

We begin by recognizing the argument as a product of three factors.

$$\log_3(30x(3x+4)) = \log_3(30 \cdot x \cdot (3x+4)).$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30 \cdot x \cdot (3x+4)) = \log_3(30) + \log_3(x) + \log_3(3x+4).$$

Finally, we need to be aware of the base and the possibility that the 30 can be factored into a power of the base and some other constant. Specifically, $30 = 3 \cdot 10$. Therefore,

$$\begin{aligned} \log_3(30x(3x+4)) &= \log_3(3 \cdot 10) + \log_3(x) + \log_3(3x+4) \\ &= \log_3(3) + \log_3(10) + \log_3(x) + \log_3(3x+4) && \text{(Laws of Logarithms: Product Law)} \\ &= 1 + \log_3(10) + \log_3(x) + \log_3(3x+4) && \text{(Laws of Logarithms: } \log_b b = 1) \end{aligned}$$

Theorem: Laws of Logarithms - Quotient Law

Let $m = \log_b(M)$, and $n = \log_b(N)$. In exponential form, these equations are $b^m = M$, and $b^n = N$. It follows that

$$\begin{aligned} \log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{(substituting)} \\ &= \log_b\left(b^{m-n}\right) && \text{(Laws of Exponents: Quotient Law)} \\ &= m - n && \text{(Laws of Logarithms: Inverse Law)} \\ &= \log_b(M) - \log_b(N) && \text{(substituting)} \end{aligned}$$

Example 12.5.2: Using the Quotient Law for Logarithms

First we note that the quotient is factored and in lowest terms, so we apply the Quotient Law.

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2(15x(x-1)) - \log_2((3x+4)(2-x)).$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the Product Law, noting that 15 is not divisible by a power of the base (2).

$$\begin{aligned} \log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= \log_2(15) + \log_2(x) + \log_2(x-1) - [\log_2(3x+4) + \log_2(2-x)] && \text{(Product Law)} \\ &= \log_2(15) + \log_2(x) + \log_2(x-1) - \log_2(3x+4) - \log_2(2-x) && \text{(distributing)} \end{aligned}$$

Theorem: Laws of Logarithms - Power Law

Let $\log_b(M) = m$.

$$\begin{aligned} \implies b^m &= M && \text{(definition of the logarithm)} \\ \implies (b^m)^p &= M^p && \text{(Laws of Exponents)} \\ \implies b^{mp} &= M^p && \text{(Laws of Exponents)} \\ \implies \log_b\left(b^{mp}\right) &= \log_b\left(M^p\right) && \text{(One-to-One Law of Logarithms)} \\ \implies \log_b\left(b^m \cdot b^p\right) &= \log_b\left(M^p\right) && \text{(Laws of Exponents)} \\ \implies \log_b\left(b^m\right) \cdot \log_b\left(b^p\right) &= \log_b\left(M^p\right) && \text{(Product Law of Logarithms)} \\ \implies m \cdot p &= \log_b\left(M^p\right) && \text{(Base Laws of Logarithms)} \\ \implies \log_b(M) \cdot p &= \log_b\left(M^p\right) && \text{(our original substitution } \log_b(M) = m\text{)} \\ \implies p\log_b(M) &= \log_b\left(M^p\right) && \text{(Commutative Property of Multiplication)} \end{aligned}$$

Example 12.5.3: Expanding a Logarithm with Powers

The argument is already written as a power, so we identify the exponent, 5, and the base, x, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2(x^5) = 5\log_2 x$$

Example 12.5.4: Rewriting an Expression as a Power before Using the Power Law

Expressing the argument as a power, we get $\log_3(25) = \log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(5^2) = 2\log_3(5)$$

Example 12.5.5: Using the Power Law in Reverse

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, x, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

Example 12.5.6: Expanding Logarithms Using Product, Quotient, and Power Laws

First, because we have a quotient of two expressions, we can use the Quotient Law:

$$\ln\left(\frac{x^4y}{7}\right) = \ln(x^4y) - \ln(7).$$

Then seeing the product in the first term, we use the Product Law:

$$\ln(x^4y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7).$$

Finally, we use the Power Law on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7).$$

Example 12.5.7: Using the Power Law for Logarithms to Simplify the Logarithm of a Radical Expression

$$\log(\sqrt{x}) = \log x^{1/2} = \frac{1}{2}\log x$$

Example 12.5.8: Expanding Complex Logarithmic Expressions

We can expand by applying the Product and Quotient Laws.

$$\begin{aligned} \log_6\left(\frac{64x^3(4x+1)}{2x-1}\right) &= \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) \\ &= \log_6 64 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1) \end{aligned}$$

Note that we *could* have rewritten $\log_6 64$ as $\log_6 2^6 = 6\log_6 2$, however, this is not any better than $\log_6 64$. When it comes to logarithms of constants, we are only concerned with the possibility that a power of the base is a factor of the constant.

Example 12.5.9: Using the Product and Quotient Laws to Combine Logarithms

Using the product and Quotient Laws

$$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40).$$

This reduces our original expression to

$$\log_3(40) - \log_3(2).$$

Then, using the Quotient Law

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20).$$

Example 12.5.10: Condensing Complex Logarithmic Expressions

We apply the Power Law first:

$$\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2((x+3)^2)=\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2((x+3)^6).$$

Next we apply the Product Law to the sum:

$$\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2((x+3)^6)=\log_2(x^2\sqrt{x-1})-\log_2((x+3)^6).$$

Finally, we apply the Quotient Law to the difference:

$$\log_2(x^2\sqrt{x-1})-\log_2((x+3)^6)=\log_2\left(\frac{x^2\sqrt{x-1}}{(x+3)^6}\right).$$

Example 12.5.11**: Rewriting as a Single Logarithm**

We apply the Power Law first:

$$2\log x-4\log(x+5)+\frac{1}{x}\log(3x+5)=\log(x^2)-\log\left[(x+5)^4\right]+\log\left[(3x+5)^{x^{-1}}\right].$$

Next, we rearrange and apply the Product Law to the sum:

$$\begin{aligned}\log(x^2)-\log\left[(x+5)^4\right]+\log\left[(3x+5)^{x^{-1}}\right] &= \log(x^2)+\log\left[(3x+5)^{x^{-1}}\right]-\log\left[(x+5)^4\right]\\ &= \log\left[x^2(3x+5)^{x^{-1}}\right]-\log\left[(x+5)^4\right]\end{aligned}$$

Finally, we apply the Quotient Law to the difference:

$$\log\left[x^2(3x+5)^{x^{-1}}\right]-\log\left[(x+5)^4\right]=\log\left[\frac{x^2(3x+5)^{x^{-1}}}{(x+5)^4}\right].$$

Example 12.5.12**: Applying of the Laws of Logs**

Suppose C, is the original concentration of hydrogen ions, and P, is the original pH of the liquid. Then $P=-\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$\text{pH}=-\log(2C).$$

Using the Product Law of logs

$$\text{pH}=-\log(2C)=-(\log(2)+\log(C))=-\log(2)-\log(C).$$

Since $P=-\log(C)$, the new pH is

$$\text{pH}=P-\log(2)\approx P-0.301.$$

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

Theorem: Change-of-Base Formula

Let $y=\log_b(M)$. By the definition of a logarithm, $b^y=M$. It follows that

$$\begin{aligned}&\log_n(b^y) &=& \log_n(M) && \text{(One-to-One Law of Logarithms)}\\ \Longrightarrow\quad &y\log_n(b) &=& \log_n(M) && \text{(Power Law of Logarithms)}\\ \Longrightarrow\quad &y &=& \frac{\log_n(M)}{\log_n(b)} && \text{(dividing both sides by } \log_n(b))\\ \Longrightarrow\quad &\log_b(M) &=& \frac{\log_n(M)}{\log_n(b)} && \text{(substituting)}\end{aligned}$$

Example 12.5.13**: Changing Logarithmic Expressions to Expressions Involving Only Natural Logs**

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n=e$.

We rewrite the log as a quotient using the Change-of-Base Formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_5 3=\frac{\ln 3}{\ln 5}.$$

Example 12.5.14**: Using the Change-of-Base Formula with a Calculator**

According to the Change-of-Base Formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

$$\log_2 10=\frac{\ln 10}{\ln 2}\approx 3.3219.$$

This page titled 12.5: Laws of Logarithms is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

12.5.1: Resources and Key Concepts

Resources

Videos

- Laws of Logarithms
  - Intermediate Algebra - Logarithms: Properties of Logarithms
  - Logarithmic Properties - A Brief Review of the Properties of Logarithms
  - Logarithmic Functions - A Brief Review of Properties of Logarithms
  - Logarithmic Properties - Using Properties of Logarithms
- Expanding and condensing logarithmic expressions
  - Logarithmic Functions - Expanding and Combining Logarithms

Key Concepts

Theorems

- **Laws of Logarithms - Base Laws**:
  - $\log_b(1) = 0$
  - $\log_b(b) = 1$
- **Laws of Logarithms - Inverse Laws**:
  - $\log_b(b^x) = x$
  - $b^{\log_b(x)} = x$, where $x > 0$
- **Laws of Logarithms - One-to-One Law**: $\log_b(M) = \log_b(N)$ if and only if $M = N$.
- **Laws of Logarithms - Product Law**: Given any real number x, and positive real numbers M and N, $\log_b(MN) = \log_b(M) + \log_b(N)$.
- **Laws of Logarithms - Quotient Law**: Given any real number x, and positive real numbers M and N, $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$.
- **Laws of Logarithms - Power Law**: For any real number p and positive real number M, $\log_b(M^p) = p\log_b(M)$.
- **Change-of-Base Formula**: Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$, $\log_b(M) = \frac{\log_n(M)}{\log_n(b)}$.

Common Mistakes

- **Incorrectly Applying Logarithm Laws**:
  - $\log_b(M+N) \neq \log_b(M) + \log_b(N)$ (Log of a sum is not sum of logs).
  - $\log_b(M-N) \neq \log_b(M) - \log_b(N)$ (Log of a difference is not difference of logs).
  - $\frac{\log_b(M)}{\log_b(N)} \neq \log_b(M) - \log_b(N)$ (Quotient of logs is not log of a quotient).
  - $(\log_b(M))^p \neq p\log_b(M)$ (Power of a log is not product of exponent and log, unless the power is on the argument).
- **Errors in Expanding Logarithms**: Not fully expanding an expression, or incorrectly applying laws (e.g., distributing a log over a sum/difference inside the argument).
- **Errors in Condensing Logarithms**:
  - Incorrectly combining terms, especially with subtraction (order matters for division).
  - Not applying the power law first to make coefficients into exponents before combining with product/quotient laws.
  - Combining logarithms with different bases.
- **Ignoring Domain Restrictions (Conceptual vs. Practical Application)**: While the "Caution: Assumptions Abound" notes that for practice with laws, well-definedness is assumed, in solving equations or other applications, the domain of each logarithmic expression must be considered.
- **Errors with Change-of-Base Formula**: Incorrectly setting up the ratio, e.g., $\frac{\log_n(b)}{\log_n(M)}$ instead of $\frac{\log_n(M)}{\log_n(b)}$.
- **Calculator Errors with Change-of-Base**: Using the wrong base on the calculator or inputting the fraction incorrectly.

This page titled 12.5.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

12.5.2: Homework

Reading Questions

1. What is the Product Law for Logarithms? State the formula.
2. What is the Quotient Law for Logarithms? State the formula.
3. What is the Power Law for Logarithms? State the formula.
4. How does the Power Law for Logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$?
5. When expanding a complex logarithmic expression, what is the general order in which you might apply the Product, Quotient, and Power Laws?
6. When condensing multiple logarithmic terms into a single logarithm, what must be true about the bases of the logarithms?
7. When condensing an expression like $2\log(x) + 3\log(y) - \log(z)$, which law should generally be applied first to the terms $2\log(x)$ and $3\log(y)$?
8. What is the Change-of-Base Formula for logarithms? Why is it particularly useful when using a calculator?
9. Rewrite $\log_5(36)$ as a quotient of common logarithms (base 10).
10. Rewrite $\log_5(3)$ as a quotient of natural logarithms (base e).
11. Can $\log_b(A + C)$ be expanded using the Product Law for Logarithms? Why or why not?
12. How does the Power Law for Logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$?
13. What does the change-of-base formula do? Why is it useful when using a calculator?

Homework

Algebraic

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

14. $\log_b(7x \cdot 2y)$
15. $\ln(3ab \cdot 5c)$
16. $\log_b\left(\frac{13}{17}\right)$
17. $\log_4\left(\frac{x/z}{w}\right)$
18. $\ln\left(\frac{1}{4^k}\right)$
19. $\log_2(y^x)$

For the following exercises, condense to a single logarithm if possible.

20. $\ln(7) + \ln(x) + \ln(y)$
21. $\log_3(2) + \log_3(a) + \log_3(11) + \log_3(b)$
22. $\log_b(28) - \log_b(7)$
23. $\ln(a) - \ln(d) - \ln(c)$
24. $-\log_b\left(\frac{1}{7}\right)$
25. $\frac{1}{3}\ln(8)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

26. $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$
27. $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$
28. $\log(\sqrt{x^3y^{-4}})$
29. $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$
30. $\log(x^2y^3\sqrt[3]{x^2y^5})$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

31. $\log(2x^4) + \log(3x^5)$
32. $\ln(6x^9) - \ln(3x^2)$
33. $2\log(x) + 3\log(x+1)$
34. $\log(x) - \frac{1}{2}\log(y) + 3\log(z)$
35. $4\log_7(c) + \frac{\log_7(a)}{3} + \frac{\log_7(b)}{3}$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

36. $\log_7(15)$ to base e
37. $\log_{14}(55.875)$ to base 10

For the following exercises, suppose $\log_5(6) = a$ and $\log_5(11) = b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b. Show the steps for solving.

38. $\log_{11}(5)$
39. $\log_6(55)$
40. $\log_{11}\left(\frac{6}{11}\right)$

Numeric

For the following exercises, use properties of logarithms to evaluate without using a calculator.

41. $\log_3\left(\frac{1}{9}\right) - 3\log_3(3)$
42. $6\log_8(2) + \frac{\log_8(64)}{3\log_8(4)}$
43. $2\log_9(3) - 4\log_9(3) + \log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

44. $\log_3(22)$
45. $\log_8(65)$
46. $\log_6(5.38)$
47. $\log_4\left(\frac{15}{2}\right)$
48. $\log_{1/2}(4.7)$

Extensions

49. Use the Product Law of Logarithms to find all x values such that $\log_{12}(2x+6) + \log_{12}(x+2) = 2$. Show the steps for solving.
50. Use the Quotient Law of Logarithms to find all x values such that $\log_6(x+2) - \log_6(x-3) = 1$. Show the steps for solving.
51. Can the Power Law of Logarithms be derived from the Power Law of Exponents using the equation $b^x = m$? If not, explain why. If so, show the derivation.
52. Prove that $\log_b(n) = \frac{1}{\log_n(b)}$ for any positive integers $b > 1$ and $n > 1$.
53. Does $\log_{81}(2401) = \log_3(7)$? Verify the claim algebraically.

This page titled 12.5.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

12.6: Exponential and Logarithmic Equations

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.



Figure 12.6.1**:** *Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the "rabbit plague." (credit: Richard Taylor, Flickr)*

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential equations.

Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall the following theorem from Algebra (the proof of which requires recalling that logarithms "undo" exponential functions).

Theorem: One-to-One Property of Exponents

For any algebraic expressions S and T, and any positive real number $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T.$$

Proof available at the end of this section.

In other words, when an exponential equation has the same base on each side, the exponents must be equal. Therefore, we can solve many exponential equations by using the Laws of Exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7} = \frac{3^{2x}}{3}$. To solve for x, we use the Quotient Rule for Exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the One-to-One Property of exponents by setting the exponents equal to one another and solving for x:

$$\begin{aligned}
& 3^{4x-7} = \frac{3^{2x}}{3} && \\
\Longrightarrow\quad & 3^{4x-7} = \frac{3^{2x}}{3^1} && (3 = 3^1) \\
\Longrightarrow\quad & 3^{4x-7} = 3^{2x-1} && (\text{Laws of Exponents: Quotient Rule}) \\
\Longrightarrow\quad & 4x - 7 = 2x - 1 && (\text{One-to-One Property of Exponents}) \\
\Longrightarrow\quad & 2x = 6 && (\text{subtracting } 2x \text{ and adding } 7 \text{ to both sides}) \\
\Longrightarrow\quad & x = 3 && (\text{dividing both sides by } 2)
\end{aligned}$$

Example 12.6.1: Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

Solution available at the end of this section.

Checkpoint 12.6.1

Solve $5^{2x} = 5^{3x+2}$.

Rewriting Equations So All Powers Have the Same Base

Sometimes the common base for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the One-to-One Property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the Laws of Exponents, along with the One-to-One Property, to solve for x:

$$\begin{aligned}
& 256 = 4^{x-5} && \\
\Longrightarrow\quad & 4^4 = 4^{x-5} && (\text{rewriting both sides in the same base}) \\
\Longrightarrow\quad & 4 = x - 5 && (\text{One-to-One Property}) \\
\Longrightarrow\quad & 9 = x && (\text{adding } 5 \text{ to both sides})
\end{aligned}$$

Example 12.6.2: Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2} = 16^{x+1}$.

Solution available at the end of this section.

Checkpoint 12.6.2

Solve $5^{2x} = 25^{3x+2}$.

Example 12.6.3: Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve $2^{5x} = \sqrt{2}$.

Solution available at the end of this section.

Checkpoint 12.6.3

Solve $5^{x} = \sqrt{5}$.

Q&A

Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?

No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.

Example 12.6.4: Solving an Equation with Positive and Negative Powers

Solve $3^{x+1} = -2$.

Solution available at the end of this section.

Checkpoint 12.6.4

Solve $2^{x} = -100$.

Solving Exponential Equations Using Logarithms

Most of the time the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

Example 12.6.5: Solving an Equation Containing Powers of Different Bases

Solve $5^{x+2} = 4^{x}$.

Solution available at the end of this section.

Checkpoint 12.6.5

Solve $2^x = 3^{x+1}$.

Q&A

Is there any way to solve $2^x = 3^x$ without logarithms?

Yes. The solution is 0.

Equations Containing e

Exponential equations involving the base e are very common in Science. This constant occurs again and again in nature, in Mathematics, in Science, in Engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

Example 12.6.6: Solve an Equation of the Form $y = Ae^{kt}$

Solve $100 = 20e^{2t}$.

Solution available at the end of this section.

Checkpoint 12.6.6

Solve $3e^{0.5t} = 11$.

Q&A

Does every equation of the form $y = Ae^{kt}$ have a solution?

No. There is a solution when $k \neq 0$, and when y and A are either both 0 or neither 0, and they have the same sign. An example of an equation with this form that has no solution is $2 = -3e^t$.

Example 12.6.7: Solving an Equation That Can Be Simplified to the Form $y = Ae^{kt}$

Solve $4e^{2x} + 5 = 12$.

Solution available at the end of this section.

Checkpoint 12.6.7

Solve $3 + e^{2t} = 7e^{2t}$.

Extraneous Solutions

Sometimes the methods used to solve an equation introduce an extraneous solution, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the

number we are evaluating in a logarithm function is negative, there is no output.

Example 12.6.8: Solving Exponential Equations in Quadratic Form

Solve $e^{2x} - e^x = 56$.

Solution available at the end of this section.

When we plan to use factoring to solve a problem, as we did in Example 12.6.8, we always get zero on one side of the equation. This is because zero has the unique property that when a product is zero, one or both of the factors must be zero. We rejected the equation $e^x = -7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

Checkpoint 12.6.8

Solve $e^{2x} = e^x + 2$.

Q&A

Does every logarithmic equation have a solution?

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the Laws of Logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use Laws of Logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x:

$$\begin{aligned}
& \log_2(2) + \log_2(3x-5) = 3 && \\
\Longrightarrow\ & \log_2(2(3x-5)) = 3 && \text{(Laws of Logarithms)} \\
\Longrightarrow\ & \log_2(6x-10) = 3 && \text{(distributing)} \\
\Longrightarrow\ & 6x - 10 = 2^3 && \text{(definition of a logarithm)} \\
\Longrightarrow\ & 6x - 10 = 8 && \text{(simplifying)} \\
\Longrightarrow\ & 6x = 18 && \text{(adding 10 to both sides)} \\
\Longrightarrow\ & x = 3 && \text{(dividing both sides by 6)}
\end{aligned}$$

As you can see, when solving a logarithmic equation, we rely on the following definition of logarithms:

$$\text{For any algebraic expression } S \text{ and real numbers } b \text{ and } c, \text{ where } b > 0 \text{ and } b \neq 1, \log_b(S) = c \text{ if and only if } b^c = S.$$

Example 12.6.9: Using Algebra to Solve a Logarithmic Equation

Solve $2\ln(x) + 3 = 7$.

Solution available at the end of this section.

Checkpoint 12.6.9

Solve $6 + \ln x = 10$.

Example 12.6.10: Using Algebra Before and After Using the Definition of the Natural Logarithm

Solve $2\ln(6x) = 7$.

Solution available at the end of this section.

Checkpoint 12.6.10

Solve $2\ln(x + 1) = 10$.

Example 12.6.11: Showcasing the Need to Check Solutions

Solve $\log_7(1 - 2x) = 1 - \log_7(3 - x)$.

Solution available at the end of this section.

Example 12.6.12: Using a Graph to Understand the Solution to a Logarithmic Equation

Solve $\ln x = 3$ and use a graph to interpret the solution.

Solution available at the end of this section.

Checkpoint 12.6.12

Use graphing technology to estimate the approximate solution to the exponential equation $2^x = 1000$ to two decimal places.

Interactive Element: Solving $2^x = c$ as an Intersection

The following Interactive Element might help you visualize how solving an exponential equation corresponds to finding where two graphs meet.

Interact: Drag the red point up and down the vertical axis to change the target height c of the line $y = c$.

Observation: The green point sits where $y = 2^x$ meets $y = c$, and its x-coordinate is the solution of $2^x = c$. Because 2^x is always positive, dragging the line to $c \leq 0$ leaves the graphs never touching — the equation has no solution.

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the One-to-One Property to solve logarithmic equations. The One-to-One Property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{ then } x-1=8.$$

So, if $x-1=8$, then we can solve for x, and we get $x=9$. To check, we can substitute $x=9$ into the original equation: $\log_2(9-1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use the Laws of Logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the One-to-One Property to solve for x:

$$\begin{aligned}
& \log(3x-2) - \log(2) = \log(x+4) && \\
\Longrightarrow \quad & \log\left(\frac{3x-2}{2}\right) = \log(x+4) && \text{(Laws of Logarithms)} \\
\Longrightarrow \quad & \frac{3x-2}{2} = x+4 && \text{(One-to-One Property)} \\
\Longrightarrow \quad & 3x-2 = 2x+8 && \text{(multiplying both sides by 2)} \\
\Longrightarrow \quad & x = 10 && \text{(subtracting } 2x \text{ and adding 2 to both sides)}
\end{aligned}$$

To check the result, substitute $x=10$ into the original equation.

$$\begin{aligned}
& \log(3(10)-2) - \log(2) \stackrel{?}{=} \log(10+4) \\
\Longrightarrow \quad & \log(28) - \log(2) \stackrel{?}{=} \log(14) \\
\Longrightarrow \quad & \log\left(\frac{28}{2}\right) \stackrel{?}{=} \log(14) \\
\Longrightarrow \quad & \log(14) \stackrel{\checkmark}{=} \log(14)
\end{aligned}$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

Example 12.6.13: Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2) = \ln(2x+3)$.

Solution available at the end of this section.

Checkpoint 12.6.13

Solve $\ln(x^2) = \ln 1$.

Miscellaneous Topics: What Are Your Top Degree Choices?

I have seen all the statistics regarding the correlation between income and education. In general, the higher your education, the higher your income.

1. Name three benefits, besides money, that a higher degree can give you.
2. What are your top three choices for degrees (e.g., English, Mathematics, Psychology)?

Success in College: Affirmations

Do you ever tell yourself (or others) that you're not good at math? This is called negative self-talk. The more you talk that way, the more likely your subconscious will believe it - and you will do poorly in math.

You can counteract years of negative self-talk by telling yourself with conviction that you're good at math. It might seem strange to state that something that hasn't happened yet is true, but it works! Such statements are called affirmations.

There are four guiding principles for getting the most out of affirmations:

I. Say affirmations that imply that the desired event is currently happening. For example, say, "I am good at algebra," not "I will be good at algebra."
II. Say affirmations that you are making progress toward the desired result. For example, say, "I am good at trigonometry, and I continue to improve at it."
II. Say affirmations in the positive rather than the negative. For example, say "I attend each class," not "I don't cut classes."
IV. Say affirmations with conviction.

1. Do you practice affirmations? If so, how?
2. After reading this, will you try to practice affirmations? Where else in life could affirmations be put to work?

Answers

Section Preview

Take care to cover the domain issues that arise when solving logarithmic equations.

Section Preview

- **Solving Equations**
  - Solving Equations Quadratic-in-Form: Example 8 (and Checkpoint 8) uses the substitution $u = e^x$ to convert $e^{2x} - e^x = 56$ into a solvable quadratic before back-substituting.

Section Preview

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the One-to-One Property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

Theorem: One-to-One Property of Exponents

Let S and T be algebraic expressions, and b be a positive real number not equal to 1.

$$\begin{aligned} & b^S = b^T \\ \leftrightarrow \quad & \ln(b^S) = \ln(b^T) && \text{(taking the natural log of both sides)} \\ \leftrightarrow \quad & S\ln(b) = T\ln(b) && \text{(Laws of Logarithms)} \\ \leftrightarrow \quad & S = T && \text{(dividing both sides by } \ln(b)\text{)} \end{aligned}$$

Example 12.6.1**: Solving an Exponential Equation with a Common Base**

$$\begin{aligned} & 2^{x-1} = 2^{2x-4} \\ \Longrightarrow \quad & x-1 = 2x-4 && \text{(One-to-One Property)} \\ \Longrightarrow \quad & 3 = x && \text{(subtracting } x \text{ and adding 4 to both sides)} \end{aligned}$$

Example 12.6.2**: Solving Equations by Rewriting Them to Have a Common Base**

$$\begin{aligned} & 8^{x+2} = 16^{x+1} \\ \Longrightarrow \quad & (2^3)^{x+2} = (2^4)^{x+1} && \text{(rewriting both sides with the same base)} \\ \Longrightarrow \quad & 2^{3(x+2)} = 2^{4(x+1)} && \text{(Laws of Exponents)} \\ \Longrightarrow \quad & 2^{3x+6} = 2^{4x+4} && \text{(distributing)} \\ \Longrightarrow \quad & 3x+6 = 4x+4 && \text{(One-to-One Property)} \\ \Longrightarrow \quad & 2 = x && \text{(subtracting } 3x \text{ and 4 from both sides)} \end{aligned}$$

Example 12.6.3**: Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base**

$$\begin{aligned} & 2^{5x} = \sqrt{2} \\ \Longrightarrow \quad & 2^{5x} = 2^{1/2} && \text{(rewriting the radical as a rational exponent)} \\ \Longrightarrow \quad & 5x = \frac{1}{2} && \text{(One-to-One Property)} \\ \Longrightarrow \quad & x = \frac{1}{10} && \text{(dividing both sides by 5)} \end{aligned}$$

Example 12.6.4**: Solving an Equation with Positive and Negative Powers**

This equation has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive.

Figure 12.6.2 shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

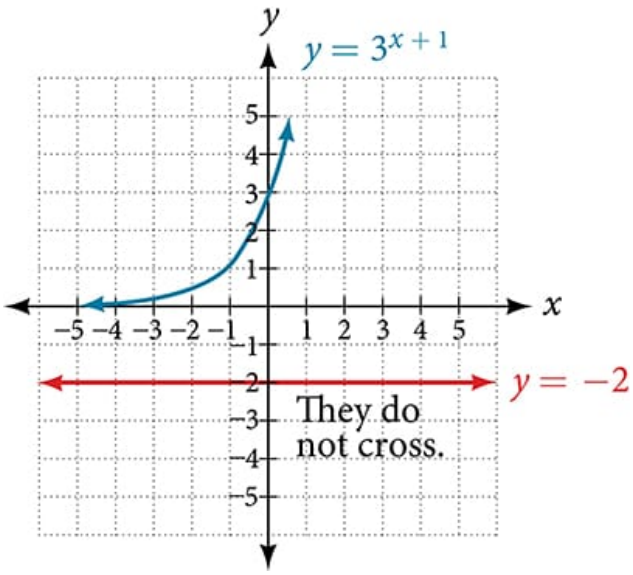


Figure 12.6.2

Example 12.6.5**: Solving an Equation Containing Powers of Different Bases**

$$\begin{aligned} & 5^{x+2} = 4^x && \text{(notice there is no common base between 5 and 4)} \\ \Longrightarrow\quad & \ln\left(5^{x+2}\right) = \ln\left(4^x\right) && \text{(take the natural log of both sides)} \\ \Longrightarrow\quad & (x+2)\ln(5) = x\ln(4) && \text{(Laws of Logarithms)} \\ \Longrightarrow\quad & x\ln(5) + 2\ln(5) = x\ln(4) && \text{(distributing)} \\ \Longrightarrow\quad & x\ln(5) - x\ln(4) = -2\ln(5) && \text{(moving all terms containing } x \text{ to one side)} \\ \Longrightarrow\quad & x(\ln(5) - \ln(4)) = -2\ln(5) && \text{(factoring out } x) \\ \Longrightarrow\quad & x = \frac{-2\ln(5)}{\ln(5) - \ln(4)} && \text{(dividing both sides by } \ln(5) - \ln(4)) \end{aligned}$$

Some could correctly argue that we could further simplify the final expression using the Laws of Logarithms.

$$x = \frac{-2\ln(5)}{\ln(5) - \ln(4)} = \frac{\ln\left(5^{-2}\right)}{\ln\left(\frac{5}{4}\right)}.$$

However, this is as correct as $\frac{-2\ln(5)}{\ln(5)-\ln(4)}$. In fact, you could go even further in your simplification to

$$x = \frac{\ln\left(5^{-2}\right)}{\ln\left(\frac{5}{4}\right)} = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}.$$

Be sure to check with your instructor as to their preference, but my guess is that any of these is acceptable as equivalent.

Example 12.6.6**: Solve an Equation of the Form** $y = Ae^{kt}$

$$\begin{aligned} & 100 = 20e^{2t} && \\ \Longrightarrow\quad & 5 = e^{2t} && \text{(dividing both sides by 20)} \\ \Longrightarrow\quad & \ln(5) = \ln\left(e^{2t}\right) && \text{(taking the natural log of both sides)} \\ \Longrightarrow\quad & \ln(5) = 2t && \text{(Laws of Logarithms)} \\ \Longrightarrow\quad & \frac{\ln(5)}{2} = t && \text{(dividing both sides by 2)} \end{aligned}$$

Using Laws of Logarithms, we can also write this answer in the form $t = \ln\sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

Example 12.6.7**: Solving an Equation That Can Be Simplified to the Form** $y = Ae^{kt}$

$$\begin{aligned} & 4e^{2x}+5 &=& 12 & \\ \Longrightarrow\quad & 4e^{2x} &=& 7 & \text{(subtracting 5 from both sides)} \\ \Longrightarrow\quad & e^{2x} &=& \frac{7}{4} & \text{(dividing both sides by 4)} \\ \Longrightarrow\quad & \ln\left(e^{2x}\right) &=& \ln\left(\frac{7}{4}\right) & \text{(taking the natural log of both sides)} \\ \Longrightarrow\quad & 2x &=& \ln\left(\frac{7}{4}\right) & \text{(Laws of Logarithms)} \\ \Longrightarrow\quad & x &=& \frac{\ln(7/4)}{2} & \text{(dividing both sides by 2)} \end{aligned}$$

Example 12.6.8**: Solving Exponential Equations in Quadratic Form**

$$\begin{aligned} & e^{2x}-e^{x} &=& 56 & \\ \Longrightarrow\quad & e^{2x}-e^{x}-56 &=& 0 & \text{(subtracting 56 from both sides)} \\ \Longrightarrow\quad & \left(e^{x}\right)^{2}-e^{x}-56 &=& 0 & \text{(Laws of Exponents)} \\ \Longrightarrow\quad & u^{2}-u-56 &=& 0 & \text{(substituting } u=e^{x}\text{)} \\ \Longrightarrow\quad & (u-8)(u+7) &=& 0 & \text{(factoring the trinomial)} \end{aligned}$$

Therefore, $u=8$ or $u=-7$. Resubstituting $u=e^{x}$, we get

$$e^{x}=8 \quad \text{or} \quad e^{x}=-7.$$

Since $e^{x}>0$, we throw out the possibility that $e^{x}=-7$. To solve $e^{x}=8$, we take the natural log of both sides to get $x=\ln(8)$.

Example 12.6.9**: Using Algebra to Solve a Logarithmic Equation**

$$\begin{aligned} & 2\ln(x)+3 &=& 7 & \\ \Longrightarrow\quad & 2\ln(x) &=& 4 & \text{(subtracting 3 from both sides)} \\ \Longrightarrow\quad & \ln(x) &=& 2 & \text{(dividing both sides by 2)} \\ \Longrightarrow\quad & x &=& e^{2} & \text{(rewriting in exponential form)} \end{aligned}$$

Example 12.6.10**: Using Algebra Before and After Using the Definition of the Natural Logarithm**

$$\begin{aligned} & 2\ln(6x) &=& 7 & \\ \Longrightarrow\quad & \ln(6x) &=& \frac{7}{2} & \text{(dividing both sides by 2)} \\ \Longrightarrow\quad & 6x &=& e^{7/2} & \text{(rewriting in exponential form)} \\ \Longrightarrow\quad & x &=& \frac{1}{6}e^{7/2} & \text{(dividing both sides by 6)} \end{aligned}$$

Example 12.6.11**: Showcasing the Need to Check Solutions**

$$\begin{aligned} & \log_{7}(1-2x) &=& 1-\log_{7}(3-x) & \\ \Longrightarrow\quad & \log_{7}(1-2x)+\log_{7}(3-x) &=& 1 & \text{(adding } \log_{7}(3-x) \text{ to both sides)} \\ \Longrightarrow\quad & \log_{7}\left((1-2x)(3-x)\right) &=& 1 & \text{(Laws of Logarithms)} \\ \Longrightarrow\quad & (1-2x)(3-x) &=& 7^{1} & \text{(rewriting in exponential form)} \\ \Longrightarrow\quad & 2x^{2}-7x+3 &=& 7 & \text{(distributing)} \\ \Longrightarrow\quad & 2x^{2}-7x-4 &=& 0 & \text{(subtracting 7 from both sides)} \\ \Longrightarrow\quad & (2x+1)(x-4) &=& 0 & \text{(factoring the trinomial)} \end{aligned}$$

Therefore, either $x=-\frac{1}{2}$ or $x=4$. We now check our candidate solutions (as we always must do when solving logarithmic equations).

Substituting $x=-\frac{1}{2}$ into the original equation, we get

$$\begin{aligned} & \log_{7}\left(1-2\left(-\frac{1}{2}\right)\right) &\stackrel{?}{=}& 1-\log_{7}\left(3-\left(-\frac{1}{2}\right)\right) & \text{(substituting)} \\ \Longrightarrow\quad & \log_{7}(1+1) &\stackrel{?}{=}& 1-\log_{7}\left(\frac{7}{2}\right) & \text{(simplifying)} \\ \Longrightarrow\quad & \log_{7}(2) &\stackrel{?}{=}& 1-\left(\log_{7}(7)-\log_{7}(2)\right) & \text{(simplifying and Laws of Logarithms)} \\ \Longrightarrow\quad & \log_{7}(2) &\stackrel{?}{=}& 1-\log_{7}(7)+\log_{7}(2) & \text{(distributing)} \\ \Longrightarrow\quad & \log_{7}(2) &\stackrel{?}{=}& 1-1+\log_{7}(2) & \text{(Laws of Logarithms)} \\ \Longrightarrow\quad & \log_{7}(2) &\stackrel{\checkmark}{=}& \log_{7}(2) & \text{(simplifying)} \end{aligned}$$

Thus, $x=-\frac{1}{2}$ is a solution.

Substituting $x=4$ into the original equation, we get the following.

$$\log_7(1-2(4)) \stackrel{?}{=} 1-\log_7(3-4)$$

$$\implies \log_7(-7) \stackrel{?}{=} \log_7(-1)$$

We can stop at this point and confidently state that $x=4$ is an extraneous solution.

WARNING: Our determination that $x=4$ is extraneous is *not* because $-7 \neq -1$. Instead, it is due to the fact that the argument of a logarithm can never be negative!

Hence, the only solution to the equation is $-\frac{1}{2}$.

Example $12.6.12$**: Using a Graph to Understand the Solution to a Logarithmic Equation**

$$\ln x = 3 \implies x = e^3.$$

Figure $12.6.3$ represents the graph of the equation. On the graph, the x-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3 \approx 20$. A calculator gives a better approximation: $e^3 \approx 20.0855$.

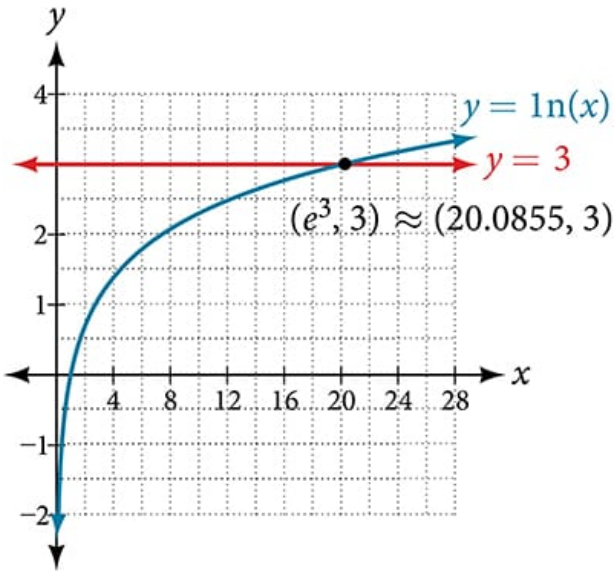


Figure $12.6.3$**:** *The graphs of* $y=\ln x$ *and* $y=3$ *cross at the point* $(e^3,3)$*, which is approximately* $(20.0855,3)$.

Example $12.6.13$**: Solving an Equation Using the One-to-One Property of Logarithms**

$$\begin{aligned} & \ln(x^2) = \ln(2x+3) & \\ \implies & x^2 = 2x+3 & \text{(One-to-One Property)} \\ \implies & x^2-2x-3 = 0 & \text{(subtracting } 2x \text{ and } 3 \text{ from both sides)} \\ \implies & (x-3)(x+1) = 0 & \text{(factoring the trinomial)} \end{aligned}$$

Therefore, $x=3$ or $x=-1$.

We leave it to the reader to check the solutions; however, it is important to note that, while the solution -1 is negative, it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

This page titled 12.6: Exponential and Logarithmic Equations is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

12.6.1: Resources and Key Concepts

Resources

Videos

- Solving exponential equations
  - Exponential and Logarithmic Equations - Solving Exponential Equations and Inequalities
- Like bases property exponents
  - Intermediate Algebra - Exponential Functions: Solving Exponential Equations (no logs)
- Using logarithms to solve exponential equations
  - Exponential and Logarithmic Equations - Solving Exponential Equations Using Logarithms
  - Intermediate Algebra - Logarithms: Solving Any Exponential Equation
  - Intermediate Algebra - Exponential and Logarithmic Functions: Solving Equations
- Solving logarithmic equations
  - Exponential and Logarithmic Equations - Solving Logarithmic Equations
  - Exponential and Logarithmic Equations - Solving Logarithmic Equations and Inequalities

Key Concepts

Theorems

- **One-to-One Property of Exponents**: For any algebraic expressions S and T, and any positive real number $b \neq 1$, $b^S = b^T$ if and only if $S = T$.

Common Mistakes

- **Incorrectly Applying the One-to-One Property of Exponents**: Attempting to equate exponents when the bases are not the same, without first rewriting them to have a common base.
- **Errors in Using Logarithms to Solve Exponential Equations**: Incorrectly applying logarithm laws, especially the power rule (e.g., $\ln(5^{x+2}) = (x+2)\ln(5)$, not $x\ln(5) + 2$).
- **Forgetting to Check for Extraneous Solutions in Logarithmic Equations**: Failing to verify that all solutions obtained make the arguments of the original logarithms positive.
- **Errors in Using the One-to-One Property of Logarithms**: Not first condensing logarithmic expressions on each side of the equation to a single logarithm before equating arguments.
- **Treating $\ln(A - B)$ as $\ln(A) - \ln(B)$ or $\frac{\ln(A)}{\ln(B)}$**: Incorrectly splitting logarithms of differences or quotients of logarithms. (Common algebraic errors with log laws)
- **Solving Equations Quadratic-in-Form Incorrectly**: Forgetting to substitute back after solving for u in an equation like $e^{2x} - e^x - 56 = 0$, or incorrectly solving the resulting exponential/logarithmic equations after substitution.

This page titled 12.6.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

12.6.2: Homework

Reading Questions

1. What is the One-to-One Property of Exponents, and how is it used to solve exponential equations?
2. If an exponential equation cannot be easily rewritten with like bases, what is the general strategy for solving it using logarithms?
3. When solving an exponential equation by taking the logarithm of both sides, does it matter which base of logarithm you use (e.g., common log vs. natural log)?
4. What is an extraneous solution in the context of solving exponential or logarithmic equations? Why do they sometimes occur?
5. How is the definition of a logarithm used to solve logarithmic equations of the form $\log_b(S) = c$?
6. What is the One-to-One Property of Logarithms, and how can it be used to solve logarithmic equations? What must be true about the bases of the logarithms?
7. Before applying the One-to-One Property of Logarithms to an equation like $\log(3x-2) - \log(2) = \log(x+4)$, what must you do first?
8. Why is it essential to check solutions when solving logarithmic equations?
9. Describe how to solve an equation like $e^{2x} - e^x = 56$ which is quadratic in form.
10. If you solve a logarithmic equation and obtain a candidate solution that makes an argument of a logarithm in the original equation negative or zero, what do you conclude about that candidate solution?
11. How can an exponential equation be solved?
12. When does an extraneous solution occur? How can an extraneous solution be recognized?
13. When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

Homework

Algebraic

For the following exercises, use like bases to solve the exponential equation.

14. $4^{-3v-2} = 4^{-v}$
15. $64 \cdot 4^{3x} = 16$
16. $3^{2x+1} \cdot 3^x = 243$
17. $2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$
18. $625 \cdot 5^{3x+3} = 125$
19. $\frac{36^{3b}}{36^{2b}} = 216^{2-b}$
20. $\left(\frac{1}{64}\right)^{3n} \cdot 8 = 2^6$

For the following exercises, use logarithms to solve.

21. $9^{x-10} = 1$
22. $2e^{6x} = 13$
23. $e^{r+10} - 10 = -42$
24. $2 \cdot 10^{9a} = 29$
25. $-8 \cdot 10^{p+7} - 7 = -24$
26. $7e^{3n-5} + 5 = -89$
27. $e^{-3k} + 6 = 44$
28. $-5e^{9x-8} - 8 = -62$
29. $-6e^{9x+8} + 2 = -74$
30. $2^{x+1} = 5^{2x-1}$
31. $e^{2x} - e^x - 132 = 0$
32. $7e^{8x+8} - 5 = -95$
33. $10e^{8x+3} + 2 = 8$
34. $4e^{3x+3} - 7 = 53$

35. $8e^{-5x-2} - 4 = -90$
36. $3^{2x+1} = 7^{x-2}$
37. $e^{2x} - e^x - 6 = 0$
38. $3e^{3-3x} + 6 = -31$

For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.

39. $\log\left(\frac{1}{100}\right) = -2$
40. $\log_{324}(18) = \frac{1}{2}$

For the following exercises, use the definition of a logarithm to solve the equation.

41. $5\log_7(n) = 10$
42. $-8\log_9(x) = 16$
43. $4 + \log_2(9k) = 2$
44. $2\log(8n + 4) + 6 = 10$
45. $10 - 4\ln(9 - 8x) = 6$

For the following exercises, use the one-to-one property of logarithms to solve.

46. $\ln(10 - 3x) = \ln(-4x)$
47. $\log_{13}(5n - 2) = \log_{13}(8 - 5n)$
48. $\log(x + 3) - \log(x) = \log(74)$
49. $\ln(-3x) = \ln(x^2 - 6x)$
50. $\log_4(6 - m) = \log_4(3m)$
51. $\ln(x - 2) - \ln(x) = \ln(54)$
52. $\log_9(2n^2 - 14n) = \log_9(-45 + n^2)$
53. $\ln(x^2 - 10) + \ln(9) = \ln(10)$

For the following exercises, solve each equation for x.

54. $\log(x + 12) = \log(x) + \log(12)$
55. $\ln(x) + \ln(x - 3) = \ln(7x)$
56. $\log_2(7x + 6) = 3$
57. $\ln(7) + \ln(2 - 4x^2) = \ln(14)$
58. $\log_8(x + 6) - \log_8(x) = \log_8(58)$
59. $\ln(3) - \ln(3 - 3x) = \ln(4)$
60. $\log_3(3x) - \log_3(6) = \log_3(77)$

Graphical

For the following exercises, solve the equation for x, if there is a solution. Then graph both sides of the equation, and observe the point of intersection (if it exists) to verify the solution.

61. $\log_9(x) - 5 = -4$
62. $\log_3(x) + 3 = 2$
63. $\ln(3x) = 2$
64. $\ln(x - 5) = 1$
65. $\log(4) + \log(-5x) = 2$
66. $-7 + \log_3(4 - x) = -6$
67. $\ln(4x - 10) - 6 = -5$
68. $\log(4 - 2x) = \log(-4x)$
69. $\log_{11}(-2x^2 - 7x) = \log_{11}(x - 2)$
70. $\ln(2x + 9) = \ln(-5x)$
71. $\log_9(3 - x) = \log_9(4x - 8)$
72. $\log(x^2 + 13) = \log(7x + 3)$
73. $\frac{3}{\log_2(10)} - \log(x - 9) = \log(44)$
74. $\ln(x) - \ln(x + 3) = \ln(6)$

For the following exercises, solve for the indicated value, and graph the situation showing the solution point.

75. An account with an initial deposit of $6,500 earns 7.25% annual interest, compounded continuously. How much will the account be worth after 20 years?
76. The formula for measuring sound intensity in decibels D is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8.3 \cdot 10^2$ watts per square meter?
77. The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach 20,000?

Technology

For the following exercises, solve each equation by rewriting the exponential expression using the indicated logarithm. Then use a calculator to approximate the variable to 3 decimal places.

78. $1000(1.03)^t = 5000$ using the common log
79. $e^{5x} = 17$ using the natural log
80. $3(1.04)^{3t} = 8$ using the common log
81. $3^{4x-5} = 38$ using the common log
82. $50e^{-0.12t} = 10$ using the natural log

For the following exercises, use a calculator to solve the equation. Unless indicated otherwise, round all answers to the nearest ten-thousandth.

83. $7e^{3x-5} + 7.9 = 47$
84. $\ln(3) + \ln(4.4x + 6.8) = 2$
85. $\log(-0.7x - 9) = 1 + 5\log(5)$
86. Atmospheric pressure P in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where x is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of 8.369 pounds per square inch? (*Hint*: there are 5,280 feet in a mile)
87. The magnitude M of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where E is the amount of energy released by the earthquake in joules and $E_0 = 10^{4.4}$ is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4 \cdot 10^{13}$ joules of energy?

Extensions

88. Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b(x)} = x$.
89. Recall the formula for continually compounding interest, $y = Ae^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.
90. Recall the compound interest formula $A = a\left(1 + \frac{r}{k}\right)^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t.
91. Newton's Law of Cooling states that the temperature T of an object at any time t can be described by the equation $T = T_s + (T_0 - T_s)e^{-kt}$, where T_s is the temperature of the surrounding environment, T_0 is the initial temperature of the object, and k is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

This page titled 12.6.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

12.7: Exponential and Logarithmic Models

Section Preview

Note to the Instructor (click to expand) available at the end of this section.

Operational Prerequisites (i.e., "Completing the Homework"): The following prerequisite topics (which have not already been listed as prerequisites in previous sections of this text) are needed to complete the homework.

(click to expand) available at the end of this section.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand) available at the end of this section.

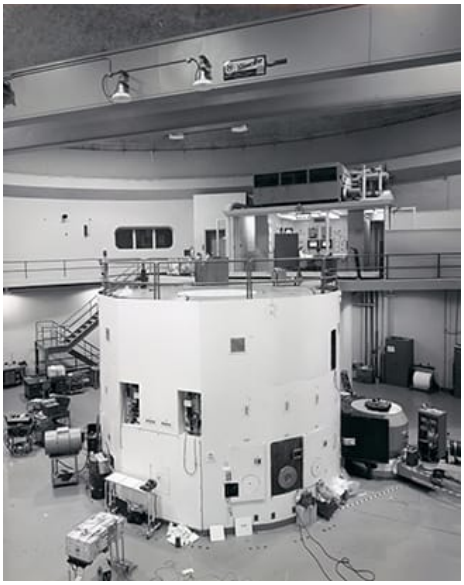


Figure 12.7.1**:** *A nuclear research reactor inside the Neely Nuclear Research Center on the Georgia Institute of Technology campus (credit: Georgia Tech Research Institute)*

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and **Newton's Law of Cooling**.

Modeling Exponential Growth and Decay

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function $P(t) = P_0 e^{kt}$, where P_0 is equal to the value at time zero, e is Euler's constant, and k is a positive constant that determines the rate (percentage) of growth. We may use the exponential growth function in applications involving **doubling time**, the time it takes for a quantity to double.

Definition: Doubling Time

The time it takes for a quantity undergoing exponential growth to double is called the quantity's **doubling time**.

Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit growth

based on a doubling time.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the exponential decay model. Again, we have the form $P(t) = P_0 e^{kt}$ where P_0 is the starting value, and e is Euler's constant. Now k is a negative constant that determines the rate of decay. We may use the exponential decay model when we are calculating **half-life**, or the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

Definition: Half-Life

The time it takes for a quantity undergoing exponential decay to decay by half is called the quantity's **half-life**.

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape, as we can see in Figure 12.7.2 and Figure 12.7.3. It is important to remember that, although parts of each of the two graphs seem to lie on the x-axis, they are really a tiny distance above the x-axis.

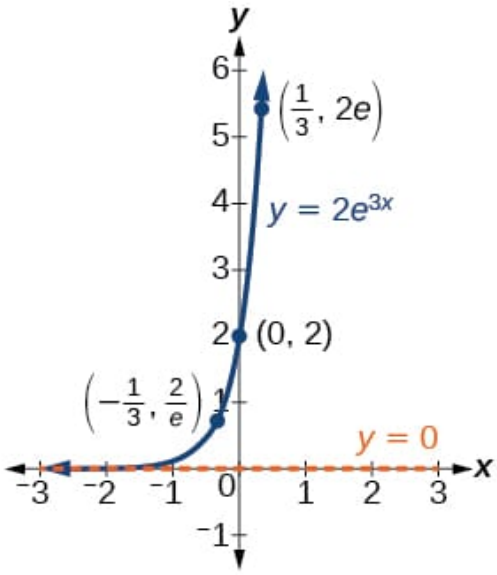


Figure 12.7.2: *A graph showing exponential growth. The equation is* $y = 2e^{3x}$.

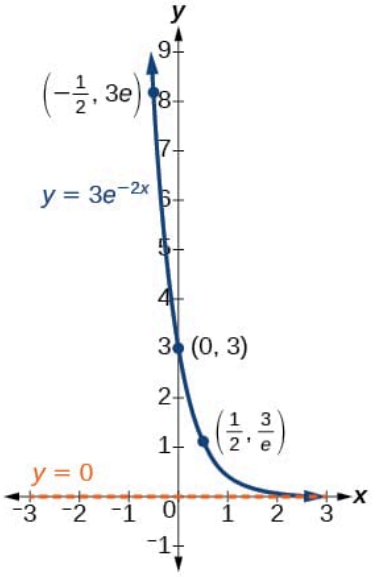


Figure 12.7.3: *A graph showing exponential decay. The equation is* $y = 3e^{-2x}$.

Exponential growth and decay often involve very large or very small numbers. To describe these numbers, we often use orders of magnitude.

Definition: Order of Magnitude

The **order of magnitude** of a number is the exponent of 10 when the number is written in scientific notation.

For example, the distance to the nearest star, Proxima Centauri, measured in kilometers, is $40,113,497,200,000$ kilometers. Expressed in scientific notation, this is $4.01134972 \times 10^{13}$. So, we could describe this number as having order of magnitude 13.

Example 12.7.1: Graphing Exponential Growth

A population of bacteria doubles every hour. If the culture started with 10 bacteria, graph the population as a function of time.

Solution available at the end of this section.

Half-Life

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its half-life. Table $12.7.1$ lists the half-life for several of the more common radioactive substances.

Table $12.7.1$

Substance	Use	Half-life
gallium-67	nuclear medicine	80 hours
cobalt-60	manufacturing	5.3 years
technetium-99m	nuclear medicine	6 hours
americium-241	construction	432 years
carbon-14	archeological dating	5,715 years
uranium-235	atomic power	703,800,000 years

To find the half-life of a function describing exponential decay, we need to know how long it takes for the initial population, P_0, to decay by half to $\frac{1}{2}P_0$. This requires us to solve the following equation:

$$\frac{1}{2}P_0 = P_0 e^{kt}.$$

We find that the half-life depends only on the constant k and not on the starting quantity P_0.

The formula is derived as follows:

$$\begin{aligned}
\frac{1}{2}P_0 &= P_0 e^{kt} \\
\Longrightarrow \quad \frac{1}{2} &= e^{kt} && \text{(dividing both sides by } P_0\text{)} \\
\Longrightarrow \quad \ln\left(\frac{1}{2}\right) &= kt && \text{(taking the natural logarithm of both sides)} \\
\Longrightarrow \quad \frac{\ln(1/2)}{k} &= t && \text{(dividing both sides by } k\text{)} \\
\Longrightarrow \quad \frac{\ln(2^{-1})}{k} &= t && \text{(Laws of Exponents)} \\
\Longrightarrow \quad -\frac{\ln(2)}{k} &= t && \text{(Laws of Logarithms)}
\end{aligned}$$

Since t, the time, is positive, k must, as expected, be negative. This gives us the half-life formula.

Theorem: Half-Life Formula

If an exponentially decaying population is modeled by the function $P(t) = P_0 e^{kt}$, then the time it takes for the population to decay by half is

$$t = -\frac{\ln(2)}{k}.$$

While most students tend to memorize this formula, you are expected to understand it and derive it, if needed.

Example 12.7.2: Finding the Function that Describes Radioactive Decay

The half-life of carbon-14 is 5,730 years. Express the amount of carbon-14 remaining as a function of time, t.

Solution available at the end of this section.

Checkpoint 12.7.2

The half-life of plutonium-244 is 80,000,000 years. Find a function that gives the amount of plutonium-244 remaining as a function of time, measured in years.

Radiocarbon Dating

The formula for exponential decay is important in radiocarbon dating, which is used to calculate the approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby, who won a Nobel Prize for his discovery. It compares the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about 1% error for plants or animals that died within the last 60,000 years.

Carbon-14 is a radioactive isotope of carbon that has a half-life of 5,730 years. It occurs in small quantities in the carbon dioxide in the air we breathe. Most of the carbon on Earth is carbon-12, which has an atomic weight of 12 and is not radioactive. Scientists have determined the ratio of carbon-14 to carbon-12 in the air for the last 60,000 years, using tree rings and other organic samples of known dates—although the ratio has changed slightly over the centuries.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Since the half-life of carbon-14 is 5,730 years, we derive a formula for the amount present at any time t as follows:

$$
\begin{aligned}
& P(t) = P_0 e^{kt} \\
\Longrightarrow \quad & \frac{1}{2} P_0 = P_0 e^{k \cdot 5730} && \text{(after 5,730 years, half the initial amount remains)} \\
\Longrightarrow \quad & \frac{1}{2} = e^{5730k} && \text{(dividing both sides by } P_0\text{)} \\
\Longrightarrow \quad & \ln\left(\tfrac{1}{2}\right) = 5730k && \text{(taking the natural logarithm of both sides)} \\
\Longrightarrow \quad & \frac{\ln(0.5)}{5730} = k && \text{(dividing both sides by 5730)}
\end{aligned}
$$

Thus, the formula for the amount of carbon-14 remaining after t years is

$$P(t) \approx P_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t},$$

where

- $P(t)$ is the amount of carbon-14 remaining at time t, and
- P_0 is the amount of carbon-14 when the plant or animal began decaying.

Again, I *highly* discourage you from memorizing this formula—it's fast to derive, and the half-life changes based on substance given. It is much preferred to derive the formula each time.

Example 12.7.3: Finding the Age of a Bone

A bone fragment is found that contains 20% of its original carbon-14. To the nearest year, how old is the bone?

Solution available at the end of this section.

In reality, the instruments that measure the percentage of carbon-14 are extremely sensitive and a scientist would need to do much more work than we did in order to be satisfied. Even so, carbon dating is only accurate to about 1%, so the age of the sample from Example 12.7.3 should be given as

$$13,301 \text{ years } \pm \text{ } 113,301 \text{ years } = 13,301 \pm 133 \text{ years.}$$

Checkpoint 12.7.3

Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

Calculating Doubling Time

For decaying quantities, we determined how long it took for half of a substance to decay. For growing quantities, we might want to find out how long it takes for a quantity to double. As we mentioned above, the time it takes for a quantity to double is called the doubling time.

Given the basic exponential growth equation $P(t) = P_0 e^{kt}$, doubling time can be found by solving for when the original quantity has doubled—that is, by solving $2P_0 = P_0 e^{kt}$. Rather than attracting you to memorize a formula, let's show how to derive it.

Example 12.7.4: Finding a Function That Describes Exponential Growth

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

Solution available at the end of this section.

Checkpoint 12.7.4

Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.

Newton's Law of Cooling

Exponential decay can also be applied to temperature. When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature. On a graph of the temperature function, the leveling off will correspond to a horizontal asymptote at the temperature of the surrounding air. Unless the room temperature is zero, this will correspond to a vertical shift of the generic exponential decay function. This translation leads to **Newton's Law of Cooling**, the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature.

Theorem: Newton's Law of Cooling

The temperature of an object, T, in a surrounding medium having temperature T_s will behave according to the formula

$$T(t) = T_s + (T_0 - T_s)e^{kt},$$

where t is the time the object has been immersed in the medium, T_0 is the temperature of the object at time 0 (usually before being immersed in the medium), and k is a constant called the **continuous rate of cooling** of the object.

Example 12.7.5: Using Newton's Law of Cooling

A cheesecake is taken out of the oven with an ideal internal temperature of 165°F and is placed into a 35°F refrigerator. After 10 minutes, the cheesecake has cooled to 150°F. If we must wait until the cheesecake has cooled to 70°F before we eat it, how long will we have to wait?

Solution available at the end of this section.

Checkpoint 12.7.5

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Using Logistic Growth Models

Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Consider an aspiring writer who writes a single line on day one and plans to double the number of lines she writes each day for a month. By the end of the month, she must write over 17 billion lines, or one-half-billion pages. It is impractical, if not impossible, for anyone to write that much in such a short period of time. Eventually, an exponential model must begin to approach some limiting value, and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an exponential growth model, though the exponential growth model is still useful over a short term, before approaching the limiting value.

Definition: Logistic Growth Model

A **logistic growth model** is any model that can be written in the form

$$P(t) = \frac{c}{1 + ae^{-bt}},$$

where a, b, and c are constants. c is called the **carrying capacity** (also known as the **limiting value**) of the population, b is a constant determined by the rate of growth, and the ratio $\frac{c}{1+a}$ is the initial value.

The logistic growth model is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, or carrying capacity. The graph in Figure $12.7.5$ shows how the growth rate changes over time. The graph increases from left to right, but the growth rate only increases until it reaches its point of maximum growth rate, at which point the rate of increase decreases.

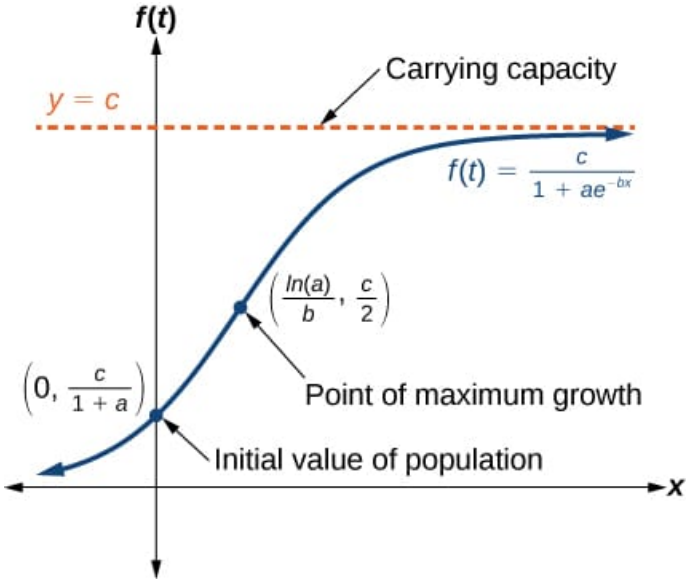


Figure $12.7.5$

Example $12.7.6$: Using the Logistic-Growth Model

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors: The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of 1,000 people who has the flu. So, in that community, at most 1,000 people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is $b = 0.6030$. Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.

Solution available at the end of this section.

Checkpoint $12.7.6$

Using the model in Example $12.7.6$, estimate the number of cases of flu on day 15.

Expressing an Exponential Model in Base e

While powers and logarithms of any base can be used in modeling, the two most common bases are 10 and e. In Science and Mathematics, the base e is often preferred. We can use Laws of Exponents and Laws of Logarithms to change any base to base e.

Given a model with the form $y = ab^x$, we change it to the form $y = A_0e^{kx}$ using the following steps.

1. Rewrite $y = ab^x$ as $y = ae^{\ln(b^x)}$.
2. Use the Power Rule of Logarithms to rewrite y as $y = ae^{x\ln(b)} = ae^{\ln(b)x}$.
3. Note that $a = A_0$ and $k = \ln(b)$ in the equation $y = A_0e^{kx}$.

Example 12.7.7: Changing to base e

Change the function $y = 2.5(3.1)^x$ so that this same function is written in the form $y = A_0e^{kx}$.

Solution available at the end of this section.

Checkpoint 12.7.7

Change the function $y = 3(0.5)^x$ to one having e as the base.

Logarithmic Scales

For quantities that vary greatly in magnitude, a standard scale of measurement is not always effective, and utilizing logarithms can make the values more manageable. For example, if the average distances from the sun to the major bodies in our solar system are listed, you see they vary greatly.

Table 12.7.2

Planet	Distance (millions of km)
Mercury	58
Venus	108
Earth	150
Mars	228
Jupiter	779
Saturn	1430
Uranus	2880
Neptune	4500

Placed on a linear scale—one with equally spaced values—these values get bunched up (see Figure 12.7.7).

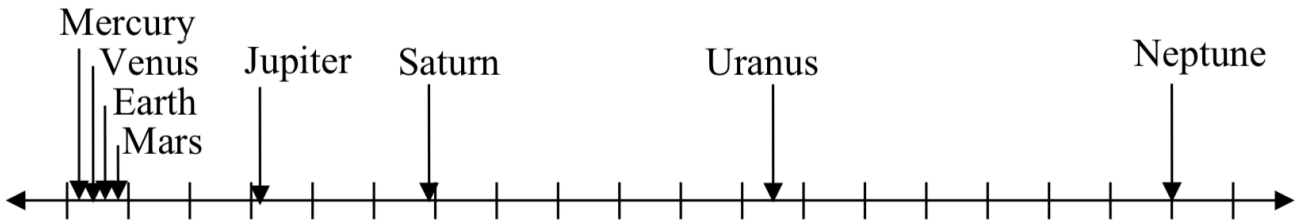


Figure 12.7.7: *Linear scale of astronomical distances in our solar system.*

However, computing the logarithm of each value and plotting these new values on a number line results in a more manageable graph, and makes the relative distances more apparent, as seen in Figure 12.7.8 below Table 12.7.3. (It is interesting to note the large gap between Mars and Jupiter on the log number line. The asteroid belt is located there, which scientists believe is a planet that never formed because of the effects of the gravity of Jupiter.)

Table 12.7.3

Planet	Distance (millions of km)	log(distance)
Mercury	58	1.76
Venus	108	2.03
Earth	150	2.18
Mars	228	2.36
Jupiter	779	2.89
Saturn	1430	3.16
Uranus	2880	3.46
Neptune	4500	3.65

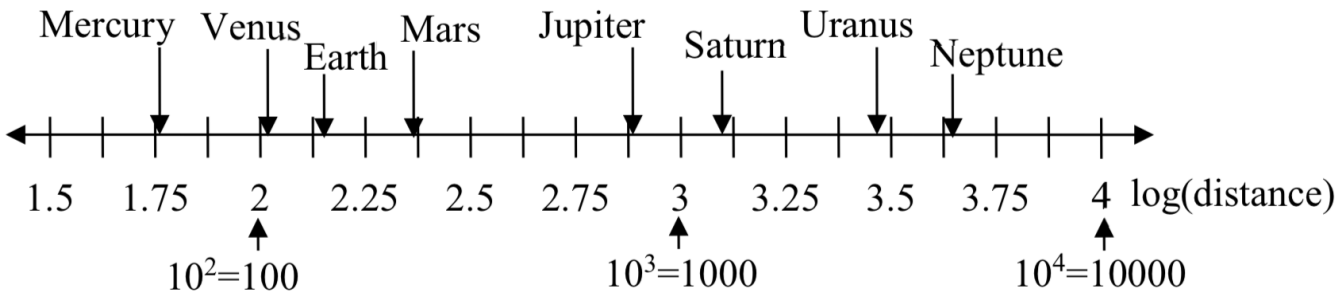


Figure 12.7.8**:** *Logarithmic scale of astronomical distances in our solar system.*

Sometimes, as shown above in Figure 12.7.8, the scale on a logarithmic number line will show the log values, but more commonly the original values are listed as powers of 10, as shown below in Figure 12.7.9.

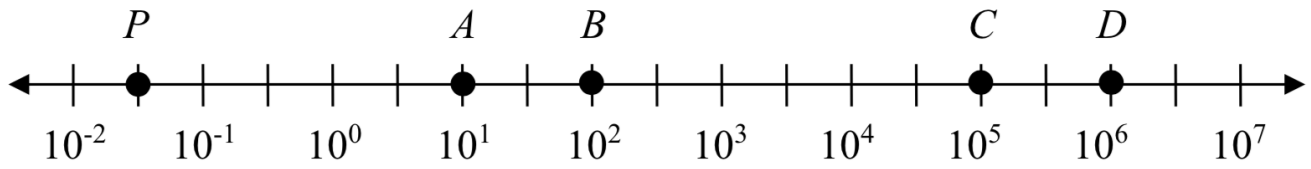


Figure 12.7.9

Example 12.7.8

Estimate the value of point P on the log scale above

Solution available at the end of this section.

Example 12.7.9

Place the number 6000 on a logarithmic scale.

Solution available at the end of this section.

Checkpoint 12.7.9

Plot the data in the table below on a logarithmic scale (From http://www.epd.gov.hk/epd/noise_educ...1/intro_5.html, retrieved Oct 2, 2010).

Table 12.7.4

Source of Sound/Noise	Approximate Sound Pressure in μ Pa (micro Pascals)
Launching of the Space Shuttle	2000,000,000
Full Symphony Orchestra	2000,000
Diesel Freight Train at High Speed at 25 m	200,000
Normal Conversation	20,000
Soft Whispering at 2 m in Library	2,000
Unoccupied Broadcast Studio	200
Softest Sound a human can hear	20

Answer available at the end of this section.

Notice that on the log scale above Figure 12.7.9, the visual distance on the scale between points A and B and between C and D is the same. When looking at the values these points correspond to, notice B is ten times the value of A, and D is ten times the value of C. A visual (linear) difference between points corresponds to a relative (ratio) change between the corresponding values.

Logarithms are useful for showing these relative changes. For example, comparing \$1,000,000 to \$10,000, the first is 100 times larger than the second.

$$\frac{1,000,000}{10,000} = 100 = 10^2.$$

Likewise, comparing \$1000 to \$10, the first is 100 times larger than the second.

$$\frac{1,000}{10} = 100 = 10^2.$$

When one quantity is roughly ten times larger than another, its exponent when written in scientific notation increases by 1. Thus, we say it is one order of magnitude larger. In both cases described above, the first number was two orders of magnitude larger than the second.

Notice that the order of magnitude can be found as the common logarithm of the ratio of the quantities. On the log scale above, B is one order of magnitude larger than A, and D is one order of magnitude larger than C.

Theorem: Determining the Order of Magnitude Between Two Numbers

Given two values A and B, the number of orders of magnitude A is greater than B is

$$\log\left(\frac{A}{B}\right).$$

Example 12.7.10

On the log scale in Figure 12.7.9, how many orders of magnitude larger is C than B?

Solution available at the end of this section.

Checkpoint 12.7.10

Using the table from Checkpoint 12.7.9, what is the difference of order of magnitude between the softest sound a human can hear and the launching of the space shuttle?

Answer available at the end of this section.

Earthquakes

An example of a logarithmic scale is the **Richter Scale** used for earthquakes. While not the only scale for measuring earthquakes, it is the most commonly known. The Richter scale compares the amplitude, A, of the earthquake's seismographic trace with the amplitude, A_0, of the smallest detectable earthquake.

Definition: Richter Scale

The **Richter scale** is a logarithmic scale used to express the magnitude of an earthquake, specifically the energy released during it. For an earthquake with **seismic trace amplitude** A (a measurement from the baseline of the seismograph), the **Richter value**, or **magnitude** of the earthquake, is defined to be

$$M = \log\left(\frac{A}{A_0}\right),$$

where A_0 is a baseline measure for the seismograph.

Example 12.7.11

If one earthquake has a Richter value of 6.0, and another has a magnitude of 8.0, how much more powerful (in terms of earth movement) is the second earthquake?

Solution available at the end of this section.

Example 12.7.12

One earthquake has magnitude of 3.0. If a second earthquake has twice as much earth movement as the first earthquake, find the magnitude of the second quake.

Solution available at the end of this section.

In fact, using the Laws of Logarithms, we could show that whenever the earth movement doubles, the Richter magnitude will increase by about 0.301:

$$\begin{aligned} M &= \log\left(\frac{2A}{A_0}\right) \\ &= \log\left(2 \cdot \frac{A}{A_0}\right) \\ &= \log(2) + \log\left(\frac{A}{A_0}\right) \\ &\approx 0.3010 + \log\left(\frac{A}{A_0}\right) \end{aligned}$$

This illustrates the most important feature of a log scale: that multiplying the quantity being considered will add to the scale value, and vice versa.

Logarithms are useful in measuring the intensities of many quantities based on senses. Earthquakes (the Richter scale), sound (decibels) and acids and bases (pH) are common examples. We now present yet a different use of the basic logarithm function, password strength.

Example 12.7.13

The information entropy H, in bits, of a randomly generated password consisting of L characters is given by

$$H = L \log_2(N),$$

where N is the number of possible symbols for each character in the password. In general, the higher the entropy, the stronger the password.

a. If a 7 character case-sensitive password is comprised of letters and numbers only, find the associated information entropy.
b. How many possible symbol options per character are required to produce a 7 character password with an information entropy of 50 bits?

Solutions available at the end of this section.

Chemical systems known as buffer solutions can adjust to small changes in acidity to maintain a range of pH values. Buffer solutions have a wide variety of applications, from maintaining a healthy fish tank to regulating the pH levels in blood.

Example 12.7.14

Blood is a buffer solution. When carbon dioxide is absorbed into the bloodstream, it produces carbonic acid and lowers the pH. The body compensates by producing bicarbonate, a weak base, to partially neutralize the acid. The equation which models blood pH in this situation is

$$\text{pH} = 6.1 + \log\left(\frac{800}{x}\right),$$

where x is the partial pressure of carbon dioxide in arterial blood, measured in torr. Find the partial pressure of carbon dioxide in arterial blood if the pH is 7.4.

Solution available at the end of this section.

Success in Mathematics: Process-Oriented Learning Preferences

Choose the statements that best describe you.

I learn better if I know why I'm learning something.	or	I don't need to know why I'm doing something before learning how to do it.
I find it annoying to learn something I don't think I will use anywhere else.	or	I believe that learning new things is always worth it because it exercises my brain.
I naturally think in steps.	or	It is hard for me to learn something step by step.

1. Do you consider your choices above good or bad for your education?
2. In specific textbooks, many procedures are summarized as a list of steps. This list is shown in a separate box. Do you refer to these boxes when you do your homework? Why or why not?

Miscellaneous Topics: How Does Math Relate to Your Field?

1. Please tell me what degree you are currently pursuing.
2. Name at least five things in your chosen field with which mathematics can help.

Answers

Section Preview

This is the culmination of the exponential and logarithmic function material—skipping this would be criminal. If any area in Mathematics deserves extra time for applications, it's this chapter.

Section Preview

- **Sets and Numbers**
  - Scientific Notation: Example 1 expresses populations such as $10{,}240$ and $10{,}485{,}760$ as 1.024×10^4 and 1.049×10^7; converting between standard and scientific form is needed to track orders of magnitude.

Section Preview

- Model exponential growth and decay.
- Use Newton's Law of Cooling.
- Use logistic-growth models.
- Express an exponential model in base e.
- Use logarithmic models.

Example $12.7.1$**: Graphing Exponential Growth**

When an amount grows at a fixed percent per unit time, the growth is exponential. To find P_0 we use the fact that P_0 is the amount at time zero, so $P_0 = 10$. To find k, use the fact that after one hour ($t = 1$) the population doubles from 10 to 20. The formula is derived as follows

$$\begin{aligned} 20 &= 10e^{k \cdot 1} && \text{(substituting)} \\ \Longrightarrow \quad 2 &= e^k && \text{(dividing both sides by 10)} \\ \Longrightarrow \quad \ln 2 &= k && \text{(applying the natural logarithm to both sides)} \end{aligned}$$

So $k = \ln(2)$. Thus, the equation we want to graph is

$$y = 10e^{(\ln 2)t} = 10\left(e^{\ln 2}\right)^t = 10 \cdot 2^t.$$

The graph is shown in Figure $12.7.4$.

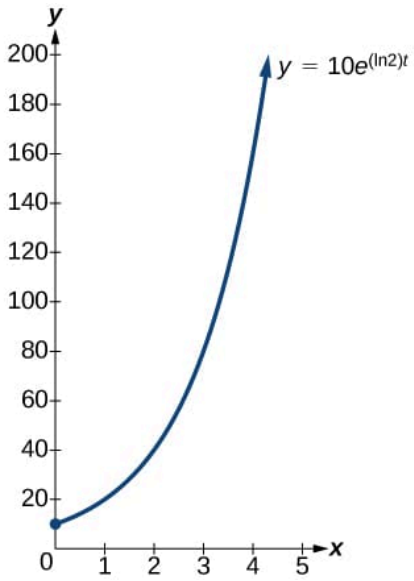


Figure $12.7.4$**:** *The graph of* $P(t) = 10e^{(\ln 2)t}$

The population of bacteria after ten hours is $10,240$. We could describe this amount as being an order of magnitude 4 because it can be written as 1.024×10^4. The population of bacteria after twenty hours is $10,485,760 \approx 1.049 \times 10^7$, which is an order of magnitude of 7, so we could say that the population has increased by three orders of magnitude in ten hours.

Example $12.7.2$**: Finding the Function that Describes Radioactive Decay**

This formula is derived as follows:

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ \Longrightarrow \quad \frac{1}{2}P_0 &= P_0 e^{k\cdot 5730} && \text{(substituting)} \\ \Longrightarrow \quad \frac{1}{2} &= e^{k\cdot 5730} && \text{(dividing both sides by } P_0\text{)} \\ \Longrightarrow \quad \ln\left(\frac{1}{2}\right) &= 5730k && \text{(taking the natural logarithm of both sides)} \\ \Longrightarrow \quad \frac{\ln(1/2)}{5730} &= k && \text{(dividing both sides by 5730)} \end{aligned}$$

Substituting this back into the original function, we get

$$P(t) = P_0 e^{\left(\frac{\ln(1/2)}{5730}\right)t}.$$

We observe that the coefficient of t, $\frac{\ln(0.5)}{5730} \approx -1.2097 \times 10^{-4}$, is negative, as expected in the case of exponential decay.

Example 12.7.3**: Finding the Age of a Bone**

We are told that 20% of an initial sample remains. That is, the current amount of carbon-14 in the sample is $0.20P_0$. Therefore, we want to solve $0.2P_0 = P_0 e^{kt}$. This gives

$$0.2 = e^{kt} \implies \ln 0.2 = kt \implies t = \frac{\ln 0.2}{k}.$$

Since we already have computed k for carbon-14 earlier, we substitute that value into this formula:

$$t = \frac{\ln 0.2}{k} = \frac{\ln 0.2}{\frac{\ln 0.5}{5730}} = \frac{5730\ln(0.2)}{\ln 0.5} \approx 13305$$

The bone fragment is about 13,305 years old.

Example 12.7.4**: Finding a Function That Describes Exponential Growth**

The formula is derived as follows:

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ \Longrightarrow \quad 2P_0 &= P_0 e^{k\cdot 2} && \text{(substituting } t = 2 \text{ and } P(2) = 2P_0\text{)} \\ \Longrightarrow \quad 2 &= e^{2k} && \text{(dividing both sides by } P_0\text{)} \\ \Longrightarrow \quad \ln 2 &= 2k && \text{(taking the natural log of both sides)} \\ \Longrightarrow \quad \frac{\ln 2}{2} &= k && \text{(dividing both sides by 2)} \end{aligned}$$

Thus, the function is

$$P(t) = P_0 e^{(\ln(2)/2)t}.$$

Example 12.7.5**: Using Newton's Law of Cooling**

Because the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially toward 35, following the equation

$$T(t) = 35 + (T_0 - 35)e^{kt}.$$

We know the initial temperature was 165, so $T(0) = T_0 = 165$. Therefore,

$$T(t) = 35 + (165 - 35)e^{k\cdot t} = 35 + 130e^{kt}.$$

We were given another data point, $T(10) = 150$, which we can use to solve for k.

$$\begin{aligned} 150 &= 35 + 130e^{k\cdot 10} && \text{(subsituting the point } (10, 150)\text{)} \\ \Longrightarrow \quad 115 &= 130e^{10k} && \text{(subtracting 35 from both sides)} \\ \Longrightarrow \quad \frac{115}{130} &= e^{10k} && \text{(dividing both sides by 130)} \\ \Longrightarrow \quad \ln\left(\tfrac{115}{130}\right) &= 10k && \text{(taking the natural log of both sides)} \\ \Longrightarrow \quad \frac{\ln\left(\frac{115}{130}\right)}{10} &= k && \text{(dividing both sides by 10)} \end{aligned}$$

Hence, $k \approx -0.0123$. This gives us the equation for the cooling of the cheesecake:

$$T(t) \approx 35 + 130e^{-0.0123t}.$$

Now we can solve for the time it will take for the temperature to cool to 70 degrees. To keep numbers as exact as possible, we wait until the end to replace k with our found value.

$$
\begin{aligned}
70 &= 35 + 130e^{kt} && \text{(substituting 70 for } T(t)\text{)} \\
\Longrightarrow \quad 35 &= 130e^{kt} && \text{(subtracting 35 from both sides)} \\
\Longrightarrow \quad \frac{35}{130} &= e^{kt} && \text{(dividing both sides by 130)} \\
\Longrightarrow \quad \ln\left(\tfrac{35}{130}\right) &= kt && \text{(taking the natural log of both sides)} \\
\Longrightarrow \quad \frac{\ln(35/130)}{k} &= t && \text{(dividing both sides by } k\text{)} \\
\Longrightarrow \quad \frac{\ln(35/130)}{\frac{\ln(115/130)}{10}} &= t && \left(\text{substituting } k = \frac{\ln\left(\frac{115}{130}\right)}{10}\right)
\end{aligned}
$$

Computing this value, we get $t \approx 106.68$. Thus, it will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool to 70°F.

Example 12.7.6**: Using the Logistic-Growth Model**

We substitute the given data into the logistic growth model

$$f(t) = \frac{c}{1 + ae^{-bt}}.$$

Because at most 1,000 people, the entire population of the community, can get the flu, we know the limiting value is $c = 1000$. To find a, we use the formula that the number of cases at time $t = 0$ is $\frac{c}{1+a} = 1$, from which it follows that $a = 999$. This model predicts that, after ten days, the number of people who have had the flu is $f(10) = \frac{1000}{1+999e^{-0.6030(10)}} \approx 293.9$. Because the actual number must be a whole number (a person has either had the flu or not) we round *down* to 293. In the long term, the number of people who will contract the flu is the limiting value, $c = 1000$.

Remember that, because we are dealing with a virus, we cannot predict with certainty the number of people infected. The model only approximates the number of people infected and will not give us exact or actual values.

The graph in Figure 12.7.6 gives a good picture of how this model fits the data.

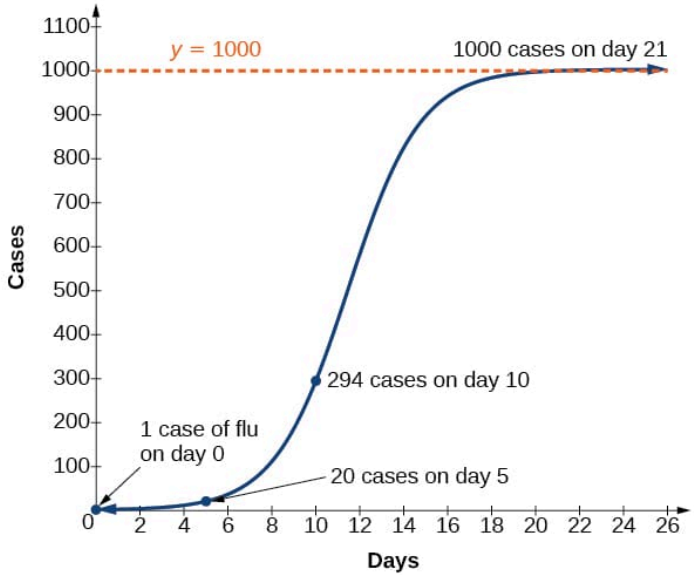


Figure 12.7.6**:** *The graph of* $f(t) = \frac{1000}{1+999e^{-0.6030t}}$.

Example 12.7.7**: Changing to base** e

The formula is derived as follows:

$$
\begin{aligned}
y &= 2.5(3.1)^x \\
&= 2.5e^{\ln(3.1^x)} && \text{(Inverse Property of Exponentials and Logarithms)} \\
&= 2.5e^{x\ln(3.1)} && \text{(Laws of Logarithms)} \\
&= 2.5e^{\ln(3.1)x} && \text{(Commutative Property of Multiplication)}
\end{aligned}
$$

Example 12.7.8

The point P appears to be half way between -2 and -1 in log value. So if V is the value of this point,

$$\log(V) \approx -1.5.$$

Rewriting in exponential form,

$$V \approx 10^{-1.5} \approx 0.0316.$$

Example 12.7.9

Since $\log(6000) \approx 3.8$, this point would belong on the log scale about here:

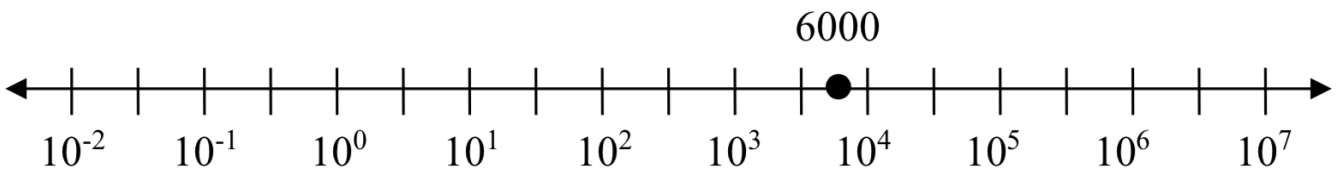


Figure 12.7.10

Checkpoint 12.7.9

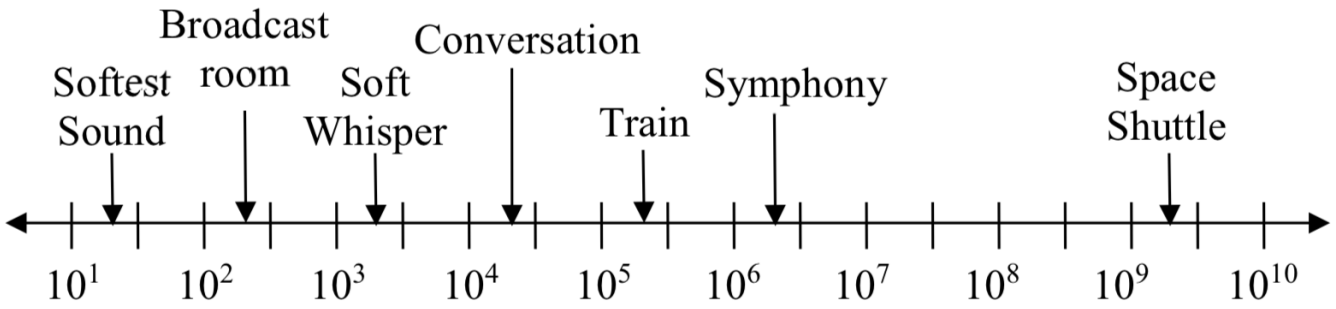


Figure 12.7.11

Example 12.7.10

The value B corresponds to $10^2 = 100$ and the value C corresponds to $10^5 = 100,000$. The relative change is $\frac{100,000}{100} = 1000 = \frac{10^5}{10^2} = 10^3$. The log of this value is 3. Therefore, C is three orders of magnitude greater than B, which can be seen on the log scale by the visual difference between the points on the scale.

Checkpoint 12.7.10

$\frac{2\times10^9}{2\times10^1} = 10^8$. The sound pressure in μ Pa created by launching the space shuttle is 8 orders of magnitude greater than the sound pressure in μ Pa created by the softest sound a human ear can hear.

Example 12.7.11

Since the first earthquake has Richter value of 6.0, we can find the amount of earth movement for that quake, which we'll denote A_1. The value of A_0 is not particularly relevant, so we will not replace it with its value.

$$\begin{aligned} 6.0 &= \log\left(\frac{A_1}{A_0}\right) \\ \Longrightarrow \quad 10^{6.0} &= \frac{A_1}{A_0} && \text{(rewriting the logarithm in exponential form)} \\ \Longrightarrow \quad 10^{6.0}A_0 &= A_1 && \text{(multiplying both sides by } A_0\text{)} \end{aligned}$$

This tells us the first earthquake has about 10^6 times more earth movement than the baseline measure.

Doing the same with the second earthquake, A_2, with a magnitude of 8.0,

$$\begin{aligned} 8.0 &= \log\left(\frac{A_2}{A_0}\right) \\ \Longrightarrow \quad 10^{8.0} &= \frac{A_2}{A_0} && \text{(rewriting the logarithm in exponential form)} \\ \Longrightarrow \quad 10^{8.0}A_0 &= A_2 && \text{(multiplying both sides by } A_0\text{)} \end{aligned}$$

Comparing the earth movement of the second earthquake to the first,

$$\frac{A_2}{A_1} = \frac{10^8 A_0}{10^6 A_0} = 10^2 = 100.$$

The second value's earth movement is 100 times as large as the first earthquake.

Example 12.7.12

Since the first quake has magnitude 3.0,

$$3.0 = \log\left(\frac{A}{A_0}\right).$$

Solving for A,

$$\begin{aligned} 3.0 &= \log\left(\frac{A}{A_0}\right) \\ \Longrightarrow \quad 10^{3.0} &= \frac{A}{A_0} && \text{(rewriting in exponential form)} \\ \Longrightarrow \quad 10^{3.0}A_0 &= A && \text{(multiplying both sides by } A_0\text{)} \end{aligned}$$

Since the second earthquake has twice as much earth movement, for the second quake we get

$$A = 2 \cdot 10^{3.0} A_0.$$

Finding the magnitude,

$$\begin{aligned} M &= \log\left(\frac{2 \cdot 10^{3.0} A_0}{A_0}\right) \\ &= \log\left(2 \cdot 10^{3.0}\right) \\ &\approx 3.3010 \end{aligned}$$

The second earthquake with twice as much earth movement will have a magnitude of about 3.3.

Example 12.7.13

a. There are 26 letters in the alphabet, 52 if upper and lower case letters are counted as different. There are 10 digits (0 through 9) for a total of $N = 62$ symbols. Since the password is to be 7 characters long, $L = 7$. Thus, $H = 7\log_2(62) = \frac{7\ln(62)}{\ln(2)} \approx 41.68$.

b. We have $L = 7$ and $H = 50$ and we need to find N. Solving the equation $50 = 7\log_2(N)$ gives $N = 2^{50/7} \approx 141.323$, so we would need 142 different symbols to choose from.

Example 12.7.14

We set $\mathrm{pH} = 7.4$ and get $7.4 = 6.1 + \log\left(\frac{800}{x}\right)$, or $\log\left(\frac{800}{x}\right) = 1.3$. Solving, we find $x = \frac{800}{10^{1.3}} \approx 40.09$. Hence, the partial pressure of carbon dioxide in the blood is about 40 torr.

This page titled 12.7: Exponential and Logarithmic Models is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

12.7.1: Resources and Key Concepts

Resources

Prerequisite Topics

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Sets and Numbers**
  - Scientific Notation: Scientific notation is often used when working with exponential models.

Videos

- Exponential growth and decay models
  - Exponential Functions - Applications Involving Exponential Functions
- Half-life and doubling time models
  - Exponential and Logarithmic Models - Doubling Time and Half-Life
  - Modeling with Exponential and Logarithmic Equations - Applications Involving Half Life
- Newton's Law of Cooling
  - Modeling with Exponential and Logarithmic Equations - Applications Involving Newton's Law of Cooling
- Logarithmic models
  - Modeling with Exponential and Logarithmic Equations - Logarithmic Applications

Key Concepts

Definitions

- **Doubling Time**: The time it takes for a quantity undergoing exponential growth to double its initial amount.
- **Half-Life**: The time it takes for a quantity undergoing exponential decay to decay to half of its original quantity.
- **Order of Magnitude**: The power of ten of a number when written in scientific notation. It describes how large or small a number is relative to powers of ten.
- **Radiocarbon Dating**: A method used to calculate the approximate date a plant or animal died by comparing the ratio of carbon-14 to carbon-12 in a sample to the known ratio in the atmosphere.
- **Continuous Rate of Cooling**: The constant k in Newton's Law of Cooling formula.
- **Logistic Growth Model**: A model for growth that is approximately exponential at first but has a reduced rate of growth as the output approaches the model's upper bound (carrying capacity). The formula is $P(t) = \frac{c}{1+ae^{-bt}}$.
- **Carrying Capacity (Limiting Value)**: The constant c in the logistic growth model, representing the upper bound or maximum sustainable population/value.
- **Richter Scale**: A logarithmic scale used to express the magnitude of an earthquake, based on the seismic trace amplitude. Defined as $M = \log\left(\frac{A}{A_0}\right)$.

Theorems

- **Half-Life Formula**: If an exponentially decaying population is modeled by $P(t) = P_0e^{kt}$, then the time it takes for the population to decay by half (half-life) is $t = -\frac{\ln(2)}{k}$ (or $t = \frac{\ln(1/2)}{k}$).
- **Newton's Law of Cooling**: The temperature of an object, T, in a surrounding medium having temperature T_s will behave according to the formula $T(t) = T_s + (T_0 - T_s)e^{kt}$, where t is time, T_0 is the initial temperature, and k is the continuous rate of cooling.
- **Determining the Order of Magnitude Between Two Numbers**: Given two values A and B, the number of orders of

magnitude A is greater than B is $\log\left(\frac{A}{B}\right)$.

Common Mistakes

- **Incorrect Sign for Rate Constant k**: Using a positive k for decay models (like half-life or Newton's cooling) or a negative k for growth models (like doubling time) in the $P(t) = P_0 e^{kt}$ formula.
- **Confusing Half-Life/Doubling Time with Rate Constant**: Mistaking the half-life or doubling time itself for the rate constant k, or vice-versa, instead of using the derived formulas (e.g., $k = -\frac{\ln(2)}{T_{half}}$) to relate them.
- **Errors in Setting Up Newton's Law of Cooling**: Incorrectly identifying T_s (ambient temperature), T_0 (initial temperature), or the temperature at a given time $T(t)$.
- **Algebraic Errors When Solving for t or k**: Making mistakes in isolating the exponential term, taking logarithms, or using logarithm properties when solving for time or the rate constant in various models.
- **Misinterpreting Logistic Growth Parameters**: Incorrectly identifying the carrying capacity c or the initial value $\frac{c}{1+a}$ from the logistic model formula.
- **Using Linear Models for Exponential Situations**: Attempting to model situations like population growth or radioactive decay with linear functions when an exponential model is appropriate.
- **Rounding Too Early**: Rounding intermediate calculations, which can lead to significant errors in the final answer, especially in exponential models.
- **Incorrectly Applying Logarithmic Scales**: Misunderstanding that a linear difference on a logarithmic scale corresponds to a multiplicative (ratio) difference in actual values (e.g., Richter scale).

This page titled 12.7.1: Resources and Key Concepts is shared under a license and was authored, remixed, and/or curated by Roy Simpson.

12.7.2: Homework

Reading Questions

1. What is the difference between "doubling time" and "half-life" in the context of exponential models? Which applies to growth and which to decay?
2. What is the general formula for exponential growth/decay using base e? Define each variable.
3. How is the half-life T related to the decay constant k in the formula $P(t) = P_0e^{kt}$?
4. Describe Newton's Law of Cooling. What does each variable in the formula $T(t) = T_s + (T_0 - T_s)e^{kt}$ represent?
5. What is a logistic growth model, and how does its behavior differ from a simple exponential growth model, especially in the long term?
6. What is the "carrying capacity" in a logistic growth model?
7. If an exponential function is given as $y = ab^x$, how can you convert it to the form $y = A_0e^{kx}$? What are A_0 and k in terms of a and b?
8. Explain what an "order of magnitude" means. If one quantity is 3 orders of magnitude larger than another, approximately how many times larger is it?
9. What is the Richter scale used to measure, and is it a linear or logarithmic scale?
10. In the context of radiocarbon dating, what does the half-life of Carbon-14 (5,730 years) signify?
11. With what kind of exponential model would half-life be associated? What role does half-life play in these models?
12. What is carbon dating? Why does it work? Give an example in which carbon dating would be useful.
13. With what kind of exponential model would *doubling time* be associated? What role does doubling time play in these models?
14. Define Newton's Law of Cooling. Then name at least three real-world situations where Newton's Law of Cooling would be applied.
15. What is an order of magnitude? Why are orders of magnitude useful? Give an example to explain.

Homework

Numeric

16. The temperature of an object in degrees Fahrenheit after t minutes is represented by the equation $T(t) = 72 + 68e^{-0.0174t}$. To the nearest degree, what is the temperature of the object after one and a half hours?

For the following exercises, use the logistic growth model $f(x) = \frac{150}{1+8e^{-2x}}$.

17. Find and interpret $f(0)$. Round to the nearest tenth.
18. Find and interpret $f(4)$. Round to the nearest tenth.
19. Find the carrying capacity.
20. Graph the model.
21. Determine whether the data from the table could best be represented as a function that is linear, exponential, or logarithmic. Then write a formula for a model that represents the data.

x	$f(x)$
–2	0.694
–1	0.833
0	1
1	1.2
2	1.44
3	1.728
4	2.074

x	$f(x)$
5	2.488

22. Rewrite $f(x) = 1.68(0.65)^x$ as an exponential equation with base e to five decimal places.

Technology

For the following exercises, enter the data from each table into a graphing calculator and graph the resulting scatter plots. Determine whether the data from the table could represent a function that is linear, exponential, or logarithmic.

23.

x	$f(x)$
1	2
2	4.079
3	5.296
4	6.159
5	6.828
6	7.375
7	7.838
8	8.238
9	8.592
10	8.908

24.

x	$f(x)$
1	2.4
2	2.88
3	3.456
4	4.147
5	4.977
6	5.972
7	7.166
8	8.6
9	10.32
10	12.383

25.

x	$f(x)$
4	9.429
5	9.972

x	$f(x)$
6	10.415
7	10.79
8	11.115
9	11.401
10	11.657
11	11.889
12	12.101
13	12.295

26.

x	$f(x)$
1.25	5.75
2.25	8.75
3.56	12.68
4.2	14.6
5.65	18.95
6.75	22.25
7.25	23.75
8.6	27.8
9.25	29.75
10.5	33.5

For the following exercises, use a graphing calculator and this scenario: the population of a fish farm in t years is modeled by the equation $P(t) = \frac{1000}{1+9e^{-0.6t}}$.

27. Graph the function.
28. What is the initial population of fish?
29. To the nearest tenth, what is the doubling time for the fish population?
30. To the nearest whole number, what will the fish population be after 2 years?
31. To the nearest tenth, how long will it take for the population to reach 900?
32. What is the carrying capacity for the fish population? Justify your answer using the graph of P.

Extensions

33. A substance has a half-life of 2.045 minutes. If the initial amount of the substance was 132.8 grams, how many half-lives will have passed before the substance decays to 8.3 grams? What is the total time of decay?
34. The formula for an increasing population is given by $P(t) = P_0e^{rt}$ where P_0 is the initial population and $r > 0$. Derive a general formula for the time t it takes for the population to increase by a factor of M.
35. Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. Show each step for solving this equation algebraically for the seismic moment S.
36. What is the y-intercept of the logistic growth model $y = \frac{c}{1+ae^{-rx}}$? Show the steps for calculation. What does this point tell us about the population?
37. Prove that $b^x = e^{xln(b)}$ for positive $b \neq 1$.

Real-World Applications

For the following exercises, use this scenario: A doctor prescribes 125 milligrams of a therapeutic drug that decays by about 30% each hour.

38. To the nearest hour, what is the half-life of the drug?
39. Write an exponential model representing the amount of the drug remaining in the patient's system after t hours. Then use the formula to find the amount of the drug that would remain in the patient's system after 3 hours. Round to the nearest milligram.
40. Using the model found in the previous exercise, find $f(10)$ and interpret the result. Round to the nearest hundredth.

For the following exercises, use this scenario: A tumor is injected with 0.5 grams of Iodine-125, which has a decay rate of 1.15 per day.

41. To the nearest day, how long will it take for half of the Iodine-125 to decay?
42. Write an exponential model representing the amount of Iodine-125 remaining in the tumor after t days. Then use the formula to find the amount of Iodine-125 that would remain in the tumor after 60 days. Round to the nearest tenth of a gram.
43. A scientist begins with 250 grams of a radioactive substance. After 250 minutes, the sample has decayed to 32 grams. Rounding to five decimal places, write an exponential equation representing this situation. To the nearest minute, what is the half-life of this substance?
44. The half-life of Radium-226 is 1590 years. What is the annual decay rate? Express the decimal result to four decimal places and the percentage to two decimal places.
45. The half-life of Erbium-165 is 10.4 hours. What is the hourly decay rate? Express the decimal result to four decimal places and the percentage to two decimal places.
46. A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. To the nearest year, about how many years old is the artifact? (The half-life of carbon-14 is 5730 years.)
47. A research student is working with a culture of bacteria that doubles in size every twenty minutes. The initial population count was 1350 bacteria. Rounding to five decimal places, write an exponential equation representing this situation. To the nearest whole number, what is the population size after 3 hours?

For the following exercises, use this scenario: A biologist recorded a count of 360 bacteria present in a culture after 5 minutes and 1000 bacteria present after 20 minutes.

48. To the nearest whole number, what was the initial population in the culture?
49. Rounding to six decimal places, write an exponential equation representing this situation. To the nearest minute, how long did it take the population to double?

For the following exercises, use this scenario: A pot of warm soup with an internal temperature of 100° Fahrenheit was taken off the stove to cool in a 69° F room. After fifteen minutes, the internal temperature of the soup was 95° F.

50. Use Newton's Law of Cooling to write a formula that models this situation.
51. To the nearest minute, how long will it take the soup to cool to 80° F?
52. To the nearest degree, what will the temperature be after 2 and a half hours?

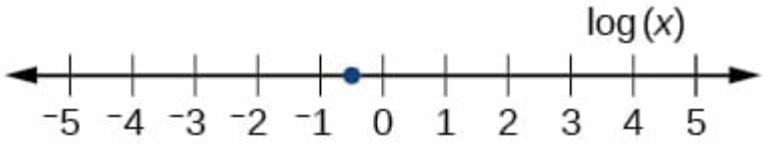
For the following exercises, use this scenario: A turkey is taken out of the oven with an internal temperature of 165° F and is allowed to cool in a 75° F room. After half an hour, the internal temperature of the turkey is 145° F.

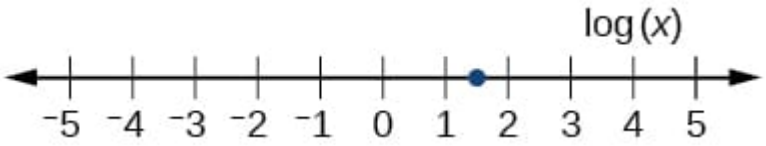
53. Write a formula that models this situation.
54. To the nearest degree, what will the temperature be after 50 minutes?
55. To the nearest minute, how long will it take the turkey to cool to 110° F?

For the following exercises, find the value of the number shown on each logarithmic scale. Round all answers to the nearest thousandth.

56.

57.





58. Plot each set of approximate values of intensity of sounds on a logarithmic scale: Whisper: $10^{-10}\frac{W}{m^2}$, Vacuum: $10^{-4}\frac{W}{m^2}$, Jet: $10^{2}\frac{W}{m^2}$.
59. Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. One earthquake has magnitude 3.9 on the MMS scale. If a second earthquake has 750 times as much energy as the first, find the magnitude of the second quake. Round to the nearest hundredth.

For the following exercises, use this scenario: The equation $N(t) = \frac{500}{1+49e^{-0.7t}}$ models the number of people in a town who have heard a rumor after t days.

60. How many people started the rumor?
61. To the nearest whole number, how many people will have heard the rumor after 3 days?
62. As t increases without bound, what value does $N(t)$ approach? Interpret your answer.

For the following exercise, choose the correct answer choice.

63. A doctor injects a patient with 13 milligrams of radioactive dye that decays exponentially. After 12 minutes, there are 4.75 milligrams of dye remaining in the patient's system. Which is an appropriate model for this situation?
    a. $f(t) = 13(0.0805)^t$
    b. $f(t) = 13e^{0.9195t}$
    c. $f(t) = 13e^{(}-0.0839t)$
    d. $f(t) = \frac{4.75}{1+13e^{-0.83925t}}$

This page titled 12.7.2: Homework is shared under a license and was authored, remixed, and/or curated by LibreTexts.

INDEX

A

absolute value equation
1.5: Absolute Value Functions

Absolute Value Function
1.1: Functions and Function Notation

absolute value inequality
1.5: Absolute Value Functions

acute angle
4.1: Angles and Basic Geometry

acute triangle
4.2: Triangles

Alternate Interior Angles
4.1: Angles and Basic Geometry

altitude
4.3: Similar Triangles

altitude (of triangle)
4.2: Triangles

Ambiguous case
11.1: The Law of Sines

amplitude
8.1: Base Graphs of the Fundamental Trigonometric Functions

angle
4.1: Angles and Basic Geometry

Angle of Declination
6.4: Applications Involving Right Triangles

angle of depression
6.4: Applications Involving Right Triangles

angle of displacement
7.2: Radian Measure, Arc Length, and Sector Area

angle of elevation
6.4: Applications Involving Right Triangles

angle of inclination
6.4: Applications Involving Right Triangles

arc
7.2: Radian Measure, Arc Length, and Sector Area

arc length
7.2: Radian Measure, Arc Length, and Sector Area

arccosecant
10.2: The Remaining Inverse Trigonometric Functions

arccosine
10.1: The Fundamental Inverse Trigonometric Functions

arccotangent
10.2: The Remaining Inverse Trigonometric Functions

arclength
7.2: Radian Measure, Arc Length, and Sector Area

arcsecant
10.2: The Remaining Inverse Trigonometric Functions

arcsine
10.1: The Fundamental Inverse Trigonometric Functions

arctangent
10.1: The Fundamental Inverse Trigonometric Functions

Area of a Circle
4.1: Angles and Basic Geometry

Arrow Notation
2.5: Rational Functions

axial angle
4.4: Circles and Angles in the Rectangular Coordinate System

axis of symmetry
2.1: Quadratic Functions

B

base angle
4.2: Triangles

bearing
6.4: Applications Involving Right Triangles

C

Cartesian coordinate plane
4.4: Circles and Angles in the Rectangular Coordinate System

Cartesian coordinate system
4.4: Circles and Angles in the Rectangular Coordinate System

Cartesian coordinates
4.4: Circles and Angles in the Rectangular Coordinate System

center of a circle
4.1: Angles and Basic Geometry

circle
4.1: Angles and Basic Geometry

circular functions
7.3: The Trigonometric Functions - Unit Circle Definition

circumference
4.1: Angles and Basic Geometry

coefficient
2.2: Power Functions and Polynomial Functions

Cofunction Identities
6.1: The Trigonometric Functions - Right Triangle Definition

cofunction theorem
6.1: The Trigonometric Functions - Right Triangle Definition

cofunctions
6.1: The Trigonometric Functions - Right Triangle Definition

common logarithms
12.3: Logarithmic Functions

complementary angle
4.1: Angles and Basic Geometry

composite function
1.3: The Arithmetic and Composition of Functions

compound interest
12.1: Exponential Functions

compound right triangle
6.4: Applications Involving Right Triangles

compressing
8.2: Non-Rigid Transformations

conditional equation
5.3: Introduction to Trigonometric Identities

congruent
4.1: Angles and Basic Geometry
4.3: Similar Triangles

constant function
1.1: Functions and Function Notation
4.4: Circles and Angles in the Rectangular Coordinate System

continuous function
2.2: Power Functions and Polynomial Functions

continuous growth formula
12.1: Exponential Functions

coordinate definition of trigonometric functions
5.1: The Trigonometric Functions - Coordinate Definition

Coordinate Trigonometry
5.1: The Trigonometric Functions - Coordinate Definition

corresponding angles
4.1: Angles and Basic Geometry

cosine curve
8.1: Base Graphs of the Fundamental Trigonometric Functions

coterminal angles
4.4: Circles and Angles in the Rectangular Coordinate System

Cube root function
1.1: Functions and Function Notation

cubic function
1.1: Functions and Function Notation

D

DD
6.2: Technology and Right Triangle Trigonometry

Decimal Degrees
6.2: Technology and Right Triangle Trigonometry

decreasing
4.4: Circles and Angles in the Rectangular Coordinate System

definition of a logarithm
12.6: Exponential and Logarithmic Equations

Definition of exponential functions
12.1: Exponential Functions

Definition of exponential growth
12.1: Exponential Functions

degree
2.2: Power Functions and Polynomial Functions

degree measure
4.1: Angles and Basic Geometry

degree mode
6.2: Technology and Right Triangle Trigonometry

dependent variable
1.1: Functions and Function Notation
5.1: The Trigonometric Functions - Coordinate Definition

diameter
4.1: Angles and Basic Geometry
difference of cubes
5.2: Algebra with Trigonometric Functions
difference of squares
5.2: Algebra with Trigonometric Functions
Distance Formula
4.4: Circles and Angles in the Rectangular Coordinate System
Division Algorithm
2.4: Dividing Polynomials
DMS
6.2: Technology and Right Triangle Trigonometry
domain
1.1: Functions and Function Notation
1.2: Domain and Range
2.5: Rational Functions
Double Angle Formulas
9.3: Double-Angle Identities
doubling time
12.7: Exponential and Logarithmic Models

E

end behavior
2.2: Power Functions and Polynomial Functions
2.3: Graphs of Polynomial Functions
equation of a circle
4.4: Circles and Angles in the Rectangular Coordinate System
equilateral triangle
4.2: Triangles
equivalent expressions
5.3: Introduction to Trigonometric Identities
evaluate
5.2: Algebra with Trigonometric Functions
even function
1.4: Transformations of Functions
expression
5.2: Algebra with Trigonometric Functions
extraneous solution
10.4: Solving Trigonometric Equations Using Identities

F

factoring by grouping
5.2: Algebra with Trigonometric Functions
function
1.1: Functions and Function Notation
fundamental trigonometric functions
8.1: Base Graphs of the Fundamental Trigonometric Functions

G

General form for the translation of the parent function
12.2: Graphs of Exponential Functions
General form for the translation of the parent logarithmic function
12.4: Graphs of Logarithmic Functions
general form of a polynomial function
2.2: Power Functions and Polynomial Functions
general form of a quadratic function
2.1: Quadratic Functions
global maximum
2.3: Graphs of Polynomial Functions
global minimum
2.3: Graphs of Polynomial Functions

H

heading
6.4: Applications Involving Right Triangles
horizontal asymptote
2.5: Rational Functions
horizontal line test
1.1: Functions and Function Notation
horizontal reflection
1.4: Transformations of Functions
horizontal shifts
1.4: Transformations of Functions
12.2: Graphs of Exponential Functions
horizontal stretch
1.4: Transformations of Functions
12.2: Graphs of Exponential Functions

I

identities
5.3: Introduction to Trigonometric Identities
identity
5.3: Introduction to Trigonometric Identities
identity function
1.1: Functions and Function Notation
increasing
4.4: Circles and Angles in the Rectangular Coordinate System
independent variable
1.1: Functions and Function Notation
5.1: The Trigonometric Functions - Coordinate Definition
initial point
4.1: Angles and Basic Geometry
initial side
4.1: Angles and Basic Geometry
initial value
4.4: Circles and Angles in the Rectangular Coordinate System
input
1.1: Functions and Function Notation
5.1: The Trigonometric Functions - Coordinate Definition
intermediate value theorem
2.3: Graphs of Polynomial Functions
interval notation
1.2: Domain and Range
invariant
6.1: The Trigonometric Functions - Right Triangle Definition
inverse cosine
6.2: Technology and Right Triangle Trigonometry
inverse function
1.6: Inverse Functions
3.1: Inverse Functions
inverse sine
6.2: Technology and Right Triangle Trigonometry
inverse tangent
6.2: Technology and Right Triangle Trigonometry
inverse trigonometric functions
10.1: The Fundamental Inverse Trigonometric Functions
6.2: Technology and Right Triangle Trigonometry
isosceles right triangle
4.2: Triangles
isosceles triangle
4.2: Triangles

K

key number
8.2: Non-Rigid Transformations

L

Law of Cosines
11.2: The Law of Cosines
Law of Sines
11.1: The Law of Sines
leading coefficient
2.2: Power Functions and Polynomial Functions
leading term
2.2: Power Functions and Polynomial Functions
logarithmic function
12.3: Logarithmic Functions
logistic growth
12.7: Exponential and Logarithmic Models
long division
2.4: Dividing Polynomials

M

midline
8.3: Rigid Transformations
minutes
6.2: Technology and Right Triangle Trigonometry
multiplicity
2.3: Graphs of Polynomial Functions

N

natural logarithm
12.3: Logarithmic Functions
natural period
8.2: Non-Rigid Transformations
negative angle
4.1: Angles and Basic Geometry
4.4: Circles and Angles in the Rectangular Coordinate System
Newton's Law of Cooling
12.7: Exponential and Logarithmic Models
nonrigid transformations
8.2: Non-Rigid Transformations

O

obtuse angle
4.1: Angles and Basic Geometry
obtuse triangle
4.2: Triangles
odd function
1.4: Transformations of Functions
oriented angle
4.1: Angles and Basic Geometry

output
1.1: Functions and Function Notation
5.1: The Trigonometric Functions - Coordinate Definition

P

parabola
2.1: Quadratic Functions
parallax
11.1: The Law of Sines
parallelogram
4.3: Similar Triangles
period
8.1: Base Graphs of the Fundamental Trigonometric Functions
8.2: Non-Rigid Transformations
periodic function
8.1: Base Graphs of the Fundamental Trigonometric Functions
periodic motion
8.1: Base Graphs of the Fundamental Trigonometric Functions
phase shift
8.3: Rigid Transformations
polynomial division
2.4: Dividing Polynomials
polynomial function
2: Polynomial and Rational Functions
2.2: Power Functions and Polynomial Functions
population growth
12.1: Exponential Functions
positive angle
4.1: Angles and Basic Geometry
4.4: Circles and Angles in the Rectangular Coordinate System
power functions
2.2: Power Functions and Polynomial Functions
power rule
12.5: Laws of Logarithms
product rule for logarithms
12.5: Laws of Logarithms
proportional
4.3: Similar Triangles
Pythagorean identities
5.3: Introduction to Trigonometric Identities
Pythagorean Identity
5.3: Introduction to Trigonometric Identities
Pythagorean Theorem
4.2: Triangles
Pythagorean Triangle
6.1: The Trigonometric Functions - Right Triangle Definition
Pythagorean triples
6.1: The Trigonometric Functions - Right Triangle Definition

Q

Quadrant
4.4: Circles and Angles in the Rectangular Coordinate System
quadrantal angle
4.4: Circles and Angles in the Rectangular Coordinate System
quadrants
4.4: Circles and Angles in the Rectangular Coordinate System
quadratic formula
2.1: Quadratic Functions
quadratic function
1.1: Functions and Function Notation
quotient rule
12.5: Laws of Logarithms

R

radian
7.2: Radian Measure, Arc Length, and Sector Area
radian measure
7.2: Radian Measure, Arc Length, and Sector Area
radian mode
6.2: Technology and Right Triangle Trigonometry
radians
7.2: Radian Measure, Arc Length, and Sector Area
radical function
3.2: Radical Functions
radicals
3.2: Radical Functions
radius
4.1: Angles and Basic Geometry
range
1.1: Functions and Function Notation
1.2: Domain and Range
rate of change
4.4: Circles and Angles in the Rectangular Coordinate System
Ratio Identities
5.3: Introduction to Trigonometric Identities
rational function
2: Polynomial and Rational Functions
2.5: Rational Functions
ray
4.1: Angles and Basic Geometry
Reciprocal function
1.1: Functions and Function Notation
Reciprocal Identities
5.3: Introduction to Trigonometric Identities
Reciprocal squared function
1.1: Functions and Function Notation
rectangular coordinate system
4.4: Circles and Angles in the Rectangular Coordinate System
reference angle
7.1: Reference Angles
reference triangle
7.1: Reference Angles
reflecting
8.2: Non-Rigid Transformations
reflection
12.2: Graphs of Exponential Functions
relation
1.1: Functions and Function Notation
right angle
4.1: Angles and Basic Geometry
right triangle
4.2: Triangles
Right Triangle Trigonometry
6.1: The Trigonometric Functions - Right Triangle Definition
rigid transformations
8.2: Non-Rigid Transformations

S

scalene triangle
4.2: Triangles
scaling
8.2: Non-Rigid Transformations
seconds
6.2: Technology and Right Triangle Trigonometry
shift
8.2: Non-Rigid Transformations
similar triangles
4.3: Similar Triangles
sine curve
8.1: Base Graphs of the Fundamental Trigonometric Functions
sinusoidal
8.1: Base Graphs of the Fundamental Trigonometric Functions
slope
4.4: Circles and Angles in the Rectangular Coordinate System
smooth curve
2.2: Power Functions and Polynomial Functions
solve a triangle
6.3: Solving Right Triangles
special angles
5.1: The Trigonometric Functions - Coordinate Definition
special triangle
4.2: Triangles
square root function
1.1: Functions and Function Notation
standard form of a quadratic function
2.1: Quadratic Functions
standard position
4.4: Circles and Angles in the Rectangular Coordinate System
step size
8.2: Non-Rigid Transformations
straight angle
4.1: Angles and Basic Geometry
stretching
8.2: Non-Rigid Transformations
subtend
7.2: Radian Measure, Arc Length, and Sector Area
Sum and Difference Formulas
9.2: Sum and Difference Identities
sum and difference formulas for cosine
9.2: Sum and Difference Identities
Sum and Difference Identities
9.2: Sum and Difference Identities
Sum and Difference Identities for Sine
9.2: Sum and Difference Identities
sum of cubes
5.2: Algebra with Trigonometric Functions
supplementary angle
4.1: Angles and Basic Geometry
Symmetric Identities
8.1: Base Graphs of the Fundamental Trigonometric Functions
synthetic division
2.4: Dividing Polynomials

T

tabular function
1.6: Inverse Functions
3.1: Inverse Functions

term of a polynomial function
2.2: Power Functions and Polynomial Functions

terminal side
4.1: Angles and Basic Geometry

transformation
8.2: Non-Rigid Transformations

transversal
4.1: Angles and Basic Geometry

triangle
4.2: Triangles

triangle inequality
4.2: Triangles

Triangle Sum
4.2: Triangles

trigonometric equation
10.3: Solving Trigonometric Equations - Algebraic Techniques

trigonometric expression
5.4: Introduction to Proofs in Trigonometry

Trigonometric Functions
5.1: The Trigonometric Functions - Coordinate Definition

turning point
2.2: Power Functions and Polynomial Functions

U

undefined
4.4: Circles and Angles in the Rectangular Coordinate System

unit circle
4.4: Circles and Angles in the Rectangular Coordinate System
7.3: The Trigonometric Functions - Unit Circle Definition

unitless measure
7.2: Radian Measure, Arc Length, and Sector Area

V

vertex
2.1: Quadratic Functions
4.1: Angles and Basic Geometry

vertex angle
4.2: Triangles

vertex form of a quadratic function
2.1: Quadratic Functions

vertex of a triangle
4.2: Triangles

vertical angles
4.1: Angles and Basic Geometry

vertical asymptote
2.5: Rational Functions

vertical compression
1.4: Transformations of Functions

vertical line test
1.1: Functions and Function Notation

vertical reflection
1.4: Transformations of Functions

vertical shift
8.3: Rigid Transformations

vertical shifts
1.4: Transformations of Functions
12.2: Graphs of Exponential Functions

vertical stretch
1.4: Transformations of Functions
12.2: Graphs of Exponential Functions

vertices
4.2: Triangles

Z

zeros of a function
2.1: Quadratic Functions

This page titled Index is shared under a license and was authored, remixed, and/or curated by LibreTexts.

GLOSSARY

S

Sample Word 1 | Sample Definition 1

This page titled Glossary is shared under a license and was authored, remixed, and/or curated by LibreTexts.

DETAILED LICENSING

Overview

Title: Math 384: Foundations for Calculus

Webpages: 173

All licenses found:

- Undeclared: 50.9% (88 pages)
- CC BY-SA 1.2: 20.2% (35 pages)
- CC BY-SA 4.0: 17.3% (30 pages)
- CC BY 4.0: 11% (19 pages)
- CC BY-SA 3.0: 0.6% (1 page)

By Page

- Math 384: Foundations for Calculus - *Undeclared*
  - Front Matter - *Undeclared*
    - TitlePage - *Undeclared*
    - InfoPage - *Undeclared*
    - Table of Contents - *Undeclared*
    - Licensing - *Undeclared*
  - 1: Functions - *CC BY-SA 4.0*
    - 1.1: Functions and Function Notation - *CC BY-SA 4.0*
      - 1.1.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 1.1.2: Homework - *CC BY-SA 4.0*
    - 1.2: Domain and Range - *CC BY-SA 4.0*
      - 1.2.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 1.2.2: Homework - *CC BY-SA 4.0*
    - 1.3: The Arithmetic and Composition of Functions - *CC BY-SA 4.0*
      - 1.3.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 1.3.2: Homework - *CC BY-SA 4.0*
    - 1.4: Transformations of Functions - *CC BY-SA 4.0*
      - 1.4.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 1.4.2: Homework - *CC BY-SA 4.0*
    - 1.5: Absolute Value Functions - *CC BY-SA 4.0*
      - 1.5.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 1.5.2: Homework - *CC BY-SA 4.0*
    - 1.6: Inverse Functions - *CC BY-SA 4.0*
      - 1.6.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 1.6.2: Homework - *CC BY-SA 4.0*
  - 2: Polynomial and Rational Functions - *CC BY 4.0*
    - 2.1: Quadratic Functions - *CC BY 4.0*
      - 2.1.1: Resources and Key Concepts - *Undeclared*
      - 2.1.2: Homework - *Undeclared*
    - 2.2: Power Functions and Polynomial Functions - *CC BY 4.0*
      - 2.2.1: Resources and Key Concepts - *Undeclared*
      - 2.2.2: Homework - *Undeclared*
    - 2.3: Graphs of Polynomial Functions - *CC BY 4.0*
      - 2.3.1: Resources and Key Concepts - *Undeclared*
      - 2.3.2: Homework - *Undeclared*
    - 2.4: Dividing Polynomials - *CC BY 4.0*
      - 2.4.1: Resources and Key Concepts - *Undeclared*
      - 2.4.2: Homework - *Undeclared*
    - 2.5: Rational Functions - *CC BY 4.0*
      - 2.5.1: Resources and Key Concepts - *Undeclared*
      - 2.5.2: Homework - *Undeclared*
  - 3: Inverse and Radical Functions - *Undeclared*
    - 3.1: Inverse Functions - *CC BY-SA 4.0*
      - 3.1.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 3.1.2: Homework - *CC BY-SA 4.0*
    - 3.2: Radical Functions - *CC BY 4.0*
      - 3.2.1: Resources and Key Concepts - *Undeclared*
      - 3.2.2: Homework - *CC BY 4.0*
    - 3.3: Radical Equations - *CC BY 4.0*
      - 3.3.1: Resources and Key Concepts - *Undeclared*
      - 3.3.2: Homework - *CC BY 4.0*
    - 3.4: Rational Exponent Expressions Encountered in Calculus - *CC BY 4.0*
      - 3.4.1: Resources and Key Concepts - *Undeclared*
      - 3.4.2: Homework - *Undeclared*

- 4: Triangles and Circles - *CC BY-SA 1.2*
  - 4.1: Angles and Basic Geometry - *CC BY-SA 1.2*
    - 4.1.1: Resources and Key Concepts - *CC BY-SA 4.0*
    - 4.1.2: Homework - *CC BY-SA 4.0*
  - 4.2: Triangles - *CC BY-SA 1.2*
    - 4.2.1: Resources and Key Concepts - *CC BY-SA 4.0*
    - 4.2.2: Homework - *CC BY-SA 4.0*
  - 4.3: Similar Triangles - *CC BY-SA 1.2*
    - 4.3.1: Resources and Key Concepts - *CC BY-SA 4.0*
    - 4.3.2: Homework - *CC BY-SA 4.0*
  - 4.4: Circles and Angles in the Rectangular Coordinate System - *CC BY-SA 1.2*
    - 4.4.1: Resources and Key Concepts - *CC BY-SA 4.0*
    - 4.4.2: Homework - *CC BY-SA 4.0*
- 5: Coordinate Trigonometry - *CC BY-SA 1.2*
  - 5.1: The Trigonometric Functions - Coordinate Definition - *CC BY-SA 1.2*
    - 5.1.1: Resources and Key Concepts - *Undeclared*
    - 5.1.2: Homework - *Undeclared*
  - 5.2: Algebra with Trigonometric Functions - *CC BY-SA 1.2*
    - 5.2.1: Resources and Key Concepts - *Undeclared*
    - 5.2.2: Homework - *Undeclared*
  - 5.3: Introduction to Trigonometric Identities - *CC BY-SA 1.2*
    - 5.3.1: Resources and Key Concepts - *Undeclared*
    - 5.3.2: Homework - *Undeclared*
  - 5.4: Introduction to Proofs in Trigonometry - *CC BY-SA 1.2*
    - 5.4.1: Resources and Key Concepts - *Undeclared*
    - 5.4.2: Homework - *Undeclared*
- 6: Right Triangle Trigonometry - *CC BY-SA 1.2*
  - 6.1: The Trigonometric Functions - Right Triangle Definition - *CC BY-SA 1.2*
    - 6.1.1: Resources and Key Concepts - *Undeclared*
    - 6.1.2: Homework - *Undeclared*
  - 6.2: Technology and Right Triangle Trigonometry - *CC BY-SA 1.2*
    - 6.2.1: Resources and Key Concepts - *Undeclared*
    - 6.2.2: Homework - *Undeclared*
  - 6.3: Solving Right Triangles - *CC BY-SA 1.2*
    - 6.3.1: Resources and Key Concepts - *Undeclared*
    - 6.3.2: Homework - *Undeclared*
  - 6.4: Applications Involving Right Triangles - *CC BY-SA 1.2*
    - 6.4.1: Resources and Key Concepts - *Undeclared*
    - 6.4.2: Homework - *Undeclared*
- 7: Radian Measure and the Circular Functions - *CC BY-SA 1.2*
  - 7.1: Reference Angles - *CC BY-SA 1.2*
    - 7.1.1: Resources and Key Concepts - *Undeclared*
    - 7.1.2: Homework - *Undeclared*
  - 7.2: Radian Measure, Arc Length, and Sector Area - *CC BY-SA 1.2*
    - 7.2.1: Resources and Key Concepts - *Undeclared*
    - 7.2.2: Homework - *Undeclared*
  - 7.3: The Trigonometric Functions - Unit Circle Definition - *CC BY-SA 1.2*
    - 7.3.1: Resources and Key Concepts - *Undeclared*
    - 7.3.2: Homework - *Undeclared*
- 8: Graphs of the Trigonometric Functions - *CC BY-SA 1.2*
  - 8.1: Base Graphs of the Fundamental Trigonometric Functions - *CC BY-SA 1.2*
    - 8.1.1: Resources and Key Concepts - *Undeclared*
    - 8.1.2: Homework - *Undeclared*
  - 8.2: Non-Rigid Transformations - *CC BY-SA 1.2*
    - 8.2.1: Resources and Key Concepts - *Undeclared*
    - 8.2.2: Homework - *Undeclared*
  - 8.3: Rigid Transformations - *CC BY-SA 1.2*
    - 8.3.1: Resources and Key Concepts - *Undeclared*
    - 8.3.2: Homework - *Undeclared*
  - 8.4: Graphs of the Remaining Trigonometric Functions - *CC BY-SA 1.2*
    - 8.4.1: Resources and Key Concepts - *Undeclared*
    - 8.4.2: Homework - *Undeclared*
- 9: Analytic Trigonometry - *CC BY-SA 1.2*
  - 9.1: Basic Trigonometric Identities and Proof Techniques - *CC BY-SA 3.0*
    - 9.1.1: Resources and Key Concepts - *Undeclared*
    - 9.1.2: Homework - *Undeclared*
  - 9.2: Sum and Difference Identities - *CC BY-SA 1.2*
    - 9.2.1: Resources and Key Concepts - *Undeclared*
    - 9.2.2: Homework - *Undeclared*
  - 9.3: Double-Angle Identities - *CC BY-SA 1.2*
    - 9.3.1: Resources and Key Concepts - *Undeclared*
    - 9.3.2: Homework - *Undeclared*
- 10: Trigonometric Equations - *CC BY-SA 1.2*
  - 10.1: The Fundamental Inverse Trigonometric Functions - *CC BY-SA 1.2*
    - 10.1.1: Resources and Key Concepts - *Undeclared*
    - 10.1.2: Homework - *Undeclared*

- 10.2: The Remaining Inverse Trigonometric Functions - *CC BY-SA 1.2*
  - 10.2.1: Resources and Key Concepts - *Undeclared*
  - 10.2.2: Homework - *Undeclared*
- 10.3: Solving Trigonometric Equations - Algebraic Techniques - *CC BY-SA 1.2*
  - 10.3.1: Resources and Key Concepts - *Undeclared*
  - 10.3.2: Homework - *Undeclared*
- 10.4: Solving Trigonometric Equations Using Identities - *CC BY-SA 1.2*
  - 10.4.1: Resources and Key Concepts - *Undeclared*
  - 10.4.2: Homework - *Undeclared*

◦ 11: Non-Right Triangle Trigonometry - *CC BY-SA 1.2*
- 11.1: The Law of Sines - *CC BY-SA 1.2*
  - 11.1.1: Resources and Key Concepts - *Undeclared*
  - 11.1.2: Homework - *Undeclared*
- 11.2: The Law of Cosines - *CC BY-SA 1.2*
  - 11.2.1: Resources and Key Concepts - *Undeclared*
  - 11.2.2: Homework - *Undeclared*

◦ 12: Exponential and Logarithmic Functions - *CC BY 4.0*
- 12.1: Exponential Functions - *CC BY 4.0*
  - 12.1.1: Resources and Key Concepts - *Undeclared*
  - 12.1.2: Homework - *Undeclared*
- 12.2: Graphs of Exponential Functions - *CC BY 4.0*
  - 12.2.1: Resources and Key Concepts - *Undeclared*
  - 12.2.2: Homework - *Undeclared*
- 12.3: Logarithmic Functions - *CC BY 4.0*
  - 12.3.1: Resources and Key Concepts - *Undeclared*
  - 12.3.2: Homework - *Undeclared*
- 12.4: Graphs of Logarithmic Functions - *CC BY 4.0*
  - 12.4.1: Resources and Key Concepts - *Undeclared*
  - 12.4.2: Homework - *Undeclared*
- 12.5: Laws of Logarithms - *CC BY 4.0*
  - 12.5.1: Resources and Key Concepts - *Undeclared*
  - 12.5.2: Homework - *Undeclared*
- 12.6: Exponential and Logarithmic Equations - *CC BY 4.0*
  - 12.6.1: Resources and Key Concepts - *Undeclared*
  - 12.6.2: Homework - *Undeclared*
- 12.7: Exponential and Logarithmic Models - *CC BY 4.0*
  - 12.7.1: Resources and Key Concepts - *Undeclared*
  - 12.7.2: Homework - *Undeclared*

◦ Back Matter - *Undeclared*
- Index - *Undeclared*
- Glossary - *Undeclared*
- Detailed Licensing - *Undeclared*
- Detailed Licensing - *Undeclared*

This page titled Math 384: Foundations for Calculus is shared under a license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College.

DETAILED LICENSING

Overview

Title: Math 384: Foundations for Calculus

Webpages: 173

All licenses found:

- Undeclared: 50.9% (88 pages)
- CC BY-SA 1.2: 20.2% (35 pages)
- CC BY-SA 4.0: 17.3% (30 pages)
- CC BY 4.0: 11% (19 pages)
- CC BY-SA 3.0: 0.6% (1 page)

By Page

- Math 384: Foundations for Calculus - *Undeclared*
  - Front Matter - *Undeclared*
    - TitlePage - *Undeclared*
    - InfoPage - *Undeclared*
    - Table of Contents - *Undeclared*
    - Licensing - *Undeclared*
  - 1: Functions - *CC BY-SA 4.0*
    - 1.1: Functions and Function Notation - *CC BY-SA 4.0*
      - 1.1.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 1.1.2: Homework - *CC BY-SA 4.0*
    - 1.2: Domain and Range - *CC BY-SA 4.0*
      - 1.2.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 1.2.2: Homework - *CC BY-SA 4.0*
    - 1.3: The Arithmetic and Composition of Functions - *CC BY-SA 4.0*
      - 1.3.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 1.3.2: Homework - *CC BY-SA 4.0*
    - 1.4: Transformations of Functions - *CC BY-SA 4.0*
      - 1.4.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 1.4.2: Homework - *CC BY-SA 4.0*
    - 1.5: Absolute Value Functions - *CC BY-SA 4.0*
      - 1.5.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 1.5.2: Homework - *CC BY-SA 4.0*
    - 1.6: Inverse Functions - *CC BY-SA 4.0*
      - 1.6.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 1.6.2: Homework - *CC BY-SA 4.0*
  - 2: Polynomial and Rational Functions - *CC BY 4.0*
    - 2.1: Quadratic Functions - *CC BY 4.0*
      - 2.1.1: Resources and Key Concepts - *Undeclared*
      - 2.1.2: Homework - *Undeclared*
    - 2.2: Power Functions and Polynomial Functions - *CC BY 4.0*
      - 2.2.1: Resources and Key Concepts - *Undeclared*
      - 2.2.2: Homework - *Undeclared*
    - 2.3: Graphs of Polynomial Functions - *CC BY 4.0*
      - 2.3.1: Resources and Key Concepts - *Undeclared*
      - 2.3.2: Homework - *Undeclared*
    - 2.4: Dividing Polynomials - *CC BY 4.0*
      - 2.4.1: Resources and Key Concepts - *Undeclared*
      - 2.4.2: Homework - *Undeclared*
    - 2.5: Rational Functions - *CC BY 4.0*
      - 2.5.1: Resources and Key Concepts - *Undeclared*
      - 2.5.2: Homework - *Undeclared*
  - 3: Inverse and Radical Functions - *Undeclared*
    - 3.1: Inverse Functions - *CC BY-SA 4.0*
      - 3.1.1: Resources and Key Concepts - *CC BY-SA 4.0*
      - 3.1.2: Homework - *CC BY-SA 4.0*
    - 3.2: Radical Functions - *CC BY 4.0*
      - 3.2.1: Resources and Key Concepts - *Undeclared*
      - 3.2.2: Homework - *CC BY 4.0*
    - 3.3: Radical Equations - *CC BY 4.0*
      - 3.3.1: Resources and Key Concepts - *Undeclared*
      - 3.3.2: Homework - *CC BY 4.0*
    - 3.4: Rational Exponent Expressions Encountered in Calculus - *CC BY 4.0*
      - 3.4.1: Resources and Key Concepts - *Undeclared*
      - 3.4.2: Homework - *Undeclared*

- 4: Triangles and Circles - *CC BY-SA 1.2*
  - 4.1: Angles and Basic Geometry - *CC BY-SA 1.2*
    - 4.1.1: Resources and Key Concepts - *CC BY-SA 4.0*
    - 4.1.2: Homework - *CC BY-SA 4.0*
  - 4.2: Triangles - *CC BY-SA 1.2*
    - 4.2.1: Resources and Key Concepts - *CC BY-SA 4.0*
    - 4.2.2: Homework - *CC BY-SA 4.0*
  - 4.3: Similar Triangles - *CC BY-SA 1.2*
    - 4.3.1: Resources and Key Concepts - *CC BY-SA 4.0*
    - 4.3.2: Homework - *CC BY-SA 4.0*
  - 4.4: Circles and Angles in the Rectangular Coordinate System - *CC BY-SA 1.2*
    - 4.4.1: Resources and Key Concepts - *CC BY-SA 4.0*
    - 4.4.2: Homework - *CC BY-SA 4.0*
- 5: Coordinate Trigonometry - *CC BY-SA 1.2*
  - 5.1: The Trigonometric Functions - Coordinate Definition - *CC BY-SA 1.2*
    - 5.1.1: Resources and Key Concepts - *Undeclared*
    - 5.1.2: Homework - *Undeclared*
  - 5.2: Algebra with Trigonometric Functions - *CC BY-SA 1.2*
    - 5.2.1: Resources and Key Concepts - *Undeclared*
    - 5.2.2: Homework - *Undeclared*
  - 5.3: Introduction to Trigonometric Identities - *CC BY-SA 1.2*
    - 5.3.1: Resources and Key Concepts - *Undeclared*
    - 5.3.2: Homework - *Undeclared*
  - 5.4: Introduction to Proofs in Trigonometry - *CC BY-SA 1.2*
    - 5.4.1: Resources and Key Concepts - *Undeclared*
    - 5.4.2: Homework - *Undeclared*
- 6: Right Triangle Trigonometry - *CC BY-SA 1.2*
  - 6.1: The Trigonometric Functions - Right Triangle Definition - *CC BY-SA 1.2*
    - 6.1.1: Resources and Key Concepts - *Undeclared*
    - 6.1.2: Homework - *Undeclared*
  - 6.2: Technology and Right Triangle Trigonometry - *CC BY-SA 1.2*
    - 6.2.1: Resources and Key Concepts - *Undeclared*
    - 6.2.2: Homework - *Undeclared*
  - 6.3: Solving Right Triangles - *CC BY-SA 1.2*
    - 6.3.1: Resources and Key Concepts - *Undeclared*
    - 6.3.2: Homework - *Undeclared*
  - 6.4: Applications Involving Right Triangles - *CC BY-SA 1.2*
    - 6.4.1: Resources and Key Concepts - *Undeclared*
    - 6.4.2: Homework - *Undeclared*
- 7: Radian Measure and the Circular Functions - *CC BY-SA 1.2*
  - 7.1: Reference Angles - *CC BY-SA 1.2*
    - 7.1.1: Resources and Key Concepts - *Undeclared*
    - 7.1.2: Homework - *Undeclared*
  - 7.2: Radian Measure, Arc Length, and Sector Area - *CC BY-SA 1.2*
    - 7.2.1: Resources and Key Concepts - *Undeclared*
    - 7.2.2: Homework - *Undeclared*
  - 7.3: The Trigonometric Functions - Unit Circle Definition - *CC BY-SA 1.2*
    - 7.3.1: Resources and Key Concepts - *Undeclared*
    - 7.3.2: Homework - *Undeclared*
- 8: Graphs of the Trigonometric Functions - *CC BY-SA 1.2*
  - 8.1: Base Graphs of the Fundamental Trigonometric Functions - *CC BY-SA 1.2*
    - 8.1.1: Resources and Key Concepts - *Undeclared*
    - 8.1.2: Homework - *Undeclared*
  - 8.2: Non-Rigid Transformations - *CC BY-SA 1.2*
    - 8.2.1: Resources and Key Concepts - *Undeclared*
    - 8.2.2: Homework - *Undeclared*
  - 8.3: Rigid Transformations - *CC BY-SA 1.2*
    - 8.3.1: Resources and Key Concepts - *Undeclared*
    - 8.3.2: Homework - *Undeclared*
  - 8.4: Graphs of the Remaining Trigonometric Functions - *CC BY-SA 1.2*
    - 8.4.1: Resources and Key Concepts - *Undeclared*
    - 8.4.2: Homework - *Undeclared*
- 9: Analytic Trigonometry - *CC BY-SA 1.2*
  - 9.1: Basic Trigonometric Identities and Proof Techniques - *CC BY-SA 3.0*
    - 9.1.1: Resources and Key Concepts - *Undeclared*
    - 9.1.2: Homework - *Undeclared*
  - 9.2: Sum and Difference Identities - *CC BY-SA 1.2*
    - 9.2.1: Resources and Key Concepts - *Undeclared*
    - 9.2.2: Homework - *Undeclared*
  - 9.3: Double-Angle Identities - *CC BY-SA 1.2*
    - 9.3.1: Resources and Key Concepts - *Undeclared*
    - 9.3.2: Homework - *Undeclared*
- 10: Trigonometric Equations - *CC BY-SA 1.2*
  - 10.1: The Fundamental Inverse Trigonometric Functions - *CC BY-SA 1.2*
    - 10.1.1: Resources and Key Concepts - *Undeclared*
    - 10.1.2: Homework - *Undeclared*

- 10.2: The Remaining Inverse Trigonometric Functions - *CC BY-SA 1.2*
  - 10.2.1: Resources and Key Concepts - *Undeclared*
  - 10.2.2: Homework - *Undeclared*
- 10.3: Solving Trigonometric Equations - Algebraic Techniques - *CC BY-SA 1.2*
  - 10.3.1: Resources and Key Concepts - *Undeclared*
  - 10.3.2: Homework - *Undeclared*
- 10.4: Solving Trigonometric Equations Using Identities - *CC BY-SA 1.2*
  - 10.4.1: Resources and Key Concepts - *Undeclared*
  - 10.4.2: Homework - *Undeclared*

◦ 11: Non-Right Triangle Trigonometry - *CC BY-SA 1.2*
- 11.1: The Law of Sines - *CC BY-SA 1.2*
  - 11.1.1: Resources and Key Concepts - *Undeclared*
  - 11.1.2: Homework - *Undeclared*
- 11.2: The Law of Cosines - *CC BY-SA 1.2*
  - 11.2.1: Resources and Key Concepts - *Undeclared*
  - 11.2.2: Homework - *Undeclared*

◦ 12: Exponential and Logarithmic Functions - *CC BY 4.0*
- 12.1: Exponential Functions - *CC BY 4.0*
  - 12.1.1: Resources and Key Concepts - *Undeclared*
  - 12.1.2: Homework - *Undeclared*
- 12.2: Graphs of Exponential Functions - *CC BY 4.0*
  - 12.2.1: Resources and Key Concepts - *Undeclared*
  - 12.2.2: Homework - *Undeclared*
- 12.3: Logarithmic Functions - *CC BY 4.0*
  - 12.3.1: Resources and Key Concepts - *Undeclared*
  - 12.3.2: Homework - *Undeclared*
- 12.4: Graphs of Logarithmic Functions - *CC BY 4.0*
  - 12.4.1: Resources and Key Concepts - *Undeclared*
  - 12.4.2: Homework - *Undeclared*
- 12.5: Laws of Logarithms - *CC BY 4.0*
  - 12.5.1: Resources and Key Concepts - *Undeclared*
  - 12.5.2: Homework - *Undeclared*
- 12.6: Exponential and Logarithmic Equations - *CC BY 4.0*
  - 12.6.1: Resources and Key Concepts - *Undeclared*
  - 12.6.2: Homework - *Undeclared*
- 12.7: Exponential and Logarithmic Models - *CC BY 4.0*
  - 12.7.1: Resources and Key Concepts - *Undeclared*
  - 12.7.2: Homework - *Undeclared*

◦ Back Matter - *Undeclared*
- Index - *Undeclared*
- Glossary - *Undeclared*
- Detailed Licensing - *Undeclared*
- Detailed Licensing - *Undeclared*

This page titled Math 384: Foundations for Calculus is shared under a license and was authored, remixed, and/or curated by Roy Simpson, Cosumnes River College.